Fuzzy Fractional Approximations by Fuzzy normalized Bell and Squashing type neural network operators

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Abstract

This paper deals with the determination of the fuzzy fractional rate of convergence to the unit to some fuzzy neural network operators, namely, the fuzzy normalized bell and "squashing" type operators. This is given through the fuzzy moduli of continuity of the involved right and left fuzzy Caputo fractional derivatives of the approximated function and they appear in the right-hand side of the associated fuzzy Jackson type inequalities.

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1 Introduction

The Cardaliaguet-Euvrard operators were studied extensively in [14], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our fuzzy "normalized bell and squashing type operators" are motivated and inspired by the "bell" and "squashing functions" of [14]. The work in [14] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [14], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of fuzzy

Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above fuzzy neural network induced operators. All involved constants there are well determined. These are poitwise estimates involving the first fuzzy moduli of continuity of the engaged right and left fuzzy Caputo fractional derivatives of the function under approximation. We give all necessary background of fuzzy and fractional calculus.

Initial work of the subject was done in [1], [8] and [13]. These works motivated the current work.

2 Fuzzy Mathematical Analysis Background

We need the following basic background

Definition 1 *(see [23]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([19]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [23]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \quad \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_{\pm}^{(r)} = (\lambda \odot u)_{\pm}^{(r)}$, respectively.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{+}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_{-}^{(r)} - v_{-}^{(r)} \right|, \left| u_{+}^{(r)} - v_{+}^{(r)} \right| \right\},$$

where

$$[v]^{r} = \left[v_{-}^{(r)}, v_{+}^{(r)} \right]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [24], [23].

Here $\sum^{*}$ stands for fuzzy summation and $\widetilde{0} : \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^{*}(f,g) := \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2 *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_{1}^{(\mathcal{F})}(f,\delta)_{X} = \sup_{x,y \in X, \ |x-y| \leq \delta} D(f(x), f(y)), \quad \delta > 0.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_{b}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 3 *([4]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_{1}^{(\mathcal{F})}(f,\delta)_{X} < \infty$, for any $\delta > 0$.*

Proposition 4 *([4]) It holds*

$$\lim_{\delta \to 0} \omega_{1}^{(\mathcal{F})}(f,\delta)_{X} = \omega_{1}^{(\mathcal{F})}(f,0)_{X} = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$.

Proposition 5 *([4]) Here $[f]^{r} = \left[f_{-}^{(r)}, f_{+}^{(r)} \right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

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Note 6 *It is clear by Propositions 4, 5, that if* $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, *then* $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ *(uniformly continuous on* $\mathbb{R}$*).*

Proposition 7 *Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. *Assume that* $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_{-}^{(r)},\delta\right)_X$, $\omega_1\left(f_{+}^{(r)},\delta\right)_X$ *are finite for any* $\delta > 0$, $r \in [0,1]$, *where X any interval of* $\mathbb{R}$.
Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_{-}^{(r)},\delta\right)_X, \omega_1\left(f_{+}^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [8]. ■

We need

Remark 8 *([2]). Here* $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1,...,m \in \mathbb{N}$. *Suppose that*

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1,...,m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \leq \sum_{i=1}^{m} \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 9 *Let* $x, y \in \mathbb{R}_{\mathcal{F}}$. *If there exists* $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, *then we call z the H-difference on x and y, denoted* $x - y$.

Definition 10 *([22]) Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, *with* $\beta > 0$. *A function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *is H-difference at* $x \in T$ *if there exists an* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits (with respect to D)*

$$\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' *the H-derivative or fuzzy derivative of f at x.*

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$.

We mention

Theorem 11 *([20]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)}\right] \subseteq \mathbb{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)}\right)', \left(f(t)_+^{(r)}\right)'\right].$$

I.e.

$$(f')_\pm^{(r)} = \left(f_\pm^{(r)}\right)', \quad \forall\, r \in [0,1].$$

Remark 12 *([3]) Let $f \in C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we obtain*

$$\left[f^{(i)}(t)\right]^r = \left[\left(f(t)_-^{(r)}\right)^{(i)}, \left(f(t)_+^{(r)}\right)^{(i)}\right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)}\right)_\pm^{(r)} = \left(f_\pm^{(r)}\right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 13 *([3]) Let $f \in C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we have $f_\pm^{(r)} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.*

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [23].

