A STUDY OF POSITIVE LINEAR OPERATORS BY THE METHOD OF MOMENTS

by

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This work is dedicated to my parents,
Angelos and Panagiota Anastassiou,
as well as to my wife, Stavroula.

Curriculum Vitae

George Anastassiou was born on November 25, 1952 in Athens, Greece. From 1970 to March 1975 he attended the University of Athens, where he was awarded a B.Sc. in Mathematics (1975). During that time he also studied political science. In the academic year 1975–76 he received a postgraduate diploma in Operations Research from the University of Southampton at Southampton, U.K.

From September 1976–March 1979 he served as an officer in the Greek air force and at the same time studied some of the connections of point-set Topology and Set theory to Functional Analysis under the guidance of Professor S. Negrepontis of the University of Athens.

On June 9, 1979 he was married to Stavroula Hondridou.

He came to the University of Rochester in Rochester, New York in the fall of 1979. He was financially supported by a fellowship from that institution and by his work as a teaching assistant in Mathematics. He obtained a M.A. in Mathematics in 1981 and worked towards a Ph.D. degree under the guidance of Professor J. H. B. Kemperman.

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Coming to the University of Rochester, I had the good fortune to meet Professor J. H. B. Kemperman, an outstanding mathematician and at the same time, a nice person. He later became my Ph.D. thesis advisor. I am greatly indebted to him for introducing me to the area in which I worked, as well as for much advice and encouragement through our many mathematical communications.

I would like also to thank my wife, Stavroula, for sustaining all the difficulties in the life of a mathematics graduate student and my parents, for their continuous support during all these years. Thanks also go to Joan Robinson for a heroic typing job using $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

And last, but not least, I would like to thank the University of Rochester, for giving me the opportunity, as well as adequate financial aid, to complete my graduate studies in a nice environment.

Abstract

Let Q a compact and convex subset of R^k, $k \geq 1$ and let $\{L_j\}_{j \in \mathbb{N}}$ be a sequence of positive linear operators from $C^n(Q)$, ($n \in \mathbb{Z}^+$) to $C(Q)$. The convergence of L_j to the identity operator I is closely related to the weak convergence of a sequence of finite measures μ_j to the unit (Dirac) measure δ_{x_0}, $x_0 \in Q$.

New estimates are given for the remainder $|\int_Q f \, d\mu_j - f(x_0)|$, where $f \in C^n(Q)$. Using moments methods, Shisha-Mond type best or nearly best upper-bounds are established for various choices of k, Q, n and given moments of μ_j. Some of them lead to attainable inequalities. The optimal functions/measures are typically spline functions and finitely supported measures. The corresponding inequalities involve various measures of smoothness of f such as the first or second modulus of continuity of $f^{(n)}$, the Peetre K-functional of f or certain modifications and generalizations.

Finally some miscellaneous sharp inequalities are obtained. These lead to Korovkin type convergence theorems relative to ratios of Fourier-Stieltjes hyperbolic coefficients.

Table of Contents

Notations and symbols

$\doteq$	Approximately equal to
$\lceil \cdot \rceil$	Ceiling of a number
$C^n(Q)$	Set of n-times continuously differentiable functions ($n \geq 0$) on $Q \subset \mathbb{R}^k$
$C_B^n(\mathbb{R})$	Set of n-times continuously differentiable and bounded functions with all nth derivatives bounded ($n \geq 0$)
$conv$	Convex hull
$d.f.$	Distribution function
δ_{x_0}	Unit (Dirac) measure carried by $\{x_0\}$
$E(X)$	Expected value
K_n	The nth K-functional of Peetre
$K_n^{(x_0)}$	The nth Reduced K-functional
L	Linear operator
γ, μ, ν	Measures (nonnegative)
$\overline{\omega}_1$	Modified first modulus of continuity
ω_1, w	First modulus of continuity
ω_2	Second modulus of continuity
$\mathbb{N}$	Natural numbers
$\mathbb{Z}^+$	Nonnegative integers
$\Vert \cdot \Vert$	Norm
$\Vert \overrightarrow{x} \Vert = \sum_{i=1}^k \vert x_i \vert$	The ℓ_1-norm of $\overrightarrow{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$
$\Vert f \Vert = \sup_{x \in Q} \vert f(x) \vert$	The supremum norm of $f \in C_B(Q), Q \subset \mathbb{R}^k$
O	The big "o"
P	Probability
X	Random variable
$\mathbb{R}^k$	The k-dimensional real vector space ($k \geq 1$)
$\mathbb{R}^+$	Nonnegative real numbers
$\xrightarrow{u}$	Uniform convergence
$Var(X)$	Variance of X

CHAPTER 1

INTRODUCTION

We start with the following definition:

Definition 1.1 Let Q be a connected compact Hausdorff space and $C(Q, \mathbb{R})$ the collection of all continuous $f : Q \to \mathbb{R}$. Let $g \in C(Q, \mathbb{R})$ be fixed and define the *g-pseudomodulus of continuity* of $f \in C(Q, \mathbb{R})$ as

$$w_g(f,h) = \sup_{x,y}\{|f(x) - f(y)| : |g(x) - g(y)| \leq h\}, \tag{1.1.1}$$

here $h \geq 0$.

Thus $w_g(g,h) \leq h$. The quantity $w_g(f,h)$ enjoys most of the basic properties of the usual modulus of continuity $\omega_1(f,h)$ (positively homogeneous as a function of f, non-decreasing nonnegative and subadditive in h). However, $w_g(f,\cdot)$ is an upper-semicontinuous function and in general not a continuous one.

Example Let

$$\begin{aligned} g(x) = 0,&\quad 0 \leq x \leq 1; \\ x - 1,&\ 1 \leq x \leq 2; \\ 1,&\quad 2 \leq x \leq 3 \end{aligned}$$

and $f(x) = x$ then

$$\begin{aligned} w_g(f,h) = 1 + h,&\ 0 \leq h < 1; \\ 3,&\quad h \geq 1. \end{aligned}$$

Obviously, $w_g(f,\cdot)$ is discontinuous.

Consider a sequence of positive linear operators $L_n : C(Q, \mathbb{R}) \to C(Q, \mathbb{R})$, such that the sequence of functions $\{L_n(1)\}_{n \in \mathbb{N}}$ is uniformly bounded. In particular, $|f| \leq \tilde{g}$ implies $|L_n(f)| \leq L_n(\tilde{g})$. The following result is an easy generalization of a result due to Shisha and Mond (1968), who took $Q = [a, b] \subset \mathbb{R}$ and $g(x) = x$.

THEOREM 1.2. *One has*

$$(1.2.1) \qquad \| L_n(f) - f \| \leq \| f \| \ \| L_n(1) - 1 \| + w_g(f, \rho_n)(1 + \| L_n(1) \|),$$

where

$$\rho_n = (\| L_n((g - g(y))^2)(y) \|)^{1/2}.$$

Further, $\| \cdot \|$ stands for the supremum norm. If $L_n(1) = 1$, then (1.2.1) simplifies to

$$\| L_n(f) - f \| \leq 2 \ w_g(f, \rho_n).$$

As an application, one has the following well-known theorem due to Korovkin (1953), see [23].

COROLLARY 1.3. *Let $Q = [a, b] \subset \mathbb{R}$ and let $\{L_n : C([a, b]) \to C([a, b])\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators. Suppose that $g \in C([a, b])$ is $1:1$ and further that $L_n(1) \xrightarrow{u} 1$, $L_n(g) \xrightarrow{u} g$ and $L_n(g^2) \xrightarrow{u} g^2$. Then $L_n(f) \xrightarrow{u} f$, for all $f \in C([a, b])$.*

PROOF: Note that

$$\rho_n^2 \leq \| L_n(g^2) - g^2 \| + 2 \| g \| \| L_n(g) - g \| + \| g \|^2 \| L_n(1) - 1 \| .$$

Now apply Theorem 1.2.∎

Z. Ditzian (1975) gave extensions of (1.2.1) to the non-compact case under suitable growth conditions for f.

Also R. A. DeVore (1972) gave analogues of (1.2.1) for $f \in C^1([a, b])$. Furthermore, E. Censor (1971) extended (1.2.1) to the multidimensional case and established related results for $f \in C^2([a, b])$. Next B. Mond (1976) gave a more flexible inequality which sometimes leads to better constants in the upper bounds for $\| L_n(f) - f \|$. This result was carried over to $f \in C^1([a, b])$, by B. Mond and R. Vasudevan (1980).

The latest result in this direction which often gives better constants as well as a higher degree of approximation for $f \in C^1([a, b])$ is due to H. H. Gonska (1983). He used the first Stekloff function of f'.

J. P. King (1975) gave a probabilistic interpretation of Korovkin's main theorem and certain pointwise inequalities analogous to (1.2.1) for $f \in C([a, b])$ or $f \in C^1([a, b])$.

We have mentioned only the papers directly related to our research. However, there is a large related literature, for instance, the significant theoretical work in Korovkin theory by: J. A. Šaškin (1966), G. G. Lorentz (1972), D. Amir and Z. Ziegler (1978), H. Bauer (1978) and most recently by K. Donner (1982).

Our method is, to reduce questions about positive linear operators to questions about finite (positive) measures. Namely, let $L : C(Q) \to C(Q)$ be a positive linear operator, where Q is a compact subset of $\mathbb{R}^k$. Then for any $x \in Q$ there is a finite measure μ_x such that

$$L(f,x) = \int f(t)\mu_x(dt), \text{ for all } f \in C(Q).$$

And many questions can be reduced to moment problems involving the measure μ_x. Using standard moment methods, see: [17], [18] and [19], we derive pointwise estimates for $|L(f,x) - f(x)|$ which sometimes imply uniform ones. The advantage of this approach is that frequently one even obtains attainable (i.e. sharp) or nearly attainable inequalities. The optimal elements f, μ_x are often spline functions and finitely supported measures, respectively. Thus, this thesis mainly deals with the quantitative study of the pointwise convergence of a sequence of positive linear operators to the identity operator through the use of moment methods.

Now we will describe some typical results of the thesis.

In Chapter 2 we estimate $|\int f\,d\mu - f(x_0)|$, $f \in C^n([a,b])$, where μ is a finite measure on $[a,b]$, $x_0 \in [a,b]$, $n \in \mathbb{Z}^+$. Let μ be of mass m and $0 \in [a,b]$ and put $c = \max(|a|, b)$. Suppose further that

$$\left(\int |t|^r \mu(dt)\right)^{1/r} = d,$$

where $r > 0$ and $d \geq 0$ are given.

Consider $f : [a,b] \to \mathbb{R}$ such that

$$|f(s) - f(t)| \leq w \quad \text{when} \quad |s-t| \leq h,\ a \leq s,t \leq b.$$

Here, $h, w > 0$ are fixed. Denote $\ell = \lceil c/h \rceil, k = \lceil d/(hm^{1/r}) \rceil$.

Then

$$(1.4) \qquad \left|\int f\,d\mu - f(0)\right| \le |m-1|\ |f(0)| + mK\,,$$

where $K = K(m, r, d, h, w, f(0))$ is the best possible constant. It is given as follows:

(i) $\qquad K = \ell w$ when $k = \ell$.

(ii) $\qquad K = \left[1 + \frac{1}{m}(d/h)^r(\ell-1)^{1-r}\right]w$

when $r \le 1$ and $k < \ell$.

(iii) $\qquad K = (k + \theta_k)w \le [1 + d/(hm^{1/r})]w$

when $r \ge 1$ and $k < \ell$. Here,

$$\theta_k = [d^r/m - (k-1)^r h^r]/[k^r h^r - (k-1)^r h^r]\,.$$

When $r = 2$, inequality (1.4) is sharper than the corresponding Shisha-Mond (1968) inequality.

The next result involves a positive linear operator $L : C^n([a,b]) \to C([a,b])$, $n \ge 1$. We put: $c_k(x) = L((t-x)^k, x)$, $(k = 0, 1, \ldots, n)$, $d_n(x) = [L(|t-x|^n, x)]^{1/n}$ and $c(x) = \max(x-a, b-x)$. Let $f \in C^n([a,b])$ be such that $\omega_1(f^{(n)}, h) \le w$. Here $w, h > 0$ are fixed, $0 < h < b-a$.

Then we have the upper bound

(1.5.1)

$$|L(f,x) - f(x)| - |f(x)|\ |c_0(x) - 1| - \left|\sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} c_k(x)\right| \le w\phi_n(c(x))(d_n(x)/c(x))^n\,,$$

where

$$\phi_n(x) = \frac{1}{n!}\left(\sum_{j=0}^{\infty}(|x| - jh)_+^n\right) \tag{1.5.2}$$

is a polynomial spline function of degree n. Inequality (1.5.1) will be proved to be sharp.

Additional upper bounds, which for important particular cases are attainable, are as follows: Let μ be a positive measure of mass $m > 0$ on $[a, b]$ and put

$$\left(\frac{1}{m}\int |t - x_0|^{n+1}\mu(dt)\right)^{1/(n+1)} = \Delta > 0, \text{ where } x_0 \in [a, b] \text{ is fixed .}$$

Consider $f \in C^n([a, b])$, $n \geq 1$ such that $\omega_1(f^{(n)}, r\Delta) \leq w$ with $r, w > 0$ given. Then

$$\begin{aligned} &\left|\int f\,d\mu - f(x_0)\right| - |m - 1|\,|f(x_0)| - \left|\sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}\int (t - x_0)^k \mu(dt)\right| \\ &\leq (mw/r)[nr^2/8 + r/2 + 1/(n + 1)](\Delta^n/n!)\,. \end{aligned} \tag{1.6}$$

In the next result we assume that $\int(t - x_0)\mu(dt) = 0$ and $(\int(t - x_0)^2\mu(dt))^{1/2} = d_2(x_0)$.

Then each $f \in C^1([a, b])$ satisfies

$$\begin{aligned} &\left|\int f\,d\mu - f(x_0)\right| - |f(x_0)|\,|m - 1| \\ &\leq \frac{1}{8r}(2 + \sqrt{m}r)^2\omega_1(f', rd_2(x_0))d_2(x_0)\,, \quad \text{if } r \leq 2/\sqrt{m}\,; \\ &\leq \sqrt{m}\omega_1(f', rd_2(x_0))d_2(x_0)\,, \qquad \text{if } r > 2/\sqrt{m}\,. \end{aligned} \tag{1.7}$$

This inequality improves most of the related results in the existing literature. It supplies a good estimate of the speed of convergence to the identity operator of most classical positive linear operators on $C^1([a, b])$. For example, let $f \in$

$C^1([0,1])$ and consider the Bernstein polynomials

$$(B_n f)(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}, t \in [0,1].$$

Then

$$\| B_n f - f \| \leq \left(\frac{0.78125}{\sqrt{n}}\right)\omega_1\left(f', \frac{1}{4\sqrt{n}}\right),$$

where $\|\cdot\|$ is the sup-norm.

In Chapter 3, by using similar techniques we obtain a variety of analogous results in the multi-dimensional case. These provide us with good upper bounds for $|\int f\,d\mu - f(x_0)|$ for important special cases. Here, Q is usually a convex compact subset of $\mathbb{R}^k$, while $f \in C^n(Q)$ $(n \geq 1)$, $x_0 \in Q$, and μ is a probability measure on Q.

For instance, let $Q = \{\vec{x} \in \mathbb{R}^k : \| \vec{x} \| \leq 1\}$, where $\|\cdot\|$ denotes the ℓ_1-norm in $\mathbb{R}^k$. Further, let $\int_Q \| \vec{x} - \vec{x}_0 \| \mu(d\vec{x}) = d^*$ and suppose that each of the nth partial derivatives f_α of f, relative to the ℓ_1 norm, has a modulus of continuity $\omega_1(f_\alpha, h) \leq w$, where $h, w > 0$ are given. Then we obtain the upper bound

$$\tag{1.8} \begin{aligned} &\left|\int_Q f\,d\mu - f(\vec{x}_0) - \sum_{j=1}^{n} \frac{1}{j!}\int_Q g_{\vec{x}}^{(j)}(0)\mu(d\vec{x})\right| \\ &\leq w d^* \phi_n(1 + \| \vec{x}_0 \|)/(1 + \| \vec{x}_0 \|), \end{aligned}$$

where $g_{\vec{x}}(t) = f(\vec{x}_0 + t(\vec{x} - \vec{x}_0))$, $t \geq 0$ and ϕ_n is as in (1.5.2).

In Chapter 4, under additional assumptions, we obtain new and attainable inequalities for the same type of errors. The first part of the chapter employs the first modulus of continuity $\omega_1(f^{(n)}, t)$, $n \geq 1$, while the second uses the second modulus of continuity $\omega_2(f^{(n)}, t)$. In both, similar results are proved. The second part of this chapter involves positive linear operators of convolution type and also contains an attainable inequality when $\omega_2(f^{(n)}, h) \le Bh^\alpha$, $(0 < \alpha \le 2)$. We make use of the function:

$$G_n(y) = \int_0^y g(t) \frac{(y-t)^{n-1}}{(n-1)!} dt\,,\; y \in [a,b], (n \ge 1) \tag{1.9.1}$$

where g is any of the above modulus functions.

The following is a typical result of the first part.

Let $[a,b] \subset \mathbb{R}$ with $0 \in (a,b)$ and $|a| \le b$, and $f \in C^n([a,b])$. We put $g_0(y) = \omega_1(f^{(n)}, [a,b]; |y|)$, $g_1(y) = \omega_1(f^{(n)}, [a,0]; |y|)$, $g_2(y) = w_1(f^{(n)}, [0,b], y)$ and consider $g = \max(g_1, g_2)$; one has $0 \le g \le g_0$. Further consider the associated function G_n as in (1.9.1). Here, μ is a probability measure on $[a,b]$. Define $c_j = \int t^j \mu(dt)$, $j = 1, \ldots, n$. Let ψ be a function on $[0,b]$ such that $\psi(0) = 0$, which is continuous and strictly increasing and let

$$d = \psi^{-1}\Big(\int \psi(|y|)\mu(dy)\Big).$$

Suppose that $\mathcal{H}_n(u) = G_n(\psi^{-1}(u))$ on $[0, \psi(b)]$ is concave. Then

$$\left| \int f\, d\mu - f(0) - \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} c_j \right| \le G_n(d)\,. \tag{1.9.2}$$

This inequality is sharp, in fact is attained by $\mu = \delta_d$ and $f = G_n$.

By employing the method of the superposition of functions, we give sufficient conditions so that $\mathcal{H}_n$ is convex/concave for particular functions ψ.

In the first part of Chapter 5, we introduce the reduced $K_n^{(x_0)}$ functional, which has all the basic properties of the usual K_n due to J. Peetre (see [5], [15] or [33]). Let $f \in C([a,b])$, $x_0 \in [a,b]$, $n \geq 1$. We define:

$$K_n^{(x_0)}(f,t) = \inf_g (\| f - g \| + t \| g^{(n)} \|).$$

Here, $t \geq 0$ and $g \in C^n([a,b])$ such that $g^{(k)}(x_0) = 0$; $k = 1, \ldots, n-1$ while $\| \cdot \|$ denotes the sup-norm. In particular $K_1^{(x_0)} = K_1$. The main result is as follows:

Let μ be a probability measure on $[a,b]$ such that $\int |y - x_0| \mu(dy) = d_1(x_0)$. Then

$$\text{(1.9.3)} \qquad \left| \int f \, d\mu - f(x_0) \right| \leq 2K_n^{(x_0)}\left(f \, ; \frac{d_1(x_0)(c(x_0))^{n-1}}{2n!} \right),$$

where $c(x_0) = \max(x_0 - a, b - x_0)$.

Furthermore, the last inequality is attained.

Finally, the second part of Chapter 5 is somewhat different from the rest of the thesis. It contains a generalization of a result due to P. P. Korovkin (1958) concerning the convergence of some ratios of the Fourier coefficients of a sequence of density functions on $[0,\pi]$. Namely, let $\ell \in \mathbb{N}$, $\ell \geq 2$. If $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of probability measures on $[0,\pi]$ with Fourier-Stieltjes coefficients p_{kn} such that $p_{1n} \neq 1$ and

$$\lim_{n \to \infty} \left(\frac{1 - p_{\ell n}}{1 - p_{1n}} \right) = \ell^2 \, , \text{ then } \lim_{n \to \infty} \left(\frac{1 - p_{kn}}{1 - p_{1n}} \right) = k^2$$

for all $k \geq 2$, $k \in \mathbb{N}$.

Several other Korovkin type results are proved. For instance, let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^+$ with existing Laplace transforms φ_n such that $\varphi_n(1) \neq 1$, all $n \in \mathbb{N}$.

Then, if

$$\lim_{n\to\infty} \frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)} = \lambda^2$$

for one value $\lambda_0 > 1$ then also for all $\lambda \geq \lambda_0$. Similarly, if

$$\lim_{n\to\infty} \frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)} = \lambda$$

for one value $\lambda_0 > 1$ then also for all $\lambda \geq \lambda_0$.

