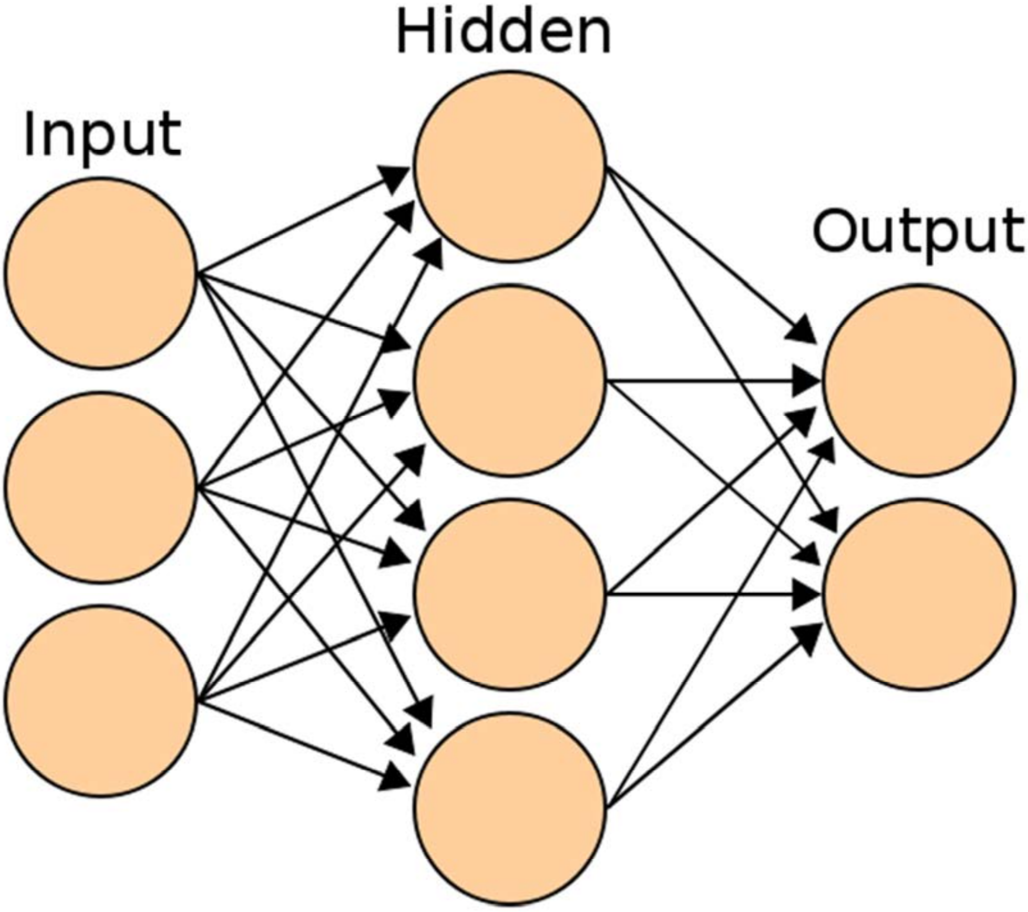
An artificial neural network is an interconnected group of nodes, akin to the vast network of neurons in the human brain.



Approximation by neural networks iterates

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Abstract

Here we study the multivariate quantitative approximation of real valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate quasi-interpolation sigmoidal and hyperbolic tangent iterated neural network operators. This approximation is derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order partial derivatives. Our multivariate iterated operators are defined by using the multidimensional density functions induced by the logarithmic sigmoidal and the hyperbolic tangent functions. The approximations are pointwise and uniform. The related feed-forward neural networks are with one hidden layer.

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Keywords and Phrases: sigmoidal and hyperbolic tangent functions, multivariate iterated neural network approximation, multivariate quasi-interpolation iterated operator, multivariate modulus of continuity.

1 Introduction

The author in [1], [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treates there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

This article is a continuation of [4], [5], [6], [7], [8].

The author here performs multivariate sigmoidal and hyperbolic tangent iterated neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$.

All convergences are with rates expressed via the multivariate modulus of continuity of the involved function or its high order partial derivatives, and given by very tight multidimensional Jackson type inequalities.

Many times for accuracy computer processes repeat themselves. We prove that the speed of the convergence of the iterated approximation remains the same, as the original, even if we increase the number of neurons per cycle.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation functions are the hyperbolic tangent and the sigmoidal. About neural networks see [9], [10], [11], [12].

2 Basics

I) Here all come from [7], [8].

We consider the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R}, \ i = 1, ..., N; \ x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [9], we consider

$$\Phi_i(x_i) := \frac{1}{2}(s_i(x_i+1) - s_i(x_i-1)), \quad x_i \in \mathbb{R}, \ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0$, $\forall \ x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1$, $\forall \ x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \ \forall \ x_i \in \mathbb{R}; n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i), \ \ x_i \geq 0$, for $i = 1, ..., N$.

We see that ([9])

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{(1+e^{x_i-1})(1+e^{-x_i-1})}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Notice that $\max \Phi_i(x_i) = \Phi_i(0) = 0.231$.
Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [8] we get

viii)

$$\sum_{\begin{cases} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{cases}}^{\infty} \Phi_i(nx_i - k_i) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.
As in [8] we obtain

ix)

$$0 < \frac{1}{\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)} < \frac{1}{\Phi_i(1)} = 5.250312578,$$

$\forall \ x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [8], we see that

$$\lim_{n \to \infty} \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \neq 1,$$

for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We will use here

$$\Phi(x_1, ..., x_N) := \Phi(x) := \prod_{i=1}^{N} \Phi_i(x_i), \quad x \in \mathbb{R}^N. \tag{1}$$

It has the properties:

(i)' $\Phi(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)'

$$\sum_{k=-\infty}^{\infty} \Phi(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1-k_1, ..., x_N-k_N) = 1, \tag{2}$$

$k := (k_1, ..., k_N)$, $\forall\, x \in \mathbb{R}^N$.

(iii)'

$$\sum_{k=-\infty}^{\infty} \Phi(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(nx_1-k_1, ..., nx_N-k_N) = 1, \tag{3}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi(x)\, dx = 1,$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil),$$
$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor).$$

In general $\|\cdot\|_\infty$ stands for the supremum norm.

We also have

(v)'

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) \le 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

(vi)'

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < (5.250312578)^N,$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

(vii)'

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \leq 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

(viii)'

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the multivariate positive linear neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$)

$$G_n(f, x_1, ..., x_N) := G_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \tag{4}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^N \Phi_i(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We need, for $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ the first multivariate modulus of continuity

$$\omega_1(f, h) := \sup_{\substack{x, y \in \left(\prod_{i=1}^N [a_i, b_i]\right) \\ \|x - y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{5}$$

Similarly it is defined for $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$). We have that $\lim_{h\to 0} \omega_1(f, h) = 0$.

When $f \in C_B\left(\mathbb{R}^N\right)$ we define

$$\overline{G}_n(f, x) := \overline{G}_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k) \tag{6}$$

$$:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^N \Phi_i(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [7]:

Theorem 1 *Let* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, n, $N \in \mathbb{N}$. *Then*

i)

$$|G_n(f,x) - f(x)| \leq (5.250312578)^N \cdot \left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984)\,\|f\|_\infty\, e^{-n^{(1-\beta)}}\right\} =: \lambda_1, \tag{7}$$

ii)

$$\|G_n(f) - f\|_\infty \leq \lambda_1. \tag{8}$$

Theorem 2 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, n, $N \in \mathbb{N}$. *Then*

i)

$$\left|\overline{G}_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984)\,\|f\|_\infty\, e^{-n^{(1-\beta)}} =: \lambda_2, \tag{9}$$

ii)

$$\left\|\overline{G}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{10}$$

II) Here we folow [5], [6].

We also consider the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\ x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\ x \in \mathbb{R}.$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 3 *([5])* $\Psi(x)$ *for* $x \geq 0$ *is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x \to -\infty} \Psi(x) = 0 = \lim_{x \to \infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 4 *([5]) We have that* $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1, \ \forall\, x \in \mathbb{R}.$

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 5 *([5]) It holds* $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1.$

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 6 *([5]) Let* $0 < \alpha < 1$ *and* $n \in \mathbb{N}$. *It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \Psi(nx-k) \leq e^{4} \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 7 *([5]) Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [5] we get that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

In this article we will use

$$\Theta(x_1,...,x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{11}$$

It has the properties:

(i) $\Theta(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1,...,x_N-k_N) = 1, \tag{12}$$

where $k := (k_1,...,k_N)$, $\forall\, x \in \mathbb{R}^N$.

(iii)

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1, ..., nx_N-k_N) = 1, \tag{13}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)

$$\int_{\mathbb{R}^N} \Theta(x)\,dx = 1,$$

that is Θ is a multivariate density function.

(v)

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k) \le e^4 \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

(vi)

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k)} < \frac{1}{(\Psi(1))^N} = (4.1488766)^N,$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \Theta(nx-k) \le e^4 \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

Also we get that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k) \neq 1,$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \le \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the multivariate positive linear neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$)

$$F_n(f, x_1, ..., x_N) := F_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k)} \tag{14}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i)\right)}.$$

When $f \in C_B\left(\mathbb{R}^N\right)$ we define,

$$\overline{F}_n(f, x) := \overline{F}_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Theta(nx - k) := \tag{15}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [6]:

Theorem 8 *Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, n, $N \in \mathbb{N}$. Then*

i)

$$|F_n(f, x) - f(x)| \leq (4.1488766)^N \cdot$$
$$\left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda_1, \tag{16}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda_1. \tag{17}$$

Theorem 9 *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, n, $N \in \mathbb{N}$. Then*

i)

$$\left|\overline{F}_n(f, x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}} =: \lambda_2, \tag{18}$$

ii)

$$\left\|\overline{F}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{19}$$

Let $r \in \mathbb{N}$, in this article we study the uniform convergence with rates to the unit operator I of the iterates G_n^r, $\overline{G}_n^r$, F_n^r and $\overline{F}_n^r$.

3 Preparatory Results

We need

Theorem 10 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $N \geq 1$. *Then* $\overline{G}_n(f) \in C_B\left(\mathbb{R}^N\right)$.

Proof. We have that

$$\left|\overline{G}_n(f,x)\right| \leq \sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{n}\right)\right|\Phi(nx-k)$$

$$\leq \|f\|_{\infty}\left(\sum_{k=-\infty}^{\infty}\Phi(nx-k)\right)\overset{(3)}{=}\|f\|_{\infty}, \ \forall\, x \in \mathbb{R}^N.$$

So that $\overline{G}_n(f)$ is bounded.

Next we prove the continuity of $\overline{G}_n(f)$. We will use the Weierstrass M test: If a sequence of positive constants $M_1, M_2, M_3, ...$, can be found such that in some interval

(a) $|u_n(x)| \leq M_n$, $n = 1, 2, 3, ...$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, ...$ are continuous in $[a,b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a,b]$, then $S(x)$ is continuous in $[a,b]$. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$. Therefore

$$\sum_{k=\lambda}^{\infty}\Phi(nx-k) = \sum_{k=\lambda}^{\infty}\Phi(k-nx) \leq \sum_{k=\lambda}^{\infty}\Phi(k-\lambda) = \sum_{k'=0}^{\infty}\Phi(k') \leq 1.$$

So for $k \geq \lambda$ we get

$$\left|f\left(\frac{k}{n}\right)\right|\Phi(nx-k) \leq \|f\|_{\infty}\Phi(k-\lambda),$$

and

$$\|f\|_{\infty}\sum_{k=\lambda}^{\infty}\Phi(k-\lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass M test we obtain that $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $f\left(\frac{k}{n}\right)\Phi(nx-k)$ is continuous in x, then $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k - nx \leq k + \lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} \Phi(k-nx) \leq \sum_{k=-\infty}^{-\lambda} \Phi(k+\lambda) = \sum_{k'=-\infty}^{0} \Phi(k') \leq 1.$$

So for $k \leq -\lambda$ we get

$$\left|f\left(\frac{k}{n}\right)\right| \Phi(nx-k) \leq \|f\|_{\infty} \Phi(k+\lambda),$$

and

$$\|f\|_{\infty} \sum_{k=-\infty}^{-\lambda} \Phi(k+\lambda) \leq \|f\|_{\infty}.$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $f\left(\frac{k}{n}\right)\Phi(nx-k)$ is continuous in x, then $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k)$ and $\sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right)\Phi(nx-k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=-\lambda+1}^{\lambda-1} f\left(\frac{k}{n}\right)\Phi(nx-k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} f\left(\frac{k}{n}\right)\Phi(nx-k) +$$

$$\sum_{k=-\lambda+1}^{\lambda-1} f\left(\frac{k}{n}\right)\Phi(nx-k) + \sum_{k=\lambda}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k)$$

we have it as a continuous function on $\mathbb{R}$. Therefore $\overline{G}_n(f)$, when $N=1$, is a continuous function on $\mathbb{R}$.

When $N=2$ we have

$$\overline{G}_n(f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1(nx_1-k_1)\Phi(nx_2-k_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \Phi_1(nx_1-k_1)\left(\sum_{k_2=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right)\Phi_2(nx_2-k_2)\right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty} \Phi_1(nx_1-k_1)\left[\sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right)\Phi_2(nx_2-k_2) +\right.$$

$$\left.\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_2\left(nx_2-k_2\right)+\sum_{k_2=\lambda_2}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_2\left(nx_2-k_2\right)\right]=$$

$$=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right)+$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right)+$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=\lambda_2}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right).$$

(for convenience call

$$F\left(k_1,k_2,x_1,x_2\right):=f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right)\ )$$

$$=\sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=-\infty}^{-\lambda_2}F\left(k_1,k_2,x_1,x_2\right)+\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\infty}^{-\lambda_2}F\left(k_1,k_2,x_1,x_2\right)+$$

$$\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2}F\left(k_1,k_2,x_1,x_2\right)+\sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1}F\left(k_1,k_2,x_1,x_2\right)+$$

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1}F\left(k_1,k_2,x_1,x_2\right)+\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1}F\left(k_1,k_2,x_1,x_2\right)+$$

$$\sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=\lambda_2}^{\infty}F\left(k_1,k_2,x_1,x_2\right)+\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=\lambda_2}^{\infty}F\left(k_1,k_2,x_1,x_2\right)+$$

$$\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=\lambda_2}^{\infty}F\left(k_1,k_2,x_1,x_2\right).$$

Notice that the finite sum of continous functions $F\left(k_1,k_2,x_1,x_2\right)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1}F\left(k_1,k_2,x_1,x_2\right)$ is a continuous function.

The rest of the summands of $\overline{G}_n\left(f,x_1,x_2\right)$ are treated all the same way and similarly to the case of $N=1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right)$ is continuous in $\left(x_1,x_2\right)\in\mathbb{R}^2$.

The continuous function

$$\left|f\left(\frac{k_1}{n},\frac{k_2}{n}\right)\right|\Phi_1\left(nx_1-k_1\right)\Phi_2\left(nx_2-k_2\right)\leq\|f\|_{\infty}\Phi_1\left(k_1-\lambda_1\right)\Phi_2\left(k_2+\lambda_2\right),$$

and

$$\|f\|_{\infty} \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \Phi_1 (k_1 - \lambda_1) \Phi_2 (k_2 + \lambda_2) =$$

$$\|f\|_{\infty} \left(\sum_{k_1=\lambda_1}^{\infty} \Phi_1 (k_1 - \lambda_1) \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \Phi_2 (k_2 + \lambda_2) \right) \leq$$

$$\|f\|_{\infty} \left(\sum_{k_1'=0}^{\infty} \Phi_1 (k_1') \right) \left(\sum_{k_2'=-\infty}^{0} \Phi_2 (k_2') \right) \leq \|f\|_{\infty} .$$

So by the Weierstrass M test we get that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1 (nx_1 - k_1) \Phi_2 (nx_2 - k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \Phi_1 (nx_1 - k_1) \Phi_2 (nx_2 - k_2)$.

Notice here that

$$\left| f\left(\frac{k_1}{n}, \frac{k_2}{n}\right) \right| \Phi_1 (nx_1 - k_1) \Phi_2 (nx_2 - k_2) \leq \|f\|_{\infty} \Phi_1 (nx_1 - k_1) \Phi_2 (k_2 + \lambda_2)$$

$$\leq \|f\|_{\infty} \Phi_1 (0) \Phi_2 (k_2 + \lambda_2) = (0.231) \|f\|_{\infty} \Phi_2 (k_2 + \lambda_2) ,$$

and

$$(0.231) \|f\|_{\infty} \left(\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} 1 \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \Phi_2 (k_2 + \lambda_2) \right) =$$

$$(0.231) \|f\|_{\infty} (2\lambda_1 - 1) \left(\sum_{k_2'=-\infty}^{0} \Phi_2 (k_2') \right) \leq (0.231) (2\lambda_1 - 1) \|f\|_{\infty} .$$

So the double series under consideration is uniformly convergent and continuous. Clearly $\overline{G}_n (f, x_1, x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similalrly reasoning one can prove easily now, but with more tedious work, that $\overline{G}_n (f, x_1, ..., x_N)$ is continuous on $\mathbb{R}^N$, for any $N \geq 1$. We choose to omit this similar extra work. ■

Theorem 11 *Let $f \in C_B\left(\mathbb{R}^N\right)$, $N \geq 1$. Then $\overline{F}_n (f) \in C_B\left(\mathbb{R}^N\right)$.*

Proof. We notice that

$$\left|\overline{F}_n (f, x)\right| \leq \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{n}\right) \right| \theta (nx - k) \leq \|f\|_{\infty} \left(\sum_{k=-\infty}^{\infty} \theta (nx - k) \right) \overset{(13)}{=} \|f\|_{\infty} ,$$

$\forall\, x \in \mathbb{R}^N$, so that $\overline{F}_n (f)$ is bounded. The continuity is proved as in Theorem 10. ■

Theorem 12 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, *then* $\|G_n(f)\|_\infty \leq \|f\|_\infty$ *and* $\|F_n(f)\|_\infty \leq \|f\|_\infty$, *also* $G_n(f),\ F_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Proof. By (4) we get

$$|G_n(f,x)| = \frac{\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)\right|}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)} \leq$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right)\right|\Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)} \leq \|f\|_\infty, \ \forall\, x \in \mathbb{R}^N,$$

so that $G_n(f)$ is bounded.

Similarly we act for the boundedness of F_n, see (14). Continuity of both is obvious. ■

We make

Remark 13 *Notice that*

$$\left\|G_n^2(f)\right\|_\infty = \|G_n(G_n(f))\|_\infty \leq \|G_n(f)\|_\infty \leq \|f\|_\infty, \ etc.$$

Therefore we get

$$\left\|G_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{20}$$

the contraction property.

Similarly we obtain

$$\left\|F_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{21}$$

Similarly by Theorems 10, 11 we obtain

$$\left\|\overline{G}_n^k(f)\right\|_\infty \leq \|f\|_\infty, \tag{22}$$

and

$$\left\|\overline{F}_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{23}$$

Infact here we have

$$\left\|G_n^k(f)\right\|_\infty \leq \left\|G_n^{k-1}(f)\right\|_\infty \leq ... \leq \|G_n(f)\|_\infty \leq \|f\|_\infty, \tag{24}$$

$$\left\|F_n^k(f)\right\|_\infty \leq \left\|F_n^{k-1}(f)\right\|_\infty \leq ... \leq \|F_n(f)\|_\infty \leq \|f\|_\infty, \tag{25}$$

$$\left\|\overline{G}_n^k(f)\right\|_\infty \leq \left\|\overline{G}_n^{k-1}(f)\right\|_\infty \leq ... \leq \left\|\overline{G}_n(f)\right\|_\infty \leq \|f\|_\infty, \tag{26}$$

and

$$\left\|\overline{F}_n^k(f)\right\|_\infty \leq \left\|\overline{F}_n^{k-1}(f)\right\|_\infty \leq ... \leq \left\|\overline{F}_n(f)\right\|_\infty \leq \|f\|_\infty. \tag{27}$$

We need

Notation 14 *Call* $L_n = G_n, \overline{G}_n, F_n, \overline{F}_n$.

Denote by

$$c_N = \begin{cases} (5.250312578)^N, & \text{if } L_n = G_n, \\ (4.1488766)^N, & \text{if } L_n = F_n, \\ 1, & \text{if } L_n = \overline{G}_n, \overline{F}_n, \end{cases} \tag{28}$$

$$\mu = \begin{cases} 6.3984, & \text{if } L_n = G_n, \overline{G}_n, \\ 2e^4, & \text{if } L_n = F_n, \overline{F}_n, \end{cases} \tag{29}$$

and

$$\gamma = \begin{cases} 1, & \text{when } L_n = G_n, \overline{G}_n, \\ 2 & \text{when } L_n = F_n, \overline{F}_n. \end{cases} \tag{30}$$

Based on the above notations Theorems 1, 2, 8, 9 can put in a unified way as follows.

Theorem 15 *Let* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $n, N \in \mathbb{N}$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ *or* $x \in \mathbb{R}^N$. *Then*

(i)

$$|L_n(f,x) - f(x)| \leq c_N \left\{ \omega_1\left(f, \frac{1}{n^\beta}\right) + \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}} \right\} =: \rho_n, \tag{31}$$

(ii)

$$\|L_n(f) - f\|_\infty \leq \rho_n. \tag{32}$$

Remark 16 *We have*

$$\left\| L_n^k f \right\|_\infty \leq \|f\|_\infty, \ \forall\, k \in \mathbb{N}, \tag{33}$$

the contraction property.

Also it holds

$$L_n 1 = 1, \tag{34}$$

and

$$L_n^k 1 = 1, \ \forall\, k \in \mathbb{N}. \tag{35}$$

Here L_n^k *are positive linear operators.*

4 Main Results

We present

Theorem 17 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $r,\ n,\ N \in \mathbb{N}$*,* $0<\beta<1$*,* $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

$$|L_n^r(f,x) - f(x)| \leq \|L_n^r f - f\|_\infty \leq r\,\|L_n f - f\|_\infty$$
$$\leq r c_N \left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + \mu\,\|f\|_\infty\, e^{-\gamma n^{(1-\beta)}}\right\}. \tag{36}$$

Proof. We observe that

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$
$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + ... + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\|L_n^r f - f\|_\infty \leq \left\|L_n^r f - L_n^{r-1} f\right\|_\infty + \left\|L_n^{r-1} f - L_n^{r-2} f\right\|_\infty +$$
$$\left\|L_n^{r-2} f - L_n^{r-3} f\right\|_\infty + ... + \left\|L_n^2 f - L_n f\right\|_\infty + \|L_n f - f\|_\infty =$$
$$\left\|L_n^{r-1}(L_n f - f)\right\|_\infty + \left\|L_n^{r-2}(L_n f - f)\right\|_\infty + \left\|L_n^{r-3}(L_n f - f)\right\|_\infty$$
$$+ ... + \|L_n(L_n f - f)\|_\infty + \|L_n f - f\|_\infty \overset{(33)}{\leq}$$
$$r\,\|L_n f - f\|_\infty \overset{(32)}{\leq} r\rho_n,$$

proving the claim. ■

More generally we have

Theorem 18 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $n,\ N,\ m_1,...,m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$*,* $0<\beta<1$*,* $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

$$\left|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1} f\right)\right)\right)(x) - f(x)\right| \leq \tag{37}$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1} f\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^{r}\|L_{m_i} f - f\|_\infty \leq$$
$$c_N \sum_{i=1}^{r}\left\{\omega_1\left(f, \frac{1}{m_i^\beta}\right) + \mu\,\|f\|_\infty\, e^{-\gamma m_i^{(1-\beta)}}\right\} \leq$$
$$r c_N \left\{\omega_1\left(f, \frac{1}{m_1^\beta}\right) + \mu\,\|f\|_\infty\, e^{-\gamma m_1^{(1-\beta)}}\right\}.$$

Clearly, we notice that the speed of convergence of the multiply iterated operator equals to the speed of L_{m_1}*.*

Proof. We write

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f=$$

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right)+$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right)+$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}f\right)\right)+...+$$
$$L_{m_r}\left(L_{m_{r-1}}f\right)-L_{m_r}f+L_{m_r}f-f=$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f-f\right)+L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f-f\right)+$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f-f\right)+...+L_{m_r}\left(L_{m_{r-1}}f-f\right)+L_{m_r}f-f.$$

Hence by the triangle inequality property of $\|\cdot\|_\infty$ we get

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\infty\leq$$

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f-f\right)\right\|_\infty+\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f-f\right)\right\|_\infty+$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f-f\right)\right\|_\infty+...+$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}f-f\right)\right\|_\infty+\left\|L_{m_r}f-f\right\|_\infty$$

(repeatedly applying (33))

$$\leq\left\|L_{m_1}f-f\right\|_\infty+\left\|L_{m_2}f-f\right\|_\infty+\left\|L_{m_3}f-f\right\|_\infty+...+$$
$$\left\|L_{m_{r-1}}f-f\right\|_\infty+\left\|L_{m_r}f-f\right\|_\infty=\sum_{i=1}^{r}\left\|L_{m_i}f-f\right\|_\infty\overset{(32)}{\leq}$$
$$c_N\sum_{i=1}^{r}\left\{\omega_1\left(f,\frac{1}{m_i^\beta}\right)+\mu\|f\|_\infty e^{-\gamma m_i^{(1-\beta)}}\right\}=:(*).$$

We have

$$\frac{1}{m_r}\leq\frac{1}{m_{r-1}}\leq...\leq\frac{1}{m_2}\leq\frac{1}{m_1},$$

and

$$\frac{1}{m_r^\beta}\leq\frac{1}{m_{r-1}^\beta}\leq...\leq\frac{1}{m_2^\beta}\leq\frac{1}{m_1^\beta}.$$

Therefore

$$\omega_1\left(f,\frac{1}{m_r^\beta}\right)\leq\omega_1\left(f,\frac{1}{m_{r-1}^\beta}\right)\leq...\leq\omega_1\left(f,\frac{1}{m_2^\beta}\right)\leq\omega_1\left(f,\frac{1}{m_1^\beta}\right).$$

Also it holds

$$\gamma m_1^{(1-\beta)}\leq\gamma m_2^{(1-\beta)}\leq...\leq\gamma m_r^{(1-\beta)},$$

and
$$e^{\gamma m_1^{(1-\beta)}} \leq e^{\gamma m_2^{(1-\beta)}} \leq ... \leq e^{\gamma m_r^{(1-\beta)}},$$
so that
$$e^{-\gamma m_r^{(1-\beta)}} \leq e^{-\gamma m_{r-1}^{(1-\beta)}} \leq ... \leq e^{-\gamma m_2^{(1-\beta)}} \leq e^{-\gamma m_1(1-\beta)}.$$
Therefore
$$(*) \leq rc_N \left\{ \omega_1 \left(f, \frac{1}{m_1^\beta} \right) + \mu \|f\|_\infty e^{-\gamma m_1^{(1-\beta)}} \right\},$$
proving the claim. ■

Next we give a partial global smoothness preservation result of operators L_n^r.

