On A System of Rational Difference Equations

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In this paper, we investigate behaviors of well-defined solutions of the following system

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$

$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{Z}^+$ the coefficients $A_1, A_2, B_1, B_2, C_1, C_2$ and the initial conditions are arbitrary real numbers.
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1 Introduction

There has been a great effort in studying periodic and asymptotic behaviors of solutions of difference equations (see e.g. [3,6,12,15,18,20-23,27,35,45,46]). Also, studying in system of difference equations has increased considerably (see, e.g. [5,7,8,16,17,19,28-30,32-34,37,38,40,43,47]).

Ozkan et al. [31] gave the solutions of the systems of the difference equations

$$\begin{aligned} x_{n+1} &= \frac{y_{n-2}}{-1 \mp y_{n-2} x_{n-1} y_n}, \\ y_{n+1} &= \frac{x_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, \\ z_{n+1} &= \frac{x_{n-2} + y_{n-2}}{-1 \mp x_{n-2} y_{n-1} x_n}, n \in \mathbb{N}_0. \end{aligned} \tag{1}$$

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In [39] it was showed that the system of difference equations, which is an extension of first and second equations of system (1) with respect to coefficients,

$$x_n = \frac{c_n y_{n-3}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3}},$$
$$y_n = \frac{\gamma_n x_{n-3}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3}}, n \in \mathbb{N}_0, \tag{2}$$

where the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial values $x_i, y_i, i \in \{1, 2, 3\}$ are real numbers, such that $c_n \neq 0, \gamma_n \neq 0, n \in \mathbb{N}_0$, can be solved in closed form, and for the case when all sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$ are constant it was described the asymptotic behavior of well-defined solutions of the system.

In [41] it was showed that an extension of system (2) with respect to indices

$$x_n = \frac{c_n y_{n-(2k-1)}}{a_n + b_n y_{n-(2k-1)} \prod_{i=1}^{k-1} y_{n-(2i-1)} x_{n-2i}},$$
$$y_n = \frac{\gamma_n x_{n-(2k-1)}}{\alpha_n + \beta_n x_{n-(2k-1)} \prod_{i=1}^{k-1} x_{n-(2i-1)} y_{n-2i}}, \tag{3}$$

where $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial conditions $x_i, y_i, i \in \{1, 2, ...2k-1\}$ are real numbers, is solved in closed form, and the behavior of its well-defined solutions when all the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$ are constant was described. Related rational difference equations are studied, e.g. in [1,2,4,9-11,13,14,24-26,31,36,42,44,48].

In this paper we consider an other extension of system (2)

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{4}$$

where $n \in \mathbb{N}_0$, k is a positive integer, the initial conditions and the coefficients A_1, A_2, B_1 , B_2, C_1, C_2 are arbitrary real numbers. We will consider only well-defined solutions, that is, $B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)} \neq 0$ and $B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)} \neq 0, n = 0, 1, 2, ...$

2 Special Cases

2.1 The case $A_1 = 0$ or $A_2 = 0$

If $A_1 = 0$, we obtain directly $x_n = 0$ for $n > 0$. By using this, we get $y_n = 0$ for $n > 3k$. If $A_2 = 0$, we obtain directly $y_n = 0$ for $n > 0$. By using this, we get $x_n = 0$ for $n > 3k$. From now on both of A_1 and A_2 will be considered a non-zero real numbers.

System (4) is equivalent to the following system

$$x_{n+1} = \frac{y_{n-(3k-1)}}{b_1 + c_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{x_{n-(3k-1)}}{b_2 + c_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{5}$$

where $n \in \mathbb{N}_0$, $b_i = \frac{B_i}{A_i}$ and $c_i = \frac{C_i}{A_i}, i = 1, 2$. So, we will consider system (5) instead of system (4).

2.2 The case $b_1 = 0$ or $b_2 = 0$

If $b_1 = 0$, from the first equation of system (5), we have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $n > 0$. Using this, we obtain directly $y_n = \alpha x_{n-3k}$, $n \geq k$, where $\alpha = \frac{c_1}{b_2 c_1 + c_2}$. From this and by the change of variables

$$z_n = \frac{y_n}{x_{n-3k}}, w_n = \frac{y_{n-3k}}{x_n}, n \geq k, \tag{6}$$

system (5) can be transformed into the system

$$w_{n+1} = c - c b_2 z_{n-(k-1)}, z_{n+1} = \alpha, n \geq k - 1, \tag{7}$$

where $c = \frac{c_1}{c_2}$. The solutions are obtained easily as $z_n = w_n = \alpha$, $n \geq k$. This means every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

If $b_2 = 0$, we get immediately $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. From the first equation in system (5) and using this, we obtain $x_n = \beta y_{n-3k}$, $n \geq k$, where $\beta = \frac{c_2}{b_1 c_2 + c_1}$. The change of variables

$$u_n = \frac{x_n}{y_{n-3k}}, t_n = \frac{x_{n-3k}}{y_n}, n \geq k, \tag{8}$$

reduces system (5) to the system

$$t_{n+1} = \bar{c} - \bar{c} b_1 u_{n-(k-1)}, u_{n+1} = \beta, n \geq 2k - 1, \tag{9}$$

where $\bar{c} = \frac{c_2}{c_1}$. The solutions of this system $t_n = u_n = \beta$, $n \geq 2k - 1$, are obtained easily. So, every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

Assume that $b_1 = 0$ and $b_2 = 0$. We have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. Then, we get immediately $x_n = \frac{c_2}{c_1} y_{n-3k}$, $y_n = \frac{c_1}{c_2} y_{n-3k}$, $n > k$. Thus, we can write $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n > 4k$.

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2.3 The case $c_1 = 0$ or $c_2 = 0$

If $c_1 = 0$, we have $x_n = \frac{1}{b_1} y_{n-3k}$, $n > 0$. From this and using the change of variables

$$v_n = \frac{1}{x_{n+3k} y_{n-k} x_n}, n > 0, \tag{10}$$

the second equation of system (5) implies the linear equation

$$v_{n+1} = b_1 b_2 v_{n-(2k-1)} + {b_1}^2 c_2, n = 0, 1, 2, ... \tag{11}$$

We can rewrite the equation (11) in the form of

$$v_{2kn+m} = b_1 b_2 v_{2k(n-1)+m} + b_1^2 c_2, \tag{12}$$

where $n \in \mathbb{N}_0$, $m = 1, 2, ..., k$. Considering the solution of a nonhomogeneous first order difference equation, we can give the solution of the equation (12) such that

$$v_{2kn+m} = (b_1 b_2)^n v_{m-2k} + b_1^2 c_2 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0. \tag{13}$$

when $b_1 b_2 \neq 1$. If $b_1 b_2 = 1$, the solution of the equation (12) can be written as

$$v_{2kn+m} = v_{m-2k} + (n+1) b_1^2 c_2, n \geq 0. \tag{14}$$

From (10), we have

$$x_{2kn+3k+m} = \frac{v_{2k(n-1)+m}}{v_{2kn+m}} x_{2kn-3k+m}.$$

Considering $x_n = \frac{1}{b_1} y_{n-3k}$, we obtain the solutions of system (5) as

$$x_{6kn+3k+m} = x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, y_{6kn+m} = b_1 x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, \tag{15}$$

$$x_{6kn+5k+m} = x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, y_{6kn++2k+m} = b_1 x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, \tag{16}$$

$$x_{6kn+7k+m} = x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, y_{6kn+4k+m} = b_1 x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, \tag{17}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that $c_2 = 0$. Then, we have $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$. From this and using the change of variable

