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A perturbation result for the layer potentials of general second order differential operators with constant coefficients

Matteo Dalla Riva & Massimo Lanza de Cristoforis

Abstract: We consider a hypersurface in Euclidean space $\mathbb{R}^n$ parametrized by a diffeomorphism of the boundary of a regular domain in $\mathbb{R}^n$ to $\mathbb{R}^n$, and a density function on the hypersurface, which we think as points in suitable Schauder spaces, and a family of second order differential operators with constant coefficients and a corresponding family of fundamental solutions depending on a parameter. Then we investigate the dependence of the corresponding layer potentials, which we also think as points in suitable Schauder spaces, upon variation of the diffeomorphism and of the density and of the parameter, and we show a real analyticity theorem for such a dependence.

Keywords: Layer potentials, second order differential operators with constant coefficients, domain perturbation, special nonlinear operators.
2000 Mathematics Subject Classification: 31B10, 47H30.

1 Introduction.

As is well known, the potential theoretic method is a powerful tool to analyze boundary value problems for elliptic differential equations and systems and can be used in particular to study boundary perturbation problems (cf. *e.g.*, Fichera [2].) Thus it is clear that it is important to understand the dependence of layer potentials both on variation of the support of integration and on data such as the integral kernel and the density (or moment.) In [5], [6], [7], those authors have considered layer potentials associated to the Laplace equation and to the Helmholtz equation. In this paper, we shall extend the methods of those papers to consider general strongly elliptic operators of second order with complex coefficients, as a preliminary step for a later analysis of the case of elliptic operators of higher order.

We fix a bounded open connected subset Ω of $\mathbb{R}^n$ with $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ connected, which we consider as a "base domain". We assume that Ω is of class $C^{m,\alpha}$ for some integer $m \geq 1$ and $\alpha \in]0,1[$. Then we consider a class of diffeomorphisms $\mathcal{A}_{\partial\Omega}$ of $\partial\Omega$ into $\mathbb{R}^n$. If $\phi \in \mathcal{A}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and unbounded open connected components of $\mathbb{R}^n \setminus \phi(\partial\Omega)$, respectively.

Next we introduce a family of differential operators. Let N denote the number of multi-indexes $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$. For each $\mathbf{a} \equiv (a_\alpha)_{|\alpha|\leq 2} \in \mathbb{C}^N$, we set

$$\mathbf{a}^{(2)} \equiv (a^{(2)}_{lj})_{l,j=1,\dots,n} \qquad \mathbf{a}^{(1)} \equiv (a_j)_{j=1,\dots,n}$$

with $a^{(2)}_{lj} \equiv a_{e_l+e_j}$ and $a_j \equiv a_{e_j}$, where $\{e_j : j = 1, \dots, n\}$ is the canonical basis of $\mathbb{R}^n$. We note that the matrix $\mathbf{a}^{(2)}$ is symmetric. Then we set

$$\mathcal{E} \equiv \left\{ \mathbf{a} \equiv (a_\alpha)_{|\alpha|\leq 2} \in \mathbb{C}^N : \inf_{\xi\in\mathbb{R}^n, |\xi|=1} \mathrm{Re} \left\{ \sum_{|\alpha|=2} a_\alpha \xi^\alpha \right\} > 0 \right\}.$$

Clearly, $\mathcal{E}$ coincides with the set of coefficients $\mathbf{a} \equiv (a_\alpha)_{|\alpha|\leq 2}$ such that the differential operator

$$P[\mathbf{a}, D] \equiv \sum_{|\alpha|\leq 2} a_\alpha D^\alpha$$

is strongly elliptic and has complex coefficients. Then we shall consider the following assumption.

$$\begin{aligned} &\text{Let } \mathcal{K} \text{ be a real Banach space. Let } \mathcal{O} \text{ be an open subset of } \mathcal{K}. \\ &\text{Let } \mathbf{a}(\cdot) \text{ be a real analytic map of } \mathcal{O} \text{ to } \mathcal{E}. \\ &\text{Let } S(\cdot,\cdot) \text{ be a real analytic map of } (\mathbb{R}^n \setminus \{0\}) \times \mathcal{O} \text{ to } \mathbb{C} \text{ such that } S(\cdot,\kappa) \\ &\text{ is a fundamental solution of } P[\mathbf{a}(\kappa), D] \text{ for all } \kappa \in \mathcal{O}. \end{aligned} \tag{1.1}$$

For all continuous functions f of $\partial\Omega$ to $\mathbb{C}$ and $\phi \in \mathcal{A}_{\partial\Omega}$, one can consider the function $f \circ \phi^{(-1)}$ defined on $\phi(\partial\Omega)$, and it makes sense to consider the simple layer potential

$$v[\phi, f, \kappa](\xi) \equiv \int_{\phi(\partial\Omega)} S(\xi - \eta, \kappa)\, f \circ \phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall \xi \in \mathbb{R}^n.$$

Then we introduce the function

$$V[\phi, f, \kappa](x) \equiv v[\phi, f, \kappa] \circ \phi(x) \qquad \forall x \in \partial\Omega. \tag{1.2}$$

We prove that the map $V[\cdot,\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ to $C^{m,\alpha}(\partial\Omega)$ which takes (ϕ, f, κ) to the function $V[\phi, f, \kappa]$ defined in (1.2) is real analytic (see Theorem 5.6.) Then we consider the functions of $\partial\Omega$ to $\mathbb{C}$ defined by

$$V_l[\phi, f, \kappa](x) \equiv \int_{\phi(\partial\Omega)} \partial_{\xi_l} S(\phi(x) - \eta, \kappa) f \circ \phi^{(-1)}(\eta)\, d\sigma_\eta, \tag{1.3}$$

$$V_*[\phi, f, \kappa](x) \equiv \int_{\phi(\partial\Omega)} D_\xi S(\phi(x) - \eta, \kappa) \mathbf{a}^{(2)}(\kappa) \nu_\phi(\phi(x)) f \circ \phi^{(-1)}(\eta) d\sigma_\eta, \tag{1.4}$$

for all $x \in \partial\Omega$, and for all $(\phi, f, \kappa) \in (C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$, and for all $l \in \{1, \ldots, n\}$, and by

$$\begin{aligned} W[\phi, f, \kappa](x) \equiv & -\int_{\phi(\partial\Omega)} D_\xi S(\phi(x) - \eta, \kappa)\mathbf{a}^{(2)}(\kappa)\nu_\phi(\eta) f \circ \phi^{(-1)}(\eta) d\sigma_\eta \\ & -\int_{\phi(\partial\Omega)} S(\phi(x) - \eta, \kappa)\nu_\phi^t(\eta)\mathbf{a}^{(1)}(\kappa) f \circ \phi^{(-1)}(\eta) d\sigma_\eta, \end{aligned} \tag{1.5}$$

for all $x \in \partial\Omega$ and for all $(\phi, f, \kappa) \in (C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}$. Here $\partial_{\xi_l} S(\cdot, \kappa)$ and $D_\xi S(\cdot, \kappa)$ denote the derivative with respect to ξ_l and the gradient of $S(\xi, \kappa)$ with respect to the first argument, respectively, and ν_ϕ denotes the exterior unit normal field to $\mathbb{I}[\phi]$. The functions V_l, V_*, W are associated to the ϕ-pull backs on $\partial\Omega$ of the derivatives of the simple layer and of the double layer potential and are well known to intervene in the integral equations associated to boundary value problems for the elliptic operator $P[\mathbf{a}(\kappa), D]$. We prove that V_l and V_* are real analytic from $(C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ to $C^{m-1,\alpha}(\partial\Omega)$ and that W is real analytic from $(C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}$ to $C^{m,\alpha}(\partial\Omega)$.

We note that Potthast [10], [11], [12] has proved a Fréchet differentiability result, at least in case f is of class $C^{0,\alpha}$ and when $P[\mathbf{a}(\kappa), D]$ is the Helmholtz operator by exploiting a different method. Our work stems from that of [5] for the Cauchy integral operator, and from that of [6] and [7] for the Laplace and for the Helmholtz operator, respectively.

The paper is organized as follows. Section 2 is a section of preliminaries. In section 3, we introduce some basics on elliptic operators and on corresponding layer potentials. In section 4, we introduce an auxiliary boundary value problem. In section 5, we prove our main results.

2 Technical preliminaries

We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [13]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. A dot "$\cdot$" denotes the inner product in $\mathbb{R}^n$, or the matrix product between matrices. Let A be a matrix. Then A^t denotes the transpose matrix of A and A_{ij} denotes the (i, j) entry of A. If A is invertible, we set $A^{-t} \equiv \left(A^{-1}\right)^t$. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\,\mathbb{D}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$,

x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$ or in $\mathbb{C}$, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{C})$, or more simply by $C^m(\Omega)$. $\mathcal{D}(\Omega)$ denotes the space of functions of $C^\infty(\Omega)$ with compact support. The dual $\mathcal{D}'(\Omega)$ denotes the space of distributions in Ω. Let $f \in (C^m(\Omega))^n$. The s-th component of f is denoted f_s, and Df denotes the gradient matrix $\left(\frac{\partial f_s}{\partial x_l}\right)_{s,l=1,\dots,n}$. Let $\eta \equiv (\eta_1,\dots,\eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \le m$ can be extended with continuity to $\mathrm{cl}\,\Omega$ is denoted $C^m(\mathrm{cl}\,\Omega)$. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [3].) The subspace of $C^m(\mathrm{cl}\,\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega\cap\mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}(\Omega \cap \mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\,\Omega)$. Let $\mathbb{D} \subseteq \mathbb{C}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : f(\mathrm{cl}\,\Omega) \subseteq \mathbb{D}\}$.

Now let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\,\Omega)$ and $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ are endowed with their usual norm and are well known to be Banach spaces (cf. *e.g.*, Troianiello [15, §1.2.1].) We say that a bounded open subset of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [3, §6.2].) For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [3] and to Troianiello [15] (see also [4, §2, Lem. 3.1, 4.26, Thm. 4.28], [6, §2].)

If M is a manifold imbedded in $\mathbb{R}^n$ of class $C^{m,\alpha}$, with $m \ge 1$, $\alpha \in]0,1[$, one can define the Schauder spaces also on M by exploiting the local parametrizations. In particular, one can consider the spaces $C^{k,\alpha}(\partial\Omega)$ on $\partial\Omega$ for $0 \le k \le m$ with Ω a bounded open set of class $C^{m,\alpha}$, and the trace operator of $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\partial\Omega)$ is linear and continuous. Moreover, for each $R > 0$ such that $\mathrm{cl}\Omega \subseteq \mathbb{B}_n(0,R)$, there exists a linear and continuous extension operator of $C^{k,\alpha}(\partial\Omega)$ to $C^{k,\alpha}(\mathrm{cl}\Omega)$, and of $C^{k,\alpha}(\mathrm{cl}\Omega)$ to $C^{k,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R))$ (cf. *e.g.*, Troianiello [15, Thm. 1.3, Lem. 1.5].)

We note that throughout the paper "analytic" means "real analytic". For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [13, p. 89]. In particular, we mention that the pointwise product in Schauder spaces is bilinear and continuous, and thus analytic, and that the map which takes a nonzero function to its reciprocal, or an invertible matrix of functions to its inverse matrix is real analytic in Schauder spaces (cf. *e.g.*, [6, pp. 141, 142].)

Now let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class C^1 such that $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ is connected. We denote by $\mathcal{A}_{\partial\Omega}$ and by $\mathcal{A}_{\mathrm{cl}\Omega}$ the sets of functions of class $C^1(\partial\Omega,\mathbb{R}^n)$ and of class $C^1(\mathrm{cl}\Omega,\mathbb{R}^n)$ which are injective and whose differential is injective at all points $x \in \partial\Omega$, and at all points $x \in \mathrm{cl}\Omega$, respectively. One can verify that $\mathcal{A}_{\partial\Omega}$ is open in $C^1(\partial\Omega,\mathbb{R}^n)$ and that $\mathcal{A}_{\mathrm{cl}\Omega}$ is open in

$C^1(\mathrm{cl}\Omega, \mathbb{R}^n)$ (cf. [4, Cor. 4.24, Prop. 4.29], [6, Lem. 2.5].) Moreover, if $\phi \in \mathcal{A}_{\partial\Omega}$, the Jordan-Leray separation theorem ensures that $\mathbb{R}^n \setminus \phi(\partial\Omega)$ has exactly two open connected components, and we denote by $\mathbb{I}[\phi]$ and $\mathbb{E}[\phi]$ the bounded and unbounded open connected components of $\mathbb{R}^n \setminus \phi(\partial\Omega)$, respectively. Then we have the following two Lemmas (cf. [7, §2].)

Lemma 2.1 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of class $C^{m,\alpha}$ of $\mathbb{R}^n$ such that both Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. Let ν_Ω denote the outward unit normal field to $\partial\Omega$. Let $\omega \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$, $|\omega(x)| = 1$, $\omega(x) \cdot \nu_\Omega(x) > 1/2$ for all $x \in \partial\Omega$. Then the following statements hold.*

(i) If $\phi \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, then $\mathbb{I}[\phi]$ is a bounded open connected set of class $C^{m,\alpha}$ and $\partial\mathbb{I}[\phi] = \phi(\partial\Omega) = \partial\mathbb{E}[\phi]$.

(ii) There exists $\delta_\Omega \in]0, +\infty[$ such that the sets

$$\begin{aligned} \Omega_{\omega,\delta} &\equiv \{x + t\omega(x) : x \in \partial\Omega, \ t \in]-\delta, \delta[\}, \\ \Omega^+_{\omega,\delta} &\equiv \{x + t\omega(x) : x \in \partial\Omega, \ t \in]-\delta, 0[\}, \\ \Omega^-_{\omega,\delta} &\equiv \{x + t\omega(x) : x \in \partial\Omega, \ t \in]0, \delta[\}, \end{aligned}$$

are connected and of class $C^{m,\alpha}$, and

$$\begin{aligned} \partial\Omega_{\omega,\delta} &= \{x + t\omega(x) : x \in \partial\Omega, \ t \in \{-\delta, \delta\}\}, \\ \partial\Omega^+_{\omega,\delta} &= \{x + t\omega(x) : x \in \partial\Omega, \ t \in \{-\delta, 0\}\}, \\ \partial\Omega^-_{\omega,\delta} &= \{x + t\omega(x) : x \in \partial\Omega, \ t \in \{0, \delta\}\}, \end{aligned}$$

and $\Omega^+_{\omega,\delta} \subseteq \Omega$, $\Omega^-_{\omega,\delta} \subseteq \mathbb{R}^n \setminus \mathrm{cl}\Omega$, for all $\delta \in]0, \delta_\Omega[$.

(iii) Let $\delta \in]0, \delta_\Omega[$. If $\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}}$, then $\phi \equiv \Phi_{|\partial\Omega} \in \mathcal{A}_{\partial\Omega}$.

(iv) If $\delta \in]0, \delta_\Omega[$, then the set $\mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}} \equiv \left\{\Phi \in \mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}} : \Phi(\Omega^+_{\omega,\delta}) \subseteq \mathbb{I}[\Phi_{|\partial\Omega}]\right\}$ is open in $\mathcal{A}_{\mathrm{cl}\Omega_{\omega,\delta}}$ and $\Phi(\Omega^-_{\omega,\delta}) \subseteq \mathbb{E}[\Phi_{|\partial\Omega}]$ for all $\Phi \in \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$.

(v) If $\delta \in]0, \delta_\Omega[$ and $\Phi \in C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, then both $\Phi(\Omega^+_{\omega,\delta})$ and $\Phi(\Omega^-_{\omega,\delta})$ are open sets of class $C^{m,\alpha}$, and

$$\partial\Phi\left(\Omega^+_{\omega,\delta}\right) = \Phi\left(\partial\Omega^+_{\omega,\delta}\right), \quad \partial\Phi\left(\Omega^-_{\omega,\delta}\right) = \Phi\left(\partial\Omega^-_{\omega,\delta}\right).$$

Then we have the following lemma (cf. *e.g.*, [7, Prop. 2.5, 2.6].)

Lemma 2.2 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let $\phi_0 \in C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$. Then the following statements hold.*

(i) There exist $\delta_0 \in]0, \delta_\Omega[$ and $\Phi_0 \in C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}}$ such that $\phi_0 \equiv \Phi_{0|\partial\Omega}$.

(ii) Let δ_0, Φ_0 be as in (i). Then there exist an open neighborhood $\mathcal{W}_0$ of ϕ_0 in $C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}$, and a real analytic operator $\mathbf{E}_0$ of $C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n)$ which maps $\mathcal{W}_0$ to $C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta_0}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta_0}}$ and such that $\mathbf{E}_0[\phi_0] = \Phi_0$ and $\mathbf{E}_0[\phi]_{|\partial\Omega} = \phi$ for all $\phi \in \mathcal{W}_0$.

3 Some basic properties of elliptic operators and of layer potentials

As is well known, the differential operator $P[\mathbf{a}, D]$ has at least a fundamental solution $S_{\mathbf{a}}(\cdot)$ for each $\mathbf{a} \in \mathcal{E}$. The membership of $\mathbf{a}$ in $\mathcal{E}$ ensures that $P[\mathbf{a}, D]$ is elliptic and that accordingly $S_{\mathbf{a}}(\cdot)$ is real analytic on $\mathbb{R}^n \setminus \{0\}$, and that any other fundamental solution of $P[\mathbf{a}, D]$ differs from $S_{\mathbf{a}}(\cdot)$ by a real analytic function defined on the whole of $\mathbb{R}^n$. We collect in the following statement some known facts on the layer potentials associated to $S_{\mathbf{a}}(\cdot)$. We find convenient to set

$$\Omega^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\Omega \,,$$

for all open subsets Ω of $\mathbb{R}^n$.

Theorem 3.1 *Let $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\mathbf{a} \in \mathcal{E}$. Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$. Then the following statements hold.*

(i) If $\mu \in C^{0,\alpha}(\partial\Omega)$, then the function $v_{S_{\mathbf{a}}}[\partial\Omega, \mu]$ of $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$v_{S_{\mathbf{a}}}[\partial\Omega, \mu](\xi) \equiv \int_{\partial\Omega} S_{\mathbf{a}}(\xi - \eta)\mu(\eta)\, d\sigma_\eta \qquad \forall \xi \in \mathbb{R}^n \,,$$

is continuous.

(ii) If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function $v^+_{S_{\mathbf{a}}}[\partial\Omega, \mu] \equiv v_{S_{\mathbf{a}}}[\partial\Omega, \mu]_{|\mathrm{cl}\Omega}$ belongs to $C^{m,\alpha}(\mathrm{cl}\Omega)$ and the operator which takes μ to $v^+_{S_{\mathbf{a}}}[\partial\Omega, \mu]$ is continuous from $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\Omega)$.

(iii) If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function $v^-_{S_{\mathbf{a}}}[\partial\Omega, \mu] \equiv v_{S_{\mathbf{a}}}[\partial\Omega, \mu]_{|\mathrm{cl}\Omega^-}$ belongs to $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^-)$. If $R \in]0, +\infty[$ and $\mathrm{cl}\Omega \subseteq \mathbb{B}_n(0, R)$, then the operator of $C^{m-1,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{B}_n(0, R) \setminus \Omega)$ which takes μ to $v^-_{S_{\mathbf{a}}}[\partial\Omega, \mu]_{|\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega}$ is continuous.

(iv) If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, $l \in \{1, \dots, n\}$, then the integral

$$v_{S_{\mathbf{a}},l}[\partial\Omega, \mu](\xi) \equiv \int_{\partial\Omega} \partial_{\xi_l} S_{\mathbf{a}}(\xi - \eta)\mu(\eta)\, d\sigma_\eta \qquad \forall \xi \in \mathbb{R}^n \,,$$

converges in the sense of Lebesgue for all $\xi \in \mathbb{R}^n \setminus \partial\Omega$ and in the sense of a principal value for all $\xi \in \partial\Omega$.

(v) Let $l \in \{1, \dots, n\}$. If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then $v_{S_{\mathbf{a}},l}[\partial\Omega, \mu]_{|\Omega}$ admits a continuous extension $v^+_{S_{\mathbf{a}},l}[\partial\Omega, \mu]$ to $\mathrm{cl}\Omega$ and $v^+_{S_{\mathbf{a}},l}[\partial\Omega, \mu] \in C^{m-1,\alpha}(\mathrm{cl}\Omega)$, and $v_{S_{\mathbf{a}},l}[\partial\Omega, \mu]_{|\Omega^-}$ admits a continuous extension $v^-_{S_{\mathbf{a}},l}[\partial\Omega, \mu]$ to $\mathrm{cl}\Omega^-$ and $v^-_{S_{\mathbf{a}},l}[\partial\Omega, \mu] \in C^{m-1,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^-)$, and

$$\begin{aligned} v^{\pm}_{S_{\mathbf{a}},l}[\partial\Omega, \mu](\xi) &= \frac{\partial}{\partial \xi_l} v^{\pm}_{S_{\mathbf{a}}}[\partial\Omega, \mu](\xi) \\ &= \mp \frac{(\nu_\Omega(\xi))_l}{2\nu_\Omega(\xi)^t \mathbf{a}^{(2)} \nu_\Omega(\xi)} \mu(\xi) + v_{S_{\mathbf{a}},l}[\partial\Omega, \mu](\xi) \,, \end{aligned}$$

$$
\begin{aligned}
&(Dv^{\pm}_{S_{\mathbf{a}}}[\partial\Omega,\mu](\xi))\mathbf{a}^{(2)}\nu_\Omega(\xi) \\
&\qquad\qquad = \mp\frac{1}{2}\mu(\xi)+\int_{\partial\Omega}(DS_{\mathbf{a}}(\xi-\eta))\mathbf{a}^{(2)}\nu_\Omega(\xi)\mu(\eta)\,d\sigma_\eta
\end{aligned}
$$

for all $\xi\in\partial\Omega$.

(vi) Let $l\in\{1,\dots,n\}$. *The operator of* $C^{m-1,\alpha}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\mathrm{cl}\Omega)$ *which takes* μ *to* $v^{+}_{S_{\mathbf{a}},l}[\partial\Omega,\mu]$ *is continuous. If* $R\in]0,+\infty[$ *and* $\mathrm{cl}\Omega\subseteq\mathbb{B}_n(0,R)$, *then the operator of* $C^{m-1,\alpha}(\partial\Omega)$ *to* $C^{m-1,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega)$ *which takes* μ *to* $v^{-}_{S_{\mathbf{a}},l}[\partial\Omega,\mu]_{|\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous.*

(vii) Let $w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]$ *be the function of* $\mathbb{R}^n$ *to* $\mathbb{C}$ *defined by*

$$
\begin{aligned}
w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}](\xi) \;\equiv\; & -\int_{\partial\Omega}(DS_{\mathbf{a}}(\xi-\eta))\mathbf{a}^{(2)}\nu_\Omega(\eta)\mu(\eta)\,d\sigma_\eta \\
& -\int_{\partial\Omega}S_{\mathbf{a}}(\xi-\eta)\nu^t_\Omega(\eta)\mathbf{a}^{(1)}\mu(\eta)\,d\sigma_\eta \qquad \forall\xi\in\mathbb{R}^n\,,
\end{aligned}
$$

for all $\mu\in C^{0,\alpha}(\partial\Omega)$. *If* $\mu\in C^{m,\alpha}(\partial\Omega)$, *then the restriction* $w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]_{|\Omega}$ *can be extended uniquely to an element* $w^{+}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]$ *of* $C^{m,\alpha}(\mathrm{cl}\Omega)$ *and the restriction* $w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]_{|\Omega^-}$ *can be extended uniquely to an element* $w^{-}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]$ *of* $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega^-)$ *and we have*

$$
\begin{aligned}
w^{+}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]-w^{-}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]=\mu \qquad & \text{on } \partial\Omega\,, \\
(Dw^{+}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}])\mathbf{a}^{(2)}\nu_\Omega-(Dw^{-}_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}])\mathbf{a}^{(2)}\nu_\Omega=0 \qquad & \text{on } \partial\Omega\,.
\end{aligned}
$$

(viii) If $\mu\in C^{0,\alpha}(\partial\Omega)$, *then we have*

$$
\begin{aligned}
w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}](\xi) \;=\; & -\sum_{j,l=1}^{n}a^{(2)}_{lj}\frac{\partial}{\partial\xi_l}\int_{\partial\Omega}S_{\mathbf{a}}(\xi-\eta)(\nu_\Omega(\eta))_j\mu(\eta)\,d\sigma_\eta \\
& -\int_{\partial\Omega}S_{\mathbf{a}}(\xi-\eta)\nu^t_\Omega(\eta)\mathbf{a}^{(1)}\mu(\eta)\,d\sigma_\eta \quad \forall\xi\in\mathbb{R}^n\setminus\partial\Omega\,.
\end{aligned}
$$

(ix) If $\mu\in C^{m,\alpha}(\partial\Omega)$ *and* U *is an open neighborhood of* $\partial\Omega$ *in* $\mathbb{R}^n$ *and* $\tilde\mu\in C^m(U)$, $\tilde\mu_{|\partial\Omega}=\mu$, *then the following equality holds*

$$
\begin{aligned}
&\frac{\partial}{\partial\xi_r}w_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}](\xi) \\
&\quad=\sum_{j,l=1}^{n}a^{(2)}_{lj}\frac{\partial}{\partial\xi_l}\biggl\{\int_{\partial\Omega}S_{\mathbf{a}}(\xi-\eta) \\
&\qquad\qquad\cdot\biggl[(\nu_\Omega(\eta))_r\frac{\partial\tilde\mu}{\partial\eta_j}(\eta)-(\nu_\Omega(\eta))_j\frac{\partial\tilde\mu}{\partial\eta_r}(\eta)\biggr]\,d\sigma_\eta\biggr\} \\
&\quad+\int_{\partial\Omega}\Bigl[(DS_{\mathbf{a}}(\xi-\eta))\mathbf{a}^{(1)}+a_0S_{\mathbf{a}}(\xi-\eta)\Bigr](\nu_\Omega(\eta))_r\mu(\eta)\,d\sigma_\eta \\
&\quad-\int_{\partial\Omega}\partial_{\xi_r}S_{\mathbf{a}}(\xi-\eta)\nu^t_\Omega(\eta)\mathbf{a}^{(1)}\mu(\eta)\,d\sigma_\eta \qquad \forall\xi\in\mathbb{R}^n\setminus\partial\Omega\,.
\end{aligned}
$$

(x) *The operator of* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\Omega)$ *which takes* μ *to* $w^+_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]$ *is continuous. If* $R\in]0,+\infty[$ *and* $\mathrm{cl}\Omega\subseteq\mathbb{B}_n(0,R)$, *then the linear operator of* $C^{m,\alpha}(\partial\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega)$ *which takes* μ *to* $w^-_{S_{\mathbf{a}}}[\partial\Omega,\mu,\mathbf{a}]_{|\mathrm{cl}\mathbb{B}_n(0,R)\setminus\Omega}$ *is continuous.*

For a proof and appropriate references of Theorem 3.1 (i)–(vi), we refer to [1]. Statements (vii)–(x) can be proved by exploiting exactly the same classical computations which can be found for example in the proof of [7, Thm. 3.4 (ii), (iii), (iv)].

Next we introduce the following result, which shows that the homogeneous equation $P[\mathbf{a},D]u=0$ has a unique solution u in $W^{1,2}_0(\Omega')$ if the volume of the domain of definition Ω' is small enough and if $\mathbf{a}$ ranges in bounded subsets of $\mathcal{E}$ which are away from its boundary. Here $W^{1,2}(\Omega')$ denotes the Sobolev space of functions of $L^2(\Omega')$ which have first order distributional derivatives in $L^2(\Omega')$ with its usual norm and $W^{1,2}_0(\Omega')$ denotes the closure of $\mathcal{D}(\Omega')$ in $W^{1,2}(\Omega')$ (cf. *e.g.*, Gilbarg and Trudinger [3, p. 153].) Also, we find convenient to set

$$\mathcal{E}(\eta)\equiv\left\{\mathbf{a}\equiv(a_\alpha)_{|\alpha|\le 2}\in\mathcal{E}:\ \inf_{\xi\in\mathbb{R}^n,|\xi|=1}\mathrm{Re}\left\{\sum_{|\alpha|=2}a_\alpha\xi^\alpha\right\}>\eta,\ \max_{|\alpha|\le 2}|a_\alpha|<\eta^{-1}\right\},$$

for all $\eta\in]0,1[$. Obviously $\mathcal{E}=\bigcup_{\eta\in]0,1[}\mathcal{E}(\eta)$ and each $\mathcal{E}(\eta)$ is open in $\mathbb{C}^N$, where N denotes the number of multi-indexes $\alpha\in\mathbb{N}^n$ with $|\alpha|\le 2$.

Lemma 3.2 *Let* $\eta\in]0,1[$. *Then there exists* $M(\eta)\in]0,+\infty[$ *such that equation*

$$P[\mathbf{a},D]u=0\,, \tag{3.3}$$

has the unique weak solution $u=0$ *in* $W^{1,2}_0(\Omega')$, *for all* $\mathbf{a}\in\mathcal{E}(\eta)$ *and for all open subsets* Ω' *of* $\mathbb{R}^n$ *such that* $\mathrm{meas}(\Omega')<M(\eta)$.

Proof. Let $u\in W^{1,2}_0(\Omega')$ solve (3.3) for some $\mathbf{a}\in\mathcal{E}(\eta)$. Then we have

$$\int_{\Omega'}(D\overline{v})\mathbf{a}^{(2)}(Du)^t-\overline{v}\mathbf{a}^{(1)}(Du)^t-a_0u\overline{v}\,dx=0\qquad\forall v\in W^{1,2}_0(\Omega')\,. \tag{3.4}$$

By exploiting the membership of $\mathbf{a}$ in $\mathcal{E}(\eta)$, we deduce that

$$\mathrm{Re}\left\{\int_{\Omega'}(D\overline{u})\mathbf{a}^{(2)}(Du)^t\,dx\right\}\ge\eta\int_{\Omega'}|Du|^2\,dx\qquad\forall u\in W^{1,2}_0(\Omega')\,.$$

Hence,

$$\begin{aligned}\mathrm{Re}&\left\{\int_{\Omega'}(D\overline{u})\mathbf{a}^{(2)}(Du)^t-\overline{u}\mathbf{a}^{(1)}(Du)^t-a_0|u|^2\,dx\right\}\\&\ge\int_{\Omega'}\eta|Du|^2-|\mathbf{a}^{(1)}||Du||u|-|a_0||u|^2\,dx\\&\ge\int_{\Omega'}(\eta-\epsilon)|Du|^2-\frac{1}{4\epsilon}|\mathbf{a}^{(1)}|^2|u|^2-|a_0||u|^2\,dx\,,\end{aligned}$$

for all $\epsilon \in]0,\eta[$. Since Ω' has finite measure, we know that there exists a constant $c_P > 0$ such that

$$\int_{\Omega'} |u|^2\,dx \le c_P(\mathrm{meas}(\Omega'))^{2/n} \int_{\Omega'} |Du|^2\,dx \qquad \forall u \in W_0^{1,2}(\Omega')\,, \tag{3.5}$$

for all open subsets Ω' of $\mathbb{R}^n$ of finite measure (cf. *e.g.*, Tartar [14, p. 50]) and thus by taking $\epsilon = \frac{1}{2}\eta$, inequality (3.5) implies that

$$\begin{aligned} \mathrm{Re}\, &\left\{\int_{\Omega'} (D\overline{u})\mathbf{a}^{(2)}(Du)^t - u\mathbf{a}^{(1)}(D\overline{u})^t - a_0|u|^2\,dx\right\} \\ &\ge \left[\frac{\eta}{2} - \left(\frac{1}{2\eta}|\mathbf{a}^{(1)}|^2 + |a_0|\right)c_P(M(\eta))^{2/n}\right]\int_{\Omega'}|Du|^2\,dx \\ &\ge \left[\frac{\eta}{2} - \left(\frac{n}{2\eta^3} + \frac{1}{\eta}\right)c_P(M(\eta))^{2/n}\right]\int_{\Omega'}|Du|^2\,dx \end{aligned} \tag{3.6}$$

for all open subsets Ω' of $\mathbb{R}^n$ of finite measure less or equal to a constant $M(\eta)$, which we can choose so small that the term in brackets in the right hand side of (3.6) is positive. Finally, by equation (3.4) and inequality (3.6), we deduce the validity of the lemma (see also (3.5).) □

Then we have the following immediate consequence of the classical elliptic theory.

Theorem 3.7 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let $\eta \in]0,1[$. Let $M(\eta) > 0$ be as in Lemma 3.2. If Ω' is a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{meas}(\Omega') < M(\eta)$, and if $\mathbf{a} \in \mathcal{E}(\eta)$, and if $(f,g) \in C^{m-2,\alpha}(\mathrm{cl}\Omega') \times C^{m,\alpha}(\partial\Omega')$, then there exists a unique $u \in C^{m,\alpha}(\mathrm{cl}\Omega')$ such that*

$$\begin{cases} P[\mathbf{a},D]u = f & \text{in } \Omega'\,, \\ u = g & \text{on } \partial\Omega'\,. \end{cases} \tag{3.8}$$

Here $C^{-1,\alpha}(\mathrm{cl}\Omega)$ denotes the space of distributions in Ω which equal the divergence of an element of class $C^{0,\alpha}(\mathrm{cl}\Omega,\mathbb{C}^n)$ endowed with the quotient norm.

Proof. As is well known, g is the trace of a function of class $C^{m,\alpha}(\mathrm{cl}\Omega')$. Then the existence of a solution of problem (3.8) follows from that of the corresponding problem for $g = 0$. Existence of a solution in $W_0^{1,2}(\Omega')$ follows by the Lax-Milgram Lemma and by inequalities (3.5), (3.6). Then by the classical Schauder regularity theory, we deduce that the solution is actually of class $C^{m,\alpha}(\mathrm{cl}\Omega')$ (cf. *e.g.*, Morrey [9, Thm. 6.4.8].) The uniqueness of problem (3.8) follows by Lemma 3.2. □

4 An auxiliary boundary value problem

For each m, α, Ω, ω, δ_Ω as in Lemma 2.1, $\delta \in]0, \delta_\Omega[$, $\Phi \in C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$, $\mathbf{a} \in \mathcal{E}$, we set

$$\begin{aligned} S_\Phi \equiv\ & C^{m-2,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m-2,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})) \times C^{m,\alpha}(\Phi(\partial\Omega)) \\ & \times C^{m-1,\alpha}(\Phi(\partial\Omega)) \times C^{m,\alpha}(\Phi((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega)) \times C^{m,\alpha}(\Phi((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega)) \end{aligned} \tag{4.1}$$

and

$$B[\mathbf{a}, \Phi](v^+, v^-) \equiv (Dv^+)_{|\Phi(\partial\Omega)} \mathbf{a}^{(2)} \nu_{\Phi_{|\partial\Omega}} - (Dv^-)_{|\Phi(\partial\Omega)} \mathbf{a}^{(2)} \nu_{\Phi_{|\partial\Omega}} \,,$$

for all $(v^+, v^-) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$. Then we have the following.

Theorem 4.2 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let $\eta \in]0,1[$. Then there exists $\delta_\eta \in]0, \delta_\Omega[$ such that if $\delta \in]0, \delta_\eta]$, and if $(\mathbf{a}, \Phi)$ belongs to $\mathcal{E}(\eta) \times \left(C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}\right)$, and if $|\det(D\Phi)| \leq \eta^{-1}$ on $\mathrm{cl}\Omega_{\omega,\delta}$, then the boundary value problem*

$$\begin{cases} P[\mathbf{a}, D]v^+ = f^+ & \text{in } \Phi\left(\Omega^+_{\omega,\delta}\right), \\ P[\mathbf{a}, D]v^- = f^- & \text{in } \Phi\left(\Omega^-_{\omega,\delta}\right), \\ v^+ - v^- = g & \text{on } \Phi\left(\partial\Omega\right), \\ B[\mathbf{a}, \Phi](v^+, v^-) = g_1 & \text{on } \Phi\left(\partial\Omega\right), \\ v^+ = h^+ & \text{on } \Phi\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right), \\ v^- = h^- & \text{on } \Phi\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right), \end{cases} \tag{4.3}$$

admits a unique solution $(v^+, v^-) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ for each given $(f^+, f^-, g, g_1, h^+, h^-)$ in S_Φ.

Proof. Let $M(\eta) > 0$ be as in Lemma 3.2. We take $\delta_\eta \in]0, \delta_\Omega[$ such that $\eta^{-1}\mathrm{meas}(\Omega_{\omega,\delta}) \leq M(\eta)$ for all $\delta \in]0, \delta_\eta]$. Then we also have $\mathrm{meas}(\Phi(\Omega_{\omega,\delta})) \leq M(\eta)$, for all $\delta \in]0, \delta_\eta]$. Now let $\delta \in]0, \delta_\eta]$ and $(f^+, f^-, g, g_1, h^+, h^-) \in S_\Phi$. We first show existence for (4.3). By Theorem 3.7, there exist $\tilde{v}^+ \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))$ and $\tilde{v}^- \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ such that

$$\begin{cases} P[\mathbf{a}, D]\tilde{v}^+ = f^+ & \text{in } \Phi\left(\Omega^+_{\omega,\delta}\right), \\ \tilde{v}^+ = g & \text{on } \Phi\left(\partial\Omega\right), \\ \tilde{v}^+ = h^+ & \text{on } \Phi\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right), \end{cases}$$

and

$$\begin{cases} P[\mathbf{a}, D]\tilde{v}^- = f^- & \text{in } \Phi\left(\Omega^-_{\omega,\delta}\right), \\ \tilde{v}^- = 0 & \text{on } \Phi\left(\partial\Omega\right), \\ \tilde{v}^- = h^- & \text{on } \Phi\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right). \end{cases}$$

Next we note that the boundary value problem

$$\begin{cases} P[\mathbf{a},D]u^+ = 0 & \text{in } \Phi\left(\Omega^+_{\omega,\delta}\right), \\ P[\mathbf{a},D]u^- = 0 & \text{in } \Phi\left(\Omega^-_{\omega,\delta}\right), \\ u^+ - u^- = 0 & \text{on } \Phi\left(\partial\Omega\right), \\ B[\mathbf{a},\Phi](u^+,u^-) = -g_1 + B[\mathbf{a},\Phi](\tilde{v}^+,\tilde{v}^-) & \text{on } \Phi\left(\partial\Omega\right), \end{cases}$$

has a solution $(u^+,u^-) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$. Indeed, we can take $u^+ \equiv v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega),\mu]$ and $u^- \equiv v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega),\mu]$ with $\mu \equiv g_1 - B[\mathbf{a},\Phi](\tilde{v}^+,\tilde{v}^-)$, where $S_{\mathbf{a}}$ is a fundamental solution of $P[\mathbf{a},D]$ (cf. Theorem 3.1.) Then boundary value problem (4.3) has a solution $(v^+,v^-) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ if and only if system

$$\begin{cases} P[\mathbf{a},D]V^+ = 0 & \text{in } \Phi\left(\Omega^+_{\omega,\delta}\right), \\ P[\mathbf{a},D]V^- = 0 & \text{in } \Phi\left(\Omega^-_{\omega,\delta}\right), \\ V^+ - V^- = 0 & \text{on } \Phi\left(\partial\Omega\right), \\ B[\mathbf{a},\Phi](V^+,V^-) = 0 & \text{on } \Phi\left(\partial\Omega\right), \\ V^+ = u^+ & \text{on } \Phi\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right), \\ V^- = u^- & \text{on } \Phi\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right), \end{cases} \tag{4.4}$$

has a solution $(V^+,V^-) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$, and in case of existence, $V^\pm = v^\pm - \tilde{v}^\pm + u^\pm$. Then we now turn to consider problem (4.4). By a standard argument based on the Green identity for $P[\mathbf{a},D]$ (cf. *e.g.*, Miranda [8, p. 12]), problem (4.4) admits a solution (V^+,V^-) if and only if problem

$$\begin{cases} P[\mathbf{a},D]V = 0 & \text{in } \Phi\left(\Omega_{\omega,\delta}\right), \\ V = u^+ & \text{on } \Phi\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right), \\ V = u^- & \text{on } \Phi\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right), \end{cases}$$

has a solution $V \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega_{\omega,\delta}))$, and in case of existence, $V^+ = V_{|\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}$ and $V^- = V_{|\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})}$. Since the existence for such a system follows by Theorem 3.7, the proof of the existence for problem (4.3) is complete.

We now turn to consider uniqueness for problem (4.3). Let the pair (v^+,v^-) in $C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta}))\times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ solve (4.3) with $(f^+,f^-,g,g_1,h^+,h^-) = 0$. Then we define a function v of $\mathrm{cl}\Phi(\Omega_{\omega,\delta})$ to $\mathbb{C}$ by setting $v = v^+$ on $\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})$ and $v = v^-$ on $\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})$. Clearly, v satisfies $P[\mathbf{a},D]v = 0$ in $\Phi(\Omega^+_{\omega,\delta})\cup\Phi(\Omega^-_{\omega,\delta})$ and is continuous on $\mathrm{cl}\Phi(\Omega_{\omega,\delta})$. Since $v^+ - v^- = 0$ and $B[\mathbf{a},\Phi](v^+,v^-) = 0$ on $\Phi(\partial\Omega)$, a standard argument based on the Green identity for $P[\mathbf{a},D]$ shows that v solves $P[\mathbf{a},D]v = 0$ in $\Phi(\Omega_{\omega,\delta})$ in the sense of distributions (cf. *e.g.*, Miranda [8, p. 12].) Since v equals 0 on $\partial\Phi(\Omega_{\omega,\delta})$, our choice of δ_η and Lemma 3.2 imply that $v = 0$. Hence, $(v^+,v^-) = (0,0)$. □

We note that if $S_{\mathbf{a}}$ is a fundamental solution of the differential operator $P[\mathbf{a}, D]$, then $(v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}, v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})})$ is the only solution of problem (4.3) with

$$
\begin{gathered}
f^+ = 0\,, \qquad f^- = 0\,, \qquad g = 0\,, \qquad g_1 = -\mu\,, \\
h^+ \equiv v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\Phi((\partial\Omega^+_{\omega,\delta})\setminus\partial\Omega)}\,, \qquad h^- \equiv v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\Phi((\partial\Omega^-_{\omega,\delta})\setminus\partial\Omega)}\,.
\end{gathered}
$$

Thus problem (4.3) with such data identifies the pair

$$(v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})}, v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu]_{|\mathrm{cl}\Phi(\Omega^-_{\omega,\delta})})\,.$$

In order to obtain a problem which identifies the pair

$$(v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu] \circ \Phi_{|\mathrm{cl}\Omega^+_{\omega,\delta}}, v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega), \mu] \circ \Phi_{|\mathrm{cl}\Omega^-_{\omega,\delta}})\,,$$

we wish to change the variable in (4.3) with the above data by means of the function Φ. However, we note that if $m = 1$, then the map Φ is only one time continuously differentiable, while the differential operator $P[\mathbf{a}, D]$ is of order 2. Thus we now follow [7] and we introduce the following Lemmas.

Lemma 4.5 *Let $m, m' \in \mathbb{N}$, $m > 0$, $m \geq m'$. Let $\alpha \in]0, 1[$. Let Ω be an open bounded subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the operator* div *of $C^{m',\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n)$ to $C^{m'-1,\alpha}(\mathrm{cl}\Omega)$ is bounded linear continuous open and surjective.*

Lemma 4.5 follows by the definition of $C^{-1,\alpha}(\mathrm{cl}\Omega)$ as the space of distributions in Ω which equal the divergence of an element of class $C^{0,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n)$ endowed with the quotient norm and by standard results in elliptic theory (cf. *e.g.*, Gilbarg and Trudinger [3, Thms. 6.14, 6.19] and Troianiello [15, Thm. 1.3, Lem. 1.5].) Then by Lemma 4.5, we have the following (see also [7, Lem. 4.5] for a proof.)

Lemma 4.6 *Let $m \in \mathbb{N}$, $\alpha \in]0, 1[$. Let Ω be an open bounded connected subset of $\mathbb{R}^n$ of class $C^{\max\{1,m\},\alpha}$. The set*

$$\mathcal{Y}^{m,\alpha}(\Omega) \equiv \left\{ w \in C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n) : \int_\Omega (D\varphi) w \, dx = 0 \ \forall \varphi \in \mathcal{D}(\Omega) \right\}$$

is a closed linear subspace of $C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n)$ and the quotient

$$\mathcal{Z}^{m,\alpha}(\Omega) \equiv C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n) / \mathcal{Y}^{m,\alpha}(\Omega)$$

is a Banach space. Moreover, if we denote by Π_Ω the canonical projection of $C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n)$ onto $\mathcal{Z}^{m,\alpha}(\Omega)$, there exists a unique homeomorphism $\widetilde{\mathrm{div}}$ of $\mathcal{Z}^{m,\alpha}(\Omega)$ onto $C^{m-1,\alpha}(\mathrm{cl}\Omega)$ such that $\mathrm{div} = \widetilde{\mathrm{div}} \circ \Pi_\Omega$.

Then we have the following lemma, which generalizes the corresponding Lemma of [6], [7].

Lemma 4.7 *Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $A[\cdot,\cdot,\cdot]$ be the map of $\mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\Omega}) \times C^{m,\alpha}(\mathrm{cl}\Omega)$ to the space $C^{m-1,\alpha}(\mathrm{cl}\Omega, \mathbb{C}^n)$ defined by*

$$A[\mathbf{a},\Phi,u] \equiv \left\{(D\Phi)^{-1}\mathbf{a}^{(2)}(D\Phi)^{-t}(Du)^t + (D\Phi)^{-1}\mathbf{a}^{(1)}u\right\} |\det D\Phi| \\ \forall (\mathbf{a},\Phi,u) \in \mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\Omega}) \times C^{m,\alpha}(\mathrm{cl}\Omega) \,.$$

Let $Q[\cdot,\cdot,\cdot]$ be the map of $\mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\Omega}) \times C^{m,\alpha}(\mathrm{cl}\Omega)$ to $\mathcal{Z}^{m-1,\alpha}(\Omega)$ defined by

$$Q[\mathbf{a},\Phi,u] \equiv \Pi_\Omega A[\mathbf{a},\Phi,u] + a_0 \widetilde{\mathrm{div}}^{(-1)}(u|\det D\Phi|) \,, \tag{4.8}$$

for all $(\mathbf{a},\Phi,u) \in \mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\Omega}) \times C^{m,\alpha}(\mathrm{cl}\Omega)$. Then we have

$$Q[\mathbf{a},\Phi,u] = \Pi_\Omega f \tag{4.9}$$

if and only if

$$P[\mathbf{a},D](u\circ\Phi^{(-1)}) = \mathrm{div}\left\{((D\Phi)f)\circ\Phi^{(-1)}|\det D\left(\Phi^{(-1)}\right)|\right\} \,, \tag{4.10}$$

in the sense of distributions in $\Phi(\Omega)$, for all f in $C^{m-1,\alpha}(\mathrm{cl}\Omega,\mathbb{C}^n)$ and for all $(\mathbf{a},\Phi,u)$ in $\mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\mathrm{cl}\Omega}) \times C^{m,\alpha}(\mathrm{cl}\Omega)$.

Proof. Since $u|\det D\Phi| \in C^{m-1,\alpha}(\mathrm{cl}\Omega)$, Lemma 4.5 ensures that there exists $g \in C^{m,\alpha}(\mathrm{cl}\Omega,\mathbb{C}^n)$ such that

$$\widetilde{\mathrm{div}}^{(-1)}(u|\det D\Phi|) = \Pi_\Omega g \,. \tag{4.11}$$

Thus equation (4.9) is equivalent to equation $\Pi_\Omega A[\mathbf{a},\Phi,u] = \Pi_\Omega(f - a_0 g)$, an equation which we rewrite in the following form

$$\int_\Omega D\varphi\left\{A[\mathbf{a},\Phi,u] + a_0 g - f\right\} dx = 0 \qquad \forall \varphi \in \mathcal{D}(\Omega) \,. \tag{4.12}$$

By equality (4.11), we have

$$\int_\Omega \varphi u|\det D\Phi|\, dx = -\int_\Omega (D\varphi) g\, dx \qquad \forall \varphi \in \mathcal{D}(\Omega) \,.$$

Hence, by changing the variables in the integral of equation (4.12), we obtain

$$\int_{\Phi(\Omega)} D(\varphi\circ\Phi^{(-1)})a^{(2)}D(u\circ\Phi^{(-1)})^t + D(\varphi\circ\Phi^{(-1)})a^{(1)}u \tag{4.13} \\ -a_0(\varphi\circ\Phi^{(-1)})(u\circ\Phi^{(-1)})\, dy \\ = \int_{\Phi(\Omega)} D(\varphi\circ\Phi^{(-1)})((D\Phi)f)\circ\Phi^{(-1)}|\det D\left(\Phi^{(-1)}\right)|\, dy \quad \forall \varphi \in \mathcal{D}(\Omega) \,.$$

By exploiting a standard argument based on the convolution with a family of mollifiers, equation (4.13) is easily seen to be equivalent to the same equation with $\varphi \circ \Phi^{(-1)}$ replaced by an arbitrary ψ of $\mathcal{D}(\Phi(\Omega))$. Hence, equation (4.13) is equivalent to equation (4.10) and thus the proof is complete. $\square$

We now transplant boundary value problem (4.3), which is defined on the pair of sets $(\Phi(\Omega^+_{\omega,\delta}), \Phi(\Omega^-_{\omega,\delta}))$ to the pair of sets $(\Omega^+_{\omega,\delta}, \Omega^-_{\omega,\delta})$ by means of the function Φ. We do so by means of the following.

Theorem 4.14 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let $\delta \in]0, \delta_\Omega[$. Let T denote the map of $\mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}) \times C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$ to the Banach space*

$$\begin{aligned}\mathcal{Z} \equiv \mathcal{Z}^{m-1,\alpha}\left(\Omega^+_{\omega,\delta}\right) \times \mathcal{Z}^{m-1,\alpha}\left(\Omega^-_{\omega,\delta}\right) \times C^{m,\alpha}(\partial\Omega) \times C^{m-1,\alpha}(\partial\Omega) \qquad (4.15)\\ \times C^{m,\alpha}\left((\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\right) \times C^{m,\alpha}\left((\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\right),\end{aligned}$$

which takes $(\mathbf{a}, \Phi, V^+, V^-)$ to

$$\begin{aligned}T[\mathbf{a}, \Phi, V^+, V^-] \equiv (Q[\mathbf{a}, \Phi, V^+], Q[\mathbf{a}, \Phi, V^-], V^+ - V^-, \qquad (4.16)\\ J[\mathbf{a}, \Phi, V^+, V^-], V^+_{|(\partial\Omega^+_{\omega,\delta})\setminus\partial\Omega}, V^-_{|(\partial\Omega^-_{\omega,\delta})\setminus\partial\Omega}),\end{aligned}$$

where we have set

$$J[\mathbf{a}, \Phi, V^+, V^-] \equiv DV^+(D\Phi)^{-1}\mathbf{a}^{(2)}\mathbf{n}[\Phi] - DV^-(D\Phi)^{-1}\mathbf{a}^{(2)}\mathbf{n}[\Phi] \qquad \text{on } \partial\Omega\,, \qquad (4.17)$$

and

$$\mathbf{n}[\Phi](x) \equiv \frac{(D\Phi(x))^{-t}\nu_\Omega(x)}{|(D\Phi(x))^{-t}\nu_\Omega(x)|} \qquad \forall x \in \partial\Omega\,.$$

Then the following statements hold.

(i) Let $(\mathbf{a}, \Phi) \in \mathcal{E} \times (C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}})$, $(F^+, F^-, G, G_1, H^+, H^-) \in \mathcal{Z}$. Then a pair (V^+, V^-) of $C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$ satisfies the equation

$$T[\mathbf{a}, \Phi, V^+, V^-] = (F^+, F^-, G, G_1, H^+, H^-) \qquad (4.18)$$

if and only if the pair $(V^+ \circ \Phi^{(-1)}, V^- \circ \Phi^{(-1)}) \in C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^+_{\omega,\delta})) \times C^{m,\alpha}(\mathrm{cl}\Phi(\Omega^-_{\omega,\delta}))$ satisfies problem (4.3) with

$$g \equiv G \circ \Phi^{(-1)}, \qquad g_1 \equiv G_1 \circ \Phi^{(-1)}, \qquad h^\pm \equiv H^\pm \circ \Phi^{(-1)}_{|\Phi((\partial\Omega^\pm_{\omega,\delta})\setminus\partial\Omega)},$$
$$f^\pm \equiv \mathrm{div}\{((D\Phi)\tilde{f}^\pm) \circ \Phi^{(-1)}|\det(D\Phi^{(-1)})|\},$$

where $\tilde{f}^\pm \in C^{m-1,\alpha}(\mathrm{cl}\Omega^\pm_{\omega,\delta}, \mathbb{C}^n)$ and $\Pi_{\Omega^\pm_{\omega,\delta}}\tilde{f}^\pm = F^\pm$.

(ii) Let $\eta \in]0, 1[$. Let δ_η be as in Theorem 4.2. If $\delta \in]0, \delta_\eta]$, $(\mathbf{a}, \Phi) \in \mathcal{E}(\eta) \times (C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}})$, $|\det(D\Phi)| \le \eta^{-1}$, then $T[\mathbf{a}, \Phi, \cdot, \cdot]$ is a linear homeomorphism of $C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$ onto $\mathcal{Z}$.

Proof. Let (V^+, V^-) satisfy equation (4.18). By Lemma 4.6, there exist $\tilde{f}^\pm$ as in statement (i). By elementary calculus, we know that

$$\mathbf{n}[\Phi](x) = \nu_{\Phi_{|\partial\Omega}} \circ \Phi(x) \qquad \forall x \in \partial\Omega\,,$$

(cf. *e.g.*, [7, Lem. 4.2].) Then Lemma 4.7 and standard calculus imply that $(V^+ \circ \Phi^{(-1)}, V^- \circ \Phi^{(-1)})$ satisfies problem (4.3). The proof of the converse is similar. Hence, statement (i) holds.

We now prove statement (ii). By continuity of the pointwise product in Schauder spaces and by elementary properties of functions in Schauder spaces (cf. *e.g.*, [4, §2]), the map $T[\mathbf{a}, \Phi, \cdot, \cdot]$ is linear and continuous. Then by the Open Mapping Theorem it suffices to show that $T[\mathbf{a}, \Phi, \cdot, \cdot]$ is a bijection. If $(F^+, F^-, G, G_1, H^+, H^-) \in \mathcal{Z}$, then elementary properties of functions in Schauder spaces imply that the sextuple $(f^+, f^-, g, g_1, h^+, h^-)$ defined as in statement (i) belongs to S_Φ (cf. (4.1).) Then Theorem 4.2 ensures that problem (4.3) admits a unique solution (v^+, v^-). Then statement (i) ensures that the pair $(V^+, V^-) \equiv (v^+ \circ \Phi, v^- \circ \Phi)$ solves problem (4.18). If $(F^+, F^-, G, G_1, H^+, H^-) = 0$, then $(f^+, f^-, g, g_1, h^+, h^-)$ must vanish and accordingly both (v^+, v^-) and $(v^+ \circ \Phi, v^- \circ \Phi)$ must vanish. □

By Theorem 4.2 and by Theorem 4.14 and by Theorem 3.1, we immediately deduce the validity of the following (see also [7, Lem. 4.2] for the form of the area element $\sigma_n[\Phi]$ below.)

Corollary 4.19 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let $\delta \in]0, \delta_\Omega[$. Let $\mathbf{a} \in \mathcal{E}$, $\Phi \in C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}$. Let $f \in C^{m-1,\alpha}(\partial\Omega)$. Let $S_{\mathbf{a}}$ be a fundamental solution of $P[\mathbf{a}, D]$. Then a pair $(V^+, V^-) \in C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$ satisfies the equation*

$$T[\mathbf{a}, \Phi, V^+, V^-] = (0, 0, 0, -f, h^+, h^-)$$

with

$$\begin{aligned} h^+(x) &\equiv \int_{\partial\Omega} S_{\mathbf{a}}(\Phi(x) - \Phi(y)) f(y) \sigma_n[\Phi](y)\, d\sigma_y \qquad \forall x \in (\partial\Omega^+_{\omega,\delta}) \setminus \partial\Omega\,, \\ h^-(x) &\equiv \int_{\partial\Omega} S_{\mathbf{a}}(\Phi(x) - \Phi(y)) f(y) \sigma_n[\Phi](y)\, d\sigma_y \qquad \forall x \in (\partial\Omega^-_{\omega,\delta}) \setminus \partial\Omega\,, \end{aligned}$$

where $\sigma_n[\Phi] \equiv |\det(D\Phi)||(D\Phi)^{-t}\nu_\Omega|$, if and only if

$$\begin{aligned} V^+ &= v^+_{S_{\mathbf{a}}}[\Phi(\partial\Omega), f \circ \Phi^{(-1)}] \circ \Phi_{|\mathrm{cl}\Omega^+_{\omega,\delta}}\,, \\ V^- &= v^-_{S_{\mathbf{a}}}[\Phi(\partial\Omega), f \circ \Phi^{(-1)}] \circ \Phi_{|\mathrm{cl}\Omega^-_{\omega,\delta}}\,. \end{aligned}$$

5 Real analyticity of layer potentials corresponding to families of fundamental solutions

In this section, we shall prove our main result, which concerns layer potentials associated to families of fundamental solutions of families of elliptic differential operators of second order. We first introduce the following Lemma, which can be proved by the same argument of [7, Lem. 4.8].

Lemma 5.1 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let $\delta \in]0, \delta_\Omega[$. Let assumption (1.1) hold. Then the map V_δ of*

$$\left(C^{m,\alpha}\left(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n\right) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}\right) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$$

to $C^{m,\alpha}(\partial\Omega_{\omega,\delta})$ defined by

$$V_\delta[\Phi, f, \kappa](x) \equiv \int_{\partial\Omega} S(\Phi(x) - \Phi(y), \kappa) f(y) \sigma_n[\Phi](y)\, d\sigma_y \qquad \forall x \in \partial\Omega_{\omega,\delta}\,,$$

for all $(\Phi, f, \kappa) \in \left(C^{m,\alpha}\left(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n\right) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}\right) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ is real analytic.

We now introduce some notation. Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. If (1.1) holds, we set

$$\begin{aligned} v^+[\phi, f, \kappa](\xi) &\equiv \int_{\phi(\partial\Omega)} S\left(\xi - \eta, \kappa\right) f \circ \phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall \xi \in \mathrm{cl}\mathbb{I}[\phi]\,, \\ v^-[\phi, f, \kappa](\xi) &\equiv \int_{\phi(\partial\Omega)} S\left(\xi - \eta, \kappa\right) f \circ \phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall \xi \in \mathrm{cl}\mathbb{E}[\phi]\,, \end{aligned}$$

for all $(\phi, f, \kappa) \in (C^{m,\alpha}(\partial\Omega, \mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$, and

$$\begin{aligned} \mathcal{U}_{\eta,\delta} &\equiv \left\{\Phi \in C^{m,\alpha}\left(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n\right) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}} :\, |\det(D\Phi)| < 1/\eta\right\}\,, \\ \mathcal{O}(\eta) &\equiv \left\{\kappa \in \mathcal{O} :\, \mathbf{a}(\kappa) \in \mathcal{E}(\eta)\right\}\,, \end{aligned}$$

for all $\eta \in]0, 1[$, $\delta \in]0, \delta_\Omega[$. Then we have the following result.

Proposition 5.2 *Let m, α, Ω, ω, δ_Ω be as in Lemma 2.1. Let assumption (1.1) hold. Let $\eta \in]0, 1[$. Let $\delta_\eta \in]0, \delta_\Omega[$ be as in Theorem 4.2. Let $\delta \in]0, \delta_\eta]$. Let $V^\pm[\Phi, f, \kappa] \equiv v^\pm[\Phi_{|\partial\Omega}, f, \kappa] \circ \Phi_{|\mathrm{cl}\Omega^\pm_{\omega,\delta}}$ on $\mathrm{cl}\Omega^\pm_{\omega,\delta}$ for all $(\Phi, f, \kappa) \in (C^{m,\alpha}\left(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n\right) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$. Then the maps of $\mathcal{U}_{\eta,\delta} \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$ to $C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta})$ and to $C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$, which take (Φ, f, κ) to $V^+[\Phi, f, \kappa]$ and to $V^-[\Phi, f, \kappa]$ are real analytic, respectively.*

Proof. First we set

$$\begin{aligned} \mathcal{X} &\equiv C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{K}\,, \\ \mathcal{Y} &\equiv C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta}) \times C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})\,, \\ \mathcal{V}_{\eta,\delta} &\equiv \mathcal{U}_{\eta,\delta} \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)\,. \end{aligned}$$

Then we consider the map Λ of $\mathcal{U} \equiv \mathcal{V}_{\eta,\delta} \times \mathcal{Y}$ to the Banach space $\mathcal{Z}$ of (4.15) defined by

$$\begin{aligned}\Lambda[\Phi, f, \kappa, V^+, V^-] \equiv\, & T[\mathbf{a}(\kappa), \Phi, V^+, V^-] \\ & -(0,0,0,-f, V^+[\Phi, f, \kappa]_{|(\partial\Omega^+_{\omega,\delta})\setminus\partial\Omega}, V^-[\Phi, f, \kappa]_{|(\partial\Omega^-_{\omega,\delta})\setminus\partial\Omega})\end{aligned}$$

for all $(\Phi, f, \kappa, V^+, V^-) \in \mathcal{U}$. By Corollary 4.19, the set of zeros of Λ in $\mathcal{U}$ coincides with the graph of the map $(V^+[\cdot,\cdot,\cdot], V^-[\cdot,\cdot,\cdot])$. Thus we can deduce the real analyticity of the operator $(V^+[\cdot,\cdot,\cdot], V^-[\cdot,\cdot,\cdot])$ by showing that we can apply the Implicit Function Theorem for real analytic operators (cf. *e.g.*, Prodi and Ambrosetti [13, Thm. 11.6]) to the equation

$$\Lambda[\Phi, f, \kappa, V^+, V^-] = 0$$

around the point $(\Phi_1, f_1, \kappa_1, V^+[\Phi_1, f_1, \kappa_1], V^-[\Phi_1, f_1, \kappa_1])$ for all (Φ_1, f_1, κ_1) in $\mathcal{V}_{\eta,\delta}$. The domain $\mathcal{U} = \mathcal{V}_{\eta,\delta} \times \mathcal{Y}$ of Λ is clearly open in $\mathcal{X} \times \mathcal{Y}$. Since $|\det(D\Phi)| \in C^{m-1,\alpha}(\mathrm{cl}\Omega_{\omega,\delta})$, the continuity of the imbedding of $C^0(\mathrm{cl}\Omega_{\omega,\delta})$ into $C^{-1,\alpha}(\mathrm{cl}\Omega_{\omega,\delta})$ in case $m = 1$ (cf. [7, Lem. 4.4]) and Lemma 4.6 imply that the operator which takes $(\Phi, V^\pm)$ in $(C^{m,\alpha}(\mathrm{cl}\Omega^\pm_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega^\pm_{\omega,\delta}}) \times C^{m,\alpha}(\mathrm{cl}\Omega^\pm_{\omega,\delta})$ to $\widetilde{\mathrm{div}}^{(-1)}(V^\pm|\det(D\Phi)|)$ in $\mathcal{Z}^{m-1,\alpha}(\Omega^\pm_{\omega,\delta})$ is bilinear and continuous. Then by the real analyticity of the map which takes an invertible matrix with Schauder functions as entries to its inverse, we conclude that both $Q[\cdot,\cdot,\cdot]$ and $J[\cdot,\cdot,\cdot,\cdot]$ are real analytic (cf. (4.8), (4.16), (4.17).) Then Lemma 5.1 and the linearity and continuity of the trace operator on the boundary, imply that Λ is real analytic. Thus it suffices to show that the differential

$$d_{(V^+,V^-)}\Lambda[\Phi_1, f_1, \kappa_1, V^+[\Phi_1, f_1, \kappa_1], V^-[\Phi_1, f_1, \kappa_1]]$$

is a homeomorphism. Now by standard rules of calculus in Banach space, such a differential coincides with $T[\mathbf{a}(\kappa_1), \Phi_1, \cdot, \cdot]$. Since $\delta \in]0, \delta_\eta]$, $\mathbf{a}(\kappa_1) \in \mathcal{E}(\eta)$, $\Phi_1 \in \mathcal{U}_{\eta,\delta}$, Theorem 4.14 (ii) ensures that $T[\mathbf{a}(\kappa_1), \Phi_1, \cdot, \cdot]$ is a linear homeomorphism of $\mathcal{Y}$ onto $\mathcal{Z}$, and thus the proof is complete. □

Corollary 5.3 *Let the assumptions of Proposition 5.2 hold. Let* $W^+[\Phi, f, \kappa]$ *and* $W^-[\Phi, f, \kappa]$ *denote the continuous extensions to* $\mathrm{cl}\Omega^+_{\omega,\delta}$ *and to* $\mathrm{cl}\Omega^-_{\omega,\delta}$ *of the functions*

$$\begin{aligned}& -\int_{\Phi(\partial\Omega)} D_\xi S(\Phi(x)-\eta, \kappa)\mathbf{a}^{(2)}(\kappa)\nu_{\Phi_{|\partial\Omega}}(\eta) f \circ \Phi^{(-1)}(\eta)\, d\sigma_\eta \\ & \qquad -\int_{\Phi(\partial\Omega)} S(\Phi(x)-\eta, \kappa)\nu^t_{\Phi_{|\partial\Omega}}(\eta)\mathbf{a}^{(1)}(\kappa) f \circ \Phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall x \in \Omega^+_{\omega,\delta}\,, \\ & -\int_{\Phi(\partial\Omega)} D_\xi S(\Phi(x)-\eta, \kappa)\mathbf{a}^{(2)}(\kappa)\nu_{\Phi_{|\partial\Omega}}(\eta) f \circ \Phi^{(-1)}(\eta)\, d\sigma_\eta \\ & \qquad -\int_{\Phi(\partial\Omega)} S(\Phi(x)-\eta, \kappa)\nu^t_{\Phi_{|\partial\Omega}}(\eta)\mathbf{a}^{(1)}(\kappa) f \circ \Phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall x \in \Omega^-_{\omega,\delta}\,,\end{aligned}$$

for all $(\Phi, f, \kappa) \in \left(C^{m,\alpha}(\mathrm{cl}\Omega_{\omega,\delta}, \mathbb{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\Omega_{\omega,\delta}}\right) \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}$*, respectively. Then the maps of* $\mathcal{U}_{\eta,\delta} \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$ *to* $C^{m,\alpha}(\mathrm{cl}\Omega^+_{\omega,\delta})$ *and to* $C^{m,\alpha}(\mathrm{cl}\Omega^-_{\omega,\delta})$ *which take* (Φ, f, κ) *to* $W^+[\Phi, f, \kappa]$ *and to* $W^-[\Phi, f, \kappa]$ *are real analytic, respectively.*

Proof. We proceed as in [6, Prop. 3.11]. We first consider $W^+[\cdot,\cdot,\cdot]$. We observe that the map Γ of $C^{m,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right)$ to $C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right) \times C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}, \mathbb{C}^n\right)$ defined by

$$\Gamma[g] \equiv (g, \partial_{x_1} g, \dots, \partial_{x_n} g) \qquad \forall g \in C^{m,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right),$$

is a linear homeomorphism of $C^{m,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right)$ onto the image space $\mathrm{Im}\,\Gamma$, a subspace of $C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right) \times C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}, \mathbb{C}^n\right)$. Thus it suffices to show that the nonlinear maps $W^+[\cdot,\cdot,\cdot]$ and $\frac{\partial}{\partial x_s} W^+[\cdot,\cdot,\cdot]$ for $s = 1, \dots, n$ are real analytic from $\mathcal{U}_{\eta,\delta} \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$ to $C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right)$. Now let $R > \sup_{x \in \Omega \cup \Omega_{\omega,\delta}} |x|$. By Troianiello [15, Thm. 1.3, Lem. 1.5], there exists a linear and continuous operator $\mathbf{F}$ of $C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\mathbb{B}_n(0,R))$ such that $\mathbf{F}[f]_{|\partial\Omega} = f$, for all $f \in C^{m,\alpha}(\partial\Omega)$. By Theorem 3.1 (viii), (ix), we have the following identities

$$\begin{aligned} &W^+[\Phi, f, \kappa] \qquad (5.4)\\ &= -\sum_{l,s,j=1}^{n} a^{(2)}_{lj}(\kappa) \frac{\partial}{\partial x_s} (V^+[\Phi, \mathbf{n}_j[\Phi] f, \kappa])((D\Phi)^{-1})_{sl} - V^+[\Phi, \mathbf{n}^t[\Phi] \mathbf{a}^{(1)} f, \kappa], \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x_s}\left(W^+[\Phi, f, \kappa]\right) \qquad (5.5)\\ &= \sum_{r=1}^{n} \frac{\partial \Phi_r}{\partial x_s} \sum_{l,j=1}^{n} a^{(2)}_{lj}(\kappa) \sum_{t=1}^{n} \frac{\partial}{\partial x_t}\left(V^+[\Phi, M_{rj}[f,\Phi], \kappa]\right)\left((D\Phi)^{-1}\right)_{tl}\\ &\quad + \sum_{r=1}^{n} \frac{\partial \Phi_r}{\partial x_s} D\left(V^+[\Phi, \mathbf{n}_r[\Phi] f, \kappa]\right)(D\Phi)^{-1} \cdot \mathbf{a}^{(1)}(\kappa)\\ &\quad + \sum_{r=1}^{n} \frac{\partial \Phi_r}{\partial x_s} a_0(\kappa) V^+[\Phi, \mathbf{n}_r[\Phi] f, \kappa] - \frac{\partial}{\partial x_s} V^+[\Phi, \mathbf{n}^t[\Phi] \mathbf{a}^{(1)} f, \kappa], \end{aligned}$$

for all $s \in \{1, \dots, n\}$, where

$$\begin{aligned} M_{rj}[f,\Phi] = &\left|(D\Phi)^{-t} \cdot \nu_\Omega\right|^{-1} \cdot\\ &\cdot \left\{\left[\sum_{l=1}^{n}\left((D\Phi)^{-1}\right)_{lr} (\nu_\Omega)_l\right]\left[\sum_{l=1}^{n} \frac{\partial(\mathbf{F}[f])}{\partial x_l}\left((D\Phi)^{-1}\right)_{lj}\right]\right.\\ &\left. - \left[\sum_{l=1}^{n}\left((D\Phi)^{-1}\right)_{lj} (\nu_\Omega)_l\right]\left[\sum_{l=1}^{n} \frac{\partial(\mathbf{F}[f])}{\partial x_l}\left((D\Phi)^{-1}\right)_{lr}\right]\right\}. \end{aligned}$$

Then by the real analyticity of the pointwise product in Schauder spaces and by the real analyticity of the map which takes an invertible matrix with Schauder functions as entries to its inverse, and by the real analyticity of the linear and continuous map $\mathbf{F}[\cdot]$ and of the trace operator, and by the real analyticity of $V^+[\cdot,\cdot,\cdot]$, and by assumption (1.1), and by identities (5.4) and (5.5), we conclude that $W^+[\cdot,\cdot,\cdot]$ and $\frac{\partial}{\partial x_s}W^+[\cdot,\cdot,\cdot]$ are real analytic from $\mathcal{U}_{\eta,\delta} \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$ to $C^{m-1,\alpha}\left(\mathrm{cl}\Omega^+_{\omega,\delta}\right)$. Similarly, we can show that $W^-[\Phi, f, \kappa]$ depends real analytically on (Φ, f, κ). □

We are now ready to prove our main result.

Theorem 5.6 *Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that both Ω and $\mathbb{R}^n \setminus \mathrm{cl}\Omega$ are connected. Let assumption (1.1) hold. Then the following statements hold.*

(i) The map $V[\cdot,\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ to the space $C^{m,\alpha}(\partial\Omega)$ defined by (1.2) is real analytic.

(ii) The map $V_l[\cdot,\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ to the space $C^{m-1,\alpha}(\partial\Omega)$ defined by (1.3) is real analytic for all $l \in \{1,\ldots,n\}$.

(iii) The map $V_*[\cdot,\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$ to the space $C^{m-1,\alpha}(\partial\Omega)$ defined by (1.4) is real analytic.

(iv) The map $W[\cdot,\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}$ to the space $C^{m,\alpha}(\partial\Omega)$ defined by (1.5) is real analytic.

Proof. We first consider statements (i)–(iii). It clearly suffices to show that if $(\phi_0, f_0, \kappa_0) \in (C^{m,\alpha}(\partial\Omega,\mathbb{R}^n) \cap \mathcal{A}_{\partial\Omega}) \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}$, then the operators of (i)–(iii) are real analytic in a neighborhood of (ϕ_0, f_0, κ_0). Now let ω, δ_0, $\mathcal{W}_0$, $\mathbf{E}_0$ be as in Lemma 2.2 for ϕ_0. Possibly shrinking $\mathcal{W}_0$, we can assume that there exists $\eta \in]0,1[$ such that

$$\sup_{\phi \in \mathcal{W}_0} \sup_{x \in \mathrm{cl}\Omega_{\omega,\delta_0}} |\det(D\mathbf{E}_0[\phi](x))| < \eta^{-1}, \qquad \kappa_0 \in \mathcal{O}(\eta).$$

Now by definition of the operators in (i)–(iii), and by Theorem 3.1, we have

$$\begin{aligned}
V[\phi,f,\kappa] &= v^+[\phi,f,\kappa] \circ \phi = V^+[\mathbf{E}_0[\phi], f, \kappa],\\
V_l[\phi,f,\kappa] &= \frac{\mathbf{n}_l[\mathbf{E}_0[\phi]]}{2\mathbf{n}^t[\mathbf{E}_0[\phi]]\mathbf{a}^{(2)}\mathbf{n}[\mathbf{E}_0[\phi]]} f + \frac{\partial}{\partial \xi_l}(v^+[\phi,f,\kappa]) \circ \phi\\
&= \frac{\mathbf{n}_l[\mathbf{E}_0[\phi]]}{2\mathbf{n}^t[\mathbf{E}_0[\phi]]\mathbf{a}^{(2)}\mathbf{n}[\mathbf{E}_0[\phi]]} f + \left((DV^+[\mathbf{E}_0[\phi], f, \kappa]) \cdot (D\mathbf{E}_0[\phi])^{-1}\right)_l,\\
V_*[\phi,f,\kappa] &= \frac{1}{2} f + \left((Dv^+[\phi,f,\kappa]) \circ \phi\right) \mathbf{a}^{(2)}(\kappa)\nu_\phi \circ \phi\\
&= \frac{1}{2} f + (DV^+[\mathbf{E}_0[\phi], f, \kappa])(D\mathbf{E}_0[\phi])^{-1} \cdot \mathbf{a}^{(2)}(\kappa) \cdot \mathbf{n}[\mathbf{E}_0[\phi]],
\end{aligned}$$

on $\partial\Omega$ for all $(\phi, f, \kappa) \in \mathcal{W}_0 \times C^{m-1,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$ where V^+ is as in Proposition 5.2 for some arbitrary $\delta \in]0, \min\{\delta_\eta, \delta_0\}]$. Hence, statements (i)–(iii) follow by Lemma 2.2 and Proposition 5.2.

In order to prove statement (iv), we just note that

$$W[\phi, f, \kappa] = -\frac{1}{2}f + W^+[\mathbf{E}_0[\phi], f, \kappa] \qquad \text{on } \partial\Omega\,,$$

for all $(\phi, f, \kappa) \in \mathcal{W}_0 \times C^{m,\alpha}(\partial\Omega) \times \mathcal{O}(\eta)$, and then we argue as above by exploiting Corollary 5.3 instead of Proposition 5.2. □

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ORTHONORMAL WAVELET SYSTEMS AND MULTIRESOLUTION ANALYSES

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Let $\mathbf{A} \in \mathbb{R}^{d\times d}$ be a matrix that preserves the lattice $\mathbb{Z}^d$ and $|a| := \det \mathbf{A}$. In [8], the author studied the properties of wavelet systems in $L^2(\mathbb{R}^d)$ of the form $\{|a|^{j/2}\psi_\ell(A^j\mathbf{t} + \mathbf{k}); j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \le \ell \le m\}$ that are associated with a multiresolution analysis of multiplicity n generated by $\mathbf{A}$. The purpose of the present paper is to extend those results to wavelet systems in $L^2(\mathbb{R}^d)$ of the form $\{|a|^{j/2}\psi_\ell(A^j\mathbf{t} + \mathbf{Bk}); j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \le \ell \le m\}$ that are associated with a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a matrix that preserves the lattice $\mathbb{Z}^d$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$ is a nonsingular matrix.

1. Introduction

In what follows, $\mathbb{Z}$ will denote the set of integers, $\mathbb{T} := [0, 1]$, and $\mathbb{T}^d$ will denote the d–dimensional torus. The underlying space will be $L^2(\mathbb{R}^d)$, where $d \ge 1$ is an integer and $\mathbb{R}$ is the set of real numbers, $\mathbf{I}$ will stand for the identity matrix, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$, $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $\mathbf{C} := (A^{-1})^T$, and $\mathbf{D} := (B^{-1})^T$. Boldface lowcase letters will denote elements of $\mathbb{R}^d$, which will be represented as column vectors; $\mathbf{x} \cdot \mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||^2 := \mathbf{x} \cdot \mathbf{x}$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g\rangle$, their bracket product with respect to $\mathbf{B}$ by $[f, g]^{\mathbf{B}}$, and the norm of f by $||f||$; thus,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$||f|| := \sqrt{\langle f, f\rangle},$$

and

$$[f, g](\mathbf{t}) := [f, g]^{\mathbf{I}}(\mathbf{t}).$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}^{\mathbf{B}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j\mathbf{t})$$

and

$$T_{\mathbf{k}}^{\mathbf{B}} f(\mathbf{t}) := f(\mathbf{t} + \mathbf{Bk}).$$

In particular,

$$T_{\mathbf{k}} f(\mathbf{t}) := T_{\mathbf{k}}^{\mathbf{I}} f(\mathbf{t}).$$

A function f will be called $\mathbf{B}\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}}^{\mathbf{B}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$. A set $S \subset L^2(\mathbb{R}^d)$ is called $\mathbf{B}$ *shift–invariant* if $f \in S$ implies that $T_{\mathbf{k}}^{\mathbf{B}} f \in S$ for every $\mathbf{k} \in \mathbb{Z}^d$. If $\mathbf{B} = \mathbf{I}$, then we speak of a $\mathbb{Z}^d$ – periodic function f and of a shift – invariant space S, omitting mention of the matrix $\mathbf{I}$.

If f is a $\mathbf{B}\mathbb{Z}^d$–periodic function and $b := \det \mathbf{B}$, then

$$f^{\mathbf{B}}(\mathbf{t}) := D_1^{\mathbf{B}} f(\mathbf{t}) = |b|^{1/2} f(\mathbf{B}\mathbf{t}) \tag{1}$$

is $\mathbb{Z}^d$–periodic.

Let $\mathbf{u} \subset L^2(\mathbb{R}^d)$; then

$$T^{\mathbf{B}}(\mathbf{u}) := \{T_{\mathbf{k}}^{\mathbf{B}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S^{\mathbf{B}}(\mathbf{u}) := \overline{\operatorname{span}}\, T^{\mathbf{B}}(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. In particular,

$$T(\mathbf{u}) := T^{\mathbf{I}}(\mathbf{u})$$

and

$$S(\mathbf{u}) := S^{\mathbf{I}}(\mathbf{u})$$

If $\mathbf{u} = \{u_1, \cdots, u_m\}$ then $S^{\mathbf{B}}(\mathbf{u})$ is called a *finitely generated* $\mathbf{B}$ *shift–invariant space* and the functions u_ℓ are called the *generators* of $S^{\mathbf{B}}(\mathbf{u})$. In this case we will also use the symbols $T^{\mathbf{B}}(u_1, \cdots, u_n)$ and $S^{\mathbf{B}}(u_1, \cdots, u_n)$ to denote $S^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{u})$ respectively.

Let $\mathbb{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$. A sequence $F = \{f_k, k \in \mathbb{Z}\} \subset \mathbb{H}$ is called a *Riesz sequence* if there are constants $0 < A \leq B$ such that for every sequence $\{c_k, k \in \mathbb{Z}\} \subset \ell^2$

$$A \sum_{\mathbf{k} \in \mathbb{Z}^d} |c_k|^2 \leq \left\| \sum_{\mathbf{k} \in \mathbb{Z}^d} c_k f_k \right\|^2 \leq B \sum_{\mathbf{k} \in \mathbb{Z}^d} |c_k|^2.$$

F is called a *Riesz basis* of $\mathbb{H}$ if it is a Riesz sequence and its linear span is dense in $\mathbb{H}$. The constants A and B are called (lower and upper) *bounds* of the Riesz basis. Clearly, every orthonormal basis is a Riesz basis with bounds $A = B = 1$. The theory of Riesz bases is discussed in, e.g., [3, 7].

Let $\Lambda \subset \mathbb{Z}$ and $\mathbf{u} = \{u_k; k \in \Lambda\} \subset S \subset L^2(\mathbb{R}^d)$. If S is a $\mathbf{B}$ shift–invariant space then $\mathbf{u}$ is called a *basis generator* of S, and we say that $\mathbf{u}$ *provides a basis* for S, if for every $f \in S$ there are $\mathbf{B}\mathbb{Z}^d$–periodic functions p_k, uniquely determined by f (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k \in \Lambda} p_k \widehat{u_k}.$$

If $\mathbf{u}$ is a finite set, the uniqueness of the functions p_k is equivalent to $G_{\mathbf{u}}^{\mathbf{B}}(\mathbf{x})$ being nonsingular for almost every $\mathbf{x} \in \mathbb{T}^d$.

The theory of basis generators has been extensively developed by De Boor, DeVore, Ron and Shen in [1, 2, 5], under the assumption that $\mathbf{B} = \mathbf{I}$. In [8] we applied some of these results to the study of *Schauder basis generators*, *Riesz basis generators* and *orthonormal basis generators*, i.e. sets $\mathbf{u}$ such that $T(\mathbf{u})$ is either a Schauder basis, a Riesz basis, or an orthonormal basis of $S(\mathbf{u})$. Note that a Riesz basis generator is a basis generator. In the following section we will extend some of the results of [8] to the case of an arbitrary lattice $\mathbf{B}\mathbb{Z}^d$, where $\mathbf{B}$ is nonsingular.

2. Some theorems on Riesz bases of translates and Linear Transformations

Given a sequence of functions $\mathbf{u} := \{u_1, \cdots, u_m\}$ in $L^2(\mathbb{R}^d)$ and $\mathbf{B} \in \mathbb{R}^{d\times d}$, by $G^{\mathbf{B}}[u_1, \cdots, u_m](\mathbf{x})$ or $G^{\mathbf{B}}_{\mathbf{u}}(\mathbf{x})$ we will denote its **B** *Gramian matrix*, viz.

$$G^{\mathbf{B}}_{\mathbf{u}}(\mathbf{x}) := \left([\widehat{u_\ell}, \widehat{u_j}]^{\mathbf{B}}(\mathbf{x})\right)_{\ell,j=1}^{m}.$$

In particular,

$$G_{\mathbf{u}}(\mathbf{x}) := G^{\mathbf{I}}_{\mathbf{u}}(\mathbf{x}).$$

If $\mathbf{u} = \{u_1, \cdots u_n\}$ and the functions $u^{\mathbf{B}}_\ell$ are defined as in (1), then

$$\mathbf{u}^{\mathbf{B}} := \{u^{\mathbf{B}}_1, \cdots, \mathbf{u}^{\mathbf{B}}_n\}. \tag{2}$$

We begin with the following simple but important result:

Lemma 1. *Let $b \in L^2(\mathbb{R}^{d\times d}$ be a nonsingular matrix, $\mathbf{u} \in L^2(\mathbb{R}^d)$, $\mathbf{D} := (\mathbf{B}^{-1})^T$, and let $u^{\mathbf{B}}(\mathbf{x})$ be given by (1). Then*

(a) *$T^{\mathbf{B}}(\mathbf{u})$ is an orthogonal basis of $S^{\mathbf{B}}(\mathbf{u})$ if and only if $T(\mathbf{u}^{\mathbf{B}})$ is an orthogonal basis of $S(\mathbf{u}^{\mathbf{B}})$.*

(b) *$T^{\mathbf{B}}(\mathbf{u})$ is a Riesz basis in $S^{\mathbf{B}}(\mathbf{u})$ if and only if $T(\mathbf{u}^{\mathbf{B}})$ is a Riesz basis in $S(\mathbf{u}^{\mathbf{B}})$. Moreover, $T^{\mathbf{B}}(\mathbf{u})$ and $T(\mathbf{u}^{\mathbf{B}})$ have the same Riesz bounds.*

Proof. Part (a) follows from a change of variables, whereas part (b) follows from the following computations:

$$||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} T^{\mathbf{B}}_{\mathbf{k}}(\mathbf{u})|| = ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{x}\cdot\mathbf{Bk}} \widehat{u}(\mathbf{x})|| = |b|^{-1/2} ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{s}\cdot\mathbf{Bk}} \widehat{u}(\mathbf{Bs})||,$$

and

$$||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} T_{\mathbf{k}}(\mathbf{u}^{\mathbf{B}})|| = |b|^{-1/2} ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{Dx}\cdot\mathbf{Bk}} \widehat{u}(\mathbf{Bx})|| = |b|^{-1/2} ||\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{s}\cdot\mathbf{Bk}} \widehat{u}(\mathbf{Bs})||.$$

□

The remaining results in this section will follow from Lemma 1 and the corresponding results in [8].

Theorem 2. *Let $\mathbf{u} := \{u_1, \cdots, u_n\}$ and $\mathbf{v} := \{v_1, \cdots, v_m\}$, and let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular. Then*

(a) *If $T^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{v})$ are Riesz bases of the same shift–invariant space $S \subset L^2(\mathbb{R}^d)$, then $n = m$.*
(b) *If $T^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{v})$ are Riesz sequences such that $n = m$ and $S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{v})$, then $T^{\mathbf{B}}(\mathbf{u})$ is a Riesz basis of $S^{\mathbf{B}}(\mathbf{v})$.*
(c) *Let $T^{\mathbf{B}}(\mathbf{u})$ and $T^{\mathbf{B}}(\mathbf{v})$ be Riesz sequences in $L^2(\mathbb{R}^d)$, and assume that $S^{\mathbf{B}}(\mathbf{u})$ is a proper subset of $S^{\mathbf{B}}(\mathbf{v})$. Then $n < m$ and there are functions $w_1, \cdots, w_{m-n}$ such that*

$$T^{\mathbf{B}}(w_1, \cdots, w_{m-n})$$

is an orthonormal basis of the orthogonal complement $S^{\mathbf{B}}(\mathbf{u})^\perp$ of $S^{\mathbf{B}}(\mathbf{u})$ in $S^{\mathbf{B}}(\mathbf{v})$, and

$$T^{\mathbf{B}}(u_1, \cdots, u_n, w_1, \cdots, w_{m-n})$$

is a Riesz basis of $S^{\mathbf{B}}(v_1, \cdots, v_m)$.

Proof. The assertion follows from [8, Theorem 1] applied to $\mathbf{u}^{\mathbf{B}}$ and $\mathbf{v}^{\mathbf{B}}$. $\square$

From the identity

$$\widehat{u^{\mathbf{B}}}(\mathbf{x}) = |b|^{-1/2}\widehat{u}(\mathbf{D}\mathbf{x}), \tag{3}$$

we obtain

Theorem 3. *Let* $\mathbf{u} := \{u_1, \cdots, u_n\} \in L^2(\mathbb{R}^d)$, $\mathbf{h} := \{h_1, \cdots, h_m\} \in L^2(\mathbb{R}^d)$, *let* $\mathbf{B} \in \mathbb{R}^{d\times d}$ *be a nonsingular matrix,* $b := \det \mathbf{B}$, $\mathbf{D} := (\mathbf{B}^{-1})^T$, *and assume that*

$$S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{h}).$$

Then there are $\mathbf{D}\mathbb{Z}^d$*–periodic functions* $q_{\ell,j}(\mathbf{x})$ *such that*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{m} q_{\ell,j}(\mathbf{x})\widehat{h_j}(\mathbf{x}) \quad a.e.; \quad \ell = 1, \cdots, n.$$

Proof. The hypotheses imply that

$$S(\mathbf{u}^{\mathbf{B}}) \subset S(\mathbf{h}^{\mathbf{B}})$$

Applying [8, Theorem F] we see that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j}(\mathbf{x})$ such that

$$\widehat{u_\ell^{\mathbf{B}}}(\mathbf{x}) = \sum_{j=1}^{m} p_{\ell,j}(\mathbf{x})\widehat{h_j^{\mathbf{B}}}(\mathbf{x}) \quad a.e.; \quad \ell = 1, \cdots, n.$$

Setting $q_{\ell,j}(\mathbf{x}) := p_{\ell,j}(\mathbf{B}^T\mathbf{x})$ and applying (3) to u_j and h_j, the assertion follows. $\square$

The $\mathbf{D}\mathbb{Z}^d$–periodic matrix

$$\mathbf{Q}^{\mathbf{D}}(\mathbf{x}) := \Big(q_{\ell,j}(\mathbf{x})\Big)_{\ell,j=1}^{n,m}$$

will be called a *transition matrix* from the sequence $T^{\mathbf{B}}(\mathbf{h})$ to the sequence $T^{\mathbf{B}}(\mathbf{u})$. If $\mathbf{h}$ is a basis generator of $S^{\mathbf{B}}(\mathbf{h})$, then $Q^{\mathbf{D}}(\mathbf{x})$ is unique (up to a set of measure 0).

Lemma 4. *Let* $\mathbf{B} \in \mathbb{R}^{d\times d}$ *be nonsingular and* $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$. *Then* $T^{\mathbf{B}}(\mathbf{u})$ *is an orthonormal sequence in* $L^2(\mathbb{R}^d)$ *if and only if* $G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x}) = |b|\mathbf{I}$ *a.e. In particular if* $n = 1$, *then* $T^{\mathbf{B}}(u)$ *is an orthonormal sequence in* $L^2(\mathbb{R}^d)$ *if and only if*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{u}(\mathbf{x} + \mathbf{D}\mathbf{k})|^2 = |b|.$$

Proof. From (3) we deduce that

$$G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x}) = |b|^{-1}G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{D}\mathbf{x}), \tag{4}$$

and the assertion follows from, e.g. [8, Lemma D]. $\square$

Lemma 5. *Let* $\mathbf{B} \in \mathbb{R}^{d\times d}$ *be nonsingular,* $\mathbf{D} := (\mathbf{B}^{-1})^T$, *and assume that* $T^{\mathbf{B}}(u)$ *and* $T^{\mathbf{B}}(v_1, \cdots, v_m)$ *are orthonormal sequences in* $L^2(\mathbb{R}^d)$, *and that there are* $\mathbf{D}\mathbb{Z}^d$*–periodic functions* p_ℓ *such that*

$$\widehat{u}(\mathbf{x}) = \sum_{\ell=1}^{m} p_\ell(\mathbf{x})\widehat{v_\ell}(\mathbf{x}) \quad a.e.$$

Then

$$\sum_{\ell=1}^{m} |p_\ell(\mathbf{x})|^2 = 1 \quad a.e.$$

Proof. The hypotheses imply that

$$\widehat{u}(\mathbf{Dx}) = \sum_{\ell=1}^{m} p_\ell(\mathbf{Dx})\widehat{v_\ell}(\mathbf{Dx}) \quad a.e.$$

Setting $q_\ell(\mathbf{x}) := p_\ell(\mathbf{Dx})$ and applying (3) to u and the functions v_ℓ we see that

$$\widehat{u}^{\mathbf{B}}(\mathbf{x}) = \sum_{\ell=1}^{m} q_\ell(\mathbf{x})\widehat{v_\ell}^{\mathbf{B}}(\mathbf{x}) \quad a.e.$$

Since $T(u^{\mathbf{B}})$ and $T(v_1^{\mathbf{B}}, \cdots, v_m^{\mathbf{B}})$ are orthonormal sequences in $L^2(\mathbb{R}^d)$, and the functions $q_\ell(\mathbf{x})$ are $\mathbb{Z}^d$–periodic, the assertion follows by an application of [8, Lemma E]. $\square$

Recall that if $\mathbf{u}$ is a finite set of functions such that $T(\mathbf{u})$ is a Riesz basis, then $G_{\mathbf{u}}$ is positive definite for almost every $\mathbf{x} \in \mathbb{T}^d$. Thus the square root of $G_{\mathbf{u}}$ (i.e. the unique positive definite matrix $H_{\mathbf{u}}$ such that $H_{\mathbf{u}}^2 = G_{\mathbf{u}}$) exists for almost every $\mathbf{x} \in \mathbb{T}^d$. Thus, (4) implies that also the square root of $G_{\mathbf{u}}^{\mathbf{D}}$ exists.

Proposition 6. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular, and assume that $\mathbf{u} := \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$ is such that $T^{\mathbf{B}}(\mathbf{u})$ is a Riesz sequence. Let*

$$\mathbf{R}^{\mathbf{D}}(\mathbf{x}) := \left[G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x})\right]^{-1/2}$$

and

$$(\widehat{h_1}(\mathbf{x}), \cdots, \widehat{h_m}(\mathbf{x}))^T := |b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{x})(\widehat{u_1}(\mathbf{x}), \cdots, \widehat{u_m}(\mathbf{x}))^T.$$

Then $T^{\mathbf{B}}(\mathbf{h})$ is an orthonormal basis of $S^{\mathbf{B}}(\mathbf{u})$.

Proof. Lemma 1 implies that $T(\mathbf{u}^{\mathbf{B}})$ is a Riesz sequence. The hypotheses and (3) imply that

$$(\widehat{h_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{h_m^{\mathbf{B}}}(\mathbf{x}))^T = |b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{Dx})(\widehat{u}_1^{\mathbf{B}}(\mathbf{x}), \cdots, \widehat{u}_m^{\mathbf{B}}(\mathbf{x}))^T.$$

But (4) implies that

$$|b|^{1/2}\mathbf{R}^{\mathbf{D}}(\mathbf{Dx}) = \left[G_{\mathbf{u}^{\mathbf{B}}}(\mathbf{x})\right]^{-1/2}.$$

Thus, [8, Proposition G] implies that $T(\mathbf{h}^{\mathbf{B}})$ is an orthonormal basis of $S(\mathbf{h}^{\mathbf{B}})$, and the assertion follows from Lemma 1. $\square$

Theorem 7. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be nonsingular, let $\mathbf{D} := (\mathbf{B}^{-1})^T$, assume that $T^{\mathbf{B}}(\mathbf{h})$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$, that $\mathbf{u}$ is a set of functions such that $S^{\mathbf{B}}(\mathbf{u}) \subset S^{\mathbf{B}}(\mathbf{h})$, and let $\mathbf{Q}^{\mathbf{D}}(\mathbf{x})$ denote the transition matrix from $T^{\mathbf{B}}(\mathbf{h})$ to $T^{\mathbf{B}}(\mathbf{u})$. Then*

$$G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x}) = |b|\mathbf{Q}^{\mathbf{D}}(\mathbf{Dx})(\mathbf{Q}^{\mathbf{D}}(\mathbf{Dx}))^* \quad a.e., \tag{5}$$

and the following statements are equivalent:

(a) *$T^{\mathbf{B}}(\mathbf{u})$ is a Riesz basis of $S^{\mathbf{B}}(\mathbf{h})$ with bounds $0 < A \le B$.*

(b) *$r = m$ and for almost every $\mathbf{x} \in \mathbb{T}^d$*

$$||G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x})|| \le |b|B \quad and \quad ||\left(G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{x})\right)^{-1}|| \le |b|^{-1}A^{-1}.$$

(c) *$r = m$ and for almost every $\mathbf{x} \in \mathbb{T}^d$*

$$||\mathbf{Q}^{\mathbf{D}}(\mathbf{x})|| \le B^{1/2} \quad and \quad ||\left(\mathbf{Q}^{\mathbf{D}}(\mathbf{x})\right)^{-1}|| \le A^{-1/2}.$$

In particular, $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal basis of $S^{\mathbf{B}}(\mathbf{h})$ if and only if $r = m$ and $G_{\boldsymbol{\psi}}^{\mathbf{D}}(\mathbf{x}) = |b|\mathbf{I}$ for almost every $\mathbf{x} \in \mathbb{T}^d$, or, equivalently, if and only if $r = m$ and $\mathbf{Q}^{\mathbf{D}}(\mathbf{x})$ is a unitary matrix for almost every $\mathbf{x} \in \mathbb{T}^d$.

Proof. The hypotheses imply that $T(\mathbf{h^B})$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$ and that $S(\mathbf{u^B}) \subset S(\mathbf{h^B})$. If $Q(\mathbf{x})$ denotes the transition matrix from $T(\mathbf{h^B})$ to $T(\mathbf{u^B})$ then [8, Theorem 5] implies that

$$G_{\mathbf{u^B}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\,(\mathbf{Q}(\mathbf{x}))^* \quad a.e.$$

By definition,

$$(\widehat{u_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{u^{\mathbf{B}}}_m(\mathbf{x}))^T = \mathbf{Q}(\mathbf{x})(\widehat{h_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{h_m^{\mathbf{B}}}(\mathbf{x}))^T$$

or, from (3),

$$(\widehat{u_1}(\mathbf{Dx}), \cdots, \widehat{u_m}(\mathbf{Dx}))^T = \mathbf{Q}(\mathbf{x})(\widehat{h_1}(\mathbf{Dx}), \cdots, \widehat{h_m}(\mathbf{Dx}))^T.$$

This implies that

$$\mathbf{Q}(\mathbf{x}) = Q^{\mathbf{D}}(\mathbf{Dx}) \tag{6}$$

and (5) follows from (4) and (6).

Assume now that (a) holds; then $T(\mathbf{u^B})$ is a Riesz basis of $S(\mathbf{u^B})$ with bounds $0 < A \le B$, and [8, Theorem 5] implies that $r = m$ and for almost every $x \in T^d$

$$||G_{\mathbf{u^B}}(\mathbf{x})|| \le B \quad \text{and} \quad ||\,(G_{\mathbf{u^B}}(\mathbf{x}))^{-1}\,|| \le A^{-1}, \tag{7}$$

and (b) follows from (4).

If (b) is satisfied, then (4) implies (7), and (c) follows from [8, Theorem 5] and (6).

Finally, if (c) is satisfied then (6) implies that $||Q(\mathbf{x})|| \le \mathbf{B}^{1/2}$ and $||(Q(\mathbf{x}))^{-1}|| \le A^{-1/2}$; thus [8, Theorem 5] implies that $T(u^{\mathbf{B}})$ is a Riesz basis of $S(\mathbf{h^B})$ with bounds A and B, and (a) follows from Lemma 1.

Let us now prove the last paragraph in the statement of the theorem: $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal basis of $S(\mathbf{h^B})$ if and only if $T(\mathbf{u^B})$ is an orthonormal basis of $S(\mathbf{h^B})$, and [8, Theorem 5] and (4) imply that this is equivalent to $G_{\mathbf{u}}^{\mathbf{D}}(\mathbf{Dx}) = |b|\mathbf{I}$. Finally, (4) and (5) imply that $T(\mathbf{u^B})$ is an orthonormal basis of $S(\mathbf{h^B})$ if and only if $Q^{\mathbf{D}}(\mathbf{x})$ is unitary. □

3. Wavelets

$\mathbf{A} \in \mathbb{R}^{d\times d}$ is called a dilation matrix preserving the lattice $\mathbb{Z}^d$ if $\mathbf{A}\mathbb{Z}^d \subset \mathbb{Z}^d$ and all its eigenvalues have modulus greater than 1. These conditions imply that $\mathbf{A} \in \mathbb{Z}^{d\times d}$, and that if $a := \det \mathbf{A}$ then $|a|$ is an integer larger than 1 (cf. Madych [4]).

Assume that $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$. A coset of $\mathbf{A}\mathbb{Z}^d$ is a set of the form

$$\mathbf{j} + \mathbf{A}\mathbb{Z}^d = \{\mathbf{j} + \mathbf{Ar}; \mathbf{r} \in \mathbb{Z}^d\},$$

where $\mathbf{j} \in \mathbb{Z}^d$. An element of a coset is called a *representative* of the coset. Any pair of cosets are either identical or disjoint, and the union of all disjoint cosets equals $\mathbb{Z}^d$. There are exactly $|a|$ disjoint cosets. (cf. Wojtaszczyk [6]). The collection of all disjoint cosets is denoted by $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. A set $\mathbf{J} \subset \mathbb{Z}^d$ is said to be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$ if it contains exactly $|a|$ elements and

$$\bigcup_{\mathbf{j}\in\mathbf{J}} (\mathbf{j} + \mathbf{A}\mathbb{Z}^d) = \mathbb{Z}^d.$$

Theorem 8. *Let $\mathbf{B} \in \mathbb{R}^{d\times d}$ be a nonsingular matrix, $\mathbf{u} = \{u_1, \cdots, u_n\} \subset L^2(\mathbb{R}^d)$, and assume that $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal sequence. Let $\mathbf{A}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, $a := \det \mathbf{A}$, $m := |a|n$, let $\mathbf{J}$ be a full collection of representatives of $\mathbb{Z}^d/\mathbf{A}\mathbb{Z}^d$. For $x > 0$ define $I(x) := [1, x] \cap \mathbb{Z}$, and let*

$$p = (p_1, p_2) : I(m) \longrightarrow I(n) \times \mathbf{J}$$

be a bijection. If

$$v_\ell(t) := |a|^{1/2} u_{p_1(\ell)}(\mathbf{A}t + \mathbf{B}p_2(\ell))$$

and $\mathbf{v} := \{v_1, \cdots, v_m\}$, then $T^{\mathbf{B}}(\mathbf{v})$ is an orthonormal basis of $S^{\mathbf{B}}(\mathbf{A}; \mathbf{u})$, and every Riesz basis generator of $S^{\mathbf{B}}(\mathbf{A}; \mathbf{u})$ has exactly m functions.

Proof. The hypotheses imply that $T(u^{\mathbf{B}})$ is an orthonormal basis of $S(u^{\mathbf{B}})$, and from [8, Theorem 3] we conclude that if

$$w_\ell(t) := |a|^{1/2} u^{\mathbf{B}}_{p_1(\ell)}(\mathbf{A}t + p_2(\ell))$$

and $\mathbf{W} := \{w_1, \cdots, w_m\}$, then $T(\mathbf{w})$ is an orthonormal basis of $S(\mathbf{A}; \mathbf{u}^{\mathbf{B}})$. Let $b := \det \mathbf{B}$ and

$$L : S^{\mathbf{B}}(A, u) \longrightarrow S(A; u^{\mathbf{B}}); \qquad Lf := f^{\mathbf{B}}.$$

Since $LT_{\mathbf{k}}v_\ell = T_{\mathbf{k}}w_\ell$, proceeding as in the proof of Lemma 1 we see that L is an isometry from $S(A, u^{\mathbf{B}})$ onto $S^{\mathbf{B}}(A; u)$, and the assertion follows. $\square$

Note. There is a typographical error in the statement of [8, Theorem 3]: The range of the function p described in that theorem is $I(n) \times \mathbf{J}$.

Let $\mathbf{A} \in \mathbb{R}^{d\times d}$ be a dilation matrix preserving the lattice $\mathbb{Z}^d$, and assume that $\mathbf{B} \in \mathbb{R}^{d\times d}$ is nonsingular. A *multiresolution analysis* (MRA) of multiplicity n in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$ and $\mathbf{B}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}$, $f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{At}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T^{\mathbf{B}}(\mathbf{u})$ is an orthonormal basis of V_0.

From Proposition 6 we deduce that the condition that $T^{\mathbf{B}}(\mathbf{u})$ be an orthonormal basis may be replaced by the condition that $T^{\mathbf{B}}(\mathbf{u})$ be a Riesz basis.

It follows from the definition of multiresolution analysis that there are $\mathbf{D}\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions u_ℓ satisfy the *scaling identity*

$$\widehat{u_\ell}(\mathbf{A}^T\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u_j}(\mathbf{x}), \quad j, \ell = 1, \cdots, n \quad a.e.,$$

The functions u_ℓ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

Assume that $\mathbf{A}$ is a dilation matrix preserving the lattice $\mathbb{Z}^d$ and that $\mathbf{B} \in \mathbb{Z}^{d\times d}$ is nonsingular. A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal or Riesz wavelet system if the affine sequence

$$\bigcup_{j\in\mathbb{Z}} T^{\mathbf{B}}(\mathbf{A}^j; \boldsymbol{\psi}) = \{D_j^{\mathbf{A}} T_{\mathbf{k}}^{\mathbf{B}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is respectively an orthonormal basis or a Riesz basis of $L^2(\mathbb{R}^d)$. If $d = 1$ we omit the word "system". If we need to emphasize the connection with the matrices $\mathbf{A}$ and $\mathbf{B}$ we will say that the wavelet system is *generated* by $\mathbf{A}$ and $\mathbf{B}$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be a Riesz wavelet system in $L^2(\mathbb{R}^d)$ generated by matrices $\mathbf{A}$ and $\mathbf{B}$; for $j \in \mathbb{Z}$ we define $P_j := S^{\mathbf{B}}(\mathbf{A}^j; \boldsymbol{\psi})$ and $V_j := \sum_{r<j} P_r$, i.e.,

$$V_j = \bigcup_{r<j} S^{\mathbf{B}}(\mathbf{A}^r; \boldsymbol{\psi}).$$

We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $M := \{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis. If this is the case, we also say that $\boldsymbol{\psi}$ is associated with M. Let W_j denote the orthogonal complement of V_j in V_{j+1}. Then $\boldsymbol{\psi}$ is an orthonormal wavelet system associated with M if and only if $P_j = W_j$ for every $j \in \mathbb{Z}$, and $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0. This implies that $\boldsymbol{\phi}$ is another orthonormal wavelet system associated with the same multiresolution analysis M if and only if $T^{\mathbf{B}}(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

Theorem 9. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n, generated by a dilation matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, and having scaling functions $u_1, \cdots, u_n$. Let $a := \det \mathbf{A}$, $m := |a|n$, $\mathbf{D} := (B^{-1})^T$, and let $\{v_1, \cdots, v_m\}$ be an orthonormal basis generator of V_1 (such as the one given in Theorem 8). The following propositions are equivalent:*

(a) *$\{w_1, \cdots w_{m-n}\}$ is an orthonormal wavelet system associated with M.*

(b) *There is an $m \times m$ matrix $\mathbf{Q}(\mathbf{x})$ of $\mathbf{D}\mathbb{Z}^d$–periodic and measurable functions, a.e. unitary on $\mathbf{D}T^d$, such that, if*

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})(\widehat{v_1}(\mathbf{x}), \cdots, \widehat{v_m}(\mathbf{x}))^T,$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n$$

and

$$y_{(\ell-1)|a|+k+1} = w_{(\ell-1)|a|+k-\ell+1}; \quad 1 \le \ell \le n, \quad 1 \le k \le |a| - 1.$$

Proof. Let $\mathbf{v} := \{v_1, \cdots, v_n\}$ be such that $T^{\mathbf{B}}(\mathbf{v})$ is an orthonormal basis of V_0, $v_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} v_\ell(\mathbf{B}\mathbf{t})$, $w_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} w_\ell(\mathbf{B}\mathbf{t})$,

$$U_j := \{f : f(\mathbf{B}^{-1}\mathbf{t}) \in V_j\},$$

and let W_0^* be the orthogonal complement of U_0 in U_1. From Lemma 1 we deduce that $N := \{U_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{I}$, with scaling functions $\mathbf{v}^{\mathbf{B}} := \{v_1^{\mathbf{B}}, \cdots, v_n^{\mathbf{B}}\}$.

Clearly (a) is equivalent to $\mathbf{w}^{\mathbf{B}} := \{w_1^{\mathbf{B}}, \cdots, w_{m-n}^{\mathbf{B}}\}$ being an orthonormal wavelet system associated with N. On the other hand, we see from (3) that (b) is equivalent to the existence of an $m \times m$ matrix $R(\mathbf{x})$ (i.e., $Q(\mathbf{D}^{-1})(\mathbf{x})$) of $\mathbb{Z}^d$–periodic and measurable functions, and a.e. unitary on $\mathbb{T}^d$, such that if

$$(\widehat{y_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{y_m^{\mathbf{B}}}(\mathbf{x}))^T := R(\mathbf{x})(\widehat{v_1^{\mathbf{B}}}(\mathbf{x}), \cdots, \widehat{v_m^{\mathbf{B}}}(\mathbf{x}))^T,$$

then

$$y^{\mathbf{B}}_{(\ell-1)|a|+1} = u^{\mathbf{B}}_{\ell}, \qquad 1 \le \ell \le n$$

and

$$y^{\mathbf{B}}_{(\ell-1)|a|+k+1} = w^{\mathbf{B}}_{(\ell-1)|a|+k-\ell+1}; \qquad 1 \le \ell \le n, 1 \le k \le |a|-1,$$

whence the assertion follows by an application of [8, Theorem 8]. □

Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$. By orthogonality we know that

$$V_1 = S^{\mathbf{B}}(A, u_1) \oplus S^{\mathbf{B}}(A, u_2) \oplus \cdots \oplus S^{\mathbf{B}}(A, u_n).$$

Theorem 8 implies that there are functions $v_{\ell,k}$ such that

$$\{v_{\ell,1}, \cdots, v_{\ell,|a|}\} \tag{8}$$

is an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u_\ell)$. It follows that

$$\{v_{\ell,k}; 1 \le \ell \le n,\ 1 \le k \le |a|\}$$

is an orthonormal basis generator of V_1.

For $k > 1$ let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{k}}$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries.

Theorem 10. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$. Let $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $m := |a|n$, $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^{|a|}$, and for $1 \le \ell \le n$ let (8) be an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u_\ell)$. Let*

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} c_{\ell,j}(\mathbf{x}) \widehat{v_{\ell,j}}(\mathbf{x}), \tag{9}$$

and define

$$w_{\ell,j}(\mathbf{t}) := |b|^{1/2} v_{\ell,j}(\mathbf{Bt}); \quad 1 \le \ell \le n, \qquad b_{\ell,j}(\mathbf{t}) := c_{\ell,j}(\mathbf{Bt}),$$

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{iArg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$

$$\mathbf{W}(\mathbf{x}) := (w_{1,1}(\mathbf{x}), \cdots, w_{1,|a|}(\mathbf{x}), \cdots, w_{n,1}(\mathbf{x}), \cdots, w_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \mathrm{diag}\,\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|\mathrm{a}|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^*/\mathbf{q}_\ell(\mathbf{x})^*\mathbf{q}_\ell(\mathbf{x})}\right].$$

Let

$$\mathbf{Q}(\mathbf{x}) = \Big(q_{\ell,k}(\mathbf{x})\Big)_{\ell,k=1}^{m}$$

be the $m \times m$ block diagonal matrix

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_n(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\mathbf{W}(\mathbf{x})$$

and

$$z_\ell(\mathbf{t}) := |b|^{-1/2} y_\ell(\mathbf{B}^{-1}\mathbf{t}), \quad 1 \le \ell \le m,$$

then

$$z_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n, \tag{10}$$

and

$$\{z_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\} \tag{11}$$

is an orthonormal wavelet system associated with M.