Theorem 19 *Same assumptions as in Theorem 17, $\delta > 0$. Then*
$$\omega_1 (L_n^r f, \delta) \leq 2rc_N \left\{ \omega_1 \left(f, \frac{1}{n^\beta} \right) + \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}} \right\} + \omega_1 (f, \delta). \tag{38}$$
In particular for $\delta = \frac{1}{n^\beta}$, we obtain
$$\omega_1 \left(L_n^r f, \frac{1}{n^\beta} \right) \leq (2rc_N + 1)\, \omega_1 \left(f, \frac{1}{n^\beta} \right) + 2rc_N \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}}. \tag{39}$$

Proof. We write
$$L_n^r f(x) - L_n^r f(y) = L_n^r f(x) - L_n^r f(y) + f(x) - f(x) + f(y) - f(y) =$$
$$(L_n^r f(x) - f(x)) + (f(y) - L_n^r f(y)) + (f(x) - f(y)).$$
Hence
$$|L_n^r f(x) - L_n^r f(y)| \leq |L_n^r f(x) - f(x)| + |L_n^r f(y) - f(y)| + |f(x) - f(y)|$$
$$\leq 2 \|L_n^r f - f\|_\infty + |f(x) - f(y)|.$$
Let $x, y \in \left(\prod_{i=1}^N [a_i, b_i] \right)$ or $x, y \in \mathbb{R}^N : |x - y| \leq \delta$, $\delta > 0$. Then
$$\omega_1 (L_n^r f, \delta) \leq 2 \|L_n^r f - f\|_\infty + \omega_1 (f, \delta).$$
That is
$$\omega_1 (L_n^r f, \delta) \overset{(36)}{\leq} 2r \|L_n f - f\|_\infty + \omega_1 (f, \delta),$$
proving the claim. ■

Notation 20 *Let $f \in C^m \left(\prod_{i=1}^N [a_i, b_i] \right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$, and $|\alpha| := \sum_{i=1}^N \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.*

We denote

$$\omega_{1,m}^{\max}\left(f_{\alpha},h\right):=\max_{\alpha:|\alpha|=m}\omega_{1}\left(f_{\alpha},h\right). \tag{40}$$

Call also

$$\left\|f_{\alpha}\right\|_{\infty,m}^{\max}:=\max_{|\alpha|=m}\left\{\left\|f_{\alpha}\right\|_{\infty}\right\}. \tag{41}$$

We discuss higher order approximation next.

We mention from [7] the following result.

Theorem 21 *Let* $f\in C^{m}\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$, $0<\beta<1$, $n,m,N\in\mathbb{N}$. *Then*

$$\left\|G_{n}\left(f\right)-f\right\|_{\infty}\leq\left(5.250312578\right)^{N}\cdot \tag{42}$$

$$\left\{\sum_{j=1}^{N}\left(\sum_{|\alpha|=j}\left(\frac{\left\|f_{\alpha}\right\|_{\infty}}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\left(3.1992\right)e^{-n^{(1-\beta)}}\right]\right)+\right.$$

$$\left.\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+\left(\frac{\left(6.3984\right)\left\|b-a\right\|_{\infty}^{m}\left\|f_{\alpha}\right\|_{\infty,m}^{\max}N^{m}}{m!}\right)e^{-n^{(1-\beta)}}\right\}=:M_{n}.$$

Using Theorem 17 we derive

Theorem 22 *Let* $f\in C^{m}\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$, $0<\beta<1$, $r,n,m,N\in\mathbb{N}$, $x\in\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$. *Then*

$$\left|G_{n}^{r}\left(f,x\right)-f\left(x\right)\right|\leq\left\|G_{n}^{r}\left(f\right)-f\right\|_{\infty}\leq r\left\|G_{n}\left(f\right)-f\right\|_{\infty}\leq rM_{n}. \tag{43}$$

One can have a similar result for the operator F_n but we omit it.

Next we specialize on Lipschitz classes of functions. We apply Theorem 18 to obtain

Theorem 23 *Let* $f\in C\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$ *or* $f\in C_{B}\left(\mathbb{R}^{N}\right)$; n, N, $m_{1},...,m_{r}\in\mathbb{N}:m_{1}\leq m_{2}\leq...\leq m_{r}$, $0<\beta<1$. *We further assume that* $\left|f\left(x\right)-f\left(y\right)\right|\leq M\left\|x-y\right\|_{\infty}^{\alpha}$, $\forall\ x,y\in\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$ *or* $x,y\in\mathbb{R}^{N}$ *(respectively)*, $0<\alpha\leq1$, $M>0$. *Then*

$$\left\|L_{m_{r}}\left(L_{m_{r-1}}\left(...L_{m_{2}}\left(L_{m_{1}}f\right)\right)\right)-f\right\|_{\infty}\leq \tag{44}$$

$$\sum_{i=1}^{r}\left\|L_{m_{i}}f-f\right\|_{\infty}\leq$$

$$c_{N}\sum_{i=1}^{r}\left\{\frac{M}{m_{i}^{\alpha\beta}}+\mu\left\|f\right\|_{\infty}e^{-\gamma m_{i}^{(1-\beta)}}\right\}.$$

Example 24 *Let* $f(x_1,...,x_N)=\sum_{i=1}^{N}\cos x_i$, $(x_1,...,x_N)\in\mathbb{R}^N$, $N\in\mathbb{N}$. *Denote* $\overline{x}=(x_1,...,x_N)$, $\overline{y}=(y_1,...,y_N)$ *and observe that*

$$\left|\sum_{i=1}^{N}\cos x_i-\sum_{i=1}^{N}\cos y_i\right|\leq\sum_{i=1}^{N}\left|\cos x_i-\cos y_i\right|$$

$$\leq\sum_{i=1}^{N}\left|x_i-y_i\right|\leq N\left\|\overline{x}-\overline{y}\right\|_{\infty}.$$

That is

$$\left|f(\overline{x})-f(\overline{y})\right|\leq N\left\|\overline{x}-\overline{y}\right\|_{\infty}.$$

Consequently by (5) we get that

$$\omega_1(f,h)\leq Nh,\quad h>0.$$

Therefore by (9) we derive

$$\left\|\overline{G}_n\left(\sum_{i=1}^{N}\cos x_i\right)-\left(\sum_{i=1}^{N}\cos x_i\right)\right\|_{\infty}\leq N\left(\frac{1}{n^{\beta}}+(6.3984)\,e^{-n^{(1-\beta)}}\right),\tag{45}$$

where $0<\beta<1$ *and* $n\in\mathbb{N}$.

Let now $m_1,...,m_r\in\mathbb{N}:m_1\leq m_2\leq...\leq m_r$. *Then by (44) we get*

$$\left\|\overline{G}_{m_r}\left(\overline{G}_{m_{r-1}}\left(...\left(\overline{G}_{m_2}\left(\overline{G}_{m_1}\left(\sum_{i=1}^{N}\cos x_i\right)\right)\right)\right)\right)-\left(\sum_{i=1}^{N}\cos x_i\right)\right\|_{\infty}\leq$$

$$\sum_{i=1}^{r}\left\|\overline{G}_{m_i}\left(\sum_{i=1}^{N}\cos x_i\right)-\left(\sum_{i=1}^{N}\cos x_i\right)\right\|_{\infty}\overset{(by\ (45))}{\leq}\tag{46}$$

$$N\sum_{i=1}^{r}\left(\frac{1}{m_i^{\beta}}+(6.3984)\,e^{-m_i^{(1-\beta)}}\right)\leq rN\left(\frac{1}{m_1^{\beta}}+(6.3984)\,e^{-m_1^{(1-\beta)}}\right).\tag{47}$$

One can give easily many other interesting applications.

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Fractional neural network approximation

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Abstract

Here we study the univariate fractional quantitative approximation of real valued functions on a compact interval by quasi-interpolation sigmoidal and hyperbolic tangent neural network operators. These approximations are derived by establishing Jackson type inequalities involving the moduli of continuity of the right and left Caputo fractional derivatives of the engaged function. The approximations are pointwise and with respect to the uniform norm. The related feed-forward neural networks are with one hidden layer. Our fractional approximation results into higher order converges better than the ordinary ones.

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1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

The author inspired by [12], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [6], [8], [9], [10], [11], by treating both the univariate and multivariate cases.

Continuation of the author's last work is this article where neural network approximation is taken at the fractional level resulting into higher rates of approximation. We involve the right and left Caputo fractional derivatives of the function under approximation and we establish tight Jackson type inequalities. An extensive background is given on fractional calculus and neural networks, all needed to expose our work. Applications are presented at the end.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general, see [16], [17], [18].

2 Background

We need

Definition 1 *Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by*

$$\omega_1(f,h) = \sup\{|f(x) - f(y); x, y \in [a,b], \ |x-y| \leq h|\} \tag{1}$$

If $h > b-a$, then we define

$$\omega_1(f,h) = \omega_1(f, b-a). \tag{2}$$

Notice here that

$$\lim_{h \to 0} \omega_1(f,h) = 0.$$

We also need

Definition 2 *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [13], pp. 49-52, [15], [19]) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{3}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, $\nu > 0$.

Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\ x \in [a,b]$.

Lemma 3 *([5]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 4 *(see also [3], [14], [15]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{4}$$

$\forall\ x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$.

Lemma 5 *([5]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$.*

Convention 6 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{5}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{6}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 7 *([5]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 8 *([5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

We further mention

Proposition 9 *([5]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{7}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 10 *([5]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) \, d\zeta, \tag{8}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

We need

Proposition 11 *([5]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t) \, dt, \text{ for } x \geq x_0, \tag{9}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

We mention

Proposition 12 *([5]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 [a,b]$. Define*

$$K(x, x_0) = \int_x^{x_0} (\zeta - x)^{c-1} g(\zeta) \, d\zeta, \quad \text{for } x \leq x_0, \tag{10}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Proposition 11, 12 we derive

Corollary 13 *([5]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We need

Theorem 14 *([5]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x, b]), \tag{11}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continous in $x \in [a,b]$.

Also it holds

Theorem 15 *([5]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continuous. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a, x]), \tag{12}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We make

Remark 16 *([5]) Let* $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. *Then we have*

$$\left| D_{*a}^{\nu} f(x) \right| \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{13}$$

Thus we observe

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left| D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y) \right|$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{14}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, *then*

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{15}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, *we find*

$$s_1(\delta) := \sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{16}$$

and

$$s_2(\delta) := \sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, \delta\right)_{[a,x_0]} \leq \frac{2\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{17}$$

By Proposition 15.114, p. 388 of [4], we get here that $D_{*x_0}^{\alpha} f \in C([x_0,b])$, *and by [7] we obtain that* $D_{x_0-}^{\alpha} f \in C([a,x_0])$. *Clearly here we have that* $s_1(\delta), s_2(\delta) \to 0$, *as* $\delta \to 0$.

We consider here the sigmoidal function of logarithmic type

$$s(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x \to +\infty} s(x) = 1$ and $\lim_{x \to -\infty} s(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks.

As in [12], we consider

$$\Phi(x) := \frac{1}{2}\left(s(x+1) - s(x-1)\right), \quad x \in \mathbb{R}. \tag{18}$$

We notice the following properties:

i) $\Phi(x) > 0, \ \forall\, x \in \mathbb{R}$,

ii) $\sum_{k=-\infty}^{\infty} \Phi(x-k) = 1, \ \forall\, x \in \mathbb{R}$,

iii) $\sum_{k=-\infty}^{\infty} \Phi(nx-k) = 1, \ \forall\, x \in \mathbb{R};\, n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi(x)\, dx = 1$,

v) Φ is a density function,

vi) Φ is even: $\Phi(-x) = \Phi(x), \ \ x \geq 0$.

We see that ([12])

$$\Phi(x) = \left(\frac{e^2-1}{2e}\right)\frac{e^{-x}}{\left(1+e^{-x-1}\right)\left(1+e^{-x+1}\right)} = \left(\frac{e^2-1}{2e^2}\right)\frac{1}{\left(1+e^{x-1}\right)\left(1+e^{-x-1}\right)}. \tag{19}$$

vii) By [12] Φ is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

viii) By [11] for $n \in \mathbb{N}$, $0 < \beta < 1$, we get

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| > n^{1-\beta} \end{cases}}^{\infty} \Phi(nx-k) < \left(\frac{e^2-1}{2}\right) e^{-n^{(1-\beta)}} = 3.1992 e^{-n^{(1-\beta)}}. \tag{20}$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

ix) By [11] it holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \frac{1}{\Phi(1)} = 5.250312578, \ \forall\, x \in [a,b]. \tag{21}$$

x) By [11] it holds $\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1$, for at least some $x \in [a,b]$.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We study further (see also [11]) the positive linear neural network operator

$$G_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \quad x \in [a,b]. \tag{22}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We study here further at fractional level the pointwise and uniform convergence of $G_n(f,x)$ to $f(x)$ with rates.

For convinience we call

$$G_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k), \tag{23}$$

that is

$$G_n(f,x) := \frac{G_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{24}$$

Thus,

$$G_n(f,x) - f(x) = \frac{G_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} - f(x)$$

$$= \frac{G_n^*(f,x) - f(x)\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{25}$$

Consequently we derive

$$|G_n(f,x) - f(x)| \leq \frac{1}{\Phi(1)} \left| G_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right|. \tag{26}$$

That is

$$|G_n(f,x) - f(x)| \leq (5.250312578) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \Phi(nx-k) \right|. \tag{27}$$

We will estimate the right hand side of (27) involving the right and left Caputo fractional derivatives of f.

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\ x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$. Furthermore it holds $\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} > 0$.

This function plays also the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\ x \in \mathbb{R}. \tag{28}$$

We easily see that$\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Here we follow [8]

Proposition 17 *$\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty}\Psi(x) = 0 = \lim_{x\to\infty}\Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is at zero, $\Psi(0) = 0.3809297$.

Theorem 18 *We have that $\sum_{i=-\infty}^{\infty}\Psi(x-i) = 1$, $\forall\ x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty}\Psi(nx-i) = 1, \quad \forall\ n \in \mathbb{N},\ \forall\ x \in \mathbb{R}.$$

Furthermore we get:

Since Ψ is even it holds $\sum_{i=-\infty}^{\infty}\Psi(i-x) = 1$, $\forall x \in \mathbb{R}$.

Hence $\sum_{i=-\infty}^{\infty}\Psi(i+x) = 1$, $\forall\ x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty}\Psi(x+i) = 1$, $\forall\ x \in \mathbb{R}$.

Theorem 19 *It holds $\int_{-\infty}^{\infty}\Psi(x)\,dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 20 *Let $0 < \beta < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}}^{\infty}\Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}}. \tag{29}$$

Theorem 21 *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Psi(nx-k)} < 4.1488766 = \frac{1}{\Psi(1)}. \tag{30}$$

Also by [8], we obtain

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

Definition 22 *Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na\rceil \leq \lfloor nb\rfloor$.*

We further study, as in [8], the positive linear neural network operator

$$F_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)}, \quad x \in [a,b]. \tag{31}$$

We study F_n similarly to G_n.

For convinience we call

$$F_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Psi(nx-k), \tag{32}$$

that is

$$F_n(f,x) := \frac{F_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)}. \tag{33}$$

Thus,

$$F_n(f,x) - f(x) = \frac{F_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)} - f(x)$$

$$= \frac{F_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)}. \tag{34}$$

Consequently we derive

$$|F_n(f,x) - f(x)| \leq \frac{1}{\Psi(1)}\left|F_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \Psi(nx-k)\right|. \tag{35}$$

That is

$$|F_n(f,x) - f(x)| \leq (4.1488766)\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)\Psi(nx-k)\right|. \tag{36}$$

We will estimate the right hand side of (36).

3 Main Results

We present our first main result

Theorem 23 *Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N([a,b])$, *with* $f^{(N)} \in L_\infty([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$. *Then*

i)

$$\left| G_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) - f(x) \right| \leq \tag{37}$$

$$\frac{(5.250312578)}{\Gamma(\alpha+1)} \cdot \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\},$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$|G_n(f,x) - f(x)| \leq \frac{(5.250312578)}{\Gamma(\alpha+1)} \cdot \tag{38}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\},$$

when $\alpha > 1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

iii)

$$|G_n(f,x) - f(x)| \leq (5.250312578) \cdot \tag{39}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j (3.1992) e^{-n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\} \right\},$$

$\forall\, x \in [a,b]$,

and

iv)

$$\|G_n f - f\|_\infty \leq (5.250312578) \cdot \tag{40}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \,(3.1992)\, e^{-n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. 3.1992 e^{-n^{(1-\beta)}} (b-a)^{\alpha} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\} \right\}.$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $G_n \to I$ the unit operator, as $n \to \infty$.

Proof. Let $x \in [a,b]$. We have that $D_{x-}^{\alpha} f(x) = D_{*x}^{\alpha} f(x) = 0$.

From [13], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{41}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x) \right) dJ,$$

for all $x \le \frac{k}{n} \le b$.

Also from [3], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{42}$$

$$\frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right) dJ,$$

for all $a \le \frac{k}{n} \le x$.

Hence we have

$$f\left(\frac{k}{n}\right) \Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \Phi(nx-k) \left(\frac{k}{n} - x\right)^j + \tag{43}$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x) \right) dJ,$$

for all $x \leq \frac{k}{n} \leq b$, iff $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$, and

$$f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} + \tag{44}$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

We have that $\lceil nx \rceil \leq \lfloor nx \rfloor + 1$.

Therefore it holds

$$\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} + \tag{45}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} + \tag{46}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ.$$

Adding the last two equalities (45) and (46) we obtain

$$G_{n}^{*}(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k) = \tag{47}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} +$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ + \right.$$

$$\left.\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}.$$

So we have derived

$$G_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right) = \tag{48}$$

$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*\left((\cdot - x)^j\right)(x) + \theta_n(x),$$

where

$$\theta_n(x) := \frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right)dJ\right.$$

$$\left. + \sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J) - D_{*x}^{\alpha}f(x)\right)dJ\right\}. \tag{49}$$

We set

$$\theta_{1n}(x) := \frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right)dJ, \tag{50}$$

and

$$\theta_{2n} := \frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J) - D_{*x}^{\alpha}f(x)\right)dJ, \tag{51}$$

i.e.

$$\theta_n(x) = \theta_{1n}(x) + \theta_{2n}(x). \tag{52}$$

We assume $b-a > \frac{1}{n^{\beta}}$, $0<\beta<1$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}}\right\rceil$. It is always true that either $\left|\frac{k}{n}-x\right| \le \frac{1}{n^{\beta}}$ or $\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na\rceil, ..., \lfloor nx\rfloor$, we consider

$$\gamma_{1k} := \left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J) - D_{x-}^{\alpha}f(x)\right)dJ\right| \tag{53}$$

$$= \left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right| \le \int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)\right|dJ$$

$$\le \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{\left(x-\frac{\kappa}{n}\right)^{\alpha}}{\alpha} \le \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}. \tag{54}$$

That is

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{55}$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left| D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right| dJ \tag{56}$$

$$\leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \omega_1 \left(D_{x-}^{\alpha} f, |J-x| \right)_{[a,x]} dJ$$

$$\leq \omega_1 \left(D_{x-}^{\alpha} f, \left| x - \frac{k}{n} \right| \right)_{[a,x]} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} dJ$$

$$\leq \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \frac{\left(x - \frac{k}{n} \right)^{\alpha}}{\alpha} \leq \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} \frac{1}{\alpha n^{a\beta}}. \tag{57}$$

That is when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$, then

$$\gamma_{1k} \leq \frac{\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]}}{\alpha n^{a\beta}}. \tag{58}$$

Consequently we obtain

$$|\theta_{1n}(x)| \leq \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \gamma_{1k} =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k) \gamma_{1k} + \sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k) \gamma_{1k} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k) \right) \frac{\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \left(\sum_{\substack{k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}}}^{\lfloor nx \rfloor} \Phi(nx-k) \right) \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha} \right\} \leq \tag{60}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{k=-\infty \\ \left\{: |nx-k|>n^{1-\beta}\right.}}^{\infty}\Phi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}\leq \tag{61}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+3.1992e^{-n^{(1-\beta)}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

So we have proved that

$$|\theta_{1n}(x)|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right. \tag{62}$$

$$\left.3.1992e^{-n^{(1-\beta)}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

Next when $k=\lfloor nx\rfloor+1,...,\lfloor nb\rfloor$ we consider

$$\gamma_{2k}:=\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right|\leq$$

$$\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left|D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right|dJ=$$

$$\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left|D_{*x}^{\alpha}f(J)\right|dJ\leq\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha}\leq \tag{63}$$

$$\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}. \tag{64}$$

Therefore when $k=\lfloor nx\rfloor+1,...,\lfloor nb\rfloor$ we get that

$$\gamma_{2k}\leq\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}. \tag{65}$$

In case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$, we get

$$\gamma_{2k}\leq\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\omega_1\left(D_{*x}^{\alpha}f,|J-x|\right)_{[x,b]}dJ\leq \tag{66}$$

$$\omega_1\left(D_{*x}^{\alpha}f,\left|\frac{k}{n}-x\right|\right)_{[x,b]}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}dJ\leq$$

$$\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha}\leq\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\frac{1}{\alpha n^{\alpha\beta}}. \tag{67}$$

So when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$ we derived that

$$\gamma_{2k}\leq\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{68}$$

Similarly we have that

$$|\theta_{2n}(x)|\leq\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma_{2k}\right)= \tag{69}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{\begin{cases}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma_{2k}+\sum_{\begin{cases}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma_{2k}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\sum_{\begin{cases}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}\right\}\leq \tag{70}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases}k=-\infty\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\infty}\Phi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}\leq \tag{71}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+3.1992e^{-n^{(1-\beta)}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}.$$

So we have proved that

$$|\theta_{2n}(x)| \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{n^{\alpha\beta}} + \right. \tag{72}$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right\}.$$

Therefore

$$|\theta_n(x)| \leq |\theta_{1n}(x)| + |\theta_{2n}(x)| \leq \tag{73}$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{n^{\alpha\beta}} + \right. \tag{74}$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}.$$

From [6], p. 15 we get that

$$\left| G_n^* \left((\cdot - x)^j \right)(x) \right| \leq \frac{1}{n^{\beta j}} + (b-a)^j (3.1992) e^{-n^{(1-\beta)}}, \tag{75}$$

for $j = 1, ..., N-1$, $\forall\ x \in [a,b]$.

Putting things together, we have established

$$\left| G_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right) \right| \leq \tag{76}$$

$$\sum_{j=1}^{N-1} \frac{\left| f^{(j)}(x) \right|}{j!} \left[\frac{1}{n^{\beta j}} + (b-a)^j (3.1992) e^{-n^{(1-\beta)}} \right] +$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\} =: A_n(x). \tag{77}$$

As a result we derive

$$|G_n(f,x) - f(x)| \leq (5.250312578) A_n(x), \tag{78}$$

$\forall\ x \in [a,b]$.