CHAPTER 2

ONE DIMENSIONAL QUANTITATIVE RESULTS FOR FINITE APPROXIMATION MEASURES OF THE UNIT MEASURE

In this chapter we study the degree of weak convergence of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}$ to the unit measure δ_{x_0}. In fact we estimate $|\int_Q f d\mu - f(x_0)|$; $f \in C^n(Q), n \in \mathbb{Z}^+$, $x_o \in Q$. Here Q is usually a compact interval of $\mathbb{R}$, sometimes $\mathbb{R}$ itself. This enables us in turn to estimate $|L(f,x) - f(x)|$, where L is a positive linear operator: $C^n(Q) \mapsto C(Q)$.

Using standard moment methods, we obtain best or nearly best upper bounds, often sharp, for different Q, n and given power moments of μ.

Our inequalities involve the first modulus of continuity $\omega_1(f^{(n)}, h)$, or a modified version of it, of the n-th derivative $f^{(n)}$ for a fixed value of the argument h. The optimal elements f, μ usually involve simple spline functions and finitely supported measures.

We present several favorable comparisons of our results to related known results, for instance, the inequality due to O. Shisha and B. Mond (1968), see [31], as well as to the latest improvement due to H. G. Gonska (1983), see [13].

I. INTRODUCTION

The following propositions 2.1, 2.2, 2.3, 2.4 are measure theoretic analogues of certain Shisha-Mond type inequalities, see O. Shisha and B. Mond [31], [32] and R. A. DeVore [8].

Proposition 2.1: Let μ be a finite measure of mass m on the subinterval I of $\mathbb{R}$ and let $x_0 \in I$. We assume that

$$h = (\int (t - x_0)^2 \mu(dt))^{1/2}$$

is finite. Let further $f : I \to \mathbb{R}$ be measurable and put $w = \omega_1(f, h) = \sup\{|f(x) - f(y)| : x, y \in I; |x - y| \le h\}$. Then

$$\left| \int f d\mu - f(x_0) \right| \le |f(x_0)|\, |m - 1| + (m + \sqrt{m})w\,. \tag{2.1.1}$$

Proof: Letting $g(t) = f(t) - f(x_0)$, one has that

$$\int f\, d\mu - f(x_0) = f(x_0)(m-1) + \int g\, d\mu$$

Further

$$|g(t)| = |f(t) - f(x_0)| \le (1 + |t - x_0|/h)w\,.$$

Thus,

$$\begin{aligned} |\int g\, d\mu| &\le mw + (w/h) \int |t - x_0| \mu(dt) \\ &\le mw + (w/h)\sqrt{mh^2} \end{aligned}$$

by Schwarz's inequality.∎

Proposition 2.2. *Let $f \in C^1([a,b])$, $x_0 \in [a,b]$, μ a finite measure on $[a,b]$ of mass m and put*

$$h = \left(\int (t-x_0)^2 \mu(dt)\right)^{1/2}$$

Denote the first modulus of continuity of f' at h by $w = \omega_1(f', h)$. Then
(2.2.1)

$$\left|\int f\,d\mu - f(x_0)\right| \leq |f(x_0)|\,|m-1| + |f'(x_0)|\,\left|\int (t-x_0)\mu(dt)\right| + (\sqrt{m} + \frac{1}{2})hw\,.$$

Proof: Let $f(t) = f(x_0) + f'(x_0)(t-x_0) + R_1(t)$. It suffices to show that

$$\left|\int R_1\,d\mu\right| \leq (\sqrt{m} + \frac{1}{2})hw\,. \tag{2.2.2}$$

Here, $R_1(t) = \int_{x_0}^t (f'(s) - f'(x_0))ds$, where

$$|f'(s) - f'(x_0)| \leq (1 + |s-x_0|/h)w\,.$$

Thus

$$|R_1(t)| \leq |t-x_0|(1 + |t-x_0|/(2h))w\,.$$

Integrating, this gives that

$$\frac{1}{w}\left|\int R_1\,d\mu\right| \leq \int |t-x_0|\mu(dt) + \frac{1}{2h}\int (t-x_0)^2\mu(dt) \leq \sqrt{h^2 m} + \frac{1}{2h}h^2 = h(\sqrt{m} + \frac{1}{2}),$$

yielding (2.2.2).∎

Proposition 2.3. *Let μ be a measure of mass m on $[-\pi, \pi]$ and put*

$$\phi = (\int \sin^2(t/2)\mu(dt))^{1/2}.$$

Then, for every $f \in C([-\pi, \pi])$ one has

$$|\int f d\mu - f(0)| \leq |f(0)|\,|m-1| + (m + \pi\sqrt{m})w_f(\phi) \tag{2.3.1}$$

where w_f is the first modulus of continuity of f.

Proof: Exactly as in the proof of Proposition 2.1, we have

$$|\int f d\mu - f(0)| \leq |f(0)|\,|m-1| + m\, w_f(\phi) + \frac{w_f(\phi)}{\phi}\int |t|\mu(dt).$$

Further, note that

$$\int |t|\mu(dt) \leq \int \pi|\sin(t/2)|\mu(dt) \leq \pi\phi\sqrt{m}. \blacksquare$$

Note: A slight variation of Proposition 2.3 would be

$$|\int f\, d\mu - f(x_0)| \leq |f(x_0)|\,|m-1| + (m + \pi\sqrt{m})w,$$

where $w = \omega_1(f, \phi)$ and $\phi = (\int \sin^2((t - x_0)/2)\mu(dt))^{1/2}$, *provided* ω_1 is defined as

$$\omega_1(f, h) = \sup\{|f(x) - f(y)| : x, y \in [-\pi, \pi], |x - y| \leq h \operatorname{mod} 2\pi\},$$

where $|x - y| \leq h \operatorname{mod} 2\pi$ means that $\min(|x - y|, 2\pi - |x - y|) \leq h$. Note that this requires that

$$|f(\pi) - f(-\pi)| \leq \omega_1(f, h).$$

Proposition 2.4. *Let μ and ϕ be as in Proposition 2.3.*

Then, for $f \in C^1[-\pi, \pi]$, letting $w = \omega_1(f', \phi)$, we have

$$\text{(2.4.1)} \qquad \begin{aligned} &|\int f d\mu - f(0)| \leq |f(0)|\,|m-1| \\ &+|f'(0)|\,|\int t\mu(dt)| + (\sqrt{m} + \pi/2)\pi\phi w. \end{aligned}$$

Proof: As in the proof of Proposition 2.2, we obtain

$$\begin{aligned} &|\int f d\mu - f(0)| \leq |f(0)|\,|m-1| + |f'(0)|\,|\int t\mu(dt)| \\ &+ w\Big[\int |t|\mu(dt) + \frac{1}{2\phi}\int t^2\mu(dt)\Big]. \end{aligned}$$

Also, note that

$$|t| \leq \pi|\sin(t/2)|\;(-\pi \leq t \leq \pi)$$

thus

$$\int |t|\mu(dt) \leq \pi \int |\sin(t/2)|\mu(dt) \leq \pi\phi\sqrt{m}$$

and

$$\frac{1}{2\phi}\int t^2\mu(dt) \leq \frac{\pi^2}{2\phi}\int \sin^2(t/2)\,\mu(dt) = (\pi^2/2)\phi. \blacksquare$$

The next result will often be used in this thesis, usually when the number of the growth conditions (2.5.1) equals one.

Theorem 2.5. *Let C be a subset of the real normed vector space $V = (V, \|\cdot\|)$ which is star-shaped relative to the fixed point x_0. Let further $\{(h_i, w_i) : i \in I\}$ be a given collection of numbers, ($h_i > 0, w_i > 0, I$ arbitrary) and consider the collection $\mathcal{F}$ of functions $f : C \to \mathbb{R}$ such that $f(x_0) = 0$ while, for each $i \in I$,*

$$\| s - t \| \le h_i \Rightarrow |f(s) - f(t)| \le w_i . \tag{2.5.1}$$

Then

$$\sup_{f \in \mathcal{F}} |f(s)| = \rho(\| s - x_0 \|) \quad (s \in C), \tag{2.5.2}$$

where

$$\rho(\| u \|) = \inf\{\sum_{i\in I} k_i w_i : \| u \| \le \sum_{i \in I} k_i h_i\}$$

where $k_i \in \mathbb{Z}^+$, $k = \sum_{i\in I} k_i < \infty$.

Proof: Obviously, $\rho(\| u \|)$ is an even subadditive function on $\mathbb{R}$ satisfying $\rho(0) = 0$ and $\rho(h_i) \le w_i$ $(i \in I)$. Moreover, $\rho(\| u \|)$ is non-decreasing on $\mathbb{R}^+$. Hence $f_0(s) = \rho(\| s - x_0 \|)$ restricted to C defines a function $f_0 \in \mathcal{F}$, showing that (2.5.2) holds with the $\ge$ sign. To prove the opposite inequality, it suffices to show that $|f(s)| \le \sum_{i\in I} k_i w_i$ as soon as $f \in \mathcal{F}$ and $\| s - x_0 \| \le \sum_{i \in I} k_i h_i$, ($k_i \in \mathbb{Z}^+$, $k = \sum_{i\in I} k_i < \infty$). This is easily done by an induction on k. The cases $k = 0$ and $k = 1$ are obvious. Let $k \ge 0$ satisfy the assertions and suppose $s \in C$ satisfies

$$\| s - x_0 \| \le \sum_{i\in I} k_i h_i + h_r, (k_i \in \mathbb{Z}^+, \Sigma k_i = k; r \in I).$$

Choosing s' on the line segment $x_0 s$ such that $\| s' - s_0 \| \le \sum_1^n k_i h_i$ and $\| s - s' \| \le h_r$, one easily sees that $|f(s)| \le \sum_1^n k_i w_i + w_r$. ∎

Corollary 2.6. *Let C and x_0 be as above and consider $f : C \to \mathbb{R}$ with the properties:*

$$(2.6.1) \qquad \begin{array}{l} f(x_0) = 0 \text{ and} \\ \| s - t \| \leq h \Rightarrow |f(s) - f(t)| \leq w; w, h > 0 \,. \end{array}$$

Then there is a maximal such function ϕ, namely

$$(2.6.2) \qquad \phi(t) = \lceil \| t - x_0 \| / h \rceil w \,.$$

2.7. An auxiliary function

Let $h > 0$ be fixed. We shall often use the even function defined by

$$(2.7.1) \qquad \phi_n(x) = \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} dt, \ (x \in \mathbb{R}) \,.$$

Equivalently,

$$(2.7.2) \qquad \phi_n(x) = \int_0^{|x|} \int_0^{x_1} \cdots \left(\int_0^{x_{n-1}} \left\lceil \frac{x_n}{h} \right\rceil dx_n \right) \ldots dx_1 \,.$$

Since $\lceil \frac{t}{h} \rceil = \sum_{j=0}^{\infty} 1_{j<th}$, the latter yields that

$$(2.7.3) \qquad \phi_n(x) = \frac{1}{n!} \left(\sum_{j=0}^{\infty} (|x| - jh)_+^n \right).$$

In particular, letting $k = \lceil |x|/h \rceil$,

$$(2.7.4) \qquad \phi_1(x) = \sum_{j=0}^{\infty} (|x| - jh)_+ = \sum_{j=0}^{k-1} (|x| - jh) = k|x| - \frac{1}{2} k(k-1)h \,.$$

Maximizing over $k \geq 0$, (attained if $k = 1/2 + |x|/h$), it follows that

(2.7.5) $$\phi_1(x) \le \Phi_{*1}(x) = \frac{1}{2h}(|x| + h/2)^2 .$$

In fact, $\phi_1(x_k) = \phi_{*1}(x_k) = \frac{1}{2} hk^2$ and $\phi_1'(x_k) = \phi_{*1}'(x_k) = k$ at $x_k = (k - \frac{1}{2})h$, $(k = 1, 2, 3, \dots)$. Further, one easily gets

(2.7.6) $$\phi_n(x) \le \phi_{*n}(x) = \left(\frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!} \right)$$

with equality only at $x = 0$. Note from (2.7.3) that ϕ_n is a (polynomial) *spline* function. In each interval $((j-1)h, jh]$ equals a polynomial of degree n. At the points $jh (j = 0, 1, \dots)$ the derivatives $D^k \varphi_n$ $(k = 0, 1, \dots, n-1)$ are continuous while the nth derivative makes an upward jump of size 1. Moreover, $\phi_n(x)$ is convex on $\mathbb{R}$ and strictly increasing on $\mathbb{R}^+$, $(n \ge 1)$. Finally

(2.7.7) $$\phi_n(x) = \int_0^x \phi_{n-1}(t) dt, \; (x \in \mathbb{R}^+ \;, \, n \ge 1)$$

provided we define $\phi_0(t) = \lceil \frac{t}{h} \rceil$.

II. BEST UPPER BOUNDS AND RELATED RESULTS

Using moment theory methods we obtain the following results.

Theorem 2.8. *Let μ be a finite measure of mass m on the interval $[a,b]$ where $0 \in [a,b]$. Let $c = \max(|a|, b)$. Suppose further that*

$$(\int |t|^r \mu(dt))^{1/r} = d\,, \tag{2.8.1}$$

where $r > 0$ and $d > 0$ are given. In order that μ exists, we also assume that $d^r \leq mc^r$. Next, consider $f : [a,b] \to \mathbb{R}$ satisfying

$$|f(s) - f(t)| \leq w \quad \text{when } s,t \in [a,b]; |s-t| \leq h\,. \tag{2.8.2}$$

Here, $h > 0$ and $w > 0$ are fixed. Then the best possible constant $K = K(m,r,d,h,w,f(0))$ in the inequality

$$|\int f d\mu - f(0)| \leq |m-1|\,|f(0)| + mK \tag{2.8.3}$$

is given as follows, (and is independent of $f(0)$). Here $n = \lceil \frac{c}{h} \rceil$, $k = \lceil \frac{d}{hm^{1/r}} \rceil$. Since $d^r \leq mc^r$ one has $1 \leq k \leq n$.

(i) $K = nw$ *when* $k = n$, *that is, when* $d/m^{1/r} > c - h$.

(ii) $K = [1 + \frac{1}{m}(\frac{d}{h})^r (n-1)^{1-r}]w$

when $r \leq 1$ *and* $k < n$.

(iii) $K = (k + \Theta_k)w \leq [1 + d/(hm^{1/r})]w$

when $r \geq 1$ and $k < n$. *Here*

$$\Theta_k = [d^r/m - (k-1)^r h^r]/[k^r h^r - (k-1)^r h^r] .$$

The equality sign in (2.8.3) is usually not attained but can be approached arbitrarily closely by the function $f(t) = f(0) + \epsilon\lceil \frac{t}{h} \rceil w$ *(with* $\epsilon = \pm 1$ *of the same sign as* $(m-1)f(0)$*) and a measure* μ *of mass* m *supported by a single point in case (i), by at most two points* 0 *and* t^* *in case (ii), (with absolute value slightly to the right of* $(n-1)h$*). And by at most two points* t_1 *and* t_2 *in case (iii), (with absolute value slightly to the right of* $(k-1)h$ *and* kh*, respectively).*

Proof: Let $g(t) = f(t) - f(0)$. From Corollary 2.6, we have

$$|g(t)| \leq \phi(t) = \lceil |t|/h \rceil w .$$

Thus

$$|\int f d\mu - f(0)| = |\int g d\mu + (m-1)f(0)| \leq \int \phi d\mu + |m-1|\,|f(0)| .$$

Here, the equality sign obtains when f is of the form $f_0(t) = f(0) + \epsilon\phi(t)$ with $\epsilon = \pm 1$ of the same sign as $(m-1)f(0)$. One easily verifies that f_0 also satisfies (2.8.2). Thus, the best constant K in (2.8.3) is given by

$$mK = \sup_{\mu} \int \phi d\mu$$

where μ ranges over the measures on $[a, b]$ of mass m which satisfy (2.8.1). Introducing the probability measure $d\nu = m^{-1} d\mu$, we have

$$K = \sup_{\nu} \int \phi(t)\nu(dt)$$

where ν ranges through the probability measures on $[a, b]$ satisfying

$$\int |t|^r \nu(dt) = d^r/m\,.$$

Note that both $\phi(t)$ and $|t|^r$ are even functions on $[a, b]$. It follows (see [17], [18]) that $K = \psi(d^r/m)$, where $\Gamma_1 = \{(u, \psi(u)) : 0 \leq u \leq c^r\}$ describes the upper boundary of the convex hull conv Γ_0 of the curve

$$\Gamma_0 = \{(t^r, \phi(t)) : 0 \leq t \leq c\}, (\text{where } c = \max(|a|, b))\,.$$

Note that Γ_0 consists of the following parts:

(i) The origin $(0, 0)$ corresponding to $t = 0$.

(ii) For $k = 1, \ldots, n - 1$, the half open horizontal line segments $(P_k, Q_k]$ where $P_k = ((k-1)^r h^r, kw)$ and $Q_k = (k^r h^r, kw)$.

(iii) The half open non-empty horizontal line segment $(P_n, Q_*]$, where $Q_* = (c^r, nw)$. It is always a part of the upper boundary Γ_1 of conv Γ_0, yielding assertion (i) of the theorem.

The line segment from P_k to P_{k+1} has a slope $wh^{-r}/[k^r - (k-1)^r]$ which is *decreasing* in k when $r \geq 1$ and *increasing* in k when $0 < r \leq 1$, (since the latter denominator has derivative $r[k^{r-1} - (k-1)^{r-1}]$). Consequently, if $0 < r \leq 1$ then Γ_1 consists of the two line segments $[P_1, P_n]$ and $[P_n, Q_*]$. Thus

$\psi(u) = w[1 + uh^{-r}(n-1)^{-r+1}]$ when $0 \le u \le (n-1)^r h^r$ while $\psi(u) = nw$ when $(n-1)^r h^r \le u \le c^r$. This yields assertion (ii). On the other hand, if $r \ge 1$ then Γ_1 is composed of the line segments $[P_k, P_{k+1}]$ $(k = 1, \ldots, n-1)$ together with the horizontal line segment $[P_n, Q_*]$. This easily yields assertion (iii). The last assertions of the theorem easily follow from the geometry of conv Γ_0. The fact that this sharp bound (2.8.3) is usually not attained derives from the fact that $[P_k, P_{k+1}]$ belongs to the closure of conv Γ_0 but not to conv Γ_0 itself.∎

Corollary 2.9. *If $r \ge 1$ then*

$$|\int f\,d\mu - f(0)| \le |m-1|\,|f(0)| + w(m + \frac{d}{h} m^{1-(1/r)}). \tag{2.9.1}$$

For $r = 2$ the latter is the inequality (2.1.1).

Note: Obviously, when $r = 2$ inequality (2.8.3) is sharper than (2.1.1).

By a similar reasoning, one obtains the following result.

Proposition 2.10. *Let $[a, b]$ be a closed finite interval containing 0 and put $c = \max(|a|, b)$ and $L = b - a$. Let $\Phi : [-L, +L] \to \mathbb{R}^+$ be an even nonnegative function, $\Phi(0) = 0$, which is subadditive on $[0, L]$ (e.g. Φ could be non-decreasing and concave). Consider*

$$\mathcal{F} = \{f : [a, b] \to \mathbb{R} \text{ and } |f(s) - f(t)| \le \Phi(s - t) \text{ for } s, t \in [a, b]\}$$

Let $\mathcal{F}_0 = \{f \in \mathcal{F} : f(0) = 0\}$.

Clearly $\Phi \in \mathcal{F}_0$ and $|f| \le \Phi$ for all $f \in \mathcal{F}_0$ thus

$$\sup_{\mathcal{F}_0} \left| \int f d\nu \right| = \int \Phi d\nu \,,$$

for each probability measure ν on $[a, b]$.

Suppose ν is further restricted by the moment condition

$$\int \Psi(t)\nu(dt) = \rho$$

where Ψ is a given continuous even and nonnegative function on $[-c, +c]$, which is strictly increasing on $[0, c]$, $\Psi(0) = 0$. Let

$$E = \sup_{\mathcal{F}_0, \nu} \left| \int f d\nu \right| .$$

(i) If $\Phi(\Psi^{-1})$ is convex on $[0, \Psi(c)]$ then $E = \rho\Phi(c)/\Psi(c)$.

(ii) If $\Phi(\Psi^{-1})$ is concave on $[0, \Psi(c)]$ then $E = \Phi(\Psi^{-1}(\rho))$.