Proof. Let $u_\ell^{\mathbf{B}}(\mathbf{t}) := |b|^{1/2} u_\ell(\mathbf{t})$ and

$$U_j := \{f : f(\mathbf{B}^{-1}\mathbf{t}) \in V_j\}.$$

Then $N := \{U_j; j \in \mathbb{Z}\}$ is a multiresolution analysis of multiplicity n generated by $\mathbf{A}$ and $\mathbf{I}$, with scaling functions $\mathbf{u}^{\mathbf{B}} := \{u_1^{\mathbf{B}}, \cdots, u_n^{\mathbf{B}}\}$. Moreover, the hypotheses imply that $\{w_{\ell,j}; 1 \le \ell \le |a|\}$ is an orthonormal basis generator of $S(\mathbf{A}, u_\ell^{\mathbf{B}})$, whereas (9) implies that

$$\widehat{u_\ell^{\mathbf{B}}}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x}) \widehat{w_j}(\mathbf{x}).$$

Applying [8, Theorem 9] we conclude that

$$y_{(\ell-1)|a|+1} = u_\ell^{\mathbf{B}}; \quad 1 \le \ell \le n, \tag{12}$$

and that

$$\{y_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\}$$

is an orthonormal wavelet system associated with N. The definitions of U_0 and $\mathbf{u}^{\mathbf{B}}$ together with (12) imply (10). Finally, if $W_0^{\mathbf{B}}$ denotes the orthogonal complement of U_0 in U_1 and W_0 denotes the orthogonal complement of V_0 in V_1, it is clear that

$$W_0^{\mathbf{B}} = \{f : f(\mathbf{B}^{-1}\cdot) \in W_0\},$$

and we conclude that (11) is an orthonormal wavelet system associated with M. $\square$

Corollary 11. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity 1, generated by a matrix $\mathbf{A}$ that preserves the lattice $\mathbb{Z}^d$ and a nonsingular matrix $\mathbf{B}$, with scaling functions u. Let $a := \det \mathbf{A}$, $b := \det \mathbf{B}$, $m := |a|$, and let $\{v_k; 1 \le k \le m\}$ be an orthonormal basis generator of $S^{\mathbf{B}}(\mathbf{A}, u)$. Let*

$$\widehat{u}(\mathbf{x}) = \sum_{j=1}^{m} c_j(\mathbf{x}) \widehat{v_j}(\mathbf{x}),$$

and define

$$w_j(\mathbf{t}) := |b|^{1/2} v_j(\mathbf{B}\mathbf{t}), \quad \delta(\mathbf{x}) := e^{i\,Arg\, c_1(\mathbf{B}\mathbf{x})},$$
$$\mathbf{q}(\mathbf{x}) := (c_1(\mathbf{B}\mathbf{x}), \cdots, b_{m-1}(\mathbf{B}\mathbf{x}), b_m(\mathbf{B}\mathbf{x}) + \delta(\mathbf{x}))^T,$$
$$\mathbf{W}(\mathbf{x}) := (w_1(\mathbf{x}), \cdots, w_m(\mathbf{x}))^T,$$

and

$$\mathbf{Q}(\mathbf{x}) := \mathrm{diag}\,\{-\delta(\mathbf{x}), 1, \cdots, 1\}_{\mathrm{m}} \left[\overline{\mathbf{I} - 2\mathbf{q}(\mathbf{x})\mathbf{q}(\mathbf{x})^*/\mathbf{q}(\mathbf{x})^*\mathbf{q}(\mathbf{x})}\right].$$

If

$$(\widehat{y_1}(\mathbf{x}), \cdots, \widehat{y_m}(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\mathbf{W}(\mathbf{x})$$

and

$$z_k(\mathbf{t}) := |b|^{-1/2} y_k(\mathbf{B}^{-1}\mathbf{t}), \quad 1 \le k \le m,$$

then $z_1 = u$, *and*

$$\{z_k; 2 \le k \le m\}$$

is an orthonormal wavelet system associated with M.

Corollary 12. *Let* M *be a multiresolution analysis of multiplicity* n, *generated by a matrix* $\mathbf{A}$ *that preserves the lattice* $\mathbb{Z}^d$ *and a nonsingular matrix* $\mathbf{B}$. *Then there is an orthogonal wanelet system of* $(|a|-1)n$ *functions associated with* M.

Theorem 13. *Let* M *be a multiresolution analysis of order* n *generated by a matrix* $\mathbf{A}$ *that preserves the lattice* $\mathbb{Z}^d$ *and a nonsingular matrix* $\mathbf{B}$, *let* $k := (|a|-1)n$, *and assume that* $\mathbf{w} := \{w_1, \cdots .w_k\}$ *is an orthonormal wavelet system associated with* M. *Let* $\mathbf{y} := \{y_1, \cdots, y_r\} \subset L^2(\mathbb{R}^d)$. *Then* $\mathbf{y}$ *is an orthonormal wavelet system associated with* M *if and only if* $r = k$ *and there is a* $k \times k$ *matrix* $P(\mathbf{x})$ *of* $\mathbf{D}\mathbb{Z}^d$*–periodic and measurable functions, a.e. unitary on* $\mathbf{D}\mathbb{T}^d$, *such that*

$$(y_1(\mathbf{x}), \cdots, y_k(\mathbf{x}))^T = P(\mathbf{x})(w_1(\mathbf{x}), \cdots, w_k(\mathbf{x}))^T.$$

Proof. $\mathbf{y}$ is an orthonormal wavelet system associated with M if and only if $T^{\mathbf{B}}(\mathbf{y})$ is an orthonormal basis of W_0, and the assertion follows from Theorem 7. $\square$

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Uniform Real and Fuzzy Estimates for Distances between Wavelet type Operators at real and fuzzy setting

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Abstract

The basic fuzzy wavelet type operators A_k, B_k, C_k, D_k, $k \in \mathbb{Z}$ were studied in [3], [5], for their pointwise and uniform convergence with rates to the fuzzy unit operator. Also they were studied in [6], in terms of estimating their fuzzy differences and giving their pointwise convergence with rates to zero.

For prior related and similar study of convergence to the unit of real analogs of these wavelet type operators see [1], section II.

Here in Section 1 we present the complete study of finding uniform estimates for the distances between the real Wavelet type operators A_k, B_k, C_k, D_k, $k \in \mathbb{Z}$.

Their differences converge to zero with rates. This is done via elegant tight Jackson type inequalities involving the modulus of continuity of the higher order derivative of the engaged real function. Based on these real analysis results in Section 2 we establish the corresponding fuzzy results regarding uniform estimates for the fuzzy differences between the fuzzy wavelet type operators. These fuzzy diferences converge to zero with rates give via fuzzy Jackson type tight inequalities. The last inequalities involve the fuzzy modulus of continuity of the higher order fuzzy derivative of the engaged fuzzy function.

The defining all these operators real scaling function is not assumed to be orthogonal and is of compact support.

Another motivation for this work is [10].

1 Estimates for Distances of Real Wavelet type Operators

The real wavelet type operators A_k, B_k, C_k, D_k, $k \in \mathbb{Z}$ we study here converge to the unit operator I, and in that respect were studied extensively in [1], section II.

We need

Definition. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function. We define the first modulus of continuity of f by

$$\omega_1(f, \delta) = \sup_{\substack{x,y \\ |x-y| \le \delta}} |f(x) - f(y)|, \delta > 0.$$

We give

Theorem 1. Let $f \in C^N(\mathbb{R})$, $N \ge 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x - j) = 1$, all $x \in \mathbb{R}$. Suppose $\varphi \ge 0$. Put

$$(B_k f)(x) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right),$$

$$(D_k f)(x) = \sum_{j=-\infty}^{\infty} \delta_{kj}(f) \varphi\left(2^k x - j\right),$$

where

$$\delta_{kj}(f) = \sum_{r=0}^{n} w_r f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right),$$

$n \in \mathbb{N}$, $w_r \ge 0$, $\sum_{r=0}^{n} w_r = 1$. Then

$$E_{1k}(x) = \left| (D_k f)(x) - (B_k f)(x) - \sum_{i=1}^{N} \frac{\left(B_k f^{(i)}\right)(x)}{2^{ki} n^i i!} \left(\sum_{r=1}^{n} w_r r^i\right) \right|$$
$$\le \frac{\sum_{r=1}^{n} w_r r^N \omega_1\left(f^{(N)}, \frac{r}{2^k n}\right)}{2^{kN} n^N N!}. \qquad (1)$$

Remark. **(i)** Given that $f^{(N)}$ is continuous and bounded or uniformly continuous we have that $\omega_1\left(f^{(N)}, \frac{r}{2^k n}\right) < \infty$, and $E_{1k}(x) \to 0$, $x \in \mathbb{R}$, as $k \to \infty$.

(ii) One also has

$$\|E_{1k}\|_{\infty} = \left\| D_k f - B_k f - \sum_{i=1}^{N} \frac{B_k f^{(i)}}{2^{ki} n^i i!} \left(\sum_{r=1}^{n} w_r r^i \right) \right\|_{\infty} \leq \frac{\sum_{r=1}^{n} w_r r^N \omega_1 \left(f^{(N)}, \frac{r}{2^k n} \right)}{2^{kN} n^N N!}.$$

Under the assumptions of (i) we get also $\|E_{1k}\|_{\infty} \to 0$, as $k \to \infty$.

(iii) By (1) we get

$$|(D_k f)(x) - (B_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left| B_k f^{(i)}(x) \right|}{2^{ki} n^i i!} \left(\sum_{r=1}^{n} w_r r^i \right) + \frac{\sum_{r=1}^{n} w_r r^N \omega_1 \left(f^{(N)}, \frac{r}{2^k n} \right)}{2^{kN} n^N N!}. \quad (2)$$

Given that $\left\| f^{(i)} \right\|_{\infty} < \infty$, for $i = 1, \ldots, N$, we obtain $\left| B_k f^{(i)}(x) \right| \leq \left\| f^{(i)} \right\|_{\infty}$, and

$$|(B_k f)(x) - (D_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left\| f^{(i)} \right\|_{\infty}}{2^{ki} n^i i!} \left(\sum_{r=1}^{n} w_r r^i \right) + \frac{\sum_{r=1}^{n} w_r r^N \omega_1 \left(f^{(N)}, \frac{r}{2^k n} \right)}{2^{kN} n^N N!}. \quad (3)$$

Clearly we get

$$|(B_k f)(x) - (D_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left\| f^{(i)} \right\|_{\infty}}{2^{ki} i!} + \frac{\omega_1 \left(f^{(N)}, \frac{1}{2^k} \right)}{2^{kN} N!} =: T_1$$

and

$$\|B_k f - D_k f\|_{\infty} \leq T_1. \quad (4)$$

So as $k \to \infty$, we have $\|B_k f - D_k f\|_{\infty} \to 0$.

Proof of Theorem1. Because $f \in C^N(\mathbb{R}), N \geq 1$ we have

$$\sum_{r=0}^{n} w_r f\left(\frac{j}{2^k} + \frac{r}{2^k n} \right) = f\left(\frac{j}{2^k} \right) + \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k} \right)}{i!} \sum_{r=0}^{n} w_r \left(\frac{r^i}{2^{ki} n^i} \right) + \sum_{r=0}^{n} w_r \int_{j/2^k}^{(j/2^k)+\frac{r}{2^k n}} \left(f^{(N)}(t) - f^{(N)}\left(j/2^k \right) \right) \frac{\left(\frac{j}{2^k} + \frac{r}{2^k n} - t \right)^{N-1}}{(N-1)!} dt.$$

Hence we get

$$\sum_{j=-\infty}^{\infty}\delta_{kj}(f)\varphi\left(2^{k}x-j\right)=\sum_{j=-\infty}^{\infty}f\left(\frac{j}{2^{k}}\right)\varphi\left(2^{k}x-j\right)$$
$$+\sum_{i=1}^{N}\frac{\sum_{j=-\infty}^{\infty}f^{(i)}\left(\frac{j}{2^{k}}\right)\varphi\left(2^{k}x-j\right)}{2^{ki}i!n^{i}}\left(\sum_{r=0}^{n}w_{r}r^{i}\right)$$
$$+\sum_{r=0}^{n}w_{r}\sum_{j=-\infty}^{\infty}\varphi\left(2^{k}x-j\right)\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\left(f^{(N)}(t)-f^{(N)}\left(j/2^{k}\right)\right)\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt.$$

So, we observe that

$$\left(D_{k}f\right)(x)-\left(B_{k}f\right)(x)-\sum_{i=1}^{N}\frac{\left(B_{k}f^{(i)}\right)(x)}{2^{ki}n^{i}i!}\left(\sum_{r=0}^{n}w_{r}r^{i}\right)=\mathcal{R}_{1},$$

where

$$\mathcal{R}_{1}=\sum_{r=1}^{n}w_{r}\sum_{j=-\infty}^{\infty}\varphi\left(2^{k}x-j\right)\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\left(f^{(N)}(t)-f^{(N)}\left(j/2^{k}\right)\right)\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt.$$

Set

$$\Gamma_{jr}=\left|\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\left(f^{(N)}(t)-f^{(N)}\left(j/2^{k}\right)\right)\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt\right|.$$

So that

$$\left|\mathcal{R}_{1}\right|\leq\sum_{r=1}^{n}w_{r}\sum_{j=-\infty}^{\infty}\varphi\left(2^{k}x-j\right)\Gamma_{jr}.$$

Next we see that

$$\Gamma_{jr}\leq\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\left|f^{(N)}(t)-f^{(N)}\left(j/2^{k}\right)\right|\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt$$
$$\leq\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\omega_{1}\left(f^{(N)},\left|t-(j/2^{k})\right|\right)\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt$$
$$\leq\omega_{1}\left(f^{(N)},\frac{r}{2^{k}n}\right)\int_{j/2^{k}}^{(j/2^{k})+\frac{r}{2^{k}n}}\frac{\left(\frac{j}{2^{k}}+\frac{r}{2^{k}n}-t\right)^{N-1}}{(N-1)!}dt$$
$$=\omega_{1}\left(f^{(N)},\frac{r}{2^{k}n}\right)\frac{\left(\frac{r}{2^{k}n}\right)^{N}}{N!}.$$

That is

$$\Gamma_{jr}\leq\omega_{1}\left(f^{(N)},\frac{r}{2^{k}n}\right)\frac{\left(\frac{r}{2^{k}n}\right)^{N}}{N!}.$$

So we have found that

$$|\mathcal{R}_1| \leq \frac{\sum_{r=1}^{n} w_r r^N \omega_1\left(f^{(N)}, \frac{r}{2^k n}\right)}{2^{kN} n^N N!}.$$

The proof of the theorem is complete. ∎

We continue with

Theorem 2. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x-j) = 1$, all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$. Put

$$(B_k f)(x) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right),$$

$$(C_k f)(x) = \sum_{j=-\infty}^{\infty} \gamma_{kj}(f) \varphi\left(2^k x - j\right),$$

where

$$\gamma_{kj}(f) = 2^k \int_{2^{-k} j}^{2^{-k}(j+1)} f(t)\, dt = 2^k \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt.$$

Then

$$E_{2k}(x) = \left| (C_k f)(x) - (B_k f)(x) - \sum_{i=1}^{N} \frac{\left(B_k f^{(i)}\right)(x)}{2^{ki}(i+1)!} \right| \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}. \tag{5}$$

Remark. **(i)** Given that $f^{(N)}$ is continuous and bounded or uniformly continuous we have that $\omega_1\left(f^{(N)}, \frac{1}{2^k}\right) < \infty$, and $E_{2k}(x) \to 0$, $x \in \mathbb{R}$, as $k \to \infty$.
(ii) One also has

$$\|E_{2k}\|_\infty = \left\| C_k f - B_k f - \sum_{i=1}^{N} \frac{B_k f^{(i)}}{2^{ki}(i+1)!} \right\|_\infty \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}.$$

Under the assumptions of (i) we get also $\|E_{2k}\|_\infty \to 0$, as $k \to \infty$.
(iii) By (5) we get

$$|(B_k f)(x) - (C_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left|\left(B_k f^{(i)}\right)(x)\right|}{2^{ki}(i+1)!} + \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}. \tag{6}$$

Given that $\left\| f^{(i)} \right\|_\infty < \infty$, for $i = 1, \ldots, N$, we obtain

$$|(B_k f)(x) - (C_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left\| f^{(i)} \right\|_\infty}{2^{ki}(i+1)!} + \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!} =: T_2,$$

and

$$\|B_k f - C_k f\|_\infty \leq T_2. \tag{7}$$

So as $k \to \infty$, we get $\|B_k f - C_k f\|_\infty \to 0$.

Proof of Theorem2. Because $f \in C^N(\mathbb{R})$, $N \geq 1$ we have

$$f\left(t + \frac{j}{2^k}\right) = f\left(\frac{j}{2^k}\right) + \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k}\right)}{i!} t^i$$
$$+ \int_{j/2^k}^{t+(j/2^k)} \left(f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds.$$

Hence we get

$$\gamma_{kj}(f) = 2^k \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt = f\left(\frac{j}{2^k}\right) + \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k}\right)}{2^{ki}(i+1)!}$$
$$+ 2^k \int_0^{2^{-k}} \left(\int_{j/2^k}^{t+(j/2^k)} \left(f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds\right) dt.$$

Hence we get

$$\sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\, \varphi\left(2^k x - j\right) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right) + \sum_{i=1}^{N} \sum_{j=-\infty}^{\infty} \frac{f^{(i)}\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right)}{2^{ki}(i+1)!}$$
$$+ \sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right) 2^k \int_0^{2^{-k}} \left(\int_{j/2^k}^{t+(j/2^k)} \left(f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds\right) dt.$$

So, we see that

$$(C_k(f))(x) - (B_k(f))(x) - \sum_{i=1}^{N} \frac{\left(B_k\left(f^{(i)}\right)\right)(x)}{2^{ki}(i+1)!} = \mathcal{R}_2,$$

where

$$\mathcal{R}_2 = \sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right) 2^k \int_0^{2^{-k}} \left(\int_{j/2^k}^{t+(j/2^k)} \left(f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds\right) dt.$$

Set

$$\Gamma_j(t) = \left| \int_{j/2^k}^{t+(j/2^k)} \left(f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds \right|.$$

So that

$$|\mathcal{R}_2| \leq \sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right) 2^k \int_0^{2^{-k}} \Gamma_j(t)\, dt.$$

Next we observe that

$$\begin{aligned}
\Gamma_j(t) &\leq \int_{j/2^k}^{t+(j/2^k)} \left| f^{(N)}(s) - f^{(N)}\left(j/2^k\right)\right| \frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!} ds \\
&\leq \int_{j/2^k}^{t+(j/2^k)} \omega_1\left(f^{(N)}, \left|s-(j/2^k)\right|\right) \frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!} ds \\
&\leq \omega_1\left(f^{(N)}, t\right) \int_{j/2^k}^{t+(j/2^k)} \frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!} ds \\
&= \omega_1\left(f^{(N)}, t\right) \frac{t^N}{N!}.
\end{aligned}$$

I.e. we get

$$\Gamma_j(t) \leq \omega_1\left(f^{(N)}, t\right) \frac{t^N}{N!},$$

and

$$\begin{aligned}
2^k \int_0^{2^{-k}} \Gamma_j(t)\, dt &\leq 2^k \int_0^{2^{-k}} \omega_1\left(f^{(N)}, t\right) \frac{t^N}{N!} dt \\
&\leq \frac{2^k}{N!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) \int_0^{2^{-k}} t^N dt \\
&= \frac{2^k}{(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) 2^{-k(N+1)} \\
&= \frac{1}{2^{kN}(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right).
\end{aligned}$$

That is we obtain

$$2^k \int_0^{2^{-k}} \Gamma_j(t)\, dt \leq \frac{1}{2^{kN}(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right).$$

It is clear that

$$|\mathcal{R}_2| \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}.$$

The proof of the theorem is now finished. ■

We continue with

Theorem 3. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x-j) = 1$, all

$x \in \mathbb{R}$. Suppose $\varphi \geq 0$. Put

$$(C_k f)(x) = \sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\, \varphi\left(2^k x - j\right),$$

where

$$\gamma_{kj}(f) = 2^k \int_{2^{-k}j}^{2^{-k}(j+1)} f(t)\, dt = 2^k \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt,$$

and

$$(D_k f)(x) = \sum_{j=-\infty}^{\infty} \delta_{kj}(f)\, \varphi\left(2^k x - j\right),$$

where

$$\delta_{kj}(f) = \sum_{r=0}^{n} w_r f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right),$$

$n \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{n} w_r = 1$. Then

$$E_{3k}(x) = \left| (C_k f)(x) - (D_k f)(x) - \sum_{i=1}^{N} \frac{\left(D_k f^{(i)}\right)(x)}{2^{ki}(i+1)!} \left[\left(1 - \frac{r}{n}\right)^{i+1} - (-1)^{i+1} \left(\frac{r}{n}\right)^{i+1}\right] \right|$$
$$\leq \frac{\left(\sum_{r=0}^{n} w_r \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right]\right)}{2^{kN}(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right). \tag{8}$$

Remark (i) Given that $f^{(N)}$ is continuous and bounded or uniformly continuous we have that $\omega_1\left(f^{(N)}, \frac{1}{2^k}\right) < \infty$, and $E_{3k}(x) \to 0, x \in \mathbb{R}$, as $k \to \infty$.

(ii) One also has

$$\|E_{3k}\|_\infty = \left\| C_k f - D_k f - \sum_{i=1}^{N} \frac{D_k f^{(i)}}{2^{ki}(i+1)!} \left[\left(1 - \frac{r}{n}\right)^{i+1} - (-1)^{i+1} \left(\frac{r}{n}\right)^{i+1}\right] \right\|_\infty$$
$$\leq \frac{\left(\sum_{r=0}^{n} w_r \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right]\right)}{2^{kN}(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right).$$

Under the assumption of (i) we get also $\|E_{3k}\|_\infty \to 0$, as $k \to \infty$.

(iii) By (8) we get

$$|(C_k f)(x) - (D_k f)(x)| \leq \sum_{i=1}^{N} \frac{\left|\left(D_k f^{(i)}\right)(x)\right|}{2^{ki-1}(i+1)!}$$
$$+ \frac{\left(\sum_{r=0}^{n} w_r \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right]\right)}{2^{kN}(N+1)!} \omega_1\left(f^{(N)}, \frac{1}{2^k}\right). \tag{9}$$

Clearly then

$$\left|\left(C_k f\right)(x)-\left(D_k f\right)(x)\right| \leq \sum_{i=1}^N \frac{\left|\left(D_k f^{(i)}\right)(x)\right|}{2^{k i-1}(i+1)!}+\frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{k N-1}(N+1)!}. \tag{10}$$

Given that $\left\|f^{(i)}\right\|_{\infty}<\infty$, for $i=1, \ldots, N$, we get

$$\left|\left(D_k f^{(i)}\right)(x)\right| \leq\left\|f^{(i)}\right\|_{\infty},$$

and

$$\left|\left(C_k f\right)(x)-\left(D_k f\right)(x)\right| \leq \sum_{i=1}^N \frac{\left\|f^{(i)}\right\|_{\infty}}{2^{k i-1}(i+1)!}+\frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{k N-1}(N+1)!}=: T_3 \tag{11}$$

and

$$\left\|\left(C_k f\right)(x)-\left(D_k f\right)(x)\right\|_{\infty} \leq T_3. \tag{12}$$

So as $k \rightarrow \infty$, we get $\left\|C_k f-D_k f\right\|_{\infty} \rightarrow 0$.

Proof of Theorem 3. Because $f \in C^N(\mathbb{R})$, $N \geq 1$ we have

$$\begin{aligned}
f\left(t+\frac{j}{2^k}\right)= & f\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)+\sum_{i=1}^N \frac{f^{(i)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)}{i!}\left(t-\frac{r}{2^k n}\right)^i \\
& +\int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)}\left(f^{(N)}(s)-f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)\right) \frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!} ds.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\gamma_{kj}(f)= & 2^k \int_0^{2^{-k}} f\left(t+\frac{j}{2^k}\right) dt=\sum_{r=0}^n w_r f\left(\frac{j}{2^k}+\frac{r}{2^k n}\right) \\
& +\sum_{i=1}^N \sum_{r=0}^n \frac{w_r f^{(i)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)}{i!} 2^k \int_0^{2^{-k}}\left(t-\frac{r}{2^k n}\right)^i dt \\
& +\sum_{r=0}^n w_r 2^k \int_0^{2^{-k}}\left(\int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)}\left(f^{(N)}(s)-f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)\right) \frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!} ds\right) dt.
\end{aligned}$$

That is we have

$$\sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\,\varphi\left(2^k x - j\right) = \sum_{j=-\infty}^{\infty} \delta_{kj}(f)\,\varphi\left(2^k x - j\right)$$
$$+\sum_{i=1}^{N} \frac{\sum_{j=-\infty}^{\infty} \delta_{kj}\left(f^{(i)}\right)\varphi\left(2^k x - j\right)}{2^{ki}(i+1)!}\left[\left(1-\frac{r}{n}\right)^{i+1} - (-1)^{i+1}\left(\frac{r}{n}\right)^{i+1}\right]$$
$$+\sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right)\sum_{r=0}^{n} w_r 2^k$$
$$\cdot\int_0^{2^{-k}}\left(\int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)}\left(f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)\right)\right.$$
$$\left.\frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!}ds\right)dt.$$

Consequently we get

$$(C_k f)(x) - (D_k f)(x) - \sum_{i=1}^{N}\frac{\left(D_k f^{(i)}\right)(x)}{2^{ki}(i+1)!}\left[\left(1-\frac{r}{n}\right)^{i+1} - (-1)^{i+1}\left(\frac{r}{n}\right)^{i+1}\right] = \mathcal{R}_3,$$

where

$$\mathcal{R}_3 = \sum_{j=-\infty}^{\infty}\varphi\left(2^k x - j\right)\sum_{r=0}^{n} w_r 2^k$$
$$\cdot\int_0^{2^{-k}}\left(\int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)}\left(f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)\right)\frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!}ds\right)dt.$$

Set

$$\Gamma_{jr}(t) = \left|\int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)}\left(f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)\right)\frac{\left(t+\frac{j}{2^k}-s\right)^{N-1}}{(N-1)!}ds\right|.$$

So that

$$|\mathcal{R}_3| \leq \sum_{j=-\infty}^{\infty}\varphi\left(2^k x - j\right)\sum_{r=0}^{n} w_r 2^k\int_0^{2^{-k}}\Gamma_{jr}(t)\,dt.$$

Next we observe that

(i) Case of $\frac{r}{2^k n} \leq t$. So we have

$$\begin{aligned}
\Gamma_{jr}(t) &\leq \int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)} \left| f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right| \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\
&\leq \int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)} \omega_1\left(f^{(N)}, \left|s - \frac{j}{2^k} - \frac{r}{2^k n}\right|\right) \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\
&\leq \omega_1\left(f^{(N)}, t - \frac{r}{2^k n}\right) \int_{(j/2^k)+\frac{r}{2^k n}}^{t+(j/2^k)} \frac{\left(t + \frac{j}{2^k} - s\right)^{N-1}}{(N-1)!} ds \\
&\leq \omega_1\left(f^{(N)}, t\right) \frac{\left(t - \frac{r}{2^k n}\right)^N}{N!} \\
&\leq \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) \frac{\left(t - \frac{r}{2^k n}\right)^N}{N!}.
\end{aligned}$$

So we obtain

$$\Gamma_{jr}(t) \leq \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) \frac{\left(t - \frac{r}{2^k n}\right)^N}{N!}.$$

(ii) Case of $\frac{r}{2^k n} \geq t$. We have

$$\begin{aligned}
\Gamma_{jr}(t) &= \left| \int_{t+(j/2^k)}^{(j/2^k)+\frac{r}{2^k n}} \left(f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right) \frac{\left(s - \left(t + \frac{j}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \right| \\
&\leq \int_{t+(j/2^k)}^{(j/2^k)+\frac{r}{2^k n}} \left| f^{(N)}(s) - f^{(N)}\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right| \frac{\left(s - \left(t + \frac{j}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\
&\leq \int_{t+(j/2^k)}^{(j/2^k)+\frac{r}{2^k n}} \omega_1\left(f^{(N)}, \left(\frac{j}{2^k} + \frac{r}{2^k n} - s\right)\right) \frac{\left(s - \left(t + \frac{j}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\
&\leq \omega_1\left(f^{(N)}, \frac{r}{2^k n} - t\right) \int_{t+(j/2^k)}^{(j/2^k)+\frac{r}{2^k n}} \frac{\left(s - \left(t + \frac{j}{2^k}\right)\right)^{N-1}}{(N-1)!} ds \\
&= \omega_1\left(f^{(N)}, \frac{r}{2^k n} - t\right) \frac{\left(\frac{r}{2^k n} - t\right)^N}{N!} \\
&\leq \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) \frac{\left(\frac{r}{2^k n} - t\right)^N}{N!}.
\end{aligned}$$

So we derive

$$\Gamma_{jr}(t) \leq \omega_1\left(f^{(N)}, \frac{1}{2^k}\right) \frac{\left(\frac{r}{2^k n} - t\right)^N}{N!}.$$

Therefore we have found

$$\Gamma_{jr}(t) \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{N!} \left| t - \frac{r}{2^k n}\right|^N.$$

Furthermore we see that

$$\begin{aligned}
\int_0^{2^{-k}} \Gamma_{jr}(t)\,dt &\leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{N!} \int_0^{2^{-k}} \left|t - \frac{r}{2^k n}\right|^N dt \\
&= \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{N!} \left[\int_0^{\frac{r}{2^k n}} \left(\frac{r}{2^k n} - t\right)^N dt + \int_{\frac{r}{2^k n}}^{2^{-k}} \left(t - \frac{r}{2^k n}\right)^N dt\right] \\
&= \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{(N+1)!} \left[\left(\frac{r}{2^k n}\right)^{N+1} + \left(\frac{1}{2^k} - \frac{r}{2^k n}\right)^{N+1}\right] \\
&= \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{(N+1)!} \frac{1}{2^{k(N+1)}} \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right].
\end{aligned}$$

Thus

$$2^k \int_0^{2^{-k}} \Gamma_{jr}(t)\,dt \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!} \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right].$$

Finally we derive that

$$|\mathcal{R}_3| \leq \left(\frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}\right) \sum_{r=0}^{n} w_r \left[\left(\frac{r}{n}\right)^{N+1} + \left(1 - \frac{r}{n}\right)^{N+1}\right],$$

proving the theorem. ∎

We continue with

Theorem 4. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, also $\left\|f^{(i)}\right\|_\infty < \infty$, $i = 1, \ldots, N$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x - j) = 1$ all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$ and φ is Lebesgue measurable (then $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$). Define

$$\varphi_{kj}(x) := 2^{k/2}\varphi\left(2^k x - j\right) \text{ all } k, j \in \mathbb{Z},$$

$$\langle f, \varphi_{kj}\rangle = \int_{-\infty}^{\infty} f(t)\,\varphi_{kj}(t)\,dt,$$

and

$$\begin{aligned}
(A_k f)(x) &= \sum_{j=-\infty}^{\infty} \langle f, \varphi_{kj}\rangle \varphi_{kj}(x) \\
&= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right) \varphi(u - j)\,du\right) \varphi\left(2^k x - j\right),
\end{aligned}$$

also define

$$(B_k f)(x) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right).$$

Then

$$\begin{aligned}\left|(A_k f)(x) - (B_k f)(x)\right| &\leq \|A_k f - B_k f\|_\infty \\ &\leq \sum_{i=1}^{N} \frac{\left\|f^{(i)}\right\|_\infty}{2^{ki} i!} a^i + \frac{a^N}{2^{kN} N!} \omega_1\left(f^{(N)}, \frac{a}{2^k}\right), \quad (13)\end{aligned}$$

$x \in \mathbb{R}$.

So as $k \to \infty$, we get $\|A_k f - B_k f\|_\infty \to 0$.

Proof. By $\sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right) = 1$ we have that $\int_{-\infty}^{\infty} \varphi(u-j)\, du = 1$.

Notice that

$$\begin{aligned}(A_k f)(x) - (B_k f)(x) &= \sum_{j=-\infty}^{\infty} \langle f, \varphi_{kj}\rangle \varphi_{kj}(x) - \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right) \\ &= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right) \varphi(u-j)\, du\right) \varphi\left(2^k x - j\right) \\ &\quad - \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right) \varphi\left(2^k x - j\right) \\ &= \sum_{j=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right) \varphi(u-j)\, du - f\left(\frac{j}{2^k}\right)\right] \varphi\left(2^k x - j\right) \\ &= \sum_{j=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right)\right) \varphi(u-j)\, du\right] \varphi\left(2^k x - j\right).\end{aligned}$$

That is

$$(A_k f)(x) - (B_k f)(x) = \sum_{j=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right)\right) \varphi(u-j)\, du\right] \varphi\left(2^k x - j\right).$$

By *supp* $\varphi \subseteq [-a, a]$ we have that $\varphi(u-j)$ is nonzero when $-a \leq u - j \leq a$, that is when $j - a \leq u \leq j + a$. Hence

$$(A_k f)(x) - (B_k f)(x) = \sum_{j=-\infty}^{\infty} \left[\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right)\right) \varphi(u-j)\, du\right] \varphi\left(2^k x - j\right).$$

Next we see that

$$f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right) = \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k}\right)}{i!} \frac{(u-j)^i}{2^{ki}} + \mathcal{R}_4,$$

where

$$\mathcal{R}_4 = \int_{j/2^k}^{u/2^k} \left(f^{(N)}(t) - f^{(N)}\left(\frac{j}{2^k}\right)\right) \frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!} dt.$$

(i) Case of $j \leq u$. We have

$$\begin{aligned}
|\mathcal{R}_4| &\leq \int_{j/2^k}^{u/2^k} \left| f^{(N)}(t) - f^{(N)}\left(\frac{j}{2^k}\right) \right| \frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!} dt \\
&\leq \int_{j/2^k}^{u/2^k} \omega_1\left(f^{(N)}, \left(t - \frac{j}{2^k}\right)\right) \frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!} dt \\
&\leq \omega_1\left(f^{(N)}, \frac{(u-j)}{2^k}\right) \int_{j/2^k}^{u/2^k} \frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!} dt \\
&= \omega_1\left(f^{(N)}, \frac{(u-j)}{2^k}\right) \frac{(u-j)^N}{2^{kN} N!}.
\end{aligned}$$

(ii) Case of $j \geq u$. We have

$$\begin{aligned}
|\mathcal{R}_4| &= \left| \int_{u/2^k}^{j/2^k} \left(f^{(N)}(t) - f^{(N)}\left(\frac{j}{2^k}\right) \right) \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \right| \\
&\leq \int_{u/2^k}^{j/2^k} \left| f^{(N)}(t) - f^{(N)}\left(\frac{j}{2^k}\right) \right| \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \\
&\leq \int_{u/2^k}^{j/2^k} \omega_1\left(f^{(N)}, \left(\frac{j}{2^k} - t\right)\right) \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \\
&\leq \omega_1\left(f^{(N)}, \frac{j-u}{2^k}\right) \int_{u/2^k}^{j/2^k} \frac{\left(t - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \\
&= \omega_1\left(f^{(N)}, \frac{j-u}{2^k}\right) \frac{(j-u)^N}{2^{kN} N!}.
\end{aligned}$$

So we have proved that

$$|\mathcal{R}_4| \leq \omega_1\left(f^{(N)}, \frac{|u-j|}{2^k}\right) \frac{|u-j|^N}{2^{kN} N!} \leq \omega_1\left(f^{(N)}, \frac{a}{2^k}\right) \frac{a^N}{2^{kN} N!},$$

i.e.

$$|\mathcal{R}_4| \leq \frac{a^N}{2^{kN} N!} \omega_1\left(f^{(N)}, \frac{a}{2^k}\right).$$

Furthermore we observe that

$$\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right) \right) \varphi(u-j)\, du = \sum_{i=1}^{N} \frac{f^{(i)}\left(j/2^k\right)}{2^{ki} i!} \int_{j-a}^{j+a} (u-j)^i \varphi(u-j)\, du + \int_{j-a}^{j+a} \mathcal{R}_4 \varphi(u-j)\, du.$$

Therefore

$$\left| \int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}\right) \right) \varphi(u-j)\, du \right| \leq \sum_{i=1}^{N} \frac{\left\| f^{(i)} \right\|_\infty}{2^{ki} i!} a^i + \frac{a^N}{2^{kN} N!} \omega_1\left(f^{(N)}, \frac{a}{2^k}\right).$$

The last proves the theorem. ■

We continue with

Theorem 5. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, also $\left\|f^{(i)}\right\|_\infty < \infty$, $i = 1, \ldots, N$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x-j) = 1$ all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$ and φ is Lebesgue measurable (then $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$). Define

$$\varphi_{kj}(x) := 2^{k/2}\varphi\left(2^k x - j\right) \text{ all } k, j \in \mathbb{Z},$$

$$\left\langle f, \varphi_{kj}\right\rangle = \int_{-\infty}^{\infty} f(t)\,\varphi_{kj}(t)\,dt,$$

and

$$\begin{aligned}(A_k f)(x) &= \sum_{j=-\infty}^{\infty} \left\langle f, \varphi_{kj}\right\rangle \varphi_{kj}(x) \\ &= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(u-j)\,du\right)\varphi\left(2^k x - j\right).\end{aligned}$$

Also define

$$(D_k f)(x) = \sum_{j=-\infty}^{\infty} \delta_{kj}(f)\,\varphi\left(2^k x - j\right),$$

where

$$\delta_{kj}(f) = \sum_{r=0}^{n} w_r f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right),$$

$n \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{n} w_r = 1$. Then

$$\begin{aligned}\left|(A_k f)(x) - (D_k f)(x)\right| &\leq \left\|A_k f - D_k f\right\|_\infty \\ &\leq \sum_{i=1}^{N} \frac{\left\|f^{(i)}\right\|_\infty}{2^{ki} i!}(a+1)^i + \frac{(a+1)^N}{N! 2^{kN}}\omega_1\left(f^{(N)}, \frac{(a+1)}{2^k}\right).\end{aligned} \tag{14}$$

So as $k \to \infty$, we get $\left\|A_k f - D_k f\right\|_\infty \to 0$.

Proof. By $\sum_{j=-\infty}^{\infty} \varphi\left(2^k x - j\right) = 1$ we have that $\int_{-\infty}^{\infty} \varphi(u-j)\,du = 1$.

Notice that

$$
\begin{aligned}
(A_k f)(x) - (D_k f)(x) &= \sum_{j=-\infty}^{\infty} \langle f, \varphi_{kj} \rangle \varphi_{kj}(x) - \sum_{j=-\infty}^{\infty} \delta_{kj}(f) \varphi\left(2^k x - j\right) \\
&= \sum_{j=-\infty}^{\infty} \left(2^{k/2} \langle f, \varphi_{kj} \rangle - \delta_{kj}(f)\right) \varphi\left(2^k x - j\right) \\
&= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right) \varphi(u-j)\, du - \sum_{r=0}^{n} w_r f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right) \\
&\quad \cdot \varphi\left(2^k x - j\right) \\
&= \sum_{j=-\infty}^{\infty} \left[\sum_{r=0}^{n} w_r \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right) \varphi(u-j)\, du - f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right)\right] \\
&\quad \cdot \varphi\left(2^k x - j\right) \\
&= \sum_{j=-\infty}^{\infty} \left[\sum_{r=0}^{n} w_r \left(\int_{-\infty}^{\infty} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right) \varphi(u-j)\, du\right)\right] \\
&\quad \cdot \varphi\left(2^k x - j\right).
\end{aligned}
$$

That is, by the compact support of φ we have

$$
\begin{aligned}
(A_k f)(x) - (D_k f)(x) &= \sum_{j=-\infty}^{\infty} \left[\sum_{r=0}^{n} w_r \left(\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right) \varphi(u-j)\, du\right)\right] \\
&\quad \cdot \varphi\left(2^k x - j\right).
\end{aligned}
$$

Next we see that

$$
f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k} + \frac{r}{2^k n}\right) = \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)}{i!} \frac{\left(u - j - \frac{r}{n}\right)^i}{2^{ki}} + \mathcal{R}_5,
$$

where

$$
\mathcal{R}_5 = \int_{\frac{j}{2^k} + \frac{r}{2^k n}}^{\frac{u}{2^k}} \left(f^{(N)}(t) - f^{(N)}\left(\frac{j}{2^k} + \frac{r}{2^k n}\right)\right) \frac{\left(\frac{u}{2^k} - t\right)^{N-1}}{(N-1)!} dt.
$$

(i) Case $j + \frac{r}{n} \leq u$. We have easily that

$$
|\mathcal{R}_5| \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\left(u - j - \frac{r}{n}\right)\right)}{N! 2^{kN}} \left(u - j - \frac{r}{n}\right)^N.
$$

(ii) Case $j + \frac{r}{n} \geq u$. We have that

$$\begin{aligned}
|\mathcal{R}_5| &= \left| \int_{\frac{u}{2^k}}^{\frac{j}{2^k}+\frac{r}{2^k n}} \left(f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right) - f^{(N)}(t) \right) \frac{\left(t-\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \right| \\
&\leq \int_{\frac{u}{2^k}}^{\frac{j}{2^k}+\frac{r}{2^k n}} \left| f^{(N)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right) - f^{(N)}(t) \right| \frac{\left(t-\frac{u}{2^k}\right)^{N-1}}{(N-1)!} dt \\
&\leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\left(j+\frac{r}{n}-u\right)\right)}{N! 2^{kN}} \left(j+\frac{r}{n}-u\right)^N .
\end{aligned}$$

So we have proved that

$$|\mathcal{R}_5| \leq \frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\left|j+\frac{r}{n}-u\right|\right)}{N! 2^{kN}} \left|j+\frac{r}{n}-u\right|^N \leq \frac{\omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right)}{N! 2^{kN}} (a+1)^N ,$$

i.e.

$$|\mathcal{R}_5| \leq \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{a+1}{2^k}\right).$$

Furthermore we observe that

$$\begin{aligned}
\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}+\frac{r}{2^k n}\right) \right) \varphi(u-j)\, du &= \sum_{i=1}^{N} \frac{f^{(i)}\left(\frac{j}{2^k}+\frac{r}{2^k n}\right)}{i! 2^{ki}} \int_{j-a}^{j+a} \left(u-j-\frac{r}{n}\right)^i \\
&\quad \cdot \varphi(u-j)\, du + \int_{j-a}^{j+a} \mathcal{R}_5 \varphi(u-j)\, du.
\end{aligned}$$

Therefore

$$\begin{aligned}
\left| \int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(\frac{j}{2^k}+\frac{r}{2^k n}\right) \right) \varphi(u-j)\, du \right| &\leq \sum_{i=1}^{N} \frac{\left\|f^{(i)}\right\|_\infty}{2^{ki} i!} (a+1)^i \\
&\quad + \frac{(a+1)^N}{N! 2^{kN}} \omega_1\left(f^{(N)}, \frac{(a+1)}{2^k}\right).
\end{aligned}$$

The last proves the theorem. ■

We continue with

Theorem 6. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, also $\left\|f^{(i)}\right\|_\infty < \infty$, $i = 1, \ldots, N$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty} \varphi(x-j) = 1$ all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$ and φ is Lebesgue measurable (then $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$). Define

$$\varphi_{kj}(x) := 2^{k/2} \varphi\left(2^k x - j\right) \text{ all } k, j \in \mathbb{Z},$$

$$\left\langle f,\varphi_{kj}\right\rangle = \int_{-\infty}^{\infty} f(t)\,\varphi_{kj}(t)\,dt,$$

and

$$\begin{aligned}(A_k f)(x) &= \sum_{j=-\infty}^{\infty} \left\langle f,\varphi_{kj}\right\rangle \varphi_{kj}(x)\\ &= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(u-j)\,du\right)\varphi\left(2^k x-j\right).\end{aligned}$$

Also define

$$(C_k f)(x) = \sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\,\varphi\left(2^k x-j\right),$$

where

$$\gamma_{kj}(f) = 2^k \int_{2^{-k}j}^{2^{-k}(j+1)} f(t)\,dt = 2^k \int_0^{2^{-k}} f\left(t+\frac{j}{2^k}\right)dt.$$

Then

$$\begin{aligned}\left|(A_k f)(x)-(C_k f)(x)\right| &\le \left\|A_k f - C_k f\right\|_\infty \\ &\le \sum_{i=1}^{N} \frac{\left\|f^{(i)}\right\|_\infty}{i!2^{ki}}(a+1)^i + \frac{(a+1)^N}{2^{kN}N!}\omega_1\left(f^{(N)},\frac{(a+1)}{2^k}\right).\end{aligned} \tag{15}$$

So as $k\to\infty$, we get $\left\|A_k f - C_k f\right\|_\infty \to 0$.

Proof. By $\sum_{j=-\infty}^{\infty} \varphi\left(2^k x-j\right)=1$ we have that $\int_{-\infty}^{\infty}\varphi(u-j)\,du=1$.

Notice that

$$(A_k f)(x)-(C_k f)(x) = \sum_{j=-\infty}^{\infty} \left\langle f,\varphi_{kj}\right\rangle \varphi_{kj}(x) - \sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\,\varphi\left(2^k x-j\right)$$

$$\begin{aligned}&= \sum_{j=-\infty}^{\infty} \left(2^{k/2}\left\langle f,\varphi_{kj}\right\rangle - \gamma_{kj}(f)\right)\varphi\left(2^k x-j\right)\\
&= \sum_{j=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(u-j)\,du - 2^k\int_0^{2^{-k}} f\left(t+\frac{j}{2^k}\right)dt\right)\varphi\left(2^k x-j\right)\\
&= \sum_{j=-\infty}^{\infty} \left[2^k\int_0^{2^{-k}}\left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(u-j)\,du\right)dt - 2^k\int_0^{2^{-k}} f\left(t+\frac{j}{2^k}\right)dt\right]\varphi\left(2^k x-j\right)\\
&= \sum_{j=-\infty}^{\infty} \left[2^k\int_0^{2^{-k}}\left[\left(\int_{-\infty}^{\infty} f\left(\frac{u}{2^k}\right)\varphi(u-j)\,du\right) - f\left(t+\frac{j}{2^k}\right)\right]dt\right]\varphi\left(2^k x-j\right)\\
&= \sum_{j=-\infty}^{\infty} \left[2^k\int_0^{2^{-k}}\left[\int_{-\infty}^{\infty}\left(f\left(\frac{u}{2^k}\right) - f\left(t+\frac{j}{2^k}\right)\right)\varphi(u-j)\,du\right]dt\right]\varphi\left(2^k x-j\right).\end{aligned}$$

That is, by the compact support of φ we have

$$(A_k f)(x) - (C_k f)(x) = \sum_{j=-\infty}^{\infty} \left[2^k \int_0^{2^{-k}} \left[\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(t + \frac{j}{2^k}\right) \right) \varphi(u-j)\, du \right] dt \right] \cdot \varphi\left(2^k x - j\right).$$

Next we see that

$$f\left(\frac{u}{2^k}\right) - f\left(t + \frac{j}{2^k}\right) = \sum_{i=1}^{N} \frac{f^{(i)}\left(t + \frac{j}{2^k}\right)}{i!} \left(\frac{u}{2^k} - t - \frac{j}{2^k}\right)^i + \mathcal{R}_6,$$

where

$$\mathcal{R}_6 = \int_{t+\frac{j}{2^k}}^{\frac{u}{2^k}} \left(f^{(N)}(s) - f^{(N)}\left(t + \frac{j}{2^k}\right) \right) \frac{\left(\frac{u}{2^k} - s\right)^{N-1}}{(N-1)!} ds.$$

(i) Case $t + \frac{j}{2^k} \le \frac{u}{2^k}$. We have easily that

$$|\mathcal{R}_6| \le \omega_1 \left(f^{(N)}, \left(\frac{u}{2^k} - t - \frac{j}{2^k} \right) \right) \frac{\left(\frac{u}{2^k} - t - \frac{j}{2^k}\right)^N}{N!}.$$

(ii) Case $t + \frac{j}{2^k} \ge \frac{u}{2^k}$. We have that

$$\begin{aligned}
|\mathcal{R}_6| &= \left| \int_{\frac{u}{2^k}}^{t+\frac{j}{2^k}} \left(f^{(N)}\left(t + \frac{j}{2^k}\right) - f^{(N)}(s) \right) \frac{\left(s - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} ds \right| \\
&\le \int_{\frac{u}{2^k}}^{t+\frac{j}{2^k}} \left| f^{(N)}\left(t + \frac{j}{2^k}\right) - f^{(N)}(s) \right| \frac{\left(s - \frac{u}{2^k}\right)^{N-1}}{(N-1)!} ds \\
&\le \omega_1 \left(f^{(N)}, t + \frac{j}{2^k} - \frac{u}{2^k} \right) \frac{\left(t + \frac{j}{2^k} - \frac{u}{2^k}\right)^N}{N!}.
\end{aligned}$$

So we have proved that

$$|\mathcal{R}_6| \le \omega_1 \left(f^{(N)}, \left| t + \frac{(j-u)}{2^k} \right| \right) \frac{\left| t + \frac{(j-u)}{2^k} \right|^N}{N!} \le \omega_1 \left(f^{(N)}, \frac{(a+1)}{2^k} \right) \frac{(a+1)^N}{2^{kN} N!}.$$

I.e. we found that

$$|\mathcal{R}_6| \le \frac{(a+1)^N}{2^{kN} N!} \omega_1 \left(f^{(N)}, \frac{(a+1)}{2^k} \right).$$

Furthermore we observe that

$$\int_{j-a}^{j+a} \left(f\left(\frac{u}{2^k}\right) - f\left(t + \frac{j}{2^k}\right) \right) \varphi(u-j)\, du = \sum_{i=1}^{N} \frac{f^{(i)}\left(t + \frac{j}{2^k}\right)}{i!} \int_{j-a}^{j+a} \left(\frac{(u-j)}{2^k} - t \right)^i \cdot \varphi(u-j)\, du + \int_{j-a}^{j+a} \mathcal{R}_6 \varphi(u-j)\, du.$$

Therefore

$$\left|\int_{j-a}^{j+a}\left(f\left(\frac{u}{2^k}\right)-f\left(t+\frac{j}{2^k}\right)\right)\varphi(u-j)\,du\right| \leq \sum_{i=1}^{N}\frac{\left\|f^{(i)}\right\|_{\infty}}{i!2^{ki}}(a+1)^i + \frac{(a+1)^N}{2^{kN}N!}\omega_1\left(f^{(N)},\frac{(a+1)}{2^k}\right).$$

The last proves the theorem. ■

We give

Theorem 7. Let $f \in C^N(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let φ be a bounded function of compact support $\subseteq [-a, a]$, $a > 0$ such that $\sum_{j=-\infty}^{\infty}\varphi(x-j) = 1$ all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$. Assume further $\left\|f^{(i)}\right\|_{\infty} < \infty$, $i = 1, \ldots, N$. Put

$$(B_k f)(x) = \sum_{j=-\infty}^{\infty} f\left(\frac{j}{2^k}\right)\varphi\left(2^k x - j\right),$$

$$(C_k f)(x) = \sum_{j=-\infty}^{\infty} \gamma_{kj}(f)\,\varphi\left(2^k x - j\right),$$

where

$$\gamma_{kj}(f) = 2^k\int_0^{2^{-k}} f\left(t+\frac{j}{2^k}\right)dt,$$

and

$$(D_k f)(x) = \sum_{j=-\infty}^{\infty} \delta_{kj}(f)\,\varphi\left(2^k x - j\right),$$

where

$$\delta_{kj}(f) = \sum_{r=0}^{n} w_r f\left(\frac{j}{2^k}+\frac{r}{2^k n}\right),$$

$n \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{n} w_r = 1$. Then

(i)

$$\begin{aligned}\left|(B_k f)(x)-(D_k f)(x)\right| &\leq \left\|B_k f - D_k f\right\|_{\infty} \\ &\leq \sum_{i=1}^{N}\frac{\left\|f^{(i)}\right\|_{\infty}}{2^{ki}i!} + \frac{\omega_1\left(f^{(N)},\frac{1}{2^k}\right)}{2^{kN}N!},\end{aligned} \tag{16}$$

(ii)

$$\begin{aligned}\left|(B_k f)(x)-(C_k f)(x)\right| &\leq \left\|B_k f - C_k f\right\|_{\infty} \\ &\leq \sum_{i=1}^{N}\frac{\left\|f^{(i)}\right\|_{\infty}}{2^{ki}(i+1)!} + \frac{\omega_1\left(f^{(N)},\frac{1}{2^k}\right)}{2^{kN}(N+1)!},\end{aligned} \tag{17}$$

(iii)

$$\begin{aligned} \left|\left(C_k f\right)(x)-\left(D_k f\right)(x)\right| &\leq \left\|C_k f-D_k f\right\|_{\infty} \\ &\leq \sum_{i=1}^N \frac{\left\|f^{(i)}\right\|_{\infty}}{2^{k i-1}(i+1)!}+\frac{\omega_1\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{k N-1}(N+1)!}. \end{aligned} \tag{18}$$

So as $k \rightarrow \infty$, we get $\left\|B_k f-D_k f\right\|_{\infty} \rightarrow 0$, $\left\|B_k f-C_k f\right\|_{\infty} \rightarrow 0$, $\left\|C_k f-D_k f\right\|_{\infty} \rightarrow 0$.

Proof. By Theorems 1-3 and especially use of (4), (7) and (12). ■

2 Estimates for Distances of Fuzzy Wavelet type Operators

We need the following background

Definition (see [7]). Let $\mu : \mathbb{R} \rightarrow [0,1]$ with the following properties

(i) is normal, i.e., $\exists x_0 \in \mathbb{R}; \mu(x_0)=1$.

(ii) $\mu(\lambda x+(1-\lambda) y) \geq \min \{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}, \forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon>0$, $\exists$ neighborhood $V(x_0): \mu(x) \leq \mu(x_0)+\varepsilon, \forall x \in V(x_0)$.

(iv) The set $\overline{\mathit{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\mathit{supp}(\mu):=\{x \in \mathbb{R}: \mu(x)>0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0<r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r:=\{x \in \mathbb{R}: \mu(x) \geq r\}$ and

$$[\mu]^0:=\overline{\{x \in \mathbb{R}: \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$.

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r=[u]^r+[v]^r, [\lambda \odot u]^r=\lambda[u]^r, \ \forall r \in [0,1],$$

where $[u]^r+[v]^r$ means the usual addition of two integrals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [7]). Notice $1 \odot u=u$ and it holds $u \oplus v=v \oplus u, \lambda \odot u=u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r=\left[u_{-}^{(r)}, u_{+}^{(r)}\right]$, where $u_{-}^{(r)} \leq u_{+}^{(r)}, u_{-}^{(r)}, u_{+}^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$. For $\lambda>0$ one has $\lambda u_{\pm}^{(r)}=(\lambda \odot u)_{\pm}^{(r)}$, respectively.

Define

$$D: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{+}$$

by

$$D(u,v) := \sup_{r\in[0,1]} \max\left\{ \left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$.

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [7].

Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) = \sup_{x\in\mathbb{R}} D(f(x), g(x)).$$

Here $\sum^*$ stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \forall u \in \mathbb{R}_{\mathcal{F}}$.

We need

Remark([4]**).** Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, \ldots, m \in \mathbb{N}$. Suppose that

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, \ldots, m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \le \sum_{i=1}^m \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right).$$

Definition. Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, we define the (first) fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{\substack{x,y\in\mathbb{R} \\ |x-y|\le\delta}} D(f(x), f(y)), \delta > 0.$$

We define $C_{\mathcal{F}}^U(\mathbb{R})$ the space of uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ the space of fuzzy continuous functions on $\mathbb{R}$.

Proposition 8 ([4])**.** Let $f \in C_{\mathcal{F}}^U(\mathbb{R})$. Then $\omega_1^{(\mathcal{F})}(f,\delta) < \infty$, any $\delta > 0$.

Proposition 9 ([4])**.** It holds $\lim_{\delta\to 0} \omega_1^{(\mathcal{F})}(f,\delta) = \omega_1^{(\mathcal{F})}(f,0) = 0$, iff $f \in C_{\mathcal{F}}^U(\mathbb{R})$.

Proposition 10 ([4])**.** Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real number valued function. Assume that $\omega_1^{(\mathcal{F})}(f,\delta)$, $\omega_1\left(f_-^{(r)}, \delta\right)$, $\omega_1\left(f_+^{(r)}, \delta\right)$ are finite, for $\delta > 0$, all $r \in [0,1]$. Then

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right), \omega_1\left(f_+^{(r)}, \delta\right)\right\}.$$

Note. It is clear from Propositions 9, 10 that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$).

Definition. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference of x and y, denoted $x - y$.

Definition ([7]). Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D) $\lim_{h \to 0+} \frac{f(x+h)-f(x)}{h}$, $\lim_{h \to 0+} \frac{f(x)-f(x-h)}{h}$ exist and are equal to $f'(x)$. We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in an neighborhood of x.

We denote by $C^N(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $N \geq 1$, the space of all $N-$times continuously fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$.

We mention

Theorem 11 ([11]). Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be $H-$fuzzy differentiable. Let $t \in [a, b]$, $0 \leq r \leq 1$. Clearly

$$[f(t)]^r = \left[(f(t))_-^{(r)}, (f(t))_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left((f(t))_-^{(r)} \right)', \left((f(t))_+^{(r)} \right)' \right].$$

I.e. $(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)'$, $\forall r \in [0, 1]$.

Remark ([4]). Let $f \in C^N(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $N \geq 1$. Then by Theorem 11 we obtain $\left[f^{(i)}(t) \right]^r = \left[\left((f(t))_-^{(r)} \right)^{(i)}, \left((f(t))_+^{(r)} \right)^{(i)} \right]$, for $i = 0, 1, 2, \ldots, N$, and in particular we have that

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)},$$

for any $r \in [0, 1]$.

Note. (i) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous, then $f_{\pm}^{(r)} : \mathbb{R} \to \mathbb{R}$ are continuous, $\forall r \in [0, 1]$.

(ii) Let $f \in C^N(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $N \geq 1$. Then by Theorem 11, we have $f_{\pm}^{(r)} \in C^N(\mathbb{R})$, for any $r \in [0, 1]$.

We need

Definition. Denote by $C_{\mathcal{F}}^{NB}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}} |$ such that all fuzzy derivatives $f^{(i)} : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, $i = 0, 1, \ldots, N$ exist and are fuzzy continuous and furthermore $D^*\left(f^{(i)}, \widetilde{0}\right) < \infty$, for $i = 1, \ldots, N\}$, $N \geq 1$.

Notice here that

$$D^*\left(f^{(i)}, \widetilde{0}\right) = \sup_{r\in[0,1]} \max\left(\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty, \left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty\right)$$
$$= \sup_{r\in[0,1]} \max\left(\left\|\left(f^{(r)}\right)_-^{(i)}\right\|_\infty, \left\|\left(f^{(r)}\right)_+^{(i)}\right\|_\infty\right), \ i = 1, \ldots, N.$$

Notice also that
$D^*\left(f^{(i)}, \widetilde{0}\right) < \infty$, implies $\left\|\left(f^{(i)}\right)_\pm^{(r)}\right\|_\infty < \infty, i = 1, \ldots, N, \forall r \in [0, 1]$.

We need

Definition ([8], p. 644). Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P{}^* (v - u) \odot f(\xi), I\right) < \varepsilon.$$

We write $I := (FR) \int_a^b f(x)\, dx$.

We mention

Theorem 12 ([9]). Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then $(FR) \int_a^b f(x)\, dx$ exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR) \int_a^b f(x)\, dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x)\, dx, (f)_+^{(r)}(x)\, dx\right],$$

$\forall r \in [0, 1]$.

Clearly $f_\pm^{(r)} : [a, b] \to \mathbb{R}$ are continuous functions.

In this section we study the fuzzy corresponding analogs of real wavelet type operators A_k, B_k, C_k, D_k, $k \in \mathbb{Z}$, of first section. For simplicity we keep the same notation at the fuzzy level. So, depending on the context we understand accordingly whether our operator is real of fuzzy, that is whether is operating on real valued functions or on fuzzy valued functions.

We present the next main fuzzy wavelet type result.

Theorem 13. Let $f \in C_{\mathcal{F}}^{NB}(\mathbb{R})$, $N \geq 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. Let φ be a bounded real valued function of compact support $\subseteq [-a, a]$, $a > 0$ such that

$\sum\limits_{j=-\infty}^{\infty} \varphi(x-j) = 1$, all $x \in \mathbb{R}$. Suppose $\varphi \geq 0$. Put

$$(B_k f)(x) = \sum_{j=-\infty}^{\infty}{}^{*} f\left(\frac{j}{2^k}\right) \odot \varphi\left(2^k x - j\right),$$

$$(C_k f)(x) = \sum_{j=-\infty}^{\infty}{}^{*} \left(2^k \odot (FR) \int_0^{2^{-k}} f\left(t + \frac{j}{2^k}\right) dt\right) \odot \varphi\left(2^k x - j\right),$$

and

$$(D_k f)(x) = \sum_{j=-\infty}^{\infty}{}^{*} \delta_{kj}(f) \odot \varphi\left(2^k x - j\right),$$

where

$$\delta_{kj}(f) = \sum_{\widetilde{r}=0}^{n}{}^{*} w_{\widetilde{r}} \odot f\left(\frac{j}{2^k} + \frac{\widetilde{r}}{2^k n}\right),$$

$n \in \mathbb{N}$, $w_{\widetilde{r}} \geq 0$, $\sum\limits_{\widetilde{r}=0}^{n}{}^{*} w_{\widetilde{r}} = 1$.

Then

(i)

$$\begin{aligned} D\left((B_k f)(x), (D_k f)(x)\right) &\leq D^*\left(B_k f, D_k f\right) \\ &\leq \sum_{i=1}^{N} \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{2^{ki} i!} + \frac{\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN} N!}, \end{aligned} \tag{19}$$

(ii)

$$\begin{aligned} D\left((B_k f)(x), (C_k f)(x)\right) &\leq D^*\left(B_k f, C_k f\right) \\ &\leq \sum_{i=1}^{N} \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{2^{ki}(i+1)!} + \frac{\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}, \end{aligned} \tag{20}$$

and

(iii)

$$\begin{aligned} D\left((C_k f)(x), (D_k f)(x)\right) &\leq D^*\left(C_k f, D_k f\right) \\ &\leq \sum_{i=1}^{N} \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{2^{ki-1}(i+1)!} + \frac{\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right)}{2^{kN-1}(N+1)!}. \end{aligned} \tag{21}$$

Note. We see that

$$\begin{aligned} D\left(f^{(N)}(x), f^{(N)}(y)\right) &\leq D\left(f^{(N)}(x), \widetilde{0}\right) + D\left(f^{(N)}(y), \widetilde{0}\right) \\ &\leq 2D^*\left(f^{(N)}, \widetilde{0}\right) < \infty. \end{aligned}$$

Thus $\omega_1^{(\mathcal{F})}\left(f, \frac{1}{2^k}\right) < \infty, \forall k \in \mathbb{Z}$.

Consequently as $k \to \infty$ we obtain $D^*\left(B_k f, D_k f\right), D^*\left(B_k f, C_k f\right), D^*\left(C_k f, D_k f\right) \to 0$ with rates.

Proof. (i) We observe the following

$$\begin{aligned}
D\left(\left(B_k f\right)(x), \left(D_k f\right)(x)\right) &= \sup_{r\in[0,1]} \max\left\{\left|\left(\left(B_k f\right)(x)\right)_-^{(r)} - \left(\left(D_k f\right)(x)\right)_-^{(r)}\right|,\right. \\
&\quad \left.\left|\left(\left(B_k f\right)(x)\right)_+^{(r)} - \left(\left(D_k f\right)(x)\right)_+^{(r)}\right|\right\} \\
&= \sup_{r\in[0,1]} \max\left\{\left|\left(B_k\left(f_-^{(r)}\right)\right)(x) - \left(D_k\left(f_-^{(r)}\right)\right)(x)\right|,\right. \\
&\quad \left.\left|\left(B_k\left(f_+^{(r)}\right)\right)(x) - \left(D_k\left(f_+^{(r)}\right)\right)(x)\right|\right\} \\
&\leq \sup_{r\in[0,1]} \max\left\{\left\|B_k\left(f_-^{(r)}\right) - D_k\left(f_-^{(r)}\right)\right\|_\infty,\right. \\
&\quad \left.\left\|B_k\left(f_+^{(r)}\right) - D_k\left(f_+^{(r)}\right)\right\|_\infty\right\} \\
&\overset{(16)}{\leq} \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^N \frac{\left\|\left(f_-^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki} i!} + \frac{\omega_1\left(\left(f_-^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN} N!},\right. \\
&\quad \left.\sum_{i=1}^N \frac{\left\|\left(f_+^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki} i!} + \frac{\omega_1\left(\left(f_+^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN} N!}\right\}
\end{aligned}$$

$$\begin{aligned}
&= \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^N \frac{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty}{2^{ki} i!} + \frac{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right)}{2^{kN} N!},\right. \\
&\quad \left.\sum_{i=1}^N \frac{\left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty}{2^{ki} i!} + \frac{\omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)}{2^{kN} N!}\right\} \\
&\leq \sum_{i=1}^N \frac{1}{2^{ki} i!} \sup_{r\in[0,1]} \max\left\{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty, \left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty\right\} \\
&\quad + \frac{1}{2^{kN} N!} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right), \omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)\right\} \\
&= \sum_{i=1}^N \frac{1}{2^{ki} i!} D^*\left(f^{(i)}, \widetilde{0}\right) + \frac{1}{2^{kN} N!} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right),
\end{aligned}$$

proving the theorem's (19).

(ii) We observe the following

$$
\begin{aligned}
D\left(\left(B_k f\right)(x), \left(C_k f\right)(x)\right) &= \sup_{r\in[0,1]} \max\left\{\left|\left(\left(B_k f\right)(x)\right)_-^{(r)} - \left(\left(C_k f\right)(x)\right)_-^{(r)}\right|,\right. \\
&\left.\left|\left(\left(B_k f\right)(x)\right)_+^{(r)} - \left(\left(C_k f\right)(x)\right)_+^{(r)}\right|\right\} \\
&= \sup_{r\in[0,1]} \max\left\{\left|\left(B_k\left(f_-^{(r)}\right)\right)(x) - \left(C_k\left(f_-^{(r)}\right)\right)(x)\right|,\right. \\
&\left.\left|\left(B_k\left(f_+^{(r)}\right)\right)(x) - \left(C_k\left(f_+^{(r)}\right)\right)(x)\right|\right\} \\
&\leq \sup_{r\in[0,1]} \max\left\{\left\|B_k\left(f_-^{(r)}\right) - C_k\left(f_-^{(r)}\right)\right\|_\infty,\right. \\
&\left.\left\|B_k\left(f_+^{(r)}\right) - C_k\left(f_+^{(r)}\right)\right\|_\infty\right\} \\
&\overset{(17)}{\leq} \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^{N} \frac{\left\|\left(f_-^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki}(i+1)!} + \frac{\omega_1\left(\left(f_-^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!},\right. \\
&\left.\sum_{i=1}^{N} \frac{\left\|\left(f_+^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki}(i+1)!} + \frac{\omega_1\left(\left(f_+^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^{N} \frac{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty}{2^{ki}(i+1)!} + \frac{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!},\right. \\
&\left.\sum_{i=1}^{N} \frac{\left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty}{2^{ki}(i+1)!} + \frac{\omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)}{2^{kN}(N+1)!}\right\} \\
&\leq \sum_{i=1}^{N} \frac{1}{2^{ki}(i+1)!} \sup_{r\in[0,1]} \max\left\{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty, \left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty\right\} \\
&\quad + \frac{1}{2^{kN}(N+1)!} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right), \omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)\right\} \\
&= \sum_{i=1}^{N} \frac{1}{2^{ki}(i+1)!} D^*\left(f^{(i)}, \widetilde{0}\right) + \frac{1}{2^{kN}(N+1)!} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right),
\end{aligned}
$$

proving the theorem's (20).

(iii) We observe the next

$$
\begin{aligned}
D\left(\left(C_k f\right)(x),\left(D_k f\right)(x)\right) &= \sup_{r\in[0,1]} \max\left\{\left|\left(\left(C_k f\right)(x)\right)_-^{(r)} - \left(\left(D_k f\right)(x)\right)_-^{(r)}\right|,\right.\\
&\qquad \left.\left|\left(\left(C_k f\right)(x)\right)_+^{(r)} - \left(\left(D_k f\right)(x)\right)_+^{(r)}\right|\right\}\\
&= \sup_{r\in[0,1]} \max\left\{\left|\left(C_k\left(f_-^{(r)}\right)\right)(x) - \left(D_k\left(f_-^{(r)}\right)\right)(x)\right|,\right.\\
&\qquad \left.\left|\left(C_k\left(f_+^{(r)}\right)\right)(x) - \left(D_k\left(f_+^{(r)}\right)\right)(x)\right|\right\}\\
&\le \sup_{r\in[0,1]} \max\left\{\left\|C_k\left(f_-^{(r)}\right) - D_k\left(f_-^{(r)}\right)\right\|_\infty,\right.\\
&\qquad \left.\left\|C_k\left(f_+^{(r)}\right) - D_k\left(f_+^{(r)}\right)\right\|_\infty\right\}\\
&\overset{(18)}{\le} \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^{N} \frac{\left\|\left(f_-^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki-1}(i+1)!} + \frac{\omega_1\left(\left(f_-^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN-1}(N+1)!},\right.\\
&\qquad \left.\sum_{i=1}^{N} \frac{\left\|\left(f_+^{(r)}\right)^{(i)}\right\|_\infty}{2^{ki-1}(i+1)!} + \frac{\omega_1\left(\left(f_+^{(r)}\right)^{(N)}, \frac{1}{2^k}\right)}{2^{kN-1}(N+1)!}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{r\in[0,1]} \max\left\{\sum_{i=1}^{N} \frac{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty}{2^{ki-1}(i+1)!} + \frac{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right)}{2^{kN-1}(N+1)!},\right.\\
&\quad \left.\sum_{i=1}^{N} \frac{\left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty}{2^{ki-1}(i+1)!} + \frac{\omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)}{2^{kN-1}(N+1)!}\right\}\\
&\le \sum_{i=1}^{N} \frac{1}{2^{ki-1}(i+1)!} \sup_{r\in[0,1]} \max\left\{\left\|\left(f^{(i)}\right)_-^{(r)}\right\|_\infty, \left\|\left(f^{(i)}\right)_+^{(r)}\right\|_\infty\right\}\\
&\quad + \frac{1}{2^{kN-1}(N+1)!} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f^{(N)}\right)_-^{(r)}, \frac{1}{2^k}\right), \omega_1\left(\left(f^{(N)}\right)_+^{(r)}, \frac{1}{2^k}\right)\right\}\\
&= \sum_{i=1}^{N} \frac{1}{2^{ki-1}(i+1)!} D^*\left(f^{(i)}, \widetilde{0}\right) + \frac{1}{2^{kN-1}(N+1)!}\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{2^k}\right),
\end{aligned}
$$

proving the theorem's (21).

Above we need that ([5])

$$\begin{aligned}(B_k f)_{\pm}^{(r)} &= B_k\left(f_{\pm}^{(r)}\right),\\ (C_k f)_{\pm}^{(r)} &= C_k\left(f_{\pm}^{(r)}\right), \text{ and}\\ (D_k f)_{\pm}^{(r)} &= D_k\left(f_{\pm}^{(r)}\right),\end{aligned}$$

$\forall r \in [0,1]$. ■

Denote by $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ the space of bounded fuzzy continuous functions on $\mathbb{R}$ with respect to metric D.

We finish with the following fuzzy wavelet type main result

Theorem 14. Let $f \in C_{\mathcal{F}}^{NB}(\mathbb{R}) \cap C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $N \geq 1$, $x \in \mathbb{R}$, and $k \in \mathbb{Z}$. Let the scaling function $\varphi(x)$ a real valued function with *supp* $\varphi(x) \subseteq [-a,a]$, $0 < a < +\infty$, φ is continuous on $[-a,a]$, $\varphi(x) \geq 0$, such that $\sum\limits_{j=-\infty}^{\infty} \varphi(x-j) = 1$ on $\mathbb{R}$ (then $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$).

Define

$$\varphi_{kj}(t) := 2^{k/2}\varphi\left(2^k t - j\right) \text{ all } k, j \in \mathbb{Z}, t \in \mathbb{R}$$

$$\left\langle f, \varphi_{kj}\right\rangle := (FR)\int_{\frac{j-a}{2^k}}^{\frac{j+a}{2^k}} f(t) \odot \varphi_{kj}(t)\,dt,$$

and the fuzzy wavelet type operator

$$(A_k f)(x) = \sum_{j=-\infty}^{\infty}{}^{*} \left\langle f, \varphi_{kj}\right\rangle \odot \varphi_{kj}(x), \ x \in \mathbb{R}.$$

The fuzzy wavelet type operators B_k, C_k, D_k are as in Theorem 13.

Then

(i)

$$\begin{aligned}D\left((A_k f)(x), (B_k f)(x)\right) &\leq D^*\left(A_k f, B_k f\right)\\ &\leq \sum_{i=1}^{N} \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{2^{ki} i!} a^i + \frac{a^N}{2^{kN} N!}\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{a}{2^k}\right),\end{aligned} \tag{22}$$

(ii)

$$D\left((A_k f)(x), (C_k f)(x)\right) \leq D^*\left(A_k f, C_k f\right)$$

$$\leq \sum_{i=1}^{N} \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{i! 2^{ki}} (a+1)^i + \frac{(a+1)^N}{2^{kN} N!}\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{(a+1)}{2^k}\right), \tag{23}$$

and

(iii)

$$D\left(\left(A_k f\right)(x),\left(D_k f\right)(x)\right) \leq D^*\left(A_k f, D_k f\right)$$
$$\leq \sum_{i=1}^N \frac{D^*\left(f^{(i)}, \widetilde{0}\right)}{2^{k i} i!}(a+1)^i+\frac{(a+1)^N}{N! 2^{k N}} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{(a+1)}{2^k}\right). \tag{24}$$

Notice that $D^*\left(A_k f, B_k f\right), D^*\left(A_k f, C_k f\right), D^*\left(A_k f, D_k f\right) \rightarrow 0$ as $k \rightarrow \infty$ with rates.

Proof. Similar to the proof of Theorem 13. Also notice here (see also [5]) that $\left(A_k f\right)_{\pm}^{(r)}=A_k\left(f_{\pm}^{(r)}\right), \forall r \in[0,1]$.

It is based on (13), (14) and (15). ■

Note. In [3] we proved, for $f \in C_{\mathcal{F}}^U(\mathbb{R})$ as $k \rightarrow \infty$, we get uniformly that A_k, B_k, C_k, $D_k \rightarrow I$ unit operator with rates in the D metric. In the case of A_k we need also f be fuzzy bounded. As related work we mention [2].

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General Over-Relaxed $A-$Proximal Point Algorithms and Applications to Nonlinear Variational Inclusions

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Abstract

Based on the notion of $A-$monotonicity and the general firm nonexpansiveness of the resolvent operator corresponding to $A-$monotonicity, the convergence analysis of the over-relaxed proximal point algorithm in the context of the approximation solvability of a class of nonlinear variational inclusions is examined. Several results on the general firm nonexpansiveness are also established. The obtained results generalize the results on firm nonexpansiveness.

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1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle .,.\rangle$. We consider the nonlinear inclusion problem: determine a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X. Rockafellar [24] generalized the algorithm of Martinet [17] in the context of convex programming, which is referred to as the proximal point algorithm in the literature. This work includes the general convergence and rate of convergence analysis for solving (1) when M is monotone. Then in [25], Rockafellar has shown as how the proximal point algorithm can be formulated in conjunction with convex programming duality theory to present a general convergence analysis for the multiplier method in convex programming. Eckstein and Bertsekas [6] introduced the relaxed proximal point algorithm and applied to the solvability of inclusion problems of the form (1). Furthermore, they demonstrated that the Douglas-Rachford splitting method [2] for convex programming was, in fact, a special case of the proximal point algorithm. Next, we state the following theorem of Eckstein and Bertsekas [6], which presents the convergence analysis for the relaxed proximal point algorithm.

Theorem 1.1. [6, Theorem 3] Let $M : X \to 2^X$ be a set-valued maximal monotone mapping on X with $0 \in range(M)$, and let the sequence $\{x^k\}$ be generated by the iterative algorithm

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k w^k \, \forall\, k \geq 0, \tag{2}$$

where w^k is such that

$$\|w^k - (I + c_k M)^{-1}(x^k)\| \leq \epsilon_k \, \forall\, k \geq 0,$$

and the scalar sequences $\{\epsilon_k\}$, $\{\alpha_k\}$ and $\{c_k\}$ satisfy

$$\Sigma_{k=0}^{\infty}\epsilon_k < \infty,\; inf\, \alpha_k > 0,\; sup\, \alpha_k < 2,\; c = inf c_k > 0.$$

Then the sequence $\{x^k\}$ converges weakly to a zero of M.

In $[32-37, 39]$, the author introduced and studied the notion of $A-$monotonicity in the context of solving variational inclusion problems based on the resolvent operator techniques. The notion of $A-$monotonicity generalizes the general theory of multivalued maximal monotone mappings, including the notion of $H-$monotonicity introduced by Fang and Huang [8], and provides a general framework to examining variational inclusion problems. Resolvent operator methods have been used in literature for a while and are still being applied to a broad spectrum of problems arising from several fields, such as equilibria problems, optimization and control theory, operations research, and mathematical programming.

In this paper, we generalize the relaxed proximal point algorithm to the case of $A-$monotone mappings, and then we apply it to the approximation solvability of a class of nonlinear inclusion problem involving $A-$ monotone set-valued mappings in a Hilbert space setting. The convergence analysis for the generalized relaxed proximal point algorithm is discussed in detail. Also, several results on the generalized firm nonexpansiveness, Lipschitz continuity of the generalized resolvent operator corresponding to $A-$ monotone mappings are included. The obtained results generalize a number of results on the general maximal monotonicity, resolvent operator technique and firm nonexpansiveness by Eckstein [5], Eckstein and Bertsekas [6], Rockafellar [24, 25], and others. For more literature, we recommend the reader [1-40].

The contents are organized as follows: section 1 deals with a historical development of the proximal point algorithm in conjunction with the maximal monotonicity, and with the approximation solvability of a class of nonlinear inclusion problems based on the proximal point algorithm. Section 2 introduces the notion of $A-$monotonicity and general firm nonexpansiveness, and then presents a number of results connecting $A-$monotonicity, general firm nonexpansiveness and generalized resolvent operator, while Section 3 presents the generalized version of the relaxed proximal point algorithm of Eckstein and Bertsekas [6] to the case of $A-$monotone mappings. Section 4 contains specializations to maximal relaxed monotone mappings and applications.

2. A-Monotonicity and Firm Nonexpansiveness

In this section we first explore some basic properties derived from the notion of $A-$ monotonicity. Then we establish some results involving $A-monotonicity$ and the firm nonexpansiveness. Let X denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $< .,. >$. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, which means, the set $\{(x,y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x,y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x,y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$dom(M) = \{x \in X : \exists y \in X : (x,y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

dom(M)=X, shall denote the full domain of M, and the range of M is defined by

$$range(M) = \{y \in X : \exists x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y, x) : (x, y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x, y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y + z) : (x, y) \in L, (x, z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) $(r)-$ strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u - v\|^2 \,\forall (u, u^*), (v, v^*) \in Graph(M).$$

(ii) $(r)-$strongly pseudomonotone if

$$\langle v^*, u - v\rangle \geq 0$$

implies

$$\langle u^*, u - v\rangle \geq r\|u - v\|^2 \,\forall (u, u^*), (v, v^*) \in Graph(M).$$

(iii) pseudomonotone if

$$\langle v^*, u - v\rangle \geq 0$$

implies

$$\langle u^*, u - v\rangle \geq 0 \,\forall (u, u^*), (v, v^*) \in Graph(M).$$

(iv) $(m)-$relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \,\forall (u, u^*), (v, v^*) \in Graph(M).$$

(v) $(c)-$ cocoercive if there is a positive constant c such that

$$\langle u^* - v^*, u - v\rangle \geq c\|u^* - v^*\|^2 \,\forall (u, u^*), (v, v^*) \in Graph(M).$$

Definition 2.2. A mapping $M : X \to 2^X$ is said to be maximal $(m)-$ relaxed monotone if

(i) M is $(m)-$relaxed monotone,

(ii) For $(u, u^*) \in X \times X$, and

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u - v\|^2 \,\forall (v, v^*) \in Graph(M),$$

we have $u^* \in M(u)$.

Definition 2.3. Let $M : X \to 2^X$ be a mapping on X. The map M is said to be:

(i) Nonexpansive if
$$\|u^* - v^*\| \leq \|u - v\| \, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(ii) Firmly nonexpansive if
$$\|u^* - v^*\|^2 \leq \langle u^* - v^*, u - v\rangle \, \forall (u, u^*), (v, v^*) \in Graph(M).$$

(iii) $(c)-$firmly nonexpansive if there exists a constant $c > 0$ such that
$$\|u^* - v^*\|^2 \leq c\langle u^* - v^*, u - v\rangle \, \forall (u, u^*), (v, v^*) \in Graph(M).$$

Definition 2.4. (Alternative) Let $A : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $A-$monotone if

(i) M is $(m)-$relaxed monotone

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Proposition 2.1. Let $A : X \to X$ be an (r)-strongly monotone single-valued mapping and let $M : X \to 2^X$ be an $A-$ monotone mapping. Then M is maximal $(m)-$relaxed monotone for $0 < \rho < \frac{r}{m}$.

Proposition 2.2. Let $A : X \to X$ be an $(r)-$strongly monotone single-valued mapping and let $M : X \to 2^X$ be an $A-$ monotone mapping. Then $(A + \rho M)$ is maximal monotone for $0 < \rho < \frac{r}{m}$.

Proof. Since A is $(r)-$strongly monotone and M is $A-$monotone, it implies that $A + \rho M$ is $(r - \rho m)-$strongly monotone. This in turn implies that $A + \rho M$ is pseudomonotone, and hence $A + \rho M$ is maximal monotone under the given conditions.

Proposition 2.3. Let $A : X \to X$ be an $(r)-$strongly monotone mapping and let $M : X \to 2^X$ be an $A-$monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Definition 2.5. Let $A : X \to X$ be an $(r)-$strongly monotone mapping and let $M : X \to 2^X$ be an $A-$monotone mapping. Then the generalized resolvent operator $J^M_{\rho,A} : X \to X$ is defined by
$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Definition 2.6. Let $A, T : X \to X$ be two mappings. Then map T is said to be:

(i) Monotone with respect to A if
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq 0 \, \forall (x, y) \in X.$$

(ii) $(r) - strongly$ monotone with respect to A if there exists a positive constant r such that
$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq r\|x - y\|^2 \, \forall (x, y) \in X.$$

(iii) (γ,α)-relaxed cocoercive with respect to A if there exist positive constants γ and α such that

$$\langle T(x)-T(y),A(x)-A(y)\rangle \geq -\gamma\|T(x)-T(y)\|^2+\alpha\|x-y\|^2$$

$\forall (x,y)\in X$.

Theorem 2.1. Let X be a real Hilbert space, and let $M : X \to 2^X$ be $(m)-$relaxed monotone. Then $M : X \to 2^X$ is $A-$ monotone if and only if

$$(A+\rho M)(X) = X \, for \, \rho > 0.$$

Proof. It follows from the definition of $A-$ monotonicity of M. □

3. $A-$Proximal Point Algorithm and Application

This section primarily deals with the relaxed $A-$proximal point algorithm and its application to approximation solvability of the inclusion problem (1). Several results connecting the generalized $A-$monotonicity and corresponding resolvent operator are established, which unify the results on the firm expansiveness from Eckstein and Bertsekas [6]. Furthermore, some auxiliary results on $A-$monotonicity, maximal relaxed monotonicity, and maximal monotonicity are obtained.

The solvability of the problem (1) depends on the equivalence between (1) and the problem of finding the fixed point of the associated generalized resolvent operator.

Lemma 3.1. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $A-$ monotone. Then the generalized resolvent operator associated with M and defined by

$$J^M_{\rho,A}(u) = (A+\rho M)^{-1}(u) \, \forall \, u \in X,$$

is $(\frac{1}{r-\rho m})-$Lipschitz continuous $r-\rho m > 0$.

Lemma 3.2. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $A-$ monotone. Then the generalized resolvent operator associated with M and defined by

$$J^M_{\rho,A}(u) = (A+\rho M)^{-1}(u) \, \forall \, u \in X,$$

is $(\frac{1}{r-\rho m})-$firmly nonexpansive for $r-\rho m > 0$.

Proof. For any $u, v \in X$, it follows from the definition of the resolvent operator $J^M_{\rho,A}$ that

$$\frac{1}{\rho}[u - A(J^M_{\rho,A}(u))] \in M(J^M_{\rho,A})(u),$$

and

$$\frac{1}{\rho}[v - A(J^M_{\rho,A}(v))] \in M(J^M_{\rho,A})(v).$$

Since M is $(m)-$relaxed monotone, we have

$$\begin{aligned} & \frac{1}{\rho}\langle u-v-[A(J^M_{\rho,A}(u))-A(J^M_{\rho,A}(v))], \\ & J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\rangle \\ \geq \; & -m\|J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\|^2. \end{aligned} \tag{3}$$

In light of (3), we have

$$\begin{aligned} & \langle u-v, J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\rangle \\ \geq\ & \langle A(J^M_{\rho,A}(u))-A(J^M_{\rho,A}(v)), \\ & J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\rangle - \\ & \rho m\|J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\|^2 \\ \geq\ & (r-\rho m)\|J^M_{\rho,A}(u)-J^M_{\rho,A}(v)\|^2. \square \end{aligned}$$

Lemma 3.3. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $A-$ monotone. Then $I - J^M_{\rho,A}$ is firmly nonexpansive for $r - \rho m > 1$.

Theorem 3.1. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $A-$monotone. Then the following statements are mutually equivalent:

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$u = J^M_{\rho,A}(A(u)),$$

where

$$J^M_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Proof. It follows from the definition of the generalized resolvent operator corresponding to M.

Theorem 3.2. Let X be a real Hilbert space, let $A : X \to X$ be $(r)-$strongly monotone and $(s)-$Lipschitz continuous, and let $M : X \to 2^X$ be $A-$monotone. For an arbitrarily chosen element x^0, let the sequence $\{x^k\}$ be generated by the relaxed $A-$proximal point algorithm

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k \; for \; k \geq 0$$

with

$$\|y^k - J^M_{\rho,A}(A(x^k))\| \leq \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$,

$$y^{k+1} = (1-\alpha_k)x^k + \alpha_k J^M_{\rho,A}(A(x^k)),$$

$$\langle x^k - x^*, J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\rangle \geq \gamma \|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2, \tag{4}$$

for $\gamma > 0$, $J^M_{\rho,A} = (A + \rho_k M)^{-1}$, and sequences $\{\delta_k\}$, $\{\alpha_k\}$ and $\{\rho_k\}$ satisfy $\alpha_k > 1$, $\sum_{k=0}^{\infty} \delta_k \leq \infty$, and $\rho_k \uparrow \rho$.

Then the sequence $\{x^k\}$ converges linearly to a unique solution x^* of (1) with rate

$$\sqrt{1 - \alpha[2(1 - \frac{s^2}{(r-\rho m)^2}) - \alpha(1 - (2\gamma - 1)\frac{s^2}{(r-\rho m)^2})]} < 1,$$

where $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$ and $\alpha = lim\, sup_{k\to\infty}\,\alpha_k$.

Proof. Applying Theorem 3.1, x^*, a solution to (1), satisfies the relaxed proximal point algorithm. It further follows from Theorem 3.1 that any solution to (1) is a fixed point of $J^M_{\rho_k,A}oA$ for all $k \geq 0$.

Next, using (4), we find the estimate

$$\begin{aligned}
& \|y^{k+1} - x^*\|^2 = \|(1-\alpha_k)x^k + \alpha_k J^M_{\rho_k,A}(A(x^k)) - \\
& [(1-\alpha_k)x^* + \alpha_k J^M_{\rho_k,A}(A(x^*))\|^2 \\
= \;& \|(1-\alpha_k)(x^k - x^*) - \alpha_k(J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*)))\|^2 \\
= \;& (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle x^k - x^*, J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\rangle + \\
& \alpha_k^2\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
\leq \;& (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\gamma\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 + \\
& \alpha_k^2\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
= \;& (1-\alpha_k)^2\|x^k - x^*\|^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\|J^M_{\rho_k,A}(A(x^k)) - J^M_{\rho_k,A}(A(x^*))\|^2 \\
\leq \;& (1-\alpha_k)^2\|x^k - x^*\|^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\|x^k - x^*\|^2 \\
= \;& \{(1-\alpha_k)^2 + [\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma]\frac{s^2}{(r-\rho_k m)^2}\}\|x^k - x^*\|^2 \\
= \;& \{1 - \alpha_k[2(1 - \frac{s^2}{(r-\rho_k m)^2}) - \alpha_k(1-(2\gamma-1)\frac{s^2}{(r-\rho_k m)^2})]\}\|x^k - x^*\|^2,
\end{aligned}$$

where $s < r - \rho_k m$, and $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$.

Therefore,

$$\|y^{k+1} - x^*\| \leq \theta_k\|x^k - x^*\|\; for\; s < r - \rho_k m,$$

where $\alpha_k^2 + 2\alpha_k(1-\alpha_k)\gamma > 0$, and

$$\theta_k = \sqrt{1 - \alpha_k[2(1 - \frac{s^2}{(r-\rho_k m)^2}) - \alpha_k(1-(2\gamma-1)\frac{s^2}{(r-\rho_k m)^2})]}.$$

Clearly, it follows that

$$\begin{aligned}
& \|x^{k+1} - y^{k+1}\| \\
= \;& \|(1-\alpha_k)x^k + \alpha_k y^k - [(1-\alpha_k)x^k + \alpha_k J^M_{\rho,A}(A(x^k))]\| \\
= \;& \|\alpha_k(y^k - J^M_{\rho,A}(A(x^k)))\| \\
\leq \;& \alpha_k\delta_k\|y^k - x^k\|.
\end{aligned}$$

Since

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k,$$

it implies that

$$\alpha_k(y^k - x^k) = x^{k+1} - x^k.$$

Now we estimate

$$
\begin{aligned}
& \|x^{k+1}-x^*\| = \|y^{k+1}-x^*\| + \|x^{k+1}-y^{k+1}\| \\
\leq\ & \|y^{k+1}-x^*\| + \|x^{k+1}-y^{k+1}\| \\
\leq\ & \|y^{k+1}-x^*\| + \alpha_k\delta_k\|y^k-x^k\| \\
=\ & \|y^{k+1}-x^*\| + \delta_k\|x^{k+1}-x^k\| \\
\leq\ & \theta_k\|x^k-x^*\| + \delta_k\|x^{k+1}-x^*\| + \delta_k\|x^k-x^*\|. \qquad (5)
\end{aligned}
$$

$$(6)$$

Therefore, we have

$$\|x^{k+1}-x^*\| \leq \frac{\theta_k+\delta_k}{1-\delta_k}\|x^k-x^*\|, \tag{7}$$

where

$$
\begin{aligned}
& lim\,sup\,\frac{\theta_k+\delta_k}{1-\delta_k} = lim\,sup\,\theta_k \\
= & \sqrt{1-\alpha[2(1-\frac{s^2}{(r-\rho m)^2})-\alpha(1-(2\gamma-1)\frac{s^2}{(r-\rho m)^2})]} < 1.
\end{aligned}
$$

Finally, to show the uniqueness of the solution, assume that x_1^* and x_2^* are two distinct solutions of (1). By Theorem 3.1, we have

$$x_1^* = J_{\rho_k,A}^M(A(x_1^*)),$$

and

$$x_2^* = J_{\rho_k,A}^M(A(x_2^*)).$$

Since $J_{\rho_k,A}^M$ is $(\frac{1}{r-\rho_k m})-$ Lipschitz continuous and A is $(s)-$ Lipschitz continuous, we arrive at

$$
\begin{aligned}
& \|x_1^*-x_2^*\| = \|J_{\rho_k,A}^M(A(x_1^*)) - J_{\rho_k,A}^M(A(x_2^*))\| \\
\leq\ & \frac{1}{r-\rho_k m}\|A(x_1^*)-A(x_2^*)\| \\
\leq\ & \frac{s}{r-\rho_k m}\|x_1^*-x_2^*\|.
\end{aligned}
$$

Therefore, we find

$$[1-\frac{s}{r-\rho_k m}]\|x_1^*-x_2^*\| \leq 0\, for\, s < r-\rho_k m.$$

It follows from this that

$$\|x_1^*-x_2^*\| = 0 for\, s < r-\rho_k m. \square$$

4. Some Applications

In this section, based on results from Sections 2 and 3, we derive some special cases of Theorem 3.2 for the $H-$ monotone mapping – introduced and studied by Fang and Huang [8].

Definition 4.1. Let $H : X \to X$ be a single-valued mapping. The map $M : X \to 2^X$ is said to be $H-$monotone if

(i) M is monotone,

(ii) $R(H + \rho M) = X$ for $\rho > 0$.

Lemma 4.1. [8] Let X be a real Hilbert space, let $H : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $H-$ monotone. Then the generalized resolvent operator associated with M and defined by

$$J_{\rho,H}^{M}(u) = (H + \rho M)^{-1}(u)\,\forall\, u \in X,$$

is $(\frac{1}{r})-$Lipschitz continuous for $r > 0$.

Lemma 4.2. Let X be a real Hilbert space, let $H : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $H-$ monotone. Then the generalized resolvent operator associated with M and defined by

$$J_{\rho,H}^{M}(u) = (H + \rho M)^{-1}(u)\,\forall\, u \in X,$$

is $(\frac{1}{r})-$firmly nonexpansive for $r > 0$.

Lemma 4.3. Let X be a real Hilbert space, let $H : X \to X$ be $(r)-$strongly monotone, and let $M : X \to 2^X$ be $H-$monotone. Then the following statements are mutually equivalent:

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$u = J_{\rho,H}^{M}(H(u)).$$

where

$$J_{\rho,H}^{M}(u) = (H + \rho M)^{-1}(u).$$

Theorem 4.1. Let X be a real Hilbert space, let $H : X \to X$ be $(r)-$strongly monotone and $(s)-$Lipschitz Continuous, and let $M : X \to 2^X$ be $H-$monotone. For an arbitrarily chosen element x^0, let the sequence $\{x^k\}$ be generated by the relaxed $H-$proximal point algorithm

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \; for\; k \geq 0$$

with

$$\|y^k - J_{\rho,H}^{M}(H(x^k))\| \leq \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$,

$$y^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_{\rho,H}^{M}(H(x^k)),$$

$$\langle x^k - x^*, J_{\rho_k,H}^{M}(H(x^k)) - J_{\rho_k,H}^{M}(H(x^*))\rangle \geq \gamma \|J_{\rho_k,H}^{M}(H(x^k)) - J_{\rho_k,H}^{M}(H(x^*))\|^2, \tag{8}$$

for $\gamma > 0$, $J_{\rho,H}^{M} = (H + \rho_k M)^{-1}$, and sequences $\{\delta_k\}$, $\{\alpha_k\}$ and $\{\rho_k\}$ satisfy $\alpha_k > 1$, $\sum_{k=0}^{\infty} \delta_k \leq \infty$, and $\rho_k \uparrow \rho$.

Then the sequence $\{x^k\}$ converges linearly to a unique solution x^* of (1) with rate

$$\sqrt{1 - \alpha^*[2(1 - \frac{s^2}{r^2}) - \alpha^*(1 - (2\gamma - 1)\frac{s^2}{r^2})]} < 1,$$

where $\alpha^* = lim\, sup_{k\to\infty}\, \alpha_k$ and $s < r$

Theorem 4.2. Let X be a real Hilbert space, and let $M : X \to 2^X$ be maximal monotone. For an arbitrarily chosen element x^0, let the sequence $\{x^k\}$ be generated by the relaxed proximal point algorithm

$$x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k)\, for\, k \geq 0$$

with

$$\|y^k - J_\rho^M(x^k)\| \leq \delta_k \|y^k - x^k\|,$$

where $\delta_k \to 0$,

$$y^{k+1} = (1 - \alpha_k)x^k + \alpha_k J_\rho^M(x^k),$$

$J_\rho^M = (I + \rho_k M)^{-1}$, and sequences $\{\delta_k\}$, $\{\alpha_k\}$ and $\{\rho_k\}$ satisfy $\alpha_k > 1$, $\sum_{k=0}^\infty \delta_k \leq \infty$, and $\rho_k \uparrow \rho$.

Then the sequence $\{x^k\}$ converges linearly to a unique solution x^* of (1) with rate

$$\sqrt{1 - \alpha^*[2 - \alpha^*]} < 1,$$

where

$$\alpha^* = lim\, sup_{k\to\infty}\, \alpha_k.$$

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Traffic Flower with n Petals

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Abstract

In this paper, some properties of solutions of a system of nonlinear ordinary differentional equationa are obtained. Above system describes a flow on the graph of "flower with n petals" type when the flow mass is constant.

1 Introduction

Let $A = (a_{ij})$ be a stochastic square matrix $n * n$, i.e. $a_{ij} \geq 0 \; \forall i, j$,

$$\sum_{j=1}^{n} a_{ij} = 1, i = 1..., n. \tag{1}$$

We assume, that $\vec{\rho} = (\rho_1..., \rho_n)$ is a vector with non-negative coordinates, $0 \leq \rho_i \leq 1$, $i = 1..., n$,

$$\|\vec{\rho}\|_{l_1^n} = \rho_1 + ... + \rho_n = C. \tag{2}$$

Let $f(x) = x\,(1-x)\,\chi_{[0,1]}\,(x)$, and $\chi_{[0,1]}$ be the characteristic function of a segment $[0, 1]$.

Then the system of ODE

$$\left\{ \begin{array}{l} \dot{\rho}_1 = (a_{11} - 1) f(\rho_1) + a_{21} f(\rho_2) + ... + a_{n1} f(\rho_n) \\ \dot{\rho}_2 = a_{12} f(\rho_1) + (a_{22} - 1)\, f(\rho_2) + ... + a_{n2} f(\rho_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{\rho}_n = a_{1n} f(\rho_1) + a_{2n} f(\rho_2) + ... + (a_{nn} - 1)\, f(\rho_n), \end{array} \right. \tag{3}$$

describes traffic on the graph, representing *"flower" - general vertex + a crossroad,* where the closed identical edges in quantity of n pieces, (Fig. 1) are united. The function $f(x)$ is

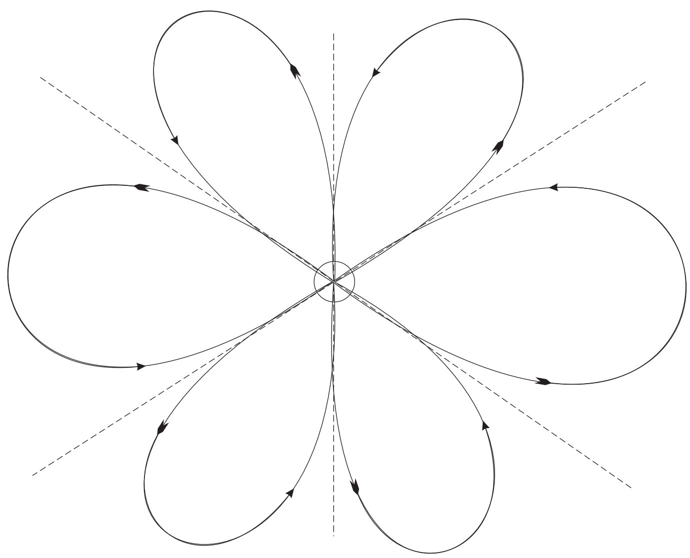

Figure 1: The traffic flower with six petals

a classic model of intensity of movement which depends on density and is called as *main diagram, [1].*

Further in (3) we will consider a value of density on the appropriate petal $\rho_i = \rho_i(t)$, $0 \leq \rho_i \leq 1$, $i = 1, ..., n$. Thus the matrix A is a matrix of hashing of flows in the vertex O. The entering flow from i petal is redistributed in the vertex O according to the shares determined in the line i of the matrix A, for this reason the condition (1) is true. Let's consider further, that in (2) C is a constant, i.e. *the flow is closed* and, hence, it is described by system (3). The purpose of work consists of research of qualitative properties of flow , i.e. solutions of (3). Thus *the stationary point* is a solution of (3) $\vec{\rho}(t) \equiv \rho(0)$, satisfying condition (2). *The solution $\vec{\rho}(t)$ such that*

$$\|\vec{\rho}\|_{l^n_\infty} = \max_{1 \leq i \leq n} |\rho_i| = 1 \tag{4}$$

is called critical. According to physics of the system it means that, at least on one edge flow density is maximal and also speed of leaving flow is equal to 0. *Hence, the edge ceases to pass a flow, "the petal falls down ".*

Thus, in case , when the system (3) with n petals is in a critical mode it is reduced to a flower with smaller number of petals and, accordingly, smaller mass. There exists different scenarios of a transformation of the matrix A to a stochastic matrix of the smaller size , i.e. different scenarios of movement control in a critical mode. We will postulate the following rules

(1) If the edge with number i has critical density, and next $(i-1)$ and $(i+1)$ edges are in an operating conditions, then line i of the matrix A shares half-and-half between $(i-1)$ (mod n) and $(i+1)$ (mod n), the column i deletes.

(2) If the critical condition i comes simultaneously with the next one, then transformation (I) is made above the sum of lines from cluster of critical edges and accordingly columns delete.

2 Flower with Two Petals

Let $n = 2$, $a_{11} = a_1$, $a_{22} = a_2$. Then the system (3) becomes

$$\begin{cases} \dot{\rho}_1 = (a_1 - 1) f(\rho_1) + (1 - a_2) f(\rho_2) \\ \dot{\rho}_2 = (1 - a_1) f(\rho_1) + (a_2 - 1) f(\rho_2) \end{cases} \qquad (5)$$

with the following conditions

$$\begin{cases} 0 \leq \rho_1 \leq 1, 0 \leq \rho_2 \leq 1, \\ \rho_1 + \rho_2 = C. \end{cases} \qquad (6)$$

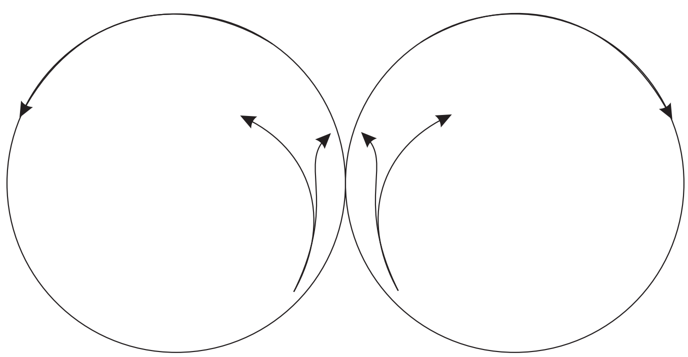

Figure 2: Double petal flower

As $\rho_2 = C - \rho_1$, then (5) - (6) are reduced to the problem (see Fig. 2)

$$\begin{cases} \dot{\rho}_1 = (a_1 - 1) f(\rho_1) + (1 - a_2) f(\rho_2) \\ 0 \leq \rho_1 \leq 1, 0 \leq C - \rho_1 \leq 1, \end{cases} , \qquad (7)$$

where $C$is a parameter, $0 < C < 2$.

Because of symmetry we will consider, that

$$1 \geq a_2 \geq a_1 \geq 0.$$

Lemma 2.1. *If $0 < a_1 = a_2 < 1$, then for any initial conditions $\rho_1(0) \neq C/2$*

(a) if $C > 1$ then the solution goes into a critical mode in finite time;

(b) if $0 < C < 1$ then the solution converges to a stationary point $\rho_1 \equiv C/2$.

Proof. If $0 \leq \rho_1 \leq 1$, and $0 \leq C - \rho_1 \leq 1$, that

$$\dot{\rho}_1 = (1 - a_2)(f(C - \rho_1) - f(\rho_1)) = (1 - a_2)(C - 2\rho_1)(1 - C),$$

whence everything also follows.

Lemma 2.2. *Let $0 < a_1 < a_2 < 1$. Then the problem (7) has the unique stationary solution.*

Proof. If $C < 1$, then $\rho_1 \in [0, C]$, and function

$$g(x) = (a_1 - 1)f(x) + (1 - a_2)f(C - x)$$

at $x = 0$ satisfies to an inequality $g(0) = (1 - a_2)f(C) > 0$, and at $x = C$ it is fair $g(C) = (a_1 - 1)f(C) < 0$. Thus, from a continuity $g(x)$ it follows that there is at least one zero. However, as

$$g(x) = (1 - a_1)f(C - x) + (a_1 - a_2)f(C - x) + (a_1 - 1)f(x) =$$

$$= (1 - a_1)(C - 2x)(1 - C) + (a_1 - a_2)f(C - x),$$

that,

first number of critical points is no more than two, i.e. $g(x)$ is parabola,

second by virtue of the above mentioned condition $g(0)g(C) < 0$ in an interval $[0, C]$ there is only one zero $g(x)$.

If $C > 1$, then $x \in [C - 1, 1]$, And on the same reasons there is exactly one critical point. The lemma 2.2 is proved.

Theorem 2.1. *If $0 < a_1 < a_2 < 1$, than for any initial conditions $\rho_1(0)$ such, that $g(\rho_1(0)) \neq 0$*

(a) If $C > 1$ then the solution (7) monotonously turns to a critical mode in finite time;

(b) If $0 < C < 1$ then the solution (7) monotonously converges to the stationary point $\rho_1(t) \equiv C_1$, $g(C_1) = 0$.

Proof. System

$$\begin{cases} \dot{\rho}_1 = (a_1 - 1)\rho_1(1-\rho_1) + (1-a_2)\rho_2(1-\rho_2) \\ \dot{\rho}_2 = (-a_1 + 1)\rho_1(1-\rho_1) + (-1+a_2)\rho_2(1-\rho_2) \end{cases} \tag{8}$$

has the following vector field (see Fig. 3).

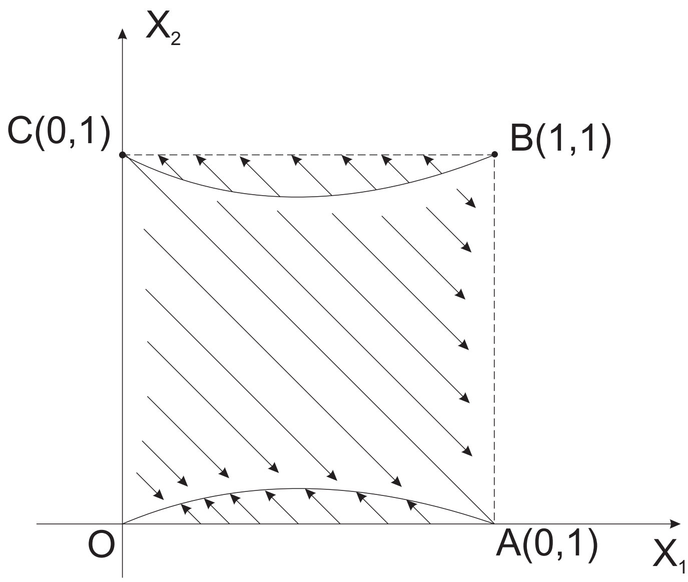

Figure 3: The vector field of system (8)

The set of stationary points (Fig. 3) consists of two fragments of the hyperbole described by the equation

$$(a_1 - 1)x_1(1-x_2) + (1-a_2)x_2(1-x_2) = 0$$

at the square $OABC$. The direction of the vector field is checked directly. The theorem is proved.

More general formulation of the problem for two petals has been considered in [3],[5].

3 General Properties of System (3)Solutions

We research a question of stationary points (3). Let $\vec{\rho}^* = (\rho_1^* ... \rho_n^*)$ be a stationary point, $\vec{f}^* = (f(\rho_1^*)...f(\rho_n^*))$,

$$(A-E)\vec{f}^* = \vec{0}. \tag{9}$$

As $\det(A-E) = 0$, then a nontrivial solution of the system $(A-E)\vec{x} = \vec{0}$, $\vec{x} \in R^n$ exists. As A^T is a stochastic matrix, for spectral radiuses $\rho(A)$ and $\rho(A^T)$ the following inequality is true

$$\rho(A^T) = 1 \leq \rho(A.) \tag{10}$$

If A is a double stochastic matrix, i.e. A and A^T are stochastic, then under condition of positivity of all elements A the Perron-Frobenius theorem [2] is true.

Lemma 3.1. *There exists and unique to within normalize a vector* $\vec{f^*} = (f_1^* ... f_n^*) \in R_+^n$ *with positive elements such, that*

$$A\vec{f^*} = \vec{f^*}. \tag{11}$$

It is obvious, that

$$\vec{f^*} = \lambda^*(1, 1, ...1). \tag{12}$$

Suppose that

$$\lambda^* = 0.25. \tag{13}$$

The equation

$$f(\rho) = a \tag{14}$$

has no more than two solutions, at $a < 0$ or at $a > \frac{1}{4}$ it is insoluble. If $0 < a < \frac{1}{4}$ then there exists numbers ρ_1, ρ_2, $0 < \rho_1 < \frac{1}{2} < \rho_2 < 1$, $\rho_1 + \rho_2 = 1$ such, that

$$f(\rho_1) = f(\rho_2) = a. \tag{15}$$

Thus, we will denote by

$$\rho_1 = f_-^{-1}(a), \rho_2 = f_+^{-1}(a.) \tag{16}$$

Let $C > 0$ is fixed, $\lambda > 0$, $\vec{f^*}$ is the solution of (12). We consider the following equation

$$\varphi_{\vec{\delta}}(\lambda) = f_{\delta_1}^{-1}(\lambda f_1^*) + f_{\delta_2}^{-1}(\lambda f_2^*) + ... + f_{\delta_n}^{-1}(\lambda f_n^*) = C. \tag{17}$$

The function $\varphi_{\vec{\delta}}(\lambda)$, where $\vec{\delta}$ is the fixed set of ± 1, $\vec{\delta} \in (\pm 1, \pm 1 ....., \pm 1)$ is continuous and determined on an interval $\lambda \in [0, 1]$. As A and A^T are stochastic matrixes, then

$$f^* = (1/4)(1, 1...1),$$

$$\varphi_{\vec{\delta}}(\lambda) = f_{\delta_1}^{-1}(\lambda/4) + f_{\delta_2}^{-1}(\lambda/4) + ... + f_{\delta_n}^{-1}(\lambda/4), \tag{18}$$

where $\delta_i = 1$ or $\delta_i = -1$. As

$$f_-^{-1}(a) + f_+^{-1}(a) = 1, \tag{19}$$

then

$$\varphi_{\vec{\delta}}(\lambda) = s + (n - 2s)\, f_{+}^{-1}\left(\lambda/4\right), \tag{20}$$

where $[\vec{\delta}] = n - 2s$ is *the signature,* excess of quantity of coordinates pluses of a vector $\vec{\delta}$ above quantity of minuses $(n \geq 2s)$, or

$$\varphi_{\vec{\delta}}(\lambda) = s + (n - 2s)\, f_{-}^{-1}\left(\lambda/4\right) \tag{21}$$

if of quantity of coordinates minuses of a vector $\vec{\delta}$ excess above quantity of pluses one.

Let us denote $\vec{\delta}_{\pm} = (\vec{\delta} \cup -\vec{\delta})$. **Lemma 3.2.** *Two-value function*

$$\varphi_{\vec{\delta}_{\pm}}(\lambda) = s + (n - 2s)\, f_{\pm}^{-1}\left(\lambda/4\right) \tag{22}$$

has the continuous plot from two monotonous components with area of values $[s, n - s]$.

Proof follows from an explicitfunction of $f(x)$.

Corollary. *For any* $C \in [s, n - s]$, $[\vec{\delta}] = n - 2s$, $n \geq 2s$ *there exists and unique* $\lambda \in [0, 1]$, *wich satisfies the equation*

$$\varphi_{\vec{\delta}_{\pm}}(\lambda) = s + (n - 2s)\, f_{\pm}^{-1}\left(\lambda/4\right) = C. \tag{23}$$

As at fixed s, that conterminous on parity with n it is possible to choose C_n^{n-s} ways of distribution of marks $\pm$ in a chain from n symbols so, that signature will be equal $n - 2s$ $(n > 2s)$, at $C \in [s, n-s]$ there is an appropriate quantity of branches of stationary points.

Theorem 3.1. *If* A *is a double stochastic matrix with positive elements, then for every* $s = 0, 1...., [n/2]$ *exists* C_n^{n-s} *branches of stationary points with a range of definition* $C \in [s, n - s]$.