We further have that

$$\|A_n\|_{\infty} \leq \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty}}{j!} \left[\frac{1}{n^{\beta j}} + (b-a)^j (3.1992) e^{-n^{(1-\beta)}} \right] + \tag{79}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left\{\sup\limits_{x\in[a,b]}\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\right)+\sup\limits_{x\in[a,b]}\left(\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)\right\}}{n^{\alpha\beta}}\right.$$

$$+3.1992e^{-n^{(1-\beta)}}(b-a)^{\alpha}\cdot$$

$$\left.\left\{\left(\sup_{x\in[a,b]}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\right)+\sup_{x\in[a,b]}\left(\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right)\right\}\right\}=:B_n.$$

Hence it holds

$$\left\|G_nf-f\right\|_{\infty}\leq(5.250312578)\,B_n. \tag{80}$$

Since $f\in AC^N([a,b])$, $N=\lceil\alpha\rceil$, $\alpha>0$, $\alpha\notin\mathbb{N}$, $f^{(N)}\in L_{\infty}([a,b])$, $x\in[a,b]$, then we get that $f\in AC^N([a,x])$, $f^{(N)}\in L_{\infty}([a,x])$ and $f\in AC^N([x,b])$, $f^{(N)}\in L_{\infty}([x,b])$.

We have

$$\left(D_{x-}^{\alpha}f\right)(y)=\frac{(-1)^N}{\Gamma(N-\alpha)}\int_y^x(J-y)^{N-\alpha-1}f^{(N)}(J)\,dJ, \tag{81}$$

$\forall\ y\in[a,x]$ and

$$\left|\left(D_{x-}^{\alpha}f\right)(y)\right|\leq\frac{1}{\Gamma(N-\alpha)}\left(\int_y^x(J-y)^{N-\alpha-1}dJ\right)\left\|f^{(N)}\right\|_{\infty} \tag{82}$$

$$=\frac{1}{\Gamma(N-\alpha)}\frac{(x-y)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty}=$$

$$\frac{(x-y)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.$$

That is

$$\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}, \tag{83}$$

and

$$\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}. \tag{84}$$

Similarly we have

$$\left(D_{*x}^{\alpha}f\right)(y)=\frac{1}{\Gamma(N-\alpha)}\int_x^y(y-t)^{N-\alpha-1}f^{(N)}(t)\,dt, \tag{85}$$

$\forall\ y\in[x,b]$.

Thus we get

$$\left|\left(D_{*x}^{\alpha}f\right)(y)\right|\leq\frac{1}{\Gamma(N-\alpha)}\left(\int_x^y(y-t)^{N-\alpha-1}dt\right)\left\|f^{(N)}\right\|_{\infty}\leq$$

$$\frac{1}{\Gamma(N-\alpha)}\frac{(y-x)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.$$

Hence

$$\|D_{*x}^{\alpha}f\|_{\infty,[x,b]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}, \tag{86}$$

and

$$\sup_{x\in[a,b]}\|D_{*x}^{\alpha}f\|_{\infty,[x,b]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}. \tag{87}$$

From (16) and (17) we get

$$\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{88}$$

and

$$\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}. \tag{89}$$

So that $B_n<\infty$.

We finally notice that

$$G_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n\left((\cdot-x)^j\right)(x)-f(x)=\frac{G_n^*(f,x)}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)}$$

$$-\frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)}\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*\left((\cdot-x)^j\right)(x)\right)-f(x)$$

$$=\frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)}\cdot \tag{90}$$

$$\left[G_n^*(f,x)-\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*\left((\cdot-x)^j\right)(x)\right)-\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)f(x)\right].$$

Therefore we get

$$\left|G_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n\left((\cdot-x)^j\right)(x)-f(x)\right|\leq(5.250312578)\cdot$$

$$\left|G_n^*(f,x)-\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*\left((\cdot-x)^j\right)(x)\right)-\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)f(x)\right|, \tag{91}$$

$\forall\ x\in[a,b]$.

The proof of the theorem is now complete. ■

We give our second main result

Theorem 24 *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N([a,b])$, with $f^{(N)} \in L_\infty([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$. Then*

i)

$$\left| F_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n\left((\cdot - x)^j\right)(x) - f(x) \right| \leq$$

$$\frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{92}$$

ii) if $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we have

$$|F_n(f,x) - f(x)| \leq \frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{93}$$

when $\alpha > 1$ notice here the extremely high rate of convergence of $n^{-(\alpha+1)\beta}$,

iii)

$$|F_n(f,x) - f(x)| \leq (4.1488766) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\} \right\}, \tag{94}$$

$\forall\, x \in [a,b]$,

and

iv)

$$\|F_n f - f\|_\infty \leq (4.1488766) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.e^4e^{-2n^{(1-\beta)}}(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}. \quad (95)$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $F_n\to I$ the unit operator, as $n\to\infty$.

Proof. Let $x\in[a,b]$. We have that $D_{x-}^{\alpha}f(x)=D_{*x}^{\alpha}f(x)=0$.

From [13], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (96)$$

$$\frac{1}{\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\le\frac{k}{n}\le b$.

Also from [3], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (97)$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a\le\frac{k}{n}\le x$.

Hence we have

$$f\left(\frac{k}{n}\right)\Psi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Psi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (98)$$

$$\frac{\Psi(nx-k)}{\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\le\frac{k}{n}\le b$, iff $\lceil nx\rceil\le k\le\lfloor nb\rfloor$, and

$$f\left(\frac{k}{n}\right)\Psi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Psi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (99)$$

$$\frac{\Psi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Psi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Psi(nx-k)\left(\frac{k}{n}-x\right)^{j}+\tag{100}$$
$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\Psi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\left(\frac{k}{n}-x\right)^{j}+\tag{101}$$
$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ.$$

Adding the last two equalities (100) and (101) obtain

$$F_{n}^{*}(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Psi(nx-k)=\tag{102}$$
$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\left(\frac{k}{n}-x\right)^{j}+$$
$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Psi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ+\right.$$
$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Psi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}.$$

So we have derived

$$F_{n}^{*}(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)=\tag{103}$$
$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_{n}^{*}\left((\cdot-x)^{j}\right)(x)+u_{n}(x),$$

where

$$u_n(x) := \frac{1}{\Gamma(\alpha)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ \right.$$

$$\left. + \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)\right) dJ \right\}. \quad (104)$$

We set

$$u_{1n}(x) := \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ, \quad (105)$$

and

$$u_{2n} := \frac{1}{\Gamma(\alpha)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Psi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x)\right) dJ, \quad (106)$$

i.e.

$$u_n(x) = u_{1n}(x) + u_{2n}(x). \quad (107)$$

We assume $b - a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x)\right) dJ \right|. \quad (108)$$

As in the proof of Theorem 23 we get

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \quad (109)$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, and when $\left|x - \frac{k}{n}\right| \leq \frac{1}{n^{\beta}}$ then

$$\gamma_{1k} \leq \frac{\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{a\beta}}. \quad (110)$$

Consequently we obtain

$$|u_{1n}(x)| \leq \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Psi(nx-k) \gamma_{1k} = \quad (111)$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Psi(nx-k)\,\gamma_{1k}+\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Psi(nx-k)\,\gamma_{1k}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Psi(nx-k)\right)\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Psi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}\right\}\leq \tag{112}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases} k=-\infty \\ :|nx-k|>n^{1-\beta}\end{cases}}^{\infty}\Psi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+e^4e^{-2n^{(1-\beta)}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

So we have proved that

$$|u_{1n}(x)|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right. \tag{113}$$

$$\left.e^4e^{-2n^{(1-\beta)}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

Next when $k=\lfloor nx\rfloor+1,\dots,\lfloor nb\rfloor$ we consider

$$\gamma_{2k}:=\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right|. \tag{114}$$

As in the proof of Theorem 23, when $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$ we get that

$$\gamma_{2k} \leq \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}, \tag{115}$$

and when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$, we derive

$$\gamma_{2k} \leq \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{116}$$

Similarly we have that

$$|u_{2n}(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Psi(nx-k)\, \gamma_{2k} \right) =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \sum_{\substack{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{array} \right.}}^{\lfloor nb \rfloor} \Psi(nx-k)\, \gamma_{2k} + \sum_{\substack{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{array} \right.}}^{\lfloor nb \rfloor} \Psi(nx-k)\, \gamma_{2k} \right\} \leq \tag{117}$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\substack{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{array} \right.}}^{\lfloor nb \rfloor} \Psi(nx-k) \right) \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \left(\sum_{\substack{\left\{ \begin{array}{l} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{array} \right.}}^{\lfloor nb \rfloor} \Psi(nx-k) \right) \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\sum_{\substack{\left\{ \begin{array}{l} k = -\infty \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{array} \right.}}^{\infty} \Psi(nx-k) \right) \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right\} \leq \tag{118}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+e^4e^{-2n^{(1-\beta)}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}.$$

So we have proved that

$$|u_{2n}(x)|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. \tag{119}$$

$$\left. e^4e^{-2n^{(1-\beta)}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}.$$

Therefore

$$|u_n(x)|\leq|u_{1n}(x)|+|u_{2n}(x)|\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. \tag{120}$$

$$\left. e^4e^{-2n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}.$$

From [8] we get that

$$\left|F_n^*\left((\cdot-x)^j\right)(x)\right|\leq\frac{1}{n^{\beta j}}+(b-a)^j e^4e^{-2n^{(1-\beta)}}, \tag{121}$$

for $j=1,...,N-1$, $\forall\ x\in[a,b]$.

Putting things together, we have established

$$\left|F_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)\right|\leq \tag{122}$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\beta j}}+(b-a)^j e^4e^{-2n^{(1-\beta)}}\right]+$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left. e^4e^{-2n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}=:\overline{A}_n(x). \tag{123}$$

As a result we derive

$$|F_n(f,x)-f(x)|\leq(4.1488766)\,\overline{A}_n(x), \tag{124}$$

$\forall\ x\in[a,b]$.

We further have that

$$\left\|\overline{A}_n\right\|_{\infty}\leq\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\beta j}}+(b-a)^j e^4e^{-2n^{(1-\beta)}}\right]+ \tag{125}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left\{\sup_{x\in[a,b]}\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\right)+\sup_{x\in[a,b]}\left(\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)\right\}}{n^{\alpha\beta}}\right.$$

$$+e^{4}e^{-2n^{(1-\beta)}}(b-a)^{\alpha}\cdot$$

$$\left.\left\{\left(\sup_{x\in[a,b]}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\right)+\sup_{x\in[a,b]}\left(\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right)\right\}\right\}=:\overline{B}_n.$$

Hence it holds

$$\|F_nf-f\|_{\infty}\leq(4.1488766)\,\overline{B}_n. \tag{126}$$

Similarly, as in the proof of Theorem 23, we can prove that $\overline{B}_n<\infty$.

We finally notice that

$$F_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n\left((\cdot-x)^j\right)(x)-f(x)=\frac{F_n^*(f,x)}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)}$$

$$-\frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)}\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*\left((\cdot-x)^j\right)(x)\right)-f(x)$$

$$=\frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)}\cdot \tag{127}$$

$$\left[F_n^*(f,x)-\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*\left((\cdot-x)^j\right)(x)\right)-\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)f(x)\right].$$

Therefore we get

$$\left|F_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n\left((\cdot-x)^j\right)(x)-f(x)\right|\leq(4.1488766)\cdot$$

$$\left|F_n^*(f,x)-\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n^*\left((\cdot-x)^j\right)(x)\right)-\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Psi(nx-k)\right)f(x)\right|, \tag{128}$$

$\forall\ x\in[a,b]$.

The proof of the theorem is now finished. ■

Next we apply Theorem 23 for $N=1$.

Corollary 25 *Let $0 < \alpha, \beta < 1$, $f \in AC([a,b])$, $f' \in L_\infty([a,b])$, $n \in \mathbb{N}$. Then*

$$\|G_n f - f\|_\infty \leq \frac{(5.250312578)}{\Gamma(\alpha+1)} \cdot \tag{129}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} (b-a)^\alpha \left(\sup_{x\in[a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \right) \right\}.$$

Also we apply Theorem 24 for $N = 1$.

Corollary 26 *Let $0 < \alpha, \beta < 1$, $f \in AC([a,b])$, $f' \in L_\infty([a,b])$, $n \in \mathbb{N}$. Then*

$$\|F_n f - f\|_\infty \leq \frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot \tag{130}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} (b-a)^\alpha \left(\sup_{x\in[a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \right) \right\}.$$

We make

Remark 27 *Let $0 < \beta < 1$, $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b])$, $n \in \mathbb{N}$. Then, by Corollary 13 and Theorems 14, 15, there exist $x_1, x_2 \in [a,b]$ depended on n, such that*

$$\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} = \omega_1 \left(D_{x_1-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x_1]}, \tag{131}$$

and

$$\sup_{x\in[a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,b]} = \omega_1 \left(D_{*x_2}^\alpha f, \frac{1}{n^\beta} \right)_{[x_2,b]}. \tag{132}$$

Clearly here we have in particular that

$$\omega_1 \left(D_{x_1-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x_1]} \to 0, \tag{133}$$

$$\omega_1 \left(D_{*x_2}^\alpha f, \frac{1}{n^\beta} \right)_{[x_2,b]} \to 0,$$

as $n \to \infty$. Notice that to each n may correspond different $x_1, x_2 \in [a,b]$.

Remark 28 *Let $0<\alpha<1$, then by (84), we get*

$$\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\leq\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|f'\right\|_{\infty}, \tag{134}$$

and by (87), we obtain

$$\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\leq\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|f'\right\|_{\infty}, \tag{135}$$

given that $f\in AC([a,b])$ and $f'\in L_{\infty}([a,b])$.

Next we specialize to $\alpha=\frac{1}{2}$.

Corollary 29 *Let $0<\beta<1$, $f\in AC([a,b])$, $f'\in L_{\infty}([a,b])$, $n\in\mathbb{N}$. Then*
i)

$$\|G_nf-f\|_{\infty}\leq\frac{(10.50062516)}{\sqrt{\pi}}\cdot$$
$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$
$$\left.3.1992e^{-n^{(1-\beta)}}\sqrt{b-a}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}, \tag{136}$$

and
ii)

$$\|F_nf-f\|_{\infty}\leq\frac{(8.2977532)}{\sqrt{\pi}}\cdot$$
$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$
$$\left.e^{4}e^{-2n^{(1-\beta)}}\sqrt{b-a}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{137}$$

We finish with

Remark 30 *(to Corollary 29) Assume that*

$$\omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} \leq \frac{K_1}{n^{\beta}}, \tag{138}$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]} \leq \frac{K_2}{n^{\beta}}, \tag{139}$$

$\forall\ x \in [a,b]$, $\forall\ n \in \mathbb{N}$, *where* $K_1, K_2 > 0$.

Then for large enough $n \in \mathbb{N}$, *by (136) and (137), we obtain*

$$\|G_n f - f\|_{\infty}, \ \|F_n f - f\|_{\infty} \leq \frac{T}{n^{\frac{3}{2}\beta}}, \tag{140}$$

for some $T > 0$.

The speed of convergence in (140) is much higher than the corresponding speeds achieved in [8], [11], which were there $\frac{1}{n^{\beta}}$.

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Fractional Approximation by Cardaliaguet-Euvrard and Squashing neural network operators

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Abstract

This article deals with the determination of the fractional rate of convergence to the unit of some neural network operators, namely, the Cardaliaguet-Euvrard and "squashing" operators. This is given through the moduli of continuity of the involved right and left Caputo fractional derivatives of the approximated function and they appear in the right-hand side of the associated Jackson type inequalities.

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1 Introduction

The Cardaliaguet-Euvrard (22) operators were first introduced and studied extensively in [7], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "squashing operator" (see [1]) (74) was motivated and isnpired by the "squashing functions" and related Theorem 6 of [7]. The work in [7] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [7], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above neural network induced operators. All involved constants there are well determined. These are pointwise, uniform and L_p, $p \geq 1$, estimates involving the first moduli of continuity of the engaged right and left Caputo fractional derivatives of the function under approximation. We give all necessary background of fractional calculus.

Initial work of the subject was done in [1], where we involved only ordinary derivatives. Article [1] motivated the current work.

2 Background

We need

Definition 1 *Let $f \in C(\mathbb{R})$ which is bounded or uniformly continuous, $h > 0$. We define the first modulus of continuity of f at h as follows*

$$\omega_1(f,h) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, \ \ |x-y| \leq h\} \tag{1}$$

Notice that $\omega_1(f,h)$ is finite for any $h > 0$, and

$$\lim_{h \to 0} \omega_1(f,h) = 0.$$

We also need

Definition 2 *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions), $\forall$ $[a,b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{2}$$

$\forall$ $x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$, $\forall$ $b > a$. We set $D_{*a}^{0} f(x) = f(x)$, $\forall$ $x \in [a,\infty)$.

Lemma 3 *([5]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall$ $a \in \mathbb{R}$.*

Definition 4 *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a,b])$, $\forall$ $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{3}$$

$\forall$ $x \leq b$. We set $D_{b-}^{0} f(x) = f(x)$, $\forall$ $x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$, $\forall$ $a < b$.

Lemma 5 *([5]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(b) = 0$, $\forall\, b \in \mathbb{R}$.

Convention 6 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \ \text{for } x < x_0, \tag{4}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{5}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 7 *(by [3]) Let* $f \in C^{n}(\mathbb{R})$, *where* $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a, \infty)$.

Also we have

Proposition 8 *(by [3]) Let* $f \in C^{m}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(x)$ *is continuous in* $x \in (-\infty, b]$.

We further mention

Proposition 9 *(by [3]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{6}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 10 *(by [3]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{7}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 11 *([5]) Let* $g \in C_b(\mathbb{R})$ *(continuous and bounded),* $0 < c < 1$, $x, x_0 \in \mathbb{R}$. *Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \ \text{for } x \geq x_0, \tag{8}$$

and $L(x, x_0) = 0$, *for* $x < x_0$.

Then L *is jointly continuous in* $(x, x_0) \in \mathbb{R}^2$.

We mention

Proposition 12 *([5]) Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$K(x, x_0) = \int_x^{x_0} (J - x)^{c-1} g(J) \, dJ, \quad \text{for } x \leq x_0, \tag{9}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Based on Propositions 11, 12 we derive

Corollary 13 *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Proposition 14 *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x, +\infty)}, \; \delta > 0, x \in \mathbb{R}. \tag{10}$$

(Here ω_1 is defined over $[x, +\infty)$ instead of $\mathbb{R}$.)

Then G is continuous on $\mathbb{R}$.

Proposition 15 *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{(-\infty, x]}, \; \delta > 0, x \in \mathbb{R}. \tag{11}$$

(Here ω_1 is defined over $(-\infty, x]$ instead of $\mathbb{R}$.)

Then H is continuous on $\mathbb{R}$.

By Propositions 14, 15 and Corollary 13 we derive

Proposition 16 *([5]) Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x \in \mathbb{R}$. Then $\omega_1 (D^\alpha_{*x} f, h)_{[x, +\infty)}$, $\omega_1 \left(D^\alpha_{x-} f, h \right)_{(-\infty, x]}$ are continuous functions of $x \in \mathbb{R}$, $h > 0$ fixed.*

We make

Remark 17 *Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\omega_1 (g, t) \leq 2 \|g\|_\infty < \infty. \tag{12}$$

Assuming that $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f \right)(t)$, are both continuous and bounded in $(x, t) \in \mathbb{R}^2$, i.e.

$$\|D^\alpha_{*x} f\|_\infty \leq K_1, \; \forall \; x \in \mathbb{R}; \tag{13}$$

$$\left\|D_{x-}^{\alpha} f\right\|_{\infty} \leq K_{2},\ \forall\ x \in \mathbb{R}, \tag{14}$$

where $K_1, K_2 > 0$, *we get*

$$\begin{gathered} \omega_1 \left(D_{*x}^{\alpha} f, \xi\right)_{[x,+\infty)} \leq 2K_1; \\ \omega_1 \left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty,x]} \leq 2K_2,\ \forall\ \xi \geq 0, \end{gathered} \tag{15}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \geq 0$,

$$\sup_{x\in\mathbb{R}} \left[\max\left(\omega_1 \left(D_{*x}^{\alpha} f, \xi\right)_{[x,+\infty)}, \omega_1 \left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty,x]}\right)\right] \leq 2\max\left(K_1, K_2\right) < \infty. \tag{16}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_{\infty} < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, *by Corollary 13 both* $\left(D_{*x}^{\alpha} f\right)(t)$, $\left(D_{x-}^{\alpha} f\right)(t)$ *are jointly continuous in* (t, x) *on* $\mathbb{R}^2$. *Assuming further that they are both bounded on* $\mathbb{R}^2$ *we get (16) valid. In particular, each of* $\omega_1 \left(D_{*x}^{\alpha} f, \xi\right)_{[x,+\infty)}$, $\omega_1 \left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty,x]}$ *is finite for any* $\xi \geq 0$.

Clearly here we have that $\sup_{x\in\mathbb{R}} \omega_1 \left(D_{*x}^{\alpha} f, \xi\right)_{[x,+\infty)} \to 0$, *as* $\xi \to 0+$, *and* $\sup_{x\in\mathbb{R}} \omega_1 \left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty,x]} \to 0$, *as* $\xi \to 0+$.

Let us now assume only that $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x \in \mathbb{R}$. *Then, by Proposition 15.114, p. 388 of [4], we find that* $D_{*x}^{\alpha} f \in C([x, +\infty))$, *and by [6] we obtain that* $D_{x-}^{\alpha} f \in C((-\infty, x])$.

We make

Remark 18 *Again let* $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1$, $\forall$ $x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. *Notice* $0 < m - \alpha < 1$. *Then*

$$D_{*x_0}^{\alpha} f(x) = \frac{(x - x_0)^{m-\alpha}}{\Gamma(m - \alpha + 1)},\ \forall\ x \geq x_0. \tag{17}$$

Let us consider $x, y \geq x_0$, then

$$\begin{aligned} \left|D_{*x_0}^{\alpha} f(x) - D_{*x_0}^{\alpha} f(y)\right| &= \frac{1}{\Gamma(m - \alpha + 1)} \left|(x - x_0)^{m-\alpha} - (y - x_0)^{m-\alpha}\right| \\ &\leq \frac{|x - y|^{m-\alpha}}{\Gamma(m - \alpha + 1)}. \end{aligned} \tag{18}$$

So it is not strange to assume that

$$\left|D_{*x_0}^{\alpha} f(x_1) - D_{*x_0}^{\alpha} f(x_2)\right| \leq K \left|x_1 - x_2\right|^{\beta}, \tag{19}$$

$K > 0$, $0 < \beta \leq 1$, $\forall$ $x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, where more generally it is $\left\|f^{(m)}\right\|_{\infty} < \infty$. Thus, one may assume

$$\omega_1 \left(D_{x-}^{\alpha} f, \xi\right)_{(-\infty,x]} \leq M_1 \xi^{\beta_1}, \text{ and} \tag{20}$$

$$\omega_1\left(D_{*x}^{\alpha}f,\xi\right)_{[x,+\infty)} \leq M_2\xi^{\beta_2},$$

where $0<\beta_1,\beta_2\leq 1$, $\forall\ \xi>0$, $M_1,M_2>0$; any $x\in\mathbb{R}$.

Setting $\beta=\min(\beta_1,\beta_2)$ and $M=\max(M_1,M_2)$, in that case we obtain

$$\sup_{x\in\mathbb{R}}\left\{\max\left(\omega_1\left(D_{x-}^{\alpha}f,\xi\right)_{(-\infty,x]},\omega_1\left(D_{*x}^{\alpha}f,\xi\right)_{[x,+\infty)}\right)\right\}\leq M\xi^{\beta}\to 0,\text{ as }\xi\to 0+. \tag{21}$$

3 Results

3.1 Fractional convergence with rates of the Cardaliaguet-Euvrard neural network operators

We need the following (see [7]).

Definition 19 *A function $b:\mathbb{R}\to\mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty,a)$ and nonincreasing on $[a,+\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T,T]$, $T>0$. Call $I:=\int_{-T}^{T}b(t)\,dt$. Note that $I>0$.*

We follow [1], [7].

Example 20 (1) *$b(x)$ can be the characteristic function over $[-1,1]$.*

(2) *$b(x)$ can be the hat function over $[-1,1]$, i.e.,*

$$b(x)=\begin{cases}1+x, & -1\leq x\leq 0,\\ 1-x, & 0<x\leq 1\\ 0, & \textit{elsewhere.}\end{cases}$$

These are centered bell-shaped functions of compact support.

Here we consider functions $f:\mathbb{R}\to\mathbb{R}$ that are continuous.

In this article we study the fractional convergence with rates over the real line, to the unit operator, of the Cardaliaguet-Euvrard neural network operators (see [7]),

$$(F_n(f))(x):=\sum_{k=-n^2}^{n^2}\frac{f\left(\frac{k}{n}\right)}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right), \tag{22}$$

where $0<\alpha<1$ and $x\in\mathbb{R}$, $n\in\mathbb{N}$. The terms in the sum (22) can be nonzero iff

$$\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right|\leq T,\ \text{ i.e. }\left|x-\frac{k}{n}\right|\leq\frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{23}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{24}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{25}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (24).

Proposition 21 *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a) - 1) \leq card(k) \leq (b-a) + 1. \tag{26}$$

Remark 22 *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 21 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \text{iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (24) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{27}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$. We call $b^* := b(0)$ the maximum of $b(x)$.

Denote by $[\cdot]$ the integral part of a number.