Definition 14 *([18], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have*

$$D\left(\sum_P^* (v-u) \odot f(\xi), I\right) < \varepsilon.$$

We write

$$I := (FR)\int_a^b f(x)\,dx. \tag{3}$$

We mention

Theorem 15 *([19]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^b f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$*, furthermore it holds*

$$\left[(FR)\int_a^b f(x)\,dx\right]^r = \left[\int_a^b (f)_-^{(r)}(x)\,dx, \int_a^b (f)_+^{(r)}(x)\,dx\right],$$

$\forall\ r \in [0,1]$.

For the definition of general fuzzy integral we follow [21] next.

Definition 16 *Let* (Ω, Σ, μ) *be a complete* σ*-finite measure space. We call* $F : \Omega \to R_{\mathcal{F}}$ *measurable iff* $\forall$ *closed* $B \subseteq \mathbb{R}$ *the function* $F^{-1}(B) : \Omega \to [0,1]$ *defined by*

$$F^{-1}(B)(w) := \sup_{x\in B} F(w)(x)\,,\ all\ w \in \Omega$$

is measurable, see [21].

Theorem 17 *([21]) For* $F : \Omega \to \mathbb{R}_{\mathcal{F}}$*,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\},$$

the following are equivalent
(1) F *is measurable,*
(2) $\forall\ r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ *are measurable.*

Following [21], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \le r \le 1 \right\} \tag{4}$$

is a fuzzy real number for each $A \in \Sigma$.
The last fact leads to

Definition 18 *([21]) A measurable function* $F : \Omega \to \mathbb{R}_{\mathcal{F}}$*,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\}$$

is integrable if for each $r \in [0,1]$, $F_\pm^{(r)}$ *are integrable, or equivalently, if* $F_\pm^{(0)}$ *are integrable.*

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \le r \le 1 \right\}.$$

By [21], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.
Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right),\ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 19 *([21]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\, r \in [0, 1].$$

We need

Definition 20 *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions), $\forall\, [a, b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [15], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x (x - t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall\, x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$, $\forall\, b > a$. We set $D_{*a}^0 f(x) = f(x)$, $\forall\, x \in [a, \infty)$.

Lemma 21 *([10]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$, $\forall\, a \in \mathbb{R}$.*

Definition 22 *(see also [7], [16], [17]) Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a, b])$, $\forall\, [a, b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (J - x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{7}$$

$\forall\, x \leq b$. We set $D_{b-}^0 f(x) = f(x)$, $\forall\, x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a, b])$ and $D_{b-}^{\alpha} f$ exists a.e. on $[a, b]$, $\forall\, a < b$.

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Lemma 23 *([10]) Let $f \in C^{m-1}(R)$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$, $\forall\, b \in \mathbb{R}$.*

Convention 24 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \quad \text{for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \quad \text{for } x > x_0, \tag{9}$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 25 *(by [5]) Let $f \in C^n(\mathbb{R})$, where $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, \infty)$.*

Also we have

Proposition 26 *(by [5]) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in (-\infty, b]$.*

We further mention

Proposition 27 *(by [5]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{10}$$

for all $x, x_0 \in \mathbb{R} : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 28 *(by [5]) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{11}$$

for all $x, x_0 \in \mathbb{R} : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Corollary 29 *([10]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

We need

Proposition 30 *([10]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x,+\infty)}, \ \delta > 0, \ x \in \mathbb{R}. \tag{12}$$

(Here ω_1 is defined over $[x, +\infty)$ instead of $\mathbb{R}$).

Then G is continuous on $\mathbb{R}$.