For $n = 3$ this result is present in [6] for partial singular case of an one-dimensional chain.

4 Traffic Stability on Double Petal Flower for One-parametrical Families of State Functions

Let's consider double petal traffic flower with state function , i.e. dependence of flow speed from density of the following type $v(x) = (1 - x)^{\gamma}, 0 \leq x \leq 1$, i.e.,

$$f(x) = f(x, \gamma) = x(1 - x)^{\gamma}, 0 \leq x \leq 1. \tag{24}$$

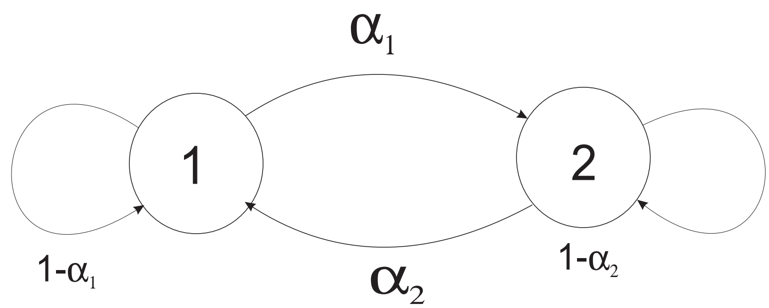

Figure 4: The dual image of a 2-flower.

In case of a stochastic matrix of transition we have

$$\alpha_1 = \alpha_2 = \alpha.$$

Thus, the equation for density, for example, on the first petal looks like

$$\dot{\rho} = -\alpha f(\rho) + \alpha f(C - \rho) = -\alpha f(\rho, \gamma) + \alpha f(C - \rho, \gamma) = g(\rho, \gamma, C) \tag{25}$$

with following conditions

$$max(0, C - 1) \leq \rho \leq min(1, C). \tag{26}$$

At $\gamma = 1$ we receive model of paragraph 2, i.e. *the solution (25) is stable in and only if* $C < 1$. If $\gamma = 0$, that

$$\dot{\rho} = -\alpha\rho + \alpha(C - \rho) = \alpha(C - 2\rho.) \tag{27}$$

The equation (27) has a stationary point $\rho^* = C/2,$, which is stable for any permitted initial conditions.

We research a problem of stability of the equation (25) - (26) at any $\gamma \in [0, \infty)$.

Lemma 4.1. *(1) Function* $f(x, \gamma) = x(1 - x)^\gamma$ *achieves a maximum in the unique critical point* $f'(x^*, \gamma) = 0$, $x^* = \frac{1}{1+\gamma}$;

(2) Function $f(x, \gamma)$ *increases on an interval* $[0, \frac{1}{1+\gamma}]$ *and also decreases on an interval* $[\frac{1}{1+\gamma}, 1]$;

(3) If $1 < C < 2(1 + \gamma)^{-1}$, *then on an interval* $[max(0, C - \frac{1}{1+\gamma}), min(\frac{1}{1+\gamma}, C)]$ *the function* $f(x)$ *increases, and function* $f(C - x)$ *decreases.*

Proof. It is checked directly.

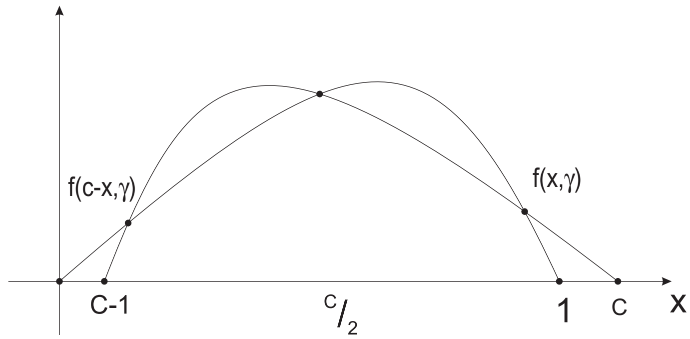

Figure 5: $0 < \gamma < 1, 1 < C < 2$.

Lemma 4.2. *(1) Function* $g(x,\gamma,C)$ *is determined on an interval* $[max(0,C-1), min(1,C)]$;

(2) $g(x,\gamma,C)$ *is symmetric concerning the point* $(C/2, g(C/2,\gamma,C) = 0)$, *i.e. is odd relative to* $x = C/2$;

(3) Let $C > 1$. *If* $C - \frac{1}{1+\gamma} < \frac{1}{1+\gamma}$, *that on the interval* $(C-1, C-\frac{1}{1+\gamma})$ *function* $g(x,\gamma,C)$ *has at least one zero.*

Proof. It is checked directly.

lemma 4.3. *If* $1 < C < \frac{2}{1+\gamma}$, *and if* $C - 1 < x < C - \frac{1}{1+\gamma}$, *then function* $g(x,\gamma,C)$ *has only one zero.*

Proof. It is directly calculated, that

$$f"(x) = (1-x)^{\gamma-2}(-2\gamma + (\gamma^2+\gamma)x).$$

Similarly,

$$g"(x) = (1-(C-x))^{\gamma-2}(-2\gamma + (\gamma^2+\gamma)(C-x)) -$$
$$-(1-x)^{\gamma-2}(-2\gamma + (\gamma^2+\gamma)x). \tag{28}$$

On an considered interval the following is true $g"(x,\gamma,C) < 0$. Therefore, if quantity of zero $g(x,\gamma,C)$ is more than one, then as this function has on the ends of an interval values of different marks, then quantity of zeroes is not less than three, otherwise from Roll's theorem $g"(x,\gamma,C)$ should not have not than one zero.

Lemma 4.4. *The quantity of zero of functions* $g(x,\gamma,C)$ *and* $g(1-x, \frac{1}{\gamma}, 2-C)$ *coincides.*

Proof. The equation $f(x,\gamma) = f(C - x,\gamma)$ is equivalent to the following

$$f^{\frac{1}{\gamma}}(x,\gamma) = f^{\frac{1}{\gamma}}(C - x,\gamma), x \in [0,1],$$

which is in turn equivalent

$$f^{\frac{1}{\gamma}}(1 - x,\gamma) = f^{\frac{1}{\gamma}}(C - 1 + x,\gamma), x \in [0,1].$$

However

$$f^{\frac{1}{\gamma}}(1 - x,\gamma) \equiv f(x,\frac{1}{\gamma}),$$

and also

$$f^{\frac{1}{\gamma}}(C - x,\gamma) \equiv f(2 - C - x,\frac{1}{\gamma}),$$

whence all follows.

Theorem 4.1. *(1) Let $0 < \gamma < 1$. Then for the equation (25) the following is true*

(1.1) If 0 $< C < 1$, then there is a unique stationary point $\rho \equiv C/2$ with area of an attraction $[0,C]$.

(1.2) If $1 \le C \le \frac{2}{\gamma+1}$, then there is a stationary point $\rho \equiv C/2$ with area of an attraction $x_1^* < \rho < x_2^*$, and unstable stationary points $\rho \equiv \rho_1^*$ with an achive in a critical mode $\rho < \rho_1^*$ and $\rho \equiv \rho_2^*$ with an achive in a critical mode at $\rho > \rho_2^*$.

(1.3) If $C > \frac{2}{\gamma}$ then there is unique and unstable stationary point $\rho \equiv C/2$ with area of achive in critical mode $[C-1, C/2)$, $(C/2, 1]$.

(2) Let $1 < \gamma < \infty$. Then

(2.1) If $1 < C < 2$, then there is a unique stationary point $\rho \equiv C/2$ with area of an achive in the critical mode $[C-1, C/2)$, $(C/2, 1]$.

(2.2) If $\frac{2}{\gamma+1} < C < 1$, then there is the unstable stationary point $\rho \equiv C/2$ and two stable stationary points $\rho \equiv \rho_1^*$ with area of an attraction $0 < \rho < C/2$, and $\rho \equiv \rho_2^*$ with area of the attraction $C/2 < \rho < C$.

(2.3) If $0 < C < \frac{2}{\gamma+1}$ then there is the unique stationary point $\rho \equiv C/2$ with area of attraction $[0,C]$.

Proof. The proof follows from Lemmas 4.1-4.4.

5 Linear Traffic Flower

Let's consider the case, when the intensity of movement linear depends on density, i.e. speed is constant.

5.1 Some Special Case

Let's consider the following traffic flower with a matrix which is not being double stochastic

$$\begin{cases} \frac{d\rho_1}{dt} = -\frac{n-1}{n}\rho_1 + \rho_2 + ... + \rho_n \\ \frac{d\rho_2}{dt} = \frac{1}{n}\rho_1 - \rho_2 \\ \frac{d\rho_3}{dt} = \frac{1}{n}\rho_1 - \rho_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{d\rho_n}{dt} = \frac{1}{n}\rho_1 - \rho_n \end{cases} \tag{29}$$

For the equations (29), since the second, linear be relative appropriate variables, solutions can be written down as

$$\rho_k(t) = \frac{1}{n}\int \rho_1(t)e^{-t}dt, k \geq 2, \tag{30}$$

and by virtue of a condition $\rho_1 + ... + \rho_n \equiv C$ the first equation (29) will become

$$\dot{\rho_1}(t) = -\frac{n-1}{n}\rho_1 + (C - \rho_1) = -\frac{2n-1}{n}\rho_1 + C.$$

It is clearly, that if

$$-\frac{2n-1}{n}\rho_1(0) + C > 0, \tag{31}$$

then the solution $\rho_1(t)$ increases until the condition $-\frac{2n-1}{n}\rho_1(t) + C = 0.$ will not be executed yet. Therefore, if

$$\frac{Cn}{2n-1} \geq 1, \tag{32}$$

the system (23) passes gets into a critical mode, as (25) in this case it is carried out automatically. If $\frac{Cn}{2n-1} < 1$, then by virtue (32) $\rho_1(0) < \frac{Cn}{2n-1}$ and $\rho_1(t)$ increases on t

$$\rho_1(t) \nearrow \frac{Cn}{2n-1},$$

and if $\rho_1(0) > \frac{Cn}{2n-1}$, then

$$\rho_1(t) \searrow \frac{Cn}{2n-1}.$$

Thus $\rho_1(t)$ does not get a critical mode.

As the obvious kind of the solution (28) looks like

$$\rho_1(t) = \frac{Cn}{2n-1} + Ae^{-\frac{2n-1}{n}t}, \tag{33}$$

that at $k \geq 2$

$$\rho_k(t) = \frac{1}{n}\int (\frac{Cn}{2n-1} + Ae^{-\frac{2n-1}{n}t})e^{-t}dt =$$

$$= -\frac{C}{2n-1}e^{-t} - \frac{A}{3n-1}e^{-\frac{3n-1}{n}t} + B_{k.} \tag{34}$$

If $A > 0$, i.e. $\rho_1(0) > \frac{Cn}{2n-1}$, then (34) decreases on t and all $\rho_k(t)$, $k \geq 2$ are noncritical.

If $\rho_1(0) < \frac{Cn}{2n-1}$, then $A = \rho_1(0) - \frac{Cn}{2n-1}$,

$$\rho_k(t) = -\frac{C}{2n-1}e^{-t} + \frac{Cn}{(2n-1)(3n-1)}e^{-\frac{3n-1}{n}t} -$$

$$-\frac{\rho_{1(0)}}{3n-1}e^{-\frac{3n-1}{n}t} + B_k < -\frac{\rho_1(0)}{3n-1}e^{-\frac{3n-1}{n}t} + B_k \searrow B_k,$$

$k \geq 2$.

Hence, all functions ρ_k also are noncritical. *Theorem 4. The system (29) gets into a critical mode if and only if* $\frac{Cn}{2n-1} > 1$.

5.2 Stochastic Flow

As before $A = (a_{ij})$ is the double stochastic matrix $n * n$, i.e. $a_{ij} > 0$, $\forall i, j$ $\sum_{j=1}^{n} a_{ij} = 1$, $i = 1, ..., n$. Therefore $\vec{\rho} = (\rho_1, ..., \rho_n)$ is the vector with not negative coordinates, $0 \leq p_i \leq 1, i = 1, ..., n$,

$$\|\vec{\rho}\|_{l_1^n} = \rho_1 + ... + \rho_n = C. \tag{35}$$

Let's consider a system

$$\begin{cases} \dot{\rho}_1 = (a_{11} - 1)\rho_1 + a_{21}\rho_2 + ... + a_{n1}\rho_n \\ \dot{\rho}_2 = a_{12}\rho_1 + (a_{22} - 1)\,\rho_2 + ... + a_{n2}\rho_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{\rho}_n = a_{1n}\rho_1 + a_{2n}\rho_2 + ... + (a_{nn} - 1)\,\rho_n, \end{cases} \tag{36}$$

Under the condition of the Perron-Frobenius theorem the system (36) has unique stationary point

$$f^* = C(1, 1, ...1)/n \tag{37}$$

Theorem 5.1.

For any allowable initial conditions $\vec{\rho}(0), \|\vec{\rho}\|_{l_\infty^n} = \max\limits_{1\leq i\leq n} |\rho_i| < 1$, *a solution(35)-(36) converges to the stationary (37) for any allowable loading* $0 < C < n$.

Proof. See also [8].

Lemma 5.1. If $\vec{\rho}(t)$ is the solution of the system (37) with initial conditions, wich satisfies (35) and (36), then

$$\lim_{t\to\infty} \rho(\vec{t}) = \vec{f^*}.$$

Lemma 5.2. If $\vec{\rho}(t)$ is the solution of the system (37), then $\vec{\rho}(t)$ is separated from critical modes $\max\limits_{1\leq i\leq n}|\rho_i| = 1$.

6 Optimization and Control on Traffic Flower

The traffic flow on a flower is characterized by a vector of density $\vec{\rho} = (\rho_1..., \rho_n)$ (pic. 4, $n = 4$) and a intensity vector $\vec{q} = (q_1 = f(\rho_1)..., q_n = f(\rho_n))$.

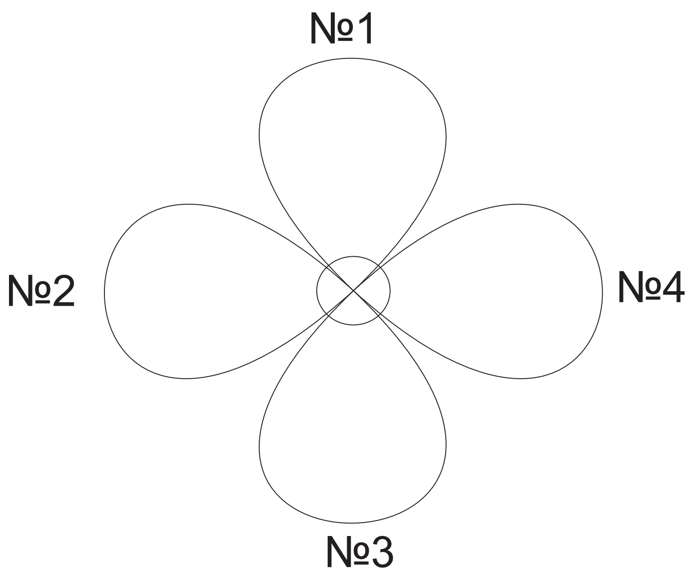

Figure 6: Traffic 4 - a flower

As the length of each edge is equal to unit, as well as weight of each particle of flow, then $q_i, i = 1..., n$ is capacity of a flow on an edge with number i, and the value

$$\Sigma_i q_i = Q$$

is the capacity of a flow on a transport flower at fixed t, and the value

$$\int_{t_0}^{t_1} Q(t)dt = A(t)$$

is the flow work in a time interval $[t_0, t_1]$. One of probable criterias of control is

$$Q(t) \to \max, \|\rho\|_{l_1} = \rho_1(t) + ... + \rho_n(t) \equiv C$$

i.e.

$$f(\rho_1) + ... + f(\rho_n) \to \max, \rho_1(t) + ... + \rho_n(t) \equiv C. \tag{38}$$

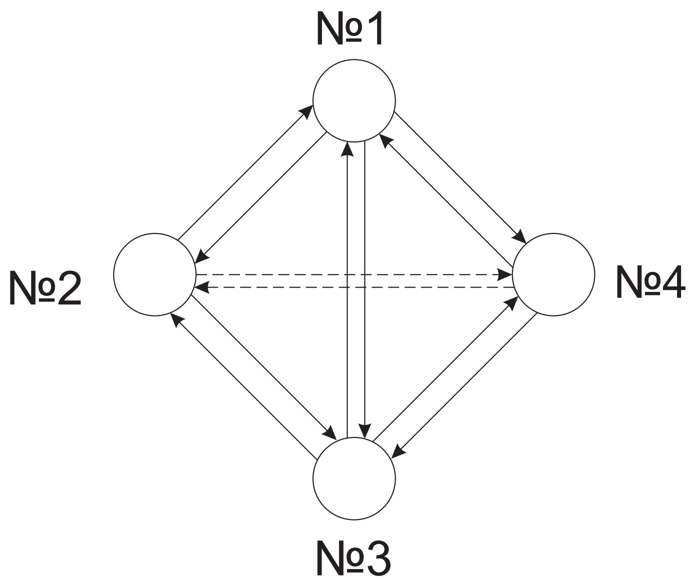

Figure 7: The dual image of a 4-flower.

Necessary conditions of extremum of a problem (38) are

$$f'(\rho_i^*) = \lambda,$$

or $\rho_i^* = 0$.

If $f'$is monotonously decreases, then within normalized critical points iof the problem (38) looks like

$$\rho_1^* = \rho_2^* = ... = \rho_m^*, \rho_{m+1} = ... = \rho_n = 0.$$

Hence, $\rho_i^* = C/m$, $1 \leq i \leq m$ and we have

$$Q_m = mf(C/m) \to \max, 1 \leq m \leq n.$$

We assume that function $H(x) = f(x)/x$, $0 \leq x \leq C$. If function $H(x)$ decreases on an interval $[0, 1]$, then a sequence Q_m grows on parameter m. So in a case $f(x) = x(1-x)$ we will receive $H(x) = (1-x)$, i.e. $H'(x) < 0$. So $\max_m(mf(C/m))$ is reached at $m = n$. Thus, it is true.

Theorem 6.1. *If $f'(x)$ and $H(x)$ monotonously decrease on an interval $[0, 1]$, then the maximal capacity at any allowable loading $0 < C < n$ it is reached for equaldistributed flow $\rho_i^* = C/n$, $i = 1.., n$.*

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ON THE RELATIVE CONTROLLABILITY AND MINIMUM ENERGY CONTROL OF SYSTEMS WITH DELAY IN STATE AND CONTROL VARIABLES

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Abstract

This research work establishes necessary and sufficient conditions for relative controllability and minimum control energy of linear time varying systems with delays in state and control variable. The aim is to use the integral equivalence of our system to deduce our controllability matrix and the reachable set. With the aid of the properties of these matrices, we achieve our results by some equivalent relationship.

Keywords: Relative controllability, control energy, linear system, controllability matrix, reachable set.

2000 Mathematics Subject Classification: Primary 93B05, Secondary 34H05

1. INTRODUCTION

The study of controllability of systems, first carried out in details by Kalman [11] has attracted lots of interest in modern research notably from Davies and Jackreece [5], Jackreece and Davies [10], Obukhovskii and Rubbioni [13] etc. because of it's direct connectivity with variety of mathematical models. Controllability of systems can be encountered in many fields of science, engineering and amongst others environmental management, industrial processes, medicine, biology, economy.

This research work is not only interested in the controllability of systems but also in reaching the targets of systems with minimum wastage of control energy. It is evident that successes in life pursuits are predicated upon our ability to direct our energies in reaching the desired target. This enigma has created the necessity for us to sort out for ways of achieving this success with minimum energy control in the shortest possible time. However, the need to steer the state of a system from an initial point to a desired target with minimum energy control poses a challenge despite the pioneering and ambitious work on the optimal control problem of single degree-of-freedom differential system of the form

$$\ddot{x} + f(x, \dot{x}) = u \qquad (1.1)$$

by Lee and Markus [2] where they found controllers for the system (1.1) which steers the initial states of the system to the equilibrium time optimally, and then indicatated how these controllers so found for each initial state can be used to construct the feedback controller synthesis. Hermes and Lasalle [3] gave the linear time optimal control problem for the ordinary differential system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad (1.2)$$

by finding a control subject to its constraints in such a way that the solution of (1.2) reaches a continuously moving target in the state space in minimum time. They further gave this minimum energy control required to transfer the system from an initial state to target and defined the energy control on the assumption that the system is controllable.

The dividend of these earlier works is in lending focus and clarity of definition to the minimum energy control problem. Recently, Iheagwam [12], Davies [4] working independently gave answers to the following questions that form the crux of the minimum energy control problem. Does a minimum energy control for the pursuit of a moving target in a context described by a differential system exist? What is the form of this energy control? Is it unique? Sebakhy and Bayoumi [9] studied the controllability of linear time-varying systems with delay in control of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)u(t-h) \qquad (1.3)$$

were they gave an expression for the control required to transfer the system (1.3) from a given state to any desired state using minimum control energy. Klamka [6], using the relative controllability matrix of a linear time varying system with distributed delays in the control of the form

$$\dot{x}(t) = A(t)x(t) + \int_{-h}^{0} [dH(t,s)]u(t+s) \qquad (1.4)$$

examined the minimum control energy and derived the control law for the system (1.4).

The objective of this research is to extend the work in Klamka [6] by establishing necessary and sufficient condition for the relative controllability and also derive control laws which establishes the minimum energy control required to transfer a class of linear time varying systems with delays in state and control variables given by

$$\dot{x}(t) = \sum_{i=0}^{h} A_i \, x(t-i) + Bu(t) \quad + \quad \sum_{i=1}^{h} D_i \, u(t-i) \qquad (1.5)$$

$t \in [t_0, t_1]$, $x(t) = \phi(t)$, $\phi \in [-h, 0]$

An example is also given

2. BASIC NOTATIONS AND PRELIMINARIES

Consider the system (1.5), where $x(t)$ is an n-vector, A_i are $n \times n$ matrices and B , D_i are $n \times m$ matrices. The control function $u(t) \in E^m$ is a measurable $m-$vector. Here $E = (-\infty, \infty)$ is the real line and E^n is the n-dimensional Euclidean space with norm $|\cdot|$. We let $C = (C[-h, 0]\ ,\ E^n)$ be the Banach space of continuous functions and the norm of an element ϕ in C by

$$\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$$

We let $L_1([a,b]\ ,\ E^m)$ be the space of Lebesgue integrable functions taking $[a,b]$ into E^n with

$$\|\phi\| = \int_a^b |\phi(s)| ds\ ,\ \phi \in L_1\left([a,b]\ ,\ E^n\right).$$

$L_2([a,b], E^m)$ is the space of square integrable functions taking $[a,b] \to E^m$

If $x \in C([a,b]\ ,\ E^n)$ for any $a < b$, then for each fixed $t \in [a,b]$, the symbol x_t denotes an element of C given by $x_t(s) = x(t+s)$. For functions $u \in L_2([a,b], E^m)$ the symbol u_t is similarly defined.

In this paper, the control space will be $L_2^{loc}([a,b], E^m)$, the space of essentially bounded measurable functions on finite intervals with values in E^m. The control constraint set will be denoted by

$U = L_2^{loc}([a,b], E^m)$, where $C^m = \{u \in E^m : |u_j| \le 1,\ j = 1, \ldots\ldots, m\}$

The above conditions on A_i , B and D_i ensure that for each initial data (t_0, ϕ), a unique solution of (1.5) exists through (t_0, ϕ) (see Hale [7], p. 142) which is continuous in (t_0, ϕ) .The solution of system (1.5) at $t = t_1$ following Sebakhy and Bayoumi [9], Manitius and Olbrot [1] will be given as

$$x(t_1,t_0,\phi,u) = X(t_1,t_0)\phi(t_0) + \sum_{i=1}^{h}\int_{-i}^{t_0} X(t_1,s+i)A_i\phi(s)\,ds$$

$$+\int_{t_0}^{t_1} X(t_1,s)\left| Bu(s) + \sum_{i=1}^{h} D_i u(s-i)\right| ds$$

or

$$x(t_1,t_0,\phi,u) = \sum_{i-1}^{h}\int_{-i}^{t_0} X(t_1,s+i)\,A_i\,\phi(s)\,ds$$

$$+ X(t_1,t_0)\left\{ \phi(t_0) + \sum_{i=1}^{h}\int_{-i}^{t_0} X(t,s+i)D_i\, u_{t_0}(s)\,ds \right\}$$

$$+\int_{t_0}^{t_1-i}\left| X(t_1,s)B + \sum_{i=1}^{h} X(t_1,s+i)\,D_i\right| u(s)\,ds$$

$$+\int_{t_1-i}^{t_1} X(t_1,s)\,Bu(s)\,ds \tag{2.1}$$

where $X(t,s)$ is the fundamental solution of (1.5) which satisfies the equation

$$\frac{\partial}{\partial t}X(t,s) = \sum_{i=0}^{h} A_i\, X(t-i,s) \ ,\ t > s \tag{2.2}$$

$$X(t,s) \ = \ \begin{cases} I \ , & t = s \\ 0 \ , & t < s \end{cases}$$

or

$$\frac{\partial}{\partial s}X(t,s) = -\sum_{i=1}^{h} X(t,s+i)\,A_i\,\phi(s) \ ,\ t > s \tag{2.3}$$

Following the methods of Dauer and Gahl [8], we define a matrix function Z by

$$Z(t_0,l,z) = \begin{cases} X(t_1,s)B + \sum_{i=1}^{h} X(t_1,s+i)\,D_i \quad for \quad t_0 \le s < t_1 - i \\ X(t_1,s)B \quad for \quad t_1 - i \le s \le t_1 \end{cases} \tag{2.4}$$

We now give some definition upon which our study hinges.

Definition 2.1

The complete state at time t of system (1.5) is said to be $z(t) = \{ x(t), \phi, u_t \}$.

Definition 2.2

Following Arstein [14], we define relative controllability of system (1.5). That, system (1.5) is relatively controllable on $[t_0, t_1]$, if for every $z(t_0)$ and every vector $x_1 \in E^n$, there exists a control $u \in B$, such that the corresponding trajectory of system (1.5) satisfies $x(t_1) = x_1$.

We now define the $n \times n$ symmetric and semi positive controllability matrix of our system as

$$W(t_0, t_1, z) = \int_{t_0}^{t_1} Z(t_0, l, z)\, Z^{T}(t_0, l, z)\, dl$$

where the symbol T denotes the matrix transpose. The controllability matrix satisfies the condition

$$x^2 W(t_0, t_1, z)x = \langle x, W(t_0, t_1, z)x \rangle = \| x \|^2_{W(t_0, t_1, z)} \geq 0 \quad \text{for all } x \in E^n$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product space in E^n

The reachable set of (1.5) at t_1 is given by

$$R(t_0, l, z) = \left\{ \int_{t_0}^{t_1} Z(t_1, l, z) u(l)\, dl : u \in L_2 \right\} \tag{2.5}$$

here $R(t_0, l, z)$ is an $m \times m$ continuous symmetric matrix defined on $[t_0, t_1]$ such that $R^{-}(t_0, l, z)$ exists for all $t \in [t_0, t_1]$. We now introduce the following notation for brevity.

$$W_1(t_0, t_1, z) = \int_{t_0}^{t_1} \left[Z(t_0, l, z)\, R^{-}(t_0, l, z) \right] . Z^{T}(t_0, l, z) dl \tag{2.6}$$

$$\bar{u} = R^{-}(t_0, l, z) Z^{T}(t_0, l, z) . W_1^{-1}(t_0, t_1, z) R(t_0, l, z) \tag{2.7}$$

3. CONTROLLABILITY RESULTS

Here sufficient conditions for the controllability of system (1.5) when L_2 control is assumed are formulated and proved.

Theorem 3.1

The following are equivalent

(i) $W(t_0, t_1, z)$ is non singular for each t

(ii) System (1.5) is proper in E^n for each interval $[t_0, t_1]$

(iii) System (1.5) is relatively controllable on each interval $[t_0, t_1]$

Proof: We shall show that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) to complete the proof. Let us show first that (i) $\Rightarrow$ (ii)

Define the operator, $K : L_2([t_0, t_1], E^{\mathrm{m}}) \to E^{\mathrm{n}}$ by

$$K(u) = \int_{t_0}^{t_1} Z(t_0, l, z)\, u(l)\, dl \qquad (3.1)$$

where K is a continuous linear operator from one Hilbert space to another. Thus, $R(K) \subseteq E^n$ is a linear subspace and its orthogonal complement satisfies the relation

$$\{R(K)\}^{\perp} = N(K^*) \qquad (3.2)$$

where K^* is the adjoint of K, $K^* : E^n \to U \subseteq L_2$ by the non-singularity of the controllability matrix $W(t_0, t_1, z)$, the symmetric operator $KK^* = W(t_0, t_1, z)$ is positive definite and hence

$$\{R(K)\}^{\perp} = \{0\} \qquad (3.3)$$

That is, $N(K^*) = \{0\}$ (3.4)

For any $c \in E^n$, $u \in L_2$, $< c, Ku > = < K^* c, u >$

$$< c, K\,u > \; = \; < c, \int_{\mathrm{t}_0}^{\mathrm{t}_1} Z(t_0, l, z)\, u(\mathrm{l})\, dl > = \int_{\mathrm{t}_0}^{\mathrm{t}_1} c^{\mathrm{T}} \left[Z(t_0, l, z) \right] u(\mathrm{l})\, dl \qquad (3.5)$$

Thus, K^* is given by $c \to c^{\mathrm{T}} \left[Z(t_0, l, z) \right]$; $l \in [\mathrm{t}_0,\ t_1]$. $N(K^*)$ is therefore the set of all $c \in E^n$ such that

$$c^{\mathrm{T}} \left[Z(t_0, l, z) \right] = 0 \qquad (3.6)$$

almost everywhere in $[t_0,\, t_1]$, since $N(K^*) = \{0\}$ all such c are equal to zero, that is $c = 0$. This establishes the properness of system (1.5).

(ii) $\Rightarrow$ (iii)

We now show that system (1.5) is relatively controllable on each interval $[t_0,\, t_1]$.

Let $c \in E^n$, if system (1.5) is proper then $c^{\mathrm{T}}\left[Z(t_0, l, z)\right] = 0$

almost everywhere such that $l \in [t_0,\, t_1]$ for each t_1 implies $c = 0$. Thus

$$\int_{t_0}^{t_1} c^{\mathrm{T}}\left[Z(t_0, l, z)\right] u(l)dl = 0$$

for $u \in L_2$. It follows that the only vector orthogonal to the set

$$R(t_0, l, z) = \left\{ \int_{t_0}^{t_1} c^T Z(t_0, l, z)\, u(l)\, dl \;:\; u \in \mathrm{L}_2 \right\}$$

is the zero vector. Hence

$$\{R(t_0, l, z)\}^{\perp} = \{0\}$$

That is, $R(t_0, l, z) = E^n$. This establishes relative controllability on $[t_0,\, t_1]$ of system (1.5).

(iii) $\Rightarrow$ (i)

We now show that if the system (1.5) is relatively controllable then $W(t_0, t_1, z)$ is non-singular. Let us assume for contradiction that W is singular, then, there exists an n vector $V \neq 0$ such that

$$VWV^T = 0 \tag{3.7}$$

Then

$$\int_{t_0}^{t_1} \left\| V[Z(t_0, l, z)] \right\|^2 dl = 0 \tag{3.8}$$

This implies that

$$\left\| V[Z(t_0, l, z)] \right\|^2 dl = 0$$

almost everywhere, hence

$$V[Z(t_0,l,z)]=0 \tag{3.9}$$

almost everywhere for $l\in[t_0,\, t_1]$. This contradicts the assumption of properness of the system since $V\neq 0$ and this completes the proof.

Corollary 3.1

System (1.5) is relatively controllable on $[t_0,t_1]$ if and only

$$rank\, W(t_0,t_1,z)=n$$

Proof: This is Corollary 1 of Sebaky and Bayoumi [9]

Theorem 3.2

System (1.5) is relatively controllable if and only if $0\in \operatorname{int} R(t_0,l,z)$ for each $t_1>t_0$

Proof: If $R(t_0,l,z)$ is a closed and convex subset of E^n, then a point y_1 on the boundary of $R(t_0,l,z)$ implies that, there is a support plane Π of $R(t_0,l,z)$ through y_1, that is $c^T(y-y_1)\leq 0$ for each $y\in R(t_0,l,z)$ where $c\neq 0$ is an outward normal to Π. If u_1 is the control corresponding to y_1, we have

$$c^T\int_{t_0}^{t_1}[Z(t_0,l,z)]u(l)dl\leq c^T\int_{t_0}^{t_1}[Z(t_0,l,z)]u_1(l)dl$$

for each $u\in U$. Since U is a unit sphere, this last inequality holds for each $u\in U$, if and only if

$$\begin{aligned}c^T\int_{t_0}^{t_1}[Z(t_0,l,z)]\,u(l)dl &\leq \int_{t_0}^{t_1}c^T[Z(t_0,l,z)]\,u_1(l)dl\\ &\leq \int_{t_0}^{t_1}\left|c^T[Z(t_0,l,z)]\right|dl\end{aligned}$$

and $u_1(t) = \operatorname{sgn} c^T Z(t_0, l, z)$ as y_1 is on the boundary. Since we always have $0 \in R(t_0, l, z)$. If 0 were not in the interior of $R(t_0, l, z)$ then 0 is on the boundary, hence, from the foregoing, this implies $0 = \int_{t_0}^{t_1} \left| c^T [Z(t_0, l, z)] \right| dl$ so that $c^T [Z(t_0, l, z)] = 0$ almost everywhere $t \in [t_0, t_1]$. This by our definition implies that the system is not proper since $c^T \neq 0$. This completes the proof.

Theorem 3.3 - If system (1.5) is relatively controllable on $[t_0, t_1]$ for each $t_1 > t_0$, then the domain of null controllability of system (1.5) contains zero in its interior

Proof - Assume that system (1.5) is relatively controllable on $[t_0, t_1]$, $t_1 > t_0$ then by Theorem 3.2, $0 \in \operatorname{int} R(t_0, l, z)$, for each $t_1 > t_0$. Since $x = 0$ is a solution of system (1.5) with $u = 0$, we have $0 \in D$. Hence, If $0 \notin \operatorname{int} D$, then there exists a sequence $\phi_m \subseteq E^n$ such that $\phi_m \to 0$ as $m \to \infty$ and no ϕ_m is in D, that is $\phi_m \neq 0$. From (2.1), we have

$$0 \neq x(t_1, t_0, \phi, u) = X(t_1, t_0)\phi(t_0) + \sum_{i=1}^{h} \int_{-i}^{t_0} X(t_1, s+i) A_i \phi(s)\, ds$$
$$+ \int_{t_0}^{t_1} X(t_1, s) \left| Bu(s) + \sum_{i=1}^{h} D_i u(s-i) \right| ds$$

for any $t_1 > t_0$ and any $u \in U$. Hence, for $u = 0$,

$$z_m \overset{def}{=} x(t_1, t_0, \phi, 0) = X(t_1, t_0)\phi(t_0) + \sum_{i=1}^{h} \int_{-i}^{t_0} X(t_1, s+i) A_i \phi(s)\, ds,$$

is not in $R(t_0, l, z)$ for any $t_1 > t_0$. Therefore the sequence $z_m \subseteq E^n$ is such that $z_m \notin R(t_0, l, z)$, $z_m \neq 0$, but $z_m \to 0$ as $m \to \infty$. Therefore, $0 \notin R(t_0, l, z)$. This is a contradiction and hence proves that $0 \in \operatorname{int} D$.

4. THE MINIMUM ENERGY CONTROL

Here we derive an explicit expression for the control that transfers system (1.5) from $z(t_0)$ to x_1 at time t_1 and examine the minimum energy control required for this transfer.

Theorem 4.1

Let $\bar{u}(t)$ be any control which transfer $z(t_0)$ with the initial control $u(t_0)$ to x_1 at time t_1 and let $u^*(t)$ be the control defined by (2.7), then

$$\int_{t_0}^{t_1} \| \bar{u}(l) \|^2_{R(t_0,l,z)} dl \geq \int_{t_0}^{t_1} \| u^*(l) \|^2_{R(t_0,l,z)} dl \qquad (4.1)$$

almost everywhere on $[t_0, t_1]$ and the minimum control energy required for the transfer (assuming the transfer is possible) is given by

$$E(u^*) = \int_{t_0}^{t_1} \| u^*(l) \|^2_{R(t_0,l,z)} dl = \| R(t_0,l,z) \|^2_{W_1^{-1}(t_0,t_1,z)} \qquad (4.2)$$

Proof: substituting (2.7) into (2.1), it is easy to verify that the control $u^*(t)$ transfers the complete state $z(t_0)$ to x_1 at time t_1. Since $\bar{u}(t)$ and $u^*(t)$ transfer $z(t_0)$ to x_1 at time t_1, we have the following equalities

$$\int_{t_0}^{t_1} Z(t_0,l,z)\bar{u}(l)dl = \int_{t_0}^{t_1} Z(t_0,l,z)u^*(l)dl \qquad (4.3)$$

subtracting both sides and using the inner product gives

$$\left\langle \int_{t_0}^{t_1} Z(t_0,l,z)(\bar{u}(l) - u^*(l))\, dl . W_1^{-1}(t_0,t_1,z)R(t_0,l,z) \right\rangle = 0 \qquad (4.4)$$

using (2.7) and the properties of the inner product, we obtain

$$\int_{t_0}^{t_1} \left\langle \bar{u}(l) - u^*(l), u^*(l) \right\rangle dl = 0 \qquad (4.5)$$

By some easy manipulation and using (4.5) we derive

$$\int_{t_0}^{t_1} \| \bar{u}(l) \|^2_{R(t_0,l,z)} dl \geq \int_{t_0}^{t_1} \| u^*(l) \|^2_{R(t_0,l,z)} dl \qquad (4.6)$$

The minimal value of the control energy in transferring $z(t_0)$ to x_1 at time t_1 by using the control $u^*(t)$ is given by $E(u^*) = \int_{t_0}^{t_1} \| u^*(l) \|^2_{R(t_0,l,z)} dl = \int_{t_0}^{t_1} \| R^{-1}(t_0,l,z) Z^T(t_0,l,z).W^{-1}(t_0,t_1,z) R(t_0,l,z) \|^2 dl\, R(t_0,l,z)$

(4.7) Since the matrix $W_1(t_1,t_0,z)$ is symmetric, by the properties of the inner product and by (4.7), we have

$$E(u^*) = \int_{t_0}^{t_1} \left\langle \left[R(t_0,l,z)W_1^{-1}(t_1,t_0,z)\right]\left[Z(t_0,l,z)R^{-1}(t_0,l,z)\right]\left[Z^T(t_0,l,z)W_1^{-1}(t_0,t_1,z)R(t_0,l,z)\right]\right\rangle dl$$

$$= \left\langle R(t_0,l,z), W_1^{-1}(t_0,t_1,z)R(t_0,l,z)\right\rangle$$

$$= \| R(t_0,l,z) \|^2_{W_1^{-1}(t_0,t_1,z)}$$

This completes the proof.

5. EXAMPLE

Consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + Bu(t) + D_1 u(t) + D_2 (t-1) \tag{5.1}$$

where

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

To verify relative controllability of (5.1), it is easily seen that, the principal fundamental matrix solution is given by

$$X(t,s) = \begin{pmatrix} e^{-t} & 0 \\ e^{1-t} - e^{1-2t} & e^{-2t} \end{pmatrix}$$

and the matrix $Z(t_0, l, z)$ as defined in (2.4) will be given by $Z(t_0, l, z) = \begin{pmatrix} 0 \\ e^{-t} \end{pmatrix}$. The controllability matrix is given by $W(t_0, l, z) = \int_{t_0}^{t_1} \begin{pmatrix} 0 & 0 \\ 0 & e^{-2t} \end{pmatrix} dt$, while $rank\ W(t_0, l, z) = 1$. This implies that system (5.1) is relatively controllable by Corollary 3.1. Furthermore, we verify the minimum control energy for system (5.1) as follows; we require the reachable set defined in (2.5) with $u = 1$ and is given by

$$R(t_0, l, z) = \left\{ \int_{t_0}^{t_1} \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix} dt \right\}$$

Hence, the minimum control energy will be given as

$$E(u^*) = \left\| \int_{t_0}^{t_1} \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix} dt \right\|^2$$

CONCLUSION

From the sequel, the relative controllability of system (1.5) has been established. Also established is the relationship between relative controllability of the system (1.5) and the domains of relative controllability. This study has been able to show that, if a system is relatively controllable then zero is in the interior of the domain of the reachable set. From the above results, the minimum control energy required to transfer system (1.5) from an initial state to a targeted state within a specified time limit in the state space has also been established.

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Global Existence of Solutions for Nonlinear integral equations of second order

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ABSTRACT

We prove the existence of mild solutions of nonlinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions in Banach spaces. The results are obtained by using the theory of strongly continuous cosine family, application of the topological transversality theorem known as Leray-Schauder alternative and rely on a priori bounds of solutions.

1. Introduction

Let X be a Banach space with norm $\|.\|$. Let $B = C([0,b], X)$ be the Banach space of all continuous functions from $[0,b]$ into X endowed with supremum norm

$$\|x\|_B = sup\{\|x(t)\| : t \in [0,b]\}.$$

This paper concerns the mixed Volterra-Fredholm integrodifferential equation of the type:

$$x''(t) = Ax(t) + f\big(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^b h(t,s,x(s))ds\big),\ t \in J = [0,b] \tag{1.1}$$

$$x(0) + g(x) = x_0, \quad x'(0) = \eta, \tag{1.2}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in Banach space X, $f : J \times X \times X \times X \to X$, $k,\ h : J \times J \times X \to X$, $g : C(J; X) \to X$ are functions, and x_0 is a given element of X.

The existence, uniqueness and other properties of solutions of these equations (1.1) or special forms with classical conditions have been studied by many authors by using different techniques, see [10, 11, 12, 13, 14, 22].

The work on nonlocal initial value problem(IVP for short) was initiated by Byszewski. In [6, 7] Byszewski using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP. For the importance of nonlocal conditions in different fields, the interesting reader is referred to [1, 2, 3, 4, 9, 20, 21, 24, 25, 26, 27] and the references are given therein. For properties of semigroup theory, we refer the reader to the books [8, 18, 19, 23]

The objective of the present paper is to study the global existence of solutions of the equations (1.1)–(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and the theory of strongly continuous cosine family. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3 deals with main result.

2. Preliminaries and Hypotheses

Before we state our main result, we list the some preliminaries and hypotheses that will be used in our subsequent discussion.

In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order systems. We will make use of some of the basic ideas from cosine family and it is useful machinery for the study of abstract second order equations. We say that a family $\{C(t) : t \in R\}$ of operators in the space $L(X)$ of bounded linear operators on X is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator);
(ii) $C(t)x$ is strongly continuous in t on R for each fixed $x \in X$;
(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t,\ s \in R$.

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in R\}$, is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in R.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in R\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_t = 0, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(R, X)\}$. For more details on strongly continuous cosine and sine families, we refer the reader to the book of Goldstein [19] and to the papers of Fattorini [16, 17] and Travis and Webb [26, 27].

Definition 2.1. *A continuous solution $x(t)$ of the integral equation*

$$\begin{aligned} x(t) &= C(t)[x_0 - g(x)] + S(t)\eta \\ &\quad + \int_0^t S(t-s)f\big(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^b h(s,\tau,x(\tau))d\tau\big)ds, \quad t \in J \end{aligned}$$

is called a mild solution of (1.1)–(1.2) *on J.*

Let us list the following hypotheses:

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ compact for $t > 0$, and there exists a constant M such that

$$\|C(t)\|_{L(X)} \le M, \quad \text{for all} \quad t \in R.$$

(H_2) There exists a constant G such that

$$\|g(x)\| \le G, \ \text{for } x \in X.$$

(H_3) There exists a continuous function $p : [0, b] \to R_+$ such that

$$\|\int_0^t k(t, s, x(s))ds\| \le p(t)\|x\|,$$

for every $t, s \in [0, b]$ and $x \in X$.

(H_4) There exists a continuous function $q : [0, b] \to R_+$ such that

$$\|\int_0^b h(t, s, x(s))ds\| \le q(t)\|x\|,$$

for every $t, s \in [0, b]$ and $x \in X$.

(H_5) There exists a continuous function $l : [0, b] \to R_+$ and a continuous increasing function $K : R_+ \to (0, \infty)$ such that

$$\|f(t, x, y, z)\| \le l(t)K(\|x\| + \|y\| + \|z\|),$$

for every $t \in [0, b]$ and $x,\ y,\ z \in X$.

(H_6) For each $t \in [0, b]$ the function $f(t, ., ., .) : [0, b] \times X \times X \times X \to X$ is continuous and for each $x,\ y,\ z \in X$ the function $f(., x, y, z) : [0, b] \times X \times X \times X \to X$ is strongly measurable.

(H_7) For each $t,\ s \in [0, b]$ the functions $k(t, s, .),\ h(t, s, .) : [0, b] \times [0, b] \times X \to X$ are continuous and for each $x \in X$ the functions $k(., ., x),\ h(., ., x) : [0, b] \times [0, b] \times X \to X$ are strongly measurable.

(H_8) For every positive integer m there exists $\alpha_m \in L^1(0, b)$ such that

$$\sup_{\|x\| \le m, \|y\| \le m, \|z\| \le m} \|f(t, x, y, z)\| \le \alpha_m(t), \text{ for } t \in [0, b] \ a.\ e.$$

In the sequel we will use the following results:

Lemma 2.2 ([27])**.** *Let $C(t)$, (resp.$S(t)$), $t \in R$ be a strongly continuous cosine (resp. sine) family on X. Then there exist constants $N^* \ge 1$and $w \ge 0$ such that*

$$\|C(t)\| \le N^* e^{|t|}, \quad \text{for all} \quad t \in R,$$

$$\|S(t_1) - S(t_2)\| \le N^* |\int_{t_1}^{t_2} e^{w|s|} ds|, \quad \text{for all} \quad t_1,\ t_2 \in R.$$

Lemma 2.3 ([15], p-61)**.** *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let $\varepsilon(F) = \{x \in S : x = \lambda Fx \quad \text{for some} \quad 0 < \lambda < 1\}$. Then either $\varepsilon(F)$ is unbounded or F has a fixed point.*

3. Existence of Mild solutions

Theorem 3.1. *Let $g : B \to X$ be a continuous function. Assume that the hypotheses $(H_1) - (H_8)$ hold and if b satisfies the following condition*

$$Mb \int_0^b l(s)[1 + p(s) + q(s)]ds < \int_c^\infty \frac{ds}{K(s)}, \tag{3.1}$$

where

$$c = M(\|x_0\| + G + b\|\eta\|).$$

Then problem (1.1)-(1.2) *has at least one mild solution on J.*

Proof. To prove the existence of a mild IVP (1.1)-(1.2), we apply Lemma 2.3. First we establish the priori bounds for the mild solutions of the parameterized problem with parameter $\lambda \in (0, 1)$ such that

$$x''(t) = Ax(t) + \lambda f\big(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^b h(t, s, x(s))ds\big), \tag{3.2}$$

$$x(0) + \lambda g(x) = \lambda x_0, \quad x'(0) = \lambda \eta, \tag{3.3}$$

and show that the solutions to this system are bounded. First it is not hard to see that system (3.2)-(3.3) has a mild solution satisfying the integral equation

$$\begin{aligned} x(t) &= \lambda C(t)[x_0 - g(x)] + \lambda S(t)\eta \\ &\quad + \lambda \int_0^t S(t - s) f\big(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau\big)ds. \end{aligned}$$

Using hypotheses $(H_1) - (H_5)$ and the fact that $\lambda \in (0,1)$, we have

$$\begin{aligned}
\|x(t)\| &\le \|\lambda[C(t)(x_0 - g(x))]\| + \|\lambda S(t)\eta\| \\
&\quad + \|\lambda \int_0^t S(t-s) f\big(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^b h(s,\tau,x(\tau))d\tau\big)ds\| \\
&\le M(\|x_0\| + G) + Mb\|\eta\| \\
&\quad + \int_0^t Mb\|f\big(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \int_0^b h(s,\tau,x(\tau))d\tau\big)\|ds \\
&\le M[\|x_0\| + G + b\|\eta\|] + Mb\int_0^t l(s)K(\|x(s)\| + p(s)\|x(s)\| + q(s)\|x(s)\|)ds \\
&\le M[\|x_0\| + G + b\|\eta\|] + Mb\int_0^t Mbl(s)[1 + p(s) + q(s)]K(\|x(s)\|)ds. \qquad (3.4)
\end{aligned}$$

Define a function $u(t)$ by right-hand side of (3.4). Using the fact that K is continuous increasing function, we obtain

$$u(t) = M[\|x_0\| + G + b\|\eta\|] + Mb\int_0^t Mbl(s)[1 + p(s) + q(s)]K(\|x(s)\|)ds.$$

Then $\|x(t)\| \le u(t)$ and $u(0) = M[\|x_0\| + G + b\|\eta\|] = c$. Therefore,

$$\begin{aligned}
u(t) &= c + Mb\int_0^t l(s)(1 + p(s) + q(s))K(\|x(s)\|)ds \\
u(t) &\le c + Mb\int_0^s l(s)(1 + p(s) + q(s))K(u(s))ds.
\end{aligned}$$

Differentiating the above inequality and using the fact that K is increasing continuous, we get

$$\begin{aligned}
u'(t) &\le Mbl(t)(1 + p(t) + q(t))K(u(t)) \\
\frac{u'(t)}{K(u(t))} &\le Mbl(t)(1 + p(t) + q(t)). \qquad (3.5)
\end{aligned}$$

Integrating (3.5) from 0 to t and using change of variables $t \to s = u(t)$ and the condition (3.1), we obtain

$$\begin{aligned}
\int_{u(0)}^{u(t)} \frac{ds}{K(s)} &\le Mb\int_0^t l(s)(1 + p(s) + q(s))ds \\
&\le Mb\int_0^b l(s)(1 + p(s) + q(s))ds < \int_c^\infty \frac{ds}{K(s)}. \qquad (3.6)
\end{aligned}$$

From inequality (3.6) and mean value theorem we observe that there exists a constant γ, independent of $\lambda \in (0,1)$ such that $u(t) \le \gamma$ for $t \in J$ and hence $\|x(t)\| \le \gamma$ for $t \in J$ and consequently, we have

$$\|x\|_B = sup\{\|x(t)\| : t \in J\} \le \gamma.$$

In order to apply Lemma 2.3, we must prove that the operator $F : B \to B$ defined for $t \in J$ by

$$\begin{aligned}
(Fy)(t) &= C(t)[x_0 - g(y(t))] + S(t)\eta \\
&\quad + \int_0^t S(t-s) f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)ds \qquad (3.7)
\end{aligned}$$

is completely continuous operator.

Let $B_m = \{y \in B : \|y\|_B \leq m\}$ for some $m \geq 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X. Let $y \in B_m$ and $t_1, t_2 \in J$. Then if $0 < \epsilon < t_1 < t_2 \leq b$, we have

$$\begin{aligned}
&\|(Fy)(t_2) - (Fy)(t_1)\| \\
&\qquad = \|C(t_2)[x_0 - g(y)] + S(t_2)\eta \\
&\qquad\quad + \int_0^{t_2} S(t_2 - s) f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)ds \\
&\qquad\quad - C(t_1)[x_0 - g(y)] - S(t_1)\eta \\
&\qquad\quad - \int_0^{t_1} S(t_1 - s) f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)ds\| \\
&\qquad \leq \|C(t_2) - C(t_1)\|_{L(X)}(\|x_0\| + G) + \|S(t_2) - S(t_1)\|_{L(X)}\|\eta\| \\
&\qquad\quad + \int_0^{t_1-\epsilon} \|S(t_2 - s) - S(t_1 - s)\|_{L(X)} \\
&\qquad\quad \times \|f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)\|ds \\
&\qquad\quad + \int_{t_1-\epsilon}^{t_1} \|S(t_2 - s) - S(t_1 - s)\|_{L(X)} \\
&\qquad\quad \times \|f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)\|ds \\
&\qquad\quad + \int_{t_1}^{t_2} \|S(t_2 - s)\|_{L(X)}\|f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)\|ds \\
&\qquad \leq \|C(t_2) - C(t_1)\|_{L(X)}(\|x_0\| + G) + \|S(t_2) - S(t_1)\|_{L(X)}\|\eta\| \\
&\qquad\quad \int_0^{t_1-\epsilon} \|S(t_2 - s) - S(t_1 - s)\|_{L(X)}\alpha_m(s)ds \\
&\qquad\quad + \int_{t_1-\epsilon}^{t_1} \|S(t_2 - s) - S(t_1 - s)\|_{L(X)}\alpha_m(s)ds \\
&\qquad\quad + \int_{t_1}^{t_2} \|S(t_2 - s)\|_{L(X)}\alpha_m(s)ds. \qquad (3.8)
\end{aligned}$$

The right hand side of (3.8) is independent of $y \in B_m$ and tends to zero as $t_2 - t_1 \to 0$ and ϵ sufficiently small, since $C(t)$, $S(t)$ are uniformly continuous for $t \in J$ and the compactness of $C(t)$, $S(t)$ for $t > 0$ imply the continuity in the uniform operator topology (see Lemma 2.2). Thus FB_m is an equicontinuous family of functions with values in X.

We next show that FB_m is uniformly bounded. From the definition of operator F and using the hypotheses $(H_1) - (H_5)$ and the fact that $\|y\|_B \leq m$, we obtain

$$\begin{aligned}
\|(Fy)(t)\| &= \|C(t)[x_0 - g(y)] + S(t)\eta \\
&\quad + \int_0^t S(t - s) f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)ds\| \\
&\leq c + \int_0^t \|S(t - s)\|_{L(X)}\alpha_m(s)ds \\
&\leq c + \int_0^t Mb\alpha_m(s)ds
\end{aligned}$$

$$\leq c + Mb \int_0^b \alpha_m(s) ds.$$

This implies that the set $\{(Fy)(t) : \|y\|_B \leq m,\ 0 \leq t \leq b\}$ is uniformly bounded in X and hence FB_m is uniformly bounded.

We have already shown that FB_m is an equicontinuous and uniformly bounded collection. To prove that F maps B_m into a precompact set in B, it is sufficient, by Arzela-Ascoli's Theorem, to show that the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X for each $t \in J$.

Let $0 < t \leq b$ be fixed and ϵ real number satisfying $0 < \epsilon < t$. For $y \in B_m$, we define

$$\begin{aligned}(F_\epsilon y)(t) &= C(t)[x_0 - g(x)] + S(t)\eta \\ &\quad + \int_0^{t-\epsilon} S(t-s) f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big) ds. \qquad (3.9)\end{aligned}$$

Since $C(t)$, $S(t)$ are compact operators, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_m\}$ is precompact in X, for every $\epsilon, 0 < \epsilon < t$. Moreover, for every $y \in B_m$, we get

$$\begin{aligned}\|(Fy)(t) - (F_\epsilon y)(t)\| &\leq \int_{t-\epsilon}^t \|S(t-s)\|_{L(X)} \\ &\qquad \times \|f\big(s, y(s), \int_0^s k(s,\tau,y(\tau))d\tau, \int_0^b h(s,\tau,y(\tau))d\tau\big)\| ds \\ &\leq Mb \int_{t-\epsilon}^t \alpha_m(s) ds.\end{aligned}$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$. Hence the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X.

It remains to show that $F : B \to B$ is continuous. Let $\{u_n\}$ be a sequence of elements of B converging to u in B. Then there exists an integer r such that $\|u_n\| \leq r$ for all n and $t \in J$. By hypotheses $(H_6) - (H_8)$, we have

$$\begin{aligned}&f\big(t, u_n(t), \int_0^t k(t,s,u_n(s))ds, \int_0^b h(t,s,u_n(s))ds\big) \\ &\quad \to f\big(t, u(t), \int_0^t k(t,s,u(s))ds, \int_0^b h(t,s,u(s))ds\big)\end{aligned}$$

for each $t \in J$, and since

$$\begin{aligned}&\|f(t, u_n(t), \int_0^t k(t,s,u_n(s))ds, \int_0^b h(t,s,u_n(s))ds) \\ &\quad - f(t, u(t), \int_0^t k(t,s,u(s))ds, \int_0^b h(t,s,u(s)))ds\| \leq 2\alpha_r(t).\end{aligned}$$

Then by by dominated convergence theorem, we have

$$\begin{aligned}\|Fu_n - Fu\|_B &= \sup_{t\in[0,b]} \|(Fu_n)(t) - (Fu)(t)\| \\ &= \sup_{t\in J} \|C(t)[g(u) - g(u_n)] \\ &\quad + \int_0^t S(t-s)\Big[f\big(s, u_n(s), \int_0^s k(s,\tau,u_n(\tau))d\tau, \int_0^b h(s,\tau,u_n(\tau))d\tau\big) \\ &\quad - f\big(s, u(s), \int_0^s k(s,\tau,u(\tau))d\tau, \int_0^b h(s,\tau,u(\tau))d\tau\big)\Big] ds\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|C(t)\|_{L(X)}\|g(u)-g(u_n)\| \\
&\quad + \int_0^t \|S(t-s)\|_{L(X)}\Big\|\Big[f\big(s,u_n(s),\int_0^s k(s,\tau,u_n(\tau))d\tau,\int_0^b h(s,\tau,u_n(\tau))d\tau\big) \\
&\quad - f\big(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau,\int_0^b h(s,\tau,u(\tau))d\tau\big)\Big]\Big\|ds \\
&\leq M\|g(u)-g(u_n)\| \\
&\quad + Mb\int_0^t \Big[\Big\|f\big(s,u_n(s),\int_0^s k(s,\tau,u_n(\tau))d\tau,\int_0^b h(s,\tau,u_n(\tau))d\tau\big) \\
&\quad - f\big(s,u(s),\int_0^s k(s,\tau,u(\tau))d\tau,\int_0^b h(s,\tau,u(\tau))d\tau\big)\Big\|\Big]ds \to 0.
\end{aligned}
$$

Thus F is continuous. This completes the proof that F is completely continuous operator.

Finally, the set $\varepsilon(F) = \{y \in B : y = \lambda Fy,\ \lambda \in (0,1)\}$ is bounded in B as we proved in the first part. Consequently, by Lemma 2.3, the operator F has a fixed point in B. This means that the IVP (1.1)-(1.2) has a solution. □

Remark 3.2. We note that in the special case, if we take (i) $Mbl(s)[1+p(s)+q(s)] = 1$ in condition (3.1) and the integral on the right side in (3.1) is assumed to diverge, then the solutions of equations $(1.1)-(1.2)$ exist for every $b < \infty$.

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Tensor Product Technique and the Degenerate Homogeneous Abstract Cauchy Problem

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Abstract

In this paper, we use the technique of tensor product of Banach spaces to study the Abstract Cauchy Problem in Banach spaces. We also introduce semi-diagonal operators on Hilbert spaces and discuss the solution of the Cauchy problem for such operators.

Key words and Phrases: Tensor product, Banach spaces, degenerate Cauchy problem.

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1 Introduction

Let I be the unit interval $[0,1]$ and $C(I)$ be the Banach space of all real valued continuous functions on I under the sup-norm, and $C^1(I)$ be the Banach space of all continuously differentiable functions under the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. If X is a Banach space, then $C^1(I, X)$ is the Banach space of all continuously differentiable functions defined on I with values in X, under the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ for all $u \in C^1(I, X)$.

One of the classical differential equations in Banach spaces is the so called Abstract Cauchy Problem. The general form of such equation is

$$Bu'(t) = Au(t) + F(t)z$$

where A, B are densely defined linear operators on the codomain of the function u , where u is continuously differentiable on $I = [0,1]$ or $[0,\infty)$ with values in the Banach space X.

If $B \neq I$, the identity operator, then the equation is called degenerate. Otherwise, it is called non-degenerate. If $f = 0$ or $z = 0$, then the equation is homogeneous, otherwise it is called nonhomogeneous.

The non-degenerate Cauchy problem has been investigated by many authors using different techniques to solve it. If $B = I$, $f = 0$ and A densely defined linear operator , the abstract Cauchy problem has been studied extensively. We refer the reader to Pazy, 1983, and the references there in.

A. Favini (1979) investigated the degenerate Cauchy problem of parabolic type in Banach space. F. Nneubrander,1994, and Bäumer, B. and F. Nneubrander 1994 used Laplace-Stieltjes transform to obtain existence and uniqueness results for exponentially bounded solutions of the homogeneous degenerate Cauchy problem where A and B are closed operators. A. Lorenzi (2001) used the projection method to derive existence and uniqueness of the solution of first-order degenerate abstract Cauchy problem. M. Alhorani (2004) studied the inverse problem of the degenerate abstract Cauchy problem where suitable hypotheses on the involved operators are made to reduce the given problem to a non-degenerate case. Thaller, B. and Thaller, S. 1995, and 1996 studied the Cauchy problem $Bu' = Au$, under six assumptions on A and B.

In this paper, we use tensor product technique to get solutions of the degenerate homogeneous abstract Cauchy problem.

2 Basics and Background

Let K be a compact Hausdorff space, X be a Banach space. $C(K, X)$ denotes the Banach space of all continuous functions from K into X with the norm $\|f\|_\infty = \sup_{t\in K} \|f(t)\|$ If $X = \mathbb{R}$ we write $C(K)$ for $C(K, \mathbb{R})$.

Let X and Y be Banach spaces with duals $X^*$and Y^*respectively and $T : D(T) \subseteq X \to Y$ be linear. T is called closed operator if its graph $G(T) = \{(x, Tx) : x \in D(T)\}$ is closed in the normed space $X \times Y$, where the norm on $X \times Y$ is defined by $\|(x, y)\| = \|x\| + \|y\|$.

. For $x \in X$ and $y \in Y$, we define the map:$x \otimes y : X^* \to Y$ with$x \otimes y\,(x^*) = \langle x, x^*\rangle y$. Clearly $x \otimes y$ is a bounded linear operator and $\|x \otimes y\| = \|x\|\,\|x\|$. We call $x \otimes y$ an **atom**. Let $K = \{x \otimes y : x \in X$ and $y \in Y\}$. We shall write $X \otimes Y$ for the span of K in $L\,(X^*, Y)$. For $T = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$, define $\|T\|_\vee = \sup \left\{ \sum_{i=1}^{n} |\langle x_i, x^*\rangle \langle y_i, y^*\rangle| : x^* \in B_1\,(X^*) \text{ and } y^* \in B_1\,(Y^*) \right\}$.So, $\|\|_\vee$ is just the operator norm on $L\,(X^*, Y)$. This is called the **injective norm** of T. The space $(X \otimes Y, \|\|_\vee)$ need not be complete. We let $X \overset{\vee}{\otimes} Y$ denote the closure of $X \otimes Y$ in $L\,(X^*, Y)$ and it is called the completed injective tensor product of X with Y.

One of the nice features of the injective tensor product of normed spaces is the fact that For any compact Hausdorff space K, and any Banach space $X, C(K, X)$ is isometrically isomorphic to $C(K)\overset{\vee}{\otimes}X$. In particular, $C(S\times K) = C(S) \overset{\vee}{\otimes} C(K)$, for S, and K are compact metric spaces.

Another norm that one can define is the **projective norm.** For $T \in X \otimes Y$ the projective norm of T is $\|T\|_\wedge = \inf \left\{ \sum_{i=1}^{n} \|x_i\|\,\|y_i\| \right\}$, where the infimum is taken over all representations of T in $X \otimes Y$. The space $X \otimes Y$ with the$(X \otimes Y, \|\|_\wedge)$ need not be complete. So, we let $X \overset{\wedge}{\otimes} Y$ to be the completion of $(X \otimes Y, \|\|_\wedge)$ .One of the nice features of the projective tensor product of two Banach spaces is the following result: Let (I, μ) be any finite measure space. Then $L^1(I,X)$ is isometrically isomorphic to $L^1(I)\overset{\wedge}{\otimes}X$. We refer to [4], for more on tensor products.

3 The Function Space

Through out this paper we let ℓ^2 be the classical Hilbert space of square summable sequences which is $\{(a_n) : \sum |a_n|^2 < \infty,\ a_n \in R\}$.$C(I,\ \ell^2)$ is the space of continuos functions on the compact interval $I = [0,1]$ with values in ℓ^2. The subspace of continuously differentiable functions on I will be denoted by $C^1(I,\ell^2)$. Any function in $C(I,\ \ell^2)$ can be written in the form $u(t) = \sum\limits_{i=1}^{\infty} u_i(t)\delta_i$, where (δ_i) is the natural basis of ℓ^2.However, we don't have guarantee that $\sum\limits_{i=1}^{\infty} \|u_i\|_\infty < \infty$. But we know that $C(I)\overset{\wedge}{\otimes}\ell^2$ and $C^1(I)\overset{\wedge}{\otimes}\ell^2$ are both in $C(I,\ell^2)$. This is because for any two normed spaces X,Y we have $X\overset{\wedge}{\otimes}Y \subset X\overset{\vee}{\otimes}Y$, and $C(I,\ell^2) = C(I)\overset{\vee}{\otimes}\ell^2$ as was pointed out in section2. So, we can introduce the following subspace of functions in $C(I,\ \ell^2)$

$$W = \left\{ u \in C^1(I,\ell^2) : u = \sum_{i=1}^{\infty} u_i \otimes \delta_i\ , \|u\|_1 = \sum_{i=1}^{\infty} \|u_i\|_\infty + \left\|u_i'\right\|_\infty < \infty \right\}.$$

We know that W is a huge space since$C^1(I) \otimes \ell^2 \subset W$. A function of the form $u = v \otimes x$ will be called an **atomic function.** A function of the form $f(t) = \sum\limits_{i=n_1}^{n_k} u_i \otimes \delta_i$ will be called a **finite rank function.** It is known that if $u_k \in C^1[a,b]$, *and* $u = \sum\limits_{i=1}^{\infty} u_i$, $\sum\limits_{i=1}^{\infty} u_i'$ *converge uniformly on* $[a,b]$ *then* $u' = \sum\limits_{i=1}^{\infty} u_i'$. The idea of the proof of the following theorem is classical and will be omitted.

Lemma 3.1 :W with the norm $\|u\|_1 = \sum\limits_{i=1}^{\infty} \|u_i\|_\infty + \|u_i'\|_\infty$ is a Banach space.

In this paper, we assume that A and B are densely defined linear operators on ℓ^2 whose domains contain the elements of the natural basis of ℓ^2,and satisfy the property $Au(t) = \sum_{i=1}^{\infty} u_i(t)A\delta_i$ and $Bu(t) = \sum_{i=1}^{\infty} u_i(t)B\delta_i$. It should be remarked that If T is a densely defined closed linear operator on ℓ^2 then $Tu(t) = \sum_{i=1}^{\infty} u_i(t)T\delta_i$.

4 Solution For The Atomic Case

In this section we solve the degenerate abstract Cauchy problem for atomic functions.

Let $A : Dom(A) \subseteq \ell^2 \to \ell^2$, $B : Dom(B) \subseteq \ell^2 \to \ell^2$, be two linear operators, and suppose that $u(t) \in Dom(A) \cap Dom(B)$ and $u'(t) \in Dom(B)$ for all $t \in I = [0, 1]$.

The homogeneous degenerate abstract Cauchy problem is

$$\begin{aligned} B\acute{u}(t) &= Au(t) \\ u(0) &= z_0 \end{aligned} \tag{P1}$$

Now, we would like to solve such a problem using the tensor product technique. The first step is to look for a solution to problem (P1) among atomic functions of the form

$$u(t) = g(t)x$$

where $g \in C^1(I)$,$x \in Dom(A) \cap Dom(B)$. This is the object of this section. The main result of this section is the following theorem:

Theorem 4.1. For $u(t) = g(t)x$, $g \in C^1(I)$ and $x \notin KerA \cap KerB$ problem (P1) has a unique solution.

Proof. There are two cases which are easy to handle..

(i) If $x \in \ker A$ and $x \notin \ker B$, then $g'(t)Bx = 0$. But since $x \notin \ker B$ then $g'(t) = 0$. Hence $g(t) = \alpha$ constant and $u(t) = \alpha x$ such that $\alpha x = z_0$ is a unique solution of problem (P1) .

(ii) If $x \in \ker B$ and $x \notin \ker A$. Then $g(t)Ax = 0$ and hence $g(t) = 0$ since $x \notin \ker A$. Thus $u(t) = 0$ is the unique solution of problem (P1) .

So the case which needs to be handled deeply is:$x \notin KerA \cup KerB$. Now, if g is a constant function , then $g(t)Ax = 0$. But since $x \notin \ker A$ we get $g(t) = 0$,$\forall t \in I$. Hence $u(t) = 0$ is the unique solution of problem (P1). Therefore, g can not be non-zero constant.

Since $u(t) = g(t)x$,we have $g'(t)Bx = g(t)Ax$. If $g(t) = 0$ on $E \subseteq I$, then

$$E^c = \{t \in I, g(t) \neq 0\} = \{t \in I : g(t) > 0\} \cup \{t \in I : g(t) < 0\}$$

Since E^c is open set, it can be written as a countable union of disjoint open intervals, say $E^c = \bigcup\limits_{i=1}^{\infty}(\alpha_i, \beta_i)$.Now, on each (α_i, β_i) , $i = 1, 2, ...,$ $g(t) \neq 0$, so we have

$$\frac{g'(t)}{g(t)} Bx = Ax \ , \forall t \in (\alpha_i, \beta_i) \ , i = 1, 2, ...$$

This implies that

$$\frac{g'(t)}{g(t)} = \lambda_i \text{ (constant) , on } (\alpha_i, \beta_i), \ i = 1, 2, ...$$

So $g'(t) - \lambda_i g(t) = 0$, on (α_i, β_i) , $i = 1, 2, ...,$ and

$$g(t) = c_i e^{\lambda_i t} \ \forall t \in (\alpha_i, \beta_i), i = 1, 2, ...$$

Now, we claim that (α_i, β_i) , $i = 1, 2, ...$are adjacent intervals. Indeed, if $g(t) = c_k e^{\lambda_k t}$ on (α_k, β_k) and if on the interval (β_k, α_{k+1}), $g(t) = 0$, then by the continuity of $g(t)$ at β_k, we have $c_k e^{\lambda_k \beta_k} = 0$ which is impossible unless $c_k = 0$. This implies $g = 0$ on (α_k, β_k) contradicting the assumption that $g(t) \neq 0$ on (α_k, β_k). Hence we can assume that

$$E^c = \bigcup_{i=1}^{\infty}(\alpha_i, \alpha_{i+1})$$

and therefore $g(t)$ can be written

$$g(t) = \begin{cases} c_1 e^{\lambda_1 t} & , t \in (\alpha_1, \alpha_2) \\ c_2 e^{\lambda_2 t} & , t \in (\alpha_2, \alpha_3) \\ \vdots & \quad \vdots \end{cases}$$

Finally, we show that $g(t) \neq 0, \forall t \in I$. In fact, since g is continuos at $t = \alpha_k, k = 2, 3, ...,$ we have $\lim_{t \to \alpha_k^-} c_k e^{\lambda_k t} = \lim_{t \to \alpha_k^+} c_{k+1} e^{\lambda_{k+1} t} = 0$. This implies that $c_k e^{\lambda_k \alpha_k} = c_{k+1} e^{\lambda_{k+1} \alpha_k} = 0$, which is impossible unless c_k, c_{k+1} both equal zero. Hence $g(t) \neq 0$, $\forall t \in I$. Therefore, $\frac{\acute{g}(t)}{g(t)} Bx = Ax$, $\forall t \in I$. Hence $\frac{\acute{g}(t)}{g(t)} = \lambda, \forall t \in I$, which implies $g(t) = ce^{\lambda t}$ $\forall t \in I$, where $g(0)x = z_0 = cx$. Thus, $u(t) = e^{\lambda t} z_{0.}$

Corollary 4.2. For $u(t) = g(t)x$, $g \in C^1(I)$ problem (P1) has a solution.

Proof If $x \notin KerA \cap KerB$ then by Theorem 4.1 has a unique solution. If x belongs to $KerA \cap KerB$, then

$$Bu'(t) = Bg'(t)x = \acute{g}(t)Bx = 0 \text{ and } Au(t) = Ag(t)x = g(t)Ax = 0$$

Thus $u(t) = g(t)x$ is a solution to (P1) for all $g \in C^1(I)$ such that $z_0 = g(0)x$. ■

5 Solution For The Finite Rank Functions Case

Here we are looking for a solution for (P1) among functions of the form $u(t) = \sum_{ik=n_1}^{n_2} u_{ik}(t)\delta_{ik}$, where $u_i \in C^1(I)$, $i = 1, 2, ..., n$. Such functions are called finite rank functions. First we assume that $B = I$, so problem (P1) becomes

$$\begin{aligned} u'(t) &= Au(t) \\ u(0) &= z \end{aligned} \qquad \text{(P2)}$$

Theorem 5.1. Problem P2 has a unique solution for any densely defined linear operator A on ℓ^2.

Proof. Let for simplicity $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$. Then $u'(t) = \sum_{i=1}^{n} \acute{u}_i(t)\delta_i$. So

$$\sum_{i=1}^{n} \acute{u}_i(t)\delta_i = \sum_{i=1}^{n} u_i(t)A\delta_i \quad \text{.....................................(1)}$$

Since $u'(t)$ is a linear combination of $\delta_1, ..., \delta_n$, then $Au(t)$ belongs to $[\delta_1, \delta_2, ..., \delta_n]$. Thus $Y = [\delta_1, \delta_2, ..., \delta_n]$ is an invariant subspace of A, and we can consider the restriction of A on $[\delta_1, \delta_2, ..., \delta_n]$, $\hat{A} : [\delta_1, \delta_2, ..., \delta_n] \to [\delta_1, \delta_2, ..., \delta_n]$. Hence $\hat{A}$ has a matrix representation (i.e $\hat{A} = [a_{ij}]$ where $a_{ij} = \langle A\delta_i, \delta_j \rangle$). Taking the inner product of δ_j with both sides of equation (1) we get

$$\sum_{i=1}^{n} u_i^{\prime}(t)\langle \delta_i, \delta_j \rangle = \sum_{i=1}^{n} u_i(t)\langle A\delta_i, \delta_j \rangle$$

Since $\{\delta_i\}$ is an orthonormal set, we have

$$u_j'(t) = \sum_{i=1}^{n} u_i(t)a_{ij} \quad \text{..(2)}$$

Then equation (2) represents a system of n linear differential equations, which can be solved by finding the spectrum of $\hat{A}$, which is a standard procedure. Further, with the initial condition in (P2), the solution is unique.

Now, we try to solve (P1) in case B is not the identity, and $u(t)$ is a finite rank function. Without loss of generality, we look for a solution of the form $u(t) = \sum_{i=1}^{n} u_i(t)\delta_i$. Further, we assume that A, B are densely defined operators on ℓ^2. Since $Bu^{\prime}(t) = Au(t)$, then $A([\delta_1, \delta_2, ..., \delta_n]) = B([\delta_1, \delta_2, ..., \delta_n])$. So to simplify the notation, we can assume that $[\delta_1, \delta_2, ..., \delta_n]$ is invariant under both A and B, and A_n and B_n are the restriction of A and B to $[\delta_1, \delta_2, ..., \delta_n]$.

With this setup, we can prove the following theorem.

Theorem 5.2. Let $B_n = B\left|_{[\delta_1, \delta_2, ..., \delta_n]}\right.$ be orthogonally diagonalizable linear operator such that $A|_{\ker B}$ is invertible. Then, problem (P1) has a unique solution.

Proof. Since B_n is diagonalizable, there exists a basis $\vartheta = \{\theta_1, ..., \theta_n\}$ such that the matrix representation of B_n with respect to ϑ is $D = diag(\lambda_1, ..., \lambda_n)$ where $\lambda_1, ..., \lambda_n$ are the eigenvalues of B and $\theta_1, ..., \theta_n$ are the corresponding eigenvectors. Further, since B is orthogonally diagonalizable, (θ_i) is orthonormal.

Now, if $\lambda_i \neq 0$,$\forall i = 1, ..., n$, then B_n is invertible and hence problem (P1) becomes $u'(t) = B_n^{-1} A_n u(t)$, which has a unique solution by Theorem 5.1.

Assume $\lambda_i \neq 0$ for $i = 1, ..., r$. Let $u(t) = \sum_{i=1}^{n} \upsilon_i(t)\theta_i$. Then $B_n u'(t) = \sum_{i=1}^{n} \upsilon_i'(t) B_n \theta_i$, and $A_n u(t) = \sum_{i=1}^{n} \upsilon_i(t) A_n \theta_i$. Hence

$$\sum_{i=1}^{n} \upsilon_i'(t) B_n \theta_i = \sum_{i=1}^{n} \upsilon_i(t) A_n \theta_i \text{..} (3)$$

Taking the inner product of θ_i with both sides of (3) we get

$$\sum_{i=1}^{n} \upsilon_i'(t) \langle B_n \theta_i, \theta_j \rangle = \sum_{i=1}^{n} \upsilon_i(t) \langle A_n \theta_i, \theta_j \rangle \text{ ....} (4)$$

Let $\tilde{A} = A_{|\ker B} = [\langle A\theta_i, \theta_j \rangle]_{i,j=r+1,\cdots,n}$.

From (3.3.5), we have

$$\begin{bmatrix} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_r & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_n' \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & \tilde{A} \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_n \end{bmatrix} \text{..........} (5)$$

Multiplying (5) from left by the matrix $\begin{bmatrix} I_r & 0 \\ 0 & \tilde{A}^{-1} \end{bmatrix}$, we get:

$$\begin{bmatrix} \lambda_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \lambda_r & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_n' \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ \tilde{A}^{-1} A_3 & I_{n-r} \end{bmatrix} \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_n \end{bmatrix}$$

So, we have

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_r' \end{bmatrix} = A_1 \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_r \end{bmatrix} + A_2 \begin{bmatrix} \upsilon_{r+1} \\ \vdots \\ \upsilon_n \end{bmatrix}$$

and

$$0 = \tilde{A}^{-1} A_3 \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_r \end{bmatrix} + I_{n-r} \begin{bmatrix} \upsilon_{r+1} \\ \vdots \\ \upsilon_n \end{bmatrix} \text{.................} (6)$$

Hence from(6) we get

$$\begin{bmatrix} \upsilon_{r+1} \\ \vdots \\ \upsilon_n \end{bmatrix} = -\tilde{A}^{-1}A_3 \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_r \end{bmatrix} \text{......................}(7)$$

Substituting (7) in (5) we get ,

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} \begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_r' \end{bmatrix} = (A_1 - A_2\tilde{A}^{-1}A_3) \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_r \end{bmatrix}$$

Let $D = diag(\lambda_1, \ldots, \lambda_r)$..Hence

$$\begin{bmatrix} \upsilon_1' \\ \vdots \\ \upsilon_r' \end{bmatrix} = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3) \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_r \end{bmatrix} \text{.................}(8)$$

Let $V_1(t) = \begin{bmatrix} \upsilon_1(t) \\ \vdots \\ \upsilon_r(t) \end{bmatrix}$, $M = D^{-1}(A_1 - A_2\tilde{A}^{-1}A_3)$. Then$V_1'(t) = MV_1(t)$.

For such M we have the following cases:

Case 1. M has $\lambda_1, \cdots, \lambda_r$ distinct eigenvalues, then the general solution of the system is of the form

$$V_1(t) = \sum_{i=1}^{r} \alpha_i E_i e^{\lambda_i t}$$

where E_i is the corresponding eigenvector

Case 2. M has $\lambda_1, \cdots, \lambda_k$ eigenvalues with multiplicity $m_1, \cdots, m_k$ $(m_1 + \cdots + m_k = r)$. For such case we have the following sub-cases

Case 2.1. For each $p = 1, \cdots, k, \lambda_p$ has m_p linearly independent eigenvectors. Hence the general solution of the system is of the form

$$V_1(t) = \sum_{p=1}^{k} (\alpha_{p1}E_{p1} + \cdots + \alpha_{pm_p}E_{pm_p})e^{\lambda_p t}$$

Case2.2. For each $p = 1, \cdots, k, \lambda_p$ has a single linearly independent eigenvector. Then

$$V_1(t) = \sum_{p=1}^{k} \left[(\alpha_{p1} E_{p1} + \alpha_{p2}(E_{p1}t + E_{p2}) \cdots + \alpha_{pm_p}(\beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p}) \right] e^{\lambda_p t}. (9)$$

Case 2.3. λ_p has $1 < m'_p < m_p$ linearly independent eigenvectors, then the solution of the system is of the form

$$V_1(t) = \sum_{p=1}^{k} (\alpha_{p1} E_{p1} + \cdots + \alpha_{pm'_p} E_{pm'_p}) e^{\lambda_p t} + \cdots +$$
$$\left[\alpha_{pm'_p+1}(Q_{p1} t + Q_{p2}) + \cdots + \alpha_{pm_p}(\beta_{p1} Q_{p1} t^{m_p - m'_p - 1} + \cdots + \beta_{p(m_p - m'_p)} Q_{p(m_p - m'_p)}) \right] e^{\lambda_p t}$$

where $\beta_{p1}, \cdots, \beta_{pm_p}, p = 1, \cdots, k$ are known constant ,Q_{p1} is a linear combination of $E_{p1}, \cdots, E_{pm'_p}$ and

$$Q_{pq} = (M - \lambda_p I) Q_{pq-1},$$

For simplicity we may assume that $V_1(0) = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$, and we consider the general solution which is given by (9), since other cases can be treated in a similar way with slight difference in notations. Then

$$\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \sum_{p=1}^{k} \alpha_{p1} E_{p1} \cdots + \alpha_{pm_p}(\beta_{pm_p} E_{pm_p}) \text{....................}(10)$$

Let $P = [E_{11} : \cdots : E_{1m_1} : \cdots : E_{k1} : \cdots : E_{km_k}]$, then

$$[c_1, \cdots, c_r]^T = P \left[\alpha_{11}, \cdots, \alpha_{1m_1}, \cdots, \alpha_{k1}, \cdots, \alpha_{km_k} \right]^T \text{..............}(11)$$

Now we can choose $E_{pq}, q = 1, \cdots, m_p, p = 1, \cdots, k$ such that

(i)$\|P^{-1}\| \leq 1$

Proof

Case 5.1 **(ii)** $|\lambda_p| \left\| \beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p} \right\| \leq 1$

(iii) $\left\| \beta_{p1}(m_p - 1) E_{p1} t^{m_p - 2} + \cdots + \beta_{pm_p - 1} E_{pm_p - 1} \right\| \leq 1$

Then from (10) we have

$$\sum_{p=1}^{k} |\alpha_{p1}| + \cdots + |\alpha_{pm_p}| \leq \sum_{i=1}^{r} |c_i| \text{........................}(12)$$

If we assume that $\Re\sigma(M) \leq \omega < \infty$, then from (9),(10) and (11) we get

$$\begin{aligned} & \sum_{p=1}^{k} \left[\|\alpha_{p1} E_{p1}\| + \cdots + \left\| \alpha_{pm_p} (\beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p}) \right\| \right] \left\| e^{\lambda_p t} \right\| \\ \leq \;& e^{\omega} \sum_{p=1}^{k} |\alpha_{p1}| \, \|E_{p1}\| + \cdots + \left| \alpha_{pm_p} \right| \left\| (\beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p}) \right\| \\ \leq \;& e^{\omega} \sum_{p=1}^{k} |\alpha_{p1}| + \cdots + \left| \alpha_{pm_p} \right| \leq e^{\omega} \sum_{i=1}^{r} |c_i| \end{aligned}$$

Also, if we differentiate $V_1(t)$ we get

$$\begin{aligned} V_1'(t) \;=\; & \sum_{p=1}^{k} \lambda_p \left[(\alpha_{p1} E_{p1} + \alpha_{p2} (E_{p1} t + E_{p2}) \cdots + \alpha_{pm_p} (\beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p}) \right] e^{\lambda_p t} + \\ & \sum_{p=1}^{k} \left[\alpha_{p2} E_{p1} \cdots + \alpha_{pm_p} (\beta_{p1} (m_p - 1) E_{p1} t^{m_p - 2} + \cdots + \beta_{pm_{p-1}} E_{pm_{p-1}}) \right] e^{\lambda_p t} \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{p=1}^{k} |\lambda_p| \left[|\alpha_{p1}| \, \|E_{p1}\| + \cdots + \left| \alpha_{pm_p} \right| \left\| (\beta_{p1} E_{p1} t^{m_p - 1} + \cdots + \beta_{pm_p} E_{pm_p}) \right\| \right] \left\| e^{\lambda_p t} \right\| + \\ & \sum_{p=1}^{k} \left[|\alpha_{p2}| \, \|E_{p1}\| + \cdots + \left| \alpha_{pm_p} \right| \left\| \beta_{p1} (m_p - 1) E_{p1} t^{m_p - 2} + \cdots + \beta_{pm_{p-1}} E_{pm_{p-1}} \right\| \right] \left\| e^{\lambda_p t} \right\| \\ \leq \;& 2e^{\omega} \sum_{p=1}^{k} |\alpha_{p1}| + \cdots + \left| \alpha_{pm_p} \right| \leq 2e^{\omega} \sum_{i=1}^{r} |c_i| \end{aligned}$$

Therefore $u \in W$. ■

Here we give an example as an application of Theorem 5.1 on $\mathbb{R}^2$ which can be considered as a finite dimensional subspace of ℓ^2.

Example 5.1. Let B ,A be two linear operators on $\mathbb{R}^2$ such that the conditions in Theorem 5.1 are satisfied and the matrix representation of B and A are

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}.$$

The eigenvalues of B are $2, 0$ and the corresponding eigenvectors are $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$.

Let $u(t) = \upsilon_1(t) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \upsilon_2(t) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$,then

$\upsilon_1'(t)B \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \upsilon_2'(t)B \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \upsilon_1(t)A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \upsilon_2(t)A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ ....(13)

.Taking the inner product of (13) with $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ to get

$$2\upsilon_1'(t) = \upsilon_1(t)\langle A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \rangle + \upsilon_2(t)\langle A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \rangle$$

and

$$0 = \upsilon_1(t)\langle A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \rangle + \upsilon_2(t)\langle A \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \rangle$$

Thus

$$\begin{aligned} 2\upsilon_1'(t) &= 3\upsilon_1(t) + 2\upsilon_2(t) \\ 0 &= -\upsilon_2(t) \end{aligned}$$

Hence $u(t) = \frac{c}{\sqrt{2}} e^{\frac{3}{2}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{c}{\sqrt{2}} e^{\frac{3}{2}t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

6 Solutions for Infinite Rank functions

In this section, we are going to solve problems (P1) and (P2) among functions of the form $u(t) = \sum_{j=1}^{\infty} u_j(t)\delta_j \in W$. We make the following assumptions

Assumption 6.1. Assume that A is a densely defined linear operator on ℓ^2 such that $[\delta_1, \delta_2, \cdots]$ is invariant under A and $\Re\lambda \leq \omega < \infty$, $\forall \lambda \in \sigma(A)$.

Assumption 6.2. Assume that $(|c_j|)$ and $(|c_j\lambda_j|)$ belong to ℓ^1, where $\{\lambda_i, \lambda_2, \cdots\}$ is the set of eigenvalues of A.

We call $A : [\delta_1, \delta_2, \cdots] \to \ell^2$ diagonal linear operator if the matrix representation of A with respect to the basis $\{\delta j\}$ is diagonal.

Theorem 6.1. Let A be a diagonal linear operator, such that **Assumption** 6.1and 2.2 are satisfied. Then problem (P2) has a unique solution.

Proof. Since $\sum\limits_{j=1}^{\infty} u_j \otimes \delta_j$and $\sum\limits_{j=1}^{\infty} u_j' \otimes \delta_j$ converge uniformly in $C(I, \ell^2)$, then $u'(t) = \sum\limits_{j=1}^{\infty} u_j'(t)\delta_j$. Also, since $Au(t) = \sum\limits_{j=1}^{\infty} u_j(t) A\delta_j$ then,

$$\sum_{j=1}^{\infty} u_j'(t)\delta_j = \sum_{j=1}^{\infty} u_j(t) A\delta_j = \sum_{j=1}^{\infty} u_j(t)\lambda_j\delta_j$$

So, $\sum\limits_{j=1}^{\infty}(u_j'(t) - u_j(t)\lambda_j)\delta_j = 0$. Hence $u_j'(t) - u_j(t)\lambda_j = 0$ for every j and $u_j(t) = c_j e^{\lambda_j t}$ where $c_j = u_j(0)$. Thus $u(t) = \sum\limits_{j=1}^{\infty} c_j e^{\lambda_j t}\delta_j$. Now, $\left|e^{\lambda_i t}\right| = e^{\Re\lambda_j t}$. But, $\Re\lambda \leq \omega$ and $t \in [0,1]$. Hence

$$\|u_i\| = \sup_{t\in I}\left|c_i e^{\lambda_i t}\right| = |c_i| \sup_{t\in I} e^{\lambda_i t} \leq \begin{cases} |c_i|\, e^{\omega} & \text{, if } \omega > 0 \\ |c_i| & \text{, if } \omega \leq 0 \end{cases}$$

and

$$\|u_i'\| = \sup_{t\in I}\left|c_i\lambda_i e^{\lambda_i t}\right| = |\lambda_i c_i| \sup_{t\in I} e^{\lambda_i t} \leq \begin{cases} |\lambda_i c_i|\, e^{\omega} & \text{, if } \omega > 0 \\ |\lambda_i c_i| & \text{, if } \omega \leq 0 \end{cases}$$

Thus,

$$\|u\| = \sum_{i=1}^{\infty} \|u_i\| + \|u_i'\| \leq \begin{cases} e^{\omega}\sum\limits_{i=1}^{\infty} |c_i| + |\lambda_i c_i| & ,if \quad \omega > 0 \\ \sum\limits_{i=1}^{\infty} |c_i| + |\lambda_i c_i| & ,if \quad \omega \leq 0 \end{cases}$$

From assumption 6.2, we see that $\|u\|_1 < \infty$, so $u \in W$.

Definition 6.2. A linear operator B defined on a Hilbert space H is called semi-diagonal if there exist orthogonal subspaces $\{W_j\}_{j=1}^{\infty}$, such that

1. $dimW_j < \infty, \forall j$
2. $B(W_j) \subseteq W_j, \forall j$
3. $H = \overset{\infty}{\underset{j=1}{\oplus}} W_j$

Theorem 6.3. Let A be semi-diagonal linear operator, such that assumptions 6.1 and 6.2 are fulfilled. Then problem (P2) has a unique solution .

Proof. since A is semi-diagonal, there exist W_1, W_2,...orthogonal subspace of ℓ^2 such that for each j $\dim W_j < \infty$, $A(W_j) \subseteq W_j$ and $\ell^2 = \overset{\infty}{\underset{j=1}{\oplus}} W_j$. With out loss of generality, according to the decomposition of ℓ^2, we may assume that $\{\delta_{n_{j-1}+1}, \cdots, \delta_{n_j}\}$where $n_0 = 0$, $j = 1, 2, \cdots$, is the corresponding basis for W_j . Since $\sum\limits_{j=1}^{\infty} u_j(t)\delta_j$ and $\sum\limits_{j=1}^{\infty} u_j'(t)\delta_j$ converge uniformly in $C(I, \ell^2)$, we have $u'(t) = \sum\limits_{j=1}^{\infty} u_j'(t)\delta_j$, and by assumption on A, we have $Au(t) = \sum\limits_{j=1}^{\infty} u_j(t)A\delta_j$. Now, $u(t) = \sum\limits_{k=1}^{\infty} u_k(t)\delta_k = \sum\limits_{j=1}^{\infty}\sum\limits_{k=n_{j-1}+1}^{n_j} u_k(t)\delta_k$ where $\sum\limits_{k=n_{j-1}+1}^{n_j} u_k(t)\delta_k \in W_j$. Hence we can write $u'(t) = Au(t)$ as

$$\sum_{j=1}^{\infty}\sum_{k=n_{j-1}+1}^{n_j} u_k'(t)\delta_k = \sum_{j=1}^{\infty}\sum_{k=n_{j-1}+1}^{n_j} u_k(t)A\delta_k$$

Since the subspaces W_1, W_2,... are orthogonal, $A(W_j) \subseteq W_j$ and $\sum\limits_{k=n_{j-1}+1}^{n_j} u_k(t)\delta_k \in W_j$ we have

$$\sum_{k=n_{j-1}+1}^{n_j} u_k'(t)\delta_k = \sum_{k=n_{j-1}+1}^{n_j} u_k(t)A\delta_k$$

This system, together with the initial condition $u(0)$ restricted to W_j,has a unique solution. So, on each W_j , we have a unique solution $w_i'(t)$. Apply the same argument as in the last part of the proof of Theorem 5.2 for $A\big|_{W_j} = M_j$ we conclude that

$$\sum_{j=1}^{\infty} \|u_j\|_{\infty} \leq e^{\omega}\sum_{j=1}^{\infty}\sum_{k=n_{j-1}+1}^{n_j} |c_k| < \infty$$

and

$$\sum_{j=1}^{\infty} \left\|u_j'\right\|_{\infty} \leq e^{\omega}\sum_{j=1}^{\infty}\sum_{k=n_{j-1}+1}^{n_j} |\lambda_k c_k| < \infty$$

Therefore, $\|u\|_1 < \infty$ and so $u \in W$.

Now we are going to solve problem (P1). But first we need to assume the following assumptions on the operators A and B in order to make sure that the desired solution belong to the space$W$

Assumption 6.3. A and B are semi-diagonal linear operators with the same decomposition (i.e there exist orthogonal subspaces $\{W_j\}_{j=1}^{\infty}$ such that

1. $dim W_j < \infty, \forall j$
2. W_j is invariant under A and B for every j
3. $\ell^2 = \bigoplus\limits_{j=1}^{\infty} W_j$

Assumption 6.4. $B_j = B\left|_{W_j}\right.$ is orthogonally diagonalizable such that $\tilde{A}_j = A_j\left|_{\ker Bj}\right.$ is invertible for every j.

From the proof of Theorem 5.2 we have on each W_j

$$M_j = D_j^{-1}(A_{j1} + A_{j2}\tilde{A}_j^{-1}A_{j3})$$

Assumption 6.5. For each j , $\Re\sigma(M_j) \leq \omega < \infty$

Theorem 6.4. Let A, B be two linear operators such that assumptions (6.3),(6.4),(6.5) are satisfied. Then for $(c_i) \in \ell^1$ problem (P4) has a unique solution.

Proof As in the proof of Theorem 6.3 we can assume that $\left\{\delta_{n_{j-1}+1}, \cdots, \delta_{n_j}\right\}$ where $n_0 = 0$, $j = 1, 2, \cdots$, is the corresponding basis for W_j . Also, since $\sum\limits_{j=1}^{\infty} u_j \otimes \delta_j$ and $\sum\limits_{j=1}^{\infty} u'_j \otimes \delta_j$ converge uniformly in $C\left(I, \ell^2\right)$, then $u'(t) = \sum\limits_{j=1}^{\infty} u'_j(t)\delta_j$. Further, by assumption on A, we have $Au(t) = \sum\limits_{j=1}^{\infty} u_j(t)A\delta_j$ and $Bu'(t) = \sum\limits_{j=1}^{\infty} u'_j(t)B\delta_j$. Now, $u(t) = \sum\limits_{k=1}^{\infty} u_k(t)\delta_k = \sum\limits_{j=1}^{\infty}\sum\limits_{k=n_{j-1}+1}^{n_j} u_k(t)\delta_k = \sum\limits_{j=1}^{\infty} w_{j(t)}$. Thus,

$$\sum_{j=1}^{\infty} Bw'_j(t) = \sum_{j=1}^{\infty} Aw_j(t)$$

Since the subspaces W_1, W_2,... are orthogonal, W_j is invariant under A and B for every j and $w_j(t) = \sum\limits_{k=n_{j-1}+1}^{n_j} u_k(t)\delta_k \in W_j$ we have

$$Bw_j'(t) = Aw_j(t), \forall j \text{ and } w_j(0) = (c_{n_{j-1}+1,}\cdots, c_{n_j}) \tag{1}$$

By Theorem5.2 problem (1) has a unique solution which satisfies

$$\sum_{k=n_{j-1}+1}^{n_j} \|u_k\| + \|u_k'\| \ \le 2e^{\omega} \sum_{k=n_{j-1}+1}^{n_j} |c_k|$$

Thus,

$$\begin{aligned} \|u\|_1 &= \sum_{k=1}^{\infty} \|u_k\| + \|u_k'\| \\ &= \sum_{j=1}^{\infty} \sum_{k=n_{j-1}+1}^{n_j} \|u_k\| + \|u_k'\| \\ &\le 2e^{\omega} \sum_{k=1}^{\infty} |c_k| \end{aligned}$$

Therefore, $u \in W$. ■

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REPRESENTATION OF QUASI QUADRATIC FUNCTIONALS BY SESQUILINEAR ONES AND JORDAN *-DERIVATIONS

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Abstract. In this study, we will mainly try to give an answer to the question of representability of quasi quadratic functional on modules over general *-ring including complex (Banach) *-algebras by using Jordan *-derivations.

1. Introduction

Let R be an *-ring with identity such that 2 is a unit in R and M be left R-module. The mapping $Q : M \to R$ is said to be quasi quadratic functional if for any $x, y \in M$ and $a \in R$ the parallelogram law

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.1}$$

and the homogeneity equation

$$Q(ax) = aQ(x)\,a^* \tag{1.2}$$

holds. A biadditive mapping $S : M \times M \to R$ satisfying

$$S(a_1x_1 + a_2x_2, y) = a_1S(x_1, y) + a_2S(x_2, y) \qquad (a_1, a_2 \in R,\ x_1, x_2, y \in M), \tag{1.3}$$

$$S(x, a_1y_1 + a_2y_2) = S(x, y_1)\,a_1^* + S(x, y_2)\,a_2^* \qquad (a_1, a_2 \in R,\ x, y_1, y_2 \in M) \tag{1.4}$$

is a sesquilinear functional. One can check that for any sesquilinear functional S the functional Q defined by $Q(x) = S(x, x)$, $x \in M$, is quasi quadratic. Now it is interesting to know that for a quasi quadratic functional Q is there a sesquilinear functional S such that $Q(x) = S(x, x)$ for any $x \in M$? In 1963, Halperin in his lecture on Hilbert spaces posed this problem for the special case that M is a vector space over $F \in \{R, C, H\}$. Here, R and C denote the field of real and complex numbers respectively and H denotes the skew-field of quaternions. In 1987 Šemrl [7] gave a positive answer to Halperin's problem for quasi-quadratic functionals defined on a vector space over a complex *-algebra with an identity. It was proved that if Q is a quasi-quadratic functional on a module over a complex *-algebra with an identity element, then the mapping S defined by

$$S(x, y) = \frac{1}{4}\left(Q(x+y) - Q(x-y)\right) + \frac{i}{4}\left(Q(x+iy) - Q(x-iy)\right)$$

is sesquilinear functional and satisfies $Q(x) = S(x, x)$. This result is an extension of the Jordan-Von Neumann theorem [14] which characterises pre-Hilbert spaces among all normed spaces.

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An additive mapping $J : R \to R$ defined on a *-ring R is called a Jordan *-derivation if $J\left(a^2\right) = aJ(a) + J(a)a^*$ for any $a \in R$. A mapping J_a on R, $a \in R$, defined by $J_a(b) = ba - ab^*$ is called an inner Jordan *-derivation. Jordan *-derivations play an important role in solution of the representability of quasi quadratic functionals by sesquilinear functionals (See [2], [3], [5], [6], [8]). Here we will use Jordan *-derivations to solve the problem of representability of quasi quadratic functionals on general *-rings including complex (Banach) *-algebras.

2. Main Results

In this section we will consider R as a *-ring with identity 1 in which 2 is a unit in R. Also M is a left R-module. Now let there exists an element $a_0 \in R$ such that

$$a_0 + a_0^* = 0, \quad a_0 a_0^* = 1 \tag{2.1}$$

$$a_0 a = a a_0 \quad (a \in R). \tag{2.2}$$

Let $J : R \to R$ be a Jordan *-derivation then $J' : R \to R$ defined by $J'(a) = (J(a))^*$ is a Jordan *-derivation. Also for each $c \in Z(R)$ the functional $J' : R \to R$ defined by $J'(a) = cJ(a)$ is a Jordan *-derivation too. So for any Jordan *-derivation $J : R \to R$ by choosing $J_1(a) = \frac{1}{2}\left(J(a) + J(a)^*\right)$ and $J_2(a) = \frac{a_0^*}{2}\left(J(a) - J(a)^*\right)$ it is seen that $J(a) = J_1(a) + a_0 J_2(a)$ which J_1 and J_2 are Jordan *-derivations and $J_1(a) = J_1(a)^*$, $J_2(a) = J_2(a)^*$. Moreover for each fixed a_0 such representation of a Jordan *-derivation is unique.

Theorem 1. *Let $J : R \to R$ be additive. The following conditions are equivalent:*

(1) *J is a Jordan *-derivation.*
(2) *For any $a, b \in R$,*

$$J(aba) = abJ(a) + aJ(b)a^* + J(a)b^*a^*. \tag{2.3}$$

Proof. (2)⇒(1); By giving $b = 1$ in (2.3) we see that J is a Jordan *-derivation.

(1)⇒(2); By giving $a + b$ replace of a in $J\left(a^2\right) = aJ(a) + J(a)a^*$, we get the identity

$$J(ab) + J(ba) = bJ(a) + aJ(b) + J(a)b^* + J(b)a^*.$$

Now let $x = J[a(ab + ba) + (ab + ba)a]$. Then

$$\begin{aligned} x &= (ab + ba)J(a) + aJ(ab + ba) + J(a)(b^*a^* + a^*b^*) + J(ab + ba)a^* \\ &= 2abJ(a) + a^2J(b) + aJ(a)b^* + 2aJ(b)a^* + baJ(a) \\ &\quad + bJ(a)a^* + 2J(a)b^*a^* + J(b)a^{*2} + J(a)a^*b^*. \end{aligned} \tag{2.4}$$

On the other hand

$$\begin{aligned} x &= J(a(ab + ba) + (ab + ba)a) \\ &= J\left(a^2b + aba + aba + ba^2\right) \\ &= 2J(aba) + J\left(a^2b\right) + J\left(ba^2\right) \\ &= 2J(aba) + bJ\left(a^2\right) + a^2J(b) + J\left(a^2\right)b^* + J(b)a^{*2} \\ &= 2J(aba) + baJ(a) + bJ(a)a^* + a^2J(b) \\ &\quad + aJ(a)b^* + J(a)a^*b^* + J(b)a^{*2}, \end{aligned}$$

This and (2.4) yields

$$J(aba) = abJ(a) + aJ(b)a^* + J(a)b^*a^*.$$

□

Corollary 1. *Each Jordan *-derivation on R is inner.*

Proof. Give $a = a_0$ in (2.3). Then

$$2J(b) = J(a_0)b^*a_0 - a_0bJ(a_0) = (a_0J(a_0))b^* - b(a_0J(a_0)),$$

so

$$J(b) = -\frac{1}{2}(b(a_0J(a_0)) - (a_0J(a_0))b^*).$$

Hence for $b_0 = -\frac{1}{2}a_0J(a_0)$, $J = J_{b_0}$. □

Theorem 2. *Let $Q : M \to R$ be a quasi quadratic functional. Then the functional $S : M \times M \to R$ defined by*

$$S(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)) + \frac{a_0}{4}(Q(x+a_0y) - Q(x-a_0y)),$$

is sesquilinear and $S(x,x) = Q(x)$.

Proof. Give $S_1(x,y) = Q(x+y) - Q(x-y)$ and define $J_1 : R \to R$ by $J_1(a) = S_1(ax,y) - aS_1(x,y)$. Then S_1 is biadditive so S is biadditive and hence J_1 is additive. Moreover

$$\begin{aligned}
aJ_1(a) + J_1(a)a^* - J_1\left(a^2\right) &= aQ(ax+y) - aQ(ax-y) - a^2Q(x+y) \\
&\quad + a^2Q(x-y) + Q(ax+y)a^* - Q(ax-y)a^* \\
&\quad - Q(ax+ay) + Q(ax-ay) - Q\left(a^2x+y\right) \\
&\quad + Q\left(a^2x-y\right) + a^2Q(x+y) - a^2Q(x-y) \\
&= aQ(ax+y) - aQ(ax-y) + Q(ax+y)a^* \\
&\quad - Q(ax-y)a^* + \frac{1}{2}[Q\left(\left(a^2+a\right)x - (a+1)y\right) \\
&\quad + Q\left(\left(a^2-a\right)x + (a-1)y\right) \\
&\quad - Q\left(\left(a^2+a\right)x + (a+1)y\right) - Q\left(\left(a^2-a\right)x - (a-1)y\right)] \\
&= aQ(ax+y) - aQ(ax-y) + Q(ax+y)a^* \\
&\quad - Q(ax-y)a^* + \frac{1}{2}[(a+1)Q(ax-y)(a^*+1) \\
&\quad + (a-1)Q(ax+y)(a^*-1) \\
&\quad - (a+1)Q(ax+y)(a^*+1) - (a-1)Q(ax-y)(a^*-1)] \\
&= aQ(ax+y) - aQ(ax-y) + Q(ax+y)a^* \\
&\quad - Q(ax-y)a^* + aQ(ax-y) + Q(ax-y)a^* \\
&\quad - aQ(ax+y) - Q(ax+y)a^* \\
&= 0.
\end{aligned}$$

Similarly for $J_2(a) = S_2(ax,y) - aS_2(x,y)$ in which $S_2(x,y) = Q(x+a_0y) - Q(x-a_0y)$ it is seen that $J_2 : R \to R$ is a Jordan *-derivation too. So the functional

$J:R\rightarrow R$ defined by $J(a)=J_1(a)+a_0J_2(a)$ is a Jordan *-derivation and hence is an inner Jordan *-derivation with $J(a)=-\frac{1}{2}\left(a\left(a_0J(a_0)\right)-\left(a_0J(a_0)\right)a^*\right)$. Since

$$Q(a_0x+y)=Q(x-a_0y)\ \text{ and }\ Q(a_0x-y)=Q(x+a_0y)$$

so

$$\begin{aligned} J(a_0) &= Q(a_0x+y)-Q(a_0x-y)-a_0Q(x+y)+a_0Q(x-y) \\ &\quad +a_0\left[Q(a_0x+a_0y)-Q(a_0x-a_0y)-a_0Q(x+a_0y)+a_0Q(x-a_0y)\right] \\ &= Q(a_0x+y)-Q(a_0x-y)-a_0Q(x+y)+a_0Q(x-y) \\ &\quad +a_0Q(x+y)-a_0Q(x-y)+Q(x+a_0y)-Q(x-a_0y) \\ &= 0. \end{aligned}$$

Thus for any $a\in R$, $J(a)=0$ and hence

$$S(ax,y)=aS(x,y). \tag{2.5}$$

Clearly for any $c\in R$ the identity $S(cx,cy)=cS(x,y)c^*$ holds and so by giving $c=a+1$;

$$S(ax,ay)+S(ax,y)+S(x,ay)+S(x,y)=aS(x,y)a^*+aS(x,y)+S(x,y)a^*+S(x,y)$$

This with (2.5) yields

$$aS(x,y)+S(x,y)a^*=S(ax,y)+S(x,ay)=aS(x,y)+S(x,ay),$$

Therefore $S(x,ay)=S(x,y)a^*$. The identity $S(x,x)=Q(x)$ is clear. □

Proposition 1. *Let R be a unital commutative *-ring with trivial involution such that 2 is a unit in R then each Jordan *-derivation on R is inner and hence zero if and only if each quasi quadratic functional on any $R-$module M be quadratic.*

Proof. If the only Jordan *-derivation on R be equal to zero then similar to Theorem 2 it is seen that for any quasi quadratic functional $Q:M\rightarrow R$ the functional $S:M\times M\rightarrow R$ defined by

$$S(x,y)=\frac{1}{4}\left(Q(x+y)-Q(x-y)\right),$$

is sesquilinear and $S(x,x)=Q(x)$ for any $x\in M$. Conversely, let there exists a Jordan *-derivation J on R which is not inner then the functional $Q:R\times R\rightarrow R$ defined by

$$Q((a,b))=J(ba)-bJ(a)-J(a)b^*=aJ(b)-bJ(a),$$

is quasi quadratic [4, theorem 2]. If there exists a sesquilinear functional S which generates Q then

$$\begin{aligned} S((a,b),(c,d)) &= aS((1,0),(1,0))c+aS((1,0),(0,1))d \\ &\quad +bS((0,1),(1,0))c+bS((0,1),(0,1))d \\ &= aS((1,0),(0,1))d+bS((0,1),(1,0))c \end{aligned}$$

So

$$-J(a)=Q((a,1))=S((a,1),(a,1))=a\left(S((1,0),(0,1))+S((0,1),(1,0))\right),$$

Now since $J(1)=0$ so we get $J=0$ which is a contradiction. □

Example 1. *Consider R the field of reals with trivial involution. Define the functional J on R by*

$$J(a) = \left\{ \begin{array}{cc} a \log |a|, & a \neq 0 \\ 0, & a = 0 \end{array} \right\}$$

*Clearly $D\left(a^2\right) = 2aD(a)$ but J is not a Jordan *-derivation because it is not additive. In fact the only continuous Jordan *-derivation on R is equal to zero since it is zero on the dense subset, rational numbers so any nontrivial Jordan *-derivation on R should be discontinuous.*

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Operations on Multipliers and Certain Spaces of Tempered Ultradistributions of Roumieu and Beurling Types for the Hankel-type Transformations

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Abstract

Having defined spaces of ultradifferentiable functions along with an assigned spaces of tempered ultradistributions, certain spaces of multipliers are investigated and various continuous mappings are obtained.

Keywords: Tempered Ultradistribution; Hankel-type transform; Multipliers; Roumieu-type ultradistributions; Beurling-type ultradistributions.

1 Introduction

The generalization of the Schwartz space of distributions D' [5] was one of the interests of many authors in the last few decades, such as, Roumieu, C. [2], Beurling, A. [1], Patbak, R.S. [9, 10], Banerji, P.K. and Al-omari, S.K. [8] and, Al-omari, S.K.Q. [12, 13] and others. Among others who contributes much to the theory, Komatsu [4] introduce a unified treatment for the ultradistributions defined in [1] and [2]. However, ultradistributions introduced here differ from that appears in [11] which are duals of test function space of Fourier transforms of functions in D(Schwartz space of test functions).

We, in [7], define certain spaces of ultradifferentiable functions of rapid descent (rapidly decreasing functions) for the Hankel-type transform so that the set of continuous linear forms are ultradistributions of slow growth. The object of finding the corresponding spaces of multipliers is fulfilled and the spaces are well-defined. In our present paper, we shall build upon analysis of [7]. Not merely we extend the analysis but we shall make ideas more precise. We devote Section 3 to necessary conditions for an infinitely smooth function (C^∞-function) to play the role of a multiplier under certain topology. In Sections 4 and 5, we establish separately continuous mappings (for definition see Treves [3]) from test function spaces into corresponding spaces of multipliers and from one space of multipliers into itself.

2 Definitions and Notations

Let N be the set of natural numbers and $N_0 = N \cup \{0\}$. Let $a_i, b_i, i, j = 0, 1, 2, ...$ be sequences of positive real numbers satisfying some of the following constraints

$$\begin{aligned} a_i^2 &\leq a_{i-1}a_{i+1}, \text{ for all } i \in N \\ b_j^2 &\leq b_{j-1}b_{j+1}, \text{ for all } j \in N. \end{aligned} \tag{2.1}$$

For some constants $S, S_1 > 0$ and $T, T_1 > 1$, we have

$$\begin{aligned} a_i &\leq S\,T^i \min\nolimits_{0\leq k\leq i}\ a_k\, a_{i-k},\ \ i \in N \\ b_j &\leq S_1\,T_1^j \min\nolimits_{0\leq k\leq j}\ b_k\, b_{j-k},\ \ j \in N. \end{aligned} \tag{2.2}$$

As a consequence of (2.1), we have [cf. [7]]

$$\begin{aligned} a_i a_k &\leq a_0 a_{i-k},\ \ i, k \in N_0 \\ b_j b_k &\leq b_0 b_{j+k},\ \ j, k \in N_0. \end{aligned} \tag{2.3}$$

For sequences above, we have the following:

Definition 2.1 **(i)** *Denote by $H_{\mu,\{a_i\},a}$ the set of all those infinitly smooth functions $\phi(x)$ such that for some positive constants $C_j^{\mu,\nu}$ and a dependent on ϕ*

$$\left|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C_j^{\mu,\nu}(a+\alpha)^i a_i, \tag{2.4}$$

holds true for arbitrary $\alpha > 0$ and $i, j \in N_0$.
On the other hand, let $H_{\mu,(a_i),a}$ be the set of all C^∞-functions $\phi(x)$ satisfying (2.4) for all $a > 0$.