Remark 2.11. *If $\Phi(t) = k|t|^\alpha$ $(0 < \alpha \leq 1, k > 0)$, $\Psi(t) = |t|^r$ $(r > 0)$ then (i), (ii) happen when $r \leq \alpha$ and $r \geq \alpha$, respectively.*

Remark 2.12. *We maintain the notations of Proposition 2.10. Let μ be a measure on $[a, b]$ with mass m and moment*

$$\int \Psi(t)\mu(dt) = m\rho \,.$$

Then each $f \in \mathcal{F}$ satisfies

$$\left| \int f \, d\mu - f(0) \right| \leq |f(0)| \, |m - 1| + m E \,.$$

This inequality is sharp and in fact is attained by $f(t) = f(0) + \epsilon\Phi(t)$ and a suitably chosen measure supported by at most two points; ($\epsilon = \pm 1$ equals the sign of $(m-1)f(0)$).

The following is well known.

Lemma 2.13. *Be given a closed finite interval $[a, b]$ and fixed $x_0 \in [a, b]$. Consider $f \in C^n([a, b])$, $n \geq 1$ and denote $\phi(x) = f^{(n)}(x) - f^{(n)}(x_0)$. Then, $(a \leq x \leq b)$*

(2.13.1)
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \int_{x_0}^{x}\Big(\int_{x_0}^{x_1}\Big(\cdots\Big(\int_{x_0}^{x_{n-1}} \phi(x_n)dx_n\Big)dx_{n-1}\Big)\cdots\Big)dx_1$$

and

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \int_{x_0}^{x} \phi(t)\frac{(x-t)^{n-1}}{(n-1)!}dt\,. \tag{2.13.2}$$

As a related result we give

Theorem 2.14. *Let μ be a measure of mass $m > 0$ on $[a, b]$ and $x_0 \in [a, b]$ fixed. Let n be a fixed positive integer and put*

$$h = [\int |t - x_0|^{n+1}\mu(dt)]^{1/(n+1)}\,. \tag{2.14.1}$$

Suppose $f \in C^n([a, b])$ satisfies

$$|f^{(n)}(s) - f^{(n)}(t)| \leq w \text{ if } a \leq s, t \leq b \text{ and } |s - t| \leq h\,, \tag{2.14.2}$$

where w is a given positive number. Then

(2.14.3)

$$|\int f d\mu - f(x_0)| \le |f(x_0)|\,|m-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(x_0)|}{k!} |\int (t-x_0)^k \mu(dt)| + \frac{wh^n}{n!}(m^{1/(n+1)} + 1/(n+1)).$$

Remark. *Note that the case $n = 1$ is precisely Proposition 2.2.*

Proof: Without loss of generality let $x_0 = 0$. From (2.14.2) $\phi(x) = f^{(n)}(x) - f^{(n)}(0)$ satisfies $|\phi(s) - \phi(t)| \le w$ when $|s-t| \le h$, therefore $|\phi(t)| \le w\lceil \frac{|t|}{h} \rceil$ by Corollary 2.6. It follows from (2.13.2) and (2.7.1) that

$$|f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k| \le w\phi_n(x).$$

From $\lceil \frac{t}{h} \rceil \le 1 + \frac{t}{h}$ and (2.7.1),

$$w\phi_n(x) \le \frac{w|x|^n}{n!}(1 + \frac{|x|}{(n+1)h}). \tag{2.14.4}$$

Integrating relative to μ and using Holder's inequality we obtain (2.14.3).∎

Proposition 2.15. *Let $f \in C^n([-\pi,\pi])$, $n \ge 1$ and μ a measure on $[-\pi,\pi]$ of mass $m > 0$. Put*

$$\beta = \left(\int (\sin \frac{|t|}{2})^{n+1} \mu(dt) \right)^{1/(n+1)} \tag{2.15.1}$$

and denote by $w = \omega_1(f^{(n)}, \beta)$ the modulus of continuity of $f^{(n)}$ at β. Then

$$\begin{aligned} |\int f d\mu - f(0)| \le |f(0)|\,|m-1| &+ \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!} |\int t^k \mu(dt)| \\ &+ w\Big[m^{1/(n+1)} + \pi/(n+1)\Big]\frac{\pi^n \beta^n}{n!}. \end{aligned} \tag{2.15.2}$$

Proof: Analogous to the proof of Theorem 2.14, using the fact $|t| \le \pi \sin(|t|/2)$.∎

Remark. *The special case $n = 1$ is precisely Proposition 2.4.*

Proposition 2.16. *With the notations and assumptions of Proposition 2.15, we obtain*

$$\begin{aligned} |\int f d\mu - f(0)| \le |f(0)|\,|m-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!} |\int t^k \mu(dt)| \\ + w\pi^n \beta^n \Big(\frac{\pi}{(n+1)!} + \frac{m^{1/(n+1)}}{2n!} + \frac{m^{2/(n+1)}}{8(n-1)!\pi} \Big) \end{aligned} \tag{2.16.1}$$

Proof: Analogous to the proof of Proposition 2.15 by using (2.7.6) instead of (2.14.4).∎

When $m \le (\frac{4\pi}{n})^{n+1}$ inequality (2.16.1) is *sharper* than (2.15.2) and in particular for the very common case of $m = 1$; $n = 1$ inequality (2.16.1) *is sharper* than inequality (2.4.1).

The analogous modification of Theorem 2.14 will be discussed in section IV.

III. SHARP INEQUALITIES

The following optimal results are obtained by using standard moment methods (see [17], [18]).

Theorem 2.17. *Let μ be a finite measure on $[a,b] \subset \mathbb{R}$ $0 \in (a,b)$ and $|a| \leq b$. Put*

$$(2.17.1) \qquad c_k = \int t^k \mu(dt),\ k = 0,1,\dots,n\,;\ d_n = \Big(\int |t|^n \mu(dt)\Big)^{1/n}$$

Let $f \in C^n([a,b])$ be such that

$$(2.17.2) \qquad |f^{(n)}(s) - f^{(n)}(t)| \leq w \ \text{ if } \ a \leq s,t \leq b \ \text{ and } \ |s-t| \leq h$$

where w, h are given positive numbers.

Then we have the following upper bound

$$(2.17.3) \quad |\int f\, d\mu - f(0)| \leq |f(0)|\,|c_0 - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}_{(0)}|}{k!}|c_k| + w\phi_n(b)\Big(\frac{d_n}{b}\Big)^n.$$

The above inequality is in a certain sense attained by the measure μ with masses $[c_0 - (d_n/b)^n]$ and $(d_n/b)^n$ at 0 and b respectively and when, moreover the optimal function is

$$(2.17.4) \qquad \tilde{f} = \begin{cases} w\phi_n\,, \text{on } [0,b]\,; \\ 0\,, \text{on } [a,0]\,. \end{cases}$$

Namely, the latter is the limit of a sequence of functions f having continuous nth derivatives satisfying (2.17.2) and $f^{(k)}(0) = 0$ $(k = 0,\dots,n)$ and such that *the difference of the two sides of (2.17.3) tends to 0. In fact,* $\lim_{N\to+\infty} f_{nN}(t) = \tilde{f}(t)$, *where* $(a \le t \le b)$

$$f_{nN}(t) = w\int_0^t \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-1}} f_{0N}(t_n)dt_n\right)\cdots\right)dt_1 . \tag{2.17.5}$$

Here, for $k = 0, 1, \ldots, \lceil b/h\rceil - 1$ *and* $N \ge 1$ f_{0N} *is the continuous function defined*

$$f_{0N}(t) = \begin{cases} 0, & \text{if } a \le t < 0\,; \\ \frac{Nwt}{2h} + kw(1 - \frac{N}{2}), & \text{if } kh \le t \le (k + \frac{2}{N})h\,; \\ (k+1)w\,, & \text{if } (k + \frac{2}{N})h < t \le (k+1)h\,; \\ \lceil b/h\rceil w\,, & (\lceil b/h\rceil - 1 + \frac{2}{N})h < t \le b\,. \end{cases} \tag{2.17.6}$$

Observe that $f_{0N}(t)$ *fulfills (2.17.2) and further*

$$\lim_{N\to+\infty} f_{0N}(t) = \lceil t/h\rceil w\,, \quad t \in [0,b]\,;$$
$$0\,, \quad t \in [a,0]\,.$$

Proof: From (2.13.2), integrating relative to μ get

$$|\int f d\mu - f(0)| \le |f(0)|\,|c_0 - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!}|c_k| + S_n$$

where

$$S_n = w\int \phi_n(t)\mu(dt)$$

We would like to maximize S_n given that μ has preassigned moments c_0 and $d_n = [\int |t|^n \mu(dt)]^{1/n}$. Since the functions on hand $|t|^n$ and $\phi_n(t)$ are both even we are essentially concerned with a measure on $[0,b]$ (using that $|a| \le b$). As usually, consider the curve defined by $u = t^n$ and $v = \phi_n(t)$, that is $v = \phi_n(u^{1/n})$ where $u \ge 0$. Here

$$\phi_n(u^{1/n}) = \frac{1}{n!}\left(\sum_{j=0}^{\infty}(u^{1/n} - jh)_+^n\right).$$

The function $(u^{1/n} - jh)_+^n$ has its first derivative equal to $(1 - jh/u^{1/n})_+^{n-1}$ which is obviously increasing in u. It follows that $\phi_n(u^{1/n})$ is convex. Consequently, the integral $\int \phi_n(t)\mu(dt)$ is maximized by a measure taking values at 0 and b only. Let μ have masses p and q at 0 and b respectively. Thus $p \geq 0$, $q \geq 0$ while $p + q = c_0$. Further, $0 + qb^n = d_n^n$, thus $q = (d_n/b)^n$. Consequently, $\max S_n = w\phi_n(b)q = w\phi_n(b)(d_n/b)^n$. ∎

Let L be a positive linear operator from $C^n([a,b])$ into $C([a,b])$. It follows from the Riesz reprsentation theorem, for every $x \in [a,b]$ there is a finite nonnegative measure μ_x such that

$$L(f,x) = \int f(t)\mu_x(dt)\,, \text{ for all } f \in C^n([a,b])\,.$$

Naturally, the converse is not true. That is, only special kernels $\mu_x(\cdot)$ will transform continuous functions into continuous functions.

Corollary 2.18. *Consider the positive linear operator*

$$L : C^n([a,b]) \to C([a,b]), n \in \mathbb{N}\,.$$

Let

$$\begin{aligned} c_k(x) &= L((t-x)^k, x),\ k = 0, 1, \ldots, n\,; \\ d_n(x) &= [L(|t-x|^n, x)]^{1/n}\,; \\ c(x) &= \max(x-a, b-x),\ (c(x) \geq (b-a)/2)\,. \end{aligned} \tag{2.18.1}$$

Let $f \in C^n([a,b])$ such that $\omega_1(f^{(n)}, h) \leq w$, where w, h are fixed positive numbers, $0 < h < b - a$. Then we have the upper bound

$$(2.18.2) \qquad |L(f,x)-f(x)| \le |f(x)|\,|c_0(x)-1| + \sum_{k=1}^{n} \frac{|f^{(k)}(x)|}{k!}|c_k(x)| + R_n\,.$$

Here

$$(2.18.3) \qquad R_n = w\phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n = \frac{w}{n!}\Theta_n(h/c(x))d_n^n(x)\,,$$

where

$$\Theta_n(h/u) = n!\phi_n(u)/u^n\,.$$

The above inequality is sharp. Analogous to Theorem 2.17, it is in a certain sense attained by $w\phi_n((t-x)_+)$ and a measure μ_x supported by $\{x,b\}$ when $x-a \le b-x$, also attained by $w\phi_n((x-t)_+)$ and a measure μ_x supported by $\{x,a\}$ when $x-a \ge b-x$: in each case with masses $c_0(x)-(d_n(x)/c(x))^n$ and $(d_n(x)/c(x))^n$ respectively.

Proof: Apply Theorem 2.17 with 0 shifted to x.∎

2.19 Explicit upper bounds for $n = 1$

Here we find simpler useful upper bounds for the remainder term of (2.18.2) *i.e.* upper bounds for $\phi_n(u)/u^n$. Typically $u = c(x)$ as above and thus satisfies $\frac{1}{2}(b-a) \le u \le b-a$.

From (2.18.3) we have

$$(2.19.1) \qquad n!\,\phi_n(u)/u^n = \sum_{j=0}^{\infty}(1-jh/u)_+^n = \Theta_n(h/u)\,,$$

where $v = h/u$ satisfies $0 < v < 2h/(b-a)$, thus v is usually small. Note

(2.19.2)
$$\Theta_n(v) = \sum_{j=0}^{\infty} (1 - jv)_+^n .$$

Let us study now the case $n = 1$. It is not difficult to show that, for v small,

(2.19.3)
$$\Theta_1(v) \le (2v)^{-1} + 2^{-1} + O(v).$$

where $O(v) \doteq v/8$. Thus taking $v = h/c(x)$, (2.18.2) with $n = 1$ yields that

(2.19.4)
$$|L(f,x) - f(x)| \le |f(x)|\,|c_0(x) - 1| + |f'(x)|\,|c_1(x)| + R_{*1},$$

where

$$R_{*1} = w d_1(x)\left[\frac{c(x)}{2h} + \frac{1}{2} + O\left(\frac{h}{c(x)}\right)\right];$$

(here, the last term $\doteq h/(8c(x))$). Hence, when $h = rd_1(x)$, $r > 0$ constant the remainder term in (2.19.4) is equal to

$$w\left[\frac{c(x)}{2r} + \frac{d_1(x)}{2} + O\left(\frac{rd_1^2(x)}{c(x)}\right)\right].$$

In many applications $f'(x)$ is Lipschitz with exponent $\alpha(0 < \alpha \le 1)$ thus $w \doteq Ah^\alpha$, with A as a positive constant. The h minimizing the right hand side of (2.19.4), is approximately given by

$$h_{opt.} = 2c(x)\left(\frac{1-\alpha}{1+\alpha}\right), \text{ for } 0 < \alpha \le 1.$$

The resulting remainder would be

$$R_{*1} \doteq A\,d_1(x)2^\alpha(c(x))^\alpha \frac{(1-\alpha)^{\alpha-1}}{(1+\alpha)^{\alpha+1}}, 0 < \alpha \le 1.$$

2.20 Explicit upper bounds for $n \geq 1$

From (2.19.2) for $(N+1)^{-1} \leq v \leq N^{-1}$, $N \in \mathbb{N}$ one has

(2.20.1) $$\Theta_n(v) = \sum_{j=0}^{N} \overline{f}(j), \text{ where } \overline{f}(x) = (1-vx)^n .$$

It follows from the Euler-MacLaurin summation formula (formula 298 in [21]) that in the same case:

(2.20.2) $$\begin{aligned}\Theta_n(v) = & \frac{1}{(n+1)v}\Big[1-(1-Nv)^{n+1}\Big] + \frac{1}{2}[1+(1-Nv)^n] \\ & + \sum_{1\leq k\leq n/2} \frac{B_{2k}}{(2k)!}\,\frac{n!}{(n-2k+1)!} v^{2k-1}\Big[1-(1-Nv)^{n+1-2k}\Big],\end{aligned}$$

where N is the integral part of v^{-1} and $0 \leq 1-Nv \leq v$. Here the Bernoulli numbers are: $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, $B_{14} = 7/6$ etc. From (2.18.2) and (2.19.1), we are interested in $\Theta_n(v)$ for values $v = h/c(x)$, where $\frac{1}{2}(b-a) \leq c(x) \leq b-a$, while h is often small, hence so is v. Note that (2.20.2) implies that

(2.20.3) $$\Theta_n(v) = \frac{1}{(n+1)v} + \frac{1}{2} + \sum_{1\leq k\leq n/2} a_k v^{2k-1} + O(v^n)$$

as $v \to 0$, where

(2.20.4) $$a_k = (n+1)^{-1}\binom{n+1}{2k} B_{2k} .$$

In particular, $\lim_{v\to 0}\Theta_n(v) = \infty$. In connection with (2.18.2), observe that from Hölder's inequality $|c_k(x)| \leq d_n^k(x)$, where $d_n(x)$ is usually small. Moreover, R_n in (2.18.2) is approximately equal to $\frac{1}{(n+1)!}\frac{w}{h}c(x)d_n^n(x)$.

If one chooses $h = rd_n(x)$ then

$$R_n \doteq \frac{c(x)}{n!r(n+1)}\omega_1(f^{(n)}, rd_n(x))(d_n(x))^{n-1},$$

assuming that $d_n(x)$ is small.

The next lemma will be used in Theorem 2.22.

LEMMA 2.21. *Be given $[a,b] \subset \mathbb{R}$ and $x_0 \in (a,b)$ fixed, consider all measures μ with prescribed moments*

(2.21.1)

$$\mu([a,b]) = c_0 > 0; \int (t - x_0)\mu(dt) = c_1(x_0), \int |t - x_0|\mu(dt) = d_1(x_0) > 0.$$

For $w, h > 0$ $(0 < h < b - a)$ as given numbers, put $M(x_0) = \sup_\mu \int \left(\frac{w}{c_0}\right)\phi_1(|t - x_0|)\mu(dt)$.

Then

(2.21.2)

$$M(x_0) = w\phi_1(b - x_0)\left(\frac{d_1(x_0) + c_1(x_0)}{2c_0(b - x_0)}\right) + w\phi_1(x_0 - a)\left(\frac{d_1(x_0) - c_1(x_0)}{2c_0(x_0 - a)}\right).$$

The optimal measure is carried by $\{a, x_0, b\}$.

PROOF: Easy.∎

The assertion of Theorem 2.17 can be improved if more is known about μ. One result in this direction is the following.

THEOREM 2.22. *Let $[a,b] \subset \mathbb{R}$, $x_0 \in (a,b)$ and consider all measures μ on $[a,b]$ such that*

(2.22.1)

$$\mu([a,b]) = c_0 > 0\,; \int (t-x_0)\mu(dt) = c_1(x_0), \int |t-x_0|\mu(dt) = d_1(x_0) > 0\,.$$

Further, consider $f \in C^1([a,b])$ with $w_1(f', h) \leq w$ where w, h are given positive numbers ($0 < h < b-a$).

Then, we get the best upper bound

$$(2.22.2) \quad \left|\int f\,d\mu - f(x_0)\right| \leq |f(x_0)|\,|c_0 - 1| + |f'(x_0)|\,|c_1(x_0)| + c_0 M(x_0)\,,$$

where $M(x_0)$ is given by (2.21.2).

Proof: Easy. ∎

IV. NEARLY SHARP INEQUALITIES

Here we establish some good inequalities with explicit constants better than those in the literature. They involve the first modulus of continuity, or its (smaller) modification, of the nth derivative of $f \in C^n([a,b])$, $n \geq 1$ evaluated at $h = r d_{n+1}(x_0)$; where $d_{n+1}(x_0) = \left(\int |t-x_0|^{n+1}\mu(dt)\right)^{1/(n+1)}$, $x_0 \in [a,b]$.

The following is a refinement of a result due to B. Mond and R. Vasudevan (see [28]).

Theorem 2.23. *Consider a closed interval $[a,b] \subset \mathbb{R}$ and a given point $x_0 \in [a,b]$, further a measure μ on $[a,b]$ of mass $m > 0$ satisfying*

$$\int (t-x_0)\mu(dt) = 0\,;\ \left(\int (t-x_0)^2\mu(dt)\right)^{1/2} = d_2(x_0) > 0\,.$$

Then for $f \in C^1([a,b])$ and $r > 0$,

$$\text{(2.23.1)}\quad \left|\int f\,d\mu - f(x_0)\right| \le |f(x_0)|\,|m-1| + (\sqrt{m}+\frac{1}{r})\omega_1(f', r d_2(x_0)) d_2(x_0)\,.$$

Proof: By the mean value theorem, there is $\xi \in (t, x_0)$ such that

$$\text{(2.23.2)}\qquad f(t) - f(x_0) = (t-x_0)f'(x_0) + (t-x_0)(f'(\xi) - f'(x_0))\,.$$

Then

$$\begin{aligned} &|f'(\xi) - f'(x_0)| \le \omega_1(f', |\xi - x_0|) \le \omega_1(f', |t-x_0|) \\ &\quad = \omega_1(f', |t-x_0|\delta^{-1}\delta) \le (1 + |t-x_0|\delta^{-1})\omega_1(f',\delta), (\text{ for all } \ \delta > 0)\,. \end{aligned}$$

Therefore $|f'(\xi) - f'(x_0)| \le (1+|t-x_0|\delta^{-1})\omega_1(f',\delta)$, for all $\delta > 0$. Multiplying by $|t-x_0|$, integrating relative to μ and applying (2.23.2) we have

$$\begin{aligned} &\left|\int f\,d\mu - mf(x_0)\right| \le |f'(x_0)|\,\left|\int (t-x_0)\mu(dt)\right| \\ &+ \left(\int |t-x_0|\mu(dt) + \left(\int (t-x_0)^2\mu(dt)\right)\delta^{-1}\right)\omega_1(f',\delta) \\ &\le |f'(x_0)|\,\left|\int (t-x_0)\mu(dt)\right| + \left[\left(\int (t-x_0)^2\mu(dt)\right)^{1/2}\sqrt{m} + \delta^{-1}\int (t-x_0)^2\mu(dt)\right]\cdot\omega_1(f',\delta)\,. \end{aligned}$$

Letting $\delta = r d_2(x_0)$, one obtains (2.23.1).∎

The following is a refinement of the main theorem of a paper due to H. H.

