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# On Stability of Quintic Functional Equations in Random Normed Spaces

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**Abstract.** In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20[f(x+y)+f(x-y)]+90f(x)$$

in random normed spaces under the minimum $t$-norm.

## 1. **Introduction**

A classical question in stability of functional equations is as follows:

*Under what conditions, is it true that a mapping which approximately satisfies a functional equation* $(\xi)$ *must be somehow close to an exact solution of* $(\xi)$?

We say the functional equation $(\xi)$ is *stable* if any approximate solution of $(\xi)$ is near to a true solution of $(\xi)$.

The study of stability problem for functional equations is related to a question of Ulam [15] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [9] for linear functional equation of Banach spaces. Subsequently, the result of Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. Cădariu and Radu [3] applied the *fixed point method* to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.

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On the other hand, the theory of *random normed spaces* (briefly, $RN$-spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an $RN$-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of passible values of this norm. The $RN$-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in $RN$-spaces [5, 6, 10, 11, 16].

In the sequel, we use the definitions and notations of a random normed space as in [1, 13, 14].

A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous, with $F(0) = 0$ and $F(+\infty) = 1$. The class of all probability distribution functions $F$ with $F(0) = 0$ is denoted by $\Lambda$. $D^+$ is a subset of $\Lambda$ consisting of all functions $F \in \Lambda$ for which $F(+\infty) = 1$, where $l^- F(x) = \lim_{t\to x^-} F(t)$. For any $a \geq 0$, $\epsilon_a$ is the element of $D^+$, which is defined by

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

**Definition 1.1.** ([13]) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a $t$-norm) if $T$ satisfies the following conditions:
(1) $T$ is commutative and associative;
(2) $T$ is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous $t$-norms are as follows:

$$T_M(a, b) = \min\{a, b\}, \quad T_P(a, b) = ab, \quad T_L(a, b) = \max\{a + b - 1, 0\}.$$

Recall that, if $T$ is a $t$-norm and $\{x_n\}$ is a sequence of numbers in $[0, 1]$, then $T^n_{i=1} x_i$ is defined recurrently by $T^1_{i=1} x_i = x_1$ and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n) = T(x_1, \cdots, x_n)$ for each $n \geq 2$ and $T^\infty_{i=n} x_n$ is defined as $T^\infty_{i=1} x_{n+i}$ ([8]).

**Definition 1.2.** ([14]) Let $X$ be a real linear space, $\mu$ be a mapping from $X$ into $D^+$ (for any $x \in X$, $\mu(x)$ is denoted by $\mu_x$) and $T$ be a continuous $t$-norm. The triple $(X, \mu, T)$ is called a random normed space (briefly $RN$-space) if $\mu$ satisfies the following conditions:
(RN1) $\mu_x(t) = \epsilon_o(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

**Example 1.1.** Every normed space $(X, \|\cdot\|)$ defines a $RN$-space $(X, \mu, T_M)$, where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ and $T_M$ is the minimum $t$-norm. This space is called the *induced random normed space.*

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**Definition 1.3.** Let $(X, \mu, T)$ be a $RN$-space.

(1) A sequence $\{x_n\}$ in $X$ is said to be *convergent* to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that

$$\mu_{x_n - x}(t) > 1 - \lambda$$

whenever $n \geq N$. In this case, $x$ is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.

(2) A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that

$$\mu_{x_n - x_m}(t) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) The $RN$-space $(X, \mu, T)$ is said to be *complete* if every Cauchy sequence in $X$ is convergent to a point in $X$.

**Theorem 1.4.** ([13]) *If $(X, \mu, T)$ is a $RN$-space and $\{x_n\}$ is a sequence of $X$ such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Cho et. al. [4] was introduced and proved the Hyers-Ulam-Rassias stability of the following quintic functional equations

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x) \tag{1.1}$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 3$ in quasi-$\beta$-normed spaces.

**Remark 1.1.** (1) If we put $x = y = 0$ in the equation (1.1), then $f(0) = 0$.

(2) $f(2^n x) = 2^{5n} f(x)$ for all $x \in X$ and $n \in \mathbb{Z}^+$.

(3) $f$ is an odd mapping.