(ii) *An infinitly smooth functions $\phi(x) \in H_\mu^{\nu,\{b_i\},b}$ (respectively, $H_\mu^{\nu,(b_i),b}$) if for some constant $b > 0$ (respectively, for all $b > 0$), there is $C_j^{\mu,\nu} > 0$ both of which depend on ϕ where*

$$\left|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C_j^{\mu,\nu}(a+\alpha)^i b_i, \tag{2.5}$$

for arbitrary $\alpha > 0$, where $D = \frac{d}{dx}$.

(iii) *Define $H_{\mu,\{a_i\},a}^{\nu,\{b_j\},b}$ (respectively, $H_{\mu,(a_i),a}^{\nu,(b_j),b}$) if for some constant $C^{\mu,\nu} > 0$ there are constants $a > 0$ and $b > 0$ (respectively, for arbitrary $a > 0$, $b > 0$) we have*

$$\left|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)\right| \leq C_j^{\mu,\nu}(a+\alpha)^i b_j (b+\beta)^j a_i b_j, \tag{2.6}$$

where α and β are arbitrary positive constants.

Spaces, so obtained, consist of ultradifferentiable functions of rapid descent which are, indeed, subspaces of $H_{\mu,\nu}$ defined in [6]. Furthermore, $H_{\mu,(a_i),a}$, $H_{\mu}^{\nu,(b_j),b}$ and $H_{\mu,(a_i),a}^{\nu,(b_j),b}$ are contained in $H_{\mu,\{a_i\},a}^{\nu}$, $H_{\mu}^{\nu,\{b_j\},b}$ and $H_{\mu,\{a_i\},a}^{\nu,\{b_j\},b}$, respectively. The set of all continuous linear forms on $H_{\mu,(a_i),a}$, $H_{\mu}^{\nu,(b_j),b}$ and $H_{\mu,(a_i),a}^{\nu,(b_j),b}$ is denoted by $\acute{H}_{\mu,(a_i),a}$, $\acute{H}_{\mu}^{\nu,(b_j),b}$ and $\acute{H}_{\mu,(a_i),a}^{\nu,(b_j),b}$, respectively and are tempered ultradistributions of Beurling-type. Similarly, $\acute{H}_{\mu,\{a_i\},a}^{\nu}$, $\acute{H}_{\mu}^{\nu,\{b_j\},b}$ and $\acute{H}_{\mu,\{a_i\},a}^{\nu,\{b_j\},b}$ are spaces of tempered (temperate) ultradistributions of Roumieu-type. It is interesting to observe that the tempered ultradistributions of Roumieu-type can be characterized as subspaces of the tempered ultra-distributions of Beurling type.
Owing to the fact that both types of tempered (slow growth) ultra-distributions assume analysis which is similar, we intend to direct the investigations to ultradistributionsof Roumieu-type. In view of above constructions we assign to $H_{\mu,\{a_i\},a}^{\nu}$, $H_{\mu}^{\nu,\{b_j\},b}$ and $H_{\mu,\{a_i\},a}^{\nu,\{b_j\},b}$ the topologies generated by the respective collections of seminorms

$$\gamma_{j,\alpha}^{\mu,\nu}(\phi) = \sup_{x\in(0,\infty)} \frac{|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)|}{(a+\alpha)^i a_i}, i \in N_0 \tag{2.7}$$

$$\gamma_{i,\beta}^{\mu,\nu}(\phi) = \sup_{x\in(0,\infty)} \frac{|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)|}{(b+\beta)^j a_j}, j \in N_0 \tag{2.8}$$

$$\gamma_{\alpha,\beta}^{\mu,\nu}(\phi) = \sup_{x\in(0,\infty)} \frac{|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}\phi(x)|}{(a+\alpha)^i (b+\beta)^j a_i b_j}, \tag{2.9}$$

for conditions already mentioned.

3 Operators for Multiplication and Necessary Conditions

Definition: For some positive constants a and b, denote by $\Phi_{\{a_i\},a}$, $\Phi^{\{b_j\},b}$ and $\Phi_{\{a_i\},a}^{\{b_j\},b}$ the set of all those C^∞-functions $\theta(x)$, over $(0,\infty)$, such that for all positive integers i, j, their respective formulae

$$\left|(x^{-1}D)^i \theta(x)\right| \leq C(1+x^2)^i a^i a_i, \tag{3.1}$$

$$\left|(x^{-1}D)^j \theta(x)\right| \leq F(1+x^2)^j b^j b_j, \tag{3.2}$$

$$\left|(x^{-1}D)^i\theta(x)\right| \leq E(1+x^2)^i a^i b^j a_i b_j, \tag{3.3}$$

hold good where C, F and E are certain positive constants. Spaces $\Phi_{\{a_i\},a}$, $\Phi^{\{b_j\},b}$ and $\Phi^{\{b_j\},b}_{\{a_i\},a}$ and the spaces $\xi_{\{a_i\},a}$, $\xi^{\{b_j\},b}$ and $\xi^{\{b_j\},b}_{\{a_i\},a}$ in [7] are quitely equivalent and are shown to be multipliers for respective spaces $H^{\nu}_{\mu,\{a_i\},a}$, $H^{\nu,\{b_j\},b}_{\mu}$ and $H^{\nu,\{b_j\},b}_{\mu,\{a_i\},a}$. For detailed analysis see [[7],Theorems 11,12,13].

Proposition 1 *(a) Let $\theta \in \Phi_{\{a_i\},a}$ and $\phi \in H_{\mu,\{a_i\},a}$, then*

$$(x^{-1}D)^i\theta(x) \in \Phi_{\{a_i\},a}$$

.

(b) Let $\theta \in \Phi^{\{b_j\},b}$ and $\phi \in H^{\nu,\{b_j\},b}_{\mu}$, then

$$(x^{-1}D)^j\theta(x) \in \Phi^{\{b_j\},b}$$

.

(c) Let $\theta \in \Phi^{\{b_j\},b}_{\{a_i\},a}$ and $\phi \in H^{\nu,\{b_j\},b}_{\mu,\{a_i\},a}$, then

$$(x^{-1}D)^j\theta(x) \in \Phi^{\{b_j\},b}_{\{a_i\},a}$$

.

Proof The proof can be easily established by induction on i, taking into account that

$$\begin{aligned}\phi(x)(x^{-1}D)\theta(x) &= x^{\mu+\nu+1}(x^{-1}D)x^{\mu+\nu+1}\theta(x)\phi(x) \\ &\quad -\theta(x)x^{\mu+\nu+1}(x^{-1}D)x^{-\mu-\nu-1}\phi(x),\end{aligned}$$

together with the fact that if ϕ belongs to either $H^{\nu}_{\mu,\{a_i\},a}$, $H^{\nu,\{b_j\},b}_{\mu}$ or $H^{\nu,\{b_j\},b}_{\mu,\{a_i\},a}$, then $x^{\mu+\nu+1}(x^{-1}D)^i x^{-\mu-\nu-1}\phi(x)$ belongs to the same space. Details are thus avoided. Therefore, the derived proposition suggests to introduce on $\Phi_{\{a_i\},a}$, $\Phi^{\{b_j\},b}$ and $\Phi^{\{b_j\},b}_{\{a_i\},a}$ the respective seperating collections of seminorms

$$\ell^{\mu,\nu,a}_{\phi,i}(\theta) = \sup_{x\in(0,\infty)} \frac{|x^{-\mu-\nu-1}\phi(x)(x^{-1}D)^i\theta(x)|}{a^i a_i}, i \in N \tag{3.4}$$

$$\ell^{\mu,\nu,b}_{\phi,j}(\theta) = \sup_{x\in(0,\infty)} \frac{|x^{-\mu-\nu-1}\phi(x)(x^{-1}D)^j\theta(x)|}{b^j b_j}, j \in N \tag{3.5}$$

$$\ell^{\mu,\nu,b}_{\phi,a}(\theta) = \sup_{x\in(0,\infty)} \frac{|x^{-\mu-\nu-1}\phi(x)(x^{-1}D)^j\theta(x)|}{a^i b^j a_i b_j}, \tag{3.6}$$

for all $\theta \in \Phi_{\{a_i\},a}$, $\Phi^{\{b_j\},b}$ and $\Phi^{\{b_j\},b}_{\{a_i\},a}$, respectively where, ϕ belongs to the respective spaces $H^{\nu}_{\mu,\{a_i\},a}$, $H^{\nu,\{b_j\},b}_{\mu}$ and $H^{\nu,\{b_j\},b}_{\mu,\{a_i\},a}$.

Theorem 1 *Let $\phi \in H^{\nu}_{\mu,\{a_i\},a}$ and the sequence a_i, $i = 1, 2, 3, \cdots$ satisfy (2.6). Then a necessary condition for a C^{∞}-function $\theta(x)$ to be in $\Phi_{\{a_i\},a}$ is*

$$\ell^{\mu,\nu,a}_{\phi,i}(\theta) < \infty,$$

for all $i \in N$.

Proof: Let $\phi \in H^{\nu}_{\mu,\{a_i\},a}$, $i, j \in N$ and $x \in (0, \infty)$. Leibnitz rule being employed leads directly to

$$\begin{aligned} &\left|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}(\theta\phi)(x)\right| \leq \\ &\textstyle\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \left|x^{-\mu-\nu-1}\phi_r(x)(x^{-1}D)^r \theta(x)\right|, \end{aligned} \tag{3.7}$$

where

$$\phi_r(x) = (1+x^2)^i x^{-\mu-\nu-1}(x^{-1}D)^{j-r} x^{-\mu-\nu-1}\phi(x). \tag{3.8}$$

The fact that $\phi(x) \in H^{\nu}_{\mu,\{a_i\},a}$, implies that $x^{-\mu-\nu-1}(x^{-1}D)^{j-r}x^{-\mu-\nu-1}\phi(x)$ and $(1+x^2)^i\phi(x)$ belong to $H^{\nu}_{\mu,\{a_i\},a}$. This together with the definition of the Hankel type integral, we can inferre that $\phi \in H^{\nu}_{\mu,\{a_i\},a}$.

Upon multiplying (3.7) by $\dfrac{1}{(a+\alpha)^i a_i}$, we obtain

$$\begin{aligned} &\frac{\left|(1+x^2)^i (x^{-1}D)^j x^{-\mu-\nu-1}(\theta\phi)(x)\right|}{(a+\alpha)^i a_i} < \\ &\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \frac{\left|x^{-\mu-\nu-1}\phi_r(x)(x^{-1}D)^r \theta(x)\right|}{a^i a_i}, \end{aligned} \tag{3.9}$$

where ϕ_r satisfies (3.8) and $\alpha > 0$. For $i > r$, we infer from (2.6) that

$$\frac{1}{a_i} \leq \frac{a_0}{a_r a_{i-r}}. \tag{3.10}$$

Using (3.10) in (3.9) and considering supremum over all $x \in (0, \infty)$, $i, r \in N$, yields

$$\begin{aligned} \gamma^{\mu,\nu}_{j,\alpha}(\theta\phi) &< \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \frac{a_0}{a_{i-r}a^{i-r}} \ell^{\mu,\nu,a}_{\phi,r}(\theta) \\ &< \sum_{r=0}^{i} \begin{pmatrix} i \\ r \end{pmatrix} \frac{a_0}{a_{i-r}a^{i-r}} \ell^{\mu,\nu,a}_{\phi,r}(\theta), \end{aligned}$$

for $i > j$. This completes the proof of the theorem.

Theorem 2 *Let $\phi \in H^{\nu,\{b_j\},b}_{\mu}$ and $j \in N$, then the necessary condition for a function $\phi(x)$ to be in $\Phi^{\{b_j\},b}$ is to be infinitly smooth and*

$$\ell^{\mu,\nu,b}_{\phi,j}(\theta) < \infty, \tag{3.11}$$

for sequences (b_j) having (2.3) imposed.

Proof: Let $\phi \in H_{\mu}^{\nu,\{b_j\},b}$ and $i, j \in N$, then analysis similar to that of the proof of Theorem 2 leads directly to the relation

$$\frac{(1+x^2)^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\phi)(x)}{(b+\beta)^j b_j} < F\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\frac{|x^{-\mu-\nu-1}\phi_r(x)(x^{-1}D)^r\theta(x)|}{b^r b_r}\frac{b_0}{b_{j-r}b^{j-r}}. \tag{3.12}$$

Once again, letting j and r traverse the set of natural numbers and considering supremum over all $x \in (0,\infty)$ we obtain

$$\gamma_{i,\beta}^{\mu,\nu}(\theta\phi) < F\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\frac{b_0}{b_{j-r}b^{j-r}}\ell_{\phi,r}^{\mu,\nu,b}(\theta).$$

Thus the proof of the theorem is completed.

Theorem 3 *Let a and b be certain positive constants and the sequences (a_i) and (b_j) satisfy (2.6). Then, it is necessary for $\theta \in C^\infty(0,\infty)$ to be in $\Phi_{\{a_i\},a}^{\{b_j\},b}$ is that*

$$\ell_{\phi,a}^{\mu,\nu,b}(\theta) < \infty,$$

for all $\phi \in H_{\nu,\{a_i\},a}^{\mu,\{b_j\},b}$.

Proof: Let $\phi \in H_{\nu,\{a_i\},a}^{\mu,\{b_j\},b}$. Analysis analogous to the proof of Theorems 1 and 2, the relation (3.7) implies

$$\frac{|(1+x^2)^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\phi)(x)|}{(a+\alpha)^i(b+\beta)^j a_i b_j} < E\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\frac{|x^{-\mu-\nu-1}\phi_r(x)(x^{-1}D)^r\theta(x)|}{a^i b^j a_i b_j}, \tag{3.13}$$

where ϕ_r have expresion (3.8).
Having (2.3) employed for (a_i) and (b_j) and allowing x to traverse the real numbers in $(0,\infty)$, then (3.13) can be put in the form

$$\gamma_{\alpha,\beta}^{\mu,\nu}(\theta\phi) < E\sum_{r=0}^{j}\left(\begin{array}{c} j \\ r \end{array}\right)\frac{a_0 b_0}{a_{i-r}a^{i-r}b_{j-r}b^{j-r}}\ell_{\phi,a}^{\mu,\nu,b}(\theta),$$

by which we complete the proof of the theorem.

4 Operators On Multipliers

The present section is devoted for operators continuous on the defined spaces that precisely constructed in such a way they consist of ultra differentiable functions of rapid decent. However, the results are established by virtue of the new equipped topology.

Theorem 4 *The mapping*

$$\Phi_{\{a_i\},a} \times \Phi_{\{a_i\},a} \to \Phi_{\{a_i\},a}$$

defined by

$$(\theta, \psi) \to \theta\psi$$

is separately continuous.

Proof: Let θ and ψ be two elements in $\Phi_{\{a_i\},a}$ and $i \in N$ be such that for some positive constants a and C we have

$$\left|(x^{-1}D)^i\theta\right| \leq C(1+x^2)^i a^i a_i, \tag{4.1}$$

for all $x \in (0, \infty)$.
Let $\phi \in H^{\nu}_{\mu,\{a_i\},a}$, then Leibnitz rule implies

$$\begin{aligned} &\left|x^{-\mu-\nu-1}\phi(x)(x^{-1}D)^i(\theta\psi)(x)\right| \leq \\ &\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \left| x^{-\mu-\nu-1}\phi_r(x)\frac{(x^{-1}D)^r\theta(x)}{(1+x^2)^r}(x^{-1}D)^{i-r}\psi(x)\right|, \end{aligned} \tag{4.2}$$

where $\phi_r(x) = (1+x^2)^r\phi(x)$ which is, in fact, a function in $H^{\nu}_{\mu,\{a_i\},a}$.
Invoking (4.1) and multiplying by $\frac{1}{a^i a_i}$ we obtain from (4.2) that

$$\begin{aligned} &\frac{\left|(x^{-1}D)\phi(x)x^{-\mu-\nu-1}(\theta\psi)(x)\right|}{a^i a_i} < \\ &C\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} a^r a_r \frac{\left|x^{-\mu-\nu-1}\phi_r(x)(x^{-1}D)^r\theta(x)\right|}{a^i a_i}, \end{aligned} \tag{4.3}$$

where C and a are certain positive constants.
Under the assumption that (3.10) holds true and allowing x to traverse the set of real numbers in $(0, \infty)$ and $i \in N$, we obtain from (4.3) that

$$\ell^{\mu,\nu,a}_{\phi,i}(\theta\psi) < C\sum_{r=0}^{i} \begin{pmatrix} i \\ r \end{pmatrix} \frac{a_0 a_i}{a_r a^r} \ell^{\mu,\nu,a}_{\phi_r,i-r}(\psi).$$

This proves the theorem.
Next, analogous to Theorem 4 we state the following:

Theorem 5 *The mapping*

$$\Phi^{\{b_j\},b} \times \Phi^{\{b_j\},b} \to \Phi^{\{b_j\},b}$$

defined by

$$(\theta, \psi) \to \theta\psi$$

is separately continuous.

Theorem 6 *The mapping*

$$\Phi^{\{b_j\},b}_{\{a_i\},a} \times \Phi^{\{b_j\},b}_{\{a_i\},a} \to \Phi^{\{b_j\},b}_{\{a_i\},a}$$

defined by

$$(\theta, \psi) \to \theta\psi$$

is separately continuous.

Theorem 7 *Let a_i, $i = 1, 2, 3, \ldots$ be a sequence imposed with (2.2). The map*

$$\Phi_{\{a_i\},a} \to \Phi_{\{a_i\},a}$$

defined by

$$\theta(x) \to (x^{-1})^k \theta(x)$$

is continuous for any k.

Proof: Owing to (2.7) together with (2.2) we have the existence of the positive constants S, T and a such that

$$\ell^{\mu,\nu,a}_{\psi,i}((x^{-1}D)^k\theta) \leq ST^{i+k}a^k \frac{a_k}{a_i} \ell^{\mu,\nu,a}_{\psi,i+k}(\theta).$$

Which proves the theorem.

Theorem 8 *Let the sequences a_i and b_j, $i, j = 1, 2, 3, \ldots$ satisfy (2.2). Then for any k*
(i) The map

$$\Phi^{\{b_j\},b} \to \Phi^{\{b_j\},b}$$

defined by

$$\theta(x) \to (x^{-1})^k \theta(x)$$

is continuous.

(ii) The map

$$\Phi^{\{b_j\},b}_{\{a_i\},a} \to \Phi^{\{b_j\},b}_{\{a_i\},a}$$

defined by

$$\theta(x) \to (x^{-1})^k \theta(x)$$

is continuous.

5 Continuous Maps Related To Ultra-Differentiable Functions And Corresponding Multiplier

In contrast to section 4, the present section deals with continuous maps related to investing spaces of testing functions together with their corresponding multipliers. However, proofs are followed analogously.

Theorem 9 *The map*

$$H^{\nu}_{\mu,\{a_i\},a} \to \Phi_{\{a_i\},a}$$

defined by

$$\phi(x) \to x^{-\mu-\nu-1}\phi(x)$$

is continuous

Proof: Let $\psi(x) \in H^{\nu}_{\mu,\{a_i\},a}$. For any $\phi(x) \in H^{\nu}_{\mu,\{a_i\},a}$ we have

$$\ell^{\mu,\nu,a}_{\psi,i}\left(x^{-\mu-\nu-1}\phi(x)\right) < \sup_{x\in(0,\infty)}\left|x^{-\mu-\nu-1}\psi(x)\right| \sup_{x\in(0,\infty)}\frac{\left|(x^{-1}D)^k x^{-\mu-\nu-1}\right|}{a_i a^i} \times \frac{(a+\alpha)^i}{(a+\alpha)^i},$$

where $\alpha > 0$, being arbitrary.
That is,

$$\ell^{\mu,\nu,a}_{\psi,i}\left(x^{-\mu-\nu-1}\phi(x)\right) \le \frac{(a+\alpha)^i}{a^i}\sup_{x\in(0,\infty)}\left|x^{-\mu-\nu-1}\psi(x)\right|\gamma^{\mu,\nu}_{a,\beta}(\phi).$$

The proof is, therefore, completed.

Theorem 10 *The mapping*

$$H^{\nu,\{b_j\},b}_{\mu} \to \Phi^{\{b_j\},b}$$

defined by

$$\phi(x) \to x^{-\mu-\nu-1}\phi(x)$$

is continuous.

Theorem 11 *The mapping*

$$H^{\nu,\{b_j\},b}_{\mu,\{a_i\},a} \to \Phi^{\{b_j\},b}_{\{a_i\},a}$$

defined by

$$\phi(x) \to x^{-\mu-\nu-1}\phi(x)$$

is continuous.

Theorems 10 and 11 concern the proof similar to that of Theorem 9 and, thus, avoided.

Definition: Let $f \in \acute{H}^{\nu}_{\mu,\{a_i\},a}$ (the dual of $H^{\nu}_{\mu,\{a_i\},a}$) and $\theta \in \Phi_{\{a_i\},a}$ then we, in view of [7], define

$$\langle \theta f, \phi\rangle = \langle f, \theta\phi\rangle, \phi \in \acute{H}^{\nu}_{\mu,\{a_i\},a}$$

which states that the space of multiplier of the ultra-distribution space $\acute{H}^{\nu}_{\mu,\{a_i\},a}$ can be considered as the space of multipliers for its predual space of ultradifferentiable functions. The previous theorems justify the case for ultra-distribution $f \in \acute{H}^{\nu,\{b_j\},b}_{\mu}$ and $\acute{H}^{\nu,\{b_j\},b}_{\mu,\{a_i\},a}$.

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ON THE NUMERICAL SOLUTION OF DELAY DIFFERENTIAL SYSTEMS

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Abstract–The objective of this study, is to present the multiquadric approximation scheme on numerical solution of delay differential systems. For this purpose, some advantages of using the method and compare it with other methods are presented. In the sequel, presented numerical solutions of some experiments, illustrate the high accuracy and the efficiency of the proposed method even where the data points are scattered.

Key-words: Multiquadric approximation scheme; Delay differential systems; Numerical comparison of the solutions; Scattered date points.
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1. Introduction

Multiquadric (MQ) approximation scheme is a useful method for the numerical solution of ordinary and partial differential equations (ODEs and PDEs). It is a grid-free spatial approximation scheme which converges exponentially for the spatial terms of ODEs and PDEs. The MQ approximation scheme was first introduced by Hardy [1] who successfully applied this method for approximating surface and bodies from field data. Hardy [2] has written a detailed review article summarizing its explosive growth in use since it was first introduced. In 1972, Franke [3] published a detailed comparison of 29 different scattered data schemes against analytic problems. Of all the techniques tested, he concluded that MQ performed the best in accuracy, visual appeal, and ease of implementation, even against various finite element schemes. The object of this paper is the development of the MQ approximation scheme on the numerical solution of delay differential systems (DDSs). A time delay phenomenon is encountered in a wide variety of scientific and engineering applications; including infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircrafts, circuit analysis, computer-aided design, real-time simulation of mechanical systems, chemical process simulation, and optimal control. Time delay arises naturally in connection with system process and information flow for different part of dynamical systems. For more information of delay systems, see Refs. ([4] and [5]). The theory of DDSs is of both theoretical and practical interest. For instance; functional differential equations are the natural models of fluctuations of voltage and current in problems arising in transmission lines. In Salamon [6], a four-dimensional linear system was derived for the current and voltage in a lossless transmission line. Also, the delay systems often appear in the study of automatic control, population dynamics, and vibrating masses attached to an elastic bar.

The organization of this paper is as follows: Section 2 is devoted to introduce the MQ approximation scheme and its preliminary concepts. In Section 3, we have applied the method for delay differential systems. In Section 4, we present some experiments in which their numerical results illustrate the accuracy and efficiency of the proposed method even where the data points are scattered and finally in Section 5, we present some conclusions.

2. MQ approximation scheme

The basic MQ approximation scheme assumes that any function can be expanded as follows as a finite series of upper hyperboloids,

$$x(t) = \sum_{j=1}^{N} a_j h(t - t_j), \qquad t \in \mathbb{R}, \tag{1}$$

where N is the total number of data centers under consideration, and

$$h(t - t_j) = ((t - t_j)^2 + R^2)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N,$$

$(t - t_j)^2$ is the square of Euclidean distances in $\mathbb{R}$ and $R^2 > 0$ is an input shape parameter. Note that, the basis function h is infinitely differentiable, and is a type of spline approximation.

The expansion coefficients a_j are found by solving a set of full linear equations as follows:

$$x(t_i) = \sum_{j=1}^{N} a_j h(t_i - t_j), i = 1, 2, \ldots, N. \tag{2}$$

The system in above sometimes is ill-conditioned, because the parameter $h(t_i - t_j)$ is the square of Euclidean distances and then produce a matrix of coefficients in which its condition number may be greater than 1 [7].
Zerroukat et al [8] found that a constant shape parameter (R^2) has achieved a better accuracy. Nam and Tranh [9] have developed new methods based on radial basis function networks (RBFNs) for the approximation of both functions and their first and higher derivatives. The so called direct RBFN (DRBFN) and indirect RBFN (IRBFN) to methods were studied and it was found that the IRBFN method yields consistently better results for both functions and their derivatives. Recently, Aminataei and Mazarei ([10] and [11]) and Mazarei and Aminataei [12] have introduced a variant of direct nd indirect RBFNs for the numerical solution of Poisson's equation. They have used transformation from Cartesian coordinates to polar ones and have used DRBFN and IRBFN methods on the basis of a MQ approximation scheme. They have experienced that the results shows better accuracy that previously known ones. Also, their new way of solution does not influence the condition number.

Micchelli [13] proved that MQ belongs to a class of conditionally positive definite RBFNs. He showed that the equation (2) is always solvable for distinct points. Madych and Nelson [14] proved that the MQ interpolation always produces a minimal semi-norm error, and that the MQ interpolant and derivative estimates converge exponentially as the density of data centers increases. However, the theoretical exponential accuracy is unreachable when solving equation (2) using standard algorithms in floating point arithmetic. It is well-known that increasing N while keeping the shape parameter constant results in severe ill-conditioning that destroys exponential accuracy and that decreasing the shape while increasing N allow destroys exponential accuracy.

The MQ interpolant is continuously differentiable over the entire domain of data centers, and the spatial derivative approximations were found to be excellent, most especially in very steep gradient regions where traditional methods fail. This excellent ability to approximate spatial derivatives is due in large part by a slight modification of the original MQ scheme by permitting the shape parameter to vary with the basis function.

Instead of using the expansion in equation (1), we have used from ([15] – [17]) the following:

$$x(t) = \sum_{j=1}^{N} a_j h(t - t_j), \qquad t \in \mathbb{R}, \tag{3}$$

where

$$h(t - t_j) = ((t - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N, \tag{4}$$

$$R_j^2 = R_{min}^2 (\frac{R_{max}^2}{R_{min}^2})^{(\frac{j-1}{N-1})}, \quad j = 1, 2, \ldots, N,$$

and

$$R_{min}^2 > 0.$$

R_{max}^2 and R_{min}^2 are two input parameters chosen so that the ratio

$$\frac{R_{max}^2}{R_{min}^2} \cong 10 \quad to \quad 10^6.$$

In [18], Madych proved that under circumstances very large values of a shape parameter are desirable. The ad-hoc formula in equation (4) is a way to have at least one very large value of a shape parameter without incurring the onset of severe ill-conditioning problems.

Spatial partial derivatives of any function are formed by differentiating the spatial basis functions. Consider a one dimensional problem. The first derivative is given in the same line by simple differentiation:

$$\dot{x}(t_i) = \sum_{j=1}^{N} \frac{a_j (t_i - t_j)}{h_{ij}}, \quad h_{ij} = ((t_i - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad i = 1, 2, \ldots, N.$$

3. Numerical solution of DDSs

In this section, we intend solve DDSs by the MQ approximation scheme. Let us consider the following DDSs,

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)x(\alpha(t)) + C(t)\dot{x}(\beta(t)) + F(t), \quad t \in [t_1, t_f], \\ &x(t) = \phi(t), \quad t \le t_1, \end{aligned} \tag{5}$$

where

$$x(t) = (x_1(t), x_2(t), \ldots x_m(t))^T, \qquad x_k(t) \in C, \;\; k = 1, 2, \ldots, m, \tag{6}$$

is the state vector, and

$$\begin{aligned} x(\alpha(t)) &= (x_1(\alpha_1(t)), x_2(\alpha_2(t)), \dots x_m(\alpha_m(t)))^T, \\ \dot{x}(\beta(t)) &= (\dot{x}_1(\beta_1(t)), \dot{x}_2(\beta_2(t)), \dots \dot{x}_m(\beta_m(t)))^T, \end{aligned} \tag{7}$$

such that $\{\alpha_k \leq t_f\}_{k=1}^m$ and $\{\beta_k \leq t_f\}_{k=1}^m$ are delay functions; $A(t), B(t)$ and $C(t)$ are m-dimensional matrices which their elements are complex functions of t and $F(t) \in C^{m\times 1}$ is the vector of continuous complex functions. Also $\phi(t) \in C^{m\times 1}$ represents the vector of initial functions or initial data points.

For the solution of equation (5), it is sufficient to suppose that the approximate solution for each $x_k(t)$ is as

$$x_k(t) = \sum_{j=1}^{N} a_{kj} h(t - t_j), \qquad t_1 \leq t \leq t_f. \tag{8}$$

Choosing $t_i,\ i = 1, 2, \dots, N$, as collocating points, we have:

$$x_k(t_i) = \sum_{j=1}^{N} a_{kj} h(t_i - t_j), \tag{9}$$

and

$$\dot{x}_k(t_i) = \sum_{j=1}^{N} \frac{a_{kj}(t_i - t_j)}{h_{ij}}, \tag{10}$$

also, for $k = 1, 2, \dots, m$,

$$x_k(\alpha_k(t_i)) = \sum_{j=1}^{N} a_{kj} h(\alpha_k(t_i) - t_j), \tag{11}$$

$$\dot{x}_k(\beta_k(t_i)) = \sum_{j=1}^{N} \frac{a_{kj}(\beta_k(t_i) - t_j)}{h_{ij}^{(\beta_k)}}, \tag{12}$$

where

$$h_{ij}^{(\beta_k)} = h(\beta_k(t_i) - t_j) = ((\beta_k(t_i) - t_j)^2 + R_j^2)^{\frac{1}{2}}. \tag{13}$$

By substituting equations (9)-(12) to equation (5), and imposing the supplementary conditions $\{x_k(t_1) = \phi_k(t_1)\}_{k=1}^m$to the problem, we gain $m(N-1)$ equations of differential forms and initial conditions to produce m equations. Hence, the system of mN equations with mN unknowns is available. Then, we must solve this system to find the unknown coefficients. Hence, we have used the Gauss elimination method with total pivoting to solve such a system.

Remark. It is noticeable that collocating points can be scattered. This is one of the most important advantages of the MQ approximation scheme. In Section 4, the numerical results demonstrate this issue easily, and the efficiency of the MQ approximation scheme in this sense, is observable.

4. Numerical experiments

In this section, two experiments are presented in which their numerical solutions illustrate some advantages of the method with high accuracy and efficiency and we compare it with the other numerical schemes.

Problem 1. Consider the following DDS,

$$\begin{cases} \dot{x}_1(t) = -2(1-cos(t))(t-sin(t))x_2(t-sin(t)) + \dot{x}_3(sin(t)) - x_3(sin(t)), & 0 \le t \le 1, \\ \dot{x}_2(t) = -2tx_2(t) + \dot{x}_1(t) + 2(1-cos(t))(t-sin(t))x_2(t-sin(t)), & 0 \le t \le 1, \\ \dot{x}_3(t) = -\dot{x}_2(\sqrt{t}) - 2\sqrt{t}x_2(\sqrt{t}) + x_3(t), & 0 \le t \le 1, \\ x_1(t) = t^2+1, \quad t \le 0, \\ x_2(t) = 1-t^3, \quad t \le 0, \\ x_3(t) = e^t, \qquad t \le 0. \end{cases}$$

The exact solution in the interval $[0, 1]$ is

$$x_1(t) = e^{-(t-sin(t))^2}, \ x_2(t) = e^{-t^2}, x_3(t) = e^t.$$

The MQ approximate solution is obtained with $R_{max} = 50, R_{min} = 1.1$ and 10 scattered data points ($N = 10$). Also, we have compared the method with 10 terms of Taylor series expansion and Chebyshev collocation method [19] with scattered data points. Then, we have found that the proposed method is more reliable and powerful than the aforesaid methods. The results are given in Tables 1-3 and Figures 1-3.

Table 1: Comparison of the Taylor series expansion, Collocation method, MQ approximation scheme, exact solution and maximum absolute error of $x_1(t)$ of problem 1

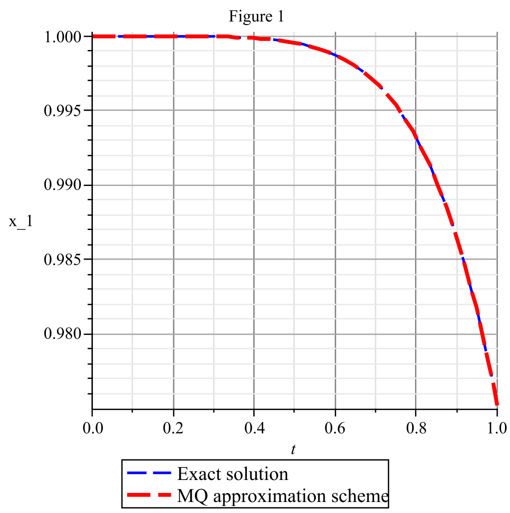
t	Taylor approximation	Collocation method	MQ approximation scheme	Exact solution	Maximum absolute error
0.000	1.0000000	1.0000000	1.0000000	1	0
0.031	1.0000010	1.0000001	1.0000000	0.9999999	1×10^{-7}
0.085	1.0000018	1.0000002	0.9999999	0.9999999	0
0.351	0.9999508	0.9999489	0.9999487	0.9999486	1×10^{-7}
0.370	0.9999318	0.9999299	0.9999296	0.9999297	1×10^{-7}
0.452	0.9997701	0.9997681	0.9997679	0.9997679	0
0.690	0.9971479	0.9971460	0.9971457	0.9971458	1×10^{-7}
0.850	0.9903040	0.9903020	0.9903018	0.9903017	1×10^{-7}
0.999	0.9753258	0.9753237	0.9753235	0.9753235	0
1.000	0.9751828	0.9751819	0.9751817	0.9751817	0

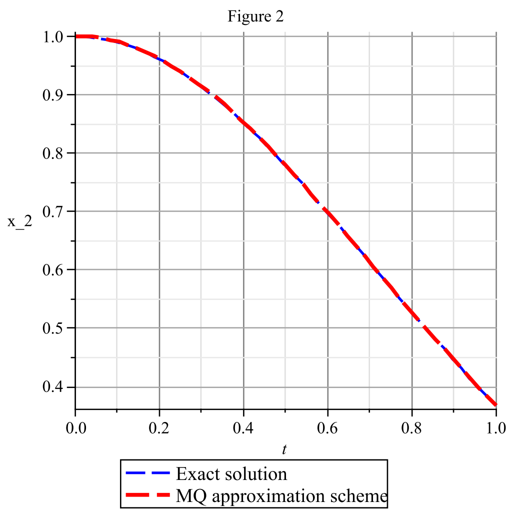
Table 2: Comparison of the Taylor series expansion, Collocation method, MQ approximation scheme, exact solution and maximum absolute error of $x_2(t)$ of problem 1

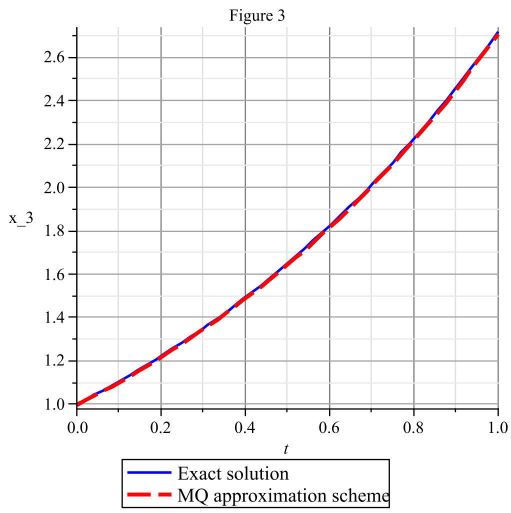
t	Taylor approximation	Collocation method	MQ approximation scheme	Exact solution	Maximum absolute error
0.000	1.0000000	1.00000000	1.0000000	1	0
0.031	0.9990388	0.9990402	0.9990394	0.9990394	0
0.085	0.9922799	0.9928022	0.9928010	0.9928010	0
0.351	0.8840848	0.8840869	0.8840859	0.8840859	0
0.370	0.8720563	0.8720584	0.8720573	0.8720574	1×10^{-7}
0.452	0.8152134	0.8152154	0.8152144	0.8152145	1×10^{-7}
0.690	0.6212005	0.6212020	0.6212013	0.6212013	0
0.850	0.4855362	0.4855374	0.4855368	0.4855368	0
0.999	0.3686145	0.3686160	0.3686155	0.3686155	0
1.000	0.3678783	0.3678799	0.3678794	0.3678794	0

Table 3: Comparison of the Taylor series expansion, Collocation method, MQ approximation scheme, exact solution and maximum absolute error of $x_3(t)$ of problem 1

t	Taylor approximation	Collocation method	MQ approximation scheme	Exact solution	Maximum absolute error
0.000	1.0000000	1.0000000	1.0000000	1	0
0.031	1.0314854	1.0314854	1.0314855	1.0314855	0
0.085	1.0887170	1.0887169	1.0887170	1.0887170	0
0.351	1.4204873	1.4204817	1.4204873	1.4204873	0
0.370	1.4477345	1.4477344	1.4477346	1.4477346	0
0.452	1.5714519	1.5714517	1.5714519	1.5714519	0
0.690	1.9937155	1.9937153	1.9937155	1.9937155	0
0.850	2.3396469	2.3396466	2.3396468	2.3396468	0
0.999	2.7155661	2.7155645	2.7155649	2.7155649	0
1.000	2.7182831	2.7182814	2.7182818	2.7182818	0







Since DDSs are depended on delay parameters, thus when we apply any collocation methods then delay parameters cause to produce scattered data and the considered method does not work well. We have observed that, this method (the MQ approximation scheme) is not dependent on collocating points. But, when we apply other methods which need collocating points, if scattered data points are used, the round off error may occurs soon. This is an other excellent advantage on the application of the MQ approximation scheme.

Problem 2. Consider the following DDS,

$$\begin{cases} \dot{x}_1(t) = tx_1(t^2) - tx_1(\frac{t}{2}) + t\dot{x}_2(\frac{t}{2}) + \frac{1}{2}x_3(t^3) - \frac{1}{2}\dot{x}_3(t^3), & 0 \le t \le 1, \\ \dot{x}_2(t) = \dot{x}_2(t^2) + x_2(t) - x_1(t^4), & 0 \le t \le 1, \\ \dot{x}_3(t) = x_1(\frac{t}{2}) - t\dot{x}_2(\frac{t}{2}) + \frac{1}{2}\dot{x}_2(\frac{t}{2}) - \frac{t}{3}x_3(\frac{t}{3}) + \dot{x}_4(t), & 0 \le t \le 1, \\ \dot{x}_4(t) = \frac{1}{3}\dot{x}_2(\frac{t}{3}) - x_2(\frac{t}{2}) + x_3(t), & 0 \le t \le 1, \\ x_1(t) = t, \qquad t \le 0, \\ x_2(t) = t^2, \qquad t \le 0, \\ x_3(t) = t^3, \qquad t \le 0, \\ x_4(t) = t^4, \qquad t \le 0. \end{cases}$$

The exact solution in the interval $[0,1]$ is

$$x_1(t) = t, x_2(t) = t^2, x_3(t) = t^3, x_4(t) = t^4.$$

Table 4: Maximum absolute error of $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ for MQ approximation scheme with different R_{max}, R_{min} and N

N	R_{max}	R_{min}	$x_1(t)$	$x_2(t)$	$x_3(t)$	$x_4(t)$
8	80	.95	2.2×10^{-4}	1.9×10^{-4}	2.5×10^{-4}	2.5×10^{-4}
10	70	.96	7.3×10^{-6}	6.5×10^{-6}	5.8×10^{-6}	5.8×10^{-6}
12	60	.97	4.7×10^{-7}	4.0×10^{-7}	5.8×10^{-7}	5.8×10^{-7}
14	50	.98	4.9×10^{-7}	4.0×10^{-7}	5.9×10^{-7}	5.9×10^{-7}
16	40	.99	4.9×10^{-8}	3.0×10^{-8}	3.0×10^{-8}	3.0×10^{-8}

Numerical results show that, the MQ approximation scheme in the required domain needs the minimum number of data points for the solution. This demonstrates one of the advantages of MQ approximation scheme in spite of its simplicity.

5. Conclusion

As shown by the above numerical results, MQ approximation scheme benefits from the following advantages:
i. being easy to implement.
ii. being independent of the collocating points in large scales.
iii. being well-applicable for the first three and the last three collocating points which have a very small metric.
iv. requiring the minimum number of data points in the required domain.

It should be noted that the computations associated in the experiments discussed above were performed by using Maple 10.

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PORTFOLIO SELECTION BASED ON A SIMULATED COPULA

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PORTFOLIO SELECTION BASED ON A SIMULATED COPULA

Abstract: In this paper, we propose a methodology to value the portfolio choices based on the prediction of future returns where the dependence structure of joint returns and the behavior of single returns are estimated separately. In particular, we assume the marginals evolve as an ARMA(0,2)-GARCH(0,2) model with stable paretian residuals and the joint distribution of residuals is estimated with an asymmetric t-copula. Then, we compare the ex-post final wealth sample paths of different strategies based on reward/risk ratios. Doing so we examine and discuss the impact of the forecasting method with respect to classic myopic portfolio strategies based on the same reward/risk ratios.
Key words: performance ratios, asymmetric t copula, stable distributions, dynamic mesures.

1 Introduction

This paper analyses and discusses the profitability of some reward risk strategies based on a forecasted evolution of returns. In particular, we examine the impact of a simulated copula on the investors choices. We first try to approximate as much as possible the joint behavior of future returns taking into account their distributional characteristics. Then we compare ex post the choices based on this approximations.

The motivations behind this paper comes from three stylized facts about real world financial markets. First, financial return series are asymmetric and heavy tailed and they cannot be approximated with a normal distribution that is symmetric and has too light tails to match market data. Second, there is volatility clustering in time series since calm periods are generally followed by highly volatile periods and vice versa. Finally, a dependence structure of multivariate distribution is needed beyond simple linear correlation. The dependence model has to be flexible enough to account for several empirical phenomena observed in the data, in particular asymmetry of dependence and dependence of the tail events (see among others Rachev and Mittnik (2000), Rachev et al. (2005), Rachev et al. (2007)). Therefore the dependence model cannot be approximated with a multivariate normal distribution that fails to describe both phenomena (i.e., the covariance is a symmetric dependence concept and the tail events are asymptotically independent). In searching for an acceptable model to describe these three stylized facts, we examine the behavior of marginals with a time series process belonging to the ARMA-GARCH family with stable paretian innovations and we suggest to model dependencies with an asymmetric t copula valued on the innovation of the marginals. As a matter of fact, the use of a copula function allows to take into account of phenomena such as clustering of the volatility effect, heavy-tails, and skewness and then separately model the dependence structure between them. By a practical point of view, the problem to fit the model to the market data can be solved in two steps, indeed marginals and copula can be estimated separately.

In order to value the impact of this methodology that simulate the joint behavior of future returns we compare the performance of several reward risk strategies based either on simulated data or on historical ones. In particular, we use the STARR ratio (see Martin et al. (2003)), and we introduce two new reward/risk ratios based on some dynamic measures recently proposed in literature (see, Rachev et al. (2008), Chekhlov, et al (2005)). As we expect the comparison confirms the better performance of strategies valued on simulated data with respect to those valued on historical data.

The remainder of the paper is organized as follows: Section 2 provides a brief description of the methodology to build scenarios based on a simulated copula. Section 3 provides a comparison among different strategies and Section 4 concludes the paper.

2 Generation of scenarios based on ARMA-GARCH and Copula Models

So far a systematic methodology to forecast, control and model portfolios in volatile markets has not been developed. If we observe the behaviour of the returns we notice several anomalies: heavy tailed distributions, volatility clustering, non Gaussian copula dependence (see, among others, Rachev and Mittnik (2000), Rachev et al. (2005), Rachev et al. (2007)). The empirical evidence remarks the opportunity and the necessity of properly considering the dependence structure of financial variables. In order to describe the dependence structure we discuss a copula approach combined with an opportune valuation of the single return series.

2.1 Methodology to model time series

Let us consider the problem of time series modeling. We propose the following model to simulate future scenarios taking into account the financial structure of the market.

- **Step 1.** Assume we have d series of size T of (closing) daily returns, i.e., $r_t^{(j)}$ is the daily return of the asset j (j=1,..., d) at day t (t=1,...,T).
- **Step 2.** Carry out maximum likelihood parameter estimation of ARMA(p,q)-GARCH(s,u) of each serie:

$$r_t^{(j)} = a_{j,0} + \sum_{i=1}^{p} a_{j,i} r_{t-i}^{(j)} + \sum_{i=1}^{q} b_{j,i} \varepsilon_{j,t-i} + \varepsilon_{j,t} \tag{1}$$

$$\varepsilon_{j,t} = \sigma_{j,t} z_{j,t} \tag{2}$$

$$\sigma_{j,t}^2 = K_j + \sum_{i=1}^{s} c_{j,i} \sigma_{j,t-i}^2 + \sum_{i=1}^{u} e_{j,i} \varepsilon_{j,t-i}^2; \tag{3}$$

$$j = 1, ..., d;\ t = 1, ..., T.$$

- **Step 3.**Approximate with α_j-stable distribution $S_{\alpha_j}(\sigma_j, \beta_j, \mu_j)$ the empirical standardized innovations

$$\widehat{z}_{j,t} = \widehat{\varepsilon}_{j,t} / \sigma_{j,t} \tag{4}$$

where the innovations

$$\widehat{\varepsilon}_{j,t} = r_t^{(j)} - a_{j,0} - \sum_{i=1}^{p} a_{j,i} r_{t-i}^{(j)} - \sum_{i=1}^{q} b_{j,i} \varepsilon_{j,t-i};\ j = 1, ..., d$$

are obtained from (1) (we refer to Samorodnitsky and Taqqu (1994) and Rachev and Mittnik (2000), for a general discussion on properies and use of stable distributions).

- **Step 4.**Simulate S stable distributed scenarios for each of the future standardized innovations series and compute the sample distribution functions of these simulated series:

$$F_{\widehat{z}}^{(j)}(x) = \frac{1}{S}\sum_{s=1}^{S} I_{\left\{\widehat{u}_{T+1}^{(j,s)} \leq x\right\}},\ x \in \Re,\ j = 1, ..., d$$

where $\widehat{u}_{T+1}^{(j,s)}(\ 1 \leq s \leq S)$ is the s-th value simulated with the fitted α_j-stable distribution for future standardized innovation (valued in $T+1$) of the j-*th* asset.

Remark: Steps 3-4 provide the marginal distributions for standardized innovations each of the d assets used to simulate the next-period returns. In the following steps we will estimate the dependence structure of the vector of standardized innovations with an asymmetric t-copula.

- **Step 5** Fit the d-dimensional vector of empirical standardized innovations

$$\widehat{z} = [\widehat{z}_1, ..., \widehat{z}_d]'$$

obtained by (4) with a d-dimensional asymmetric t-distribution[1]. So, let the d-dimensional vector $V = [V_1, ..., V_d]'$ be asymmetric t-distributed. It has the form:

$$V = \mu Y + \sigma Y \cdot Z \tag{5}$$

where μ is a constant, σ is a positive constant, $Y \cdot Z = [Y^{(1)}Z^{(1)}, ..., Y^{(d)}Z^{(d)}]'$, and $Y = [Y^{(1)}, ..., Y^{(d)}]'$ is a d-dimensional vector with positive components distributed as: $Y^{(j)} \overset{d}{=} \frac{1}{\sqrt{\chi^2(\upsilon_j)}}$, where $\chi^2(\upsilon_j)$ is chi-squared distributed random variable with υ_j degrees of freedom. We assume that the components $Y^{(j)}$ $(j = 1, ..., d)$ are independent and vector Y is independent of the vector $Z = [Z^{(1)}, ..., Z^{(d)}]'$. The vector Z is normally distributed with zero mean and covariance matrix Σ. We use the maximum likelihood method to estimate all the parameters of the asymmetric t-distribution $F_V(x_1, ..., x_d)$ for $\widehat{z}$, given by (4). Thus, the asymmetric t-copula is given by:

$$C(t_1, ..., t_d) = F_V(F_{V_1}^{-1}(t_1), ..., F_{V_d}^{-1}(t_d)); 0 \leq t_i \leq 1; 1 \leq i \leq d \tag{6}$$

where $F_{V_i}^{-1}(t_i)$ is the left inverse of the i-*th* marginal distribution of F_V.

- **Step 6.** Since we have estimated all the parameters of Y and Z as well as the constants μ and σ we can generate S scenarios for Y and, independently, S scenarios for Z, and using (5) we obtain S scenarios for the vector of standardized innovations z, that is asymmetric t-distributed. Denote these scenarios $(V_1^{(s)}, ..., V_d^{(s)})$, $s = 1, ..., S$ and denote the marginal

[1] Clearly many other possible heavy tailed, asymmetric multivariate distributions can be considered in this step.

distributions $F_{V_j}(x)$, $1 \leq j \leq d$ of the estimated d-dimensional asymmetric t- distribution:

$$F_V(x_1, ..., x_d) = P(V_1 \leq x_1, ..., V_d \leq x_d).$$

Then considering $U_j^{(s)} = F_{V_j}(V_j^{(s)})$, $1 \leq j \leq d, 1 \leq s \leq S$, we can generate S scenarios $(U_1^{(s)}, ..., U_d^{(s)})$, $s = 1, ..., S$ of the uniform random vector $(U_1, ..., U_d)$ that has support on the d-dimensional unit cube and whose distribution is given by the copula $C(t_1, ..., t_d)$ of formula (6).

Remark: Steps 5-6 serve to estimate the dependence structure among the innovations with an asymmetric t-copula. The next step combines the marginal distributions and the scenarios for the copula into scenarios for the vector of returns.

- **Step 7.** On one side, we have determined the stable distributed marginal sample distribution function of the j-th standardized innovation $F_z^{(j)}(x)$, $j = 1, ..., d$ (see step 4), on the other side we have the scenarios $U_j^{(s)}$ for $1 \leq j \leq d; 1 \leq s \leq S$ (see step 6). Then we generate S scenario of the vector of standardized innovations $z_{T+1}^{(s)} = (z_{T+1}^{(1,s)}, ..., z_{T+1}^{(d,s)}), s = 1, ..., S$ valued at time $T + 1$ (taking into account of the dependence structure of the vector) assuming

$$z_{T+1}^{(j,s)} = \left(F_z^{(j)}\right)^{-1} \left(U_j^{(s)}\right); 1 \leq j \leq d; 1 \leq s \leq S,$$

  Doing so, we generate the vector of standardized innovation assuming that the marginal distributions are $\alpha_j -$stable distributions and considering the copula dependence, given by (6).

- **Step 8** Once we have described the multivariate behavior of standardized innovation at time $T+1$ using relation (2) we can generate S scenario of the vector of innovation $\varepsilon_{T+1}^{(s)} = \left(\varepsilon_{T+1}^{(1,s)}, ..., \varepsilon_{T+1}^{(d,s)}\right) = \left(\sigma_{1,T+1} z_{T+1}^{(1,s)}, ..., \sigma_{d,T+1} z_{T+1}^{(d,s)}\right)$, $s = 1, ..., S$ where $\sigma_{j,T+1}$ are defined by (3). Thus, using relation (1), we can generate S scenario of the vector of returns

$$r_{T+1,s} = [r_{T+1}^{(1,s)}, ..., r_{T+1}^{(d,s)}]'.$$

Observe steps 4-7 can be always used to generate a distribution with some given marginals and a given dependence structure (see among others Rachev et al. (2005), Sun et al. (2008a-2008b), Biglova et al. (2008), Cherubini et al. (2004) for the definition of some classical copula used in finance literature)..

3 An empirical comparison among some reward/risk ratios

Let us consider the optimal portfolio choice problem among d risky assets assets with log-returns $r_t = [r_t^{(j)}, ..., r_t^{(d)}]'$ and assume that there exist a risk-free benchmark with log-return $r_{t,b}$ all valued at time t. When no short selling is allowed, every portfolio of returns is a convex combination of the risky log-returns $r_t^{(j)}$ i.e., $x'r_t = \sum_{i=1}^{d} x_i r_t^{(i)}$ where $x = (x_1, ..., x_d)' \in C = \left\{ y \in \Re^d \middle| y_i \geq 0, \sum_{i=1}^{d} y_i = 1 \right\}$ is the vector of weights. In order to value the impact of the previous model, we provide an empirical ex-post comparison among several strategies based on simulated data and on historical one. As initial data we use the daily return series of the benchmark three months treasury bill and n =5 daily returns on US stock indexes [2] from 12/14/1992 till 2/27/2008 for a total of 3831 observations.

First of all we decide the type of ARMA-GARCH model should be used for the simulation of future scenarios. From this preliminary analysys we deduce that the above asset returns are well approximated by an ARMA(2,0)-GARCH(0,2) model. That is, for each series ($j = 1, ..., 5$) the formulas (1, 2, 3) are represented by:

$$r_t^{(j)} = C + AR(1) r_{t-1}^{(j)} + AR(2) r_{t-2}^{(j)} + \varepsilon_t$$

$$\varepsilon_{j,t} = \sigma_{j,t} z_{j,t}$$
$$\sigma_{j,t}^2 = K + ARCH(1)\varepsilon_{j,t-1}^2 + ARCH(2)\varepsilon_{j,t-2}^2.$$

Then we suppose that decision makers invest their wealth purchasing the *market portfolio* taking into account that each investor has a diverse reward/risk perception. In the last years, several performance measures have been proposed and used in portfolio theory to capture the different perception of reward and risk (see Biglova et al. (2004) (2008)). Among these we recall the STARR ratio and we propose two other alternative performance measures. Thus, we assume that the market portfolio is determined by maximizing one of the following performance measures.

3.1 Performance measures

STARR ratio ($STARR$). The STARR ratio (see Martin et al. (2003)) serves to value the expected excess return for unity of risk represented by the Expected Tail Loss (ETL).i.e.:

$$STARR(x'r_{T+1}, \alpha) = \frac{E(x'r_{T+1} - r_{T+1,b})}{ETL_\alpha(x'r_{T+1} - r_{T+1,b})}.$$

The Expected Tail Loss , also known as Conditional Value-at-Risk (CVaR) , is defined as

$$ETL_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR_q(X) dq,$$

where $VaR_q(X) = -F_X^{-1}(q) = -\inf\{x \,| P(X \leq x) > q\}$ is the Value-at-Risk (VaR) of the random return X. If we assume a continuous distribution for the

[2] The indexes are: Dow Jones Industrials, NYSE, Major Market Index, DJ composite, DJAIG Commodity.

probability law of X, then $ETL_\alpha(X) = -E(X\,|X \leq -VaR_\alpha(X))$ and thus, ETL can be interpreted as the average loss beyond VaR. In our comparison we consider the $STARR$ ratio with parameters $\alpha = 0.01,\ 0.05$.

Rachev maximum drawup/down ratio (*R-maxdud*) With this dynamic performance ratio we introduce the concept of drawup measure and we suggest the use of the drawdown in a reward/risk plane. The absolute drawdown process $dd(x) = \{dd_t(x)\}_{t=1,...,T}$ of a portfolio x has been proposed in portfolio literature by Grossman and Zohu (1993), Cvitanic and Karatzas (1995) (see also Chechkov et al. (2005)) as function of the uncompunded cumulative excess portfolio rate $w_t(x)$ valued at time t, that is:

$$w_t(x) = \sum_{s=1}^{t} x' r_s - r_{t,b},\ \ t = 1, ..., T.$$

Analogously we can define the absolute drawup process $du(x) = \{du_t(x)\}_{t=1,...,T}$. Thus we call drawup and drawdown of the portfolio x respectively the processes given by:

$$du_t(x) = w_t(x) - \min_{s=1,...,t} w_s(x)$$

$$dd_t(x) = \max_{s=1,...,t} w_s(x) - w_t(x)$$

for $t = 1, ..., T$. Then the Rachev maximum drawup/down ratio is a performance ratio between the maximum drawup and the maximum drawdown that is

$$R - maxdud(x' r_{T+1}) = \frac{\max_{t=1,...,T} du_t(x)}{\max_{t=1,...,T} dd_t(x)}$$

Rachev average drawup/down ratio (*R-avedud*) The Rachev average drawup/down ratio is still defined as function of the absolute drawdown and drawup processes $dd(x)$, $du(x)$. However, instead of the maximum we consider the empirical mean, that is:

$$R - avedud(x' r_{T+1}) = \frac{\frac{1}{T}\sum_{t=1}^{T} du_t(x)}{\frac{1}{T}\sum_{t=1}^{T} dd_t(x)}$$

Moreover as in the Chechkov et al. (2005) analysis we can introduce futher ratios that value the ETL ratio of absolute drawdown and drawup processes, but these further performance measures will be object of future empirical analysis. Next we summarize the empirical comparison among different portfolio strategies.

3.2 A first ex-post empirical comparison among different portfolio strategies

Let us summarize the empirical ex-post comparison between historical and simulated data and among different portfolio strategies. For any optimal portfolio chosen on a daily basis, when we use the dynamic performance measures R-avedud and R-maxdud we consider a window of two years $T = 500$ of historical obervations, while we adopt a window of one year $T = 250$ when we use the $STARR$ ratio with parameters $\alpha = 0.01$, 0.05. We use these observations:

a) to compute the performance measures if we use historical data, and

b) to generate $S = T$ future scenarios of returns according to the algorithm proposed in the previous section when we use simulated data.

Therefore, for any reward/risk criterion measure $\rho(x'r)$, we can compute the optimal portfolio as solution the following optimization problem:

$$\begin{array}{c} \max\limits_{x \in C} \rho(x'r) \\ \text{subject to} \\ \sum_{i=1}^{n} x_i = 1; x_i \geq 0; i = 1, ..., n \end{array} \tag{7}$$

Since we assume that decision makers invest their wealth in the market portfolio (solution of (7)), we consider the sample path of the final wealth and of the cumulative return obtained from the different approaches. Then, we compare the efficiency of alternative performance measures. We assume that the investor has an initial wealth W_0 equal to 1 and an initial cumulative return CR_0 equal to 0 (at the date 12/5/1994 when we use $T = 500$ and at the date 12/8/1993 when we use $T = 250$) and at the k-th recalibration ($k = 0, 1, 2, ...$), three main steps are performed to compute the ex-post final wealth and cumulative return:

Step 1 Choose a performance ratio. Generate scenarios with the algorithm of the previous section if we use simulated data. Determine the market portfolio $x_M^{(k)}$ that maximizes the performance ratio $\rho(.)$ associated to the strategy, i.e. the solution of optimization problem of (7).

Step 2 The ex-post final wealth is given by:

$$W_{k+1} = W_k \left(\left(x_M^{(k)} \right)' (1 + r_{kT+1}) \right), \tag{8}$$

where r_{kT+1} is the vector of observed returns between kT and kT+1. The ex-post cumulative return is given by:

$$CR_{k+1} = CR_k + \left(x_M^{(k)} \right)' r_{kT+1}.$$

Step 3 The optimal portfolio $x_M^{(k)}$ is the new starting point for the $(k+1)$-th optimization problem (7).

Steps 1, 2 and 3 are repeated until there are observations available and for all the performance ratios.

The output of this analysis is represented in Figures 1,2,3,4,5,6. All figures show a greater performance for the reward/risk ratios based on simulated data. STARR ratio with $\alpha = 0.05$ presents better performance with respect to STARR ratio with $\alpha = 0.01$ (see Figures 1,2). Moreover the most relevant impact is due

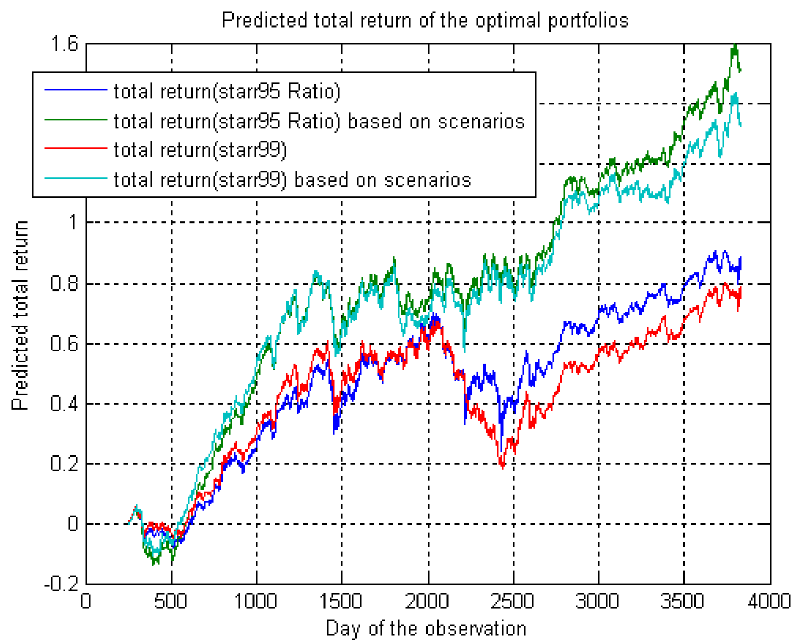

Figure 1: This figure compares the ex-post cumulative return obtained maximizing the STARR ratio (with $1-\alpha = 99\%, 95\%$) and using either real data or simulated data.

to the use of simulated data with respect to the historical observations. Figures 3,4 reports the ex-post final wealth and cumulative return processes when we use the Rachev average drawup/down ratio and even in this case we observe an higher final wealth and cumulative return using simulated data. This difference is a little bit greater (see Figures 5,6) when we use the Rachev maximum drawup/down ratio that presents better performance even with respect to the Rachev average drawup/down ratio.

4 Conclusion

The fundamental contribution of this paper consists in the methodology to solve dynamic portfolio strategies considering realistic assumptions on the returns. In particular, we examine the impact of simulating a copula with opportune marginals in optimal portfolio choices. We first describe how to generate scenarios that take into account of return anomalies: heavy tailed distributions, volatility clustering, non Gaussian copula dependence. Then, we discuss the use of reward/risk criteria to select optimal portfolios and we propose the use of the STARR ratio and two new dynamic performance measures. Finally, we propose

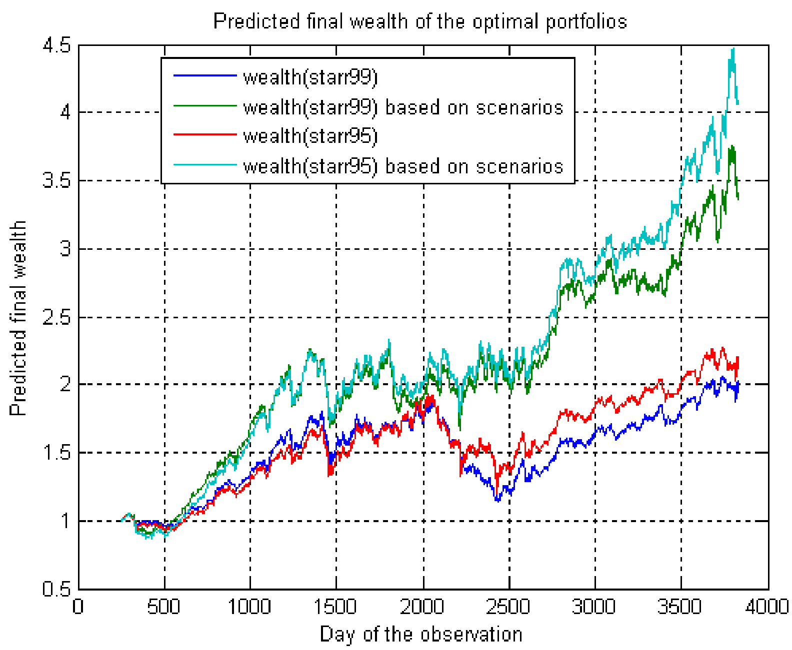

Figure 2: This figure compares the ex-post final wealth obtained maximizing the STARR ratio (with $1 - \alpha = 99\%, 95\%$) and using either real data or simulated data.

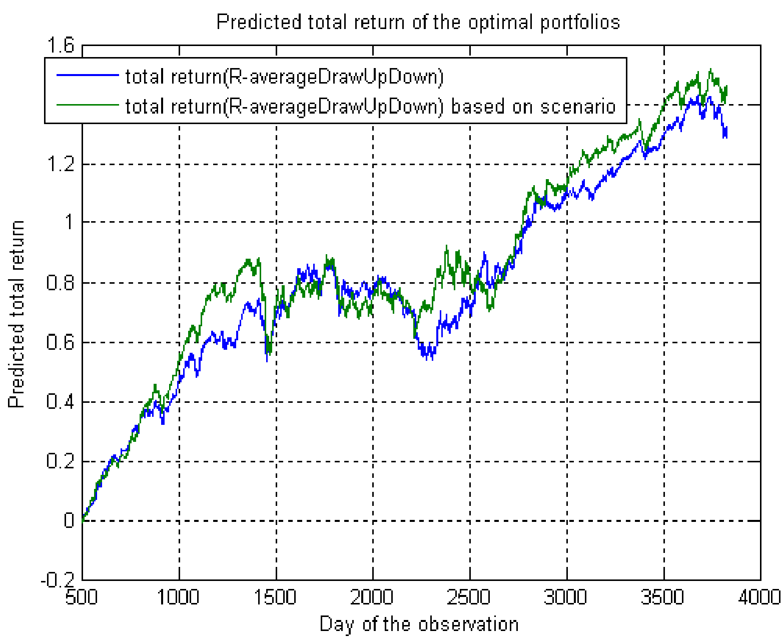

Figure 3: This figure compares the ex-post cumulative return obtained maximizing the R-averagedrawup/down ratio and using either real data or simulated data.

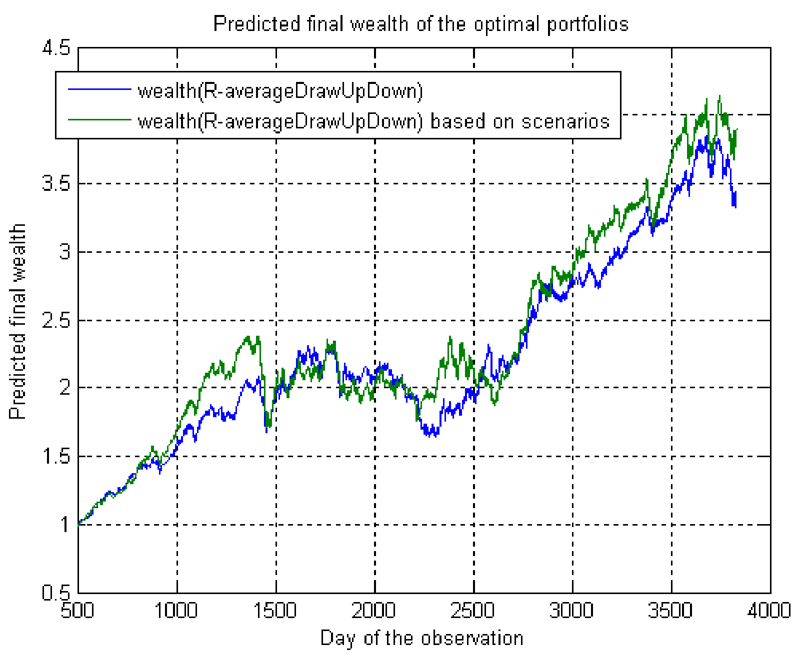

Figure 4: This figure compares the ex-post final wealth obtained maximizing the R-average drawundown ratio and using either real data or simulated data.

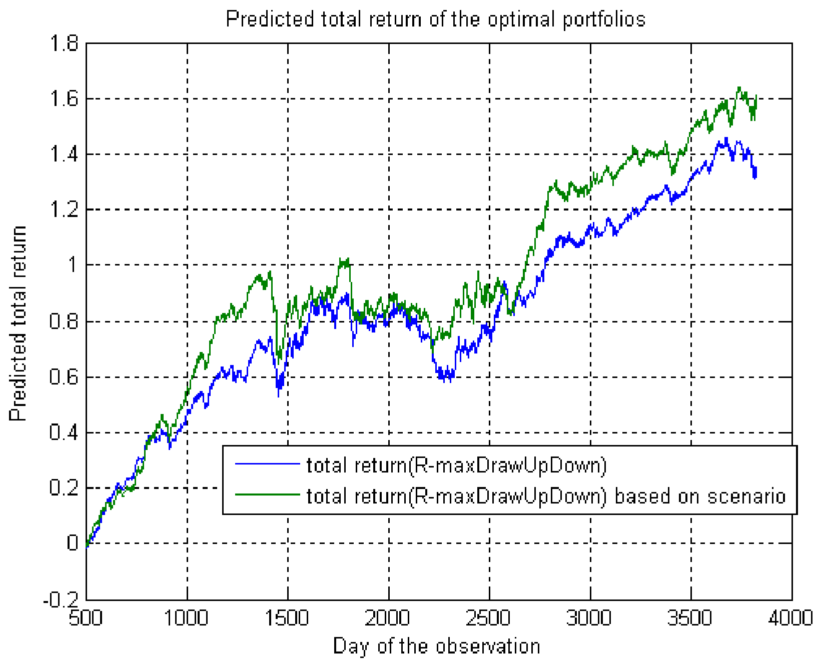

Figure 5: This figure compares the ex-post cumulative return obtained maximizing the R-maximum drawup/down ratio and using either real data or simulated data.

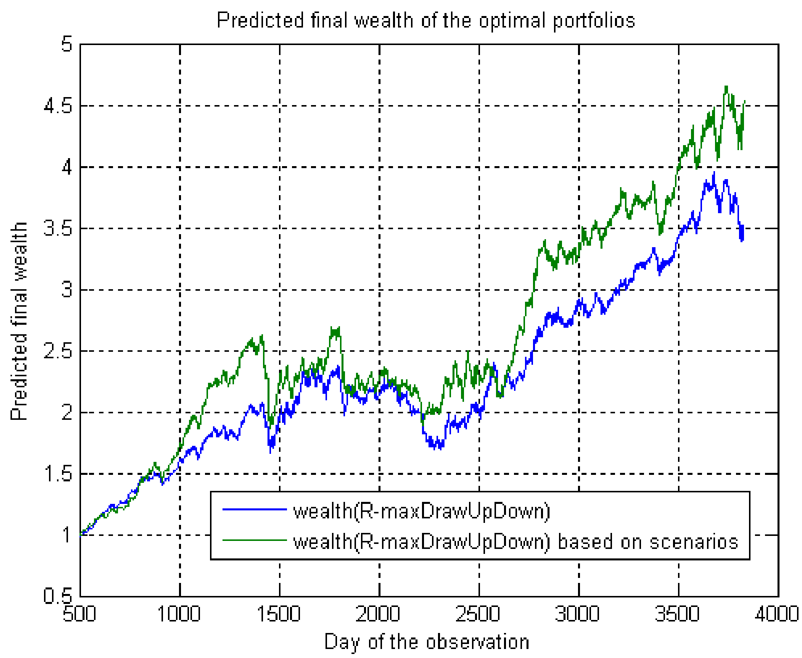

Figure 6: This figure compares the ex-post final wealth obtained maximizing the R-maximum drawup/down ratio and using either real data or simulated data.

an empirical comparison among final wealth and cumulative return processes obtained either using historical observations or using the simulated data. As we expect the ex-post empirical comparison among classic myopic approaches based on historical observations and those based on simulated data shows the greater predictable capacity of the latter.

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A NOTE ON THE IMPACT OF NON LINEAR REWARD AND RISK MEASURES

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A NOTE ON THE IMPACT OF NON LINEAR REWARD AND RISK MEASURES

Abstract: In this note, we examine the impact of non linear reward and risk measures on portfolio selection. In particular, we compare the ex-post final wealth sample paths of strategies based on the Sharpe ratio and strategies based on non-linear reward/risk measures. As suggested by the recent literature, we model dependencies with an asymmetric t copula estimated on the innovations of the marginals that follow an ARMA-GARCH type model. Therefore, we first simulate future scenarios, on the basis of which allocation decisions are made, and then we compare the ex-post final wealth obtained with non-linear risk and reward strategies and the wealth obtained with classic portfolio strategies .
Key words: Sharpe ratio, portfolio choice, asymmetric t copula.

1 Introduction

This paper critically reviews the non-linearity property of risk and reward measures in order to determine its effects on portfolio strategies. In particular, we show with an ex-post empirical analysis how important it is to model it appropriately.

In portfolio-choice theory, a consistency between risk measures and investor's preferences is viewed as desirable and has been explored in the literature. See Rachev et al (2008) and the reference therein for a classification of basic portfolio selection problems with respect to types of investor's behavior. In particular, Ortobelli et al. (2008b) discuss how to compute optimal choices consistent with any admissible preference ordering. Clearly, in portfolio selection theory the criteria of choice represent the factor with highest impact for the investors. Therefore, we should guarantee that these criteria are consistent with orderings of preferences that consider all intuitive characteristics of investors' attitude.

It is well known that a measure of uncertainty is not necessarily adequate in measuring risk which is asymmetric as it is related to downside outcomes only. In particular, any realistic way of optimizing risk should maximize upside potential outcomes and minimize the downside outcomes. Artzner et al. (1999) define some intuitive characteristics of investment risk in the concept of a coherent risk measure. However, as observed by Balzer (2001), Rachev et al. (2008) even coherent risk measures cannot consider exhaustively all investment characteristics. According to recent studies (see Balzer (2001), Okuyama and Francis (2007), Farinelli et al. (2008)), investor's attitude is non-linear with respect to different sources of risk . Thus, even the celebrated expected shortfall, which is a coherent risk measure, does not take into account that most investors perceive a low probability of a large loss to be far more risky than a high probability of a small loss. Therefore, investors perceive risk to be non-linear (see Olsen (1997)).

In this note, we value the impact of non-linearity of risk perception and we propose a new different reward/risk ratio that takes into account some standardized moments of upside and downside outcomes. Moreover, in order to value correctly the evolution of wealth, we generate the future multivariate returns as suggested by Biglova et al. (2008) (see also Sun et al. (2008) and Ortobelli et al. (2008a)). Thus, we approximate the behavior of the corresponding marginals with an ARMA(2,0)-GARCH(0,2) model with stable paretian innovations and we approximate the dependencies of the innovation of the marginals with an asymmetric t copula. Finally, we compute the ex post final wealth sample paths obtained with the new ratios and with the Sharpe ratio (see Sharpe (1994)). The empirical analysis demonstrates the superiority of the new reward/risk measures with respect to the classical mean variance analysis.

Section 2 describes the empirical comparison and we summarize our principal findings in the last Section.

2 On the portfolio selection problem: choices consistent with investor behavior and empirical evidence

Suppose we have a frictionless market in which no short selling is allowed and all investors act as price takers. Given a risk-free benchmark with log-return r_b and n risky securities with a vector of log-returns $r = [r_1, ..., r_n]'$, the classical portfolio selection problem in the reward-risk plane consists of minimizing a given risk measure ρ provided that the expected reward v is constrained by some minimal value m, i.e.:

$$\begin{gathered}\min_x \rho\left(x'r - r_b\right) \\ s.t. \\ v\left(x'r - r_b\right) \geq m \\ x_i \geq 0, \sum_{i=1}^{n} x_i = 1\end{gathered}$$

where the vector notation $x'r = \sum_{i=1}^{n} x_i r_i$ stands for the returns of a portfolio with composition $x = (x_1, ..., x_n)'$. Along the efficient choices obtained by varying the value of the constraint m, there is a portfolio (often called the *market portfolio*) that provides the maximum expected reward v per unit of risk ρ. So, assuming that the reward and risk are both positive the *market portfolio* is obtained as the solution of the optimization problem

$$\begin{array}{c}\max_{x} \frac{v(x'r-r_b)}{\rho(x'r-r_b)} \\ s.t. \\ x_i \geq 0, \sum_{i=1}^{n} x_i = 1\end{array} \tag{1}$$

Starting from the original Markowitz' analysis, Sharpe suggestedthat investors should maximize the so called Sharpe ratio (see Sharpe (1994) and the reference therein).

Sharpe ratio (SR). The Sharpe ratio computes the expected excess return for unity of risk, i.e.:

$$SR(x'r) = \frac{E(x'r - r_b)}{STD(x'r - r_b)}.$$

In the Sharpe ratio, risk is proxied by the standard deviation $STD(x'r - r_b)$ of excess returns. Thus, maximizing the Sharpe ratio, we get a market portfolio that is not dominated in the sense of second-order stochastic dominance and therefore it should be optimal for non-satiable risk averse investors.

Almost in contrast with the tendency of these first studies, behavioral finance (see Friedman and Savage (1948), Markowitz (1952), Tversky and Kahneman (1992), Levy and Levy (2002), Ortobelli et al. (2008b)) suggests that most investors are clearly non-satiable but they are neither risk averse nor risk loving. To identify optimal portfolios selected by these investors, Rachev et al. (2008) propose to maximize reward-risk ratios $G(X) = \dfrac{v\,(X)}{\rho\,(X)}$, (where v is a positive reward measure, ρ is a positive generic risk measure for all X) isotonic with non-satiable investors' preferences (i.e. if $X \geq Y$, then $G(X) \geq G(Y)$), and that are not isotonic neither with risk averse investors' choices (that is $G(X+Y)$ is not greater or equal to $G(X)+G(Y)$ for all admissible X and Y) nor with risk lover investors' preferences (that is $G(X+Y)$ is not smaller or equal to $G(X)+G(Y)$ for all admissible X and Y) (see also Ortobelli et al. (2008b) and Bauerle and Müller (2006) for a classification of risk measures consistent with orderings).

Next we propose a new type of reward/risk ratio that takes into account the non-linearity of reward and risk measures according to the evidence on investors' risk perception (see Balzer (2001), Okuyama and Francis (2007), Olsen (1997)).

Rachev High Moments Ratio ($RHMR$).With this performance ratio, we propose a reward-risk ratio isotonic with the preferences of non-satiable investors who are neither risk averse nor risk loving. Moreover, we suggest to approximate the non-linearity attitude to risk of decision makers considering the first four moments of the standardized tails of the return distribution. Rachev high moments ratio is given by:

$$RHMR(x'r) = \frac{v_1(x'r - r_b)}{\rho_1(x'r - r_b)}$$

where

$$v_1(x'r - r_b) = E(x'r - r_b / x'r - r_b > F^{-1}_{x'r-r_b}(p_1)) +$$

$$+\sum_{i=2}^{4} a_i E\left(\left(\frac{x'r - r_b}{\sigma_{x'r-r_b}}\right)^i \left| x'r - r_b > F^{-1}_{x'r-r_b}(p_i)\right.\right);$$

$$\rho_1(x'r - r_b) = -E(x'r - r_b / x'r - r_b < F^{-1}_{x'r-r_b}(q_1)) -$$

$$-\sum_{i=2}^{4} b_i E\left(\left(\frac{x'r - r_b}{\sigma_{x'r-r_b}}\right)^i \left| x'r - r_b < F^{-1}_{x'r-r_b}(q_i)\right.\right),$$

$F_X^{-1}(q) = \inf\{x \,|P(X \leq x) > q\}$; $\sigma_{x'r-r_b}$ is the standard deviation of $x'r - r_b$, a_i, $b_i \in \mathbb{R}$ and $p_i, q_i \in (0,1)$. As we can observe from the definition, the Rachev high moments ratio is very versatile and depends on many parameters. To simplify our analysis in the following empirical comparison, we assume $a_i = b_i = 2$ for $i = 2, 3$, $a_4 = b_4 = 1$; $p_1 = 0.99$; $p_2 = 0.97$; $p_3 = 0.95$; $p_4 = 0.5$; and $q_i = 0.5$, $i = 1, 2, 3, 4$ (here the values of q_i are big enough to guarantee that the risk measure $\rho_1(x'r - r_b)$ is positive for all portfolios).

2.1 An empirical comparison between two different portfolio strategies

In order to value the impact of non-linear reward-risk measures, we provide an empirical ex-post comparison among the above strategies. The data includes the daily return series of the three-months treasury bill used as benchmark and the daily returns of the following US stock indexes: DJIA, NYSE, Major Market Index, DJ composite, DJAIG Commodity. As observed by Ortobelli et al. (2008a), we can generate future scenarios by fitting to these marginal series an ARMA(2,0)-GARCH(0,2) model with stable innovations and then estimating the dependence structure for the innovations with an asymmetric t-copula. Using the fitted model, we generate future scenarios for the vector of returns $r_{T+1,s} = [r_{T+1}^{(1,s)}, ..., r_{T+1}^{(n,s)}]'$ (where $r_{T+1}^{(i,s)}$ is the s-th scenario of the i-th return at time $T+1$). In this way, we consider realistic models for marginals and the dependence structure (see also, Sun et al. (2008), Biglova et al. (2008)).

As a next step, we suppose that decision makers invest their wealth purchasing the *market portfolio* determined by maximizing either the Sharpe ratio or the Rachev high moments ratio. For any optimal portfolio chosen on a daily basis, we consider a window of one year $T = 250$ of historical observations. We use these observations to generate $S = T$ future scenarios of returns according to the algorithm proposed by Ortobelli et al. (2008a). We assume that the investor has an initial wealth W_0 equal to 1 and an initial cumulative return CR_0 equal to 0 at the date 12/8/1993.

Therefore, for both ratios (Sharpe ratio and Rachev high moments ratio), we can compute the optimal portfolio as a solution of the optimization problem (1). Since we want to compare the ex post sample path of the final wealth and of the cumulative return obtained from the two approaches, we assume that investors recalibrate their portfolio every day investing their wealth in the market portfolio obtained from solving (1) on the simulated data. Therefore, after k days we compute the ex-post final wealth and cumulative return determining first the

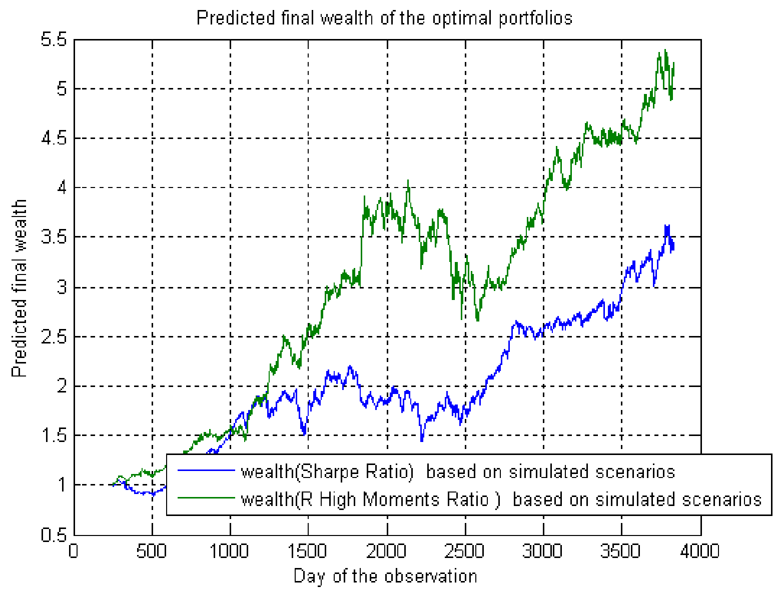

Figure 1: This figure compares the ex-post final wealth obtained by maximizing the RHM ratio and the Sharpe ratio.

market portfolio $x_M^{(k)}$ that maximizes one of the two performance ratios (the solution of problem (1)). The ex-post final wealth is given by:

$$W_{k+1} = W_k \left(\left(x_M^{(k)} \right)' (1 + r_{k+1}) \right),$$

and the ex-post cumulative return is given by:

$$CR_{k+1} = CR_k + \left(x_M^{(k)} \right)' r_{k+1}.$$

where r_{k+1} is the vector of observed returns at $(k+1)$-th day. We repeat this computation for both ratios (Sharpe ratio and Rachev high moments ratio) till the end of the period.

The output of this analysis is represented in Figures 1,2. The two figures show the superiority of the approach based on Rachev high moments ratio with respect to the classic approach. Therefore, we can conclude that the application of non-linear reward and risk measures which are consistent with realistic investors' preferences[1] has an important impact in portfolio selection theory.

[1] We implicity consider non-satiable investors who are neither risk averse nor risk loving.

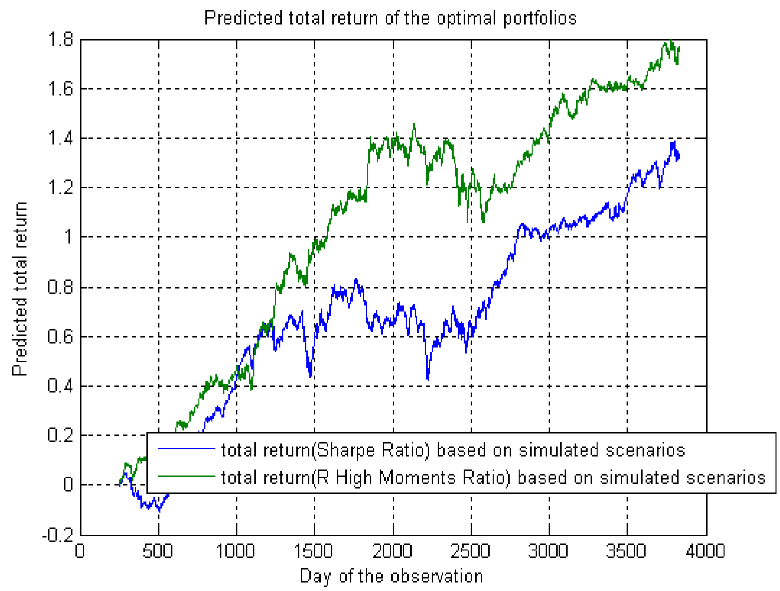

Figure 2: This figure compares the ex-post cumulative return obtained by maximizing the RHM ratio and the Sharpe ratio.

3 Conclusions

This paper examines and shows the impact of non-linear reward, risk measures in portfolio selection theory. In particular, we first discuss the use of opportune reward/risk criteria to select optimal portfolios. Then, we simulate realistic future scenarios using a copula approach and, finally, we compare the ex-post final wealth and cumulative return processes obtained using either a new reward/risk ratio or the classical Sharpe ratio. As anticipated, the ex-post empirical comparison shows the greater predictable capacity of the non-linear reward and risk measures.

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DEFINITE INTEGRALS OF A CLASS OF RATIONAL FUNCTIONS

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We establish a general theorem on definite integrals of a large class of rational functions. Several definite integral evaluations are proposed as challenging examples.

Keywords: Residue theorem; Contour integral; Resultant between two polynomials.

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Recently, Glasser [1] proposed the following definite integral for evaluation.

Problem. Show that

$$\int_0^{+\infty} \frac{z^8-4z^6+9z^4-5z^2+1}{z^{12}-10z^{10}+37z^8-42z^6+26z^4-8z^2+1}\mathrm{d}z \;=\; \frac{\pi}{2}.$$

When trying to work out a solution, the author establishes a general theorem on definite integrals of a large class of rational functions.

For a given polynomial $p(z)$, denote by $\mathrm{LC}(p)$ the leading coefficient of $p(z)$ and by $\bar{p}(z)$ the conjugate polynomial of $p(z)$ with the coefficients being replaced by their conjugate ones.

Theorem. *Let $h(z)$ be a monic polynomial of degree $n+1$ with distinct zeros $\{\alpha_k\}_{k=0}^n$ having positive imaginary parts and $w(z)$ a polynomial of $\deg(w) \le n$. When $\deg(w)=n$, we suppose further that the leading coefficient of $w(z)$ is purely imaginary. Further define two rational functions by*

$$g(z)=\frac{w(z)}{h(z)}+\frac{\bar{w}(z)}{\bar{h}(z)} \quad \textit{and} \quad f(z)=g(z)+g(-z).$$

Then there holds the following definite integral formula:

$$\int_0^{+\infty} f(z)dz \;=\; \int_{-\infty}^{+\infty} g(z)dz \;=\; \begin{cases} 2\pi i \times LC(w), & \deg(w)=n; \\ 0, & \deg(w)<n. \end{cases}$$

Proof. According to the definition of $g(z)$, we have

$$g(z) = \frac{w(z)}{h(z)} + \frac{\bar{w}(z)}{\bar{h}(z)} = \frac{w(z)\bar{h}(z) + \bar{w}(z)h(z)}{h(z)\bar{h}(z)}.$$

We see that $g(z)$ is a rational function with the numerator degree being less than the denominator degree by at least 2. Further the denominator polynomial has no real zeros. Therefore both definite integrals displayed in the theorem are convergent. Denote by Γ_M the contour consisting of the real axis from $-M$ to $+M$ and the semicircle over the real axis with its center at the origin and the radius equal to M. Then the residue theorem (cf. [3, §3.11] for example) affirms that

$$\begin{aligned}\int_0^{+\infty} f(z)\mathrm{d}z &= \int_{-\infty}^{+\infty} g(z)\mathrm{d}z = \lim_{M\to+\infty}\oint_{\Gamma_M} g(z)\mathrm{d}z \\ &= 2\pi i\sum_{k=0}^{n} \operatorname{Res}\left\{g(z)\right\}_{z=\alpha_k}.\end{aligned}$$

Note that the residue of $g(z)$ at the single pole $z = \alpha_k$

$$\begin{aligned}\operatorname*{Res}_{z=\alpha_k}\left\{g(z)\right\} &= \lim_{z\to\alpha_k}(z-\alpha_k)g(z) \\ &= \lim_{z\to\alpha_k}(z-\alpha_k)\frac{w(z)}{h(z)} \\ &= \frac{w(\alpha_k)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(\alpha_k-\alpha_j)}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=0}^{n}\operatorname*{Res}_{z=\alpha_k}\left\{g(z)\right\}_{z=\alpha_k} &= \sum_{k=0}^{n}\frac{w(\alpha_k)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(\alpha_k-\alpha_j)} \\ &= \Delta[\alpha_0,\alpha_1,\cdots,\alpha_n]w(z)\end{aligned}$$

where $\Delta[\alpha_0,\alpha_1,\cdots,\alpha_n]w(z)$ stands for the divided differences of $w(z)$ at the uneven grids $\{\alpha_k\}_{k=0}^{n}$.

Recall that the last divided differences (cf. [2, §2.3] for example) results in zero for $\deg(w) < n$ and the leading coefficients of $w(z)$ for $\deg(w) = n$, i.e.,

$$\Delta[\alpha_0,\alpha_1,\cdots,\alpha_n]w(z) = \begin{cases}\mathrm{LC}(w), & \deg(w) = n;\\ 0, & \deg(w) < n.\end{cases}$$

We establish the integral formula stated in the theorem. □

By means of the theorem, we are now ready to present a solution to the problem announced at the beginning of the paper.

Let $f(z)$ stand for the integrand:

$$f(z) = \frac{z^8 - 4z^6 + 9z^4 - 5z^2 + 1}{z^{12} - 10z^{10} + 37z^8 - 42z^6 + 26z^4 - 8z^2 + 1}.$$

According to its denominator factorization

$$\begin{aligned} & z^{12} - 10z^{10} + 37z^8 - 42z^6 + 26z^4 - 8z^2 + 1 \\ = \; & (1 - 2z^2 + 3z^4 + z^6)^2 - (2z - 4z^3 + 4z^5)^2 \\ = \; & (1 + 2z - 2z^2 - 4z^3 + 3z^4 + 4z^5 + z^6) \\ \times \; & (1 - 2z - 2z^2 + 4z^3 + 3z^4 - 4z^5 + z^6) \end{aligned}$$

we have the following partial fraction decomposition:

$$f(z) = g(z) + g(-z) \text{ where } g(z) = \frac{(1 + 2z + z^2)/2}{1 + 2z - 2z^2 - 4z^3 + 3z^4 + 4z^5 + z^6}.$$

Reformulate further the denominator of $g(z)$ as

$$\begin{aligned} & 1 + 2z - 2z^2 - 4z^3 + 3z^4 + 4z^5 + z^6 \\ & = (1 + z - 2z^2 - z^3)^2 + (z + z^2)^2 \\ & = \big\{(z^3 + 2z^2 - z - 1) - i(z + z^2)\big\} \\ & \times \big\{(z^3 + 2z^2 - z - 1) + i(z + z^2)\big\}. \end{aligned}$$

From the second line just displayed we can check without difficulty that $g(z)$ has no real poles. All the poles of $g(z)$ are zeros of two conjugate polynomials displayed in the last two lines.

Now we show further that the polynomial defined by

$$h(z) := (z^3 + 2z^2 - z - 1) - i(z + z^2)$$

has three distinct zeros with positive imaginary parts.

First, $h(z)$ has no multiple zeros because the resultant between $h(z)$ and $h'(z)$ does not vanish:

$$\rho[h(z), h'(z)/3] = \frac{12i - 35}{27}.$$

Replacing z by $x - iy$, we can write $h(x - iy) = u(x, y) + iv(x, y)$ by separating the real and imaginary parts:

$$\begin{aligned} u(x, y) &= x^3 + 2x^2 - x(1 + 2y + 3y^2) - (1 + y + 2y^2) \\ v(x, y) &= x^2(-1 - 3y) - x(1 + 4y) + (y + y^2 + y^3). \end{aligned}$$

Then for any fixed $y > 0$, two polynomials $u(x, y)$ and $v(x, y)$ in real variable x have no common zeros because their resultant is negative:

$$\rho\Big[u(x, y), \frac{v(x, y)}{-1 - 3y}\Big] = \frac{-1}{(1 + 3y)^2}\left\{\begin{smallmatrix} 1+16y+85y^2+241y^3+468y^4 \\ +644y^5+624y^6+448y^7+192y^8+64y^9 \end{smallmatrix}\right\}.$$

Therefore, all the three zeros of $h(z)$ have positive imaginary parts.

According to the denominator factorization $h(z)\bar{h}(z)$ of $g(z)$, we can further decompose $g(z)$ into partial fractions:

$$g(z) \;=\; \frac{w(z)}{h(z)} + \frac{\bar{w}(z)}{\bar{h}(z)}$$

where

$$\begin{aligned} h(z) &= (z^3 + 2z^2 - z - 1) - i(z + z^2) \\ w(z) &= \frac{-i}{4}\big\{(z^2 + 2z - i(1+z)\big\}. \end{aligned}$$

Then $h(z)$ and $w(z)$ satisfy all the conditions stated in the theorem. Hence the integral in question can be evaluated as

$$\int_0^{+\infty} f(z)\mathrm{d}z \;=\; 2\pi i \times \mathrm{LC}(w) \;=\; 2\pi i \times \frac{-i}{4} \;=\; \frac{\pi}{2}.$$

As a byproduct, we have also the following integral evaluation

$$\int_{-\infty}^{+\infty} \frac{1 + 2z + z^2}{1 + 2z - 2z^2 - 4z^3 + 3z^4 + 4z^5 + z^6}\mathrm{d}z \;=\; \pi. \tag{1}$$

This proves the formula stated in the **Problem** proposed by Glasser. □

As exercises, we invite the reader to check the following challenging examples.

Example 1. *For two polynomials defined by*

$$h(z) \;=\; 1 + i - (5+i)z + z^2 \quad \textit{and} \quad w(z) \;=\; 1 - 2iz$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{1 - 7z + 3z^2}{2 - 12z + 28z^2 - 10z^3 + z^4}dz \;=\; 2\pi \tag{2a}$$

$$\int_{0}^{+\infty} \frac{2 - 50z^2 + 15z^4 + 3z^6}{4 - 32z^2 + 548z^4 - 44z^6 + z^8}dz \;=\; \pi. \tag{2b}$$

Example 2. *For two polynomials defined by*

$$h(z) \;=\; 1 + i - (5+i)z + z^2 \quad \textit{and} \quad w(z) \;=\; 1$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{1 - 5z + z^2}{2 - 12z + 28z^2 - 10z^3 + z^4}dz \;=\; 0 \tag{3a}$$

$$\int_{0}^{+\infty} \frac{2 - 30z^2 - 21z^4 + z^6}{4 - 32z^2 + 548z^4 - 44z^6 + z^8}dz \;=\; 0. \tag{3b}$$

Example 3. *For two polynomials defined by*

$$h(z) \;=\; -1 - (1+i)z + (3-i)z^2 + z^3 \quad \textit{and} \quad w(z) \;=\; 1 + 2z - 3iz^2$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{-1-3z+z^2+10z^3+5z^4}{1+2z-4z^2-6z^3+8z^4+6z^5+z^6}dz \;=\; 3\pi \tag{4a}$$

$$\int_{0}^{+\infty} \frac{-1+11z^2-45z^4+65z^6-19z^8+5z^{10}}{1-12z^2+56z^4-122z^6+128z^8-20z^{10}+z^{12}}dz \;=\; \frac{3}{2}\pi. \tag{4b}$$

Example 4. *For two polynomials defined by*

$$h(z) \;=\; -1-(1+i)z+(3-i)z^2+z^3 \quad \textit{and} \quad w(z) \;=\; 1+2z$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{-1-3z+z^2+7z^3+2z^4}{1+2z-4z^2-6z^3+8z^4+6z^5+z^6}dz \;=\; 0 \tag{5a}$$

$$\int_{0}^{+\infty} \frac{-1+11z^2-42z^4+59z^6-25z^8+2z^{10}}{1-12z^2+56z^4-122z^6+128z^8-20z^{10}+z^{12}}dz \;=\; 0. \tag{5b}$$

Example 5. *For two polynomials defined by*

$$\begin{aligned} h(z) &\;=\; 5+7(1+i)z-5(2-i)z^2-3(1+i)z^3+z^4 \\ w(z) &\;=\; 1+2z+3z^2-4iz^3 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{5+17z+19z^2-2z^3-63z^4-27z^5+15z^6}{25+70z-2z^2-100z^3+51z^4+44z^5-2z^6-6z^7+z^8}dz \;=\; 4\pi \tag{6a}$$

$$\int_{0}^{+\infty} \frac{125-725z^2+482z^4+2402z^6-5786z^8+2086z^{10}-255z^{12}+15z^{14}}{625-5000z^2+16554z^4-16464z^6+12299z^8-3344z^{10}+634z^{12}-40z^{14}+z^{16}}dz \;=\; 2\pi. \tag{6b}$$

Example 6. *For two polynomials defined by*

$$\begin{aligned} h(z) &\;=\; 5+7(1+i)z-5(2-i)z^2-3(1+i)z^3+z^4 \\ w(z) &\;=\; 1+2z+3z^2 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{5+17z+19z^2-2z^3-35z^4-7z^5+3z^6}{25+70z-2z^2-100z^3+51z^4+44z^5-2z^6-6z^7+z^8}dz \;=\; 0 \tag{7a}$$

$$\int_{0}^{+\infty} \frac{125-725z^2+1182z^4+646z^6-2334z^8+538z^{10}-83z^{12}+3z^{14}}{625-5000z^2+16554z^4-16464z^6+12299z^8-3344z^{10}+634z^{12}-40z^{14}+z^{16}}dz \;=\; 0. \tag{7b}$$

Example 7. *For two polynomials defined by*

$$\begin{aligned} h(z) &\;=\; (1+4i)-(18+14i)z+(28+12i)z^2-(10+2i)z^3+z^4 \\ w(z) &\;=\; 1+2z+3z^2-4iz^3 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{1-16z-5z^2-24z^3+121z^4-76z^5+11z^6}{17-148z+672z^2-1380z^3+1346z^4-644z^5+160z^6-20z^7+z^8}dz \;=\; 4\pi \tag{8a}$$

$$\int_{0}^{+\infty} \frac{17-1781z^2-25589z^4+20257z^6+48803z^8-15263z^{10}+361z^{12}+11z^{14}}{289+944z^2+88868z^4-280560z^6+243430z^8-37872z^{10}+2532z^{12}-80z^{14}+z^{16}}dz \;=\; 2\pi. \tag{8b}$$

Example 8. *For two polynomials defined by*

$$\begin{aligned} h(z) &= (1+4i)-(18+14i)z+(28+12i)z^2-(10+2i)z^3+z^4) \\ w(z) &= 1+iz+z^2 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{1-14z+15z^2-16z^3+27z^4-10z^5+z^6}{17-148z+672z^2-1380z^3+1346z^4-644z^5+160z^6-20z^7+z^8}\,dz = 0 \tag{9a}$$

$$\int_{0}^{+\infty} \frac{17-1145z^2-9803z^4+5935z^6+15031z^8-1079z^{10}-13z^{12}+z^{14}}{289+944z^2+88868z^4-280560z^6+243430z^8-37872z^{10}+2532z^{12}-80z^{14}+z^{16}}\,dz = 0. \tag{9b}$$

Example 9. *For two polynomials defined by*

$$\begin{aligned} h(z) &= (-2-i)+(5-i)z+(7+8i)z^2-(16-2i)z^3-(2+2i)z^4+z^5 \\ w(z) &= 1+2z+3z^2+4z^3-5iz^4 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{-2+z+11z^2+5z^3+12z^4-18z^5-108z^6-15z^7+14z^8}{5-18z-18z^2+114z^3-39z^4-212z^5+210z^6+70z^7-24z^8-4z^9+z^{10}}\,dz = 5\pi \tag{10a}$$

$$\int_{0}^{+\infty} \frac{-10+109z^2-84z^4-2287z^6+6676z^8+3762z^{10}-25403z^{12}+6522z^{14}-504z^{16}+14z^{18}}{25-504z^2+4038z^4-17124z^6+44577z^8-76554z^{10}+76528z^{12}-16754z^{14}+1556z^{16}-64z^{18}+z^{20}}\,dz = \frac{5}{2}\pi. \tag{10b}$$

Example 10. *For two polynomials defined by*

$$\begin{aligned} h(z) &= -i+(5-i)z+(7+8i)z^2-(16-2i)z^3-(2+2i)z^4+z^5 \\ w(z) &= 1+2z+3z^2+4z^3-5iz^4 \end{aligned}$$

the corresponding integral evaluations read as

$$\int_{-\infty}^{+\infty} \frac{z(5+17z+13z^2+12z^3-18z^4-108z^5-15z^6+14z^7)}{1+2z+10z^2+50z^3-47z^4-208z^5+210z^6+70z^7-24z^8-4z^9+z^{10}}\,dz = 5\pi \tag{11a}$$

$$\int_{0}^{+\infty} \frac{z^2(7-94z^2-361z^4+5224z^6+3444z^8-25417z^{10}+6522z^{12}-504z^{14}+14z^{16})}{1+16z^2-194z^4-2188z^6+26881z^8-70466z^{10}+75896z^{12}-16738z^{14}+1556z^{16}-64z^{18}+z^{20}}\,dz = \frac{5}{2}\pi. \tag{11b}$$

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Mixed Boundary Value Problems in the Upper Half Plane

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Boundary value problem arising from the combination of (n-1) - Dirichlet and a Neumann, (n-1) - Dirichlet and a Schwarz boundary conditions for the inhomogeneous polyanalytic equation in the upper half plane $\mathbb{H}$ are investigated. We also give explicit solution to the Schwarz problem for Poisson equation in the upper half plane.

Keywords: Dirichlet problem, Neumann and Schwarz boundary value problems, Cauchy-Pompeiu representation, inhomogeneous polyanalytic equation.

AMS 2000 Mathematics Subject classification: 32A30, 30G20, 31A30.

1. Introduction

In order to treat boundary value problem for higher order complex partial differential equations, special kernel functions have to be constructed. The complex form of the Gauss theorem and Cauchy- Pompeiu formula in the upper half plane can be obtained as the limiting case of the corresponding formula for the regular domains [1], [8]. There are three basic boundary value problems for complex partial differential equations, namely, Schwarz, Dirichlet and Neumann problems. To find the solutions in explicit form they are investigated for particular domains e.g. the unit disc, half planes, quarter planes [2, 3, 4, 6, 7, 8, 10]. Let $\mathcal{F}_k$ be the space of functions w in $W^{k,1}(\mathbb{H},\mathbb{C})$ for which $\lim\limits_{R\to\infty} R^{\nu}M(\partial_{\bar z}^{\nu}w,R) = 0, 0 \le \nu \le k-1$, where $M(\partial_{\bar z}^{\nu}w,R) = \max\limits_{\substack{|z|=R\\ 0\le Imz}} |\partial_{\bar z}^{\nu}w(z)|$ and $\bar z^{k-2}\partial_{\bar z}^{k}w \in L^1(\mathbb{H},\mathbb{C})$.

Any $w \in \mathcal{F}_k$ can be represented as

$$\begin{aligned} w(z) &= \sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\nu!}\frac{\overline{(z-\zeta)}^{\nu}}{(\zeta-z)}\partial_{\bar\zeta}^{\nu}w(\zeta)d\zeta \\ &\quad -\frac{1}{\pi}\frac{1}{(k-1)!}\int_{\mathbb{H}}\frac{\overline{(z-\zeta)}^{k-1}}{(\zeta-z)}\partial_{\bar z}^{k}w(\zeta)d\xi d\eta \end{aligned} \tag{1}$$

for $z \in \mathbb{H}$. Moreover,

$$\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int\limits_{\mathbb{H}} \partial_{\overline{\zeta}}^n w(\zeta)\left(\overline{\zeta}-z\right)^{n-1}\frac{d\xi d\eta}{(\zeta-\overline{z})}$$
$$= -\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int\limits_{-\infty}^{\infty}\partial_{\overline{\zeta}}^\lambda w(t)\,(t-z)^\lambda\frac{dt}{(t-\overline{z})}. \tag{2}$$

In the following sections, we investigate mixed boundary value problem arising from (n-1) Dirichlet and a Neumann, (n-1) Dirichlet and a Schwarz boundary value condition. Our proofs are independent of n- Dirichlet, Neumann, Schwarz problems for the inhomogeneous Cauchy Riemann equation and do not utilize any iteration process. Last section deals with the Schwarz problem for the Poisson equation on the upper half plane.

2. Dirichlet - Neumann problem

In this section, we consider the inhomogeneous polyanalytic equation with first (n-1) factors having Dirichlet boundary conditions and the last factor with Neumann condition. Basic techniques used are the representation formulae (1), (2), Cauchy Pompeiu formula, Gauss theorem and higher order Pompeiu operators [5, 7].

Theorem 1:- For $w \in \mathcal{F}_n$, the mixed $(n-1)$ Dirichlet and a Neumann problem for the inhomogeneous polyanalytic equation in the upper half plane $\mathbb{H}$,

$$\partial_{\overline{z}}^n w = f \text{ in } \overline{\mathbb{H}},\ \partial_{\overline{z}}^\lambda w = \gamma_\lambda \text{ on } \mathbb{R},\ 0 \le \lambda \le \text{n} - 2$$
$$\partial_y(\partial_{\overline{z}}^{n-1} w) = i\gamma_{n-1} \text{ on } \mathbb{R},\ \partial_{\overline{z}}^{n-1} w(i) = c \tag{3}$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H},\mathbb{C}) \cap C(\overline{\mathbb{H}},\mathbb{C})$, $\text{p} \ge 2$,

$t^\lambda \gamma_\lambda(t) \in L^p(\mathbb{R},\mathbb{C}) \cap \text{C}(\mathbb{R},\mathbb{C})$, $0 \le \lambda \le n-1$ if and only if for $0 \le \nu \le n-2$,

$$\frac{(-1)^{n-\nu}}{(n-\nu-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{(\overline{\zeta}-z)^{n-\nu-1}}{\zeta-z}-\frac{z^{n-\nu-1}}{\zeta-i}\right)d\xi d\eta$$
$$+\frac{(-1)^{n-\nu-1}}{(n-\nu-1)!}z^{n-\nu-1}c+\sum_{\lambda=\nu}^{n-2}\frac{1}{2\pi i}\frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!}\int\limits_{-\infty}^{\infty}\gamma_\lambda(t)\frac{(t-z)^{\lambda-\nu}}{t-\overline{z}}dt$$
$$+\frac{1}{2\pi i}\frac{(-1)^{n-\nu}}{(n-\nu-1)!}\int\limits_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))(h_\nu(t,z)-z^{n-\nu-1}\log(t-i))dt = 0 \tag{4}$$

where

$$h_\nu(t,z) = (t-z)^{n-1-\nu}\log(t-\bar{z}) + \sum_{\mu=2}^{n-\nu}(-1)^{\mu-1}(n-\nu-1)\dots(n-\nu-\mu+1)(t-z)^{n-\nu-\mu}I_{\mu-1}(t,z)$$

$$I_\mu(t,z) = \frac{(t-\bar{z})^\mu}{\mu!}\left(\log(t-\bar{z}) - \sum_{r=1}^{\mu}\frac{1}{r}\right) \tag{5}$$

and $$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\gamma_{n-1}(t)+f(t)\}\frac{dt}{t-\bar{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} = 0. \tag{6}$$

The solution then is given by

$$\begin{aligned} w(z) &= \frac{c(-\bar{z})^{n-1}}{(n-1)!} + \sum_{\lambda=0}^{n-2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\lambda!}\frac{(\overline{z-t})^\lambda}{t-\lambda}\gamma_\lambda(t)dt \\ &- \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))(h_0(t,\bar{z})) - \bar{z}^{n-1}\log(t-i))dt \\ &+ \frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{(\overline{\zeta-z})^{n-1}}{\zeta-z} - \frac{\bar{z}^{n-1}}{\zeta-i}\right)d\xi d\eta. \end{aligned} \tag{7}$$

Proof. Applying the representation formula (1) and the Cauchy-Pompeiu formula, we obtain

$$\begin{aligned} &\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{(\overline{\zeta-z})^{n-1}}{\zeta-z} - \frac{\bar{z}^{n-1}}{\zeta-i}\right)d\xi d\eta \\ &= -\sum_{\lambda=0}^{n-2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\nu!}\frac{(\overline{z-t})^\lambda}{t-z}\gamma_\lambda(t)dt + w(z) - \frac{c(-\bar{z})^{n-1}}{(n-1)!} - I_1 + \frac{(-\bar{z})^{n-1}}{(n-1)!}I_2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} I_1 &= \frac{(-1)^{n-1}}{2\pi i}\frac{1}{(n-1)!}\int_{-\infty}^{\infty}\frac{(\overline{t-z})^{n-1}}{t-z}\partial_{\bar{\zeta}}^{n-1}w(t)dt \\ \text{and } I_2 &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial_{\bar{\zeta}}^{n-1}w(t)}{t-i}dt. \end{aligned}$$

Let $h_0(t,\bar z)$ be as in (5) above so that $\frac{d}{dt}(h_0(t,\bar z)) = \frac{(\overline{t-z})^{n-1}}{t-z}$.
Integrating by parts I_1 and using regularity conditions of $\partial_{\bar\zeta}^{n-1}w$, we have

$$\begin{aligned} I_1 &= \frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta\partial_{\bar\zeta}^{n-1}w(t)+\partial_{\bar\zeta}^{n}w(t))h_0(t,\bar z)dt \\ &= \frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))h_0(t,\bar z)dt \end{aligned}$$

and

$$I_2 = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))\log(t-i)dt.$$

Substituting these values in (8), we obtain (7).For $0\leq \nu \leq$n-2, using (2) and Cauchy- Pompeiu formula, we have

$$\frac{(-1)^{n-\nu}}{(n-\nu-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{(\overline{\zeta}-z)^{n-\nu-1}}{\zeta-\overline{z}}-\frac{z^{n-\nu-1}}{\zeta-i}\right)d\xi d\eta$$

$$\begin{aligned} = \; & -\sum_{\lambda=\nu}^{n-2}\frac{1}{2\pi i}\frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!}\int_{-\infty}^{\infty}\gamma_\lambda(t)\frac{(t-z)^{\lambda-\nu}}{t-\bar z}dt \\ & -\frac{(-1)^{n-1-\nu}}{(n-1-\nu)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\bar\zeta}^{n-1}w(t)\frac{(t-z)^{n-1-\nu}}{t-\bar z}dt \\ & -\frac{(-1)^{n-\nu}}{(n-\nu-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial_{\bar\zeta}^{n-1}w(t)}{t-i}dt+\partial_{\bar\zeta}^{n-1}w(i). \end{aligned} \tag{9}$$

Again integrating by parts and noting that $\frac{d}{dt}h_\nu(t,z) = \frac{(t-z)^{n-1-\nu}}{t-\bar z}$, we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\bar\zeta}^{n-1}w(t)\frac{(t-z)^{n-1-\nu}}{(t-\bar z)}dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))h_\nu(t,z)dt$$

and $$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial_{\bar\zeta}^{n-1}w(t)}{t-i}dt = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t)+\gamma_{n-1}(t))\log(t-i)dt.$$

Substituting these values in (9), we obtain (4). Using Gauss theorem, we have

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{H}} {}_{\zeta}f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial_\zeta(\partial_{\bar\zeta}^{n-1}w)}{\zeta-\bar{z}}d\zeta \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(f(t)+\gamma_{n-1}(t))\frac{dt}{t-\bar{z}}.
\end{aligned}$$

Hence we have the solvability conditions (5).