Gonska (1983), see [13].

Gonska's result is the latest improvement in this type of inequalities. Here $\mu(f) = \int f\, d\mu$.

THEOREM 2.24. *Let μ a measure on $[a, b]$ of mass $m > 0$. Consider $0 < h < b-a$ and $x_0 \in [a, b]$ and let $f \in C^1([a, b])$.*

Then

$$(2.24.1)\qquad \begin{aligned} &|\int f\, d\mu - f(x_0)| \leq |f(x_0)|\, |m-1| \\ &+ \omega_1(f', h)\{\mu(|t-x_0|) + \frac{1}{2h}\mu((t-x_0)^2)\} \\ &+ |\frac{1}{2h}\int_{-h}^{h} f'_*(x_0+u)du|\, |\mu(t-x_0)|, \end{aligned}$$

where

$$f'_*(t) = \begin{cases} f'(a), & t < a \\ f'(t), & t \in [a, b] \\ f'(b), & t > b\,. \end{cases}$$

REMARK. *Observe that $|\frac{1}{2h}\int_{-h}^{h} f'_*(x_0+u)du| \leq \| f' \|$, where $\| \cdot \|$ denotes the sup-norm over $[a, b]$.*

PROOF: If $g \in C^2([a, b])$, then clearly

$$(2.24.2)\quad |\int g\, d\mu - mg(x_0)| \leq |g'(x_0)|\, |\int (t-x_0)\mu(dt)| + \frac{\| g'' \|}{2}\int (t-x_0)^2 \mu(dt)\,.$$

Further,

$$|\mu(f) - f(x_0)m| \leq |\mu(f-g) - m(f-g)(x_0)| + |\mu(g) - g(x_0)m|\,.$$

Hence,

$$(2.24.3) \qquad \begin{aligned} |\mu(f) - f(x_0)m| &\leq \| (f-g)' \| \, \mu(|t - x_0|) \\ &+ |g'(x_0)| \, |\mu(t-x_0)| + \frac{\| g'' \|}{2} \mu((t-x_0)^2)\,. \end{aligned}$$

Let f'_h be the so-called *first Stekloff function* of f'; that is,

$$(f'_h)(t) = \frac{1}{2h} \int_{-h}^{h} f'_*(t+u)du, \ (a \leq t \leq b)\,.$$

By a well-known theory [35],

$$\| f' - (f')_h \| \leq \omega_1(f', h) \text{ and } \| (f')'_h \| \leq h^{-1}\omega_1(f', h)\,.$$

It is easy to find $g \in C^2([a,b])$ such that $g' = (f')_h$. Applying (2.24.3), one obtains

$$|\mu(f) - f(x_0)m| \leq \omega_1(f', h)\{\mu(|t-x_0|) + \frac{1}{2h}\mu((t-x_0)^2)\} + |\frac{1}{2h}\int_{-h}^{h} f'_*(x_0+u)du| \, |\mu(t-x_0)|\,.$$

∎

Corollary 2.25. *Let the closed interval $[a,b] \subset \mathbb{R}$ and $x_0 \in [a,b]$. Also consider a measure μ on $[a,b]$ of mass $m > 0$, such that*

$$\int (t-x_0)\mu(dt) = 0\,; \ (\int (t-x_0)^2 \mu(dt))^{1/2} = d_2(x_0) > 0\,.$$

Let $r > 0$ and $f \in C^1([a,b])$. Then

$$(2.25.1) \ |\int f \, d\mu - f(x_0)| \leq |f(x_0)| \, |m-1| + (\sqrt{m} + \frac{1}{2r})\omega_1(f', r\, d_2(x_0))d_2(x_0)\,.$$

Observe that (2.25.1) is sharper that (2.23.1).

Proof: By Schwarz's inequality $\mu(|t-x_0|) \leq (\mu((t-x_0)^2))^{1/2}\sqrt{m}$. Now apply (2.24.1) with $h = rd_2(x_0)$.∎

Using (2.7.5), we obtain the following result.

Theorem 2.26. *Let $f \in C^1([a,b])$ and μ be a measure on $[a,b]$ of mass $m > 0$ with given moments*

$$\int (t-x_0)\mu(dt) = 0\,; \left(\int (t-x_0)^2\mu(dt)\right)^{1/2} = d_2(x_0) > 0\,,$$

where $x_0 \in [a,b]$. Consider $r > 0$. Then

(2.26.1)

$$|\int f\,d\mu - f(x_0)| \leq |f(x_0)|\,|m-1| + \frac{1}{8r}(2+\sqrt{m}r)^2\omega_1(f', rd_2(x_0))d_2(x_0)\,.$$

Note. *If $x_0 = 0$ we could get a sharper inequality by using the modified modulus of continuity $\overline{\omega}_1$ instead of ω_1; where*

(2.26.2) $$\overline{\omega}_1(f',h) = \sup\{|f'(x)-f'(y)| : x\cdot y \geq 0\,,\ |x-y| \leq h\}\,.$$

Obviously, $\overline{\omega}_1 \leq \omega_1$.

Corollary 2.27. *In the special case of $m=1$, $x_0 = 0$ we have*

(2.27.1) $$|\int f\,d\mu - f(0)| \leq \overline{\omega}_1(f', r\,d_2)\frac{(2+r)^2}{8r}d_2\,,$$

(where $d_2 = d_2(0)$).

Proof of Theorem 2.26: Integrating (2.13.2) relative to μ we get

$$\int f\,d\mu - mf(x_0) = f'(x_0)\int (t-x_0)\mu(dt) + \int K_1(t\,,\,x_0)\mu(dt)\,, \tag{2.26.3}$$

where

$$K_1(t,x_0) = \int_{x_0}^{t} \phi(x)dx\,;\ \phi(x) = f'(x) - f'(x_0)\,.$$

Note that $\phi(x_0) = 0$ and $|\phi(x)| \leq \omega_1(f',h)\lceil\frac{|x-x_0|}{h}\rceil$ for all $h > 0$.

Hence,

$$|K_1(t,x_0)| \leq M_1(t,x_0) = \omega_1(f',h)\int_{x_0}^{t}\lceil\frac{x-x_0}{h}\rceil dx\,,$$

for all $t, x_0 \in [a,b]$. By (2.7.5), we obtain

$$|K_1(t,x_0)| \leq M_1(t,x_0) \leq \omega_1(f',h)\left[\frac{(t-x_0)^2}{2h} + \frac{|t-x_0|}{2} + \frac{h}{8}\right] \tag{2.26.4}$$

Now integrating (2.26.4) against μ, using Schwarz's inequality and setting $h = rd_2(x_0)$, we find:

$$\int |K_1(t,x_0)|\mu(dt) \leq \frac{1}{8r}(2+\sqrt{m}r)^2\omega_1(f',rd_2(x_0))d_2(x_0)\,. \tag{2.26.5}$$

Finally, from (2.26.5) and (2.26.3)

$$|\int f d\mu - mf(x_0)| \leq \frac{1}{8r}(2+\sqrt{m}r)^2\omega_1(f',rd_2(x_0))d_2(x_0)\,.$$

Since

$$|\int f d\mu - f(x_0)| \leq |f(x_0)|\,|m-1| + |\int f d\mu - f(x_0)m|$$

the theorem follows.∎

Remark 2.28. *When* $\omega_1(f',h) = Ah^{\alpha}$, $0 < \alpha \leq 1$ *and* $A > 0$ *constant, the value* *of $r > 0$ minimizing the right hand side of (2.26.1) is given by*

$$r = 2(1-\alpha)/(\sqrt{m}(1+\alpha)). \tag{2.28.1}$$

If $\alpha = 1$ then letting $r \downarrow 0$ one obtains that

$$\left|\int f\,d\mu - f(x_0)\right| \le |f(x_0)|\,|m-1| + \frac{1}{2}Ad_2^2(x_0), \tag{2.28.2}$$

where

$$A = \sup_{s \neq t}\{|f'(s) - f'(t)|/|s-t|\}.$$

When $|a| = b$ and $x_0 = 0$, the last inequality is attained by $f(x) = x^2$ and a measure μ with masses $m/2$ at $\pm b$; (both sides are then equal mb^2).

REMARK 2.29. *When $m = 1$, $|a| = b$, $x_0 = 0$, $r = 2$ and α is small, the inequality (2.26.1) is nearly attained by $f(x) = |x|^{1+\alpha}$ and μ with mass $1/2$ at $\pm b$.*

REMARK 2.30. *With $m = 1$, $|a| = b$, $x_0 = 0$, the measure μ having mass $1/2$ at $\pm b$ and $f(x) = |x|^{1+\alpha}$ $(0 < \alpha \le 1)$, the left hand side of (2.26.1) equals $a^{1+\alpha}$, while the right hand side equals $(1+\alpha)\frac{r^{\alpha-1}}{8}(2+r)^2 a^{1+\alpha}$. Minimizing over r the right hand side becomes $C(\alpha)a^{1+\alpha}$, where $C(\alpha) = 2^\alpha(1-\alpha)^{-1+\alpha}(1+\alpha)^{-\alpha}$. The quantity $\ell n C(\alpha)$ is a concave function of α taking its largest value at $\alpha = 0.580332$ and there $C(\alpha) = 1.650485$. Further*

$$\begin{array}{llll} C(.01) & = 1.016923 & C(.6) & = 1.649385 \\ C(.05) & = 1.084313 & C(.7) & = 1.607942 \\ C(.1) & = 1.167200 & C(.8) & = 1.501667 \\ C(.2) & = 1.324023 & C(.9) & = 1.318405 \\ C(.3) & = 1.46069 & C(.95) & = 1.189863 \\ C(.4) & = 1.56700 & C(.99) & = 1.052338 \\ C(0.5) & = 1.63299 & C(.999) & = 1.0074349. \end{array}$$

So we see that (2.26.1) is never far off, in that it is attained up to a factor 1.65 at most.

Note. *(i) If $0 < r \leq 4/\sqrt{m}$, then inequality (2.26.1) is sharper than inequality (2.25.1). (ii) If $r \geq 4/\sqrt{m}$, then (2.25.1) is sharper than (2.26.1). Because of (2.28.1) case (i) is probably more interesting.*

In terms of best constants, in this type of inequalities, the next result improves all the related results we are aware of.

Corollary 2.31. *Let $x_0 \in [a,b]$ and the measure μ on $[a,b]$ of mass $m > 0$ satisfy the moment conditions*

$$\int (t - x_0)\mu(dt) = 0 \text{ and } \acute{d}_2(x_0) = \left(\int (t - x_0)^2 \mu(dt)\right)^{1/2}.$$

Consider $r > 0$ and $f \in C^1([a,b])$. Then

$$\text{(2.31.1)} \quad \left|\int f\,d\mu - f(x_0)\right| - |f(x_0)|\,|m-1| \leq \frac{1}{8r}(2+\sqrt{m}r)^2 \omega_1(f', r d_2(x_0)) d_2(x_0), \quad \text{if } r \leq 2/\sqrt{m};$$
$$\leq \sqrt{m}\,\omega_1(f', r d_2(x_0)) d_2(x_0), \quad \text{if } r > 2/\sqrt{m}.$$

When $x_0 = 0$, we get a sharper estimate by replacing ω_1 by $\overline{\omega}_1$, (see (2.26.2)).

Proof: Note that the first part of (2.31.1) follows from (2.26.1). If $r \geq$

$2/\sqrt{m}$ then apply (2.26.1) with r replaced by $r_1 = 2/\sqrt{m}$ and note that $\frac{1}{8r_1}(2+\sqrt{m}\,r_1)^2 = \sqrt{m}$.∎

Taking $r = 1/2$ in (2.26.1) one obtains:

Theorem 2.32. *Let the random variable Y have distribution μ, $E(Y) = x_0$ and $Var(Y) = \sigma^2$. Consider $f \in C_B^1(\mathbb{R})$. Then*

$$(2.32.1) \qquad |Ef(Y) - f(x_0)| = |\int f\,d\mu - f(x_0)| \le (1.5625)\omega_1(f', \frac{1}{2}\sigma)\sigma\,.$$

The last inequality is stronger than the corresponding results by B. Mond–R. Vasudevan [28] and J. P. King [20].

2.33 Application

Consider X_j real i.i.d. random variables and put $S_n = \sum_{j=1}^{n} X_j$, $n \ge 1$. Let $x_0 = E(X)$, $\sigma^2 = \mathrm{Var}(X)$ thus $E(S_n/n) = x_0$ and $\mathrm{Var}(S_n/n) = \sigma^2/n$. Denote F_{S_n} the d.f. of S_n. Then (2.32.1) yields

(2.33.1)

$$|Ef(S_n/n) - f(x_0)| = |\int_{\mathbb{R}} f(t/n)dF_{S_n}(t) - f(x_0)| \le (1.5625)\omega_1(f', \frac{\sigma}{2\sqrt{n}})\frac{\sigma}{\sqrt{n}}\,.$$

The following Corollaries 2.34, 2.35, 2.36 and 2.37 are applications of (2.33.1) to well-known positive linear operators arising from probability theory. The corollaries about the Baskakov, Szász-Mirakjan and Weierstrass operators are improvements of corresponding results contained in Z. Ditzian [9] and S. P. Sing [30].

We start with the classical Bernstein polynomials.

Corollary 2.34. *For any $f \in C^1([0,1])$ consider $(B_n f)(t) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} t^k(1-t)^{n-k}, t \in [0,1]$. Then*

$$|(B_n f)(t) - f(t)| \leq (1.5625)\omega_1\left(f', \frac{1}{2}\sqrt{\frac{t(1-t)}{n}}\right)\sqrt{\frac{t(1-t)}{n}} \leq \left(\frac{0.78125}{\sqrt{n}}\right)\omega_1\left(f', \frac{1}{4\sqrt{n}}\right).$$

Proof: Consider $(X_j)_{j\in\mathbb{N}}$ Bernoulli (i.i.d) random variables such that $P(X_j = 0) = 1-t$, $P(X_j = 1) = t; t \in (0,1)$, then $E(X) = t$ and $\mathrm{Var}(X) = t(1-t)$. Now apply (2.33.1) with $x_0 = t$. Further, note that $\max_{0\leq t\leq 1}(t(1-t)) = 1/4$ at $t = 1/2$.∎

For $t \geq 0$ and $f \in C_B(\mathbb{R}^+)$ the Szász-Mirakjan operator is defined as

$$(M_n f)(t) = e^{-nt} \sum_{k=0}^{\infty} f(\frac{k}{n})\frac{(nt)^k}{k!}$$

while the Baskakov type operator is defined as

$$(V_n f)(t) = \sum_{k=0}^{\infty} f(\frac{k}{n})\binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

Both operators are of the form $E(S_n/n)$ above. Namely, X there has the distribution

$$P_X = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}\delta_k \text{ and } \acute{P}_X = \sum_{k=0}^{\infty}\left(\frac{1}{1+t}\right)\left(\frac{t}{1+t}\right)^k\delta_k,$$

(Poisson and geometric), respectively. In both cases, $E(X) = t$ while $\mathrm{Var}(X) = t$ and $\mathrm{Var}(X) = (t+t^2)$, respectively.

Thus (2.33.1) implies:

Corollary 2.35. *With the above notations, we have*

$$|(M_nf)(t) - f(t)| \le (1.5625)\omega_1\left(f', \frac{1}{2}\left(\frac{t}{n}\right)^{1/2}\right)\left(\frac{t}{n}\right)^{1/2}$$

and

$$|(V_nf)(t) - f(t)| \le (1.5625)\omega_1\left(f', \frac{1}{2}\left(\frac{t+t^2}{n}\right)^{1/2}\right)\left(\frac{t+t^2}{n}\right)^{1/2},$$

for all $f \in C_B^1(\mathbb{R}^+)$.

The Weierstrass operator is defined by

$$(W_nf)(t) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(x)e^{-n(x-t)^2}dx :$$

It agrees with $Ef(S_n/n)$ *when* X *has the normal distribution* $(t, 1/2)$ *with density* $\frac{1}{\sqrt{\pi}}e^{-(x-t)^2}$.

Corollary 2.36. *For all* $f \in C_B^1(\mathbb{R})$ *we have*

$$\| W_n(f) - f \| \le (1.5625)\omega_1\left(f', \frac{1}{2\sqrt{2n}}\right)\frac{1}{\sqrt{2n}},$$

where $\|\cdot\|$ *the sup-norm.*

As our last illustration, let X *have an exponential density* $e^{-x/t}$ *on* $\mathbb{R}^+$ *so that* $E(X) = t$, $\mathrm{Var}(X) = t^2$. *Then* S_n *has a gamma density with parameters* n *and* t^{-1}, *so that* S_n/n *has a gamma density with parameters* n *and* n/t.

This leads to the operator (see [12], p. 219)

$$(H_nf)(t) = \frac{n^n}{(n-1)!t^n}\int_0^{\infty} f(x)x^{n-1}e^{-nx/t}dx, \; t > 0.$$

Corollary 2.37. *For* $f \in C_B^1(\mathbb{R}^+)$, $t > 0$ *we find*

$$|(H_n f)(t) - f(t)| \le (1.5625)\omega_1\left(f', \frac{t}{2\sqrt{n}}\right)\frac{t}{\sqrt{n}}.$$

In the following we establish similar inequalities for higher derivatives ($n \ge 1$).

Theorem 2.38. *Let μ be a positive measure of mass $m > 0$ on the closed interval $[a, b] \subset \mathbb{R}$, for which we assume that $(\frac{1}{m}\int |t - x_0|^{n+1}\mu(dt))^{1/(n+1)} = d_{n+1} > 0$, where $x_0 \in [a, b]$ is fixed. Consider $f \in C^n([a, b])$, $n \ge 1$ with $\omega_1(f^{(n)}, rd_{n+1}) \le w$, where r, w are given positive numbers. Then*

$$\text{(2.38.1)} \quad \left|\int f d\mu - f(x_0)\right| \le |m - 1|\,|f(x_0)| + \sum_{k=1}^{n} \frac{|f^{(k)}(x_0)|}{k!}\left|\int (t - x_0)^k \mu(dt)\right| + \frac{mw}{rn!}\left[\frac{nr^2}{8} + \frac{r}{2} + \frac{1}{(n+1)}\right] d_{n+1}^n .$$

Note.

(i) *When $x_0 = 0 \in [a, b]$ then (2.38.1) is also true when ω_1 is replaced by $\overline{\omega}_1$.*

(ii) *In applications r is usually small.*

(iii) *Inequality (2.38.1) on $[-b, b]$ for $m = 1$, $r \downarrow 0$ and $x_0 = 0$ is attained by $f(x) = |x|^{n+1}$ and μ with mass $1/2$ at $\pm b$.*

Proof: Exactly as the proof of Theorem 2.14 except that we use the bound (2.7.6) for ϕ_n and take $h = rd_{n+1}$ instead. ∎

When $m = 1$, $|a| = b$, $x_0 = 0$ and α small, inequality (2.38.1) for no value of $r > 0$ can be nearly attained by $|x|^{n+\alpha}(n > 1)$ and μ with mass 1/2 at $\pm b$. In this direction we give the following:

THEOREM 2.39. *Let a probability measure μ on $[-b, b]$, $b > 0$ with $(\int |t|^{n+1}\mu(dt))^{1/(n+1)} = d_{n+1} > 0$, $(n \geq 1)$ and consider $f \in C^n([-b, b])$ so that $\overline{\omega}_1(f^{(n)}, h) = w$; h, w given positive numbers. We have that*

$$|\int f\, d\mu - f(0)| \leq \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!} |\int t^k \mu(dt)| + S_n \text{, where} \tag{2.39.1}$$

$$S_n = w \int \phi_n(t)\mu(dt). \tag{2.39.2}$$

If $h = r d_{n+1}$, $r > 0$ and $w = Ah^\alpha$, $0 < \alpha \leq 1$, $A > 0$ constant, then (for $n = 2, 3, 4, 5$ respectively):

$$(2.39.3)_2 \quad S_2 < \frac{A}{2} d_3^{2+\alpha} \left(\frac{r^{\alpha-1}}{3} + \frac{r^\alpha}{2} + \frac{r^{\alpha+1}}{6} + (0.0160375) r^{\alpha+2} \right);$$

$$(2.39.3)_3 \quad S_3 < \frac{A}{12} d_4^{3+\alpha} \left(\frac{r^{\alpha-1}}{2} + r^\alpha + \frac{r^{\alpha+1}}{2} \right);$$

$$(2.39.3)_4 \quad S_4 < A\, d_5^{4+\alpha} \left(\frac{r^{\alpha-1}}{120} + \frac{r^\alpha}{48} + \frac{r^{\alpha+1}}{72} + (0.0002038) r^{4+\alpha} \right);$$

$$(2.39.3)_5 \quad S_5 < A\, d_6^{5+\alpha} \left(\frac{r^{\alpha-1}}{720} + \frac{r^\alpha}{240} + \frac{5}{1440} r^{\alpha+1} + (0.0000651) r^{5+\alpha} \right).$$

The inequality which results from combining (2.39.1), (2.39.3) is almost attained, up to a factor 1.0000002, 1, 1.01, 1.027, by $f(x) = |x|^{n+\alpha}$, when α is small and r takes the values 1.26794926, 1, 0.7745967, 0.6324555 (respectively), while the probability measure μ has mass 1/2 at $\pm b$.