Throughout this paper, let $X$ be a real linear space, $(Z, \mu', T_M)$ be an $RN$-space and $(Y, \mu, T_M)$ be a complete $RN$-space. For any mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x, y) \\ &= 2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) - 20[f(x+y) + f(x-y)] - 90f(x) \end{aligned}$$

for all $x, y \in X$. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x)$$

in random normed spaces under the minimum $t$-norm.

## 2. **Random stability of the functional equation (1.1)**

In this section, we investigate the generalized Hyers-Ulam stability problem of the quintic functional equation (1.1) in $RN$-spaces in the sense of Scherstnev under the minimum $t$-norm $T_M$.

**Theorem 2.1.** *Let $\phi : X^2 \to Z$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{2.1}$$

*and* $\lim_{n\to\infty}\mu'_{\phi(2^n x,2^n y)}(2^{5n}t)=1$ *for all* $x,y\in X$ *and* $t>0$*. If* $f:X\to Y$ *is a mapping with* $f(0)=0$ *such that*

$$\mu_{Df(x,y)}(t)\ge \mu'_{\phi(x,y)}(t) \tag{2.2}$$

*for all* $x,y\in X$ *and* $t>0$*, then there exists a unique quintic mapping* $Q:X\to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t)\ge \mu'_{\phi(x,0)}\Big(2^2(2^5-\alpha)t\Big) \tag{2.3}$$

*for all* $x\in X$ *and* $t>0$*.*

*Proof.* Letting $y=0$ in (2.2), we get

$$\mu_{\frac{f(2x)}{2^5}-f(x)}(t)\ge \mu'_{\phi(x,0)}(128t) \tag{2.4}$$

for all $x\in X$ and $t>0$. Replacing $x$ by $2^n x$ in (2.4), we get

$$\mu_{\frac{f(2^{n+1}x)}{2^{5(n+1)}}-\frac{f(2^n x)}{2^{5n}}}(t)\ge \mu'_{\phi(x,0)}\Big(\Big(\frac{2^5}{\alpha}\Big)^n 128t\Big)$$

for all $x\in X$ and $t>0$. Since $\frac{f(2^n x)}{2^{5n}}-f(x)=\sum_{j=0}^{n-1}\Big(\frac{f(2^{j+1}x)}{2^{5(j+1)}}-\frac{f(2^j x)}{2^{5j}}\Big)$,

$$\mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}\Big(\sum_{j=0}^{n-1}\frac{1}{128}\Big(\frac{\alpha}{2^5}\Big)^j t\Big)\ge T_{M\,j=0}^{\,n-1}(\mu'_{\phi(x,0)}(t))=\mu'_{\phi(x,0)}(t) \tag{2.5}$$

for all $x\in X$ and $t>0$. Substituting $x$ by $2^m x$ in (2.5), we get

$$\mu_{\frac{f(2^{n+m}x)}{2^{5(n+m)}}-\frac{f(2^m x)}{2^{5m}}}(t)\ge \mu'_{\phi(x,0)}\Big(\frac{t}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{2^5})^j}\Big) \tag{2.6}$$

for all $x\in X$ and $m,n\in\mathbb{Z}$ with $n>m\ge 0$. Since $\alpha<k^3$, the sequence $\{\frac{f(2^n x)}{2^{5n}}\}$ is a Cauchy sequence in the complete $RN$-space $(Y,\mu,T_M)$ and so it converges to some point $Q(x)\in Y$. Fix $x\in X$ and put $m=0$ in (2.6). Then we get

$$\mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}(t)\ge \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right),$$

and so, for any $\delta>0$,

$$\begin{aligned}
&\mu_{Q(x)-f(x)}(\delta+t)\\
&\ge T_M\Big(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta),\mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}(t)\Big)\\
&\ge T_M\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta),\mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right)\right)
\end{aligned} \tag{2.7}$$

for all $x\in X$ and $t>0$. Taking the limit as $n\to\infty$ in (2.7), we get

$$\mu_{Q(x)-f(x)}(\delta+t)\ge \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right) \tag{2.8}$$

Since $\delta$ is arbitrary, by taking $\delta\to 0$ in (2.8), we have

$$\mu_{Q(x)-f(x)}(t)\ge \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right) \tag{2.9}$$

for all $x\in X$ and $t>0$. Therefore, we conclude that the condition (2.3) holds.