Following [1] we have

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} 1$$

$$\leq \frac{b^*}{I \cdot n^{\alpha}} \cdot (2Tn^{\alpha} + 1) = \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}}\right). \tag{28}$$

We will use

Lemma 23 *It holds that*

$$S_n(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I\cdot n^{\alpha}} \cdot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \to 1, \tag{29}$$

pointwise, as $n\to+\infty$, where $x\in\mathbb{R}$.

Remark 24 *Clearly we have that*

$$nx - Tn^{\alpha} \le nx \le nx + Tn^{\alpha}. \tag{30}$$

We prove in general that

$$nx - Tn^{\alpha} \le [nx] \le nx \le \lceil nx\rceil \le nx + Tn^{\alpha}. \tag{31}$$

Indeed we have that, if $[nx] < nx - Tn^{\alpha}$, then $[nx] + Tn^{\alpha} < nx$, and $[nx] + [Tn^{\alpha}] \le [nx]$, resulting into $[Tn^{\alpha}] = 0$, which for large enough n is not true. Therefore $nx - Tn^{\alpha} \le [nx]$. Similarly, if $\lceil nx\rceil > nx + Tn^{\alpha}$, then $nx + Tn^{\alpha} \ge nx + [Tn^{\alpha}]$, and $\lceil nx\rceil - [Tn^{\alpha}] > nx$, thus $\lceil nx\rceil - [Tn^{\alpha}] \ge \lceil nx\rceil$, resulting into $[Tn^{\alpha}] = 0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx - Tn^{\alpha} \le [nx] \le nx \le \lceil nx\rceil \le nx + Tn^{\alpha}. \tag{32}$$

Hence $\lceil nx - Tn^{\alpha}\rceil \le [nx]$ and $\lceil nx\rceil \le [nx + Tn^{\alpha}]$. Also if $[nx] \ne \lceil nx\rceil$, then $\lceil nx\rceil = [nx]+1$. If $[nx] = \lceil nx\rceil$, then $nx\in\mathbb{Z}$; and by assuming $n \ge T^{-\frac{1}{\alpha}}$, we get $Tn^{\alpha} \ge 1$ and $nx + Tn^{\alpha} \ge nx+1$, so that $[nx+Tn^{\alpha}] \ge nx+1 = [nx]+1$.

We present our first main result

Theorem 25 *We consider $f:\mathbb{R}\to\mathbb{R}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in AC^N([a,b])$, $\forall\ [a,b]\subset\mathbb{R}$, with $f^{(N)}\in L_{\infty}(\mathbb{R})$. Let also $x\in\mathbb{R}$, $T>0$, $n\in\mathbb{N}: n\ge\max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta}f$, $D_{x-}^{\beta}f$ are uniformly continuous functions or continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$|F_n(f)(x) - f(x)| \le |f(x)| \cdot \tag{33}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}\right.$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0}\cdot=0$,

2)

$$\left|(F_n(f))(x)-\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(F_n\left((\cdot-x)^j\right)\right)(x)\right|\leq \tag{34}$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}=:\lambda_n(x),$$

3) assume further that $f^{(j)}(x)=0$, *for* $j=0,1,...,N-1$, *we get*

$$|F_n(f)(x)|\leq\lambda_n(x), \tag{35}$$

4) in case of $N=1$, *we obtain*

$$|F_n(f)(x)-f(x)|\leq|f(x)|\cdot \tag{36}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with rates the pointwise convergence of $(F_n(f))(x)\to f(x)$, *as* $n\to\infty$, $x\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$. We have that

$$D_{x-}^{\beta}f(x)=D_{*x}^{\beta}f(x)=0. \tag{37}$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+ \tag{38}$$

$$\frac{1}{\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq x+Tn^{\alpha-1}$, iff $\lceil nx\rceil\leq k\leq[nx+Tn^{\alpha}]$, where $k\in\mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \tag{39}$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ,$$

for all $x-Tn^{\alpha-1}\leq\frac{k}{n}\leq x$, iff $\lceil nx-Tn^{\alpha}\rceil\leq k\leq[nx]$, where $k\in\mathbb{Z}$.

Notice that $\lceil nx\rceil\leq[nx]+1$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}+ \tag{40}$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}+ \tag{41}$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}= \tag{42}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}= \tag{43}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \left(\frac{k}{n}-x\right)^j b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} +$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

We notice here that

$$(F_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) = \tag{44}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right).$$

Adding the two equalities (42) and (43) we obtain

$$(F_n(f))(x) =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^j b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}} \right) + \theta_n(x), \tag{45}$$

where

$$\theta_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{46}$$

We call

$$\theta_{1n}(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{47}$$

and

$$\theta_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}\Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ. \tag{48}$$

I.e.

$$\theta_n(x) = \theta_{1n}(x) + \theta_{2n}(x). \tag{49}$$

We further have

$$(F_n(f))(x) - f(x) = f(x)\left(\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^\alpha} - 1\right) + \tag{50}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\left(\frac{k}{n}-x\right)^j b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^\alpha}\right) + \theta_n(x),$$

and

$$|(F_n(f))(x) - f(x)| \leq |f(x)|\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right| +$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\left|x-\frac{k}{n}\right|^j b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^\alpha}\right) + |\theta_n(x)| \leq$$

$$|f(x)|\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right| + \tag{51}$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{T^j}{n^{(1-\alpha)j}}\left(\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^\alpha}\right) + |\theta_n(x)| =: (*).$$

But we have

$$\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \leq \frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right), \tag{52}$$

by (28).

Therefore we obtain

$$|(F_n(f))(x) - f(x)| \leq |f(x)|\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right| +$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}\right) + |\theta_n(x)|. \tag{53}$$

Next we see that

$$\gamma_{1n} := \frac{1}{\Gamma(\beta)}\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J) - D_{x-}^{\beta}f(x)\right)dJ\right| \leq \tag{54}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\beta - 1} \left| D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x) \, dJ \right| \leq$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\beta - 1} \omega_1 \left(D_{x-}^{\beta} f, |J - x| \right)_{(-\infty, x]} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1 \left(D_{x-}^{\beta} f, \left| x - \frac{k}{n} \right| \right)_{(-\infty, x]} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\beta - 1} dJ \leq$$

$$\frac{1}{\Gamma(\beta)} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]} \frac{\left(x - \frac{k}{n} \right)^{\beta}}{\beta} \leq$$

$$\frac{1}{\Gamma(\beta + 1)} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]} \frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta + 1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]}. \quad (55)$$

Furthermore

$$|\theta_{1n}(x)| \leq \sum_{k = \lceil nx - Tn^{\alpha} \rceil}^{[nx]} \frac{b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{I n^{\alpha}} \gamma_{1n} \leq \quad (56)$$

$$\left(\sum_{k = \lceil nx - Tn^{\alpha} \rceil}^{[nx]} \frac{b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{I n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta + 1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]} \leq$$

$$\left(\sum_{k = \lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{I n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta + 1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]} \leq$$

$$\frac{b^*}{I} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta + 1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]}.$$

So that

$$|\theta_{1n}(x)| \leq \frac{b^*}{I} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta + 1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty, x]}. \quad (57)$$

Similarly we derive

$$\gamma_{2n} := \frac{1}{\Gamma(\beta)} \left| \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta - 1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right) dJ \right| \leq \quad (58)$$

$$\frac{1}{\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta - 1} \left| D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right| dJ \leq$$

$$\frac{\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma(\beta+1)}\left(\frac{k}{n}-x\right)^{\beta}\leq$$

$$\frac{\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}}{\Gamma(\beta+1)}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is

$$\gamma_{2n}\leq\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}. \tag{59}$$

Consequently we find

$$|\theta_{2n}(x)|\leq\left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{In^{\alpha}}\right)\cdot$$

$$\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \tag{60}$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

So we have proved that

$$|\theta_n(x)|\leq\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{61}$$

$$\left\{\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D^{\beta}_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Combining (53) and (61) we have (33). ∎

As an application of Theorem 25 we give

Theorem 26 *Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Let also $T>0$, $n\in\mathbb{N}:n\geq\max\left(2T,T^{-\frac{1}{\alpha}}\right)$. We further assume that $D^{\beta}_{*x}f(t)$, $D^{\beta}_{x-}f(t)$ are both bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|F_n(f)-f\|_{\infty,[-T,T]}\leq\|f\|_{\infty,[-T,T]}\cdot \tag{62}$$

$$\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]}+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty,[-T,T]}T^j}{j!n^{(1-\alpha)j}}\right)+\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{x-} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|F_n(f) - f\|_{\infty,[-T,T]} \leq \|f\|_{\infty,[-T,T]} \cdot \tag{63}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right) - 1 \right\|_{\infty,[-T,T]} +$$

$$\frac{b^*}{I} \left(2T + \frac{1}{n^{\alpha}} \right) \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{x-} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Assuming further that $\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0$, as $n \to \infty$, we get fractionally with rates the uniform convergence of $F_n(f) \to f$, as $n \to \infty$.

Proof. From (33), (36) of Theorem 25, and by Remark 17.

Also by

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right) \leq \frac{b^*}{I}(2T+1), \tag{64}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right) - 1 \right\|_{\infty,[-T,T]} \leq \left(\frac{b^*}{I}(2T+1) + 1 \right). \tag{65}$$

■

One can also apply Remark 18 to the last Theorem 26, to get interesting and simplified results.

We make

Remark 27 *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T > 0$. Let $x \in [-T^*, T^*]$, $T^* > 0$, and $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$. Consider $p \geq 1$.*

Clearly we get here that

$$\left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right) - 1 \right|^p \leq \left(\frac{b^*}{I}(2T+1) + 1 \right)^p, \tag{66}$$

for all $x \in [-T^*, T^*]$, *for any* $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$.

By Lemma 23, we obtain that

$$\lim_{n\to\infty}\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|^p=0, \tag{67}$$

all $x \in [-T^*, T^*]$.

Now it is clear, by the bounded convergence theorem, that

$$\lim_{n\to\infty}\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}=0. \tag{68}$$

Let $\beta > 0$, $N = \lceil\beta\rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, $f^{(N)} \in L_\infty(\mathbb{R})$. *Here both* $D^\alpha_{*x}f(t)$, $D^\alpha_{x-}f(t)$ *are bounded in* $(x,t) \in \mathbb{R}^2$.

By Theorem 25 we have

$$|F_n(f)(x)-f(x)| \leq \|f\|_{\infty,[-T^*,T^*]}\cdot \tag{69}$$

$$\left|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}\right)$$

$$+\frac{b^*}{I}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Applying to the last inequality (69) the monotonicity and subadditive property of $\|\cdot\|_p$, *we derive the following* L_p, $p \geq 1$, *interesting result.*

Theorem 28 *Let* $b(x)$ *be a centered bell-shaped continuous function on* $\mathbb{R}$ *of compact support* $[-T,T]$, $T > 0$. *Let* $x \in [-T^*, T^*]$, $T^* > 0$, *and* $n \in \mathbb{N}$: $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$, $p \geq 1$. *Let* $\beta > 0$, $N = \lceil\beta\rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, *with* $f^{(N)} \in L_\infty(\mathbb{R})$. *Here both* $D^\beta_{*x}f(t)$, $D^\beta_{x-}f(t)$ *are bounded in* $(x,t) \in \mathbb{R}^2$. *Then*

1)

$$\|F_nf-f\|_{p,[-T^*,T^*]} \leq \|f\|_{\infty,[-T^*,T^*]}\cdot \tag{70}$$

$$\left\|\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{1}{In^\alpha}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{p,[-T^*,T^*]}T^j}{j!n^{(1-\alpha)j}}\right)+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) When $N=1$, we derive

$$\|F_nf-f\|_{p,[-T^*,T^*]}\leq\|f\|_{\infty,[-T^*,T^*]}\cdot \tag{71}$$

$$\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{In^{\alpha}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{p,[-T^*,T^*]}+$$

$$\frac{2^{\frac{1}{p}}T^{*\frac{1}{p}}b^*}{I}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

By (70), (71) we derive the fractional L_p, $p\geq1$, convergence with rates of F_nf to f.

3.2 The "Squashing operators" and their fractional convergence to the unit with rates

We need (see also [1], [7]).

Definition 29 *Let the nonnegative function $S:\mathbb{R}\to\mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty,T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function".*

Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. Assume that

$$I^*:=\int_{-T}^{T}S(t)\,dt>0. \tag{72}$$

Obviously

$$\max_{x\in[-T,T]}S(x)=S(T). \tag{73}$$

For $x \in \mathbb{R}$ we define the "squashing operator" ([1])

$$\left(G_n(f)\right)(x) := \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{74}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$\left(G_n(f)\right)(x) = \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{f\left(\frac{k}{n}\right)}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right). \tag{75}$$

Here we study the fractional convergence with rates of $\left(G_n f\right)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Notice that

$$\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} 1 \leq \left(2Tn^\alpha + 1\right). \tag{76}$$

From [1] we need

Lemma 30 *It holds that*

$$D_n(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \to 1, \tag{77}$$

pointwise, as $n \to +\infty$, where $x \in \mathbb{R}$.

We present our second main result

Theorem 31 *We consider $f : \mathbb{R} \to \mathbb{R}$. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $\forall$ $[a,b] \subset \mathbb{R}$, with $f^{(N)} \in L_\infty(\mathbb{R})$. Let also $x \in \mathbb{R}, T > 0, n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^\beta f$, $D_{x-}^\beta f$ are uniformly continuous functions or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively.*

Then

1)

$$\left|G_n(f)(x) - f(x)\right| \leq |f(x)| \cdot \tag{78}$$

$$\left|\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I^* n^\alpha} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) - 1\right| +$$

$$\frac{S(T)}{I^*}\left(2T + \frac{1}{n^\alpha}\right)\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right)$$

$$+\frac{S(T)}{I^*}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0}\cdot=0,$

2)

$$\left|(G_n(f))(x)-\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(G_n\left((\cdot-x)^j\right)\right)(x)\right|\leq \tag{79}$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}=:\lambda_n^*(x),$$

3) assume further that $f^{(j)}(x)=0$, *for* $j=0,1,...,N-1$, *we get*

$$|G_n(f)(x)|\leq\lambda_n^*(x), \tag{80}$$

4) in case of $N=1$, *we obtain*

$$|G_n(f)(x)-f(x)|\leq|f(x)|\cdot \tag{81}$$

$$\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^*n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with rates the pointwise convergence of $(G_n(f))(x)\to f(x)$, *as* $n\to\infty$, $x\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$. We have that

$$D_{x-}^{\beta}f(x)=D_{*x}^{\beta}f(x)=0.$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+ \tag{82}$$

$$\frac{1}{\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq x+Tn^{\alpha-1}$, iff $\lceil nx\rceil\leq k\leq[nx+Tn^{\alpha}]$, where $k\in\mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \tag{83}$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ,$$

for all $x-Tn^{\alpha-1}\leq\frac{k}{n}\leq x$, iff $\lceil nx-Tn^{\alpha}\rceil\leq k\leq[nx]$, where $k\in\mathbb{Z}$.

Hence we have

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{84}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+ \tag{85}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{86}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}= \tag{87}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\left(\frac{k}{n}-x\right)^{j}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}+$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)} \int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

Adding the two equalities (86) and (87) we obtain

$$(G_n(f))(x)=$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+M_n(x), \quad (88)$$

where

$$M_n(x):=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \quad (89)$$

We call

$$M_{1n}(x):=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ, \quad (90)$$

and

$$M_{2n}(x):=\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}\Gamma(\beta)}\cdot$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \quad (91)$$

I.e.

$$M_n(x)=M_{1n}(x)+M_{2n}(x). \quad (92)$$

We further have

$$(G_n(f))(x)-f(x)=f(x)\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}-1\right)+ \quad (93)$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^{*}n^{\alpha}}\right)+M_n(x),$$

and

$$\left|\left(G_n(f)\right)(x)-f(x)\right| \leq |f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+ \tag{94}$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left|x-\frac{k}{n}\right|^j S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}}\right)+\left|M_n(x)\right| \leq$$

$$|f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{T^j}{n^{(1-\alpha) j}}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}}\right)+\left|M_n(x)\right|=:(*) . \tag{95}$$

Therefore we obtain

$$\left|\left(G_n(f)\right)(x)-f(x)\right| \leq |f(x)|\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right|+$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha) j}}\right)+\left|M_n(x)\right| . \tag{96}$$

We call

$$\gamma_{1n}:=\frac{1}{\Gamma(\beta)}\left|\int_{\frac{k}{n}}^x\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta} f(J)-D_{x-}^{\beta} f(x)\right) dJ\right| . \tag{97}$$

As in the proof of Theorem 25 we have

$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha) \beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty, x]} . \tag{98}$$

Furthermore

$$\left|M_{1n}(x)\right| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}} \gamma_{1n} \leq \tag{99}$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha) \beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty, x]} \leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^* n^{\alpha}}\right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha) \beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty, x]} \leq$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}.$$

So that

$$|M_{1n}(x)|\leq\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}. \tag{100}$$

We also call

$$\gamma_{2n}:=\frac{1}{\Gamma(\beta)}\left|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D^\beta_{*x}f(J)-D^\beta_{*x}f(x)\right)dJ\right|. \tag{101}$$

As in the proof of Theorem 25 we get

$$\gamma_{2n}\leq\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}. \tag{102}$$

Consequently we find

$$|M_{2n}(x)|\leq\left(\sum_{k=[nx]+1}^{[nx+Tn^\alpha]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{I^*n^\alpha}\right)\cdot$$

$$\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \tag{103}$$

$$\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

So we have proved that

$$|M_n(x)|\leq\frac{S(T)}{I^*}\left(2T+\frac{1}{n^\alpha}\right)\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{104}$$

$$\left\{\omega_1\left(D^\beta_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D^\beta_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Combining (96) and (104) we have (78). ■

As an application of Theorem 31 we give

Theorem 32 *Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Let also $T>0$, $n\in\mathbb{N}:n\geq\max\left(2T,T^{-\frac{1}{\alpha}}\right)$. We further assume that $D^\beta_{*x}f(t)$, $D^\beta_{x-}f(t)$ are both bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|G_n(f)-f\|_{\infty,[-T,T]}\leq\|f\|_{\infty,[-T,T]}. \tag{105}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} +$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\left(\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty,[-T,T]}T^{j}}{j!n^{(1-\alpha)j}}\right)+$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T,T]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T,T]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) in case of $N=1$, we obtain

$$\left\|G_n(f)-f\right\|_{\infty,[-T,T]}\leq\left\|f\right\|_{\infty,[-T,T]}\cdot \tag{106}$$

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]} +$$

$$\frac{S(T)}{I^{*}}\left(2T+\frac{1}{n^{\alpha}}\right)\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{x\in[-T,T]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T,T]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

An interesting case is when $\beta=\frac{1}{2}$.

Assuming further that $\left\|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^{*}n^{\alpha}}S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]}\to 0$, as $n\to\infty$, we get fractionally with rates the uniform convergence of $G_n(f)\to f$, as $n\to\infty$.

Proof. From (78), (81) of Theorem 31, and by Remark 17.
Also by

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\leq\frac{S(T)}{I^{*}}(2T+1), \tag{107}$$

we get that

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^{*}n^{\alpha}} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-1\right\|_{\infty,[-T,T]}\leq\left(\frac{S(T)}{I^{*}}(2T+1)+1\right). \tag{108}$$

■

One can also apply Remark 18 to the last Theorem 32, to get interesting and simplified results.

Note 33 *The maps F_n, G_n, $n \in \mathbb{N}$, are positive linear operators.*

We finish with

Remark 34 *The condition of Theorem 26 that*

$$\left\| \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{In^{\alpha}} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-T,T]} \to 0, \tag{109}$$

as $n \to \infty$, is not uncommon.

We give an example related to that.

We take as $b(x)$ the characteristic function over $[-1,1]$, that is $\chi_{[-1,1]}(x)$. Here $T = 1$ and $I = 2$, $n \geq 2$, $x \in [-1,1]$.

We get that

$$\sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} \chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) \overset{(23)}{=} \sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} =$$

$$\frac{1}{2n^{\alpha}}\left(\sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} 1\right) = \frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha} \rceil + 1)}{2n^{\alpha}}. \tag{110}$$

But we have

$$[nx+n^{\alpha}] - \lceil nx-n^{\alpha} \rceil + 1 \leq 2n^{\alpha} + 1,$$

hence

$$\frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha} \rceil + 1)}{2n^{\alpha}} \leq 1 + \frac{1}{2n^{\alpha}}. \tag{111}$$

Also it holds

$$[nx+n^{\alpha}] - \lceil nx-n^{\alpha} \rceil + 1 \geq 2n^{\alpha} - 2 + 1 = 2n^{\alpha} - 1,$$

and

$$\frac{([nx+n^{\alpha}] - \lceil nx-n^{\alpha} \rceil + 1)}{2n^{\alpha}} \geq 1 - \frac{1}{2n^{\alpha}}. \tag{112}$$

Consequently we derive that

$$-\frac{1}{2n^{\alpha}} \leq \left(\sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} \chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1\right) \leq \frac{1}{2n^{\alpha}}, \tag{113}$$

for any $x \in [-1,1]$ and for any $n \geq 2$.

Hence we get

$$\left\| \sum_{k=\lceil nx-n^{\alpha} \rceil}^{[nx+n^{\alpha}]} \frac{1}{2n^{\alpha}} \chi_{[-1,1]}\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - 1 \right\|_{\infty,[-1,1]} \to 0, \text{ as } n \to \infty. \tag{114}$$

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Rate of Convergence of Some Neural Network Operators to the Unit-Univariate Case, Revisited

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Abstract

This article deals with the determination of the rate of convergence to the unit of some neural network operators, namely, "the normalized bell and squashing type operators". This is given through the modulus of continuity of the involved function or its derivative and that appears in the right-hand side of the associated Jackson type inequalities.

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1 Introduction

The Cardaliaguet-Euvrard operators were first introduced and studied extensively in [2], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "normalized bell and squashing type operators" (1) and (18) were motivated and inspired by [2]. The work in [2] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [2], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of inequalities giving close upper bounds to the errors in approximating the unit operator by the above neural network induced operators. All involved constants there are well determined. These are mainly pointwise estimates involving the first modulus of continuity of the engaged continuous function or some of its derivatives. This work is a continuation and simplification of our earlier work in [1].

2 Convergence with rates of the normalized bell type neural network operators

We need the following (see [2]).

Definition 1 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

Example 2 (1) *$b(x)$ can be the characteristic function over $[-1, 1]$.*

(2) *$b(x)$ can be the hat function over $[-1, 1]$, i.e.,*

$$b(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 < x \le 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

In this article we study the pointwise convergence with rates over the real line, to the unit operator, of the "normalized bell type neural network operators",

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{1}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$. The terms in the ratio of sums (1) can be nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right) \right| \le T, \text{ i.e. } \left| x - \frac{k}{n} \right| \le \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \le k \le nx + Tn^{\alpha}. \tag{2}$$

In order to have the desired order of numbers

$$-n^2 \le nx - Tn^{\alpha} \le nx + Tn^{\alpha} \le n^2, \tag{3}$$

it is sufficient enough to assume that

$$n \geq T + |x| \,. \tag{4}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (3).

Proposition 3 *(see [1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let card (k) (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \leq card\,(k) \leq (b-a)+1.$$

Note 4 *We would like to establish a lower bound on card (k) over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 3 we get that*

$$card\,(k) \geq \max(2Tn^{\alpha} - 1, 0) \,.$$

We obtain card $(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \textit{iff}\ n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (3) and card $(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x| \,, T^{-\frac{1}{\alpha}}\right). \tag{5}$$

Also notice that card $(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling. Here comes our first main result.

Theorem 5 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x| \,, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(H_n(f))(x) - f(x)| \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right), \tag{6}$$

where ω_1 is the first modulus of continuity of f.

Proof. Call

$$V(x) := \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Clearly we obtain

$$\Delta := |(H_n(f))(x) - f(x)|$$

$$= \left| \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} - f(x) \right|.$$

The last comes by the compact support $[-T,T]$ of b and (2).

Hence it holds

$$\Delta = \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{\left(f\left(\frac{k}{n}\right)-f(x)\right)}{V(x)} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|$$

$$\leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{\omega_1\left(f,\left|\frac{k}{n}-x\right|\right)}{V(x)} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right).$$

Thus

$$|(H_n(f))(x)-f(x)| \leq$$

$$\omega_1\left(f,\frac{T}{n^{1-\alpha}}\right)\frac{\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right)}{V(x)} = \omega_1\left(f,\frac{T}{n^{1-\alpha}}\right),$$

proving the claim. ∎

Our second main result follows.

Theorem 6 *Let $x\in\mathbb{R}$, $T>0$ and $n\in\mathbb{N}$ such that $n\geq\max\left(T+|x|,T^{-\frac{1}{\alpha}}\right)$. Let $f\in C^N(\mathbb{R})$, $N\in\mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(H_n(f))(x)-f(x)| \leq \left(\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|T^j}{n^{j(1-\alpha)}j!}\right)+ \tag{7}$$

$$\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\cdot\frac{T^N}{N!n^{N(1-\alpha)}}.$$

Notice that as $n\to\infty$ we have that R.H.S.(7)$\to 0$, therefore L.H.S.(7)$\to 0$, i.e., (7) gives us with rates the pointwise convergence of $(H_n(f))(x)\to f(x)$, as $n\to+\infty$, $x\in\mathbb{R}$.