Let $T_{0,n}$ be higher order Cauchy Pompeiu operators on $\mathbb{H}$ [5], [7]. Since $\partial_{\bar{z}}T_{0,n}w = T_{0,n-1}w$, it follows that (7) indeed satisfy $\partial_{\bar z}^n w = f$. For a fix k, $0 \leq k \leq n-2$, note that $\partial_{\bar z}^k h_0(t,\bar z) = h_k(t, z)$. Subtracting (4) for $\nu = k$ from $\partial_{\bar z}^k w$, we have

$$\begin{aligned}
\partial_{\bar z}^k w(z) &= \frac{(-1)^{n-k-1}}{(n-k-1)!}c[\bar z^{n-k-1}-z^{n-k-1}] \\
&+\sum_{\lambda=k}^{n-2}\frac{1}{2\pi i}\frac{(-1)^{\lambda-k}}{(\lambda-k)!}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left[\frac{(t-\bar z)^{\lambda-k}}{t-z}-\frac{(t-z)^{\lambda-k}}{t-\bar z}\right]dt \\
&-\frac{(-1)^{n-k-1}}{(n-k-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}(2f(t) + \gamma_{n-1}(t))\log(t-i)(z^{n-k-1}-\bar z^{n-k-1})dt \\
&+\frac{(-1)^{n-k}}{(n-k-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{(\overline{\zeta-z})^{n-k-1}}{\zeta-z}-\frac{\bar z^{n-k-1}}{\zeta-i}\right. \\
&\qquad\left.-\frac{(\bar\zeta-z)^{n-k-1}}{\zeta-z}+\frac{z^{n-k-1}}{\zeta-i}\right)d\xi d\eta.
\end{aligned}$$

$$\text{Since } \lim_{z\to t_0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\gamma_\lambda(t)y}{|t-z|^2}dt = \gamma_\lambda(t_0) \quad \text{so} \quad \partial_{\bar z}^k w = \gamma_k \quad \text{on } \mathbb{R}.$$

Also $\partial_z(\partial_{\bar z}^{n-1}w)(z) - \partial_{\bar z}(\partial_{\bar z}^{n-1}w)(z)$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_{n-1}(t)+2f(t))\frac{dt}{t-z} - \frac{1}{\pi}\int_{\mathbb{H}}\frac{f(\zeta)}{(\zeta-z)^2}d\xi d\eta - f(z).$$

So subtracting (6) and using Plemelj-Sokhotzki formula [9], we obtain $\partial_y(\partial_{\bar z}^{n-1}w) = \gamma_{n-1}$ on $\mathbb{R}$. □

3. Dirichlet-Schwarz Problem

This section deals with the first (n-1) Dirichlet and the last component as Schwarz boundary condition.

Theorem 2. For $w \in \mathcal{F}_n$, the mixed (n - 1)-Dirichlet and a Schwarz problem for the inhomogeneous polyanalytic equation in the upper half plane $\mathbb{H}$,

$$\begin{array}{l} \partial_{\bar z}^{n} w = f \text{ in } \mathbb{H},\ \partial_{\bar z}^{\lambda} w = \gamma_\lambda \text{ on } \mathbb{R},\ 0 \le \lambda \le \mathrm{n}-2 \\ Re(\partial_{\bar z}^{n-1} w) = \gamma_{n-1} \text{ on } \mathbb{R},\ Im\, \partial_{\bar z}^{n-1} w(i) = c \end{array} \tag{10}$$

is uniquely solvable for f$\in$L$_{p,2}$ ($\mathbb{H}$,$\mathbb{C}$), p > 2, $t^\lambda \gamma_\lambda(t) \in L^p(\mathbb{R},\mathbb{C}) \cap C(\mathbb{R},\mathbb{C})$, $0 \le \lambda \le n-1$ if and only if for $0 \le \nu \le n-2$.

$$\begin{aligned} &\frac{(z-i)^{n-\nu-1}}{(n-\nu-1)!} ic + \sum_{\lambda=\nu}^{n-2} \frac{1}{2\pi i} \frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!} \int_{-\infty}^{\infty} \gamma_\lambda(t)(t-\bar z)^{\lambda-\nu-1} dt \\ + \quad & \frac{(-1)^{n-\nu-1}}{(n-\nu-1)!} \frac{1}{\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \left(\frac{(t-\bar z)^{n-\nu-1}}{t-z} - (i-z)^{n-\nu-1} \frac{t}{t^2+1} \right) dt \\ + \quad & \frac{(-1)^{n-\nu}}{(n-\nu-1)!} \frac{1}{\pi} \int_{\mathbb{H}} [f(\zeta) \left(\frac{(\overline{\zeta - z})^{n-1-\nu}}{\zeta - \bar z} - (i-z)^{n-1-\nu} \frac{\zeta}{\zeta^2+1} \right) \\ - \quad & \overline{f(\zeta)} \left(\frac{(\overline{\zeta - z})^{n-\nu-1}}{\overline{\zeta - z}} - \frac{\overline{\zeta}}{\overline{\zeta}^2+1} (i-z)^{n-\nu-1} \right)] d\xi d\eta = 0. \end{aligned} \tag{11}$$

The solution then is given by

$$\begin{aligned} w(z) \quad = \quad & \frac{(\bar z - i)^{n-1} ic}{(n-1)!} + \sum_{\lambda=0}^{n-2} \frac{1}{2\pi i} \frac{1}{\lambda!} \int_{-\infty}^{\infty} \gamma_\lambda(t) \frac{(\bar z - t)^\lambda}{t-z} dt \\ & + \frac{1}{(n-1)!} \frac{1}{\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \left(\frac{(\bar z - t)^{n-1}}{t-z} - \frac{(\bar z - i)^{n-1} t}{t^2+1} \right) dt \\ & + \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{H}} [f(\zeta) \left(\frac{(\overline{\zeta - z})^{n-1}}{\zeta - z} - \frac{\zeta (i - \bar z)^{n-1}}{\zeta^2+1} \right) \\ & - \overline{f(\zeta)} \left(\frac{(\overline{\zeta - z})^{n-1}}{\overline{\zeta} - z} - \frac{\overline{\zeta}(i-\bar z)^{n-1}}{\overline{\zeta}^2+1} \right)] d\xi d\eta. \end{aligned} \tag{12}$$

Proof: Using (1) and the Gauss theorem, we have

$$\begin{aligned}
&\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\Big(\frac{f(z)}{\zeta-z}-\frac{\overline{f(z)}}{\bar{\zeta}-z}\Big)(\overline{\zeta-z})^{n-1}d\xi d\eta\\
=\ & w(z)-\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int_{-\infty}^{\infty}\partial_{\bar{\zeta}}^{\lambda}w(t)\frac{(t-\bar{z})^\lambda}{t-z}dt\\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\zeta}^{n-1}\bar{w}(t)\frac{(t-z)^{n-1}}{t-z}dt. \qquad (13)
\end{aligned}$$

Repeated applications of Gauss theorem and Cauchy Pompeiu formula give

$$\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{f(\zeta)\zeta}{\zeta^2+1}-\frac{\overline{f(\zeta)\zeta}}{\bar{\zeta}^2+1}\right)d\xi d\eta$$

$$\begin{aligned}
=\ & \frac{1}{2}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{\partial_{\bar{\zeta}}^{n-1}w(t)}{t+i}+\frac{\partial_{\bar{\zeta}}^{n-1}w(t)}{t-i}+\frac{\overline{\partial_{\zeta}^{n-1}w(t)}}{t-i}+\frac{\overline{\partial_{\zeta}^{n-1}w(t)}}{t+i}\right)dt\right.\\
&\left.-\partial_{\bar{z}}^{n-1}w(i)+\partial_{z}^{n-1}\bar{w}(i)\right]\\
=\ & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_n(t)\frac{t}{t^2+1}dt-ic. \qquad (14)
\end{aligned}$$

Combining (13) and (14), we have

$$\begin{aligned}
&\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\left[f(\zeta)\left(\frac{\overline{\zeta-z}^{n-1}}{\zeta-z}-\frac{\zeta(i-\bar{z})^{n-1}}{\zeta^2+1}\right)\right.\\
&\qquad\left.-\overline{f(\zeta)}\left(\frac{\overline{\zeta-z}^{n-1}}{\bar{\zeta}-z}-\frac{\bar{\zeta}(i-\bar{z})^{n-1}}{\bar{\zeta}^2+1}\right)\right]d\xi d\eta
\end{aligned}$$

$$
\begin{aligned}
= \quad & w(z) - \sum_{\lambda=0}^{n-1} \frac{1}{2\pi i} \frac{(-1)^\lambda}{\lambda!} \int_{-\infty}^{\infty} \partial_{\bar\zeta}^{\lambda} w(t) \frac{(t-\bar z)^\lambda}{t-z} dt \\
& - \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \partial_{\zeta}^{n-1} \bar w(t) \frac{(t-\bar z)^{n-1}}{t-z} dt \\
& + \frac{(-1)^{n-1}}{(n-1)!} \frac{(i-\bar z)^{n-1}}{2\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \frac{t}{t^2+1} dt - \frac{(-1)^{n-1}}{(n-1)!} (i-\bar z)^{n-1} ic \\
& = w(z) - \frac{(\bar z - i)^{n-1}}{(n-1)!} c - \sum_{\lambda=0}^{n-2} \frac{1}{2\pi i} \frac{(-1)^\lambda}{\lambda!} \int_{-\infty}^{\infty} \gamma_\lambda(t) \frac{(t-\bar z)^\lambda}{(t-z)} dt \\
& \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \left(\frac{(t-\bar z)^{n-1}}{t-z} - \frac{(i-\bar z)^{n-1} t}{t^2+1} \right) dt.
\end{aligned}
$$

Hence (12) follows. In order to obtain (11), we apply Gauss theorem successively and (2) to obtain

$$
\begin{aligned}
& \frac{(-1)^{n-\nu}}{(n-\nu-1)!} \frac{1}{\pi} \int_{\mathbb{H}} \frac{f(\zeta)(\overline{\zeta - z})^{n-\nu-1}}{\zeta - \bar z} d\xi d\eta \\
& = \sum_{\lambda=\nu}^{n-1} \frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \partial_{\bar\zeta}^{\lambda} w(t) (t-\bar z)^{\lambda-\nu-1} dt,
\end{aligned} \tag{15}
$$

$$
-\frac{1}{\pi} \int_{\mathbb{H}} \frac{\overline{f(\zeta)}(\overline{\zeta - z})^{n-\nu-1}}{\zeta - \bar z} d\xi d\eta = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \overline{\partial_{\bar\zeta}^{n-1} w(t)} (t-\bar z)^{n-\nu-2} dt, \tag{16}
$$

and

$$
\frac{1}{\pi} \int_{\mathbb{H}} \left(\frac{f(\zeta)\zeta}{\zeta^2+1} - \frac{\overline{f(\zeta)}\bar\zeta}{\bar\zeta^2+1} \right) d\xi d\eta = \frac{1}{\pi i} \int_{-\infty}^{\infty} \gamma_{n-1}(t) \frac{t}{t^2+1} dt - ic. \tag{17}
$$

Combining (15), (16), (17), we obtain (11). The verification part is similar to that in Theorem 1. $\square$

4. Schwarz Problem

In this section we explicitly represent the solution of the Schwarz problem for the Poisson equation. The following lemma is a simple application of usual complex analytic methods. We leave the details for the reader.

Lemma 1. For $w, w_\zeta \in \mathcal{F}_1$ and $\zeta = \xi + \mathrm{i}\eta$, we have the following

$$(i) \quad I_1 = \frac{1}{\pi}\int_{\mathbb{H}} \partial_{\bar{\zeta}\zeta} w(\zeta) \log|\zeta - z|^2 d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_\zeta w(t) \log|t - z|^2 dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)}{t - \bar{z}} dt + w(z)$$

$$(ii) \quad I_2 = \frac{1}{\pi}\int_{\mathbb{H}} \partial_{\bar{\zeta}\zeta} w(\xi) \log|\zeta^2 + 1|^2 d\xi d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_\zeta w(t) \log(t^2 + 1) dt + \frac{1}{\pi i}\int_{-\infty}^{\infty} w(t)\frac{t}{t^2 + 1} dt + w(i)$$

$$(iii) \quad I_3 = \frac{1}{\pi}\int_{\mathbb{H}} \partial_{\bar{\zeta}\zeta} w(\zeta) \log|\overline{\zeta} - z|^2 d\xi d\eta$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \overline{\partial_\zeta w(t)} \log|t - z|^2 dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)}{t - \bar{z}} dt$$

□

Theorem 3: Let $w, w_\zeta \in \mathcal{F}_1$, the Schwarz problem for the Poisson equation in the upper half plane$\mathbb{H}$.

$$\partial_{\bar{z}z} w = f \text{ in } \mathbb{H}, \ \mathrm{Re}\, w = \gamma_0, \ \ Re\, w_z = \gamma_1 \ on\ \mathbb{R}$$
$$Im\, w(i) = c_0, Im\, w_z(i) = c_1$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H}, \mathbb{C})$, $\gamma_0(t), t\gamma_1(t) \in L_p(\mathbb{H}, \mathbb{C})$, p$>$ 2 and the solution is given by

$$\begin{aligned}
w(z) &= ic_0 + i(z+\bar{z})c_1 - \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\left(\frac{\gamma_0(t)}{t-\bar{z}} - \gamma_0(t)\frac{t}{t^2+1}\right)dt \\
&\quad - \frac{1}{\pi i}\int\limits_{-\infty}^{\infty}\left\{\frac{(z+\bar{z})t}{t^2+1} + \log|t-z|^2 - \log(t^2+1)\right\}\gamma_1(t)dt \\
&\quad + \frac{(z+\bar{z})}{\pi}\int_{\mathbb{H}}\left(\frac{f(\zeta)}{\zeta^2+1} - \frac{\overline{\zeta f(\zeta)}}{\bar{\zeta}^2+1}\right)d\xi d\eta \\
&\quad + \frac{1}{\pi}\int_{\mathbb{H}}\left(f(\zeta)\log\frac{|\zeta-z|^2}{|\zeta^2+1|} - \overline{f(\zeta)}\log\frac{|\bar{\zeta}-z|^2}{|\bar{\zeta}^2+1|}\right)d\xi d\eta. \qquad (18)
\end{aligned}$$

Proof: Express the last two area integrals in (18) as

$$\begin{aligned}
& \frac{z+\bar{z}}{2\pi}\int_{\mathbb{H}}\left(\frac{1}{\zeta+i}+\frac{1}{\zeta-i}\right)\partial_{\bar{\zeta}}(\partial_\zeta w(\zeta))d\xi d\eta \\
- & \frac{z+\bar{z}}{2\pi}\int_{\mathbb{H}}\left(\frac{1}{\bar{\zeta}+i}+\frac{1}{\bar{\zeta}-i}\right)\partial_{\zeta}(\partial_{\bar{\zeta}}\bar{w}(\zeta))d\xi d\eta \\
+ & \frac{1}{\pi}\int_{\mathbb{H}}\left(\log|\zeta-z|^2 - \log|\zeta^2+1|\right)\partial_{\bar{\zeta}\zeta}w(\zeta)d\xi d\eta \\
- & \frac{1}{\pi}\int_{\mathbb{H}}\left(\log|\zeta-z|^2 - \log|\zeta^2+1|\right)\partial_{\bar{\zeta}\zeta}\overline{w}(\zeta)d\xi d\eta \\
= & \frac{z+\bar{z}}{2}[I_4+I_5] - \frac{z+\bar{z}}{2}[\bar{I}_4+\bar{I}_5] + \left(I_1 - \frac{1}{2}I_2\right) - \left(I_3 - \frac{1}{2}\bar{I}_2\right), \qquad (19)
\end{aligned}$$

where I_1, I_2, I_3 are as in Lemma 1 and

$$I_4 = \frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial_{\bar{\zeta}\zeta}w(\zeta)}{\zeta+i}d\xi d\eta = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\partial_\zeta w(t)}{t+i}dt$$

and

$$I_5 = \frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial_{\bar{\zeta}\zeta}w(\zeta)}{\zeta-i}d\xi d\eta = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\partial_\zeta w(t)}{t-i}dt - w_z(i). \qquad (20)$$

Using Lemma 1 and (20),(19) can be expressed as

$$\frac{z+\bar{z}}{2}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta w(t)+\overline{\partial_\zeta w(t)})\frac{dt}{t+i}+\overline{w_z(i)}\right.$$
$$\left.+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta w(t)+\overline{\partial_\zeta w(t)})\frac{dt}{t-i}-w_z(i)\right]$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta w(t)+\overline{\partial_\zeta w(t)})\log|t-z|^2dt$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{w(t)+\overline{w(t)}}{t-\bar{z}}dt+w(z)-\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\partial_\zeta w(t)+\overline{\partial_\zeta w(t)})\log(t^2+1)dt$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty}(w(t)+\overline{w(t)})\frac{t}{t^2+1}dt+\frac{1}{2}(\overline{w(i)}-w(i)).$$

Using the boundary values, we obtain (18). Verification of (18) to be indeed a solution involves simple computations. □

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ON $A-$ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENT SEQUENCES

AYHAN ESI

Abstract. In this paper we present the following new definitions which is natural combination of the definition for $A-$asymptotically equivalence and $[S]_{A,\theta}$-statistically convergence . Using these definitions we have proved the st-$[S]_{A,\theta}^{L}$ $-$asymptotically equivalence analogues of Fridy and Orhan's theorems in [3] .

1.Introduction

The idea of statistical convergence was introduced by Fast [2] and studied by Fridy [4] ,Connor [5] ,Salat [9] and many others. A sequence $x=(x_k)$ is said to be statistically convergent to number L if for every $\varepsilon>0$

$$\lim_n \frac{1}{n}\left|\{k\leq n:\ \ |x_k-L|\geq\varepsilon\}\right|=0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S-\lim x=L$ or $x_k\rightarrow L\,(S)$ and S denotes the set of all statistically convergent sequences.

Let $A=(a_{ik})$ $(i,k=1,2,...)$ be an infinite matrix of complex numbers. A complex number sequence $x=(x_k)$ is said to be $A-$statistically convergent to the number L [1] if for every $\varepsilon>0$

$$\lim_n \frac{1}{n}\left|\{i\leq n:\ \ |A_i\,(x)-L|\geq\varepsilon\}\right|=0,$$

in this case, we write $S_A-\lim x=L$ or $x_k\rightarrow L\,(S_A)$ and S_A denotes the set of all statistically convergent sequences, where

$$Ax=(A_i\,(x))\ \text{ and } A_i\,(x)=\sum_k a_{ik}x_k \text{ converges for each } i.$$

By a lacunary sequence $\theta=(k_r)$; $r=0,1,2,...$where $k_0=0$, we shall mean an increasing sequence of nonnegative integers with $k_r-k_{r-1}\rightarrow\infty$ as $r\rightarrow\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and $h_r=k_r-k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

In 1993, Marouf [6] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [7] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices.

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In 2006, Patterson and Savaş [8] extended these definitions by using lacunary sequences.

2. Definitions and Notations

Definition 2.1. (Marouf [6]) Two nonnegative sequences x, y are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1 \text{ (denoted by } x \sim y).$$

Definition 2.2. (Patterson [7]) Two nonnegative sequences x, y are said to be asymptotically statistical equivalent of multiple L provided that for $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0 \text{ (denoted by } x \overset{S_L}{\sim} y)$$

and simply asymptotically statistical equivalent, if $L = 1$.

Definition 2.3. (Patterson and Savaş [8]) Let $\theta = (k_r)$ be a lacunary sequence, the two nonnegative sequences x and y are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0 \text{ (denoted by } x \overset{S_\theta^L}{\sim} y)$$

and simply asymptotically lacunary statistical equivalent, if $L = 1$.

Definition 2.4. (Patterson and Savaş [8]) Let $\theta = (k_r)$ be a lacunary sequence, the two nonnegative sequences x and y are said to be strong asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = 0 \text{ (denoted by } x \overset{N_\theta^L}{\sim} y)$$

and simply strong asymptotically lacunary equivalent, if $L = 1$.

In this paper we present the following new definitions which is natural combination of the definition for $A-$asymptotically equivalence and $[S]_{A,\theta}$-statistically convergence .

Throughout the paper $A = (a_{ik})$ $(i, k = 1, 2, ...)$ be an infinite matrix of nonnegative complex numbers such that

$$\|A\| = \sup_i \sum_k a_{ik} < \infty.$$

Definition 2.5. Two nonnegative sequences x and y are said to be $A-$asymptotically equivalent if

$$\lim_i \frac{A_i(x)}{A_i(y)} = 1 \text{ (denoted by } x \overset{A}{\sim} y),$$

Let $[w]_A$ denotes the set of all x and y such that $x \overset{A}{\sim} y$.

Definition 2.6. Two nonnegative sequences x and y are said to be $st-A-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ i \leq n : \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \varepsilon \right\} \right| = 0 \text{ (denoted by } x \overset{st-[S]_A^L}{\sim} y)$$

and simply $[S]_A$ $-$asymptotically statistical equivalent, if $L = 1$. Let $[S]_A$ denotes the set of all x and y such that $x \overset{st-[S]_A}{\sim} y$.

Definition 2.7. Let $\theta = (k_r)$ be a lacunary sequence, the two nonnegative sequences x and y are said to be $st - A_\theta-$asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ i \in I_r : \ \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \varepsilon \right\} \right| = 0 \text{ (denoted by } x \overset{st-[S]^L_{A_\theta}}{\sim} y)$$

and simply $[S]_{A_\theta}$ $-$asymptotically statistical equivalent, if $L = 1$. Furthermore, let $[S]_{A_\theta}$ denotes the set of all x and y such that $x \overset{st-[S]_{A_\theta}}{\sim} y$.

Definition 2.8. Let θ be a lacunary sequence, the two nonnegative sequences x and y are strong $A-$asymptotically lacunary equivalent to multiple L provided that

$$\lim_r \frac{1}{h_r} \sum_{i \in I_r} \left| \frac{A_i(x)}{A_i(y)} - L \right| = 0 \text{ (denoted by } x \overset{N^L_{A_\theta}}{\sim} y)$$

and simply strong $A-$asymptotically lacunary equivalent, if $L = 1$. In addition, let N_{A_θ} denotes the set of all x and y such that $x \overset{N_{A_\theta}}{\sim} y$.

If we take $A = I$ (Identity matrix), then $[w]_{\Delta^m}$ -asymptotically equivalence, st-$[S]_A$ $-$asymptotically equivalence, st-$[S]_{A_\theta}$ $-$asymptotically lacunary equivalence and $N_{A_\theta}-$asymptotically equivalence reduce to ordinary asymptotically equivalence [6],asymptotically statistical equivalence [7], asymptotically lacunary statistical equivalence and strong asymptotically lacunary equivalence [8],respectively.

3. Main Results

Theorem 3.1. Let $\theta = (k_r)$ be a lacunary sequence, then

(a) If $x \overset{N^L_{A_\theta}}{\sim} y$ then $x \overset{st-[S]^L_{A_\theta}}{\sim} y$,

(b) If $x, y \in l_\infty$ and $x \overset{st-[S]^L_{A_\theta}}{\sim} y$ then $x \overset{N^L_{A_\theta}}{\sim} y$,

(c) $[S]^L_{A_\theta} \cap l_\infty = N^L_{A_\theta} \cap l_\infty$.

Proof.(a) If $\varepsilon > 0$ and $x \overset{N^L_{A_\theta}}{\sim} y$ then

$$\sum_{i \in I_r} \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \sum_{\substack{i \in I_r \\ \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \varepsilon}} \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \varepsilon \left| \left\{ i \in I_r : \ \left| \frac{A_i(x)}{A_i(y)} - L \right| \geq \varepsilon \right\} \right|.$$

Therefore $x \overset{st-[S]^L_{A_\theta}}{\sim} y$.

(b) Suppose that $x, y \in l_\infty$ and $x \overset{st-[S]^L_{A_\theta}}{\sim} y$. Then we can assume that

$$\left|\frac{A_i(x)}{A_i(y)} - L\right| \le M \text{ for all } i.$$

Given $\varepsilon > 0$

$$\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{A_i(x)}{A_i(y)} - L\right| = \frac{1}{h_r}\sum_{\substack{i\in I_r \\ \left|\frac{A_i(x)}{A_i(y)} - L\right| \ge \varepsilon}}\left|\frac{A_i(x)}{A_i(y)} - L\right| + \frac{1}{h_r}\sum_{\substack{i\in I_r \\ \left|\frac{A_i(x)}{A_i(y)} - L\right| < \varepsilon}}\left|\frac{A_i(x)}{A_i(y)} - L\right|$$

$$\le \frac{M}{h_r}\left|\left\{i \in I_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \ge \varepsilon\right\}\right| + \varepsilon.$$

Therefore $x \overset{N^L_{A_\theta}}{\sim} y$.

(c) This immediately follows from (a) and (b).

In order to show that the converse of Theorem 3.1.(a) is not generally true, we now give the following simple example.

Example. Let $A = I$ (Identity matrix) and the sequence x be define as follows x_k to be $1, 2, ..., \left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r and zero otherwise. $y_k = 1$ for all k. These two satisfies $x \overset{st-[S]^L_{A_\theta}}{\sim} y$, but it fails $x \overset{N^L_{A_\theta}}{\sim} y$.

Theorem 3.2. Let $\theta = (k_r)$ be a lacunary sequence with $\liminf q_r > 1$,

$$x \overset{st-[S]^L_{A}}{\sim} y \ \text{ implies } x \overset{st-[S]^L_{A_\theta}}{\sim} y \ .$$

Proof. Suppose that $\liminf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \ge 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \ge \frac{\delta}{1+\delta}.$$

If $x \overset{st-[S]^L_{\Delta^m}}{\sim} y$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{i \le k_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \ge \varepsilon\right\}\right| \ge \frac{1}{k_r}\left|\left\{i \in I_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \ge \varepsilon\right\}\right|$$

$$\ge \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{i \in I_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \ge \varepsilon\right\}\right|;$$

this completes the proof.

Theorem 3.3. Let $\theta = (k_r)$ be a lacunary sequence with $limsupq_r < \infty$,

$$x \overset{st-[S]^L_{A_\theta}}{\sim} y \ \text{ implies } x \overset{st-[S]^L_{A}}{\sim} y \ .$$

Proof. Suppose that $limsupq_r < \infty$, then there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$.Let $x \overset{st-[S]_{A_\theta}^L}{\sim} y$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geq R$

$$A_j = \frac{1}{h_j}\left|\left\{i \in I_j : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right| < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, ...$. Now let n be any integer with $k_{r-1} < n < k_r$, where $r > R$.Then

$$\frac{1}{n}\left|\left\{i \leq n : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right| \leq \frac{1}{k_{r-1}}\left|\left\{i \leq k_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$
$$= \frac{1}{k_{r-1}}\left|\left\{i \in I_1 : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{i \in I_2 : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$
$$+ ... + \frac{1}{k_{r-1}}\left|\left\{i \in I_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{k_1}{k_{r-1}k_1}\left|\left\{i \in I_1 : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right| + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\left\{i \in I_2 : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$
$$+ ... + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}\left|\left\{i \in I_R : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$
$$+ ... + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}\left|\left\{i \in I_r : \ \left|\frac{A_i(x)}{A_i(y)} - L\right| \geq \varepsilon\right\}\right|$$
$$= \frac{k_1}{k_{r-1}k_1}A_1 + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}A_2$$
$$+ ... + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}A_R + ... + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}A_r$$
$$\leq \left(\sup_{j \geq 1} A_j\right)\frac{k_R}{k_{r-1}} + \left(\sup_{j \geq R} A_j\right)\frac{k_r - k_R}{k_{r-1}}$$
$$\leq K\frac{k_R}{k_{r-1}} + \varepsilon B.$$

This completes the proof.

Theorem 3.4. Let θ be a lacunary sequence with $1 < \liminf q_r \leq limsupq_r < \infty$,then

$$x \overset{st-[S]_{A_\theta}^L}{\sim} y \Leftrightarrow x \overset{st-[S]_A^L}{\sim} y \ .$$

Proof. This immediately follows from Theorem 3.2. and Theorem 3.3.

4. Conclusion

Taking special matrices, we may obtain some classes of sequences of asymptotically equivalence that we have defined. The most of the results proved in the previous section will be true for these classes.

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Boundedness and stability of solutions to a kind of nonlinear third order differential equations

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Abstract: In this paper, a class of nonlinear third order vector differential equations is considered. It is obtained some sufficient conditions for solutions of these equations to be uniform stable and bounded, which substantially extend and improve some important results in the literature. Two examples are also introduced throughout the paper for illustrations of the topic.

1. Introduction

By a recent paper, which was published in 2007, Yan [3] studied the stability of solutions to nonlinear third order scalar differential equations:

$$x''' + ax'' + f(x') + g(x) = 0$$

and

$$x''' + f(x'') + bx' + g(x) = 0 ,$$

where a and b are some positive constants.

More recently, Tunç [5] investigated the stability and boundedness of solutions of non-linear vector differential equation:

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + B\dot{X} + cX = P(t) ,$$

when $P(t) = 0$ and $P(t) \neq 0$, respectively.

This paper concerns with the problems of stability and boundedness of solutions to nonlinear third order vector equations of the form:

$$\dddot{X} + F(t, X, \dot{X}, \ddot{X})\ddot{X} + G(\dot{X}) + cX = P(t, X, \dot{X}, \ddot{X}) . \quad \textbf{(1)}$$

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The associated system of (1) is

$$\begin{cases} \dot{X} = Y, \dot{Y} = Z \\ \dot{Z} = -F(t, X, Y, Z)Z - G(Y) - cX + P(t, X, Y, Z), \end{cases} \tag{2}$$

which was obtained from (1) by setting $\dot{X} = Y$, $\ddot{X} = Z$, where $t \in \Re^+$, $\Re^+ = [0, \infty)$ and $X \in \Re^n$; c is a positive constant, F is an $n \times n$- continuous symmetric matrix function for the arguments displayed explicitly; $G : \Re^n \to \Re^n$, $G(0) = 0$ and $P : \Re^+ \times \Re^n \times \Re^n \times \Re^n \to \Re^n$; the dots indicate differentiation with respect to t. It is also assumed that the functions G and P are continuous for the arguments displayed explicitly. Moreover, the existence and the uniqueness of the solutions of equation (1) will be assumed (see Ahmad and Rama Mohana Rao [1]). Let $J_G(Y)$ denote the linear operator from the vector $G(Y)$ to the matrix

$$J_G(Y) = \left(\frac{\partial g_i}{\partial y_j} \right), \ (i,\ j = 1, 2, ..., n),$$

where $(y_1, y_2, ..., y_n)$ and (g_i) are components of Y and G, respectively. Besides, throughout the paper, it is also assumed as basic that $J_G(Y)$ exists and is symmetric and continuous.

The motivation for the present work comes from the papers of Yan [3], Tunç [5] and Tunç and Ateş [6]. The aim of this paper is to prove two theorems for uniform stability of zero solution of (1) when $P \equiv 0$ and boundedness of all solutions of the same equation when $P \neq 0$, respectively. We also introduce two examples for the illustrations of the subject. It is also worth mentioning that Yan [3] only studied the stability of solutions to nonlinear third order scalar differential equations aforementioned, without giving any example on the topic. Further, it is well known that nearly all the papers published in the literature on the stability and boundedness of solutions of nonlinear third order scalar and vector differential equations do not include an any example on the subject (see in particular the book of Reissig et al.[4]). It should be noted that by this way our results include and improve the stability results obtained by Yan [3] when $g(x) = cx$ ($c-$constant), which are related to the scalar nonlinear third order differential equations aforementioned, to the stability and boundedness of solutions of nonlinear and non-autonomous third order vector differential equations considered here. In the special case, our assumptions are also less restrictive than those in Tunç and Ateş [6, Theorem 1 and Theorem 2], and our results extend the results of Tunç [5, Theorem 1 and Theorem 2]. The equation considered, the assumptions and Liapunov function will be established here are completely different than that mentioned in the literature.

Notation:

Throughout this paper, the symbol $\langle X, Y \rangle$ will be used to denote the usual scalar product in $\Re^n$ for given any vectors X, Y in $\Re^n$, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X, X \rangle$. It should be noted that the matrix A is said to be negative-definite, when $\langle AX, X\rangle < 0$ for all non-zero X in $\Re^n$, and $\lambda_i(A)$, $(i = 1,2,...,n)$, are eigenvalues of the $n\times n$- matrix A.

2. Preliminaries

Before introducing our main results, we state some basic information which will be required in future.

Consider the non-autonomous differential system

$$\frac{dx}{dt} = F(t,x), \tag{3}$$

where x is an n-vector, $t\in[0,\infty)$. Suppose that $F(t,x)$ is continuous in (t,x) on $I\times D$, where I denotes the interval $0\le t<\infty$ and D is a connected open set in $\Re^n$, $\Re^n$ denotes Euclidean n-space.

Now, we shall dispose of the following theorems and the lemmas which will be required in the proof of our main results.

Theorem 1. Suppose that $F(t,0) = 0$ in (3) and there exists a Lyapunov function $V = V(t,x)$ defined on $0\le t<\infty$, $\|x\| < H$, $H > 0$, which satisfies the following conditions;

(i) $V(t,0) = 0$,

(ii) $a(\|x\|) \le V(t,x) \le b(\|x\|)$, where $a(r)\in CIP$ and $b(r)\in CIP$ (CIP denotes the families of continuous increasing and positive definite functions).

(iii) $\dot{V}(t,x)\le 0$.

Then the solution $x(t)\equiv 0$ of system (3) is uniform stable.

Proof: See Yoshizawa [7].

Lemma. Let A be a real symmetric $n\times n$-matrix. Then for any $X\in\Re^n$

$$\delta_a\|X\|^2 \le \langle AX, X\rangle \le \Delta_a\|X\|^2 ,$$

where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.

Proof. See Bellman [2].

3. Main Results

In the case $P(t,X,Y,Z)\equiv 0$, we introduce the following theorem:

Theorem 2. Let all the basic assumptions imposed on F, G and c hold. We further assume that there are positive constants a, b, c and η such that the following conditions hold:

$F(t, X, Y, Z)$ and $J_G(Y)$ are symmetric, $\eta + b \geq \lambda_i(J_G(Y)) \geq b$

and

$$0 \leq \lambda_i((F(t, X, Y, Z) - aI) < \frac{4(ab-c)}{a^2} \text{ with } ab - c > 0.$$

Then, the zero solution of system (2) is uniform stable.

Proof. We introduce a Liapunov function $V = V(t, X, Y, Z)$, which is defined as the following:

$$V = \frac{1}{2}\langle acX, X\rangle + \int_0^1 \langle G(\sigma Y), Y\rangle d\sigma + \langle cX, Y\rangle + \frac{1}{2}\langle aY + Z, aY + Z\rangle. \tag{4}$$

Now, it is clear from (4) that

$$V(t,0,0,0) = 0.$$

On the other hand, since

$$G(0) = 0, \ \frac{\partial}{\partial \sigma_1} G(\sigma_1 Y) = J_G(\sigma_1 Y)Y,$$

then we have

$$G(Y) = \int_0^1 J_G(\sigma_1 Y) Y d\sigma_1.$$

Hence,

$$\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma = \int_0^1\int_0^1 \sigma_2 \langle J_G(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_1 d\sigma_2.$$

By using the assumption $\lambda_i(J_G(Y)) \geq b$, it follows that

$$\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma = \int_0^1\int_0^1 \sigma_2 \langle J_G(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_1 d\sigma_2$$

$$\geq \int_0^1\int_0^1 \sigma_2 \langle bY, Y\rangle d\sigma_1 d\sigma_2 \geq \frac{b}{2}\|Y\|^2.$$

In view of the above discussion, one can conclude the following:

$$V \geq \frac{1}{2}ac\|X\|^2 + \frac{b}{2}\|Y\|^2 - c\|X\|\,\|Y\| + \frac{1}{2}\|aY + Z\|^2$$

$$= \frac{b}{2}\left\|Y - cb^{-1}X\right\|^2 + \frac{1}{2}(ac - b^{-1}c^2)\|X\|^2 + \frac{1}{2}\|aY + Z\|^2$$

$$\geq D_1\|X\|^2 + D_2\|Y\|^2 + D_3\|Z\|^2, \tag{5}$$

where D_1, D_2 and D_3 are sufficiently small positive constants.

In addition to the above discussion, subject to the assumptions of Theorem 2, one can easily show that the function $V(t,X,Y,Z)$ satisfies the right hand side of the assumption (ii) of Theorem 1, $V(t,x) \leq b(\|x\|)$. That is, subject to the assumptions of Theorem 2, it is obvious that there are positive constants D_4, D_5 and D_6 such that

$$V(t,X,Y,Z) \leq D_4\|X\|^2 + D_5\|Y\|^2 + D_6\|Z\|^2.$$

Now, let $(X,Y,Z) = (X(t),Y(t),Z(t))$ be any solution of the differential system (2). Differentiating the function $V(t,X,Y,Z)$ with respect to t along system (2), we obtain

$$\begin{aligned}\frac{d}{dt}V(t,X,Y,Z) &= \langle cY,Y\rangle - \langle F(t,X,Y,Z)Z,Z\rangle + \langle aZ,Z\rangle \\ &\quad - \langle F(t,X,Y,Z)Z,aY\rangle + \langle aZ,aY\rangle - \langle G(Y),Z\rangle \\ &\quad - \langle G(Y),aY\rangle + \frac{d}{dt}\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma.\end{aligned} \tag{6}$$

Now, we recall

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle G(\sigma Y),Y\rangle d\sigma &= \int_0^1 \sigma\langle J_G(\sigma Y)Z,Y\rangle d\sigma + \int_0^1 \langle G(\sigma Y),Z\rangle d\sigma \\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle G(\sigma Y),Z\rangle d\sigma + \int_0^1 \langle G(\sigma Y),Z\rangle d\sigma \\ &= \sigma\langle G(\sigma Y),Z\rangle\Big|_0^1 = \langle G(Y),Z\rangle.\end{aligned}$$

Substituting aforementioned estimate into (6), we get

$$\begin{aligned}\frac{d}{dt}V(t,X,Y,Z) &= \langle cY,Y\rangle - \langle F(t,X,Y,Z)Z,Z\rangle + \langle aZ,Z\rangle \\ &\quad - \langle F(t,X,Y,Z)Z,aY\rangle + \langle aZ,aY\rangle - \langle G(Y),aY\rangle.\end{aligned} \tag{7}$$

Now, since

$$G(Y) = \int_0^1 J_G(\sigma_1 Y)Y d\sigma_1 \quad\text{and}\quad \lambda_i(J_G(Y)) \geq b$$

then we have

$$-\langle G(Y), aY\rangle = -< \int_0^1 J_G(\sigma_1 Y)Y d\sigma_1, aY > \le -ab\|Y\|^2 .$$

Subject to the assumptions of Theorem 2, it is also clear the following:

$$-\langle F(t,X,Y,Z)Z,Z\rangle + \langle aZ,Z\rangle = -\langle (F(t,X,Y,Z)-aI)Z,Z\rangle ,$$

$$-\langle F(t,X,Y,Z)Z,aY\rangle + \langle aZ,aY\rangle = -\langle (F(t,X,Y,Z)-aI)Z,aY\rangle .$$

In view of aforementioned relations and (7), we have

$$\frac{d}{dt}V(t,X,Y,Z) \le -\langle (ab-c)Y,Y\rangle - \langle (F(t,X,Y,Z)-aI)Z,Z\rangle$$

$$-\langle (F(t,X,Y,Z)-aI)Z,aY\rangle .$$

Now, since

$$0 \le \lambda_i((F(t,X,Y,Z)-aI) < \frac{4(ab-c)}{a^2}$$

one can write

$$\frac{a^2}{4}\{\lambda_i((F(t,X,Y,Z)-aI)\} < ab-c . \qquad \textbf{(8)}$$

Making use of (8), we obtain

$$\frac{d}{dt}V(t,X,Y,Z) \le -\frac{a^2}{4}\langle (F(t,X,Y,Z)-aI)Y,Y\rangle - \langle (F(t,X,Y,Z)-aI)Z,Z\rangle$$

$$-\langle (F(t,X,Y,Z)-aI)Z,aY\rangle$$

$$\le -(F(t,X,Y,Z)-aI)\left[2^{-1}a\|Y\| + \|Z\|\right]^2 \le 0 . \qquad \textbf{(9)}$$

The whole discussion shows that the zero solution of equation (1) is uniform stable.

Example 1. As a special case of the system (2), let us take for $n=2$ that

$$F(t,X,Y,Z) = \begin{bmatrix} 10 + \dfrac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2} & 2 \\ 2 & 10 + \dfrac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2} \end{bmatrix},$$

$$G(Y)=\begin{bmatrix} 8y_1+arctgy_1 \\ 8y_2+arctgy_2 \end{bmatrix}$$

and $c=2$.
Clearly, $F(t,X,Y,Z)$ is a symmetric matrix. Now, by an easy calculation, we have

$$J_G(Y)=\begin{bmatrix} 8+\dfrac{1}{1+y_1^2} & 0 \\ 0 & 8+\dfrac{1}{1+y_2^2} \end{bmatrix}$$

and obtain eigenvalues of the matrices $F(t,X,Y,Z)$ and $J_G(Y)$ as the following:

$$\lambda_1(F(t,X,Y,Z))=8+\frac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2},$$

$$\lambda_2(F(t,X,Y,Z))=12+\frac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2},$$

$$\lambda_1(G(Y))=8+\frac{1}{1+y_1^2}$$

and

$$\lambda_2(G(Y))=8+\frac{1}{1+y_2^2}.$$

Next, it is also clear the following:

$$\lambda_i(F(t,X,Y,Z))\geq 8>6=a,$$

$$9\geq \lambda_i(G(Y)\geq 8=b,\ (i=1,\ 2)$$

$$ab-c=46>0,$$

$$0\leq \lambda_i((F(t,X,Y,Z)-6I)<\frac{46}{9}.$$

Thus, all the assumptions of Theorem 2 hold. When we take into account the above choices, then the corresponding Liapunov function $V=V(t,X,Y,Z)$ can also be rearranged as in (5) and (9). These facts show that the zero solution of the system corresponding to aforementioned choices is uniform stable.

In the case $P(t,X,Y,Z)\neq 0$, the second and last main result of this paper is the following theorem.

Theorem 3. Let us assume that all the assumptions of Theorem 2 hold. In addition, we assume the following:

$$\|P(t,X,Y,Z)\| \le \theta(t) \text{ for all } t \ge 0, \text{ and } \quad \theta \in L^1(0,\infty),$$

where $L^1(0,\infty)$ is space of integrable Lebesgue functions.
Then there exists a constant $D > 0$ such that any solution $(X(t),Y(t),Z(t))$ of system (2) determined by

$$X(0) = X_0,\ Y(0) = Y_0,\ Z(0) = Z_0$$

satisfies

$$\|X(t)\| \le D,\ \|Y(t)\| \le D,\ \|Z(t)\| \le D$$

for all $t \in [0,\infty)$..

Proof. Our main tool for the proof of Theorem 3 is also the Liapunov function V, which is defined in (4). Then, under the assumptions of Theorem 3, we have

$$V \ge D_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2),$$

where $D_7 = \min\{D_1, D_2, D_3\}$, and since $P(t,X,Y,Z) \ne 0$, it is also clear from (4), (2) and (9) that

$$\frac{d}{dt}V(t,X,Y,Z) \le \langle aY + Z, P(t,X,Y,Z)\rangle.$$

Now, taking into account the assumptions of Theorem 3 and using Schwarz's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}V(t,X,Y,Z) &\le (a\|Y\| + \|Z\|)\times\|P(t,X,Y,Z)\| \\ &\le (a\|Y\| + \|Z\|)\times\theta(t) \\ &\le D_8(\|Y\| + \|Z\|)\times\theta(t),\end{aligned}$$

where $D_8 = \max\{1,a\}$..

The rest of the proof is similar to that of Tunç [5, Theorem 2]. Therefore, we omit the details of the proof.

Example 2. We assume that $F(t,X,Y,Z)$, $G(Y)$ and c is given as the same as in Example 1. Further, we choose

$$P(t) = \begin{bmatrix} \dfrac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2} \\ \dfrac{1}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2} \end{bmatrix}.$$

Clearly, it follows that

$$\|P(t)\| = \frac{2}{1+t^2+x_1^2+x_2^2+y_1^2+y_2^2+z_1^2+z_2^2} \leq \frac{2}{1+t^2} = \theta(t)$$

and

$$\int_0^\infty \frac{2}{1+t^2}dt = \pi ,$$

respectively. That is,

$$\theta \in L^1(0,\infty) .$$

Thus, all the conditions of Theorem 3 hold. These facts show that all solutions of the corresponding system to the above choices are bounded.

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ESSENTIAL NORMS OF WEIGHTED COMPOSITION OPERATORS BETWEEN HARDY SPACES IN THE UNIT BALL

ZHONG-SHAN FANG AND ZE-HUA ZHOU *

ABSTRACT. Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of B_n and $\psi(z)$ a holomorphic function on B_n, and $H(B_n)$ the class of all holomorphic functions on B_n, where B_n is the unit ball of C^n, the weight composition operator $W_{\psi,\varphi}$ is defined by $W_{\psi,\varphi} = \psi f(\varphi)$ for $f \in H(B_n)$. In this paper we estimate the essential norm for the weighted composition operator $W_{\psi,\varphi}$ acting from the Hardy space H^p to H^q $(0 < p, q \leq \infty)$. When $p = \infty$ and $q = 2$, we give an exact formula for the essential norm. As their applications, we also obtain some sufficient and necessary conditions for the bounded weighted composition operator to be compact from H^p to H^q.

1. Introduction

Let B_n be the unit ball of C^n with boundary ∂B_n, σ the normalized rotation invariant measure on ∂B_n. The class of all holomorphic functions on domain B_n will be denoted by $H(B_n)$. Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of B_n and $\psi(z)$ is in $H(B_n)$. Multiplication operator, Composition operator and weighted composition operator are defined as follows:

$$M_\psi(f)(z) = \psi(z) \cdot f(z);$$
$$C_\varphi(f)(z) = f(\varphi(z));$$
$$W_{\psi,\varphi}(f)(z) = \psi(z) \cdot f(\varphi(z))$$

for any $f \in H(B_n)$ and $z \in B_n$.

If let $\psi \equiv 1$, then $W_{\psi,\phi} = C_\phi$; if let $\phi = Id$, then $W_{\psi,\phi} = M_\psi$. So we can regard weighted composition operator as a generalization of a multiplication operator and a composition operator. It is easy to show that C_ϕ and $W_{\psi,\phi}$ take $H(B_n)$ into itself. Shapiro's monograph [Shap1] gives an interesting account of these developments. See also Cowen and MacCluer's book [CowMac] for a comprehensive treatment of these and other related problems with composition operators.

In the recent years, boundedness and compactness of composition operators between several spaces of holomorphic functions have been studied by many authors: by Smith [Smi1] between Bergman and Hardy spaces, by Jarchow and Ried [JarR] between generalized Bloch-type spaces and Hardy spaces, between Bloch spaces and Besov spaces and BMOA and VMOA in Tian's thesis [JarR], on BMOA by Simth [Smi2], and by Simth and Zhao [SmiZ] from Bergman and Hardy spaces and Bloch space into Q_p spaces. All of papers above focus on studying the composition operators in function spaces for 1-dimensional case.

More recently, there have been many papers focused on studying the same problems for n-dimensional case : by Luo and Shi [LS1] between Hardy spaces on the unit ball,[LS2] weighted Bergman spaces on bounded symmetric domains, by Zhou and

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Shi[ZS1][ZS2][ZS3] on the Bloch space in polydisk or classical symmetric domains, Gorkin and MacCluer [GorM] between hardy spaces in the unit ball, and Lipschitz space in polydisc by Zhou [Zho]. In all these works the main goal is to relate function theoretic properties of ϕ to boundedness and compactness of C_ϕ.

The essential norm of an operator T is by definition its distance to the compact operators; that is

$$||T||_e := \inf\{||T-K|| : K \;\; compact\}.$$

Notice that $\|T\|_e = 0$ if and only if T is compact, so that estimates on $\|T\|_e$ lead to the conditions for T to be compact.

In general, there is no easy way to determine the essential norms of composition operator or weighted composition operator.

Let f be in $H(B_n)$. For $0 < p < \infty$, f is said to be in the Hardy space $H^p(B_n)$ provided that

$$||f||_p^p = \sup_{0<r<1} \int_{\partial B_n} |f(r\xi)|^p d\sigma(\xi) < \infty.$$

The Banach space of bounded holomorphic functions on B_n in the sup norm is donated by H^∞.

When $f \in H^p$, then f has radial limits at almost every ([dσ]) point of ∂B_n, and its H^p norm is also given by the $L^p(d\sigma)$ norm of its radial limit function f^*. That is

$$||f||_p^p = \int_{\partial B_n} |f^*(\xi)|^p d\sigma(\xi).$$

Typically we continue to write $f(\xi)$ for the radial limit; occasionally for clarity we use the special notation $f^*(\xi)$ for $\lim_{r\to 1} f(r\xi)$. In the whole of paper, $E = \{\xi \in \partial B_n : |\varphi^*(\xi)| = 1\}$, which we call it the extreme set of φ.

It is well known that C_φ is always bounded on $H^p(D)$ for $0 < p \le \infty$, this is a consequence of a theorem of J. Littlewood, see [CowMac], where $D = B_1$ is an unit disk. In 1987, J.Shapiro [Shap2] determined precisely when C_φ acts compactly on $H^p(D)$, for $p < \infty$, and gave a formula for the essential norm of C_φ acting on $H^2(D)$ in terms of the Nevanlinna counting function for φ. In 2002, L. Zheng [Zhe] proved the essential norm of C_φ acting on $H^\infty(D)$ is 1 whenever C_φ is not compact on $H^\infty(D)$ (equivalently, whenever $\|\varphi\|_\infty = 1$); it is also true when D is replaced by the unit ball [GorMS]. For $\infty \ge p > q > 0$, C_φ acting from $H^p(D)$ to $H^q(D)$ will of course be bounded. H. Jarchow [JarR] and T. Goebeler [Goe] shew independently that C_φ is compact if and only if $|E| = 0$.

It seems reasonable to expect the essential norm to be given by a formula that involves $|E|$. In fact, P.Gorkin and B.MacCluer [GorM] pointed out the essential norm of C_φ acting from $H^\infty(D)$ to $H^2(D)$ is precisely $|E|^{\frac{1}{2}}$, and they have obtained the same results in the setting of Hardy spaces $H^p(B_n)$ (we write it H^p in the following) and also gave some simple estimates for the essential norm of a composition operator acting from H^∞ to H^q for $q \neq 2$ and for $q < p < \infty$, from H^p to H^q under a natural additional condition. Here the additional condition is that there exists $0 < p < \infty$ such that $C_\varphi : H^p \to H^p$ is bounded, which is naturally satisfied in the case $n = 1$. This assumption has two properties of interest to us:

(1) No set of positive measure in ∂B_n is mapped by φ^* to a set of measure 0 in ∂B_n (see Corollary 3.38 of [GorM]);

(2) If $f \in H^p(B_n)$,then for a.e. $[d\sigma]\xi \in \partial B_n$, $(f \circ \varphi)^*(\xi) = f^*(\varphi^*(\xi))$ (see Lemma 1.6 in [Mac]).

In our paper, in addition to extend corresponding cases in [GorM] to the weighted composition operator, we also get the lower estimates for the essential norm of a weighted composition operator from H^p to H^q for $1 < p \le q \le \infty$.

The remainder of the present paper is assembled as follows: In section 2, we refer the reader some Lemmas which needs in next sections. In section 3, we will show that the essential norm of the bounded weighted composition operator $W_{\psi,\varphi}$ is precisely $(\mu_{\psi,\varphi,2}(\varphi(E)))^{1/2}$ for the case $p=\infty, q=2$ (Theorem 3.1), and give a estimate for the case $p=\infty, q\neq 2$ (Theorem 3.2). In section 4, we give the upper estimate for the case $1<p<\infty$ (Theorem 4.1) and lower estimate for the case $1<q<p<\infty$ (Theorem 4.2). The fundamental ideas of the proof are those used by Gorkin and MacCluerin in [GorM], but some new techniques are still used in this section because of the citation of the new measure induced by ψ and ϕ and the difference between weighted composition operator and composition operator. If $\psi=1$, $W_{\psi,\varphi}=C_\varphi$, we can completely the corresponding results in [GorM].

In sections 5 and 6 (not be considered in Gorkin and MacCluerin's paper), using different methods, we obtain some estimates for the essential norms of the weighted operator acting from H^p to H^∞ for $p>1$ (Theorem 5.2) and from H^p to H^q for $1<p\leq q<\infty$ (Theorem 6.2).

All of them are done under the same additional condition. As their applications, we also obtained some sufficient and necessary conditions for the weighted composition operator to be compact from H^p to H^q for the above cases. For convenience, we always abbreviate $H^p(B_n)$ to H^p.

2. Some Lemmas

Lemma 2.1. *Let φ is holomorphic self-map of B_n and $\psi\in H^p$, where $0<p<\infty$. For any measurable subset E of ∂B_n, denote $\mu_{\psi,\varphi,p}(E)=\int_{\varphi^{-1}(E)\cap\partial B_n}|\psi|^p d\sigma$. Then*

$$\int_{\overline{B}_n} g d\mu_{\psi,\varphi,p}=\int_{\partial B_n}|\psi|^p(g\circ\varphi)d\sigma,$$

where g is an arbitrary measurable positive function in $\overline{B}_n$.

Proof If g is a measurable simple function defined on $\overline{B}_n$ given by $g=\sum\limits_{i=1}^n \alpha_i\chi_{E_i}$, then

$$\begin{aligned}\int_{\overline{B}_n} g d\mu_{\psi,\varphi,p} &= \sum_{i=1}^n \alpha_i\mu_{\psi,\varphi,p}(E_i)=\sum_{i=1}^n\alpha_i\int_{\varphi^{-1}(E_i)\cap\partial B_n}|\psi|^p d\sigma\\ &= \int_{\partial B_n}|\psi|^p(\sum_{i=1}^n\alpha_i\chi_{\varphi^{-1}(E_i)\cap\partial B_n})d\sigma\\ &= \int_{\partial B_n}|\psi|^p(g\circ\varphi)d\sigma.\end{aligned}$$

Now, if g is a measurable positive function in $\overline{B}_n$, then we can take an increasing sequence $\{g_m\}$ of positive and simple functions such that $g_m(z)\to g(z)$ for all $z\in\overline{B}_n$, it follows that

$$\int_{\overline{B}_n} g_m d\mu_{\psi,\varphi,p}\to\int_{\overline{B}_n} g d\mu_{\psi,\varphi,p}.$$

On the other hand, $|\psi|^p(g_m\circ\varphi)$ is an increasing sequence such that

$$|\psi(z)|^p(g_m(\varphi(z))\to|\psi(z)|^p(g(\varphi(z))$$

for all $z\in\overline{B}_n$, so

$$\int_{\overline{B}_n} g_m d\mu_{\psi,\varphi,p}=\int_{\partial B_n}|\psi|^p(g_m\circ\varphi)d\sigma\to\int_{\partial B_n}|\psi|^p(g\circ\varphi)d\sigma.$$

And the conclusion follows by the uniqueness of the limit.

Lemma 2.2. *(See p116 in* [Zhu2]*) Suppose* $0 < p < \infty$ *and* $f \in H^p$*. Then* $|f(z)| \leq \frac{||f||_p}{(1-|z|^2)^{n/p}}$ *for all* $z \in B_n$*.*

Lemma 2.3. *Let* Ω *be a domain in* C^n*,* $f \in H(\Omega)$*. If a compact set* K *and its neighborhood* G *satisfy* $K \subset G \subset\subset \Omega$ *and* $\rho = dist(K, \partial G) > 0$*, then*

$$\sup_{z\in K} |\frac{\partial f}{\partial z_j}(z)| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in G} |f(z)|.$$

Proof Since $\rho = dist(K, \partial G) > 0$, for any $a \in K$, the polydisc

$$P_a = \left\{(z_1, \cdots, z_n) \in C^n : |z_j - a_j| < \frac{\rho}{\sqrt{n}}, j = 1, \cdots, n\right\}$$

is contained in G. Using Cauchy inequality, we have

$$\left|\frac{\partial f}{\partial z_j}(a)\right| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in\partial_0 P_a} |f(z)| \leq \frac{\sqrt{n}}{\rho} \sup_{z\in G} |f(z)|.$$

So the Lemma follows.

Lemma 2.4. *For fixed* $0 < \delta < 1$*, let* $G = \{z \in B_n : |z| \leq 1 - \delta\}$*. Then*

$$\lim_{r\to 1} \sup_{z\in G} |f(z) - f(rz)| = 0$$

for any $f \in H^p(B_n)$*.*

Proof

$$\begin{aligned}
\sup_{z\in G} |f(z) - f(rz)| &= \sup_{z\in G} |\sum_{j=1}^{n} (f(rz_1, rz_2, \cdots, rz_{j-1}, z_j, \cdots, z_n) \\
&- f(rz_1, rz_2, \cdots, rz_j, z_{j+1}, \cdots, z_n))| \\
&\leq \sup_{z\in G} \sum_{j=1}^{n} |\int_r^1 |z_j \frac{\partial f}{\partial z_j}(rz_1, rz_{j-1}, tz_j, z_{j+1}, \cdots, z^n) dt| \\
&\leq (1-r) n \sup_{z\in G} |\frac{\partial f}{\partial z_j}(z)|.
\end{aligned}$$

Define $G_1 = \{z \in B_n : |z| \leq 1 - \frac{\delta}{2}\}$, then $G \subset G_1$ and $dist(G, \partial G_1) = \frac{\delta}{2}$.
It follows from Lemma 2.3 that

$$\sup_{z\in G} |\frac{\partial f}{\partial z_j}(z)| \leq \frac{2\sqrt{n}}{\delta} \sup_{z\in G_1} |f(z)|.$$

If $p = \infty$, then

$$\sup_{z\in G} |f(z) - f(rz)| \leq \frac{2(1-r)n\sqrt{n}}{\delta} ||f||_\infty.$$

For $0 < p < \infty$, it follows from Lemma 2.2 that

$$\begin{aligned}
\sup_{z\in G} |f(z) - f(rz)| &\leq \frac{2(1-r)n\sqrt{n}}{\delta} \sup_{z\in G_1} \frac{||f||_p}{(1-|z|^2)^{n/p}} \\
&\leq \frac{2(1-r)n\sqrt{n}}{\delta} \sup_{z\in G_1} \frac{||f||_p}{(1-|z|)^{n/p}} \\
&\leq \frac{2(1-r)n\sqrt{n}}{\delta} \frac{||f||_p}{(\frac{\delta}{2})^{n/p}}.
\end{aligned}$$

Let $r \to 1$, the conclusion follows.

Lemma 2.5. *(See corollary 1.3 in* [CowMac]*) A sequence in a reflexive functional Banach space converges weakly if and only if it is bounded and converges point-wise.*

Lemma 2.6. *Assume $\{f_m\}$ is a bounded sequence in $H^p(B_n)(p>1)$, and $\{f_m\}$ converges weakly to 0, then for any compact operator K from $H^p(B_n)$ to Y (Y is a normalized linear space), we have $\|Kf_m\|_Y \to 0$.*

Proof This is easily followed by Lemma 2.5 and the property of compact operator.

3. From H^∞ to H^q

Case 1. $p=\infty, q=2$

It is well known that for any $f \in H(B_n)$, f has homogeneous expansion $f(z) = \sum_{s=0}^{\infty} F_s(z)$, where $F_s(z)$ is the homogeneous polynomial $\sum_{|\alpha|=s} c(\alpha)z^\alpha$, $z^\alpha = z^{\alpha_1}\cdots z^{\alpha_n}$, $\alpha=(\alpha_1,\cdots,\alpha_n)$, and $|\alpha| = \alpha_1+\cdots\alpha_n$.

If $f \in H^2(B_n)$, then

$$||f||_2^2 = \sum_\alpha |c(\alpha)|^2 ||z^\alpha||_2^2,$$

where

$$||z^\alpha||_2^2 = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!},$$

where $\{\frac{z^\alpha}{||z^\alpha||_2}\}$ is an orthonormal basis for $H^2(B_n)$, and $c(\alpha) = D^\alpha f(0)/\alpha!$ with $\alpha! = \alpha_1!\cdots\alpha_n!$. If necessary, we refer the reader to see [Rud].

For m a positive integer, define the operators from $H^2(B_n)$ to itself:

$$R_m(\sum_{s=0}^{\infty} F_s) = \sum_{s=m+1}^{\infty} F_s$$

and

$$Q_m = I - R_m.$$

It is easy to show that R_m is compact and $\|R_m\| = 1$.

Lemma 3.1. $W_{\psi,\varphi}: H^\infty \to H^q$, $0<q<\infty$ *is bounded if and only if* $\psi \in H^q$.

Proof If $W_{\psi,\varphi}$ is bounded, let $f=1$, then $W_{\psi,\varphi}f = \psi f(\varphi) = \psi \in H^q$. Conversely, apparently we have $||W_{\psi,\varphi}f||_q \le ||\psi||_q||f||_\infty$ for any $f \in H^\infty$, that is, $||W_{\psi,\varphi}|| \le ||\psi||_q$.

Using the same methods as that of Gorkin-MacCluer in [GorM], with minor modifications, we can obtain the following Lemmas 3.2 and 3.3. But for the reader's convenience, we give still the detail proof for the results.

Lemma 3.2. *If* $W_{\psi,\varphi}: H^\infty \to H^2$ *and* $\psi \in H^2$, *then*

$$||W_{\psi,\varphi}||_e = \lim_{m\to\infty} ||R_m W_{\psi,\varphi}||.$$

Proof On one hand, by hypothesis and Lemma 3.1, we know $W_{\psi,\varphi}$ is bounded, so the compactness of Q_m implies that $Q_m W_{\psi,\varphi}$ is also compact,

$$\|W_{\psi,\varphi}\|_e = \|(R_m+Q_m)W_{\psi,\varphi}\|_e = \|R_m W_{\psi,\varphi}\|_e \le \|R_m W_{\psi,\varphi}\|,$$

it follows that

$$\|W_{\psi,\varphi}\|_e \le \liminf_{n\to\infty} \|R_m W_{\psi,\varphi}\|.$$

On the other hand, let $K: H^\infty \to H^2$ be compact. Since $\|R_m\|=1$,

$$\begin{aligned}||W_{\psi,\varphi}-K|| &\ge ||R_m(W_{\psi,\varphi}-K)||\\ &= ||R_m W_{\psi,\varphi} - R_m K|| \ge ||R_m W_{\psi,\varphi}|| - ||R_m K||.\end{aligned}$$

Note that K is compact, the image of the unit ball in H^∞ under K has compact closure in H^2. Since $||R_m|| = 1$ and R_mK tends to 0 point-wise in H^2, R_mK tends to 0 uniformly on the unit ball of H^∞, that is $||R_mK|| \to 0$ as $n \to \infty$. It follows that

$$||W_{\psi,\varphi}||_e \geq \limsup_{m\to\infty} ||R_m W_{\psi,\varphi}||,$$

this completes the proof.

Lemma 3.3. *For $W_{\psi,\varphi} : H^\infty \to H^2$ and $\psi \in H^2$, if k is fixed positive integer and g is any non-constant holomorphic function on B_n with $||g||_\infty \leq 1$, then $||Q_k W_{\psi,\varphi}(g^m)||_2 \to 0$ as $m \to \infty$.*

Proof If α is a multi-index with $|\alpha| \leq k$, then

$$||z^\alpha||_2^2 = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} \leq (k!)^n \equiv c(n,k).$$

Since $\overline{D}^n(0, \frac{1}{2n}) \subseteq B_n$ and Cauchy's estimates, for any holomorphic function F in B_n, we have

$$\frac{D^\alpha F(0)}{\alpha!} \leq (2n)^{|\alpha|} ||F||_{\infty, \overline{D}^n(0,\frac{1}{2n})}$$

where$||F||_{\infty,\overline{D}^n(0,\frac{1}{2n})}$ denotes the maximum modulus of F on the polydisc $\overline{D}^n(0, \frac{1}{2n})$. Since the series coefficients for F are $c(\alpha) = \frac{D^\alpha F(0)}{\alpha!}$, we get the series coefficients for $\psi \cdot g^m \circ \varphi$ are bounded above by

$$(2n)^{|\alpha|} ||\psi \cdot g^m \circ \varphi||_{\infty,\overline{D}^n(0,\frac{1}{2n})}.$$

Let $c = max|\psi|$ and $s = max|g \circ \varphi|$ on $\overline{D}^n(0, \frac{1}{2n})$, then $s < 1$ by hypothesis. This implies that $||\psi \cdot g^m \circ \varphi||_{\infty,\overline{D}^n(0,\frac{1}{2n})} \leq cs^m$, which tends to 0 as $m \to \infty$. For fixed k, $||Q_k W_{\psi,\varphi}(g^m)||_2^2 = \sum_{|\alpha|\leq k} |c(\alpha)|||z^\alpha||_2^2$, where $c(\alpha)$ is the coefficients of z^α in the expansion of $\psi \cdot (g \circ \varphi)^m$. By the above estimate, we have

$$||Q_k W_{\psi,\varphi}(g^m)||_2^2 \leq \sum_{|\alpha|\leq k} ((2n)^k cs^m)^2 c(n,k) \leq c'(n,k)s^{2m}.$$

For fixed k, the last expression tends to 0 as $m \to \infty$.

Lemma 3.4. *Let $\epsilon > 0$, set $E_\epsilon = \{\xi \in \partial B_n : |\varphi(\xi)| \geq 1 - \epsilon\}$ and let E_ϵ^c denote its complement in ∂B_n, $\psi \in H^2$. Define an operator $K : H^\infty \to H^2$ by $K(f) = P(\chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi))$, where P is the orthogonal projection of L^2 onto H^2 (where we identify a function in H^2 with its radial limit function). Then K is compact from H^p to H^2, for any $2 < p \leq \infty$.*

Proof Let $\{f_m\}$ be a sequence from the unit ball of H^p. By Lemma 2.4, $\{f_m\}$ is a normal family when $2 < p < \infty$, and this is obviously true for $p = \infty$. So there is a subsequence which converges uniformly on compact subset of B_n, to say f. For simplicity we still denote this subsequence as $\{f_m\}$. Clearly $f \in H^p$. So

$$\begin{aligned} ||Kf_m - Kf||_2^2 &\leq ||P||^2 ||\chi_{E_\epsilon^c}\psi \cdot (f_m \circ \varphi) - \chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi)||_2^2 \\ &\leq \int_{\partial B_n} |\chi_{E_\epsilon^c}\psi \cdot (f_m \circ \varphi) - \chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi)|^2 d\sigma \\ &= \int_{E_\epsilon^c} |\psi \cdot (f_m \circ \varphi) - \psi \cdot (f \circ \varphi)|^2 d\sigma. \end{aligned}$$

Since $\{f_m\}$ are uniformly bounded on E_ϵ^c and $\psi \in H^2$, the above expression tends to 0 as $n \to \infty$ by Lebesgue's dominated convergence theorem. This verifies the compactness of K.

Theorem 3.1. *For $W_{\psi,\varphi} : H^\infty \to H^2$ and $\psi \in H^2$, then $||W_{\psi,\varphi}||_e = (\mu_{\psi,\varphi,2}(\varphi(E)))^{1/2}$, where $E = \{\xi \in \partial B_n : |\varphi^*(\xi)| = 1\}$.*

Proof we consider the lower estimate first.

Let g be a non-constant inner function on B_n and set $h = g^m$ for a positive integer m, then

$$\begin{aligned}||W_{\psi,\varphi}(g^m)||_2^2 &= \int_{\partial B_n} |\psi^* \cdot (h^* \circ \varphi^*)|^2 d\sigma = \int_{\overline{B}_n} |h^*| d\mu_{\psi,\varphi,2} \\ &\geq \int_{\varphi(E)} |h^*| d\mu_{\psi,\varphi,2} \geq \mu_{\psi,\varphi,2}(\varphi(E))\end{aligned}$$

where the last inequality follows by the fact that $|h^*| = 1$ a.e $[d\mu]$ on $\varphi(E)$, this is true that h is inner and the restriction of $\mu_{\psi,\varphi,2}$ to ∂B_n is absolutely continuous with respect to σ.

In fact, for any measurable subset E of ∂B_n,

$$\mu_{\psi,\varphi,2}(E) = \int_{\varphi^{-1}(E)\cap\partial B_n} |\psi|^2 d\sigma,$$

by hypothesis of C_φ, if $\sigma(E) = 0$, then $\sigma(\varphi^{-1}(E)) = 0$, and $\mu_{\psi,\varphi,2}(E) = 0$ follows. So

$$\begin{aligned}||R_k W_{\psi,\varphi}|| &\geq ||R_k W_{\psi,\varphi}(g^m)|| \geq ||W_{\psi,\varphi}(g^m)|| - ||Q_k W_{\psi,\varphi}(g^m)|| \\ &\geq \mu_{\psi,\varphi,2}(\varphi(E)) - ||Q_k W_{\psi,\varphi}(g^m)||.\end{aligned}$$

for all m.

Fix k and let $m \to \infty$ and apply Lemma 3.2 we obtain

$$||R_k W_{\psi,\varphi}|| \geq (\mu_{\psi,\varphi,2}(\varphi(E)))^{1/2}$$

for any k. Now let $k \to \infty$, by Lemma 3.1 we have the desired lower estimate on $||W_{\psi,\varphi}||_e$.

Now we turn to the upper estimate.

Take K as in Lemma 3.3, for any $g \in H^\infty$ with $||g||_\infty = 1$, we have

$$\begin{aligned}||W_{\psi,\varphi}(g) - K(g)||_2 &= ||\psi \cdot g \circ \varphi - P(\chi_{E_\epsilon^c} \psi \cdot (f \circ \varphi))||_2 \\ &= ||P(\chi_{E_\epsilon} \psi \cdot (f \circ \varphi))||_2 \leq ||\chi_{E_\epsilon^c} \psi \cdot (f \circ \varphi)||_2 \\ &= (\int_{E_\epsilon} |\psi \cdot g \circ \varphi|^2 d\sigma)^{\frac{1}{2}} \leq ||g \circ \varphi||_\infty (\int_{E_\epsilon} |\psi|^2 d\sigma)^{\frac{1}{2}} \\ &\leq ||g \circ \varphi||_\infty (\int_{\varphi^{-1}(\varphi(E_\epsilon))\cap\partial B_n} |\psi|^2 d\sigma)^{\frac{1}{2}} \\ &= ||g \circ \varphi||_\infty (\mu_{\psi,\varphi,2}(\varphi(E_\epsilon)))^{1/2}.\end{aligned}$$

Let $\epsilon_m \downarrow 0$ and K_m the corresponding operator defined by

$$K_m(f) = P(\chi_{E_{\epsilon_m}^c} \psi \cdot (f \circ \varphi)).$$

For $p = \infty$ we have

$$||W_{\psi,\varphi}||_e \leq ||W_{\psi,\varphi} - K_m|| \leq (\mu_{\psi,\varphi,2}(\varphi(E_{\epsilon_m})))^{1/2}$$

for all m, and let $m \to \infty$, as desired.

Corollary 3.1. *$W_{\psi,\varphi} : H^\infty \to H^2$ is compact if and only if $\psi \in H^2$ and $\sigma(E) = 0$.*

Proof If $W_{\psi,\varphi}$ is compact, it is obviously bounded, it follows from Lemma 3.1 that $\psi \in H^2$. From Theorem 3.1, the compactness of $W_{\psi,\varphi}$ implies $\mu_{\psi,\varphi,2}(\varphi(E)) = 0$, so $\sigma(\varphi^{-1}(\varphi(E))\cap\partial B_n) = 0$ (see 5.5.9 in [Rud]), therefore $0 \leq \sigma(E) \leq \sigma(\varphi^{-1}(\varphi(E))\cap\partial B_n) = 0$, $\sigma(E) = 0$.

On the other hand, if $\psi \in H^2$, from the proof of theorem 3.1, it follows that

$$||W_{\psi,\varphi}||_e \leq (\int_{E_\epsilon} |\psi|^2 d\sigma)^{\frac{1}{2}}$$

when $\epsilon \to 0$ and since $\sigma(E) = 0$, we get $||W_{\psi,\varphi}||_e = 0$, so $W_{\psi,\varphi}$ is compact.

In the above proof, set $\psi = 1 \in H^2$, then $||W_{1,\varphi}||_e = ||C_\varphi||_e \le \sigma(E)^{1/2}$. And if set $\psi = 1$ in theorem 3.1, then

$$||C_\varphi||_e \ge (\mu_{1,\varphi,2}(E))^{1/2} = \sigma(\varphi^{-1}(\varphi(E)))^{1/2},$$

so $\sigma(\varphi^{-1}(\varphi(E))) = \sigma(E)$, we have the following Corollary

Corollary 3.2. *(Theorem 1 in*[GorM]*)* $C_\varphi : H^\infty \to H^2$ *is bounded and*

$$||C_\varphi||_e = \sigma(E)^{1/2}.$$

Case 2. $p = \infty, q \ne 2$

Theorem 3.2. *Suppose* $W_{\psi,\varphi} : H^\infty \to H^q$ $(q > 1)$*, and* $\psi \in H^q$*, then*

$$\frac{1}{2}(\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q} \le ||W_{\psi,\varphi}||_e \le 2(\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q}.$$

Proof We consider upper estimate first. Obviously $W_{\psi,r\varphi}$ is compact for any fixed $0 < r < 1$. Let $E_\epsilon = \{\xi \in \partial B_n : |\varphi(\xi)| \ge 1 - \epsilon\}$ and let E_ϵ^c denote its complement in ∂B_n. So

$$\begin{aligned} ||W_{\psi,\varphi} - W_{\psi,r\varphi}|| &= \sup_{||f||_\infty = 1} ||(W_{\psi,\varphi} - W_{\psi,r\varphi})f||_q \\ &= \sup_{||f||_\infty = 1} (\int_{\partial B_n} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma)^{1/q} \\ &= \sup_{||f||_\infty = 1} (\int_{E_\epsilon} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma)^{1/q} \\ &+ \sup_{||f||_\infty = 1} (\int_{E_\epsilon^c} |\psi(f \circ \varphi) - \psi(f \circ (r\varphi))|^q d\sigma)^{1/q}. \end{aligned}$$

Apply Lemma 2.4, we can choose r sufficiently close to 1 to make the second term less than $\epsilon||\varphi||_q$. For the first term, the triangle inequality yields

$$|f \circ \varphi(\xi) - f \circ (r\varphi)(\xi)| \le 2$$

So, the first term is less than

$$2(\int_{E_\epsilon} |\psi|^q d\sigma)^{1/q} \le 2(\int_{\varphi^{-1}(\varphi(E_\epsilon)) \cap \partial B_n} |\psi|^q d\sigma)^{1/q} = 2(\mu_{\psi,\varphi,q}(\varphi(E_\epsilon)))^{1/q}.$$

Let $\epsilon_m \downarrow 0$, and $E_{\epsilon_m} = \{\xi \in \partial B_n : |\varphi(\xi)| \ge 1 - \epsilon_m\}$, then $\mu_{\psi,\varphi,q}(\varphi(E_{\epsilon_m})) \to \mu_{\psi,\varphi,q}(\varphi(E))$, the upper estimate follows.

Now we turn to lower estimate. Let f be a non-constant inner function in B_n, K is any compact operator. For any positive integer m, the sequence $\{f^m\}$ are in the unit ball of H^∞, So there exists a subsequence $\{f^{m_k}\}$ such that $\{K(f^{m_k})\}$ converges in norm. Therefore, given $\epsilon > 0$, there exists M such that $||K(f^{m_k}) - K(f^{m_l})||_q < \epsilon$ for any $k, l > M$. Fix $k > M$, there exists r with $0 < r < 1$ such that $(\psi(f \circ \varphi)^{m_k})_r(z) = \psi(rz)(f \circ \varphi(rz))^{m_k}$ satisfies

$$||(\psi(f \circ \varphi)^{m_k})_r||_q \ge ||(\psi(f \circ \varphi)^{m_k})|| - \epsilon.$$

Thus, for $m \ge M$

$$\begin{aligned} ||W_{\psi,\varphi} - K|| &\ge ||(W_{\psi,\varphi} - K)\frac{f^{m_k} - f^{m_l}}{2}||_q \\ &\ge (1/2)||(\psi(f \circ \varphi)^{m_k}) - (\psi(f \circ \varphi)^{m_l})||_q - \epsilon/2 \\ &\ge (1/2)(||(\psi(f \circ \varphi)^{m_k})||_q - ||(\psi(f \circ \varphi)^{m_l})||_q) - \epsilon/2 \\ &\ge (1/2)(||(\psi(f \circ \varphi)^{m_k})||_q - ||(\psi(f \circ \varphi)^{m_l})_r||_q) - \epsilon. \end{aligned}$$

letting $l \to \infty$ and $h = f^{m_k}$, we have

$$\begin{aligned}
||W_{\psi,\varphi} - K|| &\geq (1/2)(||(\psi(f \circ \varphi)^{m_k})||_q - \epsilon \\
&= (1/2)(\int_{\partial B_n} |\psi^* \cdot (h^* \circ \varphi^*)|^q d\sigma)^{1/q} - \epsilon \\
&= (1/2)(\int_{\overline{B}_n} |h^*|^q d\mu_{\psi,\varphi,q})^{1/q} - \epsilon \\
&\geq (1/2)(\int_{\varphi(E)} |h^*|^q d\mu_{\psi,\varphi,q})^{1/q} - \epsilon \\
&\geq (1/2)(\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q} - \epsilon
\end{aligned}$$

Now letting $\epsilon \to 0$ yields the result.

Corollary 3.3. *$W_{\psi,\varphi} : H^\infty \to H^q$ is compact if and only if $\psi \in H^q$ and $\sigma(E) = 0$.*

Proof Combining Lemma 3.1 and Theorem 3.2, the corollary follows.

Corollary 3.4. *(Theorems 2 and 3* [GorM] *$C_\varphi : H^\infty \to H^q$ is bounded and*

$$\frac{1}{2}\sigma(E)^{1/q} \leq ||C_\varphi||_e \leq 2\sigma(E)^{1/q}.$$

Proof Let $\psi = 1 \in H^q$, then $W_{\psi,\varphi} = C_\varphi$, the corollary follows by Theorem 3.2.

4. From H^p to H^q for $1 < q < p < \infty$

Theorem 4.1. *Assume $W_{\psi,\varphi} : H^p \to H^q$ $(1 < p < \infty)$ is bounded, then $||W_{\psi,\varphi}||_e \geq (\mu_{\psi,\varphi,q}(\varphi(E)))^{1/q}$.*

Proof Let g be a non-constant inner function on B_n and set $h = g^m$ for a positive integer m. Then $||g^m||_p = 1$ for any m, and g^m converges weakly to 0 as $m \to \infty$, thus $||Kf_w|| \to 0$ for any compact operator from H^P to H^q when $|w| \to 1$. Like in Theorem 3.1, we have

$$\begin{aligned}
||W_{\psi,\varphi} - K|| &\geq \limsup_{m\to\infty} ||(W_{\psi,\varphi} - K)(g^m)||_q \\
&\geq \limsup_{m\to\infty} ||W_{\psi,\varphi}(g^m)||_q - \limsup_{m\to\infty} ||K(g^m)||_q \\
&= \limsup_{m\to\infty} ||W_{\psi,\varphi}(g^m)||_q = \limsup_{m\to\infty} (\int_{\partial B_n} |\psi^* \cdot (h^* \circ \varphi^*)|^q d\sigma)^{1/q} \\
&= \limsup_{m\to\infty} (\int_{\overline{B}_n} |h^*| d\mu_{\psi,\varphi,q})^{1/q} \geq \limsup_{m\to\infty} (\int_{\varphi(E)} |h^*| d\mu_{\psi,\varphi,q})^{1/q} \\
&\geq (\mu_{\psi,\varphi,2}(\varphi(E)))^{1/q}.
\end{aligned}$$

This ends the proof.

Corollary 4.1. *Assume $W_{\psi,\varphi} : H^p \to H^q, p > 1, 0 < q < \infty$ is compact, then $\sigma(E) = 0$.*

Remark 1. We will show that when $0 < p < q < \infty$ and $W_{\psi,\varphi} : H^p \to H^q$ is bounded, then $\mu_{\psi,\varphi,q}(\varphi(E)) = 0$ (see Corollary 6.1), So the above estimate is useless.

Theorem 4.2. *Suppose $1 < q < p < \infty$ and there exists $r > q$ such that $W_{\psi,\varphi} : H^p \to H^r$ $(1 < p < \infty)$ is bounded, then*

$$||W_{\psi,\varphi}||_e \leq ||P|| \cdot ||W_{\psi,\varphi}||_{p,r} \cdot \sigma(E)^{\frac{r-q}{qr}}$$

where P is the Szegö projection of $L^q(\sigma)$ onto H^q.

Proof We consider the operator $K : H^p \to H^q$ defined by

$$K(f) = P(\chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi)),$$

where P is the Szegö projection of $L^q(\sigma)$ onto H^q. Like in Lemma 3.3, K is compact operator from H^p to H^q. So for any $g \in H^p$ with $||g||_p = 1$, we have

$$\begin{aligned}
||W_{\psi,\varphi}(g) - K(g)||_q &= ||\psi \cdot g \circ \varphi - P(\chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi))||_q \\
&= ||P(\chi_{E_\epsilon}\psi \cdot (f \circ \varphi))||_q \\
&\leq ||P|| \cdot ||\chi_{E_\epsilon^c}\psi \cdot (f \circ \varphi)||_q \\
&= ||P|| \cdot (\int_{E_\epsilon} |\psi \cdot g \circ \varphi|^q d\sigma)^{\frac{1}{q}} \\
&\leq ||P|| \cdot (\int_{\partial B_n} \chi_{E_\epsilon^c} |\psi \cdot g \circ \varphi|^q d\sigma)^{\frac{1}{q}} \\
&\leq ||P|| \cdot ||W_{\psi,\varphi}(g)||_r \sigma(E_\epsilon)^{\frac{r-q}{qr}} \\
&\leq ||P|| \cdot ||W_{\psi,\varphi}||_{p,r} \cdot \sigma(E_\epsilon)^{\frac{r-q}{qr}}.
\end{aligned}$$

Letting $\epsilon \to 0$ yields the conclusion.

5. From H^p to H^∞

Theorem 5.1. *For $W_{\psi,\varphi} : H^p \to H^\infty$, and $0 < p < \infty$, then $W_{\psi,\varphi}$ is bounded if and only if* $\sup\limits_{z\in B_n} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}} < \infty$.

Proof " $\Rightarrow$ " For any $w \in B_n$, define $f_w(z) = \frac{(1-|w|^2)^{n/p}}{(1-<z,w>)^{2n/p}}$, and it is easy to check $||f_w||_p = 1$. So

$$\begin{aligned}
C \geq ||W_{\psi,\varphi}|| &= \sup_{||f||_p=1} ||W_{\psi,\varphi} f||_\infty \geq \sup_{z\in B_n} ||W_{\psi,\varphi} f_w||_\infty \\
&= \sup_{w\in B_n} \sup_{z\in B_n} |\psi(z)||f_w(\varphi(z))|
\end{aligned}$$

setting $w = \varphi(z)$, as desired.

" $\Leftarrow$ "

$$\begin{aligned}
||W_{\psi,\varphi}|| &= \sup_{||f||_p=1} ||W_{\psi,\varphi}||_\infty = \sup_{||f||_p=1} \sup_{z\in B_n} |\psi(z) f(\varphi(z))| \\
&\leq \sup_{||f||_p=1} \sup_{z\in B_n} |\psi(z)| \frac{||f||_p}{(1-|\varphi(z)|^2)^{n/p}} = \sup_{z\in B_n} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}}
\end{aligned}$$

Theorem 5.2. *For $W_{\psi,\varphi} : H^p \to H^\infty$ $(p > 1)$, and $W_{\psi,\varphi}$ is bounded, then*

$$\lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial B_n)<\delta} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}} \leq ||W_{\psi,\varphi}||_e$$

$$\leq 2 \lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial B_n)<\delta} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}}.$$

Proof We consider the upper estimate first.

For any fixed $0 < r < 1$, it is easy to check that $W_{\psi,r\varphi}$ is compact. Thus

$$||W_{\psi,\varphi}||_e \leq ||W_{\psi,\varphi} - W_{\psi,r\varphi}||.$$

Now for any $0 < \delta < 1$

$$\begin{aligned}
||W_{\psi,\varphi} - W_{\psi,r\varphi}|| &= \sup_{||f||_p=1} ||(W_{\psi,\varphi} - W_{\psi,r\varphi})f||_\infty \\
&= \sup_{||f||_p=1} \sup_{z\in B_n} |\psi(z)| \cdot |f(\varphi(z)) - f(r\varphi(z))| \\
&\leq ||\psi||_\infty \sup_{||f||_p=1} \sup_{dist(\varphi(z),\partial B_n)\geq\delta} |f(\varphi(z)) - f(r\varphi(z))| \\
&+ \sup_{||f||_p=1} \sup_{dist(\varphi(z),\partial B_n)<\delta} |\psi(z)| \cdot |f(\varphi(z)) - f(r\varphi(z))|.
\end{aligned}$$

From Lemma 2.4, we can choose r sufficiently close to 1 such that the first term of the right hand side is less than any given ϵ. And we denote the second term by I. Then,

$$\begin{aligned}
I &\leq \sup_{||f||_p=1} \sup_{dist(\varphi(z),\partial B_n)<\delta} |\psi(z)| \cdot (|f(\varphi(z))| + |f(r\varphi(z))|) \\
&\leq \sup_{||f||_p=1} \sup_{dist(\varphi(z),\partial B_n)<\delta} |\psi(z)|(\frac{||f||_p}{(1-|\varphi(z)|^2)^{n/p}} + \frac{||f||_p}{(1-|r\varphi(z)|^2)^{n/p}}) \\
&\leq 2 \sup_{dist(\varphi(z),\partial B_n)<\delta} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}}.
\end{aligned}$$

Now let $r \to 1$ first, then let $\delta \to 0$, we get the desired upper estimate.

We now turn to the lower estimate.

Let K be any compact operator from H^p to H^∞. For any $w \in B_n$ define $f_w(z) = \frac{(1-|w|^2)^{n/p}}{(1-<z,w>)^{2n/p}}$, it is easy to check $||f_w||_p = 1$ and f_w converge weakly to 0 as $|w| \to 1$, thus $||Kf_w|| \to 0$ when $|w| \to 1$.

So for any $0 < \delta < 1$

$$\begin{aligned}
||W_{\psi,\varphi} - K|| &\geq \limsup_{|w|\to 1} ||(W_{\psi,\varphi} - K)f_w||_\infty \\
&\geq \limsup_{|w|\to 1} ||W_{\psi,\varphi}f_w||_\infty - \limsup_{|w|\to 1} ||Kf_w||_\infty \\
&= \limsup_{|w|\to 1} \sup_{z\in B_n} |\psi(z)||f_w(\varphi(z))| \\
&\geq \limsup_{|w|\to 1} \sup_{dist(\varphi(z),\partial B_n)<\delta} |\psi(z)||f_w(\varphi(z))|
\end{aligned}$$

Let $\delta \to 0$ then $|\varphi(z)| \to 1$ and set $w = \varphi(z)$, we obtain the lower estimate of $||W_{\psi,\varphi}||_e$.

Corollary 5.1. *Assume $W_{\psi,\varphi} : H^p \to H^\infty$ is bounded, then it is compact if and only if*

$$\lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial B_n)<\delta} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}} = 0.$$

Remark 2. If $||\varphi||_\infty < 1$, then $E = \{z \in \overline{B}_n | \varphi(z) = 1\} = \emptyset$, without the loss of generality, we set

$$\lim_{\delta\to 0} \sup_{dist(\varphi(z),\partial B_n)<\delta} \frac{|\psi(z)|}{(1-|\varphi(z)|^2)^{n/p}} = 0.$$

6. From H^p to H^q for $1 < p \leq q < \infty$

Definition Let $\beta \geq 1$. A finite and positive measure μ on is called a $\beta - Carleson$ measure. If there is a constant $M \leq \infty$ such that $\mu(S_h(\xi)) \geq Mh^{n\beta}$ for all $\xi \in \partial B_n$ and $0 < h < 2$, and it is called vanishing $\beta - Carleson$ measure if $\lim_{h\to 0} \sup_{\xi\in\partial B_n} \frac{\mu(S_h(\xi))}{h^{n\beta}} = 0$.

Lemma 6.1. *(see corollary 2 in* [LS1]*). Let μ be a finite and positive measure on $\overline{B}_n$, and $0 < p \le q < \infty$, then the following statement are equivalent:*
(i) μ is a bounded $\frac{q}{p}-Carleson$ measure.
(ii) There is a constant $C < \infty$ so that

$$\int_{\overline{B}_n} |f|^p d\mu \le C||f||_p^q$$

for all f in $\overline{B}_n$.

Lemma 6.2. *(*[X]*) Suppose that $0 < p \le q < \infty$ and $W_{\psi,\varphi} : H^p \to H^q$ is bounded, then the following conditions are equivalent:*
(i) $\mu_{\psi,\varphi,q}$ is vanishing $\frac{q}{p}-Carleson$ measure
(ii) $W_{\psi,\varphi} : H^p \to H^q$ is compact operator.

Theorem 6.1. *For Fixed $0 \le p \le q < \infty$, then the following statement are equivalent:*
(i) $\mu_{\psi,\varphi,q}$ is a bounded $\frac{q}{p}-Carleson$ measure.
(ii) $W_{\psi,\varphi} : H^p \to H^q$ is bounded.
(iii)

$$\sup_{z\in B_n} \int_{\overline{B}_n} \frac{(1-|z|^2)^{nq/p}}{|1-<w,z>|^{2nq/p}} d\mu_{\psi,\varphi,q}(w) < \infty.$$

Proof $(i) \Rightarrow (ii)$
From Lemma 3.4 , if $\mu_{\psi,\varphi,q}$ is bounded $\frac{q}{p}-Carleson$ measure, then there exists constant C such that

$$\int_{\overline{B}_n} |f|^p d\mu_{\psi,\varphi,q} \le C||f||_p^q$$

for any $f \in H^p(B_n)$. Apply Lemma 2.1, and put $g = |f|^q$, we have

$$\int_{\overline{B}_n} |f|^q d\mu_{\psi,\varphi,q} = \int_{\partial B_n} |\psi|^q |f\circ\varphi|^q d\sigma = ||W_{\psi,\varphi} f||_q^q.$$

So

$$||W_{\psi,\varphi}(f)||_q \le C^{1/q}||f||_p$$

for any $f \in H^p(B_n)$. That is, $W_{\psi,\varphi} : H^p \to H^q$ is bounded.
$(ii) \Rightarrow (iii)$
For any $z \in B_n$, set $f_z(w) = \frac{(1-|z|^2)^{n/p}}{(1-<w,z>)^{2n/p}}$, then $||f_w||_p = 1$

$$\begin{aligned} C &\ge ||W_{\psi,\varphi}||^q = \sup_{||f||_p=1} ||W_{\psi,\varphi} f||_q^q \ge \sup_{z\in B_n} ||W_{\psi,\varphi} f_z||_q^q \\ &= \sup_{z\in B_n} (\int_{\partial B_n} |\psi|^p |f_z\circ\varphi|^q d\sigma) = \sup_{z\in B_n} \int_{\overline{B}_n} |f_z|^q d\mu_{\psi,\varphi,q} \\ &= \sup_{z\in B_n} \int_{\overline{B}_n} \frac{(1-|z|^2)^{nq/p}}{|1-<w,z>|^{2nq/p}} d\mu_{\psi,\varphi,q}(w) \end{aligned}$$

$(iii) \Rightarrow (i)$
Assume that

$$M = \sup_{z\in B_n} \int_{\overline{B}_n} \frac{(1-|z|^2)^{nq/p}}{|1-<w,z>|^{2nq/p}} d\mu_{\psi,\varphi,q}(w) < \infty$$

we show that $\mu_{\psi,\varphi,q}$ is a bounded $\frac{q}{p}-Carleson$ measure.

First let $z = 0$, then $\mu_{\psi,\varphi,q}(\overline{B}_n) \le M$. Thus $\mu_{\psi,\varphi,q}$ is finite and hence $\mu_{\psi,\varphi,q}(S_h(\xi)) \le M \le 4Mh^{nq/p}$ for all $\xi \in \partial B_n$ and $h \ge (\frac{1}{4})^{\frac{1}{nq/p}}$. Suppose $h \le (\frac{1}{4})^{\frac{1}{nq/p}}$ and $\xi \in \partial B_n$. Let

$\xi_0 = (1-\frac{h}{2})\xi$, then for any $w \in S_h(\xi)$,

$$\begin{aligned} |1- < w, \xi_0 > | &= |1 - \frac{h}{2} + \frac{h}{2} - < w, \xi_0 > | \\ &= |(1-\frac{h}{2})(1- < w, \xi >) + \frac{h}{2}| \\ &\leq |(1-\frac{h}{2})h| + \frac{h}{2} \leq \frac{3h}{2} \end{aligned}$$

and $1 - |\xi_0|^2 = (1-|\xi_0|)(1+|\xi_0|) \geq (1-|\xi_0|)$, we have

$$\frac{(1-|\xi_0|^2)^{nq/p}}{|1- < w, \xi_0 > |^{2nq/p}} \geq \frac{(1-|\xi_0|)^{nq/p}}{\frac{3h}{2}} = \frac{c}{h^{nq/p}}.$$

So

$$\begin{aligned} M &\geq \int_{\overline{B}_n} \frac{(1-|\xi_0|^2)^{nq/p}}{|1- < w, \xi_0 > |^{2nq/p}} d\mu_{\psi,\varphi,q}(w) \\ &\geq \int_{S_h(\xi)} \frac{c}{h^{nq/p}} d\mu_{\psi,\varphi,q} \geq \frac{c\mu_{\psi,\varphi,q}(S_h(z))}{h^{nq/p}}. \end{aligned}$$

Therefore, $\mu_{\psi,\varphi,q}$ is bounded $\frac{q}{p} - Carleson$ measure.

Corollary 6.1. *If $0 < p < q < \infty$ and $W_{\psi,\varphi} : H^p \to H^q$ is bounded, then $\mu_{\psi,\varphi,q}(\varphi(E)) = 0$.*

Proof Denote g the $Radon - Nikodým$ derivative of $\mu_{\psi,\varphi,q}|\partial B_n$ with respect to σ, $\mu_{\psi,\varphi,q}$ is absolutely continuous with respect to σ on ∂B_n, so it follows that

$$g(b) = \lim_{h\to 0} \frac{1}{\sigma(S_h(b))} \int_{S_h(b)} g d\sigma = \lim_{h \to 0} \frac{\mu_{\psi,\varphi,q}(S_h(b))}{\sigma(S_h(b))} \geq \lim_{h\to 0} C h^{nq/p-n} = 0$$

almost everywhere in ∂B_n. Where the penultimate inequality uses the fact that $\sigma(S_h(b))$ is roughly proportional to h^n(see P67 in [Rud]). Now we have $\mu_{\psi,\varphi,q}|\partial B_n$=0, the corollary is proved.

Theorem 6.2. *For fixed $1 < p \leq q < \infty$ and weighted composition operator $W_{\psi,\varphi}$: $H^p \to H^q$ is bounded, then*

$$||W_{\psi,\varphi}||_e \geq \lim_{|w|\to 1} \int_{\overline{B}_n} \frac{(1-|w|^2)^{nq/p}}{|1- < z, w > |^{2nq/p}} d\mu_{\psi,\varphi,q}(z).$$

Proof Let K be any compact operator from H^p to H^∞. For any $w \in B_n$ define $f_w(z) = \frac{(1-|w|^2)^{n/p}}{(1-<z,w>)^{2n/p}}$, it is easy to check $||f_w||_p = 1$ and f_w converge weakly to 0 as $|w| \to 1$, thus $||Kf_w|| \to 0$ when $|w| \to 1$. So for any $0 < \delta < 1$,

$$\begin{aligned} ||W_{\psi,\varphi} - K|| &\geq \limsup_{|w|\to 1} ||(W_{\psi,\varphi} - K)f_w||_q \\ &\geq \limsup_{|w|\to 1} ||W_{\psi,\varphi} f_w||_q - \limsup_{|w|\to 1} ||Kf_w||_q \\ &= \limsup_{|w|\to 1} \int_{\partial B_n} |\psi(z)|^q \frac{(1-|w|^2)^{nq/p}}{|1- < \varphi(z), w > |^{2nq/p}} d\sigma(z) \\ &\geq \limsup_{|w|\to 1} \int_{\overline{B}_n} \frac{(1-|w|^2)^{nq/p}}{|1- < z, w > |^{2nq/p}} d\mu_{\psi,\varphi,q}(z) \end{aligned}$$

The conclusion follows.

We cannot give the upper estimate in the above form, but we have the following theorem.

Theorem 6.3. *Assume $1 < p \leq q < \infty$ and $W_{\psi,\varphi} : H^p \to H^q$ is bounded, then $W_{\psi,\varphi} : H^p \to H^q$ is compact if and only if*

$$\lim_{|w|\to 1} \int_{\overline{B}_n} \frac{(1-|w|^2)^{nq/p}}{|1-<z,w>|^{2nq/p}} d\mu_{\psi,\varphi,q}(z) = 0.$$

Proof The necessary condition follows by theorem 6.2. We consider the sufficient condition. By Lemma 6.2, we only have to show $\mu_{\psi,\varphi,q}$ is vanishing $\frac{q}{p}-Carleson$ measure. From the proof of $(iii) \Rightarrow (i)$ in theorem 6.1, for any $z \in \partial B_n$, set $|z_0| = 1 - \frac{h}{2}$. Suppose

$$\lim_{|w|\to 1} \int_{\overline{B}_n} \frac{(1-|w|^2)^{nq/p}}{|1-<z,w>|^{2nq/p}} d\mu_{\psi,\varphi,q}(z) = 0.$$

That is , $\forall \epsilon > 0, \exists 1 > r > 0$, when $|w| > r$ we have

$$|\int_{\overline{B}_n} \frac{(1-|w|^2)^{nq/p}}{|1-<z,w>|^{2nq/p}} d\mu_{\psi,\varphi,q}(z)| < \epsilon.$$

When $h < 2(1-r)$, for any $z \in \partial B_n$, the corresponding $|z_0| > r$, so

$$\begin{aligned}
\epsilon &> \int_{\overline{B}_n} \frac{(1-|z_0|^2)^{nq/p}}{|1-<w,z_0>|^{2nq/p}} d\mu_{\psi,\varphi,q}(w) \\
&\geq \int_{S_h(z)} \frac{c}{h^{nq/p}} d\mu_{\psi,\varphi,q} \\
&\geq \frac{c\mu_{\psi,\varphi,q}(S_h(z))}{h^{nq/p}}.
\end{aligned}$$

This is true for any $z \in \partial B_n$. So $\mu_{\psi,\varphi,q}$ is vanishing $\frac{q}{p} - Carleson$ measure.

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THE ABSTRACT WAVELET TRANSFORM

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Abstract. Given an abstract locally compact topological group, the abstract wavelet transform is defined so that the inversion formula is used to prove that the image of the wavelet transform is a reproducing kernel Hilbert space, and it is also shown that any function in this space can be reconstructed by its sampled values.

Key words and phrases: unitary representation, admissible function, wavelet transform, inversion formula, sampled values.

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1. Introduction

The continuous wavelet transform for a given signal f is a time-frequency localization method and it is considered as an alternative of the windowed Fourier transform. The wavelet transform determines better than the windowed Fourier transform the localization of high and low-frequencies w for a specific time t [1].

The continuous wavelet transform with respect to an "admissible" function has been used to detect singularities of functions in the Hilbert space $L^2(\mathbb{R})$. For example, by using a reconstruction formula given by Grossmann-Morlet-Paul [4], and the locally compact topological group

$$G = \{ (a,b) \mid a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \}, \tag{1}$$

and a group representation U acting on the Hilbert space $L^2(\mathbb{R})$ given by

$$U(a,b)h\,(x) \;=\; \frac{1}{\sqrt{|a|}} h\left(\frac{x-b}{a}\right), \tag{2}$$

it was proved that for a given function f in $L^2(\mathbb{R})$, the singularities of the continuous wavelet transform $(L_h f)(a,b)$ are the singularities of f [7].

In two dimensions, the continuous wavelet transform has been used to study singularities of distributions u in $\mathcal{S}'(\mathbb{R}^2)$. In this case, the wavelet transform of u yields a function on phase space whose high-frequency singularities are precisely the elements in the wave front set of u [9].

For n dimensions [6], one option is to consider a radially symmetric function h. In this case, the group is given by

$$G = \{(a,b) \mid a > 0 \text{ and } b \in \mathbb{R}^n\} \tag{3}$$

and the representation U of G acting on $L^2(\mathbb{R}^n)$ is taken as $(U(a,b)h)(x) = \frac{1}{a^{\frac{n}{2}}} h(\frac{x-b}{a})$. As before, the singularities of f in $L^2(\mathbb{R}^n)$ are the singularities of $(L_h f)(a,b)$ [8].

In [10], we consider a second option for h, with just two conditions; h in $C_0^\infty(\mathbb{R}^n)$ and $\widehat{h}(0) = 0$. That is, if we drop the condition that h is radially symmetric, then we need extra parameters besides a in $\mathbb{R}^+$ and $b \in \mathbb{R}^n$ to define a locally compact topological group G so that the representation U of G acting on $L^2(\mathbb{R}^n)$ satisfies the condition that as $a \to 0$, then $(U(g)h)(x)$ with g in G concentrates around $x = b$. In this case, the locally compact topological group is taken as

$$G = \left\{(a,b,\theta) \mid a > 0,\ b \in \mathbb{R}^2, \quad \text{and} \quad \theta \in [0, 2\pi)\right\}, \tag{4}$$

and then, it is proved again that the singularities of f are the singularities of $(L_h f)(a,b)$.

For a product of two locally compact groups, the wavelet transform as well as its inverse were studied by means of a special homogeneous space [2].

In this paper, we follow the locally compact topological groups point of view, to define the wavelet transform, where our group G is given as the product of $n+1$ locally compact topological groups $A_1, A_2, \ldots, A_n, B$, by means of a square integrable, irreducible, and unitary representation acting on the Hilbert space $L^2(\prod_{i=1}^n A_i)$, where this representation depends of $n+1$ parameters $a_i \in A_i$, with $i = 1, 2, \ldots, n$, and $b \in B$ so that the inversion formula is obtained for a given function $f \in L^2(\prod_{i=1}^n A_i)$. A reproducing kernel Hilbert space on $L^2(G)$ is given as well as a sampling formula for functions in the image of the wavelet transform. Moreover, we study different characterizations of the wavelet transform by using the Fourier transform, and the convolution of two functions.

2. Notations and definitions

Let us begin by defining a homomorphism for an Abelian locally compact topological group. So, consider $n+1$ locally compact topological groups $A_1, A_2, \ldots, A_n$ and B where

$A_1, A_2, \ldots, A_n$ are Abelian. Now, for each $b \in B$ consider the map $\Gamma_b : A \to A$, where $A = \prod_{i=1}^n A_i$ such that for $a = (a_1, a_2, \ldots, a_n) \in A$,

$$a \to \Gamma_b(a) = (\Gamma_b(a_1), \Gamma_b(a_2), \ldots, \Gamma_b(a_n))$$

is a homeomorphism. Note that for $i = 1, 2, \ldots, n$ the homomorphism Γ from B into the group of all automorphisms of A_i given by $(a_i, b) \to \Gamma_b(a_i)$ is continuous on $A_i \times B$ to A_i.

On the other hand, for $a = (a_1, a_2, \ldots, a_n)$ and $a' = (a_1', a_2', \ldots, a_n')$ in A define

$$aa' = (a_1, a_2, \ldots, a_n)(a_1', a_2', \ldots, a_n') = (a_1 a_1', a_2 a_2', \ldots, a_n a_n').$$

So, A is an Abelian group, where $e_A = (e_1, e_2, \ldots, e_n)$ is the identity with e_i the identity in A_i, and $a^{-1} = (a_1, a_2, \ldots, a_n)^{-1} = (a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$ is the inverse with a_i^{-1} the inverse in A_i, for $i = 1, 2, \ldots, n$.

Definition 1. *Define G as the product of A and B. That is, consider $G = A \times B = \{(a, b) \mid a \in A, \text{ and } b \in B\}$. In G define*

$$(a, b)(a', b') = (a\Gamma_b(a'), bb'). \tag{5}$$

Then, with this product G becomes a group, where $e_G = (e_A, e_B)$ is the identity (e_A is de identity in A and e_B is the identity in B), and where $(a, b)^{-1} = (\Gamma_{b^{-1}}(a^{-1}), b^{-1})$ is the inverse.

Note also that $G = A \times B$ is a locally compact topological group. Then we will denote by $d\mu_G(a, b)$ the left Haar measure on G, the left Haar measure on A by $d\mu_A(a)$ and the left Haar measure in B by $d\mu_B(b)$.

Definition 2. *Given a group G and a set E, an action of G on E is a map $(s, x) \to sx$ of $G \times E \to E$ such that*

1) $ex = x$ for any x in E, and where e is the identity in G

2) $s(tx) = (st)x$ for any x in E, and where s, t are in G.

Then we have the following Lemma.

Lemma 1. *The function $\cdot : G \times A \to A$ given by $(a, b) \cdot x = a\Gamma_b(x)$ is an action of G on A where $(a, b) \in G$ and $x \in A$.*

Proof.

1) Let $e_G = (e_A, e_B)$ be the identity in G, and let x be in A. Then

$$e_G \cdot x = (e_A, e_B) \cdot x = e_A \Gamma_{e_B}(x) = \Gamma_{e_B}(x) = x.$$

2) Let (a, b) and (a', b') be in G, and let x be in A. Then

$$\begin{aligned}(a, b)\left[(a', b') \cdot x\right] &= (a, b)\left[a'\Gamma_{b'}(x)\right] = a\Gamma_b\left[a'\Gamma_{b'}(x)\right] \\ &= a\Gamma_b(a')\Gamma_{bb'}(x) = (a\Gamma_b(a'), bb') \cdot x = \left[(a, b)(a', b')\right] \cdot x.\end{aligned}$$

□

Definition 3. *Let G be a locally compact topological group. The support of the function $f : G \to \mathbb{C}$ denoted by $supp(f)$ is defined as the closure of $\{x \in G | f(x) \neq 0\}$, and $C_0(G)$ is defined as the set of continuous functions $f : G \to \mathbb{C}$ such that $supp(f)$ is compact.*

Definition 4. *Let G be a locally compact topological group, H a closed subgroup of G. A measure $\mu \neq 0$ on G/H is said to be relatively invariant under G if for each $s \in G$ there is a function $\chi : G \to (0, \infty)$ such that for every function $h \in C_0(G/H)$, we have*

$$\int_{G/H} h(s^{-1}z)d\mu(z) = \chi(s) \int_{G/H} h(z)d\mu(z)$$

In this case, the function $\chi : G \to (0, \infty)$ is a continuous homomorphism such that $\chi(z) = \frac{\Delta_H(z)}{\Delta_G(z)}$ with $z \in H$ and where Δ_H and Δ_G are the modular functions on H and G respectively.

Lemma 2. *Let G be a locally compact topological group, and let H be a closed subgroup of G. If there is a continuous positive homomorphism χ on G satisfying $\chi(g) = \frac{\Delta_H(g)}{\Delta_G(g)}$ with $g \in H$, then there exists on G/H a relatively invariant measure μ* [11].

In our case, for $G = A \times B$, there is a positive continuous homomorphism $\chi : G \to (0, \infty)$ that satisfies Lemma 2 for h in $C_0(A)$ [2]. That is, for any $h \in C_0(A)$ we have

$$\int_A h\left((a, b)^{-1} \cdot x\right) d\mu_A(x) = \chi(a, b) \int_A h(x)d\mu_A(x). \tag{6}$$

Definition 5. *For $1 \leq p < \infty$ and for a complex valued function defined on the locally compact topological group A, define*

$$L^p(A) = \left\{h : A \to \mathbb{C} \mid \int_A |h(x)|^p d\mu_A(x) < \infty\right\}$$

where $d\mu_A(x)$ is the left Haar measure on A.

Formula (6) can be extended for all $h \in L^1(A)$ [11]. So, from now on consider $\chi : G \to (0, \infty)$ satisfying (6) for $h \in L^1(A)$.

3. Unitary operators

Definition 6. *For $h \in L^2(A)$ define the following operators*

$$1) \quad (J_a h)(x) = \frac{1}{\sqrt{\chi(a, e_B)}} h[(a, e_B)^{-1} \cdot x],$$

where $(a, e_B) \in G, x \in A$, and $a \in A$.

$$2) \quad (T_b h)(x) = \frac{1}{\sqrt{\chi(e_A, b)}} h[(e_A, b)^{-1} \cdot x],$$

where $(e_A, b) \in G, x \in A$, and $b \in B$.

Lemma 3. *For the operators J_a and T_b,*

1) $J_a J_{a'} = J_{aa'}$, *where* $a, a' \in A$.
2) $T_b T_{b'} = T_{bb'}$, *where* $b, b' \in B$.
3) $T_b J_a = J_{\Gamma_b(a)}, T_b$, *and* $J_a T_b = T_b J_{\Gamma_{b^{-1}}(a)}$, *where* $a \in A$, *and* $b \in B$.

Proof.