Since the proof of the theorem for all cases $n = 2, 3, 4, 5$ is exactly the same, we present only the one for $n = 5$.

Proof: Recall from (2.19.1) that

$$\phi_5(t) = \frac{t^5}{120}\Theta_5(h/t),\ t > 0$$

where $\Theta_5(v) = \sum_{j=0}^{\infty}(1 - jv)_+^5$, with $v = h/t$. If $v \geq 1$ then $\Theta_5(v) = 1$. Otherwise $(N+1)^{-1} \leq v \leq N^{-1}$ for some integer $N \geq 1$. It follows from (2.20.2) that

$$\Theta_5(v) = \frac{1}{6v}(1 - z^6) + \frac{1}{2}(1 + z^5) + \frac{5}{12}v(1 - z^4) - \frac{v^3}{12}(1 - z^2)$$

where $z = 1 - Nv$. Thus, letting

$$\tau = z/v = v^{-1} - N,\ (0 \leq \tau \leq 1)$$

one has

$$\Theta_5(v) = \frac{1}{6v} + \frac{1}{2} + \frac{5}{12}v - \frac{v^3}{12} + v^5\, g(\tau)\,.$$

Here,

$$g(\tau) = -\frac{\tau^6}{6} + \frac{\tau^5}{2} - \frac{5}{12}\tau^4 + \frac{1}{12}\tau^2\,.$$

Selecting τ so as to maximize $g(\tau)$, one obtains an upper bound on $\Theta_5(v)$ which happens to be valid also when $v \geq 1$. Substituting the resulting upper bound on $\phi_5(t)$ into (2.39.2) and using Hölder's inequality, one obtains $(2.39.3)_5$. Let us apply (2.39.1), (2.39.3) with $f(x) = |x|^{5+\alpha}$ with α small, but not h. Thus

$$w = \overline{\omega}_1(f^{(5)}, h) = (5+\alpha)(4+\alpha)(3+\alpha)(2+\alpha)(1+\alpha)h^\alpha \doteq 120\,.$$

Let $b = 1$ and μ with mass 1/2 at ± 1 so that $d_6 = 1$ and $h = r$. One obtains

$$1 = |\int f \, d\mu| \leq 120\Big(\frac{1}{720r} + \frac{1}{240} + \frac{5}{1440}r + 0.0000651\, r^5\Big)$$

The value $r = 0.6324555$ minimizes the right hand side resulting us in an upper bound equal to 1.027. ∎

CHAPTER 3

MULTI-DIMENSIONAL QUANTITATIVE RESULTS FOR PROBABILITY MEASURES APPROXIMATING THE UNIT MEASURE

Here we continue the study of the degree of weak convergence of a sequence of finite measures $\{\mu_j\}_{n\in\mathbb{N}}$, now on $\mathbb{R}^k$, to the unit measure δ_{x_0}, analogous to Chapter 2. In this chapter, Q denotes a compact subset of $\mathbb{R}^k$, while $f \in C^n(Q), n \geq 1$ and μ is a probability measure on Q. Using standard moment methods, we obtain nearly best or best upper bounds, for different Q, k, n and subclasses of f and given power moments of μ. Our inequalities involve, for a fixed value of $h > 0$, the (first) modulus of continuity (relative to the ℓ_1 norm in $\mathbb{R}^k$) of the nth partial derivatives of f.

Definition 3.1 For f a continuous real valued function on a compact subset Q of $\mathbb{R}^k$, we define its (first) modulus of continuity

$$\omega_1(f,h) = \sup\{|f(\overrightarrow{x}) - f(\overrightarrow{y})| : \text{ all } \overrightarrow{x}, \overrightarrow{y} \in Q, \| \overrightarrow{x} - \overrightarrow{y} \| \leq h\}$$

where $\|\cdot\|$ is the chosen norm in $\mathbb{R}^k$.

We need the following:

Lemma 3.2. *Take Q a compact and convex subset of $\mathbb{R}^k$ and let $\vec{x}_0 = (x_{01}, \ldots, x_{0k}) \in Q$ be fixed and let μ be a probability measure on Q. Let $f \in C^n(Q)$ and suppose that each nth partial derivative $f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$ and $|\alpha| = \sum_{i=1}^k \alpha_i = n$, has, relative to Q and the ℓ_1-norm, a modulus of continuity $\omega_1(f_\alpha, h) \le w$. Here h and w are fixed positive numbers. Then*

(3.2.1)

$$\left|\int_Q f\, d\mu - f(\vec{x}_0)\right| \le \left|\sum_{j=1}^n \frac{1}{j!} \int_Q g^{(j)}_{\vec{x}}(0)\mu(d\vec{x})\right| + w\int_Q \phi_n(\| \vec{x} - \vec{x}_0 \|)\mu(d\vec{x}),$$

where $g_{\vec{x}}(t) = f(\vec{x}_0 + t(\vec{x} - \vec{x}_0)), t \ge 0$.

Proof: The jth derivative of $g_{\vec{z}}(t) = f(\vec{x}_0 + t(\vec{z} - \vec{x}_0))$ is given by

$$(3.2.2)\quad g^{(j)}_{\vec{z}}(t) = \left[\left(\sum_{i=1}^k (z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), \ldots, x_{0k} + t(z_k - x_{0k}))$$

Consequently

$$(3.2.3)\qquad f(z_1, \ldots, z_k) = g_{\vec{z}}(1) = \sum_{j=0}^n \frac{g^{(j)}_{\vec{z}}(0)}{j!} + \mathcal{R}_n(\vec{z}, 0)$$

where

$$\mathcal{R}_n(\vec{z}, 0) = \int_0^1 (\int_0^{t_1} \cdots (\int_0^{t_{n-1}} (g^{(n)}_{\vec{z}}(t_n) - g^{(n)}_{\vec{z}}(0))dt_n)\ldots)dt_1 .$$

By Corollary 2.6 we get

$$|f_\alpha(\vec{x}_0 + t(\vec{z} - \vec{x}_0)) - f_\alpha(\vec{x}_0)| \le w\left\lceil \frac{t \| \vec{z} - \vec{x}_0 \|}{h} \right\rceil, \text{all } t \ge 0 .$$

For $\| \vec{z} - \vec{x}_0 \| \ne 0$ it follows from (3.2.2) using (2.7.2) that $|\mathcal{R}_n(\vec{z}, 0)| \le$

$\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} (\sum_{|\alpha|=n} \frac{n!}{\alpha_1! \ldots \alpha_k!} |z_1 - x_{01}|^{\alpha_1} \ldots |z_k - x_{0k}|^{\alpha_k} \cdot w \lceil \frac{t_n \| \vec{z} - \vec{x}_0 \|}{h} \rceil dt_n) \ldots) dt_1$
$= \sum_{|\alpha|=n} \frac{n!}{\alpha_1! \ldots \alpha_k!} \frac{\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i}}{\| \vec{z} - \vec{x}_0 \|^n} w\phi_n(\| \vec{z} - \vec{x}_0 \|) = w\phi_n(\| \vec{z} - \vec{x}_0 \|)$, since $\| \vec{z} - \vec{x}_0 \| = \sum_{i=1}^k |z_i - x_{0i}|$. Therefore

$$(3.2.4) \qquad |\mathcal{R}_n(\vec{z}, 0)| \leq w\phi_n(\| \vec{z} - \vec{x}_0 \|), \quad \text{for all} \quad \vec{z} \in Q.$$

Also $g_{\vec{z}}(0) = f(\vec{x}_0)$. Integrating (3.2.3) relative to μ and using (3.2.4) we conclude our claim.∎

Analogously to Theorem 2.17 we have for the n-dimensional case:

Theorem 3.3. *Take Q of the form $Q = \{\vec{x} \in \mathbb{R}^k : \| \vec{x} \| \leq 1\}$, where $\| \cdot \|$ the ℓ_1-norm in $\mathbb{R}^k$ and let $\vec{x}_0 = (x_{01}, \ldots, x_{0k}) \in Q$ be fixed. Let the measure μ satisfy $\mu(Q) = 1$ and $\int_Q \| \vec{x} - \vec{x}_0 \| \mu(d\vec{x}) = d^*$. Also, let $f \in C^n(Q)$ and suppose that each of its nth partial derivatives has a modulus of continuity $\omega_1(h) \leq w$, where h and w are fixed positive numbers. Then we have the upper bound*

(3.3.1)
$$\left| \int_Q f\, d\mu - f(\vec{x}_0) \right| \leq \left| \sum_{j=1}^n \frac{1}{j!} \int_Q g_{\vec{x}}^{(j)}(0)\, \mu(d\vec{x}) \right| + wd^* \phi_n(1 + \| \vec{x}_0 \|)/(1 + \| \vec{x}_0 \|)$$

where $g_{\vec{x}}(t) = f(\vec{x}_0 + t(\vec{x} - \vec{x}_0))$, $t \geq 0$.

Remark 3.4. *For $n = 1$, this yields the inequality, $(x_0 \in Q)$:*

$$\left| \int_Q f\, d\mu - f(\vec{x}_0) \right| \leq \left| \sum_{i=1}^k \frac{\partial f}{\partial x_i}(\vec{x}_0) \int_Q (x_i - x_{0i}) \mu(d\vec{x}) \right| + wd^* \phi_1(1 + \| \vec{x}_0 \|)/(1 + \| \vec{x}_0 \|).$$

Proof of Theorem 3.3: Let ν be the probability measure induced by μ and the mapping $x \to \| x - x_0 \|$ into the interval $I = (0, 1 + \| x_0 \|)$. Then $\int y\, d\nu = d^*$ and we need an upper bound of $\int \phi_n(y)\nu(dy)$. Here

$$\frac{\phi_n(y)}{y} \leq \frac{\phi_n(1 + \| x_0 \|)}{1 + \| x_0 \|},$$

thus $\int \phi_n(y)\nu(dy) \leq d^* \phi_n(1 + \| x_0 \|)/(1 + \| x_0 \|)$.∎

Next we give some explicit upper bounds.

Theorem 3.5. *Let μ be a probability measure on a convex subset Q of $\mathbb{R}^k$. Further, assume that $\frac{1}{(n+1)}(\int \| \vec{x} - \vec{x}_0 \|^{n+1} \mu(d\vec{x}))^{1/(n+1)} = h > 0$, where $\vec{x}_0$ is a fixed point of Q and $\| \cdot \|$ denotes the ℓ_1-norm in $\mathbb{R}^k$. Also, let $f \in C^n_B(Q)$ and suppose that each nth partial derivative has at h a modulus of continuity $\omega_1(h) \leq w$, where w is a given positive number. Then*

$$\text{(3.5.1)} \quad \left| \int_Q f\, d\mu - f(\vec{x}_0) \right| \leq \left| \sum_{j=1}^{n} \frac{1}{j!} \int_Q g^{(j)}_{\vec{x}}(0) \mu(d\vec{x}) \right| + wh^n \left[\frac{3}{2} \frac{(n+1)^n}{n!} + \frac{(n+1)^{n-1}}{8(n-1)!} \right],$$

where $g_{\vec{x}}(t) = f(\vec{x}_0 + t(\vec{x} - \vec{x}_0))$, $t \geq 0$.

Proof: Similar to Theorem 2.38; here we make use of (3.2.1) and (2.7.6), etc.∎

As a related result we have:

Corollary 3.6. *Let the random vector $(X_1, \ldots, X_k)$ take values in a convex subset Q of $\mathbb{R}^k$, with distribution function μ and expectations $E(X_i) = x_{0i}$, $i = 1, \ldots, k$. Thus $\vec{x}_0 = (x_{01}, \ldots, x_{0k})$ is a fixed point of Q. Further, let $\left(\int \| \vec{x} - \vec{x}_0 \|^2 \mu(d\vec{x})\right)^{1/2} = \sigma$, where $\| \cdot \|$ denotes the ℓ_1 norm in $\mathbb{R}^k$. Be* *given $f \in C^1_B(Q)$ and suppose that each first partial derivative f_i has a modulus of continuity $\omega_1(f_i, \frac{1}{2}\sigma) \le w$; $i = 1, \dots, k$ where w is a given positive number. Then*

$$\left| \int_Q f\, d\mu - f(\overrightarrow{x}_0) \right| \le \frac{25}{16} w\sigma = (1.5625) w\sigma . \tag{3.6.1}$$

Proof: Apply Theorem 3.5 with $n = 1$ and $h = \sigma/2$. ∎

The following Lemma 3.8 is needed in order to establish a similar inequality over a rectangle $K \subset \mathbb{R}^2$ when we know only the expectations $\int (x_i - x_{0i}) \mu(d\overrightarrow{x})$; $i = 1, 2$ where $\overrightarrow{x}_0 = (x_{01}, x_{02}) \in K$.

3.7 Preliminaries

Let $K = [a, b] \times [c, d]$ and let $\varphi : K \to \mathbb{R}$ be a continuous and *convex* function and be given $y = (y_1, y_2) \in K$. Consider probability measures μ on K satisfying $\int_K x_i \mu(dx) = y_i$, $i = 1, 2$. For later use, we would like to find

$$U(y) = U(\varphi, y) = \sup_\mu \int_K \varphi(x_1, x_2) \mu(dx) \tag{3.7.1}$$

Denote $\xi_1 = (b, d)$, $\xi_2 = (b, c)$, $\xi_3 = (a, c)$, $\xi_4 = (a, d)$,

$$D = [(\varphi(\xi_1) + \varphi(\xi_3)) - (\varphi(\xi_2) + \varphi(\xi_4))] \text{ and}$$
$$B = (b - y_1)/(b - a),\ C = (d - y_2)/(d - c) .$$

Further

$$T = [(1 - B)\varphi(\xi_2) + (B + C - 1)\varphi(\xi_3) + (1 - C)\varphi(\xi_4)] ,$$
$$\gamma = \min(1 - B, 1 - C) \text{ and } \delta = \max(0, 1 - B - C) .$$

LEMMA 3.8. *We have*

$$(3.8.1) \qquad U(y) = \begin{cases} D\gamma + T, & \text{if } D > 0\,; \\ D\delta + T, & \text{if } D \le 0\,. \end{cases}$$

PROOF: By well-known theory: (Proposition I.4.6 and Lemma I.4.7, pp.35–36 of [1]), the optimal measure is carried by the four extreme points of K. Let z denote the mass at the vertex ξ_1, $0 \le z \le 1$. Then one can compute the masses p, q, r at ξ_2, ξ_3, ξ_4, respectively, from the given moment conditions:

$$(*) \quad (z + p + q + r = 1\,; zb + pb + qa + ra = y_1\,; zd + pc + qc + rd = y_2)\,.$$

Solving this system (*) we obtain

$$p = -z + (1 - B)\,; q = z + ((B + C) - 1)\,; r = -z + (1 - C)\,.$$

In order $p, q, r \ge 0$ we need that $\delta \le z \le \gamma$. Moreover, $\Theta = \int_{\{\xi_1,\xi_2,\xi_3,\xi_4\}} \varphi(\xi)\mu(d\xi) = Dz + T$.

Hence,

$$U(y) = \sup_z \Theta = \sup_z (Dz + T) = \begin{cases} D\gamma + T, & \text{if } D > 0\,; \\ D\delta + T, & \text{if } D \le 0\,. \end{cases}$$

∎

Now we are ready to state:

PROPOSITION 3.9. *Let $K = [a, b] \times [c, d] \subset \mathbb{R}^2$ and a probability measure μ on K be such that $\int_K (x_i - x_{0i})\mu(d\vec{x}) = y_i$, $i = 1, 2$ where the points $\vec{y} = (y_1, y_2)$, $\vec{x}_0 = (x_{01}, x_{02}) \in K$ are given. Also, let $f \in C^n(K)$ and suppose* *that each of its nth partial derivatives has, relative to K and the ℓ_1-norm, a modulus of continuity $\omega_1(h) \le w$. Here h and w are fixed positive numbers. Then*

$$\left|\int_K f\,d\mu - f(\vec{x}_0)\right| \le \left|\sum_{j=1}^{n} \frac{1}{j!}\int_K g^{(j)}_{\vec{x}}(0)\mu(d\vec{x})\right| + U(\varphi, \vec{x}_0 + \vec{y}), \tag{3.9.1}$$

where $g_{\vec{x}}(t) = f(\vec{x}_0 + t(\vec{x} - \vec{x}_0))$, $t \ge 0$ and $\varphi(\vec{x}) = w\phi_n(\| \vec{x} - \vec{x}_0 \|)$.

Proof: Immediate from Lemma 3.2 and the definition (3.7.1) of $U(\varphi, y)$.∎

As a complementary result we give:

Corollary 3.10. *Let $Q = \{\vec{x} \in \mathbb{R}^k : -\alpha \le x_i \le \alpha, i = 1, \ldots, k\}$, $\alpha > 0$ and μ a probability measure on Q. Also, let $f \in C^n(Q)$ and suppose that the nth order partial derivatives for $h > 0$ have modulus of continuity (relative to the ℓ_1-norm) at most $w > 0$. Denote $g_{\vec{x}}(t) = (\vec{x}_0 + t(\vec{x} - \vec{x}_0))$ for all $t \ge 0$.*

Then

$$\left|\int_Q f\,d\mu - f(\vec{x}_0)\right| \le \left|\sum_{j=1}^{n} \frac{1}{j!}\int_Q g^{(j)}_{\vec{x}}(0)\mu(d\vec{x})\right| + w\phi_n(\| \vec{x}_0 \| + k\alpha). \tag{3.10.1}$$

Proof: By Lemma 3.2 noting that $\phi_n(\| x - x_0 \|) \le \phi_n(\| x_0 \| + k\alpha)$.∎

3.11 Preliminaries

Let Q be a compact subset of $\mathbb{R}^k$ with the ℓ_1-norm. Consider the set

$$L_\alpha^{(n)} = \{f \in C^n(Q) : \omega_1\left(\frac{\partial^\beta f}{\partial x^\beta}, h\right) \le Ah^\alpha;\ \text{all } |\beta| = n\},$$

where h is a positive *variable*, $0 < \alpha \leq 1$ and A is a positive constant. As a related result, we have:

Theorem 3.12. *Let $f \in L_\alpha^{(n)}$ and n even. Consider a probability measure μ on Q such that $\left(\int_Q \| z \|^{n+1} \mu(dz)\right)^{1/(n+1)} = D_{n+1} > 0$. Then*

$$\left|\int_Q f\, d\mu - f(\overrightarrow{0})\right| \leq \left|\sum_{j=1}^{n} \frac{1}{j!} \int_Q g_{\overrightarrow{z}}^{(j)}(0)\mu(d\overrightarrow{z})\right| + c(D_{n+1})^{n+\alpha}, \tag{3.12.1}$$

where $c = A/[(1+\alpha)(2+\alpha)\dots(n+\alpha)]$ and $g_{\overrightarrow{z}}(t) = f(t\overrightarrow{z})$, $t \geq 0$. The above inequality is attained by the function $r(z) = c \| z \|^{n+\alpha}$ and the probability measure μ with mass $1/2$ at $z_0, -z_0$ and also by the measure with mass 1 at z_0. Here, we assume that $\| z_0 \| = D_{n+1}$, $\pm z_0 \in Q$ and
$Q \subset \Delta = \{(z_1, \dots, z_1) : z_1 \geq 0 \text{ or } z_1 < 0\}$.

Proof: We have

(3.12.2)
$$f(z_1, \dots, z_k) = g(1) = \sum_{j=0}^{n} \frac{g^{(j)}(0)}{j!} + \int_0^1 (g^{(n)}(u) - g^{(n)}(0))\frac{(1-u)^{n-1}}{(n-1)!} du\,.$$

Since $\omega_1(f_\beta, h) \leq Ah^\alpha$; ($|\beta| = n$), one has $|f_\beta(t\overrightarrow{z}) - f_\beta(\overrightarrow{0})| \leq At^\alpha \| \overrightarrow{z} \|^\alpha$, for all $t \geq 0$.