Proof. Note that b here is of compact support $[-T,T]$ and all assumptions are as earlier. By Taylor's formula we have that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+$$

$$\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Call

$$V(x):=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right).$$

Hence

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n}-x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} +$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt.$$

Thus

$$(H_n(f))(x) - f(x) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} - f(x)$$

$$= \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \right) + R,$$

where

$$R := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt. \tag{8}$$

So that

$$|(H_n(f))(x) - f(x)| \leq$$

$$\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{T^{j}}{n^{j(1-\alpha)}} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \right) + |R|.$$

And hence

$$|(H_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N} \frac{T^{j}\left|f^{(j)}(x)\right|}{n^{j(1-\alpha)} j!} \right) + |R|. \tag{9}$$

Next we estimate

$$|R| = \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left| \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot \gamma \leq (*),$$

where

$$\gamma := \left| \int_x^{\frac{k}{n}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left| \frac{k}{n} - t \right|^{N-1}}{(N-1)!} dt \right|, \tag{10}$$

and

$$(*) \leq \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot \varphi = \varphi, \tag{11}$$

where

$$\varphi := \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{T^N}{n! n^{N(1-\alpha)}}. \tag{12}$$

The last part of inequality (11) comes from the following:

(i) Let $x \leq \frac{k}{n}$, then

$$\gamma = \int_x^{\frac{k}{n}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left| \frac{k}{n} - t \right|^{N-1}}{(N-1)!} dt$$

$$\leq \int_x^{\frac{k}{n}} \omega_1\left(f^{(N)}, |t-x|\right) \frac{\left| \frac{k}{n} - t \right|^{N-1}}{(N-1)!} dt$$

$$\leq \omega_1\left(f^{(N)}, \left| x - \frac{k}{n} \right|\right) \int_x^{\frac{k}{n}} \frac{\left(\frac{k}{n} - t \right)^{N-1}}{(N-1)!} dt$$

$$\leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{\left(\frac{k}{n} - x\right)^N}{N!} \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{T^N}{N! n^{N(1-\alpha)}};$$

i.e., when $x \leq \frac{k}{n}$ we get

$$\gamma \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \frac{T^N}{N! n^{N(1-\alpha)}}. \tag{13}$$

(ii) Let $x \geq \frac{k}{n}$, then

$$\gamma = \left| \int_{\frac{k}{n}}^x \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left| t - \frac{k}{n} \right|^{N-1}}{(N-1)!} dt \right| =$$

$$\int_{\frac{k}{n}}^x \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left(t - \frac{k}{n} \right)^{N-1}}{(N-1)!} dt \leq$$

$$\int_{\frac{k}{n}}^x \omega_1\left(f^{(N)}, |t-x|\right) \frac{\left(t - \frac{k}{n} \right)^{N-1}}{(N-1)!} dt \leq$$

$$\omega_1\left(f^{(N)}, \left| x - \frac{k}{n} \right|\right) \int_{\frac{k}{n}}^x \frac{\left(t - \frac{k}{n} \right)^{N-1}}{(N-1)!} dt =$$

$$\omega_1\left(f^{(N)},\left|x-\frac{k}{n}\right|\right)\frac{\left(x-\frac{k}{n}\right)^N}{N!}\leq\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\frac{T^N}{N!n^{N(1-\alpha)}}.$$

Thus in both cases we have

$$\gamma\leq\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\frac{T^N}{N!n^{N(1-\alpha)}}. \tag{14}$$

Consequently from (11), (12) and (14) we obtain

$$|R|\leq\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\frac{T^N}{N!n^{N(1-\alpha)}}. \tag{15}$$

Finally from (15) and (9) we conclude inequality (7). ■

Corollary 7 *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}$ be such that $n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p\geq 1$. Then*

$$\|H_n(f)-f\|_{p,[-T^*,T^*]}\leq\omega_1\left(f,\frac{T}{n^{1-\alpha}}\right)\cdot 2^{\frac{1}{p}}\cdot T^{*\frac{1}{p}}. \tag{16}$$

From (16) we get the L_p convergence of $H_n(f)$ to f with rates.

Corollary 8 *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}$ be such that $n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p\geq 1$. Then*

$$\|H_n(f)-f\|_{p,[-T^*,T^*]}\leq \tag{17}$$

$$\left(\sum_{j=1}^{N}\frac{T^j\cdot\left\|f^{(j)}\right\|_{p,[-T^*,T^*]}}{n^{j(1-\alpha)}j!}\right)+\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\frac{2^{\frac{1}{p}}T^NT^{*\frac{1}{p}}}{N!n^{N(1-\alpha)}},$$

where $N\geq 1$.

Here from (17) we get again the L_p convergence of $H_n(f)$ to f with rates.

Proof. Inequality (17) now comes by integration of (7) and the properties of the L_p-norm. ■

2.1 The "normalized Squashing type operators" and their convergence to the unit with rates

We need

Definition 9 *Let the nonnegative function $S:\mathbb{R}\to\mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty,T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function" (see also [2]).*

Let $f : \mathbb{R} \to \mathbb{R}$ be either uniformly continuous or continuous and bounded.

For $x \in \mathbb{R}$ we define the "normalized squashing type operator"

$$(K_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}, \tag{18}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$(K_n(f))(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{W(x)}, \tag{19}$$

where

$$W(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).$$

Here we give the pointwise convergence with rates of $(K_n f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

Theorem 10 *Under the above terms and assumptions we obtain*

$$|K_n(f)(x) - f(x)| \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right). \tag{20}$$

Proof. As in Theorem 5. ■

We also give

Theorem 11 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(K_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{j(1-\alpha)}}\right) + \tag{21}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! n^{N(1-\alpha)}}.$$

So we obtain the pointwise convergence of $K_n(f)$ to f with rates.

Proof. As similar to Theorem 6 is omitted. ■

Note 12 *The maps H_n, K_n are positive linear operators reproducing constants, in particular*

$$H_n(1) = K_n(1) = 1. \tag{22}$$

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Rate of convergence of some multivariate neural network operators to the unit, revisited

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Abstract

This paper deals with the determination of the rate of convergence to the unit of some multivariate neural network operators, namely the the normalized "bell" and "squashing" type operators. This is given through the multidimensional modulus of continuity of the involved multivariate function or its partial derivatives of specific order that appear in the right-hand side of the associated multivariate Jackson type inequalitiy.

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1 Introduction

The multivariate Cardaliaguet-Euvrard operators were first introduced and studied thoroughly in [3], where the authors among many other interesting things proved that these multivariate operators converge uniformly on compacta, to the unit over continuous and bounded multivariate functions. Our multivariate normalized "bell" and "squashing" type operators (1) and (16) were motivated and inspired by the "bell" and "squashing" functions of [3].

The work in [3] is qualitative where the used multivariate bell-shaped function is general. However, though our work is greatly motivated by [3], it is quantitative and the used multivariate "bell-shaped" and "squashing" functions are of compact support.

This paper is the continuation and simplification of [1] and [2], in the multidimensional case.We produce a set of multivariate inequalities giving close upper bounds to the errors in approximating the unit operator by the above multidimensional neural network induced operators. All appearing constants there are well determined. These are mainly pointwise estimates involving the first multivariate modulus of continuity of the engaged multivariate continuous function or its partial derivatives of some fixed order.

2 Convergence with rates of multivariate neural network operators

We need the following (see [3]) definitions.

Definition 1 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.*

Definition 2 *(see [3]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,*

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 3 *(from [3]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function.*

Assumption 4 *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function or a uniformly continuous function.*

In this paper, we study the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the "normalized bell" multivariate neural network operators

$$M_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}, \tag{1}$$

where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (1) can be nonzero iff simultaneously

$$\left| n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i, \quad \text{all } i = 1, ..., d,$$

i.e., $\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\alpha}}$, all $i = 1, ..., d$, iff

$$nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha, \quad \text{all } i = 1, ..., d. \tag{2}$$

To have the order

$$-n^2 \leq nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha \leq n^2, \tag{3}$$

we need $n \geq T_i + |x_i|$, all $i = 1, ..., d$. So (3) is true when we take

$$n \geq \max_{i \in \{1,...,d\}} (T_i + |x_i|). \tag{4}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (3) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Consider

$$\widetilde{I}_i := [nx_i - T_i n^\alpha, nx_i + T_i n^\alpha], \; i = 1, ..., d, \; n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^\alpha$. By Proposition 1 of [1], we get that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card(k_i) \geq \max(2T_i n^\alpha - 1, 0)$, any $i \in \{1, ..., d\}$. In order to have $card(k_i) \geq 1$, we need $2T_i n^\alpha - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\alpha}}$, any $i \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (3) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, ..., d$ is that

$$n \geq \max_{i \in \{1,...,d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\alpha}}\right\}. \tag{5}$$

Clearly as $n \to +\infty$ we get that $card(k_i) \to +\infty$, all $i = 1, ..., d$. Also notice that $card(k_i)$ equals to the cardinality of integers in $[\lceil nx_i - T_i n^\alpha \rceil, [nx_i + T_i n^\alpha]]$ for all $i = 1, ..., d$. Here, $[\cdot]$ denotes the integral part of the number while. $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this article we will assume (5). Furthermore it holds

$$(M_n(f))(\overrightarrow{x}) = \frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right)}{V(\overrightarrow{x})} \cdot \tag{6}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)$$

all $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, where

$$V(\overrightarrow{x}) :=$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right).$$

Denote by $\|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, ..., d$, we get that

$$\left\| \overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right\|_\infty \leq \frac{T^*}{n^{1-\alpha}},$$

where $\overrightarrow{k} := (k_1, ..., k_d)$.

Definition 5 *Let $f : \mathbb{R}^d \to \mathbb{R}$. We call*

$$\omega_1(f, h) := \sup_{\substack{all\ \overrightarrow{x}, \overrightarrow{y}: \\ \|\overrightarrow{x} - \overrightarrow{y}\|_\infty \leq h}} |f(\overrightarrow{x}) - f(\overrightarrow{y})|, \tag{7}$$

where $h > 0$, the first modulus of continuity of f.

Here is our first main result.

Theorem 6 *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{8}$$

Inequality (8) is attained by constant functions.

Inequality (8) gives $M_n(f)(\overrightarrow{x}) \to f(\overrightarrow{x})$, pointwise with rates, as $n \to +\infty$, where $\overrightarrow{x} \in \mathbb{R}^d$, $d \geq 1$.

Proof. Next, we estimate

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \overset{(6)}{=}$$

$$\left| \sum_{k_1 = \lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} ... \sum_{k_d = \lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) \cdot \right.$$

$$\left. \frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V(\overrightarrow{x})} - f(\overrightarrow{x}) \right| =$$

$$\left| \frac{\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} \left(f\left(\frac{\overrightarrow{k}}{n}\right) - f(\overrightarrow{x})\right) b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} \right| \leq$$

$$\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} \frac{\left|f\left(\frac{\overrightarrow{k}}{n}\right) - f(\overrightarrow{x})\right|}{V(\overrightarrow{x})} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right) \leq$$

$$\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]} \frac{\omega_1\left(f, \left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_{\infty}\right)}{V\left(\overrightarrow{x}\right)} b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right).$$

That is

$$\left|\left(M_n\left(f\right)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)\right| \leq \frac{\omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right)}{V\left(\overrightarrow{x}\right)} \cdot$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)$$

$$=\omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right), \tag{9}$$

proving the claim. ■

Our second main result follows.

Theorem 7 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$\left|\left(M_n\left(f\right)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)\right| \leq \tag{10}$$

$$\left\{\sum_{j=1}^{N} \frac{\left(T^*\right)^j}{j! n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\overrightarrow{x}\right)\right)\right\}+$$

$$\frac{\left(T^*\right)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (10) is attained by constant functions. Also, (10) gives us with rates the pointwise convergence of $M_n(f) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Proof. Set

$$g_{\frac{\overrightarrow{k}}{n}}(t) := f\left(\overrightarrow{x}+t\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right)\right), \quad 0 \leq t \leq 1.$$

Then

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(t) =$$

$$\left[\left(\sum_{i=1}^{d}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^j f\right]\left(x_1+t\left(\frac{k_1}{n}-x_1\right), ..., x_d+t\left(\frac{k_d}{n}-x_d\right)\right)$$

and $g_{\frac{\overrightarrow{k}}{n}}(0) = f\left(\overrightarrow{x}\right)$. By Taylor's formula, we get

$$f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) = g_{\frac{\overrightarrow{k}}{n}}(1) = \sum_{j=0}^{N} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} + R_N\left(\frac{\overrightarrow{k}}{n}, 0\right),$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n},0\right)=\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left(g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N)-g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0)\right)dt_N\right)...\right)dt_1.$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \alpha_i\in\mathbb{Z}^+,$$

$i=1,...,d$, such that $|\widetilde{\alpha}|:=\sum_{i=1}^d\alpha_i=N$. Thus,

$$\frac{f\left(\frac{\overrightarrow{k}}{n}\right)b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}=$$

$$\sum_{j=0}^N\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}+\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\cdot R_N\left(\frac{\overrightarrow{k}}{n},0\right).$$

Therefore

$$(M_n(f))(\overrightarrow{x})-f(\overrightarrow{x})=$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{V(\overrightarrow{x})}b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)-f(\overrightarrow{x})=$$

$$\sum_{j=1}^N\frac{1}{j!}\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right)+R^*,$$

where

$$R^*:=\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\cdot R_N\left(\frac{\overrightarrow{k}}{n},0\right).$$

Consequently, we obtain

$$|(M_n(f))(\overrightarrow{x})-f(\overrightarrow{x})|\leq$$

$$\sum_{j=1}^N\frac{1}{j!}\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\right|b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right)+|R^*|=:\Theta.$$

Noyice that

$$\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\right|\leq\left(\frac{T^*}{n^{1-\alpha}}\right)^j\left(\left(\sum_{i=1}^d\left|\frac{\partial}{\partial x_i}\right|\right)^jf(\overrightarrow{x})\right)$$

and

$$\Theta \leq \left\{ \sum_{j=1}^{N} \frac{1}{j!} \left(\frac{T^*}{n^{1-\alpha}} \right)^j \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} + |R^*| . \tag{11}$$

That is, by (11), we get

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq$$

$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} + |R^*| . \tag{12}$$

Next, we need to estimate $|R^*|$. For that, we observe $(0 \leq t_N \leq 1)$

$$\left| g^{(N)}_{\frac{\overrightarrow{k}}{n}}(t_N) - g^{(N)}_{\frac{\overrightarrow{k}}{n}}(0) \right| =$$

$$\left| \left(\sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f\left(\overrightarrow{x} + t_N \left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right) \right) - \right.$$

$$\left. \left(\sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f(\overrightarrow{x}) \right|$$

$$\leq \frac{(T^*)^N d^N}{n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Thus,

$$\left| R_N \left(\frac{\overrightarrow{k}}{n}, 0 \right) \right| \leq \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{N-1}} \left| g^{(N)}_{\frac{\overrightarrow{k}}{n}}(t_N) - g^{(N)}_{\frac{\overrightarrow{k}}{n}}(0) \right| dt_N \right) ... \right) dt_1$$

$$\leq \frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Therefore,

$$|R^*| \leq \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{\lfloor n\overrightarrow{x} + \overrightarrow{T} n^\alpha \rfloor} \frac{b\left(n^{1-\alpha} \left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right)}{V(\overrightarrow{x})} \left| R_N \left(\frac{\overrightarrow{k}}{n}, 0 \right) \right|$$

$$\leq \frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right). \tag{13}$$

By (12) and (13) we get (10). ■

Corollary 8 *Here, additionally assume that b is continuous on $\mathbb{R}^d$. Let*

$$\Gamma := \prod_{i=1}^{d} [-\gamma_i, \gamma_i] \subset \mathbb{R}^d, \ \ \gamma_i > 0,$$

and take

$$n \geq \max_{i \in \{1,\ldots,d\}} \left(T_i + \gamma_i, T_i^{-\frac{1}{\alpha}} \right).$$

Consider $p \geq 1$. Then,

$$\|M_n f - f\|_{p,\Gamma} \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{14}$$

attained by constant functions. From (14), we get the L_p convergence of $M_n f$ to f with rates.

Proof. By (8). ■

Corollary 9 *Same assumptions as in Corollary 8. Then*

$$\|M_n f - f\|_{p,\Gamma} \leq \left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left\| \left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f \right\|_{p,\Gamma} \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{15}$$

attained by constants. Here, from (15), we get again the L_p convergence of $M_n(f)$ to f with rates.

Proof. By the use of (10). ■

3 The multivariate "normalized squashing type operators" and their convergence to the unit with rates

We give the following definition

Definition 10 *Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d} (-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [3]).*

Example 11 *Let $\widehat{S}$ as above when $d=1$. Then,*

$$S\left(\overrightarrow{x}\right) := \widehat{S}\left(x_1\right)...\widehat{S}\left(x_d\right), \quad \overrightarrow{x} := \left(x_1, ..., x_d\right) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f : \mathbb{R}^d \to \mathbb{R}$ be either uniformly continuous or continuous and bounded function.

For $\overrightarrow{x} \in \mathbb{R}^d$, we define the multivariate "normalized squashing type operator",

$$L_n\left(f\right)\left(\overrightarrow{x}\right) :=$$
$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{W\left(\overrightarrow{x}\right)}, \tag{16}$$

where $0 < \alpha < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1,...,d\}} \left\{T_i + \left|x_i\right|, T_i^{-\frac{1}{\alpha}}\right\}, \tag{17}$$

and

$$W\left(\overrightarrow{x}\right) := \sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{18}$$

Obviously L_n is a positive linear operator. It is clear that

$$\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]} \frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{\Phi\left(\overrightarrow{x}\right)} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{19}$$

where

$$\Phi\left(\overrightarrow{x}\right) := \sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right). \tag{20}$$

Here, we study the pointwise convergence with rates of $\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) \to f\left(\overrightarrow{x}\right)$, as $n \to +\infty$, $\overrightarrow{x} \in \mathbb{R}^d$.

This is given by the next result.

Theorem 12 *Under the above terms and assumptions, we find that*

$$\left|\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{21}$$

Inequality (21) is attained by constant functions.

Proof. Similar to (8). ∎

We also give

Theorem 13 *Let* $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, *such that all of its partial derivatives* $f_{\widetilde{\alpha}}$ *of order* N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, *are uniformly continuous or continuous are bounded. Then,*

$$|(L_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \tag{22}$$

$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Inequality (22) is attained by constant functions. Also, (22) gives us with rates the pointwise convergence of $L_n(f) \to f$ *over* $\mathbb{R}^d$, *as* $n \to +\infty$.

Proof. Similar to (10). ∎

Note 14 *We see that*

$$M_n(1) = L_n(1) = 1.$$

References

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Fractional Approximation by normalized bell and Squashing type neural network operators

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Abstract

This article deals with the determination of the fractional rate of convergence to the unit of some neural network operators, namely, the normalized bell and "squashing" type operators. This is given through the moduli of continuity of the involved right and left Caputo fractional derivatives of the approximated function and they appear in the right-hand side of the associated Jackson type inequalities.

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Keywords and Phrases: Neural network fractional approximation, bell function, squashing function, fractional derivative, modulus of continuity.

1 Introduction

The Cardaliaguet-Euvrard operators were studied extensively in [7], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "normalized bell and squashing type operators" (see (22), (63)) were motivated and inspired by the "bell" and "squashing functions" of [7]. The work in [7] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [7], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above neural network induced operators. All involved constants there are well determined. These are pointwise, uniform and L_p, $p \geq 1$, estimates involving the first moduli of continuity of the engaged right and left

Caputo fractional derivatives of the function under approximation. We give all necessary background of fractional calculus.

Initial work of the subject was done in [1], where we involved only ordinary derivatives. Article [1] motivated the current work.

2 Background

We need

Definition 1 *Let $f \in C(\mathbb{R})$ which is bounded or uniformly continuous, $h > 0$. We define the first modulus of continuity of f at h as follows*

$$\omega_1(f,h) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, \ |x-y| \le h\} \tag{1}$$

Notice that $\omega_1(f,h)$ is finite for any $h > 0$, and

$$\lim_{h \to 0} \omega_1(f,h) = 0.$$

We also need

Definition 2 *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu \ge 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions), $\forall$ $[a,b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{2}$$

$\forall$ $x \ge a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$, $\forall$ $b > a$. We set $D_{*a}^0 f(x) = f(x)$, $\forall$ $x \in [a,\infty)$.

Lemma 3 *([5]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall$ $a \in \mathbb{R}$.*

Definition 4 *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a,b])$, $\forall$ $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{3}$$

$\forall$ $x \le b$. We set $D_{b-}^0 f(x) = f(x)$, $\forall$ $x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$, $\forall$ $a < b$.

Lemma 5 *([5]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D^\alpha_{b-} f(b) = 0$, $\forall\, b \in \mathbb{R}$.

Convention 6 *We assume that*

$$D^\alpha_{*x_0} f(x) = 0, \text{ for } x < x_0, \tag{4}$$

and

$$D^\alpha_{x_0-} f(x) = 0, \text{ for } x > x_0, \tag{5}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 7 *(by [3]) Let* $f \in C^n(\mathbb{R})$, *where* $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D^\nu_{*a} f(x)$ *is continuous in* $x \in [a, \infty)$.

Also we have

Proposition 8 *(by [3]) Let* $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D^\alpha_{b-} f(x)$ *is continuous in* $x \in (-\infty, b]$.

We further mention

Proposition 9 *(by [3]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^\alpha_{*x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{6}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D^\alpha_{*x_0} f(x)$ *is continuous in* x_0.

Proposition 10 *(by [3]) Let* $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D^\alpha_{x_0-} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{7}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D^\alpha_{x_0-} f(x)$ *is continuous in* x_0.

Proposition 11 *([5]) Let* $g \in C_b(\mathbb{R})$ *(continuous and bounded),* $0 < c < 1$, $x, x_0 \in \mathbb{R}$. *Define*

$$L(x, x_0) = \int_{x_0}^x (x-t)^{c-1} g(t)\, dt, \text{ for } x \geq x_0, \tag{8}$$

and $L(x, x_0) = 0$, *for* $x < x_0$.

Then L *is jointly continuous in* $(x, x_0) \in \mathbb{R}^2$.

We mention

Proposition 12 *([5]) Let $g \in C_b(\mathbb{R})$, $0 < c < 1$, $x, x_0 \in \mathbb{R}$. Define*

$$K(x, x_0) = \int_x^{x_0} (J - x)^{c-1} g(J)\, dJ, \quad \text{for } x \leq x_0, \tag{9}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous in $(x, x_0) \in \mathbb{R}^2$.

Based on Propositions 11, 12 we derive

Corollary 13 *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Proposition 14 *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta)_{[x, +\infty)}, \ \delta > 0, x \in \mathbb{R}. \tag{10}$$

(Here ω_1 is defined over $[x, +\infty)$ instead of $\mathbb{R}$.)

Then G is continuous on $\mathbb{R}$.

Proposition 15 *([5]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$H(x) = \omega_1(f(\cdot, x), \delta)_{(-\infty, x]}, \ \delta > 0, x \in \mathbb{R}. \tag{11}$$

(Here ω_1 is defined over $(-\infty, x]$ instead of $\mathbb{R}$.)

Then H is continuous on $\mathbb{R}$.

By Propositions 14, 15 and Corollary 13 we derive

Proposition 16 *([5]) Let $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x \in \mathbb{R}$. Then $\omega_1(D^\alpha_{*x} f, h)_{[x, +\infty)}$, $\omega_1\left(D^\alpha_{x-} f, h\right)_{(-\infty, x]}$ are continuous functions of $x \in \mathbb{R}$, $h > 0$ fixed.*

We make

Remark 17 *Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\omega_1(g, t) \leq 2 \|g\|_\infty < \infty. \tag{12}$$

Assuming that $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f\right)(t)$, are both continuous and bounded in $(x, t) \in \mathbb{R}^2$, i.e.

$$\|D^\alpha_{*x} f\|_\infty \leq K_1, \ \forall \ x \in \mathbb{R}; \tag{13}$$

$$\left\| D_{x-}^{\alpha} f \right\|_{\infty} \leq K_2, \ \forall \ x \in \mathbb{R}, \tag{14}$$

where $K_1, K_2 > 0$, *we get*

$$\omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x,+\infty)} \leq 2K_1;$$
$$\omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty,x]} \leq 2K_2, \ \forall \ \xi \geq 0, \tag{15}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \geq 0$,

$$\sup_{x \in \mathbb{R}} \left[\max \left(\omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x,+\infty)}, \omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty,x]} \right) \right] \leq 2 \max \left(K_1, K_2 \right) < \infty. \tag{16}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\| f^{(m)} \right\|_{\infty} < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, *by Corollary 13 both* $\left(D_{*x}^{\alpha} f \right)(t)$, $\left(D_{x-}^{\alpha} f \right)(t)$ *are jointly continuous in* (t, x) *on* $\mathbb{R}^2$. *Assuming further that they are both bounded on* $\mathbb{R}^2$ *we get (16) valid. In particular, each of* $\omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x,+\infty)}$, $\omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty,x]}$ *is finite for any* $\xi \geq 0$.

