Univariate Hardy type fractional inequalities

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Abstract

Here we present integral inequalities for convex and increasing functions applied to products of functions. As applications we derive a wide range of fractional inequalities of Hardy type. They involve the left and right Riemann-Liouville fractional integrals and their generalizations, in particular the Hadamard fractional integrals. Also inequalities for left and right Riemann-Liouville, Caputo, Canavati and their generalizations fractional derivatives. These application inequalities are of L_p type, $p \geq 1$, and exponential type, as well as their mixture.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.
Key words and phrases: Jensen inequality, fractional integral, fractional derivative, Hardy fractional inequality, Hadamard fractional integral.

1 Introduction

We start with some facts about fractional derivatives needed in the sequel, for more details see, for instance [1], [9].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann-Liouville fractional integrals, see [12]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$.

The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1}$$

$$\left(I_{b-}^{\alpha} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b), \tag{2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$, see also [13]. The first result yields that the fractional integral operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\left\| I_{a+}^{\alpha} f \right\|_p \leq K \|f\|_p \quad , \quad \left\| I_{b-}^{\alpha} f \right\|_p \leq K \|f\|_p , \tag{3}$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}. \tag{4}$$

Inequality (3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [10]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we follow [11].

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative measurable function, $k(x, \cdot)$ measurable on Ω_2 and

$$K(x) = \int_{\Omega_2} k(x,y) \, d\mu_2(y), \quad x \in \Omega_1. \tag{5}$$

We suppose that $K(x) > 0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y) \, d\mu_2(y), \tag{6}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

Theorem 1 *([11]) Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (5). Assume that the function $x \mapsto u(x) \frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define ν on Ω_2 by*

$$\nu(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{7}$$

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If $\Phi : [0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x) \Phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} \nu(y) \Phi(|f(y)|) d\mu_2(y) \tag{8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that:

(i) $f, \Phi(|f|)$ are both $k(x, y) d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\nu(y) \Phi(|f|)$ is μ_2 -integrable, and for all corresponding functions g given by (6).

Important assumptions (i) and (ii) are missing from Theorem 2.1. of [11].

In this article we generalize Theorem 1 for products of several functions and we give wide applications to Fractional Calculus.

2 Main Results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x, y) d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{9}$$

$i = 1, ..., m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x, y) f_i(y) d\mu_2(y), \tag{10}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., m$.

Here u stands for a weight function on Ω_1.

The first introductory result is proved for $m = 2$.

Theorem 2 *Assume that the function $x \mapsto \left(\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}\right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_2 on Ω_2 by*

$$\lambda_2(y) := \int_{\Omega_1} \frac{u(x) k_1(x, y) k_2(x, y)}{K_1(x) K_2(x)} d\mu_1(x) < \infty. \tag{11}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right) \Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) \leq$$

$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|) d\mu_2(y)\right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|) \lambda_2(y) d\mu_2(y)\right), \tag{12}$$

true for all measurable functions, $i = 1, 2$, $f_i : \Omega_2 \to \mathbb{R}$ such that

(i) f_i, $\Phi_i(|f_i|)$, are both $k_i(x,y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_2 \Phi_1(|f_1|)$, $\Phi_2(|f_2|)$, are both μ_2 -integrable,

and for all corresponding functions g_i given by (10).

Proof. Notice here that Φ_1, Φ_2 are continuous functions. Here we use Jensen's inequality, Fubini's theorem, and that Φ_i are increasing. We have

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)d\mu_1(x) =$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,f_1(y)\,d\mu_2(y)\right|\right)\cdot \tag{13}$$

$$\Phi_2\left(\left|\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,f_2(y)\,d\mu_2(y)\right|\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,|f_1(y)|\,d\mu_2(y)\right)\cdot$$

$$\Phi_2\left(\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,|f_2(y)|\,d\mu_2(y)\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\frac{1}{K_1(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)\cdot$$

$$\frac{1}{K_2(x)}\left(\int_{\Omega_2} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)d\mu_1(x) =$$

(calling $\gamma_1(x) := \int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)$)

$$\int_{\Omega_1}\int_{\Omega_2} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\,d\mu_1(x) =$$

$$\int_{\Omega_2}\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_1(x)\,d\mu_2(y) = \tag{14}$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\left(\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left(\int_{\Omega_1} \frac{u(x)\,k_2(x,y)}{K_1(x)\,K_2(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y) = \tag{15}$$

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$$\left(\int_{\Omega_2} \Phi_2\left(|f_2(y)|\right) d\mu_2(y)\right) \cdot$$
$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\, k_1(x,y)\, k_2(x,y)}{K_1(x)\, K_2(x)} \Phi_1\left(|f_1(y)|\right) d\mu_2(y)\right) d\mu_1(x)\right] =$$
$$\left(\int_{\Omega_2} \Phi_2\left(|f_2(y)|\right) d\mu_2(y)\right) \cdot$$
$$\left[\int_{\Omega_2}\left(\int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)}{K_1(x)\, K_2(x)} \Phi_1\left(|f_1(y)|\right) d\mu_1(x)\right) d\mu_2(y)\right] =$$
$$\left(\int_{\Omega_2} \Phi_2\left(|f_2(y)|\right) d\mu_2(y)\right) \cdot$$
$$\left[\int_{\Omega_2} \Phi_1\left(|f_1(y)|\right)\left(\int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)}{K_1(x)\, K_2(x)} d\mu_1(x)\right) d\mu_2(y)\right] = \tag{16}$$
$$\left(\int_{\Omega_2} \Phi_2\left(|f_2(y)|\right) d\mu_2(y)\right)\left[\int_{\Omega_2} \Phi_1\left(|f_1(y)|\right) \lambda_2(y)\, d\mu_2(y)\right],$$

proving the claim. ■

When $m = 3$, the corresponding result follows.

Theorem 3 *Assume that the function $x \mapsto \left(\frac{u(x)k_1(x,y)k_2(x,y)k_3(x,y)}{K_1(x)K_2(x)K_3(x)}\right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_3 on Ω_2 by*

$$\lambda_3(y) := \int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)\, k_3(x,y)}{K_1(x)\, K_2(x)\, K_3(x)} d\mu_1(x) < \infty. \tag{17}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, 3,$ are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq \tag{18}$$
$$\left(\prod_{i=2}^{3} \int_{\Omega_2} \Phi_i\left(|f_i(y)|\right) d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1\left(|f_1(y)|\right) \lambda_3(y)\, d\mu_2(y)\right),$$

true for all measurable functions, $i = 1, 2, 3,$ $f_i : \Omega_2 \to \mathbb{R}$ such that

(i) f_i, $\Phi_i(|f_i|)$, are both $k_i(x,y)\, d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_3 \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|)$, are all μ_2 -integrable,

and for all corresponding functions g_i given by (10).

Proof. Here we use Jensen's inequality, Fubini's theorem, and that Φ_i are increasing. We have

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) =$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i \left(\left| \frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) f_i(y) \, d\mu_2(y) \right| \right) d\mu_1(x) \leq \tag{19}$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) |f_i(y)| \, d\mu_2(y) \right) d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\frac{u(x)}{\prod_{i=1}^{3} K_i(x)} \right) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

(calling $\theta(x) := \frac{u(x)}{\prod_{i=1}^{3} K_i(x)}$)

$$\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) = \tag{20}$$

$$\int_{\Omega_1} \theta(x) \left[\int_{\Omega_2} \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$
$$\left. k_3(x,y) \Phi_3(|f_3(y)|) \, d\mu_2(y) \right] d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\int_{\Omega_2} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$
$$\left. k_3(x,y) \Phi_3(|f_3(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_2} \left(\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$
$$\left. k_3(x,y) \Phi_3(|f_3(y)|) \, d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left(\int_{\Omega_1} \theta(x) k_3(x,y) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right. \tag{21}$$
$$\left. d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left[\int_{\Omega_1} \theta(x) k_3(x,y) \left(\int_{\Omega_2} \left\{ \int_{\Omega_2} k_1(x,y) \Phi_1(|f_1(y)|) \, d\mu_2(y) \right\} \cdot \right. \right.$$
$$\left. \left. k_2(x,y) \Phi_2(|f_2(y)|) \, d\mu_2(y) \right) d\mu_1(x) \right] d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3 (|f_3 (y)|) \left[\int_{\Omega_1} \left(\int_{\Omega_2} \theta (x) k_2 (x,y) k_3 (x,y) \Phi_2 (|f_2 (y)|) \cdot \right.\right. \tag{22}$$

$$\left.\left. \left\{ \int_{\Omega_2} k_1 (x,y) \Phi_1 (|f_1 (y)|) d\mu_2 (y) \right\} d\mu_2 (y) \right) d\mu_1 (x) \right] d\mu_2 (y) =$$

$$\left(\int_{\Omega_2} \Phi_3 (|f_3 (y)|) d\mu_2 (y) \right) \left[\int_{\Omega_1} \left(\int_{\Omega_2} \theta (x) k_2 (x,y) k_3 (x,y) \Phi_2 (|f_2 (y)|) \cdot \right.\right.$$

$$\left.\left. \left\{ \int_{\Omega_2} k_1 (x,y) \Phi_1 (|f_1 (y)|) d\mu_2 (y) \right\} d\mu_2 (y) \right) d\mu_1 (x) \right] =$$

$$\left(\int_{\Omega_2} \Phi_3 (|f_3 (y)|) d\mu_2 (y) \right) \left[\int_{\Omega_2} \left(\int_{\Omega_1} \theta (x) k_2 (x,y) k_3 (x,y) \Phi_2 (|f_2 (y)|) \cdot \right.\right.$$

$$\left.\left. \left\{ \int_{\Omega_2} k_1 (x,y) \Phi_1 (|f_1 (y)|) d\mu_2 (y) \right\} d\mu_1 (x) \right) d\mu_2 (y) \right] = \tag{23}$$

$$\left(\int_{\Omega_2} \Phi_3 (|f_3 (y)|) d\mu_2 (y) \right) \left[\int_{\Omega_2} \Phi_2 (|f_2 (y)|) \left(\int_{\Omega_1} \theta (x) k_2 (x,y) k_3 (x,y) \cdot \right.\right.$$

$$\left.\left. \left(\int_{\Omega_2} k_1 (x,y) \Phi_1 (|f_1 (y)|) d\mu_2 (y) \right) d\mu_1 (x) \right) d\mu_2 (y) \right] =$$

$$\left(\int_{\Omega_2} \Phi_3 (|f_3 (y)|) d\mu_2 (y) \right) \left[\int_{\Omega_2} \Phi_2 (|f_2 (y)|) \left\{ \int_{\Omega_1} \left(\int_{\Omega_2} \theta (x) \prod_{i=1}^{3} k_i (x,y) \cdot \right.\right.\right.$$

$$\Phi_1 (|f_1 (y)|) d\mu_2 (y)) d\mu_1 (x)\} d\mu_2 (y)] =$$

$$\left(\int_{\Omega_2} \Phi_3 (|f_3 (y)|) d\mu_2 (y) \right) \left(\int_{\Omega_2} \Phi_2 (|f_2 (y)|) d\mu_2 (y) \right) \cdot$$

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} \theta (x) \prod_{i=1}^{3} k_i (x,y) \Phi_1 (|f_1 (y)|) d\mu_2 (y) \right) d\mu_1 (x) \right) = \tag{24}$$

$$\left(\prod_{i=2}^{3} \int_{\Omega_2} \Phi_i (|f_i (y)|) d\mu_2 (y) \right) \cdot$$

$$\left(\int_{\Omega_2} \left(\int_{\Omega_1} \theta (x) \prod_{i=1}^{3} k_i (x,y) \Phi_1 (|f_1 (y)|) d\mu_1 (x) \right) d\mu_2 (y) \right) =$$

$$\left(\prod_{i=2}^{3} \int_{\Omega_2} \Phi_i (|f_i (y)|) d\mu_2 (y) \right) \cdot$$

$$\left(\int_{\Omega_2} \Phi_1 (|f_1 (y)|) \left(\int_{\Omega_1} \theta (x) \prod_{i=1}^{3} k_i (x,y) d\mu_1 (x) \right) d\mu_2 (y) \right) =$$

$$\left(\prod_{i=2}^{3} \int_{\Omega_2} \Phi_i (|f_i (y)|) d\mu_2 (y) \right) \left(\int_{\Omega_2} \Phi_1 (|f_1 (y)|) \lambda_3 (y) d\mu_2 (y) \right), \tag{25}$$

proving the claim. ■

For general $m \in \mathbb{N}$, the following result is valid.

Theorem 4 *Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{26}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K_i(x)} \right| \right) d\mu_1(x) \leq \tag{27}$$

$$\left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|) \, \lambda_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y) \, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\lambda_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), ..., \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (10).*

When $k(x,y) = k_1(x,y) = k_2(x,y) = ... = k_m(x,y)$, then $K(x) := K_1(x) = K_2(x) = ... = K_m(x)$. Then from Theorem 4 we get:

Corollary 5 *Assume that the function* $x \mapsto \left(\frac{u(x) k^m(x,y)}{K^m(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* U_m *on* Ω_2 *by*

$$U_m(y) := \int_{\Omega_1} \left(\frac{u(x) k^m(x,y)}{K^m(x)} \right) d\mu_1(x) < \infty. \tag{28}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{29}$$

$$\left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|) \, U_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k(x,y) \, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $U_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), ..., \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (10).*

When $m = 2$ from Corollary 5 we obtain

Corollary 6 *Assume that the function* $x \mapsto \left(\frac{u(x)k^2(x,y)}{K^2(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* U_2 *on* Ω_2 *by*

$$U_2(y) := \int_{\Omega_1} \left(\frac{u(x)\, k^2(x,y)}{K^2(x)}\right) d\mu_1(x) < \infty. \tag{30}$$

Here $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x)\, \Phi_1\left(\left|\frac{g_1(x)}{K(x)}\right|\right) \Phi_2\left(\left|\frac{g_2(x)}{K(x)}\right|\right) d\mu_1(x) \leq \tag{31}$$

$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|)\, d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_1(|f_1(y)|)\, U_2(y)\, d\mu_2(y)\right),$$

true for all measurable functions, $f_1, f_2 : \Omega_2 \to \mathbb{R}$ *such that*

(i) f_1, f_2, $\Phi_1(|f_1|)$, $\Phi_2(|f_2|)$ *are all* $k(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $U_2\Phi_1(|f_1|), \Phi_2(|f_2|)$, *are both* μ_2 *-integrable,*

and for all corresponding functions g_1, g_2 *given by (10).*

For $m \in \mathbb{N}$, the following more general result is also valid.

Theorem 7 *Let* $j \in \{1, ..., m\}$ *be fixed. Assume that the function* $x \mapsto \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right) d\mu_1(x) < \infty. \tag{32}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *are convex and increasing functions.*

Then

$$I := \int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq \tag{33}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|)\, d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_j(|f_j(y)|)\, \lambda_m(y)\, d\mu_2(y)\right) := I_j,$$

true for all measurable functions, $i = 1, ..., m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\lambda_m \Phi_j(|f_j|); \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), ..., \widehat{\Phi_j(|f_j|)}, ..., \Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (10). Above* $\widehat{\Phi_j(|f_j|)}$ *means missing item.*

We make

Remark 8 *In the notations and assumptions of Theorem 7, replace assumption (ii) by the assumption,*

(iii) $\Phi_1(|f_1|), ..., \Phi_m(|f_m|); \lambda_m \Phi_1(|f_1|), ..., \lambda_m \Phi_m(|f_m|)$, *are all* μ_2 *-integrable functions.*

Then, clearly it holds,

$$I \leq \frac{\sum_{j=1}^{m} I_j}{m}. \tag{34}$$

An application of Theorem 7 follows.

Theorem 9 *Let* $j \in \{1, ..., m\}$ *be fixed. Assume that the function* $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ *is integrable on* Ω_1*, for each* $y \in \Omega_2$*. Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{35}$$

Then

$$\int_{\Omega_1} u(x)\, e^{\sum_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|} d\mu_1(x) \leq \tag{36}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} e^{|f_i(y)|} d\mu_2(y) \right) \left(\int_{\Omega_2} e^{|f_j(y)|} \lambda_m(y)\, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m,$ $f_i : \Omega_2 \to \mathbb{R}$ *such that*

(i) f_i, $e^{|f_i|}$, *are both* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\lambda_m e^{|f_j|}; e^{|f_1|}, e^{|f_2|}, e^{|f_3|}, ..., \widehat{e^{|f_j|}}, ..., e^{|f_m|}$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (10). Above* $\widehat{e^{|f_j|}}$ *means absent item.*

Another application of Theorem 7 follows.

Theorem 10 *Let $j \in \{1, ..., m\}$ be fixed, $\alpha \geq 1$. Assume that the function $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m on Ω_2 by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{37}$$

Then

$$\int_{\Omega_1} u(x) \left(\prod_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|^{\alpha} \right) d\mu_1(x) \leq \tag{38}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} |f_i(y)|^{\alpha} d\mu_2(y) \right) \left(\int_{\Omega_2} |f_j(y)|^{\alpha} \lambda_m(y) d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_i : \Omega_2 \to \mathbb{R}$ such that

(i) $|f_i|^{\alpha}$ is $k_i(x,y) d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_m |f_j|^{\alpha}; |f_1|^{\alpha}, |f_2|^{\alpha}, |f_3|^{\alpha}, ..., \widehat{|f_j|^{\alpha}}, ..., |f_m|^{\alpha}$, are all μ_2 -integrable,

and for all corresponding functions g_i given by (10). Above $\widehat{|f_j|^{\alpha}}$ means absent item.

We make

Remark 11 *Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1, ..., m$, e.g. when $f_i \in L_{\infty}(a,b)$.*

Consider

$$g_i(x) = \left(I_{a+}^{\alpha_i} f_i\right)(x), \quad x \in (a,b),\ i = 1, ..., m, \tag{39}$$

we remind

$$\left(I_{a+}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x-t)^{\alpha_i - 1} f_i(t)\, dt.$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+}^{\alpha_i} f\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t)\, dt, \tag{40}$$

where χ stands for the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, ..., m. \tag{41}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & a < y \le x, \\ 0, & x < y < b. \end{cases} \tag{42}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)} dy = \frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i+1)}, \tag{43}$$

$a < x < b$, $i = 1, ..., m$.

Notice that

$$\prod_{i=1}^{m} \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^{m} \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i+1)}{(x-a)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^{m} \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}\alpha_i}{(x-a)^{\alpha_i}} \right) = \frac{\chi_{(a,x]}(y)(x-y)^{\left(\sum_{i=1}^{m}\alpha_i - m\right)} \left(\prod_{i=1}^{m}\alpha_i\right)}{(x-a)^{\left(\sum_{i=1}^{m}\alpha_i\right)}}. \tag{44}$$

Calling

$$\alpha := \sum_{i=1}^{m} \alpha_i > 0, \quad \gamma := \prod_{i=1}^{m} \alpha_i > 0, \tag{45}$$

we have that

$$\prod_{i=1}^{m} \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)(x-y)^{\alpha-m}\gamma}{(x-a)^{\alpha}}. \tag{46}$$

Therefore, for (32), we get for appropiate weight u that

$$\lambda_m(y) = \gamma \int_y^b u(x) \frac{(x-y)^{\alpha-m}}{(x-a)^{\alpha}} dx < \infty, \tag{47}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing functions. Then by (33) we obtain

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i+1) \right) dx \le$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b \Phi_i\left(|f_i(x)|\right)dx\right)\left(\int_a^b \Phi_j\left(|f_j(x)|\right)\lambda_m(x)\,dx\right), \tag{48}$$

with $j\in\{1,...,m\}$, *true for measurable* f_i *with* $I_{a+}^{\alpha_i}(|f_i|)$ *finite* $(i=1,...,m)$ *and with the properties:*

(i) $\Phi_i(|f_i|)$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $\lambda_m\Phi_j(|f_j|);\Phi_1(|f_1|),\Phi_2(|f_2|),...,\widehat{\Phi_j(|f_j|)},...,\Phi_m(|f_m|)$ *are all Lebesgue integrable functions,*

where $\widehat{\Phi_j(|f_j|)}$ *means absent item.*

Let now

$$u(x)=(x-a)^{\alpha},\quad x\in(a,b). \tag{49}$$

Then

$$\lambda_m(y)=\gamma\int_y^b (x-y)^{\alpha-m}\,dx=\frac{\gamma(b-y)^{\alpha-m+1}}{\alpha-m+1}, \tag{50}$$

$y\in(a,b)$, *where* $\alpha>m-1$.

Hence (48) becomes

$$\int_a^b (x-a)^{\alpha}\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\left(I_{a+}^{\alpha_i}f_i\right)(x)}{(x-a)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx\leq$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b \Phi_i\left(|f_i(x)|\right)dx\right)\left(\int_a^b (b-x)^{\alpha-m+1}\Phi_j\left(|f_j(x)|\right)dx\right)\leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\int_a^b \Phi_i\left(|f_i(x)|\right)dx\right), \tag{51}$$

where $\alpha>m-1$, f_i *with* $I_{a+}^{\alpha_i}(|f_i|)$ *finite,* $i=1,...,m$, *under the assumptions (i), (ii) following (48).*

If $\Phi_i=id$, *then (51) turns to*

$$\int_a^b \prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|dx\leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m}\Gamma(\alpha_i+1)\right)(\alpha-m+1)}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b |f_i(x)|\,dx\right)\cdot$$

$$\left(\int_a^b (b-x)^{\alpha-m+1}|f_j(x)|\,dx\right)\leq$$

$$\left(\frac{\gamma \left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m} \Gamma\left(\alpha_i+1\right)\right)\left(\alpha-m+1\right)} \right) \left(\prod_{i=1}^{m} \int_a^b \left|f_i\left(x\right)\right| dx \right), \tag{52}$$

where $\alpha > m-1$, f_i with $I_{a+}^{\alpha_i}\left(|f_i|\right)$ finite and f_i Lebesgue integrable, $i = 1, ..., m$.

Next let $p_i > 1$, and $\Phi_i\left(x\right) = x^{p_i}$, $x \in \mathbb{R}_+$. These Φ_i are convex, increasing and continuous on $\mathbb{R}_+$.

Then, by (48), we get

$$I_1 := \int_a^b \left(x-a\right)^{\alpha} \prod_{i=1}^{m} \left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{\left(x-a\right)^{\alpha_i}} \right|^{p_i} dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m} \left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)} \right) \left(\prod_{\substack{i=1 \\ i\neq j}}^{m} \int_a^b \left|f_i\left(x\right)\right|^{p_i} dx \right) \cdot$$

$$\left(\int_a^b \left(b-x\right)^{\alpha-m+1} \left|f_j\left(x\right)\right|^{p_j} dx \right) \leq$$

$$\left(\frac{\gamma \left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m} \left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)} \right) \left(\prod_{i=1}^{m} \int_a^b \left|f_i\left(x\right)\right|^{p_i} dx \right). \tag{53}$$

Notice that $\sum_{i=1}^{m} \alpha_i p_i > \alpha$, thus $\beta := \alpha - \sum_{i=1}^{m} \alpha_i p_i < 0$. Since $0 < x-a < b-a$ $(x \in (a,b))$, then $(x-a)^{\beta} > (b-a)^{\beta}$.

Therefore

$$I_1 := \int_a^b \left(x-a\right)^{\beta} \prod_{i=1}^{m} \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \geq$$

$$\left(b-a\right)^{\beta} \int_a^b \prod_{i=1}^{m} \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx. \tag{54}$$

Consequently, by (53) and (54), it holds

$$\int_a^b \prod_{i=1}^{m} \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \leq \tag{55}$$

$$\left(\frac{\gamma(b-a)^{\left(\left(\sum_{i=1}^{m}\alpha_i p_i\right)-m+1\right)}}{\left(\prod_{i=1}^{m}(\Gamma(\alpha_i+1))^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^{m}\int_a^b|f_i(x)|^{p_i}\,dx\right),$$

where $p_i > 1$, $i = 1,..,m$, $\alpha > m-1$, *true for measurable* f_i *with* $I_{a+}^{\alpha_i}(|f_i|)$ *finite, with the properties* ($i = 1,...,m$):

(i) $|f_i|^{p_i}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $|f_i|^{p_i}$ *is Lebesgue integrable on* (a,b).

If $p = p_1 = p_2 = ... = p_m > 1$, *then by (55), we get*

$$\left\|\prod_{i=1}^{m}\left(I_{a+}^{\alpha_i}f_i\right)\right\|_{p,(a,b)} \leq \tag{56}$$

$$\left(\frac{\gamma^{\frac{1}{p}}(b-a)^{\left(\alpha-\frac{m}{p}+\frac{1}{p}\right)}}{\left(\prod_{i=1}^{m}(\Gamma(\alpha_i+1))\right)(\alpha-m+1)^{\frac{1}{p}}}\right)\left(\prod_{i=1}^{m}\|f_i\|_{p,(a,b)}\right),$$

$\alpha > m-1$, *true for measurable* f_i *with* $I_{a+}^{\alpha_i}(|f_i|)$ *finite, and such that* ($i = 1,...,m$),

(i) $|f_i|^p$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $|f_i|^p$ *is Lebesgue integrable on* (a,b).

Using (ii) and if $\alpha_i > \frac{1}{p}$, *by Hölder's inequality we derive that* $I_{a+}^{\alpha_i}(|f_i|)$ *is finite on* (a,b). *If we set* $p = 1$ *to (56) we get (52).*

If $\Phi_i(x) = e^x$, $x \in \mathbb{R}_+$, *then from (51) we get*

$$\int_a^b (x-a)^{\alpha} e^{\sum_{i=1}^{m}\left(\left|\frac{\left(I_{a+}^{\alpha_i}f_i\right)(x)}{(x-a)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)}dx \leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\left(\int_a^b e^{|f_i(x)|}dx\right)\right), \tag{57}$$

where $\alpha > m-1$, f_i *with* $I_{a+}^{\alpha_i}(|f_i|)$ *finite,* $i = 1,...,m$, *under the assumptions,*

(i) $e^{|f_i|}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $e^{|f_i|}$ *is Lebesgue integrable on* (a,b).

We continue with

Remark 12 *Let* f_i *be Lebesgue measurable functions* : $(a,b) \to \mathbb{R}$, *such that* $I_{b-}^{\alpha_i}(|f_i|)(x) < \infty$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1,...,m$, *e.g. when* $f_i \in L_\infty(a,b)$.

Consider

$$g_i(x) = \left(I_{b-}^{\alpha_i} f_i\right)(x), \quad x \in (a,b), \, i = 1, ..., m, \tag{58}$$

we remind

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b f_i(t)(t-x)^{\alpha_i - 1} dt, \tag{59}$$

($x < b$).

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \int_a^b \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t) \, dt. \tag{60}$$

So, we pick here

$$k_i(x,t) := \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, ..., m. \tag{61}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & x \leq y < b, \\ 0, & a < y < x. \end{cases} \tag{62}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} dy = \frac{(b-x)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{63}$$

$a < x < b$, $i = 1, ..., m$.

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + 1)}{(b-x)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1} \alpha_i}{(b-x)^{\alpha_i}} \right) = \chi_{[x,b)}(y) \frac{(y-x)^{\left(\sum_{i=1}^m \alpha_i - m\right)} \left(\prod_{i=1}^m \alpha_i\right)}{(b-x)^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{64}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \quad \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{65}$$

we have that

$$\prod_{i=1}^{m}\frac{k_i(x,y)}{K_i(x)}=\frac{\chi_{[x,b)}(y)(y-x)^{\alpha-m}\gamma}{(b-x)^{\alpha}}. \tag{66}$$

Therefore, for (32), we get for appropiate weight u that

$$\lambda_m(y)=\gamma\int_a^y u(x)\frac{(y-x)^{\alpha-m}}{(b-x)^{\alpha}}dx<\infty, \tag{67}$$

for all $a<y<b$.

Let $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,...,m$, be convex and increasing functions. Then by (33) we obtain

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\left(I_{b-}^{\alpha_i}f_i\right)(x)}{(b-x)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx\le$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\Phi_i(|f_i(x)|)\,dx\right)\left(\int_a^b\Phi_j(|f_j(x)|)\,\lambda_m(x)\,dx\right), \tag{68}$$

with $j\in\{1,...,m\}$,

true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite ($i=1,...,m$) and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$,

(ii) $\lambda_m\Phi_j(|f_j|);\Phi_1(|f_1|),...,\widehat{\Phi_j(|f_j|)},...,\Phi_m(|f_m|)$ are all Lebesgue integrable functions,

where $\widehat{\Phi_j(|f_j|)}$ means absent item.

Let now

$$u(x)=(b-x)^{\alpha},\quad x\in(a,b). \tag{69}$$

Then

$$\lambda_m(y)=\gamma\int_a^y(y-x)^{\alpha-m}\,dx=\frac{\gamma(y-a)^{\alpha-m+1}}{\alpha-m+1}, \tag{70}$$

$y\in(a,b)$, where $\alpha>m-1$.

Hence (68) becomes

$$\int_a^b(b-x)^{\alpha}\prod_{i=1}^{m}\Phi_i\left(\frac{\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|}{(b-x)^{\alpha_i}}\Gamma(\alpha_i+1)\right)dx\le$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\Phi_i(|f_i(x)|)\,dx\right)\left(\int_a^b(x-a)^{\alpha-m+1}\Phi_j(|f_j(x)|)\,dx\right)\le$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\int_a^b\Phi_i(|f_i(x)|)\,dx\right), \tag{71}$$

where $\alpha > m-1$, f_i *with* $I_{b-}^{\alpha_i}(|f_i|)$ *finite,* $i = 1, ..., m$, *under the assumptions (i), (ii) following (68).*

If $\Phi_i = id$, *then (71) turns to*

$$\int_a^b \prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m \Gamma(\alpha_i + 1) \right)(\alpha - m + 1)} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b |f_i(x)| \, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha - m + 1} |f_j(x)| \, dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\left(\prod_{i=1}^m \Gamma(\alpha_i + 1) \right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)| \, dx \right), \tag{72}$$

where $\alpha > m-1$, f_i *with* $I_{b-}^{\alpha_i}(|f_i|)$ *finite and* f_i *Lebesgue integrable,* $i = 1, ..., m$.

Next let $p_i > 1$, *and* $\Phi_i(x) = x^{p_i}$, $x \in \mathbb{R}_+$.

Then, by (68), we get

$$I_2 := \int_a^b (b-x)^{\alpha} \frac{\left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right)}{(b-x)^{\sum_{i=1}^m \alpha_i p_i}} dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m (\Gamma(\alpha_i + 1))^{p_i} \right)(\alpha - m + 1)} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b |f_i(x)|^{p_i} \, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha - m + 1} |f_j(x)|^{p_j} \, dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\left(\prod_{i=1}^m (\Gamma(\alpha_i + 1))^{p_i} \right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} \, dx \right). \tag{73}$$

Notice here that $\beta := \alpha - \sum_{i=1}^{m} \alpha_i p_i < 0$. *Since* $0 < b - x < b - a$ $(x \in (a,b))$, *then* $(b-x)^{\beta} > (b-a)^{\beta}$.

Therefore

$$I_2 := \int_a^b (b-x)^{\beta} \left(\prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right) dx \geq$$

$$(b-a)^{\beta} \int_a^b \left(\prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right) dx. \tag{74}$$

Consequently, by (73) and (74), it holds

$$\int_a^b \prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{75}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^{m} \alpha_i p_i \right) - m + 1 \right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i + 1) \right)^{p_i} \right) (\alpha - m + 1)} \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $p_i > 1$, $i = 1, .., m$, $\alpha > m - 1$,

true for measurable f_i *with* $I_{b-}^{\alpha_i}(|f_i|)$ *finite, with the properties* $(i = 1, ..., m)$:

(i) $|f_i|^{p_i}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $|f_i|^{p_i}$ *is Lebesgue integrable on* (a,b).

If $p := p_1 = p_2 = ... = p_m > 1$, *then by (75), we get*

$$\left\| \prod_{i=1}^{m} \left(I_{b-}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{76}$$

$$\left(\frac{\gamma^{\frac{1}{p}} (b-a)^{\left(\alpha - \frac{m}{p} + \frac{1}{p} \right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i + 1) \right) \right) (\alpha - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \|f_i\|_{p,(a,b)} \right),$$

$\alpha > m - 1$,

true for measurable f_i *with* $I_{b-}^{\alpha_i}(|f_i|)$ *finite, and such that* $(i = 1, ..., m)$,

(i) $|f_i|^{p}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $|f_i|^{p}$ *is Lebesgue integrable on* (a,b).

Using (ii) and if $\alpha_i > \frac{1}{p}$, *by Hölder's inequality we derive that* $I_{b-}^{\alpha_i}(|f_i|)$ *is finite on* (a,b).

If we set $p = 1$ *to (76) we obtain (72).*

If $\Phi_i(x) = e^x$, $x \in \mathbb{R}_+$, *then from (71) we obtain*

$$\int_a^b (b-x)^\alpha e^{\sum_{i=1}^m \left(\left| \frac{\left(I_{b-}^{\alpha_i} f_i\right)(x)}{(b-x)^{\alpha_i}} \right| \Gamma(\alpha_i+1) \right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1} \right) \left(\prod_{i=1}^m \left(\int_a^b e^{|f_i(x)|} dx \right) \right), \tag{77}$$

where $\alpha > m-1$, f_i *with* $I_{b-}^{\alpha_i}(|f_i|)$ *finite,* $i = 1, ..., m$, *under the assumptions,*

(i) $e^{|f_i|}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $e^{|f_i|}$ *is Lebesgue integrable on* (a,b).

We mention

Definition 13 *([1], p. 448) The left generalized Riemann-Liouville fractional derivative of* f *of order* $\beta > 0$ *is given by*

$$D_a^\beta f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx} \right)^n \int_a^x (x-y)^{n-\beta-1} f(y)\,dy, \tag{78}$$

where $n = [\beta] + 1$, $x \in [a,b]$.

For $a, b \in \mathbb{R}$, *we say that* $f \in L_1(a,b)$ *has an* L_∞ *fractional derivative* $D_a^\beta f$ $(\beta > 0)$ *in* $[a,b]$, *if and only if*

(1) $D_a^{\beta-k} f \in C([a,b])$, $k = 2, ..., n = [\beta] + 1$,

(2) $D_a^{\beta-1} f \in AC([a,b])$

(3) $D_a^\beta f \in L_\infty(a,b)$.

Above we define $D_a^0 f := f$ *and* $D_a^{-\delta} f := I_{a+}^\delta f$, *if* $0 < \delta \leq 1$.

From [1, p. 449] and [9] we mention and use

Lemma 14 *Let* $\beta > \alpha \geq 0$ *and let* $f \in L_1(a,b)$ *have an* L_∞ *fractional derivative* $D_a^\beta f$ *in* $[a,b]$ *and let* $D_a^{\beta-k} f(a) = 0$, $k = 1, ..., [\beta] + 1$, *then*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^\beta f(y)\,dy, \tag{79}$$

for all $a \leq x \leq b$.