Consequently, (compare the proof of Lemma 3.2),

$$|g^{(n)}(t) - g^{(n)}(0)| \leq At^\alpha \| z \|^\alpha \left(\sum_{i=1}^{k} |z_i|\right)^n = At^\alpha \| z \|^{n+\alpha},$$

since $\| z \| = \sum_{i=1}^k |z_i|$. Denoting the remainder in (3.12.2) by $\mathcal{R}_n(\overrightarrow{z}, 0)$, we get:

$$|\mathcal{R}_n(\overrightarrow{z}, 0)| \leq c \| \overrightarrow{z} \|^{n+\alpha}\,. \tag{3.12.3}$$

Integrating (3.12.2) relative to μ and using (3.12.3), one obtains (3.12.1). Here we used that $D_{n+\alpha} = \left(\int \| z \|^{n+\alpha} \mu(dz)\right)^{1/(n+\alpha)} \leq D_{n+1}$. Further, note that the function $\tau(z) = c \| z \|^{n+\alpha} = c\left(\sum_{i=1}^{k} |z_i|\right)^{n+\alpha}$ is convex and since n is even, any nth order single partial derivative has the form $A \| z \|^{\alpha}$. Moreover,

$$|A \| x \|^{\alpha} - A \| y \|^{\alpha} | \leq A(| \, \| x \| - \| y \| \, |)^{\alpha} \leq A(\| x - y \|)^{\alpha},$$

thus $\tau \in L_{\alpha}^{(n)}$ when $Q \subset \Delta$. Also, note that $g_{\overrightarrow{z}}(t) = \tau(t\overrightarrow{z}) = t^{n+\alpha} \| z \|^{n+\alpha}$ has $g_{\overrightarrow{z}}^{(j)}(0) = 0$ for all $0 \leq j \leq n$. These remarks earlier imply the last assertion.∎

CHAPTER 4

GENERALIZED ATTAINABLE INEQUALITIES RELATED TO THE WEAK CONVERGENCE OF FINITE MEASURES TO THE UNIT MEASURE

In this chapter we still study the degree of weak convergence of a sequence of finite measures $\{\mu_j\}_{j\in\mathbb{N}}$ on $\mathbb{R}$ to the unit measure δ_{x_0}, stressing the role of convexity/concavity of the main function involved.

We estimate $|\int f\,d\mu - f(x_0)|$ for functions $f \in C^n([a,b])$, $n \geq 1$, $x_0 \in [a,b]$, as well as

$$\left|\int \left(\frac{f(x_0+y)+f(x_0-y)}{2}\right)\mu(dy) - f(x_0)\right|.$$

In both cases μ is a probability measure. The first type of inequalities involves the first modulus of continuity ω_1 of $f^{(n)}$ or a generalization of it. The second type of inequalities involves the second modulus of continuity ω_2 of $f^{(n)}$. Our inequalities are proved by the help of standard moment methods and are attained by conveniently chosen n-times continuously differentiable functions/finitely (usually 1-point) supported probability measures.

To our knowledge the inequalities established below have not been considered before in the literature.

I. ATTAINABILITY: FOR A FIRST MODULUS OF CONTINUITY WITH A FREELY VARYING ARGUMENT

(Without loss of generality, we take $x_0 = 0$).

4.1 Preliminaries

Let $[a, b] \subset \mathbb{R}$ with $0 \in (a, b)$ and $|a| \leq b$, and $f \in C^n([a, b]); n \in \mathbb{N}$. We put

(4.1.1)

$$g_0(y) = \omega_1(f^{(n)}, [a, b]; y) = \sup\{|f^{(n)}(x) - f^{(n)}(x')| : |x - x'| \leq |y|\} \, ; x, x' \in [a, b] \, .$$

Note that $g_0(-y) = g_0(y)$, g_0 is increasing in $y \geq 0$, $g_0(y) \geq g_0(0) = 0$ and $g_0(y + y') \leq g_0(y) + g_0(y')$, also g_0 is continuous in y. Any function with these properties will be called a first modulus of continuity. Namely, for any such function g_0 we have $\omega_1(g_0, [a, b], y) = g_0(y)$. Note that we also have $g_0(y) = g_0(b - a)$ if $|y| \geq b - a$. Let further

$$(4.1.2) \qquad g_1(y) = \omega_1(f^{(n)}, [a, 0]; y), \; g_2(y) = \omega_1(f^{(n)}, [0, b]; y)$$

Observe that given two moduli of continuity g_1 and g_2 on the intervals $[a, 0]$ and $[0, b]$, respectively, we have for both $c = \pm 1$ that (4.1.2) holds for the following function $f^{(n)}$:

$$(4.1.3) \qquad f^{(n)}(y) = \begin{cases} c g_1(y) & , \text{ if } a \leq y \leq 0 \, ; \\ g_2(y) & , \text{ if } 0 \leq y \leq b \, . \end{cases}$$

Also, note that $g_0(y) \geq \max(g_1(y), g_2(y))$ for $|y| \leq a$, which need not be an equality.

Example. Choose $c = -1$ and $g_1(y) = g_2(y) = A|y|^{\alpha}$; $0 < \alpha \leq 1$, $A > 0$.

Then (4.1.3) becomes

$$f^{(n)}(y) = \begin{cases} -A|y|^\alpha & \text{, if } a \le y \le 0; \\ A|y|^\alpha & \text{, if } 0 \le y \le b. \end{cases} \tag{4.1.4}$$

In this case $f^{(n)}(h) - f^{(n)}(-h) = 2Ah^\alpha \ge A(2h)^\alpha$, $(0 \le h \le |a|)$. If $\alpha < 1$ then the strict inequality sign holds, hence the function (4.1.4) *does not* have a modulus of continuity $A|y|^\alpha$.

Let μ be a nonnegative measure on $[a, b]$ of mass c_0 and with central moments $c_j = \int t^j \mu(dt)$, $j = 1, \ldots, n$. By integrating the Taylor's formula with Cauchy remainder relative to μ and using Fubini's theorem one obtains

$$\int f\,d\mu - f(0) = f(0)(c_0 - 1) + \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} c_j + U_n\,, \tag{4.1.5}$$

where

$$\begin{aligned} U_n = (-1)^n & \int_a^0 \left(f^{(n)}(t) - f^{(n)}(0)\right)\left[\int_a^t \frac{(t-y)^{n-1}}{(n-1)!}\mu(dy)\right]dt \\ & + \int_0^b \left(f^{(n)}(t) - f^{(n)}(0)\right)\left[\int_t^b \frac{(y-t)^{n-1}}{(n-1)!}\mu(dy)\right]dt\,. \end{aligned} \tag{4.1.6}$$

It follows from (4.1.2) that

$$|U_n| \le V_n\,, \tag{4.1.7}$$

where

(4.1.8)

$$V_n = \int_a^0 g_1(t)\left(\int_a^t \frac{(t-y)^{n-1}}{(n-1)!}\mu(dy)\right)dt + \int_0^b g_2(t)\left(\int_t^b \frac{(y-t)^{n-1}}{(n-1)!}\mu(dy)\right)dt\,.$$

The equality sign in (4.1.7) is *assumed* by the function $f^{(n)}$ defined by (4.1.3) with $c = (-1)^n$; it also satisfies (4.1.2).

Introducing the nonnegative function

$$(4.1.9)\qquad G_j^*(y) = \begin{array}{ll} \int_0^y g_2(t)\frac{(y-t)^{j-1}}{(j-1)!}dt, & \text{if } y \geq 0\,; \\ \int_y^0 g_1(t)\frac{(t-y)^{j-1}}{(j-1)!}dt, & \text{if } y < 0 \end{array}$$

one may write (4.1.8) as

$$(4.1.10)\qquad V_n = \int G_n^*(y)\mu(dy)\,.$$

Observe that

$$(4.1.11)\qquad G_j^*(y) = (sgny)\int_0^y G_{j-1}^*(z)dz = \int_0^y G_{j-1}^*(z)|dz|$$

for $j = 1, 2, \ldots, n$ provided we put

$$(4.1.12)\qquad G_0^*(y) = \begin{cases} g_2(y), & \text{if } y \geq 0\,; \\ g_1(y), & \text{if } y < 0\,. \end{cases}$$

It follows from (4.1.11) since g_1, g_2 are continuous functions that

$$(4.1.13)\qquad G_j^{*\prime}(y) = G_{j-1}^*(y) \quad \text{all } j \geq 1\,.$$

It will be convenient to introduce the modulus of continuity (which depends on $f^{(n)}$) defined by the even function

$$(4.1.14)\qquad g = \max(g_1, g_2)\,, \quad \text{thus} \quad 0 \leq g \leq g_0\,.$$

Let further G_j be the even function such that

$$(4.1.15)\qquad G_j(y) = \int_0^y g(t)\frac{(y-t)^{j-1}}{(j-1)!}dt \quad \text{if } y \geq 0\,.$$

Clearly $G_j^*(y) \le G_j(y)$. Also $G_n^{(j)}(0) = 0$ for all $j \le n$.

Remark 4.2 Note that for the function $\tilde{f} = G_n$ one has $\tilde{f}^{(n)}(y) = (sgn\, y)^n g(y)$ so that $\tilde{g}_1 = \tilde{g}_2 = \tilde{g} = g$ and $\tilde{G}_j = G_j$. If n is even then $\tilde{g}_0 = g$ but not when n is odd.

Employing the previous notations and assumptions we have:

Theorem 4.3. *Assume $c_0 = 1$ and let ψ be a function on $[0,b]$ such that $\psi(0) = 0$, which is continuous and strictly incraesing and let*

$$\psi^{-1}\Big(\int \psi(|y|)\mu(dy)\Big) = d. \tag{4.3.1}$$

Suppose that

$$\mathcal{H}_n(u) = G_n(\psi^{-1}(u)) \text{ on } [0, \psi(b)] \tag{4.3.2}$$

is concave. Then

$$\left|\int f\, d\mu - f(0) - \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} c_j\right| \le G_n(d). \tag{4.3.3}$$

The above inequality is attained (i.e. it is sharp) by the probability measure $\mu = \delta_d$ and with f replaced by the function $\tilde{f}$ introduced in Remark 4.2.

Proof: In view of (4.1.5), (4.1.7), (4.1.10) and $G_n^* \le G_n$, we only need to prove that $\int G_n(y)\mu(dy) \le G_n(d)$. Note that both $G_n(y)$ and $\psi(|y|)$ are even functions on $[a,b]$, where $|a| \le b$. It follows (see [17], [18]) that $\sup_\mu \int G_n(y)\mu(dy) = G_n(d)$, since by concavity of $\mathcal{H}_n$ the set $\Gamma_1 = \{(u, \mathcal{H}_n(u)) : 0 \le u \le \psi(b)\}$ describes the upper boundary of the convex hull conv Γ_0 of the curve

$$\Gamma_0 = \{(\psi(y), G_n(y)) : 0 \leq y \leq b\}.$$

∎

The next generalizes the last theorem.

Theorem 4.4. *Assume $c_0 = 1$ and consider the upper concave envelope $\mathcal{H}_n^*(u)$ of $\mathcal{H}_n(u)$; the latter is defined in (4.3.2).*

Then

$$\left| \int f \, d\mu - f(0) - \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} c_j \right| \leq \mathcal{H}_n^*(\psi(d)). \tag{4.4.1}$$

When $\mathcal{H}_n$ is concave in u then the right hand side of (4.4.1) equals $G_n(d)$ and the inequality is attained as in Theorem 4.3. Otherwise (4.4.1) is still (non-trivially) attained by the same function $\tilde{f}$ introduced in Remark 4.2 and a two points supported measure μ.

Proof: Similar to Theorem 4.3.∎

Note 4.5 One could also use an arbitrary integrable nonnegative function

$$g : [0, b-a] \to \mathbb{R}^+; \;\; g(0) = 0, \tag{4.5.1}$$

not necessarily continuous such that

$$\left| f^{(n)}(x) - f^{(n)}(y) \right| \leq g(|x-y|), \;\; \text{all } x, y \in [a,b] \;\; \text{with } x \cdot y \geq 0 \tag{4.5.2}$$

If g is subadditive then (4.5.2) holds for $\tilde{f} = G_n$ in so far $\tilde{f}^{(n)}$ exists, which is true at each continuity point of g. In this situation Theorems 4.3, 4.4 and also

Theorem 4.6 remain valid, except that the function $\tilde{f}$ attaining equality is not of class C^n.

For the particular case of $\psi(|y|) = |y|^s$ with $n < s \leq n+1$ the next result gives a sufficient condition in order that $\mathcal{H}_n(u) = G_n(\psi^{-1}(u)) = G_n(u^{1/s})$, $0 \leq u \leq b^s$ be concave or convex. This is proved by the method of *superposition of functions.*

THEOREM 4.6. *Let $s > n-1$.*

(i) If $g(y)/y^{s-n}$ is decreasing on $[0,b]$ then

$$\mathcal{H}_n(u) = G_n(u^{1/s})\,;0 \leq u \leq b^s\,,\ n \in \mathbb{N} \quad \textit{is concave}\ . \tag{4.6.1}$$

(ii) If $g(y)/y^{s-n}$ is increasing on $[0,b]$ then

$$\mathcal{H}_n(u) = G_n(u^{1/s});0 \leq u \leq b^s,\ n \in \mathbb{N} \quad \textit{is convex}\ . \tag{4.6.2}$$

PROOF: (i) Assume that $g(y)/y^{s-n}$ is decreasing. There exists a positive measure γ on $[0,b]$ such that

$$g(y)/y^{s-n} = \int_y^b \gamma(dz)\,. \tag{4.6.3}$$

The γ-mass c at b would contribute a term cy^{s-n} to g. Formula (4.6.3) says that $g(y)$ is a *superposition of functions*

$$s_0(y;z) = \begin{cases} y^{s-n}\,, & \text{if } 0 < y \leq z\,; \\ 0\,, & \text{if } y > z\,. \end{cases} \tag{4.6.4}$$

Here $0 < z \leq b$.

The function G_n associated to $s_0(\cdot\,;z)$ is of the form

(4.6.5)

$$s_n(y;z) = \int_0^y s_0(t;z)\frac{(y-t)^{n-1}}{(n-1)!}dt = \int_0^{min(y,z)} t^{s-n}\frac{(y-t)^{n-1}}{(n-1)!}dt, (y,z \geq 0)\,.$$

Thus

$$(4.6.6)\qquad \begin{aligned} s_n(y;z) &= \int_0^y t^{s-n}\frac{(y-t)^{n-1}}{(n-1)!}dt = \frac{\Gamma(s-n+1)}{\Gamma(s+1)}y^s, \quad \text{if } y \leq z\,; \\ &= \int_0^z t^{s-n}\frac{(y-t)^{n-1}}{(n-1)!}dt\,, \quad \text{if } y \geq z\,. \end{aligned}$$

We want to show that $\mathcal{H}_n(u) = G_n(u^{1/s})$ is concave. It suffices to show that $s_n(u^{1/s};z)$ is concave in u for each fixed z. For $u^{1/s} \leq z$ it is linear from (4.6.6). If $u^{1/s} \geq z$ (z fixed) then

$$s_n(u^{1/s};z) = \frac{1}{(n-1)!}\int_0^z t^{s-n}(u^{1/s}-t)^{n-1}dt\,.$$

We have from an elementary calculation that

$$\left(\frac{d}{du}\right)^2 s_n(u^{1/s};z) = \frac{1}{(n-1)!}\,\frac{n-1}{s^2}u^{-2+(1/s)}I$$

where

$$I = \int_0^z t^{s-n}(u^{1/s}-t)^{n-3}[(s-1)t-(s+1-n)u^{1/s}]dt\,.$$

Here, the integrand equals the derivative relative to t of the function $-(u^{1/s}-t)^{n-2}t^{s-n+1}$. Hence, $I = -(u^{1/s}-z)^{n-2}z^{s-n+1} \leq 0$, whenever $u^{1/s} \geq z$. Therefore, $s_n(u^{1/s};z)$ is concave in u when $u^{1/s} \geq z$. Since $s_n(u^{1/s};z)$ is a continuous function in u, also concave in each of the branches $u \leq z^s$ and $u \geq z^s$ and at $u = z^s$ the left and right hand derivatives agree, we conclude that it is concave on the whole of $[0,b]$.

(ii) Assume that $g(y)/y^{s-n}$ is increasing. One may write

$$g(y)/y^{s-n} = \int_0^y \gamma(dz) \tag{4.6.7}$$

with γ a positive measure.

The γ-mass c at 0 would contribute a term cy^{s-n} to g. Formula (4.6.7) says that $g(y)$ is a *superposition of functions*

$$\overline{s}_0(y;z) = \begin{cases} 0, & \text{if } y \le z\,; \\ y^{s-n}, & \text{if } y > z\,. \end{cases} \tag{4.6.8}$$

The associated function G_n is

$$\overline{s}_n(y;z) = \begin{cases} 0, & \text{if } y \le z\,; \\ \int_z^y t^{s-n}\frac{(y-t)^{n-1}}{(n-1)!}\,dt, & \text{if } y \ge z\,. \end{cases} \tag{4.6.9}$$

We would like to show that $\overline{s}_n(u^{1/s};z)$ is convex in u. This is obvious in so far $u^{1/s} \le z$. It is easy to establish the convexity for $u^{1/s} \ge z$. Namely, from (4.6.6) and (4.6.9) we see that

$$\overline{s}_n(u^{1/s};z) + s_n(u^{1/s};z) = \int_0^y t^{s-n}\frac{(y-t)^{n-1}}{(n-1)!}\,dt = qy^s = qu$$

is linear in u, (q is a constant).

Since $s_n(u^{1/s};z)$ is concave in u, it follows that $\overline{s}_n(u^{1/s};z)$ will be convex in u with $u^{1/s} \ge z$. ∎

Note 4.7 By Theorem 4.6 we have that if $g(y)/y^k$ ($k \ge 1$ real) is *decreasing* (respectively *increasing*) on $[0,b]$ then the function

$$\mathcal{H}_n(u) = G_n(u^{1/(n+k)}), (n \in \mathbb{N}) \tag{4.7.1}$$

is *concave* (respectively *convex*) on $[0, b^{n+k}]$.

Remark 4.8 (to Theorem 4.3)

(i) If $\psi(|y|) = |y|^{n+k}$, $k \geq 1$ real and $\mathcal{H}_n(u)$ (as in (4.7.1)) is concave on $[0, b^{n+k}]$, we have the attainable inequality:

$$\left| \int f \, d\mu - f(0) - \sum_{j=1}^{n} \frac{f^{(j)}(0)}{j!} c_j \right| \leq G_n(d_{n+k}), \tag{4.8.1}$$

where $d_{n+k} = (\int |y|^{n+k} \mu(dy))^{1/(n+k)}$.

(ii) If $n = 1$ and $g(y) = A|y|$: $A > 0$ constant, then $G_1(y) = Ay^2/2$ so that $G_1(d_2) = Ad_2^2/2$. Consequently, the sharp result (2.28.2) (when $x_0 = 0$, $m = c_0 = 1$, $c_1 = 0$ and $|a| = b$) is a special case of (4.8.1).

(iii) For $c_0 = 1$, $c_1 = 0$, $n = 1$ we obtain (from (4.8.1)):

$$\left| \int f \, d\mu - f(0) \right| \leq G_1(d_2) = \int_0^{d_2} g(t) dt, \tag{4.8.2}$$

provided that $g(y)$ is concave in $y > 0$ or (even weaker) that $g(y)/y$ is decreasing in $y > 0$.

(iv) From G. Lorentz [25] (p.45, Theorem 8) we know that, for every first modulus of continuity $\omega_1(y)$ there exists a *concave* one $\tilde{\omega}_1(y)$ such that $\omega_1(y) \leq \tilde{\omega}_1(y) \leq 2\omega_1(y)$, thus the bound $2\,G_n(d_{n+k})$ in (4.8.1) is always true.

Still referring to Theorem 4.4, where g could be as general as in (4.5.1) and ψ a moment inducing function as in that theorem. We find sufficient conditions, Theorem 4.9, in order that $\mathcal{H}_n$ be convex which in turn implies that

$$\mathcal{H}_n^*(\psi(d)) = [\psi(d)/\psi(b)]\,\mathcal{H}_n(\psi(b))\,.$$

Theorem 4.9. *(i) If $n > 1$ and there exists ψ'' (second derivative of ψ) so that $\psi''(y) \leq 0$, $(0 < y \leq b)$, then $\mathcal{H}_n$ is convex on $[0, \psi(b)]$.*

(ii) Suppose that ψ has a continuous derivative ψ'. If $g(y)/\psi'(y)$ is increasing in $y\,(0 \leq y \leq b)$, then $\mathcal{H}_1$ is convex on $[0, \psi(b)]$.