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$$u_n = \frac{1}{y_{n+3k}y_{n+k}y_{n-k}}, n > 0, \tag{18}$$

the first equation of system (5) implies the linear equation

$$u_{n+1} = b_1 b_2 u_{n-(2k-1)} + b_2^2 c_1, n \geq 0. \tag{19}$$

By similar processes just as we did, we can rewrite the equation (19) as

$$u_{2kn+m} = b_1 b_2 u_{2k(n-1)+m} + b_2^2 c_1, n \geq 0, \tag{20}$$

where $m = 1, 2, ..., k$. We obtain the solution of the equation (20)

$$u_{2kn+m} = (b_1 b_2)^n u_{m-2k} + b_2^2 c_1 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0, \tag{21}$$

when $b_1 b_2 \neq 1$. When $b_1 b_2 = 1$, the solution of the equation (20)

$$u_{2kn+m} = u_{m-2k} + (n+1) b_2^2 c_1, n \geq 0. \tag{22}$$

From (18), we have

$$y_{n+3k} = \frac{1}{u_n y_{n+k} y_{n-k}}, n > 0,$$

and

$$y_{2kn+3k+m} = \frac{u_{2k(n-1)+m}}{u_{2kn+m}} y_{2kn-3k+m}.$$

Considering $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$, we obtain the solutions of system (5) as

$$x_{6kn+m} = b_2 y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, y_{6kn+3k+m} = y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, \tag{23}$$

$$x_{6kn++2k+m} = b_2 y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, y_{6kn+5k+m} = y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, \tag{24}$$

$$x_{6kn+4k+m} = b_2 y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, y_{6kn+7k+m} = y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, \tag{25}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that both c_1 and c_2 are equal to zero. We get immediately $x_{n+1} = \frac{1}{b_1} y_{n-(3k-1)}$, $y_{n+1} = \frac{1}{b_2} x_{n-(3k-1)}$, $n \geq 0$. From this result, we obtain $x_{n+1} = \frac{1}{b_1 b_2} x_{n-(6k-1)}, y_{n+1} = \frac{1}{b_1 b_2} y_{n-(6k-1)}$, $n \geq 3k$. So, we have $x_{6kn+3k+m} = \frac{1}{b_1 b_2}$ $x_{6kn-3k+m}$, $y_{6kn+3k+m} = \frac{1}{b_1 b_2} y_{6kn-3k+m}$ and from this $x_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1}$ x_{-3k+m}, $y_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1} y_{-3k+m}$, $n \geq 0$, $m = 1, 2, ..., 6k$.

3 Main Case

In this section, we will need the following results, given in the reference [16], in the proofs of our results.

Consider the first order Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, n = 0, 1, ..., \tag{26}$$

where the parameters and the initial condition x_0 are arbitrary real numbers.

Theorem 1 *The followings are true:*

1) *Eq.*(26) *has a prime period-2 solution if and only if* $b + c = 0$.

2) *Suppose* $b + c = 0$. *Then every solution* $\{x_n\}$ *of Eq.* (26) *with* $x_0 \neq 0$ *is periodic with period 2.*

Theorem 2 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} < \frac{1}{4}$. *Then the forbidden set* F *of Eq.*(26) *is given as follows:*

$$F = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\} .$$

For any well-defined solution $\{x_n\}$ *of Eq.* (26), *we have*

$$x_n = \frac{b+c}{d} \left(\frac{c_1 \lambda_1^{n+1} - c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n - c_2 \lambda_2^n} \right) - \frac{c}{d},$$

for $n = 0, 1, ...,$ *where* $\lambda_1 = \frac{1-\sqrt{1-4R}}{2}$, $\lambda_2 = \frac{1+\sqrt{1-4R}}{2}$, $c_1 = \frac{\lambda_2(b+c)-(dx_0+c)}{(\lambda_2-\lambda_1)(b+c)}$ *and* $c_2 = \frac{(dx_0+c)-\lambda_1(b+c)}{(\lambda_2-\lambda_1)(b+c)}$.

Corollary 1 *Assume that the conditions in Theorem2 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq.* (26). *Then*

$$lim_{n\to\infty} x_n = \frac{\lambda_2(b+c)-c}{d}.$$

Theorem 3 *Assume that* $d \neq 0, bc - ad \neq 0,\ b + c \neq 0$ *and* $R = \frac{bc-ad}{(b+c)^2} = \frac{1}{4}$. *Then the forbidden set* F *of Eq.*(26) *is given as follows:*

$$F = \left\{ \frac{n(b-c)-(b+c)}{2dn} : n \geq 1 \right\}.$$

For any well-defined solution $\{x_n\}$ *of Eq.* (26), *we have*

$$x_n = \frac{b+c}{d} \left(\frac{(b+c)+(n+1)(2dx_0+(c-b))}{2(b+c)+2n(2dx_0+(c-b))} \right) - \frac{c}{d},$$

for $n = 0, 1, ...$

Corollary 2 *Assume that the conditions in Theorem3 hold. Let* $\{x_n\}$ *be a well-defined solution of Eq.*(26). *Then*

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$$lim_{n\to\infty} x_n = \tfrac{b-c}{2d}.$$

Now we consider the system (5) with b_1, b_2, c_1, c_2 parameters and the initial conditions are non-zero real numbers. By the change of variables (6), the system (5) reduces to

$$z_{n+1} = \frac{w_{n-(k-1)}}{\frac{1}{\alpha} w_{n-(k-1)} - \gamma_2}, w_{n+1} = \frac{1}{\beta} - \gamma_1 z_{n-(k-1)}, n \geq k, \tag{27}$$

where $\gamma_1 = \frac{c_1 b_2}{c_2}, \gamma_2 = \frac{c_2 b_1}{c_1}, \alpha = \frac{c_1}{b_2 c_1 + c_2}, \beta = \frac{c_2}{b_1 c_2 + c_1}$. We can rewrite the system (27) such that

$$z_{n+1} = \frac{\frac{1}{\beta} - \gamma_1 z_{n-(2k-1)}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{n-(2k-1)}}, w_{n+1} = \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{n-(2k-1)} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{n-(2k-1)} - \gamma_2}, \tag{28}$$

$n \geq 2k$. Each of the equation in (28)is a $2k$th order Riccati difference equation. Furthermore, the equations in (28) can be rewritten such that

$$\begin{aligned} z_{2kn+1+i} &= \frac{\frac{1}{\beta} - \gamma_1 z_{2k(n-1)+1+i}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{2k(n-1)+1+i}}, \\ w_{2kn+1+i} &= \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{2k(n-1)+1+i} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{2k(n-1)+1+i} - \gamma_2}, \end{aligned} \tag{29}$$

$n > 0$, $i = 0, 1, ..., (2k-1)$. Note that the equations in (29) are first order Riccati difference equation in variables z_{2kn+i}, w_{2kn+i}, for $i = 1, 2, ..., 2k$.

Theorem 4 *Assume that $b_1 b_2 = -1$ and $\{x_n, y_n\}$ is a well-defined solution of system (5). Then,*

$$\begin{aligned} x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}}, \\ y_{2kn+1+i} &= \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}, \end{aligned}$$

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

***Proof* 1** *Consider system (29) and suppose that $b_1 b_2 = -1$. Then, we have*

$$\begin{aligned} \frac{1}{\alpha\beta} - \gamma_1 - \gamma_2 &= \frac{1}{\frac{c_1}{b_2 c_1 + c_2} \frac{c_2}{b_1 c_2 + c_1}} - \frac{c_1 b_2}{c_2} - \frac{c_2 b_1}{c_1} \\ &= \frac{b_1 b_2 c_1 c_2 + c_1^2 b_2 + c_2^2 b_1 + c_1 c_2 - c_1^2 b_2 - c_2^2 b_1}{c_1 c_2} \\ &= \frac{c_1 c_2 (b_1 b_2 + 1)}{c_1 c_2} \\ &= 0. \end{aligned}$$

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So, from Theorem1(2) we conclude that every solution of each equation in system (29) is periodic with period $4k$, that is,

$$z_{2kn+1+i} = z_{2k(n-2)+1+i}, w_{2kn+1+i} = w_{2k(n-2)+1+i}, \tag{30}$$

for $n \geq 2$, $i = 0, 1, ..., (2k-1)$.