Clearly here we have that $\sup_{x \in \mathbb{R}} \omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x,+\infty)} \to 0$, *as* $\xi \to 0+$, *and* $\sup_{x \in \mathbb{R}} \omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty,x]} \to 0$, *as* $\xi \to 0+$.

Let us now assume only that $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x \in \mathbb{R}$. *Then, by Proposition 15.114, p. 388 of [4], we find that* $D_{*x}^{\alpha} f \in C([x, +\infty))$, *and by [6] we obtain that* $D_{x-}^{\alpha} f \in C((-\infty, x])$.

We make

Remark 18 *Again let* $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1$, $\forall$ $x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. *Notice* $0 < m - \alpha < 1$. *Then*

$$D_{*x_0}^{\alpha} f(x) = \frac{(x - x_0)^{m-\alpha}}{\Gamma(m - \alpha + 1)}, \ \forall \ x \geq x_0. \tag{17}$$

Let us consider $x, y \geq x_0$, then

$$\left| D_{*x_0}^{\alpha} f(x) - D_{*x_0}^{\alpha} f(y) \right| = \frac{1}{\Gamma(m - \alpha + 1)} \left| (x - x_0)^{m-\alpha} - (y - x_0)^{m-\alpha} \right|$$
$$\leq \frac{|x - y|^{m-\alpha}}{\Gamma(m - \alpha + 1)}. \tag{18}$$

So it is not strange to assume that

$$\left| D_{*x_0}^{\alpha} f(x_1) - D_{*x_0}^{\alpha} f(x_2) \right| \leq K |x_1 - x_2|^{\beta}, \tag{19}$$

$K > 0$, $0 < \beta \leq 1$, $\forall$ $x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, where more generally it is $\left\| f^{(m)} \right\|_{\infty} < \infty$. Thus, one may assume

$$\omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty,x]} \leq M_1 \xi^{\beta_1}, \text{ and} \tag{20}$$

$$\omega_1\left(D_{*x}^{\alpha}f,\xi\right)_{[x,+\infty)} \leq M_2\xi^{\beta_2},$$

where $0<\beta_1,\beta_2\leq 1$, $\forall\ \xi>0$, $M_1,M_2>0$; any $x\in\mathbb{R}$.

Setting $\beta=\min\left(\beta_1,\beta_2\right)$ and $M=\max\left(M_1,M_2\right)$, in that case we obtain

$$\sup_{x\in\mathbb{R}}\left\{\max\left(\omega_1\left(D_{x-}^{\alpha}f,\xi\right)_{(-\infty,x]},\omega_1\left(D_{*x}^{\alpha}f,\xi\right)_{[x,+\infty)}\right)\right\}\leq M\xi^{\beta}\to 0,\text{ as }\xi\to 0+. \tag{21}$$

3 Results

3.1 Fractional convergence with rates of the normalized bell type neural network operators

We need the following (see [7]).

Definition 19 *A function $b:\mathbb{R}\to\mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty,a)$ and nonincreasing on $[a,+\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T,T]$, $T>0$.*

We follow [1], [7].

Example 20 (1) *$b(x)$ can be the characteristic function over $[-1,1]$.*

(2) *$b(x)$ can be the hat function over $[-1,1]$, i.e.,*

$$b(x)=\begin{cases}1+x, & -1\leq x\leq 0,\\ 1-x, & 0<x\leq 1\\ 0, & \textit{elsewhere.}\end{cases}$$

These are centered bell-shaped functions of compact support.

Here we consider functions $f:\mathbb{R}\to\mathbb{R}$ that are continuous.

In this article we study the fractional convergence with rates over the real line, to the unit operator, of the "normalized bell type neural network operators",

$$\left(H_n(f)\right)(x):=\frac{\sum_{k=-n^2}^{n^2}f\left(\frac{k}{n}\right)\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)}, \tag{22}$$

where $0<\alpha<1$ and $x\in\mathbb{R}$, $n\in\mathbb{N}$. The terms in the ratio of sums (22) can be nonzero iff

$$\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right|\leq T,\ \text{ i.e. }\left|x-\frac{k}{n}\right|\leq\frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{23}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{24}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{25}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (24).

Proposition 21 *Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a) - 1) \leq card(k) \leq (b-a) + 1. \tag{26}$$

Remark 22 *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 21 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \text{iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (24) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{27}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number.

We make

Remark 23 *Clearly we have that*

$$nx - Tn^{\alpha} \leq nx \leq nx + Tn^{\alpha}. \tag{28}$$

We prove in general that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{29}$$

Indeed we have that, if $[nx] < nx - Tn^{\alpha}$, then $[nx] + Tn^{\alpha} < nx$, and $[nx] + [Tn^{\alpha}] \leq [nx]$, resulting into $[Tn^{\alpha}] = 0$, which for large enough n is not true. Therefore $nx - Tn^{\alpha} \leq [nx]$. Similarly, if $\lceil nx \rceil > nx + Tn^{\alpha}$, then $nx + Tn^{\alpha} \geq$

$nx + [Tn^{\alpha}]$, and $\lceil nx \rceil - [Tn^{\alpha}] > nx$, thus $\lceil nx \rceil - [Tn^{\alpha}] \geq \lceil nx \rceil$, resulting into $[Tn^{\alpha}] = 0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{30}$$

Hence $\lceil nx - Tn^{\alpha} \rceil \leq [nx]$ and $\lceil nx \rceil \leq [nx + Tn^{\alpha}]$. Also if $[nx] \neq \lceil nx \rceil$, then $\lceil nx \rceil = [nx] + 1$. If $[nx] = \lceil nx \rceil$, then $nx \in \mathbb{Z}$; and by assuming $n \geq T^{-\frac{1}{\alpha}}$, we get $Tn^{\alpha} \geq 1$ and $nx + Tn^{\alpha} \geq nx + 1$, so that $[nx + Tn^{\alpha}] \geq nx + 1 = [nx] + 1$.

We present our first main result

Theorem 24 *We consider $f : \mathbb{R} \to \mathbb{R}$. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $\forall$ $[a,b] \subset \mathbb{R}$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$. Let also $x \in \mathbb{R}$, $T > 0$, $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f$, $D_{x-}^{\beta} f$ are uniformly continuous functions or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively.*

Then

1)

$$|H_n(f)(x) - f(x)| \leq \left(\sum_{j=1}^{N-1} \frac{\left| f^{(j)}(x) \right| T^j}{j! n^{(1-\alpha)j}} \right) + \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{31}$$

$$\left\{ \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

above $\sum_{j=1}^{0} \cdot = 0$,

2)

$$\left| (H_n(f))(x) - \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(H_n\left((\cdot - x)^j \right) \right)(x) \right| \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{32}$$

$$\left\{ \omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\} =: \lambda_n(x),$$

3) assume further that $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we get

$$|H_n(f)(x) - f(x)| \leq \lambda_n(x), \tag{33}$$

4) in case of $N = 1$, we obtain again

$$|H_n(f)(x) - f(x)| \leq \lambda_n(x). \tag{34}$$

Here we get fractionally with rates the pointwise convergence of $(H_n(f))(x) \to f(x)$, as $n \to \infty$, $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. We have that

$$D_{x-}^{\beta} f(x) = D_{*x}^{\beta} f(x) = 0. \tag{35}$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{36}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

for all $x \leq \frac{k}{n} \leq x + Tn^{\alpha-1}$, iff $\lceil nx \rceil \leq k \leq [nx + Tn^{\alpha}]$, where $k \in \mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \tag{37}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ,$$

for all $x - Tn^{\alpha-1} \leq \frac{k}{n} \leq x$, iff $\lceil nx - Tn^{\alpha} \rceil \leq k \leq [nx]$, where $k \in \mathbb{Z}$.

Notice that $\lceil nx \rceil \leq [nx] + 1$.

Call

$$V(x) := \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} + \tag{38}$$

$$\frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} + \tag{39}$$

$$\frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} = \tag{40}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right)+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}= \tag{41}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\left(\frac{k}{n}-x\right)^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}+$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ.$$

We notice here that

$$\left(H_{n}(f)\right)(x):=\frac{\sum_{k=-n^{2}}^{n^{2}}f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^{2}}^{n^{2}}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{42}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}f\left(\frac{k}{n}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}.$$

Adding the two equalities (40) and (41) we obtain

$$\left(H_{n}(f)\right)(x)=$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right)+\theta_{n}(x), \tag{43}$$

where

$$\theta_{n}(x):=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \tag{44}$$

We call

$$\theta_{1n}(x):=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ, \tag{45}$$

and

$$\theta_{2n}(x):=\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)\Gamma(\beta)}\cdot$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \tag{46}$$

I.e.

$$\theta_n(x)=\theta_{1n}(x)+\theta_{2n}(x). \tag{47}$$

We further have

$$(H_n(f))(x)-f(x)= \tag{48}$$

$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right)+\theta_n(x),$$

and

$$|(H_n(f))(x)-f(x)|\leq$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left|x-\frac{k}{n}\right|^{j}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right)+|\theta_n(x)|\leq \tag{49}$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{T^{j}}{n^{(1-\alpha)j}}+|\theta_n(x)|=:(*).$$

Next we see that

$$\gamma_{1n}:=\frac{1}{\Gamma(\beta)}\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ\right|\leq \tag{50}$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left|D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\,dJ\right|\leq$$

$$\frac{1}{\Gamma(\beta)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\omega_1\left(D_{x-}^{\beta}f,|J-x|\right)_{(-\infty,x]}dJ\leq$$

$$\frac{1}{\Gamma(\beta)}\omega_1\left(D_{x-}^{\beta}f,\left|x-\frac{k}{n}\right|\right)_{(-\infty,x]}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}dJ\leq$$

$$\frac{1}{\Gamma(\beta)}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\frac{\left(x-\frac{k}{n}\right)^{\beta}}{\beta}\leq$$

$$\frac{1}{\Gamma(\beta+1)}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is
$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}. \tag{51}$$

Furthermore
$$|\theta_{1n}(x)| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \gamma_{1n} \leq \tag{52}$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \leq$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} =$$

$$\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}.$$

So that
$$|\theta_{1n}(x)| \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}. \tag{53}$$

Similarly we derive
$$\gamma_{2n} := \frac{1}{\Gamma(\beta)} \left| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right) dJ \right| \leq \tag{54}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta-1} \left| D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right| dJ \leq$$

$$\frac{\omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}}{\Gamma(\beta+1)} \left(\frac{k}{n} - x \right)^{\beta} \leq$$

$$\frac{\omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}}{\Gamma(\beta+1)} \frac{T^{\beta}}{n^{(1-\alpha)\beta}}.$$

That is
$$\gamma_{2n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}. \tag{55}$$

Consequently we find
$$|\theta_{2n}(x)| \leq \left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \right).$$

$$\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\leq \tag{56}$$
$$\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}.$$

So we have proved that

$$|\theta_n(x)|\leq\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}. \tag{57}$$
$$\left\{\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Combining (49) and (57) we have (31). ■

As an application of Theorem 24 we give

Theorem 25 *Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^{N}(\mathbb{R})$, with $f^{(N)}\in L_{\infty}(\mathbb{R})$. Let also $T>0$, $n\in\mathbb{N}:n\geq\max\left(2T,T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta}f(t)$, $D_{x-}^{\beta}f(t)$ are both bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|H_n(f)-f\|_{\infty,[-T,T]}\leq \tag{58}$$
$$\left(\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty,[-T,T]}T^{j}}{j!n^{(1-\alpha)j}}\right)+\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}.$$
$$\left\{\sup_{x\in[-T,T]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T,T]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) in case of $N=1$, we obtain

$$\|H_n(f)-f\|_{\infty,[-T,T]}\leq\frac{T^{\beta}}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}. \tag{59}$$
$$\left\{\sup_{x\in[-T,T]}\omega_1\left(D_{*x}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T,T]}\omega_1\left(D_{x-}^{\beta}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

An interesting case is when $\beta=\frac{1}{2}$.

Here we get fractionally with rates the uniform convergence of $H_n(f)\to f$, as $n\to\infty$.

Proof. From (31), (34) of Theorem 24, and by Remark 17. ■

One can also apply Remark 18 to the last Theorem 25, to get interesting and simplified results.

We make

Remark 26 *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T>0$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}: n\geq \max\left(T+T^*, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$. Consider $p\geq 1$.*

Let also $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, $f^{(N)}\in L_\infty(\mathbb{R})$. Here both $D^{\alpha}_{*x}f(t)$, $D^{\alpha}_{x-}f(t)$ are bounded in $(x,t)\in\mathbb{R}^2$.

By Theorem 24 we have

$$|H_n(f)(x)-f(x)|\leq \tag{60}$$
$$\left(\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}\right)+\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$
$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Applying to the last inequality (60) the monotonicity and subadditive property of $\|\cdot\|_p$, we derive the following L_p, $p\geq 1$, interesting result.

Theorem 27 *Let $b(x)$ be a centered bell-shaped continuous function on $\mathbb{R}$ of compact support $[-T,T]$, $T>0$. Let $x\in[-T^*,T^*]$, $T^*>0$, and $n\in\mathbb{N}$: $n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $p\geq 1$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C^N(\mathbb{R})$, with $f^{(N)}\in L_\infty(\mathbb{R})$. Here both $D^{\beta}_{*x}f(t)$, $D^{\beta}_{x-}f(t)$ are bounded in $(x,t)\in\mathbb{R}^2$. Then*

1)

$$\|H_nf-f\|_{p,[-T^*,T^*]}\leq \tag{61}$$
$$\left(\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{p,[-T^*,T^*]}T^j}{j!n^{(1-\alpha)j}}\right)+2^{\frac{1}{p}}T^{*\frac{1}{p}}\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot$$
$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

2) When $N=1$, we derive

$$\|H_nf-f\|_{p,[-T^*,T^*]}\leq 2^{\frac{1}{p}}T^{*\frac{1}{p}}\frac{T^\beta}{\Gamma(\beta+1)\,n^{(1-\alpha)\beta}}\cdot \tag{62}$$
$$\left\{\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{*x}f,\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\sup_{x\in[-T^*,T^*]}\omega_1\left(D^{\beta}_{x-}f,\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

By (61), (62) we derive the fractional L_p, $p\geq 1$, convergence with rates of H_nf to f.

3.2 The "normalized Squashing type operators" and their fractional convergence to the unit with rates

We need (see also [1], [7]).

Definition 28 *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support $[-T, T]$, $T > 0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T, T]$. S can have jump discontinuities. We call S the "squashing function".*

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

For $x \in \mathbb{R}$ we define the "normalized squashing type operator"

$$(K_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}, \tag{63}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. It is clear that

$$(K_n(f))(x) = \frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)}. \tag{64}$$

Here we study the fractional convergence with rates of $(K_n f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbb{R}$.

We present our second main result

Theorem 29 *We consider $f : \mathbb{R} \to \mathbb{R}$. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a, b])$, $\forall$ $[a, b] \subset \mathbb{R}$, with $f^{(N)} \in L_{\infty}(\mathbb{R})$. Let also $x \in \mathbb{R}, T > 0, n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta} f$, $D_{x-}^{\beta} f$ are uniformly continuous functions or continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively.*

Then

1)

$$|K_n(f)(x) - f(x)| \leq \tag{65}$$

$$\left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right) + \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot$$

$$\left\{\omega_1\left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

above $\sum_{j=1}^{0} \cdot = 0$,

2)

$$\left|\left(K_n(f)\right)(x) - \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(K_n\left((\cdot - x)^j\right)\right)(x)\right| \leq \frac{T^\beta}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot \tag{66}$$

$$\left\{\omega_1\left(D^\beta_{*x} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega_1\left(D^\beta_{x-} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\} =: \lambda^*_n(x),$$

3) assume further that $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we get

$$\left|K_n(f)(x) - f(x)\right| \leq \lambda^*_n(x), \tag{67}$$

4) in case of $N = 1$, we obtain also

$$\left|K_n(f)(x) - f(x)\right| \leq \lambda^*_n(x). \tag{68}$$

Here we get fractionally with rates the pointwise convergence of $(K_n(f))(x) \to f(x)$, as $n \to \infty$, $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. We have that

$$D^\beta_{x-} f(x) = D^\beta_{*x} f(x) = 0.$$

From [8], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \tag{69}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left(D^\beta_{*x} f(J) - D^\beta_{*x} f(x)\right) dJ,$$

for all $x \leq \frac{k}{n} \leq x + Tn^{\alpha-1}$, iff $\lceil nx \rceil \leq k \leq [nx + Tn^\alpha]$, where $k \in \mathbb{Z}$.

Also from [2], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} - x\right)^j + \tag{70}$$

$$\frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^x \left(J - \frac{k}{n}\right)^{\beta-1} \left(D^\beta_{x-} f(J) - D^\beta_{x-} f(x)\right) dJ,$$

for all $x - Tn^{\alpha-1} \leq \frac{k}{n} \leq x$, iff $\lceil nx - Tn^\alpha \rceil \leq k \leq [nx]$, where $k \in \mathbb{Z}$.

Call

$$W(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n}-x\right)^{j} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} + \tag{71}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n}-x\right)^{j} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} + \tag{72}$$

$$\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

Therefore we obtain

$$\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} = \tag{73}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)}\right) +$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ,$$

and

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} = \tag{74}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} \left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)} +$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \int_{\frac{k}{n}}^{x} \left(J-\frac{k}{n}\right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ.$$

Adding the two equalities (73) and (74) we obtain

$$(K_n(f))(x) =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)}\right) + M_n(x), \tag{75}$$

where

$$M_n(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ+$$

$$\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \quad (76)$$

We call

$$M_{1n}(x) := \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \cdot$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ, \quad (77)$$

and

$$M_{2n}(x) := \sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]}\frac{S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)\Gamma(\beta)} \cdot$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ. \quad (78)$$

I.e.

$$M_n(x) = M_{1n}(x) + M_{2n}(x). \quad (79)$$

We further have

$$(K_n(f))(x) - f(x) = \quad (80)$$

$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\frac{k}{n}-x\right)^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)}\right) + M_n(x),$$

and

$$|(K_n(f))(x) - f(x)| \leq$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left|x-\frac{k}{n}\right|^{j} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)}\right) + |M_n(x)| \leq$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{T^{j}}{n^{(1-\alpha)j}}\left(\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{W(x)}\right) + |M_n(x)| =: (*). \quad (81)$$

Therefore we obtain

$$|(K_n(f))(x) - f(x)| \leq \left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}} \right) + |M_n(x)|. \tag{82}$$

We call

$$\gamma_{1n} := \frac{1}{\Gamma(\beta)} \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x) \right) dJ \right|. \tag{83}$$

As in the proof of Theorem 24 we have

$$\gamma_{1n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}. \tag{84}$$

Furthermore

$$|M_{1n}(x)| \leq \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{W(x)} \gamma_{1n} \leq \tag{85}$$

$$\left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx]} \frac{S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{W(x)} \right) \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \leq$$

$$\frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}.$$

So that

$$|M_{1n}(x)| \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]}. \tag{86}$$

We also call

$$\gamma_{2n} := \frac{1}{\Gamma(\beta)} \left| \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right) dJ \right|. \tag{87}$$

As in the proof of Theorem 24 we get

$$\gamma_{2n} \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}. \tag{88}$$

Consequently we find

$$|M_{2n}(x)| \leq \left(\sum_{k=[nx]+1}^{[nx+Tn^{\alpha}]} \frac{S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{W(x)} \right).$$

$$\frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} \leq \tag{89}$$

$$\frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)}.$$

So we have proved that

$$|M_n(x)| \leq \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot \tag{90}$$

$$\left\{ \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1 \left(D^{\beta}_{x-} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

Combining (82) and (90) we have (65). ∎

As an application of Theorem 29 we give

Theorem 30 *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(N)} \in L_\infty(\mathbb{R})$. Let also $T > 0$, $n \in \mathbb{N} : n \geq \max\left(2T, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D^{\beta}_{*x} f(t)$, $D^{\beta}_{x-} f(t)$ are both bounded in $(x,t) \in \mathbb{R}^2$. Then*

1)

$$\|K_n(f) - f\|_{\infty,[-T,T]} \leq \tag{91}$$

$$\left(\sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty,[-T,T]} T^j}{j!\, n^{(1-\alpha)j}} \right) + \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{x-} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

2) in case of $N = 1$, we obtain

$$\|K_n(f) - f\|_{\infty,[-T,T]} \leq \frac{T^{\beta}}{\Gamma(\beta+1)\, n^{(1-\alpha)\beta}} \cdot \tag{92}$$

$$\left\{ \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{*x} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \sup_{x\in[-T,T]} \omega_1 \left(D^{\beta}_{x-} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\}.$$

An interesting case is when $\beta = \frac{1}{2}$.

Here we get fractionally with rates the uniform convergence of $K_n(f) \to f$, as $n \to \infty$.

Proof. From (65), (68) of Theorem 29, and by Remark 17. ∎

One can also apply Remark 18 to the last Theorem 30, to get interesting and simplified results.

Note 31 *The maps H_n, K_n, $n \in \mathbb{N}$, are positive linear operators reproducing constants.*

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Voronovskaya type asymptotic expansions for multivariate quasi-interpolation neural network operators

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Abstract

Here we study further the multivariate quasi-interpolation of sigmoidal and hyperbolic tangent types neural network operators of one hidden layer. We derive multivariate Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1 Background

Here we follow [5], [6].

We consider here the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R},\ i = 1, ..., N;\ x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [7], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R},\ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0, \ \forall \, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1, \ \forall \, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \ \forall \, x_i \in \mathbb{R}; \, n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i) \, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i), \ x_i \geq 0$, for $i = 1, ..., N$.

We see that ([7])

$$\Phi_i(x_i) = \left(\frac{e^2 - 1}{2e^2}\right) \frac{1}{(1 + e^{x_i - 1})(1 + e^{-x_i - 1})}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [6] we get

viii)

$$\sum_{\begin{cases} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{cases}}^{\infty} \Phi_i(nx_i - k_i) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

As in [6] we obtain

ix)
$$0 < \frac{1}{\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)} < \frac{1}{\Phi_i(1)} = 5.250312578,$$

$\forall \, x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [6], we see that

$$\lim_{n \to \infty} \sum_{k_i = \lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \neq 1,$$

for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We will use here

$$\Phi(x_1, ..., x_N) := \Phi(x) := \prod_{i=1}^{N} \Phi_i(x_i), \quad x \in \mathbb{R}^N. \tag{1}$$

It has the properties:

(i)' $\Phi(x) > 0$, $\forall\ x \in \mathbb{R}^N$,

We see that

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1 - k_1, x_2 - k_2, ..., x_N - k_N) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} \Phi_i(x_i - k_i) = \prod_{i=1}^{N} \left(\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) \right) = 1.$$

That is

(ii)'

$$\sum_{k=-\infty}^{\infty} \Phi(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1 - k_1, ..., x_N - k_N) = 1,$$

$k := (k_1, ..., k_N)$, $\forall\ x \in \mathbb{R}^N$.

(iii)'

$$\sum_{k=-\infty}^{\infty} \Phi(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(nx_1 - k_1, ..., nx_N - k_N) = 1,$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi(x)\, dx = 1,$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil),$$
$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor).$$

For $0<\beta<1$ and $n\in\mathbb{N}$, fixed $x\in\mathbb{R}^N$, have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)=$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)+\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k).$$

In the last two sums the counting is over disjoint vector of k's, because the condition $\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^\beta}$, $r\in\{1,...,N\}$.

It holds

(v)'

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Phi(nx-k)\leq 3.1992e^{-n^{(1-\beta)}},$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$.

Furthermore it holds

(vi)'

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}<(5.250312578)^N,$$

$\forall\ x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$, $n\in\mathbb{N}$.

It is clear also that

(vii)'

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\infty}\Phi(nx-k)\leq 3.1992e^{-n^{(1-\beta)}},$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$.

By (x) we obviously see that

(viii)'

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\neq 1$$

for at least some $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i=1,...,N$.

We define the multivariate positive linear neural network operator ($x := (x_1,...,x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$$G_n(f,x_1,...,x_N) := G_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Phi(nx-k)} \tag{2}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor}\sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor}\cdots\sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi_i(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\Phi_i(nx_i-k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i=1,...,N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i=1,...,N$.

Notice here that for large enough $n \in \mathbb{N}$ we get that

$$e^{-n^{(1-\beta)}} < n^{-\beta j}, \;\; j=1,...,m \in \mathbb{N},\; 0<\beta<1.$$

Thus be given fixed $A, B > 0$, for the linear combination $\left(An^{-\beta j} + Be^{-n^{(1-\beta)}}\right)$ the (dominant) rate of convergence to zero is $n^{-\beta j}$. The closer β is to 1 we get faster and better rate of convergence to zero.

By $AC^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N \in \mathbb{N}$, we denote the space of functions such that all partial derivatives of order $(m-1)$ are coordinatewise alsolutely continuous functions, also $f \in C^{m-1}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f \in AC^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i=1,...,N$, and $|\alpha| := \sum_{i=1}^{N}\alpha_i = l$, where $l=0,1,...,m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is order l.