Here $D_a^\alpha f \in AC([a,b])$ *for* $\beta - \alpha \geq 1$, *and* $D_a^\alpha f \in C([a,b])$ *for* $\beta - \alpha \in (0,1)$.

Notice here that

$$D_a^\alpha f(x) = \left(I_{a+}^{\beta-\alpha} \left(D_a^\beta f \right) \right)(x), \quad a \leq x \leq b. \tag{80}$$

We give

Theorem 15 *Let $f_i \in L_1(a,b)$, $\alpha_i, \beta_i : \beta_i > \alpha_i \geq 0$, $i = 1, ..., m$. Here (f_i, α_i, β_i) fulfill terminology and assumptions of Definition 13 and Lemma 14. Let $\overline{\alpha} := \sum_{i=1}^{m} (\beta_i - \alpha_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\beta_i - \alpha_i)$, assume $\overline{\alpha} > m-1$, and $p \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(D_a^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{81}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p} \right)}}{\left(\prod_{i=1}^{m} \left(\Gamma\left(\beta_i - \alpha_i + 1 \right) \right) \right) \left(\overline{\alpha} - m + 1 \right)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| D_a^{\beta_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (52) and (56). ■

We continue with

Theorem 16 *All here as in Theorem 15. Then*

$$\int_a^b (x-a)^{\overline{\alpha}} e^{\sum_{i=1}^{m} \left(\left| \frac{\left(D_a^{\alpha_i} f_i \right)(x)}{(x-a)^{\left(\beta_i - \alpha_i \right)}} \right| \Gamma\left(\beta_i - \alpha_i + 1 \right) \right)} dx \leq$$

$$\left(\frac{\overline{\gamma} (b-a)^{\overline{\alpha} - m + 1}}{\overline{\alpha} - m + 1} \right) \left(\prod_{i=1}^{m} \left(\int_a^b e^{\left| \left(D_a^{\beta_i} f_i \right)(x) \right|} dx \right) \right). \tag{82}$$

Proof. By (57); assumptions there (i) and (ii) are easily fulfilled. ■

We need

Definition 17 *([6], p. 50, [1], p. 449) Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then the left Caputo fractional derivative is given by*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) \, dt$$
$$= \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{83}$$

and it exists almost everywhere for $x \in [a,b]$, in fact $D_{*a}^{\nu} f \in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.

We also need

Theorem 18 *([4]) Let $\nu \geq \rho + 1$, $\rho > 0$, $\nu, \rho \notin \mathbb{N}$. Call $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. Assume $f \in AC^n([a,b])$, such that $f^{(k)}(a) = 0$, $k = m^*, m^*+1, ..., n-1$, and $D_{*a}^{\nu} f \in L_{\infty}(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)}\right)(x)$), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\, dt$$
$$= \left(I_{a+}^{\nu-\rho}\left(D_{*a}^{\nu} f\right)\right)(x), \tag{84}$$

$\forall\, x \in [a,b]$.

We give

Theorem 19 *Let (f_i, ν_i, ρ_i), $i = 1, ..., m$, $m \geq 2$, as in the assumptions of Theorem 18. Set $\overline{\alpha} := \sum_{i=1}^{m} (\nu_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\nu_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_i\right) \right\|_{p,(a,b)} \leq \tag{85}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p}\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\nu_i - \rho_i + 1)\right)\right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| D_{*a}^{\nu_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (52) and (56), see here $\overline{\alpha} \geq m > m-1$. ■

We also give

Theorem 20 *Here all as in Theorem 19, let $p_i \geq 1$, $i = 1, ..., l$; $l < m$. Then*

$$\int_a^b (x-a)^{\left(\overline{\alpha} - \sum_{i=1}^{l} p_i(\nu_i - \rho_i)\right)} \left(\prod_{i=1}^{l} \left| D_{*a}^{\rho_i} f_i(x) \right|^{p_i} \right) \cdot$$

$$e^{\left(\sum_{i=l+1}^{m} \left| D_{*a}^{\rho_i} f_i(x) \right| \left(\frac{\Gamma(\nu_i - \rho_i + 1)}{(x-a)^{(\nu_i - \rho_i)}} \right) \right)} dx \leq$$

$$\left(\frac{\overline{\gamma}\, (b-a)^{\overline{\alpha} - m + 1}}{\left(\prod_{i=1}^{l} \left(\Gamma(\nu_i - \rho_i + 1)\right)^{p_i} \right) (\overline{\alpha} - m + 1)} \right) \left(\prod_{i=1}^{l} \int_a^b \left| D_{*a}^{\nu_i} f_i(x) \right|^{p_i} dx \right) \cdot \tag{86}$$

$$\left(\prod_{i=l+1}^{m} \int_a^b e^{\left| D_{*a}^{\nu_i} f_i(x) \right|} dx \right).$$

Proof. By (51). ■

We need

Definition 21 *([2], [7], [8]) Let $\alpha \geq 0$, $n := \lceil \alpha \rceil$, $f \in AC^n([a,b])$. We define the right Caputo fractional derivative of order $\alpha \geq 0$, by*

$$\overline{D}_{b-}^{\alpha} f(x) := (-1)^n I_{b-}^{n-\alpha} f^{(n)}(x), \tag{87}$$

we set $\overline{D}_{-}^{0} f := f$, i.e.

$$\overline{D}_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (J-x)^{n-\alpha-1} f^{(n)}(J)\, dJ. \tag{88}$$

Notice that $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$, $n \in \mathbb{N}$.

We need

Theorem 22 *([4]) Let $f \in AC^n([a,b])$, $\alpha > 0$, $n \in \mathbb{N}$, $n := \lceil \alpha \rceil$, $\alpha \geq \rho + 1$, $\rho > 0$, $r = \lceil \rho \rceil$, $\alpha, \rho \notin \mathbb{N}$. Assume $f^{(k)}(b) = 0$, $k = r, r+1, ..., n-1$, and $\overline{D}_{b-}^{\alpha} f \in L_\infty([a,b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho} \left(\overline{D}_{b-}^{\alpha} f \right) \right)(x) \in AC([a,b]), \tag{89}$$

that is

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f \right)(t)\, dt, \tag{90}$$

$\forall\, x \in [a,b]$.

We give

Theorem 23 *Let (f_i, α_i, ρ_i), $i = 1, ..., m$, $m \geq 2$, as in the assumptions of Theorem 22. Set $\overline{\alpha} := \sum_{i=1}^{m} (\alpha_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\alpha_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^{m} \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{91}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p}\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i - \rho_i + 1) \right) \right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| \overline{D}_{b-}^{\nu_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (72) and (76), see here $\overline{\alpha} \geq m > m - 1$. ■

We make

Remark 24 *Let* $r_1, r_2 \in \mathbb{N}$*;* $A_j > 0$*,* $j = 1, ..., r_1$*;* $B_j > 0$*,* $j = 1, ..., r_2$*;* $x \geq 0$*,* $p \geq 1$*. Clearly* $e^{A_j x^p}, e^{B_j x^p} \geq 1$*, and* $\sum_{j=1}^{r_1} e^{A_j x^p} \geq r_1$*,* $\sum_{j=1}^{r_2} e^{B_j x^p} \geq r_2$*. Hence* $\varphi_1(x) := \ln\left(\sum_{j=1}^{r_1} e^{A_j x^p}\right)$*,* $\varphi_2(x) := \ln\left(\sum_{j=1}^{r_2} e^{B_j x^p}\right) \geq 0$*. Clearly here* $\varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *are increasing, convex and continuous.*

We give

Theorem 25 *Let* (f_i, α_i, ρ_i)*,* $i = 1, 2$*, as in the assumptions of Theorem 22. Set* $\overline{\alpha} := \sum_{i=1}^{2} (\alpha_i - \rho_i)$*,* $\overline{\gamma} := \prod_{i=1}^{2} (\alpha_i - \rho_i)$*. Here* $a, b \in \mathbb{R}$*,* $a < b$*, and* φ_1, φ_2 *as in Remark 24. Then*

$$\int_a^b (b-x)^{\overline{\alpha}} \prod_{i=1}^{2} \varphi_i \left(\frac{\left| \overline{D}_{b-}^{\rho_i} f_i(x) \right|}{(b-x)^{(\alpha_i - \rho_i)}} \Gamma(\alpha_i - \rho_i + 1) \right) dx \leq \tag{92}$$

$$\left(\frac{\overline{\gamma} (b-a)^{\overline{\alpha}-1}}{\overline{\alpha} - 1} \right) \left(\prod_{i=1}^{2} \int_a^b \varphi_i \left(\left| \overline{D}_{b-}^{\alpha_i} f_i(x) \right| \right) dx \right),$$

under the assumptions $(i = 1, 2)$*:*

(i) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i(t)\right|\right)$ *is* $\left(\chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - \rho_i - 1}}{\Gamma(\alpha_i - \rho_i)} dt\right)$ *-integrable, a.e. in* $x \in (a, b)$*,*

(ii) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i\right|\right)$ *is Lebesgue integrable on* (a, b)*.*

We make

Remark 26 *(i) Let now* $f \in C^n([a, b])$*,* $n = \lceil \nu \rceil$*,* $\nu > 0$*. Clearly* $C^n([a, b]) \subset AC^n([a, b])$*. Assume* $f^{(k)}(a) = 0$*,* $k = 0, 1, ..., n-1$*. Given that* $D_{*a}^{\nu} f$ *exists, then there exists the left generalized Riemann-Liouville fractional derivative* $D_a^{\nu} f$*, see (78), and* $D_{*a}^{\nu} f = D_a^{\nu} f$*, see also [6], p. 53. In fact here* $D_{*a}^{\nu} f \in C([a, b])$*, see [6], p. 56.*

So Theorems 19, 20 can be true for left generalized Riemann-Liouville fractional derivatives.

(ii) Let also $\alpha > 0$*,* $n := \lceil \alpha \rceil$*, and* $f \in C^n([a, b]) \subset AC^n([a, b])$*. From [2] we derive that* $\overline{D}_{b-}^{\alpha} f \in C([a, b])$*. By [2], we obtain that the right Riemann-Liouville fractional derivative* $D_{b-}^{\alpha} f$ *exists on* $[a, b]$*. Furthermore if* $f^{(k)}(b) = 0$*,* $k = 0, 1, ..., n-1$*, we get that* $\overline{D}_{b-}^{\alpha} f(x) = D_{b-}^{\alpha} f(x)$*,* $\forall\, x \in [a, b]$*, hence* $D_{b-}^{\alpha} f \in C([a, b])$*.*

So Theorems 23, 25 can be valid for right Riemann-Liouville fractional derivatives. To keep article short we avoid details.

We give

Definition 27 *Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$ $(0 \leq \alpha < 1)$. Let $a, b \in \mathbb{R}$, $a \leq x \leq b$, $f \in C([a,b])$. We consider $C_a^\nu([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$. For $f \in C_a^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as*

$$\Delta_a^\nu f := \left(I_{a+}^{1-\alpha} f^{(n)}\right)', \tag{93}$$

see [1], p. 24, and Canavati derivative in [5].

Notice here $\Delta_a^\nu f \in C([a,b])$.

So that

$$(\Delta_a^\nu f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t)\, dt, \tag{94}$$

$\forall\ x \in [a,b]$.

Notice here that

$$\Delta_a^n f = f^{(n)}, \quad n \in \mathbb{Z}_+. \tag{95}$$

We need

Theorem 28 *([4]) Let $f \in C_a^\nu([a,b])$, $n = [\nu]$, such that $f^{(i)}(a) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, with $0 < \rho < \nu$. Then*

$$(\Delta_a^\rho f)(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} (\Delta_a^\nu f)(t)\, dt, \tag{96}$$

i.e.

$$(\Delta_a^\rho f) = I_{a+}^{\nu-\rho} (\Delta_a^\nu f) \in C([a,b]). \tag{97}$$

Thus $f \in C_a^\rho([a,b])$.

We present

Theorem 29 *Let (f_i, ν_i, ρ_i), $i = 1, ..., m$, as in Theorem 28 and fractional derivatives as in Definition 27. Let $\alpha := \sum_{i=1}^m (\nu_i - \rho_i)$, $\gamma := \prod_{i=1}^m (\nu_i - \rho_i)$, $p_i \geq 1$, $i = 1, ..., m$, assume $\alpha > m-1$. Then*

$$\int_a^b \prod_{i=1}^m |\Delta_a^{\rho_i} f_i(x)|^{p_i}\, dx \leq \tag{98}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^m (\nu_i - \rho_i) p_i\right) - m + 1\right)}}{\left(\prod_{i=1}^m (\Gamma(\nu_i - \rho_i + 1))^{p_i}\right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |\Delta_a^{\nu_i} f_i(x)|^{p_i}\, dx \right).$$

Proof. By (52) and (55). ■

We continue with

Theorem 30 *Let all here as in Theorem 29. Consider λ_i, $i = 1, ..., m$, distinct prime numbers. Then*

$$\int_a^b (x-a)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_a^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i-\rho_i+1)}{(x-a)^{(\nu_i-\rho_i)}}\right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\alpha-m+1}\right) \left(\prod_{i=1}^{m} \int_a^b \lambda_i^{\left|\Delta_a^{\nu_i} f_i(x)\right|} dx\right). \tag{99}$$

Proof. By (51). ■

We need

Definition 31 *([2]) Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C([a,b])$. Consider*

$$C_{b-}^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b])\}. \tag{100}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1} \left(I_{b-}^{1-\alpha} f^{(n)}\right)'. \tag{101}$$

We set $\Delta_{b-}^{0} f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (J-x)^{-\alpha} f^{(n)}(J)\, dJ, \tag{102}$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 32 *([4]) Let $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. Assume $f^{(i)}(b) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, $n := [\nu]$. Then*

$$\Delta_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_x^b (J-x)^{\nu-\rho-1} \left(\Delta_{b-}^{\nu} f\right)(J)\, dJ, \tag{103}$$

$\forall\ x \in [a,b]$, i.e.

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu-\rho} \left(\Delta_{b-}^{\nu} f\right) \in C([a,b]), \tag{104}$$

and $f \in C_{b-}^{\rho}([a,b])$.

We give

Theorem 33 *Let (f_i, ν_i, ρ_i), $i = 1, ..., m$, and fractional derivatives as in Theorem 32 and Definition 31. Let $\alpha := \sum_{i=1}^{m} (\nu_i - \rho_i)$, $\gamma := \prod_{i=1}^{m} (\nu_i - \rho_i)$, $p_i \geq 1$, $i = 1, ..., m$, and assume $\alpha > m - 1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left|\Delta_{b-}^{\rho_i} f_i(x)\right|^{p_i} dx \leq \tag{105}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) p_i\right) - m + 1\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\nu_i - \rho_i + 1)\right)^{p_i}\right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^{m} \int_a^b \left|\Delta_{b-}^{\nu_i} f_i(x)\right|^{p_i} dx \right).$$

Proof. By (72) and (75). ■

We continue with

Theorem 34 *Let all here as in Theorem 33. Consider λ_i, $i = 1, ..., m$, distinct prime numbers. Then*

$$\int_a^b (b-x)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_{b-}^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i - \rho_i + 1)}{(b-x)^{(\nu_i - \rho_i)}}\right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^{m} \int_a^b \lambda_i^{\left|\Delta_{b-}^{\nu_i} f_i(x)\right|} dx \right). \tag{106}$$

Proof. By (71). ■

We make

Definition 35 *[12, p. 99] The fractional integrals of a function f with respect to given function g are defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is an increasing function on $[a, b]$ and $g \in C^1([a, b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a, b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)\, dt}{(g(x) - g(t))^{1-\alpha}}, \quad x > a, \tag{107}$$

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)\, dt}{(g(t) - g(x))^{1-\alpha}}, \quad x < b, \tag{108}$$

respectively.

We make

Remark 36 *Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+;g}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1,...,m$.*

Consider

$$g_i(x) := \left(I_{a+;g}^{\alpha_i} f_i\right)(x), \quad x \in (a,b), \ i = 1,...,m, \tag{109}$$

where

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x \frac{g'(t) f_i(t)\, dt}{(g(x) - g(t))^{1-\alpha_i}}, \quad x > a. \tag{110}$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)\, g'(t)\, f_i(t)}{\Gamma(\alpha_i)(g(x) - g(t))^{1-\alpha_i}} dt, \tag{111}$$

where χ is the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t)\, g'(t)}{\Gamma(\alpha_i)(g(x) - g(t))^{1-\alpha_i}}, \quad i = 1,...,m. \tag{112}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}, & a < y \le x, \\ 0, & x < y < b. \end{cases} \tag{113}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{(a,x]}(y)\, g'(y)}{\Gamma(\alpha_i)(g(x) - g(y))^{1-\alpha_i}} dy =$$

$$\int_a^x \frac{g'(y)}{\Gamma(\alpha_i)(g(x) - g(y))^{1-\alpha_i}} dy = \frac{1}{\Gamma(\alpha_i)} \int_a^x (g(x) - g(y))^{\alpha_i - 1} dg(y) = \tag{114}$$

$$\frac{1}{\Gamma(\alpha_i)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha_i - 1} dz = \frac{(g(x) - g(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)}.$$

So for $a < x < b$, $i = 1,...,m$, we get

$$K_i(x) = \frac{(g(x) - g(a))^{\alpha_i}}{\Gamma(\alpha_i + 1)}. \tag{115}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)\, g'(y)}{\Gamma(\alpha_i)(g(x) - g(y))^{1-\alpha_i}} \cdot \frac{\Gamma(\alpha_i + 1)}{(g(x) - g(a))^{\alpha_i}} \right) =$$

$$\frac{\chi_{(a,x]}(y)\left(g(x)-g(y)\right)^{\left(\sum_{i=1}^{m}\alpha_i-m\right)}\left(g'(y)\right)^m\left(\prod_{i=1}^{m}\alpha_i\right)}{\left(g(x)-g(a)\right)^{\left(\sum_{i=1}^{m}\alpha_i\right)}}. \tag{116}$$

Calling

$$\alpha:=\sum_{i=1}^{m}\alpha_i>0,\quad \gamma:=\prod_{i=1}^{m}\alpha_i>0, \tag{117}$$

we have that

$$\prod_{i=1}^{m}\frac{k_i(x,y)}{K_i(x)}=\frac{\chi_{(a,x]}(y)\left(g(x)-g(y)\right)^{\alpha-m}\left(g'(y)\right)^m\gamma}{\left(g(x)-g(a)\right)^{\alpha}}. \tag{118}$$

Therefore, for (32), we get for appropiate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y)=\gamma\left(g'(y)\right)^m\int_y^b u(x)\frac{\left(g(x)-g(y)\right)^{\alpha-m}}{\left(g(x)-g(a)\right)^{\alpha}}dx<\infty, \tag{119}$$

for all $a<y<b$.

Let $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,...,m$, be convex and increasing functions. Then by (33) we obtain

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\left(I_{a+;g}^{\alpha_i}f_i\right)(x)}{\left(g(x)-g(a)\right)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx\leq$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\Phi_i\left(|f_i(x)|\right)dx\right)\left(\int_a^b\Phi_j\left(|f_j(x)|\right)\lambda_m^g(x)\,dx\right), \tag{120}$$

with $j\in\{1,...,m\}$,

true for measurable f_i with $I_{a+;g}^{\alpha_i}(|f_i|)$ finite, $i=1,...,m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$,

(ii) $\lambda_m^g\Phi_j(|f_j|)$; $\Phi_1(|f_1|),\Phi_2(|f_2|),...,\widehat{\Phi_j(|f_j|)},...,\Phi_m(|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j(|f_j|)}$ means absent item.

Let now

$$u(x)=\left(g(x)-g(a)\right)^{\alpha}g'(x),\quad x\in(a,b). \tag{121}$$

Then

$$\lambda_m^g(y)=\gamma\left(g'(y)\right)^m\int_y^b\left(g(x)-g(y)\right)^{\alpha-m}g'(x)\,dx=$$

$$\gamma\left(g'(y)\right)^m\int_{g(y)}^{g(b)}\left(z-g(y)\right)^{\alpha-m}dz= \tag{122}$$

$$\gamma\left(g'(y)\right)^{m}\frac{\left(g(b)-g(y)\right)^{\alpha-m+1}}{\alpha-m+1},$$

with $\alpha>m-1$. *That is*

$$\lambda_{m}^{g}(y)=\gamma\left(g'(y)\right)^{m}\frac{\left(g(b)-g(y)\right)^{\alpha-m+1}}{\alpha-m+1}, \tag{123}$$

$\alpha>m-1$, $y\in(a,b)$.

Hence (120) becomes

$$\int_{a}^{b}g'(x)\left(g(x)-g(a)\right)^{\alpha}\prod_{i=1}^{m}\Phi_{i}\left(\left|\frac{\left(I_{a+;g}^{\alpha_i}f_i\right)(x)}{\left(g(x)-g(a)\right)^{\alpha_i}}\right|\Gamma\left(\alpha_i+1\right)\right)dx\leq$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_{a}^{b}\Phi_{i}\left(\left|f_i(x)\right|\right)dx\right)\cdot$$

$$\left(\int_{a}^{b}\left(g'(x)\right)^{m}\left(g(b)-g(x)\right)^{\alpha-m+1}\Phi_{j}\left(\left|f_j(x)\right|\right)dx\right)\leq \tag{124}$$

$$\left(\frac{\gamma\left(g(b)-g(a)\right)^{\alpha-m+1}\left\|g'\right\|_{\infty}^{m}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\int_{a}^{b}\Phi_{i}\left(\left|f_i(x)\right|\right)dx\right),$$

where $\alpha>m-1$, f_i *with* $I_{a+;g}^{\alpha_i}\left(\left|f_i\right|\right)$ *finite,* $i=1,...,m$, *under the assumptions:*

(i) $\Phi_i\left(\left|f_i\right|\right)$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $\Phi_i\left(\left|f_i\right|\right)$ *is Lebesgue integrable on* (a,b).

If $\Phi_i(x)=x^{p_i}$, $p_i\geq1$, $x\in\mathbb{R}_{+}$, *then by (124), we have*

$$\int_{a}^{b}g'(x)\left(g(x)-g(a)\right)^{\left(\alpha-\sum_{i=1}^{m}p_i\alpha_i\right)}\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx\leq \tag{125}$$

$$\left(\frac{\gamma\left(g(b)-g(a)\right)^{\alpha-m+1}\left\|g'\right\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^{m}\int_{a}^{b}\left|f_i(x)\right|^{p_i}dx\right),$$

but we see that

$$\int_{a}^{b}g'(x)\left(g(x)-g(a)\right)^{\left(\alpha-\sum_{i=1}^{m}p_i\alpha_i\right)}\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx\geq$$

$$\left(g(b)-g(a)\right)^{\left(\alpha-\sum_{i=1}^{m}p_i\alpha_i\right)}\int_{a}^{b}g'(x)\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx. \tag{126}$$

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By (125) and (126) we get

$$\int_a^b g'(x) \prod_{i=1}^m \left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{127}$$

$$\left(\frac{\gamma \left(g(b) - g(a) \right)^{\left(\sum_{i=1}^m p_i \alpha_i - m + 1 \right)} \|g'\|_\infty^m}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right)^{p_i} \right) (\alpha - m + 1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} dx \right),$$

$\alpha > m - 1$, f_i with $I_{a+;g}^{\alpha_i}(|f_i|)$ finite, $i = 1, ..., m$, under the assumptions:
(i) $|f_i|^{p_i}$ is $k_i(x, y) dy$ -integrable, a.e. in $x \in (a, b)$,
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a, b).

We need

Definition 37 *([11]) Let $0 < a < b < \infty$, $\alpha > 0$. The left- and right-sided Hadamard fractional integrals of order α are given by*

$$\left(J_{a+}^\alpha f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y} \right)^{\alpha - 1} \frac{f(y)}{y} dy, \quad x > a, \tag{128}$$

and

$$\left(J_{b-}^\alpha f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x} \right)^{\alpha - 1} \frac{f(y)}{y} dy, \quad x < b, \tag{129}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Above f is a Lebesgue measurable function from (a, b) into $\mathbb{R}$, such that $\left(J_{a+}^\alpha(|f|) \right)(x)$ and/or $\left(J_{b-}^\alpha(|f|) \right)(x) \in \mathbb{R}$, $\forall\, x \in (a, b)$.

We give

Theorem 38 *Let (f_i, α_i), $i = 1, ..., m$; $J_{a+}^{\alpha_i} f_i$ as in Definition 37. Set $\alpha := \sum_{i=1}^m \alpha_i$, $\gamma := \prod_{i=1}^m \alpha_i$; $p_i \geq 1$, $i = 1, ..., m$, assume $\alpha > m - 1$. Then*

$$\int_a^b \prod_{i=1}^m \left| \left(J_{a+}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{130}$$

$$\left(\frac{b\gamma \left(\ln\left(\frac{b}{a}\right) \right)^{\left(\sum_{i=1}^m p_i \alpha_i - m + 1 \right)}}{a^m (\alpha - m + 1) \left(\prod_{i=1}^m \left(\Gamma(\alpha_i + 1) \right)^{p_i} \right)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ *is finite,* $i=1,...,m$, *under the assumptions:*

(i) $|f_i(y)|^{p_i}$ *is* $\left(\frac{\chi_{(a,x]}(y)dy}{\Gamma(\alpha_i)y\left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_i}}\right)$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $|f_i|^{p_i}$ *is Lebesgue integrable on* (a,b).

We also present

Theorem 39 *Let all as in Theorem 38. Consider* $p:=p_1=p_2=...=p_m\geq 1$. *Then*

$$\left\|\prod_{i=1}^{m}\left(J_{a+}^{\alpha_i}f_i\right)\right\|_{p,(a,b)}\leq \tag{131}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}}\left(\ln\left(\frac{b}{a}\right)\right)^{\left(\alpha-\frac{m}{p}+\frac{1}{p}\right)}}{a^{\frac{m}{p}}(\alpha-m+1)^{\frac{1}{p}}\left(\prod_{i=1}^{m}\left(\Gamma(\alpha_i+1)\right)\right)}\right)\left(\prod_{i=1}^{m}\|f_i\|_{p,(a,b)}\right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ *is finite,* $i=1,...,m$, *under the assumptions:*

(i) $|f_i(y)|^{p}$ *is* $\left(\frac{\chi_{(a,x]}(y)dy}{\Gamma(\alpha_i)y\left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_i}}\right)$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $|f_i|^{p}$ *is Lebesgue integrable on* (a,b).

We make

Remark 40 *Let* f_i *be Lebesgue measurable functions from* (a,b) *into* $\mathbb{R}$, *such that* $\left(I_{b-;g}^{\alpha_i}(|f_i|)\right)(x)\in\mathbb{R}$, $\forall\ x\in(a,b)$, $\alpha_i>0$, $i=1,...,m$.

Consider

$$g_i(x):=\left(I_{b-;g}^{\alpha_i}f_i\right)(x),\quad x\in(a,b),\ i=1,...,m, \tag{132}$$

where

$$\left(I_{b-;g}^{\alpha_i}f_i\right)(x)=\frac{1}{\Gamma(\alpha_i)}\int_x^b\frac{g'(t)f(t)\,dt}{(g(t)-g(x))^{1-\alpha_i}},\quad x<b. \tag{133}$$

Notice that $g_i(x)\in\mathbb{R}$ *and it is Lebesgue measurable.*

We pick $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, *the Lebesgue measure.*

We see that

$$\left(I_{b-;g}^{\alpha_i}f_i\right)(x)=\int_a^b\frac{\chi_{[x,b)}(t)\,g'(t)\,f(t)\,dt}{\Gamma(\alpha_i)\,(g(t)-g(x))^{1-\alpha_i}}, \tag{134}$$

where χ *is the characteristic function.*

So, we pick here

$$k_i(x,y):=\frac{\chi_{[x,b)}(y)\,g'(y)}{\Gamma(\alpha_i)\,(g(y)-g(x))^{1-\alpha_i}},\quad i=1,...,m. \tag{135}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}}, & x \le y < b, \\ 0, & a < y < x. \end{cases} \tag{136}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{[x,b)}(y)\, g'(y)\, dy}{\Gamma(\alpha_i)\,(g(y)-g(x))^{1-\alpha_i}} =$$

$$\frac{1}{\Gamma(\alpha_i)} \int_x^b g'(y)\,(g(y)-g(x))^{\alpha_i-1}\, dy = \tag{137}$$

$$\frac{1}{\Gamma(\alpha_i)} \int_{g(x)}^{g(b)} (z-g(x))^{\alpha_i-1}\, dg(y) = \frac{(g(b)-g(x))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

So for $a < x < b$, $i = 1, ..., m$, we get

$$K_i(x) = \frac{(g(b)-g(x))^{\alpha_i}}{\Gamma(\alpha_i+1)}. \tag{138}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_i)\,(g(y)-g(x))^{1-\alpha_i}} \cdot \frac{\Gamma(\alpha_i+1)}{(g(b)-g(x))^{\alpha_i}} \right) =$$

$$\frac{\chi_{[x,b)}(y)\,(g'(y))^m\,(g(y)-g(x))^{\left(\sum_{i=1}^m \alpha_i - m\right)} \prod_{i=1}^m \alpha_i}{(g(b)-g(x))^{\sum_{i=1}^m \alpha_i}}. \tag{139}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \quad \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{140}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{[x,b)}(y)\,(g'(y))^m\,(g(y)-g(x))^{\alpha-m}\,\gamma}{(g(b)-g(x))^{\alpha}}. \tag{141}$$

Therefore, for (32), we get for appropiate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y) = \gamma\,(g'(y))^m \int_a^y u(x)\, \frac{(g(y)-g(x))^{\alpha-m}}{(g(b)-g(x))^{\alpha}}\, dx < \infty, \tag{142}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing functions. Then by (33) we obtain

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{b-;g}^{\alpha_i} f_i \right)(x)}{(g(b) - g(x))^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i (|f_i(x)|) \, dx \right) \left(\int_a^b \Phi_j (|f_j(x)|) \, \lambda_m^g(x) \, dx \right), \tag{143}$$

with $j \in \{1, ..., m\}$,

true for measurable f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i = 1, ..., m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x, y)\, dy$ -integrable, a.e. in $x \in (a, b)$,

(ii) $\lambda_m^g \Phi_j (|f_j|) ; \Phi_1 (|f_1|), ..., \widehat{\Phi_j(|f_j|)}, ..., \Phi_m (|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j(|f_j|)}$ means absent item.

Let now

$$u(x) = (g(b) - g(x))^{\alpha} g'(x), \quad x \in (a, b). \tag{144}$$

Then

$$\lambda_m^g(y) = \gamma (g'(y))^m \int_a^y g'(x) (g(y) - g(x))^{\alpha - m} dx =$$

$$\gamma (g'(y))^m \int_a^y (g(y) - g(x))^{\alpha - m} dg(x) = \gamma (g'(y))^m \int_{g(a)}^{g(y)} (g(y) - z)^{\alpha - m} dz = \tag{145}$$

$$\gamma (g'(y))^m \frac{(g(y) - g(a))^{\alpha - m + 1}}{\alpha - m + 1},$$

with $\alpha > m - 1$. That is

$$\lambda_m^g(y) = \gamma (g'(y))^m \frac{(g(y) - g(a))^{\alpha - m + 1}}{\alpha - m + 1}, \tag{146}$$

$\alpha > m - 1$, $y \in (a, b)$.

Hence (143) becomes

$$\int_a^b g'(x) (g(b) - g(x))^{\alpha} \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{b-;g}^{\alpha_i} f_i \right)(x)}{(g(b) - g(x))^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\left(\frac{\gamma}{\alpha - m + 1} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i (|f_i(x)|) \, dx \right) \cdot$$

$$\left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right)\left(g'(x)\right)^m (g(x)-g(a))^{\alpha-m+1} dx\right) \leq \tag{147}$$

$$\left(\frac{\gamma\,(g(b)-g(a))^{\alpha-m+1}\left\|g'\right\|_\infty^m}{\alpha-m+1}\right)\left(\prod_{i=1}^m \int_a^b \Phi_i\left(\left|f_i(x)\right|\right) dx\right),$$

where $\alpha > m-1$, f_i *with* $I_{b-;g}^{\alpha_i}\left(\left|f_i\right|\right)$ *finite,* $i=1,...,m$, *under the assumptions:*

(i) $\Phi_i\left(\left|f_i\right|\right)$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\Phi_i\left(\left|f_i\right|\right)$ *is Lebesgue integrable on* (a,b).

If $\Phi_i(x) = x^{p_i}$, $p_i \geq 1$, $x \in \mathbb{R}_+$, *then by (147), we have*

$$\int_a^b g'(x)\,(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)} \prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \leq \tag{148}$$

$$\left(\frac{\gamma\,(g(b)-g(a))^{\alpha-m+1}\left(\left\|g'\right\|_\infty\right)^m}{(\alpha-m+1)\prod_{i=1}^m\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}}\right)\left(\prod_{i=1}^m \int_a^b \left|f_i(x)\right|^{p_i} dx\right),$$

but we see that

$$\int_a^b g'(x)\,(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)} \prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \geq$$

$$(g(b)-g(a))^{\left(\alpha-\sum_{i=1}^m \alpha_i p_i\right)} \int_a^b g'(x) \prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx. \tag{149}$$

Hence by (148) and (149) we derive

$$\int_a^b g'(x) \prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \leq \tag{150}$$

$$\left(\frac{\gamma\,(g(b)-g(a))^{\left(\sum_{i=1}^m p_i\alpha_i-m+1\right)}\left\|g'\right\|_\infty^m}{\left(\prod_{i=1}^m\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^m \int_a^b \left|f_i(x)\right|^{p_i} dx\right),$$

$\alpha > m-1$, f_i *with* $I_{b-;g}^{\alpha_i}\left(\left|f_i\right|\right)$ *finite,* $i=1,...,m$, *under the assumptions:*

(i) $\left|f_i\right|^{p_i}$ *is* $k_i(x,y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\left|f_i\right|^{p_i}$ *is Lebesgue integrable on* (a,b).

We give

Theorem 41 *Let (f_i, α_i), $i = 1, ..., m$; $J_{b-}^{\alpha_i} f_i$ as in Definition 37. Set $\alpha := \sum_{i=1}^{m} \alpha_i$, $\gamma := \prod_{i=1}^{m} \alpha_i$; $p_i \geq 1$, $i = 1, ..., m$, assume $\alpha > m - 1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left| \left(J_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{151}$$

$$\left(\frac{b\gamma \left(\ln \left(\frac{b}{a} \right) \right)^{\left(\sum_{i=1}^{m} p_i \alpha_i - m + 1 \right)}}{a^m \left(\alpha - m + 1 \right) \left(\prod_{i=1}^{m} \left(\Gamma \left(\alpha_i + 1 \right) \right)^{p_i} \right)} \right) \left(\prod_{i=1}^{m} \int_a^b \left| f_i(x) \right|^{p_i} dx \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i = 1, ..., m$, under the assumptions:

(i) $|f_i(y)|^{p_i}$ is $\left(\frac{\chi_{[x,b)}(y) dy}{\Gamma(\alpha_i) y \left(\ln \left(\frac{y}{x} \right) \right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a, b)$,

(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a, b).