From (6), we have

$x_n = \frac{z_{n-3k}}{w_n} x_{n-6k}, y_n = \frac{z_n}{w_{n-3k}} y_{n-6k}$, (31)

for $n \geq 4k$. *System* (31) *can be written such that*

$$x_{2kn+1+i} = \frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} x_{2kn+1+i-6k}, y_{2kn+1+i} = \frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} y_{2kn+1+i-6k} \tag{32}$$

for $n \geq 2$, $i = 0, 1, ..., (2k-1)$. *From* (6), (30) *and* (32), *we get*

$$\begin{aligned} x_{2kn+1+i} &= \frac{z_{2k(n-2)+1+i-3k}}{w_{2k(n-2)+1+i}} x_{2kn+1+i-6k} \\ &= \frac{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i-6k}}}{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i}}} x_{2kn+1+i-6k} \\ x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}} \end{aligned} \tag{33}$$

and similarly

$y_{2kn+1+i} = \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}$ (34)

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

Theorem 5 *Assume that* $\{x_n, y_n\}$ *is a well-defined solution of system* (5). *Then the followings are true:*

i) *Assume that* $b_1 b_2 = 1$. *Then every solution converges to a periodic solution with period* $6k$.

ii) *Assume that* $b_1 b_2 \neq 1$. *Then,*

a) *If* $b_1 b_2 < -1$ *or* $b_1 b_2 > 1$, *then*

$$lim_{n\to\infty} \frac{x_n}{x_{n-6k}} = lim_{n\to\infty} \frac{y_n}{y_{n-6k}} = \frac{b_2 c_1 + c_2}{b_2 c_2 (b_1 b_2 c_1 + c_2 b_1 + c_1)}.$$

b) *If $-1 < b_1 b_2 < 1$, then every solution converges to a periodic solution with period $6k$.*

Proof 2

i) Consider system (29) with $\gamma_1 = \frac{c_1 b_2}{c_2}, \gamma_2 = \frac{c_2 b_1}{c_1}, \alpha = \frac{c_1}{b_2 c_1 + c_2}, \beta = \frac{c_2}{b_1 c_2 + c_1}$. Suppose that $b_1 b_2 = 1$. Then, we have

$$\begin{aligned} \frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} &= \frac{\gamma_1\gamma_2}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} \\ &= \frac{\frac{c_1 b_2}{c_2}\frac{c_2 b_1}{c_1}}{\left(-\frac{c_1 b_2}{c_2}+\frac{1}{\frac{c_1}{b_2 c_1+c_2}\frac{c_2}{b_1 c_2+c_1}}-\frac{c_2 b_1}{c_1}\right)^2} \\ &= \frac{b_1 b_2}{(b_1 b_2+1)^2} \\ &= \frac{1}{4}. \end{aligned} \tag{35}$$

Similarly, it can be seen that

$$\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} = \frac{\gamma_1\gamma_2}{(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2)^2} = \frac{1}{4}. \tag{36}$$

So, from (31), (32) and Theorem3, we obtain

$$\begin{aligned} \lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1, \end{aligned}$$

$i = 0, 1, ..., (2k-1)$. *Thus, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-6k}$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n-6k}$. So, theproof of (i) is finished.*

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ii)a) Assume that $b_1b_2 < -1$ *or* $b_1b_2 > 1$. *From* (35) *and* (36), *we get that*
$-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right))\frac{1}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2}<\frac{1}{4}$
(36)

and
$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha})\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}$.
So, from (31), (32) and Theorem2, we obtain that

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}}$$

$$=\frac{\frac{\frac{1+\sqrt{1-4\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2}}}{2}\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\frac{1+\sqrt{1-4\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}}}{2}\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)-(-\gamma_2)}{\frac{1}{\alpha}}}$$

$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)} \tag{37}$$

$$=\frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$$

and

$\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}}$
$=(b_2c_1+c_2)\frac{1}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$,
$i=0,1,...,(2k-1)$. *Thus, we have* $\lim_{n\to\infty}\frac{x_n}{x_{n-6k}}=\lim_{n\to\infty}\frac{y_n}{y_{n-6k}}=\frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$.

b) Assume that $-1 < b_1b_2 < 1$. *From (35) and (36), we get that*
(
-$\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha}))\frac{1}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2}<\frac{1}{4}$
and
(

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$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}\right)\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}$. *So, from (31), (32), (37) and Theorem2, we obtain*

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}$$

$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2+\frac{c_1b_2}{c_2}}{\frac{c_1b_2}{c_2}\left(1+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2c_2+c_1b_2}{b_1b_2c_2+c_1b_2}=1,$$

$i=0,1,...,(2k-1)$. *So, we get immediately* $\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{n-6k}$ *and* $\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{n-6k}$ *and then the proof of is finished.*

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A Numerical Approach Based on Subdivision Schemes for Solving Non-Linear Fourth Order Boundary Value Problems

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Abstract

This paper presents an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems involving ordinary differential equations. Numerical evidence suggests that the scheme converges to a smooth approximate solution of non-linear fourth order boundary value problem. The convergence of the approach is also discussed. Main purpose of this article is to explore and seek the applications of subdivision schemes in the field of physics and engineering.

Keywords: Boundary value problem, Subdivision schemes, Collocation algorithm, Approximation, interpolation

AMS Classification: 30E25; 65D07; 97N50

1 Introduction

Boundary value problems arise in several branches of physics and engineering. In recent years, there has been significant progress in solving problems associated with system of linear and nonlinear partial and ordinary differential equations involving boundary conditions. Two point nonlinear boundary value problems often cannot be solved by analytical methods. With increasing interest in finding solutions to nonlinear boundary value problems has come an increasing need for solution techniques.

In this paper, we consider the following type of nonlinear boundary value problem

$$y^{(iv)} = f(x, y, y^{'}) \tag{1.1}$$

with the boundary conditions

$$\begin{cases} y(a) = \alpha_1, & y'(a) = \alpha_2, \\ y(b) = \alpha_3, & y'(b) = \alpha_4, \end{cases} \tag{1.2}$$

where $\alpha_i, i = 1, 2, 3, 4$ are constants. We assume that the problem is well-posed.

A variety of methods have been introduced to solve these problem e.g., shooting methods, splines methods [2, 3, 4, 5, 6], finite difference methods, finite element methods, the collocation methods and other approximation methods. For discrete methods, like shooting and finite differences methods, only discrete approximate values of the unknown $y(x)$ can be obtained. For fitting curve to data we need further data processing techniques. For the case of spline interpolation or approximation methods the unknown function $y(x)$ is assumed to be piecewise polynomial which requires at least piecewise higher order differentiability of the function $f(x, y, y^{'})$. To overcome these disadvantages, Qu and Agarwal

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[8, 9, 10] introduced the subdivision based algorithm for the solution of two point second order boundary value problems. Mustafa and Ejaz [14] solved third order boundary value problem by using subdivision technique. Higher order nonlinear problems have not been solved by subdivision techniques. This motivates us to solve fourth order boundary value problems by subdivision schemes based collocation iterative method. This paper introduces a numerical method based on subdivision technique for the solution of fourth order nonlinear boundary value problem.

In Section 2, some results about subdivision algorithms and basis function are given. In Section 3, a numerical method to solve (1.1) using refinable basis functions is formulated and its convergence properties studied. Error properties are given in Section 4. Numerical examples illustrating the feasibility of our proposed algorithm are given in Section 5.

2 Basis Functions and their Derivatives

Some useful results for the solution of non-linear boundary value problem are discussed in this section. Introduction to the basis functions of subdivision schemes that are used to construct the approximate solutions of proposed problem (1.1) is also part of this section.

2.1 Interpolating subdivision scheme

A mathematical formulation of binary subdivision scheme is defined as:

$$\begin{cases} P_{2i}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{-2j} P_{i+j}^{k} \\ \\ P_{2i+1}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{i-2j} P_{i+j}^{k} \end{cases} \tag{2.1}$$

The scheme is a stepwise interpolatory scheme if and only if the coefficient a_i satisfy $a_{2i} = \delta_i \quad \forall\ i \in \mathbb{Z}$. We consider the following binary interpolating subdivision scheme [7, 12, 13].