We denote

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m}\{\|f_\alpha\|_\infty\}, \tag{3}$$

where $\|\cdot\|_\infty$ is the supremum norm.

We assume here that $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Next we follow [3], [4].

We consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}(\tanh(x+1) - \tanh(x-1)) > 0, \quad \forall\, x \in \mathbb{R}.$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 1 *([3]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty}\Psi(x) = 0 = \lim_{x\to\infty}\Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 2 *([3]) We have that $\sum_{i=-\infty}^{\infty}\Psi(x-i) = 1,\ \forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty}\Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty}\Psi(x+i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 3 *([3]) It holds $\int_{-\infty}^{\infty}\Psi(x)\,dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 4 *([3]) Let $0 < \alpha < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 5 *([3]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [3] we get that

$$\lim_{n\to\infty}\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

In this article we will use

$$\Theta(x_1, ..., x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{4}$$

It has the properties:

(i)* $\Theta(x) > 0, \ \forall \, x \in \mathbb{R}^N$,

(ii)*

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1 - k_1, ..., x_N - k_N) = 1,$$

where $k := (k_1, ..., k_N)$, $\forall \, x \in \mathbb{R}^N$.

(iii)*

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(nx_1 - k_1, ..., nx_N - k_N) = 1,$$

$\forall \, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)*

$$\int_{\mathbb{R}^N} \Theta(x) \, dx = 1,$$

that is Θ is a multivariate density function.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \prod_{i=1}^{N} \Psi(nx_i - k_i) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \prod_{i=1}^{N} \Psi(nx_i - k_i) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i) \right).$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \Theta(nx-k).$$

In the last two sums the counting is over disjoint vector of k's, because the condition $\left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^{\beta}}$, $r \in \{1, ..., N\}$.

Il holds

(v)*

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \Theta\left(nx - k\right) \leq e^{4} \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Also it holds

(vi)*

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta\left(nx - k\right)} < \frac{1}{\left(\Psi\left(1\right)\right)^{N}} = \left(4.1488766\right)^{N},$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is clear that

(vii)*

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \Theta\left(nx - k\right) \leq e^{4} \cdot e^{-2n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^{N}$.

Also we get

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta\left(nx - k\right) \neq 1,$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i] \right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We define the multivariate positive linear neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$)

$$F_n\left(f, x_1, ..., x_N\right) := F_n\left(f, x\right) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta\left(nx - k\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta\left(nx - k\right)} \tag{5}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \ldots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i)\right)}.$$

Our considered neural networks here are of one hidden layer.

In this article we find Voronovskaya type asymptotic expansions for the above described neural networks quasi-interpolation normalized operators $G_n(f,x)$, $F_n(f,x)$, where $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$ is fixed but arbitrary. For other neural networks related work, see [2], [3], [4], [5], [6] and [7]. For neural networks in general, see [8], [9] and [10].

Next we follow [1], pp. 284-286.

About Taylor formula -Multivariate Case and Estimates

Let Q be a compact convex subset of $\mathbb{R}^N$; $N \geq 2$; $z := (z_1, \ldots, z_N)$, $x_0 := (x_{01}, \ldots, x_{0N}) \in Q$.

Let $f : Q \to \mathbb{R}$ be such that all partial derivatives of order $(m-1)$ are coordinatewise absolutely continuous functions, $m \in \mathbb{N}$. Also $f \in C^{m-1}(Q)$. That is $f \in AC^m(Q)$. Each m^{th} order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$ and $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right] (x_{01} + t(z_1 - x_{01}), \ldots, x_{0N} + t(z_N - x_{0N})), \tag{6}$$

for all $j = 0, 1, 2, \ldots, m$.

Example 6 *Let $m = N = 2$. Then*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in \mathbb{R},$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \tag{7}$$

Setting

$$(*) = (x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})) = (x_0 + t(z - x_0)),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) +$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*). \tag{8}$$

Similarly, we have the general case of $m, N \in \mathbb{N}$ for $g_z^{(m)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 7 *Under the above assumptions we have*

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + R_m(z, 0), \tag{9}$$

where

$$R_m(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{m-1}} g_z^{(m)}(t_m)\, dt_m \right) ... \right) dt_1, \tag{10}$$

or

$$R_m(z, 0) = \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \tag{11}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 8 *Assume here that*

$$\|f_\alpha\|_{\infty,Q,m}^{\max} := \max_{|\alpha|=m} \|f_\alpha\|_{\infty,Q} < \infty.$$

Then

$$\left\| g_z^{(m)} \right\|_{\infty,[0,1]} = \left\| \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^m f \right] (x_0 + t(z - x_0)) \right\|_{\infty,[0,1]} \leq \tag{12}$$

$$\left(\sum_{i=1}^{N} |z_i - x_{0i}| \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max},$$

that is

$$\left\| g_z^{(m)} \right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max} < \infty. \tag{13}$$

Hence we get by (11) that

$$|R_m(z, 0)| \leq \frac{\left\| g_z^{(m)} \right\|_{\infty,[0,1]}}{m!} < \infty. \tag{14}$$

And it holds

$$|R_m(z, 0)| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,Q,m}^{\max}, \tag{15}$$

$\forall\ z, x_0 \in Q$.

Inequality (15) will be an important tool in proving our main results.

2 Main Results

We present our first main result

Theorem 9 *Let* $0 < \beta < 1$, $x \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$ *large enough,* $f \in AC^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$. *Assume further that* $\|f_\alpha\|_{\infty,m}^{\max} < \infty$. *Then*

$$G_n(f,x) - f(x) =$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)G_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right), \tag{16}$$

where $0 < \varepsilon \leq m$.

If $m = 1$, *the sum in (16) collapses.*

The last (16) implies that

$$n^{\beta(m-\varepsilon)}\left[G_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)G_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right] \tag{17}$$

$$\to 0, \quad as\ n \to \infty,\ 0 < \varepsilon \leq m.$$

When $m = 1$, *or* $f_\alpha(x) = 0$, *for* $|\alpha| = j$, $j = 1, ..., m-1$, *then we derive that*

$$n^{\beta(m-\varepsilon)}\left[G_n(f,x) - f(x)\right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq m$.

Proof. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \prod_{i=1}^{N}[a_i, b_i]$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{18}$$

for all $j = 0, 1, ..., m$.

By (9) we have the multivariate Taylor's formula

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1}\frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!}\int_0^1(1-\theta)^{m-1}g_z^{(m)}(\theta)\,d\theta. \tag{19}$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, ..., m-1$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=j}}\left(\frac{j!}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(z_i - x_{0i})^{\alpha_i}\right)f_\alpha(x_0). \tag{20}$$

Furthermore

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i}\right) f_\alpha\left(x_0 + \theta\left(z - x_0\right)\right), \tag{21}$$

$0 \le \theta \le 1$.

So we treat $f \in AC^m\left(\prod_{i=1}^N [a_i, b_i]\right)$ with $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Thus, by (19) we have for $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ that

$$f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) - f(x) = \sum_{j=1}^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i\right)^{\alpha_i}\right) f_\alpha(x) + R, \tag{22}$$

where

$$R := m\int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right) \cdot$$

$$\left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i\right)^{\alpha_i}\right) f_\alpha\left(x + \theta\left(\frac{k}{n} - x\right)\right) d\theta. \tag{23}$$

By (15) we obtain

$$|R| \le \frac{\left(\left\|x - \frac{k}{n}\right\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{24}$$

Notice here that

$$\left\|\frac{k}{n} - x\right\|_\infty \le \frac{1}{n^\beta} \Leftrightarrow \left|\frac{k_i}{n} - x_i\right| \le \frac{1}{n^\beta},\ i = 1, \ldots, N. \tag{25}$$

So, if $\left\|\frac{k}{n} - x\right\|_\infty \le \frac{1}{n^\beta}$ we get that $\left\|x - \frac{k}{n}\right\|_{l_1} \le \frac{N}{n^\beta}$, and

$$|R| \le \frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{26}$$

Also we see that

$$\left\|x - \frac{k}{n}\right\|_{l_1} = \sum_{i=1}^N \left|x_i - \frac{k_i}{n}\right| \le \sum_{i=1}^N (b_i - a_i) = \|b - a\|_{l_1},$$

therefore in general it holds

$$|R| \leq \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max} . \tag{27}$$

Call

$$V(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) .$$

Hence we have

$$U_n(x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)} = \tag{28}$$

$$\frac{\sum_{\left\{\begin{array}{c} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{array}\right.}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)} + \frac{\sum_{\left\{\begin{array}{c} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{array}\right.}^{\lfloor nb \rfloor} \Phi(nx-k) R}{V(x)} .$$

Consequently we obtain

$$|U_n(x)| \leq \left(\frac{\sum_{\left\{\begin{array}{c} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{array}\right.}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \right) \left(\frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} \right) +$$

$$\frac{1}{V(x)} \left(\sum_{\left\{\begin{array}{c} k=\lceil na \rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{array}\right.}^{\lfloor nb \rfloor} \Phi(nx-k) \right) \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max} \overset{\text{(by (v)', (vi)')}}{\leq}$$

$$\frac{N^m}{n^{m\beta} m!} \|f_\alpha\|_{\infty,m}^{\max} + (5.250312578)^N (3.1992) e^{-n^{(1-\beta)}} \frac{\left(\|b-a\|_{l_1}\right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max} . \tag{29}$$

Therefore we have found

$$|U_n(x)| \leq \frac{\|f_\alpha\|_{\infty,m}^{\max}}{m!} \left\{ \frac{N^m}{n^{m\beta}} + (5.250312578)^N (3.1992) e^{-n^{(1-\beta)}} \left(\|b-a\|_{l_1}\right)^m \right\} . \tag{30}$$

For large enough $n \in \mathbb{N}$ we get

$$|U_n(x)| \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \left(\frac{1}{n^{m\beta}} \right) . \tag{31}$$

That is
$$|U_n(x)| = O\left(\frac{1}{n^{m\beta}}\right), \tag{32}$$
and
$$|U_n(x)| = o(1). \tag{33}$$
And, letting $0 < \varepsilon \leq m$, we derive
$$\frac{|U_n(x)|}{\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right)} \leq \left(\frac{2\|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}\right)\frac{1}{n^{\beta\varepsilon}} \to 0, \tag{34}$$
as $n \to \infty$.

I.e.
$$|U_n(x)| = o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right). \tag{35}$$
By (22) we observe that
$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} - f(x) =$$
$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\right)\right)\frac{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\prod_{i=1}^N\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)}{V(x)} + \tag{36}$$
$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)R}{V(x)}.$$
The last says
$$G_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\right)G_n\left(\prod_{i=1}^N(\cdot - x_i)^{\alpha_i}, x\right)\right) = U_n(x). \tag{37}$$
The proof of the theorem is complete. ■

We present our second main result

Theorem 10 *Let $0 < \beta < 1$, $x \in \prod_{i=1}^N [a_i,b_i]$, $n \in \mathbb{N}$ large enough, $f \in AC^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $m, N \in \mathbb{N}$. Assume further that $\|f_\alpha\|_{\infty,m}^{\max} < \infty$. Then*
$$F_n(f,x) - f(x) =$$
$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\right)F_n\left(\prod_{i=1}^N(\cdot - x_i)^{\alpha_i}, x\right)\right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right), \tag{38}$$
where $0 < \varepsilon \leq m$.

If $m = 1$, the sum in (38) collapses.

The last (38) implies that

$$n^{\beta(m-\varepsilon)}\left[F_n(f,x)-f(x)-\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!}\right)F_n\left(\prod_{i=1}^N(\cdot-x_i)^{\alpha_i},x\right)\right)\right] \tag{39}$$

$$\to 0, \quad as\ n\to\infty,\ 0<\varepsilon\le m.$$

When $m=1$, *or* $f_\alpha(x)=0$, *for* $|\alpha|=j$, $j=1,...,m-1$, *then we derive that*

$$n^{\beta(m-\varepsilon)}\left[F_n(f,x)-f(x)\right]\to 0,$$

as $n\to\infty$, $0<\varepsilon\le m$.

Proof. Similar to Theorem 9, using the properties of $\Theta(x)$, see (4), (i)*-(vii)* and (5). ∎

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Fractional Voronovskaya type asymptotic expansions for quasi-interpolation neural network operators

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Abstract

Here we study further the quasi-interpolation of sigmoidal and hyperbolic tangent types neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1 Background

We need

Definition 1 *Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [13], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Definition 2 *(see also [3], [14], [15]). Let* $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *The right Caputo fractional derivative of order* $\alpha > 0$ *is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{2}$$

$\forall\ x \in [a,b]$. *We set* $D_{b-}^{0} f(x) = f(x)$. *Notice* $D_{b-}^{\alpha} f \in L_1([a,b])$ *and* $D_{b-}^{\alpha} f$ *exists a.e.on* $[a,b]$.

Convention 3 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \quad \text{for } x < x_0, \tag{3}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{4}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 4 *(by [5]) Let* $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a,b]$.

Also we have

Proposition 5 *(by [5]) Let* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(x)$ *is continuous in* $x \in [a,b]$.

Theorem 6 *([5]) Let* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. *Then* $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a,b]^2$ *into* $\mathbb{R}$.

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 7 *([13], p. 54) Let* $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\, dJ, \tag{5}$$

$\forall\ x \geq x_0;\ x, x_0 \in [a,b]$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 8 *([3]) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_x^{x_0} (J - x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\, dJ, \quad (6)$$

$\forall\ x \leq x_0;\ x, x_0 \in [a,b]$.

For more on fractional calculus related to this work see [2], [4] and [7].

We consider here the sigmoidal function of logarithmic type

$$s(x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x \to +\infty} s(x) = 1$ and $\lim_{x \to -\infty} s(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks.

As in [12], we consider

$$\Phi(x) := \frac{1}{2}\left(s(x+1) - s(x-1)\right), \quad x \in \mathbb{R}. \quad (7)$$

We notice the following properties:

i) $\Phi(x) > 0,\ \forall\ x \in \mathbb{R}$,

ii) $\sum_{k=-\infty}^{\infty} \Phi(x - k) = 1,\ \forall\ x \in \mathbb{R}$,

iii) $\sum_{k=-\infty}^{\infty} \Phi(nx - k) = 1,\ \forall\ x \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi(x)\, dx = 1$,

v) Φ is a density function,

vi) Φ is even: $\Phi(-x) = \Phi(x),\ x \geq 0$.

We see that ([12])

$$\Phi(x) = \left(\frac{e^2 - 1}{2e}\right) \frac{e^{-x}}{\left(1 + e^{-x-1}\right)\left(1 + e^{-x+1}\right)} = \left(\frac{e^2 - 1}{2e^2}\right) \frac{1}{\left(1 + e^{x-1}\right)\left(1 + e^{-x-1}\right)}. \quad (8)$$

vii) By [12] Φ is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

viii) By [11] for $n \in \mathbb{N}$, $0 < \beta < 1$, we get

$$\sum_{\substack{k = -\infty \\ : |nx - k| > n^{1-\beta}}}^{\infty} \Phi(nx - k) < \left(\frac{e^2 - 1}{2}\right) e^{-n^{(1-\beta)}} = 3.1992 e^{-n^{(1-\beta)}}. \quad (9)$$

Denote by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

ix) By [11] it holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \frac{1}{\Phi(1)} = 5.250312578, \ \forall\, x \in [a,b]. \tag{10}$$

x) By [11] it holds $\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1$, for at least some $x \in [a,b]$.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We study further (see also [11]) the quasi-interpolation positive linear neural network operator

$$G_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \quad x \in [a,b]. \tag{11}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$. Furthermore it holds $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} > 0$.

This function plays also the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{12}$$

We easily see that$\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Here we follow [8]

Proposition 9 *$\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x \to -\infty} \Psi(x) = 0 = \lim_{x \to \infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is at zero, $\Psi(0) = 0.3809297$.

Theorem 10 *We have that* $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1, \ \forall\, x \in \mathbb{R}$.

Thus
$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N}, \ \forall\, x \in \mathbb{R}.$$

Furthermore we get:
Since Ψ is even it holds $\sum_{i=-\infty}^{\infty} \Psi(i-x) = 1, \ \forall x \in \mathbb{R}$.
Hence $\sum_{i=-\infty}^{\infty} \Psi(i+x) = 1, \ \forall\, x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \ \forall\, x \in \mathbb{R}$.

Theorem 11 *It holds* $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$.

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 12 *Let* $0 < \beta < 1$ *and* $n \in \mathbb{N}$. *It holds*

$$\sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}\right.}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}}. \tag{13}$$

Theorem 13 *Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < 4.1488766 = \frac{1}{\Psi(1)}. \tag{14}$$

Also by [8], we obtain

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1, \tag{15}$$

for at least some $x \in [a,b]$.

Definition 14 *Let* $f \in C([a,b])$ *and* $n \in \mathbb{N}$ *such that* $\lceil na \rceil \leq \lfloor nb \rfloor$.

We further study, as in [8], the quasi-interpolation positive linear neural network operator

$$F_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}, \quad x \in [a,b]. \tag{16}$$

We find here fractional Voronovskaya type asymptotic expansions for $G_n(f,x)$ and $F_n(f,x)$, $x \in [a,b]$.

For related work on neural networks also see [1], [6], [9] and [10]. For neural networks in general see [16], [17] and [18].

2 Main Results

We present our first main result

Theorem 15 *Let* $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ *large enough. Assume that* $\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}, \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \leq M$, $M > 0$. *Then*

$$G_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \quad (17)$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, *the sum in (17) collapses.*

The last (17) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[G_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) \right] \to 0, \quad (18)$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, *or* $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, *then we derive that*

$$n^{\beta(\alpha-\varepsilon)} \left[G_n(f,x) - f(x) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From [13], p. 54; (5), we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_x^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ, \quad (19)$$

for all $x \leq \frac{k}{n} \leq b$.

Also from [3]; (6), using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{k}{n} - x\right)^j + \frac{1}{\Gamma(\alpha)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ, \quad (20)$$

for all $a \leq \frac{k}{n} \leq x$.

We call

$$V(x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx - k). \quad (21)$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\Phi(nx-k)}{V(x)} \left(\frac{k}{n} - x\right)^j + \quad (22)$$

$$\frac{\Phi(nx-k)}{V(x)\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ,$$

all $x \leq \frac{k}{n} \leq b$, iff $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^j + \tag{23}$$

$$\frac{\Phi(nx-k)}{V(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ,$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

We have that $\lceil nx \rceil \leq \lfloor nx \rfloor + 1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\Phi(nx-k)\left(\frac{k}{n}-x\right)^j}{V(x)}+$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)}{V(x)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right), \tag{24}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor} f\left(\frac{k}{n}\right)\frac{\Phi(nx-k)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^j + \tag{25}$$

$$\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\Phi(nx-k)}{V(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right).$$

Adding the last two equalities (24) and (25) we obtain

$$G_n(f,x) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\frac{\Phi(nx-k)}{V(x)} = \tag{26}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\Phi(nx-k)}{V(x)}\left(\frac{k}{n}-x\right)^j +$$

$$\frac{1}{\Gamma(\alpha)V(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ+\right.$$

$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)\right)dJ\right\}.$$

So we have derived

$$T(x) := G_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) = \theta_n^*(x), \tag{27}$$

where

$$\theta_n^*(x) := \frac{1}{\Gamma(\alpha) V(x)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right.$$

$$\left. + \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right\}. \tag{28}$$

We set

$$\theta_{1n}^*(x) := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)}{V(x)} \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right), \tag{29}$$

and

$$\theta_{2n}^* := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k)}{V(x)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right), \tag{30}$$

i.e.

$$\theta_n^*(x) = \theta_{1n}^*(x) + \theta_{2n}^*(x). \tag{31}$$

We assume $b - a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right| \leq \tag{32}$$

$$\int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left| D_{x-}^{\alpha} f(J) \right| dJ$$

$$\leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \tag{33}$$

That is

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{34}$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)\right| dJ \tag{35}$$

$$\leq \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} \frac{\left(x-\frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} \frac{1}{n^{\alpha\beta}\alpha}.$$

So that, when $\left(x-\frac{k}{n}\right) \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} \frac{1}{\alpha n^{a\beta}}. \tag{36}$$

Therefore

$$\left|\theta_{1n}^{*}(x)\right| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)}{V(x)}\gamma_{1k}\right) = \frac{1}{\Gamma(\alpha)}\cdot$$

$$\left\{\frac{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)}{V(x)}\gamma_{1k} + \frac{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)}{V(x)}\gamma_{1k}\right\}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left\{\left(\frac{\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)}{V(x)}\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left.\frac{1}{V(x)}\left(\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}\right\} \overset{(\text{by }(9),\,(10))}{\leq}$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}} + (5.250312578)(3.1992)\,e^{-n^{(1-\beta)}}(x-a)^{\alpha}\right\}. \tag{37}$$

Therefore we proved

$$\left|\theta_{1n}^{*}(x)\right| \leq \frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}} + (16.7968)\,e^{-n^{(1-\beta)}}(x-a)^{\alpha}\right\}. \tag{38}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left|\theta_{1n}^{*}(x)\right| \leq \frac{2\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)\,n^{\alpha\beta}}. \tag{39}$$

Similarly we have

$$\gamma_{2k} := \left|\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right| \leq$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left|D_{*x}^{\alpha}f(J)\right|dJ \leq$$

$$\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha} \leq \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}. \tag{40}$$

That is

$$\gamma_{2k} \leq \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}, \tag{41}$$

for $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$.

Also we have in case of $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{2k} \leq \frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \tag{42}$$

Consequently it holds

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)}{V(x)}\gamma_{2k}\right) =$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\frac{\sum_{\begin{cases} k=\lfloor nx \rfloor+1 \\ :\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb \rfloor}\Phi(nx-k)}{V(x)}\right)\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{V(x)}\left(\sum_{\begin{cases} k=\lfloor nx \rfloor+1 \\ :\left|\frac{k}{n}-x\right| > \frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb \rfloor}\Phi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}\right\} \leq$$

$$\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}+(16.7968)\,e^{-n^{(1-\beta)}}(b-x)^{\alpha}\right\}. \tag{43}$$

That is

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{\left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}+(16.7968)\, e^{-n^{(1-\beta)}}(b-x)^{\alpha}\right\}. \tag{44}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left|\theta_{2n}^{*}(x)\right| \leq \frac{2\left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{45}$$

Since $\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]}, \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} \leq M,\ M>0$, we derive

$$\left|\theta_{n}^{*}(x)\right| \leq\left|\theta_{1n}^{*}(x)\right|+\left|\theta_{2n}^{*}(x)\right| \overset{\text{(by (39), (45))}}{\leq} \frac{4M}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{46}$$

That is for large enough $n \in \mathbb{N}$ we get

$$|T(x)|=\left|\theta_{n}^{*}(x)\right| \leq\left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\alpha\beta}}\right), \tag{47}$$

resulting to

$$|T(x)|=O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{48}$$

and

$$|T(x)|=o(1). \tag{49}$$

And, letting $0<\varepsilon \leq \alpha$, we derive

$$\frac{|T(x)|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq\left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \rightarrow 0, \tag{50}$$

as $n \rightarrow \infty$.

I.e.

$$|T(x)|=o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{51}$$

proving the claim. ■

We present our second main result

Theorem 16 *Let $\alpha>0,\ N \in \mathbb{N},\ N=\lceil\alpha\rceil,\ f \in AC^{N}([a,b]),\ 0<\beta<1$, $x \in[a,b]$, $n \in \mathbb{N}$ large enough. Assume that $\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]}, \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} \leq M$, $M>0$. Then*

$$F_{n}(f, x)-f(x)=\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_{n}\left((\cdot-x)^{j}\right)(x)+o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{52}$$

where $0<\varepsilon \leq \alpha$.

If $N=1$, the sum in (52) collapses.

The last (52) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[F_n(f,x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}F_n\left((\cdot-x)^j\right)(x)\right]\to 0, \tag{53}$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$.

When $N=1$, *or* $f^{(j)}(x)=0$, $j=1,...,N-1$, *then we derive that*

$$n^{\beta(\alpha-\varepsilon)}\left[F_n(f,x)-f(x)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

Proof. Similar to Theorem 15, using (13) and (14). ∎

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Multivariate Voronovskaya type asymptotic expansions for normalized bell and squashing type neural network operators

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Abstract

Here we introduce the multivariate normalized bell and squashing type neural network operators of one hidden layer. We derive multivariate Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1 Background

In [6] the authors presented for the fist time approximation of functions by specific completely described neural network operators. However their approach was only qualitative. The author in [1], [2] continued the work of [6] by presenting for the first time quantitative approximation by determining the rate of convergence and involving the modulus of continuity of the function under approximation. In this work we engage very flexible neural network operators for the first time that derive by normalization of operators of [6], so we are able to produce asymptotic expansions of Voronovkaya type regarding the approximation of these operators to the unit operator.

We use the following (see [6]).

Definition 1 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing* *on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.*

Definition 2 *(see [6]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,*

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 3 *(from [6]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function.*

Assumption 4 *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there.*

Let $f \in C(\mathbb{R}^d)$.