We finish with

Theorem 42 *Let all as in Theorem 41. Take $p := p_1 = p_2 = ... = p_m \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{152}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}} \left(\ln \left(\frac{b}{a} \right) \right)^{\left(\alpha - \frac{m}{p} + \frac{1}{p} \right)}}{a^{\frac{m}{p}} \left(\alpha - m + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \left(\Gamma \left(\alpha_i + 1 \right) \right) \right)} \right) \left(\prod_{i=1}^{m} \left\| f_i \right\|_{p,(a,b)} \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i = 1, ..., m$, under the properties:

(i) $|f_i(y)|^{p}$ is $\left(\frac{\chi_{[x,b)}(y) dy}{\Gamma(\alpha_i) y \left(\ln \left(\frac{y}{x} \right) \right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a, b)$,

(ii) $|f_i|^{p}$ is Lebesgue integrable on (a, b).

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Fractional Integral Inequalities involving Convexity

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Abstract

Here we present general integral inequalities involving convex and increasing functions applied to products of functions. As specific applications we derive a wide range of fractional inequalities of Hardy type. These involve the left and right: Erdélyi-Kober fractional integrals, mixed Riemann-Liouville fractional multiple integrals. Next we produce multivariate Poincaré type fractional inequalitites involving left fractional radial derivatives of Canavati type, Riemann-Liouville and Caputo types. The exposed inequalities are of L_p type, $p \geq 1$, and exponential type.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.

Key words and phrases: fractional integral, fractional radial derivative, Hardy fractional inequality, Poincaré fractional inequality, Erdélyi-Kober fractional integrals.

1 Introduction

We start with some facts about fractional derivatives needed in the sequel, for more details see, for instance [1], [10].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann-Liouville fractional integrals, see [13]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1}$$

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt, \quad (x < b), \tag{2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$, see also [16]. The first result yields that the fractional integral operators $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ are bounded in $L_p(a,b)$, $1 \leq p \leq \infty$, that is

$$\left\|I_{a+}^{\alpha} f\right\|_p \leq K \|f\|_p \quad , \quad \left\|I_{b-}^{\alpha} f\right\|_p \leq K \|f\|_p , \tag{3}$$

where

$$K = \frac{(b-a)^{\alpha}}{\alpha \Gamma(\alpha)}. \tag{4}$$

Inequality (3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [11]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we follow [12].

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative measurable function, $k(x, \cdot)$ measurable on Ω_2 and

$$K(x) = \int_{\Omega_2} k(x,y)\, d\mu_2(y), \quad x \in \Omega_1. \tag{5}$$

We suppose that $K(x) > 0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y), \tag{6}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function.

Theorem 1 *([12]) Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (5). Assume that the function $x \mapsto u(x) \frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define ν on Ω_2 by*

$$\nu(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{7}$$

If $\Phi : [0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\, \Phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} \nu(y)\, \Phi(|f(y)|)\, d\mu_2(y) \tag{8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that:

(i) $f, \Phi(|f|)$ are both $k(x,y)\, d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\nu\Phi(|f|)$ is μ_2 -integrable,

and for all corresponding functions g given by (6).

Important assumptions (i) and (ii) are missing from Theorem 2.1. of [12].

In this article we use and generalize Theorem 1 for products of several functions and we give wide applications to Fractional Calculus.

2 Main Results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\, d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{9}$$

$i = 1, ..., m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y)\, f_i(y)\, d\mu_2(y), \tag{10}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., m$.

Here u stands for a weight function on Ω_1.

The first introductory result is proved for $m = 2$.

Theorem 2 *Assume that the functions ($i = 1, 2$) $x \mapsto \left(u(x)\, \frac{k_i(x,y)}{K_i(x)}\right)$ are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define u_i on Ω_2 by*

$$u_i(y) := \int_{\Omega_1} u(x)\, \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{11}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let the functions $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, be convex and increasing.

Then

$$\int_{\Omega_1} u(x)\, \Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right) \Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) \leq$$

$$\left(\int_{\Omega_2} u_1(y)\,\Phi_1\left(|f_1(y)|\right)^p d\mu_2(y)\right)^{\frac{1}{p}}\left(\int_{\Omega_2} u_2(y)\,\Phi_2\left(|f_2(y)|\right)^q d\mu_2(y)\right)^{\frac{1}{q}}, \quad (12)$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ *(* $i = 1, 2$*) such that*

(i) f_1, $\Phi_1(|f_1|)^p$ *are both* $k_1(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $f_2, \Phi_2(|f_2|)^q$ *are both* $k_2(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(iii) $u_1\Phi_1(|f_1|)^p$, $u_2\Phi_2(|f_2|)^q$*, are both* μ_2*-integrable,*

and for all corresponding functions g_i $(i = 1, 2)$ *given by (10).*

Proof. Notice that Φ_1, Φ_2 are continuous functions. Here we use Hölder's inequality. We have

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) =$$

$$\int_{\Omega_1} u(x)^{\frac{1}{p}}\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right) u(x)^{\frac{1}{q}}\,\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) \leq \quad (13)$$

$$\left(\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)^p d\mu_1(x)\right)^{\frac{1}{p}} \cdot$$

$$\left(\int_{\Omega_1} u(x)\,\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)^q d\mu_1(x)\right)^{\frac{1}{q}} \leq$$

(notice here that Φ_1^p, Φ_2^q are convex, increasing and continuous nonnegative functions, and by Theorem 1 we get)

$$\left(\int_{\Omega_2} u_1(y)\,\Phi_1\left(|f_1(y)|\right)^p d\mu_2(y)\right)^{\frac{1}{p}}\left(\int_{\Omega_2} u_2(y)\,\Phi_2\left(|f_2(y)|\right)^q d\mu_2(y)\right)^{\frac{1}{q}}. \quad (14)$$

■

The general result follows

Theorem 3 *Assume that the functions* $(i = 1, 2, ..., m \in \mathbb{N})$ $x \mapsto \left(u(x)\,\frac{k_i(x,y)}{K_i(x)}\right)$ *are integrable on* Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x)\,\frac{k_i(x,y)}{K_i(x)}\,d\mu_1(x) < \infty. \quad (15)$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$*. Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$*, be convex and increasing.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m}\left(\int_{\Omega_2} u_i(y)\,\Phi_i(|f_i(y)|)^{p_i}\,d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{16}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ ($i = 1, ..., m$) such that

(i) f_i, $\Phi_i(|f_i|)^{p_i}$ are both $k_i(x, y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$, $i = 1, ..., m$,

(ii) $u_i\Phi_i(|f_i|)^{p_i}$ is μ_2-integrable, $i = 1, ..., m$,

and for all corresponding functions g_i ($i = 1, ..., m$) given by (10).

Proof. Notice that $\Phi_i, i = 1, ..., m$, are continuous functions. Here we use the generalized Hölder's inequality. We have

$$\int_{\Omega_1} u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x) =$$

$$\int_{\Omega_1}\prod_{i=1}^{m}\left(u(x)^{\frac{1}{p_i}}\,\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)\right)d\mu_1(x) \leq \tag{17}$$

$$\prod_{i=1}^{m}\left(\int_{\Omega_1} u(x)\,\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)^{p_i}d\mu_1(x)\right)^{\frac{1}{p_i}} \leq$$

(notice here that $\Phi_i^{p_i}$, $i = 1, ..., m$, are convex, increasing and continuous, non-negative functions, and by Theorem 1 we get)

$$\prod_{i=1}^{m}\left(\int_{\Omega_2} u_i(y)\,\Phi_i(|f_i(y)|)^{p_i}\,d\mu_2(y)\right)^{\frac{1}{p_i}}. \tag{18}$$

proving the claim. ∎

When $k(x, y) := k_1(x, y) = k_2(x, y) = ... = k_m(x, y)$, then $K(x) := K_1(x) = K_2(x) = ... = K_m(x)$, we get by Theorems 2, 3 the following:

Corollary 4 *Assume that the function $x \mapsto \left(u(x)\,\frac{k(x,y)}{K(x)}\right)$ is integrable on Ω_1, for each fixed $y \in \Omega_2$. Define U on Ω_2 by*

$$U(y) := \int_{\Omega_1} u(x)\,\frac{k(x, y)}{K(x)}d\mu_1(x) < \infty. \tag{19}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let the functions $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, be convex and increasing.

Then

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K(x)}\right|\right)d\mu_1(x) \leq$$

$$\left(\int_{\Omega_2} U(y)\,\Phi_1(|f_1(y)|)^{p}\,d\mu_2(y)\right)^{\frac{1}{p}}\left(\int_{\Omega_2} U(y)\,\Phi_2(|f_2(y)|)^{q}\,d\mu_2(y)\right)^{\frac{1}{q}}, \tag{20}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ *(* $i = 1, 2$*) such that*

(i) f_1, f_2, $\Phi_1 (|f_1|)^p$, $\Phi_2 (|f_2|)^q$ *are all* $k (x, y) d\mu_2 (y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $U\Phi_1 (|f_1|)^p$, $U\Phi_2 (|f_2|)^q$, *are both* μ_2*-integrable,*

and for all corresponding functions g_i *(*$i = 1, 2$*) given by (10).*

Corollary 5 *Assume that the function* $x \mapsto \left(u(x) \frac{k(x,y)}{K(x)}\right)$ *is integrable on* Ω_1, *for each fixed* $y \in \Omega_2$. *Define* U *on* Ω_2 *by*

$$U(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{21}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K(x)} \right| \right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} U(y) \Phi_i (|f_i(y)|)^{p_i} d\mu_2(y) \right)^{\frac{1}{p_i}}, \tag{22}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, ..., m$, *such that*

(i) f_i, $\Phi_i (|f_i|)^{p_i}$ *are both* $k(x, y) d\mu_2 (y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, *for all* $i = 1, ..., m$,

(ii) $U\Phi_i (|f_i|)^{p_i}$ *is* μ_2*-integrable,* $i = 1, ..., m$,

and for all corresponding functions g_i *(*$i = 1, ..., m$*) given by (10).*

Next we give two applications of Theorem 3.

Theorem 6 *Assume that the functions (*$i = 1, 2, ..., m \in \mathbb{N}$*)* $x \mapsto \left(u(x) \frac{k_i(x,y)}{K_i(x)}\right)$ *are integrable on* Ω_1, *for each fixed* $y \in \Omega_2$. *Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{23}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$; $\alpha_i \geq 1$, $i = 1, ..., m$.

Then

$$\int_{\Omega_1} u(x) \left(\prod_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|^{\alpha_i} \right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i(y) |f_i(y)|^{\alpha_i p_i} d\mu_2(y) \right)^{\frac{1}{p_i}}, \tag{24}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, ..., m$, *such that*

(i) f_i, $|f_i|^{\alpha_i p_i}$ *are* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, $i = 1, ..., m$,

(ii) $u_i |f_i|^{\alpha_i p_i}$ *is* μ_2*-integrable,* $i = 1, ..., m$,

and for all corresponding functions g_i $(i = 1, ..., m)$ *given by (10).*

Theorem 7 *Assume that the functions* $(i = 1, 2, ..., m \in \mathbb{N})$ $x \mapsto \left(u(x) \frac{k_i(x,y)}{K_i(x)}\right)$ *are integrable on* Ω_1, *for each fixed* $y \in \Omega_2$. *Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{25}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1.$

Then

$$\int_{\Omega_1} u(x) \left(e^{\sum_{i=1}^{m} \left| \frac{g_i(x)}{K_i(x)} \right|} \right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i(y)\, e^{p_i |f_i(y)|} d\mu_2(y) \right)^{\frac{1}{p_i}}, \tag{26}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, ..., m$, *such that*

(i) f_i, $e^{p_i |f_i|}$ *are* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, $i = 1, ..., m$,

(ii) $u_i e^{p_i |f_i|}$ *is* μ_2*-integrable,* $i = 1, ..., m$,

and for all corresponding functions g_i $(i = 1, ..., m)$ *given by (10).*

We need

Definition 8 *([16]) Let* (a, b), $0 \leq a < b < \infty$; $\alpha, \sigma > 0$. *We consider the left- and right-sided fractional integrals of order* α *as follows:*

1) for $\eta > -1$, *we define*

$$\left(I^{\alpha}_{a+;\sigma,\eta} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)\, dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}}, \tag{27}$$

2) for $\eta > 0$, *we define*

$$\left(I^{\alpha}_{b-;\sigma,\eta} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\, dt}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}. \tag{28}$$

These are the Erdélyi-Kober type fractional integrals.

We remind the Beta function

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1}\, dt, \tag{29}$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, and the Incomplete Beta function

$$B(x;\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \tag{30}$$

where $0 < x \leq 1$; $\alpha, \beta > 0$.

We make

Remark 9 *Regarding (27) we have*

$$k(x,y) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\chi_{(a,x]}(y)\frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}}, \tag{31}$$

$x, y \in (a,b)$, *χ stands for the characteristic function.*

Here

$$K(x) = \int_a^b k(x,t)\,dt = \left(I^\alpha_{a+;\sigma;\eta}1\right)(x)$$

$$= \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x \frac{t^{\sigma\eta+\sigma-1}}{(x^\sigma - t^\sigma)^{1-\alpha}}dt \tag{32}$$

(setting $z = \frac{t}{x}$)

$$= \frac{\sigma}{\Gamma(\alpha)}\int_{\frac{a}{x}}^1 z^{\sigma\left((\eta+1)-\frac{1}{\sigma}\right)}(1-z^\sigma)^{\alpha-1}\,dz$$

(setting $\lambda = z^\sigma$)

$$= \frac{1}{\Gamma(\alpha)}\int_{\left(\frac{a}{x}\right)^\sigma}^1 \lambda^\eta(1-\lambda)^{\alpha-1}\,d\lambda. \tag{33}$$

Hence

$$K(x) = \frac{1}{\Gamma(\alpha)}\int_{\left(\frac{a}{x}\right)^\sigma}^1 \lambda^\eta(1-\lambda)^{\alpha-1}\,d\lambda. \tag{34}$$

Indeed it is

$$K(x) = \left(I^\alpha_{a+;\sigma;\eta}(1)\right)(x) \tag{35}$$

$$= \frac{B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma;\eta+1,\alpha\right)}{\Gamma(\alpha)}.$$

We also make

Remark 10 *Regarding (28) we have*

$$k(x,y) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\chi_{[x,b)}(y)\frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^\sigma - x^\sigma)^{1-\alpha}}, \tag{36}$$

$x, y \in (a,b)$.

Here

$$K(x) = \int_a^b k(x,t)\,dt = \left(I^\alpha_{b-;\sigma;\eta}1\right)(x)$$

$$= \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1}}{(t^\sigma - x^\sigma)^{1-\alpha}} dt \tag{37}$$

(setting $z = \frac{t}{x}$)

$$= \frac{\sigma}{\Gamma(\alpha)} \int_1^{\left(\frac{b}{x}\right)} (z^\sigma - 1)^{\alpha-1} z^{\sigma(1-\eta-\alpha)-1} dz$$

(setting $\lambda = z^\sigma$, $1 \le \lambda < \left(\frac{b}{x}\right)^\sigma$)

$$= \frac{1}{\Gamma(\alpha)} \int_1^{\left(\frac{b}{x}\right)^\sigma} (\lambda - 1)^{\alpha-1} \lambda^{-\eta-\alpha} d\lambda \tag{38}$$

$$= \frac{1}{\Gamma(\alpha)} \int_1^{\left(\frac{b}{x}\right)^\sigma} \frac{1}{\lambda^{\eta+1}} \left(1 - \frac{1}{\lambda}\right)^{\alpha-1} d\lambda$$

(setting $w := \frac{1}{\lambda}$, $0 < \left(\frac{x}{b}\right)^\sigma < w \le 1$)

$$= \frac{1}{\Gamma(\alpha)} \int_{\left(\frac{x}{b}\right)^\sigma}^1 w^{\eta-1} (1-w)^{\alpha-1} dw \tag{39}$$

$$= \frac{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma; \eta, \alpha\right)\right)}{\Gamma(\alpha)}.$$

That is

$$K(x) = \left(I^\alpha_{b-;\sigma;\eta}(1)\right)(x) \tag{40}$$

$$= \frac{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma; \eta, \alpha\right)\right)}{\Gamma(\alpha)}.$$

We give

Theorem 11 *Assume that the function*

$$x \mapsto \left(u(x) \frac{\chi_{(a,x]}(y)\, \sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha} \left[B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha\right)\right]} \right) \tag{41}$$

is integrable on (a,b), for each $y \in (a,b)$. Here $\alpha, \sigma > 0$, $\eta > -1$, $0 \le a < b < \infty$. Define u_1 on (a,b) by

$$u_1(y) := \sigma y^{\sigma\eta+\sigma-1} \int_y^b \frac{u(x)\, x^{-\sigma(\alpha+\eta)} (x^\sigma - y^\sigma)^{\alpha-1}}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha\right)\right)} dx < \infty. \tag{42}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing.

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| I_{a+;\sigma;\eta}^{\alpha} f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1, \alpha \right) \right)} \right) dx \leq$$

$$\prod_{i=1}^m \left(\int_a^b u_1(y) \Phi_i (|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}}, \tag{43}$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, ..., m$, *such that*

(i) f_i, $\Phi_i(|f_i|)^{p_i}$ *are both* $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1} dy}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}$ *-integrable, a.e. in* $x \in (a,b)$, *for all* $i = 1, ..., m$,

(ii) $u_1 \Phi_i(|f_i|)^{p_i}$ *is Lebesgue integrable,* $i = 1, ..., m$,

Proof. By Corollary 5. ■

Remark 12 *In (42), if we choose*

$$u(x) = x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\alpha}; \eta+1, \alpha \right) \right), \quad x \in (a,b), \tag{44}$$

then

$$u_1(y) = \sigma y^{\sigma\eta+\sigma-1} \int_y^b x^{\sigma-1} (x^{\sigma} - y^{\sigma})^{\alpha-1} dx$$

(setting $w := x^{\sigma}$, $\frac{dw}{dx} = \sigma x^{\sigma-1}$, $dx = \frac{dw}{\sigma x^{\sigma-1}}$*)*

$$= y^{\sigma\eta+\sigma-1} \int_{y^{\sigma}}^{b^{\sigma}} (w - y^{\sigma})^{\alpha-1} dw = y^{\sigma\eta+\sigma-1} \frac{(b^{\sigma} - y^{\sigma})^{\alpha}}{\alpha}. \tag{45}$$

That is

$$u_1(y) = y^{\sigma\eta+\sigma-1} \frac{(b^{\sigma} - y^{\sigma})^{\alpha}}{\alpha}, \quad y \in (a,b). \tag{46}$$

Based on the above, (43) becomes

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1, \alpha \right) \right) \cdot$$

$$\prod_{i=1}^m \Phi_i \left(\frac{\left| I_{a+;\sigma;\eta}^{\alpha} f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1, \alpha \right) \right)} \right) dx \leq$$

$$\frac{1}{\alpha} \prod_{i=1}^m \left(\int_a^b y^{\sigma\eta+\sigma-1} (b^{\sigma} - y^{\sigma})^{\alpha} \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}} \leq \tag{47}$$

$$\frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha} \prod_{i=1}^m \left(\int_a^b y^{\sigma(\eta+1)-1} \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) following (43), and

*(ii)** $y^{\sigma(\eta+1)-1} \Phi_i(|f_i(y)|)^{p_i}$ *is Lebesgue integrable on* (a,b), $i = 1, ..., m$.

Corollary 13 *Let* $0 \leq a < b$; $\alpha, \sigma > 0$, $\eta > -1$; $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$; $\beta_i \geq 1$, $i = 1, ..., m$.

Then

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1, \alpha \right) \right)^{\left(1-\sum_{i=1}^{m}\beta_i\right)} \cdot$$

$$\left(\prod_{i=1}^{m} \left| I_{a+;\sigma;\eta}^{\alpha} f_i(x) \right|^{\beta_i} \right) dx \leq \frac{1}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m}\beta_i}} \cdot$$

$$\prod_{i=1}^{m} \left(\int_a^b y^{\sigma\eta+\sigma-1} (b^{\sigma} - y^{\sigma})^{\alpha} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}} \leq \tag{48}$$

$$\left(\frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m}\beta_i}} \right) \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(\eta+1)-1} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, ..., m$ *such that*

(i) $|f_i|^{\beta_i p_i}$ *is* $\left(\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1} dy}{(x^{\sigma}-y^{\sigma})^{1-\alpha}} \right)$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $y^{\sigma(\eta+1)-1} |f_i(y)|^{\beta_i p_i}$ *is Lebesgue integrable on* (a,b); $i = 1, ..., m$.

Proof. By Theorem 11 and (47). ■

Corollary 14 *Let* $0 \leq a < b$; $\alpha, \sigma > 0$, $\eta > -1$; $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$.

Then

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1, \alpha \right) \right) \cdot$$

$$e^{\frac{\Gamma(\alpha)\left(\sum_{i=1}^{m} \left| I_{a+;\sigma;\eta}^{\alpha} f_i(x) \right| \right)}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^{\sigma}; \eta+1,\alpha \right) \right)}} dx \leq$$

$$\frac{1}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(\eta+1)-1} (b^{\sigma} - y^{\sigma})^{\alpha} e^{p_i |f_i(y)|} dy \right)^{\frac{1}{p_i}} \leq$$

$$\frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(\eta+1)-1} e^{p_i |f_i(y)|} dy \right)^{\frac{1}{p_i}}, \tag{49}$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, ..., m$ *such that*

(i) f_i, $e^{p_i|f_i|}$ *are both* $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1} dy}{(x^\sigma - y^\sigma)^{1-\alpha}}$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $y^{\sigma(\eta+1)-1} e^{p_i|f_i(y)|}$ *is Lebesgue integrable on* (a,b); $i = 1, ..., m$.

Proof. By Theorem 11 and (47). ■

We present

Theorem 15 *Assume that the function*

$$x \mapsto \left(u(x) \frac{\sigma x^{\sigma\eta} \chi_{[x,b)}(y) y^{\sigma(1-\eta-\alpha)-1}}{(y^\sigma - x^\sigma)^{1-\alpha} \left[B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma ; \eta, \alpha \right) \right]} \right)$$

is integrable on (a,b), *for each* $y \in (a,b)$. *Here* $\alpha, \sigma, \eta > 0$, $0 \le a < b < \infty$. *Define* u_2 *on* (a,b) *by*

$$u_2(y) := \sigma y^{\sigma(1-\eta-\alpha)-1} \int_a^y \frac{u(x) x^{\sigma\eta} (y^\sigma - x^\sigma)^{\alpha-1} dx}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma ; \eta, \alpha \right) \right)} < \infty. \tag{50}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing.*

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| I^\alpha_{b-;\sigma;\eta} f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma ; \eta, \alpha \right) \right)} \right) dx \le$$

$$\prod_{i=1}^m \left(\int_a^b u_2(y) \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}}, \tag{51}$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, ..., m$, *such that*

(i) f_i, $\Phi_i(|f_i|)^{p_i}$ *are both* $\left(\frac{\sigma x^{\sigma\eta} \chi_{[x,b)}(y) y^{\sigma(1-\eta-\alpha)-1} dy}{\Gamma(\alpha)(y^\sigma - x^\sigma)^{1-\alpha}} \right)$ *-integrable, a.e. in* $x \in (a,b)$, *for all* $i = 1, ..., m$,

(ii) $u_2 \Phi_i(|f_i|)^{p_i}$ *is Lebesgue integrable on* (a,b), $i = 1, ..., m$.

Proof. By Corollary 5. ■

Remark 16 *Here* $0 < a < b < \infty$; $\alpha, \sigma, \eta > 0$.

In (50), if we choose

$$u(x) = x^{\sigma(1-\eta)-1} \left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\alpha ; \eta, \alpha \right) \right), \quad x \in (a,b), \tag{52}$$

then

$$u_2(y) = \sigma y^{\sigma(1-\eta-\alpha)-1} \int_a^y x^{\sigma-1} (y^\sigma - x^\sigma)^{\alpha-1} dx$$

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(setting $w := x^{\sigma}$, $dx = \frac{dw}{\sigma x^{\sigma-1}}$*)*

$$= y^{\sigma(1-\eta-\alpha)-1} \int_{a^{\sigma}}^{y^{\sigma}} (y^{\sigma} - w)^{\alpha-1} dw = y^{\sigma(1-\eta-\alpha)-1} \frac{(y^{\sigma} - a^{\sigma})^{\alpha}}{\alpha}. \tag{53}$$

That is

$$u_2(y) = y^{\sigma(1-\eta-\alpha)-1} \frac{(y^{\sigma} - a^{\sigma})^{\alpha}}{\alpha}, \quad y \in (a,b). \tag{54}$$

Based on the above, (51) becomes

$$\int_a^b x^{\sigma(1-\eta)-1} \left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^{\sigma}; \eta, \alpha\right)\right) \cdot$$

$$\prod_{i=1}^{m} \Phi_i \left(\frac{\left| I_{b-;\sigma;\eta}^{\alpha} f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^{\sigma}; \eta, \alpha\right)\right)} \right) dx \leq$$

$$\frac{1}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} (y^{\sigma} - a^{\sigma})^{\alpha} \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}} \leq \tag{55}$$

$$\frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\alpha(1-\eta-\alpha)-1} \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) following (51), and

*(ii)** $y^{\sigma(1-\eta-\alpha)-1} \Phi_i(|f_i(y)|)^{p_i}$ *is Lebesgue integrable on* (a,b), $i = 1, ..., m$.

Corollary 17 *Let* $0 < a < b < \infty$; $\alpha, \sigma, \eta > 0$; $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$; $\beta_i \geq 1$, $i = 1, ..., m$.

Then

$$\int_a^b x^{\sigma(1-\eta)-1} \left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^{\sigma}; \eta, \alpha\right)\right)^{\left(1-\sum_{i=1}^{m} \beta_i\right)} \cdot$$

$$\left(\prod_{i=1}^{m} \left| I_{b-;\sigma;\eta}^{\alpha} f_i(x) \right|^{\beta_i} \right) dx \leq \frac{1}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m} \beta_i}} \cdot$$

$$\prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} (y^{\sigma} - a^{\sigma})^{\alpha} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}} \leq \tag{56}$$

$$\frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m} \beta_i}} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $|f_i|^{\beta_i p_i}$ *is* $\left(\frac{\sigma x^{\sigma\eta}\chi_{[x,b)}(y)y^{\sigma(1-\eta-\alpha)-1}dy}{\Gamma(\alpha)(y^\sigma-x^\sigma)^{1-\alpha}}\right)$ *-integrable, a.e. in* $x \in (a,b)$*, for all* $i = 1,...,m$,

(ii) $y^{\sigma(1-\eta-\alpha)-1}|f_i(y)|^{\beta_i p_i}$ *is Lebesgue integrable on* (a,b), $i = 1,...,m$.

Proof. By Theorem 15 and (55). ∎

Corollary 18 *Let* $0 < a < b < \infty$*;* $\alpha, \sigma, \eta > 0$*;* $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$.

Then

$$\int_a^b x^{\sigma(1-\eta)-1}\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha\right)\right)\cdot e^{\frac{\Gamma(\alpha)\left(\sum_{i=1}^m \left|I^\alpha_{b-;\sigma;\eta}f_i(x)\right|\right)}{\left(B(\eta,\alpha)-B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha\right)\right)}}dx \leq$$

$$\frac{1}{\alpha}\prod_{i=1}^m\left(\int_a^b y^{\sigma(1-\eta-\alpha)-1}(y^\sigma - a^\sigma)^\alpha e^{p_i|f_i(y)|}dy\right)^{\frac{1}{p_i}} \leq$$

$$\frac{(b^\sigma - a^\sigma)^\alpha}{\alpha}\prod_{i=1}^m\left(\int_a^b y^{\sigma(1-\eta-\alpha)-1}e^{p_i|f_i(y)|}dy\right)^{\frac{1}{p_i}}, \tag{57}$$

under the assumptions:

(i) f_i, $e^{p_i|f_i|}$ *are both* $\left(\frac{\sigma x^{\sigma\eta}\chi_{[x,b)}(y)y^{\sigma(1-\eta-\alpha)-1}dy}{\Gamma(\alpha)(y^\sigma-x^\sigma)^{1-\alpha}}\right)$ *-integrable, a.e. in* $x \in (a,b)$, $i = 1,...,m$,

(ii) $y^{\sigma(1-\eta-\alpha)-1}e^{p_i|f_i(y)|}$ *is Lebesgue integrable on* (a,b); $i = 1,...,m$.

Proof. By Theorem 15 and (55). ∎

We make

Remark 19 *Let* $\prod_{i=1}^N (a_i,b_i) \subset \mathbb{R}^N$, $N > 1$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$*. Let* $\alpha_i > 0$, $i = 1,...,N$*;* $f \in L_1\left(\prod_{i=1}^N (a_i,b_i)\right)$*, and set* $a = (a_1,...,a_N)$, $b = (b_1,...,b_N)$, $\alpha = (\alpha_1,...,\alpha_N)$, $x = (x_1,...,x_N)$, $t = (t_1,...,t_N)$.

We define the left mixed Riemann-Liouville fractional multiple integral of order α *(see also [14]):*

$$\left(I^\alpha_{a+}f\right)(x) := \frac{1}{\prod_{i=1}^N \Gamma(\alpha_i)}\int_{a_1}^{x_1}...\int_{a_N}^{x_N}\prod_{i=1}^N (x_i - t_i)^{\alpha_i - 1} f(t_1,...,t_N)\,dt_1...dt_N, \tag{58}$$

with $x_i > a_i,\ i = 1, ..., N.$

We also define the right mixed Riemann-Liouville fractional multiple integral of order α *(see also [12]):*

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} ... \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_N)\, dt_1 ... dt_N, \tag{59}$$

with $x_i < b_i,\ i = 1, ..., N.$

Notice $I_{a+}^{\alpha}(|f|),\ I_{b-}^{\alpha}(|f|)$ *are finite if* $f \in L_\infty\left(\prod_{i=1}^{N}(a_i, b_i)\right).$

One can rewrite (58) and (59) as follows:

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i, b_i)} \chi_{\prod_{i=1}^{N}(a_i, x_i]}(t) \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f(t)\, dt, \tag{60}$$

with $x_i > a_i,\ i = 1, ..., N,$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i, b_i)} \chi_{\prod_{i=1}^{N}[x_i, b_i)}(t) \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f(t)\, dt, \tag{61}$$

with $x_i < b_i,\ i = 1, ..., N.$

The corresponding $k(x, y)$ *for* $I_{a+}^{\alpha},\ I_{b-}^{\alpha}$ *are*

$$k_{a+}(x, y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i, x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1}, \tag{62}$$

$\forall\ x, y \in \prod_{i=1}^{N}(a_i, b_i),$

and

$$k_{b-}(x, y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}[x_i, b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1}, \tag{63}$$

$\forall\ x, y \in \prod_{i=1}^{N}(a_i, b_i).$

The corresponding $K(x)$ for I_{a+}^{α} is:

$$K_{a+}(x) = \int_{\prod_{i=1}^{N}(a_i,b_i)} k_{a+}(x,y)\,dy = \left(I_{a+}^{\alpha}1\right)(x) =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{a_1}^{x_1} \dots \int_{a_N}^{x_N} \prod_{i=1}^{N}(x_i - t_i)^{\alpha_i - 1}\,dt_1 \dots dt_N =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \int_{a_i}^{x_i} (x_i - t_i)^{\alpha_i - 1}\,dt_i = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \frac{(x_i - a_i)^{\alpha_i}}{\alpha_i}$$

$$= \prod_{i=1}^{N}\left(\frac{(x_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right),$$

that is

$$K_{a+}(x) = \prod_{i=1}^{N}\frac{(x_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{64}$$

$\forall\ x \in \prod_{i=1}^{N}(a_i, b_i)$.

Similarly the corresponding $K(x)$ for I_{b-}^{α} is:

$$K_{b-}(x) = \int_{\prod_{i=1}^{N}(a_i,b_i)} k_{b-}(x,y)\,dy = \left(I_{b-}^{\alpha}1\right)(x) =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{x_1}^{b_1} \dots \int_{x_N}^{b_N} \prod_{i=1}^{N}(t_i - x_i)^{\alpha_i - 1}\,dt_1 \dots dt_N =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \int_{x_i}^{b_i} (t_i - x_i)^{\alpha_i - 1}\,dt_i = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \frac{(b_i - x_i)^{\alpha_i}}{\alpha_i}$$

$$= \prod_{i=1}^{N}\frac{(b_i - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)},$$

that is

$$K_{b-}(x) = \prod_{i=1}^{N}\frac{(b_i - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{65}$$

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$\forall\ x \in \prod_{i=1}^{N} (a_i, b_i)$.

Next we form

$$\frac{k_{a+}(x,y)}{K_{a+}(x)} = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1} \prod_{i=1}^{N} \frac{\Gamma(\alpha_i + 1)}{(x_i - a_i)^{\alpha_i}}$$

$$= \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \left(\prod_{i=1}^{N} \alpha_i\right) \left(\prod_{i=1}^{N} \frac{(x_i - y_i)^{\alpha_i - 1}}{(x_i - a_i)^{\alpha_i}}\right),$$

that is

$$\frac{k_{a+}(x,y)}{K_{a+}(x)} = \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \left(\prod_{i=1}^{N} \alpha_i\right) \left(\prod_{i=1}^{N} \frac{(x_i - y_i)^{\alpha_i - 1}}{(x_i - a_i)^{\alpha_i}}\right), \tag{66}$$

$\forall\ x, y \in \prod_{i=1}^{N} (a_i, b_i)$.

Similarly we form

$$\frac{k_{b-}(x,y)}{K_{b-}(x)} = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1} \prod_{i=1}^{N} \frac{\Gamma(\alpha_i + 1)}{(b_i - x_i)^{\alpha_i}}$$

$$= \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \left(\prod_{i=1}^{N} \alpha_i\right) \left(\prod_{i=1}^{N} \frac{(y_i - x_i)^{\alpha_i - 1}}{(b_i - x_i)^{\alpha_i}}\right),$$

that is

$$\frac{k_{b-}(x,y)}{K_{b-}(x)} = \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \left(\prod_{i=1}^{N} \alpha_i\right) \left(\prod_{i=1}^{N} \frac{(y_i - x_i)^{\alpha_i - 1}}{(b_i - x_i)^{\alpha_i}}\right), \tag{67}$$

$\forall\ x, y \in \prod_{i=1}^{N} (a_i, b_i)$.

We choose the weight function $u_1(x)$ *on* $\prod_{i=1}^{N} (a_i, b_i)$ *such that the function* $x \mapsto \left(u_1(x) \frac{k_{a+}(x,y)}{K_{a+}(x)}\right)$ *is integrable on* $\prod_{i=1}^{N} (a_i, b_i)$, *for each fixed* $y \in \prod_{i=1}^{N} (a_i, b_i)$.