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ \\ p_{2i+1}^{k+1} = \frac{35}{65536}\left(p_{i-4}^k + p_{i+5}^k\right) - \frac{405}{65536}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{567}{16384}\left(p_{i-2}^k + p_{i+3}^k\right) \\ \\ \qquad\qquad - \frac{2205}{16384}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{19845}{32768}\left(p_i^k + p_{i+1}^k\right). \end{cases} \tag{2.2}$$

The scheme (2.2) is C^4-continuous, having support length $(-9, 9)$ and approximation order is ten.

2.2 Basis functions

The basis functions is the limit function resulting from cardinal data, where all vertices of the polygon have value zero except for one. Let $\phi(x)$, $x \in \mathbb{R}$ be the fundamental solution of (2.2) and satisfies the two scale equations

$$\begin{aligned} \phi(x) = \phi(2x) + \frac{1}{65536}&[39690\{\phi(2x-1) + \phi(2x+1)\} - 8820\{\phi(2x-3) \\ &+\phi(2x+3)\} + 2268\{\phi(2x-5) + \phi(2x+5)\} - 405\{\phi(2x-7) \\ &+\phi(2x+7)\} + 35\{\phi(2x-9) + \phi(2x+9)\}], \quad x \in \mathbb{R} \end{aligned} \tag{2.3}$$

and

$$\phi(x) \in C^4, \quad \phi(x) = 0, \quad x \in]-8, 8[, \quad \phi(i) = \delta_0, \quad i \in \mathbb{Z}. \tag{2.4}$$

Furthermore, first derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(i)}(0) \quad = 0, & \phi^{(i)}(\pm1) = \mp\frac{1914621952}{1159104017}, & \phi^{(i)}(\pm2) = \pm\frac{530452796}{1159104017}, \\ \\ \phi^{(i)}(\pm3) = \mp\frac{1470464}{13780629}, & \phi^{(i)}(\pm4) = \pm\frac{17297069}{1159104017}, & \phi^{(i)}(\pm5) = \mp\frac{2772992}{5795520085}, \\ \\ \phi^{(i)}(\pm6) = \mp\frac{1127636}{10431936153}, & \phi^{(i)}(\pm7) = \mp\frac{4096}{8113728119}, & \phi^{(i)}(\pm8) = \mp\frac{5}{9272832136}. \end{cases} \tag{2.5}$$

Second derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(ii)}(0) = -\frac{2370618501415}{309077185968}, & \phi^{(ii)}(\pm 1) = \frac{3265310153216}{676106344305}, & \phi^{(ii)}(\pm 2) = -\frac{878265102572}{676106344305}, \\ \phi^{(ii)}(\pm 3) = \frac{734063059456}{2028319032915}, & \phi^{(ii)}(\pm 4) = -\frac{80883901277}{1352212688610}, & \phi^{(ii)}(\pm 5) = \frac{214899200}{135221268861}, \\ \phi^{(ii)}(\pm 6) = \frac{297875188}{405663806583}, & \phi^{(ii)}(\pm 7) = \frac{64000}{19317324123}, & \phi^{(ii)}(\pm 8) = \frac{4375}{618154371936}. \end{cases} \tag{2.6}$$

Third derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iii)}(0) = 0, & \phi^{(iii)}(\pm 1) = \pm\frac{43317515008}{15295995855}, & \phi^{(iii)}(\pm 2) = \mp\frac{121530512357}{61183983420}, \\ \phi^{(iii)}(\pm 3) = \pm\frac{240606976}{566518365}, & \phi^{(iii)}(\pm 4) = \mp\frac{5285889107}{244735933680}, & \phi^{(iii)}(\pm 5) = \mp\frac{37414144}{3059199171}, \\ \phi^{(iii)}(\pm 6) = \pm\frac{1090169}{453214692}, & \phi^{(iii)}(\pm 7) = \mp\frac{21760}{437028453}, & \phi^{(iii)}(\pm 8) = \mp\frac{2975}{13984910496}. \end{cases} \tag{2.7}$$

Fourth derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iv)}(0) = \frac{33869667}{457408}, & \phi^{(iv)}(\pm 1) = -\frac{5295054752}{89730585}, & \phi^{(iv)}(\pm 2) = \frac{10404741119}{358922340}, \\ \phi^{(iv)}(\pm 3) = -\frac{74879584}{9970065}, & \phi^{(iv)}(\pm 4) = \frac{295020869}{2871378720}, & \phi^{(iv)}(\pm 5) = \frac{9238624}{17946117}, \\ \phi^{(iv)}(\pm 6) = -\frac{900187}{7976052}, & \phi^{(iv)}(\pm 7) = \frac{71840}{17946117}, & \phi^{(iv)}(\pm 8) = \frac{11225}{328157568}. \end{cases} \tag{2.8}$$

The above derivative values are found by using the left eigenvectors of the subdivision process (2.2). The detailed description about these left eigenvectors and derivatives can be found in [8, 11, 14]. The graphical representations of above mentioned derivatives are shown in Figure 1.

3 Description of Iterative Numerical Method

This section describes the method for the numerical solution of nonlinear boundary value problem (1.1). The detail of the method is given below:

3.1 The collocation method

In this subsection, the collocation method is constructed based on the interpolating subdivision scheme (2.2). Our numerical approach for nonlinear fourth order boundary value problem using collocation method based on subdivision scheme is to seek an approximate solution as

$$Z(x) = \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.1}$$

where N is the positive integer $N \geqslant 8$, $h = 1/N$ and $x_i = i/N = ih$ and $\{z_i\}$ are the unknowns to be determined for the solution of (1.1). In order to solve the problem, a collocation method $Z(x)$ is considered to be the solution of the above differential equation at $x = x_j$ and we substitute equation (3.1) into equation (1.1). This leads to

$$Z^{(iv)}(x_j) = f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N \tag{3.2}$$

and boundary conditions

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4 \tag{3.3}$$

From (3.1), we get

$$Z^{(iv)}(x) = \frac{1}{h^4} \sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.4}$$