Also, replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (2.2), respectively, we have

$$\mu_{\frac{Df(2^n x,2^n y)}{2^{5n}}}(t)\ge \mu'_{\phi(2^n x,2^n y)}(2^{5n}t)$$

for all $x, y \in X$ and $t > 0$. It follows from $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ that $Q$ satisfies the equation (1.1), which implies that $Q$ is a quintic mapping.

To prove the uniqueness of the quintic mapping $Q$, let us assume that there exists another mapping $\widetilde{Q} : X \to Y$ which satisfies (2.3). Fix $x \in X$. Then $Q(2^n x) = 2^{5n}Q(x)$ and $\widetilde{Q}(2^n x) = 2^{5n}\widetilde{Q}(x)$ for all $n \in \mathbb{Z}^+$. Thus it follows from (2.3) that

$$\begin{aligned}
&\mu_{Q(x)-\widetilde{Q}(x)}(t) \\
&= \mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}(t) \\
&\geq T_M\left(\mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{f(2^n x)}{2^{5n}}}\left(\frac{t}{2}\right), \mu_{\frac{f(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}\left(\frac{t}{2}\right)\right) \\
&\geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)\left(\frac{2^5}{\alpha}\right)^n t\right).
\end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty}\left(2^2(2^5-\alpha)\left(\frac{2^5}{\alpha}\right)^n t\right) = \infty$, we have $\mu_{Q(x)-\widetilde{Q}(x)}(t) = 1$ for all $t > 0$. Thus the quintic mapping $Q$ is unique. This completes the proof. □

**Theorem 2.2.** *Let $\phi : X^2 \to Z$ be a function such that, for some $2^5 < \alpha$,*

$$\mu'_{\phi(\frac{x}{2}, \frac{y}{2})}(t) \geq \mu'_{\phi(x,y)}(\alpha t) \tag{2.11}$$

*and $\lim_{n\to\infty} \mu'_{2^{5n}\phi(\frac{x}{2^n}, \frac{y}{2^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.2), then there exists a unique cubic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.12}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* It follows from (2.2) that

$$\mu_{f(x)-2^5 f(\frac{x}{2})}(t) \geq \mu'_{\phi(x,0)}\left(2^2\alpha t\right) \tag{2.13}$$

for all $x \in X$. Applying the triangle inequality and (2.13), we have

$$\mu_{f(x)-2^{5n}f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{2^2\alpha t}{\sum_{j=m}^{n+m-1}\left(\frac{2^5}{\alpha}\right)^j}\right) \tag{2.14}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Then the sequence $\{2^{5n}f(\frac{x}{2^n})\}$ is a Cauchy sequence in the complete $RN$-space $(Y, \mu, T_M)$ and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 2^{5n} f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Then the mapping $Q$ satisfies (1.1) and (2.12). The remaining assertion follows the similar proof method in Theorem 2.1. This complete the proof. □

**Corollary 2.3.** *Let $\theta$ be a nonnegative real number and $z_0$ be a fixed unit point of $Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\theta z_0}(t) \tag{2.15}$$

*for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $C : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\theta z_0}(124t) \tag{2.16}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = \theta z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof. □

**Corollary 2.4.** *Let $p, q \in \mathbb{R}$ be positive real numbers with $p, q < 5$ and $z_0$ be a fixed unit point of $Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^q)z_0}(t) \tag{2.17}$$

*for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\|x\|^p z_0}\Big(2^2(2^5 - 2^p)t\Big) \tag{2.18}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the proof follows from Theorem 2.1 by $\alpha = 2^p$. This completes the proof. □

Now, we give an example to illustrate that the quintic functional equation (1.1) is not stable for $r = 5$ in Corollary 2.4