We define w_1 *on* $\prod_{i=1}^{N}(a_i,b_i)$ *by*

$$w_1(y) := \int_{\prod_{i=1}^{N}(a_i,b_i)} u_1(x)\frac{k_{a+}(x,y)}{K_{a+}(x)}dx < \infty. \tag{68}$$

We have that

$$w_1(y) = \left(\prod_{i=1}^{N}\alpha_i\right)\int_{y_1}^{b_1}...\int_{y_N}^{b_N} u_1(x_1,...,x_N)\left(\prod_{i=1}^{N}\frac{(x_i-y_i)^{\alpha_i-1}}{(x_i-a_i)^{\alpha_i}}\right)dx_1...dx_N, \tag{69}$$

$\forall\ y \in \prod_{i=1}^{N}(a_i,b_i)$.

We also choose the weight function $u_2(x)$ *on* $\prod_{i=1}^{N}(a_i,b_i)$ *such that the function* $x \mapsto \left(u_2(x)\frac{k_{b-}(x,y)}{K_{b-}(x)}\right)$ *is integrable on* $\prod_{i=1}^{N}(a_i,b_i)$, *for each fixed* $y \in \prod_{i=1}^{N}(a_i,b_i)$. *We define* w_2 *on* $\prod_{i=1}^{N}(a_i,b_i)$ *by*

$$w_2(y) := \int_{\prod_{i=1}^{N}(a_i,b_i)} u_2(x)\frac{k_{b-}(x,y)}{K_{b-}(x)}dx < \infty. \tag{70}$$

We have that

$$w_2(y) = \left(\prod_{i=1}^{N}\alpha_i\right)\int_{a_1}^{y_1}...\int_{a_N}^{y_N} u_2(x_1,...,x_N)\left(\prod_{i=1}^{N}\frac{(y_i-x_i)^{\alpha_i-1}}{(b_i-x_i)^{\alpha_i}}\right)dx_1...dx_N, \tag{71}$$

$\forall\ y \in \prod_{i=1}^{N}(a_i,b_i)$.

If we choose as

$$u_1(x) = u_1^*(x) := \prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}, \tag{72}$$

then

$$w_1^*(y) := w_1(y) = \left(\prod_{i=1}^{N}\alpha_i\right)\int_{y_1}^{b_1}...\int_{y_N}^{b_N}\left(\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}\right)dx_1...dx_N$$

$$= \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \int_{y_i}^{b_i} (x_i - y_i)^{\alpha_i - 1} dx_i \right)$$

$$= \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(b_i - y_i)^{\alpha_i}}{\alpha_i} \right) = \prod_{i=1}^{N} (b_i - y_i)^{\alpha_i}.$$

that is

$$w_1^* (y) = \prod_{i=1}^{N} (b_i - y_i)^{\alpha_i}, \quad \forall y \in \prod_{i=1}^{N} (a_i, b_i). \tag{73}$$

If we choose as

$$u_2 (x) = u_2^* (x) := \prod_{i=1}^{N} (b_i - x_i)^{\alpha_i}, \tag{74}$$

then

$$w_2^* (y) := w_2 (y) = \left(\prod_{i=1}^{N} \alpha_i \right) \int_{a_1}^{y_1} ... \int_{a_N}^{y_N} \left(\prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1} \right) dx_1 ... dx_N$$

$$= \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \int_{a_i}^{y_i} (y_i - x_i)^{\alpha_i - 1} dx_i \right)$$

$$= \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(y_i - a_i)^{\alpha_i}}{\alpha_i} \right) = \prod_{i=1}^{N} (y_i - a_i)^{\alpha_i}.$$

That is

$$w_2^* (y) = \prod_{i=1}^{N} (y_i - a_i)^{\alpha_i}, \quad \forall y \in \prod_{i=1}^{N} (a_i, b_i). \tag{75}$$

Here we choose $f_j : \prod_{i=1}^{N} (a_i, b_i) \to \mathbb{R}$, $j = 1, ..., m$, *that are Lebesgue measurable and* $I_{a+}^{\alpha} (|f_j|)$, $I_{b-}^{\alpha} (|f_j|)$ *are finite a.e., one or the other, or both.*

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$ *and the functions* $\Phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, ..., m$, *to be convex and increasing.*

Then by (22) we obtain

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} u_1 (x) \prod_{j=1}^{m} \Phi_j \left(\frac{\left| I_{a+}^{\alpha} (f_j) (x) \right| \prod_{i=1}^{N} \Gamma (\alpha_i + 1)}{\prod_{i=1}^{N} (x_i - a_i)^{\alpha_i}} \right) dx \leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} w_1(y)\,\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}}, \tag{76}$$

under the assumptions:

(i) $f_j, \Phi_j(|f_j|)^{p_j}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}dy$ *-integrable,*

a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$, *for all* $j=1,...,m$,

(ii) $w_1\Phi_j(|f_j|)^{p_j}$ *is Lebesgue integrable,* $j=1,...,m$.

Similarly, by (22), we obtain

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} u_2(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|I_{b-}^{\alpha}(f_j)(x)\right|\prod_{i=1}^{N}\Gamma(\alpha_i+1)}{\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}}\right)dx\leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} w_2(y)\,\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}}, \tag{77}$$

under the assumptions:

(i) $f_j, \Phi_j(|f_j|)^{p_j}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}dy$ *-integrable,*

a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$, *for all* $j=1,...,m$,

(ii) $w_2\Phi_j(|f_j|)^{p_j}$ *is Lebesgue integrable,* $j=1,...,m$.

Using (72) and (73) we rewrite (76), as follows

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|I_{a+}^{\alpha}(f_j)(x)\right|\prod_{i=1}^{N}\Gamma(\alpha_i+1)}{\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}}\right)dx\leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}}\leq \tag{78}$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) following (76)

and

*(ii)** $\Phi_j\left(|f_j|\right)^{p_j}$ *is Lebesgue integrable,* $j=1,...,m$.

Similarly, using (74) and (75) we rewrite (77),

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}\right)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|I_{b-}^{\alpha}(f_j)(x)\right|\prod_{i=1}^{N}\Gamma(\alpha_i+1)}{\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}}\right)dx\leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(y_i-a_i)^{\alpha_i}\right)\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}}\leq \tag{79}$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) following (77),

and

*(ii)** $\Phi_j\left(|f_j|\right)^{p_j}$ *is Lebesgue integrable,* $j=1,...,m$.

Let now $\beta_j\geq 1$, $j=1,...,m$.

Then, by (78), we obtain

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}\right)dx\leq$$

$$\left(\frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right).$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}} \leq \tag{80}$$

$$\left(\frac{\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}}.$$

But it holds

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}\right)dx \geq$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}dx\right). \tag{81}$$

So by (80) and (81) we derive

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}dx \leq \tag{82}$$

$$\left(\prod_{i=1}^{N}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)^{\sum_{j=1}^{m}\beta_j}\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $|f_j|^{p_j\beta_j}$ *is* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$, *for all* $j=1,...,m$,

(ii) $|f_j|^{p_j\beta_j}$ *is Lebesgue integrable,* $j=1,...,m$.

We also have, by (78), that

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right) e^{\left(\sum_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|\right)\left(\prod_{i=1}^{N}\frac{\Gamma(\alpha_i+1)}{(x_i-a_i)^{\alpha_i}}\right)} dx \leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}} \leq \tag{83}$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) f_j, $e^{p_j|f_j|}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}\,dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$, *for all* $j=1,...,m$,

(ii) $e^{p_j|f_j|}$ *is Lebesgue integrable,* $j=1,...,m$.

From (79) we get

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\prod_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|^{\beta_j}\right)dx \leq$$

$$\left(\frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)\cdot$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(y_i-a_i)^{\alpha_i}\right)|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}} \leq \tag{84}$$

$$\left(\frac{\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i}}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i + 1) \right)^{\sum_{j=1}^{m} \beta_j}} \right) \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(y)|^{\beta_j p_j} \, dy \right)^{\frac{1}{p_j}}.$$

But it holds

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} \left(\prod_{i=1}^{N} (b_i - x_i)^{\alpha_i} \right)^{\left(1 - \sum_{j=1}^{m} \beta_j\right)} \left(\prod_{j=1}^{m} \left| I_{b-}^{\alpha} (f_j)(x) \right|^{\beta_j} \right) dx \geq$$

$$\left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right)^{\left(1 - \sum_{j=1}^{m} \beta_j\right)} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} \left(\prod_{j=1}^{m} \left| I_{b-}^{\alpha} (f_j)(x) \right|^{\beta_j} \right) dx \right). \tag{85}$$

So by (84) and (85) we obtain

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} \left(\prod_{j=1}^{m} \left| I_{b-}^{\alpha} (f_j)(x) \right|^{\beta_j} \right) dx \leq \tag{86}$$

$$\left(\prod_{i=1}^{N} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right)^{\sum_{j=1}^{m} \beta_j} \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(y)|^{\beta_j p_j} \, dy \right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $|f_j|^{p_j \beta_j}$ *is* $\frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N} [x_i, b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1} dy$ *-integrable, a.e. in* $x \in \prod_{i=1}^{N} (a_i, b_i)$, *for all* $j = 1, ..., m$,

(ii) $|f_j|^{p_j \beta_j}$ *is Lebesgue integrable,* $j = 1, ..., m$.

We also have, by (79), that

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} \left(\prod_{i=1}^{N} (b_i - x_i)^{\alpha_i} \right) e^{\left(\sum_{j=1}^{m} \left| I_{b-}^{\alpha} (f_j)(x) \right| \right) \left(\prod_{i=1}^{N} \frac{\Gamma(\alpha_i + 1)}{(b_i - x_i)^{\alpha_i}} \right)} dx \leq$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(y_i-a_i)^{\alpha_i}\right)e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}} \leq \tag{87}$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) f_j, $e^{p_j|f_j|}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}\,dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$, *for all* $j=1,...,m$,

(ii) $e^{p_j|f_j|}$ *is Lebesgue integrable,* $j=1,...,m$.

Background 20 *In order to apply Theorem 1 to the case of a spherical shell we need:*

Let $N\geq 2$, $S^{N-1}:=\{x\in\mathbb{R}^N:|x|=1\}$ *the unit sphere on* $\mathbb{R}^N$, *where* $|\cdot|$ *stands for the Euclidean norm in* $\mathbb{R}^N$. *Also denote the ball* $B(0,R):=\{x\in\mathbb{R}^N:|x|<R\}\subseteq\mathbb{R}^N$, $R>0$, *and the spherical shell*

$$A:=B(0,R_2)-\overline{B(0,R_1)},\ \ 0<R_1<R_2. \tag{88}$$

For the following see [15, pp. 149-150], and [17, pp. 87-88].

For $x\in\mathbb{R}^N-\{0\}$ *we can write uniquely* $x=r\omega$, *where* $r=|x|>0$, *and* $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$.

Clearly here

$$\mathbb{R}^N-\{0\}=(0,\infty)\times S^{N-1}, \tag{89}$$

and

$$\overline{A}=[R_1,R_2]\times S^{N-1}. \tag{90}$$

We will be using

Theorem 21 *[1, p. 322] Let* $f:A\to\mathbb{R}$ *be a Lebesgue integrable function. Then*

$$\int_A f(x)\,dx=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}f(r\omega)\,r^{N-1}dr\right)d\omega. \tag{91}$$

So we are able to write an integral on the shell in polar form using the polar coordinates (r,ω).

We need

Definition 22 *[1, p. 458] Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $f \in C^n\left(\overline{A}\right)$, and A is a spherical shell. Assume that there exists function $\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} \in C\left(\overline{A}\right)$, given by*

$$\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\alpha} \frac{\partial^n f(t\omega)}{\partial r^n} dt \right), \tag{92}$$

where $x \in \overline{A}$; that is $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We call $\frac{\partial^{\nu}_{R_1} f}{\partial r^{\nu}}$ the left radial Canavati-type fractional derivative of f of order ν. If $\nu = 0$, then set $\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} := f(x)$.

Based on [1, p. 288], and [5] we have

Lemma 23 *Let $\gamma \geq 0$, $m := [\gamma]$, $\nu > 0$, $n := [\nu]$, with $0 \leq \gamma < \nu$. Let $f \in C^n\left(\overline{A}\right)$ and there exists $\frac{\partial^{\nu}_{R_1} f(x)}{\partial r^{\nu}} \in C\left(\overline{A}\right)$, $x \in \overline{A}$, A a sperical shell. Further assume that $\frac{\partial^j f(R_1\omega)}{\partial r^j} = 0$, $j = m, m+1, ..., n-1$, $\forall\ \omega \in S^{N-1}$. Then there exists $\frac{\partial^{\gamma}_{R_1} f(x)}{\partial r^{\gamma}} \in C\left(\overline{A}\right)$ such that*

$$\frac{\partial^{\gamma}_{R_1} f(x)}{\partial r^{\gamma}} = \frac{\partial^{\gamma}_{R_1} f(r\omega)}{\partial r^{\gamma}} = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \frac{\partial^{\nu}_{R_1} f(t\omega)}{\partial r^{\nu}} dt, \tag{93}$$

$\forall\ \omega \in S^{N-1}$; all $R_1 \leq r \leq R_2$, indeed $f(r\omega) \in C^{\gamma}_{R_1}([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$.

We make

Remark 24 *In the settings and assumptions of Theorem 1 and Lemma 23 we have*

$$k(r,t) = \frac{1}{\Gamma(\nu-\gamma)} \chi_{[R_1,r]}(t) (r-t)^{\nu-\gamma-1}, \tag{94}$$

and

$$K(r) = \frac{(r-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}, \tag{95}$$

$r, t \in [R_1, R_2]$.

Furthermore we get

$$\frac{k(r,t)}{K(r)} = (\nu-\gamma) \chi_{[R_1,r]}(t) \frac{(r-t)^{\nu-\gamma-1}}{(r-R_1)^{\nu-\gamma}}, \tag{96}$$

and by choosing

$$u(r) := (r-R_1)^{\nu-\gamma}, \quad r \in [R_1, R_2], \tag{97}$$

we find

$$U(t) = (\nu-\gamma) \int_t^{R_2} (r-t)^{\nu-\gamma-1} dr = (R_2-t)^{\nu-\gamma}, \tag{98}$$

$t \in [R_1, R_2]$.

Then by (8) for $p \geq 1$ *we find*

$$\int_{R_1}^{R_2} (r - R_1)^{\nu-\gamma} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p \frac{(\Gamma(\nu-\gamma+1))^p}{(r-R_1)^{(\nu-\gamma)p}} dr \leq$$

$$\int_{R_1}^{R_2} (R_2 - r)^{\nu-\gamma} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr, \tag{99}$$

and

$$\int_{R_1}^{R_2} (r - R_1)^{(\nu-\gamma)(1-p)} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq$$

$$\frac{1}{(\Gamma(\nu-\gamma+1))^p} \int_{R_1}^{R_2} (R_2 - r)^{\nu-\gamma} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr \leq$$

$$\frac{(R_2 - R_1)^{\nu-\gamma}}{(\Gamma(\nu-\gamma+1))^p} \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr. \tag{100}$$

But it holds

$$\int_{R_1}^{R_2} (r - R_1)^{(\nu-\gamma)(1-p)} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \geq$$

$$(R_2 - R_1)^{(\nu-\gamma)(1-p)} \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr. \tag{101}$$

Consequently we derive

$$\int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr, \tag{102}$$

$\forall\ \omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, *and* $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, *and* $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$.

From (102) we have

$$R_2^{1-N} \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq$$

$$\int_{R_1}^{R_2} r^{1-N} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq$$

$$\left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} r^{1-N} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr \leq$$

$$R_1^{1-N} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr. \tag{103}$$

So we get

$$\int_{R_1}^{R_2} r^{N-1}\left|\frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}}\right|^p dr \leq$$

$$\left(\frac{R_2}{R_1}\right)^{N-1}\left(\frac{(R_2-R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right)^p \int_{R_1}^{R_2} r^{N-1}\left|\frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}}\right|^p dr, \tag{104}$$

$\forall\ \omega \in S^{N-1}$.

Hence

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left|\frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}}\right|^p dr\right) d\omega \leq$$

$$\left(\frac{R_2}{R_1}\right)^{N-1}\left(\frac{(R_2-R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right)^p \int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left|\frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}}\right|^p dr\right) d\omega. \tag{105}$$

By Theorem 21, equality (91), we obtain

$$\int_A\left|\frac{\partial_{R_1}^{\gamma} f(x)}{\partial r^{\gamma}}\right|^p dx \leq \left(\frac{R_2}{R_1}\right)^{N-1}\left(\frac{(R_2-R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right)^p \int_A\left|\frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}}\right|^p dx. \tag{106}$$

We have proved the following fractional Poincaré type inequalities on the shell.

Theorem 25 *Here all as in Lemma 23, $p \geq 1$.*

It holds

1)

$$\left\|\frac{\partial_{R_1}^{\gamma} f}{\partial r^{\gamma}}\right\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)}\left(\frac{(R_2-R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right)\left\|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right\|_{p,A}, \tag{107}$$

2) When $\gamma = 0$, we have

$$\|f\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)}\left(\frac{(R_2-R_1)^{\nu}}{\Gamma(\nu+1)}\right)\left\|\frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}}\right\|_{p,A}. \tag{108}$$

See the related, and proof, results in [1, pp. 458-459] with different constants and proof in the corresponding inequalities.

Similar results can be produced for the right radial Canavati type fractional derivative.

We choose to omit it.

We make

Remark 26 *(from [1], p. 460) Here we denote $\lambda_{\mathbb{R}^N}(x) \equiv dx$ the Lebesgue measure on $\mathbb{R}^N$, $N \geq 2$, and by $\lambda_{S^{N-1}}(\omega) = d\omega$ the surface measure on S^{N-1},* *where* $\mathcal{B}_X$ *stands for the Borel class on space* X*. Define the measure* R_N *on* $\left((0,\infty),\mathcal{B}_{(0,\infty)}\right)$ *by*

$$R_N(B)=\int_B r^{N-1}dr,\quad \text{any } B\in\mathcal{B}_{(0,\infty)}.$$

Now let $F\in L_1(A)=L_1\left([R_1,R_2]\times S^{N-1}\right).$

Call

$$K(F):=\{\omega\in S^{N-1}:F(\cdot\omega)\notin L_1\left([R_1,R_2],\mathcal{B}_{[R_1,R_2]},R_N\right)\}. \tag{109}$$

We get, by Fubini's theorem and [17], pp. 87-88, that

$$\lambda_{S^{N-1}}(K(F))=0.$$

Of course

$$\theta(F):=[R_1,R_2]\times K(F)\subset A,$$

and

$$\lambda_{\mathbb{R}^N}(\theta(F))=0.$$

Above $\lambda_{S^{N-1}}$ *is defined as follows: let* $A\subset S^{N-1}$ *be a Borel set, and let*

$$\widetilde{A}:=\{ru:0<r<1,u\in A\}\subset\mathbb{R}^N;$$

we define

$$\lambda_{S^{N-1}}(A):=N\lambda_{\mathbb{R}^N}\left(\widetilde{A}\right).$$

We have that

$$\lambda_{S^{N-1}}\left(S^{N-1}\right)=\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},$$

the surface area of S^{N-1}*.*

See also [15, pp. 149-150], [17, pp. 87-88] and [1], p. 320.

Following [1, p. 466] we define the left Riemann-Liouville radial fractional derivative next.

Definition 27 *Let* $\beta>0$*,* $m:=[\beta]+1$*,* $F\in L_1(A)$*, and* A *is the spherical shell. We define*

$$\frac{\overline{\partial}_{R_1}^{\beta}F(x)}{\partial r^{\beta}}:=\begin{cases}\frac{1}{\Gamma(m-\beta)}\left(\frac{\partial}{\partial r}\right)^m\int_{R_1}^r(r-t)^{m-\beta-1}F(t\omega)\,dt, & \\ \qquad\qquad \text{for } \omega\in S^{N-1}-K(F), & \\ 0,\quad \text{for } \omega\in K(F), & \end{cases} \tag{110}$$

where $x=r\omega\in A$*,* $r\in[R_1,R_2]$*,* $\omega\in S^{N-1}$*;* $K(F)$ *as in (109).*

If $\beta=0$*, define*

$$\frac{\overline{\partial}_{R_1}^{\beta}F(x)}{\partial r^{\beta}}:=F(x).$$

We need the following important representation result for left Riemann-Liouville radial fractional derivatives, by [1, p. 466].

Theorem 28 *Let $\nu \geq \gamma+1$, $\gamma \geq 0$, $n := [\nu]$, $m := [\gamma]$, $F : \overline{A} \to \mathbb{R}$ with $F \in L_1(A)$. Assume that $F(\cdot\omega) \in AC^n([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$, and that $\frac{\overline{\partial}_{R_1}^{\nu} F(\cdot\omega)}{\partial r^{\nu}}$ is measurable on $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$. Also assume $\exists\ \frac{\overline{\partial}_{R_1}^{\nu} F(r\omega)}{\partial r^{\nu}} \in \mathbb{R}$, $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$, and $\frac{\overline{\partial}_{R_1}^{\nu} F(x)}{\partial r^{\nu}}$ is measurable on $\overline{A}$. Suppose $\exists$ $M_1 > 0$:*

$$\left|\frac{\overline{\partial}_{R_1}^{\nu} F(r\omega)}{\partial r^{\nu}}\right| \leq M_1, \quad \forall\ (r,\omega) \in [R_1, R_2] \times S^{N-1}. \tag{111}$$

We suppose that $\frac{\overline{\partial}^{j} F(R_1\omega)}{\partial r^{j}} = 0$, $j = m, m+1, ..., n-1$; $\forall\ \omega \in S^{N-1}$.

Then

$$\frac{\overline{\partial}_{R_1}^{\gamma} F(x)}{\partial r^{\gamma}} = \overline{D}_{R_1}^{\gamma} F(r\omega) = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \left(\overline{D}_{R_1}^{\nu} F\right)(t\omega)\, dt, \tag{112}$$

valid $\forall\ x \in \overline{A}$; that is, true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Here

$$\overline{D}_{R_1}^{\gamma} F(\cdot\omega) \in AC([R_1, R_2]), \tag{113}$$

$\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Furthermore

$$\frac{\overline{\partial}_{R_1}^{\gamma} F(x)}{\partial r^{\gamma}} \in L_{\infty}(A), \ \gamma > 0. \tag{114}$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\nu)} \int_{R_1}^{r} (r-t)^{\nu-1} \left(\overline{D}_{R_1}^{\nu} F\right)(t\omega)\, dt, \tag{115}$$

true $\forall\ x \in \overline{A}$; that is, true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$, and

$$F(\cdot\omega) \in AC([R_1, R_2]), \quad \forall\ \omega \in S^{N-1}. \tag{116}$$

We give also the following fractional Poincaré type inequalities on the spherical shell.

Theorem 29 *Here all as in Theorem 28, $p \geq 1$. Then*

1)

$$\left\|\frac{\overline{\partial}_{R_1}^{\gamma} F}{\partial r^{\gamma}}\right\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2-R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)}\right) \left\|\frac{\overline{\partial}_{R_1}^{\nu} F}{\partial r^{\nu}}\right\|_{p,A}, \tag{117}$$

2) When $\gamma = 0$, we have

$$\|F\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2-R_1)^{\nu}}{\Gamma(\nu+1)}\right) \left\|\frac{\overline{\partial}_{R_1}^{\nu} F}{\partial r^{\nu}}\right\|_{p,A}. \tag{118}$$

Proof. As in Theorem 25, based on Theorem 28. ∎

See also similar results in [1, p. 468].

We also need (see [1], p. 421).

Definition 30 *Let $F : \overline{A} \to \mathbb{R}$, $\nu \geq 0$, $n := \lceil \nu \rceil$ such that $F(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$.*

We call the left Caputo radial fractional derivative the following function

$$\frac{\partial^{\nu}_{*R_1} F(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(n-\nu)} \int_{R_1}^{r} (r-t)^{n-\nu-1} \frac{\partial^n F(t\omega)}{\partial r^n} dt, \tag{119}$$

where $x \in \overline{A}$, i.e. $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

Clearly

$$\frac{\partial^{0}_{*R_1} F(x)}{\partial r^{0}} = F(x), \tag{120}$$

$$\frac{\partial^{\nu}_{*R_1} F(x)}{\partial r^{\nu}} = \frac{\partial^{\nu} F(x)}{\partial r^{\nu}}, \quad \text{if } \nu \in \mathbb{N}.$$

Above function (119) exists almost everywhere for $x \in \overline{A}$, see [1], p. 422.

We mention the following fundamental representation result (see [1], p. 422-423 and [5]).

Theorem 31 *Let $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := \lceil \nu \rceil$, $m := \lceil \gamma \rceil$, $F : \overline{A} \to \mathbb{R}$ with $F \in L_1(A)$. Assume that $F(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$, and that $\frac{\partial^{\nu}_{*R_1} F(\cdot\omega)}{\partial r^{\nu}} \in L_\infty(R_1, R_2)$ for all $\omega \in S^{N-1}$.*

Further assume that $\frac{\partial^{\nu}_{*R_1} F(x)}{\partial r^{\nu}} \in L_\infty(A)$. More precisely, for these $r \in [R_1, R_2]$, for each $\omega \in S^{N-1}$, for which $D^{\nu}_{*R_1} F(r\omega)$ takes real values, there exists $M_1 > 0$ such that $\left| D^{\nu}_{*R_1} F(r\omega) \right| \leq M_1$.

We suppose that $\frac{\partial^j F(R_1\omega)}{\partial r^j} = 0$, $j = m, m+1, ..., n-1$; for every $\omega \in S^{N-1}$.

Then

$$\frac{\partial^{\gamma}_{*R_1} F(x)}{\partial r^{\gamma}} = D^{\gamma}_{*R_1} F(r\omega) = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \left(D^{\nu}_{*R_1} F \right)(t\omega)\, dt, \tag{121}$$

valid $\forall\ x \in \overline{A}$; i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Here

$$D^{\gamma}_{*R_1} F(\cdot\omega) \in AC([R_1, R_2]), \tag{122}$$

$\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Furthermore

$$\frac{\partial^{\gamma}_{*R_1} F(x)}{\partial r^{\gamma}} \in L_\infty(A), \ \gamma > 0. \tag{123}$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\nu)} \int_{R_1}^{r} (r-t)^{\nu-1} \left(D^{\nu}_{*R_1} F \right)(t\omega)\, dt, \tag{124}$$

true $\forall\ x \in \overline{A}$; *i.e. true* $\forall\ r \in [R_1, R_2]$ *and* $\forall\ \omega \in S^{N-1}$, *and*

$$F(\cdot\omega) \in AC([R_1, R_2]), \quad \forall\ \omega \in S^{N-1}. \tag{125}$$

We finish with the following Poincaré type inequalities involving left Caputo radial fractional derivatives.

Theorem 32 *Here all as in Theorem 31, $p \geq 1$. Then*
1)

$$\left\| \frac{\partial^{\gamma}_{*R_1} F}{\partial r^{\gamma}} \right\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu - \gamma + 1)}\right) \left\| \frac{\partial^{\nu}_{*R_1} F}{\partial r^{\nu}} \right\|_{p,A}, \tag{126}$$

2) When $\gamma = 0$, we have

$$\|F\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu + 1)}\right) \left\| \frac{\partial^{\nu}_{*R_1} F}{\partial r^{\nu}} \right\|_{p,A}. \tag{127}$$

Proof. As in Theorem 25, based on Theorem 31. ■
See also similar results in [1, p. 464].

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Rational Inequalities for Integral Operators under convexity

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Abstract

Here we present integral inequalities for convex and increasing functions applied to products of ratios of functions and other important mixtures. As applications we derive a wide range of fractional inequalities of Hardy type. They involve the left and right Riemann-Liouville fractional integrals and their generalizations, in particular the Hadamard fractional integrals. Also inequalities for Riemann-Liouville, Caputo, Canavati and their generalizations fractional derivatives. These application inequalities are of L_p type, $p \geq 1$, exponential type and of other general forms.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.
Key words and phrases: integral operator, fractional integral, fractional derivative, Hardy fractional inequality, Hadamard fractional integral.

1 Introduction

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2 and

$$K_i(x) = \int_{\Omega_2} k_i(x, y)\, d\mu_2(y), \text{ for any } x \in \Omega_1, \tag{1}$$

$i = 1, ..., m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x, y)\, f_i(y)\, d\mu_2(y), \tag{2}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., m$.

Here u stands for a weight function on Ω_1.

In [4] we proved the following general result.

Theorem 1 *Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} K_i(x)} \right)$$

is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define λ_m on Ω_2 by

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{3}$$

If $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K_i(x)} \right| \right) d\mu_1(x) \leq \tag{4}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_j(|f_j(y)|) \, \lambda_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_i : \Omega_2 \to \mathbb{R}$ such that:

(i) $f_i, \Phi_i(|f_i|)$ are both $k_i(x, y) \, d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_m \Phi_j(|f_j|) ; \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), ..., \widehat{\Phi_j(|f_j|)}, ..., \Phi_m(|f_m|)$ are all μ_2 -integrable,

and for all corresponding functions g_i given by (2). Above $\widehat{\Phi_j(|f_j|)}$ means missing item.

Here $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$. Let $\varphi : \mathbb{R}^{*2} \to \mathbb{R}^*$ be a Borel measurable function. Let $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ be measurable functions, $i = 1, ..., m$.

The function $\varphi(f_{1i}(y), f_{2i}(y))$, $y \in \Omega_2$, $i = 1, ..., m$, is Σ_2-measurable. In this article we assume that $0 < \varphi(f_{1i}(y), f_{2i}(y)) < \infty$, a.e., $i = 1, ..., m$.

We consider

$$f_{3i}(y) := \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))}, \tag{5}$$

$i = 1, ..., m$, $y \in \Omega_2$, which is a measurable function.

We also consider here

$$k_i^*(x,y) := k_i(x,y)\,\varphi(f_{1i}(y), f_{2i}(y)), \tag{6}$$

$y \in \Omega_2$, $i = 1, ..., m$, which is a nonnegative a.e. measurable function on $\Omega_1 \times \Omega_2$.

We have that $k_i^*(x,\cdot)$ is measurable on Ω_2, $i = 1, ..., m$.

Denote by

$$K_i^*(x) := \int_{\Omega_2} k_i^*(x,y)\,d\mu_2(y) \tag{7}$$

$$= \int_{\Omega_2} k_i(x,y)\,\varphi(f_{1i}(y), f_{2i}(y))\,d\mu_2(y), \quad i = 1, ..., m.$$

We assume that $K_i^*(x) > 0$, a.e. on Ω_1.

So here the function

$$g_{1i}(x) = \int_{\Omega_2} k_i(x,y)\,f_{1i}(y)\,d\mu_2(y)$$

$$= \int_{\Omega_2} k_i(x,y)\,\varphi(f_{1i}(y), f_{2i}(y))\left(\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))}\right)d\mu_2(y)$$

$$= \int_{\Omega_2} k_i^*(x,y)\,f_{3i}(y)\,d\mu_2(y), \quad i = 1, ..., m. \tag{8}$$

A typical example is when

$$\varphi(f_{1i}(y), f_{2i}(y)) = f_{2i}(y), \quad i = 1, ..., m,\ y \in \Omega_2. \tag{9}$$

In that case we have that

$$f_{3i}(y) = \frac{f_{1i}(y)}{f_{2i}(y)}, \ \ i = 1, ..., m, \ \ y \in \Omega_2. \tag{10}$$

Tha latter case was studied in [13], for $i = 1$, which is an article with interesting ideas however containing several mistakes.

In the special case (10) we get that

$$K_i^*(x) = g_{2i}(x) := \int_{\Omega_2} k_i(x,y)\,f_{2i}(y)\,d\mu_2(y), \quad i = 1, ..., m. \tag{11}$$

In this article we get first general results by applying Theorem 1 for (f_{3i}, g_{1i}), $i = 1, ..., m$, and on other various important settings, then we give wide applications to Fractional Calculus.

2 Main Results

We present

Theorem 2 *Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y) \varphi(f_{1i}(y), f_{2i}(y))}{\prod_{i=1}^{m} K_i^*(x)} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m^* on Ω_2 by

$$\lambda_m^*(y) := \left(\prod_{i=1}^{m} \varphi(f_{1i}(y), f_{2i}(y)) \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} K_i^*(x)} \right) d\mu_1(x) < \infty. \tag{12}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{K_i^*(x)} \right| \right) d\mu_1(x) \leq \tag{13}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i \left(\left| \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right| \right) d\mu_2(y) \right) \cdot$$

$$\left(\int_{\Omega_2} \Phi_j \left(\left| \frac{f_{1j}(y)}{\varphi(f_{1j}(y), f_{2j}(y))} \right| \right) \lambda_m^*(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ such that

(i) $\left(\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right)$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right| \right)$ are both $k_i(x, y) \varphi(f_{1i}(y), f_{2i}(y))$ $d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_m^* \Phi_j \left(\left| \frac{f_{1j}(y)}{\varphi(f_{1j}(y), f_{2j}(y))} \right| \right)$ and $\Phi_i \left(\left| \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right| \right)$, for $i \in \{1, ..., m\} - \{j\}$ are all μ_2 -integrable

and for all corresponding functions g_{1i} given by (8).

Proof. Direct application of Theorem 1 on the setting described at introduction. ■

In the special case of (9), (10), (11) we derive

Theorem 3 *Here $0 < f_{2i}(y) < \infty$, a.e., $i = 1, ..., m$. Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y) f_{2i}(y)}{\prod_{i=1}^{m} g_{2i}(x)} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m^{**} on Ω_2 by

$$\lambda_m^{**}(y) := \left(\prod_{i=1}^{m} f_{2i}(x) \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} g_{2i}(x)} \right) d\mu_1(x) < \infty.$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{g_{2i}(x)} \right| \right) d\mu_1(x) \leq \tag{14}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right) d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right) \lambda_m^{**}(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ such that

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$ are both $k_i(x, y) f_{2i}(y) d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\lambda_m^{**} \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right)$, and $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$, for $i \in \{1, ..., m\} - \{j\}$, are all μ_2-integrable

and for all corresponding functions g_{1i} given by (8), and g_{2i} given by (11).

Proof. By Theorem 2. ■

Theorem 3 generalizes and fixes Theorem 1.2 of [13], which inspired the current article.

Next we consider the case of $\varphi(s_i, t_i) = |a_{1i} s_i + a_{2i} t_i|^r$, where $r \in \mathbb{R}$; $s_i, t_i \in \mathbb{R}^*$; $a_{1i}, a_{2i} \in \mathbb{R}$, $i = 1, ..., m$. We assume here that

$$0 < |a_{1i} f_{1i}(y) + a_{2i} f_{2i}(y)|^r < \infty, \tag{15}$$

a.e., $i = 1, ..., m$.