1)

$$(J_a J_{a'} h)(x) = [J_a(J_{a'} h)](x) = \frac{1}{\sqrt{\chi(a, e_B)}}(J_{a'} h)[(a, e_B)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi(a, e_B)}} \frac{1}{\sqrt{\chi(a', e_B)}} h[(a', e_B)^{-1}(a, e_B)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi\,[(a, e_B)(a', e_B)]}} h\left[((a, e_B)(a', e_B))^{-1} \cdot x\right] = \frac{1}{\sqrt{\chi(aa', e_B)}} h[(aa', e_B)^{-1} \cdot x] = (J_{aa'} h)(x).$$

2)

$$(T_b T_{b'} h)(x) = [T_b(T_{b'} h)](x) = \frac{1}{\sqrt{\chi(e_A, b)}}(T_{b'} h)[(e_A, b)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi(e_A, b)}} \frac{1}{\sqrt{\chi(e_A, b')}} h[(e_A, b')^{-1}(e_A, b)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi[(e_A, b)(e_A, b')]}} h\left[((e_A, b)(e_A, b'))^{-1} \cdot x\right] = \frac{1}{\sqrt{\chi(e_A, bb')}} h[(e_A, bb')^{-1} \cdot x] = (T_{bb'} h)(x).$$

3) On one hand,

$$(T_b J_a h)(x) = [T_b(J_a h)](x) = \frac{1}{\sqrt{\chi(e_A, b)}}(J_a h)[(e_A, b)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi(e_A, b)}} \frac{1}{\sqrt{\chi(a, e_B)}} h[(a, e_B)^{-1}(e_A, b)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi[(e_A, b)(a, e_B)]}} h\left[((e_A, b)(a, e_B))^{-1} \cdot x\right] = \frac{1}{\sqrt{\chi(\Gamma_b(a), b)}} h[(\Gamma_b(a), b)^{-1} \cdot x].$$

On the other hand,

$$(J_{\Gamma_b(a)} T_b h)(x) = \frac{1}{\sqrt{\chi(\Gamma_b(a), e_B)}}(T_b h)[(\Gamma_b(a), e_B)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi(\Gamma_b(a), e_B)}} \frac{1}{\sqrt{\chi(e_A, b)}} h[(e_A, b)^{-1}(\Gamma_b(a), e_B)^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi[(\Gamma_b(a), e_B)(e_A, b)]}} h[((\Gamma_b(a), e_B)(e_A, b))^{-1} \cdot x]$$
$$= \frac{1}{\sqrt{\chi(\Gamma_b(a)\Gamma_{e_B}(e_A), e_B b)}} h[(\Gamma_b(a)\Gamma_{e_B}(e_A), e_B b)^{-1} \cdot x] = \frac{1}{\sqrt{\chi(\Gamma_b(a), b)}} h[(\Gamma_b(a), b)^{-1} \cdot x].$$

In a similar way, it can be proved that $J_a T_b = T_b J_{\Gamma_{b^{-1}}(a)}$. □

Lemma 4. *For $a \in A$ and $b \in B$, the operators J_a and T_b are unitary operators.*

Proof. Let h be in $L^2(A)$. Then by (6),

1)

$$\begin{aligned}
||J_a h||^2 &= \int_A |(J_a h)(x)|^2\, d\mu_A(x) = \int_A (J_a h)(x)\overline{(J_a h)(x)} d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\chi(a,e_B)}} h((a,e_B)^{-1}\cdot x) \frac{1}{\sqrt{\chi(a,e_B)}} \overline{h}((a,e_B)^{-1}\cdot x) d\mu_A(x) \\
&= \frac{1}{\chi(a,e_B)} \int_A (h\overline{h})((a,e_B)^{-1}\cdot x) d\mu_A(x) \\
&= \frac{1}{\chi(a,e_B)} \chi(a,e_B) \int_A (h\overline{h})(x) d\mu_A(x) = \int_A |h(x)|^2 d\mu_A(x) = ||h||^2.
\end{aligned}$$

2)

$$\begin{aligned}
||T_b h||^2 &= \int_A |(T_b h)(x)|^2 d\mu_A(x) = \int_A (T_b h)(x)\overline{(T_b h)(x)} d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\chi(e_A,b)}} h((e_A,b)^{-1}\cdot x) \frac{1}{\sqrt{\chi(e_A,b)}} \overline{h}((e_A,b)^{-1}\cdot x) d\mu_A(x) \\
&= \frac{1}{\chi(e_A,b)} \int_A (h\overline{h})((e_A,b)^{-1}\cdot x) d\mu_A(x) \\
&= \frac{1}{\chi(e_A,b)} \chi(e_A,b) \int_A (h\overline{h})(x) d\mu_A(x) = \int_A |h(x)|^2 d\mu_A(x) = ||h||^2.
\end{aligned}$$

Moreover, since $J_a^* = J_a^{-1} = J_{a^{-1}}$, and $T_b^* = T_b^{-1} = T_{b^{-1}}$, it follows that both, J_a and T_b are unitary. $\square$

4. Fourier transform

Definition 7. *Let G be a locally compact topological Abelian group, and let $\mathbf{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. We say that the function $\rho : G \to \mathbf{T}$ is a character on G if ρ is a continuous homomorphism.*

Definition 8. *Given a locally compact topological Abelian group G, we define the dual group of G as*

$$\widehat{G} = \{\rho : G \to \mathbf{T} \mid \rho \text{ is a character}\}$$

In this case we denote $\rho(g) = \langle g, \rho\rangle$ where $g \in G$ and $\rho \in \widehat{G}$.

Note that $\widehat{G}$ is clearly an Abelian group under pointwise multiplication $(\rho_1\rho_2)(g) = \rho_1(g)\rho_2(g)$. Its identity element is the constant function $\mathbf{1}$ and the inverse element is $\rho^{-1}(g) = \overline{\rho(g)} = \rho(g^{-1})$.

The dual group of a locally compact topological Abelian group is used to define an abstract version of the Fourier transform.

Definition 9. *Given $h \in L^1(G)$, the Fourier transform of h is the function $\widehat{h} : \widehat{G} \to \mathbb{C}$ defined by*

$$\widehat{h}(\rho) = \int_G h(g)\overline{\rho(g)} d\mu_G(g), \tag{7}$$

where the integral is relative to the left Haar measure on G.

Definition 10. *For a function $h \in L^1(\widehat{G})$, the inverse Fourier transform of h is the function $\check{h} : G \to \mathbb{C}$ defined as*

$$\check{h}(g) = \int_{\widehat{G}} h(\rho)\rho(g)d\mu_{\widehat{G}}(\rho), \tag{8}$$

where $d\mu_{\widehat{G}}(\rho)$ is the left Haar measure on $\widehat{G}$.

Then for $h \in L^1(G)$ and $\widehat{h} \in L^1(\widehat{G})$,

$$h(g) = \int_{\widehat{G}} \widehat{h}(\rho)\rho(g)d\mu_{\widehat{G}}(\rho).$$

Lemma 5. *For $h \in C_0(A)$ we have*

$$\begin{aligned} &1) \quad \widehat{(J_a h)}(\rho) = \sqrt{\chi(a, e_B)} \quad \overline{\rho(a)} \quad \widehat{h}(\rho) \\ &2) \quad \widehat{(T_b h)}(\rho) = \sqrt{\chi(e_A, b)} \quad \widehat{h}(\rho \circ \Gamma_b), \end{aligned}$$

where $\rho \in \widehat{A}$, $a \in A$, and $b \in B$.

Proof.

1) Note that since

$$\widehat{(J_a h)}(\rho) = \int_A (J_a h)(x)\overline{\rho(x)}d\mu_A(x) = \int_A \frac{1}{\sqrt{\chi(a, e_B)}} h((a, e_B)^{-1} \cdot x)\overline{\rho(x)}d\mu_A(x),$$

and by Lemma 1,

$$\rho(x) = \rho[aa^{-1}\Gamma_{e_B}(x)] = \rho(a)\rho[(a^{-1}, e_B) \cdot x] = \rho(a)\rho[(a, e_B)^{-1} \cdot x],$$

it follows from (6) that

$$\begin{aligned} \widehat{(J_a h)}(\rho) &= \int_A \frac{1}{\sqrt{\chi(a, e_B)}} h((a, e_B)^{-1} \cdot x)\, \overline{\rho(a)} \quad \overline{\rho((a, e_B)^{-1} \cdot x)} d\mu_A(x) \\ &= \frac{1}{\sqrt{\chi(a, e_B)}} \overline{\rho(a)} \int_A (h\overline{\rho})((a, e_B)^{-1} \cdot x) d\mu_A(x) \\ &= \frac{1}{\sqrt{\chi(a, e_B)}} \overline{\rho(a)} \chi(a, e_B) \int_A (h\overline{\rho})(x) d\mu_A(x) \\ &= \sqrt{\chi(a, e_B)} \quad \overline{\rho(a)} \int_A h(x)\overline{\rho(x)}d\mu_A(x) = \sqrt{\chi(a, e_B)} \quad \overline{\rho(a)} \quad \widehat{h}(\rho). \end{aligned}$$

2) Similarly, since

$$\widehat{(T_b h)}(\rho) = \int_A (T_b h)(x)\overline{\rho(x)}d\mu_A(x) = \int_A \frac{1}{\sqrt{\chi(e_A, b)}} h((e_A, b)^{-1} \cdot x)\overline{\rho(x)}d\mu_A(x),$$

and

$$\begin{aligned} \rho(x) &= \rho\,[\Gamma_{bb^{-1}}(x)] = \rho\,[\Gamma_b(\Gamma_{b^{-1}}(x))] = \rho\,[\Gamma_b(e_A \Gamma_{b^{-1}}(x))] \\ &= \rho\left[\Gamma_b((e_A, b^{-1}) \cdot x)\right] = \rho\left[\Gamma_b((e_A, b)^{-1} \cdot x)\right] = (\rho \circ \Gamma_b)((e_A, b)^{-1} \cdot x), \end{aligned}$$

it follows from (6) that

$$\begin{aligned}\widehat{(T_b h)}(\rho) &= \int_A \frac{1}{\sqrt{\chi(e_A,b)}} h((e_A,b)^{-1}\cdot x)\overline{(\rho\circ\Gamma_b)((e_A,b)^{-1}\cdot x)}d\mu_A(x)\\ &= \frac{1}{\sqrt{\chi(e_A,b)}}\int_A (h\cdot\overline{\rho\circ\Gamma_b})((e_A,b)^{-1}\cdot x)d\mu_A(x)\\ &= \frac{1}{\sqrt{\chi(e_A,b)}}\chi(e_A,b)\int_A (h\cdot\overline{\rho\circ\Gamma_b})(x)d\mu_A(x)\\ &= \sqrt{\chi(e_A,b)}\quad\int_A h(x)\overline{(\rho\circ\Gamma_b)(x)}d\mu_A(x) = \sqrt{\chi(e_A,b)}\quad\widehat{h}(\rho\circ\Gamma_b).\end{aligned}$$

□

Corollary 1. *Let h be in $L^1(A)$. Then for a in A*

$$\widehat{(J_a T_b h)}(\rho) = \sqrt{\chi(a,b)}\quad\overline{\rho(a)}\quad\widehat{h}(\rho\circ\Gamma_b).$$

Proof. It comes from Lemma 5. □

Lemma 6. *The function $\cdot : G\times\widehat{A}\to\widehat{A}$ given by*

$$[(a,b)\cdot\rho](x) = \rho[(a,b)\cdot x]$$

is an action of G on $\widehat{A}$ where $x\in A$.

Proof.

1) $(e_G\cdot\rho)(x) = [(e_A,e_B)\cdot\rho](x) = \rho[(e_A,e_B)\cdot x] = \rho(x)$.

2) One one hand,

$$\begin{aligned}(a,b)\left[(a',b')\cdot\rho\right](x) &= ((a,b)\cdot\rho)\left[(a',b')\cdot x\right] = [(a,b)\cdot\rho]\left(a'\Gamma_{b'}(x)\right)\\ &= \rho\left[(a,b)\cdot a'\Gamma_{b'}(x)\right] = \rho\left[a\Gamma_b(a'\Gamma_{b'}(x))\right] = \rho\left[a\Gamma_b(a')\Gamma_{bb'}(x)\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}\left[(a,b)(a',b')\cdot\rho\right](x) &= \left[(a\Gamma_b(a'),bb')\cdot\rho\right](x)\\ &= \rho\left[(a\Gamma_b(a'),bb')\cdot x\right] = \rho\left[a\Gamma_b(a')\Gamma_{bb'}(x)\right].\end{aligned}$$

This proves Lemma 6. □

Corollary 2. *Let h be in $L^1(A)$. Then for $a\in A$,*

$$\begin{aligned}&1)\quad \widehat{(J_a h)}(\rho) = (J_{a^{-1}}\widehat{h})(\rho)\\ &2)\quad \widehat{(T_b h)}(\rho) = (T_{b^{-1}}\widehat{h})(\rho)\\ &3)\quad \widehat{(J_a T_b h)}(\rho) = (J_{a^{-1}}T_{b^{-1}}\widehat{h})(\rho).\end{aligned}$$

Proof. Note that from Lemma 6,

1)

$$(J_{a^{-1}}\widehat{h})(\rho) = \frac{1}{\sqrt{\chi(a^{-1},e_B)}}\widehat{h}((a^{-1},e_B)^{-1}\cdot\rho) = \sqrt{\chi(a,e_B)}\int_A h(x)\overline{[(a,e_B)\cdot\rho]\,(x)}d\mu_A(x)$$
$$= \sqrt{\chi(a,e_B)}\int_A h(x)\overline{\rho[(a,e_B)\cdot x]}d\mu_A(x) = \sqrt{\chi(a,e_B)}\int_A h(x)\overline{\rho[a\Gamma_{e_B}(x)]}d\mu_A(x)$$
$$= \sqrt{\chi(a,e_B)}\;\overline{\rho(a)}\int_A h(x)\overline{\rho(x)}d\mu_A(x) = \sqrt{\chi(a,e_B)}\;\overline{\rho(a)}\,\widehat{h}(\rho) = \widehat{(J_a h)}(\rho).$$

2)

$$(T_{b^{-1}}\widehat{h})(\rho) = \frac{1}{\sqrt{\chi(e_A,b^{-1})}}\widehat{h}((e_A,b^{-1})^{-1}\cdot\rho) = \sqrt{\chi(e_A,b)}\int_A h(x)\overline{[(e_A,b)\cdot\rho]\,(x)}d\mu_A(x)$$
$$= \sqrt{\chi(e_A,b)}\int_A h(x)\overline{\rho[(e_A,b)\cdot x]}d\mu_A(x) = \sqrt{\chi(e_A,b)}\int_A h(x)\overline{\rho[(e_A)\Gamma_b(x)]}d\mu_A(x)$$
$$= \sqrt{\chi(e_A,b)}\;\int_A h(x)\overline{\rho(\Gamma_b(x))}d\mu_A(x) = \sqrt{\chi(e_A,b)}\;\widehat{h}(\rho\circ\Gamma_b) = \widehat{(T_b h)}(\rho).$$

3)

$$\widehat{(J_aT_bh)}(\rho) = (J_{a^{-1}}\widehat{T_bh})(\rho) = (J_{a^{-1}}T_{b^{-1}}\widehat{h})(\rho).$$

□

5. Unitary representation

Definition 11. *For (a,b) in $G = A\times B$, define the $n+1$ parameter family of operators $U(a,b) = J_aT_b$. Note that $U(a,b)$ acts on the Hilbert space $L^2(A)$ by:*

$$(U(a,b)h)(x) = (J_aT_bh)(x) = (J_a(T_bh))(x) = \frac{1}{\sqrt{\chi(a,e_B)}}(T_bh)\left[(a,e_B)^{-1}\cdot x\right]$$
$$= \frac{1}{\sqrt{\chi(a,e_B)}}\frac{1}{\sqrt{\chi(e_A,b)}}h\left[(e_A,b)^{-1}(a,e_B)^{-1}\cdot x\right]$$
$$= \frac{1}{\sqrt{\chi((a,e_B)(e_A,b))}}h\left[((a,e_B)(e_A,b))^{-1}\cdot x\right] = \frac{1}{\sqrt{\chi(a,b)}}h\left[(a,b)^{-1}\cdot x\right].$$

Lemma 7. *$U(a,b) = J_aT_b$ is a unitary representation of G acting on the Hilbert space $L^2(A)$.*

Proof. Note that since the operators J_a and T_b are unitary (Lemma 4), it follows that $U(a,b)$ is unitary.

Now, let us prove that $U(a,b)$ is a representation of G acting on $L^2(A)$.

On one hand, from Lemma 3,

$$U\left[(a,b)(a',b')\right] = U(a\Gamma_b(a'),bb') = J_{a\Gamma_b(a')}T_{bb'}$$
$$= J_aJ_{\Gamma_b(a')}T_bT_{b'} = J_aT_bJ_{a'}T_{b'} = U(a,b)U(a',b').$$

On the other hand, since $U(e_A, e_B) = J_{e_A} T_{e_B} = I$, where I is the identity operator, it follows that $U(a,b)$ is a representation of G acting on $L^2(A)$. □

Lemma 8. *The left Haar measure on $G = A \times B$ is given by :*

$$d(a,b) = \frac{1}{\chi(a,b)} d\mu_A(a) d\mu_B(b).$$

Proof. Let h be in $L^1(G)$, then by (6)

$$\int_G h\left[(a',b')^{-1}(a,b)\right] d\mu_A(a) d\mu_B(b) = \chi(a',b') \int_G h(a,b) d\mu_A(a) d\mu_B(b).$$

Now, replacing h by $\frac{h}{\chi}$ we have

$$\int_G \frac{h\left[(a',b')^{-1}(a,b)\right]}{\chi\left[(a',b')^{-1}(a,b)\right]} d\mu_A(a) d\mu_B(b) = \chi(a',b') \int_G \frac{h(a,b)}{\chi(a,b)} d\mu_A(a) d\mu_B(b).$$

Then

$$\int_G h\left[(a',b')^{-1}(a,b)\right] \frac{1}{\chi(a,b)} d\mu_A(a) d\mu_B(b) = \int_G h(a,b) \frac{1}{\chi(a,b)} d\mu_A(a) d\mu_B(b).$$

That is

$$\int_G h\left[(a',b')^{-1}(a,b)\right] d(a,b) = \int_G h(a,b) d(a,b).$$

This shows that $d(a,b)$ is a left Haar measure on G. □

6. Admissibility condition

Definition 12. *A function h in $L^2(A)$ is said to be admissible if*

$$\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) < \infty.$$

Lemma 9. *Let h be in $L^1(A) \cap L^2(A)$. If $\mu(B) < \infty$, then*

$$C_h \equiv \int_B |\widehat{h}(\rho \circ \Gamma_b)|^2 d\mu_B(b)$$

is uniformly bounded for $\rho \in \widehat{A}$.

Proof. Note that since

$$\widehat{h}(\rho \circ \Gamma_b) = \int_A h(x) \overline{(\rho \circ \Gamma_b)(x)} d\mu_A(x),$$

it follows that

$$|\widehat{h}(\rho \circ \Gamma_b)| \le \int_A |h(x)|\, |\rho(\Gamma_b(x))| d\mu_A(x) = \int_A |h(x)| d\mu_A(x) = ||h||_1.$$

Hence, $C_h \le ||h||_1^2\, \mu(B) < \infty$. □

Lemma 10. *Let h be in $L^1(A) \cap L^2(A)$ if*

$$0 < C_h \equiv \int_B |\widehat{h}(\rho \circ \Gamma_b)|^2 d\mu_B(b)$$

is uniformly bounded for $\rho \in \widehat{A}$, then h is admissible.

Proof. Since $0 < C_h < \infty$ for any $\rho \in \widehat{A}$ and h is in $L^2(A)$, it follows that

$$\begin{aligned} R &\equiv ||h||^2 C_h = ||\widehat{h}||^2 C_h \\ &= \left(\int_{\widehat{A}} |\widehat{h}(\rho)|^2 d\mu_{\widehat{A}}(\rho) \right) \left(\int_B |\widehat{h}(\rho \circ \Gamma_b)|^2 d\mu_B(b) \right) < \infty. \end{aligned}$$

Then R can be written as

$$\begin{aligned} R &= \int_B \int_{\widehat{A}} \left| \widehat{h}(\rho)|^2 \quad |\widehat{h}(\rho \circ \Gamma_b) \right|^2 d\mu_{\widehat{A}}(\rho) d\mu_B(b) \\ &= \int_B \left(\int_{\widehat{A}} \left| \widehat{h}(\rho) \overline{\sqrt{\chi(e_A, b)} \quad \widehat{h}(\rho \circ \Gamma_b)} \right|^2 d\mu_{\widehat{A}}(\rho) \right) \frac{1}{\chi(e_A, b)} d\mu_B(b). \end{aligned}$$

Then by Corollary 1,

$$\begin{aligned} R &= \int_B \left(\int_{\widehat{A}} |\widehat{h}(\rho) \overline{\widehat{T_b h}(\rho)}|^2 d\mu_{\widehat{A}}(\rho) \right) \frac{1}{\chi(e_A, b)} d\mu_B(b) \\ &= \int_B \left\| \widehat{h} \overline{\widehat{T_b h}} \right\|^2_{L^2(\widehat{A})} \frac{1}{\chi(e_A, b)} d\mu_B(b) \\ &= \int_B \left\| \widecheck{\left(\widehat{h} \overline{\widehat{T_b h}} \right)} \right\|^2_{L^2(A)} \frac{1}{\chi(e_A, b)} d\mu_B(b) \\ &= \int_B \left(\int_A \left| \widecheck{\left(\widehat{h} \overline{\widehat{T_b h}} \right)}(a) \right|^2 d\mu_A(a) \right) \frac{1}{\chi(e_A, b)} d\mu_B(b) \\ &= \int_B \left(\int_A \left| \int_{\widehat{A}} (\widehat{h} \overline{\widehat{T_b h}})(\gamma)\, \gamma(a) d\mu_{\widehat{A}}(\gamma) \right|^2 d\mu_A(a) \right) \frac{1}{\chi(e_A, b)} d\mu_B(b) \\ &= \int_B \int_A \left| \int_{\widehat{A}} \widehat{h}(\gamma) \overline{\sqrt{\chi(a, e_B)}\ \overline{\gamma(a)}\ \widehat{T_b h}(\gamma)} d\mu_{\widehat{A}}(\gamma) \right|^2 \frac{1}{\chi(a, b)} d\mu_A(a) d\mu_B(b) \\ &= \int_G \left| \int_{\widehat{A}} \widehat{h}(\gamma) \overline{(J_{(a)^{-1}} \widehat{T_b h})(\gamma)} d\mu_{\widehat{A}}(\gamma) \right|^2 d(a, b) \\ &= \int_G \left| \left\langle \widehat{h}, \widehat{J_a T_b h} \right\rangle_{L^2(\widehat{A})} \right|^2 d(a, b) \\ &= \int_G \left| \langle h, J_a T_b h \rangle_{L^2(A)} \right|^2 d(a, b) \\ &= \int_G |\langle h, U(a, b) h \rangle|^2 d(a, b). \end{aligned}$$

That is

$$\int_G |\langle h, U(b) h \rangle|^2 d(a, b) = C_h ||h||^2 < \infty, \tag{9}$$

which means that h is admissible. $\square$

7. The abstract wavelet transform

Definition 13. *Given (a,b) in $G = A \times B$ and h admissible in $L^2(A)$, the abstract wavelet transform with respect to h is defined as the linear operator*

$$L_h(a,b) : L^2(A, d\mu_A) \to L^2(G, d(a,b))$$

such that for any f in $L^2(A)$ we have

$$(L_h f)(a,b) = \langle f, U(a,b)h \rangle_{L^2(A)}.$$

That is,

$$(L_h f)(a,b) = \int_A f(x)\overline{(U(a,b)h)\,(x)}d\mu_A(x) = \int_A f(x)\frac{1}{\sqrt{\chi(a,b)}}\overline{h((a,b)^{-1}\cdot x)}d\mu_A(x).$$

Now, in order to get back the function f from the abstract wavelet transform $(L_h f)(a,b)$, we will apply the Grossmann-Morlet-Paul theorem [4], where the hypotheses for the representation $U(a,b)$ are: unitary, irreducible and strongly continuous. In our case, our representation is unitary, so the following lemmas will show that $U(a,b) = J_a T_b$ is irreducible and strongly continuous.

Lemma 11. *The representation $U(a,b)$ of the group $G = A \times B$ acting on $L^2(A)$ is irreducible.*

Proof. Suppose $h \in L^2(A) \setminus \{0\}$ and suppose $f \in L^2(A)$ is such that $\langle f, U(a,b)h \rangle = 0$ for all (a,b) in G. To show the representation $U(a,b)$ is irreducible [5], we will show that $f = 0$ in $L^2(A)$.

Under the assumptions we have

$$\int_G |\langle f, U(a,b)h \rangle|^2 d(a,b) = 0,$$

but by (9),

$$0 = \int_G |\langle f, U(a,b)h \rangle|^2 d(a,b) = C_h ||f||^2.$$

Since h is not identically zero, it follows that $||f|| = 0$. Thus $f = 0$. □

Definition 14. *For a function $h : A \to \mathbb{C}$, define the left and right translations of h by*

$$(I_a h)(x) = h(a^{-1}x) \quad \text{and} \quad (D_a h)(x) = h(xa), \quad \text{where} \quad a, x \in A.$$

Definition 15. *For a function $h : A \to \mathbb{C}$, we say that:*

a) h is left uniformly continuous if $||I_a h - h||_\infty \to 0$ as $a \to e_A$

b) h is right uniformly continuous if $||D_a h - h||_\infty \to 0$ as $a \to e_A$,

where $||\quad||_\infty$ is the uniform norm.

Lemma 12. *If $h \in C_0(A)$, then h is left and right uniformly continuous* [3].

Lemma 13. *If h is in $C_0(A)$, then* *a)* $||J_a h - h||_\infty \to 0$ *as* $a \to e_A$

b) $||T_b h - h||_\infty \to 0$ *as* $b \to e_B$.

Proof.

a) Note that

$$|(J_a h)(x) - h(x)| = \left| \frac{1}{\sqrt{\chi(a, e_B)}} h((a, e_B)^{-1} \cdot x) - h(x) \right| = \left| \frac{1}{\sqrt{\chi(a, e_B)}} h(a^{-1}x) - h(x) \right|$$

$$\leq \frac{1}{\sqrt{\chi(a, e_B)}} |h(a^{-1}x) - h(x)| + \left| \frac{1}{\sqrt{\chi(a, e_B)}} - 1 \right| |h(x)|.$$

Then, since $\chi(a, e_B)$ is continuous at (e_A, e_B), it follows from Lemma 12 that $||J_a h - h||_\infty \to 0$ as $a \to e_A$.

b) Similarly, since

$$|(T_b h)(x) - h(x)| = \left| \frac{1}{\sqrt{\chi(e_A, b)}} h((e_A, b)^{-1} \cdot x) - h(x) \right|$$

$$\leq \frac{1}{\sqrt{\chi(e_A, b)}} |\left(h \circ \Gamma_{b^{-1}}\right)(x) - \left(h \circ \Gamma_{e_B}\right)(x)| + \left| \frac{1}{\sqrt{\chi(e_A, b)}} - 1 \right| |h(x)|$$

and $\chi(e_A, b)$ is continuous at (e_A, e_B), it follows that $||T_b h - h||_\infty \to 0$ as $b \to e_B$. □

Lemma 14. *Let* h *be in* $L^2(A)$. *Then*

1) $||J_a h - h||_2 \to 0$ *as* $a \to e_A$

2) $||T_b h - h||_2 \to 0$ *as* $b \to e_B$.

Proof. Let $\epsilon > 0$ be given. Since $C_0(A)$ is dense in $L^2(A)$, there is $l \in C_0(A)$ such that $||h - l||_2 < \frac{\epsilon}{3}$.

Now, since l is uniformly continuous, it follows from Lemma 13 that there is a neighborhood V of e_A such that $||J_a l - l||_\infty < \frac{\epsilon}{3} \frac{1}{\sqrt{m(S)}}$ for all $a \in V$ and where $S = supp(l)$.

Then

$$||J_a l - l||_2^2 = \int_S |(J_a l)(x) - l(x)|^2 d\mu_A(x) \leq ||J_a l - l||_\infty^2 m(S).$$

Thus, $||J_a l - l||_2 \leq ||J_a l - l||_\infty \sqrt{m(S)}$. Hence, $||J_a l - l||_2 < \frac{\epsilon}{3}$.

Therefore, for $a \in V$ we have from Lemma 4,

$$||J_a h - h||_2 \leq ||J_a h - J_a l||_2 + ||J_a l - l||_2 + ||h - l||_2$$

$$= ||J_a(h - l)||_2 + ||J_a l - l||_2 + ||h - l||_2$$

$$= ||h - l||_2 + ||J_a l - l||_2 + ||h - l||_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

2) In a similar way, it can be proved that $||T_b h - h||_2 \to 0$ as $b \to e_B$. □

Lemma 15. *Let* h *be in* $L^2(A)$. *Then* $||U(a, b)h - h||_2 \to 0$ *as* $(a, b) \to (e_A, e_B)$.

Proof. By Lemma 12,

$$\begin{aligned}||U(a,b)h-h||_2 &= ||J_aT_bh-h||_2 = ||J_aT_bh-J_ah+J_ah-h||_2\\ &\le ||J_a(T_bh-h)||_2+||J_ah-h||_2 = ||T_bh-h||_2+||J_ah-h||_2 \to 0\end{aligned}$$

as $a \to e_A$ and $b \to e_B$. □

Lemma 16. *The representation $U(a,b)$ is strongly continuous.*

Proof. Let us prove that $||U(a,b)h - U(a',b')h||_2 \to 0$ as $(a,b) \to (a',b')$ for any h in $L^2(A)$.

Consider a, a' in A. Then for b, b' in B, and Lemma 3

$$\begin{aligned}J_aT_bh &= (J_aT_bT_{(b')^{-1}}J_{(a')^{-1}})(J_{a'}T_{b'}h) = (J_aT_{b(b')^{-1}}J_{(a')^{-1}})(J_{a'}T_{b'}h)\\ &= J_a(J_{\Gamma_{b(b')^{-1}}((a')^{-1})}T_{b(b')^{-1}})(J_{a'}T_{b'}h) = (J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}T_{b(b')^{-1}})(J_{a'}T_{b'}h).\end{aligned}$$

Then for $u = J_{a'}T_{b'}h$,

$$\begin{aligned}&J_aT_bh - J_{a'}T_{b'}h = J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}T_{b(b')^{-1}}u - u\\ &= J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}T_{b(b')^{-1}}u - J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}u + J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}u - u\\ &= J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}\left[T_{b(b')^{-1}}u - u\right] + \left[J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}u - u\right].\end{aligned}$$

Then by Lemma 14,

$$\begin{aligned}&||J_aT_bh - J_{a'}T_{b'}h||_2\\ &\le ||J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}\left[T_{b(b')^{-1}}u - u\right]||_2 + ||J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}u - u||_2\\ &= ||T_{b(b')^{-1}}u - u||_2 + ||J_{a\Gamma_{b(b')^{-1}}((a')^{-1})}u - u||_2 \to 0\\ &\quad \text{as} \quad (a(a')^{-1}, b(b')^{-1}) \to (e_A, e_B).\end{aligned}$$

Thus, $||J_aT_bh - J_{a'}T_{b'}h||_2 \to 0$ as $(a,b) \to (a',b')$. □

8. Inversion formula

Lemma 17. *For any f, g in $L^2(A)$ and an admissible non-zero function h in $L^2(A)$, we have the following identity in the weak sense*

$$f = \frac{1}{C_h}\int_G (L_h f)(a,b)\, U(a,b)h\, d(a,b).$$

Proof. The representation $U(a,b)$ is a strongly continuous irreducible unitary representation of the locally compact topological group $G = A \times B$ acting on the Hilbert space $L^2(A)$. So, if there is a non-zero admissible vector h in $L^2(A)$, then by the Grossmann-Morlet-Paul theorem [4], for f, g in $L^2(A)$,

$$\int_G \langle f, U(a,b)h\rangle \overline{\langle g, U(a,b)h\rangle} d(a,b) = C_h \langle f, g\rangle . \tag{10}$$

Hence,

$$f = \frac{1}{C_h}\int_G (L_h f)(a,b)\, U(a,b)h\, d(a,b)$$

in the weak sense. □

9. Isometry

Lemma 18. *Given f, g in $L^2(A)$,*

$$\langle f, g\rangle_{L^2(A)} = \frac{1}{C_h}\langle L_h f, L_h g\rangle_{L^2(G)} \tag{11}$$

for any non-zero admissible function h in $L^2(A)$.

Proof. It comes from (10). □

Lemma 19. *For f, g in $L^2(A)$,*

$$||f||^2_{L^2(A)} = \frac{1}{C_h}||L_h f||^2_{L^2(G)}. \tag{12}$$

Proof. If $f = g$, then from Lemma 18,

$$\langle f, f\rangle_{L^2(A)} = \frac{1}{C_h}\langle L_h f, L_h f\rangle_{L^2(G)} .$$

That is,

$$||f||^2_{L^2(A)} = \frac{1}{C_h}||L_h f||^2_{L^2(G)}.$$

□

Note that from Lemma 19,

$$\int_A |f(a)|^2 d\mu_A(a) = \frac{1}{C_h}\int_G |(L_h f)(a,b)|^2 d(a,b).$$

10. Co-variance properties

In this section, the co-variance of wavelet transforms is given as well as with respect to admissible functions.

Lemma 20. *If h in $L^2(A)$ is admissible, then for f in $L^2(A)$ and for a' in A and b' in B,*

1) $(L_h T_{b'} f)(a,b) = (L_h f)(\Gamma_{(b')^{-1}}(a), (b')^{-1}b)$

2) $(L_h J_{a'} f)(a,b) = (L_h f)((a')^{-1}a, b)$.

Proof. From Lemma 3,

1)

$$(L_h T_{b'} f)(a,b) = \langle T_{b'} f, J_a T_b h\rangle = \left\langle f, T_{(b')^{-1}} J_a T_b h\right\rangle$$
$$= \left\langle f, J_{\Gamma_{(b')^{-1}}(a)} T_{(b')^{-1}} T_b h\right\rangle = \left\langle f, J_{\Gamma_{(b')^{-1}}(a)} T_{(b')^{-1}b} h\right\rangle = (L_h f)(\Gamma_{(b')^{-1}}(a), (b')^{-1}b).$$

2)
$$\begin{aligned}(L_h J_{a'} f)(a,b) &= \langle J_{a'} f, J_a T_b h\rangle \\ &= \left\langle f, J_{(a')^{-1}} J_a T_b h\right\rangle = \left\langle f, J_{(a')^{-1}a} T_b h\right\rangle = (L_h f)((a')^{-1}a, b).\end{aligned}$$

□

Corollary 3. *If $h \in L^2(A)$ is admissible, then for $f \in L^2(A)$ and for $a' \in A$ and $b' \in B$,*

$$(L_h J_{a'} T_{b'} f)(a,b) = (L_h f)(\Gamma_{(b')^{-1}}((a')^{-1}a), (b')^{-1}b). \tag{13}$$

Proof. It comes from Lemma 20. □

Lemma 21. *If $f \in L^2(A)$ and h is admissible in $L^2(A)$, then for $a' \in A$ and $b' \in B$, we have $J_{a'}h$ and $T_{b'}h$ are admissible. Moreover*

$$1)\quad (L_{J_{a'}h} f)(a,b) = (L_h f)(a\Gamma_b(a'), b)$$
$$2)\quad (L_{T_{b'}h} f)(a,b) = (L_h f)(a, bb').$$

Proof. First, we will show that $J_{a'}$ and $T_{b'}$ are admissible. It is clear that $J_{a'}h$ and $T_{b'}h$ are in $L^2(A)$.

By Lemma 10, since

$$\begin{aligned}C_{J_{a'}h} &\equiv \int_B |\widehat{(J_{a'}h)}(\rho\circ\Gamma_b)|^2 d\mu_B(b) \\ &= \int_B \left|\sqrt{\chi(a', e_B)}\quad \overline{(\rho\circ\Gamma_b)(a')}\quad \widehat{h}(\rho\circ\Gamma_b)\right|^2 d\mu_B(b) = \chi(a', e_B)\int_B |\widehat{h}(\rho\circ\Gamma_b)|^2 d\mu_B(b) < \infty,\end{aligned}$$

it follows that $J_{a'}h$ is admissible.

Also, since

$$\begin{aligned}C_{T_{b'}h} &\equiv \int_B |\widehat{(T_{b'}h)}(\rho\circ\Gamma_b)|^2 d\mu_b(b) \\ &= \int_B \left|\sqrt{\chi(e_A, b')}\,\widehat{h}(\rho\circ\Gamma_b\circ\Gamma_{b'})\right|^2 d\mu_B(b) = \chi(e_A, b')\int_B |\widehat{h}(\rho\circ\Gamma_{bb'})|^2 d\mu_B(b) < \infty,\end{aligned}$$

it follows that $T_{b'}h$ is admissible.

Now, note that

1)
$$(L_{J_{a'}h} f)(a,b) = \langle f, J_a T_b J_{a'}h\rangle = \left\langle f, J_a J_{\Gamma_b(a')} T_b h\right\rangle = (L_h f)(a\Gamma_b(a'), b),$$

and

2)
$$(L_{T_{b'}h} f)(a,b) = \langle f, J_a T_b T_{b'}h\rangle = \langle f, J_a T_{bb'}h\rangle = (L_h f)(a, bb').$$

□

Corollary 4. *If $f \in L^2(A)$ and h is admissible in $L^2(A)$, then*

$$(L_{J_{a'}T_{b'}h} f)(a,b) = (L_h f)(a\Gamma_b(a'), bb'). \tag{14}$$

Proof. It comes from Lemma 21. □

Corollary 5. *If $f \in L^2(A)$ and h is admissible in $L^2(A)$, then*

$$(L_{J_{a'}T_{b'}h} J_{a'}T_{b'}f)(a,b) = (L_h f)(\Gamma_{(b')^{-1}}((a')^{-1}a\Gamma_b(a')), (b')^{-1}bb'). \tag{15}$$

Proof. It comes from (13) and (14). □

11. Main Results

We now give a characterization of the image of the wavelet transform by a reproducing kernel. Note that not every function $F(a,b)$ in $L^2(G)$ is the wavelet transform of some function f in $L^2(A)$. That is,

$$Im(L_h(a,b)) = \left\{F(a,b) \,|\, (L_h f)(a,b) = F(a,b) \text{ for some } f \in L^2(A)\right\}$$

is a proper subspace of $L^2(G)$.

To see this, note that for $F(a,b) = (L_h f)(a,b)$, we have $F(a,b)$ is bounded since

$$|F(a,b)| = |(L_h f)(a,b)| = |\,\langle f, U(a,b)h\rangle\,| \leq ||f||_2||h||_2.$$

Hence, any unbounded and square integrable function $F(a,b)$ is not in $Im(L_h(a,b))$. Then we have the following result.

Theorem 1. *The image of the wavelet transform with respect to an admissible function h in $L^2(A)$ is the closed subspace of functions $F(a,b)$ in $L^2(G)$ that satisfy*

$$F(a,b) = \frac{1}{C_h}\int_G F(a',b')K(a,b\,;a',b')d(a',b')$$

where

$$K(a,b\,;a',b') = \overline{[L_h U(a,b)h](a',b')}$$

is the reproducing kernel associated with h.

Proof. If F is in $Im(L_h(a,b))$, there is $f \in L^2(A)$ such that $(L_h f)(a,b) = F(a,b)$. Then by (11) with $g = U(a,b)h$,

$$\begin{aligned} F(a,b) = (L_h f)(a,b) = \langle f, U(a,b)h\rangle &= \frac{1}{C_h}\,\langle L_h f, L_h g\rangle \\ &= \frac{1}{C_h}\int_G (L_h f)(a',b')\,\overline{(L_h g)(a',b')}\,d(a',b'). \end{aligned}$$

Now by taking

$$K(a,b\,;a',b') = \overline{(L_h g)(a',b')},$$

we have

$$F(a,b) = \frac{1}{C_h}\int_G F(a',b')K(a,b\,;a',b')d(a',b'). \tag{16}$$

This shows that the image of $L_h(a,b)$ is a reproducing kernel Hilbert space embedded as a close subspace of $L^2(G, \frac{1}{C_h}d(a,b))$, where by Corollary 3,

$$K(a,b\,;a',b') = \overline{[L_h J_a T_b h]\,(a',b')} = \overline{(L_h h)(\Gamma_{b^{-1}}(a^{-1}a'), b^{-1}b')}$$

is the reproducing kernel. □

Now, we will develop a sampling formula for functions in the reproducing Hilbert Space $Im(L_h(a,b))$. Thus, for $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ in A, consider a countable set $\{(a^{(k)}, b^{(k)})\}_{k=1,2,\ldots}$ in $G = A \times B$ such that $\{U(a^{(k)}, b^{(k)})h\}_{k=1,2,\ldots}$ is an orthonormal basis of $L^2(A)$ where h is admissible in $L^2(A)$. Then since the wavelet transform is an isometry (11), it follows that $\{L_h U(a^{(k)}, b^{(k)})h\}_{k=1,2,\ldots}$ is an orthonormal basis for $Im(L_h(a,b))$. Then we have the following expression for the L^2 norm for $L_h U(a,b)h$.

Lemma 22. *For (a,b) in G and an admissible function h in $L^2(A)$, set $g = U(a,b)h$, and for a given countable set $\{(a^{(k)}, b^{(k)})\}_{k=1,2,\ldots}$ in G, set $g_k = U(a^{(k)}, b^{(k)})h$. Then*

$$||L_h g||^2_{L^2(G)} = C_h^2 \sum_{k=1}^{\infty} |(L_h g_k)(a,b)|^2.$$

Proof. Since $\{L_h g_k\}_{k=1,2,\ldots}$ is an orthonormal basis for $Im(L_h(a,b))$ and $(L_h g)(a,b) \in Im(L_h(a,b))$, it follows that

$$(L_h g)(a,b) = \sum_{k=1}^{\infty} d_k (L_h g_k)(a,b), \tag{17}$$

where $d_k = \langle L_h g, L_h g_k \rangle$.

Note that from (11),

$$\begin{aligned} d_k &= \langle L_h g, L_h g_k \rangle = C_h \langle g, g_k \rangle = C_h \overline{\langle g_k, g \rangle} \\ &= C_h \overline{\langle g_k, U(a,b)h \rangle} = C_h \overline{(L_h g_k)(a,b)}. \end{aligned} \tag{18}$$

Then by (17) and (18),

$$\begin{aligned} ||L_h g||^2 &= \langle L_h g, L_h g \rangle = \left\langle \sum_{k=1}^{\infty} d_k L_h g_k, \sum_{k=1}^{\infty} d_k L_h g_k \right\rangle \\ &= \sum_{k=1}^{\infty} \overline{d_k} \left\langle \sum_{k=1}^{\infty} d_k L_h g_k, L_h g_k \right\rangle = \sum_{k=1}^{\infty} \overline{d_k} d_k \,\langle L_h g_k, L_h g_k \rangle \\ &= \sum_{k=1}^{\infty} \overline{d_k} d_k = \sum_{k=1}^{\infty} |d_k|^2 = C_h^2 \sum_{k=1}^{\infty} |(L_h g_k)(a,b)|^2 . \end{aligned}$$

□

Lemma 23. *Suppose that h in $L^2(A)$ is admissible. Then the series representation for $L_h g$ in $Im(L_h(a,b))$, where $g = U(a,b)h$ is absolutely convergent in G.*

Proof. Since $\{L_h g_k\}_{k=1,2,\ldots}$ is an orthonormal basis for $Im(L_h(a,b))$, it follows from (17) that,

$$(L_h g)(a,b) = \sum_{k=1}^{\infty} d_k (L_h g_k)(a,b),$$

where $d_k = C_h \overline{(L_h g_k)(a,b)}$. So, the series converges in $L^2(G)$.

Then from Lemma 22,

$$\begin{aligned}\sum_{k=1}^{\infty} |d_k\ (L_h g_k)(a,b)| &= \sum_{k=1}^{\infty} \left| C_h \overline{(L_h g_k)(a,b)} (L_h g_k)(a,b) \right| \\ &= C_h \sum_{k=1}^{\infty} |(L_h g_k)(a,b)|^2 = C_h \frac{1}{C_h^2} ||L_h g||^2 = \frac{1}{C_h} ||L_h g||^2 < \infty.\end{aligned}$$

Thus, the series converges absolutely in G. □

Now, we will give our second main result.

Theorem 2. *For a wavelet h in $L^2(A)$ we have that any F in $Im(L_h(a,b))$ can be reconstructed in terms of its sampled values $\{F(a^{(k)}, b^{(k)})\}_{k=1,2,\ldots}$ by the following interpolation formula*

$$F(a,b) = C_h \sum_{k=1}^{\infty} F(a^{(k)}, b^{(k)}) (L_h g_k)(a,b), \tag{19}$$

where $g_k = U(a^{(k)}, b^{(k)})h$.

Proof. Since $\{L_h g_k\}_{k=1,2,\ldots}$ is an orthonormal basis for $Im(L_h(a,b))$, it follows that for F in $Im(L_h(a,b))$,

$$F(a,b) = \sum_{k=1}^{\infty} \langle F, L_h g_k \rangle_{L^2(G)} (L_h g_k)(a,b).$$

First note that from Lemma 23, this series representation for F is absolutely convergent. Moreover,

$$F(a,b) = \sum_{k=1}^{\infty} \left[\int_G F(a',b') \overline{(L_h g_k)(a',b')} d(a',b') \right] (L_h g_k)(a,b).$$

Then by Theorem 1,

$$\begin{aligned}F(a,b) &= \sum_{k=1}^{\infty} \left[\int_G F(a',b') K(a^{(k)}, b^{(k)}; a', b')\, d(a',b') \right] (L_h g_k)(a,b) \\ &= \sum_{k=1}^{\infty} C_h F(a^{(k)}, b^{(k)})\, (L_h g_k)(a,b).\end{aligned}$$

□

12. Characterization of the wavelet transform

In this section, we will see that the wavelet transform can be written in different ways by using the co-variance properties, the Fourier transform, and the convolution of two functions.

Lemma 24. *Suppose $f \in L^2(A)$ and h is admissible in $L^2(A)$, then*

$$1)\quad (L_h f)(a,b) = (L_{J_{a'}h} J_{a'} f)(a' a \Gamma_b((a')^{-1}), b)$$
$$2)\quad (L_h f)(a,b) = (L_{T_{b'}h} T_{b'} f)(\Gamma_{b'}(a), b' b (b')^{-1}),$$

where (a,b) and (a',b') are in G.

Proof. It comes from Lemmas 20 and 21. □

Corollary 6. *Suppose $f \in L^2(A)$ and h in $L^2(A)$ is admissible, then*

$$(L_h f)(a,b) = (L_{J_{a'} T_{b'} h} J_{a'} T_{b'} f)(a' \Gamma_{b'}(a) \Gamma_b((a')^{-1}), b' b (b')^{-1}).$$

Proof. It comes from Corollary 5. □

Lemma 25. *If $f \in L^2(A)$ and h in $L^2(A)$ is admissible, then*

$$(L_h f)(a,b) = (L_{\widehat{h}} \widehat{f})(a^{-1}, b^{-1}).$$

Proof. By Corollary 2,

$$(L_{\widehat{h}} \widehat{f})(a^{-1}, b^{-1}) = \left\langle \widehat{f}, J_{a^{-1}} T_{b^{-1}} \widehat{h} \right\rangle = \left\langle \widehat{f}, \widehat{J_a T_b h} \right\rangle = \langle f, J_a T_b h \rangle = (L_h f)(a,b).$$

□

Lemma 26. *Suppose f and h in $L^2(A)$ are admissible, then*

$$(L_h f)(a,b) = (L_{\overline{f}} \overline{h})(a,b)^{-1}.$$

Proof. By Definition 13 and Lemma 3,

$$(L_h f)(a,b) = \langle f, J_a T_b h \rangle = \langle T_{b^{-1}} J_{a^{-1}} f, h \rangle = \left\langle J_{\Gamma_{b^{-1}}(a^{-1})} T_{b^{-1}} f, h \right\rangle$$
$$= \left\langle \overline{h}, J_{\Gamma_{b^{-1}}(a^{-1})} T_{b^{-1}} \overline{f} \right\rangle = (L_{\overline{f}} \overline{h})(\Gamma_{b^{-1}}(a^{-1}), b^{-1}) = (L_{\overline{f}} \overline{h})(a,b)^{-1}.$$

□

Definition 16. *If $f, g \in L^1(G)$, then the convolution of f and g is defined as the function*

$$(f * g)(x) = \int_G f(y) g(y^{-1}x) d\mu_G(y),$$

where $x, y \in G$

Lemma 27. *If $h \in C_0(A)$ is admissible and $f \in L^2(A)$, then*

$$(L_h f)(a,b) = \frac{1}{\sqrt{\chi(a, e_B)}} \left[f * (T_b \overline{h})^{\sim} \right](a), \tag{20}$$

where $\sim$ means $\psi^{\sim}(x) = \psi(x^{-1})$.

Proof.

$$\begin{aligned}
&\frac{1}{\sqrt{\chi(a,e_B)}}\left[f*(T_b\overline{h})^{\sim}\right](a)\\
&=\frac{1}{\sqrt{\chi(a,e_B)}}\int_A f(x)(T_b\overline{h})^{\sim}(x^{-1}a)d\mu_A(x)\\
&=\frac{1}{\sqrt{\chi(a,e_B)}}\int_A f(x)(T_b\overline{h})(a^{-1}x)d\mu_A(x)\\
&=\frac{1}{\sqrt{\chi(a,e_B)}}\int_A f(x)(T_b\overline{h})((a,e_B)^{-1}\cdot x)d\mu_A(x)\\
&=\frac{1}{\sqrt{\chi(a,e_B)}}\int_A f(x)\frac{1}{\sqrt{\chi(e_A,b)}}\overline{h}((e_A,b)^{-1}(a,e_B)^{-1}\cdot x)d\mu_A(x)\\
&=\int_A f(x)\frac{1}{\sqrt{\chi(a,b)}}\overline{h}((a,b)^{-1}\cdot x)d\mu_A(x)\\
&=(L_h f)(a,b).
\end{aligned}$$

□

Lemma 28. *If $f\in L^1(A)$ and $g\in C_0(A)$ is admissible, then $f*g$ is admissible.*

Proof. Note that since $f\in L^1(A)$ and $g\in L^2(A)$, it follows that $f*g\in L^2(A)$ and $||f*g||_2\le ||f||_1||g||_2$, [3]. Also since $\widehat{f*g}=\widehat{f}\widehat{g}$, [11], then

$$\begin{aligned}
C_{f*g}&\equiv\int_B\left|\widehat{f*g}(\rho\circ\Gamma_b)\right|^2 d\mu_B(b)=\int_B\left|\widehat{f}(\rho\circ\Gamma_b)\widehat{g}(\rho\circ\Gamma_b)\right|^2 d\mu_B(b)\\
&=\int_B|\widehat{f}(\rho\circ\Gamma_b)|^2\quad|\widehat{g}(\rho\circ\Gamma_b)|^2 d\mu_B(b)\le ||f||_1^2\int_B|\widehat{g}(\rho\circ\Gamma_b)|^2 d\mu_B(b)<\infty.
\end{aligned}$$

□

Corollary 7. *Suppose $h\in L^2(A)$ is admissible. If $f\in C_0(A)$ and $g\in C_0(A)$ is admissible, then*

$$(L_h(f*g))(a,b)=(L_{\overline{f}*\overline{g}}\overline{h})(a,b)^{-1} \tag{21}$$

Proof. By Lemma 29, $f*g$ is admissible. Then, the result comes from Lemma 26. □

13. Example

We will give now an example related to the definition of the abstract wavelet transform. According to section 2, let us consider the additive group $\mathbb{R}^n$ with identity $e_{\mathbb{R}^n}=(0,0,\dots,0)$ and the multiplicative group $\mathbb{R}^+$ with identity $e_{\mathbb{R}^+}=1$.

Now, let us take the homomorphism Γ from $\mathbb{R}^+$ to the group of all automorphisms of $\mathbb{R}^n$. That is, for each $s\in\mathbb{R}^+$, the map $\Gamma_s:\mathbb{R}^n\to\mathbb{R}^n$ is defined as $\Gamma_s(k_1,k_2,\dots,k_n)=(\Gamma_s(k_1),\Gamma_s(k_2),\dots,\Gamma_s(k_n))=(sk_1,sk_2,\dots,sk_n)$, where $(k_1,k_2,\dots,k_n)\in\mathbb{R}^n$.

In this case, the product in

$$G=\mathbb{R}^n\times\mathbb{R}^+=\left\{(k_1,k_2,\dots,k_n,s)|(k_1,k_2,\dots,k_n)\in\mathbb{R}^n\quad\text{and}\quad s\in\mathbb{R}^+\right\}$$

is defined as

$$\begin{aligned}(k_1, k_2, \ldots, k_n, s)(k'_1, k'_2, \ldots, k'_n, s') &= (k_1 + \Gamma_s(k'_1), k_2 + \Gamma_s(k'_2), \ldots, k_n + \Gamma_s(k'_n), ss') \\ &= (k_1 + sk'_1, k_2 + sk'_2, \ldots, k_n + sk'_n, ss').\end{aligned}$$

Note that with this product, G becomes a locally compact topological group where the identity is $e_G = (e_{\mathbb{R}^n}, e_{\mathbb{R}^+}) = (0, 0, \ldots, 0, 1)$ and

$$\begin{aligned}(k_1, k_2, \ldots, k_n, s)^{-1} &= (\Gamma_{s^{-1}}(-k_1), \Gamma_{s^{-1}}(-k_2), \ldots, \Gamma_{s^{-1}}(-k_n), s^{-1}) \\ &= (-s^{-1}k_1, -s^{-1}k_2, \ldots, -s^{-1}k_n, s^{-1}).\end{aligned}$$

Moreover, G acts on $\mathbb{R}^n$ with the following action $\cdot : G \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\begin{aligned}(k_1, k_2, \ldots, k_n, s) \cdot (x_1, x_2, \ldots, x_n) &= (k_1 + \Gamma_s(x_1), k_2 + \Gamma_s(x_2), \ldots, k_n + \Gamma_s(x_n)) \\ &= (k_1 + sx_1, k_2 + sx_2, \ldots, k_n + sx_n),\end{aligned}$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

On the other hand, the function $\chi : G \to (0, \infty)$ satisfying

$$\int_{\mathbb{R}^n} h((k_1, k_2, \ldots, k_n, s)^{-1} \cdot x) d\mu_{\mathbb{R}^n}(x) = \chi(k_1, k_2, \ldots, k_n, s) \int_{\mathbb{R}^n} h(x) d\mu_{\mathbb{R}^n}(x)$$

is given by $\chi(k_1, k_2, \ldots, k_n, s) = s^n$, where $h \in L^1(\mathbb{R}^n)$.

Also note that $d\mu_{\mathbb{R}^n}(x) = dx$, and

$$\begin{aligned}&(k_1, k_2, \ldots, k_n, s)^{-1} \cdot x = (-s^{-1}k_1, -s^{-1}k_2, \ldots, -s^{-1}k_n) \cdot (x_1, x_2, \ldots, x_n) \\ &= (-s^{-1}k_1 + \Gamma_{s^{-1}}(x_1), -s^{-1}k_2 + \Gamma_{s^{-1}}(x_2), \ldots, -s^{-1}k_n + \Gamma_{s^{-1}}(x_n)) \\ &(-s^{-1}k_1 + s^{-1}x_1, -s^{-1}k_2 + s^{-1}x_2, \ldots, -s^{-1}k_n + s^{-1}x_n) \\ &= \left(\frac{x_1 - k_1}{s}, \frac{x_2 - k_2}{s}, \ldots, \frac{x_n - k_n}{s}\right).\end{aligned}$$

Now following Section 5, for $(k_1, k_2, \ldots, k_n, s) \in G$ define the family of two operators $U(k_1, k_2, \ldots, k_n, s) = J_{(k_1, k_2, \ldots, k_n)} T_s$. Then for $h \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}&(J_{(k_1, k_2, \ldots, k_n)}h)(x) = \frac{1}{\sqrt{\chi(k_1, k_2, \ldots, k_n, 1)}} h\left((k_1, k_2, \ldots, k_n, 1)^{-1} \cdot x\right) \\ &= \frac{1}{\sqrt{(1)^n}} h\left(\frac{x_1 - k_1}{1}, \frac{x_2 - k_2}{1}, \ldots, \frac{x_n - k_n}{1}\right) \\ &= h(x_1 - k_1, x_2 - k_2, \ldots, x_n - k_n),\end{aligned}$$

where $x \in \mathbb{R}^n$ and $(k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$, and

$$\begin{aligned}(T_s h)(x) &= \frac{1}{\sqrt{\chi(0, 0, \ldots, 0, s)}} h\left((0, 0, \ldots, 0, s)^{-1} \cdot x\right) \\ &= \frac{1}{\sqrt{(s)^n}} h\left(\frac{x_1 - 0}{s}, \frac{x_2 - 0}{s}, \ldots, \frac{x_n - 0}{s}\right) = \frac{1}{(s)^{\frac{n}{2}}} h\left(\frac{x_1}{s}, \frac{x_2}{s}, \ldots, \frac{x_n}{s}\right),\end{aligned}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and $s \in \mathbb{R}^+$.

Then $U(k_1, k_2, \ldots, k_n, s)$ is a unitary representation of G acting on $L^2(\mathbb{R}^n)$ by

$$[U(k_1, k_2, \ldots, k_n, s)h](x) = (J_{(k_1,k_2,\ldots,k_n)} T_s h)(x)$$
$$= (T_s h)(x_1 - k_1, x_2 - k_2, \ldots, x_n - k_n) = \frac{1}{s^{\frac{n}{2}}} h\left(\frac{x_1 - k_1}{s}, \frac{x_2 - k_2}{s}, \ldots, \frac{x_n - k_n}{s}\right).$$

Moreover since the left Haar measure on $\mathbb{R}^+$ is $d\mu_{\mathbb{R}^+} = \frac{1}{s} ds$, it follows from Lemma 8, that the left Haar measure on G is

$$d(k_1, k_2, \ldots, k_n, s) = \frac{1}{\chi(k_1, k_2, \ldots, k_n, s)} d\mu_{\mathbb{R}^n}(k_1, k_2, \ldots, k_n)\, d\mu_{\mathbb{R}^+}(s)$$
$$= \frac{1}{s^n} d(k_1, k_2, \ldots, k_n)\, \frac{1}{s} ds = \frac{1}{s^{n+1}} d(k_1, k_2, \ldots, k_n) ds.$$

Thus, the admissibility condition for a radially symmetric function $h \in L^2(\mathbb{R}^n)$ (Lemma 10) becomes

$$C_h \equiv \int_{\mathbb{R}^+} \left|\widehat{h}(\rho \circ \Gamma_s)\right|^2 d\mu_{\mathbb{R}^+}(s) = \int_{\mathbb{R}^+} |\widehat{h}(\rho s)|^2 \frac{1}{s} ds,$$

where $\rho \in \mathbb{R}^n$, and $\widehat{h}(r) = \widehat{\eta}(|r|)$. Then, if we apply the change of variable $y = |\rho s|$, we get

$$C_h \equiv \int_{\mathbb{R}^+} |\widehat{\eta}(y)|^2 \frac{1}{y} dy.$$

Hence, for $(k_1, k_2, \ldots, k_n, s) \in G$ and h admissible in $L^2(\mathbb{R}^n)$, the continuous wavelet transform for $f \in L^2(\mathbb{R}^n)$ with respect to h is given by

$$(L_h f)(k_1, k_2, \ldots, k_n, s) = \left\langle f, J_{(k_1,k_2,\ldots,k_n)} T_s h \right\rangle$$
$$= \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n) \frac{1}{s^{\frac{n}{2}}} \overline{h\left(\frac{x_1 - k_1}{s}, \frac{x_2 - k_2}{s}, \ldots, \frac{x_n - k_n}{s}\right)} d(x_1, x_2, \ldots, x_n)$$
$$= \int_{\mathbb{R}^n} f(x) \frac{1}{s^{\frac{n}{2}}} \overline{h\left(\frac{x - k}{s}\right)} dx,$$

which agrees with the one given in [1].

Also, the inverse formula according to Lemma 17 is given by

$$f = \frac{1}{C_h} \int_G (L_h f)(k_1, k_2, \ldots, k_n, s)[U(k_1, k_2, \ldots, k_n, s)h] d(k_1, k_2, \ldots, k_n, s)$$

in the weak sense.

That is,

$$
\begin{aligned}
f &= \frac{1}{C_h} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} \langle f, U(k_1, k_2, \ldots, k_n, s)h \rangle \, U(k_1, k_2, \ldots, k_n, s)h \, d(k_1, k_2, \ldots, k_n) \frac{ds}{s^{n+1}} \\
&= \frac{1}{C_h} \int_G \langle f, U(k, s)h \rangle \, U(k, s)h \, d(k, s),
\end{aligned}
$$

which also agrees with the one given in [8].

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On the Green Function of the Operator $(\boxplus_B + m^6)^k$ Related to the Bessel Helmholtz Operator and the Bessel Klein-Gordon Operator

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Abstract

In this article, we study the Green function of the form $(\boxplus_B + m^6)^k$ which is iterated k-times and is defined by

$$(\boxplus_B + m^6)^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^3 + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^3 + m^6\right]^k$$

where $p+q=n$ is the dimension of $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$, m is a positive real number and k is a positive integer. At first, we study the Green function or elementary solution of the operator $(\boxplus_B + m^6)^k$. Moreover, the operator $(\boxplus_B + m^6)^k$ can be related to the Bessel Helmholtz operator $(\triangle_B + m^2)^k$ and the Bessel ultra-hyperbolic Klein-Gordon operator $(\square_B + m^2)^k$. After that, we apply such a Green function to solve the solution of the equation $(\boxplus_B + m^6)^k u(x) = f(x)$ where f is a generalized function and $u(x)$ is an unknown function for $x \in \mathbb{R}^n$.

Keywords: Green function, Bessel Helmholtz operator, Bessel Ultra-hyperbolic Klein-Gordon operator

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1 Introduction

The operator $\boxplus_B$ can be expressed in the form

$$
\begin{aligned}
\boxplus_B &= \left(\sum_{i=1}^{p} B_{x_i}\right)^3 + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^3 \\
&= \left[\sum_{i=1}^{p} B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i}\right]\left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \sum_{i=1}^{p} B_{x_i}\sum_{i=p+1}^{p+q} B_{x_i} + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^2\right] \\
&= \triangle_B\left[\triangle_B^2 - \frac{3}{4}\left(\triangle_B + \Box_B\right)\left(\triangle_B - \Box_B\right)\right] \\
&= \frac{3}{4}\triangle_B\Box_B^2 + \frac{1}{4}\triangle_B^3 \qquad (1.1)
\end{aligned}
$$

where $p+q=n$ is the dimension of $\mathbb{R}_n^+$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$, $\triangle_B$ is the Laplace-Bessel operator which is defined by

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_n}, \qquad (1.2)$$

and $\Box$ is the Bessel ultra-hyperbolic operator which is defined by

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}. \qquad (1.3)$$

Furthermore, Yildirim et al. [8] first introduced the Bessel diamond operator $\Diamond_B$ which is defined by

$$\Diamond_B = \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + B_{x_{p+2}} + \cdots + B_{x_{p+q}}\right)^2. \qquad (1.4)$$

The Bessel diamond operator can also be expressed in the form $\Diamond_B = \triangle_B\Box_B = \Box_B\triangle_B$, from (1.1) we have

$$\boxplus_B = \frac{3}{4}\Diamond_B\Box_B + \frac{1}{4}\triangle_B^3. \qquad (1.5)$$

Later, Bunpog and Kananthai [2] have studied the elementary solution or Green function of the operator $(\Diamond_B + m^4)^k$ which related to the Bessel Helmholtz operator $\triangle_B + m^2$ and the Bessel Klein-Gordon operator $\Box_B + m^2$ and obtained the function

$$G(x) = \left(T_{2k}(x) * W_{2k}(x)\right) * \left(C^{*k}(x)\right)^{-1} \qquad (1.6)$$

which is a Green function for the operator

$$(\Diamond_B + m^4)^k = \left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^2 + m^4\right]^k, \qquad (1.7)$$

where the symbol $*k$ denotes the convolution of itself k-times and the symbol $*-1$ is an inverse of the convolution algebra, $T_{2k}(x)$ is the elementary solution of the Bessel Helmholtz operator $(\triangle_B + m^2)^k$ iterated k-times, that is $T_{2k}(x)$ satisfy the equation

$$(\triangle_B + m^2)^k u(x) = \delta(x), \tag{1.8}$$

$W_{2k}(x)$ is the elementary solution of the Bessel Klein-Gordon operator $(\Box_B + m^2)^k$ iterated k-times, that is $W_{2k}(x)$ satisfy the equation

$$(\Box_B + m^2)^k u(x) = \delta(x) \tag{1.9}$$

and $C(x)$ is defined by

$$C(x) = \delta(x) - m^2\left(T_2(x) + W_2(x)\right) + 2m^4\left(T_2(u) * W_2(x)\right). \tag{1.10}$$

The purpose of this work is to find the elementary solution or Green function of the operator $(\boxplus_B + m^6)^k$, that is

$$(\boxplus_B + m^6)^k G(x) = \delta(x), \tag{1.11}$$

where $G(x)$ is the Green function, δ is the Direc-delta distribution, m is a positive real number, k is a nonnegative integer and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$. We then find the solution of the equation $(\boxplus_B + m^6)^k u(x) = f(x)$ where f is a given generalized function and $u(x)$ is an unknown function.

2 Preliminaries

Before reaching the main results, the following definitions and the basic concepts are needed. At first, the generalized shift operator T_x^y has the following form [5],

$$T_x^y = C_v^* \int_0^{\pi} \cdots \int_0^{\pi} \varphi(s_1, \ldots, s_n)\left(\prod_{i=1}^{n} \sin^{2v_i - 1}\theta_i\right) d\theta_1 \cdots d\theta_n,$$

where $s_i^2 = x_i^2 + y_i^2 - 2x_i y_i \cos\theta_i$, $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^{n} \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [5],

$$\begin{aligned} \frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i}\frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i}\frac{d\varphi}{dy_i}, \\ \varphi(x_i, 0) &= f(x), \\ \varphi_{y_i}(x_i, 0) &= 0, \end{aligned}$$

where $x_i, y_i \in \mathbb{R}_n^+$ for $i = 1, 2, \ldots, n$. The convolution operator determined by the T_x^y is as follows

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy. \tag{2.1}$$

Convolution in (2.1) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator,

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i} \right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows [7],

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \; C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma \left(v_i + \frac{1}{2} \right) \right)^{-1},$$

where $J_{v_i - \frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are following equalities for Fourier-Bessel transformation [7],

$$F_B \delta(x) = 1 \quad \text{and} \quad F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Lemma 2.1 *There is a following equality for Fourier-Bessel transformation*

$$F_B(|x|^{-\alpha}) = 2^{n+2|v|-2\alpha}\Gamma\left(\frac{n+2|v|-\alpha}{2}\right)\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1}|x|^{\alpha-n-2|v|},$$

where $|v| = v_1 + v_2 + \cdots + v_n$.

Proof. See [7]. □

Lemma 2.2 *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B^k$ *is the Laplace-Bessel operator iterated* k*-times defined by* (1.2). *Then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* $\triangle_B^k$, *where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k}\Gamma(\frac{n+2|v|-2k}{2})}{\Pi_{i=1}^n 2^{v_i-\frac{1}{2}}\Gamma(v_i+\frac{1}{2})\Gamma(k)}|x|^{2k-n-2|v|}. \tag{2.2}$$

Proof. See [8]. □

Lemma 2.3 *Given the equation* $\Box_B^k u(x) = \delta(x)$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \cdots, x_n > 0$ *and* $V > 0\}$, *where* $\Box_B^k$ *is the Bessel-ultra-hyperbolic operator iterated* k*-times defined by* (1.3). *Then* $u(x) = R_{2k}(x)$ *is an elementary solution of the operator* $\Box_B^k$, *where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-2|v|}{2}}}{K_n(2k)} \tag{2.3}$$

for

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma(\frac{2+2k-n-2|v|}{2})\Gamma(\frac{1-2k}{2})\Gamma(2k)}{\Gamma(\frac{2+2k-p-2|v|}{2})\Gamma(\frac{p-2k}{2})}$$

Proof. See [8]. □

Lemma 2.4 *The functions* $S_{2k}(x)$ *and* $R_{2k}(x)$ *are homogeneous distributions of order* $(2k-n-2|v|)$ *for* $Re(2k) < n+2|v|$. *In particular, the* B*-convolution* $S_{2k}(x) * R_{2k}(x)$ *exists and is a tempered distribution.*

Proof. See [8]. □

Lemma 2.5 (The elementary solution of Bessel Helmholtz operator)

Given the equation $(\triangle_B + m^2)^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B$ *is defined by* (1.2). *Then* $u(x) = T_{2k}(x,m)$ *is an elementary solution of the operator* $(\triangle_B + m^2)^k$ *where*

$$T_{2k}(x,m) = \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r(-1)^{k+r}S_{2k+2r}(x) \tag{2.4}$$

for S_{2k+2r} *is defined by* (2.2).

Proof. Since the operator $\triangle_B$ is a linearly continuous and have $1-1$ mapping, it has an inverse. By Lemma 2.2, we obtain

$$T_{2k}(x,m) = \sum_{r=0}^{\infty} \left(\begin{array}{c} -k \\ r \end{array} \right) (m^2)^r \triangle_B^{-k-r}\delta(x) = (\triangle_B + m^2)^{-k}\delta(x),$$

where $(\triangle_B + m^2)^{-k}$ is the inverse operator of the operator $(\triangle_B + m^2)^k$. By applying the operator $(\triangle_B + m^2)^k$ to both sides of the above equation, we have

$$(\triangle_B + m^2)^k T_{2k}(x,m) = (\triangle_B + m^2)^k (\triangle_B + m^2)^{-k}\delta(x).$$

Therefore,

$$(\triangle_B + m^2)^k T_{2k}(x,m) = \delta(x).$$

This completes the proof. □

Lemma 2.6 (The elementary solution of Bessel Klein-Gordon operator)
Given the equation $(\square_B + m^2)^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\square_B$ is defined by (1.3)*. Then $u(x) = W_{2k}(x,m)$ is an elementary solution of the operator $(\square_B + m^2)^k$ where*

$$W_{2k}(x,m) = \sum_{r=0}^{\infty} \left(\begin{array}{c} -k \\ r \end{array} \right) (m^2)^r R_{2k+2r}(x) \tag{2.5}$$

for R_{2k+2r} is defined by (2.3)*.*

Proof. The proof of Lemma 2.6 is similar to the proof of lemma 2.5. □

Lemma 2.7 (The B-convolution of tempered distribution)
Let k and r be nonnegative integer.

(a) *Let $T_{2k}(x,m)$ and $T_{2r}(x,m)$ be defined by* (2.4)*, then*

$$T_{2k}(x,m) * T_{2r}(x,m) = T_{2k+2r}(x,m).$$

(b) *Let $W_{2k}(x,m)$ and $W_{2r}(x,m)$ be defined by* (2.5)*, then*

$$W_{2k}(x,m) * W_{2r}(x,m) = W_{2k+2r}(x,m).$$

(c) *Let $S_{2k}(x)$ and $S_{2r}(x)$ be defined by* (2.2)*, then*

$$S_{2k}(x) * S_{2r}(x) = S_{2k+2r}(x).$$

(d) *Let $R_{2k}(x)$ and $R_{2r}(x)$ be defined by* (2.3)*, then*

$$R_{2k}(x) * R_{2r}(x) = R_{2k+2r}(x).$$

Proof. (a) From the equation $(\triangle_B+m^2)^{k+r}G(x)=\delta(x)$, we obtain $G(x)=T_{2k+2r}(u,m)$ by Lemma 2.5. For any nonnegative integer r, we write

$$(\triangle_B+m^2)^{k+r}G(x)=(\triangle_B+m^2)^k(\triangle_B+m^2)^rG(x)=\delta(x),$$

then by Lemma 2.5 again we have the following equation

$$(\triangle_B+m^2)^rG(x)=T_{2k}(u,m).$$

Convoluing both sides of the above equation by $W_{2r}(u,m)$, we obtain

$$T_{2r}(u,m)*(\triangle_B+m^2)^rG(x)=T_{2r}(u,m)*T_{2k}(u,m)$$

or

$$(\triangle_B+m^2)^rT_{2r}(u,m)*G(x)=T_{2r}(u,m)*T_{2k}(u,m).$$

Hence, by Lemma 2.5 we have

$$\delta(x)*G(x)=T_{2r}(u,m)*T_{2k}(u,m).$$

It follows that

$$G(x)=T_{2r}(u,m)*T_{2k}(u,m).$$

From the fact that $G(x)=T_{2k+2r}(u,m)$, we obtain

$$T_{2k}(u,m)*T_{2r}(u,m)=T_{2k+2r}(u,m).$$

The proof of (b), (c) and (d) are similar to (a). □

Lemma 2.8 *Let k and r be nonnegative integer.*

(a) *Let $S_{2r}(x)$ and $S_{2r-2k}(x)$ be defined by (2.2), then $\triangle_B^kS_{2r}(x)=(-1)^kS_{2r-2k}(x)$.*

(b)*Let $R_{2r}(x)$ and $R_{2r-2k}(x)$ be defined by (2.3), then $\square_B^kR_{2r}(x)=R_{2r-2k}(x)$.*

Proof. (a) By Lemma 2.7 (c), we obtain

$$S_{2k}(x)*S_{2r-2k}(x)=\delta(x)*S_{2r}(x).$$

By Lemma 2.3 we have

$$S_{2k}(x)*S_{2r-2k}(x)=\triangle_B^k(-1)^kS_{2k}(x)*S_{2r}(x)=S_{2k}(x)*\triangle_B^k(-1)^kS_{2r}(x).$$

Therefore,

$$\triangle_B^kS_{2r}(x)=(-1)^kS_{2r-2k}(x).$$

(b) The proof is similar to (a). □

Lemma 2.9 *Let k be nonnegative integer.*

(a) *Let $T_{2k}(x,m)$ and $S_{-2k}(x)$ be defined by* (2.4) *and* (2.2) *respectively, then*

$$\triangle_B^k T_{2k}(x,m) = T_{2k}(x,m) * (-1)^k S_{-2k}(x).$$

(b) *Let $W_{2k}(x,m)$ and $R_{-2k}(x)$ be defined by* (2.5) *and* (2.3) *respectively, then*

$$\Box_B^k W_{2k}(x,m) = W_{2k}(x,m) * R_{-2k}(x).$$

Proof. (a) From (2.4) we have

$$\triangle_B^k T_{2k}(x,m) = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} \triangle_B^k S_{2k+2r}(x).$$

By Lemma 2.8 (a), we have

$$\triangle_B^k T_{2k}(x,m) = T_{2k}(x,m) * (-1)^k S_{-2k}(x).$$

(b) The proof is similar to (a). □

Lemma 2.10 (The existence of the convolution $T_{6k}(x,m) * W_{4k}(x,m)$) *The convolution $T_{6k}(x,m) * W_{4k}(x,m)$ exists and is a tempered distribution where $T_{6k}(x,m) = T_{2k}(x,m) * T_{2k}(x,m) * T_{2k}(x,m)$ and $W_{4k}(x,m) = W_{2k}(x,m) * W_{2k}(x,m)$ such that $T_{2k}(x,m)$ and $W_{2k}(x,m)$ are defined by* (2.4) *and* (2.5), *respectively.*

Proof. From (2.4) and (2.5), we have

$$\begin{aligned}
&T_{2k}(x,m) * W_{2k}(x,m) \\
&\quad = \left(\sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} S_{2k+2r}(x) \right) * \left(\sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R_{2k+2r}(x) \right) \\
&\quad = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \binom{-k}{r} \binom{-k}{s} (m^2)^{r+s} (-1)^{k+r} S_{2k+2r}(x) * R_{2k+2s}(x).
\end{aligned}$$

By Lemma 2.6, the B-convolution of $S_{2k+2r}(x) * R_{2k+2r}(x)$ exists and is also a tempered distribution. Then $T_{2k}(x,m) * W_{2k}(x,m)$ exists and is also a tempered distribution. Since $T_{2k}(x,m)$, $W_{2k}(x,m)$ and $T_{2k}(x,m) * W_{2k}(x,m)$ exists and is also a tempered distribution, by Donoghue [2, p. 152] we obtain $T_{6k}(x,m) * W_{4k}(x,m)$ exists and is also a tempered distribution. □

Lemma 2.11 *Let $T_6(x,m)$ and $W_4(x,m)$ be defined by* (2.4) *and* (2.5) *with $k = 3$ and $k = 2$, respectively. Then*

(a) $(\triangle_B + m^2)(\Box_B + m^2)^2 (T_6(x,m) * W_4(x,m)) = T_4(x,m)$

(b) $(\triangle_B + m^2)^3 (T_6(x,m) * W_4(x,m)) = W_4(x,m)$

(c) $(\triangle_B + \Box_B)(T_6(x,m) * W_4(x,m)) = (T_6(x,m) * W_4(x,m)) * (R_{-2}(x) - S_{-2}(x))$

(d) $(\triangle_B + \Box_B)^2 (T_6(x,m) * W_4(x,m))$

$$= (T_6(x,m) * W_4(x,m)) * (S_{-4}(x) - 2S_{-2}(x) * R_{-2}(x) + R_{-4}(x))$$

where $S_{-2}(x), S_{-4}(x)$ are defined by (2.2) and $R_{-2}(v), R_{-4}(x)$ are defined by (2.3).

Proof. (a) We have

$$\begin{aligned}
&\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 (T_6(x,m) * W_4(x,m)) \\
&\qquad = \left(\triangle_B + m^2\right) T_2(x,m) * T_4(x,m) * \left(\Box_B + m^2\right)^2 W_4(x,m) \\
&\qquad = \delta(x) * T_4(x,m) * \delta(x), \quad \text{by Lemma 2.5 and 2.6,} \\
&\qquad = T_4(x,m).
\end{aligned}$$

(b) We get

$$\begin{aligned}
\left(\triangle_B + m^2\right)^3 (T_6(x,m) * W_4(x,m)) &= \left(\triangle_B + m^2\right)^3 T_6(x,m) * W_4(x,m) \\
&= \delta(x) * W_4(x,m), \quad \text{by Lemma 2.5,} \\
&= W_4(x,m).
\end{aligned}$$

(c) We obtain

$$\begin{aligned}
&(\triangle_B + \Box_B)(T_6(x,m) * W_4(x,m)) \\
&= \triangle_B T_2(x,m) * T_4(x,m) * W_4(x,m) + T_6(x,m) * \Box_B W_2(x,m) * W_2(x,m) \\
&= T_6(x,m) * W_4(x,m) * (-1)S_{-2}(x) + T_6(x,m) * W_4(x,m) * R_{-2}(x), \text{ by Lemma 2.9} \\
&= (T_6(x,m) * W_4(x,m)) * (R_{-2}(x) - S_{-2}(x)).
\end{aligned}$$

(d) We have

$$\begin{aligned}
&(\triangle_B + \Box_B)^2 (T_6(x,m) * W_4(x,m)) \\
&= \left(\triangle_B^2 + 2\triangle_B \Box_B + \Box_B^2\right)(T_6(x,m) * W_4(x,m)) \\
&= \triangle_B^2 T_4(x,m) * T_2(x,m) * W_4(x,m) + 2\triangle_B T_2(x,m) * T_4(x,m) * \Box_B W_2(x,m) * W_2(x,m) \\
&\quad + T_6(x,m) * \Box_B^2 W_4(x,m) \\
&= T_6(x,m) * W_4(x,m) * (-1)^2 S_{-4}(x) + 2T_6(x,m) * W_4(x,m) * (-1)S_{-2}(x) * R_{-2}(x) \\
&\quad + T_6(x,m) * W_4(x,m) * R_{-4}(x), \text{ by Lemma 2.9} \\
&= (T_6(x,m) * W_4(x,m)) * (S_{-4}(x) - 2S_{-2}(x) * R_{-2}(x) + R_{-4}(x)).
\end{aligned}$$

$\Box$

3 Main results

Theorem 3.1 *Given the equation*

$$(\boxplus_B + m^6)^k G(x) = \delta(x), \tag{3.1}$$

then $G(x) = T_{6k}(x,m) * W_{4k}(x,m) * \left(C^{*k}(x)\right)^{*-1}$ *is a Green function for the operator* $(\boxplus_B + m^6)^k$ *iterated k-times where* $\boxplus_B$ *is defined by* (1.1), δ *is the Direc-delta distribution,* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, k *is a nonnegative integer,* m *is a nonnegative real number and*

$$\begin{aligned} C(x) = \frac{3}{4}T_4(x,m) + \frac{1}{4}W_4(x,m) - \frac{3m^4}{2}\left(T_4(x,m) * W_6(x,m)\right) * \left(R_{-2}(x) - S_{-2}(x)\right) \\ - \frac{3m^2}{4}\left(T_4(x,m) * W_6(x,m)\right) * \left(S_{-4}(x) - 2S_{-2}(x) * R_{-2}(x) + R_{-4}(x)\right). \end{aligned} \tag{3.2}$$

$C^{*k}(x)$ *denotes the convolution of* C *itself k-times,* $\left(C^{*k}(x)\right)^{*-1}$ *denotes the inverse of* $C^{*k}(x)$ *in the convolution algebra. Moreover* $C^{*k}(x)$ *is a tempered distribution.*

Proof. Since

$$\boxplus_B + m^6 = \frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3 - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right) - \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2,$$

by (3.1) we have

$$\begin{aligned} \delta(x) &= (\boxplus_B + m^6)(\boxplus_B + m^6)^{k-1}G(x) \\ &= \left[\frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3 - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right)\right. \\ &\quad \left. - \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2\right]\left[\frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3\right. \\ &\quad \left. - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right) - \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2\right]^{k-1} G(x). \end{aligned} \tag{3.3}$$

By Lemma 2.10 with $k = 1$, we have $T_6(x,m) * W_4(x,m)$ exists and is a tempered distribution. Convolving both sides of (3.3) by $T_6(x,m) * W_4(x,m)$, we obtain

$$\begin{aligned} &\left[\frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3 - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right) - \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2\right] \\ &T_6(x,m) * W_4(x,m) * \left[\frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3 - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right)\right. \\ &\left. - \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2\right]^{k-1} G(x) = \left(T_6(x,m) * W_4(x,m)\right) * \delta(x). \end{aligned}$$

By Lemma 2.11, we have

$$\begin{aligned} C(x) * \bigg[&\frac{3}{4}\left(\triangle_B + m^2\right)\left(\Box_B + m^2\right)^2 + \frac{1}{4}\left(\triangle_B + m^2\right)^3 - \frac{3m^4}{2}\left(\triangle_B + \Box_B\right) \\ &- \frac{3m^2}{4}\left(\triangle_B + \Box_B\right)^2\bigg]^{k-1} G(x) = T_4(x,m) * W_6(x,m). \end{aligned}$$

Keeping on convolving both sides of the above equation by $T_6(x,m) * W_4(x,m)$ up to $k-1$ times, we have

$$C^{*k}(x) * G(x) = \left(T_6(x,m) * W_4(x,m)\right)^{*k},$$

where the symbol $*k$ denotes the convolution of itself k-times. By Tellez [6], we have

$$\left(T_6(x,m) * W_4(x,m)\right)^{*k} = T_{6k}(x,m) * W_{4k}(x,m),$$

and so

$$C^{*k}(x) * G(x) = T_{6k}(x,m) * W_{4k}(x,m). \tag{3.4}$$

Now, consider the function $C^{*k}(x)$, since $W_4(x,m)$, $T_4(x,m)$, $T_6(x,m) * W_4(x,m)$, $R_{-2}(x) - S_{-2}(x)$ and $S_{-4}(x) - 2S_{-2}(x) * R_{-2}(x) + R_{-4}(x)$ are lies in S' where S' is a space of tempered distribution, $C(x) \in S'$. By Donoghue [2, p. 152], we obtain $C^{*k}(x) \in S'$. Since $T_{6k}(x,m) * W_{4k}(x,m) \in S'$, choose $S' \subset D'_R$ where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $T_{6k}(x,m) * W_{4k}(x,m) \in D'_R$, it follows that $T_{6k}(x,m) * W_{4k}(x,m)$ is an element of the convolution algebra. Hence, by Zemanian [10, p. 150-151], the equation (3.4) has an unique solution

$$G(x) = T_{6k}(x,m) * W_{4k}(x,m) * \left(C^{*k}(x)\right)^{*-1}$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra, $G(x)$ is called the Green function of the operator $(\boxplus_B + m^6)^k$. Since $T_{6k}(x,m) * W_{4k}(x,m)$ and $\left(C^{*k}(x)\right)^{*-1}$ are tempered distribution, then by Donoghue [2, p. 152], we obtain $T_{6k}(x,m) * W_{4k}(x,m) * \left(C^{*k}(x)\right)^{*-1}$ is a tempered distribution. It follows that $G(x)$ is a tempered distribution. □

Theorem 3.2 (An application of Green function) *Given the equation*

$$(\boxplus_B + m^6)^k u(x) = f(x) \tag{3.5}$$

where f is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = G(x) * f(x)$$

is an unique solution of (3.5) where $G(x)$ is a Green function for $(\boxplus_B + m^6)^k$.

Proof. Convolving both sides of the equation (3.5) by $G(x)$ where $G(x)$ is a Green function for the operator $(\boxplus_B + m^6)^k$ in Theorem 3.1, we obtain

$$G(x) * f(x) = G(x) * (\boxplus_B + m^6)^k u(x) = (\boxplus_B + m^6)^k G(x) * u(x).$$

Applying the Theorem 3.1, we have

$$G(x) * f(x) = \delta(x) * u(x) = u(x).$$

Since $G(x)$ is an unique, $u(x)$ is an unique solution of the equation (3.5). □

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NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH SHIFT IN THE GENERALIZED HÖLDER SPACES

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Abstract.
This paper concerns sufficient conditions for the convergence of Newton-Kantorovich majorant method for the solution of a certain class of nonlinear singular integral equations with shift. A criterion for the Noetherity of a correspondence singular integral functional operator of second order with Carleman shift preserving orientation is obtained and the index formula is given.

Key words: Nonlinear singular integral equations, Newton-Kantorovich majorant method, Carleman shift, Generalized Hölder spaces.

0. Introduction

There is a large Literature on the classical theory of nonlinear singular integral equations (NSIE) (see [4],[6,7],[13],[15,16] and others).The development of the theory of singular integral equations (SIE) naturally stimulated the study of singular integral equations with shift (SIES).The Noether theory of singular integral operators with shift (SIOS) is developed for a closed and open contour (see [9],[10],[11] and others). Existence results and approximate solutions have been studied for (NSIE) and the nonlinear singular integral equations with shift (NSIES) by authors (see [1-3],[5],[8].[12]). In this paper, our aim is to apply the Newton-Kantorovich majorant method to a class of (NSIES) under certain conditions.

Consider the following nonlinear singular integral equation with Carleman shift(NSIES)

$$\begin{aligned}(\mathrm{T}(u))(t)=a(t)u(t)+b(t)u(\alpha(t))+c(t)u(\alpha_2(t))+\frac{d(t)}{\pi i}\int_L \frac{u(\tau)}{\tau-t}d\tau+\frac{e(t)}{\pi i}\int_L \frac{u(\tau)}{\tau-\alpha(t)}d\tau+ \\ +\frac{f(t)}{\pi i}\int_L \frac{u(\tau)}{\tau-\alpha_2(t)}d\tau-\frac{1}{\pi i}\int_L\left\{\frac{\Psi_1(\tau,u(\tau))}{\tau-t}+\frac{\Psi_2(\tau,u(\tau))}{\tau-\alpha(t)}+\frac{\Psi_3(\tau,u(\tau))}{\tau-\alpha_2(t)}\right\}d\tau=0,\end{aligned} \tag{0.1}$$

is a simple closed Lyapunov contour, dividing the complex plane into interior L where domain D^+ is unknown function and the homeomorphism $D^-, u(t)$ and exterior domain $\alpha: L\to L$ is a shift preserving orientation, satisfying the Carleman condition

$$\alpha_3(t)=\alpha(\alpha(\alpha(t)))=t, \qquad t\in L \tag{0.2}$$

The functions $\alpha'(t)\neq 0 \quad \forall\ t\in L$.satisfies usual Hölder condition, $\alpha'(t)$ whose derivative $a(t), b(t), c(t), d(t), e(t)$ and $f(t)$ belong to the generalized Hölder space $\mathrm{H}_{\phi,m}(L)$. Assume that the functions $\Psi_1(t,u(t)), \Psi_2(t,u(t))$ and $\Psi_3(t,u(t))$ have partial derivatives up to $(m-1)$ order, and satisfy the following conditions:

$$\left|\frac{\partial^k \Psi_1(t_1,u_1)}{\partial t^i\,\partial u^j}-\frac{\partial^k\,\Psi_1(t_2,u_2)}{\partial t^i\,\partial u^j}\right|\le c_k(r)\left\{\phi(\left|t_1-t_2\right|)+\left|u_1-u_2\right|\right\}, \tag{0.3}$$

$$\left|\frac{\partial^k \Psi_2(t_1,u_1)}{\partial t^i\,\partial u^j}-\frac{\partial^k\,\Psi_2(t_2,u_2)}{\partial t^i\,\partial u^j}\right|\le c'_k(r)\left\{\phi_1(\left|t_1-t_2\right|)+\left|u_1-u_2\right|\right\}, \tag{0.4}$$

and

$$\left|\frac{\partial^k \Psi_3(t_1,u_1)}{\partial t^i\,\partial u^j}-\frac{\partial^k\,\Psi_3(t_2,u_2)}{\partial t^i\,\partial u^j}\right|\le c''_k(r)\left\{\phi_2(\left|t_1-t_2\right|)+\left|u_1-u_2\right|\right\}, \tag{0.5}$$

where $\phi,\phi_1,\phi_2\in\Phi$, $i+j=k$, $k=0,1,...,m-1$ and $c_k(r),c'_k(r),c''_k(r)$ are positive increasing functions.
The functions $\Psi_1(t,u(t)),\Psi_2(t,u(t))$ and $\Psi_3(t,u(t))$ for any $\mathrm{H}_{\phi,m}(L)$ belong to the space $u\in\mathrm{H}_{\phi,m}(L)$, [14].

1. Formulation of the problem

Let $\mathrm{T}:\overline{S}(u_0,R)\subset\mathrm{X}\to\mathrm{Y}$ be a nonlinear operator defined on the closure of a ball $S(u_0,R)=\{u:u\in\mathrm{X},\|u-u_0\|<R\}$ in a Banach space X into a Banach space Y. We give new conditions to ensure the convergence of Newton-Kantorovich approximations toward a solution of $\mathrm{T}(u)=0$, under the hypothesis that T is Frechet differentiable in $S(u_0,R)$, and that its derivative T' satisfies the local Lipschitz condition:

$$\|\mathrm{T}'(u_1)-\mathrm{T}'(u_2)\|\le k(r)\|u_1-u_2\|,\qquad u_1,u_2\in\overline{S}(u_0,r),0<r<R \tag{1.1}$$

where $k(r)$ is a non-decreasing function on the interval [0,R] and

$$k(r)=\sup\left\{\frac{\|\mathrm{T}'(u_1)-\mathrm{T}'(u_2)\|}{\|u_1-u_2\|}:u_1,u_2\in\overline{S}(u_0,r),u_1\ne u_2\right\}. \tag{1.2}$$

Define a scalar function $\psi:[0,R]\to[0,\infty)$ by

$$\psi(r)=\gamma+\beta\int_0^r\omega(t)\,dt-r, \tag{1.3}$$

where the function

$$\omega(r)=\int_0^r k(t)\,dt, \tag{1.4}$$

and

$$\gamma=\left\|\mathrm{T}'(u_0)^{-1}\,\mathrm{T}(u_0)\right\|\ ,\ \beta=\left\|\mathrm{T}'(u_0)^{-1}\right\|. \tag{1.5}$$

Theorem 1.1 [17]. Suppose that the function ψ has a unique positive root r_* in $[0,R]$ and $\psi(R)\le 0$. Then the equation $\mathrm{T}(u)=0$ has a unique solution u_* in $S(u_0,R)$ and the Newton-Kantorovich approximations

$$u_n=u_{n-1}-\mathrm{T}'(u_{n-1})^{-1}\,\mathrm{T}(u_{n-1}),\ n\in\mathrm{N}, \tag{1.6}$$

are defined for all $n\in\mathrm{N}$, belong to $S(u_0,r_*)$ and converge to u_*. Moreover the following estimate holds

$$\| u_{n+1}-u_n \| \le r_{n+1}-r_n \quad , \quad \| u_* - u_n \| \le r_* - r_n \,, \tag{1.7}$$

where the sequence $(r_n)_{n\in\mathrm{N}}$, converges to r_*, is defined by the recurrence formula.

$$r_0=0 \;,\; r_{n+1}=r_n-\frac{\psi(r_n)}{\psi'(r_n)} \;,\; n\in\mathrm{N}. \tag{1.8}$$

In this paper, we investigate some sufficient conditions, which ensure that the class of (NSIES) (0.1) verifies the hypotheses of theorem 1.1.

2. Some auxiliary results

Definition 2.1

defined ϕ the class of all continuous almost increasing function Φ 1) We denote by on $(0,\ell/2]$ such that $\phi(t)>0, \lim\limits_{t\to 0^+}\phi(t)=0$, where ℓ L.is the length of the curve

implies $0<t_1<t_2$ such that $\phi\in\Phi$ the class of all functions Φ^m 2) We denote by $t_1^m\,\phi(t_2)\le c(m)\,t_2^m\,\phi(t_1)$ where m is a natural number.

3) We denote by $c(L)$ with the norm L the space of all continuous functions defined on

$$\| u \|_{c(L)} = \max_{t\in L} | u(t) |.$$

4) For a natural number m we define the generalized Hölder space

$$\mathrm{H}_{\phi,m}=\{u\in c(L): \omega_u^m(\delta)=o(\phi(\delta)), \phi\in\Phi^m\},$$

where $\omega_u^m(\delta)$ is the modulus of continuity of order m of the function u defined as follows:

$$\omega_u^m(\delta)=\sup_{0\le h\le\delta,\delta>0} \left| \Delta_h^m(u;x) \right| \text{and } \Delta_h^m(u;x)=\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}u(x+i\,h)$$

is the $m-$ difference of the function $u(x)$ with step h.

5) We denote by $\mathrm{H}\,\Phi^m$ the class of all functions defined as follows

$$\mathrm{H}\,\Phi^m=\left\{\phi\in\Phi^m: \int_0^\delta \frac{\phi(\xi)}{\xi}d\xi+\delta^m\int_\delta^\ell \frac{\phi(\xi)}{\xi^{m+1}}d\xi \le \tilde{c}(m)\phi(\delta)\right\},$$

where $\tilde{c}(m)$ is a positive constant.

6) for $u\in\mathrm{H}_{\phi,m}(L)$ we define the norm:

$$\| u \|_{\phi,m}=\| u \|_c+\sup_{\delta>0}\frac{\omega_u^m(\delta)}{\phi(\delta)}. \tag{2.1}$$

7) Consider the following operators on the space $\mathrm{H}_{\phi,m}(L)$

$$(i)(W^i u)(t)=u(\alpha_i(t)), \quad i=0,1,\ldots,m-1 \quad \text{and} \quad W^3=\mathrm{I}. \tag{2.2}$$

Under the assumption that $\alpha(t)$ homeomorphically maps L into itself with preservation of orientation and satisfies the carleman condition

$$\alpha_m(t)=t \quad , \quad \alpha_i(t)\neq t \quad , \quad 1\leq i\leq m-1 \quad , m\geq 3$$

where

$$\alpha_i(t)=\alpha(\alpha_{i-1}(t)) \quad , \quad \alpha_0(t)=t.$$

(ii) The inverse operator W^{-1} is defined by

$$(W^{-1}u)(t)=u(\beta(t)),$$

where $\beta(t)$ is the inverse of $\alpha(t)$.

(iii) The singular operator

$$(Su)(t)=\frac{1}{\pi i}\int_L \frac{u(\tau)}{\tau-t}d\tau. \qquad (2.3)$$

(iv) The complementary projection operators

$$\mathrm{P}_{\pm}=\frac{1}{2}(\mathrm{I}\pm S) \quad , \quad S^2=\mathrm{I}, \qquad (2.4)$$

where I is the identity operator.

Lemma 2.1. Let the function $v(t)=(Su)(t)$ be defined for all $t\in L$ and has derivatives up to $(m-1)$ order and satisfy the following condition:

$$\left|v^{(k)}(t_1)-v^{(k)}(t_2)\right|\leq l_k\,\{\phi(\left|t_1-t_2\right|)\}, \qquad (2.5)$$

where $k=0,1,...,m-1$, $t_1,t_2\in L$, $u(t)\in \mathrm{H}_{\phi,m}(L)$,l_k is a positive constant and $\phi\in\Phi$. $v(t)\in \mathrm{H}_{\phi,m}(L)$. Then

Proof. for $m=1$, we have

$$\Delta_h^1 Su(t)=Su(t+h)-Su(t),$$

from condition (2.5)

$$\left|\Delta_h^1 Su(t)\right|=\left|Su(t+h)-Su(t)\right|\leq l_0\ (\phi(\delta)).$$

Then,

$$\omega_{Su}^1(\delta)\leq l_0\ (\phi(\delta)) \qquad (2.6)$$

for $m=2$, we have

$$\Delta_h^2 Su(t)=Su(t+2h)-2Su(t+h)+Su(t).$$

From (2.3) we have

$$\Delta_h^2 Su(t)=\frac{1}{\pi i}\int_L \frac{u(\tau+2h)-2u(\tau+h)+u(\tau)}{\tau-t}d\tau.$$

Using Lagrange's formula we have

$$\Delta_h^2 Su(t)=\frac{1}{\pi i}\int_L \frac{\int_0^1 (u'(\tau+h+\theta h)-u'(\tau+\theta h))h\,d\theta}{\tau-t}d\tau,$$

using condition (2.5) then we have

$$\omega^2_{Su}(\delta) \le l_1 \ \delta(\phi(\delta)) = o(\phi(\delta)). \tag{2.7}$$

Thus, the lemma is proved at $m = 1,2$.
Now, we prove that the lemma is true for any m by induction we see that

$$\Delta^m_h Su(t) = h\int_0^1 \Delta^{m-1}_h Su'(t+\theta h)\,d\theta. \tag{2.8}$$

Suppose that this equality is true at $m = 1,2$. The equality (2.8) is true at $m = n-1$, that is

$$\Delta^{n-1}_h Su(t) = h\int_0^1 \Delta^{n-2}_h Su'(t+\theta h)\,d\theta.$$

Now, we prove that (2.8) is true at $m = n$, thus

$$\Delta^n_h Su(t) = \Delta^1_h(\Delta^{n-1}_h Su(t)) = \Delta^1_h\Big(h\int_0^1 \Delta^{n-1}_h Su'(t+\theta h)\,d\theta\Big) = h\int_0^1 \Delta^{n-1}_h Su'(t+\theta h)\,d\theta.$$

Consequently,

$$\left|\Delta^m_h Su(t)\right| = \left| h\int_0^1 \Delta^{m-1}_h Su'(t+\theta h)\,d\theta\right| \le l_{m-1}\,\delta^{m-1}(\phi(\delta)) = o(\phi(\delta)).$$

Hence,

$$\omega^m_{Su}(\delta) = o(\phi(\delta)). \tag{2.9}$$

Therefore, from (2.6),(2.7) and (2.9) we have $v(t) \in \mathrm{H}_{\phi,m}(L)$. Thus the lemma is valid.

Lemma 2.2. Let the condition (2.5) be satisfied for the shift operator W, and $u(t) \in \mathrm{H}_{\phi,m}(L)$. Then the function $(W^i u)(t) \in \mathrm{H}_{\phi,m}(L).\ \ , i = 0,1,2,...,m-1.$

Proof. From Lemma 2.1 and the properties of the shift $\alpha(t)$ the proof is immediate.

Lemma 2.3. [14] Let the function $u(t) \in c(L)$ and $\int_0^{\ell} \frac{\omega^m_u(\xi)}{\xi} d\xi < \infty$. Then the following inequalities:

$$\| Su \|_c \le c_1(m)\Big(\int_0^{\delta} \frac{\omega^m_u(\xi)}{\xi} d\xi + \| u \|_c\Big) \tag{2.10}$$

and

$$\omega^m_{Su}(\delta) \le c_2(m)\Big(\int_0^{\delta} \frac{\omega^m_u(\xi)}{\xi} d\xi + \delta^m \int_{\delta}^{\ell} \frac{\omega^m_u(\xi)}{\xi^{m+1}} d\xi\Big), \tag{2.11}$$

are valid, where $c_1(m)$ and $c_2(m)$ are constants.

then the singular operator $\phi \in \mathrm{H}\Phi^m$, **Lemma 2.4.** Let S is bounded operator on the space $\mathrm{H}_{\phi,m}(L)$.
Proof.

From lemma 2.1, we have $Su(t)\in \mathrm{H}_{\phi,m}(L)$, for any $u\in \mathrm{H}_{\phi,m}(L)$. Hence

$$\|Su\|_{\phi,m}=\|Su\|_c+\sup_{\delta>0}\frac{\omega^m_{Su}(\delta)}{\phi(\delta)}.$$

Using (2.10) and (2.11) we have

$$\|Su\|_{\phi,m}\le c_1(m)(\int_0^{\delta}\frac{\omega^m_u(\xi)}{\xi}d\xi+\|u\|_c)+c_2(m)\sup_{\delta>0}\frac{1}{\phi(\delta)}(\int_0^{\delta}\frac{\omega^m_u(\xi)}{\xi}d\xi+\delta^m\int_{\delta}^{\ell}\frac{\omega^m_u(\xi)}{\xi^{m+1}}d\xi).$$

By using equality (2.1) we have

$$\|Su\|_{\phi,m}\le c_1(m)\|u\|_{\phi,m}(\int_0^{\delta}\frac{\phi(\xi)}{\xi}d\xi+1)+c_2(m)\|u\|_{\phi,m}\sup_{\delta>0}\frac{1}{\phi(\delta)}(\int_0^{\delta}\frac{\phi(\xi)}{\xi}d\xi+\delta^m\int_{\delta}^{\ell}\frac{\phi(\xi)}{\xi^{m+1}}d\xi).$$

Since

$$\phi\in \mathrm{H}\Phi^m,\quad \text{then } \|Su\|_{\phi,m}\le \rho_0\|u\|_{\phi,m}, \tag{2.12}$$

is a constant, defined as follows ρ_0 where

$$\rho_0=c_1(m)\int_0^{\delta}\frac{\phi(\xi)}{\xi}d\xi+c_1(m)+c_2(m)\widetilde{c}(m).$$

Therefore the singular operator S is bounded in generalized Hölder space $\mathrm{H}_{\phi,m}(L)$.

Lemma 2.5. The shift operator W is a linear bounded continuously invertible operator on the space $\mathrm{H}_{\phi,m}(L)$.

Proof.

where $u(t),\widetilde{u}(t)\in \mathrm{H}_{\phi,m}(L)$, Since

$$\widetilde{u}(t)=(Wu)(t)=u(\alpha(t)).$$

Therefore,

$$\begin{aligned}\|Wu\|_{\phi,m}&=\max_{t\in L}|\widetilde{u}(t)|+\sup_{\delta>0}\frac{\omega^m_{\widetilde{u}}(\delta)}{\phi(\delta)}\le \max_{t\in L}|u(t)|+\alpha_0\sup_{\delta>0}\frac{\omega^m_u(\delta)}{\phi(\delta)}\\&\le \gamma_0\|u\|_{\phi,m},\end{aligned} \tag{2.13}$$

where $\gamma_0=\max\{1,\alpha_0\}$ and α_0 is a constant given by $\alpha_0=\sup_{\delta>0}\frac{\omega^m_{\widetilde{u}}(\delta)}{\omega^m_u(\delta)}$. Then the shift operator W is bounded on the space $\mathrm{H}_{\phi,m}(L)$. The continuous invertibility of the operator W on $\mathrm{H}_{\phi,m}(L)$ follows from the properties of the homeomorphism $\alpha(t)$.

Lemma 2.6. Let the functions $\Psi_1(\tau,u(\tau)),\Psi_2(\tau,u(\tau))$ and $\Psi_3(\tau,u(\tau))$ satisfy the conditions (0.3),(0.4) and (0.5) respectively. Then the operator $\mathrm{T}(u)$ is Frechet differentiable at every fixed point $u\in \mathrm{H}_{\phi,m}(L)$. and its derivative is given by:

$$\begin{aligned}\mathrm{T}'(u)h&=a(t)h(t)+b(t)h(\alpha(t))+c(t)h(\alpha_2(t))+\frac{d(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-t}d\tau+\frac{e(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-\alpha(t)}d\tau+\\&+\frac{f(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-\alpha_2(t)}d\tau-\frac{1}{\pi i}\int_L\Big\{\frac{\Psi_{1u}(\tau,u(\tau))}{\tau-t}+\frac{\Psi_{2u}(\tau,u(\tau))}{\tau-\alpha(t)}+\frac{\Psi_{3u}(\tau,u(\tau))}{\tau-\alpha_2(t)}\Big\}h(\tau)d\tau,\end{aligned} \tag{2.14}$$

moreover it satisfies Lipschitz condition:

$$\| \mathrm{T}'(u_1) - \mathrm{T}'(u_2) \|_{\phi,m} \le k(r) \, \| u_1 - u_2 \|_{\phi,m}, \qquad \forall u_1, u_2 \in S(u_0, r) \text{ and } 0 < r < R \tag{2.15}$$

where $k(r)$ is local Lipschitz constant given by

$$k(r) = (\rho_0 \, c_1(r) + \rho_1 \, c_1'(r) + \rho_2 \, c_1''(r)) \,, \rho_1 = \rho_0 \, \gamma_0 \text{ and } \rho_2 = \rho_1 \, \gamma_0 \,.$$

Proof.

Let $u(t)$ be a fixed element and $h(t)$ be an arbitrary element in the space $\mathrm{H}_{\phi,m}(L)$. Then we have

$$\mathrm{T}(u+h) - \mathrm{T}(u) = \mathrm{T}'(u)\, h + \lambda(u,h),$$

where

$$\mathrm{T}'(u)\, h = a(t)\, h(t) + b(t)\, h(\alpha(t)) + c(t)\, h(\alpha_2(t)) + \frac{d(t)}{\pi i} \int_L \frac{h(\tau)}{\tau - t} d\tau + \frac{e(t)}{\pi i} \int_L \frac{h(\tau)}{\tau - \alpha(t)} d\tau +$$

$$+ \frac{f(t)}{\pi i} \int_L \frac{h(\tau)}{\tau - \alpha_2(t)} d\tau - \frac{1}{\pi i} \int_L \left\{ \frac{\Psi_{1u}(\tau, u(\tau))}{\tau - t} + \frac{\Psi_{2u}(\tau, u(\tau))}{\tau - \alpha(t)} + \frac{\Psi_{3u}(\tau, u(\tau))}{\tau - \alpha_2(t)} \right\} h(\tau)\, d\tau,$$

and

$$\lambda(u,h) = -\frac{1}{\pi i} \int_L \frac{1}{\tau - t} \left\{ \int_0^1 (1-\theta)\, \Psi_{1uu}(\tau, u(\tau) + \theta\, h(\tau))\, h^2(\tau)\, d\theta \right\} d\tau$$

$$- \frac{1}{\pi i} \int_L \frac{1}{\tau - \alpha(t)} \left\{ \int_0^1 (1-\theta)\, \Psi_{2uu}(\tau, u(\tau) + \theta\, h(\tau))\, h^2(\tau)\, d\theta \right\} d\tau$$

$$- \frac{1}{\pi i} \int_L \frac{1}{\tau - \alpha_2(t)} \left\{ \int_0^1 (1-\theta)\, \Psi_{3uu}(\tau, u(\tau) + \theta\, h(\tau))\, h^2(\tau)\, d\theta \right\} d\tau.$$

From the inequalities (2.12) and (2.13) we have

$$\left\| \frac{1}{\pi} \int_L \frac{h(\tau)}{\tau - \alpha(t)} d\tau \right\|_{\phi,m} \le \rho_1 \, \| h \|_{\phi,m}, \tag{2.16}$$

where

$\rho_1 = \rho_0 \, \gamma_0$ is a constant. Therefore we have

$$\frac{\| \lambda(u,h) \|_{\phi,m}}{\| h \|_{\phi,m}} \le (\rho_0 \, \| \omega_1 \|_{\phi,m} + \rho_1 \, \| \omega_2 \|_{\phi,m} + \rho_2 \, \| \omega_3 \|_{\phi,m}) \| h \|_{\phi,m},$$

where $\rho_2 = \rho_1 \, \gamma_0$,

$$\omega_1(\tau) = \int_0^1 (1-\theta)\, \Psi_{1uu}(\tau, u(\tau) + \theta\, h(\tau))\, d\theta,$$

$$\omega_2(\tau) = \int_0^1 (1-\theta)\, \Psi_{2uu}(\tau, u(\tau) + \theta\, h(\tau))\, d\theta,$$

and

$$\omega_3(\tau) = \int_0^1 (1-\theta)\, \Psi_{3uu}(\tau, u(\tau) + \theta\, h(\tau))\, d\theta.$$

Hence we have $$\lim_{\|h\|_{\phi,m}\to 0}\frac{\|\lambda(u,h)\|_{\phi,m}}{\|h\|_{\phi,m}}=0.$$

Thus the Frechet's derivative is given by (2.14).Moreover T' satisfies Lipschitz condition, using conditions (0.3) - (0.5) and inequalities (2.13), (2.14) and (2.16) we have

$$\|\mathrm{T}'(u_1)-\mathrm{T}'(u_2)\|_{\phi,m}\le k(r)\,\|u_1-u_2\|_{\phi,m},$$

where

$$k(r)=(\rho_0\,c_1(r)+\rho_1\,c_1'(r)+\rho_2\,c_1''(r)).$$

Thus the lemma is valid.