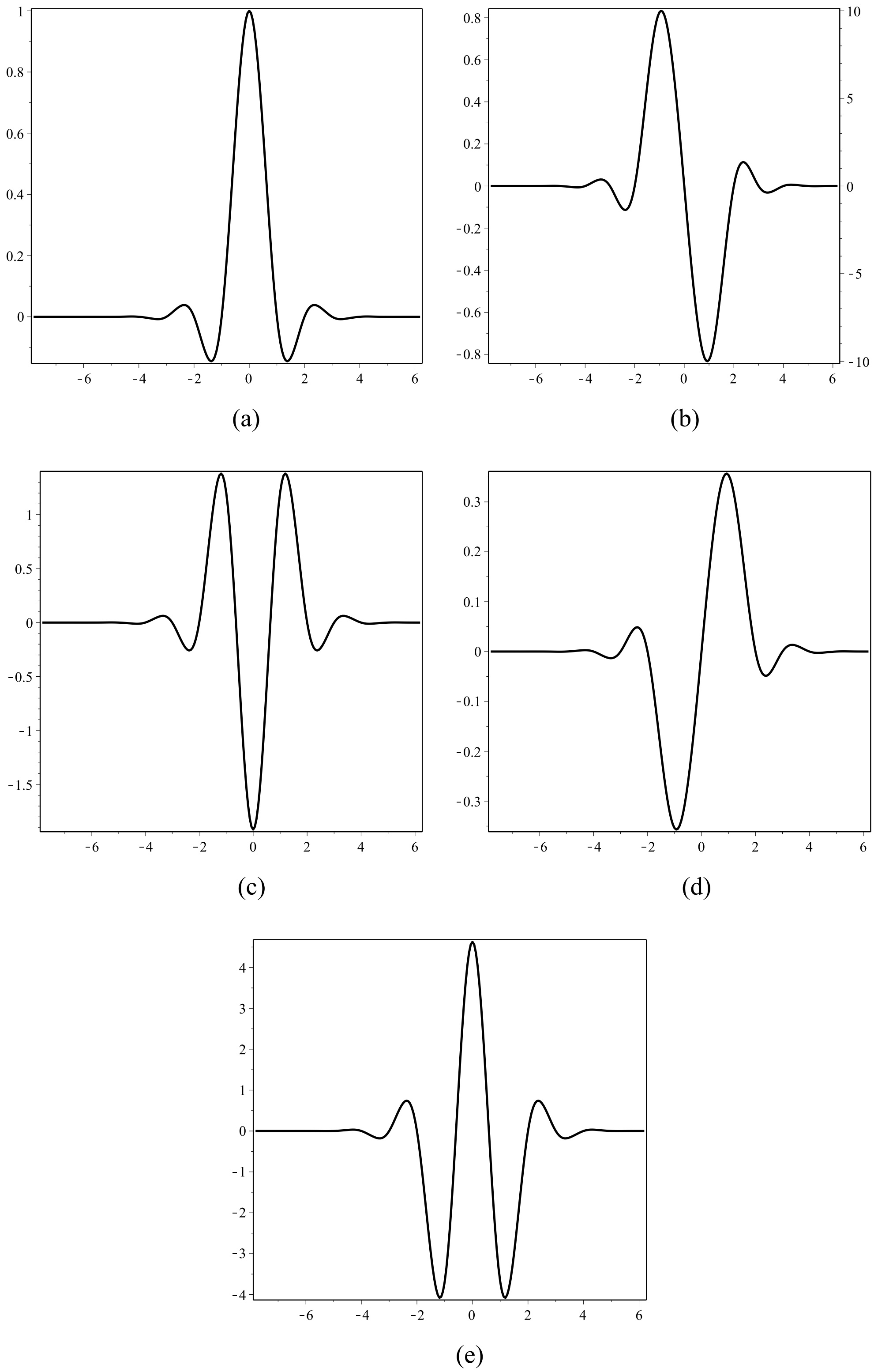

Figure 1: Graphical representation of: (a) basis functions, (b) first, (c) second, (d) third, (e) fourth derivatives of basis function

Substituting (3.4) into (3.2), we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x_j - x_i}{h}\right) = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N.$$

This can be written as:

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_{j-i} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N$$

Since $\phi^{(iv)}_i = \phi^{(iv)}_{-i}$, the above system of equations becomes

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_{i-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N. \tag{3.5}$$

The nonlinear system of equations (3.5) can be simply in following Theorems 1 and 2.

Theorem 1. *The nonlinear system of equations (3.5) for $j = 0$ becomes*

$$\sum_{i=-8}^{8} z_i \phi^{(iv)}_i = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \tag{3.6}$$

Proof. By expanding (3.5) for $j = 0$, we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)}_i = h^4 f(x_0, Z(x_0), Z^{'}(x_0))$$

$$z_{-8}\phi^{iv}_{-8} + z_{-7}\phi^{iv}_{-7} + z_{-6}\phi^{iv}_{-6} + \cdots + z_7\phi^{iv}_7 + z_8\phi^{iv}_8 + z_9\phi^{iv}_9 + \cdots + z_{N+7}\phi^{iv}_{N+7}$$
$$+ z_{N+8}\phi^{iv}_{N+8} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

Since $\phi^{iv}(i)$ exists only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation can be written as

$$z_{-8}\phi^{iv}_{-8} + z_{-7}\phi^{iv}_{-7} + z_{-6}\phi^{iv}_{-6} + \cdots + z_7\phi^{iv}_7 + z_8\phi^{iv}_8 = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

□

Theorem 2. *For $j = 1, 2, \cdots, N$, the nonlinear system of equations (3.5) becomes*

$$\sum_{i=-8+j}^{8+j} z_i \phi^{(iv)}_{i-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \tag{3.7}$$

Proof. By expanding (3.5), for $j = 1, 2, 3, \cdots, N$, we get

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-j} + z_{-7}\phi^{iv}_{-7-j} + z_{-6}\phi^{iv}_{-6-j} + \cdots + z_7\phi^{iv}_{7-j} + z_8\phi^{iv}_{8-j} + z_9\phi^{iv}_{9-j} + z_{10}\phi^{iv}_{10-j} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-j} + z_{N+7}\phi^{iv}_{N+7-j} + z_{N+8}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.8}$$

Substituting $j = 1$ in (3.8), it becomes

$$z_{-8}\phi^{iv}_{-8-1} + z_{-7}\phi^{iv}_{-7-1} + z_{-6}\phi^{iv}_{-6-1} + \cdots + z_7\phi^{iv}_{7-1} + z_8\phi^{iv}_{8-1} + z_9\phi^{iv}_{9-1} + z_{10}\phi^{iv}_{10-1}$$
$$+ \cdots + z_{N+6}\phi^{iv}_{N+6-1} + z_{N+7}\phi^{iv}_{N+7-1} + z_{N+8}\phi^{iv}_{N+8-1} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)).$$

This implies

$$z_{-8}\phi^{iv}_{-9} + z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 + z_{10}\phi^{iv}_9 + \cdots$$
$$+ z_{N+6}\phi^{iv}_{N+5} + z_{N+7}\phi^{iv}_{N+6} + z_{N+8}\phi^{iv}_{N+7} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)).$$

Since $\phi^{iv}(i)$ is non-zero only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation becomes

$$z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \tag{3.9}$$

For j=2, (3.8) becomes

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-2} + z_{-7}\phi^{iv}_{-7-2} + z_{-6}\phi^{iv}_{-6-2} + \cdots + z_7\phi^{iv}_{7-2} + z_8\phi^{iv}_{8-2} + z_9\phi^{iv}_{9-2} + z_{10}\phi^{iv}_{10-2} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-2} + z_{N+7}\phi^{iv}_{N+7-2} + z_{N+8}\phi^{iv}_{N+8-2} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

This implies

$$\begin{aligned} &z_{-8}\phi^{iv}_{-10} + z_{-7}\phi^{iv}_{-9} + z_{-6}\phi^{iv}_{-8} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+4} + z_{N+7}\phi^{iv}_{N+5} + z_{N+8}\phi^{iv}_{N+6} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

By using the definition of ϕ^{iv}_i given in (2.8), above equation yields

$$z_{-6}\phi^{iv}_{-8} + z_{-5}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \tag{3.10}$$

By using the similar pattern for $j = 1, 2$, we can find the expression for $j = 3, 4, \cdots N$

$$\begin{aligned} &z_{-8+j}\phi^{iv}_{-8-j} + z_{-7+j}\phi^{iv}_{-7-j} + z_{-6+j}\phi^{iv}_{-6-j} + \cdots + z_{7+j}\phi^{iv}_{7-j} + z_{8+j}\phi^{iv}_{8-j} + z_{9+j}\phi^{iv}_{9-j} \\ &+ \cdots + z_{N+6+j}\phi^{iv}_{N+6-j} + z_{N+7+j}\phi^{iv}_{N+7-j} + z_{N+8+j}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.11}$$

□

The nonlinear system of equations (3.5) is equivalent to the following non-linear system of $N + 1$ equations with (N+17) unknowns $\{z_i\}$.

$$AZ = F(z) \tag{3.12}$$

where A is banded matrix of order $(N + 1) \times (N + 17)$, Z is the unknown vector of order $N + 17$ and $F(z)$ is the vector of order $N + 1$ depends on z. The matrix A, vectors Z and $F(z)$ are given explicitly by

$$A = [\phi^{iv}_{pq}(q - p - 8)]_{(N+1)\times(N+17)} \tag{3.13}$$

where $p = 1, 2, 3 \cdots, N + 1$ and $q = 1, 2, 3, \cdots, N + 17$ represent the row and column respectively.

$$F(z) = \left(h^4 f(x_0, Z(x_0), Z^{'}(x_0)), \cdots, h^4 f(x_N, Z(x_N), Z^{'}(x_N))\right)^T \tag{3.14}$$

$$Z = (z_{-8}, z_{-7}, z_{-6}, \cdots, z_{N+6}, z_{N+7}, z_{N+8})^T \tag{3.15}$$

$$Z^{'}(x_j) = \sum_{i=-8}^{N+8} z_j \phi^{'}\left(\frac{x_j - x_i}{h}\right)$$

where $\phi'(i)$ is already defined in (2.5) with $\phi(i) = \phi_i$.