We further assume that

$$\overline{K_i^*}(x) := \int_{\Omega_2} k_i(x, y) |a_{1i} f_{1i}(y) + a_{2i} f_{2i}(y)|^r d\mu_2(y) > 0, \tag{16}$$

a.e. on Ω_1, $i = 1, ..., m$.

Here we have

$$f_{3i}(y) = \frac{f_{1i}(y)}{|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r}, \tag{17}$$

$i = 1, ..., m$, $y \in \Omega_2$.

Denote by

$$\overline{k_i^*}(x, y) := k_i(x, y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r, \tag{18}$$

$i = 1, ..., m$.

By Theorem 2 we obtain

Theorem 4 *Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r}{\prod_{i=1}^{m} \overline{K_i^*}(x)} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define $\overline{\lambda_m^*}$ on Ω_2 by

$$\overline{\lambda_m^*}(y) := \left(\prod_{i=1}^{m} |a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} \overline{K_i^*}(x)} \right) d\mu_1(x) < \infty. \tag{19}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions. Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left| \frac{g_{1i}(x)}{\overline{K_i^*}(x)} \right| \right) d\mu_1(x) \leq \tag{20}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \Phi_i\left(\left| \frac{f_{1i}(y)}{(a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y))^r} \right| \right) d\mu_2(y) \right) \cdot$$

$$\left(\int_{\Omega_2} \Phi_j\left(\left| \frac{f_{1j}(y)}{(a_{1j}f_{1j}(y) + a_{2j}f_{2j}(y))^r} \right| \right) \overline{\lambda_m^*}(y)\, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ such that

(i) $\left(\frac{f_{1i}(y)}{|a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y)|^r} \right)$, $\Phi_i\left(\left| \frac{f_{1i}(y)}{(a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y))^r} \right| \right)$ are both $k_i(x, y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r\, d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $\overline{\lambda_m^*}\Phi_j\left(\left| \frac{f_{1j}(y)}{(a_{1j}f_{1j}(y)+a_{2j}f_{2j}(y))^r} \right| \right)$ and $\Phi_i\left(\left| \frac{f_{1i}(y)}{(a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y))^r} \right| \right)$, for $i \in \{1, ..., m\} - \{j\}$ are all μ_2 -integrable

and for all corresponding functions g_{1i} given by (8).

In Theorem 4 of great interest is the case of $r \in \mathbb{Z} - \{0\}$ and $a_{1i} = a_{2i} = 1$, all $i = 1, ..., m$; or $a_{1i} = 1$, $a_{2i} = -1$, all $i = 1, ..., m$.

Another interesting case arises when

$$\varphi\left(f_{1i}(y), f_{2i}(y)\right) := |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2}, \tag{21}$$

$i = 1, ..., m$, where $r_1, r_2 \in \mathbb{R}$. We assume that

$$0 < |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2} < \infty, \text{ a.e., } i = 1, ..., m. \tag{22}$$

In this case

$$f_{3i}(y) = \frac{f_{1i}(y)}{|f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2}}, \tag{23}$$

$i = 1, ..., m$, $y \in \Omega_2$, also

$$k_i^*(x, y) = k_{pi}^*(x, y) := k_i(x, y) |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2}, \tag{24}$$

$y \in \Omega_2$, $i = 1, ..., m$.

We have

$$K_i^*(x) = K_{pi}^*(x) := \int_{\Omega_2} k_i(x, y) |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2} d\mu_2(y), \tag{25}$$

$i = 1, ..., m$.

We assume that $K_{pi}^* > 0$, a.e. on Ω_1.

By Theorem 2 we derive

Theorem 5 *Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y) |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2}}{\prod_{i=1}^{m} K_{pi}^*(x)} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_{pm}^* on Ω_2 by

$$\lambda_{pm}^*(y) := \left(\prod_{i=1}^{m} |f_{1i}(y)|^{r_1} |f_{2i}(y)|^{r_2} \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} K_{pi}^*(x)} \right) d\mu_1(x) < \infty. \tag{26}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{K_{pi}^*(x)} \right| \right) d\mu_1(x) \leq \tag{27}$$

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$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_{\Omega_2}\Phi_i\left(|f_{1i}(y)|^{1-r_1}|f_{2i}(y)|^{-r_2}\right)d\mu_2(y)\right)\cdot$$

$$\left(\int_{\Omega_2}\Phi_j\left(|f_{1j}(y)|^{1-r_1}|f_{2j}(y)|^{-r_2}\right)\lambda^*_{pm}(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i=1,...,m,\ f_{1i},f_{2i}:\Omega_2\to\mathbb{R}$ *such that*

(i) $\left(\frac{f_{1i}(y)}{|f_{1i}(y)|^{r_1}|f_{2i}(y)|^{r_2}}\right)$, $\Phi_i\left(|f_{1i}(y)|^{1-r_1}|f_{2i}(y)|^{-r_2}\right)$ *are both* $k_i(x,y)|f_{1i}(y)|^{r_1}|f_{2i}(y)|^{r_2}d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$,

(ii) $\lambda^*_{pm}\Phi_j\left(|f_{1j}(y)|^{1-r_1}|f_{2j}(y)|^{-r_2}\right)$ *and* $\Phi_i\left(|f_{1i}(y)|^{1-r_1}|f_{2i}(y)|^{-r_2}\right)$, *for* $i\in\{1,...,m\}-\{j\}$ *are all* μ_2 *-integrable*

and for all corresponding functions g_{1i} *given by (8).*

In Theorem 5 of interest will be the case of $r_1=1-n$, $r_2=-n$, $n\in\mathbb{N}$. In that case $|f_{1i}(y)|^{1-r_1}|f_{2i}(y)|^{-r_2}=|f_{1i}(y)f_{2i}(y)|^n$, etc.

Next we apply Theorem 2 for specific convex functions.

Theorem 6 *Let* $j\in\{1,...,m\}$ *be fixed. Assume that the function*

$$x\mapsto\left(\frac{u(x)\prod_{i=1}^{m}k_i(x,y)\varphi(f_{1i}(y),f_{2i}(y))}{\prod_{i=1}^{m}K_i^*(x)}\right)$$

is integrable on Ω_1, *for each* $y\in\Omega_2$. *Define* λ^*_m *on* Ω_2 *by*

$$\lambda^*_m(y):=\left(\prod_{i=1}^{m}\varphi(f_{1i}(y),f_{2i}(y))\right)\int_{\Omega_1}\left(\frac{u(x)\prod_{i=1}^{m}k_i(x,y)}{\prod_{i=1}^{m}K_i^*(x)}\right)d\mu_1(x)<\infty. \tag{28}$$

Then

$$\int_{\Omega_1}u(x)\,e^{\sum_{i=1}^{m}\left|\frac{g_{1i}(x)}{K_i^*(x)}\right|}d\mu_1(x)\le \tag{29}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_{\Omega_2}e^{\left|\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}\right|}d\mu_2(y)\right)\left(\int_{\Omega_2}e^{\left|\frac{f_{1j}(y)}{\varphi(f_{1j}(y),f_{2j}(y))}\right|}\lambda^*_m(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i=1,...,m,\ f_{1i},f_{2i}:\Omega_2\to\mathbb{R}$ *such that*

(i) $\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}$, $e^{\left|\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}\right|}$ *are both* $k_i(x,y)\varphi(f_{1i}(y),f_{2i}(y))\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$,

(ii) $\lambda_m^* e^{\left|\frac{f_{1j}(y)}{\varphi(f_{1j}(y),f_{2j}(y))}\right|}$ *and* $e^{\left|\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}\right|}$, *for* $i \in \{1,...,m\} - \{j\}$ *are all* μ_2 *-integrable*

and for all corresponding functions g_{1i} *given by (8).*

We continue with

Theorem 7 *Let* $j \in \{1,...,m\}$ *be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y))}{\prod_{i=1}^{m} K_i^*(x)} \right)$$

is integrable on Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m^* *on* Ω_2 *by*

$$\lambda_m^*(y) := \left(\prod_{i=1}^{m} \varphi(f_{1i}(y), f_{2i}(y)) \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i^*(x)} \right) d\mu_1(x) < \infty. \tag{30}$$

Let $p_i \geq 1$, $i = 1, ..., m$.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \left| \frac{g_{1i}(x)}{K_i^*(x)} \right|^{p_i} d\mu_1(x) \leq \tag{31}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_{\Omega_2} \left| \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right|^{p_i} d\mu_2(y) \right) \cdot$$

$$\left(\int_{\Omega_2} \left| \frac{f_{1j}(y)}{\varphi(f_{1j}(y), f_{2j}(y))} \right|^{p_j} \lambda_m^*(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ *such that*

(i) $\left|\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}\right|^{p_i}$ *is* $k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y)) d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\lambda_m^* \left|\frac{f_{1j}(y)}{\varphi(f_{1j}(y),f_{2j}(y))}\right|^{p_j}$ *and* $\left|\frac{f_{1i}(y)}{\varphi(f_{1i}(y),f_{2i}(y))}\right|^{p_i}$, *for* $i \in \{1,...,m\} - \{j\}$ *are all* μ_2 *-integrable*

and for all corresponding functions g_{1i} *given by (8).*

We continue as follows:

Choosing $r_1 = 0, r_2 = -1, i = 1, ..., m$, on (21) we have that

$$\varphi(f_{1i}(y), f_{2i}(y)) = |f_{2i}(y)|^{-1}. \tag{32}$$

We assume that

$$0 < |f_{2i}(y)|^{-1} < \infty, \text{ a.e., } i = 1, ..., m, \tag{33}$$

which is the same as

$$0 < |f_{2i}(y)| < \infty, \text{ a.e., } i = 1, ..., m. \tag{34}$$

In this case

$$f_{3i}(y) = f_{1i}(y) |f_{2i}(y)|, \tag{35}$$

$i = 1, ..., m$, $y \in \Omega_2$, also it is

$$k^*_{pi}(x, y) = \overline{k^*_{pi}}(x, y) := \frac{k_i(x, y)}{|f_{2i}(y)|}, \tag{36}$$

$y \in \Omega_2$, $i = 1, ..., m$.

We have that

$$K^*_{pi}(x) = \overline{K^*_{pi}}(x) := \int_{\Omega_2} \frac{k_i(x, y)}{|f_{2i}(y)|} d\mu_2(y), \tag{37}$$

$i = 1, ..., m$.

We assume that $\overline{K^*_{pi}}(x) > 0$, a.e. on Ω_1.

By Theorem 5 we obtain

Corollary 8 *Let $j \in \{1, ..., m\}$ be fixed. Assume that the function*

$$x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} \overline{K^*_{pi}}(x) |f_{2i}(y)|} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define $\overline{\lambda^*_{pm}}$ on Ω_2 by

$$\overline{\lambda^*_{pm}}(y) := \left(\frac{1}{\prod_{i=1}^{m} |f_{2i}(y)|} \right) \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x, y)}{\prod_{i=1}^{m} \overline{K^*_{pi}}(x)} \right) d\mu_1(x) < \infty. \tag{38}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{\overline{K^*_{pi}}(x)} \right| \right) d\mu_1(x) \leq \tag{39}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_{\Omega_2}\Phi_i\left(\left|f_{1i}(y)\,f_{2i}(y)\right|\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_j\left(\left|f_{1j}(y)\,f_{2j}(y)\right|\right)\overline{\lambda_{pm}^{*}}(y)\,d\mu_2(y)\right)$$

true for all measurable functions, $i=1,...,m$, $f_{1i},f_{2i}:\Omega_2\to\mathbb{R}$ such that

(i) $f_{1i}(y)\,f_{2i}(y)$, $\Phi_i\left(\left|f_{1i}(y)\,f_{2i}(y)\right|\right)$ are both $\frac{k_i(x,y)}{|f_{2i}(y)|}d\mu_2(y)$ -integrable, μ_1 -a.e. in $x\in\Omega_1$,

(ii) $\overline{\lambda_{pm}^{*}}\Phi_j\left(\left|f_{1j}(y)\,f_{2j}(y)\right|\right)$ and $\Phi_i\left(\left|f_{1i}(y)\,f_{2i}(y)\right|\right)$, for $i\in\{1,...,m\}-\{j\}$ are all μ_2 -integrable

and for all corresponding functions g_{1i} given by (8).

To keep exposition short, in the rest of this article we give only applications of Theorem 3 to Fractional Calculus. We need the following:

Let $a<b$, $a,b\in\mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)}\in AC([a,b])$. For any $\alpha\in\mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k\le\alpha<k+1$), and $\lceil\alpha\rceil$ is the ceiling of α ($\min\{n\in\mathbb{N},\ n\ge\alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b)\subset L_1(a,b)$

We give the definition of the Riemann-Liouville fractional integrals, see [14]. Let $[a,b]$, $(-\infty<a<b<\infty)$ be a finite interval on the real axis $\mathbb{R}$.

The Riemann-Liouville fractional integrals $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$ are defined by

$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)\,(x-t)^{\alpha-1}\,dt,\quad (x>a), \tag{40}$$

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b f(t)\,(t-x)^{\alpha-1}\,dt,\quad (x<b), \tag{41}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals.

Let f_{1i},f_{2i} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+}^{\alpha_i}(|f_{1i}|)\right)(x)$, $\left(I_{a+}^{\alpha_i}(|f_{2i}|)\right)(x)\in\mathbb{R}$, $\forall\ x\in(a,b)$, $\alpha_i>0$, $i=1,...,m$, e.g. when $f_{1i},f_{2i}\in L_\infty(a,b)$.

Assume $0<f_{2i}(y)<\infty$, a.e., $i=1,...,m$.

Consider

$$g_{1i}(x)=\left(I_{a+}^{\alpha_i}f_{1i}\right)(x), \tag{42}$$

$$g_{2i}(x)=\left(I_{a+}^{\alpha_i}f_{2i}\right)(x), \tag{43}$$

$x\in(a,b)$, $i=1,...,m$.

Notice that $g_{1i}(x), g_{2i}(x) \in \mathbb{R}$ and they are Lebesgue measurable. We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+}^{\alpha_i} f_{1i}\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{1i}(t)\,dt, \tag{44}$$

$$\left(I_{a+}^{\alpha_i} f_{2i}\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{2i}(t)\,dt, \tag{45}$$

where χ stands for the characteristic function, $x > a$.

So here it is

$$k_i(x,t) := \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, ..., m. \tag{46}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & a < y \leq x, \\ 0, & x < y < b. \end{cases} \tag{47}$$

Let $j \in \{1, ..., m\}$ be fixed.

Assume that the function

$$x \to \frac{\left(\prod_{i=1}^m f_{2i}(y)\right) u(x)\, \chi_{(a,x]}(y)\,(x-y)^{\left(\sum_{i=1}^m \alpha_i\right) - m}}{\left(\prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_{2i}\right)(x)\right)\left(\prod_{i=1}^m \Gamma(\alpha_i)\right)} \tag{48}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$\lambda_m^{**}(y) = \theta_m(y) := \left(\prod_{i=1}^m \left(\frac{f_{2i}(y)}{\Gamma(\alpha_i)}\right)\right) \int_y^b u(x) \left(\frac{(x-y)^{\left(\sum_{i=1}^m \alpha_i\right) - m}}{\prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_{2i}\right)(x)}\right) dx < \infty, \tag{49}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

We get

Proposition 9 *Here all as above. It holds*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i\left(\left|\frac{I_{a+}^{\alpha_i} f_{1i}(x)}{I_{a+}^{\alpha_i} f_{2i}(x)}\right|\right) dx \leq \tag{50}$$

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$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b \Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)dy\right)\left(\int_a^b \Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)\theta_m(y)\,dy\right),$$

true for all measurable functions, $i=1,...,m,$ $f_{1i},f_{2i}:(a,b)\to\mathbb{R}$ *such that*

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$ *are both* $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $\theta_m(y)\,\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)$; *and* $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$, *for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Corollary 10 *It holds*

$$\int_a^b u(x)\,e^{\left(\sum_{i=1}^{m}\left|\frac{I_{a+}^{\alpha_i}f_{1i}(x)}{I_{a+}^{\alpha_i}f_{2i}(x)}\right|\right)}dx\leq \tag{51}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}dy\right)\left(\int_a^b e^{\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}\theta_m(y)\,dy\right),$$

true for all measurable functions, $i=1,...,m,$ $f_{1i},f_{2i}:(a,b)\to\mathbb{R}$ *such that*

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ *are both* $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $\theta_m(y)\,e^{\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$; *and* $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$, *for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Corollary 11 *Let* $p_i\geq 1$, $i=1,...,m$. *It holds*

$$\int_a^b u(x)\left(\prod_{i=1}^{m}\left|\frac{I_{a+}^{\alpha_i}f_{1i}(x)}{I_{a+}^{\alpha_i}f_{2i}(x)}\right|^{p_i}\right)dx\leq \tag{52}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i}dy\right)\left(\int_a^b\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{p_j}\theta_m(y)\,dy\right),$$

true for all measurable functions, $i=1,...,m,$ $f_{1i},f_{2i}:(a,b)\to\mathbb{R}$ *such that*

(i) $\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i}$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $\theta_m(y)\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{p_j}$; *and* $\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i}$, *for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Let us assume that $0 < f_{2i}(x) < \infty$, a.e. in $x \in (a,b)$, and we choose $u(x) = \prod_{i=1}^{m} \left(I_{a+}^{\alpha_i} f_{2i}\right)(x)$. Then

$$\theta_m(y) = \left(\prod_{i=1}^{m} \left(\frac{f_{2i}(y)}{\Gamma(\alpha_i)}\right)\right) \frac{(b-y)^{\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)}}{\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)}, \tag{53}$$

given that $\sum_{i=1}^{m} \alpha_i > m - 1$.

Corollary 12 *Let $p_i \geq 1$, $i = 1, ..., m$, and $\sum_{i=1}^{m} \alpha_i > m-1$. Assume $0 < f_{2i}(x) < \infty$, a.e., $i = 1, ..., m$. It holds*

$$\int_a^b \prod_{i=1}^{m} \left(I_{a+}^{\alpha_i} f_{1i}(x)\right)^{p_i} \left(I_{a+}^{\alpha_i} f_{2i}(x)\right)^{1-p_i} dx \leq \tag{54}$$

$$\frac{1}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i)\right)\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i} dy\right) \cdot$$

$$\left(\int_a^b (f_{1j}(y))^{p_j} (f_{2j}(y))^{1-p_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} f_{2i}(y)\right) (b-y)^{\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)} dy\right) \leq$$

$$\frac{(b-a)^{\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)}}{\left(\sum_{i=1}^{m} \alpha_i - m + 1\right)\left(\prod_{i=1}^{m} \Gamma(\alpha_i)\right)} \cdot$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i} dy\right) \left(\int_a^b (f_{1j}(y))^{p_j} (f_{2j}(y))^{1-p_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^{m} f_{2i}(y)\right) dy\right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : (a,b) \to \mathbb{R}$ such that

(i) $\left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i}$ is $\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{2i}(y)\, dy$ -integrable, a.e. in $x \in (a,b)$, $i = 1, ..., m$,

(ii) $(f_{1j}(y))^{p_j}(f_{2j}(y))^{1-p_j}\left(\prod_{\substack{i=1\\i\neq j}}^{m} f_{2i}(y)\right)$*; and* $\left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i}$*, for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Let f_{1i}, f_{2i} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{b-}^{\alpha_i}(|f_{1i}|)\right)(x)$, $\left(I_{b-}^{\alpha_i}(|f_{2i}|)\right)(x)\in\mathbb{R}$, $\forall\, x\in(a,b)$, $\alpha_i>0$, $i=1,...,m$, e.g. when $f_{1i}, f_{2i}\in L_\infty(a,b)$.

Assume $0<f_{2i}(y)<\infty$, a.e., $i=1,...,m$.

Consider

$$g_{1i}(x)=\left(I_{b-}^{\alpha_i}f_{1i}\right)(x), \tag{55}$$

$$g_{2i}(x)=\left(I_{b-}^{\alpha_i}f_{2i}\right)(x), \tag{56}$$

$x\in(a,b)$, $i=1,...,m$.

Notice that $g_{1i}(x), g_{2i}(x)\in\mathbb{R}$ and they are Lebesgue measurable. We pick $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, the Lebesgue measure.

We see that

$$\left(I_{b-}^{\alpha_i}f_{1i}\right)(x)=\int_a^b\frac{\chi_{[x,b)}(t)(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{1i}(t)\,dt, \tag{57}$$

$$\left(I_{b-}^{\alpha_i}f_{2i}\right)(x)=\int_a^b\frac{\chi_{[x,b)}(t)(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(t)\,dt, \tag{58}$$

$x<b$.

So here it is

$$k_i(x,t):=\frac{\chi_{[x,b)}(t)(t-x)^{\alpha_i-1}}{\Gamma(\alpha_i)},\quad i=1,...,m. \tag{59}$$

In fact here

$$k_i(x,y)=\begin{cases}\frac{(y-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}, & x\le y<b,\\ 0, & a<y<x.\end{cases} \tag{60}$$

Let $j\in\{1,...,m\}$ be fixed.

Assume that the function

$$x\to\frac{\left(\prod_{i=1}^{m}f_{2i}(y)\right)u(x)\,\chi_{[x,b)}(y)\,(y-x)^{\left(\sum_{i=1}^{m}\alpha_i\right)-m}}{\left(\prod_{i=1}^{m}\left(I_{b-}^{\alpha_i}f_{2i}\right)(x)\right)\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\right)} \tag{61}$$

is integrable on (a,b), for each $y\in(a,b)$.

Here we have

$$\lambda_m^{**}(y) = \psi_m(y) := \left(\prod_{i=1}^{m}\left(\frac{f_{2i}(y)}{\Gamma(\alpha_i)}\right)\right)\int_a^y u(x)\left(\frac{(y-x)^{\left(\sum_{i=1}^{m}\alpha_i\right)-m}}{\prod_{i=1}^{m}\left(I_{b-}^{\alpha_i}f_{2i}\right)(x)}\right)dx < \infty, \tag{62}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m$, are convex and increasing functions.

We get

Proposition 13 *Here all as above. It holds*

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{I_{b-}^{\alpha_i}f_{1i}(x)}{I_{b-}^{\alpha_i}f_{2i}(x)}\right|\right)dx \leq \tag{63}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_a^b\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)dy\right)\left(\int_a^b\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)\psi_m(y)\,dy\right),$$

true for all measurable functions, $i = 1,...,m$, $f_{1i}, f_{2i} : (a,b) \to \mathbb{R}$ such that

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$ are both $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ -integrable, a.e. in $x \in (a,b)$,

(ii) $\psi_m(y)\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)$; and $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$, for $i \in \{1,...,m\}-\{j\}$ are all Lebesgue integrable.

Corollary 14 *It holds*

$$\int_a^b u(x)\,e^{\left(\sum_{i=1}^{m}\left|\frac{I_{b-}^{\alpha_i}f_{1i}(x)}{I_{b-}^{\alpha_i}f_{2i}(x)}\right|\right)}dx \leq \tag{64}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_a^b e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}dy\right)\left(\int_a^b e^{\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}\psi_m(y)\,dy\right),$$

true for all measurable functions, $i = 1,...,m$, $f_{1i}, f_{2i} : (a,b) \to \mathbb{R}$ such that

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ are both $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ -integrable, a.e. in $x \in (a,b)$,

(ii) $\psi_m(y)\,e^{\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$; and $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$, for $i \in \{1,...,m\}-\{j\}$ are all Lebesgue integrable.

Corollary 15 *Let $p_i \geq 1$, $i = 1, ..., m$. It holds*

$$\int_a^b u(x) \left(\prod_{i=1}^m \left| \frac{I_{b-}^{\alpha_i} f_{1i}(x)}{I_{b-}^{\alpha_i} f_{2i}(x)} \right|^{p_i} \right) dx \leq \tag{65}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|^{p_i} dy \right) \left(\int_a^b \left| \frac{f_{1j}(y)}{f_{2j}(y)} \right|^{p_j} \psi_m(y)\, dy \right),$$

true for all measurable functions, $i = 1, ..., m$, $f_{1i}, f_{2i} : (a, b) \to \mathbb{R}$ such that

(i) $\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|^{p_i}$ is $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{2i}(y)\, dy$ -integrable, a.e. in $x \in (a, b)$,

(ii) $\psi_m(y) \left| \frac{f_{1j}(y)}{f_{2j}(y)} \right|^{p_j}$; and $\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|^{p_i}$, for $i \in \{1, ..., m\} - \{j\}$ are all Lebesgue integrable.

Let us again assume $0 < f_{2i}(x) < \infty$, a.e. in $x \in (a, b)$, and we choose $u(x) = \prod_{i=1}^m \left(I_{b-}^{\alpha_i} f_{2i} \right)(x)$. Then

$$\psi_m(y) = \left(\prod_{i=1}^m \left(\frac{f_{2i}(y)}{\Gamma(\alpha_i)} \right) \right) \frac{(y-a)^{\left(\sum_{i=1}^m \alpha_i - m + 1 \right)}}{\left(\sum_{i=1}^m \alpha_i - m + 1 \right)}, \tag{66}$$

given that $\sum_{i=1}^m \alpha_i > m - 1$.

Corollary 16 *Let $p_i \geq 1$, $i = 1, ..., m$, and $\sum_{i=1}^m \alpha_i > m - 1$. Assume $0 < f_{2i}(x) < \infty$, a.e., $i = 1, ..., m$. It holds*

$$\int_a^b \prod_{i=1}^m \left(I_{b-}^{\alpha_i} f_{1i}(x) \right)^{p_i} \left(I_{b-}^{\alpha_i} f_{2i}(x) \right)^{1-p_i} dx \leq \tag{67}$$

$$\frac{1}{\left(\prod_{i=1}^m \Gamma(\alpha_i) \right) \left(\sum_{i=1}^m \alpha_i - m + 1 \right)} \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \left(\frac{f_{1i}(y)}{f_{2i}(y)} \right)^{p_i} dy \right) \cdot$$

$$\left(\int_a^b (f_{1j}(y))^{p_j} (f_{2j}(y))^{1-p_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^m f_{2i}(y) \right) (y-a)^{\left(\sum_{i=1}^m \alpha_i - m + 1 \right)} dy \right) \leq$$

$$\frac{(b-a)^{\left(\sum_{i=1}^{m}\alpha_i-m+1\right)}}{\left(\sum_{i=1}^{m}\alpha_i-m+1\right)\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\right)}\cdot$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i}dy\right)\left(\int_a^b(f_{1j}(y))^{p_j}(f_{2j}(y))^{1-p_j}\left(\prod_{\substack{i=1\\i\neq j}}^{m}f_{2i}(y)\right)dy\right),$$

true for all measurable functions, $i=1,...,m$, $f_{1i},f_{2i}:(a,b)\to\mathbb{R}$ such that

(i) $\left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i}$ is $\frac{\chi_{[x,b)}(y)(y-x)^{\alpha_i-1}}{\Gamma(\alpha_i)}f_{2i}(y)\,dy$ -integrable, a.e. in $x\in(a,b)$, $i=1,...,m$,

(ii) $(f_{1j}(y))^{p_j}(f_{2j}(y))^{1-p_j}\left(\prod_{\substack{i=1\\i\neq j}}^{m}f_{2i}(y)\right)$; and $\left(\frac{f_{1i}(y)}{f_{2i}(y)}\right)^{p_i}$, for $i\in\{1,...,m\}-\{j\}$ are all Lebesgue integrable.

We mention

Definition 17 *([1], p. 448) The left generalized Riemann-Liouville fractional derivative of f of order $\beta>0$ is given by*

$$D_a^{\beta}f(x)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n\int_a^x(x-y)^{n-\beta-1}f(y)\,dy, \tag{68}$$

where $n=[\beta]+1$, $x\in[a,b]$.

For $a,b\in\mathbb{R}$, we say that $f\in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\beta}f$ ($\beta>0$) in $[a,b]$, if and only if

(1) $D_a^{\beta-k}f\in C([a,b])$, $k=2,...,n=[\beta]+1$,

(2) $D_a^{\beta-1}f\in AC([a,b])$

(3) $D_a^{\beta}f\in L_\infty(a,b)$.

Above we define $D_a^0f:=f$ and $D_a^{-\delta}f:=I_{a+}^{\delta}f$, if $0<\delta\leq1$.

From [1, p.449] and [11] we mention and use

Lemma 18 *Let $\beta>\alpha\geq0$ and let $f\in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta}f$ in $[a,b]$ and let $D_a^{\beta-k}f(a)=0$, $k=1,...,[\beta]+1$, then*

$$D_a^{\alpha}f(x)=\frac{1}{\Gamma(\beta-\alpha)}\int_a^x(x-y)^{\beta-\alpha-1}D_a^{\beta}f(y)\,dy, \tag{69}$$

for all $a\leq x\leq b$.

Here $D_a^{\alpha}f\in AC([a,b])$ for $\beta-\alpha\geq1$, and $D_a^{\alpha}f\in C([a,b])$ for $\beta-\alpha\in(0,1)$.

Notice here that

$$D_a^{\alpha}f(x)=\left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta}f\right)\right)(x),\quad a\leq x\leq b. \tag{70}$$

For more on the last, see [5].

Let $f_{1i}, f_{2i} \in L_1(a,b)$; $\alpha_i, \beta_i : \beta_i > \alpha_i \geq 0$, $i = 1, ..., m$. Here $(j = 1, 2)$ $(f_{ji}, \alpha_i, \beta_i)$ fulfill terminology and assumptions of Definition 17 and Lemma 18. Indeed we have

$$D_a^{\alpha_i} f_{1i}(x) = \left(I_{a+}^{\beta_i - \alpha_i}\left(D_a^{\beta_i} f_{1i}\right)\right)(x), \tag{71}$$

and

$$D_a^{\alpha_i} f_{2i}(x) = \left(I_{a+}^{\beta_i - \alpha_i}\left(D_a^{\beta_i} f_{2i}\right)\right)(x), \tag{72}$$

$a \leq x \leq b$, $i = 1, ..., m$.

Assume $0 < D_a^{\beta_i} f_{2i}(x) < \infty$, a.e., $i = 1, ..., m$.

Let $j \in \{1, ..., m\}$ be fixed. Assume that the function

$$x \to \frac{\left(\prod_{i=1}^{m} f_{2i}(y)\right) u(x) \chi_{(a,x]}(y) (x-y)^{\left(\left(\sum_{i=1}^{m}(\beta_i - \alpha_i)\right) - m\right)}}{\left(\prod_{i=1}^{m} \left(D_a^{\alpha_i} f_{2i}\right)(x)\right)\left(\prod_{i=1}^{m} \Gamma(\beta_i - \alpha_i)\right)} \tag{73}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$\theta_m^*(y) := \left(\prod_{i=1}^{m}\left(\frac{f_{2i}(y)}{\Gamma(\beta_i - \alpha_i)}\right)\right).$$

$$\int_y^b u(x)\left(\frac{(x-y)^{\left(\sum_{i=1}^{m}(\beta_i - \alpha_i)\right) - m}}{\prod_{i=1}^{m}\left(D_a^{\alpha_i} f_{2i}\right)(x)}\right) dx < \infty, \tag{74}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Proposition 19 *Here all as above. It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{D_a^{\alpha_i} f_{1i}(x)}{D_a^{\alpha_i} f_{2i}(x)}\right|\right) dx \leq \tag{75}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m} \int_a^b \Phi_i\left(\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|\right) dy\right)\left(\int_a^b \Phi_j\left(\left|\frac{D_a^{\beta_j} f_{1j}(y)}{D_a^{\beta_j} f_{2j}(y)}\right|\right) \theta_m^*(y)\, dy\right),$$

under the properties: $(i = 1, ..., m)$

(i) $\Phi_i\left(\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|\right)$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{(\beta_i - \alpha_i - 1)}}{\Gamma(\beta_i - \alpha_i)}\left(D_a^{\beta_i} f_{2i}(y)\right) dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\theta_m^*(y)\,\Phi_j\left(\left|\frac{D_a^{\beta_j} f_{1j}(y)}{D_a^{\beta_j} f_{2j}(y)}\right|\right)$; *and* $\Phi_i\left(\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|\right)$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 9. ■

Corollary 20 *It holds*

$$\int_a^b u(x)\, e^{\left(\sum_{i=1}^m \left|\frac{D_a^{\alpha_i} f_{1i}(x)}{D_a^{\alpha_i} f_{2i}(x)}\right|\right)} dx \leq \tag{76}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b e^{\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|} dy\right)\left(\int_a^b e^{\left|\frac{D_a^{\beta_j} f_{1j}(y)}{D_a^{\beta_j} f_{2j}(y)}\right|}\theta_m^*(y)\,dy\right),$$

under the properties: $(i = 1, ..., m)$

(i) $e^{\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|}$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{(\beta_i-\alpha_i-1)}}{\Gamma(\beta_i-\alpha_i)}\left(D_a^{\beta_i} f_{2i}(y)\right) dy$ *-integrable, a.e. in* $x \in (a, b)$,

(ii) $\theta_m^*(y)\, e^{\left|\frac{D_a^{\beta_j} f_{1j}(y)}{D_a^{\beta_j} f_{2j}(y)}\right|}$; *and* $e^{\left|\frac{D_a^{\beta_i} f_{1i}(y)}{D_a^{\beta_i} f_{2i}(y)}\right|}$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

We need

Definition 21 *([8], p. 50, [1], p. 449) Let* $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a, b])$. *Then the left Caputo fractional derivative is given by*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt$$
$$= \left(I_{a+}^{n-\nu} f^{(n)}\right)(x), \tag{77}$$

and it exists almost everywhere for $x \in [a, b]$, *in fact* $D_{*a}^{\nu} f \in L_1(a, b)$, *([1], p. 394).*

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.

We also need

Theorem 22 *([3] and [6]) Let* $\nu > \rho > 0$, $\nu, \rho \notin \mathbb{N}$. *Call* $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. *Assume* $f \in AC^n([a, b])$, *such that* $f^{(k)}(a) = 0$, $k = m^*, m^*+1, ..., n-1$, *and* $D_{*a}^{\nu} f \in L_\infty(a, b)$. *Then* $D_{*a}^{\rho} f \in C([a, b])$ *if* $\nu - \rho \in (0, 1)$, *and* $D_{*a}^{\rho} f \in AC([a, b])$, *if* $\nu - \rho \geq 1$ *(where* $D_{*a}^{\rho} f = I_{a+}^{m^*-\rho} f^{(m^*)}(x)$*), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\,dt$$
$$= \left(I_{a+}^{\nu-\rho}\left(D_{*a}^{\nu} f\right)\right)(x), \tag{78}$$

$\forall\, x \in [a, b]$.

For more on the last, see [6].