Proposition 31 *([10]) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be jointly continous. Consider*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{(-\infty,x]}, \ \delta > 0, \ x \in \mathbb{R}. \tag{13}$$

(Here ω_1 is defined over $(-\infty, x]$ instead of $\mathbb{R}$).

Then H is continuous on $\mathbb{R}$.

By Propositions 30, 31 and Corollary 29 we derive

Proposition 32 *([10]) Let $f \in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, $x \in \mathbb{R}$. Then $\omega_1 (D^\alpha_{*x} f, h)_{[x,+\infty)}$, $\omega_1 \left(D^\alpha_{x-} f, h\right)_{(-\infty,x]}$ are continuous functions of $x \in \mathbb{R}$, $h > 0$ fixed.*

We make

Remark 33 *Let g be continuous and bounded from $\mathbb{R}$ to $\mathbb{R}$. Then*

$$\omega_1 (g, t) \leq 2 \|g\|_\infty < \infty. \tag{14}$$

Assuming that $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f\right)(t)$, are both continuous and bounded in $(x, t) \in \mathbb{R}^2$, i.e.

$$\|D^\alpha_{*x} f\|_\infty \leq K_1, \ \ \forall \ x \in \mathbb{R}; \tag{15}$$

$$\left\|D^\alpha_{x-} f\right\|_\infty \leq K_2, \ \ \forall \ x \in \mathbb{R}; \tag{16}$$

where $K_1, K_2 > 0$, we get

$$\begin{aligned} \omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)} &\leq 2K_1; \\ \omega_1 \left(D^\alpha_{x-} f, \xi\right)_{(-\infty,x]} &\leq 2K_2, \ \ \forall \ \xi \geq 0, \end{aligned} \tag{17}$$

for each $x \in \mathbb{R}$.

Therefore, for any $\xi \geq 0$,

$$\sup_{x \in \mathbb{R}} \left[\max\left(\omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)}, \omega_1 \left(D^\alpha_{x-} f, \xi\right)_{(-\infty,x]}\right)\right] \leq 2 \max(K_1, K_2) < \infty. \tag{18}$$

So in our setting for $f \in C^m(\mathbb{R})$, $\left\|f^{(m)}\right\|_\infty < \infty$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$, by Corollary 29 both $(D^\alpha_{*x} f)(t)$, $\left(D^\alpha_{x-} f\right)(t)$ are jointly continuous in (t, x) on $\mathbb{R}^2$. Assuming further that they are both bounded on $\mathbb{R}^2$ we get (18) valid. In particular, each of $\omega_1 (D^\alpha_{*x} f, \xi)_{[x,+\infty)}$, $\omega_1 \left(D^\alpha_{x-} f, \xi\right)_{(-\infty,x]}$ is finite for any $\xi \geq 0$.

Let us now assume only that $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x \in \mathbb{R}$. Then, by Proposition 15.114, p. 388 of [6], we find that $D^\alpha_{*x} f \in C([x, +\infty))$, and by [9] we obtain that $D^\alpha_{x-} f \in C((-\infty, x])$.

We make

Remark 34 *Again let* $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$; $f^{(m)}(x) = 1$, $\forall$, $x \in \mathbb{R}$; $x_0 \in \mathbb{R}$. *Notice* $0 < m - \alpha < 1$. *Then*

$$D_{*x_0}^{\alpha} f(x) = \frac{(x - x_0)^{m-\alpha}}{\Gamma(m - \alpha + 1)}, \quad \forall\ x \geq x_0. \tag{19}$$

Let us consider $x, y \geq x_0$, *then*

$$\left| D_{*x_0}^{\alpha} f(x) - D_{*x_0}^{\alpha} f(y) \right| = \frac{1}{\Gamma(m - \alpha + 1)} \left| (x - x_0)^{m-\alpha} - (y - x_0)^{m-\alpha} \right|$$

$$\leq \frac{|x - y|^{m-\alpha}}{\Gamma(m - \alpha + 1)}. \tag{20}$$

So it is not strange to assume that

$$\left| D_{*x_0}^{\alpha} f(x_1) - D_{*x_0}^{\alpha} f(x_2) \right| \leq K |x_1 - x_2|^{\beta}, \tag{21}$$