3.2 Boundary conditions at end points

For unique solution of the nonlinear system (3.5), we need sixteen more conditions. Four conditions can be attained from given boundary conditions for the nonlinear system of equations and remaining conditions are attained by using some extrapolation method. The details of the given boundary conditions and extrapolation method are given below:

3.2.1 Boundary Conditions

The given boundary conditions are

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4$$

The approximation of derivative conditions at ends point is defined as:

$$\begin{aligned} Z'(0) = \left(\frac{N}{2520}\right)\{&-7381z_0 + 25200z_1 - 56700z_2 + 100800z_3 - 132300z_4 + 127008z_5 \\ &-88200z_6 + 43200z_7 - 14175z_8 + 28800z_9 - 252z_{10}\} + O(h^{10}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Z'(N) = \left(\frac{N}{2520}\right)\{&7381z_N - 25200z_{N-1} + 56700z_{N-2} - 100800z_{N-3} + 132300z_{N-4} \\ &-127008z_{N-5} + 88200z_{N-6} - 43200z_{N-7} + 14175z_{N-8} - 28800z_{N-9} \\ &+252z_{N-10}\} + O(h^{10}). \end{aligned} \tag{3.17}$$

3.2.2 Extrapolation Method

The remaining twelve conditions for the nonlinear systems (3.5) to obtain stable systems for the solution of (1.1) are obtained by using the following extrapolation method.
We define six conditions at left end points and six conditions at the right end points. Since subdivision scheme (2.2) reproduces nine degree (i.e. tenth order) polynomials, so we define boundary conditions of order ten for the solution of (3.5). For simplicity only left end points $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ are discussed and the values of right end points $z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}, z_{N+6}, z_{N+7}$ can be treated similarly.

The values $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ can be determined by the polynomial $q(x)$ interpolating (x_i, z_i), $2 \leq i \leq 7$. Precisely, we have

$$z_{-i} = q(-x_i),\ i = 2, 3, 4, 5, 6, 7$$

where

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} Z(x_{i-j}).$$

From (3.1), $Z_1(x_i) = z_i$, $i = 2, 3, 4, 5, 6, 7$ and replacing x_i by $-x_i$, we have

$$q(-x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{-i+j}.$$

Hence the following boundary conditions can be employed at the left end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{-i+j} = 0,\ i = 7, 6, 5, 4, 3, 2. \tag{3.18}$$

Similarly for the right end, we can define $z_i = q(-x_i)$, $i = N+2,\ N+3,\ N+4,\ N+5,\ N+6,\ N+7$ and

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{i-j}.$$

So we have the following boundary conditions at the right end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{i-j} = 0,\ i = N+2, N+3, N+4, N+5, N+6, N+7. \tag{3.19}$$

Finally, we obtain a new system of $(N+17)$ linear equations with $(N+17)$ unknowns $\{z_i\}$. The $N+1$ equations are obtained from (3.5), four equations from boundary conditions (3.3) and twelve from boundary conditions (3.18) and (3.19) for the numerical solution of proposed problem.
Hence the stable nonlinear system of equations is defined as:

$$BZ = R(z) \tag{3.20}$$

where the matrix B is given by

$$B = (C_0^T, A^T, C_1^T)^T \tag{3.21}$$

A is defined in (3.13), C_0, C_1 and the vector R(z) is defined as

$$C_0 = \left(\begin{matrix}
0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\
0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 \\
0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 \\
0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7381N}{2520} & \frac{25200N}{2520} & -\frac{56700N}{2520} & \frac{100800N}{2520} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{matrix}\right.$$

$$\left.\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-10 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{132300N}{2520} & \frac{127008N}{2520} & -\frac{88200N}{2520} & \frac{43200N}{2520} & -\frac{14175N}{2520} & \frac{2800N}{2520} & -\frac{252N}{2520} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{matrix}\right) \tag{3.22}$$

The first six rows of C_0 are obtained from (3.18), second last row is obtained from (3.16) and last row is taken from given boundary conditions $Z_1(0)$ which is defined in (3.3) and

$$C_1 = \left(\begin{matrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & \frac{N}{10} & -\frac{10N}{9} & \frac{45N}{8} & -\frac{120N}{7} & 35N & -\frac{252N}{5} & \frac{105N}{2} & -40N & \frac{45N}{2} & -10N \\
0 & 0 & \cdots & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45
\end{matrix}\right.$$

$$\left.\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7381N}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 \\
210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 \\
-252 & 210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 \\
-120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 & 0
\end{matrix}\right), \tag{3.23}$$

First row of C_1 is obtained from $Z_1(N)$ which is defined in (3.3), second row is obtained from (3.17) and the last six rows are obtained from (3.19), Z which is defined in (3.15) and R_1 is defined as

$$R(z) = (0,0,0,0,0,0,Z'(0),Z(1),F^T(z),Z(1),Z'(1),0,0,0,0,0,0)^T, \tag{3.24}$$

where $F(z)$ is defined by (3.14).

3.2.3 Non-singularity of a matrix

We can check the non-singularity of coefficient matrix B defined in (3.21) by different methods. We observe that the determinant of matrix B is non-zero for $N \leq 500$. Hence the non linear system of equations have a solution for $N \leq 500$. We also check the non singularity of matrix by finding eigenvalues up to $N \leq 500$ and we observe that all the eigenvalues are non-zero. Hence by [15] we conclude that the B is non-singular. For large $N > 500$ the matrix may or may not be singular.

3.3 Iterative algorithm and its convergence

An iterative algorithm and its convergence are described in this section.

3.3.1 Iterative algorithm based on basis function

The iterative algorithm based on basis function of the subdivision scheme (2.2) are as defined in the following three steps.

First step: Initial approximation

The initial approximation is important because the numerical solution depends on the initial approximation. We define the process for finding the initial approximation as follows:
Let initial approximate solution Z^0 be the solution of the following linear system

$$BZ^0 = F^0 \tag{3.25}$$

where

$$\begin{cases} F^0 = (0,0,0,0,0,0,y'(a),y(a),f_0,f_1,f_2,\cdots,f_N,y(b),y'(b),0,0,0,0,0,0)^T, \\ f_i = h^4 f(x_i, L_i, D), \quad i = 0,1,2,\cdots N \\ L_i = y(0) + ih\left(\frac{y(b)-y(a)}{b-a}\right) \\ D = y(b) - y(a). \end{cases} \tag{3.26}$$

F^0 is the initial linear approximation of the non-linear vector $R(z)$.

Second step: Numerical solution

The numerical solutions Z^* of the nonlinear system are obtained by using the simple iterative scheme

$$BZ^{(m+1)} = R(Z^m), \quad m = 0,1,2,3,\cdots \tag{3.27}$$

Third step: Stopping condition

The above iterative processes will terminate when the following condition is satisfied

$$||z^{(m)} - z^{(m-1)}|| \leq tol \tag{3.28}$$

where tolerance is supposed value i.e. $tol = 10^{-6}$. The convergence of the above iterative algorithm is guaranteed by the following proposition.