**Example 2.1.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^5, & \text{for} \quad |x| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^{5n}}$$

for all $x \in \mathbb{R}$. Then $f$ satisfies the functional inequality

$$\begin{aligned} &|2f(2x + y) + 2f(2x - y) + f(x + 2y) + f(x - 2y) - 20[f(x + y) + f(x - y)] - 90f(x)| \\ &\leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big) \end{aligned} \tag{2.19}$$

for all $x, y \in X$, but there do not exist a quintic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that

$$|f(x) - Q(x)| \leq d|x|^5$$

for all $x \in \mathbb{R}$. In fact, it is clear that $f$ is bounded by $\frac{32}{31}$ on $\mathbb{R}$. If $|x|^5 + |y|^5 = 0$, then (2.19) is trivial. If $|x|^5 + |y|^5 \geq \frac{1}{32}$, then

$$|Df(x, y)| \leq \frac{136 \cdot 32}{31} \leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big).$$

Now, suppose that $0 < |x|^5 + |y|^5 < \frac{1}{32}$. Then there exists a positive integer $k \in Z^+$ such that

$$\frac{1}{32^{k+2}} \leq |x|^5 + |y|^5 < \frac{1}{32^{k+1}}$$

and so

$$32^k|x|^5 < \frac{1}{32}, \quad 32^k|y|^5 < \frac{1}{32},$$

$$2^n(2x + y),\ 2^n(2x - y),\ 2^n(x + 2y),\ 2^n(x - 2y),\ 2^n(x - y),\ 2^n x \in (-1, 1)$$

and

$$\begin{aligned}&\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\quad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)\\&=0\end{aligned}$$

for all $n=0,1,\cdots,k-1$. Thus we obtain

$$\begin{aligned}&|Df(x,y)|\\&\le\sum_{n=0}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\sum_{n=k}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\frac{136\cdot 32^2}{31}\Big(|x|^5+|y|^5\Big).\end{aligned}$$

Therefore, $f$ satisfies (2.19).

Now, we claim that the quintic functional equation (1.1) is not stable for $r=5$ in Corollary 2.4. Suppose on the contrary that there exists a quintic mapping $Q:\mathbb{R}\to\mathbb{R}$ and constant $d>0$ such that

$$|f(x)-Q(x)|\le d|x|^5$$

for all $x\in\mathbb{R}$. Since $f$ is bounded and continuous for all $x\in\mathbb{R}$, $Q$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, $Q$ must have $Q(x)=cx^5$ for all $x\in\mathbb{R}$. So, we obtain

$$|f(x)|\le(d+|c|)|x|^5 \tag{2.20}$$

for all $x\in\mathbb{R}$. Let $m\in\mathbb{Z}^+$ such that $m+1>d+|c|$.

If $x$ is in $(0,2^{-m})$, then $2^nx\in(0,1)$ for $n=0,1,\cdots,m$. For this $x$, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(2^n)}{2^{5n}}\ge\sum_{n=0}^{m}\frac{(2^nx)^5}{2^{5n}}=(m+1)x^5>(d+|c|)|x|^5,$$

which contradiction (2.20).

**Remark 2.1.** In Corollary 2.4, if we assume that

$$\phi(x,y)=\|x\|^r\|y\|^r z_0$$

or

$$\phi(x,y)=(\|x\|^r\|y\|^s+\|x\|^{r+s}+\|y\|^{r+s})z_0,$$

then we have Ulam-Gavuta-Rassias product stability and JMRassias mixed product-sum stability, respectively.

Next, we apply a fixed point method for the generalized Hyer-Ulam stability of the functional equation (1.1) in $RN$-spaces. The following Theorem will be used in the proof of Theorem 2.6.

**Theorem 2.5.** ([7]) *Suppose that $(\Omega, d)$ is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n \geq 0$ or there exists a natural number $n_0$ such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y*$ of $J$;*
(3) *$y*$ is the unique fixed point of $J$ in the set $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