In this article we find a multivariate Voronovskaya type asymptotic expansion for the multivariate normalized bell type neural network operators,

$$M_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}, \quad (1)$$

where $0 < \beta < 1$ and $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (1) can be nonzero iff simultaneously

$$\left| n^{1-\beta}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i, \quad \text{all } i = 1, ..., d$$

i.e., $\left| x_i - \frac{k_i}{n} \right| \leq \frac{T_i}{n^{1-\beta}}$, all $i = 1, ..., d$, iff

$$nx_i - T_i n^{\beta} \leq k_i \leq nx_i + T_i n^{\beta}, \quad \text{all } i = 1, ..., d. \quad (2)$$

To have the order

$$-n^2 \leq nx_i - T_i n^{\beta} \leq k_i \leq nx_i + T_i n^{\beta} \leq n^2, \quad (3)$$

we need $n \geq T_i + |x_i|$, all $i = 1, ..., d$. So (3) is true when we consider

$$n \geq \max_{i \in \{1,...,d\}} (T_i + |x_i|). \quad (4)$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (3) it is enough to suppose that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Take

$$\widetilde{I}_i := \left[nx_i - T_i n^\beta, nx_i + T_i n^\beta\right], \ \ i = 1, ..., d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^\beta$. By Proposition 2.1, p. 61 of [3], we obtain that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card\,(k_i) \geq \max\left(2T_i n^\beta - 1, 0\right)$, any $i \in \{1, ..., d\}$. In order to have $card\,(k_i) \geq 1$ we need $2T_i n^\beta - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\beta}}$, any $i \in \{1, ..., d\}$.

Therefore, a sufficient condition for causing the order (3) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, ..., d$ is that

$$n \geq \max_{i \in \{1,...,d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\beta}}\right\}. \tag{5}$$

Clearly as $n \to +\infty$ we get that $card\,(k_i) \to +\infty$, all $i = 1, ..., d$. Also notice that $card\,(k_i)$ equals to the cardinality of integers in $\left[\lceil nx_i - T_i n^\beta\rceil, \left[nx_i + T_i n^\beta\right]\right]$ for all $i = 1, ..., d$.

Here we denote by $\lceil\cdot\rceil$ the ceiling of the number, and by $[\cdot]$ we denote the integral part.

From now on in this article we assume (5). Therefore

$$\left(M_n\,(f)\right)(\overrightarrow{x}) = \tag{6}$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^\beta\rceil}^{\left[nx_1 + T_1 n^\beta\right]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\beta\rceil}^{\left[nx_d + T_d n^\beta\right]} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=\lceil nx_1 - T_1 n^\beta\rceil}^{\left[nx_1 + T_1 n^\beta\right]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\beta\rceil}^{\left[nx_d + T_d n^\beta\right]} b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}$$

all $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$.

In brief we write

$$\left(M_n\,(f)\right)(\overrightarrow{x}) = \frac{\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x} - \overrightarrow{T} n^\beta\right\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^\beta\right]} f\left(\frac{\overrightarrow{k}}{n}\right) b\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x} - \overrightarrow{T} n^\beta\right\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^\beta\right]} b\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}, \tag{7}$$

all $\overrightarrow{x} \in \mathbb{R}^d$.

Denote by $\|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\beta}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, ..., d$, we find that

$$\left\|\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right\|_\infty \leq \frac{T^*}{n^{1-\beta}}, \tag{8}$$

where $\overrightarrow{k} := (k_1, ..., k_d)$.

We also need

Definition 5 *Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing for each coordinate. S can be continuous only on either $\prod_{i=1}^{d}(-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [6]).*

Example 6 *Let $\widehat{S}$ as above when $d = 1$. Then*

$$S(\overrightarrow{x}) := \widehat{S}(x_1) \ldots \widehat{S}(x_d), \quad \overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f \in C\left(\mathbb{R}^d\right)$.

For $\overrightarrow{x} \in \mathbb{R}^d$ we define also the "multivariate normalized squashing type neural network operators",

$$L_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) S\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{9}$$

We also here find a multivariate Voronovskaya type asymptotic expansion for $(L_n(f))(\overrightarrow{x})$.

Here again $0 < \beta < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1, \ldots, d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\beta}} \right\},$$

and L_n is a positive linear operator. It is clear that

$$(L_n(f))(\overrightarrow{x}) = \frac{\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\beta} \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^{\beta}]} f\left(\frac{\overrightarrow{k}}{n}\right) S\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\beta} \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^{\beta}]} S\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{10}$$

For related articles on neural networks approximation, see [1], [2], [3] and [5]. For neural networks in general, see [7], [8] and [9].

Next we follow [4], pp. 284-286.

About Multivariate Taylor formula and estimates.

Let $\mathbb{R}^d$; $d \geq 2$; $z := (z_1, \ldots, z_d)$, $x_0 := (x_{01}, \ldots, x_{0d}) \in \mathbb{R}^d$. We consider the space of functions $AC^N\left(\mathbb{R}^d\right)$ with $f : \mathbb{R}^d \to \mathbb{R}$ be such that all partial derivatives of order $(N-1)$ are coordinatewise absolutely continuous functions on compacta, $N \in \mathbb{N}$. Also $f \in C^{N-1}\left(\mathbb{R}^d\right)$. Each N^{th} order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., d$ and $|\alpha| := \sum_{i=1}^d \alpha_i = N$. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^d (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right] (x_{01} + t(z_1 - x_{01}), ..., x_{0d} + t(z_N - x_{0d})), \tag{11}$$

for all $j = 0, 1, 2, ..., N$.

Example 7 *Let $d = N = 2$. Then*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in \mathbb{R},$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \tag{12}$$

Setting

$$(*) = (x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})) = (x_0 + t(z - x_0)),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) +$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*). \tag{13}$$

Similarly, we have the general case of $d, N \in \mathbb{N}$ for $g_z^{(N)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 8 *Under the above assumptions we have*

$$f(z_1, ..., z_d) = g_z(1) = \sum_{j=0}^{N-1} \frac{g_z^{(j)}(0)}{j!} + R_N(z, 0), \tag{14}$$

where

$$R_N(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{N-1}} g_z^{(N)}(t_N)\, dt_N\right) ...\right) dt_1, \tag{15}$$

or

$$R_N(z, 0) = \frac{1}{(N-1)!} \int_0^1 (1 - \theta)^{N-1} g_z^{(N)}(\theta)\, d\theta. \tag{16}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 9 *Assume here that*

$$\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max} := \max_{|\alpha|=N} \|f_\alpha\|_{\infty,\mathbb{R}^d} < \infty.$$

Then

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} = \left\|\left[\left(\sum_{i=1}^{d}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^N f\right](x_0 + t(z - x_0))\right\|_{\infty,[0,1]} \leq \tag{17}$$

$$\left(\sum_{i=1}^{d}|z_i - x_{0i}|\right)^N \|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max},$$

that is

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1}\right)^N \|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty. \tag{18}$$

Hence we get by (16) that

$$|R_N(z,0)| \leq \frac{\left\|g_z^{(N)}\right\|_{\infty,[0,1]}}{N!} < \infty. \tag{19}$$

And it holds

$$|R_N(z,0)| \leq \frac{\left(\|z - x_0\|_{l_1}\right)^N}{N!}\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}, \tag{20}$$

$\forall\ z, x_0 \in \mathbb{R}^d$.

Inequality (20) will be an important tool in proving our main results.

2 Main Results

We present our first main result.

Theorem 10 *Let $f \in AC^N(\mathbb{R}^d)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, with $\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. Here $n \geq \max_{i\in\{1,\ldots,d\}}\left\{T_i + |x_i|, T_i^{-\frac{1}{\beta}}\right\}$, where $\overrightarrow{x} \in \mathbb{R}^d$, $0 < \beta < 1$, $n \in \mathbb{N}$, $T_i > 0$. Then*

$$(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) =$$

$$\sum_{j=1}^{N-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!}\right) M_n\left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x}\right)\right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right), \tag{21}$$

where $0 < \varepsilon \leq N$.

If $N = 1$, the sum in (21) collapses.

The last (21) implies that

$$n^{(N-\varepsilon)(1-\beta)}\left[(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) - \right. \tag{22}$$

$$\left. \sum_{j=1}^{N-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!} \right) M_n \left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x} \right) \right) \right] \to 0, \quad \text{as } n \to \infty,$$

$0 < \varepsilon \leq N$.

When $N = 1$, or $f_\alpha(\overrightarrow{x}) = 0$, all $\alpha : |\alpha| = j = 1, ..., N-1$, then we derive

$$n^{(N-\varepsilon)(1-\beta)}\left[(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. Put

$$g_{\frac{\overrightarrow{k}}{n}}(t) := f\left(\overrightarrow{x} + t\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right) \right), \quad 0 \leq t \leq 1.$$

Then

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(t) =$$

$$\left[\left(\sum_{i=1}^d \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right] \left(x_1 + t\left(\frac{k_1}{n} - x_1 \right), ..., x_d + t\left(\frac{k_d}{n} - x_d \right) \right), \tag{23}$$

and $g_{\frac{\overrightarrow{k}}{n}}(0) = f(\overrightarrow{x})$. By Taylor's formula (14), (16) we obtain

$$f\left(\frac{k_1}{n}, ..., \frac{k_d}{n} \right) = g_{\frac{\overrightarrow{k}}{n}}(1) = \sum_{j=0}^{N-1} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} + R_N\left(\frac{\overrightarrow{k}}{n}, 0 \right), \tag{24}$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n}, 0 \right) = \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta)\, d\theta. \tag{25}$$

More precisely we can rewrite

$$f\left(\frac{\overrightarrow{k}}{n} \right) - f(\overrightarrow{x}) =$$

$$\sum_{j=1}^{N-1} \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_d),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,d,\ |\alpha|:=\sum_{i=1}^d \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha(\overrightarrow{x}) + R_N\left(\frac{\overrightarrow{k}}{n}, 0 \right), \tag{26}$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n},0\right)=N\int_0^1(1-\theta)^{N-1}\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_d),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\dots,d,\ |\alpha|:=\sum_{i=1}^d\alpha_i=N}}\left(\frac{1}{\prod_{i=1}^d\alpha_i!}\right)\cdot$$

$$\left(\prod_{i=1}^d\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)f_\alpha\left(\overrightarrow{x}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right)\right)d\theta. \tag{27}$$

By (20) we get

$$\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right|\leq\frac{\left(\left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{l_1}\right)^N}{N!}\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{28}$$

So, since here it holds

$$\left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_\infty\leq\frac{T^*}{n^{1-\beta}},$$

then

$$\left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_{l_1}\leq\frac{dT^*}{n^{1-\beta}},$$

and

$$\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right|\leq\frac{d^NT^{*N}}{n^{N(1-\beta)}N!}\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}, \tag{29}$$

for all $\overrightarrow{k}\in\left\{\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil,\dots,\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]\right\}$.

Call

$$V(\overrightarrow{x}):=\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]}b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right). \tag{30}$$

We observe for

$$U_n(\overrightarrow{x}):=\frac{\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]}R_N\left(\frac{\overrightarrow{k}}{n},0\right)b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, \tag{31}$$

that

$$|U_n(\overrightarrow{x})|\overset{\text{(by (29))}}{\leq}\frac{d^NT^{*N}}{n^{N(1-\beta)}N!}\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{32}$$

That is

$$|U_n(\overrightarrow{x})|=O\left(\frac{1}{n^{N(1-\beta)}}\right), \tag{33}$$

and

$$\left|U_n\left(\overrightarrow{x}\right)\right| = o\left(1\right). \tag{34}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{\left|U_n\left(\overrightarrow{x}\right)\right|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right)} \leq \left(\frac{d^N T^{*N} \left\|f_\alpha\right\|_{\infty,\mathbb{R}^d,N}^{\max}}{N!}\right) \frac{1}{n^{\varepsilon(1-\beta)}} \to 0, \tag{35}$$

as $n \to \infty$.

I.e.

$$\left|U_n\left(\overrightarrow{x}\right)\right| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right). \tag{36}$$

By (26) we get

$$\frac{\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]} f\left(\frac{\overrightarrow{k}}{n}\right) b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} - f\left(\overrightarrow{x}\right) =$$

$$\sum_{j=1}^{N-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,d,\ |\alpha|:=\sum_{i=1}^d \alpha_i=j}} \left(\frac{f_\alpha\left(\overrightarrow{x}\right)}{\prod_{i=1}^d \alpha_i!}\right). \tag{37}$$

$$\frac{\left(\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]} \left(\prod_{i=1}^d \left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right) b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} + U_n\left(\overrightarrow{x}\right).$$

The last says

$$\left(M_n\left(f\right)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right) -$$

$$\sum_{j=1}^{N-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha\left(\overrightarrow{x}\right)}{\prod_{i=1}^d \alpha_i!}\right) M_n\left(\prod_{i=1}^d \left(\cdot - x_i\right)^{\alpha_i}, \overrightarrow{x}\right)\right) = U_n\left(\overrightarrow{x}\right). \tag{38}$$

The proof of the theorem is complete. ■

We present our second main result

Theorem 11 *Let $f \in AC^N\left(\mathbb{R}^d\right)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, with $\left\|f_\alpha\right\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. Here $n \geq \max_{i\in\{1,\ldots,d\}}\left\{T_i + |x_i|, T_i^{-\frac{1}{\beta}}\right\}$, where $\overrightarrow{x} \in \mathbb{R}^d$, $0 < \beta < 1$, $n \in \mathbb{N}$, $T_i > 0$. Then*

$$\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right) =$$

$$\sum_{j=1}^{N-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha\left(\overrightarrow{x}\right)}{\prod_{i=1}^d \alpha_i!}\right) L_n\left(\prod_{i=1}^d \left(\cdot - x_i\right)^{\alpha_i}, \overrightarrow{x}\right)\right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right), \tag{39}$$

where $0 < \varepsilon \leq N$.

If $N=1$, the sum in (39) collapses.

The last (39) implies that

$$n^{(N-\varepsilon)(1-\beta)}\left[(L_n(f))(\overrightarrow{x})-f(\overrightarrow{x})-\right. \tag{40}$$

$$\left.\sum_{j=1}^{N-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!}\right)L_n\left(\prod_{i=1}^d(\cdot-x_i)^{\alpha_i},\overrightarrow{x}\right)\right)\right]\rightarrow 0, \quad as\ n\rightarrow\infty,$$

$0<\varepsilon\leq N$.

When $N=1$, or $f_\alpha(\overrightarrow{x})=0$, all $\alpha:|\alpha|=j=1,...,N-1$, then we derive that

$$n^{(N-\varepsilon)(1-\beta)}\left[(L_n(f))(\overrightarrow{x})-f(\overrightarrow{x})\right]\rightarrow 0,$$

as $n\rightarrow\infty$, $0<\varepsilon\leq N$.

Proof. As similar to Theorem 10 is omitted. ■

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Fractional Voronovskaya type asymptotic expansions for bell and squashing type neural network operators

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Abstract

Here we introduce the normalized bell and squashing type neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1 Background

We need

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), such that $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions), $\forall$ $[a,b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

$\forall$ $x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$, $\forall$ $b > a$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall$ $x \in [a,+\infty)$.

We also need

Definition 2 *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{2}$$

$\forall\ x \leq b$. We set $D_{b-}^0 f(x) = f(x)$, $\forall\ x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$, $\forall\ a < b$.

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 3 *([8], p. 54) Let $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\, dJ, \tag{3}$$

$\forall\ x \geq x_0$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 4 *([2]) Let $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J-x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\, dJ, \tag{4}$$

$\forall\ x \leq x_0$.

Convention 5 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \ \text{for } x < x_0,$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 6 *(by [3]) i) Let $f \in C^n(\mathbb{R})$, where $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, \infty)$.*

ii) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in (-\infty, b]$.

We also mention

Theorem 7 *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D_{*x_0}^\alpha f(x)$, $D_{x_0-}^\alpha f(x)$ are jointly continuous in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

For more see [4], [6].

We need the following (see [7]).

Definition 8 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

Example 9 (1) *$b(x)$ can be the characteristic function over $[-1, 1]$.*

(2) *$b(x)$ can be the hat function over $[-1, 1]$, i.e.,*

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

Here we consider functions $f \in C(\mathbb{R})$.

We study the following "normalized bell type neural network operators" (see also related [1], [7])

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{5}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$.

We find a fractional Voronovskaya type asymptotic expansion for $H_n(f)(x)$.

The terms in $H_n(f)(x)$ are nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right) \right| \leq T, \text{ i.e. } \left| x - \frac{k}{n} \right| \leq \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^\alpha \leq k \leq nx + Tn^\alpha. \tag{6}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^\alpha \leq nx + Tn^\alpha \leq n^2, \tag{7}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{8}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (7).

Proposition 10 *(see [1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \leq card(k) \leq (b-a)+1. \tag{9}$$

Remark 11 *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 10 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \text{iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (7) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{10}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number.

Remark 12 *Clearly we have that*

$$nx - Tn^{\alpha} \leq nx \leq nx + Tn^{\alpha}. \tag{11}$$

We prove in general that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{12}$$

Indeed we have that, if $[nx] < nx - Tn^{\alpha}$, then $[nx] + Tn^{\alpha} < nx$, and $[nx] + [Tn^{\alpha}] \leq [nx]$, resulting into $[Tn^{\alpha}] = 0$, which for large enough n is not true. Therefore $nx - Tn^{\alpha} \leq [nx]$. Similarly, if $\lceil nx \rceil > nx + Tn^{\alpha}$, then $nx + Tn^{\alpha} \geq nx + [Tn^{\alpha}]$, and $\lceil nx \rceil - [Tn^{\alpha}] > nx$, thus $\lceil nx \rceil - [Tn^{\alpha}] \geq \lceil nx \rceil$, resulting into $[Tn^{\alpha}] = 0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{13}$$

Hence $\lceil nx - Tn^{\alpha} \rceil \leq [nx]$ and $\lceil nx \rceil \leq [nx + Tn^{\alpha}]$. Also if $[nx] \neq \lceil nx \rceil$, then $\lceil nx \rceil = [nx] + 1$. If $[nx] = \lceil nx \rceil$, then $nx \in \mathbb{Z}$; and by assuming $n \geq T^{-\frac{1}{\alpha}}$, we get $Tn^{\alpha} \geq 1$ and $nx + Tn^{\alpha} \geq nx + 1$, so that $[nx + Tn^{\alpha}] \geq nx + 1 = [nx] + 1$.

We need also

Definition 13 *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support $[-T, T]$, $T > 0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T, T]$, S can have jump discontinuites. We call S the "squashing function", see [1], [7].*

Let $f \in C(\mathbb{R})$. For $x \in \mathbb{R}$ we define the following "normalized squashing type neural network operators" (see also related [1])

$$\left(K_n(f)\right)(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{14}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$.

It is clear that

$$\left(K_n(f)\right)(x) := \frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{15}$$

We find a fractional Voronovskaya type asymptotic expansion for $\left(K_n(f)\right)(x)$.

2 Main Results

We present our first main result.

Theorem 14 *Let $\beta > 0$, $N \in \mathbb{N}$, $N = \lceil \beta \rceil$, $f \in AC^N([a,b])$, $\forall$ $[a.b] \subset \mathbb{R}$, with $\left\|D_{x_0-}^{\beta} f\right\|_{\infty}$, $\left\|D_{*x_0}^{\beta} f\right\|_{\infty} \leq M$, $M > 0$, $x_0 \in \mathbb{R}$. Let $T > 0$, $n \in \mathbb{N} : n \geq \max\left(T + |x_0|, T^{-\frac{1}{\alpha}}\right)$ Then*

$$\left(H_n(f)\right)(x_0) - f(x_0) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0) + o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \tag{16}$$

where $0 < \varepsilon \leq \beta$.

If $N = 1$, the sum in (16) disappears.

The last (16) implies that

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[\left(H_n(f)\right)(x_0) - f(x_0) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0)\right] \to 0, \tag{17}$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$.

When $N = 1$, or $f^{(j)}(x_0) = 0$, $j = 1, ..., N-1$, then we derive

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[\left(H_n(f)\right)(x_0) - f(x_0)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$. Of great interest is the case of $\beta = \frac{1}{2}$.

Proof. From [8], p. 54; (3), we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j + \frac{1}{\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ, \tag{18}$$

for all $x_0 \leq \frac{k}{n} \leq x_0 + Tn^{\alpha-1}$, iff $\lceil nx_0 \rceil \leq k \leq [nx_0 + Tn^{\alpha}]$, where $k \in \mathbb{Z}$.

Also from [2]; (4), using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j + \frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ, \tag{19}$$

for all $x_0 - Tn^{\alpha-1} \leq \frac{k}{n} \leq x_0$, iff $\lceil nx_0 - Tn^{\alpha} \rceil \leq k \leq [nx_0]$, where $k \in \mathbb{Z}$. Notice that $\lceil nx_0 \rceil \leq [nx_0] + 1$.

Call

$$V(x_0) := \sum_{k=\lceil nx_0 - Tn^{\alpha} \rceil}^{[nx_0 + Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right).$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{20}$$
$$\frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{21}$$
$$\frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ,$$

Therefore we obtain

$$\frac{\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} =$$
$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} \left(\frac{k}{n} - x_0\right)^j b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)}\right) + \tag{22}$$
$$\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ,$$

and

$$\frac{\sum_{k=\lceil nx_0 - Tn^{\alpha} \rceil}^{[nx_0]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!} \frac{\sum_{k=\lceil nx_0 - Tn^{\alpha}\rceil}^{[nx_0]} \left(\frac{k}{n} - x_0\right)^j b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{23}$$

$$\frac{\sum_{k=\lceil nx_0 - Tn^{\alpha}\rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ.$$

We notice here that

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} \tag{24}$$

$$= \frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \quad \forall\, x \in \mathbb{R}.$$

Adding the two equalities (22), (23) and rewriting it, we obtain

$$T(x_0) := (H_n(f))(x_0) - f(x_0) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0) = \theta_n^*(x_0), \tag{25}$$

where

$$\theta_n^*(x_0) := \frac{\sum_{k=\lceil nx_0 - Tn^{\alpha}\rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ$$

$$+ \sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ. \tag{26}$$

We observe that

$$|\theta_n^*(x_0)| \le \frac{1}{V(x_0)\Gamma(\beta)} \cdot$$

$$\left\{ \sum_{k=\lceil nx_0 - Tn^{\alpha}\rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} \left|D_{x_0-}^{\beta} f(J)\right| dJ \right. \tag{27}$$

$$\left. + \sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left|D_{*x_0}^{\beta} f(J)\right| dJ \right\} \le$$

$$\frac{M}{V(x_0)\Gamma(\beta)} \left\{ \sum_{k=\lceil nx_0 - Tn^{\alpha}\rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \frac{\left(x_0 - \frac{k}{n}\right)^{\beta}}{\beta} + \right.$$

$$\left. \sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \frac{\left(\frac{k}{n} - x_0\right)^{\beta}}{\beta} \right\} \le$$

$$\frac{M}{V(x_0)\Gamma(\beta+1)}\left\{\left(\sum_{k=\lceil nx_0-Tn^{\alpha}\rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0-\frac{k}{n}\right)\right)\right)\left(\frac{T}{n^{1-\alpha}}\right)^{\beta}+\right.$$

$$\left.\left(\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x_0-\frac{k}{n}\right)\right)\right)\left(\frac{T}{n^{1-\alpha}}\right)^{\beta}\right\}=\frac{M}{\Gamma(\beta+1)}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}. \quad (28)$$

So we have proved that

$$|T(x_0)|=|\theta_n^*(x_0)|\leq\left(\frac{MT^{\beta}}{\Gamma(\beta+1)}\right)\left(\frac{1}{n^{(1-\alpha)\beta}}\right), \quad (29)$$

resulting to

$$|T(x_0)|=O\left(\frac{1}{n^{(1-\alpha)\beta}}\right), \quad (30)$$

and

$$|T(x_0)|=o(1). \quad (31)$$

And, letting $0<\varepsilon\leq\beta$, we derive

$$\frac{|T(x_0)|}{\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right)}\leq\frac{MT^{\beta}}{\Gamma(\beta+1)}\left(\frac{1}{n^{(1-\alpha)\varepsilon}}\right)\to 0, \quad (32)$$

as $n\to\infty$.

I.e.

$$|T(x_0)|=o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \quad (33)$$

proving the claim. ■

Our second main result follows

Theorem 15 *Same assumptions as in Theorem 14. Then*

$$(K_n(f))(x_0)-f(x_0)=\sum_{j=1}^{N-1}\frac{f^{(j)}(x_0)}{j!}K_n\left((\cdot-x_0)^j\right)(x_0)+o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \quad (34)$$

where $0<\varepsilon\leq\beta$.

If $N=1$*, the sum in (34) disappears.*

The last (34) implies that

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[(K_n(f))(x_0)-f(x_0)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x_0)}{j!}K_n\left((\cdot-x_0)^j\right)(x_0)\right]\to 0, \quad (35)$$

as $n\to\infty$, $0<\varepsilon\leq\beta$.

When $N=1$*, or* $f^{(j)}(x_0)=0$, $j=1,...,N-1$*, then we derive*

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[(K_n(f))(x_0)-f(x_0)\right]\to 0 \quad (36)$$

as $n\to\infty$, $0<\varepsilon\leq\beta$*. Of great interest is the case of* $\beta=\frac{1}{2}$.

Proof. As in Theorem 14. ■

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