Theorem 3. *The successive solutions $\{Z^{(m)}\}$ generated by the iterative algorithm (3.27) linearly converges to the solution Z^* of the non-linear solution of the system (3.20) provided that the M_0 and M_1 are Lipschitz constants and step size h is small.*
i.e.

$$\|B^{-1}\| \leq \left(M_0 h^4 + \frac{4994220330463}{1460471061420} M_1 h^3\right). \tag{3.29}$$

Proof. Let Z^* and $Z^{(m)}$ be the solutions of the nonlinear system (3.20). Then by definition, for small h we have

$$BZ^* = R(Z^*), \tag{3.30}$$
$$BZ^{m+1} = R(Z^m). \tag{3.31}$$

Let the error vector be defined as $e^{(k)} = Z^k - Z^*$ at kth iteration which satisfies

$$\begin{aligned} BZ^{(m+1)} - BZ^* &= R(Z^k) - R(Z^*), \\ B(Z^{(m+1)} - Z^*) &= R(Z^k) - R(Z^*), \\ Be^{(k+1)} &= R(Z^k) - R(Z^*). \end{aligned} \tag{3.32}$$

For $i = 0, 1, 2, \cdots, N$

$$D_4 e_i^{(k+1)} = (F(Z^k) - F(Z^*))_i.$$

By mean value theorem, which is stated as "If a function $f(x, y, z)$ is continuously differentiable in an open set of $\mathbb{R}^3$ containing points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the line segment connecting them, then an equation

$$f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = f_x^{'}(r, s, t)(x_2 - x_1) + f_y^{'}(r, s, t)(y_2 - y_1) + f_z^{'}(r, s, t)(z_2 - z_1)$$

is valid for the interior point (a, b, c) of the segment.", we have

$$D_4 e_i^{(k+1)} = f(x_i, Z_i^{(k)}, Z'^{(k)}) - f(x_i, Z_i^{(*)}, Z'^{(*)}).$$

The above equation can be written as (by using mean value theorem)

$$D_4 e_i^{(k+1)} = f_x^*(x_i - x_i) + f_y^*(Z_i^{(k)} - Z_i^{(*)}) + f_{y'}^*(Z'^{(k)} - Z'^{(*)})$$

by using the definition of error vector, we have

$$\begin{aligned} D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* e'^{(k)}, \\ D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* D_1 e^{(k)} \end{aligned}$$

where D_4 and D_1 are the derivative difference operators defined as

$$\begin{aligned} D_1 f_i = \frac{1}{2920942122840h} & [1575(f_{i-8} - f_{i+8}) + 1474560(f_{i-7} - f_{i+7}) \\ & +315738080(f_{i-6} - f_{i+6}) + 1397587968(f_{i-5} - f_{i+5}) \\ & -43588613880(f_{i-4} - f_{i+4}) + 311679549440(f_{i-3} - f_{i+3}) \\ & -1336741045920(f_{i-2} - f_{i+2}) + 4824847319040(f_{i-1} - f_{i+1})] \end{aligned}$$

$$\begin{aligned} D_4 f_i = \frac{1}{183768238080h^4} & [392875(f_{i+8} - f_{i-8}) + 45977600(f_{i+7} - f_{i-7}) \\ & -1296269280(f_{i+6} - f_{i-6}) + 5912719360(f_{i+5} - f_{i-5}) \\ & +1180083476(f_{i+4} - f_{i-4}) - 86261280768(f_{i+3} - f_{i-3}) \\ & +332951715808(f_{i+2} - f_{i-2}) - 677767008256(f_{i+1} - f_{i-1}) \\ & +850467338370 f_i]\,. \end{aligned}$$

This implies

$$D_4 e_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}.$$

Since $e_i = e_{N-i} = 0, \quad i = 0, -1, -2, \cdots, -8$, we have

$$Be_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}$$

This can be written as

$$e_i^{(k+1)} = B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}).$$

By taking norm on both sides, we get

$$\|e_i^{(k+1)}\| = \|B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

This implies

$$\|e_i^{(k+1)}\| = \|B^{-1}\|\|(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

By using the definition of Lipschitz condition, we get

$$\|e^{(k+1)}\| \leq h^4 M_0(b-a)\|B^{-1}\|\|e^k\| + h^3 M_1\|D_1\|\|e^{(k)}\|.$$

This implies

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \leq \|B^{-1}\| \left(h^4 M_0(b-a) + h^3 M_1\|D_1\|\right),$$

which is equivalent to

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h^3 M_1\|B^{-1}\|\|D_1\| \leq hM_1\|B^{-1}\|\|D_1\|,$$

i-e

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx hM_1\|B^{-1}\|\|D_1\|.$$

The results follows immediately from this inequality and the following fact

$$\|D_1\| = \frac{4994220330463}{1460471061420}. \tag{3.33}$$

A simple approximation of condition by omitting the quatric term is

$$h \leq \frac{1460471061420}{4994220330463} M_1^{-1} \left\|B^{-1}\right\|^{-1}. \tag{3.34}$$

This complete the proof. □

4 Error Estimation

From the approximation properties of the basis function $\phi(x)$, it is shown that the collocation method (3.1) with nonic precision treatments at the end points has at least power of approximation $O(h^3)$. Here we present our main results for error estimation. Proof of these results are similar to the proof of Proposition [14, 8].

Theorem 4. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (3.20) then absolute error by interpolating collocation algorithm is*

$$||err(x)||_\infty = ||Z^{(l)}(x) - y^{(l)}(x)||_\infty = O(h^{3-l}),\ l = 0, 1, 2, 3.$$

where l denotes the order of derivative.

Proof. Since the order of approximation of subdivision scheme (2.2) is ten so by direct calculation (fourth left eigenvector), we can find derivative of smooth function $y(x)$ as

$$y^{iv}(x_j) = \frac{2^4}{183768238080h^4}\{392875y(x_j-8h)+45977600y(x_j-7h)$$

$$-1296269280y(x_j-6h)+5912719360y(x_j-5h)+1180083476y(x_j-4h)$$

$$-86261280786y(x_j-3h)+332951715808y(x_j-2h)-677767008256y(x_j-h)$$

$$+850467338370y(x_j)-677767008256y(x_j+h)+332951715808y(x_j+2h)$$

$$-86261280786y(x_j+3h)+1180083476y(x_j+4h)+5912719360y(x_j+5h)$$

$$-1296269280y(x_j+6h)+45977600y(x_j+7h)+392875y(x_j+8h)\}+O(h^{10}).$$

This can be written as

$$y_j^{iv} = \frac{2^4}{183768238080h^4}\{392875y_{j-8}+45977600y_{j-7}-1296269280y_{j-6}$$

$$+5912719360y_{j-5}+1180083476y_{j-4}-86261280786y_{j-3}+332951715808y_{j-2}$$

$$-677767008256y_{j-1}+850467338370y_j-677767008256y_{j+1}+332951715808y_{j+2}$$

$$-86261280786y_{j+3}+1180083476y_{j+4}+5912719360y_{j+5}-1296269280y_{j+6}$$

$$+45977600y_{j+7}+392875y_{j+8}\}+O(h^{10}). \tag{4.1}$$

Similarly, we have

$$Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875z_{j-8}+45977600z_{j-7}-1296269280z_{j-6}$$

$$+5912719360z_{j-5}+1180083476z_{j-4}-86261280786z_{j-3}+332951715808z_{j-2}$$

$$-677767008256z_{j-1}+850467338370z_j-677767008256z_{j+1}+332951715808z_{j+2}$$

$$-86261280786z_{j+3}+1180083476z_{j+4}+5912719360z_{j+5}-1296269280z_{j+6}$$

$$+45977600z_{j+7}+392875z_{j+8}\}+O(h^{10}). \tag{4.2}$$

If we define error function $e(x)=Z(x)-y(x)$ and error vectors at the nodes by

$$e(x_j)=Z(x_j)-y(x_j+jh),\ -8\le j\le N+8,$$

or equivalently $e_j=Z_j-y_j,\ -8\le j\le N+8$, This implies

$$\left\{\begin{array}{l} e'_j = Z'_j - y'_j,\\ e''_j = Z''_j - y''_j,\\ e'''_j = Z'''_j - y'''_j\\ e^{iv}_j = Z^{iv}_j - y^{iv}_j. \end{array}\right. \tag{4.3}$$

By subtracting (4.2) from (4.1), we get

$$y_j^{iv} - Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875(y_{j-8} - z_{j-8}) + 45977600(y_{j-7} - z_{j-7})$$

$$-1296269280(y_{j-6} - z_{j-6}) + 5912719360(y_{j-5} - z_{j-5}) + 1180083476(y_{j-4} - z_{j-4})$$

$$-86261280786(y_{j-3} - z_{j-3}) + 332951715808(y_{j-2} - z_{j-2}) - 677767008256(y_{j-1} - z_{j-1})$$

$$+850467338370(y_j - z_j) - 677767008256(y_{j+1} - z_{j+1}) + 332951715808(y_{j+2} - z_{j+2})$$

$$-86261280786(y_{j+3} - z_{j+3}) + 1180083476(y_{j+4} - z_{j+4}) + 5912719360(y_{j+5} - z_{j+5})$$

$$-1296269280(y_{j+6} - z_{j+6}) + 45977600(y_{j+7} - z_{j+7}) + 392875(y_{j+8} - z_{j+8})\} + O(h^{10}).$$