Let $\nu_i > \rho_i > 0$, $\nu_i, \rho_i \notin \mathbb{N}$, $n_i := \lceil \nu_i \rceil$, $m_i^* := \lceil \rho_i \rceil$, $i = 1, ..., m$. Assume $f_{1i}, f_{2i} \in AC^{n_i}([a,b])$, such that $(j = 1,2)$ $f_{ji}^{(k_i)}(a) = 0$, $k_i = m_i^*, m_i^*+1, ..., n_i - 1$, and $D_{*a}^{\nu_i} f_{ji} \in L_\infty(a,b)$. Based on Definition 21 and Theorem 22 we get that

$$D_{*a}^{\rho_i} f_{ji}(x) = \left(I_{a+}^{\nu_i - \rho_i} \left(D_{*a}^{\nu_i} f_{ji} \right) \right)(x) \in \mathbb{R}, \tag{79}$$

$\forall\, x \in [a,b]$; $j = 1,2$; $i = 1, ..., m$.

Assume $0 < D_{*a}^{\nu_i} f_{2i}(x) < \infty$, a.e., $i = 1, ..., m$.

Let $j \in \{1, ..., m\}$ be fixed. Assume that the function

$$x \to \frac{\left(\prod_{i=1}^{m} D_{*a}^{\nu_i} f_{2i}(y) \right) u(x)\, \chi_{(a,x]}(y)\, (x-y)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) \right) - m \right)}}{\left(\prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_{2i} \right)(x) \right) \left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) \right)} \tag{80}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$\psi_m^*(y) := \left(\prod_{i=1}^{m} \left(\frac{D_{*a}^{\nu_i} f_{2i}(y)}{\Gamma(\nu_i - \rho_i)} \right) \right) \int_y^b u(x) \left(\frac{(x-y)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) \right) - m \right)}}{\prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_{2i} \right)(x)} \right) dx < \infty, \tag{81}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Proposition 23 *It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{D_{*a}^{\rho_i} f_{1i}(x)}{D_{*a}^{\rho_i} f_{2i}(x)} \right| \right) dx \leq \tag{82}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \Phi_i \left(\left| \frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)} \right| \right) dy \right) \left(\int_a^b \Phi_j \left(\left| \frac{D_{*a}^{\nu_j} f_{1j}(y)}{D_{*a}^{\nu_j} f_{2j}(y)} \right| \right) \psi_m^*(y)\, dy \right),$$

under the properties: $(i = 1, ..., m)$

(i) $\Phi_i \left(\left| \frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)} \right| \right)$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{(\nu_i - \rho_i - 1)}}{\Gamma(\nu_i - \rho_i)} \left(D_{*a}^{\nu_i} f_{2i}(y) \right) dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\psi_m^*(y)\, \Phi_j \left(\left| \frac{D_{*a}^{\nu_j} f_{1j}(y)}{D_{*a}^{\nu_j} f_{2j}(y)} \right| \right)$; *and* $\Phi_i \left(\left| \frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)} \right| \right)$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 9. ■

Corollary 24 *It holds*

$$\int_a^b u(x) e^{\left(\sum_{i=1}^m \left|\frac{D_{*a}^{\rho_i} f_{1i}(x)}{D_{*a}^{\rho_i} f_{2i}(x)}\right|\right)} dx \leq \tag{83}$$

$$\left(\prod_{\substack{i=1 \\ i\neq j}}^m \int_a^b e^{\left|\frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)}\right|} dy\right) \left(\int_a^b e^{\left|\frac{D_{*a}^{\nu_j} f_{1j}(y)}{D_{*a}^{\nu_j} f_{2j}(y)}\right|} \psi_m^*(y)\, dy\right),$$

under the properties: $(i = 1, ..., m)$

(i) $e^{\left|\frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)}\right|}$ is $\frac{\chi_{(a,x]}(y)(x-y)^{(\nu_i-\rho_i-1)}}{\Gamma(\nu_i-\rho_i)} \left(D_{*a}^{\nu_i} f_{2i}(y)\right) dy$ -integrable, a.e. in $x \in (a, b)$,

(ii) $\psi_m^*(y) e^{\left|\frac{D_{*a}^{\nu_j} f_{1j}(y)}{D_{*a}^{\nu_j} f_{2j}(y)}\right|}$; and $e^{\left|\frac{D_{*a}^{\nu_i} f_{1i}(y)}{D_{*a}^{\nu_i} f_{2i}(y)}\right|}$, for $i \in \{1, ..., m\} - \{j\}$ are all Lebesgue integrable.

We need

Definition 25 *([2], [9], [10]) Let $\alpha \geq 0$, $n := \lceil \alpha \rceil$, $f \in AC^n([a, b])$. We define the right Caputo fractional derivative of order $\alpha \geq 0$, by*

$$\overline{D}_{b-}^{\alpha} f(x) := (-1)^n I_{b-}^{n-\alpha} f^{(n)}(x), \tag{84}$$

we set $\overline{D}_{-}^{0} f := f$, i.e.

$$\overline{D}_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (J - x)^{n-\alpha-1} f^{(n)}(J)\, dJ. \tag{85}$$

Notice that $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$, $n \in \mathbb{N}$.

We need

Theorem 26 *([3]) Let $f \in AC^n([a, b])$, $n \in \mathbb{N}$, $n := \lceil \alpha \rceil$, $\alpha > \rho > 0$, $r = \lceil \rho \rceil$, $\alpha, \rho \notin \mathbb{N}$. Assume $f^{(k)}(b) = 0$, $k = r, r+1, ..., n-1$, and $\overline{D}_{b-}^{\alpha} f \in L_\infty([a, b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho}\left(\overline{D}_{b-}^{\alpha} f\right)\right)(x) \in C([a, b]), \tag{86}$$

if $\alpha - \rho \in (0, 1)$, and $\overline{D}_{b-}^{\rho} f \in AC([a, b])$, if $\alpha - \rho \geq 1$, that is

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t - x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f\right)(t)\, dt, \tag{87}$$

$\forall\ x \in [a, b]$.

Here $i = 1, ..., m$. Let $\alpha_i > \rho_i > 0$, $\alpha_i, \rho_i \notin \mathbb{N}$, $n_i = \lceil \alpha_i \rceil$, $r_i = \lceil \rho_i \rceil$. Take $f_{1i}, f_{2i} \in AC^{n_i}([a,b])$, such that $(j = 1,2)$ $f_{ji}^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1, ..., n_i - 1$. Furthermore assume that $\overline{D}_{b-}^{\alpha_i} f_{1i}, \overline{D}_{b-}^{\alpha_i} f_{2i} \in L_\infty(a,b)$. Then by Theorem 26 we get that $(j = 1,2)$:

$$\overline{D}_{b-}^{\rho_i} f_{ji}(x) = \left(I_{b-}^{\alpha_i - \rho_i} \left(\overline{D}_{b-}^{\alpha_i} f_{ji} \right) \right)(x) \in C([a,b]), \quad \forall\, x \in [a,b]. \tag{88}$$

Assume $0 < \overline{D}_{b-}^{\alpha_i} f_{2i}(x) < \infty$, a.e., $i = 1, ..., m$.

Let $j \in \{1, ..., m\}$ be fixed. Assume that the function

$$x \to \frac{\left(\prod_{i=1}^m \overline{D}_{b-}^{\alpha_i} f_{2i}(y) \right) u(x)\, \chi_{[x,b)}(y)\, (y-x)^{\left(\left(\sum_{i=1}^m (\alpha_i - \rho_i) \right) - m \right)}}{\left(\prod_{i=1}^m \left(\overline{D}_{b-}^{\rho_i} f_{2i} \right)(x) \right) \left(\prod_{i=1}^m \Gamma(\alpha_i - \rho_i) \right)} \tag{89}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$T_m(y) := \left(\prod_{i=1}^m \left(\frac{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)}{\Gamma(\alpha_i - \rho_i)} \right) \right) \int_a^y u(x) \left(\frac{(y-x)^{\left(\left(\sum_{i=1}^m (\alpha_i - \rho_i) \right) - m \right)}}{\prod_{i=1}^m \left(\overline{D}_{b-}^{\rho_i} f_{2i} \right)(x)} \right) dx < \infty, \tag{90}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

We get

Proposition 27 *Here all as above. It holds*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\overline{D}_{b-}^{\rho_i} f_{1i}(x)}{\overline{D}_{b-}^{\rho_i} f_{2i}(x)} \right| \right) dx \leq \tag{91}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i \left(\left| \frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)} \right| \right) dy \right) \left(\int_a^b \Phi_j \left(\left| \frac{\overline{D}_{b-}^{\alpha_j} f_{1j}(y)}{\overline{D}_{b-}^{\alpha_j} f_{2j}(y)} \right| \right) T_m(y)\, dy \right),$$

under the properties: $(i = 1, ..., m)$

(i) $\Phi_i \left(\left| \frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)} \right| \right)$ *is* $\frac{\chi_{[x,b)}(y)(y-x)^{(\alpha_i - \rho_i - 1)}}{\Gamma(\alpha_i - \rho_i)} \left(\overline{D}_{b-}^{\alpha_i} f_{2i}(y) \right) dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $T_m(y)\, \Phi_j \left(\left| \frac{\overline{D}_{b-}^{\alpha_j} f_{1j}(y)}{\overline{D}_{b-}^{\alpha_j} f_{2j}(y)} \right| \right)$; *and* $\Phi_i \left(\left| \frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)} \right| \right)$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 13. ■

Corollary 28 *It holds*

$$\int_a^b u(x) e^{\left(\sum_{i=1}^m \left|\frac{\overline{D}_{b-}^{\rho_i} f_{1i}(x)}{\overline{D}_{b-}^{\rho_i} f_{2i}(x)}\right|\right)} dx \leq \tag{92}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b e^{\left|\frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)}\right|} dy\right) \left(\int_a^b e^{\left|\frac{\overline{D}_{b-}^{\alpha_j} f_{1j}(y)}{\overline{D}_{b-}^{\alpha_j} f_{2j}(y)}\right|} T_m(y) dy\right),$$

under the properties: $(i = 1, ..., m)$

(i) $e^{\left|\frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)}\right|}$ *is* $\frac{\chi_{[x,b)}(y)(y-x)^{(\alpha_i-\rho_i-1)}}{\Gamma(\alpha_i-\rho_i)} \left(\overline{D}_{b-}^{\alpha_i} f_{2i}(y)\right) dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $T_m(y) e^{\left|\frac{\overline{D}_{b-}^{\alpha_j} f_{1j}(y)}{\overline{D}_{b-}^{\alpha_j} f_{2j}(y)}\right|}$*; and* $e^{\left|\frac{\overline{D}_{b-}^{\alpha_i} f_{1i}(y)}{\overline{D}_{b-}^{\alpha_i} f_{2i}(y)}\right|}$*, for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 27. ■

We give

Definition 29 *Let* $\nu > 0$*,* $n := [\nu]$*,* $\alpha := \nu - n$ $(0 \leq \alpha < 1)$*. Let* $a, b \in \mathbb{R}$*,* $a \leq x \leq b$*,* $f \in C([a,b])$*. We consider* $C_a^\nu([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$*. For* $f \in C_a^\nu([a,b])$*, we define the left generalized* ν*-fractional derivative of* f *over* $[a,b]$ *as*

$$\Delta_a^\nu f := \left(I_{a+}^{1-\alpha} f^{(n)}\right)', \tag{93}$$

see [1], p. 24, and Canavati derivative in [7].

Notice here $\Delta_a^\nu f \in C([a,b])$.

So that

$$(\Delta_a^\nu f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t) dt, \tag{94}$$

$\forall\ x \in [a,b]$.

Notice here that

$$\Delta_a^n f = f^{(n)}, \quad n \in \mathbb{Z}_+. \tag{95}$$

We need

Theorem 30 *([3]) Let* $f \in C_a^\nu([a,b])$*,* $n = [\nu]$*, such that* $f^{(i)}(a) = 0$*,* $i = r, r+1, ..., n-1$*, where* $r := [\rho]$*, with* $0 < \rho < \nu$*. Then*

$$(\Delta_a^\rho f)(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} (\Delta_a^\nu f)(t) dt, \tag{96}$$

i.e.

$$\left(\Delta_a^{\rho} f\right) = I_{a+}^{\nu-\rho}\left(\Delta_a^{\nu} f\right) \in C\left([a,b]\right). \tag{97}$$

Thus $f \in C_a^{\rho}([a,b])$.

Let $\nu_i > \rho_i > 0$, $n_i := [\nu_i]$, $r_i := [\rho_i]$, $i = 1,...,m$. Let $f_{1i}, f_{2i} \in C_a^{\nu_i}([a,b])$, such that $(j = 1,2)$ $f_{ji}^{(k_i)}(a) = 0$, $k_i = r_i, r_i + 1, ..., n_i - 1$. Notice here $\Delta_a^{\nu_i} f_{ji} \in C([a,b])$, and $\Delta_a^{\rho_i} f_{ji} \in C([a,b])$. Based on Definition 29 and Theorem 30 we get

$$\Delta_a^{\rho_i} f_{ji}(x) = \left(I_{a+}^{\nu_i-\rho_i}\left(\Delta_a^{\nu_i} f_{ji}\right)\right)(x), \tag{98}$$

$\forall\, x \in [a,b]$; $j = 1,2$; $i = 1,...,m$.

Assume $\Delta_a^{\nu_i} f_{2i}(x) > 0$, $\forall\, x \in [a,b]$.

Let $j \in \{1,...,m\}$ be fixed. Assume that the function

$$x \to \frac{\left(\prod_{i=1}^{m} \Delta_a^{\nu_i} f_{2i}(y)\right) u(x)\, \chi_{(a,x]}(y)\,(x-y)^{\left(\left(\sum_{i=1}^{m}(\nu_i-\rho_i)\right)-m\right)}}{\left(\prod_{i=1}^{m}\left(\Delta_a^{\rho_i} f_{2i}\right)(x)\right)\left(\prod_{i=1}^{m}\Gamma\left(\nu_i-\rho_i\right)\right)} \tag{99}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$W_m(y) := \left(\prod_{i=1}^{m}\left(\frac{\Delta_a^{\nu_i} f_{2i}(y)}{\Gamma(\nu_i-\rho_i)}\right)\right)\int_y^b u(x)\left(\frac{(x-y)^{\left(\left(\sum_{i=1}^{m}(\nu_i-\rho_i)\right)-m\right)}}{\prod_{i=1}^{m}\left(\Delta_a^{\rho_i} f_{2i}\right)(x)}\right)dx < \infty, \tag{100}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m$, are convex and increasing functions.

Proposition 31 *It holds*

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\Delta_a^{\rho_i} f_{1i}(x)}{\Delta_a^{\rho_i} f_{2i}(x)}\right|\right)dx \leq \tag{101}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_a^b \Phi_i\left(\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|\right)dy\right)\left(\int_a^b \Phi_j\left(\left|\frac{\Delta_a^{\nu_j} f_{1j}(y)}{\Delta_a^{\nu_j} f_{2j}(y)}\right|\right)W_m(y)\,dy\right),$$

under the properties: $(i = 1,...,m)$

(i) $\Phi_i\left(\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|\right)$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{(\nu_i-\rho_i-1)}}{\Gamma(\nu_i-\rho_i)}\left(\Delta_a^{\nu_i} f_{2i}(y)\right)dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $W_m(y)\,\Phi_j\left(\left|\frac{\Delta_a^{\nu_j} f_{1j}(y)}{\Delta_a^{\nu_j} f_{2j}(y)}\right|\right)$; *and* $\Phi_i\left(\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|\right)$, *for* $i \in \{1,...,m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 9. ■

Corollary 32 *Let τ be a fixed prime number. It holds*

$$\int_a^b u(x)\,\tau^{\left(\sum_{i=1}^m \left|\frac{\Delta_a^{\rho_i} f_{1i}(x)}{\Delta_a^{\rho_i} f_{2i}(x)}\right|\right)} dx \leq \tag{102}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b \tau^{\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|} dy\right)\left(\int_a^b \tau^{\left|\frac{\Delta_a^{\nu_j} f_{1j}(y)}{\Delta_a^{\nu_j} f_{2j}(y)}\right|} W_m(y)\,dy\right),$$

under the properties: $(i = 1,...,m)$

(i) $\tau^{\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|}$ *is* $\frac{\chi_{(a,x]}(y)(x-y)^{(\nu_i-\rho_i-1)}}{\Gamma(\nu_i-\rho_i)}\left(\Delta_a^{\nu_i} f_{2i}(y)\right) dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $W_m(y)\,\tau^{\left|\frac{\Delta_a^{\nu_j} f_{1j}(y)}{\Delta_a^{\nu_j} f_{2j}(y)}\right|}$; *and* $\tau^{\left|\frac{\Delta_a^{\nu_i} f_{1i}(y)}{\Delta_a^{\nu_i} f_{2i}(y)}\right|}$, *for* $i \in \{1,...,m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 31. ■

We need

Definition 33 *([2]) Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C([a,b])$. Consider*

$$C_{b-}^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b])\}. \tag{103}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1}\left(I_{b-}^{1-\alpha} f^{(n)}\right)'. \tag{104}$$

We set $\Delta_{b-}^0 f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (J-x)^{-\alpha} f^{(n)}(J)\,dJ, \tag{105}$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 34 *([3]) Let $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. Assume $f^{(i)}(b) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, $n := [\nu]$. Then*

$$\Delta_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_x^b (J-x)^{\nu-\rho-1} \left(\Delta_{b-}^{\nu} f\right)(J)\, dJ, \tag{106}$$

$\forall\ x \in [a,b]$, i.e.

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu-\rho} \left(\Delta_{b-}^{\nu} f\right) \in C([a,b]), \tag{107}$$

and $f \in C_{b-}^{\rho}([a,b])$.

Let $\nu_i > \rho_i > 0$, $n_i := [\nu_i]$, $r_i := [\rho_i]$, $i = 1, ..., m$. Let $f_{1i}, f_{2i} \in C_{b-}^{\nu_i}([a,b])$, such that $(j = 1, 2)$ $f_{ji}^{(k_i)}(b) = 0$, $k_i = r_i, r_i + 1, ..., n_i - 1$. Notice here $\Delta_{b-}^{\nu_i} f_{ji} \in C([a,b])$, and $\Delta_{b-}^{\rho_i} f_{ji} \in C([a,b])$. Based on Definition 33 and Theorem 34 we get

$$\Delta_{b-}^{\rho_i} f_{ji}(x) = \left(I_{b-}^{\nu_i-\rho_i}\left(\Delta_{b-}^{\nu_i} f_{ji}\right)\right)(x), \tag{108}$$

$\forall\ x \in [a,b]$; $j = 1, 2$; $i = 1, ..., m$.

Assume $\Delta_{b-}^{\nu_i} f_{2i}(x) > 0$, $\forall\ x \in [a,b]$.

Let $j \in \{1, ..., m\}$ be fixed. Assume that the function

$$x \to \frac{\left(\prod_{i=1}^{m} \Delta_{b-}^{\nu_i} f_{2i}(y)\right) u(x)\, \chi_{[x,b)}(y)\, (y-x)^{\left(\left(\sum_{i=1}^{m}(\nu_i-\rho_i)\right)-m\right)}}{\left(\prod_{i=1}^{m}\left(\Delta_{b-}^{\rho_i} f_{2i}\right)(x)\right)\left(\prod_{i=1}^{m}\Gamma(\nu_i-\rho_i)\right)} \tag{109}$$

is integrable on (a,b), for each $y \in (a,b)$.

Here we have

$$W_m^*(y) := \left(\prod_{i=1}^{m}\left(\frac{\Delta_{b-}^{\nu_i} f_{2i}(y)}{\Gamma(\nu_i-\rho_i)}\right)\right) \int_a^y u(x) \left(\frac{(y-x)^{\left(\left(\sum_{i=1}^{m}(\nu_i-\rho_i)\right)-m\right)}}{\prod_{i=1}^{m}\left(\Delta_{b-}^{\rho_i} f_{2i}\right)(x)}\right) dx < \infty, \tag{110}$$

for any $y \in (a,b)$.

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Proposition 35 *It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{\Delta_{b-}^{\rho_i} f_{1i}(x)}{\Delta_{b-}^{\rho_i} f_{2i}(x)}\right|\right) dx \leq \tag{111}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_{a}^{b}\Phi_{i}\left(\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|\right)dy\right)\left(\int_{a}^{b}\Phi_{j}\left(\left|\frac{\Delta_{b-}^{\nu_{j}}f_{1j}(y)}{\Delta_{b-}^{\nu_{j}}f_{2j}(y)}\right|\right)W_{m}^{*}(y)\,dy\right),$$

under the properties: ($i=1,...,m$)

(i) $\Phi_{i}\left(\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|\right)$ *is* $\frac{\chi_{[x,b)}(y)(y-x)^{(\nu_{i}-\rho_{i}-1)}}{\Gamma(\nu_{i}-\rho_{i})}\left(\Delta_{b-}^{\nu_{i}}f_{2i}(y)\right)dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $W_{m}^{*}(y)\,\Phi_{j}\left(\left|\frac{\Delta_{b-}^{\nu_{j}}f_{1j}(y)}{\Delta_{b-}^{\nu_{j}}f_{2j}(y)}\right|\right)$; *and* $\Phi_{i}\left(\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|\right)$, *for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 13. ■

Corollary 36 *Let τ be a fixed prime number. It holds*

$$\int_{a}^{b}u(x)\,\tau^{\left(\sum_{i=1}^{m}\left|\frac{\Delta_{b-}^{\rho_{i}}f_{1i}(x)}{\Delta_{b-}^{\rho_{i}}f_{2i}(x)}\right|\right)}dx\leq \tag{112}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_{a}^{b}\tau^{\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|}dy\right)\left(\int_{a}^{b}\tau^{\left|\frac{\Delta_{b-}^{\nu_{j}}f_{1j}(y)}{\Delta_{b-}^{\nu_{j}}f_{2j}(y)}\right|}W_{m}^{*}(y)\,dy\right),$$

under the properties: ($i=1,...,m$)

(i) $\tau^{\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|}$ *is* $\frac{\chi_{[x,b)}(y)(y-x)^{(\nu_{i}-\rho_{i}-1)}}{\Gamma(\nu_{i}-\rho_{i})}\left(\Delta_{b-}^{\nu_{i}}f_{2i}(y)\right)dy$ *-integrable, a.e. in* $x\in(a,b)$,

(ii) $W_{m}^{*}(y)\,\tau^{\left|\frac{\Delta_{b-}^{\nu_{j}}f_{1j}(y)}{\Delta_{b-}^{\nu_{j}}f_{2j}(y)}\right|}$; *and* $\tau^{\left|\frac{\Delta_{b-}^{\nu_{i}}f_{1i}(y)}{\Delta_{b-}^{\nu_{i}}f_{2i}(y)}\right|}$, *for* $i\in\{1,...,m\}-\{j\}$ *are all Lebesgue integrable.*

Proof. By Proposition 35. ■

We need

Definition 37 *([14], p. 99) The fractional integrals of a function f with respect to given function g are defined as follows:*

Let $a,b\in\mathbb{R}$, $a<b$, $\alpha>0$. Here g is an increasing function on $[a,b]$, and $g\in C^{1}([a,b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\,f(t)\,dt}{(g(x)-g(t))^{1-\alpha}},\quad x>a, \tag{113}$$

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\,f(t)\,dt}{(g(t)-g(x))^{1-\alpha}},\quad x<b, \tag{114}$$

respectively.

We make

Remark 38 *Let f_{1i}, f_{2i} be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+;g}^{\alpha_i}(|f_{ji}|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1,...,m$, $j = 1,2$.*

Consider

$$g_{ji}(x) = \left(I_{a+;g}^{\alpha_i}(f_{ji})\right)(x), \tag{115}$$

$x \in (a,b)$, $i = 1,...,m$, $j = 1,2$.

Assume $0 < f_{2i}(y) < \infty$, a.e., $i = 1,...,m$.

Notice that $g_{ji}(x) \in \mathbb{R}$ and it is Lebesgue measurable. We pick again $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

Here we have

$$k_i(x,y) = \frac{\chi_{(a,x]}(y)\, g'(y)}{\Gamma(\alpha_i)\,(g(x) - g(y))^{1-\alpha_i}}, \quad i = 1,...,m. \tag{116}$$

Let $j \in \{1,...,m\}$ be fixed.

Assume that the function

$$x \to \left(\frac{u(x)\left(\prod_{i=1}^m f_{2i}(y)\right)\chi_{(a,x]}(y)\,(g'(y))^m\,(g(x)-g(y))^{\left(\sum_{i=1}^m \alpha_i\right)-m}}{\left(\prod_{i=1}^m \Gamma(\alpha_i)\right)\left(\prod_{i=1}^m \left(I_{a+;g}^{\alpha_i} f_{2i}\right)(x)\right)}\right) \tag{117}$$

is integrable on (a,b), for each $y \in (a,b)$.

Define ρ_m^g on (a,b) by

$$\rho_m^g(y) = \frac{\left(\prod_{i=1}^m f_{2i}(y)\right)(g'(y))^m}{\left(\prod_{i=1}^m \Gamma(\alpha_i)\right)} \int_y^b \frac{u(x)\,(g(x)-g(y))^{\left(\sum_{i=1}^m \alpha_i\right)-m}}{\prod_{i=1}^m \left(I_{a+;g}^{\alpha_i} f_{2i}\right)(x)}\,dx < \infty, \tag{118}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m$, are convex and increasing functions.

Theorem 39 *Here all are as in Remark 38. It holds*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I_{a+;g}^{\alpha_i}(f_{1i})\right)(x)}{\left(I_{a+;g}^{\alpha_i}(f_{2i})\right)(x)}\right|\right) dx \leq \tag{119}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right) dy\right)\left(\int_a^b \Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)\rho_m^g(y)\,dy\right),$$

true for all measurable functions $f_{1i}, f_{2i} : (a,b) \to \mathbb{R}$ *such that* $\left(I_{a+;g}^{\alpha_i}(|f_{ji}|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1, ..., m$; $j = 1, 2$, *with:*

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$ *are both* $\frac{\chi_{(a,x]}(y)g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}} f_{2i}(y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\rho_m^g(y)\,\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)$; *and* $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all integrable.*

Proof. By Theorem 3. ■

We make

Remark 40 *Let* f_{1i}, f_{2i} *be Lebesgue measurable functions from* (a,b) *into* $\mathbb{R}$, *such that* $\left(I_{b-;g}^{\alpha_i}(|f_{ji}|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1, ..., m$, $j = 1, 2$.

Consider now

$$g_{ji}(x) = \left(I_{b-;g}^{\alpha_i}(f_{ji})\right)(x), \tag{120}$$

$x \in (a,b)$, $i = 1, ..., m$, $j = 1, 2$.

Assume $0 < f_{2i}(y) < \infty$, *a.e.*, $i = 1, ..., m$.

Notice that $g_{ji}(x) \in \mathbb{R}$ *and it is Lebesgue measurable.*

Here we have

$$k_i(x,y) = \frac{\chi_{[x,b)}(y)\,g'(y)}{\Gamma(\alpha_i)\,(g(y)-g(x))^{1-\alpha_i}}, \quad i = 1, ..., m. \tag{121}$$

Let $j \in \{1, ..., m\}$ *be fixed.*

Assume that the function

$$x \to \left(\frac{u(x)\left(\prod_{i=1}^{m} f_{2i}(y)\right)\chi_{[x,b)}(y)\,(g'(y))^m\,(g(y)-g(x))^{\left(\sum_{i=1}^{m}\alpha_i\right)-m}}{\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\right)\left(\prod_{i=1}^{m}\left(I_{b-;g}^{\alpha_i}(f_{2i})\right)(x)\right)}\right) \tag{122}$$

is integrable on (a,b), *for each* $y \in (a,b)$.

Define $\overline{\rho}_m^g$ *on* (a,b) *by*

$$\overline{\rho}_m^g(y) = \frac{\left(\prod_{i=1}^{m} f_{2i}(y)\right)(g'(y))^m}{\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\right)}\int_a^y \frac{u(x)\,(g(y)-g(x))^{\left(\sum_{i=1}^{m}\alpha_i\right)-m}}{\prod_{i=1}^{m}\left(I_{b-;g}^{\alpha_i}(f_{2i})\right)(x)}dx < \infty, \tag{123}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *are convex and increasing functions.*

Theorem 41 *Here all are as in Remark 40. It holds*

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{b-;g}^{\alpha_i}(f_{1i})\right)(x)}{\left(I_{b-;g}^{\alpha_i}(f_{2i})\right)(x)} \right| \right) dx \leq \tag{124}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right) dy \right) \left(\int_a^b \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right) \overline{\rho}_m^g(y)\, dy \right),$$

true for all measurable functions $f_{1i}, f_{2i} : (a,b) \to \mathbb{R}$ *such that* $\left(I_{b-;g}^{\alpha_i}(|f_{ji}|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$, $\alpha_i > 0$, $i = 1, ..., m$; $j = 1, 2$, *under the properties:*

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$ *are both* $\frac{\chi_{[x,b)}(y) g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}} f_{2i}(y)\, dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\overline{\rho}_m^g(y)\, \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right)$; *and* $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$, *for* $i \in \{1, ..., m\} - \{j\}$ *are all Lebesgue integrable.*

Proof. By Theorem 3. ■

We need

Definition 42 *([12]) Let* $0 < a < b < \infty$, $\alpha > 0$. *The left- and right-sided Hadamard fractional integrals of order* α *are given by*

$$\left(J_{a+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y} \right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x > a, \tag{125}$$

and

$$\left(J_{b-}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x} \right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x < b, \tag{126}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g(x) = \ln x$ on $[a,b]$, $0 < a < b < \infty$.

Above f is a Lebesgue measurable function from (a,b) into $\mathbb{R}$, such that $\left(J_{a+}^\alpha(|f|)\right)(x)$, $\left(J_{b-}^\alpha(|f|)\right)(x) \in \mathbb{R}$, $\forall\ x \in (a,b)$.

We make

Remark 43 *Let* $(f_{1i}, f_{2i}, \alpha_i)$, $i = 1, ..., m$, *and* $\left(J_{a+}^{\alpha_i} f_{ji}\right)$, $j = 1, 2$, *all as in Definition 42.*

Assume $0 < f_{2i}(y) < \infty$, *a.e.,* $i = 1, ..., m$.

Let $j \in \{1, ..., m\}$ *be fixed.*

Assume that the function

$$x \to \left(\frac{u(x) \left(\prod_{i=1}^{m} f_{2i}(y) \right) \chi_{(a,x]}(y) \ln \left(\frac{x}{y} \right)^{\left(\sum_{i=1}^{m} \alpha_i \right) - m}}{y^m \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right) \left(\prod_{i=1}^{m} \left(J_{a+}^{\alpha_i} f_{2i} \right)(x) \right)} \right) \tag{127}$$

is integrable on (a,b), for each $y \in (a,b)$.

Define γ_m^g on (a,b) by

$$\gamma_m^g(y) = \frac{\left(\prod_{i=1}^{m} f_{2i}(y) \right)}{y^m \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right)} \int_y^b \frac{u(x) \left(\ln \left(\frac{x}{y} \right) \right)^{\left(\sum_{i=1}^{m} \alpha_i \right) - m}}{\prod_{i=1}^{m} \left(J_{a+}^{\alpha_i} f_{2i} \right)(x)} dx < \infty, \tag{128}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Theorem 44 *Here all are as in Remark 43. It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{\left(J_{a+}^{\alpha_i} (f_{1i}) \right)(x)}{\left(J_{a+}^{\alpha_i} (f_{2i}) \right)(x)} \right| \right) dx \leq \tag{129}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right) dy \right) \left(\int_a^b \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right) \gamma_m^g(y) \, dy \right),$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$ are both $\frac{\chi_{(a,x]}(y)}{y \Gamma(\alpha_i) \left(\ln \left(\frac{x}{y} \right) \right)^{1-\alpha_i}} f_{2i}(y) \, dy$ -integrable, a.e. in $x \in (a,b)$,

(ii) $\gamma_m^g(y) \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right)$; and $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$, for $i \in \{1, ..., m\} - \{j\}$ are all integrable.

Proof. By Theorem 39. ■

We make

Remark 45 *Let $(f_{1i}, f_{2i}, \alpha_i)$, $i = 1, ..., m$, and $\left(J_{b-}^{\alpha_i} f_{ji} \right)$, $j = 1, 2$, all as in Definition 42.*

Assume $0 < f_{2i}(y) < \infty$, a.e., $i = 1, ..., m$.

Let $j \in \{1, ..., m\}$ be fixed.

Suppose that the function

$$x \to \left(\frac{u(x) \left(\prod_{i=1}^{m} f_{2i}(y) \right) \chi_{[x,b)}(y) \ln\left(\frac{y}{x}\right)^{\left(\sum_{i=1}^{m} \alpha_i \right) - m}}{y^m \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right) \left(\prod_{i=1}^{m} \left(J_{b-}^{\alpha_i}(f_{2i}) \right)(x) \right)} \right) \tag{130}$$

is integrable on (a,b), for each $y \in (a,b)$.

Define $\overline{\gamma}_m^g$ on (a,b) by

$$\overline{\gamma}_m^g(y) = \frac{\left(\prod_{i=1}^{m} f_{2i}(y) \right)}{y^m \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right)} \int_a^y \frac{u(x) \left(\ln\left(\frac{y}{x}\right) \right)^{\left(\sum_{i=1}^{m} \alpha_i \right) - m}}{\prod_{i=1}^{m} \left(J_{b-}^{\alpha_i}(f_{2i}) \right)(x)} dx < \infty, \tag{131}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, are convex and increasing functions.

Theorem 46 *Here all as in Remark 45. It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{\left(J_{b-}^{\alpha_i}(f_{1i}) \right)(x)}{\left(J_{b-}^{\alpha_i}(f_{2i}) \right)(x)} \right| \right) dx \leq \tag{132}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right) dy \right) \left(\int_a^b \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right) \overline{\gamma}_m^g(y)\, dy \right),$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$ are both $\frac{\chi_{[x,b)}(y)}{y\Gamma(\alpha_i)\left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} f_{2i}(y)\, dy$ -integrable, a.e. in $x \in (a,b)$,

(ii) $\overline{\gamma}_m^g(y)\, \Phi_j \left(\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right| \right)$; and $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)$, for $i \in \{1, ..., m\} - \{j\}$ are all integrable.