**Theorem 2.6.** *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(2x,2y)}(\alpha t) \tag{2.21}$$

*for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that*

$$\mu_{D(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.22}$$

*for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,y)}\big(2^2(2^5 - \alpha)t\big) \tag{2.23}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* It follows from (2.22) that

$$\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.24}$$

for all $x \in X$ and $t > 0$. Let $\Omega = \{g : X \to Y, g(x) = 0\}$ and the mapping $d$ defined on $\Omega$ by

$$d(g,h) = \inf\{c \in [0,\infty) : \mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t),\ \forall x \in X\}$$

where, as usual, $\inf \emptyset = -\infty$. Then $(\Omega, d)$ is a generalized complete metric space (see [10]). Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) = \frac{1}{2^5} g(2x)$$

for all $g \in \Omega$ and $x \in X$. Let $g, h$ in $\Omega$ and $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) < c$. Then $\mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t)$ for all $x \in X$ and $t > 0$ and so

$$\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha ct}{2^5}\right) = \mu_{g(2x)-h(2x)}(\alpha ct) \geq \mu'_{\phi(x,0)}(t) \tag{2.25}$$

for all $x \in X$ and $t > 0$. Hence we have

$$d(Jg, Jh) \leq \frac{\alpha c}{2^5} \leq \frac{\alpha}{2^5} d(g,h)$$

for all $g, h \in \Omega$. Then $J$ is a contractive mapping on $\Omega$ with the Lipschitz constant $L = \frac{\alpha}{2^5} < 1$. Thus it follows from Theorem 2.5 that there exists a mapping $Q : X \to Y$, which is a unique fixed point of $J$ in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{5n}}$$

for all $x \in X$ since $\lim_{n\to\infty} d(J^n f, Q) = 0$. Also, from $\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t)$, it follows that $d(f, Jf) \leq \frac{1}{128}$. Therefore, using Theorem 2.5 again, we get

$$d(f,Q) \leq \frac{1}{1-L} d(f,Jf) \leq \frac{1}{2^2(2^5 - \alpha)}.$$

This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right)$$

for all $x \in X$ and $t > 0$.

Also, replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (2.22), respectively, we have

$$\mu_{DQ(x,y)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = \lim_{n\to\infty} \mu'_{\phi(x,y)}\left(\left(\frac{2^5}{\alpha}\right)^n t\right) = 1$$

for all $x, y \in X$ and $t > 0$. By (RN1), the mapping $Q$ is quintic.

To prove the uniqueness, let us assume that there exists a quintic mapping $Q' : X \to Y$ which satisfies (2.23). Then $Q'$ is a fixed point of $J$ in $\Omega_1$. However, it follows from Theorem 2.5 that $J$ has only one fixed point in $\Omega_1$. Hence $Q = Q'$. This completes the proof. □

**Theorem 2.7.** *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < 2^5 < \alpha$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(\frac{x}{2},\frac{y}{2})}(\alpha t) \tag{2.26}$$

*for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.22), then there exists a unique quintic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha-2^5)t\right) \tag{2.27}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* By a modification in the proofs of Theorem 2.2 and 2.6, we can easily obtain the desired results. This completes the proof. □

Now, we present a corollary that is an application of Theorem 2.6 and 2.7 in the classical case.

**Corollary 2.8.** *Let $X$ be a Banach space, $\epsilon$ and $p$ be positive real numbers with $p \neq 5$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies*

$$\|Df(x,y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

*for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ such that*

$$\|Q(x) - f(x)\| \leq \frac{\epsilon\|x\|^p}{2^2|2^5 - 2^p|}$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Define $\mu : X \times \mathbb{R} \to \mathbb{R}$ by

$$\mu_x(t) = \begin{cases} \frac{t}{t+\|x\|}, & \text{if } t > 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then $(X, \mu, T_M)$ is a complete $RN$-space. Denote $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $t > 0$. It follows from $\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$ that

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, where $\mu' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\mu'_x(t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a random norm on $\mathbb{R}$. Then all the conditions of Theorems 2.6 and 2.7 hold and so there exists a unique quintic mapping $Q : X \to X$ such that

$$\begin{aligned} \frac{t}{t + \|Q(x) - f(x)\|} &= \mu_{Q(x)-f(x)}(t) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2|2^5 - \alpha|t\Big) = \frac{2^2|2^5 - \alpha|t}{2^2|2^5 - \alpha|t + \epsilon\|x\|^p}. \end{aligned}$$

Therefore, we obtain the desired result, where $\alpha = 2^p$. This completes the proof. □

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