This implies

$$e_j^{iv} = \frac{2^4}{183768238080h^4}\{392875e_{j-8} + 45977600e_{j-7} - 1296269280e_{j-6}$$

$$+5912719360e_{j-5} + 1180083476e_{j-4} - 86261280786e_{j-3} + 332951715808e_{j-2}$$

$$-677767008256e_{j-1} + 850467338370e_j - 677767008256e_{j+1} + 332951715808e_{j+2}$$

$$-86261280786e_{j+3} + 1180083476e_{j+4} + 5912719360e_{j+5} - 1296269280e_{j+6}$$

$$+45977600e_{j+7} + 392875e_{j+8}\} + O(h^{10}). \tag{4.4}$$

From (1.1), (3.1), (4.3) and by assuming the tenth order boundary treatments at the end points, we have

$$e_j^{iv} = a_j e_j + b_j e_j^{'}, \quad 0 \le i \le N \tag{4.5}$$

and

$$e_j = \begin{cases} \max\limits_{0\le k\le 7}\{|e_k|\}O(h^{10}), & -8 \le i \le 0 \\ \max\limits_{N-3\le k\le N}\{|e_k|\}O(h^{10}), & N \le i \le N+8 \end{cases} \tag{4.6}$$

where $j = 0, 1, \cdots N$

$$a_j = f_y(t_j, y_j^*, y_j^{'*}), \quad b_j = f_{y'}(t_j, y_j^*, y_j^{'*}),$$

and

$$y_j^* = y_j + \theta_j e_j, \quad y_j^{'*} = y_j' + \theta_j e_j^{'}, \quad 0 \le \theta_j \le 1.$$

Using the results (4.4) and

$$\begin{aligned}&[1575(z_{i-8} - z_{i+8}) + 1474560(z_{i-7} - z_{i+7}) + 315738080(z_{i-6} - z_{i+6}) + 1397587968\\ &(z_{i-5} - z_{i+5}) - 43588613880(z_{i-4} - z_{i+4}) + 311679549440(z_{i-3} - z_{i+3}) - 1336741045920\\ &(z_{i-2} - z_{i+2}) + 4824847319040(z_{i-1} - z_{i+1})] = 2920942122840hZ^{'} + O(h^{10}),\end{aligned} \tag{4.7}$$

It can be conclude that relation (4.5)and (4.6) is equivalent to

$$(B + O(h^8) - O(h^4) - D_1 O(h^3))E = O(h^{10})\|E\|,$$

where $E = (e_{-8}, e_{-7}, \cdots, e_7, e_8)$.

Hence for small h, the coefficient matrix $B + O(h)$, will be invertible, thus using the standard result from algebra and effect of $\|B^{-1}\|$, we have the following estimate

$$\|E\| \le \frac{\|B^{-1}\|}{1 - O(h)} O(h^{10}) = O(h^3). \tag{4.8}$$

□

5 Results and Discussions

In this section, we test the proposed method on some nonlinear problems. Numerical results for each of the problems are presented in the tables. These values are very close to the true solutions and the values of the errors are also given in the table.

Example 1. *Consider the following non-linear boundary value problem [1]*

$$y^{iv} - 6\exp(-4y) = -12(1+x)^{-4}, \tag{5.1}$$

with boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = ln(2) = y'(1) = 0.5.$$

The exact solution of the problem (5.1) is $y = \ln(1+x)$. *Using the collocation method described in Section 3 for* $N = 10$, $h = 10^{-1}$ *and* $tol = 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 1. The maximum absolute error obtained by the proposed method is* 1.78×10^{-3}. *The graphical comparison between exact and approximate solutions is shown in Figure 2.*

Table 1: Numerical results of Example 1

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.0953101798	0.0950147533	0.0002954265
0.2	0.1823215568	0.1814496227	0.0008719341
0.3	0.2623642645	0.2609546573	0.0014096072
0.4	0.3364722366	0.3347370220	0.0017352146
0.5	0.4054651081	0.4036840381	0.0017810699
0.6	0.4700036292	0.4684459279	0.0015577013
0.7	0.5306282511	0.5294932609	0.0011349902
0.8	0.5877866649	0.5871580370	0.0006286279
0.9	0.6418538862	0.6416636708	0.0001902154
1.0	0.6931471806	0.6931471806	0

Example 2. *Consider the non-linear boundary value problem [1]*

$$y^{(iv)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \tag{5.2}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y(1) = y'(1) = 1.$$

Using the collocation method described in Section 3 for $N = 10$, $h = 10^{-1}$ *and* $tol = 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 2. The maximum absolute error obtained by the proposed method is* 1.73×10^{-2}. *The graphical comparison between exact and approximate solutions is shown in Figure 3.*

6 Conclusion

This study has presented a numerical approach based on subdivision collocation algorithm for solving the numerical solution of nonlinear fourth order boundary value problems. The proposed iterative method

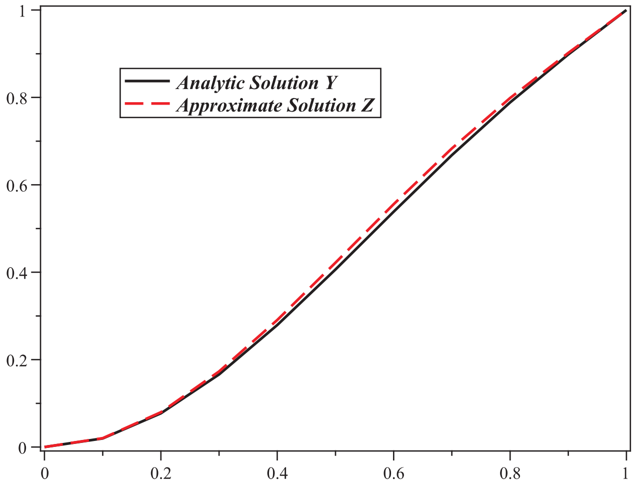

Figure 2: *Comparison of the analytic and approximate solution of Example 1.*

Table 2: Numerical results of Example 2

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.01981	0.0202195	0.0004095
0.2	0.07712	0.0796952	0.0025752
0.3	0.16623	0.1728732	0.0066432
0.4	0.27904	0.2905995	0.0115595
0.5	0.40625	0.4219208	0.0156708
0.6	0.53856	0.5558846	0.0173246
0.7	0.66787	0.6833406	0.0154706
0.8	0.78848	0.7987412	0.0102612
0.9	0.89829	0.9019417	0.0036517
1.0	1.00000	1.0000000	0

has been applied on different nonlinear fourth order boundary value problems. Numerical results show that the accuracy of the approximate solution is $O(h^3)$. We have also observed that the accuracy of the solution can be improved by choosing different subdivision schemes with the proper adjustment of boundary conditions.

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Competing interests

The authors declare that they have no competing interests.

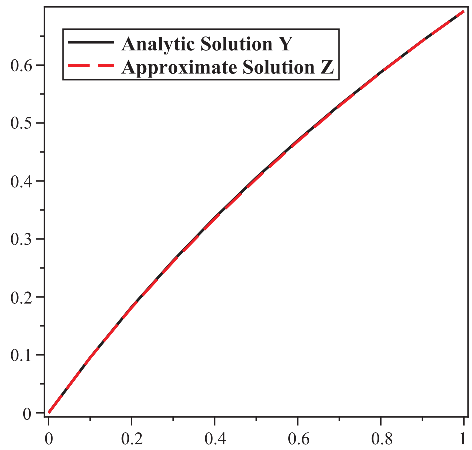

Figure 3: *Comparison of the analytic and approximate solution of Example 2.*

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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