Proof. By Theorem 41. ■

Corollary 47 *(to Theorem 44) It holds*

$$\int_a^b u(x)\, e^{\sum_{i=1}^{m} \left| \frac{\left(J_{a+}^{\alpha_i}(f_{1i}) \right)(x)}{\left(J_{a+}^{\alpha_i}(f_{2i}) \right)(x)} \right|} dx \leq \tag{133}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b e^{\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|} dy \right) \left(\int_a^b e^{\left| \frac{f_{1j}(y)}{f_{2j}(y)} \right|} \gamma_m^g(y)\, dy \right),$$

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under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ *are both* $\frac{\chi_{(a,x]}(y)}{y\Gamma(\alpha_i)\left(\ln\left(\frac{x}{y}\right)\right)^{1-\alpha_i}} f_{2i}(y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\gamma_m^g(y)\,e^{\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$; *and* $e^{\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$, *for* $i \in \{1,...,m\} - \{j\}$ *are all integrable.*

Corollary 48 *(to Theorem 46) Let* $p_i \geq 1$. *It holds*

$$\int_a^b u(x)\left(\prod_{i=1}^m \left|\frac{\left(J_{b-}^{\alpha_i}(f_{1i})\right)(x)}{\left(J_{b-}^{\alpha_i}(f_{2i})\right)(x)}\right|^{p_i}\right)dx \leq \tag{134}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b \left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i} dy\right)\left(\int_a^b \left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{p_j} \overline{\gamma}_m^g(y)\,dy\right),$$

under the assumptions

(i) $\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i}$ *is* $\frac{\chi_{[x,b)}(y)}{y\Gamma(\alpha_i)\left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} f_{2i}(y)\,dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\overline{\gamma}_m^g(y)\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{p_j}$; *and* $\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{p_i}$, *for* $i \in \{1,...,m\} - \{j\}$ *are all integrable.*

3 Appendix

In this work we used a lot the following

Proposition 49 *Let* $f : [0,\infty) \to \mathbb{R}$ *be convex and increasing. Then* f *is continuous on* $[0,\infty)$.

Proof. Fact: f is continuous on $(0,\infty)$, it is known. We want to prove that f is continuous at $x = 0$. Let $\nu > 0$ be fixed. Consider the line (l) through $(0, f(0))$ and $(\nu, f(\nu))$. It has slope $\frac{f(\nu)-f(0)}{\nu} \geq 0$, and equation $y = l(x) = \left(\frac{f(\nu)-f(0)}{\nu}\right)x + f(0)$. We can always pick up $\nu : f(\nu) > f(0)$, otherwise if for all $\nu > 0$ it is $f(\nu) = f(0)$, we have the trivial case of continuity.

By convexity of f we have that for any $0 < x < \nu$, it is $f(x) \leq l(x)$, equivalently,

$$f(x) \leq \left(\frac{f(\nu)-f(0)}{\nu}\right)x + f(0),$$

equivalently,

$$0 \leq f(x) - f(0) \leq \left(\frac{f(\nu)-f(0)}{\nu}\right)x;$$

here $\left(\frac{f(\nu)-f(0)}{\nu}\right) > 0$.

Let $x \to 0$, then $f(x) - f(0) \to 0$. That is $\lim_{x\to 0} f(x) = f(0)$, proving continuity of f at $x = 0$. ■

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Separating Rational L_p Inequalities for Integral Operators

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Abstract

Here we present L_p, $p > 1$, integral inequalities for convex and increasing functions applied to products of ratios of functions and other important mixtures. As applications we derive a wide range of fractional inequalities of Hardy type. They involve the left and right Erdélyi-Kober fractional integrals and left and right mixed Riemann-Liouville fractional multiple integrals. Also we give inequalities for Riemann-Liouville, Caputo, Canavati radial fractional derivatives. Some inequalities are of exponential type.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.

Key words and phrases: integral operator, mixed fractional integral, radial fractional derivative, Erdélyi-Kober fractional integral, Hardy fractional inequality.

1 Introduction

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2 and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\, d\mu_2(y), \text{ for any } x \in \Omega_1, \tag{1}$$

$i = 1, ..., m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y)\, f_i(y)\, d\mu_2(y), \tag{2}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, ..., m$.

Here u stands for a weight function on Ω_1.

In [3] we proved the following general result.

Theorem 1 *Assume that the functions* $(1 = 1, 2, ..., m \in \mathbb{N})$ $x \mapsto \left(u(x) \frac{k_i(x,y)}{K_i(x)}\right)$ *are integrable on* Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{3}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$*. Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$*,* $i = 1, ..., m$*, be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq \tag{4}$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i(y) \Phi_i(|f_i(y)|)^{p_i} d\mu_2(y)\right)^{\frac{1}{p_i}},$$

for all measurable functions, $f_i : \Omega_2 \to \mathbb{R}$ $(i = 1, ..., m)$ *such that*

(i) $f_i, \Phi_i(|f_i|)^{p_i}$ *are both* $k_i(x,y) d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$*,* $i = 1, ..., m$*,*

(ii) $u_i \Phi_i(|f_i|)^{p_i}$ *is* μ_2 *-integrable,* $i = 1, ..., m$*,*

and for all corresponding functions g_i $(i = 1, ..., m)$ *given by (2).*

Here $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$. Let $\varphi : \mathbb{R}^{*2} \to \mathbb{R}^*$ be a Borel measurable function. Let $f_{1i}, f_{2i} : \Omega_2 \to \mathbb{R}$ be measurable functions, $i = 1, ..., m$.

The function $\varphi(f_{1i}(y), f_{2i}(y))$, $y \in \Omega_2$, $i = 1, ..., m$, is Σ_2-measurable. In this article we assume that $0 < \varphi(f_{1i}(y), f_{2i}(y)) < \infty$, a.e., $i = 1, ..., m$.

We consider

$$f_{3i}(y) := \frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))}, \tag{5}$$

$i = 1, ..., m$, $y \in \Omega_2$, which is a measurable function.

We also consider here

$$k_i^*(x,y) := k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y)), \tag{6}$$

$y \in \Omega_2$, $i = 1, ..., m$, which is a nonnegative a.e. measurable function on $\Omega_1 \times \Omega_2$.

We have that $k_i^*(x, \cdot)$ is measurable on Ω_2, $i = 1, ..., m$.

Denote by

$$K_i^*(x) := \int_{\Omega_2} k_i^*(x,y) d\mu_2(y) \tag{7}$$

$$= \int_{\Omega_2} k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y)) d\mu_2(y), \quad i = 1, ..., m.$$

We assume that $K_i^*(x) > 0$, a.e. on Ω_1.

So here the function

$$g_{1i}(x) = \int_{\Omega_2} k_i(x,y) f_{1i}(y) d\mu_2(y)$$

$$= \int_{\Omega_2} k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y)) \left(\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right) d\mu_2(y)$$

$$= \int_{\Omega_2} k_i^*(x,y) f_{3i}(y) d\mu_2(y), \quad i = 1, ..., m. \tag{8}$$

A typical example is when

$$\varphi(f_{1i}(y), f_{2i}(y)) = f_{2i}(y), \quad i = 1, ..., m, \ y \in \Omega_2. \tag{9}$$

In that case we have that

$$f_{3i}(y) = \frac{f_{1i}(y)}{f_{2i}(y)}, \ \ i = 1, ..., m, \ \ y \in \Omega_2. \tag{10}$$

The latter case was studied in [6], for $m = 1$, which is an article with interesting ideas however containing several mistakes.

In the special case (10) we get that

$$K_i^*(x) = g_{2i}(x) := \int_{\Omega_2} k_i(x,y) f_{2i}(y) d\mu_2(y), \quad i = 1, ..., m. \tag{11}$$

In this article we get first general L_p, $p > 1$, results by applying Theorem 1 for (f_{3i}, g_{1i}), $i = 1, ..., m$, and on other various important settings, then we give wide related application to Fractional Calculus.

2 Main Results

We present

Theorem 2 *Assume that the functions* $(i = 1, ..., m \in \mathbb{N})$

$$x \mapsto \left(u(x) \frac{k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y))}{K_i^*(x)} \right)$$

are integrable on Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i^* *on* Ω_2 *by*

$$u_i^*(y) := \varphi(f_{1i}(y), f_{2i}(y)) \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i^*(x)} d\mu_1(x) < \infty, \tag{12}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{K_i^*(x)} \right| \right) d\mu_1(x) \leq \tag{13}$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i^*(y) \left(\Phi_i \left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right) \right)^{p_i} d\mu_2(y) \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left(\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right), \Phi_i \left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right)^{p_i}$ *are both* $k_i(x, y) \varphi(f_{1i}(y), f_{2i}(y))$ $d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $u_i^*(y) \Phi_i \left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right)^{p_i}$ *is* μ_2 *-integrable,* $i = 1, ..., m$.

In the special case of (9), (10), (11) we derive

Theorem 3 *Here* $0 < f_{2i}(y) < \infty$, *a.e.,* $i = 1, ..., m$. *Assume that the functions* $(i = 1, ..., m \in \mathbb{N})$

$$x \mapsto \left(\frac{u(x) k_i(x, y) f_{2i}(y)}{g_{2i}(x)} \right)$$

are integrable on Ω_1, *for each fixed* $y \in \Omega_2$; *with* $g_{2i}(x) > 0$, *a.e. on* Ω_1.

Define ψ_i *on* Ω_2 *by*

$$\psi_i(y) := f_{2i}(y) \int_{\Omega_1} u(x) \frac{k_i(x, y)}{g_{2i}(x)} d\mu_1(x) < \infty, \tag{14}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{g_{2i}(x)} \right| \right) d\mu_1(x) \leq \tag{15}$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} \psi_i(y) \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i} d\mu_2(y) \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}, \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i}$ *are both* $k_i(x, y) f_{2i}(y) d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\psi_i(y) \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i}$ *is* μ_2*-integrable,* $i = 1, ..., m$.

Next we consider the case of $\varphi(s_i, t_i) = |a_{1i}s_i + a_{2i}t_i|^r$, where $r \in \mathbb{R}$; $s_i, t_i \in \mathbb{R}^*$; $a_{1i}, a_{2i} \in \mathbb{R}$, $i = 1, ..., m$. We assume here that

$$0 < |a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r < \infty, \tag{16}$$

a.e., $i = 1, ..., m$.

We further assume that

$$\overline{K_i^*}(x) := \int_{\Omega_2} k_i(x,y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r\,d\mu_2(y) > 0, \tag{17}$$

a.e. on Ω_1, $i = 1, ..., m$.

Here we have

$$f_{3i}(y) = \frac{f_{1i}(y)}{|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r}, \tag{18}$$

$i = 1, ..., m$, $y \in \Omega_2$.

Denote by

$$\overline{k_i^*}(x,y) := k_i(x,y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r, \tag{19}$$

$i = 1, ..., m$.

By Theorem 2 we obtain

Theorem 4 *Assume that the functions ($i = 1, ..., m \in \mathbb{N}$)*

$$x \mapsto \left(u(x)\,\frac{k_i(x,y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r}{\overline{K_i^*}(x)}\right)$$

are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define $\overline{u_i^*}$ on Ω_2 by

$$\overline{u_i^*}(y) := |a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r \int_{\Omega_1} u(x)\,\frac{k_i(x,y)}{\overline{K_i^*}(x)}\,d\mu_1(x) < \infty. \tag{20}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing. Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_{1i}(x)}{\overline{K_i^*}(x)}\right|\right) d\mu_1(x) \leq \tag{21}$$

$$\prod_{i=1}^{m}\left(\int_{\Omega_2} \overline{u_i^*}(y)\left(\Phi_i\left(\frac{|f_{1i}(y)|}{(a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y))^r}\right)\right)^{p_i} d\mu_2(y)\right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left(\frac{f_{1i}(y)}{|a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y)|^r}\right)$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{(a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y))^r}\right|\right)^{p_i}$ *are both* $k_i(x,y)\,|a_{1i}f_{1i}(y) + a_{2i}f_{2i}(y)|^r\,d\mu_2(y)$ *-integrable, μ_1 -a.e. in $x \in \Omega_1$,*

(ii) $\overline{u_i^*}(y)\left(\Phi_i\left(\left|\frac{f_{1i}(y)}{(a_{1i}f_{1i}(y)+a_{2i}f_{2i}(y))^r}\right|\right)\right)^{p_i}$ *is μ_2 -integrable, $i = 1, ..., m$.*

In Theorem 4 of great interest is the case of $r \in \mathbb{Z} - \{0\}$ and $a_{1i} = a_{2i} = 1$, all $i = 1, ..., m$; or $a_{1i} = 1$, $a_{2i} = -1$, all $i = 1, ..., m$.

Another interesting case arises when

$$\varphi\left(f_{1i}(y), f_{2i}(y)\right) := \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2}, \tag{22}$$

$i = 1, ..., m$, where $r_1, r_2 \in \mathbb{R}$. We assume that

$$0 < \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2} < \infty, \text{ a.e., } i = 1, ..., m. \tag{23}$$

In this case

$$f_{3i}(y) = \frac{f_{1i}(y)}{\left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2}}, \tag{24}$$

$i = 1, ..., m$, $y \in \Omega_2$, also

$$k_i^*(x, y) = k_{pi}^*(x, y) := k_i(x, y) \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2}, \tag{25}$$

$y \in \Omega_2$, $i = 1, ..., m$.

We have

$$K_i^*(x) = K_{pi}^*(x) := \int_{\Omega_2} k_i(x, y) \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2} d\mu_2(y), \tag{26}$$

$i = 1, ..., m$.

We assume that $K_{pi}^* > 0$, a.e. on Ω_1.

By Theorem 2 we derive

Theorem 5 *Assume that the functions ($i = 1, ..., m \in \mathbb{N}$)*

$$x \mapsto \left(u(x) \frac{k_i(x, y) \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2}}{K_{pi}^*(x)}\right)$$

are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define u_{pi}^* on Ω_2 by

$$u_{pi}^*(y) := \left|f_{1i}(y)\right|^{r_1} \left|f_{2i}(y)\right|^{r_2} \int_{\Omega_1} u(x) \frac{k_i(x, y)}{K_{pi}^*(x)} d\mu_1(x) < \infty, \tag{27}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing. Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_{1i}(x)}{K_{pi}^*(x)}\right|\right) d\mu_1(x) \le \tag{28}$$

$$\prod_{i=1}^{m}\left(\int_{\Omega_2} u_{pi}^{*}(y)\left(\Phi_i\left(\frac{|f_{1i}(y)|^{1-r_1}}{|f_{2i}(y)|^{r_2}}\right)\right)^{p_i} d\mu_2(y)\right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left(\frac{f_{1i}(y)}{|f_{1i}(y)|^{r_1}|f_{2i}(y)|^{r_2}}\right)$, $\left(\Phi_i\left(\frac{|f_{1i}(y)|^{1-r_1}}{|f_{2i}(y)|^{r_2}}\right)\right)^{p_i}$ *are both* $k_i(x,y)|f_{1i}(y)|^{r_1}|f_{2i}(y)|^{r_2} d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $u_{pi}^{*}(y)\left(\Phi_i\left(\frac{|f_{1i}(y)|^{1-r_1}}{|f_{2i}(y)|^{r_2}}\right)\right)^{p_i}$ *is* μ_2 *-integrable,* $i = 1,...,m$.

In Theorem 5 of interest will be the case of $r_1 = 1-n$, $r_2 = -n$, $n \in \mathbb{N}$. In that case $|f_{1i}(y)|^{1-r_1}|f_{2i}(y)|^{-r_2} = |f_{1i}(y) f_{2i}(y)|^n$, etc.

Next we apply Theorem 2 for specific convex functions.

Theorem 6 *Assume that the functions* $(i = 1,...,m \in \mathbb{N})$

$$x \mapsto \left(u(x)\frac{k_i(x,y)\varphi(f_{1i}(y), f_{2i}(y))}{K_i^{*}(x)}\right)$$

are integrable on Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i^{*} *on* Ω_2 *by*

$$u_i^{*}(y) := \varphi(f_{1i}(y), f_{2i}(y))\int_{\Omega_1} u(x)\frac{k_i(x,y)}{K_i^{*}(x)} d\mu_1(x) < \infty, \tag{29}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m}\frac{1}{p_i} = 1$*. Then*

$$\int_{\Omega_1} u(x) e^{\sum_{i=1}^{m}\left|\frac{g_{1i}(x)}{K_i^{*}(x)}\right|} d\mu_1(x) \le \tag{30}$$

$$\prod_{i=1}^{m}\left(\int_{\Omega_2} u_i^{*}(y)\left(e^{\frac{p_i|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))}}\right) d\mu_2(y)\right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))}$, $e^{\frac{p_i|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))}}$ *are both* $k_i(x,y)\varphi(f_{1i}(y), f_{2i}(y)) d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $u_i^{*}(y) e^{\frac{p_i|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))}}$ *is* μ_2 *-integrable,* $i = 1,...,m$.

Theorem 7 *Assume that the functions* $(i = 1,...,m \in \mathbb{N})$

$$x \mapsto \left(u(x)\frac{k_i(x,y)\varphi(f_{1i}(y), f_{2i}(y))}{K_i^{*}(x)}\right)$$

are integrable on Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i^{*} *on* Ω_2 *by*

$$u_i^{*}(y) := \varphi(f_{1i}(y), f_{2i}(y))\int_{\Omega_1} u(x)\frac{k_i(x,y)}{K_i^{*}(x)} d\mu_1(x) < \infty, \tag{31}$$

a.e. on Ω_2. *Let* $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$; $\beta_i \geq 1$, $i = 1, ..., m$.

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \left| \frac{g_{1i}(x)}{K_i^*(x)} \right|^{\beta_i} d\mu_1(x) \leq \tag{32}$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i^*(y) \left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right)^{p_i \beta_i} d\mu_2(y) \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left(\frac{f_{1i}(y)}{\varphi(f_{1i}(y), f_{2i}(y))} \right)$, $\left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right)^{p_i \beta_i}$ *are both* $k_i(x,y) \varphi(f_{1i}(y), f_{2i}(y))$ $d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $u_i^*(y) \left(\frac{|f_{1i}(y)|}{\varphi(f_{1i}(y), f_{2i}(y))} \right)^{p_i \beta_i}$ *is* μ_2 *-integrable,* $i = 1, ..., m$.

We continue as follows:

Choosing $r_1 = 0, r_2 = -1, i = 1, ..., m$, on (22) we have that

$$\varphi(f_{1i}(y), f_{2i}(y)) = |f_{2i}(y)|^{-1}. \tag{33}$$

We assume that

$$0 < |f_{2i}(y)|^{-1} < \infty, \ \text{a.e.}, \ i = 1, ..., m, \tag{34}$$

equivalently,

$$0 < |f_{2i}(y)| < \infty, \ \text{a.e.}, \ i = 1, ..., m. \tag{35}$$

In this case

$$f_{3i} = f_{1i}(y) |f_{2i}(y)|, \tag{36}$$

$i = 1, ..., m$, $y \in \Omega_2$, also it is

$$k_{pi}^*(x,y) = \overline{k_{pi}^*}(x,y) := \frac{k_i(x,y)}{|f_{2i}(y)|}, \tag{37}$$

$y \in \Omega_2$, $i = 1, ..., m$.

We have that

$$K_{pi}^*(x) = \overline{K_{pi}^*}(x) := \int_{\Omega_2} \frac{k_i(x,y)}{|f_{2i}(y)|} d\mu_2(y), \tag{38}$$

$i = 1, ..., m$.

We assume that $\overline{K_{pi}^*}(x) > 0$, a.e. on Ω_1.

By Theorem 5 we obtain

Corollary 8 *Assume that the functions* ($i = 1, ..., m \in \mathbb{N}$)

$$x \mapsto \left(\frac{u(x) k_i(x,y) |f_{2i}(y)|^{-1}}{\overline{K^*_{pi}}(x)} \right)$$

are integrable on Ω_1*, for each fixed* $y \in \Omega_2$*. Define* ψ^*_{pi} *on* Ω_2 *by*

$$\psi^*_{pi}(y) := |f_{2i}(y)|^{-1} \int_{\Omega_1} u(x) \frac{k_i(x,y)}{\overline{K^*_{pi}}(x)} d\mu_1(x) < \infty, \tag{39}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, *be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_{1i}(x)}{\overline{K^*_{pi}}(x)} \right| \right) d\mu_1(x) \leq \tag{40}$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} \psi^*_{pi}(y) \left(\Phi_i \left(|f_{1i}(y) f_{2i}(y)| \right) \right)^{p_i} d\mu_2(y) \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $f_{1i}(y) f_{2i}(y)$, $(\Phi_i(|f_{1i}(y) f_{2i}(y)|))^{p_i}$ *are both* $k_i(x,y) |f_{2i}(y)|^{-1} d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $\psi^*_{pi}(y) (\Phi_i(|f_{1i}(y) f_{2i}(y)|))^{p_i}$ *is* μ_2 *-integrable,* $i = 1, ..., m$.

To keep exposition short, in the next big part of this article we give only applications of Theorem 3 to Fractional Calculus. We need the following:

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We need

Definition 9 *([9]) Let* (a,b), $0 \leq a < b < \infty$; $\alpha, \sigma > 0$. *We consider the left and right-sided fractional integrals of order* α *as follows:*

1) for $\eta > -1$, *we define*

$$\left(I^\alpha_{a+;\sigma,\eta} f \right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t) \, dt}{(x^\sigma - t^\sigma)^{1-\alpha}}, \tag{41}$$

2) for $\eta > 0$, we define

$$\left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\,dt}{(t^\sigma - x^\sigma)^{1-\alpha}}. \tag{42}$$

These are the Erdélyi-Kober type fractional integrals.

We make

Remark 10 *Regarding (41) we have all*

$$k_i(x,y) = k_1(x,y) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}}, \tag{43}$$

$x, y \in (a,b)$, *χ stands for the characteristic function.*

In this case

$$g_{1i}(x) = \left(I_{a+;\alpha,\eta}^{\alpha} f_{1i}\right)(x) = \int_a^b k_1(x,y) f_{1i}(y)\,dy, \tag{44}$$

and

$$g_{2i}(x) = \left(I_{a+;\sigma,\eta}^{\alpha} f_{2i}\right)(x) = \int_a^b k_1(x,y) f_{2i}(y)\,dy, \tag{45}$$

$i = 1, ..., m$.

We assume $\left(I_{a+;\sigma,\eta}^{\alpha} f_{2i}\right)(x) > 0$, *a.e. on* (a,b), *and* $0 < f_{2i}(y) < \infty$, *a.e.*, $i = 1, ..., m$.

We also make

Remark 11 *Regarding (42) we have all*

$$k_i(x,y) = k_2(x,y) := \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \chi_{[x,b)}(y) \frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^\sigma - x^\sigma)^{1-\alpha}}, \tag{46}$$

$x, y \in (a,b)$.

In this case

$$g_{1i}(x) = \left(I_{b-;\alpha,\eta}^{\alpha} f_{1i}\right)(x) = \int_a^b k_2(x,y) f_{1i}(y)\,dy, \tag{47}$$

and

$$g_{2i}(x) = \left(I_{b-;\sigma,\eta}^{\alpha} f_{2i}\right)(x) = \int_a^b k_2(x,y) f_{2i}(y)\,dy, \tag{48}$$

$i = 1, ..., m$.

We assume $\left(I_{b-;\sigma,\eta}^{\alpha} f_{2i}\right)(x) > 0$, *a.e. on* (a,b), *and* $0 < f_{2i}(y) < \infty$, *a.e.*, $i = 1, ..., m$.

Next we apply Theorem 3.

Theorem 12 *Here all as in Remark 10. Assume that the functions ($i = 1, ..., m \in \mathbb{N}$)*

$$x \mapsto \left(\frac{u(x)\,\sigma x^{-\sigma(\alpha+\eta)}}{\left(I_{a+;\sigma,\eta}^{\alpha_i} f_{2i}\right)(x)\,\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1} f_{2i}(y)}{(x^\sigma - y^\sigma)^{1-\alpha}} \right)$$

are integrable on (a,b), for each fixed $y \in (a,b)$.

Define ψ_i^+ on (a,b) by

$$\psi_i^+(y) := \frac{\sigma f_{2i}(y)\, y^{\sigma\eta+\sigma-1}}{\Gamma(\alpha)} \int_y^b u(x) \frac{x^{-\sigma(\alpha+\eta)} (x^\sigma - y^\sigma)^{\alpha-1}}{\left(I_{a+;\sigma,\eta}^{\alpha} f_{2i}\right)(x)} dx < \infty, \tag{49}$$

a.e. on (a,b). Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left|\left(I_{a+;\sigma,\eta}^{\alpha} f_{1i}\right)(x)\right|}{\left(I_{a+;\sigma,\eta}^{\alpha} f_{2i}\right)(x)} \right) dx \leq \tag{50}$$

$$\prod_{i=1}^m \left(\int_a^b \psi_i^+(y)\, \Phi_i \left(\frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)^{p_i}$ are both $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha}} f_{2i}(y)\, dy$ -integrable, a.e. in $x \in (a,b)$,

(ii) $\psi_i^+(y)\, \Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)^{p_i}$ is integrable, $i = 1, ..., m$.

Corollary 13 *(to Theorem 12) It holds*

$$\int_a^b u(x)\, e^{\sum_{i=1}^m \frac{\left|\left(I_{a+;\sigma,\eta}^{\alpha} f_{1i}\right)(x)\right|}{\left(I_{a+;\sigma,\eta}^{\alpha} f_{2i}\right)(x)}} dx \leq \tag{51}$$

$$\prod_{i=1}^m \left(\int_a^b \psi_i^+(y)\, e^{\frac{p_i |f_{1i}(y)|}{f_{2i}(y)}} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $e^{p_i\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ are both $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y)\, y^{\sigma\eta+\sigma-1} (x^\sigma - y^\sigma)^{\alpha-1} f_{2i}(y)$ dy -integrable, a.e. in $x \in (a,b)$,

(ii) $\psi_i^+(y)\, e^{p_i\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$ is integrable, $i = 1, ..., m$.

Corollary 14 *(to Theorem 12) Let $\beta_i \geq 1$, $i = 1, ..., m$. It holds*

$$\int_a^b u(x) \prod_{i=1}^m \left(\frac{\left| I_{a+;\sigma,\eta}^{\alpha} f_{1i}(x) \right|}{I_{a+;\sigma,\eta}^{\alpha} f_{2i}(x)} \right)^{\beta_i} dx \leq \tag{52}$$

$$\prod_{i=1}^m \left(\int_a^b \psi_i^+(y) \left(\frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|^{\beta_i p_i}$ *is* $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}} f_{2i}(y)\, dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\psi_i^+(y) \left| \frac{f_{1i}(y)}{f_{2i}(y)} \right|^{\beta_i p_i}$ *is integrable,* $i = 1, ..., m$.

We also give

Theorem 15 *Here all as in Remark 11. Assume that the functions ($i = 1, ..., m \in \mathbb{N}$)*

$$x \mapsto \left(\frac{u(x)\, \sigma x^{\sigma\eta}}{\left(I_{b-;\sigma,\eta}^{\alpha} f_{2i} \right)(x)} \chi_{[x,b)}(y) \frac{y^{\sigma(1-\eta-\alpha)-1} f_{2i}(y)}{(y^{\sigma} - x^{\sigma})^{1-\alpha} \Gamma(\alpha)} \right)$$

are integrable on (a,b), for each fixed $y \in (a,b)$.

Define ψ_i^- on (a,b) by

$$\psi_i^-(y) := \frac{\sigma f_{2i}(y)\, y^{\sigma(1-\eta-\alpha)-1}}{\Gamma(\alpha)} \int_a^y u(x) \frac{x^{\sigma\eta} (y^{\sigma} - x^{\sigma})^{\alpha-1}}{\left(I_{b-;\sigma,\eta}^{\alpha} f_{2i} \right)(x)} dx < \infty, \tag{53}$$

a.e. on (a,b). Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, ..., m$, be convex and increasing. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left(I_{b-;\sigma,\eta}^{\alpha} f_{1i} \right)(x) \right|}{\left(I_{b-;\sigma,\eta}^{\alpha} f_{2i} \right)(x)} \right) dx \leq \tag{54}$$

$$\prod_{i=1}^m \left(\int_a^b \psi_i^-(y)\, \Phi_i \left(\frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i}$ *are both* $\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \chi_{[x.b)}(y) \frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^{\sigma}-x^{\sigma})^{1-\alpha}} f_{2i}(y)\, dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\psi_i^-(y)\, \Phi_i \left(\left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i}$ *is integrable,* $i = 1, ..., m$.

Proof. By Theorem 3. ∎

Corollary 16 *(to Theorem 15) It holds*

$$\int_a^b u(x)\, e^{\sum_{i=1}^{m} \frac{\left|\left(I_{b-;\sigma,\eta}^{\alpha} f_{1i}\right)(x)\right|}{\left(I_{b-;\sigma,\eta}^{\alpha} f_{2i}\right)(x)}} dx \leq \tag{55}$$

$$\prod_{i=1}^{m} \left(\int_a^b \psi_i^{-}(y)\, e^{\frac{p_i \left|f_{1i}(y)\right|}{f_{2i}(y)}} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $e^{p_i \left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ *are both* $\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \chi_{[x.b)}(y) \frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^{\sigma}-x^{\sigma})^{1-\alpha}} f_{2i}(y)\, dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\psi_i^{-}(y)\, e^{p_i \left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|}$ *is integrable,* $i = 1, ..., m$.

Corollary 17 *(to Theorem 15) Let* $\beta_i \geq 1$, $i = 1, ..., m$. *It holds*

$$\int_a^b u(x) \prod_{i=1}^{m} \left(\frac{\left| I_{b-;\sigma,\eta}^{\alpha} f_{1i}(x) \right|}{I_{b-;\sigma,\eta}^{\alpha} f_{2i}(x)} \right)^{\beta_i} dx \leq \tag{56}$$

$$\prod_{i=1}^{m} \left(\int_a^b \psi_i^{-}(y) \left(\frac{\left|f_{1i}(y)\right|}{f_{2i}(y)} \right)^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{\beta_i p_i}$ *is* $\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \chi_{[x.b)}(y) \frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^{\sigma}-x^{\sigma})^{1-\alpha}} f_{2i}(y)\, dy$ *-integrable, a.e. in* $x \in (a,b)$,

(ii) $\psi_i^{-}(y) \left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|^{\beta_i p_i}$ *is integrable,* $i = 1, ..., m$.

We make

Remark 18 *Let* $\prod_{i=1}^{N} (a_i, b_i) \subset \mathbb{R}^N$, $N > 1$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$. *Let* $\alpha_i > 0$, $i = 1, ..., N$; $f \in L_1 \left(\prod_{i=1}^{N} (a_i, b_i) \right)$, *and set* $a = (a_1, ..., a_N)$, $b = (b_1, ..., b_N)$, $\alpha = (\alpha_1, ..., \alpha_N)$, $x = (x_1, ..., x_N)$, $t = (t_1, ..., t_N)$.

We define the left mixed Riemann-Liouville fractional multiple integral of order α *(see also [7]):*

$$\left(I_{a+}^{\alpha} f\right)(x) := \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f(t_1, ..., t_N)\, dt_1 ... dt_N, \tag{57}$$

with $x_i > a_i,\ i = 1, ..., N$.

We also define the right mixed Riemann-Liouville fractional multiple integral of order α *(see also [5]):*

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} ... \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_N)\, dt_1 ... dt_N, \tag{58}$$

with $x_i < b_i,\ i = 1, ..., N$.

Notice $I_{a+}^{\alpha}(|f|)$, $I_{b-}^{\alpha}(|f|)$ *are finite if* $f \in L_{\infty}\left(\prod_{i=1}^{N}(a_i, b_i)\right)$.

One can rewrite (57) and (58) as follows:

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i, b_i)} \chi_{\prod_{i=1}^{N}(a_i, x_i]}(t) \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f(t)\, dt, \tag{59}$$

with $x_i > a_i,\ i = 1, ..., N$,

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i, b_i)} \chi_{\prod_{i=1}^{N}[x_i, b_i)}(t) \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f(t)\, dt, \tag{60}$$

with $x_i < b_i,\ i = 1, ..., N$.

The corresponding $k(x, y)$ *for* I_{a+}^{α}, I_{b-}^{α} *are*

$$k_{a+}(x, y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i, x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1}, \tag{61}$$

$\forall\ x, y \in \prod_{i=1}^{N}(a_i, b_i)$,

and

$$k_{b-}(x, y) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}[x_i, b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1}, \tag{62}$$

$\forall\ x, y \in \prod_{i=1}^{N}(a_i, b_i)$.

We make

Remark 19 *In the case of (57) we choose*

$$g_{1j}(x) = \left(I_{a+}^{\alpha} f_{1j}\right)(x) \tag{63}$$

and

$$g_{2j}(x) = \left(I_{a+}^{\alpha} f_{2j}\right)(x), \tag{64}$$

$\forall\ x \in \prod_{i=1}^{N} (a_i, b_i)$.

We assume $\left(I_{a+}^{\alpha} f_{2j}\right)(x) > 0$, *a.e. on* $\prod_{i=1}^{N} (a_i, b_i)$, *and* $0 < f_{2j}(y) < \infty$, *a.e.*, $j = 1, ..., m$.

Above the functions $f_{1j}, f_{2j} \in L_1\left(\prod_{i=1}^{N} (a_i, b_i)\right)$, $j = 1, ..., m$.

We make

Remark 20 *In the case of (58) we choose*

$$g_{1j}(x) = \left(I_{b-}^{\alpha} f_{1j}\right)(x) \tag{65}$$

and

$$g_{2j}(x) = \left(I_{b-}^{\alpha} f_{2j}\right)(x), \tag{66}$$

$\forall\ x \in \prod_{i=1}^{N} (a_i, b_i)$.

We assume $\left(I_{b-}^{\alpha} f_{2j}\right)(x) > 0$, *a.e. on* $\prod_{i=1}^{N} (a_i, b_i)$, *and* $0 < f_{2j}(y) < \infty$, *a.e.*, $j = 1, ..., m$.

Above the functions $f_{1j}, f_{2j} \in L_1\left(\prod_{i=1}^{N} (a_i, b_i)\right)$, $j = 1, ..., m$.

We present

Theorem 21 *Here all as in Remark 19. Assume that the functions* ($j = 1, ..., m \in \mathbb{N}$)

$$x \mapsto \left(\frac{u(x) f_{2j}(x) \chi_{\prod_{i=1}^{N}(a_i, x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1}}{\left(I_{a+}^{\alpha} f_{2j}\right)(x) \prod_{i=1}^{N} \Gamma(\alpha_i)} \right)$$

are integrable on $\prod_{i=1}^{N}(a_i,b_i)$, *for each fixed* $y\in\prod_{i=1}^{N}(a_i,b_i)$. *We define* W_j *on* $\prod_{i=1}^{N}(a_i,b_i)$ *by*

$$W_j(y):=\frac{f_{2j}(y)}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\int_{y_1}^{b_1}\ldots\int_{y_N}^{b_N}\frac{u(x_1,\ldots,x_N)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}}{\left(I_{a+}^{\alpha}f_{2j}\right)(x_1,\ldots,x_N)}dx_1\ldots dx_N<\infty, \tag{67}$$

a.e.

Let $p_j>1:\sum_{j=1}^{m}\frac{1}{p_j}=1$. *Let the functions* $\Phi_j:\mathbb{R}_+\to\mathbb{R}_+$, $j=1,\ldots,m$, *be convex and increasing. Then*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\left(I_{a+}^{\alpha}f_{1j}\right)(x)\right|}{\left(I_{a+}^{\alpha}f_{2j}\right)(x)}\right)dx\leq \tag{68}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}W_j(y)\,\Phi_j\left(\frac{|f_{1j}(y)|}{f_{2j}(y)}\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $\frac{f_{1j}(y)}{f_{2j}(y)}$, $\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)^{p_j}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}$ $f_{2j}(y)\,dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$,

(ii) $W_j(y)\,\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)^{p_j}$ *is integrable,* $j=1,\ldots,m$.