3. Criterion of Noetherity for (SIOS):

Using equations (2.2), (2.3) and (2.14) we obtain the following (SIES), for the unknown function $h(t)$:

$$\begin{aligned}\mathrm{T}'(u_0)h=a(t)h(t)+b(t)(Wh)(t)+c(t)(W^2h)(t)+(d(t)-\Psi_{1u}(t,u_0(t)))(Sh)(t)+\\+(e(t)-\Psi_{2u}(\alpha(t),u_0(\alpha(t))))(WSh)(t)+\\(f(t)-\Psi_{3u}(\alpha_2(t),u_0(\alpha_2(t))))(W^2Sh)(t)+\frac{1}{\pi i}\int_L R(t,\tau)h(\tau)\,d\tau=g(t),\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}R(t,\tau)=\frac{\Psi_{1u}(t,u_0(t))-\Psi_{1u}(\tau,u_0(\tau))}{\tau-t}+\frac{\Psi_{2u}(\alpha(t),u_0(\alpha(t)))-\Psi_{2u}(\tau,u_0(\tau))}{\tau-\alpha(t)}+\\+\frac{\Psi_{3u}(\alpha_2(t),u_0(\alpha_2(t)))-\Psi_{3u}(\tau,u_0(\tau))}{\tau-\alpha_2(t)}.\end{aligned}$$

According to the assumption

$$\begin{aligned}a^*(t)&=d(t)-\Psi_{1u}(t,u_0(t)),\\b^*(t)&=e(t)-\Psi_{2u}(\alpha(t),u_0(\alpha(t))),\\c^*(t)&=f(t)-\Psi_{3u}(\alpha_2(t),u_0(\alpha_2(t))),\end{aligned}\tag{3.2}$$

the dominant equation of equation (3.1) can be written in the following operator form:

$$\begin{aligned}\mathrm{N}(u_0)h=a(t)h(t)+b(t)h(\alpha(t))+c(t)h(\alpha_2(t))+\frac{a^*(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-t}d\tau+\\+\frac{b^*(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-\alpha(t)}d\tau+\frac{c^*(t)}{\pi i}\int_L\frac{h(\tau)}{\tau-\alpha_2(t)}d\tau=J(t),\end{aligned}\tag{3.3}$$

where

$$J(t)=g(t)-\mathrm{M}(t),\qquad \mathrm{M}(t)=\int_L R(t,\tau)h(\tau)\,d\tau.$$

By using equality (2.4) equation (3.3) reduces to the following (SIOS)

$$(\mathrm{N}h)(t)=\{[(a(t)+a^*(t))\mathrm{I}+(b(t)+b^*(t))W+(c(t)+c^*(t))W^2]\mathrm{P}_+ + \\ +[(a(t)-a^*(t))\mathrm{I}+(b(t)-b^*(t))W+(c(t)-c^*(t))W^2]\mathrm{P}_-\}h(t)=J(t),$$

or

$$(\mathrm{N}h)(t)=(\mathrm{A}\mathrm{P}_+ +\mathrm{B}\mathrm{P}_-)h(t)=J(t),$$

where

$$\mathrm{A}=(a(t)+a^*(t))\mathrm{I}+(b(t)+b^*(t))W+(c(t)+c^*(t))W^2$$

and

$$\mathrm{B}=(a(t)-a^*(t))\mathrm{I}+(b(t)-b^*(t))W+(c(t)-c^*(t))W^2.$$

From the theory of singular integral operators with shift [10]. the Noether condition for the operator N is given by:

$$\Delta_1(t)=\begin{vmatrix} a(t)+a^*(t) & b(t)+b^*(t) & c(t)+c^*(t) \\ c(\alpha(t))+c^*(\alpha(t)) & a(\alpha(t))+a^*(\alpha(t)) & b(\alpha(t))+b^*(\alpha(t)) \\ b(\alpha_2(t))+b^*(\alpha_2(t)) & c(\alpha_2(t))+c^*(\alpha_2(t)) & a(\alpha_2(t))+a^*(\alpha_2(t)) \end{vmatrix}\neq 0, \tag{3.4}$$

$$\Delta_2(t)=\begin{vmatrix} a(t)-a^*(t) & b(t)-b^*(t) & c(t)-c^*(t) \\ c(\alpha(t))-c^*(\alpha(t)) & a(\alpha(t))-a^*(\alpha(t)) & b(\alpha(t))-b^*(\alpha(t)) \\ b(\alpha_2(t))-b^*(\alpha_2(t)) & c(\alpha_2(t))-c^*(\alpha_2(t)) & a(\alpha_2(t))-a^*(\alpha_2(t)) \end{vmatrix}\neq 0. \tag{3.5}$$

Moreover the index formula of the operator N has the form:

$$\operatorname{ind}\mathrm{N}=\frac{1}{6\pi}\left\{\arg\frac{\Delta_2(t)}{\Delta_1(t)}\right\}_L. \tag{3.6}$$

4. Solution of linear singular integral equation with shift

Now, we prove that $\mathrm{T}'(u_0)$ has inverse. For this aim, we investigate the solvability of the linear singular integral equation

$$\begin{aligned}\mathrm{T}'(u_0)h=a(t)h(t)+b(t)h(\alpha(t))+c(t)h(\alpha_2(t))+\frac{1}{\pi i}(d(t)-\Psi_{1u}(t,u_0(t)))\int_L\frac{h(\tau)}{\tau-t}d\tau+ \\ +\frac{1}{\pi i}(e(t)-\Psi_{2u}(\alpha(t),u_0(\alpha(t))))\int_L\frac{h(\tau)}{\tau-\alpha(t)}d\tau+ \\ +\frac{1}{\pi i}(f(t)-\Psi_{3u}(\alpha_2(t),u_0(\alpha_2(t))))\int_L\frac{h(\tau)}{\tau-\alpha_2(t)}d\tau+\frac{1}{\pi i}\int_L R(t,\tau)h(\tau)d\tau=g(t),\end{aligned} \tag{4.1}$$

by using equations (2.2), (2.3), equation (4.1) takes the following operator form:

$$\begin{aligned}(\mathrm{N}h)(t)=a(t)h(t)+b(t)(Wh)(t)+c(t)(W^2h)(t)+a^*(t)(Sh)(t)+b^*(t)(WSh)(t) \\ +c^*(t)(W^2Sh)(t)+(\mathrm{M}h)(t)=g(t).\end{aligned} \tag{4.2}$$

To solve equation (4.2), we apply the operators W, W^2, S, WS and $W^2 S$ successively to both sides of equation (4.2), hence we obtain the following system:

$$
\begin{aligned}
& a(t)h(t)+b(t)(Wh)(t)+c(t)(W^2 h)(t)+a^*(t)(Sh)(t)+b^*(t)(WSh)(t)+ \\
&\qquad +c^*(t)(W^2 Sh)(t)+(\mathrm{M}h)(t)=g(t), \\
& c(\alpha(t))h(t)+a(\alpha(t))(Wh)(t)+b(\alpha(t))(W^2 h)(t)+c^*(\alpha(t))(Sh)(t)+ \\
&\qquad +a^*(\alpha(t))(WSh)(t)+b^*(\alpha(t))(W^2 Sh)(t)+(W\mathrm{M}h)(t)=(Wg)(t), \\
& b(\alpha_2(t))h(t)+c(\alpha_2(t))(Wh)(t)+a(\alpha_2(t))(W^2 h)(t)+b^*(\alpha_2(t))(Sh)(t)+ \\
&\qquad +c^*(\alpha_2(t))(WSh)(t)+a^*(\alpha_2(t))(W^2 Sh)(t)+(W^2\mathrm{M}h)(t)=(W^2 g)(t), \\
& a^*(t)h(t)+b^*(t)(Wh)(t)+c^*(t)(W^2 h)(t)+a(t)(Sh)(t)+b(t)(WSh)(t)+ \\
&\qquad +c(t)(W^2 Sh)(t)+(\mathrm{M}_1 h)(t)=(Sg)(t), \\
& c^*(\alpha(t))h(t)+a^*(\alpha(t))(Wh)(t)+b^*(\alpha(t))(W^2 h)(t)+c(\alpha(t))(Sh)(t)+ \\
&\qquad +a(\alpha(t))(WSh)(t)+b(\alpha(t))(W^2 Sh)(t)+(\mathrm{M}_2 h)(t)=(WSg)(t), \\
& b^*(\alpha_2(t))h(t)+c^*(\alpha_2(t))(Wh)(t)+a^*(\alpha_2(t))(W^2 h)(t)+b(\alpha_2(t))(Sh)(t)+ \\
&\qquad +c(\alpha_2(t))(WSh)(t)+a(\alpha_2(t))(W^2 Sh)(t)+(\mathrm{M}_3 h)(t)=(W^2 Sg)(t),
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
\mathrm{M}_1 &= S(a(t)\mathrm{I}+b(t)W+c(t)W^2+a^*(t)S+b^*(t)WS+c^*(t)W^2 S+\mathrm{M}) \\
&\quad -(a^*(t)\mathrm{I}+b^*(t)W+c^*(t)W^2+a(t)S+b(t)WS+c(t)W^2 S), \\
\mathrm{M}_2 &= W\mathrm{M}_1, \quad \mathrm{M}_3 = W\mathrm{M}_2 = W^2\mathrm{M}_1 .
\end{aligned}
$$

No solutions are lost when W, W^2, S, WS and $W^2 S$ are applied to equation (4.2), hence all solutions of (4.2) are solution of the system (4.3) and conversely.

Let D be a closed subspace defined by

$$D=\{(h, Wh, W^2 h, Sh, WSh, W^2 Sh), h\in \mathrm{H}_{\phi,m}\},$$

and let $\overline{C}$ be the linear operator from D into $\mathrm{H}_{\phi m}(L)$ defined by

$$\overline{C}\,\mathrm{H}(t)=C(t)\mathrm{H}(t).$$

Moreover if we put

$$
\mathrm{K}=\begin{pmatrix}
\mathrm{M} & 0 & 0 & 0 & 0 & 0 \\
0 & W\mathrm{M}W^2 & 0 & 0 & 0 & 0 \\
0 & 0 & W^2\mathrm{M}W & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{M}_1 S & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{M}_2 SW^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{M}_3 SW
\end{pmatrix}, \quad
\mathrm{H}=\begin{pmatrix} h \\ Wh \\ W^2 h \\ Sh \\ WSh \\ W^2 Sh \end{pmatrix}, \quad
G=\begin{pmatrix} g \\ Wg \\ W^2 g \\ Sg \\ WSg \\ W^2 Sg \end{pmatrix},
$$

then the system (4.3) can be written as the following form:

$$\overline{C}\,\mathrm{H}+\mathrm{K}\,\mathrm{H}=G, \quad \mathrm{H}\in D, \tag{4.4}$$

where

$$C=\begin{pmatrix} a & b & c & a* & b* & c* \\ W(c) & W(a) & W(b) & W(c*) & W(a*) & W(b*) \\ W^2(b) & W^2(c) & W^2(a) & W^2(b*) & W^2(c*) & W^2(a*) \\ a* & b* & c* & a & b & c \\ W(c*) & W(a*) & W(b*) & W(c) & W(a) & W(b) \\ W^2(b*) & W^2(c*) & W^2(a*) & W^2(b) & W^2(c) & W^2(a) \end{pmatrix}, \tag{4.5}$$

is a matrix of functions from the space $\mathrm{H}_{\phi,m}(L)$ corresponding to the operator $\overline{C}$.

Theorem 4.1. Let the hypotheses of lemma 2.6 be satisfied and assume that

(1) $\det C(t)\neq 0, \quad \forall t\in L$ (4.6)

(2) $\left\|\overline{C}^{-1}\,\mathrm{K}\right\|<1.$ (4.7)

Then the operator $\mathrm{N}(u_0)$ is invertible, moreover

$$\left\|\mathrm{N}(u_0)^{-1}\right\|\leq \frac{\|C*\|}{1-\left\|\overline{C}^{-1}\,\mathrm{K}\right\|}\left(\frac{1}{m}+\frac{\|\det C\|}{m^2}\right), \tag{4.8}$$

where

$$m=\min_{t\in L}\left|\det C(t)\right|,$$

and

$C*$ be the adjoint matrix of C.

Proof.

It is well known that the condition (4.6) is necessary and sufficient for the invertibility moreover equation (4.4) is equivalent to the equation $\overline{C}$ on D, of the operator

$$\mathrm{H}=\overline{C}^{-1}G-\overline{C}^{-1}\,\mathrm{K}\,\mathrm{H}, \quad \mathrm{H}\in D.$$

The problem of the invertibility of the operator $\overline{C}+\mathrm{K}$ can be reduced to the following fixed point problem

$$\mathrm{H}=\mathrm{P}\,\mathrm{H}, \;\; \mathrm{P}\,\mathrm{H}=\overline{C}^{-1}G-\overline{C}^{-1}\,\mathrm{K}\,\mathrm{H}, \quad \mathrm{H}\in D,$$

where

$$\left\|\mathrm{P}\,\mathrm{H}_1-\mathrm{P}\,\mathrm{H}_2\right\|\leq \left\|\overline{C}^{-1}\,\mathrm{K}\right\|\left\|\mathrm{H}_1-\mathrm{H}_2\right\|.$$

From condition (4.7) and the contraction mapping theorem, it follows that for every $G\in D$, the operator P has a unique fixed point. Then the operator $\overline{C}+\mathrm{K}$ and therefore $(\mathrm{N}(u_0))$ is invertible and

$$(\overline{C}+\mathrm{K})^{-1}=(\overline{C}\,(\mathrm{I}+\overline{C}^{-1}\,\mathrm{K}))^{-1}=(\mathrm{I}+\overline{C}^{-1}\,\mathrm{K})^{-1}\,\overline{C}^{-1}.$$

Thus, we have

$$\left\| \mathrm{N}(u_0)^{-1} \right\| \le \left\| (\mathrm{I}+\overline{C}^{-1}\,\mathrm{K})^{-1}\,\overline{C}^{-1} \right\| \le \frac{\left\| \overline{C}^{-1} \right\|}{1-\left\| \overline{C}^{-1}\,\mathrm{K} \right\|},$$

since C^* be the adjoint matrix of C, we have

$$\overline{C}^{-1}\,G = \frac{C^* G}{\det C},$$

moreover

$$\left\| \overline{C}^{-1} \right\| = \max_{\|G\|\le 1} \left\| \frac{C^* G}{\det C} \right\| \le \max_{\|G\|\le 1} \left\| C^* \right\| \left\| \frac{1}{\det C} \right\| \left\| G \right\| \le \left\| C^* \right\| \left(\frac{1}{m} + \frac{\left\| \det C \right\|}{m^2} \right),$$

hence, we get

$$\left\| \mathrm{N}(u_0)^{-1} \right\| \le \frac{\left\| C^* \right\|}{1-\left\| \overline{C}^{-1}\,\mathrm{K} \right\|} \left(\frac{1}{m} + \frac{\left\| \det C \right\|}{m^2} \right).$$

Thus the theorem is proved.

Assume that

$$\begin{aligned}\gamma \le \beta \big(&\| a(t) \|_{\Phi,m} \| u_0(t) \|_{\phi,m} + \gamma_0 \| b(t) \|_{\phi,m} \| u_0(t) \|_{\phi,m} + \gamma_0^2 \| c(t) \|_{\phi,m} \| u_0(t) \|_{\phi,m} + \\ &+ \rho_0 \| d(t) \|_{\phi,m} \| u_0(t) \|_{\phi,m} + \rho_1 \| e(t) \|_{\phi,m} \| u_0(t) \|_{\phi,m} + \rho_2 \| f(t) \|_{\phi,m} \| u_0(t) \|_{\phi,m} + \\ &+ \rho_0 \| \Psi_1(t,u(t)) \|_{\phi,m} + \rho_1 \| \Psi_2(t,u(t)) \|_{\phi,m} + \rho_2 \| \Psi_3(t,u(t)) \|_{\phi,m} \big),\end{aligned}$$

and

$$\beta \le \frac{\left\| C^* \right\|}{1-\left\| \overline{C}^{-1}\,\mathrm{K} \right\|} \left(\frac{1}{m} + \frac{\left\| \det C \right\|}{m^2} \right).$$

Therefore, the following theorem is valid.

Theorem 4.2. Suppose that the hypotheses of theorem 4.1 are satisfied, moreover the scalar function $\psi(r)$ defined by (1.3), (1.4) has a unique positive root r_* in $[0,R]$ and $\psi(R)\le 0$. Then the equation (0.1) has a unique solution u_* in $S(u_0,R)$ and the Newton-Kantorovich approximations

$$u_n = u_{n-1} - \mathrm{T}'(u_{n-1})^{-1}\,\mathrm{T}(u_{n-1}),\; n\in \mathrm{N},$$

belong to $S(u_0,r_*)$ and satisfy the following estimate

$$\left\| u_{n+1} - u_n \right\| \le r_{n+1} - r_n \quad , \quad \left\| u_* - u_n \right\| \le r_* - r_n \,,$$

where the sequence $(r_n)_{n\in \mathrm{N}}$, converges to r_*, is defined by the recurrence formula.

$$r_0 = 0 \;,\; r_{n+1} = r_n - \frac{\psi(r_n)}{\psi'(r_n)} \;,\; n\in \mathrm{N}.$$

We will illustrate theorem 4.2 by the following examples.

Example (1) Consider the nonlinear function

$$f(u)=\frac{u^3}{3}+u$$

with derivative

$$f'(u)=u^2+1$$

since

$$\| f'(u_1)-f'(u_2)\|=\left\| u_1^2-u_2^2 \right\| \le \| u_1-u_2 \| \, \| u_1+u_2 \|.$$

From inequality (1.2) we take

$$k(r)=\sup_{\|u_1\|,\|u_2\|\le r}\left\{\frac{\| f'(u_1)-f'(u_2)\|}{\| u_1-u_2 \|}: u_1,u_2\in \overline{S}(0,r), 0<r<R\right\},$$

it is clear that

$$\frac{\| f'(u_1)-f'(u_2)\|}{\| u_1-u_2 \|}\le \| u_1+u_2 \| \le 2r,$$

therefore we get

$k(r)=2r$ and $\omega(r)=r^2$.

Obviously, the scalar function (1.3) takes the form:

$$\psi(r)=\gamma+\beta\int_0^r t^2\,dt-r=\gamma+\frac{1}{3}\beta r^3-r.$$

Consider the scalar equation $\psi(r)=0$ we have

$$r^3-\frac{3}{\beta}r+3\left(\frac{\gamma}{\beta}\right)=0, \tag{4.9}$$

equation (4.9) has a unique positive solution r_* in $[0,R]$ if and only if

$$\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3>0,$$

where

$$p=-\frac{3}{\beta} \quad \text{and} \quad q=3\left(\frac{\gamma}{\beta}\right)$$

Hence, the equation $f(u)=0$ has a unique solution u_* in $S(u_0,R)$ and the assumptions of theorem 1.1 are valid.

Example (2) Consider the nonlinear function

$$f(u)=u^3+2u^2+u+3$$

with derivative

$$f'(u)=3u^2+4u+1$$

since

$$\| f'(u_1)-f'(u_2)\|=\left\| 3(u_1^2-u_2^2)+4(u_1-u_2)\right\| \le [3\| u_1+u_2 \|+4]\,\| u_1-u_2 \|.$$

From inequality (1.2) we take

$$k(r)=\sup_{\|u_1\|,\|u_2\|\le r}\left\{\frac{\| f'(u_1)-f'(u_2)\|}{\| u_1-u_2 \|}: u_1,u_2\in \overline{S}(0,r), 0<r<R\right\},$$

it is clear that

$$\frac{\| f'(u_1)-f'(u_2)\|}{\| u_1-u_2 \|} \leq 3\| u_1+u_2 \| +4 \leq 6r+4,$$

therefore we get

$$k(r)=6r+4 \qquad \text{and} \qquad \omega(r)=3r^2+4r.$$

Obviously, the scalar function (1.3) takes the form

$$\psi(r)=\gamma+\beta r^3+2\beta r^2-r.$$

Consider the scalar equation $\psi(r)=0$ we have

$$r^3+2r^2-\frac{1}{\beta}r+(\frac{\gamma}{\beta})=0. \tag{4.10}$$

Let $r=y-\frac{2}{3}$, then

$$y^3-(\frac{4}{3}+\frac{1}{\beta})y+(\frac{2}{3\beta}+\frac{\gamma}{\beta}-\frac{8}{27})=0, \tag{4.11}$$

equation (4.11) has a unique positive solution r_* in $[0,R]$ if and only if

$$(\frac{q}{2})^2+(\frac{p}{3})^3>0,$$

$$p=-(\frac{4}{3}+\frac{1}{\beta}) \text{ and } q=\frac{2}{3\beta}+\frac{\gamma}{\beta}-\frac{8}{27}.$$

Hence, the equation $f(u)=0$ has a unique solution u_* in $S(u_0,R)$ and the assumptions of theorem 1.1 are valid.

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Some problems of random operator equations in the Z-C-X Space

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Abstract: In this paper, the new concept of φ-cone is introduced, several important inequalities are proved, and the existences of random solutions of random semi-closed 1-set-contractive operator equations with different boundary conditions are investigated in the Z-C-X space, some new results are obtained.

Keywords: Z-C-X Space, random semi-closed 1-set-contractive operator, random operator equation, random solution.

AMS(2000): 60H25, 47H10 **Chinese Library Classification :** O177.91, O211.63

1. Introduction and preliminaries

Let (Ω, U, γ) be a complete probability measure space, $\gamma(\Omega)=1$, let E be a separable real Banach space, (E, B) be a measurable space, where B denotes the $\sigma-$algebra of generating by all subset in E, and let X be a closed convex subset of E, and let D be a bounded open set in X, $\overline{D}$ and ∂D the closure and boundary of D in X, respectively.

Definition1. [1] Let E be a separable real Banach space, which satisfies the following conditions:

(H_1) E be an algebra over the real number field R, that has

（1） E is closed to multiplication, that is, for every $x, y \in E$, we have $x \cdot y \in E$;

（2） for every $\alpha \in R, x, y \in E$, we have $(\alpha x) \cdot y = x \cdot (\alpha y) = \alpha (x \cdot y)$;

(H_2) E hasn't nilpotent element, that is, for every $x \in E, n \in N$, if $x \neq \theta$, we have $x^n \neq \theta$.

Then E is called the Z-C-X Space.

Obviously, because of E is algebra over the real number field R, we obtain:

（3） for every $\alpha, \lambda \in R, x, y \in E$, we have $\alpha x \cdot \lambda y = (\alpha\lambda)(x \cdot y)$.

In the Z-C-X Space E, let $\underbrace{x \cdot x \cdots x}_{n} = x^n$, where $x \in E, n$ is natural number.

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Definition2. Let X be a cone of E, and suppose that linear functional $\varphi: X \to [0,+\infty)$, which satisfies the following conditions:

(a) for every $x, y \in X$, if $\theta < x < y$, then $0 < \varphi(x) < \varphi(y)$;

(b) X is closed to multiplication, that is, for every $x, y \in X$, we have $x \cdot y \in X$.

Then X is called the φ-cone.

In the paper, we suppose that "$\le$" is the derived partial ordering by φ-cone X in E.

By the definition2, we also obtain: for every $x, y \in X$, if $\theta \le x \le y$, then $0 \le \varphi(x) \le \varphi(y)$. That is because of when $\theta = x = y$, we naturally have $\varphi(x) = \varphi(y) = 0$.

Lemma1.1.[2] Let X be a closed convex subset of E, and let D be a bounded open set in X, and $\theta \in D$, $\mu \ge 1$, suppose that A: $\Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, such that $A(\omega, x) \ne \alpha x$, for every $(\omega, x) \in \Omega \times \partial D, \alpha \ge \mu$ and α is variable. Then the random operator equation $A(\omega, x) = \mu x$ has a random solution in D.

2. Main results

Lemma2.1. when $\alpha, \mu \ge 0, p, q > 0, p + q = 1$, we have

(i) $p\alpha^k + q\mu^k \ge (p\alpha + q\mu)^k$, where $k > 1$;

(ii) $p\alpha^k + q\mu^k \le (p\alpha + q\mu)^k$, where $0 < k < 1$.

Proof. when $\alpha = \mu$, the both sides of (i) and (ii) are equal to α^k, hence the equalities are true. Suppose $\alpha \ne \mu$, and $\alpha < \mu$. Let $\beta = p\alpha + q\mu$, because $p + q = 1$, then $\beta = \alpha + q(\mu - \alpha) > \alpha$, in like matter, we have $\beta < \mu$, that is $\alpha < \beta < \mu$. In (i) the inequality can be overwritten to $p\alpha^k + q\mu^k \ge (p+q)\beta^k$,

that is
$$q(\mu^k - \beta^k) \ge p(\beta^k - \alpha^k) \tag{1}$$

Let $f(x) = qx^k, g(x) = px^k$, then we have $f'(x) = qkx^{k-1}, g'(x) = pkx^{k-1}$.

Using the Lagrange value theorem to them in $[\beta, \mu]$ and $[\alpha, \beta]$, respectively, we have

$$q(\mu^k-\beta^k)=qk\xi_1^{k-1}(\mu-\beta)=pqk\xi_1^{k-1}(\mu-\alpha),$$

$$p(\beta^k-\alpha^k)=pk\xi_2^{k-1}(\beta-\alpha)=pqk\xi_2^{k-1}(\mu-\alpha),\ \text{where } 0\le\alpha<\xi_2<\beta<\xi_1<\mu.$$

Because $k>1$, then $k-1>0$, thus $\xi_1^{k-1}>\xi_2^{k-1}$. Therefore, $q(\mu^k-\beta^k)>p(\beta^k-\alpha^k)$, which proves that the inequality (1)is true. Hence the sign of inequality of(i)is true. When $0<k<1$, then $k-1<0$, we remark the $\xi_1^{k-1}<\xi_2^{k-1}$ in the above process of proving, then we know that the sign of inequality of(ii)is true.

Theorem2.1. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta\in D$, $\mu\ge 1$, suppose that A: $\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator, such that for every $(\omega,x)\in\Omega\times\partial D$, $p,q>0,\ p+q=1$, which satisfies to one of the following conditions:

(Y_1) $p(A(\omega,x))^k+q(\mu x)^k<(pA(\omega,x)+q\mu x)^k$, where $k>1$,

(Y_1') $p(A(\omega,x))^k+q(\mu x)^k>(pA(\omega,x)+q\mu x)^k$, where $0<k<1$,

Then the random operator equation $A(\omega,x)=\mu x$ has a random solution in D.

Proof. By the virtue of Lemma1.1, we only prove

$$A(\omega,x)\ne\alpha x,\ \text{for every } (\omega,x)\in\Omega\times\partial D,\alpha\ge\mu\ge 1. \tag{2}$$

In fact, suppose (2) is not true, that is there exists a $\alpha_0\ge\mu\ge 1$ and an $(\omega_0,x_0)\in\Omega\times\partial D$ such that $A(\omega_0,x_0)=\alpha_0x_0$.

Inserting $A(\omega_0,x_0)=\alpha_0x_0$ into (Y_1), we obtain

$$p(\alpha_0x_0)^k+q(\mu x_0)^k<(p\alpha_0x_0+q\mu x_0)^k,\ k>1,$$

that is, $(p\alpha_0^k+q\mu^k)x_0^k<(p\alpha_0+q\mu)^kx_0^k$.

Because X is the φ-cone, by the definition2, we have

$$\varphi((p\alpha_0^k+q\mu^k)x_0^k)<\varphi((p\alpha_0+q\mu)^kx_0^k),$$

that is $$(p\alpha_0^k+q\mu^k)\varphi(x_0^k)<(p\alpha_0+q\mu)^k\varphi(x_0^k) \tag{3}$$

This is because $x_0\in\partial D, x_0\ne\theta$, and E is the Z-C-X Space, which hasn't nilpotent element, hence $x_0^n\ne\theta,\varphi(x_0^n)>0$.By (3) , we have $p\alpha_0^k+q\mu^k<(p\alpha_0+q\mu)^k$.

This is contradiction to（ⅰ）in Lemma2.1.

Thus, we know that $A(\omega, x) \neq \alpha x$, for every $(\omega, x) \in \Omega \times \partial D, \alpha \geq \mu \geq 1$.

Then, by the virtue of Lemma1.1, the random operator equation $A(\omega, x) = \mu x$ has a random solution in D.

We can use the same method to prove the random operator equation $A(\omega, x) = \mu x$ has a random solution in D, when the operator A satisfies to the boundary condition of $(Y_1^{'})$.

Corollary1. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta \in D$, $\mu \geq 1$, suppose that A: $\Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, such that for every $(\omega, x) \in \Omega \times \partial D$, which satisfies to one of the following conditions:

(Y_2) $2^{k-1}[(A(\omega, x))^k + (\mu x)^k] < (A(\omega, x) + \mu x)^k$, where $k > 1$,

$(Y_2^{'})$ $2^{k-1}[(A(\omega, x))^k + (\mu x)^k] > (A(\omega, x) + \mu x)^k$, where $0 < k < 1$,

Then the random operator equation $A(\omega, x) = \mu x$ has a random solution in D.

Proof. We only let $p = q = \frac{1}{2}$ in the theorem2.3, and the conclusion is true.

Lemma2.2. $\frac{k}{2}(2t+1)^n - \frac{\varepsilon}{2}(2t-1)^n > (k-\varepsilon)nt$, where $0 < t \leq 1$, $k > \varepsilon > 0$, $n \geq 1$.

Proof. Let $f(t) = \frac{k}{2}(2t+1)^n - \frac{\varepsilon}{2}(2t-1)^n - (k-\varepsilon)nt$, where $0 < t \leq 1$, $k > \varepsilon > 0$, $n \geq 1$,

then $f^{'}(t) = kn[(2t+1)^{n-1} - 1] + \varepsilon n[1-(2t-1)^{n-1}]$.

When $0 < t \leq 1$, $n \geq 1$, we have $(2t+1)^{n-1} - 1 > 0$, $1-(2t-1)^{n-1} \geq 0$, thus $f^{'}(t) > 0$.

Therefore $f(t)$ is a strictly monotone increasing function in $(0,1]$.

That is, when $t \in (0,1]$, we have

$$f(t) = \frac{k}{2}(2t+1)^n - \frac{\varepsilon}{2}(2t-1)^n - (k-\varepsilon)nt > f(0) = \frac{k}{2} - \frac{\varepsilon}{2}(-1)^n > 0.$$

That is $\frac{k}{2}(2t+1)^n - \frac{\varepsilon}{2}(2t-1)^n > (k-\varepsilon)nt$.

Theorem2.2. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta \in D$, $\mu \geq 1$, suppose that A: $\Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, such that

(Y_3) $\frac{k}{2}(2\mu x + A(\omega,x))^n - \frac{\varepsilon}{2}(2\mu x - A(\omega,x))^n \le (k-\varepsilon)n\mu x(A(\omega,x))^{n-1}$,

for every $(\omega,x) \in \Omega \times \partial D$, $n \ge 1$, $k > \varepsilon > 0$,

Then the random operator equation $A(\omega,x) = \mu x$ has a random solution in D.

Proof. By the virtue of Lemma1.1, we only prove

$A(\omega,x) \ne \alpha x$, for every $(\omega,x) \in \Omega \times \partial D, \alpha \ge \mu \ge 1$. (4)

In fact, suppose (4) is not true, that is there exists a $\alpha_0 \ge \mu \ge 1$ and an $(\omega_0, x_0) \in \Omega \times \partial D$ such that $A(\omega_0, x_0) = \alpha_0 x_0$.

Inserting $A(\omega_0, x_0) = \alpha_0 x_0$ into (Y_1), we obtain

$$\frac{k}{2}(2\mu x_0 + \alpha_0 x_0)^n - \frac{\varepsilon}{2}(2\mu x_0 - \alpha_0 x_0)^n \le (k-\varepsilon)n\mu x_0(\alpha_0 x_0)^{n-1},\ n \ge 1,\ k > \varepsilon > 0,$$

That is, $[\frac{k}{2}(2\mu + \alpha_0)^n - \frac{\varepsilon}{2}(2\mu - \alpha_0)^n]x_0^n \le (k-\varepsilon)n\mu\alpha_0^{n-1}x_0^n$

Because X is the φ-cone, by the definition2, we have

$$\varphi([\frac{k}{2}(2\mu + \alpha_0)^n - \frac{\varepsilon}{2}(2\mu - \alpha_0)^n]x_0^n) \le \varphi((k-\varepsilon)n\mu\alpha_0^{n-1}x_0^n),$$

That is, $[\frac{k}{2}(2\mu + \alpha_0)^n - \frac{\varepsilon}{2}(2\mu - \alpha_0)^n]\varphi(x_0^n) \le (k-\varepsilon)n\mu\alpha_0^{n-1}\varphi(x_0^n)$ (5)

This is because $x_0 \in \partial D, x_0 \ne \theta$, and E is the Z-C-X Space, which hasn't nilpotent element, hence $x_0^n \ne \theta, \varphi(x_0^n) > 0$. By (5), we have $\frac{k}{2}(2\mu + \alpha_0)^n - \frac{\varepsilon}{2}(2\mu - \alpha_0)^n \le (k-\varepsilon)n\mu\alpha_0^{n-1}$.

By dividing $\alpha_0^n (> 0)$ on the both sides of the inequality, we have

$$\frac{k}{2}(2\frac{\mu}{\alpha_0}+1)^n - \frac{\varepsilon}{2}(2\frac{\mu}{\alpha_0}-1)^n < (k-\varepsilon)n\frac{\mu}{\alpha_0} \tag{6}$$

Let $\frac{\mu}{\alpha_0} = t$, by $\alpha_0 \ge \mu \ge 1$, we obtain $0 < t \le 1$.

Hence (6) is that $\frac{k}{2}(2t+1)^n - \frac{\varepsilon}{2}(2t-1)^n < (k-\varepsilon)nt$.

This is contradiction to Lemma2.2.

Thus, we know that $A(\omega,x) \ne \alpha x$, for every $(\omega,x) \in \Omega \times \partial D, \alpha \ge \mu \ge 1$.

Then, by the virtue of Lemma1.1, the random operator equation $A(\omega,x) = \mu x$ has a random solution in D.

Lemma2.3. $k(t+1)^n+\varepsilon(t-1)^n>(k-\varepsilon)nt$, where $\varepsilon\in[0,\frac{1}{2}), k+\varepsilon=1, 0<t\le 1$, $n\ge 1$.

Proof. Let $f(t)=k(t+1)^n+\varepsilon(t-1)^n-(k-\varepsilon)nt$, $\varepsilon\in[0,\frac{1}{2}), k+\varepsilon=1, 0<t\le 1$, $n\ge 1$,

then $f'(t)=kn(t+1)^{n-1}+\varepsilon n(t-1)^{n-1}-(k-\varepsilon)n=kn[(t+1)^{n-1}-1]+\varepsilon n[1+(t-1)^{n-1}]$,

when $0<t\le 1$, $n\ge 1$, we have $(t+1)^{n-1}-1>0, 1+(t-1)^{n-1}>0$, then $f'(t)>0$.

Therefore $f(t)$ is a strictly monotone increasing function in $(0,1]$.

That is, when $t\in(0,1]$, we have

$$f(t)=k(t+1)^n-\varepsilon(1-t)^n-(k-\varepsilon)nt>f(0)=k-\varepsilon(-1)^n>0.$$

That is $k(t+1)^n-\varepsilon(1-t)^n>(k-\varepsilon)nt$.

Theorem2.3. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta\in D$, $\mu\ge 1$, suppose that A: $\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator, if there is $\varepsilon\in[0,\frac{1}{2}), k+\varepsilon=1$, $n\ge 1$ such that

$$(Y_4)\quad k(\mu x+A(\omega,x))^n+\varepsilon(\mu x-A(\omega,x))^n\le(k-\varepsilon)n\mu x(A(\omega,x))^{n-1},$$

for every $(\omega,x)\in\Omega\times\partial D$

Then the random operator equation $A(\omega,x)=\mu x$ has a random solution in D.

Proof. By the virtue of Lemma1.1, we only prove

$$A(\omega,x)\ne\alpha x, \text{ for every } (\omega,x)\in\Omega\times\partial D, \alpha\ge\mu\ge 1. \quad (7)$$

In fact, suppose (7) is not true, that is there exists a $\alpha_0\ge\mu\ge 1$ and an $(\omega_0,x_0)\in\Omega\times\partial D$ such that $A(\omega_0,x_0)=\alpha_0 x_0$.

Inserting $A(\omega_0,x_0)=\alpha_0 x_0$ into (Y_4), we obtain

$$k(\mu x_0+\alpha_0 x_0)^n+\varepsilon(\mu x_0-\alpha_0 x_0)^n\le(k-\varepsilon)n\mu x_0(\alpha_0 x_0)^{n-1},\ n\ge 1,$$

That is, $[k(\mu+\alpha_0)^n+\varepsilon(\mu-\alpha_0)^n]x_0^n\le(k-\varepsilon)n\mu(\alpha_0)^{n-1}x_0^n$,

Because X is the φ-cone, by the definition2, we have

$$\varphi([k(\mu+\alpha_0)^n+\varepsilon(\mu-\alpha_0)^n]x_0^n)\le\varphi((k-\varepsilon)n\mu(\alpha_0)^{n-1}x_0^n),$$

That is, $[k(\mu+\alpha_0)^n+\varepsilon(\mu-\alpha_0)^n]\varphi(x_0^n)\le(k-\varepsilon)n\mu(\alpha_0)^{n-1}\varphi(x_0^n)$ (8)

This is because $x_0\in\partial D, x_0\ne\theta$, and E is the Z-C-X Space, which hasn't nilpotent element,

hence $x_0^n\ne\theta,\varphi(x_0^n)>0$.By (8) , we have $k(\mu+\alpha_0)^n+\varepsilon(\mu-\alpha_0)^n\le(k-\varepsilon)n\mu(\alpha_0)^{n-1}$.

By dividing $\alpha_0^n(>0)$ on the both sides of the inequality, we have

$$[k(\frac{\mu}{\alpha_0}+1)^n+\varepsilon(\frac{\mu}{\alpha_0}-1)^n]\le(k-\varepsilon)n\frac{\mu}{\alpha_0} \tag{9}$$

Let $\frac{\mu}{\alpha_0}=t$, by $\alpha_0\ge\mu\ge1$, we obtain $0<t\le1$.

Hence (9) is that $k(t+1)^n+\varepsilon(t-1)^n\le(k-\varepsilon)nt$.

This is contradiction to Lemma2.3.

Thus, we know that $A(\omega,x)\ne\alpha x$, for every $(\omega,x)\in\Omega\times\partial D,\alpha\ge\mu\ge1$.

Then, by the virtue of Lemma1.1, the random operator equation $A(\omega,x)=\mu x$ has a random solution in D.

Remark: when E only is a separable real Banach space, the above theorems can be overwritten to the following theorems, correspondingly.

Theorem2.4. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta\in D$, $\mu\ge1$, suppose that A: $\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator, such that for every $(\omega,x)\in\Omega\times\partial D$, $p,q>0$, $p+q=1$, which satisfies to one of the following conditions:

(Y_5) $p\|A(\omega,x)\|^k+q\|\mu x\|^k<\|pA(\omega,x)+q\mu x\|^k$, where $k>1$,

$(Y_5^{'})$ $p\|A(\omega,x)\|^k+q\|\mu x\|^k>\|pA(\omega,x)+q\mu x\|^k$, where $0<k<1$,

Then the random operator equation $A(\omega,x)=\mu x$ has a random solution in D.

Theorem2.5. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta\in D$, $\mu\ge1$, suppose that A: $\Omega\times\overline{D}\to X$ is a random semi-closed 1-set-contractive operator, such that

(Y_6) $\frac{k}{2}\|2\mu x+A(\omega,x)\|^n-\frac{\varepsilon}{2}\|2\mu x-A(\omega,x)\|^n\le(k-\varepsilon)n\|\mu x\|\cdot\|A(\omega,x)\|^{n-1}$,

for every $(\omega,x)\in\Omega\times\partial D$, $n\ge1$, $k>\varepsilon>0$, Then the random operator equation $A(\omega, x) = \mu x$ has a random solution in D.

Theorem2.6. Let E be the Z-C-X Space, X be a cone of E, and let D be a bounded open set in X, and $\theta \in D$, $\mu \geq 1$, suppose that A: $\Omega \times \overline{D} \to X$ is a random semi-closed 1-set-contractive operator, if there is $\varepsilon \in [0, \frac{1}{2}), k + \varepsilon = 1,\ n \geq 1$ such that

$$(Y_7)\quad k\left\|\mu x + A(\omega, x)\right\|^n + \varepsilon\left\|\mu x - A(\omega, x)\right\|^n \leq (k - \varepsilon)n\left\|\mu x\right\|\left\|A(\omega, x)\right\|^{n-1}$$

for every $(\omega, x) \in \Omega \times \partial D$

Then the random operator equation $A(\omega, x) = \mu x$ has a random solution in D.

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Topological Sequence Entropy of Operators on Function Spaces*

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Abstract

We study sequence entropy of actions on function spaces with the focus on Markov operators on Compact spaces. We defined the natural definition of topological sequence entropy for Markov operators on C(X). Firstly, we prove that the three are equal. Secondly, It is proved that $h_A(T) = h_A(S)$ If $Tf = f \circ S$ is an operate generated by a continuous map: $S : X \to X$ and A is an increasing integer sequence. Finally, It is proved that If for every continuous f there exists an invariant function φ_f such that $\lim\limits_{n\to\infty} \sup\limits_{x\in X} |T^n f(x) - \varphi_f(x)| = 0$ then $h_A(T) = 0$ for every increasing integer sequence A.

Keyword: k-Warsaw circle; Pointwise recurrent; Equicontinuity;

AMS(2000) Subject Classification: 58F10, 54H20

1 Introduction

Let $C_1(X)$ denote the set of all continuous functions $f : X \to [0,1]$. In this paper, X is a compact Hausdorff space and T denotes a Markov operators actin on $C(X)$.

For a continuous f let us define

$U^\varepsilon_{<f} = \{(x,t) \in X \times [0,1] : t < f(x) + \varepsilon\}$.

$U^\varepsilon_{>f} = \{(x,t) \in X \times [0,1] : t > f(x) + \varepsilon\}$.

$U^\varepsilon_f = U^\varepsilon_{<f} \cap U^\varepsilon_{>f}$.

Given a finite collection $\mathcal{F} \subset C_1(X)$ we obtain a finite open cover of $X \times [0,1]$ by the formula

$$\mathcal{U}^\varepsilon_{\mathcal{F}} = \bigvee_{f\in\mathcal{F}} \mathcal{U}^\varepsilon_f$$

If $\mathcal{V}$ is a finite open cover of the unit interval then we let

$$\mathcal{F}^{-1}\mathcal{V} = \bigvee_{f\in\mathcal{F}} f^{-1}(\mathcal{V}).$$

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If $Tf = f \circ S$, where $S : X \to X$ is a continuous transformation and $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ is a integer sequence, we denote $\mathcal{F}_A^n = \bigsqcup_{k=0}^{n-1} T^{a_k} f$.

It is easy to prove the following Lemma 1, in which (i) and (ii) can be found in [2].

Lemma 1 *Let $\mathcal{F}$,$\mathcal{G}$ be finite subsets of $C_1(X)$, $\mathcal{V}$ a finite open cover of the unit interval and ε a positive number. Then*

(i) $\mathcal{U}^\varepsilon_{\mathcal{F}\cup\mathcal{G}} = \mathcal{U}^\varepsilon_{\mathcal{F}} \bigvee \mathcal{U}^\varepsilon_{\mathcal{G}}$

(ii) $(\mathcal{F} \cup \mathcal{G})^{-1}(\mathcal{V}) = \mathcal{F}^{-1}(\mathcal{V}) \bigvee \mathcal{G})^{-1}(\mathcal{V})$; (iii) *if $Tf = f \circ S$, where $S : X \to X$ is a continuous transformation then $\mathcal{U}^\varepsilon_{\mathcal{F}_A^n} = \bigvee_{i=0}^{n-1} (S \times Id)^{-a_i}(\mathcal{U}^\varepsilon_{\mathcal{F}})$ and $(\mathcal{F}_A^n)^{-1}(\mathcal{V}) = \bigvee_{i=0}^{n-1} S^{-a_i}(\mathcal{F}^{-1}(\mathcal{V}))$.*

Recall that, for any open cover $\mathcal{U}$, the symbol $N(\mathcal{U})$ denotes the minimal cardinality of a subcover chosen from $\mathcal{U}$.

Definition 1 Let $\mathcal{F} \subset C_1(X)$ be a finite collection of functions, $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ be a integer sequence and $\varepsilon > 0$. We define

(i) $H_1(\mathcal{F}, \varepsilon) = \log N(\mathcal{U}^\varepsilon_{\mathcal{F}})$,

(ii) $h_1(T, \mathcal{F}, A, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} H_1(\mathcal{F}_A^n, \varepsilon)$,

(iii) $h_1(T, A) = \sup_{\mathcal{F}} \sup_{\varepsilon} h_1(T, \mathcal{F}, A, \varepsilon)$.

Definition 2 Let $\mathcal{F} \subset C_1(X)$ be a finite collection of functions, $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ be a integer sequence and $\mathcal{V}$ be a cover of interval $[0, 1]$. We define

(i) $H_2(\mathcal{F}, \mathcal{V}) = \log N(\mathcal{F}^{-1}\mathcal{V})$,

(ii) $h_2(T, \mathcal{F}, A, \mathcal{V}) = \limsup_{n\to\infty} \frac{1}{n} H_2(\mathcal{F}_A^n, \mathcal{V})$,

(iii)$h_2(T, A) = \sup_{\mathcal{F}} \sup_{\mathcal{V}} h_2(T, \mathcal{F}, A, \mathcal{V})$.

Let $d_{\mathcal{F}}$ and $s(d_{\mathcal{F}}, \varepsilon)$ be the same as in [Tomasz]

Definition 3 Let $\mathcal{F} \subset C_1(X)$ be a finite collection of functions, $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ be a integer sequence and $\varepsilon > 0$. We define

(i) $H_3(\mathcal{F}, \varepsilon) = \log s(d_{\mathcal{F}}, \varepsilon)$,

(ii) $h_3(T, \mathcal{F}, A, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} H_3(\mathcal{F}_A^n, \varepsilon)$,

(iii) $h_3(T, A) = \sup_{\mathcal{F}} \sup_{\varepsilon} h_3(T, \mathcal{F}, A, \varepsilon)$.

In [2], the authors proved the following Theorems A and B:

Theorem A *For every Markov operate T holds $h_1(T) = h_2(T) = h_3(T)$.*

Theorem B *If $Tf = f \circ S$ is an operate generated by a continuous map: $S : X \to X$, then $h_1(T)$ is equal to the classic topological entropy of S.*

In this paper, we defined the topological sequence entropy of a Markov operator T and proved the following Theorems 1, 2 and 3:

Theorem 1 *For every Markov operate T and integer sequence $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ hold $h_1(T, A) = h_2(T, A) = h_3(T, A)$.*

Theorem 2 *If $Tf = f\circ S$ is an operate generated by a continuous map: $S : X \to X$ and $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ is a integer sequence , then $h_A(T)$ is equal to the classic topological entropy $h_A(S)$ of S.*

Theorem 3. *If for every continuous f there exists an invariant function φ_f such that $\lim\limits_{n\to\infty}\sup\limits_{x\in X}|T^n f(x) - \varphi_f(x)| = 0$ then $h_A(T) = 0$ for every integer sequence $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$.*

2 Proofs of Theorems 1, 2 and 3

Theorem 1 *For every Markov operate T and integer sequence $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ hold $h_1(T, A) = h_2(T, A) = h_3(T, A)$.*

Proof. Firstly, we prove that $h_1(T, A) \le h_2(T, A)$. Choose $\varepsilon > 0$ and let $\mathcal{V}$ be a finite open cover of interval $[0, 1]$ of sets with diameters not greater than ε. Let $\mathcal{W}_n^A = \{U \times V : U \in (\mathcal{F}_{A}^{\setminus})^{-1}(\mathcal{V}, v \in \mathcal{V})$.

For each $U \times V \in \mathcal{W}_n^A$, let $\mathcal{F}'_A = \{f \in \mathcal{F}_A^n : f(x) \ge \inf V$ for each $x \in U\}$. It is not difficult to see that $U \times V \subset \bigcap\limits_{f\in\mathcal{F}'_A} U^{\varepsilon}_{<f} \cap \bigcap\limits_{f\in\mathcal{F}_A^n-\mathcal{F}'_A} U^{\varepsilon}_{>f} \in \mathcal{U}^{\varepsilon}_{\mathcal{F}_A^n}$. It follows that $\mathcal{W}_n^A$ is in scribed in $\mathcal{U}^{\varepsilon}_{\mathcal{F}_A^n}$. Thus, $N(\mathcal{U}^{\varepsilon}_{\mathcal{F}_A^n}) \le N(\mathcal{W}_n^A) leq N((\mathcal{F}_{A}^{\setminus})^{-1}(\mathcal{V}) \cdot N(\mathcal{V})$. Since $N(\mathcal{V})$ is independent of n. It follows that $h_1(T, \mathcal{F}, A, \varepsilon) \le h_2(T, \mathcal{F}, A, \mathcal{V})$. Thus, $h_1(T, A) \le h_2(T, A)$.

Secondly, we will prove that $h_2(T, A) \le h_3(T, A)$. et $\mathcal{V}$ be a finite open cover of interval $[0, 1]$. Denote its Lebesgue number by δ and let E be a maximal $(d_{\mathcal{F}_A^n}, \varepsilon)$-separated set in X. It follows from the maximality of E that the collection $\{B(x, \frac{\delta}{2}) : x \in E\}$ of balls constitutes a finite open cover of X. for every $f \in \mathcal{F}_A^n$ and $x \in E$, the interval $(f(x) - \frac{\delta}{2}, f(x) + \frac{\delta}{2})$ is contained in some element $V(f, x)$ of $\mathcal{V}$. Hence $B(x, \frac{\delta}{2}) = \bigcup\limits_{f\in\mathcal{F}_A^n} f^{-1}(f(x) - \frac{\delta}{2}, f(x) + \frac{\delta}{2}) \subset \bigcup\limits_{f\in\mathcal{F}_A^n} f^{-1}(V(f, x)) \in (\mathcal{F}_A^n)^{-1}(\mathcal{V})$ and $N((\mathcal{F}_A^n)^{-1}(\mathcal{V})) \le N\{B(x, \frac{\delta}{2}) : x \in E = N(E) = s(d_{\mathcal{F}_A^n}, \frac{\delta}{2})$. It follows that $h_2(T, A) \le h_3(T, A)$.

Now we will show that $h_3(T, A) \le h_1(T, A)$.

Let $D \subset X$ be a $(d_{\mathcal{F}}, \varepsilon)$-separated set of maximal cardinality. Put $\gamma = \frac{\varepsilon}{6}$ and define $\widetilde{\mathcal{F}} = \{\frac{1}{2}f + i\gamma : f \in \mathcal{F}, i \in \mathbf{Z}, 0 \le i \le \frac{1}{2\gamma}\}$. Then, by the proof of Theorem 4.2 in [2], we have $s(d_{\mathcal{F}}, \varepsilon) \le N(\mathcal{U}^{\gamma}_{\widetilde{\mathcal{F}}})$. Recall that T, as a Markov operator, is linear and preserves constants. This implies that $\widetilde{\mathcal{F}_A^n} = \widetilde{\mathcal{F}}_A^n$. So we can replace $\mathcal{F}$ with $\mathcal{F}_A^n$ and obtain $s(d_{\mathcal{F}_{A}^{\setminus}}, \varepsilon) \le N(\mathcal{U}^{\gamma}_{\mathcal{F}_{A}^{\setminus}})$. $h_3(T, A) \le h_1(T, A)$ holds by taking upper limits and superma.

The proof of Theorem 1 is completed.

In the following we will use the symbol $h_A(T)$ to denote the common value of $h_1(T, A)$, $h_2(T, A)$ and $h_3(T, A)$.

Theorem 2 *If $Tf = f \circ S$ is an operate generated by a continuous map: $S : X \to X$ and $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ is a integer sequence , then $h_A(T)$ is equal to the classic topological entropy $h_A(S)$ of S.*

Proof. Firstly, by Lemma 1(iii), we have $(\mathcal{F}_A^n)^{-1}(\mathcal{V}) = \bigvee\limits_{i=0}^{n-1} S^{-a_i}(\mathcal{F}^{-1}(\mathcal{V}))$. Denote $\mathcal{B}_{\mathcal{V}} = \mathcal{F}^{-1}(\mathcal{V})$.

Then, $(\mathcal{F}_A^n)^{-1}(\mathcal{V}) = \bigvee_{i=0}^{n-1} S^{-a_i}(\mathcal{B})$. Therefore,

$$h_2(T,\mathcal{F},A,\mathcal{V}) = \limsup_{n\to\infty} \tfrac{1}{n} H_2(\mathcal{F}_A^n,\mathcal{V}) = \limsup_{n\to\infty} \tfrac{1}{n}\log N((\mathcal{F}_A^n)^{-1}\mathcal{V}) = \limsup_{n\to\infty}\tfrac{1}{n}\log N(\mathcal{B}_\mathcal{V}).$$

It follows that

$$h_2(T,A) = \sup_{\mathcal{F}}\sup_{\mathcal{V}} \tfrac{1}{n} h_2(T,\mathcal{F},A,\mathcal{V}) = \sup_{\mathcal{F}}\sup_{\mathcal{V}}\limsup_{n\to\infty}\tfrac{1}{n}\log N(\mathcal{B}_\mathcal{V}) = \sup_{\mathcal{V}}\limsup_{n\to\infty}\tfrac{1}{n}\log N(\mathcal{B}_\mathcal{V}) \le h_A(S).$$

Let $\mathcal{W}$ be a finite open cover of X. Choose a minimal subcover $\{W_1, W_2, \cdots, W_r\} \subset \mathcal{W}$. Using the Tietze−Urysohn Theorem(see [4], Theorem 2.1.8), we can construct $\mathcal{F} = \{f_1, f_2, \cdots, f_r\}$ consisting of continuous functions on X such that if $x \in W_i^c$ then $f_i(x) = 0$. Consequently, if $x \in \bigcap_{j\neq i}(W_j^c)$, i.e. x is covered exclusively by W_i, then $f_i(x) = 1 (i = 1, ..., r)$. Fix $0 < \varepsilon < \frac{1}{2r}$. Every member of $\mathcal{U}^\varepsilon_{\mathcal{F}_A^n}$ is of the form $\bigcap_{g\in\mathcal{F}_A^n} U_g = \bigcap_{k=0}^{n-1}\bigcap_{g\in T^{a_k}\mathcal{F}} U_g$, where $U_g \in \mathcal{U}_g^\varepsilon$. We will prove that each subcover $\mathcal{U}'$ chosen from $\mathcal{U}^\varepsilon_{\mathcal{F}_A^n}$ determines a subcover of $\mathcal{W}_A^n$ of the same or smaller cardinality, where $\mathcal{W}_A^n = \bigvee_{i=0}^{n-1} S^{-a_i}(\mathcal{W})$. Suppose that an element of $\mathcal{U}'$ satisfies the following condition

$$\text{For each } k < n, \text{ there exists } g_k \in T^{a_k}\mathcal{F} \text{ such that } U_{g_k} = U^\varepsilon_{<g_k} \tag{1}$$

Denote by $\widetilde{W_k}$ the element of $T^{-a_k}\mathcal{W}$ such that $g_k = 1$ on the set covered exclusively by $\widetilde{W_k}$ and vanishes on $\widetilde{W_k}$. The set $\bigcap_{k=0}^{n-1}\widetilde{W_k}$ belongs to $\mathcal{W}_A^n$. Pick any $x \in X$. The point $(x, \frac{1}{2r})$ does not belong to $\bigcap_{g\in T^{a_k}\mathcal{F}} U^\varepsilon_{>g}$ for any $k = 0, 1, \cdots, n-1$, since one cam always find $g_k \in T^{a_k}\mathcal{F}$ such that $g_k(x) \ge \frac{1}{r}$. Thus, if an element of $\mathcal{U}'$ contains $(x, \frac{1}{2r})$, it satisfies the condition (1) and determines some set $\bigcap_{k=0}^{n-1}\widetilde{W_k} \in \mathcal{W}_A^n$. This set contains x, because $g_k(x) \ge \frac{1}{r}$ for each $k = 0, 1, \cdots, n-1$. Since x is arbitrary, the sets of form $\bigcap_{k=0}^{n-1}\widetilde{W_k}$ constitute a cover of X and hence $N(\mathcal{W}_A^n) \le N((U)^\varepsilon_{\mathcal{F}_A^n})$. It follows that

$$\begin{aligned}
h_A(S) &= \sup_{\mathcal{W}}\limsup_{n\to\infty}\tfrac{1}{n}\log N(\mathcal{W}_A^n)\\
&\le \sup_{\mathcal{W}}\limsup_{n\to\infty}\tfrac{1}{n}\log N((U)^\varepsilon_{\mathcal{F}_A^n}) = \sup_{\mathcal{W}} h_1(T,\mathcal{F},A,\varepsilon)\\
&\le \sup_{\mathcal{F}}\sup_{\varepsilon} h_1(T,\mathcal{F},A,\varepsilon)\\
&= h_1(T,A)\\
&= h_A(T).
\end{aligned}$$

The proof of Theorem 2 is completed.

Corollary 1. *If $Tf = f\circ S$ is an operate generated by a continuous map: $S : X \to X$, where $X = [0,1]$, then S is chaotic in the sense of Li-York if and only if there is some integer sequence $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ such that $h_A(T) > 0$.*

Theorem 3. *If for every continuous f there exists an invariant function φ_f such that $\lim_{n\to\infty}\sup_{x\in X}|T^n f(x) - \varphi_f(x)| = 0$ then $h_A(T) = 0$ for every integer sequence $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$.*

Proof. The proof of Theorem 3 is analogous to the proof of Theorem 4.7 in [2], so we omitted it.

It is clear that Theorem 3 is a promotion of Theorem 4.7 in [2].

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Numerical simulation of the generalized Huxley equation by homotopy analysis method

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Abstract: In this paper we present the homotopy analysis method (HAM) for obtaining the numerical solution of the generalized Huxley equation. Convergence of the solution and effects for the method are discussed. Comparisons are made among Adomian's decomposition method (ADM), homotopy perturbation method (HPM), variational iteration method (VIM), the exact solution and the homotopy analysis method. The results reveal that the proposed method is very effective, simple and also suggest that both the HPM and ADM are special case of the HAM.

Key words: Nonlinear PDE; Huxley equation; Homotopy analysis method; Homotopy perturbation method; Adomian decomposition method.

1. Introduction

Nonlinear partial differential equations (NPDEs) are encountered in such various fields as physics, chemistry, biology, mathematics and engineering,. Most nonlinear models of real life problems are still very difficult to solve, either numerically or theoretically. The generalized Huxley equation

$$u_t - u_{xx} = \beta u(1-u^{\delta})(u^{\delta}-\gamma), \qquad 0 \le x \le 1,\ t \ge 0, \tag{1}$$

with the initial condition of

$$u\,(x,0) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\,(\sigma\gamma x)]^{\frac{1}{\delta}} \tag{2}$$

describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals. The exact solution of this equation was derived by Wang et al. [1],using nonlinear transformations and is given by

$$u\,(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma(x + \left\{\frac{(1+\delta-\gamma)\,\rho}{2\,(1+\delta)}\right\} t\right))]^{\frac{1}{\delta}}, \tag{3}$$

where $\sigma = \frac{\delta\rho}{4(1+\delta)}$ and $\rho = \sqrt{4\beta\,(1+\delta)}$.

Many researchers have used various numerical methods to solve equation (1) numerically. Hashim et al. investigated the generalized Huxley equation, using Adomian decomposition method (ADM) [2] and Wazwaz studied the generalized forms of Burgers, Burgers-Kdv and Burgers-Huxley equations [3]. Hashem et al. studied the generalized Burger's-Huxley equation [4], and Estevez investigated non-classical symmetries and the singular modified Burger's

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and Burger's-Huxley equation [5]. Hashemi et al. solved the generalized Huxley equation by He's homotopy perturbation method (HPM) [6]. Batiha et al. solved the generalized Huxley equation by variational iteration method (VIM) [7].

In this paper, we solve the generalized Huxley equation by Homotopy analysis method (HAM) [8-14]. The results are compared with the available exact solutions and those were obtained by the (ADM) [2], homotopy perturbation method (HPM) [6] and variational iteration method (VIM) [7].

2. The homotopy analysis method (HAM)

We apply the HAM [8-14] to Huxley equation with initial conditions. We consider the following differential equation

$$N[u(x,t)] = 0, \tag{4}$$

where N is a nonlinear operator for this problem, x and t denote independent variables, $u(x,t)$ is an unknown function. By means of the HAM, one first construct zero-order deformation equation

$$(1-q)\mathcal{L}(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)], \tag{5}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{6}$$

Liao [8-14] expanded $\phi(x,t;q)$ in Taylor series with respect to the embedding parameter q, as follows:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{7}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial^m q}\mid_{q=0} \tag{8}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are selected such that the series (7) is convergent at $q = 1$, then we have from (7)

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{9}$$

Let us define the vector

$$u_n^{\rightarrow}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}. \tag{10}$$

Differentiating (5) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the mth-order deformation equation

$$\pounds(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}), \tag{11}$$

where

$$\mathcal{R}_m(\vec{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;q)]}{\partial^{m-1}q}\;|_{q=0}, \tag{12}$$

and

$$\varkappa_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{13}$$

The mth-order deformation Eq. (11) is linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MathLab.

3. Analysis of the method by the HAM

To solve Eq. (1) with an initial condition (2) by means of HAM, we choose the linear operator

$$\mathcal{L}[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t}, \tag{14}$$

with property $\mathcal{L}[c] = 0$, where c is a constant. We define a nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} - \frac{\partial^2\phi(x,t;q)}{\partial x^2} - \beta(1+\gamma)\phi^{\delta+1}(x,t;q) + \beta\phi^{2\delta+1}(x,t;q) + \beta\gamma\phi(x,t;q). \tag{15}$$

We construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)].$$

For $q = 0$ and $q = 1$, we can write

$$\begin{aligned} \phi(x,t;0) &= u_0(x,t) = u(x,0), \\ \phi(x,t;1) &= u(x,t). \end{aligned} \tag{16}$$

Thus, we obtain the mth-order deformation equations

$$\pounds(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}),$$

where

$$\begin{aligned} \mathcal{R}_m(\vec{u_{m-1}}) &= \frac{\partial\phi_{m-1}(x,t;q)}{\partial t} - \frac{\partial^2\phi_{m-1}(x,t;q)}{\partial x^2} + \beta\phi_{m-1}^{2\delta+1}(x,t;q) - \beta(1+\gamma)\phi_{m-1}^{\delta+1}(x,t;q) \\ &\quad + \beta\gamma\phi_{m-1}(x,t;q). \end{aligned} \tag{17}$$

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [12], the auxiliary function can be determined uniquely $H(t) = 1$.

Now the solution of the mth-order deformation equations (17) for $m \geq 1$ become

$$u_m(x,t) = \varkappa_m u_{m-1}(x,t) + h\pounds^{-1}\mathcal{R}_m(\vec{u}_{m-1}). \tag{18}$$

So, a few terms of series solution are as follows:

$$u_0\left(x,t\right) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}}, \tag{19}$$

$$\begin{aligned} u_1\left(x,t\right) \;=\;& \frac{-h}{\delta^2}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right) + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2 \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta - 2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right) \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 - 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t). \end{aligned} \tag{20}$$

According to the HAM, we can conclude that

$$u\left(x,t\right) = u_0\left(x,t\right) + u_1\left(x,t\right) + \ldots\ldots \tag{21}$$

Therefore, substituting the values of $u_0\left(x,t\right)$ and $u_1\left(x,t\right)$ from Eqs. (19), (20) into. Eq. (21) yields:

$$\begin{aligned} u\left(x,t\right) \;=\;& \frac{-h}{\delta^2}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right) + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2 \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta - 2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right) \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 - 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t) + [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}}. \end{aligned} \tag{22}$$

When $h=-1$, we obtain

$$\begin{aligned} u\left(x,t\right) \;=\;& \frac{1}{\delta^2}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right) + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2 \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta - 2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right) \\ & +2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 - 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta + 2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2 \\ & -2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t) + [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}}, \end{aligned} \tag{23}$$

which the same as the solution obtained by [6]. Then we find at $h=-1$,

$$u\left(x,t\right)_{\mathrm{HAM}} = u\left(x,t\right)_{\mathrm{ADM}} = u\left(x,t\right)_{\mathrm{HPM}}.$$

4. Numerical results and discussion

We shall illustrate the accuracy and efficiency of HAM applied to Eq. (1) compared to the ADM [2], HPM[6] and VIM [7]. For this purpose, we consider the same parameter values for the generalized Huxley equation (1) as considered specifically in [2], we take $\beta = 1$, $\gamma = 0.001$ and $\delta = 1, 2, 3$. We present in Tables 1–3, the values of exact solution, five-term approximate of ADM , 1-iteration VIM and 2-term of HAM.

Table 1 numerical solutions for $\beta = 1$, $\gamma = 0.001$ and $\delta = 1$

$h = -0.6$						
x	t	Exact	ADM [2]	HPM [6]	VIM [7]	HAM
0.1	0.05	5.000302E-4	5.000052E-4	5.000052E-4	5.000052E 04	5.000100E-4
	0.1	5.000427E-4	4.999927E-4	4.999927E-4	4.999927E 04	5.000030E-4
	1	5.002676E-4	4.997678E-4	4.997678E-4	4.997678E 04	4.998680E-4
0.5	0.05	5.001009E-4	5.000759E-4	5.000759E-4	5.000759E 04	5.000810E-4
	0.1	5.001134E-4	5.000634E-4	5.000634E-4	5.000634E 04	5.000730E-4
	1	5.003383E-4	4.998385E-4	4.998385E-4	4.998385E 04	4.999380E-4
0.9	0.05	5.001716E-4	5.001466E-4	5.001466E-4	5.001466E 04	5.001520E-4
	0.1	5.001841E-4	5.001341E-4	5.001341E-4	5.001341E 04	5.001440E-4
	1	5.004090E-4	4.999092E-4	4.999092E-4	4.999092E 04	5.000090E-4

Table 2 numerical solutions for $\beta = 1$, $\gamma = 0.001$ and $\delta = 2$

$h = -0.6$						
x	t	Exact	ADM [2]	HPM [6]	VIM [7]	HAM
0.1	0.05	2.236188E-2	2.236077E-2	2.236077E-2	2.236077E -2	2.236100E-2
	0.1	2.236244E-2	2.236021E-2	2.236021E-2	2.236023E -2	2.236070E-2
	1	2.237250E-2	2.235015E-2	2.235015E-2	2.235015E -2	2.223546E-2
0.5	0.05	2.236447E-2	2.236335E-2	2.236335E-2	2.236335E-2	2.236360E-2
	0.1	2.236502E-2	2.236279E-2	2.236279E-2	2.236279E-2	2.236320E-2
	1	2.237508E-2	2.235273E-2	2.235273E-2	2.235273E -2	2.235720E-2
0.9	0.05	2.236705E-2	2.236593E-2	2.236593E-2	2.236593E -2	2.236620E-2
	0.1	2.236761E-2	2.236537E-2	2.236537E-2	2.236537E -2	2.236580E-2
	1	2.237766E-2	2.235531E-2	2.235531E-2	2.235531E -2	2.235980E-2

Table 3 numerical solutions for $\beta = 1$, $\gamma = 0.001$ and $\delta = 3$

$h = -0.6$						
x	t	Exact	ADM [2]	HPM [6]	VIM [7]	HAM
0.1	0.05	7.937402E-2	7.937005E-2	7.937005E-2	7.937005E-2	7.937080E-2
	0.1	7.937600E-2	7.936807E-2	7.936807E-2	7.936807E -2	7.936970E-2
	1	7.941169E-2	7.933234E-2	7.933234E-2	7.933236E -2	7.934820E-2
0.5	0.05	7.938196E-2	7.937799E-2	7.937799E-2	7.937799E-2	7.937880E-2
	0.1	7.938394E-2	7.937601E-2	7.937601E-2	7.937601E-2	7.937760E-2
	1	7.941962E-2	7.934029E-2	7.934029E-2	7.934031E -2	7.935620E-2
0.9	0.05	7.93899E-2	7.938989E-2	7.938989E-2	7.938592E -2	7.938670E-2
	0.1	7.939187E-2	7.939187E-2	7.939187E-2	7.938394E -2	7.938550E-2
	1	7.942754E-2	7.934823E-2	7.934823E-2	7.934825E -2	7.936410E-2

The results clearly show that HAM is more efficient than the ADM, HPM, and VIM. The HAM avoids the needs for calculating the Adomian polynomials which can be difficult in some cases.

5. Conclusion

In this paper, we propose HAM to solve the generalized Huxley equation. The solution is also, given by ADM, HPM and VIM. We reveal the relationship between HAM and other methods, That is ADM and HPM are special case of HAM for this problem.

Compared with ADM, HPM and VIM, this illustrative problem shows that HAM has the following advantages. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

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BASIS PROPERTY IN $L_p(0,1)$ OF THE ROOT FUNCTIONS CORRESPONDING TO A BOUNDARY-VALUE PROBLEM

H. MENKEN AND KH. R. MAMEDOV

Abstract. The non-self adjoint Sturm-Liouville operators with periodic and anti-periodic boundary conditions are studied. The basis property in the space $L_p(0,1)$ ($p > 1$) of the root functions of these operators is proved. For the basisness in $L_p(0,1)$ the inequality in the F. Riesz's theorem is used.

1. Introduction

Let us consider differential operators generated by the differential equation

$$\ell(y) \equiv y'' + q(x)y = \lambda y \tag{1.1}$$

and the periodic boundary conditions

$$y(0) = y(1), \quad y'(0) = y'(1), \tag{1.2}$$

or the anti-periodic boundary conditions

$$y(0) = -y(1), \quad y'(0) = -y'(1), \tag{1.3}$$

where the potential $q(x)$ is an complex-valued function defined on $[0,1]$. We note that the boundary conditions (1.2) and (1.3) in Birkhoff classification are regular, but not strongly regular (see [14], p.71).

In this work we study the basis properties of the root functions for boundary value problem (1.1), (1.2) and (1.1), (1.3) which correspond to periodic and antiperiodic problems. This problem is important for the study of Sturm-Liouville operators with periodic complex potential on the whole real line, which is called the Hill operator, and it is of independent interest. A recent progress in the study of the Hill operator is presented in a paper by Djakov and Mityagin [2] and references therein.

It is well known that the basisness of the root functions of linear differential operators depends on regularity of boundary conditions generating the given differential operators. The basisness in the space $L_2(0,1)$ of the root functions of a linear differential operator of order n with strongly regular boundary conditions was shown by Mikhailov (1962) [13], Kesel'man (1964) [7], and in the monagraph of Dunford and Schwartz [3]. Some examples of differential operators with regular but not strongly regular boundary conditions whose root functions do not form a basis in $L_2(0,1)$ were given by Kesel'man (1964) [7] and Walker (1977) [17].

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Ionkin (1976) [4] studied one non-classical heat conduction problem in homogeneous rod, which is reduced to following boundary-value problem by method of partition

$$y'' + \lambda y = 0, \quad y'(0) - y'(1) = 0, \quad y(0) = 0$$

whose boundary conditions are regular, but not strongly regular. All the eigenvalues of this problem starting with the second one are double, the general number of associated functions is infinite. However, it was established that the chosen specially system of root functions forms an unconditional basis in $L_2(0,1)$.

Shkalikov (1979, 1982) [15, 16] proved that the eigenfunctions and associated functions of an ordinary differential operator of $2n$ order with not strongly regular boundary conditions do not form a Riesz basis, they form a basis with parentheses.

Kerimov and Mamedov (1998) [6] proved that the root functions of the boundary value problems (1.1), (1.2) and (1.1), (1.3) form a Riesz basis in $L_2(0,1)$ when $q(x) \in C^{(4)}[0,1]$ is complex-valued functions satisfying the condition $q(0) \neq q(1)$. Kurbanov (2006) [8] (by using the results in [6]) obtained the basis property in the space $L_p(0,1)$ of the root functions of the boundary value problems(1.1), (1.2) and (1.1), (1.3).

Dernek and Veliev (2005) [1] and Makin (2006) [9] obtained some results on the basis properties of the root functions in terms of the Fourier coefficient of the potential $q(x)$. Moreover, some theorems for determining whether the root functions form a Riesz basis in $L_2(0,1)$ or not were given in [9]. Makin (2006) [10] established a classification on the boundary conditions for a second order differential equation under which the root functions form a Riesz basis.

Mamedov and Menken (2008) [11] proved that the root functions of the boundary problems (1.1), (1.2) and (1.1), (1.3) form a Riesz basis in $L_2(0,1)$ when $q(x) \in C^{(4)}[0,1]$ satisfying the conditions $q(0) = q(1)$ and $q'(0) \neq q'(1)$. In the present work under the same conditions we prove the basis property in the space $L_p(0,1)$ $(1 < p < \infty)$ of the root functions the boundary problems (1.1), (1.2) and (1.1), (1.3).

The basis property in $L_p(0,1)$ of the root functions depends on the inequality at the following theorem in the book of Kashin and Saakyan (see [5], Section I).

Theorem 1. [5] *A system* $\{\varphi_j\}_{j=1}^{\infty}$ *is a basis in the Banach space* X *if only if the following conditions are satisfied:*

a) $\{\varphi_j\}_{j=1}^{\infty}$ *is complete in* X*,*
b) $\{\varphi_j\}_{j=1}^{\infty}$ *is minimal,*
c) there exists a number $M > 0$ *such that for each* $f \in X$*, the inequality*

$$\left\| \sum_{j=1}^{N} (f, \psi_j)\varphi_j \right\| \leq M \|x\|, \quad N = 1, 2, \cdots$$

where the sequence $\{\psi_j\}_{j=1}^{\infty}$ *is the biorthogonal adjoint system to* $\{\varphi_j\}_{j=1}^{\infty}$.

We use also the inequality in F. Riesz theorem which was given in [18].

We need the following asymptotic formulas which were obtained in [11, 12].

Lemma 1. *Let* $q(x) \in C^{(4)}[0,1]$*,* $q(0) = q(1)$ *and* $q'(0) \neq q'(1)$*. Then the following assertions hold:*

a) All eigenvalues of the boundary-value problem (1.1), (1.2), starting from some number, are simple and form two infinite sequences $\lambda_{k,1},\ \lambda_{k,2},\ k=N,N+1,\cdots$, where N is a positive integer and

$$\lambda_{k,1}=-(2k\pi)^2-\frac{q'(1)-q'(0)+\int\limits_0^1 q^2(t)dt}{(4k\pi)^2}+O(\tfrac{1}{k^3})\quad(k\geq N),\tag{1.4}$$
$$\lambda_{k,2}=-(2k\pi)^2+\frac{q'(1)-q'(0)-\int\limits_0^1 q^2(t)dt}{(4k\pi)^2}+O(\tfrac{1}{k^3})\quad(k\geq N);$$

and the corresponding eigenfunctions are of the form

$$y_{k,1}(x)\ =\ \sin 2k\pi x+O(\frac{1}{k})\quad(k\geq N),\tag{1.5}$$

$$y_{k,2}(x)\ =\ \cos 2k\pi x+O(\frac{1}{k})\quad(k\geq N);\tag{1.6}$$

b) All eigenvalues of the boundary value problem (1.1), (1.3), starting from some number, are simple and form two infinite sequence $\lambda_{k,1},\ \lambda_{k,2},\ k=N,N+1,\cdots$, where N is a positive integer and

$$\lambda_{k,1}=-\left[(2k+1)\pi\right]^2+\frac{q'(1)-q'(0)-\int\limits_0^1 q^2(t)dt}{[2(2k+1)\pi]^2}+O(\tfrac{1}{k^3})\quad(k\geq N),$$
$$\lambda_{k,2}=-\left[(2k+1)\pi\right]^2-\frac{q'(1)-q'(0)+\int\limits_0^1 q^2(t)dt}{[2(2k+1)\pi]^2}+O(\tfrac{1}{k^3})\quad(k\geq N);$$

and the corresponding eigenfunctions are of the form

$$y_{k,1}(x)\ =\ \sin(2k+1)\pi x+O(\frac{1}{k})\quad(k\geq N),\tag{1.7}$$

$$y_{k,2}(x)\ =\ \cos(2k+1)\pi x+O(\frac{1}{k})\quad(k\geq N).\tag{1.8}$$

Theorem 2. [11] *Assume that the conditions of Lemma 1 are satisfied. Then, the system of the root functions of the boundary problem (1.1), (1.2) forms a Riesz basis in $L_2(0,1)$.*

It can be easily showed that the system of the root functions of the boundary problem (1.1), (1.2) has a biorthogonal system consisting of the root functions of the adjoint operator

$$\begin{aligned}\ell^*(v)\ &=\ v''+\overline{q(x)}v,\\ v(1)\ &=\ v(0),\quad v'(1)=v'(0),\end{aligned}$$

and the eigenfunctions of the adjoint operator have of the form

$$\upsilon_{k,1}(x)=2\sin 2k\pi x+O(\frac{1}{k}),\tag{1.9}$$

$$\upsilon_{k,2}(x)=2\cos 2k\pi x+O(\frac{1}{k}).\tag{1.10}$$

2. Main Results

In the present work we obtain the following main results.

Theorem 3. *Let $q(x) \in C^{(4)}[0,1]$, $q(0) = q(1)$ and $q'(0) \neq q'(1)$. Then, the system of the root functions of the boundary problem (1.1), (1.2) forms a basis in the space $L_p(0,1)$ $(1 < p < \infty)$.*

Proof. Let $1 < p < 2$ be fixed. According the Theorem 2, the system of the root functions $\{y_k\}_{k=1}^{\infty} = \{y_{k,1}(x), y_{k,2}(x)\}_{k=1}^{\infty}$ is a basis in $L_2(0,1)$ and $\{\upsilon_n\}_{n=1}^{\infty} = \{\upsilon_{k,1}(x), \upsilon_{k,2}(x)\}_{k=1}^{\infty}$ is biorthogonal to the system$\{y_k\}_{k=1}^{\infty}$. Thus, this system is complete in $L_p(0,1)$. For basisness in $L_p(0,1)$ of the system it is sufficient to show to existence of a constant $M > 0$ such that

$$\left\| \sum_{n=1}^{N} (f, \upsilon_n) y_n \right\|_p \leq M \left\| f \right\|_p, \ (N = 1, 2, \cdots), \tag{2.1}$$

for all $f \in L_p(0,1)$, where $\|\cdot\|_p$ denotes the norm in $L_p(0,1)$. (see Theorem 1). Let $\psi_1(x) = 1$, $\psi_{2n+1}(x) = \sqrt{2}\sin 2n\pi x$, $\psi_{2n}(x) = \sqrt{2}\cos 2n\pi x, (n = 1, 2, \cdots)$. The asymptotic formulas

$$y_n(x) = \frac{1}{\sqrt{2}}\psi_n(x) + O(\frac{1}{n}), \quad \upsilon_n(x) = \sqrt{2}\psi_n(x) + O(\frac{1}{n}) \tag{2.2}$$

are also valid for sufficiently large n where $\{\psi_n(x)\}_{n=1}^{\infty} = \left\{\psi_1(x), \psi_{2n}(x), \psi_{2n+1}(x)\right\}_{n=1}^{\infty}$ is a basis in the space $L_p(0,1)$. This follows immediately from (1.5), (1.6) and (1.9), (1.10). By (2.2)

$$\begin{aligned} \left\| \sum_{n=1}^{N} (f, \upsilon_n) y_n \right\|_p \ \leq \ & \left\| \sum_{n=1}^{N} (f, \psi_n) \psi_n \right\|_p + \left\| \sum_{n=1}^{N} (f, \psi_n) O(\frac{1}{n}) \right\|_p + \\ & \left\| \sum_{n=1}^{N} (f, O(\frac{1}{n})) \psi_n \right\|_p + \left\| \sum_{n=1}^{N} (f, O(\frac{1}{n})) O(\frac{1}{n}) \right\|_p . \end{aligned} \tag{2.3}$$

We shall now prove that all the summands on the right side of (2.3) are bounded from above by constant $\|f\|_p$.

Since $\{\psi_n\}_{n=1}^{\infty}$ is a basis of $L_p(0,1)$, applying Theorem 1 , we have

$$\left\| \sum_{n=1}^{N} (f, \psi_n) \psi_n \right\|_p \leq C \left\| f \right\|_p, \quad C = const \ (N = 1, 2, \cdots) \tag{2.4}$$

for all $f \in L_p(0,1)$. Thus, by the inequality in F. Riesz theorem (see [18], p.154, the inequality (2.10))

$$\begin{aligned} \left\| \sum_{n=1}^{N} (f, \psi_n) O(\frac{1}{n}) \right\|_p \ & \leq \ C \sum_{n=1}^{N} (f, \psi_n) \frac{1}{n} \\ & \leq \ C \left(\sum_{n=1}^{N} |(f, \psi_n)|^q \right)^{\frac{1}{q}} \cdot \left(\sum_{n=1}^{N} \frac{1}{n^p} \right)^{\frac{1}{p}} \leq C \left\| f \right\|_p , \end{aligned} \tag{2.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the Parseval's equality we have

$$\begin{aligned}\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))\psi_n\right\|_p &\le \left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))\psi_n\right\|_2 = \left(\sum_{n=1}^{N}\left|(f,O(\frac{1}{n}))\right|^2\right)^{\frac{1}{2}}\\ &\le C\left\|f\right\|_1 \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}} \le C\left\|f\right\|_p, \end{aligned}\tag{2.6}$$

$$\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))O(\frac{1}{n})\right\|_p \le C\left\|f\right\|_1\left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}} \le C\left\|f\right\|_p.\tag{2.7}$$

Using the inequalities (2.4)-(2.6) in the estimate (2.3) we have (2.1), i.e. the basisness of the system $\{y_n(x)\}_{n=1}^{\infty}$ in the space $L_p(0,1)$ at $1<p<2$ is proved.

Now let $2<p<\infty$. It is clear that the adjoint system $\{\upsilon_n(x)\}_{n=1}^{\infty}$ is a basis of the space $L_p(0,1)$. Consequently, this system is complete in the space $L_q(0,1)$, where $\frac{1}{p}+\frac{1}{q}=1$. Note that $1<q<2$.

By means of absolute analogous discussion used above the basisness in $L_q(0,1)$ of the system $\{\upsilon_n(x)\}_{n=1}^{\infty}$ is proved. Hence it follows the basisness in $L_p(0,1)$ $(2<p<\infty)$ of the system of $\{y_n(x)\}_{n=1}^{\infty}$. Thus, the theorem is proved. □

Similarly, the following result is proved for the boundary problem (1.1), (1.3).

Theorem 4. *Let $q(x)\in C^{(4)}[0,1]$, $q(0)=q(1)$ and $q'(0)\neq q'(1)$. Then, the system of the root functions of the boundary problem (1.1), (1.3) forms a basis in the space $L_p(0,1)$ $(1<p<\infty)$. Moreover, if $p=2$ this system is a Riesz basis in $L_2(0,1)$.*

We note that the smoothness condition on $q(x)$ follows from the asymptotic formulas which were obtained in [11, 12]. This smoothness condition may be reduced, but it does not play important role in the proof of the basis property.

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Dimensions of bivariate C^1 cubic spline spaces over unconstricted triangulations with valence six

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Abstract. In this paper, we consider the open problem of dimensions of bivariate C^1 cubic spline spaces. Firstly, by introducing two new operators, element-3 and element-4, in constructing a triangulation, a kind of unconstricted triangulations with valence 6 is defined, which is an extension of the unconstricted triangulation introduced by Farin in [*Dimensions of spline spaces over unconstricted triangulations*, J. Comput. Appl. Math., 192(2006), pp.320-327] and some well-known triangulations such as the Morgan-Scott triangulation and the Robbins triangulation are therefore included. Then, by using the technique of minimal determining set, the dimension of bivariate C^1 cubic spline spaces over the unconstricted triangulation with valence six is determined and the Lagrange interpolation on all vertices in the unconstricted triangulation with valence 6 is considered.

Key words: bivariate C^1 cubic spline space, unconstricted triangulations with valence six, dimension, Lagrange interpolation

AMS Subject Classifications. 65D07, 41A15, 41A63

1 Introduction

Let Δ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$, i.e., Δ is a set of closed triangles whose union coincides with Ω such that the intersection of any two triangles in Δ is either empty, a common edge or a vertex. Let $V(\Delta)$, $V_I(\Delta)$, $V_B(\Delta)$, $E(\Delta)$, $E_I(\Delta)$, $E_B(\Delta)$ and $F(\Delta)$ denote the sets of vertices, interior vertices, boundary vertices, edges, interior edges, boundary edges and triangles in Δ, respectively, and we omit the triangulation and simply use the notations V, V_I, V_B, E, E_I, E_B and F if statements apply to any triangulation or there is otherwise no confusion.

For two given integers n and r with $0 \le r \le n-1$, the space of bivariate splines of degree n and smoothness order r with respect to Δ is defined by

$$S_n^r(\Delta) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_n, \forall T \in F\}, \tag{1.1}$$

where $\mathcal{P}_n$ is the space of bivariate polynomials of total degree being at most n.

The dimension problem of bivariate spline spaces was initiated with a conjecture by Strang [12]. The first result was given by Morgan and Scott [9] for bivariate spline space $S_n^1(\Delta)$ with $n \geq 5$. Later, Schumaker [11] gave a lower bound formula for the dimension of the spaces $S_n^r(\Delta)$. Alfeld and Schumaker [3] and Wang and Lu [13] independently proved that Schumaker's lower bound is in fact the dimension of $S_n^r(\Delta)$ for $n \geq 4r+1$. By carefully working with the smoothness conditions in terms of B-net representation of spline functions, Hong [8] determined the dimension for the space $S_n^r(\Delta)$ when $n \geq 3r+2$. Alfeld, Piper and Schumaker [2] also extended Morgan and Scott's results [9] to the space $S_4^1(\Delta)$. Alfeld and Schumaker [4] determined the dimension of $S_{3r+1}^r(\Delta)$ for a nondegenerate triangulation.

However, when $n \leq 3r$, the dimension of $S_n^r(\Delta)$ is poorly understood as it may depend on the geometric shape of Δ, see [10]. Especially, the dimension of $S_3^1(\Delta)$ is still open and the following two conjectures provide extremely challenging research problems, see [1].

Conjecture 1.1. Let $\sigma(\Delta)$ be the number of singular vertices in triangulation Δ, then

$$\dim S_3^1(\Delta) = 2|V_I| + 3|V_B| + 1 + \sigma, \tag{1.2}$$

where the so-called singular vertex v is the intersection point of the two diagonals of a quadrilateral.

Conjecture 1.2. Given a triangulation Δ with its all vertices being v_i $(i = 1, ..., |V|)$ and numbers z_i, $i = 1, 2, ..., |V|$, there exists a function $s \in S_3^1(\Delta)$ such that

$$s(v_i) = z_i, \quad i = 1, ..., |V|. \tag{1.3}$$

Billera [5] and Whiteley [14] gave the generic dimension of $S_n^1(\Delta)$ for $n \geq 2$, where the so-called generic dimension is such that if $\dim S_n^1(\Delta)$ does not equal it then there is an arbitrary small perturbation in the location of the vertices that will cause $\dim S_n^1(\Delta)$ to equal the generic value, also see [1]. According to [5], the generic dimension of $S_3^1(\Delta)$ is $2|V_I| + 3|V_B| + 1$. This means that the related generic triangulation does not contain any singular vertex.

Recently, Farin [7] introduced a flap-and-pair manner to construct a triangulation and determined the dimension of $S_3^1(\Delta)$ over a kind of special triangulations, called the unconstricted triangulations.

In this paper, by defining two kinds of operators in triangulation construction, called element-3 and element-4, we shall first extend the unconstricted triangulations to a kind of new triangulations, called unconstricted triangulations with valence 6. The new triangulations include some well-known difficult cases such as the Morgan-Scott triangulation and the Robbins triangulation. Then by using the technique of minimal determining set, dimensions of bivariate C^1 cubic spline spaces over the unconstricted triangulations with valence 6 are given. At the end of this paper, the Conjecture 1.2 is also partially answered.

2 Preliminaries

Let $T := [u, v, w]$ be a triangle in Δ. As we know, every polynomial $p \in \mathcal{P}_3$ defined on T can be written uniquely in the Bernstein-Bézier (B-net) representation as follows

$$p(x, y) = p(\alpha, \beta, \gamma) = \sum_{i+j+k=3} c^T_{ijk} \frac{3!}{i!j!k!} \alpha^i \beta^j \gamma^k, \tag{2.4}$$

where (α, β, γ) are the barycentric coordinates of (x, y) with respect to T and $\{c^T_{ijk}\}_{i+j+k=3}$ are the Bernstein-Bézier (B-net) coefficients of $p(\alpha, \beta, \gamma)$. Clearly, each c^T_{ijk} associates with the domain point

$$\xi^T_{ijk} := \left(\frac{i}{3}, \frac{j}{3}, \frac{k}{3}\right),$$

thus the B-net coefficients of p on T can be indexed with the set

$$\mathcal{D}_T := \left\{\xi^T_{ijk}\right\}_{i+j+k=3},$$

and the spline space $S^0_3(\Delta)$ is in one-to-one correspondence with the set of domain points

$$\mathcal{D}_\Delta := \bigcup_{T \in \Delta} \mathcal{D}_T.$$

Following [3], for any vertex $v \in V$, we define the m-th ring around v to be the set

$$R_m(v) := \{\text{domain points which are distance } m \text{ from } v\}. \tag{2.5}$$

A related concept is the m-th disk around v defined by

$$D_m(v) := \bigcup_{i=0}^{m} R_i(v). \tag{2.6}$$

For each $\xi \in \mathcal{D}_\Delta$, let λ_ξ be the linear functional such that for any spline $s \in S^0_3(\Delta)$,

$$\lambda_\xi s = \text{the B-net coefficient } c_\xi \text{ of } s \text{ associated with domain point } \xi. \tag{2.7}$$

A subset $M \subseteq \mathcal{D}_\Delta$ is said to be a determining set for $S^1_3(\Delta)$ if for $\forall s \in S^1_3(\Delta)$, we have

$$\lambda_\xi s = 0 \ \text{ for all } \xi \in M, \quad \text{implies} \quad s \equiv 0. \tag{2.8}$$

Furthermore, M is called a minimal determining set for S if there is no other determining set for S with the cardinality being smaller than $|M|$. Obviously, if M is a determining set for S, we then have $\dim S \leq |M|$, and furthermore, if M is a minimal determining set for S, then $\dim S = |M|$, see [3].

Definition 2.1. Suppose e_{j-1}, e_j, e_{j+1} are three consecutive edges attached to a vertex v, then edge e_j is called to be degenerate at v if two edges e_{j-1} and e_{j+1} are collinear, otherwise e_j is called to be nondegenerate at v.

The following result on C^1 smoothness of $s(x,y)$ across two adjacent triangles is a special case of the general smoothness condition given in [6].

Lemma 2.1. Let $T^{(1)} = [v_0, v_1, v_2]$, $T^{(2)} = [v_0, v_1, v_3]$ be two adjacent triangles in Δ sharing a common edge $[v_0, v_1]$. Suppose $s(x,y)$ agrees with $p_1(x,y) \in \mathcal{P}_3$ in $T^{(1)}$ and with $p_2(x,y) \in \mathcal{P}_3$ in $T^{(2)}$. Assume that

$$p_1(x,y) = \sum_{i+j+k=3} c^{(1)}_{ijk} \frac{3!}{i!j!k!} \alpha_1^i \beta_1^j \gamma_1^k, \qquad p_2(x,y) = \sum_{i+j+k=3} c^{(2)}_{ijk} \frac{3!}{i!j!k!} \alpha_2^i \beta_2^j \gamma_2^k, \tag{2.9}$$

where $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are barycentric coordinates of (x,y) with respect to triangles $T^{(1)}$ and $T^{(2)}$, respectively. Then $s(x,y) \in C^1(T^{(1)} \bigcup T^{(2)})$ if and only if the following smoothness conditions

$$c^{(2)}_{i,j,0} = c^{(1)}_{i,j,0}, \quad i+j=3, \tag{2.10}$$

$$c^{(2)}_{i,j,1} = \overline{\alpha} c^{(1)}_{i+1,j,0} + \overline{\beta} c^{(1)}_{i,j+1,0} + \overline{\gamma} c^{(1)}_{i,j,1}, \quad i+j=2, \tag{2.11}$$

hold, where $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ are the barycentric coordinates of v_3 with respect to triangle $T^{(1)}$.

Corollary 2.1. Under the condition of Lemma 2.1, if the common edge $[v_0, v_1]$ is degenerate at v_0, then the above Eq.(2.11) degenerates into

$$c^{(2)}_{i,j,1} = \overline{\alpha} c^{(1)}_{i+1,j,0} + \overline{\gamma} c^{(1)}_{i,j,1}, \quad i+j=2. \tag{2.12}$$

3 The unconstricted triangulations

Given a triangulation Δ, let $v \in V$, the number d of all edges emanating from v is called the valence of v, denoted by $val(v)$.

Definition 3.1. A subtriangulation Δ' of a triangulation Δ is a triangulation satisfying

$$T \in F(\Delta') \Longrightarrow T \in F(\Delta). \tag{3.13}$$

Definition 3.2. For a triangulation Δ, if $val(v) \geq d$, $\forall v \in V_b(\Delta)$, then we call Δ a constricted triangulation with valence d.

Definition 3.3. Given a triangulation Δ, if it does not contain any constricted subtriangulation with valence d, then we call Δ an unconstricted triangulation with valence d. The set of all unconstricted triangulations with valence d is denoted by $\mathcal{A}_d$.

It is clear that

$$\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \emptyset, \tag{3.14}$$

$$\mathcal{A}_3 \subset \mathcal{A}_4 \subset \cdots \subset \mathcal{A}_k \subset \mathcal{A}_{k+1} \subset \cdots. \tag{3.15}$$

According to above Definition 3.3, the unconstricted triangulation Δ introduced by Farin in [7] is in fact an unconstricted triangulation with valence 4, i.e., $\Delta \in \mathcal{A}_4$. In this paper, we are interested in the unconstricted triangulation with valence 6, i.e., $\Delta \in \mathcal{A}_6$. In addition, it is easy to check by using the following Theorem 3.1 that both the Morgan-Scott triangulation and the Robbins triangulation (see Figure 1) are in $\mathcal{A}_6$ but $\mathcal{A}_4$.

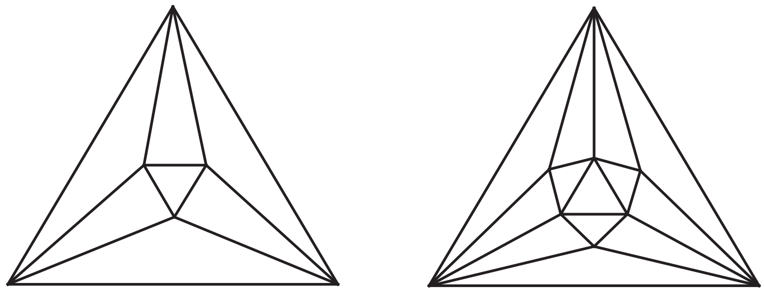

Figure 1: Left: the Morgan-Scott triangulation, Right: the Robbins triangulation.

To give the construction manner of a triangulation $\Delta \in \mathcal{A}_6$, we first recall the two definitions of a flap and a pair of triangles, which are introduced in [7].

Definition 3.4. The triangle formed by one boundary edge of Δ and a point outside Δ is called a flap to Δ. The point outside Δ is called the expansion vertex of the flap.

Definition 3.5. The two triangles formed by two adjacent boundary edges of Δ and a point outside Δ is called a pair of triangles to Δ. The point outside Δ is called the expansion vertex of the pair of triangles.

Besides, we need introduce two new concepts.

Definition 3.6. A set of three triangles formed by a point outside Δ and three adjacent boundary edges of Δ is called an element-3 to Δ . The point outside Δ is called the expansion vertex of the element-3.

Definition 3.7. A set of four triangles formed by a point outside Δ and four adjacent boundary edges of Δ is called an element-4 to Δ. The point outside Δ is called the expansion vertex of the element-4.

Examples of both element-3 and element-4 are shown by the dash lines in Figure 2.

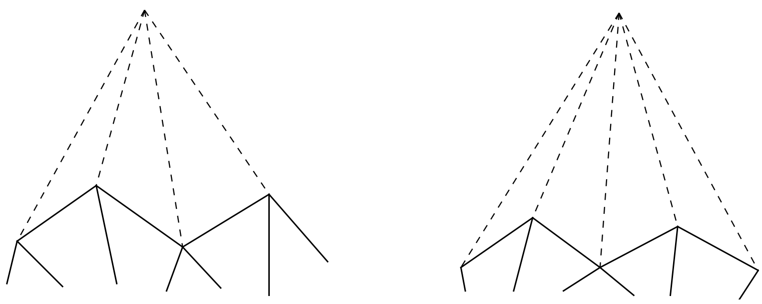

Figure 2: Left: an element-3, Right: an element-4.

It is noted that, for a given triangulation Δ, it is difficult to judge whether $\Delta \in \mathcal{A}_6$ or not by using Definitions 3.1 and 3.2. To make the judgement easier, we have

Theorem 3.1. If $\Delta \in \mathcal{A}_6$, then Δ can be constructed in the following manner.

Start: one triangle.

Recursive step: assuming a subtriangulation Δ has been constructed, we may extend the triangulation Δ to a new triangulation Δ' using four operations on boundary vertices:

1) Adding a flap;

2) Adding a pair of triangles;

3) Adding an element-3;

4) Adding an element-4.

Proof. Assume that $\Delta \in \mathcal{A}_6$. Set $\Delta^{(0)} := \Delta$ and let v_0 be a boundary vertex such that it has the minimal valence among all the boundary vertices in Δ. It is clear that $2 \leq val(v_0) \leq 5$. Then we go to step 1.

Step 1. If $val(v_0) = 2$, we remove the flap in which v_0 is the expansion vertex. It is clear that the resulting triangulation is still an unconstricted triangulation with valence 6, otherwise, the original triangulation Δ would not be an unconstricted triangulation with valence 6, which is in contradiction with the assumption. By repeating the procedure of removing a flap, the triangulation $\Delta^{(0)}$ becomes a new triangulation which is either a single triangle (if the procedure of removing a flap continues, then nothing would be left) or a triangulation with the valences of all boundary vertices being greater than 2. In the latter case, we denote the resulting triangulation by $\Delta^{(0)}$ again and let v_0 still be a boundary vertex such that it has the minimal valence among all the boundary vertices in $\Delta^{(0)}$. It is clear that $3 \leq val(v_0) \leq 5$. Then we go to step 2.

Step 2. If $val(v_0) = 3$, we remove the pair of triangles in which v_0 is the expansion vertex. It is clear that the resulting triangulation is still an unconstricted triangulation with valence 6. By repeating the procedures of removing a flap and a pair of triangles, the triangulation $\Delta^{(0)}$ becomes a new triangulation which is either a single triangle or a triangulation with the valences of all boundary vertices being greater than 3. In the latter case, we denote the resulting triangulation by $\Delta^{(0)}$ again and let v_0 still be a boundary vertex such that it has the minimal valence among all the boundary vertices in $\Delta^{(0)}$. It is clear that $4 \leq val(v_0) \leq 5$. Then we go to step 3.

Step 3. If $val(v_0) = 4$, we remove the element-3 in which v_0 is the expansion vertex. It is clear that the resulting triangulation is still an unconstricted triangulation with valence 6. By repeating the procedures of removing a flap, a pair of triangles and an element-3, the triangulation $\Delta^{(0)}$ becomes a new triangulation which is either a single triangle or a triangulation with the valences of all boundary vertices being greater than 5. In the latter case, we denote the resulting triangulation by $\Delta^{(0)}$ again and let v_0 still be a boundary vertex such that it has the minimal valence among all the boundary vertices in $\Delta^{(0)}$. It is clear that $val(v_0) = 5$. Then we go to step 4.

Step 4. We remove the element-4 in which v_0 is the expansion vertex. The resulting triangulation is still an unconstricted triangulation with valence 6. By repeating the procedures of removing a flap, a pair of triangles, an element-3 and an element-4, the triangulation $\Delta^{(0)}$ becomes a new triangulation which is a single triangle.

By reversing the above procedures of removing a flap, a pair of triangles, an element-3 and an element-4, we obtain the construction procedures of adding a flap, a pair of triangles, an element-3 and an element-4. That is to say, any triangulation $\Delta \in \mathscr{A}_6$ can be constructed by recursively adding a flap, a pair of triangles, an element-3 and an element-4 only.

As an example, the construction procedure of the Robbins triangulation is shown in Figure 3.

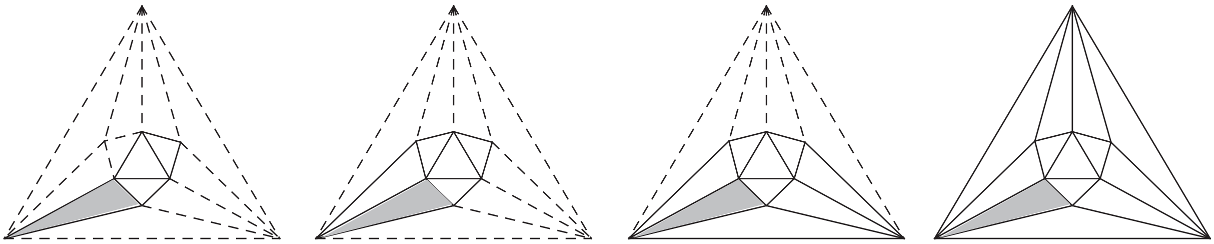

Figure 3: The construction of the Robbins triangulation, from left to right: adding three flaps, adding a pair of triangles, adding an element-3 and adding an element-4.

4 $\dim S_3^1(\Delta)$ over the unconstricted triangulation with valence six

In [7], Farin proved that the conjecture (1.2) is correct in the case of the unconstricted triangulation with valence four, but in his paper the singular vertex is excluded. In this section, we shall prove that the conjecture (1.2) is still correct when the triangulation Δ is in $\mathscr{A}_6$ with some singular vertices being also included.

Theorem 4.1. Let Δ be an unconstricted triangulation with valence 6, i.e., $\Delta \in \mathscr{A}_6$. Assume that there is no degenerate edge at any nonsingular vertex in Δ, then

$$\dim S_3^1(\Delta) = 2|V_I| + 3|V_B| + 1 + \sigma. \tag{4.16}$$

Proof. It is clear that Eq.(4.16) holds for Δ being one single triangle.

For an inductive proof, assume that Eq. (4.16) holds for a subtriangulation $\Delta^{(k)}$ of Δ. Let $\Delta^{(k+1)}$ be the triangulation after we add a flap, a pair of triangles, an element-3 or an element-4 to $\Delta^{(k)}$.

1) When $\Delta^{(k+1)} = \Delta^{(k)} \bigcup \{\text{a flap}\} = \Delta^{(k)} \bigcup [v, v_1, v_2]$, where v is the expansion point of the added flap, it is clear that

$$\left|V_I\left(\Delta^{(k+1)}\right)\right| = \left|V_I\left(\Delta^{(k)}\right)\right|, \quad \left|V_B\left(\Delta^{(k+1)}\right)\right| = \left|V_B\left(\Delta^{(k)}\right)\right| + 1, \quad \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right). \tag{4.17}$$

By using Lemma 2.1, all the B-net coefficients associated with the domain points in $\mathcal{D}_{\Delta^{(k+1)}} \setminus D_1(v)$ can be determined by the B-net coefficients associated with the minimal determining set for $S_3^1\left(\Delta^{(k)}\right)$, see Figure

4. Hence a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be constructed by adding all three domain points in the first disk around v, $D_1(v)$, to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$. This means

$$\begin{aligned}\dim S_3^1\left(\Delta^{(k+1)}\right) &\le \dim S_3^1\left(\Delta^{(k)}\right)+3\\ &= 2\left|V_I\left(\Delta^{(k)}\right)\right|+3\left|V_B\left(\Delta^{(k)}\right)\right|+1+\sigma\left(\Delta^{(k)}\right)+3\\ &= 2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \end{aligned} \tag{4.18}$$

Eq. (4.18) together with Schumaker's lower bound [11] leads to

$$\dim S_3^1\left(\Delta^{(k+1)}\right)=2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \tag{4.19}$$

This is consistent with Eq. (4.16).

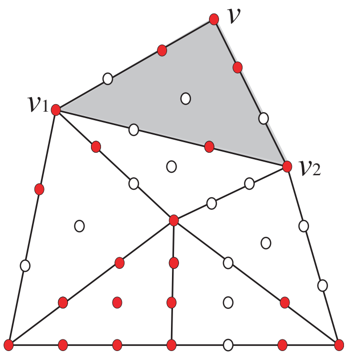

Figure 4: The variation of the minimal determining set when a flap is added.

2) When $\Delta^{(k+1)} = \Delta^{(k)}\bigcup\{\text{a pair of triangles}\} = \Delta^{(k)}\bigcup[v,v_1,v_2]\bigcup[v,v_2,v_3]$, where v is the expansion point of the added pair of triangles, it is clear that

$$\left|V_I\left(\Delta^{(k+1)}\right)\right|=\left|V_I\left(\Delta^{(k)}\right)\right|+1,\quad \left|V_B\left(\Delta^{(k+1)}\right)\right|=\left|V_B\left(\Delta^{(k)}\right)\right|. \tag{4.20}$$

By using Lemma 2.1, all the B-net coefficients associated with the domain points in $\mathcal{D}_{\Delta^{(k+1)}}\setminus D_1(v)$ can be determined by the B-net coefficients associated with the minimal determining set for $S_3^1\left(\Delta^{(k)}\right)$.

The construction of a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ depends on whether the new interior vertex v_2 is singular or not. If v_2 is nonsingular, i.e., $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)$, see Figure 5(1), then the edge $[v,v_2]$ is nondegenerate at v_2, hence the B-net coefficient associated with the domain point in $R_1(v)\bigcap[v,v_2]$ can be further determined by using Lemma 2.1, and a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be constructed by adding two other domain points in $D_1(v)$ to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$. If v_2 is singular, see Figure 5(2), i.e., $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+1$, then the edge $[v,v_2]$ is degenerate at v_2. By using Corollary 2.1, the B-net coefficient associated with the domain point in $R_1(v)\bigcap[v,v_2]$ cannot be determined, so three domain points in $D_1(v)$ have to be added to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$ to form a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$. Hence, we have

$$\dim S_3^1\left(\Delta^{(k+1)}\right) \le \begin{cases} \dim S_3^1\left(\Delta^{(k)}\right)+2, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right) \\ \dim S_3^1\left(\Delta^{(k)}\right)+3, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)+1 \end{cases}$$
$$= \begin{cases} 2\left|V_I\left(\Delta^{(k)}\right)\right|+3\left|V_B\left(\Delta^{(k)}\right)\right|+1+\sigma\left(\Delta^{(k)}\right)+2, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right) \\ 2\left|V_I\left(\Delta^{(k)}\right)\right|+3\left|V_B\left(\Delta^{(k)}\right)\right|+1+\sigma\left(\Delta^{(k)}\right)+3, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)+1 \end{cases}$$
$$= 2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \tag{4.21}$$

Eq. (4.21) together with Schumaker's lower bound [11] leads to

$$\dim S_3^1\left(\Delta^{(k+1)}\right) = 2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \tag{4.22}$$

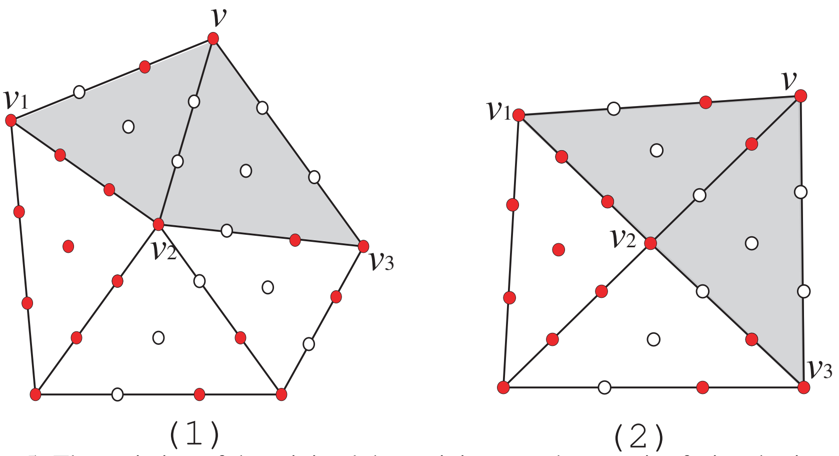

Figure 5: The variation of the minimal determining set when a pair of triangles is added.

3) When $\Delta^{(k+1)} = \Delta^{(k)} \bigcup \{\text{an element-3}\} = \Delta^{(k)} \bigcup [v, v_1, v_2] \bigcup [v, v_2, v_3] \bigcup [v, v_3, v_4]$, where v is the expansion point of the added element-3, it is clear that

$$\left|V_I\left(\Delta^{(k+1)}\right)\right| = \left|V_I\left(\Delta^{(k)}\right)\right|+2, \quad \left|V_B\left(\Delta^{(k+1)}\right)\right| = \left|V_B\left(\Delta^{(k)}\right)\right|-1. \tag{4.23}$$

By using Lemma 2.1, all the B-net coefficients associated with the domain points in $\mathcal{D}_{\Delta^{(k+1)}} \setminus D_1(v)$ can be determined by the B-net coefficients associated with the minimal determining set for $S_3^1\left(\Delta^{(k)}\right)$.

On one hand, if both v_2 and v_3 are nonsingular, i.e., $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)$, see Figure 6(1), then edges $[v, v_2]$ and $[v, v_3]$ are nondegenerate at v_2 and v_3, respectively, therefore by using Lemma 2.1, two B-net coefficients associated with two domain points in $R_1(v) \bigcap ([v, v_2] \bigcup [v, v_3])$ can be further determined, and a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be constructed by adding the expansion vertex v to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$.

On the other hand, if one of two vertices v_2 and v_3, for example v_2, is singular, see Figure 6(2), then $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+1$, and the edge $[v, v_2]$ is degenerate at v_2 but the edge $[v, v_3]$ is nondegenerate at v_3, hence only the B-net coefficient associated with the domain point in $R_1(v)\bigcap[v, v_3]$ can be further determined, so two more domain points in $D_1(v)$ have to be added. We actually have

$$\begin{aligned}\dim S_3^1\left(\Delta^{(k+1)}\right) &\le \begin{cases}\dim S_3^1\left(\Delta^{(k)}\right)+1, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)\\ \dim S_3^1\left(\Delta^{(k)}\right)+2, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)+1\end{cases}\\ &= \begin{cases}2\left|V_I\left(\Delta^{(k)}\right)\right|+3\left|V_B\left(\Delta^{(k)}\right)\right|+1+\sigma\left(\Delta^{(k)}\right)+1, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)\\ \\ 2\left|V_I\left(\Delta^{(k)}\right)\right|+3\left|V_B\left(\Delta^{(k)}\right)\right|+1+\sigma\left(\Delta^{(k)}\right)+2, & \text{if } \sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)+1\end{cases}\\ &= 2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \end{aligned} \tag{4.24}$$

Eq. (4.24) together with Schumaker's lower bound [11] leads to

$$\dim S_3^1\left(\Delta^{(k+1)}\right)=2\left|V_I\left(\Delta^{(k+1)}\right)\right|+3\left|V_B\left(\Delta^{(k+1)}\right)\right|+1+\sigma\left(\Delta^{(k+1)}\right). \tag{4.25}$$

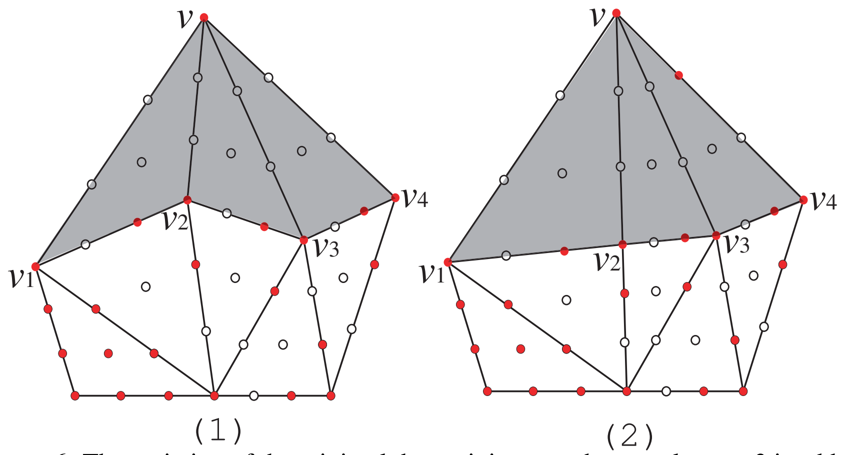

Figure 6: The variation of the minimal determining set when an element-3 is added.

4) When $\Delta^{(k+1)} = \Delta^{(k)}\bigcup\{\text{an element-4}\} = \Delta^{(k)}\bigcup[v, v_1, v_2]\bigcup[v, v_2, v_3]\bigcup[v, v_3, v_4]\bigcup[v, v_4, v_5]$, where v is the expansion point of the added element-4, it is clear that

$$\left|V_I\left(\Delta^{(k+1)}\right)\right|=\left|V_I\left(\Delta^{(k)}\right)\right|+3,\quad \left|V_B\left(\Delta^{(k+1)}\right)\right|=\left|V_B\left(\Delta^{(k)}\right)\right|-2. \tag{4.26}$$

By using Lemma 2.1, all the B-net coefficients associated with the domain points in $\mathcal{D}_{\Delta^{(k+1)}}\setminus D_1(v)$ can be determined by the B-net coefficients associated with the minimal determining set for $S_3^1\left(\Delta^{(k)}\right)$. The construction of a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ depends on the following cases.

i) All the three interior vertices v_2, v_3 and v_4 are nonsingular. Thus edges $[v, v_2]$, $[v, v_3]$ and $[v, v_4]$ are nondegenerate at v_2, v_3 and v_4, respectively, see Figure 7(1). In this case, $\sigma\left(\Delta^{(k+1)}\right)=\sigma\left(\Delta^{(k)}\right)$, and three B-net coefficients associated with three domain points in $R_1(v)\bigcap([v, v_2]\bigcup[v, v_3]\bigcup[v, v_4])$ can be further determined, therefore a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be the same to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$.

ii) One of the three interior vertices v_2, v_3 and v_4 is singular, for example say v_2, see Figure 7(2). Thus edges $[v, v_3]$ and $[v, v_4]$ are nondegenerate at v_3 and v_4, respectively. In this case, $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right) + 1$, and two B-net coefficients associated with domain points in $R_1(v) \bigcap ([v, v_3] \bigcup [v, v_4])$ can be further determined, therefore a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be constructed by adding the expansion vertex v to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$.

iii) Two of the three interior vertices v_2, v_3 and v_4 are singular. Obviously, v_3 cannot be singular, i.e., the edge $[v, v_3]$ is nondegenerate at v_3, see Figure 7(3). In this case, $\sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right) + 2$, and the B-net coefficient associated with the domain point in $R_1(v) \bigcap [v, v_3]$ can be further determined, therefore a determining set for $S_3^1\left(\Delta^{(k+1)}\right)$ can be constructed by adding other two domain points in $D_1(v)$ to the minimal determining set for the space $S_3^1\left(\Delta^{(k)}\right)$.