Proof: (i) From $\psi' > 0$, $\psi'' \leq 0$ on $(0, b]$ we get that $(\psi^{-1}(u))'' \geq 0$ on $(0, \psi(b)]$, therefore $\psi^{-1}(u)$ is convex on $[0, \psi(b)]$ and since $G_n(y)$, $(0 \leq y \leq b)$ is always convex for $n > 1$, we obtain easily that $\mathcal{H}_n(u)$ is convex $(n > 1)$.

(ii) Assume that $g(y)/\psi'(y)$ is *increasing* in y. One may write

$$g(y)/\psi'(y) = \int_0^y \gamma(dz),\ (0 \leq y \leq b) \tag{4.9.1}$$

where γ is a positive measure. The γ-mass c at 0 would contribute a term $c\psi'(y)$ to $g(y)$. Formula (4.9.1) says that $g(y)$ is a superposition of the functions:

$$\bar{s}_0(y; z) = \begin{cases} 0, & \text{if } y < z \\ \psi'(y), & \text{if } y \geq z\,. \end{cases} \tag{4.9.2}$$

The associated function G_1 is

$$\bar{s}_1(y; z) = \begin{cases} 0, & \text{if } y \leq z \\ \int_z^y \psi'(t)dt, & \text{if } y \geq z\,. \end{cases} \tag{4.9.3}$$

We would like to show that $\gamma_z(u) = \bar{s}_1(\psi^{-1}(u); z)$ is convex in u (for then $\mathcal{H}_1(u)$ will be convex in u). This is obvious in so far $\psi^{-1}(u) \leq z$. For $\psi^{-1}(u) \geq z$ we get $\gamma_z(u) = u - \psi(z)$, $(0 \le u \le \psi(b))$, which is linear in u. Therefore, $\gamma_z(u)$ is convex in u.∎

II. ATTAINABILITY: FOR A SECOND MODULUS OF CONTINUITY WITH A FREELY VARYING ARGUMENT. CONNECTION TO POSITIVE LINEAR CONVOLUTION OPERATORS.

The positive linear convolution type operators we consider are given by:

Definition 4.10 Let $f \in C([-2a, 2a])$, $a > 0$ and μ be a probability measure on $[-a, a]$. For any $x \in [-a, a]$ we introduce

$$L(f, x) = \int_{[-a,a]} \left(\frac{f(x+y) + f(x-y)}{2} \right) \mu(dy) . \tag{4.10.1}$$

Observe that L is a *positive linear convolution operator* from $C([-2a, 2a])$ into $C([-a, a])$.

Several authors such as R. DeVore, see [8], have studied the above operators in detail. In this section we also study this type of operators, however, from the point of view of attainable inequalities similar to those in Section I of this chapter.

The next Proposition 4.14 will be useful in the Preliminaries 4.15 following, as well as in Proposition 4.16, to prove it we need:

LEMMA 4.11. *For all $x, t \in \mathbb{R}$ obtain*

$$\big|\, |x+t|^\alpha + |x-t|^\alpha - 2|x|^\alpha \big| \le 2|t|^\alpha , \quad 0 \le \alpha \le 2 . \tag{4.11.1}$$

Proof: It is left as an exercise. ∎

Corollary 4.12. *For all $x, t \in \mathbb{R}$ get*

$$\text{(4.12.1)} \qquad \big|\, |x+2t|^\alpha - 2|x+t|^\alpha + |x|^\alpha \big| \leq 2|t|^\alpha, \ \ 0 \leq \alpha \leq 2\,.$$

Proof: Apply (4.11.1) with x replaced by $x+t$.

Here we would like to mention (see [25]):

Definition 4.13 For $f \in C([a,\beta])$, $[a,\beta] \subset \mathbb{R}$, its *second modulus of continuity* is given by

$$\text{(4.13.1)} \quad \omega_2(f,h) = \sup_{\substack{x,t \\ |t| \leq h}} |f(x) - 2\,f(x+t) + f(x+2t)|, \ 0 \leq h \leq \frac{\beta - a}{2}\,.$$

Proposition 4.14. *The function $|t|^\alpha$, $(0 \leq \alpha \leq 2)$ with domain $[-2a, 2a]$, $a > 0$ has second modulus of continuity*

$$\text{(4.14.1)} \qquad \omega_2(|t|^\alpha, h) = 2h^\alpha\,;\ 0 < h \leq 2a\,.$$

Proof: Apply (4.12.1), equality holds for $x = -h$, $t = h$. ∎

4.15 Preliminaries In this section we consider $f \in C^n([-2a, 2a])$, $a > 0$, n is *even* and we study inequalities involving $\omega_2(f^{(n)}, h)$ over $[-2a, 2a]$.

For fixed $x_0 \in [-a, a]$, by using Taylor's formula with Cauchy remainder for

$f(x_0+y)$, $f(x_0-y)$ we get, $(n=2m)$:

$$(4.15.1) \qquad (\Delta_y^2 f)(x_0) = 2\sum_{\rho=1}^{m} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} y^{2\rho} + \int_0^y (\Delta_t^2 f^{(n)})(x_0) \frac{(y-t)^{n-1}}{(n-1)!}\, dt\,,$$

where

$$(4.15.2) \qquad (\Delta_y^2 f)(x_0) = f(x_0+y) + f(x_0-y) - 2f(x_0)$$

and similarly

$$(4.15.3) \qquad (\Delta_t^2 f^{(n)})(x_0) = f^{(n)}(x_0+t) + f^{(n)}(x_0-t) - 2f^{(n)}(x_0)$$

are the so-called central second order differences. Now we integrate (4.15.1) relative to the variable y and a probability measure μ over $[-a,a]$. We further use Fubini's theorem, plus the fact that n is even. Also we denote

$$(4.15.4) \qquad c_{2\rho} = \int y^{2\rho} \mu(dy)\,.$$

We have found:

$$(4.15.5) \qquad \begin{aligned} L(f,x_0) - f(x_0) &= \frac{1}{2}\int (\Delta_y^2 f)(x_0)\mu(dy) \\ &= \sum_{\rho=1}^{m} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} + \tilde{U}_n\,, \end{aligned}$$

where

$$(4.15.6) \qquad \begin{aligned} \tilde{U}_n &= \frac{1}{2}\int_{-a}^{0} (\Delta_t^2 f^{(n)})(x_0) \left[\int_{-a}^{t} \frac{(t-y)^{n-1}}{(n-1)!}\mu(dy)\right] dt \\ &+ \frac{1}{2}\int_0^a \left(\Delta_t^2 f^{(n)}\right)(x_0) \left[\int_t^a \frac{(y-t)^{n-1}}{(n-1)!}\mu(dy)\right] dt\,. \end{aligned}$$

Using (4.13.1) it follows that

$$(4.15.7) \qquad |\tilde{U}_n| \le V_n\,,$$

where

$$\tilde{V}_n = \frac{1}{2}\int_{-a}^{0} \omega_2(f^{(n)}, |t|)\left[\int_{-a}^{t} \frac{(t-y)^{n-1}}{(n-1)!}\mu(dy)\right]dt + \frac{1}{2}\int_{0}^{a} \omega_2(f^{(n)}, t)\left[\int_{t}^{a} \frac{(y-t)^{n-1}}{(n-1)!}\mu(dy)\right]dt. \tag{4.15.8}$$

Note that $\tilde{U}_n = \tilde{V}_n$ when $x_0 = 0$ and $f^{(n)}(y) = |y|^\alpha$, with $0 \le \alpha \le 2$; (by Proposition 4.1.4).

It is convenient to introduce

$$w_2(t) = \omega_2(f^{(n)}, |t|),\ 0 \le |t| \le 2a \tag{4.15.9}$$

and further the even function which for $y \ge 0$ is defined by

$$\tilde{G}_j(y) = \int_0^y w_2(t)\frac{(y-t)^{j-1}}{(j-1)!}dt,\ (j = 1, \ldots, n). \tag{4.15.10}$$

Further let $\tilde{G}_0(y) = w_2(y)$. Observe that

$$\tilde{V}_n = \frac{1}{2}\int_{[-a,a]} \tilde{G}_n(y)\mu(dy) \tag{4.15.11}$$

and $\tilde{G}'_j(y) = \tilde{G}_{j-1}(y)$ all $j \ge 1$, since $\omega_2(f^{(n)}, h)$ is a continuous function of h.

Employing the previous notations and assumptions we have:

Proposition 4.16. *Let probability measure μ on $[-a, a]$, $a > 0$ and define $d_r = [\int |y|^r \mu(dy)]^{1/r}, r > 0$. Let $n = 2m, m \in \mathbb{N}$ and consider all $f \in C^n([-2a, 2a])$ such that*

$$\omega_2(f^{(n)}, |y|) \le 2A|y|^\alpha,\ (0 \le |y| \le 2a); 0 < \alpha \le 2, A > 0.$$

Then

$$
(4.16.1)\quad \begin{aligned}
&\left|L(f,x_0)-f(x_0)-\sum_{\rho=1}^{m}\frac{f^{(2\rho)}(x_0)}{(2\rho)!}c_{2\rho}\right|=\\
&=\left|\int_{[-a,a]}\left(\frac{f(x_0+y)+f(x_0-y)}{2}\right)\mu(dy)-f(x_0)-\sum_{\rho=1}^{m}\frac{f^{(2\rho)}(x_0)}{(2\rho)!}c_{2\rho}\right|\\
&\leq A\,d_{n+2}^{n+\alpha}/[(\alpha+1)(\alpha+2)\ldots(\alpha+n)],\quad \text{where } x_0\in[-a,a]\,.
\end{aligned}
$$

The last inequality is attained (i.e. it is sharp) when $x_0 = 0$ by the function

$$
(4.16.2)\qquad f_*(y)=A|y|^{\alpha+n}/[(\alpha+1)(\alpha+2)\ldots(\alpha+n)]
$$

and the associated probability measure μ having mass 1/2 at the two points $\pm d_{n+2}$.

PROOF: We easily find that

$$
(4.16.3)\qquad \tilde{G}_n(y)\leq 2A|y|^{\alpha+n}/[(\alpha+1)\ldots(\alpha+n)],(0\leq|y|\leq 2a)
$$

thus

$$
(4.16.4)\qquad \tilde{V}_n\leq A d_{n+\alpha}^{n+\alpha}/[(\alpha+1)\ldots(\alpha+n)]\,.
$$

Because of $d_{n+\alpha}\leq d_{n+2}$ and $|\tilde{U}_n|\leq\tilde{V}_n$ we get

$$
(4.16.5)\qquad |\tilde{U}_n|\leq A\,d_{n+2}^{n+\alpha}/[(\alpha+1)\ldots(\alpha+n)]\,.
$$

Using (4.15.5), we finally obtain (4.16.1). Also, note that $f_*^{(n)}(y)=A|y|^\alpha$ hence by Proposition 4.14 $\omega_2(f_*^{(n)},|y|)=2A|y|^\alpha$. Also $f_*^{(k)}(0)=0$ all $k=0,\ldots,n$. This proves the last assertion.∎

The next result assumes that $f \in C^n([-2a, 2a])$, $a > 0$ and that $\omega_2(f^{(n)}, |t|) \le w_2(t)$, $(0 \le |t| \le 2a)$, where $w_2(t)$ is a given arbitrary even positive function. Let $\tilde{G}_n$ be defined as in (4.15.10) also for the above $w_2(t)$.

Theorem 4.17. *Let ψ be a function on $[0, a]$ such that $\psi(0) = 0$, which is continuous and strictly increasing. Let a probability measure μ on $[-a, a]$ with*

$$\psi^{-1}\Big(\int \psi(|y|)\mu(dy)\Big) = d\,. \tag{4.17.1}$$

Suppose that, $(n = 2m)$:

$$\tilde{\mathcal{H}}_n(u) = \tilde{G}_n(\psi^{-1}(u)) \tag{4.17.2}$$

is *concave* on $[0, \psi(a)]$. Then

(4.17.3)

$$\begin{aligned}
E(x_0) &= \left| L(f, x_0) - f(x_0) - \sum_{\rho=1}^{m} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} \right| \\
&= \left| \int_{[-a,a]} \left(\frac{f(x_0 + y) + f(x_0 - y)}{2} \right) \mu(dy) - f(x_0) - \sum_{\rho=1}^{m} \frac{f^{(2\rho)}(x_0)}{(2\rho)!} c_{2\rho} \right| \\
&\le \frac{1}{2} \tilde{G}_n(d), \quad \text{where } x_0 \in [-a, a]\,.
\end{aligned}$$

The above inequality is *attained* (*i.e.*it is *sharp*) when $x_0 = 0$ and $\omega_2(f^{(n)}, |t|) = w_2(t)$ and moreover $f(y) = \frac{1}{2}\tilde{G}_n(y)$, while $\mu = \delta_d$. If $w_2(t) = 2A|t|^\alpha$, $(0 < \alpha \le 2)$, $A > 0$, one could chose f as in (4.16.2).

Proof: Similar to Theorem 4.3 by using the Preliminaries 4.15. ∎

The next generalizes the last theorem.

Theorem 4.18. *Let μ be a probability measure on $[-a,a]$ and consider the upper concave envelope $\tilde{\mathcal{H}}_n^*(u)$ of $\tilde{\mathcal{H}}_n(u)$; the latter is defined in (4.17.2), ($n = 2m$). Also, consider the error $E(x_0)$ defined in (4.17.3).*

Then

$$E(x_0) \leq \frac{1}{2}\tilde{\mathcal{H}}_n^*(\psi(d)), \quad \text{where } x_0 \in [-a,a]. \tag{4.18.1}$$

When $\tilde{\mathcal{H}}_n$ is concave in u then the right hand side of (4.18.1) equals $(1/2)\tilde{G}_n(d)$. If, moreover, $\omega_2(\frac{1}{2}w_2, |t|) = w_2(t)$, the inequality is attained as in Theorem 4.17. Otherwise (4.18.1) is still (non-trivially, that is with $\mu \neq \delta_{x_0}$) attained, when $x_0 = 0$ and $\omega_2(\frac{1}{2}w_2, |t|) = w_2(t)$, by the same function $f(y) = \frac{1}{2}\tilde{G}_n(y)$ and a two points supported probability measure μ.

Proof: As in Theorem 4.17.∎

Remark 4.19 Theorems 4.6 and 4.9 are also useful in the present situation where $g(y)$ is replaced by an upper bound $w_2(y)$ on $\omega_2(f^{(n)}, y)$.

CHAPTER 5

MISCELLANEOUS SHARP INEQUALITIES AND RELATED CONVERGENCE THEOREMS FOR PROBABILITY MEASURES

The first section deals with an attainable inequality of the sort presented in the previous chapters which involves a new measure of smoothness, the reduced K-functional.

The second section contains a generalization of a theorem of P. P. Korovkin, see [22], concerning the Fourier coefficients of a sequence of density functions. Also several related results are given as well as applications.

I. A "K-ATTAINABLE" INEQUALITY

We introduce the following notion:

Definition 5.1 Let $f \in C([a,b])$. Let $n \in \mathbb{N}$ and $x_0 \in [a,b]$ be fixed. We define the *reduced K-functional* as:

$$K_n^{(x_0)}(f,t) = \inf(\| f - g \| + t \| g^{(n)} \|).$$

Here, $t \geq 0$ while g ranges through the functions $g \in C^n([a,b])$ satisfying $g^{(k)}(x_0) = 0$; $k = 1, \ldots, n-1$ where $\| \cdot \|$ denotes the sup-norm.

It is similar to a functional $K_n(f,t)$ employed by Peetre [33], where one does not impose the conditions $g^{(k)}(x_0) = 0$.

The quantity $K_n^{(x_0)}$ measures how well f can be approximated by a smooth function g. It is the right sort of measure for the approximation of small functions f possessing large derivatives in sup-norm.

Remark 5.2 The reduced K-functional has the following properties. Namely it is:

(i) Subadditive in terms of f.

(ii) Continuous, nonnegative, monotonely increasing and concave in t (and hence subadditive).

(iii) $K_n^{(x_0)}(f, 0) = 0$.

I.e. it has the basic properties of the usual Peetre functional $K_n(f, t)$. When $n = 1$, then $K_1^{(x_0)} = K_1$. Obviously

$$K_n(f, t) \le K_n^{(x_0)}(f, t)$$

with equality when f is a constant function.

Now we give the main result of this section:

Theorem 5.3. *Let μ be a probability measure on $[a,b]$ with prescribed absolute moment:*

$$d_1(x_0) = \int |y - x_0|\,\mu(dy)\,, \tag{5.3.1}$$

where x_0 is a fixed point of $[a,b]$. Consider $f \in C([a,b])$.

Then

$$\left|\int f\,d\mu - f(x_0)\right| \le 2K_n^{(x_0)}\left(f;\,\frac{d_1(x_0)(c(x_0))^{n-1}}{2n!}\right), \tag{5.3.2}$$

where $c(x_0) = \max(x_0 - a, b - x_0)$, $n \in \mathbb{N}$.

The above inequality is attained (i.e. it is sharp) by the function $f(x) = (x - x_0)^n$ and the probability measure μ_0 carried by $\{x_0, a\}$ if $x_0 - a \ge b - x_0$, as well as by $\{x_0, b\}$ if $x_0 - a \le b - x_0$: in each case having mass $[1 - (d_1(x_0)/c(x_0))]$ and $d_1(x_0)/c(x_0)$, respectively.

Proof: For $f \in C([a,b])$ and $g \in C^n([a,b])$ we have $f(y) - f(x_0) = (f(y) - g(y)) + (g(y) - g(x_0)) + (g(x_0) - f(x_0))$. Integrating relative to μ obtain

$$\begin{aligned}\left|\int f\,d\mu - f(x_0)\right| &\le \left|\int (f(y) - g(y))\mu(dy)\right| + |g(x_0) - f(x_0)| \\ &+ \left|\int (g(y) - g(x_0))\mu(dy)\right| \le 2\,\|\,f - g\,\| + \int |g(y) - g(x_0)|\mu(dy)\,,\end{aligned}$$

where $\|\cdot\|$ denotes the supremum norm.

Now assume that $g^{(k)}(x_0) = 0$, $k = 1, \ldots, n-1$. By Taylor's theorem we have, $\gamma \in (x_0, y)$):

$$|g(y) - g(x_0)| = \left|\frac{g^{(n)}(\gamma)}{n!}(y - x_0)^n\right| \le \frac{\|\,g^{(n)}\,\|}{n!}|y - x_0|^n\,.$$

Thus

$$\int |g(y)-g(x_0)|\mu(dy) \leq \frac{\| g^{(n)} \|}{n!} \int |y-x_0|^n \mu(dy) .$$

Letting $d_n(x_0) = (\int |y-x_0|^n \mu(dy))^{1/n}$ we then have

$$\left| \int f\, d\mu - f(x_0) \right| \leq 2 \| f-g \| + \frac{\| g^{(n)} \|}{n!} d_n^n(x_0) .$$

Now by applying Definition 5.1 we conclude:

$$\left| \int f\, d\mu - f(x_0) \right| \leq 2K_n^{(x_0)}\left(f;\ \frac{d_n^n(x_0)}{2n!} \right).$$

Since clearly $d_n^n(x_0) \leq d_1(x_0)(c(x_0))^{n-1}$ we have proved (5.3.2).

Furthermore, consider $f(x) = (x-x_0)^n$ this satisfies $f^{(i)}(x_0) = 0;\ i = 1,\ldots,n-1$. Then, integrating relative to the probability measure μ_0 described in the theorem one easily verifies the last assertion. Note that (taking $g = f$ in the definition of $K_n^{(x_0)}$) $K_n^{(x_0)}(f,t) \leq t \| f^{(n)} \| = tn!$.∎

Corollary 5.4. *We get the attainable inequality:*

$$\left| \int f\, d\mu - f(x_0) \right| \leq 2\, K_1\left(f; \frac{d_1(x_0)}{2} \right), \tag{5.4.1}$$

(where K_1 is the usual K-functional).

Now by using the language of positive linear operators one obtains:

Corollary 5.5. *(Pointwise approximation) Let $f \in C([a,b])$ and let L be a positive linear operator acting on $C([a,b])$ satisfying:*

$$L(1,x) = 1 \quad \text{all} \quad x \in [a,b] .$$

Then, we have the sharp inequality:

$$|L(f,x)-f(x)| \le 2\,K_1(f;\frac{1}{2}L(|y-x|,x)), \tag{5.5.1}$$

all $x \in [a,b]$.

Proof: By Riesz representation theorem there exists a probability measure μ_x such that $L(f,x) = \int f(t)\mu_x(dt)$. And then (5.5.1) is just another way of writing (5.3.2) with $n = 1$. ∎

Remark It may be interesting to note (see [33]) that $K_1(f,t) \le M\omega_1(f,t)$, where M is a constant depending only on the space $C([a,b])$ with the supremum norm.