Corollary 22 *(to Theorem 21) It holds*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}u(x)\,e^{\sum_{j=1}^{m}\frac{\left|\left(I_{a+}^{\alpha}f_{1j}\right)(x)\right|}{\left(I_{a+}^{\alpha}f_{2j}\right)(x)}}dx\leq \tag{69}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}W_j(y)\,e^{\frac{p_j|f_{1j}(y)|}{f_{2j}(y)}}dy\right)^{\frac{1}{p_j}},$$

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under the assumptions:

(i) $\frac{f_{1j}(y)}{f_{2j}(y)}$, $e^{p_j\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}f_{2j}(y)\,dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$,

(ii) $W_j(y)\,e^{p_j\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$ *is integrable,* $j=1,...,m$.

Corollary 23 *(to Theorem 21) Let* $\beta_j\geq 1$, $j=1,...,m$. *It holds*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}u(x)\prod_{j=1}^{m}\left(\frac{\left|I_{a+}^{\alpha}f_{1j}(x)\right|}{I_{a+}^{\alpha}f_{2j}(x)}\right)^{\beta_j}dx\leq \tag{70}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}W_j(y)\left(\frac{|f_{1j}(y)|}{f_{2j}(y)}\right)^{\beta_jp_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{\beta_jp_j}$ *is* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}f_{2j}(y)\,dy$ *-integrable, a.e. in* $x\in\prod_{i=1}^{N}(a_i,b_i)$,

(ii) $W_j(y)\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{\beta_jp_j}$ *is integrable,* $j=1,...,m$.

We also give

Theorem 24 *Here all as in Remark 20. Assume that the functions* ($j=1,...,m\in\mathbb{N}$)

$$x\mapsto\left(\frac{u(x)\,f_{2j}(y)\,\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}}{\left(I_{b-}^{\alpha}f_{2j}\right)(x)\prod_{i=1}^{N}\Gamma(\alpha_i)}\right)$$

are integrable on $\prod_{i=1}^{N}(a_i,b_i)$, *for each fixed* $y\in\prod_{i=1}^{N}(a_i,b_i)$.

Define M_j *on* $\prod_{i=1}^{N}(a_i,b_i)$ *by*

$$M_j(y) := \frac{f_{2j}(y)}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{a_1}^{y_1} \dots \int_{a_N}^{y_N} \frac{u(x_1,\dots,x_N)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}}{\left(I_{b-}^{\alpha}f_{2j}\right)(x_1,\dots,x_N)} dx_1\dots dx_N < \infty, \tag{71}$$

a.e.

Let $p_j > 1 : \sum_{j=1}^{m}\frac{1}{p_j} = 1$. *Let the functions* $\Phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j=1,\dots,m$, *be convex and increasing. Then*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\left(I_{b-}^{\alpha}f_{1j}\right)(x)\right|}{\left(I_{b-}^{\alpha}f_{2j}\right)(x)}\right)dx \leq \tag{72}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} M_j(y)\,\Phi_j\left(\frac{|f_{1j}(y)|}{f_{2j}(y)}\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $\frac{f_{1j}(y)}{f_{2j}(y)}$, $\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)^{p_j}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}$ $f_{2j}(y)\,dy$ *-integrable, a.e. in* $x \in \prod_{i=1}^{N}(a_i,b_i)$,

(ii) $M_j(y)\,\Phi_j\left(\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|\right)^{p_j}$ *is integrable,* $j=1,\dots,m$.

Corollary 25 *(to Theorem 24) It holds*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} u(x)\,e^{\sum_{j=1}^{m}\frac{\left|\left(I_{b-}^{\alpha}f_{1j}\right)(x)\right|}{\left(I_{b-}^{\alpha}f_{2j}\right)(x)}}dx \leq \tag{73}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} M_j(y)\,e^{\frac{p_j|f_{1j}(y)|}{f_{2j}(y)}}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $\frac{f_{1j}(y)}{f_{2j}(y)}, e^{p_j\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$ *are both* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}f_{2j}(y)\,dy$

-integrable, a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$,

(ii) $M_j(y)\,e^{p_j\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|}$ *is integrable,* $j=1,...,m$.

Corollary 26 *(to Theorem 24) Let* $\beta_j\geq 1$, $j=1,...,m$. *It holds*

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}u(x)\prod_{j=1}^{m}\left(\frac{\left|I_{b-}^{\alpha}f_{1j}(x)\right|}{I_{b-}^{\alpha}f_{2j}(x)}\right)^{\beta_j}dx\leq \tag{74}$$

$$\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}M_j(y)\left(\frac{|f_{1j}(y)|}{f_{2j}(y)}\right)^{\beta_jp_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{\beta_jp_j}$ *is* $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}f_{2j}(y)\,dy$ *-integrable,*

a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$,

(ii) $M_j(y)\left|\frac{f_{1j}(y)}{f_{2j}(y)}\right|^{\beta_jp_j}$ *is integrable,* $j=1,...,m$.

We make

Remark 27 *Next we follow [6] and our introduction.*

Let $(\Omega_1,\Sigma_1,\mu_1)$ *and* $(\Omega_2,\Sigma_2,\mu_2)$ *be measure spaces with positive* σ*-finite measures, and let* $k:\Omega_1\times\Omega_2\to\mathbb{R}$ *be a nonnegative measurable function,* $k(x,\cdot)$ *measurable on* Ω_2 *and*

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y),\ \ \text{for any } x\in\Omega_1. \tag{75}$$

We assume $K(x)>0$, *a.e. on* Ω_1, *and the weight functions are nonnegative functions on the related set. We consider measurable functions* $g_1,g_2:\Omega_1\to\mathbb{R}$ *with the representation*

$$g_i(x)=\int_{\Omega_2}k(x,y)\,f_i(y)\,d\mu_2(y), \tag{76}$$

where $f_1, f_2 : \Omega_2 \to \mathbb{R}$ *are measurable,* $i = 1, 2$. *Here* u *stands for a weight function on* Ω_1.

The next theorem comes from [6], but it is fixed by us, the terms come from Remark 27.

Theorem 28 *Here* $0 < f_2(y) < \infty$, *a.e. on* Ω_2, *and* $g_2(x) > 0$, *a.e. on* Ω_1. *Assume that the function*

$$x \mapsto u(x) \frac{f_2(y) k(x,y)}{g_2(x)}$$

is integrable on Ω_1, *for each fixed* $y \in \Omega_2$.

Define v *on* Ω_2 *by*

$$v(y) := f_2(y) \int_{\Omega_1} \frac{u(x) k(x,y)}{g_2(x)} d\mu_1(x) < \infty, \tag{77}$$

a.e. on Ω_2.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ *be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \Phi\left(\frac{|g_1(x)|}{g_2(x)}\right) d\mu_1(x) \leq \tag{78}$$

$$\int_{\Omega_2} v(y) \Phi\left(\frac{|f_1(y)|}{f_2(y)}\right) d\mu_2(y),$$

under the further assumptions:

(i) $\frac{f_1(y)}{f_2(y)}$, $\Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)$ *are both* $k(x,y) f_2(y) d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $v(y) \Phi\left(\left|\frac{f_1(y)}{f_2(y)}\right|\right)$ *is* μ_2*-integrable.*

Next we deal with the spherical shell

Background 29 *We need:*

Let $N \geq 2$, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ *the unit sphere on* $\mathbb{R}^N$, *where* $|\cdot|$ *stands for the Euclidean norm in* $\mathbb{R}^N$. *Also denote the ball* $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $R > 0$, *and the spherical shell*

$$A := B(0,R_2) - \overline{B(0,R_1)}, \quad 0 < R_1 < R_2. \tag{79}$$

For the following see [8, pp. 149-150], and [10, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$, *where* $r = |x| > 0$, *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Clearly here

$$\mathbb{R}^N - \{0\} = (0,\infty) \times S^{N-1}, \tag{80}$$

and

$$\overline{A} = [R_1, R_2] \times S^{N-1}. \tag{81}$$

We will be using

Theorem 30 *[1, p. 322] Let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then*

$$\int_A f(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(r\omega)\, r^{N-1} dr \right) d\omega. \tag{82}$$

So we are able to write an integral on the shell in polar form using the polar coordinates (r,ω).

We need

Definition 31 *[1, p. 458] Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $f \in C^n(\overline{A})$, and A is a spherical shell. Assume that there exists function $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} \in C(\overline{A})$, given by*

$$\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\alpha} \frac{\partial^n f(t\omega)}{\partial r^n} dt \right), \tag{83}$$

where $x \in \overline{A}$; that is $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We call $\frac{\partial^\nu_{R_1} f}{\partial r^\nu}$ the left radial Canavati-type fractional derivative of f of order ν. If $\nu = 0$, then set $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} := f(x)$.

Based on [1, p. 288], and [2] we have

Lemma 32 *Let $\gamma \geq 0$, $m := [\gamma]$, $\nu > 0$, $n := [\nu]$, with $0 \leq \gamma < \nu$. Let $f \in C^n(\overline{A})$ and there exists $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} \in C(\overline{A})$, $x \in \overline{A}$, A a spherical shell. Further assume that $\frac{\partial^j f(R_1\omega)}{\partial r^j} = 0$, $j = m, m+1, ..., n-1$, $\forall\, \omega \in S^{N-1}$. Then there exists $\frac{\partial^\gamma_{R_1} f(x)}{\partial r^\gamma} \in C(\overline{A})$ such that*

$$\frac{\partial^\gamma_{R_1} f(x)}{\partial r^\gamma} = \frac{\partial^\gamma_{R_1} f(r\omega)}{\partial r^\gamma} = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \frac{\partial^\nu_{R_1} f(t\omega)}{\partial r^\nu} dt, \tag{84}$$

$\forall\, \omega \in S^{N-1}$; all $R_1 \leq r \leq R_2$, indeed $f(r\omega) \in C^\gamma_{R_1}([R_1, R_2])$, $\forall\, \omega \in S^{N-1}$.

We make

Remark 33 *In the settings and assumptions of Theorem 28 and Lemma 32 we have*

$$k(r,t) = k_*(r,t) := \frac{1}{\Gamma(\nu-\gamma)} \chi_{[R_1,r]}(t)\,(r-t)^{\nu-\gamma-1}, \tag{85}$$

$r, t \in [R_1, R_2]$.

Let f_1, f_2 as in Lemma 32. Assume $\frac{\partial^\nu_{R_1} f_2}{\partial r^\nu} > 0$ on $\overline{A}$. We take

$$g_1(r,\omega) := \frac{\partial^\gamma_{R_1} f_1(r\omega)}{\partial r^\gamma} \tag{86}$$

and

$$g_2(r,\omega) := \frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}}, \tag{87}$$

for every $(r,\omega) \in [R_1, R_2] \times S^{N-1}$.

We further assume that $g_2(r,\omega) > 0$ *for every* $(r,\omega) \in [R_1, R_2] \times S^{N-1}$.

We choose here $u(r,\omega) = g_2(r,\omega)$. *Then for a fixed* $\omega \in S^{N-1}$, *the function* $r \mapsto \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(t,\omega) k_*(r,t)$ *is continuous, hence integrable on* $[R_1, R_2]$, *for each fixed* $t \in [R_1, R_2]$. *So we have*

$$\begin{aligned} v(t) &= \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(t\omega) \int_t^{R_2} \frac{1}{\Gamma(\nu-\gamma)} (r-t)^{\nu-\gamma-1} dr \\ &= \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(t\omega) \frac{(R_2-t)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} < \infty, \end{aligned} \tag{88}$$

for every $t \in [R_1, R_2]$.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ *be convex and increasing. Then by (78) and (88), we get*

$$\int_{R_1}^{R_2} \left(\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}} \right) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\gamma} f_1(r\omega)}{\partial r^{\gamma}} \right|}{\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}}} \right) dr \leq \tag{89}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \int_{R_1}^{R_2} \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(t\omega) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\nu} f_1(t\omega)}{\partial r^{\nu}} \right|}{\frac{\partial_{R_1}^{\nu} f_2(t\omega)}{\partial r^{\nu}}} \right) dt =$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \int_{R_1}^{R_2} \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(r\omega) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\nu} f_1(r\omega)}{\partial r^{\nu}} \right|}{\frac{\partial_{R_1}^{\nu} f_2(r\omega)}{\partial r^{\nu}}} \right) dr,$$

true $\forall\ \omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, *and* $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, *and* $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$. *So by (89) and* $r^{N-1} r^{1-N} = 1$, *we have*

$$\int_{R_1}^{R_2} \left(\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}} \right) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\gamma} f_1(r\omega)}{\partial r^{\gamma}} \right|}{\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}}} \right) r^{N-1} dr \leq \tag{90}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)} \left(\frac{R_2}{R_1} \right)^{N-1} \int_{R_1}^{R_2} \frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}} f_2(r\omega) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\nu} f_1(r\omega)}{\partial r^{\nu}} \right|}{\frac{\partial_{R_1}^{\nu} f_2(r\omega)}{\partial r^{\nu}}} \right) r^{N-1} dr.$$

Hence we get

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} \left(\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}} \right) \Phi \left(\frac{\left| \frac{\partial_{R_1}^{\gamma} f_1(r\omega)}{\partial r^{\gamma}} \right|}{\frac{\partial_{R_1}^{\gamma} f_2(r\omega)}{\partial r^{\gamma}}} \right) r^{N-1} dr \right) d\omega \leq \tag{91}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_{S^{N-1}}\left(\int_{R_1}^{R_2}\frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}}f_2(r\omega)\,\Phi\left(\frac{\left|\frac{\partial_{R_1}^{\nu}f_1(r\omega)}{\partial r^{\nu}}\right|}{\frac{\partial_{R_1}^{\nu}f_2(r\omega)}{\partial r^{\nu}}}\right)r^{N-1}dr\right)d\omega.$$

Using Theorem 30 we obtain

$$\int_A\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}\Phi\left(\frac{\left|\frac{\partial_{R_1}^{\gamma}f_1(x)}{\partial r^{\gamma}}\right|}{\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}}\right)dx\leq \tag{92}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A\frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}}f_2(x)\,\Phi\left(\frac{\left|\frac{\partial_{R_1}^{\nu}f_1(x)}{\partial r^{\nu}}\right|}{\frac{\partial_{R_1}^{\nu}f_2(x)}{\partial r^{\nu}}}\right)dx.$$

We have proved the following result.

Theorem 34 *Let all and f_1,f_2 as in Lemma 32, $\nu>\gamma\geq 0$. Assume $\frac{\partial_{R_1}^{\nu}f_2}{\partial r^{\nu}}>0$ and $\frac{\partial_{R_1}^{\gamma}f_2}{\partial r^{\gamma}}>0$ on $\overline{A}$. Let $\Phi:\mathbb{R}_+\to\mathbb{R}$ be convex and increasing. Then*

$$\int_A\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}\Phi\left(\frac{\left|\frac{\partial_{R_1}^{\gamma}f_1(x)}{\partial r^{\gamma}}\right|}{\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}}\right)dx\leq \tag{93}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A\frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}}f_2(x)\,\Phi\left(\frac{\left|\frac{\partial_{R_1}^{\nu}f_1(x)}{\partial r^{\nu}}\right|}{\frac{\partial_{R_1}^{\nu}f_2(x)}{\partial r^{\nu}}}\right)dx.$$

Corollary 35 *(to Theorem 34) It holds*

$$\int_A\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}e^{\frac{\left|\frac{\partial_{R_1}^{\gamma}f_1(x)}{\partial r^{\gamma}}\right|}{\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}}}dx\leq \tag{94}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A\frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}}f_2(x)\,e^{\frac{\left|\frac{\partial_{R_1}^{\nu}f_1(x)}{\partial r^{\nu}}\right|}{\frac{\partial_{R_1}^{\nu}f_2(x)}{\partial r^{\nu}}}}dx.$$

Corollary 36 *(to Theorem 34) Let $p\geq 1$. It holds*

$$\int_A\left(\frac{\partial_{R_1}^{\gamma}f_2(x)}{\partial r^{\gamma}}\right)^{1-p}\left|\frac{\partial_{R_1}^{\gamma}f_1(x)}{\partial r^{\gamma}}\right|^p dx\leq \tag{95}$$

$$\frac{(R_2-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\left(\int_A\frac{\partial_{R_1}^{\nu}}{\partial r^{\nu}}f_2(x)\right)^{1-p}\left|\frac{\partial_{R_1}^{\nu}f_1(x)}{\partial r^{\nu}}\right|^p dx.$$

Similar results can be produced for the right radial Canavati type fractional derivative. We omit this treatment.

We make

Remark 37 *(from [1], p. 460) Here we denote* $\lambda_{\mathbb{R}^N}(x) \equiv dx$ *the Lebesgue measure on* $\mathbb{R}^N$, $N \geq 2$, *and by* $\lambda_{S^{N-1}}(\omega) = d\omega$ *the surface measure on* S^{N-1}, *where* $\mathcal{B}_X$ *stands for the Borel class on space* X. *Define the measure* R_N *on* $\left((0,\infty), \mathcal{B}_{(0,\infty)}\right)$ *by*

$$R_N(B) = \int_B r^{N-1} dr, \quad \textit{any } B \in \mathcal{B}_{(0,\infty)}.$$

Now let $F \in L_1(A) = L_1\left([R_1, R_2] \times S^{N-1}\right)$.

Call

$$K(F) := \{\omega \in S^{N-1} : F(\cdot\omega) \notin L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right)\}. \tag{96}$$

We get, by Fubini's theorem and [10], pp. 87-88, that

$$\lambda_{S^{N-1}}(K(F)) = 0.$$

Of course

$$\theta(F) := [R_1, R_2] \times K(F) \subset A,$$

and

$$\lambda_{\mathbb{R}^N}(\theta(F)) = 0.$$

Above $\lambda_{S^{N-1}}$ *is defined as follows: let* $A \subset S^{N-1}$ *be a Borel set, and let*

$$\widetilde{A} := \{ru : 0 < r < 1, u \in A\} \subset \mathbb{R}^N;$$

we define

$$\lambda_{S^{N-1}}(A) := N \lambda_{\mathbb{R}^N}\left(\widetilde{A}\right).$$

We have that

$$\lambda_{S^{N-1}}\left(S^{N-1}\right) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},$$

the surface area of S^{N-1}.

See also [8, pp. 149-150], [10, pp. 87-88] and [1], p. 320.

Following [1, p. 466] we define the left Riemann-Liouville radial fractional derivative next.

Definition 38 *Let* $\beta > 0$, $m := [\beta] + 1$, $F \in L_1(A)$, *and* A *is the spherical shell. We define*

$$\frac{\overline{\partial}_{R_1}^{\beta} F(x)}{\partial r^{\beta}} := \begin{cases} \frac{1}{\Gamma(m-\beta)} \left(\frac{\partial}{\partial r}\right)^m \int_{R_1}^r (r-t)^{m-\beta-1} F(t\omega)\, dt, \\ \qquad\qquad \textit{for } \omega \in S^{N-1} - K(F), \\ 0, \quad \textit{for } \omega \in K(F), \end{cases} \tag{97}$$

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where $x = r\omega \in A$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$; $K(F)$ as in (96).

If $\beta = 0$, define

$$\frac{\overline{\partial}_{R_1}^{\beta} F(x)}{\partial r^{\beta}} := F(x).$$

Definition 39 *([1], p. 327) We say that $f \in L_1(a, w)$, $a < w$; $a, w \in \mathbb{R}$ has an L_∞ left Riemann-Liouville fractional derivative $\overline{D}_a^{\beta} f$ ($\beta > 0$) in $[a, w]$, iff*

(1) $\overline{D}_a^{\beta-k} f \in C([a, w])$, $k = 1, ..., m := [\beta] + 1$;

(2) $\overline{D}_a^{\beta-1} f \in AC([a, w])$; and

(3) $\overline{D}_a^{\beta} f \in L_\infty(a, w)$.

Define $\overline{D}_a^{0} f := f$ and $\overline{D}_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \le 1$; here $I_{a+}^{\delta} f$ is the left univariate Riemann-Liouville fractional integral of f, see (57).

We need the following representation result.

Theorem 40 *([1, p.331]) Let $\beta > \alpha > 0$ and $F \in L_1(A)$. Assume that $\frac{\overline{\partial}_{R_1}^{\beta} F(x)}{\partial r^{\beta}} \in L_\infty(A)$. Further assume that $\overline{D}_{R_1}^{\beta} F(r\omega)$ takes real values for almost all $r \in [R_1, R_2]$, for each $\omega \in S^{N-1}$, and for these $\left|\overline{D}_{R_1}^{\beta} F(r\omega)\right| \le M_1$ for some $M_1 > 0$. For each $\omega \in S^{N-1} - K(F)$, we assume that $F(\cdot\omega)$ have an L_∞ fractional derivative $\overline{D}_{R_1}^{\beta} F(\cdot\omega)$ in $[R_1, R_2]$, and that*

$$\overline{D}_{R_1}^{\beta-k} F(R_1\omega) = 0, \quad k = 1, ..., [\beta] + 1.$$

Then

$$\frac{\overline{\partial}_{R_1}^{\alpha} F(x)}{\partial r^{\alpha}} = \left(\overline{D}_{R_1}^{\alpha} F\right)(r\omega) = \frac{1}{\Gamma(\beta-\alpha)} \int_{R_1}^{r} (r-t)^{\beta-\alpha-1} \left(\overline{D}_{R_1}^{\beta} F\right)(t\omega)\, dt, \tag{98}$$

is true for all $x \in A$; i.e. true for all $r \in [R_1, R_2]$ and for all $\omega \in S^{N-1}$.

Here

$$\left(\overline{D}_{R_1}^{\alpha} F\right)(\cdot\omega) \in AC([R_1, R_2]), \quad \text{for } \beta - \alpha \ge 1$$

and

$$\left(\overline{D}_{R_1}^{\alpha} F\right)(\cdot\omega) \in C([R_1, R_2]), \quad \text{for } \beta - \alpha \in (0, 1),$$

for all $\omega \in S^{N-1}$. Furthermore

$$\frac{\overline{\partial}_{R_1}^{\alpha} F(x)}{\partial r^{\alpha}} \in L_\infty(A).$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\beta)} \int_{R_1}^{r} (r-t)^{\beta-1} \left(\overline{D}_{R_1}^{\beta} F\right)(t\omega)\, dt, \tag{99}$$

for all $r \in [R_1, R_2]$ *and* $\omega \in S^{N-1} - K(F)$; $x = r\omega$, *and*

$$F(\cdot\omega) \in AC([R_1, R_2]), \quad \text{for } \beta \geq 1$$

and

$$F(\cdot\omega) \in C([R_1, R_2]), \quad \text{for } \beta \in (0,1),$$

for all $\omega \in S^{N-1} - K(F)$.

Similarly to Theorem 34 we obtain

Theorem 41 *Let all here and* F_1, F_2 *as in Theorem 40,* $\beta > \alpha \geq 0$. *Assume that* $0 < \frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}} < \infty$, *a.e. on* A *and* $\frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\beta}} > 0$, *a.e. on* A. *Let* $\Phi : \mathbb{R}_+ \to \mathbb{R}$ *be convex and increasing. Then*

$$\int_A \frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\alpha}} \Phi\left(\frac{\left|\frac{\overline{\partial}_{R_1}^{\alpha} F_1(x)}{\partial r^{\alpha}}\right|}{\frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\alpha}}}\right) dx \leq \tag{100}$$

$$\frac{(R_2 - R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} \left(\frac{R_2}{R_1}\right)^{N-1} \int_A \frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}} \Phi\left(\frac{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(x)}{\partial r^{\beta}}\right|}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}}}\right) dx,$$

under the assumptions: for every $\omega \in S^{N-1}$ *we have*

(i) $\frac{\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}}$, $\Phi\left(\frac{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}\right|}\right)$ *are both* $\frac{1}{\Gamma(\beta-\alpha)} \chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1}$ $\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}} dt$*-integrable in* $t \in [R_1, R_2]$, *a.e. in* $r \in [R_1, R_2]$,

(ii) $\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}} \Phi\left(\frac{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}\right|}\right)$ *is integrable in* t *on* $[R_1, R_2]$.

Corollary 42 *(to Theorem 41) It holds*

$$\int_A \frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\alpha}} e^{\frac{\left|\frac{\overline{\partial}_{R_1}^{\alpha} F_1(x)}{\partial r^{\alpha}}\right|}{\frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\alpha}}}} dx \leq \tag{101}$$

$$\frac{(R_2 - R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} \left(\frac{R_2}{R_1}\right)^{N-1} \int_A \frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}} e^{\frac{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(x)}{\partial r^{\beta}}\right|}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}}}} dx,$$

under the assumptions: for every $\omega \in S^{N-1}$ we have

(i) $\frac{\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}}, e^{\left|\frac{\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}}\right|}$ *are both* $\frac{1}{\Gamma(\beta-\alpha)}\chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1}\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}dt$*-integrable in* $t \in [R_1, R_2]$*, a.e. in* $r \in [R_1, R_2]$*,*

(ii) $\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}} e^{\left|\frac{\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}}{\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}}\right|}$ *is integrable in* t *on* $[R_1, R_2]$*.*

Corollary 43 *(to Theorem 41) Let $p \geq 1$. It holds*

$$\int_A \left(\frac{\overline{\partial}_{R_1}^{\alpha} F_2(x)}{\partial r^{\alpha}}\right)^{1-p} \left|\frac{\overline{\partial}_{R_1}^{\alpha} F_1(x)}{\partial r^{\alpha}}\right|^p dx \leq \tag{102}$$

$$\frac{(R_2-R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A \left(\frac{\overline{\partial}_{R_1}^{\beta} F_2(x)}{\partial r^{\beta}}\right)^{1-p} \left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(x)}{\partial r^{\beta}}\right|^p dx,$$

under the assumptions: for every $\omega \in S^{N-1}$ we have

(i) $\left(\frac{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}\right|}\right)^p$ *is* $\frac{1}{\Gamma(\beta-\alpha)}\chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1}\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}dt$*-integrable in* $t \in [R_1, R_2]$*, a.e. in* $r \in [R_1, R_2]$*,*

(ii) $\left(\left|\frac{\overline{\partial}_{R_1}^{\beta} F_2(t\omega)}{\partial r^{\beta}}\right|\right)^{1-p}\left|\frac{\overline{\partial}_{R_1}^{\beta} F_1(t\omega)}{\partial r^{\beta}}\right|^p$ *is integrable in* t *on* $[R_1, R_2]$*.*

We also need (see [1], p. 421).

Definition 44 *Let $F : \overline{A} \to \mathbb{R}$, $\nu \geq 0$, $n := \lceil \nu \rceil$ such that $F(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$.*

We call the left Caputo radial fractional derivative the following function

$$\frac{\partial_{*R_1}^{\nu} F(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(n-\nu)}\int_{R_1}^{r}(r-t)^{n-\nu-1}\frac{\partial^n F(t\omega)}{\partial r^n}dt, \tag{103}$$

where $x \in \overline{A}$, i.e. $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

Clearly

$$\frac{\partial_{*R_1}^{0} F(x)}{\partial r^{0}} = F(x), \tag{104}$$

$$\frac{\partial_{*R_1}^{\nu} F(x)}{\partial r^{\nu}} = \frac{\partial^{\nu} F(x)}{\partial r^{\nu}}, \quad \text{if } \nu \in \mathbb{N}. \tag{105}$$

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Above function (103) exists almost everywhere for $x \in \overline{A}$, see [1], p. 422.

We mention the following fundamental representation result (see [1], p. 422-423, [2] and [4]).

Theorem 45 *Let $\beta > \alpha \geq 0$, $n := \lceil \beta \rceil$, $m := \lceil \alpha \rceil$, $F : \overline{A} \to \mathbb{R}$ with $F \in L_1(A)$. Assume that $F(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$, and that $\frac{\partial^{\beta}_{*R_1} F(\cdot\omega)}{\partial r^{\beta}} \in L_\infty(R_1, R_2)$ for all $\omega \in S^{N-1}$.*

Further assume that $\frac{\partial^{\beta}_{*R_1} F(x)}{\partial r^{\beta}} \in L_\infty(A)$. More precisely, for these $r \in [R_1, R_2]$, for each $\omega \in S^{N-1}$, for which $D^{\beta}_{*R_1} F(r\omega)$ takes real values, there exists $M_1 > 0$ such that $\left| D^{\beta}_{*R_1} F(r\omega) \right| \leq M_1$.

We suppose that $\frac{\partial^j F(R_1\omega)}{\partial r^j} = 0$, $j = m, m+1, ..., n-1$; for every $\omega \in S^{N-1}$.

Then

$$\frac{\partial^{\alpha}_{*R_1} F(x)}{\partial r^{\alpha}} = D^{\alpha}_{*R_1} F(r\omega) = \frac{1}{\Gamma(\beta - \alpha)} \int_{R_1}^{r} (r-t)^{\beta-\alpha-1} \left(D^{\beta}_{*R_1} F \right)(t\omega)\, dt, \tag{106}$$

valid $\forall\ x \in \overline{A}$; i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$; $\alpha > 0$.

Here

$$D^{\alpha}_{*R_1} F(\cdot\omega) \in C([R_1, R_2]),$$

$\forall\ \omega \in S^{N-1}$; $\alpha > 0$.

Furthermore

$$\frac{\partial^{\alpha}_{*R_1} F(x)}{\partial r^{\alpha}} \in L_\infty(A),\ \alpha > 0. \tag{107}$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\beta)} \int_{R_1}^{r} (r-t)^{\beta-1} \left(D^{\beta}_{*R_1} F \right)(t\omega)\, dt, \tag{108}$$

true $\forall\ x \in \overline{A}$; i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$, and

$$F(\cdot\omega) \in C([R_1, R_2]), \quad \forall\ \omega \in S^{N-1}. \tag{109}$$

Similarly to Theorem 34 we obtain

Theorem 46 *Let all here and F_1, F_2 as in Theorem 45, $\beta > \alpha \geq 0$. Assume that $0 < \frac{\partial^{\beta}_{*R_1} F_2(x)}{\partial r^{\beta}} < \infty$, a.e. on A and $\frac{\partial^{\alpha}_{*R_1} F_2(x)}{\partial r^{\beta}} > 0$, a.e. on A. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be convex and increasing. Then*

$$\int_A \frac{\partial^{\alpha}_{*R_1} F_2(x)}{\partial r^{\alpha}} \Phi \left(\frac{\left| \frac{\partial^{\alpha}_{*R_1} F_1(x)}{\partial r^{\alpha}} \right|}{\frac{\partial^{\alpha}_{*R_1} F_2(x)}{\partial r^{\alpha}}} \right) dx \leq \tag{110}$$

$$\frac{(R_2-R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A \frac{\partial^{\beta}_{*R_1}F_2(x)}{\partial r^{\beta}}\Phi\left(\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(x)}{\partial r^{\beta}}\right|}{\frac{\partial^{\beta}_{*R_1}F_2(x)}{\partial r^{\beta}}}\right)dx,$$

under the assumptions: for every $\omega\in S^{N-1}$ we have

(i) $\frac{\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}}{\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}}$, $\Phi\left(\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}\right|}\right)$ *are both* $\frac{1}{\Gamma(\beta-\alpha)}\chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1}$ $\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}$*dt-integrable in $t\in[R_1,R_2]$, a.e. in $r\in[R_1,R_2]$,*

(ii) $\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}\Phi\left(\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}\right|}\right)$ *is integrable in t on $[R_1,R_2]$.*

Corollary 47 *(to Theorem 46) It holds*

$$\int_A \frac{\partial^{\alpha}_{*R_1}F_2(x)}{\partial r^{\alpha}}e^{\frac{\left|\frac{\partial^{\alpha}_{*R_1}F_1(x)}{\partial r^{\alpha}}\right|}{\frac{\partial^{\alpha}_{*R_1}F_2(x)}{\partial r^{\alpha}}}}dx\le \tag{111}$$

$$\frac{(R_2-R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A \frac{\partial^{\beta}_{*R_1}F_2(x)}{\partial r^{\beta}}e^{\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(x)}{\partial r^{\beta}}\right|}{\frac{\partial^{\beta}_{*R_1}F_2(x)}{\partial r^{\beta}}}}dx,$$

under the assumptions: for every $\omega\in S^{N-1}$ we have

(i) $\frac{\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}}{\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}}$, $e^{\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}\right|}}$ *are both* $\frac{1}{\Gamma(\beta-\alpha)}\chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1}\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}$ *dt-integrable in $t\in[R_1,R_2]$, a.e. in $r\in[R_1,R_2]$,*

(ii) $\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}e^{\frac{\left|\frac{\partial^{\beta}_{*R_1}F_1(t\omega)}{\partial r^{\beta}}\right|}{\left|\frac{\partial^{\beta}_{*R_1}F_2(t\omega)}{\partial r^{\beta}}\right|}}$ *is integrable in t on $[R_1,R_2]$.*

We finish with

Corollary 48 *(to Theorem 46) Let $p\ge 1$. It holds*

$$\int_A\left(\frac{\partial^{\alpha}_{*R_1}F_2(x)}{\partial r^{\alpha}}\right)^{1-p}\left|\frac{\partial^{\alpha}_{*R_1}F_1(x)}{\partial r^{\alpha}}\right|^p dx\le \tag{112}$$

$$\frac{(R_2-R_1)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\left(\frac{R_2}{R_1}\right)^{N-1}\int_A\left(\frac{\partial^{\beta}_{*R_1}F_2(x)}{\partial r^{\beta}}\right)^{1-p}\left|\frac{\partial^{\beta}_{*R_1}F_1(x)}{\partial r^{\beta}}\right|^p dx,$$

under the assumptions: for every $\omega \in S^{N-1}$ *we have*

(i) $\left(\frac{\left| \frac{\partial^{\beta}_{*R_1} F_1(t\omega)}{\partial r^{\beta}} \right|}{\left| \frac{\partial^{\beta}_{*R_1} F_2(t\omega)}{\partial r^{\beta}} \right|} \right)^p$ *is* $\frac{1}{\Gamma(\beta-\alpha)} \chi_{[R_1,r]}(t)(r-t)^{\beta-\alpha-1} \frac{\partial^{\beta}_{*R_1} F_2(t\omega)}{\partial r^{\beta}}$*dt-integrable in* $t \in [R_1, R_2]$, *a.e. in* $r \in [R_1, R_2]$,

(ii) $\left(\left| \frac{\partial^{\beta}_{*R_1} F_2(t\omega)}{\partial r^{\beta}} \right| \right)^{1-p} \left| \frac{\partial^{\beta}_{*R_1} F_1(t\omega)}{\partial r^{\beta}} \right|^p$ *is integrable in* t *on* $[R_1, R_2]$.

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