Hence, we have

$$
\begin{aligned}
\dim S_3^1\left(\Delta^{(k+1)}\right) &\leq \begin{cases} \dim S_3^1\left(\Delta^{(k)}\right), & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right) \\ \dim S_3^1\left(\Delta^{(k)}\right) + 1, & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+1 \\ \dim S_3^1\left(\Delta^{(k)}\right) + 2, & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+2 \end{cases} \\
&= \begin{cases} 2\left|V_I\left(\Delta^{(k)}\right)\right| + 3\left|V_B\left(\Delta^{(k)}\right)\right| + 1 + \sigma\left(\Delta^{(k)}\right), & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right) \\ 2\left|V_I\left(\Delta^{(k)}\right)\right| + 3\left|V_B\left(\Delta^{(k)}\right)\right| + 1 + \sigma\left(\Delta^{(k)}\right) + 1, & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+1 \\ 2\left|V_I\left(\Delta^{(k)}\right)\right| + 3\left|V_B\left(\Delta^{(k)}\right)\right| + 1 + \sigma\left(\Delta^{(k)}\right) + 2, & \text{if } \sigma\left(\Delta^{(k+1)}\right) = \sigma\left(\Delta^{(k)}\right)+2 \end{cases} \\
&= 2\left|V_I\left(\Delta^{(k+1)}\right)\right| + 3\left|V_B\left(\Delta^{(k+1)}\right)\right| + 1 + \sigma\left(\Delta^{(k+1)}\right). \qquad (4.27)
\end{aligned}
$$

Eq. (4.27) together with Schumaker's lower bound [11] leads to

$$
\dim S_3^1\left(\Delta^{(k+1)}\right) = 2\left|V_I\left(\Delta^{(k+1)}\right)\right| + 3\left|V_B\left(\Delta^{(k+1)}\right)\right| + 1 + \sigma\left(\Delta^{(k+1)}\right). \qquad (4.28)
$$

This completes the proof of the theorem.

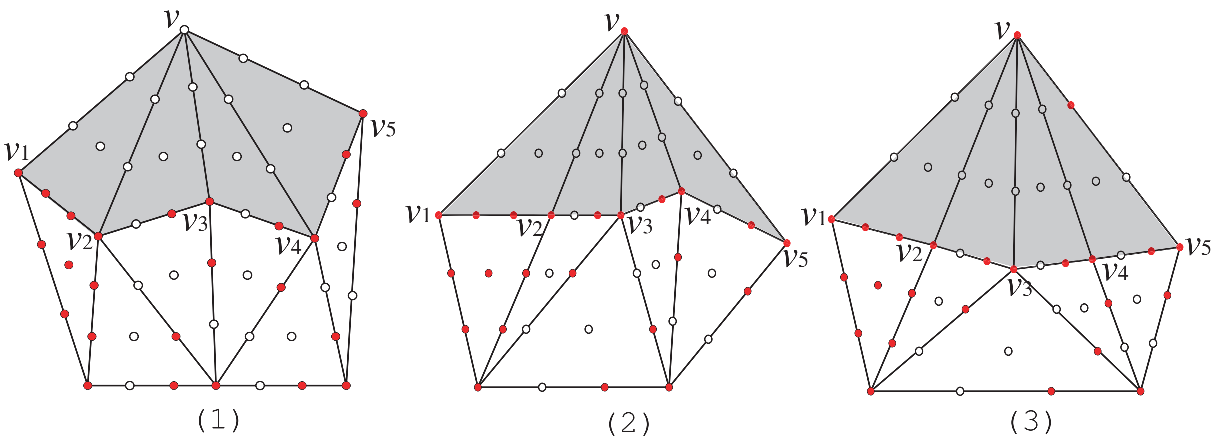

Figure 7: The variation of the minimal determining set when an element-4 is added.

From the proof of Theorem 4.1, it can be seen that when a flap or a pair of triangles or an element-3 is added in every expansion step from $\Delta^{(k)}$ to $\Delta^{(k+1)}$, the expansion vertex can be always chosen as a domain point in the minimal determining set for $S_3^1(\Delta^{(k+1)})$; however, when an element-4 is added, the expansion vertex can be also chosen as a domain point in the minimal determining set for $S_3^1(\Delta^{(k+1)})$ only if there is at least one singular vertex among the three added interior vertices. Hence we actually have proved the existence of the Lagrange interpolation by $S_3^1(\Delta)$ on all the vertices in Δ and have partially answered the Conjecture 1.2.

Theorem 4.2. Let $\Delta \in \mathcal{A}_6$ such that there is at least one singular vertex among the three added interior vertices when every element-4 is added in the construction of Δ. If there is no degenerate edge at any nonsingular vertex in Δ, then for arbitrary numbers z_i, $i = 1, 2, ..., |V(\Delta)|$, there exists a function $s \in S_3^1(\Delta)$ to satisfy that

$$s(v_i) = z_i, \quad i = 1, ..., |V(\Delta)|, \tag{4.29}$$

where v_i, $i = 1, ..., |V(\Delta)|$, are all the vertices in Δ.

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Existence of solutions to a model of long range diffusion involving flux

By

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Abstract

A model for insect dispersal has been considered, existence and uniqueness of solutions to the long range diffusion involving flux for such model has been shown in $L^{p,q}$ space.

1. Introduction

The dynamics of population has been described using mathematical models which have been very successful in giving good effect in the study of animal and human populations. Fife [4] considered reaction and diffusion systems which are distributed in 3-dimensional spaces or on a surface rather than on a line. Abualrub [1] studied diffusion in two dimensional spaces for which diffusion is more realistic and applicable in life. Also he talked about long range diffusion with population pressure in Plankton-Herbivore populations.In this paper we include long range diffusion involving flux for insect population and then talk about the existence and uniqueness of solutions to our model in the $L^{p,q}$ space. But we are going to find the required p and q in similar approach used in [2] .

2. Long Range Diffusion Involving Flux

Here we consider long range diffusion to a modified insect dispersal model in two dimensions as follows

$$u_t - D\Delta^{(2)}u = \alpha_1 u + \alpha_2 u^2 + \alpha_3 u_x + \alpha_4 \Delta\left(u^{\alpha+1}\right) \tag{1}$$

$$u\left(x,0\right) = f\left(x\right) \tag{2}$$

;where $u = u\left(x,t\right)$ is the insect population density and $x \in R^2$. Here Δ

represents the laplacian operator and

$$\Delta^{(2)} = \sum_{i,j=1}^{2} \frac{\partial^4}{\partial x_i^2 \partial x_j^2}. \tag{3}$$

u_t is the rate of change of the insect population density, $D\Delta^{(2)}$u is the long range diffusion term; where D is a small constant, and α, α_4 are positive constants. But u^2 is the interaction between the Males and Females of the insect population, u_x is the instantaneous flux in the x direction due to molecular diffusion. $\Delta\left(u^{\alpha+1}\right)$ is the regular diffusion of the insect population.
We now want to discuss existence and uniqueness of solutions to equation (1) togother with condition (2) in the $L^{p,q}$ space which is the function space consisting of Lebesgue measurable functions $u\left(x,t\right)$ such that $\|u\|_{p,q} < \infty$; where $\|\cdot\|_{p,q}$ is the norm in $L^{p,q}$, p is taken in the t variable, and q is taken in the x variable. We want to find the appropriate values for p and q in the next section. In addition, we will consider large values of time since we are talking about long range diffusion.

3. Existence and Uniqueness of Solutions

First of all, we have to prove the following Lemma for the initial data:

Lemma 1 *Assume that u satisfies equations* (1) *and* (2). *If* $f\left(x\right) \in L^P\left(R^2\right)$, *and*

$\left|K\left(x,t\right)\right| \leq \frac{D}{\left(|x|+t^{\frac{1}{4}}\right)^2}$; $t > 0$, *and D is a constant. Then,* $K * f \in L^{3q}$

;where $*$ represents the convolution in space only.

PROOF. As we did in [1]; we will use the following estimate for the Kernel; namely

$$\left|K\left(x,t\right)\right| \leq \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^2}; t > 0. \tag{4}$$

Now, if $f\ \left(x\right) \in L^P\left(R^2\right)$we have

$$K * f \leq \int_{R^2} \frac{Df\left(y\right)dy}{\left(|x-y| + t^{\frac{1}{4}}\right)^2}$$

We first take the p norm in t; namely

$$\|K * f\|_p \leq \left\| \int_{R^2} \frac{Df\left(y\right)dy}{\left(|x-y| + t^{\frac{1}{4}}\right)^2} \right\|_p$$

Applying Minkowski's integral inequality on the right hand side of the above inequality to obtain

$$\begin{aligned}
\|K * f\|_p &\le D \int_{R^2} |f(y)| \left(\int_{R^+} \frac{dt}{\left(|x-y| + t^{\frac{1}{4}}\right)^{2p}} \right) dy \\
&\le D\alpha \int_{R^2} |f(y)| \left(\frac{1}{\left(|x-y| + t^{\frac{1}{4}}\right)^{2p-4}} \right)^{\frac{1}{p}} dy \\
&= D\alpha \int_{R^2} \frac{|f(y)|\, dy}{\left(|x-y| + t^{\frac{1}{4}}\right)^{2-\frac{4}{P}}}
\end{aligned}$$

;where α is a constant. We now take the q norm in x of the above inequality to obtain

$$\|K * f\|_{p,q} \le D\alpha \left\| \int_{R^2} \frac{|f(y)|\, dy}{\left(|x-y| + t^{\frac{1}{4}}\right)^{2-\frac{4}{P}}} \right\|_q$$

The right hand side of the above inequality is less than or equal to constant$\cdot \|f\|_q$, if $\frac{1}{p} = \frac{1}{q} - \frac{4}{2p} = \frac{1}{q} - \frac{2}{p}$ (using the Benedek-Panzone Potential Theorem [3] ,see Appendix). This implies that $p = 3q$ and hence $K * f \in L^{3q}$, this concludes the proof for the initial data.

Theorem 2 *The solution $u(x,t)$ of (1) and (2) exists and it is unique in the space $L^{3\alpha,\alpha}$ for $\alpha > \frac{1}{2}$; whenever the initial data $f(x)$ is small enough in the norm of its space and if $e^{-\alpha_1 t} u_x(x,t) \in L^{3\alpha,\alpha}$.*

PROOF. We begin by eliminating the linear term $\alpha_1 u$ of (1), so we let

$$u(x,t) = e^{\alpha_1 t} w(x,t).$$

Then we get,

$$w_t - D\Delta^{(2)} w = \alpha_2 e^{\alpha_1 t} w^2 + \alpha_3 w_x + \alpha_4 \Delta\left(w^{\alpha+1}\right), \tag{5}$$

$$w(x,0) = f(x); \text{ where } x \in R^2. \tag{6}$$

We may assume population pressure in the first and second terms in the right hand side of equation (5).Therefore, we write

$$\alpha_2 = c_1 w^\beta \text{ and } \alpha_3 = c_2 (w_x)^\gamma$$

then (5) becomes:

$$w_t - D\Delta^{(2)}w = c_1 e^{\alpha_1 t} w^{\beta+2} + c_2 \left(w_x\right)^{\gamma+1} + \alpha_4 \Delta \left(w^{\alpha+1}\right), \tag{7}$$

$$w(x,0) = f(x);\ \text{where } x \in R^2. \tag{8}$$

Now $w(x,t)$ can be obtained by solving the integral equation:

$$w = \int_0^t \int_{R^2} K\left(x-y, t-\tau\right)\left[c_1 e^{\alpha_1 t} w^{\beta+2} + c_2 \left(w_x\right)^{\gamma+1} + \alpha_4 \Delta \left(w^{\alpha+1}\right)\right] dy d\tau + \int_{R^2} K\left(x-y, t\right) f(y) dy \tag{9}$$

;where K is the fundamental solution to the homogeneous problem of (7), in two dimensions,namely

$$K(x,t) = \frac{1}{2\pi t} e^{\frac{-|x|^2}{2t}},\ \ |x| = \left(x_1^2 \ + x_2^2\right)^{\frac{1}{2}},\ \text{and } x \in R^2.$$

Also K can be approximated by (4).We will now rewrite (9) simply as

$$w = K \otimes \left[c_1 e^{\alpha_1 t} w^{\beta+2} + c_2 \left(w_x\right)^{\gamma+1} + \alpha_4 \Delta \left(w^{\alpha+1}\right)\right] + K * f \tag{10}$$

;where $\otimes$ represents the convolution in space and time; and $w(x,t)$ is a weak solution of (9) provided that the integrals in (10) exist in the Lebesque sense. Using integration by parts on the term $K \otimes \alpha_4 \Delta \left(w^{\alpha+1}\right)$, and set $w_x = H$ in (10), we obtain:

$$w = K \otimes \left[c_1 e^{\alpha_1 t} w^{\beta+2} + c_2 H^{\gamma+1}\right] + \alpha_4 \Gamma \otimes w^{\alpha+1} + K * f \tag{11}$$

;where

$$\Gamma = \Delta K = \sum_{i\,=1}^{2} \frac{\partial^2 K}{\partial x_i^2}$$

Now for the first, second, third and fourth terms on the right hand side of (11), we shall use exponents r, s, p,q, respectively, when considering the L^p norm. For the first term in (11) we have:

$$\begin{aligned} |K| &\leq \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^2} \\ &= \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^{2+4-4}} \end{aligned}$$

So,

$$\frac{1}{q} = \frac{\beta+2}{r} - \frac{4}{2+4} = \frac{\beta+2}{r} - \frac{2}{3};\text{where } 1 < \frac{r}{\beta+2} < \frac{3}{2} \tag{12}$$

Setting $r = q$ we get:

$$r = \frac{3}{2}(\beta + 1) \tag{13}$$

Using $(12), (13)$ we have $\beta + 2 < \frac{3}{2}(\beta + 1) < \frac{3}{2}(\beta + 2)$, which gives $\beta > 1$.

For the second term in (11), we have:

$$\frac{1}{q} = \frac{\gamma + 1}{s} - \frac{1}{2+1} = \frac{\gamma + 1}{s} - \frac{1}{3}; \text{where } 1 < \frac{s}{\gamma + 1} < 3 \tag{14}$$

Using

$$\begin{aligned} |K| &\leq \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^2} \\ &= \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^{2+1-1}}. \end{aligned}$$

Setting $s = q$ in (14) we get:

$$s = 3\gamma \tag{15}$$

Using $(14), (15)$ we have: $\gamma + 1 < 3\gamma < 3(\gamma + 1)$,which gives $\gamma > \frac{1}{2}$.

For the third term in (11), we have:

$$\begin{aligned} |\Gamma| &= |\Delta K| \\ &\leq \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^4} \\ &= \frac{D}{\left(|x| + t^{\frac{1}{4}}\right)^{4+2-2}} \end{aligned}$$

So,

$$\frac{1}{q} = \frac{\alpha + 1}{p} - \frac{2}{4+2} = \frac{\alpha + 1}{p} - \frac{1}{3}; \text{where } 1 < \frac{p}{\alpha + 1} < 3 \tag{16}$$

Setting $p = q$ we get:

$$p = 3\alpha \tag{17}$$

Using (16), (17) we have: $\alpha + 1 < 3\alpha < 3(\alpha + 1)$ which gives:

$$\alpha > \frac{1}{2}. \tag{18}$$

Now to get a contraction mapping

$$L^p\left(R^2 \times R^+\right) \longrightarrow L^p\left(R^2 \times R^+\right)$$

in (11), we have to equate all exponents in $K * f \in L^{3q}$,$(13), (15)$ and (17). That is

$$3q = \frac{3}{2}(\beta + 1) = 3\gamma = 3\alpha.$$

In view of $(13), (15)$ and (17) and the above relationship we arrive at:

$$p = 3\alpha, q = \alpha, r = 3\alpha, \alpha = \gamma \tag{19}$$

Consequently, the following relationship exists between α and β; namely

$$\beta = 2\alpha - 1$$

Hence,

$$w(x,t) \in L^{3\alpha,\alpha};\ \text{where } \alpha > \frac{1}{2}.$$

Now, it is enough to show the uniqueness of the solution. Our mapping in (11) will be:

$$\|T(\cdot)\|_{3\alpha} \leq C(\alpha) \|(\cdot)\|_{3\alpha} + \|(*)\|_{3\alpha}.$$

;where C is a constant depends on α. That is if we apply the mapping T to (11) we have:

$$T(w) = K \otimes G + \alpha_4 \Gamma \otimes w^{\alpha+1} + K * f \tag{20}$$

;where

$$G = c_1 e^{\alpha_1 t} w^{\beta+2} + c_2 H^{\gamma+1},$$

then

$$\|T(w)\|_{3\alpha} \leq C(\alpha) \|w\|_{3\alpha}^{\alpha+1} + \|g\|_{3\alpha}, \tag{21}$$

;where g is an auxiliary function which is the sum of the first and the third terms in the right hand side of equation (20).We are going now to compare (21) with the following mapping:

$$y = \delta x^{\alpha+1} + \eta; \tag{22}$$

where δ, η are positive constants and $x \geq 0$. Now $\delta x^{\alpha+1}$ increases faster than a linear function and it is convex.

For $\eta = 0$ we have only one non-zero root of (22) because the graph of

$y = \delta x^{\alpha+1}$ and $y = x$ will intersect in only one non-zero point.

For the same reason if $0 < \eta < \epsilon$ (where ϵ is sufficiently small), we have two roots, say $\widetilde{x_1}, \widetilde{x_2}$. Let $\widetilde{x_1}$ be the smallest root, therefore if $\widetilde{x_1}$ is small enough then the mapping T will be a contraction mapping which maps the ball of radius $\widetilde{x_1}$ into itself. This implies that the solution to the equation $w = T(w)$, in (20), exists and its unique in the ball of radius $\widetilde{x_1}$.Here $\widetilde{x_1}$ depends on the size of the initial data. This completes the proof of Theorem 2.

Remark 3 *We have shown in Theorem 2 that α must be greater than $\frac{1}{2}$ to guarantee the existence and uniquness of solutions for equation(1) togother with condition (2). Let us now take $\alpha = 1$. Therefore, equation* (1) *becomes:*

$$u_t - D\Delta^{(2)}u = \alpha_1 u + \alpha_2 u^2 + \alpha_3 u_x + \alpha_4 \Delta \left(u^2\right) \tag{23}$$

We will work on (23) *in future research.*

Appendix:

Benedek-Panzone Potential Theorem: Let X=E^n (the nth dimensional Euclidean space), and $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be an n-tuple of real numbers, $0 < \lambda_i < 1$. If P and Q are such that $\frac{1}{P} - \frac{1}{Q} = \Lambda$, $1 < P < \frac{1}{\Lambda}$, then $\left\| f * |x|^{\lambda - n} \right\|_Q \leq c \left\| f \right\|_P$ holds for every $f \in L^P$; where $\lambda = \sum_{i=1}^{n} \lambda_i$and $c = c(\Lambda, P)$.

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On Best $N-$Simultaneous Approximation

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Abstract

Let X and Y be Banach Spaces, $L(X,Y)$ be the space of bounded linear operators from X into Y, and $X \overset{\alpha}{\otimes} Y$ be the tensor product of X and Y with a uniform cross norm α. Given a natural number n, let N be a monotonus norm on R^n. A subspace G is called $N-$simultaneuosly proximinal if for every $x_1, x_2, ..., x_n \in X$ there exists $y \in G$ such that :

$$N(\| x_1-y \|, \| x_2-y \|, ..., \| x_n-y \|) \leq N(\| x_1-g \|, \| x_2-g \|, ..., \| x_n-g \|),$$

for all $g \in G$. In this paper, we discuss simultaneous approximation in $L(X,Y)$ and $X \overset{\alpha}{\otimes} Y$.

Key Words : Simultaneuos approximation
Classification Numbers : Primiry : 41A28, secondary : 41A65

1 Introduction

Throughout this section, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach Spaces, X^* denotes the dual of X and $L(X,Y)$ is the Banach space of all bounded linear operator from X into Y, endowed with usual norm $\| f \| = \sup_{\|x\|=1} \| f(x) \|$, for every $f \in L(X,Y)$. we denote $X \overset{\vee}{\otimes} Y$ and $X \overset{\wedge}{\otimes} Y$ to the injective and the projective tensor products of X and Y respectively, $X \overset{\wedge}{\otimes} Y$ is a subspace of the space of the nuclear operators from X^* into Y, [8].

We wish to consider here the problem of simultaneous approximation in $L(X,Y)$ in the sense of Fathi, Hussien, and Khalil [10].

Let G be a closed subspace of Y. Given n points $y_1, y_2, ..., y_n$ in Y, there are several ways of simultaneuosly approximatig them by an element g in G (see [10], [2]), throughout this paper we will use the definition introduced by Fathi, Hussien, and Khalil in [10]. Namely, we say that a norm N in R^n is monotonus if for every $t = (t_i)_{1 \leq i \leq n}$, $s = (s_i)_{1 \leq i \leq n} \in R^n$ such that $| t_i | \leq | s_i |$ for $i = 1, 2, ..., n$ we have $N(t) \leq N(s)$. Notice that all the usual norms in R^n are monotonus.

Definition 1.1. ([3,Definition 1]). we say that $g_0 \in G$ is a best $N-$simultaneuos approximation from G of vectors $y_1, y_2, ..., y_n \in Y$ if

$$N(\| y_1 - g_0 \|, \| y_2 - g_0 \|, ..., \| y_n - g_0 \|) \leq N(\| y_1 - g \|, \| y_2 - g \|, ..., \| y_n - g \|)$$

for all $g \in G$.

If every n-tuple of vectors $y_1, y_2, ..., y_n \in Y$ admits a best $N-$simultaneuos approximation from G, then G is said to be $N-$simultaneuosly proximinal in Y.

Of course, for $n = 1$ the preceding concepts are just best approximation and proximinality. For an example of a Banach space which has a proper subspace that is proximinal but not $N-$simultaneously prximinal see [6].

The problem of best $N-$simultaneuos approximation had been studied by some authers : Tijani Pakhrou [14], studied the problem on $L^{\infty}(\mu, Y)$, the Banach space of all essensially bounded $\mu-$Bochner integrable $Y-$valued functions on a finite measure space (Ω, Σ, μ). Fathi, Hussien, and Khalil [10], studied the problem on $L^p(I, Y)$, the Banach space of all $p-$integrable $Y-$valued functions on the interval $I = [0, 1]$ with lebesque measure. J. Mendoza and Tijani [6], studied the problem on $L^1(\mu, Y)$, the Banach space of all Bochner $\mu-$integrable $Y-$valued functions on a finite measure space (Ω, Σ, μ). Little works have been done $L(X, Y)$. E. Abu-Sirhan and R. Khalil [5], studied the problem on $L(X, Y)$ in a particular case : N is an ℓ^1-norm on R^2, $N(a, b) = |a| + |b|$, $a, b \in R$. The object of this paper is to study the subspase of $L(X, Y)$ which are $N-$simultaneuosly proximinal in $L(X, Y)$. In this paper we will show that if G is a reflexive subspace of Y, then $L(X, G)$ is $N-$simultaneuosly proximinal in $L(X, Y)$. Result on other spaces, Tensor produt spaces, $L^1(\mu, Y)$, and $L^{\infty}(\mu, Y)$ are presented.

2 Preliminary

The uncomplete tensor product of two Banach spaces X and Y is the set of all finite sums of the form $\sum x_i \otimes y_i$ with $x_i \in X$ and $y_i \in Y$. An equivalence relation is introduce by stipulating that $\sum x_i \otimes y_i$ is (equivalent to) 0 when $\sum \langle f, x_i \rangle y_i$ for all $f \in X^*$, X^* is the dual space of X. A norm α on $X \otimes Y$ is termed a cross-norm if $\alpha(x \otimes y) = \|x\| \|y\|$ for all $x \in X$ and all $y \in Y$. A cross-norm α is said to be uniform cross-norn if

$$\alpha\left(\sum Ax_i \otimes By_i\right) \leq \|A\| \|B\| \alpha\left(\sum x_i \otimes y_i\right)$$

for any bounded linear operators A and B. The completion of the normed linear space $X \otimes Y$ with a cross-norm α is denoted by $X \underset{\alpha}{\otimes} Y$. The projective norm is a special uniform cross-norm defined by the equation

$$\|z\|_{\wedge} = \inf\left\{\sum \|x_i\| \|y_i\| : x_i \in X,\ y_i \in Y,\ z = \sum x_i \otimes y_i\right\},$$

and the completion of $X \otimes Y$ under such norm is denoted by $X \overset{\wedge}{\otimes} Y$. The injective norm is a special uniform cross-norm defined by the equation

$$\left\| \sum x_i \otimes y_i \right\|_{\vee} = \sup_{f \in X^*} \left\| \sum \langle f, x_i \rangle \, y_i \right\|,$$

and the completion of $X \otimes Y$ under such norm is denoted by $X \overset{\vee}{\otimes} Y$. for given Banach spaces X and Y, we have the following equation

$$X^* \overset{\vee}{\otimes} Y^* \subset L(X, Y^*) = \left(X \overset{\wedge}{\otimes} Y \right)^*.$$

The idetifications made here are as follows. With an element $\sum \phi_i \otimes \psi_i$ in $X^* \otimes Y^*$ (uncompleted tensor product) we associate an operator $A \in L(X, Y^*)$ whose defining equation is $Ax = \sum \langle \phi_i, x \rangle \, \psi_i$. With an arbitrary B in $L(X, Y^*)$ we associate a funcional Φ in $\left(X \overset{\wedge}{\otimes} Y \right)^*$ by putting $\Phi \left(\sum x_i \otimes y_i \right) = \sum \langle B x_i, y_i \rangle$.

The weak$^*-$topology in $L(X, Y^*)$ is the weak topology induced by the duality of $X \overset{\wedge}{\otimes} Y$ with $\left(X \overset{\wedge}{\otimes} Y \right)^*$. Convergence of a net A_α to A in this topology means $\langle A_\alpha x, y \rangle = \langle A x, y \rangle$ for all $x \in X$ and $y \in Y$.

It is known, [7], that $L \left(\ell^p, X \right)$ is isometrically isomorphic to

$$\ell^{p^*} (X) = \left\{ (x_n) : \sum \| x_n \|^{p^*} < \infty \right\}.$$

Where p^* is the conjugate of p. Hence $L \left(\ell^1, X \right) = \ell^\infty (X)$.

For any measure space (S, Σ, μ) and any Banach space X, it is known that $L^1 (\mu, X) = L^1 (\mu) \overset{\wedge}{\otimes} X$. In particular $\ell^1 (X) = \ell^1 \overset{\wedge}{\otimes} X$. Thus, for any Banach space X, we have

$$\ell^\infty (X^*) = L \left(\ell^1, X^* \right) = \left(\ell^1 \overset{\wedge}{\otimes} X \right)^* = \left(\ell^1 (X) \right)^*.$$

Remark 2.1 For a Banach Space X, $\ell^\infty \overset{\vee}{\otimes} X$ can be identified with a closed subspace of $\ell^\infty (X)$. The identification as follows :

With each element $z = \sum_{i=1}^n F_i \otimes x_i$ in $\ell^\infty \otimes X$, $F_i = \left(a_k^{(i)} \right)_{k=1}^\infty \in \ell^\infty$, we associate an element F_z in $\ell^\infty (X)$ to be defining by

$$F_z = \sum_{i=1}^n \left(a_k^{(i)} x_i \right)_{k=1}^\infty.$$

Observe that

$$\begin{aligned}
\|z\|_{\vee} &= \sup_{\psi\in X^*,\|\psi\|=1} \left\|\Sigma_{i=1}^{n}\psi(x_i)F_i\right\|_{\infty} \\
&= \sup_{\psi\in X^*,\|\psi\|=1} \sup_{k} \left|\Sigma_{i=1}^{n}\psi(x_i)a_k^{(i)}\right| \\
&= \sup_{k} \sup_{\psi\in X^*,\|\psi\|=1} \left|\Sigma_{i=1}^{n}\psi(x_i)a_k^{(i)}\right| \\
&= \sup_{k} \left\|\Sigma_{i=1}^{n}x_i a_k^{(i)}\right\| \\
&= \|F_z\|.
\end{aligned}$$

Thus the linear map $z \mapsto F_z$, after being extended by continuity is an isometry of $\ell^{\infty} \overset{\vee}{\otimes} X$ into $\ell^{\infty}(X)$.

3 Main Result

Throughout this section, n is agiven natural number and N is a monotonous norm in R^n.

First, we present the following two lemmas needed to prove our main result.

Lemma 3.1. Let X be a Banach Space. If G is a reflixive subspace of a Banach Y, then $L(X,G)$ is w^*- closed in $L(X,Y^{**})$.

Proof. Now, we have

$$L(X,G) = L(X,G^{**}) \subset L(X,Y^{**}) = (X \overset{\wedge}{\otimes} Y^*)^*.$$

So, let $(F_\alpha) \subset L(X,G)$ be a net converging to $F \in (X \overset{\wedge}{\otimes} Y^*)^*$ in the w^*-topology on $(X \overset{\wedge}{\otimes} Y^*)^*$. Let $x \in X$ and $y^* \in Y^*$. Then

$$\lim_{\alpha} F_\alpha(x \otimes y^*) = F(x \otimes y^*).$$

Thus

$$\lim_{\alpha} y^*(F_\alpha(x)) = y^*(F(x)).$$

Since $y^* \in Y^*$ was arbitrary, then $(F_\alpha(x)) \subset G = G^{**}$ is a net converging to $F(x)$ in the w^*-topology on Y^{**}. Since G is w^*- closed in Y^{**} and $x \in X$ was arbitrary, then $F(x) \in G$ for all $x \in X$ and $F \in L(X,G)$.

Lemma 3.2. ([14, Lemma 2.2]). Let X be a Banach space and let $A \subset X^*$. If A is w^*- closed, then A is N-simultaneuosly proximinal in X^*.

Theorem 3.3. Let X be a Banach space. If G is a reflexive subspace of a Banach Y, then $L(X,G)$ is N-simultaneuosly proximinal in $L(X,Y)$.

Proof. Let $f_1, f_2, ..., f_n \in L(X,Y) \subset L(X,Y^{**})$. Since $L(X,G)$ is w^*-closed in $L(X,Y^{**})$ (Lemma 3.1), then it is N-simultaneuosly proximinal in $L(X,Y^{**})$ (Lemma 3.2). Therefore, there exists $g_0 \in L(X,G)$, a best $N-$simultaneous approximation from $L(X,G)$ of the vectors $f_1, f_2, ..., f_n \in L(X,Y)$. Of course, this means that $L(X,G)$ is N-simultaneuosly proximinal in $L(X,Y)$.

Theorem 3.4. Let X be a Banach space G be a closed subspace of a Banach Y. If $L(X,G)$ is N-simultaneuosly proximinal in $L(X,Y)$, then G is N-simultaneuosly proximinal in Y.

Proof. Let $y_1, y_2, ..., y_n \in Y$. Let $x_0 \in X$, $\|x_0\| = 1$. Choose $x^* \in X^*$, $\|x^*\| = 1$, such that $< x_0, x^* >= 1$. The elements $x^* \otimes y_i : X \to Y$, $i = 1, 2, ..., n$, defined by $x^* \otimes y_i(x) = x^*(x) y_i$, are in $L(X,Y)$. Since $L(X,G)$ is N-simultaneously proximinal in $L(X,Y)$, there exists $g_0 \in L(X,G)$ such that

$$N\begin{pmatrix} \|x^* \otimes y_1 - g_0\|, \|x^* \otimes y_2 - g_0\|, ... \\ , \|x^* \otimes y_n - g_0\| \end{pmatrix} \leq N\begin{pmatrix} \|x^* \otimes y_1 - g\|, \|x^* \otimes y_2 - g\|, ... \\ , \|x^* \otimes y_n - g\| \end{pmatrix},$$

for all $g \in L(X,G)$. In particular this inequality is valid if we choose $g = x^* \otimes z$, for some $z \in G$. So,

$$\begin{aligned} N\begin{pmatrix} \|x^* \otimes y_1 - g_0\|, \|x^* \otimes y_2 - g_0\|, ... \\ , \|x^* \otimes y_n - g_0\| \end{pmatrix} &\leq N\begin{pmatrix} \|x^* \otimes y_1 - x^* \otimes z\|, \|x^* \otimes y_2 - x^* \otimes z\|, \\ ..., \|x^* \otimes y_n - x^* \otimes z\| \end{pmatrix} \\ &\leq \|x^*\| N(\|y_1 - z\|, \|y_2 - z\|, ..., \|y_n - z\|) \\ &= N(\|y_1 - z\|, \|y_2 - z\|, ..., \|y_n - z\|). \end{aligned}$$

Hence, for any $x \in X$, $\|x\| = 1$, we have

$$N\begin{pmatrix} \|x^* \otimes y_1(x) - g_0(x)\|, \|x^* \otimes y_2(x) - g_0(x)\|, ... \\ , \|x^* \otimes y_n(x) - g_0(x)\| \end{pmatrix} \leq N(\|y_1 - z\|, \|y_2 - z\|, ..., \|y_n - z\|).$$

Choose x to equal x_0. However, $(x^* \otimes y_i)(x_0) = y_i$, $i = 1, 2, ..., n$. Thus

$$N(\|y_1 - g_0(x_0)\|, \|y_2 - g_0(x_0)\|, ..., \|y_n - g_0(x_0)\|) \leq N(\|y_1 - z\|, \|y_2 - z\|, ..., \|y_n - z\|).$$

for all $z \in G$. Consequently, $g_0(x_0)$ is a best $N-$simultaneous approximation of the vectors $y_1, y_2, ..., y_n \in Y$. Hence G is $N-$simultaneously proximinal in Y.

Theorem 3.5. Let X and Y be Banach spaces. If P is a projecton on X, Q is a projecton on Y, and Q^* is the adjoint operator of Q, then the following subspaces are $N-$simultaneously proximinal in $L(X, Y^*)$.

1. $M = \{AP : A \in L(X, Y^*)\}$.
2. $N = \{Q^*B : B \in L(X, Y^*)\}$.
3. $W = \{AP + Q^*B : A, B \in L(X, Y^*)\}$.

Proof. The proof follows from Lemma 3.2 and the fact that the subspaces N, M, and W are w^*-closed in $L(X, Y^*)$, [8].

Defnition 3.6. A subspace G of a Banach space X is called p-summand, $1 \leq p < \infty$, if there exists a closed subspace $W \subset X$, such that $X = G \oplus W$, and for $x = g + w$, one has $\|x\| = (\|g\|^p + \|w\|^p)^{\frac{1}{p}}$. We write $X = G \oplus_p W$.

If G is 1-summand in X, then the projection $P : X \longrightarrow G$, $P(g + w) = g$, is called an L^1- projection of X onto G, then G is called $1-$ complemented in X. In case $\|x\| = \max\{\|g\|, \|w\|\}$, we call G an $\infty-$ summand. We refer to [1] for more on contractive projections.

Lemma 3.7. Let X and Y be Banach spaces. If G is 1-summand of X,

$X = G \oplus_1 W$, then

$$L(X,\ Y) = L(G,\ Y) \oplus_\infty L(W,\ Y).$$

Proof. For $f_1 + f_2 \in L(G,\ Y) \oplus_\infty L(W,\ Y)$, define $f : G \oplus_1 W \to Y$ by $f(g + w) = f_1(g) + f_2(w)$. Now, define

$$\psi :\ L(G,\ Y) \oplus_\infty L(W,\ Y) \to L(G \oplus_1 W,\ Y)$$

by $\psi(f_1 + f_2) = f$. Let $g + w \in G \oplus_1 W$ with $\|g + w\| = 1$, then

$$\begin{aligned}
\|f(g + w)\| &\leq \|f_1(g)\| + \|f_2(w)\| \\
&\leq \|f_1\| \|g\| + \|f_2\| \|w\| \\
&\leq \max\{\|f_1\|, \|f_2\|\} (\|g\| + \|w\|) \\
&= \max\{\|f_1\|, \|f_2\|\} \|g + w\| \\
&= \|f_1 + f_2\|.
\end{aligned}$$

Since $g + w \in X$, $\|g + w\| = 1$ was arbitrary, then $\|f\| \leq \|f_1 + f_2\|$. For the convers inequality,

$$\begin{aligned}
\|f\| &= \sup_{\|g\|+\|w\|=1} \|f_1(g) + f_2(w)\| \\
&\geq \sup_{\|g\|=1} \|f_1(g)\| = \|f_1\|.
\end{aligned}$$

Similarly, $\|f\| \geq \|f_2\|$, and hence $\|f\| \geq \max\{\|f_1\|, \|f_2\|\}$. Thus $\|f\| = \max\{\|f_1\|, \|f_2\|\}$. Then one can easly check that ψ is onto isometry.

Theorem 3.8. Let X and Y be Banach spaces. If G is 1-summand of X, then $L(G, Y^*)$ is $N-$simultaneously proximinal in $L(X, Y^*)$.

Proof. Assume that $X = G \oplus_1 W$. Let $P_1 : X \to G$ be the L^1-Projection. Using Lemma 3.7, one can easly show that $M = \{AP_1 : A \in L(X, Y^*)\}$ is isometrically isomorphic to $L(G, Y^*)$. The result follows from Theorem 3.5.

Lemma 3.9. Let X be a reflexive Banach space with the approximation property, and G be a reflexive subspace of a Banach space Y. Then $X \overset{\wedge}{\otimes} G$ is w^*-closed in $X \overset{\wedge}{\otimes} Y^{**}$.

Proof. Since X is reflexive, then X has the Radon Nikodym property, [4]. Hence by assumption on X, $X \overset{\wedge}{\otimes} Y^{**} = (X^* \overset{\vee}{\otimes} Y^*)^*$ [4]. Let S be a w^*-limit point of $X \overset{\wedge}{\otimes} G$ in $X \overset{\wedge}{\otimes} Y^{**}$. Then, we can assume the existence of a net (S_α) in $X \overset{\wedge}{\otimes} G$ such that $\lim_\alpha S_\alpha = S$ in the w^*-topology on $X \overset{\wedge}{\otimes} Y^{**}$. Let $x^* \in X^*$ and $y^* \in Y^*$. Then,

$$\begin{aligned} \lim_\alpha S_\alpha(x^* \otimes y^*) &= S(x^* \otimes y^*), \\ \lim_\alpha \langle y^*, S_\alpha(x^*) \rangle &= \langle y^*, S(x^*) \rangle . \end{aligned}$$

Since $y^* \in Y^*$ is an arbitrary, then $(S_\alpha(x^*))$ is a net in $G^{**} = G$ that converging to $S(x^*)$ in the w^*-topology on Y^{**}. Since G is w^*-closed in Y^{**}, then $S(x^*) \in G$. Since $x^* \in X^*$ was an arbitrary, then $S(x^*) \in G$ for all $x^* \in X^*$ and $S \in X \overset{\wedge}{\otimes} G$. Thus $X \overset{\wedge}{\otimes} G$ is w^*-closed in $X \overset{\wedge}{\otimes} Y^{**}$.

Theorem 3.10. Let X be a reflexive Banach space with the approximation property, n be a natural number, N be a monotonous norm in R^n, and G be a reflexive subspace of a Banach space Y. Then $X \overset{\wedge}{\otimes} G$ is $N-$simultaneously proximinal in $X \overset{\wedge}{\otimes} Y$.

Proof. Let $f_1, f_2, ..., f_n \in X \overset{\wedge}{\otimes} Y \subset X \overset{\wedge}{\otimes} Y^{**}$. Since $X \overset{\wedge}{\otimes} G$ is w^*- closed in $X \overset{\wedge}{\otimes} Y^{**}$ (Lemma 3.9), then it is N-simultaneuosly proximinal in $X \overset{\wedge}{\otimes} Y^{**}$(Lemma 3.2). Therefore, there exists $g_0 \in X \overset{\wedge}{\otimes} G$, a best $N-$simultaneous approximation from $X \overset{\wedge}{\otimes} G$ of the vectors $f_1, f_2, ..., f_n \in X \overset{\wedge}{\otimes} Y$. Of course, this means that $X \overset{\wedge}{\otimes} G$ is N-simultaneuosly proximinal in in $X \overset{\wedge}{\otimes} Y$.

Lemma 3.11. Let G be a w^*-closed subspace of a dual space X. Then $L(\ell^1, G)$ is w^*-closed in $\left(\ell^1(X^*)\right)^*$.

Proof. Let $(x_n) \in \left(\ell^1(X^*)\right)^*$ be a w^*-limit piont of $L(\ell^1, G) = \ell^\infty(G)$. Then, we can assume the existence of a sequence (H_n) in $\ell^\infty(G)$ such that $\lim\limits_n H_n = (x_n)$ in the w^*-topology on $\left(\ell^1(X^*)\right)^*$. Assume that $H_n = (h_m^n)_{m=1}^\infty$ for all n. Let $(s_n^*) \in \ell^1(X^*)$, then

$$\begin{aligned} \lim_{m\to\infty} \left\langle (s_n^*), H_m \right\rangle &= \left\langle (s_n^*), (x_n) \right\rangle, \\ \lim_{m\to\infty} \left(\sum_{n=1}^\infty \left\langle h_n^m, s_n^* \right\rangle \right) &= \left(\sum_{n=1}^\infty \left\langle s_n^*, x_n \right\rangle \right). \end{aligned}$$

Take $(s_n^*) = (0, 0, ..., x^*_{n^{th}}, 0, ...)$, then $\lim\limits_{m\to\infty} x^*\left(h_n^m\right) = x^*\left(x_n\right)$ and this is true for all $x^* \in X^*$. Thus $h_n^m \xrightarrow{w^*} x_n$ for all n. Since G is w^*-closed, then $x_n \in G$ for all n, then $(x_n) \in \ell^\infty(G)$. This ends the proof.

Theorem 3.12. Let G be a w^*-closed subspace of a dual space X, n be a natural number, and N be a monotonous norm in R^n. Then $L(\ell^1, G)$ is $N-$simultaneously proximinal in $\ell^\infty(X)$.

Proof. By Lemma 3.2 and Lemma 3.11, $L(\ell^1, G)$ is $N-$simultaneously proximinal in $\left(\ell^1(X^*)\right)^*$. Since

$$L(\ell^1, G) = \ell^\infty(G) \subset \ell^\infty(X) \subset \left(\ell^1(X^*)\right)^*,$$

then $L(\ell^1, G)$ is $N-$simultaneously proximinal in $\ell^\infty(X)$.

From Remark (2.1), we deduce the following

$$\ell^\infty \overset{\vee}{\otimes} G \subset \ell^\infty \overset{\vee}{\otimes} X \subset \ell^\infty(X) \subset \left(\ell^1(X^*)\right)^*.$$

Thus we have the following result.

Theorem 3.13. Let G be a closed subspace of a Banach space X. If $\ell^\infty \overset{\vee}{\otimes} G$ is w^*-closed in $\left(\ell^1(X^*)\right)^*$, then $\ell^\infty \overset{\vee}{\otimes} G$ is $N-$simultaneously proximinal in $\ell^\infty \overset{\vee}{\otimes} X$.

Proof. The result follows from Lemma(3.2).

In [14] it is shown that if G is a reflexive subspace of a Banach space Y, then $L^\infty(\mu, G)$ is $N-$simultaneously proximinal in $L^\infty(\mu, Y)$. As a result on $L^\infty(\mu, X)$, we have :

Theorem 3.14. If $L^{\infty}(\mu, G)$ is $N-$simultaneously proximinal in $L^{\infty}(\mu, Y)$, then G is $N-$simultaneously proximinal in Y.

Proof. Let $y_1, y_2, ..., y_n \in Y$. Let 1 be the constant function on Ω. Then $1 \otimes y_i : \Omega \to Y$, $i = 1, 2, ..., n$, defined by $1 \otimes y_i(s) = y_i$, are elements in $L^{\infty}(\mu, Y)$. Since $L^{\infty}(\mu, G)$ is N-simultaneously proximinal in $L^{\infty}(\mu, Y)$, there exists $g_0 \in L^{\infty}(\mu, G)$ such that

$$N\left(\begin{array}{c} \|1 \otimes y_1 - g_0\|, \|1 \otimes y_2 - g_0\|, ... \\ , \|1 \otimes y_2 - g_0\| \end{array}\right) \leq N\left(\begin{array}{c} \|1 \otimes y_1 - g\|, \|1 \otimes y_2 - g\|, ... \\ , \|1 \otimes y_2 - g\| \end{array}\right),$$

for all $g \in L^{\infty}(\mu, G)$. In particular this inequality is valid if we choose $g = 1 \otimes z$, for some $z \in G$. So

$$\begin{aligned} N\left(\begin{array}{c} \|1 \otimes y_1 - g_0\|, \|1 \otimes y_2 - g_0\|, ... \\ , \|1 \otimes y_2 - g_0\| \end{array}\right) &\leq N\left(\begin{array}{c} \|1 \otimes y_1 - 1 \otimes z\|, \|1 \otimes y_2 - 1 \otimes z\|, ... \\ , \|1 \otimes y_n - 1 \otimes z\| \end{array}\right) \\ &= N(\|y_1 - z\|, \|y_2 - z\|, ..., \|y_n - z\|). \end{aligned}$$

There exists $s_0 \in \Omega$ such that

$$\begin{aligned} N\left(\begin{array}{c} \|1 \otimes y_1(s_0) - g_0(s_0)\|, \|1 \otimes y_2(s_0) - g_0(s_0)\|, ... \\ , \|1 \otimes y_n(s_0) - g_0(s_0)\| \end{array}\right) &\leq N\left(\begin{array}{c} \|y_1 - z\|, \|y_2 - z\|, ... \\ , \|y_n - z\| \end{array}\right), \\ N\left(\begin{array}{c} \|y_1 - g_0(s_0)\|, \|y_2 - g_0(s_0)\|, ... \\ , \|y_n - g_0(s_0)\| \end{array}\right) &\leq N\left(\begin{array}{c} \|y_1 - z\|, \|y_2 - z\|, ... \\ , \|y_n - z\| \end{array}\right). \end{aligned}$$

Since $z \in G$ was arbitrary, then $g_0(s_0) \in G$ is a best $N-$simultaneous approximation for $y_1, y_2, ..., y_n \in Y$ and G is $N-$simultaneously proximinal in Y.

Theorem 3.15. Let G be a closed subspace of the Banach space Y. If $X \overset{\vee}{\otimes} G$ is $N-$simultaneously proximinal in $X \overset{\vee}{\otimes} Y$, then G is $N-$simultaneously proximinal in Y.

Proof. Let y_1, $y_2, ..., y_n \in Y$. Let $x \in X$, $\|x\| = 1$. Choose $x^* \in X^*$, $\|x^*\| = 1$, such that $< x, x^* >= 1$. The elements $x \otimes y_1$, $x \otimes y_2, ..., x \otimes y_n$ are in $X \overset{\vee}{\otimes} Y$. Since $X \overset{\vee}{\otimes} G$ is N-simultaneously proximinal in $X \overset{\vee}{\otimes} Y$, there exists $z_0 \in X \overset{\vee}{\otimes} G$ such that

$$N\left(\begin{array}{c} \|x \otimes y_1 - z_0\|, \|x \otimes y_2 - z_0\|, ..., \\ \|x \otimes y_n - z_0\| \end{array}\right) \leq N\left(\begin{array}{c} \|x \otimes y_1 - z\|, \|x \otimes y_2 - z\|, ..., \\ \|x \otimes y_n - z\| \end{array}\right),$$

for all $z \in X \overset{\vee}{\otimes} G$. In particular this inequality is valid if we choose $z = x \otimes g$, for some $g \in G$. So

$$N\left(\begin{array}{c}\|x\otimes y_1-z_0\|,\|x\otimes y_2-z_0\|,...,\\ \|x\otimes y_n-z_0\|\end{array}\right) \leq N\left(\begin{array}{c}\|x\otimes y_1-x\otimes g\|,\|x\otimes y_2-x\otimes g\|,...\\ ,\|x\otimes y_n-x\otimes g\|\end{array}\right)$$
$$\leq \|x\| N\left(\begin{array}{c}\|y_1-g\|,\|y_2-g\|,...\\ ,\|y_n-g\|\end{array}\right).$$

Hence, for any $v^*\in X^*$, $\|v^*\|=1$, we have

$$N\left(\begin{array}{c}\|(x\otimes y_1)(v^*)-z_0(v^*)\|,\|(x\otimes y_2)(v^*)-z_0(v^*)\|,...\\ ,\|(x\otimes y_n)(v^*)-z_0(v^*)\|\end{array}\right)\leq N\left(\begin{array}{c}\|y_1-g\|,\|y_2-g\|,...\\ ,\|y_n-g\|\end{array}\right).$$

Choose v^* to equal x^*. However, $(x\otimes y_i)(x^*)=y_i$, $i=1,2,...,n$. Thus

$$N\left(\begin{array}{c}\|y_1-z_0(v^*)\|,\|y_2-z_0(v^*)\|,...\\ ,\|y_n-z_0(v^*)\|\end{array}\right)\leq N\left(\begin{array}{c}\|y_1-g\|,\|y_2-g\|,...\\ ,\|y_n-g\|\end{array}\right),$$

for all $g\in G$. Consequently, $z_0(x^*)$ is a best N-simultaneous approximation of y_1, $y_2,...,y_n\in Y$. Hence G is N-simultaneously proximinal in Y.

Let S be a compact Hausdorff space, X be a Banach space. we denote $C(S,X)$ to the Banach space of all $X-$valued contuous functions on S equipped with supremum norm. If $X=R$, then we write $C(S)$ instead of $C(S,R)$.

Corollary 2.16. If $C(S,G)$ is N-simultaneously proximinal in $C(S,X)$, then G is N-simultaneously proximinal in X.

Proof. The proof follows from Theorem 2.15 and the fact that $C(S,X)=C(S)\overset{\vee}{\otimes}X$, [8].

Theorem 3.17. If $L^1(\mu)\overset{\wedge}{\otimes}G$ is $N-$simultaneously proximinal in $L^1(\mu)\overset{\wedge}{\otimes}Y$, then G is $N-$simultaneously proximinal in Y.

Proof. The proof is similar to that given in Theorem 3.15.

In [10] it is shown that if G is a reflexive subspace of a Banach space Y, then $L^1(\mu,G)$ is $N-$simultaneously proximinal in $L^1(\mu,Y)$. As a result on $L^1(\mu,Y)$, we have :

Theorem 3.18. If $L^1(\mu,G)$ is $N-$simultaneously proximinal in $L^1(\mu,Y)$, then G is $N-$simultaneously proximinal in Y.

Proof. The proof follows from Theorem 3.17 and the fact that $L^1(\mu, Y) = L^1(\mu) \overset{\wedge}{\otimes} Y$, [8].

Let S, T be compact Hausdorff spaces, then $C(S \times T) = C(S) \overset{\vee}{\otimes} C(T)$, [8]. Now we have the following result

Theorem 3.19. If G is a subspace of $C(S)$ such that $G \overset{\vee}{\otimes} C(T)$ is $N-$simultaneously proximinal in $C(S \times T)$, then G is $N-$simultaneously proximinal in $C(S)$.

Proof. Assume that $G \overset{\vee}{\otimes} C(T)$ is $N-$simultaneously proximinal in $C(S \times T)$. Let $x_1, x_2, ..., x_n \in C(S)$. Put $x_i^{'}(s,t) = x_i(s)$ for all $(s,t) \in S \times T$ and $i = 1, 2, ..., n$. Let z be a best $N-$simultaneous approximation for $x_1^{'}, x_2^{'}, ..., x_n^{'}$ from $G \overset{\vee}{\otimes} C(T)$. Note that for any $g \in G$,

$$\begin{aligned} N\begin{pmatrix} \left\|x_1^{'} - z\right\|, \left\|x_2^{'} - z\right\|, ... \\ , \left\|x_n^{'} - z\right\| \end{pmatrix} &\leq N\begin{pmatrix} \left\|x_1^{'} - g \otimes 1\right\|, \left\|x_2^{'} - g \otimes 1\right\|, ... \\ , \left\|x_n^{'} - g \otimes 1\right\| \end{pmatrix} \\ &= N\left(\|x_1 - g\|, \|x_2 - g\|, ..., \|x_n - g\|\right). \end{aligned}$$

Let $\tau \in T$ and put $h(s) = z(s,t)$. Then $h \in G$ and

$$N\left(\|x_1 - h\|, \|x_2 - h\|, ..., \|x_n - h\|\right) \leq N\left(\|x_1 - g\|, \|x_2 - g\|, ..., \|x_n - g\|\right)$$

for all $g \in G$. Hence h is a best $N-$simultaneous approximation for $x_1, x_2, ..., x_n$ from G.

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A Rank-One Fitting Method for Solving Symmetric Nonlinear Equations *

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Abstract. In this paper, a rank-one updated method for solving symmetric nonlinear equations is proposed. This method possesses some features: (1) The updated matrix is positive definite whatever line search technique is used; (2) The search direction is descent for the norm function; Under reasonable conditions, we establish its global convergence. Numerical results show that the presented method is competitive to the normal BFGS method for the test problems.

Key Words. rank-one update; global convergence; nonlinear equations.

AMS subject classifications. 90C26.

1. Introduction

Consider the following system of nonlinear equations:

$$F(x) = 0,\ x \in \Re^n, \tag{1.1}$$

where $F : \Re^n \to \Re^n$ is continuously differentiable and the Jaconbian $\nabla F(x)$ of $F(x)$ is symmetric for all $x \in \Re^n$. Let θ be the norm function defined by $\theta(x) = \frac{1}{2}\|F(x)\|^2$. Then the nonlinear equation (1.1) is equivalent to the following global optimization problem

$$\min \theta(x),\ \ x \in \Re^n. \tag{1.2}$$

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The following iterative method is used for solving (1.1)

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.3}$$

where x_k is the current iterative point, d_k is a search direction, and α_k is a positive step-size.

It is well known that there are many methods [9, 11, 23, 24, 26, 29] for the unconstrained optimization problems $\min_{x\in\Re^n} f(x)$ (UOP), where the BFGS method is one of the most effective quasi-Newton methods [2, 3, 4, 6, 28, 30, 31]. These years, lots of modified BFGS methods (see [13, 14, 17, 19, 32]) have been proposed for UOP. Especially, many efficient attempts have been made to modify the usual quasi-Newton methods using both the gradient and function values information (e.g. [18, 32]). Lately, in order to get a higher order accuracy in approximating the second curvature of the objective function, Wei, Yu, Yuan, and Lian [19] proposed a new BFGS-type method for UOP, and the reported numerical results showed that the average performance is better than that of the standard BFGS method. The superlinear convergence of this modified has been established for uniformly convex function. Its global convergence is established by Wei, Li, and Qi [18]. Motivated by their ideas, Yuan and Wei [31] presented a modified BFGS method which can ensure that the update matrices are positive definite for the general convex functions. Moreover, the global convergence is proved for the general convex functions. For general functions, it is now known that the BFGS method may fail for non-convex functions with inexact line search [4], and Mascarenhas [15] showed that the nonconvergence of the standard BFGS method even with exact line search. In order to obtain a global convergence of BFGS method without convexity assumption on the objective function, Li and Fukushima [13, 14] made a slight modification to the standard BFGS method. Different from above techniques, Xu [20] presented a rank-one fitting algorithm for UOP, and the numerical examples are very interesting. Motivated by their ideas, we give a new rank-one fitting algorithm for (1.1) which possesses the global convergence. The method can ensure that the updated matrices are positive definite without carrying out any line search, the search direction is descent for the normal function, and the numerical results is competitive to the BFGS method.

For nonlinear equations, the global convergence is due to Griewank [10] for Broyden's rank one method. Fan [8], Yuan [21], Yuan, Lu and Wei [27], and Zhang [33] presented the trust region algorithms for nonlinear equations. Zhu [34] gave a family of nonmonotone backtracking inexact quasi-Newton algorithms for solving smooth nonlinear equations. In particular, a Gauss-Newton-based BFGS method is proposed by Li and Fukushima [12] for solving symmetric nonlinear equations, and the modified methods [22, 25] are studied.

We all know that the BFGS method for solving (1.1) is to generate a sequence of iterates $\{x_k\}$ by letting $x_{k+1} = x_k + \alpha_k d_k$, where α_k is the steplength and d_k is a solution of the system of linear equation

$$B_k d_k + F(x_k) = 0, \tag{1.4}$$

where B_k is generated by BFGS update formula

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{s_k^T y_k}, \tag{1.5}$$

where $s_k = x_{k+1} - x_k$, and $y_k = F(x_{k+1}) - F(x_k)$. In the following, we briefly review some line search technique to obtain the stepsize α_k.

Brown and Saad [1] proposed the following line search method:

$$\theta(x_k + \alpha_k d_k) - \theta(x_k) \le \sigma \alpha_k \nabla \theta(x_k)^T d_k, \tag{1.6}$$

where $\theta(x_k)^T d_k = F(x_k)^T \nabla F(x_k) d_k$, $\sigma \in (0,1)$, $\alpha_k = r^{i_k}$, $r \in (0,1)$, and i_k is the smallest nonnegative integer i such that (1.6). Based on this technique, Zhu [34] gave the nonmonotone line search technique:

$$\theta(x_k + \alpha_k d_k) - \theta(x_{l(k)}) \le \sigma \alpha_k \nabla \theta(x_k)^T d_k,$$

$\|\theta(x_{l(k)})\| = \max_{0\leq j\leq m(k)}\{\|\theta(x_{k-j})\|\}$, $m(0) = 0$, $m(k) = \min\{m(k-1)+1, M\}$, $k \geq 1$, and M is a nonnegative integer. From these two techniques, it is easy to see that the Jacobian matric $\nabla F(x_k)$ must be computed at every iteration, which will increase the workload especially for large-scale problems or this matric is expensive. Considering these points, Yuan and Lu [25] presented a new backtracking inexact technique to obtain the stepsize α_k :

$$\|F(x_k+\alpha_k d_k)\|^2 \leq \|F(x_k)\|^2 + \delta\alpha_k^2 F(x_k)^T d_k, \tag{1.7}$$

where $\delta \in (0,1)$ is a constant, and d_k is a solution of the system of linear equations (1.4). Li and Fukashima [12] give a line search technique to determine a positive step-size α_k satisfying

$$\|F(x_k+\alpha_k d_k)\|^2 - \|F(x_k)\|^2 \leq -\delta_1\|\alpha_k F(x_k)\|^2 - \delta_2\|\alpha_k d_k\|^2 + \varepsilon_k\|F(x_k)\|^2, \tag{1.8}$$

where δ_1 and δ_2 are positive constants, and $\{\varepsilon_k\}$ is a positive sequence such that

$$\sum_{k=0}^{\infty}\varepsilon_k < \infty. \tag{1.9}$$

The formula (1.8) means that $\{F(x_k)\}$ is norm descent when k is sufficiently large. In this paper, we also use the formula (1.8) as line search to find the step-size α_k.

Normally, the update matrix is defined by formula (1.5). Is there another way to determine the update formula? Accordingly the search direction d_k is determined by the way. In this paper, we give a positive answer. The updated matrix B_k is generated by the following rank-one updated formulas

$$B_{k+1} = B_k + v_k v_k^T, \tag{1.10}$$

$$H_{k+1} = H_k - \frac{H_k v_k v_k^T H_k}{1 + v_k^T H_k v_k}, \tag{1.11}$$

where, as $k = 0$, B_0 is the given symmetric positive definite matrix, $B_k^{-1} = H_k$, and $v_k = \delta_0\alpha_k F(x_k)$, here δ_0 is a positive constant. Then we use the following formula to get the search direction,

$$B_k d + q_k(\alpha_{k-1}) = 0, \tag{1.12}$$

where

$$q_k(\alpha_{k-1}) = \frac{F(x_k + \alpha_{k-1}F_k) - F(x_k)}{\alpha_{k-1}}, \tag{1.13}$$

B_k follows (1.10), α_{k-1} is the steplength used at the previous iteration, and the equation (1.13) is inspired by [12]. Throughout the paper, we use these notations: $\|\cdot\|$ is the Euclidean norm, and $F(x_k)$ and $F(x_{k+1})$ are replaced by F_k and F_{k+1}, respectively.

This paper is organized as follows. In the next section, the algorithm is stated. The global convergence convergence are established in Section 3. The numerical results are reported in Section 4.

2. The algorithm

In this section, we state our new algorithm based on formulas (1.3), (1.8), (1.10), (1.11) and (1.12) for solving (1.1).

Rank-One Updated Algorithm (ROUA).

Step 0: Choose an initial point $x_0 \in \Re^n$, constants $r \in (0,1)$, $0 < \delta_0,\ \delta_1,\ \delta_2 < 1$, $\alpha_{-1} > 0$, a positive sequence $\{\varepsilon_k\}$ satisfying (1.9), symmetric positive definite matrices B_0 and $B_0^{-1} = H_0$. Let: k = 0;

Step 1: If $\|F_k\| = 0$, stop. Otherwise, solving linear equations (1.12) to get d_k;

Step 2: Find a α_k is the largest number of $\{1, r, r^2, r^3, \cdots\}$ such that (1.8);

Step 3: Let the next iterative point be $x_{k+1} = x_k + \alpha_k d_k$;

Step 4: Update B_{k+1} and H_{k+1} by the formula (1.10) and (1.11) respectively;

Step 5: Set $k := k + 1$. Go to step 1.

In this paper, we also give the normal BFGS method for solving (1.1), and the algorithm which has the same conditions to ROUA is stated as follows.

BFGS Algorithm(BFGSA).

In ROUA, the step 4 is replaced by: Update B_{k+1} by the formula (1.5).

Remark 1. (a) By the step 0 of ROUA, there should exist constants $\lambda_1 \geq \lambda_0 > 0$ such that

$$\lambda_1 \|d\|^2 \geq d^T B_0 d \geq \lambda_0 \|d\|^2, \ \frac{1}{\lambda_0}\|d\|^2 \geq d^T H_0 d \geq \frac{1}{\lambda_1}\|d\|^2, \ \forall \ d \in R^n. \tag{2.1}$$

(b) By the step 4 of ROUA, it is easy to deduce that the updated matrices are symmetric.

3. Convergence Analysis

In this section, we establish the global convergence of ROUA. Let Ω be the level set defined by

$$\Omega = \{x | \|F(x)\| \leq e^{\frac{\varepsilon}{2}} \|F(x_0)\|\}, \tag{3.1}$$

where ε is a constant satisfying

$$\sum_{k=0}^{\infty} \varepsilon_k \leq \varepsilon. \tag{3.2}$$

Then the following lemma is satisfied (see [12]), here we also give its proof.

Lemma 3.1 *Let $\{x_k\}$ be generated by ROUA. Consider the line search (1.8). Then $\{x_k\} \subset \Omega$, moreover, $\{\|F(x)\|\}$ converges.*

Proof. By line search (1.8), we have

$$\|F_{k+1}\| \leq (1+\varepsilon_k)^{\frac{1}{2}} \|F_k\| \leq (1+\varepsilon_k)\|F_k\|.$$

Since ε_k satisfies (3.2), from Lemma 3.3 in [5], we conclude that $\{\|F_k\|\}$ converges. Moreover, we have for all k

$$\begin{aligned}
\|F_{k+1}\| &\leq (1+\varepsilon_k)^{\frac{1}{2}}\|F_k\| \\
&\leq \cdots \\
&\leq \prod_{i=0}^{k}(1+\varepsilon_i)^{\frac{1}{2}}\|F_0\| \\
&\leq \|F(x_0)\|[\frac{1}{k+1}\sum_{i-0}^{k}(1+\varepsilon_i)]^{\frac{k+1}{2}} \\
&= \|F(x_0)\|[1+\frac{1}{k+1}\sum_{i-0}^{k}\varepsilon_i]^{\frac{k+1}{2}} \\
&\leq \|F(x_0)\|[1+\frac{\varepsilon}{k+1}]^{\frac{k+1}{2}} \\
&\leq e^{\frac{\varepsilon}{2}}\|F(x_0)\|,
\end{aligned}$$

which implies that $\{x_k\} \subset \Omega$. The proof is complete.

In order to get the global convergence, the following assumptions are needed [12, 22].

Assumption A (i) F is continuously differentiable on an open convex set Ω_1 containing Ω.

(ii) The Jaconbian of F is symmetric, bounded and uniformly nonsingular on Ω_1, i.e., there exist constants $M \geq m > 0$ such that

$$\|\nabla F(x)\| \leq M, \quad \forall x \in \Omega_1, \tag{3.3}$$

and

$$\|\nabla F(x)d\| \geq m\|d\|, \quad \forall x \in \Omega_1,\ d \in \Re^n. \tag{3.4}$$

Remark 2. Assumption $A(ii)$ implies that

$$m\|d\| \leq \|\nabla F(x)d\| \leq M\|d\|,\ \forall\ x \in \Omega_1, d \in \Re^n, \tag{3.5}$$

$$m\|x-y\| \leq \|F(x)-F(y)\| \leq M\|x-y\|,\ \forall\ x, y \in \Omega_1. \tag{3.6}$$

In particular, for all $x \in \Omega_1$, we have

$$m\|x - x^*\| \leq \|F(x)\| = \|F(x) - F(x^*)\| \leq M\|x - x^*\|, \tag{3.7}$$

where x^* stands for the unique solution of (1.1) in Ω_1.

Lemma 3.2 *Let Assumption A hold and $\{\alpha_k, d_k, x_{k+1}, F_k\}$ be generated by ROUA. Then we have*

$$\sum_{k=0}^{\infty} \|\alpha_k F_k\|^2 < \infty, \tag{3.8}$$

and

$$\sum_{k=0}^{\infty} \|\alpha_k d_k\|^2 < \infty. \tag{3.9}$$

Proof. By the line search (1.8), we get

$$\delta_1\|\alpha_k F_k\|^2 + \delta_2\|\alpha_k d_k\|^2 \leq \|F_k\|^2 - \|F_{k+1}\|^2 + \varepsilon_k\|g_k\|^2. \tag{3.10}$$

Since $\{\varepsilon_k\}$ satisfies (3.2) and $\{\|F_k\|\}$ is bounded, we obtain (3.8) and (3.9) by summing these inequalities (3.10) for k from 0 to ∞. The proof is complete.

Lemma 3.3 *Let Assumption A hold. Consider ROUA. Then $\{\|B_k\|\}$ converges, and for any $d \in \Re^n$, then there exist constants m_0 and M_0 such that*

$$M_0\|d\|^2 \geq d^T B_k d \geq m_0\|d\|^2, \ for \ all \ k, \ \forall \ d \in R^n, \tag{3.11}$$

and

$$\frac{1}{m_0}\|d\|^2 \geq d^T H_k d \geq \frac{1}{M_0}\|d\|^2, \ for \ all \ k, \ \forall \ d \in R^n, \tag{3.12}$$

which means that the updated matrices are all positive by ROUA.

Proof. By the updated formula (1.10), we have

$$\begin{aligned} \|B_{k+1}\| &= \|B_k + v_k v_k^T\| \\ &\leq \|B_k\| + \|v_k\|^2 \\ &= \|B_k\| + \delta_0^2 \|\alpha_k F_k\|^2 \\ &\leq \|B_0\| + \delta_0^2 \sum_{i=0}^{k} \|\alpha_i F_i\|^2. \end{aligned} \tag{3.13}$$

By (3.8), we know that $\sum_{i=0}^{k} \|\alpha_i F_i\|^2$ is convergent. Then we can deduce that $\{\|B_k\|\}$ is convergent. So there exists a constant M_0 such that

$$\|B_k\| \leq M_0, \ for \ all \ k. \tag{3.14}$$

Accordingly, we get the left side of (3.11). Then, we prove the right side of (3.11). By ROUA, we know that the initial matrix B_0 is symmetric positive, then we have (2.1). Using (1.10), for $k \geq 1$, we have

$$\begin{aligned} d^T B_k d &= d^T B_{k-1} d + d^T v_k v_k^T d \\ &= d^T B_{k-1} d + (d^T v_k)^2 \\ &\geq d^T B_{k-1} d \\ &\geq \cdots \\ &\geq d^T B_0 d \geq \lambda_0 \|d\|^2, \end{aligned} \tag{3.15}$$

let $m_0 = \lambda_0$, thus we get (3.11).
By (3.11) and the Remark 1(b), we can deduce that the updated matrices are all symmetric and positive definite. Consider $H_k = B_k^{-1}$, we obtain (3.12) immediately. The proof is complete.

Since B_k is positive definite, then d_k which is determined by (1.12) has the unique solution.

Lemma 3.4 *Let Assumption A hold. If x_k is not a stationary point of (1.2), then there exists a constant $\alpha' > 0$ depending on k such that when $\alpha_{k-1} \in (0, \alpha')$, and the unique solution $d(\alpha_{k-1})$ of (1.12) such that*

$$\nabla\theta(x_k) d(\alpha_{k-1}) < 0. \tag{3.16}$$

Proof. By (1.13), we can deduce that

$$\lim_{\alpha_{k-1}\to 0} q_k(\alpha_{k-1}) = \nabla F(x_k)F(x_k). \tag{3.17}$$

From (1.12), we get

$$\begin{aligned}\lim_{\alpha_{k-1}\to 0^+} \nabla\theta(x_k)d(\alpha_{k-1}) &= -\lim_{\alpha_{k-1}\to 0^+} F(x_k)^T\nabla F(x_k)B_k^{-1}q_k(\alpha_{k-1}) \\ &= -F(x_k)^T\nabla F(x_k)B_k^{-1}\nabla F(x_k)F(x_k).\end{aligned} \tag{3.18}$$

Since x_k is not a stationary point of (1.2), we have $\nabla F(x_k)F(x_k) \neq 0$. By $\nabla F(x_k)$ is symmetric and B_k is positive, we obtain (3.16). The proof is complete.

By (3.11) and (3.14), we have

$$\|q_k(\alpha_{k-1})\| = \|B_kd_k\| \leq M_0\|d_k\|, \ \|d_k\| \leq \frac{1}{m_0}\|q_k(\alpha_{k-1})\|. \tag{3.19}$$

Now we establish the global convergence theorem of ROUA.

Theorem 3.1 *Let Assumption A hold and* $\{\alpha_k, d_k, x_{k+1}, g_{k+1}\}$ *be generated by ROUA. Then the sequence* $\{x_k\}$ *converges to the unique solution* x^* *of (1.1) in sense of*

$$\lim_{k\to\infty} \|F_k\| = 0. \tag{3.20}$$

Proof. By Lemma 3.1, we know that $\{\|F_k\|\}$ converges. By Lemma 3.2, we get

$$\lim_{k\to\infty} \|\alpha_kF_k\| = 0, \tag{3.21}$$

then, we have

$$\lim_{k\to\infty} \|F_k\| = 0 \tag{3.22}$$

or

$$\lim_{k\to\infty} \alpha_k = 0. \tag{3.23}$$

Therefore, we only discuss the case of (3.23). In this case, for all k sufficiently large and $\alpha'_k = \frac{\alpha_k}{r}$, by (1.8), we obtain

$$\begin{aligned}\|F(x_k + \alpha'_k d_k)\|^2 - \|F(x_k)\|^2 &> -\delta_1\|\alpha'_k F(x_k)\|^2 - \delta_2\|\alpha'_k d_k\|^2 + \varepsilon_k\|F(x_k)\|^2 \\ &> -\delta_1\|\alpha'_k F(x_k)\|^2 - \delta_2\|\alpha'_k d_k\|^2. \qquad (3.24)\end{aligned}$$

By Lemma 3.1, we know that $\{x_k\} \subset \Omega$ is bounded. Considering (3.19), it is easy to deduce that $\{q_k(\alpha_{k-1})\}$ and $\{d_k\}$ are bounded. Let $\{x_k\}$ and $\{d_k(\alpha)\}$ converge to x^* and d^*, respectively. Then we have

$$\lim_{k\to\infty} q_k(\alpha_{k-1}) = \nabla\theta(x^*). \qquad (3.25)$$

Let both sides of (3.24) be divided by α'_k and take limits as $k \to \infty$, we obtain

$$\theta(x^*)d^* \geq 0. \qquad (3.26)$$

By (3.11) and (1.12), we have

$$0 = d_k^T B_k d_k + q_k(\alpha_{k-1})^T d_k \geq m_0\|d_k\|^2 + q_k(\alpha_{k-1})^T d_k. \qquad (3.27)$$

As $k \to \infty$, taking limits in both of (3.27) yields

$$\nabla\theta(x^*)d^* \leq -m_0\|d^*\|^2.$$

This together with (3.26) implies $d^* = 0$. From (3.19), we have $\lim_{k\to\infty} q_k(\alpha_{k-1}) = 0$, which together with (3.25), we obtain

$$\nabla\theta(x^*) = 0. \qquad (3.28)$$

By $\nabla\theta(x^*) = F(x^*)\nabla F(x^*)$ and using that $\nabla F(x^*)$ is nonsingular, we have $F(x^*) = 0$. This implies (3.20). The proof is complete.

4. Numerical Results

In this section, we report results of some numerical experiments with ROUA and BFGSA.

The discretized two-point boundary value problem is similar to the problem in [16]

$$F(x) = Ax + \frac{1}{(n+1)^2}T(x) = 0,$$

where A is the $n \times n$ tridiagonal matrix given by

$$A = \begin{bmatrix} 8 & -1 & & & & \\ -1 & 8 & -1 & & & \\ & -1 & 8 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 8 \end{bmatrix},$$

and $T(x) = (T_1(x), T_2(x), ..., T_n(x))^T$ with $T_i(x) = sinx_i - 1, \quad i = 1, 2, ..., n.$ In the experiments, the parameters in ROUA and BFGSA were chosen as $r = 0.1$, $\delta_0 = 10^{-4}$, $\delta_1 = \delta_2 = 10^{-3}$, $\varepsilon_k = \frac{1}{k^2}$, and k is the number of iteration. The program was coded in MATLAB 6.5.1. We stopped the iteration when the condition $\|F(x)\| \le 10^{-6}$ was satisfied. The detailed numerical results are listed on the web site

$$http://210.36.18.9:8018/publication.asp?id = 34337.$$

Dolan and Moré [7] gave a new tool to analyze the efficiency of Algorithms. They introduced the notion of a performance profile as a means to evaluate and compare the performance of the set of solvers S on a test set P. Assuming that there exist n_s solvers and n_p problems, for each problem p and solver s, they defined

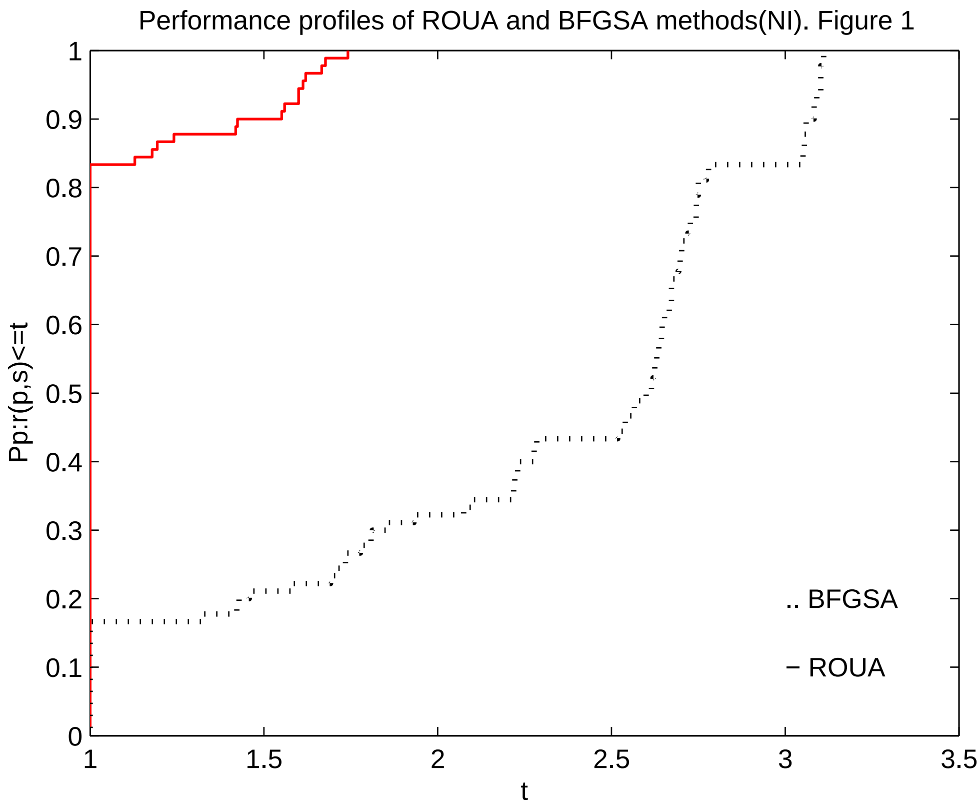
$t_{p,s}$ = computing time (the number of function evaluations or others) required to solve

problem p by solver s.

Requiring a baseline for comparisons, they compared the performance on problem p by solver s with the best performance by any solver on this problem; that is, using the performance ratio

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}.$$

Suppose that a parameter $r_M \geq r_{p,s}$ for all p, s is chosen, and $r_{p,s} = r_M$ if and only if solver s does not solve problem p.



The performance of solver s on any given problem might be of interest, but we would like to obtain an overall assessment of the performance of the solver, then they defined

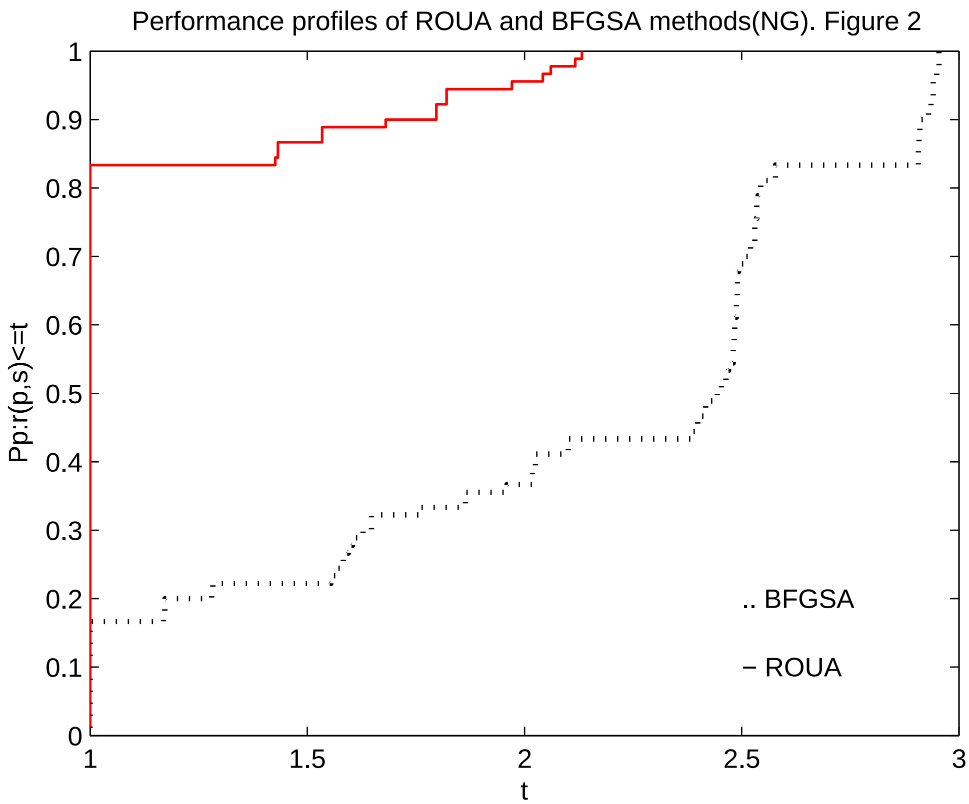
$$\rho_s(t) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq t\},$$

thus $\rho_s(t)$ was the probability for solver $s \in S$ that a performance ratio

$r_{p,s}$ was within a factor $t \in R$ of the best possible ration. Then function ρ_s was the (cumulative) distribution function for the performance ratio. The performance profile $\rho_s : R \mapsto [0,1]$ for a solver was a nondecreasing, piecewise constant function, and continuous from the right at each breakpoint. The value of $\rho_s(1)$ was the probability that the solver would win over the rest of the solvers.

According to the above rules, we know that one solver whose performance profile plot is on top right will win over the rest of the solvers.

Figures 1-2 show that the performance of these methods is relative to the total number of iterations (NI) and the number of the function evaluations (NG), respectively.



From Figures 1 and 2, it is easy to see that these two are successful for solving this problem, and the proposed method is competitive to the normal BFGS method. These two figures show that ROUA and BFGSA can completely solve the test problems.

5. Conclusion

In this paper, we propose a rank-one updated method for symmetric nonlinear equations. Numerical results show that the method is successful. The search direction of this method is descent for the norm function, and the updated matrix is positive definite without carrying out any line search. In fact, this method can be also explanted to the normal nonlinear equations. We hope that the method can be an important topic of further research for symmetric nonlinear equations.

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MINIMUM CONTROL ENERGY PROBLEM FOR INFINITE NEUTRAL DIFFERENTIAL SYSTEMS

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Abstract

This paper is aimed at establishing the minimum control energy for an infinite neutral differential system of the form

$$\frac{d}{dt}D(t)x_t = L(t, x, u) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta$$

when the controls are essentially bounded measurable functions on finite intervals, with values in a compact subset U of an m-dimensional Euclidean space with zero in its interior. By exploiting the properties of the reachable set, and the attainable set; complete controllability conditions were stated and proved. The existence, form and uniqueness of the minimum control energy problem for our system were also established. An example is also given.

Keywords: Controllability, neutral system, Infinite delay, control energy.

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1. INTRODUCTION

The controllability of neutral functional differential equation have been studied by several authors see Chukwu [2-3], Fu [14], Gahl [11], Underwood and Chukwu

[13] and independent results obtained. The control equations of linear neutral systems have applications in the study of electrical networks containing lossless transmission lines; electrodynamics, variation problems etc. (see Onwuatu [9]). These studies have been extended to the controllability of infinite neutral functional differential equations which has applications in ecology, epidemics, population growth and in some complex economic studies etc. Several authors Onwuatu [9], Balachandran and Anandhi [10], Davies [6] etc. have studied the controllability of such systems using different approaches. For example, Onwuatu [9] studied a class of nonlinear infinite neutral system, where he developed sufficient condition for the null controllability of such systems. Davies [6] studied the Euclidean null controllability of infinite neutral differential systems and established sufficient computable criteria for the Euclidean null controllability of such systems.

We are not only interested in the controllability of infinite neutral functional differential equations in this research, but also to reach our target with minimum wastage of control energy. That is, if the systems are controllable, there exist admissible control energy such that, the states of the systems are found in the target. The establishment of this control energy is the main objective of this research. Furthermore, we shall present the existence, form and uniqueness of this control energy.

Our results extend that in Davies [6-7], and other known results on the minimum control energy problem to infinite neutral functional differential equation. An example is given to illustrate the obtained results.

2. BASIC NOTATIONS, PRELIMINARIES AND DEFINITIONS

Suppose $h > 0$ is a given number, $E = (-\infty, \infty)$, E^n is a real $n-$ dimensional Euclidean space with norm $|\cdot|$ $C = C([-h,0], E^n)$ is the space of continuous function mapping the interval $[-h,0]$ into E^n with the norm $\|\cdot\|$ where $\|\phi\| = \sup_{-h<s\leq 0} |\phi(s)|$, for $\phi \in C$. Let $\tau \in E$, $a > 0$ and $x \in C([\tau - h, \tau + a], E^n)$, then given $t \in [\tau, \tau + a]$, we define the symbol x_t by $x_t(s) = x(t+s)$, $-h \leq s < 0$. Let g be a bounded linear operator taking $[\tau, \infty] \times C \to E^n$, we define the functional difference operator $D(\cdot):[\tau,\infty] \times C \to E^n$ by

$$D(t)\phi = \phi(0) - g(t,\phi) \tag{2.1}$$

for $t \in [\tau, \infty]$, $\phi \in C$

We now define a neutral functional differential equation to be a system of the form

$$\frac{d}{dt} D(t)\, x_t = f(t, x_t) \tag{2.2}$$

where $x_t \in C$, and f is a continuous function from $(\tau, \infty) \times C$ into E^n, we say that x is a solution of (2.2) with initial value ϕ at σ if there exists $a \in [\tau, \infty]$, $a > 0$ such that $x \in C([\sigma - h, \sigma + a], E^n)$, $x_0 = \phi$, $D(t)x_t$ is continuously differentiable on $(\sigma, \sigma + a)$ and (2.2) is satisfied on $(\sigma, \sigma + a)$.

We shall consider control systems of the form

$$\frac{d}{dt}D(t)x_t = L(t,x,u) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta \tag{2.3}$$

its linear base control system

$$\frac{d}{dt}D(t)x_t = L(t,x,u) \tag{2.4}$$

and its free system

$$\frac{d}{dt}D(t)x_t = L(t,x,0) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta \tag{2.5}$$

where $D(t)x_t = x(t) - Ax(t-1)$, $L(t,x,u) = Gx(t) + Bx(t-1) + Fu(t) + Hu(t-h)$.

A, B, G are $n \times n$ matrices, F, H are $n \times m$ matrices. $A(\theta)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty, 0]$.

Let $X(t)$ be the unique $n \times n$ constant matrix function with the following properties

a) $X(t) = 0$, for $t < 0$

b) $X(t) = I$ the identity matrix

c) $X(t) - AX(t-1)$ is continuous on $[0,\infty)$

d) $X(t)$ satisfies $\dot{X}(t) - A\dot{x}(t-1) = L(t,x,0)$

for $t \in (0,\infty) - S_2$, where S_2 is the set of non-negative integers.

Then a unique solution of (2.4) exist on $[1, t]$ satisfying $x_L(t,u) = \phi(t)$ for $t \in [0,1]$ and by Gahl [11], this solution is given by

$$x_L(t,u)=X(t-1)\phi(1)-X(t-2)\phi(1)+\int_1^t X(t-s-1)[A\phi(s)+B\phi(s)]ds$$
$$+\int_1^t X(t-s)[Fu(s)+Hu(s-h)]ds \tag{2.6}$$

for all $t\in[1,t_1]$. $x_L(t,u)$ is a continuous function which satisfies (2.4) on $[1,t_1]$ except for a finite number of points which are contained in the set

$$S_3=S_2\{t:t=t_1\pm h+I;\ t\neq k\ or\ h\neq I,\ \ for\ k\in S_2\}$$

Define the matrix functions Z by

$$Z(t,s)=X(t-s)F+X(t-s-h)H \tag{2.7}$$

Then it follows immediately that

$$x_L(t,u)=x_L(t,0)+\int_1^t Z(t,s)u(s)ds \tag{2.8}$$

Since $X(t)$ is continuous and bounded on $[a,b]-S_2$

$$\frac{\partial}{\partial t}Z(t,s)=\dot{X}(t-s)F+\dot{X}(t-s-h)H$$

is continuous and bounded on $[a,b]-S_2$. In a similar manner, any solution of system (2.3) following the methods of Gahl [11] and Sinha [1] will be given by

$$x(t,\phi,u)=x_L(t,u)+\int_1^t X(t-s)\int_{-\infty}^0 A(\theta)x(t+\theta)d\theta ds$$

Or

$$x(t,\phi,u)=x_L(t,0)+\int_1^t Z(t,s)u(s)ds+\int_1^t X(t-s)\int_{-\infty}^0 A(\theta)x(t+\theta)d\theta ds \tag{2.9}$$

In this paper, the control space will be

$L_2^{loc}\left([0,\infty), E^m\right)$ the space of essentially bounded measurable functions on finite intervals with values in E^m. The control constraint set will be denoted by

$U = L_2^{loc}\left([0,\infty), C^m\right)$ where $C^m = \left\{u \in E^m : \left|u_j\right| \le 1, \quad j = 1,2,...,m\right\}$

We now give some definitions upon which our study hinges.

Definition 2.1

The complete state of system (2.3) at time t is given by $y(t) = \{x(t), \phi(t), u(t)\}$

Definition 2.2

System (2.3) is said to be relatively controllable on J if for any function $\phi \in C$, and each $x_1 \in E^n$, there is an admissible control $u \in U$ such that the solution $x(t,\phi,u)$ of (2.3) satisfies $x_{t_0}(\cdot, \phi, u) = \phi$, $x(t_1,\phi,u) = x_1$.

Definition 2.3

The reachable set of (2.3) is a subset of E^m given by

$$P(t_1,t_0) = \left\{\int_1^t Z(t, s)u(s)ds : \ u \in L_2\left([1, t_1], E^m\right)\right\}$$

If the controls are in $L_2\left([1, t_1], C^m\right)$, we define the constraint reachable set by

$$R(t_1,t_0) = \left\{\int_1^t Z(t,s)\ u(s)ds : \ u \in L_2\left([1, t_1], C^m\right)\right\}$$

Note that $P(t_1,t_0)$ is a subset of E^m which is symmetric about zero.

Definition 2.4

The attainable set (2.3) is given by

$$A(t) = \{x(t,\phi,u) : u \in U\}$$

Definition 2.5

The target set for system (2.3) is given by $T(t) = \{x(t, \phi, u)\}: \ t_1 > t_0, \ u \in U$

Corollary 2.1

Consider system (2.5), with all its assumptions. If there exists v>0 such that $|A(\theta)| \leq M \exp(v\theta) \leq M, \quad \theta \in (-\infty, 0]$ and if

$$B(\lambda) = \left\{ \operatorname{Re} \lambda \geq 0, \ \det \left[\lambda(I - Ae^{-\lambda}) - G - Be^{-\lambda} + \int_{-\infty}^{0} \exp(\lambda\theta)[A(\theta)d\theta] = 0 \right] \right\} = \phi$$

Then the solutions of (2.5) is uniformly asymptotically stable such that

$\| x_t(t_0, \phi) \| \leq K \| \phi \| \exp\left[-\alpha(t - s) \right], \quad t \geq t_0$ for some $\alpha > 0, \ \ K > 0$

Proof: The proof can be observed from Sinha [1] and Onwuatu [9]

RELATIONSHIP BETWEEN THE REACHABLE SET AND THE ATTAINABLE SET

Here we establish the relationship between the two set functions reachable and attainable sets. However, we shall first establish a relationship between the sets, to enable us see that once the properties have been proved for one set then they are applicable to the other.

From equation (2.9)

$$A(t) = X(t - s)\left[\phi(t_0) + R(t_1, t_0)\right] \text{ for } u \in U, \ t \in [t_0, t_1]$$

This means that the attainable set is the transition of the reachable set through $\eta \in E^n$, where

$$\eta = \left[\phi(t_0) + R(t_1, t_0)\right]$$

We shall then use the attainable set to establish that the two set functions posses the following properties: convexity, closedness and compactness

Theorem 2.1

The attainable set $A(t)$ is convex and compact

Proof: The convexity of $A(t)$ follows trivially from the convexity of the constrained set *U*. We now show that, $A(t)$ is compact. Assume S to be a convex, compact subset of the space of continuous functions C then $x(t,S,0)$ is bounded. Also, since $Z(t,s)$ is integrably bounded, by the analyticity of X and an assumption that F and H are of bounded variation, coupled with the fact that $u \in U$, the attainable set $A(t)$ is bounded in E^n. From the weak compactness argument and the assumed compactness property on S, $A(t)$ is closed in E^n (Chukwu [3]). Having established boundedness and closedness properties for $A(t)$, we concluded that $A(t)$ is compact in E^n. The convexity and compactness of the reachable set $R(t_1,t_0)$ follows from the convexity and compactness of the attainable set. Also the convexity and compactness of the target set $T(t)$ follows trivially from the convexity and compactness of the constrained control set U.

Corollary 2.2

The reachable set $R(t_1,t_0)$ is a continuous set function on $[t_0,\infty)$ to the metric space of compact subsets of E^n.

Proof: For $t \geq 0,$ we set $t_0 = 0$ and let $Z(t,u) = Z(t,s)u(t)$ so that,

$$|Z(t,u) - Z(t_0,u)| = \left| \int_1^t Z(t,s)u(s)\,ds \right|$$

since u is admissible, we have

$$|Z(t,u) - Z(t_0,u)| \leq \int_1^t \|Z(t,s)\|\,ds$$

by the definition of metric d,

$d(R(t),R(t_0)) \le \left| \int_1^t \| Z(t,s) \| ds \right|$ since $\int_1^t \| Z(t,s) \| ds$ is absolutely continuous, the reachable set is continuous, and by the transition property of both attainable set and the reachable set we conclude that, the attainable set $A(t)$ is also continuous.

3. CONTROLLABILITY RESULTS

Here we state and prove theorems that summarizes our result on the controllability of system (2.3)

Definition 3.1

System (2.3) is controllable if $A(t) \cap T(t) \neq \psi.$ for $t \in [t_0, t_1]$.

We now introduce computational criteria for system (2.3) following the methods of (Gabasov and Kirillova [12]; Chukwu [4]) to check when the system (2.3) is relatively controllable by introducing the following notation

$$Q_k(s) = GQ_{k-1}(s) + BQ_{k-1}(s-1) + AQ_k(s-1),$$
$$k = 0, 1, 2, \cdots; \quad s = 0, 1, 2, \cdots$$

$$Q_k(s) = \begin{cases} I, & k = 0,\ s = 0 \\ 0, & k = 0,\ s < 0 \ \ or \ \ k < 0,\ s = 0 \end{cases}$$

and

$$\Omega(t_1) = \{ Q_k(s)F, Q_k(s)H, \ \ k = 0, 1, \cdots n-1, \ \ s \in [t_0, t_1] \}$$

We define the rank of $\Omega(t_1)$ as the rank of the block matrix composed of all matrices from the set $\Omega(t_1)$.

Theorem 3.1

(Chukwu [4]; Klamka [8], for every $t_1 \in (0, \infty)$ the following conditions are equivalent:

(i) If $c^T Z(t,s) = 0$ for $t \in J$ and $c \in E^n$, then $c = 0$

(ii) $rank\ \Omega(t_1) = n$

(iii) The system (2.3) without constraints on the control is (globally) relatively controllable in J

Remark 3.1

To prove the above theorem, we use the fact that $Z(t,s) = X(t-s)F + X(t-s-h)H$ and proceeds as in the cited proofs. The proof of conditions (i) and (ii) follows the same pattern as that shown in Chukwu [4], whereas that of condition (iii) follows the same pattern as that shown in the monograph Klamka [8].

The next theorems are direct consequences of Theorem 3.1 and Corollary 2.1

Theorem 3.2

The system (2.3) is completely controllable in the time interval $[t_0, t_1]$ if:

(i) the system (2.5) is uniformly asymptotically stable

(ii) the $(n \times n)$- dimensional controllability matrix $W(t_1)$ of the system

(2.3) given by $W(t_1) = \int_1^t Z(t,s) Z^T(t,s) ds$ satisfies $rank\ W(t_1) = n$ and

$W^{-1}(t_1) \in E^{n \times n}$, $Z^T(t,s) \in E^{m \times n}$ for every $t \in [1,\ t_1]$ and $\left(x_1 - x(t,\ 0)\right) \in E^n$

Proof: Let the assumptions of Theorem 3.1 be satisfied, and the solutions of system (2.5) uniformly asymptotically stable. We let

$y_0 = \left(x(0), x_0\right) \in E^n \times L_2\left([t_0, t_1], E^n\right)$ be any initial complete state of the system (2.3) and $x_1 \in E^n$ be any vector. We shall prove that the admissible control function $u \in L_2\left([t_0, t_1], E^m\right)$ of the form

$$u(t) = Z^T(t,s)W^{-1}(t_1)\big(x - x(t,0)\big) \tag{3.1}$$

for $t \in [t_0, t_1]$ steers the system (2.3) from the initial state y_0 to the state $x(t_1, u) = x_1$. Substituting (3.1) into (2.9) for $t = t_1$, we get

$$\begin{aligned} x(t_1,u) &= x_L(t_1,0) + \int_1^t \Big(Z(t,s)Z^T(t,s) \times W^{-1}(t_1)\big(x_1 - x(t_1,0)\big)\Big)ds \\ &= x(t_1,0) + W(t_1)W^{-1}(t_1)\big(x_1 - x(t_1,0)\big) = x_1 \end{aligned}$$

Since y_0 and x_1 were arbitrary, the system (2.3) is completely controllable in the interval $[t_0, t_1]$.

4. THE MINIMUM CONTROL ENERGY

The minimum control energy problem is best understood in the context of capture problem or rescue effort (see Chukwu [3] and references therein). Emphasis here is on the minimum control energy to reach the target or intercept it. However the need to first determine the existence of control for pursuit is evident. If the intersection of the attainable set and the reachable set is non-empty, the target could be reached using appropriate control energy. The next theorem establishes the existence of such control efforts required to capture a moving target- either moving point function or a compact set function

Theorem 4.1

In system (2.3), $A(t) \cap T(t) \neq \psi$, if and only if, there exists an admissible control such that weapon for the capture of a target satisfies system (2.3) on the interval $[t_0, t_1]$.

Proof: Suppose the state $y(t)$ of the weapon for the capture or rescue mission satisfies the given system then, $y(t) \in T(t)$. We are to show that there exist $x(t,\phi,u) \in A(t)$ such that $y(t) = x(t,\phi,u)$ for some u. Let $\{u^n\}$ be a sequence in U, since U is compact, $\lim_{n\to\infty} u^n = u$. Now $x(t,\phi,u_n) \in A(t)$ and from (2.9).

$$x(t_1,\phi,u_n) = x_L(t_1,0) + \int_1^{t_1} Z(t_1,s)u^n(s)ds + \int_1^{t_1} X(t_1-s)\int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds \qquad (4.1)$$

taking limits on both sides of (4.1), we have

$$\lim_{n\to\infty} x(t_1,\phi,u_n) = x_L(t_1,0) + \int_1^{t_1} Z(t_1,s)\lim_{n\to\infty} u^n(s)ds + \int_1^{t_1} X(t_1-s)\int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds \qquad (4.2)$$

(4.2) implies

$$\lim_{n\to\infty} x(t_1,\phi,u_n) = x_L(t_1,0) + \int_1^{t_1} Z(t_1,s)u^n(s)ds + \int_1^{t_1} X(t_1-s)\int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds$$
$$= x(t,\phi,u)$$

Since $A(t)$ is compact, $\lim_{n\to\infty} x(t_1,\phi,u_n) \in A(t)$. That is, $x(t,\phi,u) \in A(t)$. Thus, there exists $u \in U$ such that $x(t,\phi,u) = y(t)$. Since $y(t) \in T(t)$ and $A(t)$, $A(t) \cap A(t) \neq \psi$ or $A(t) \cap T(t) \neq \psi$. Conversely, suppose $A(t) \cap T(t) \neq \psi$. There is, $y(t) \in A(t)$ such that $y(t) \in T(t)$. This implies that $y(t) = x(t,\phi,u)$ being an element of the attainable set. Thus, establishing that, the state of the weapon for capture of a target satisfies systems (2.3). This completes the proof.

EXISTENCE OF MINIMUM CONTROL ENERGY

Theorem 4.2

In systems (2.3), suppose the system is controllable using a admissible control at time *t*, then there exists a minimum control energy

Proof: Let $y(t) \in T(t)$ and by the controllability of (4.1), $y(t) \in A(t)$, since $A(t)$ is the translation of the reachable set through η, $y(t) \in R(t_1,t_0)$ for $t_1 > t_0$. Let $t^* = \inf\{t, y(t) \in R(t_1,t_0)\}$. Now, $0 \le t^* \le t_1$ and there is a non-increasing sequence of time t_n converging to the minimum time t^* and a sequence of controls $u_n \in U$. Let $y(t_n) = z(t,u) \in R(t_1,t_0)$. Also

$$\left| y(t^*) - z(t^*,u_n) \right| \le \left| y(t^*) - y(t_n) \right| + \left| y(t_n) - z(t^*,u_n) \right|$$
$$\le \left| y(t^*) - y(t_n) \right| + \left| z(t_n,u_n) - z(t^*,u^*) \right|$$

$$\leq \left| y(t^*) - y(t_n) \right| + \int_{t^*}^{t_n} \| z(s) \| ds$$

By continuity of $y(t)$ and the integrability of $\| z(t) \|$, it follows that $z(t^*, u_n) \to W(t^*)$ as $n \to \infty$ where $W(t^*) = z(t^*, u^*)$ for some $u^* \in U$. Since $R(t_1, u_n)$ contains $z(t^*, u_n)$ for each n and $R(t_1^*, t_0)$ is closed then $W(t^*) = z(t^*, u^*) \in R(t_1, t_0)$ for some $u^* \in U$ and by definition of t^*, u^* is the required minimum control energy. This establishes the existence of minimum control energy for system (2.3).

FORM OF THE MINIMUM CONTROL ENERGY

We shall in this section present the form of the minimum control energy for system (2.3), and this can be seen in the following theorem.

Theorem 4.3

In system (2.3) u^* is the minimum control energy if and only if u^* is of the form.

$$u^*(t) = \text{sgn}[k^T Z(t, s)] \text{ where } k \in E^n$$

Proof: Suppose u^* is the minimum control energy for system (2.3) then it maximizes the rate of change of $Z(t, u) = Z(t, s)u(t)$ in the direction of k, that is, we want to minimize $k^T Z(t, s)$ since $u(t)$ are admissible controls, that is, they are constrained to lie in a unit sphere, we have

$$k^T Z(t, s)u(t) \leq k^T Z(t, s)$$

$$\leq \left| k^T Z(t, s) \text{sgn}[k^T Z(t, s)] \right| \leq \left| k^T Z(t, s) u^*(t) \right|$$

This shows that the minimum control u^* has the form $u^* = \text{sgn}\ [k^T Z(t, s)]$.

Conversely, let $u^* = \text{sgn}[k^T Z(t, s)]$ then for admissible controls $u \in U$

$$k^T \int_1^{t_1} Z(t_1, s)u(s)\,ds \le \int_1^{t_1} \left| k^T Z(t_1, s)\operatorname{sgn} \int_1^{t_1} \left[k^T Z(t_1, s)\right] ds \right|$$

$$\le \int_1^{t_1} \left[k^T Z(t_1, s)\right] ds$$

$$\le \int_1^{t_1} k^T \left[Z(t_1, s)\right] u^*(s) ds$$

This shows that u^* maximizes $k^T[Z(t,s)]$ over all admissible controls u, hence it is the minimum control energy for system (2.3).

REALIZATION FROM THEOREM 4.3

Let

$$z^* = z(t^*, u^*) = \int_1^{t^*} Z(t,s)u^*(s)\,ds \text{ and } z = z(t,u) = \int_1^{t} Z(t,s)u(s)\,ds$$

From the result in Theorem 4.3, $k^T z \le k^T z^* \Rightarrow k^T(z - z^*) \le 0$, for each $z \in R(t_1, t_0)$. Since the reachable set is closed, convex subset of E^n , there is a support plane η of $R(t_1, t_0)$ through z^* with $k \ne 0$ an outward normal to η at z^* and hence z^* is in the boundary of $R(t_1, t_0)$. Thus, showing that, if u^* be the minimum control energy then the target is on the boundary of the reachable set. The above realization is now stated below as a theorem.

Theorem 4.4

Let u^* be the minimum control energy for system (2.3), with t^* the minimum time, then the target $x(t^*) = x(t^*, \phi, u^*)$ is in the boundary of the attainable set $A(t)$ i.e. $y(t) \in \partial A(t)$ (∂ symbolizes boundary).

Proof: Suppose u^* is the minimum control energy, then $x(t^*, \phi, u^*) = X(t^* - s)[\eta + z^*]$, $z^* \in R(t_1^*, t_0)$. Therefore $x(t^*, \phi, u^*) \in A(t^*)$ suppose $x(t^*, \phi, u^*)$ is not on the boundary of $A(t^*)$ then $x(t^*, \phi, u^*)$ is in the interior of $A(t^*)$; $t^* > t_0$. Therefore, there is a ball, $B(x(t^*, \phi, u^*), r) \in A(t)$. Because $A(t)$ is a continuous set function of t, we can preserve the above inclusion for t near t^*. if we reduce the size of the ball $B(x(t^*,u^*),r)$; that is, if there is an $\varepsilon > 0$ such that $B\left(x(t^*,\phi,u^*),\frac{r}{2}\right) \subset A(t_1)$ for $t^* - \varepsilon \le t_1 \le t^*$. Thus $x(t^*,\phi,u^*) \in A(t_1)$ for $t^* - \varepsilon \le t_1 \le t^*$. This contradicts the optimality of t^* and u^* as the minimum control energy. The contradiction, however, proves that $x(t^*,\phi,u^*)$ is on the boundary of the attainable set i.e. $x(t^*,\phi,u^*) \in \partial A(t^*)$.

UNIQUENESS OF MINIMUM CONTROL ENERGY

Theorem 4.5

Consider systems (2.3) with its basic assumption, if u^* is the minimum control, then it is unique

Proof: Let u^* and v^* be minimum controls for system (2.3), then both u^* and v^* maximize $k^T[Z(t,s)]$ for $t \in J$ over all admissible controls u, and so we have

$$k^T \int_1^{t^*} Z(t,s)u(s)\,ds \le \int_1^{t^*} \left| k^T Z(t,s) \right| u^*(s)\,ds \tag{4.3}$$

Also using v^* we have

$$k^T \int_1^{t^*} Z(t,s)u(s)\,ds \le \int_1^{t^*} \left| k^T Z(t,s) \right| v^*(s)\,ds \tag{4.4}$$

Clearly,

$$\max_{-1\le u\le 1}\left\{ k^T \int_1^{t^*} [Z(t,s)]u(s)\,ds \right\} = \int_1^{t^*} \left| k^T Z(t,s) \right| u^*(s)\,ds \tag{4.5}$$

implies

$$\max_{-1\le u\le 1}\left\{ k^T \int_1^{t^*} [Z(t,s)]u(s)\,ds \right\} = \int_1^{t^*} \left| k^T Z(t,s) \right| v^*(s)\,ds \tag{4.6}$$

Subtracting equation (4.5) from equation (4.6) gives $v^*(t) - u^*(t) = 0$, for all $t \in J$ implies $v^* = u^*$, proving the uniqueness of the minimum control energy.

Theorem 4.6

Consider systems (2.3) with its basic assumption. Suppose

(i) (2.5) is uniformly asymptotically stable

(ii) $rank\ \Omega(t_1) = n$

(iii) $u^*(t) = \text{sgn}[k^T Z(t,s)]$

Then there exist a unique minimum control energy that drives the system to target.

Proof: Immediately from Theorems 3.1, 4.3 and 4.5

5. EXAMPLE

Here, we give numerical examples to illustrate the theoretical analysis.

Consider the neutral system

$$\begin{aligned}\frac{d}{dt}\left(x(t) - A_{-1}x(t-1)\right) &= A_1 x(t-1) + A_0 x(t) + B_0 u(t) \\ &+ B_1 u(t-1) + C_0 \int_{-\infty}^{0} \exp(v\theta)\, x(t+\theta)d\theta\end{aligned} \tag{5.1}$$

where

$$A_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

The characteristic root of the homogenous equation

$$\dot{x}(t) = A_{-1}\dot{x}(t-1) + A_1 x(t-1) + A_0 x(t) + \int_{-\infty}^{0} \exp(v\theta) x(t+\theta)d\theta$$

is

$$\lambda^2 + 3\lambda + 1 + (3\lambda - \lambda^2)e^{-2\lambda} + (2-3\lambda)e^{-\lambda} + (\lambda+1)\int_{-\infty}^{0} \exp\left[(\lambda+v)\theta\right]d\theta = 0 \tag{5.2}$$

and every root of (5.2) has negative real part. Hence by Corollary 2.1, the system (5.1) with $u = 0$ is uniformly asymptotically stable. Furthermore, we shall show that condition (ii) of Theorem 3.1 is satisfied i.e. $rank\ \Omega(t_1) = n$, $n = 2$. We require all matrices belonging to the set $\Omega(t_1)$:

$$Q_0(0) = IB_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } Q_0(s) = 0 \text{ for } s < 0$$

$$Q_0(1) = A_{-1}B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q_0(2) = A_{-1}^2 B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q_1(0) = A_0 B_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$Q_1(1) = A_0 A_{-1} B_0 + A_1 B_0 + A_{-1} A_0 B_0 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$Q_1(2) = A_0 A_{-1}^2 B_0 + A_1 A_{-1} B_0 + A_{-1} B_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Q_0(0) = IB_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } Q_0(s) = 0 \text{ for } s < 0$$

$$Q_0(1) = A_{-1}B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q_0(2) = A_{-1}^2 B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q_1(0) = A_0 B_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$Q_1(1) = A_0 A_{-1} B_1 + A_1 B_1 + A_{-1} A_0 B_1 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q_1(2) = A_0 A_{-1}^2 B_1 + A_1 A_{-1} B_1 + A_{-1} B_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\Omega(t_1) = \{Q_0(0), Q_0(1), Q_0(2), Q_1(0), Q_1(1), Q_1(2), Q_0(0), Q_0(1), Q_0(2), Q_1(0), Q_1(1), Q_1(2)\},$$

$n = 2$

$$rank\ \Omega(t_1) = rank \begin{bmatrix} 1 & 0 & 1 & -1 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -3 & 1 & 1 & 0 & 1 & -2 & 1 & -2 \end{bmatrix} = 2$$

From the sequel, condition (ii) of Theorem 3.1 is satisfied in any interval $[t_0, t_1]$ and since the system (5.1) with $u = 0$ is uniformly asymptotically stable, we conclude that system (5.1) is completely controllable.

To see the minimum control energy, when complete controllability is established, it is easily verified (Chukwu [5], Pp 60-64) that, the principal fundamental matrix solution of (5.1) (with $u = 0$) is given by

$$X(t) = \begin{pmatrix} te^{-1} & \dfrac{-1+9t}{6} \\ \dfrac{-1+9t}{6} & te^{-2} \end{pmatrix}$$

Then by Theorem 4.3, the minimum control energy is of the form

$$\begin{aligned} u^*(t) &= \operatorname{sgn}[k^T Z(t,s)] \\ &= \operatorname{sgn}\left[k^T \left(X(t-s)B + X(t-s-h)B_1 \right)\right] \end{aligned} \qquad (5.3)$$

$$u^*(t) = \operatorname{sgn}\left\{ (k_1, k_2) \begin{pmatrix} te^{-1} & \dfrac{-1+9t}{6} \\ \dfrac{-1+9t}{6} & te^{-2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (k_1, k_2) \begin{pmatrix} te^{-1} & \dfrac{-1+9t}{6} \\ \dfrac{-1+9t}{6} & te^{-2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= \operatorname{sgn}\left((6te^{-1} + 9t - 1)\frac{k_1}{6} + (6te^{-2} + 9t - 1)\frac{k_2}{6} \right)$$

Hence, the minimum control energy that drives the system state to the target is unique and is given by the equation (5.3).

CONCLUSION

We have established the relationship that exists between the minimum control energy problem and the complete controllability of systems (2.3). Namely, if system (2.3) is completely controllable on J then there exists minimum control energy for the system. The establishment of the existence, form, and uniqueness of this control energy for infinite neutral differential systems is one of the major results of this research.

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