II. MISCELLANEOUS SHARP INEQUALITIES AND KOROVKIN TYPE CONVERGENCE THEOREMS INVOLVING SEQUENCES OF PROBABILITY MEASURES

The following is the basis for the next convergence results.

Lemma 5.6. *For* $k,\ell \ge 2$, $k,\,\ell \in \mathbb{N}$ *there exists a positive constant* $C(k,\ell) \ge [k^2(k^2-1)]/[\ell^2(\ell^2-1)]$ *such that:*

(5.6.1)

$$[k^2(1-\cos t)-(1-\cos kt)] \le C(k,\ell)[\ell^2(1-\cos t)-(1-\cos \ell t)], \quad \text{all } t \in [0,\pi].$$

Proof: Since $|\sin nt| \le n|\sin t|, n \in \mathbb{N}$ we have that for $t \in (0,\pi]$:

$$n^2(1-\cos t)-(1-\cos nt) = \left(n^2 - \frac{\sin^2 \frac{nt}{2}}{\sin^2 \frac{t}{2}}\right)(1-\cos t) \ge 0\,.$$

The function $\varphi(t) = \frac{k^2(1-\cos t)-(1-\cos kt)}{\ell^2(1-\cos t)-(1-\cos \ell t)}$ on $[0,\pi]$ satisfies $\lim_{t\to 0}\varphi(t) = \frac{k^2(k^2-1)}{\ell^2(\ell^2-1)} > 0$, so that $\varphi(t)$ is strictly positive and continuous, therefore bounded.∎

Remark We conjecture that

$$C(k,\ell) = k^2(k^2-1)\ell^{-2}(\ell^2-1)^{-1} \text{ if } 2 \le \ell \le k\,.$$

It is correct for $k \le 5$ and the corresponding inequality is sharp.

Definition 5.7. *Let μ be a probability measure on $[0,\pi]$. Its Fourier-Stieltjes coefficients are defined as*

$$(5.7.1) \qquad p_k = \int_0^\pi \cos kt\ \mu(dt),\ (k = 0,1,2,\cdots)\,.$$

If $p_1 = 1$, then $\mu = \delta_0$.

Lemma 5.8. *Let μ be a probability measure on $[0,\pi]$ with Fourier-Stieltjes coefficients $p_k, k \in \mathbb{Z}^+$ and $p_1 \neq 1$.*

Then

$$(5.8.1) \qquad \left[k^2 - \left(\frac{1-p_k}{1-p_1}\right)\right] \le C(k,\ell)\left[\ell^2 - \left(\frac{1-p_\ell}{1-p_1}\right)\right],$$

where $k,\ell \ge 2$, k, $\ell \in \mathbb{N}$.

Proof: Integrate (5.6.1) relative to μ and divide both sides by $(1-p_1)$.∎

Theorem 5.9. *Let $\ell \in \mathbb{N}$, $\ell \ge 2$. If $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of probability measures on $[0,\pi]$ with Fourier-Stieltjes coefficients p_{kn} such that $p_{1n} \neq 1$ and*

$\lim_{n\to\infty}\left(\frac{1-p_{\ell n}}{1-p_{1n}}\right)=\ell^2$, *then* $\lim_{n\to\infty}\left(\frac{1-p_{kn}}{1-p_{1n}}\right)=k^2$ *for all* $k\geq 2$, $k\in\mathbb{N}$.

Proof: Use (5.8.1).∎

Note A special case of the last theorem is due to P. P. Korovkin (see [22]). He assumed that the probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ are absolutely continuous with respect to the Lebesque measure and further that $k\geq\ell=2$.

Remark 5.10 In the sequel $k,\ell\in\mathbb{N}$ and $k,\ell\geq 2$. Let $g(s)$ be the characteristic function (Fourier transform) of a probability measure μ on $[0,\pi]$. We have that

$$Re(1-g(s))=\int_0^\pi(1-\cos st)\mu(dt),\quad s\in\mathbb{R}\,.$$

Then by applying (5.6.1) we get

$$[k^2(1-Reg(1))-(1-Reg(k))]\leq C(k,\ell)[\ell^2(1-Reg(1))-(1-Reg(\ell))]\,.$$

As an illustration, let $g(s)=|f(s)|^2$, where f is a characteristic function. Then also g is a characteristic function. It follows that

$$[k^2(1-|f(1)|^2)-(1-|f(k)|^2)]\leq C(k,\ell)[\ell^2(1-|f(1)|^2)-(1-|f(\ell)|^2)]$$

Consequently, if a sequence $\{f_n\}_{n\in\mathbb{N}}$ of characteristic functions with $|f_n(1)|<1$ satisfies

$$\lim_{n\to\infty}\left(\frac{1-|f_n(k)|^2}{1-|f_n(1)|^2}\right)=k^2\quad\text{for one }\ k\in\mathbb{N}\,,\ k\geq 2$$

then for all such k.

Now we proceed to a similar type of result.

Lemma 5.11. *Let $k \geq \ell > 1$, $B(k,\ell) = \ell^2(\ell^2-1)k^{-2}(k^2-1)^{-1}$. Then*
(5.11.1)

$$\Big[(\operatorname{Cos} h\,\ell t - 1) - \ell^2(\operatorname{Cos} h\,t - 1)\Big] \leq B(k,\ell)\Big[(\operatorname{Cos} h\,kt - 1) - k^2(\operatorname{Cos} h\,t - 1)\Big]$$

for all $t \in \mathbb{R}$.

Here the constant $B(k,\ell)$ cannot be improved, i.e. (5.11.1) is sharp.

Proof: Easy, namely by writing (5.11.1) as

$$\sum_{j=2}^{\infty} \frac{t^{2j}}{(2j)!}(\ell^{2j} - \ell^2) \leq B(k,\ell) \sum_{j=2}^{\infty} \frac{t^{2j}}{(2j)!}(k^{2j} - k^2).$$

The last assertion follows by

$$A_\gamma(t) = [(\operatorname{Cos} h\gamma t - 1) - \gamma^2(\operatorname{Cos} ht - 1)] \geq 0, \quad \gamma > 1 \quad \text{and} \quad \lim_{t \to 0}(A_\ell / A_k) = B(k,\ell).$$

∎

Definition 5.12. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ such that the following integrals exist*

$$\tilde{p}_{k,n} = \int_{\mathbb{R}} \operatorname{Cos} hkt\, \mu_n(dt), \quad (k \in \mathbb{R}). \tag{5.12.1}$$

We shall call the numbers $\tilde{p}_{k,n}$ hyperbolic coefficients of the measure μ_n.

Now we have:

Theorem 5.13. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ with $\tilde{p}_{k,n} < \infty$ and $\tilde{p}_{1,n} \neq 1$. Let $k > 1$ and suppose that:*

$$\lim_{n\to\infty} \frac{\tilde{p}_{k,n}-1}{\tilde{p}_{1,n}-1} = k^2, \quad \text{then} \lim_{n\to\infty} \frac{\tilde{p}_{\ell,n}-1}{\tilde{p}_{1,n}-1} = \ell^2 \text{ for all } 1 < \ell \leq k\,.$$

PROOF: Immediate from an integration of (5.11.1) with respect to μ_n.∎

The next result is a characterization of the convexity of functions and leads to some applications as it is Proposition 5.16.

LEMMA 5.14. *If $f : (0,\infty) \to \mathbb{R}$ then $h(x) = f(x)/x$ is convex iff $\varphi(k) = (k^2 f(x) - f(kx))/(k^2-k)$ is non-increasing in $k > 1$, for each $x > 0$. Equivalently if $\psi(k) = (kf(x) - f(kx))/(k^2-k)$ is non-increasing in $k > 1$, for each $x > 0$.*

PROOF: Let us first assume that h is convex. Consider $1 < \lambda < k$. We have

$$\lambda x = \Big(\frac{k-\lambda}{k-1}\Big)x + \Big(\frac{\lambda-1}{k-1}\Big)kx\,, \quad \text{with} \quad \frac{k-\lambda}{k-1} + \frac{\lambda-1}{k-1} = 1\,,$$

where both $\frac{k-\lambda}{k-1}, \frac{\lambda-1}{k-1} > 0$. Since h is convex, one has

$$h(\lambda x) \leq \Big(\frac{k-\lambda}{k-1}\Big)h(x) + \Big(\frac{\lambda-1}{k-1}\Big)h(kx)\,. \tag{5.14.1}$$

Substituting $h(x) = f(x)/x$ one obtains precisely $\varphi(\lambda) \geq \varphi(k)$. Next, assume that φ is non-increasing for each $x > 0$. This is equivalent to (5.14.1). Now suppose $0 < x_1 < x_2$ and let $\alpha = x_2/x_1 > 1$. Applying (5.14.1) with $\lambda = (1+\alpha)/2$, $k = \alpha$ and $x = x_1$ one has $h\big(\frac{x_1+x_2}{2}\big) = h\,\Big(\big(\frac{1+\alpha}{2}\big)x_1\Big) \leq \Big(\frac{\alpha-(\frac{1+\alpha}{2})}{\alpha-1}\Big)\,h(x_1) + \Big(\frac{(\frac{1+\alpha}{2})-1}{\alpha-1}\Big)h(\alpha x_1) = \frac{1}{2}h(x_1) + \frac{1}{2}h(x_2)$. Therefore h is convex.

Next we prove that $\varphi(k)$ being non-increasing in $k > 1$ for each $x > 0$ is equivalent to $\psi(k)$ being non-increasing. In other words, we want to show that for $1 < \lambda < k$:

$$\frac{[\lambda^2 f(x) - f(\lambda x)]}{(\lambda^2 - \lambda)} - \frac{[k^2 f(x) - f(kx)]}{(k^2 - k)} \geq 0$$

is equivalent to

$$\frac{[\lambda f(x) - f(\lambda x)]}{(\lambda^2 - \lambda)} - \frac{[k f(x) - f(kx)]}{(k^2 - k)} \geq 0 .$$

In fact, the difference between the two left hand sides equals

$$\frac{(\lambda^2 - \lambda) f(x)}{(\lambda^2 - \lambda)} - \frac{(k^2 - k) f(x)}{(k^2 - k)} = 0 .$$

∎

Remark 5.15 Let $f : (0, \infty) \to \mathbb{R}$ such that $f(x)/x$ is convex. Then by Lemma 5.14 one has for $1 < \lambda < k$, all $x > 0$, the two equivalent inequalities:

$$(k^2 f(x) - f(kx)) \leq \left(\frac{k^2 - k}{\lambda^2 - \lambda}\right)(\lambda^2 f(x) - f(\lambda x)) \tag{5.15.1}$$

and

$$(k f(x) - f(kx)) \leq \left(\frac{k^2 - k}{\lambda^2 - \lambda}\right)(\lambda f(x) - f(\lambda x)) . \tag{5.15.2}$$

Quantities such as $[\lambda f(x) - f(\lambda x)]$ can be regarded as a measure of linearity for f.

Proposition 5.16. *For $k > \lambda > 1$ and $x > 0$ obtain the following two equivalent sharp inequalities:*

$$(5.16.1)\quad k^2(1-e^{-x})-(1-e^{-kx}) < \left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda^2(1-e^{-x})-(1-e^{-\lambda x}))$$

and

$$(5.16.2)\quad k(1-e^{-x})-(1-e^{-kx}) < \left(\frac{k^2-k}{\lambda^2-\lambda}\right)(\lambda(1-e^{-x})-(1-e^{-\lambda x})).$$

PROOF: Note that the function $f(x)=1-e^{-x}$, $x\in(0,+\infty)$ satifies:

$$\frac{f(x)}{x}=\int_0^1 e^{-\theta x}d\theta$$

showing that $f(x)/x$ is convex. By Lemma 5.14 we have (5.15.1) and (5.15.2). These are precisely (5.16.1) and (5.16.2). The sharpness follows from

$$\lim_{x\to 0}\frac{k^2(1-e^{-x})-(1-e^{-kx})}{\lambda^2(1-e^{-x})-(1-e^{-\lambda x})}=\lim_{x\to 0}\frac{k(1-e^{-x})-(1-e^{-kx})}{\lambda(1-e^{-x})-(1-e^{-\lambda x})}=\left(\frac{k^2-k}{\lambda^2-\lambda}\right).\blacksquare$$

Some applications of the last proposition are Theorem 5.19 and Proposition 5.21 following.

DEFINITION 5.17. *Let μ be a probability measure on $\mathbb{R}^+$. For $\lambda\geq 0$ its Laplace transform is defined as:*

$$(5.17.1)\qquad \varphi(\lambda)=\int_0^\infty e^{-\lambda t}\mu(dt).$$

LEMMA 5.18. *Let $k\geq\lambda>1$ and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^+$ with existing Laplace transforms φ_n such that $\varphi_n(1)\neq 1$ all $n\in\mathbb{N}$. Then one has the equivalent inequalities:*

(5.18.1)
$$\left[k^2 - \left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)}\right)\right] \le \left(\frac{k^2-k}{\lambda^2-\lambda}\right)\left[\lambda^2 - \left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right)\right]$$

and

(5.18.2)
$$\left[k - \left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)}\right)\right] \le \left(\frac{k^2-k}{\lambda^2-\lambda}\right)\left[\lambda - \left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right)\right].$$

Proof: Integrate (5.16.1) and (5.16.2) relative to μ_n and divide by $(1-\varphi_n(1)) > 0$. ∎

Theorem 5.19. *Let $\lambda > 1$. Then*

$$\lim_{n\to\infty}\left(\frac{1-\varphi_n(\lambda)}{1-\varphi_n(1)}\right) = \lambda^2, \text{ (respectively } \lambda)$$

implies that

$$\lim_{n\to\infty}\left(\frac{1-\varphi_n(k)}{1-\varphi_n(1)}\right) = k^2, \text{(respectively } k) \text{ for all } k \ge \lambda.$$

Proof: Apply (5.18.1), (respectively (5.18.2)). ∎

Recall:

Definition 5.20. *The Weierstrass operator is the positive linear operator defined by*

$$(W_n f)(t) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} f(x)e^{-n(t-x)^2}dx, \text{ all } n \in \mathbb{N}.$$

where $f \in C_B(\mathbb{R})$.

One has $W_n f \xrightarrow{u} f$ as $n \to \infty$.

Proposition 5.21. *Let $f \in C_B(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |f(x)|^n dx = c_n \in (0, \infty)$; all $n \in \mathbb{N}$. Consider $k \geq \lambda$ with $k, \lambda \in \mathbb{N}$.*

$$\text{If} \quad \lim_{n \to \infty} \left[\frac{1 - \sqrt{\frac{\pi}{\lambda}} W_\lambda\left(\frac{|f|^n}{c_n}, t \right)}{1 - \sqrt{\pi} W_1\left(\frac{|f|^n}{c_n}, t \right)} \right] = \lambda^2, \text{(respectively } \lambda)$$

$$\text{then} \quad \lim_{n \to \infty} \left[\frac{1 - \sqrt{\frac{\pi}{k}} W_k\left(\frac{|f|^n}{c_n}, t \right)}{1 - \sqrt{\pi} W_1\left(\frac{|f|^n}{c_n}, t \right)} \right] = k^2, (\text{ respectively } k).$$

Proof: Apply (5.16.1) (respectively (5.16.2)) with x replaced by $(t - x)^2$, where t is fixed. Afterwards multiply by $|f|^n$ and integrate over $\mathbb{R}$. ∎

References

[1] Alfsen, E. M. (1971). *Compact Convex Sets and Boundary Integrals.* Springer-Verlag, Berlin-New York.

[2] Amir, D. and Ziegler, Z. (1978). *Korovkin shadows and Korovkin systems in $C(S)$-spaces.* Journal of Math. Analysis and Appl. **62**, 640–675.

[3] Apostol, T. M. (1969). *Mathematical Analysis.* Addison-Wesley, London.

[4] Bauer, H. (1978). *Approximation and Abstract Boundaries.* Ann. Math. *Monthly* **85**, 632–647.

[5] Butzer, P. L. and Berens, H. (1967). *Semi-Groups of Operators and Approximation.* Springer-Verlag, Berlin-New York.

[6] Butzer, P. L. and Hahn, L. (1978). *General theorems on rates of convergence in distribution of random variables. I. General limit theorems.* Journal of Multivariate Analysis **8**, 181–201.

[7] Censor, E. (1971). *Quantitative Results for Positive Linear Approximation Operators.* Journal of Approximation Theory **4**, 442–450.

[8] DeVore, R. A. (1972). *The Approximation of Continuous Function by Positive Linear Operators.* Lecture Notes in Mathematics No. 293, Springer-Verlag, Berlin-New York.

[9] Ditzian, Z. (1975) *Convergence of Sequences of Linear Positive Operators: Remarks and Applications.* Journal of Approximation Theory **14**, 296–301.

[10] Donner, K. (1982). *Extension of Positive Operators and Korovkin Theorems.* Lecture Notes in Mathematics No. 904, Springer-Verlag, Berlin-New York.

[11] Dunford, N. and Schwartz, J. T. (1957). *Linear Operators, part I.* Interscience, New York.

[12] Feller, W. (1966). *An Introduction to Probability Theory and its Applications, Vol. II.* Wiley, New York.

[13] Gonska, H. H. (1983). *On Approximation of Continuously differentiable functions by Positive Linear Operators.* Bull. Austral. Math. Soc. Vol. **27**, 73–81.

[14] Hahn, L. (1982). *Stochastic Methods in connection with Approximation theorems for Positive Linear Operators.* Pacific Journal of Mathematics, *Vol. 101, No. 2,* 307–319.

[15] Johnen, H. (1972). *Inequalities connected with the Moduli of Smoothness. Mat. Vestnik, Vol. 24,* 289–303.

[16] Karlin, S. and Studden, W. J. (1966). *Tchebycheff Systems: with Applications in Analysis and Statistics.* Interscience, New York.

[17] Kemperman, J. H. B. (1968). *The General Moment Problem, a geometric approach.* The Annals of Mathematical Statistics, *Vol. 39, No. 1,* 93–122.

[18] Kemperman, J. H. B. (1983). *Moment theory.* Class notes, University of

Rochester, New York.

[19] Kemperman, J. H. B. (1983). *On the role of duality in the theory of Moments*. Semi-Infinite Progr. and Appl. An international symposium at Austin, Texas 1981. Lecture notes in Economics and Math. Systems No. 215, 63–92. Springer-Verlag, Berlin-New York.

[20] King, J. P. (1975). *Probability and Positive Linear Operators. Rev. Roum. Math. Pures et Appl. Tome XX, No. 3*, 325–327.

[21] Knopp, K. (1928). *Theory and application of infinite series*, (translation). Blackie and son ltd., London and Glasgow.

[22] Korovkin, P. P. (1958). *An asymptotic property of positive methods of summation of Fourier Series and best approximations of functions of Class Z_2 by linear positive polynomial operators*. (Russian) Uspehi Mat. Nauk 13, No. 6 (84), 99–103.

[23] Korovkin, P. P. (1960). *Linear Operators and Approximation Theory*. Hindustan Publ. Corp., Delhi, India.

[24] Lorentz, G. G. (1953). *Bernstein Polynomials*. Univ. of Toronto Press, Toronto.

[25] Lorentz, G. G. (1966). *Approximation of functions*. Holt, Rinehart and Winston, New York.

[26] Lorentz, G. G. (1972). *Korovkin Sets*. Lecture notes: Sept. 1972, U. C. Riverside, Center for Numerical Analysis, the University of Texas at Austin.

[27] Mond, B. (1976). *On the Degree of Approximation by Linear Positive Operators.*. *Journal of Approximation theory* **18**, 304–306.

[28] Mond, B. and Vasudevan, R. (1980). *On Approximation by Linear Positive Operators*. *Journal of Approximation Theory* **30**, 334–336.

[29] Šaškin, J. A. (1966). *Korovkin Systems in spaces of continuous functions*. American Mathematical Society translations on Analysis, Series 2, Vol. 54, 125–144.

[30] Singh, S. P. (1981). *On the degree of approximation by Szàsz Operators. Bull. Austral. Math. Soc. Vol. 24*, 221–225.

[31] Shisha, O. and Mond, B. (1968). *The degree of convergence of sequences of linear positive operators.*. *Nat. Acad. of Sc. U.S. (60)*, 1196–1200.

[32] Shisha, O. and Mond, B. (1968). *The degree of approximation to periodic functions by linear positive operators*. *Journal of Approximation Theory* **1**, 335–339.

[33] Schumaker, L. L. (1981). *Spline functions. Basic theory*. John Wiley & Sons, New York.

[34] Stark, E. L. (1972). *An extension of a theorem of P. P. Korovkin to singular integrals with not necessarily positive kernels*. *Akad. Van Wetensc. Proc. A. Math. Sci.* **75**, 227–235.

[35] Timan, A. (1963). *The theory of approximation of functions of a real variable*, (translation from Russian). Pergammon Press, Oxford-New York.