$K > 0$, $0 < \beta \leq 1$, $\forall\ x_1, x_2 \in \mathbb{R}$, $x_1, x_2 \geq x_0 \in \mathbb{R}$, *where more generally it is* $\left\| f^{(m)} \right\|_{\infty} < \infty$. *Thus, one may assume*

$$\omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty, x]} \leq M_1 \xi^{\beta_1}, \quad \text{and} \tag{22}$$

$$\omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x, +\infty)} \leq M_2 \xi^{\beta_2},$$

where $0 < \beta_1, \beta_2 \leq 1$, $\forall\ \xi > 0$, $M_1, M_2 > 0$; *any* $x \in \mathbb{R}$.

Setting $\beta = \min(\beta_1, \beta_2)$ *and* $M = \max(M_1, M_2)$, *in that case we obtain*

$$\sup_{x \in \mathbb{R}} \left\{ \max \left(\omega_1 \left(D_{x-}^{\alpha} f, \xi \right)_{(-\infty, x]}, \omega_1 \left(D_{*x}^{\alpha} f, \xi \right)_{[x, +\infty)} \right) \right\} \leq M \xi^{\beta} \to 0, \text{ as } \xi \to 0+. \tag{23}$$

We need

Definition 35 *([11]) Let* $f \in C_{\mathcal{F}}([a, b])$ *(fuzzy continuous on* $[a, b] \subset \mathbb{R}$*),* $\nu > 0$.

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x - t)^{\nu - 1} \odot f(t)\, dt, \quad x \in [a, b], \tag{24}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t - x)^{\nu - 1} \odot f(t)\, dt, \quad x \in [a, b], \tag{25}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 36 *([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ for every finite, pairwise disjoint, family*

$$(c_k, d_k)_{k=1}^{n} \subseteq (a,b) \quad \textit{with} \quad \sum_{k=1}^{n} (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^{n} D\left(f\left(d_k\right), f\left(c_k\right)\right) < \epsilon. \tag{26}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 37 *([11]) $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 38 *([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_{-}^{(r)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_{+}^{(r)}(t)\, dt \right) | 0 \le r \le 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt \right) | 0 \le r \le 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt \right) \right], \quad 0 \le r \le 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\,dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [15], [6].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\ x \in [a,b]$, *in short*

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{30}$$

We need

Lemma 39 *([11]) $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 40 *([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$\begin{aligned}
D_{b-}^{\nu\mathcal{F}} f(x) &:= \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\,dt \\
&= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{-}^{(r)}(t)\,dt, \right.\right. \\
&\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{+}^{(r)}(t)\,dt \right) | 0 \le r \le 1 \right\} \\
&= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\,dt, \right.\right. \\
&\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\,dt \right) | 0 \le r \le 1 \right\}.
\end{aligned} \tag{31}$$

We get

$$\begin{aligned}
\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r &= \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\,dt, \right.\right. \\
&\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\,dt \right) \right], \quad 0 \le r \le 1.
\end{aligned}$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{32}$$

$\forall\ x \in [a,b]$, *in short*

$$\left(D_{b-}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right) \leq D_{b-}^{\nu}\left(f_{+}^{(r)}\right), \quad \forall\ r \in [0,1].$$

We need

Lemma 41 *([11]) $D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

3 Fractional convergence with rates of real normalized bell type neural network operators ([13])

We need the following (see [14]).

Definition 42 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

We follow [14], [1], [8], [13].

Example 43 *(1) $b(x)$ can be the characteristic function over $[-1,1]$.*

(2) $b(x)$ can be the hat function over $[-1,1]$, i.e.,

$$b(x) = \begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

These are centered bell-shaped functions of compact support.

Here we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous.

In [13] we studied the fractional convergence with rates over the real line, to the unit operator, of the real "normalized bell type neural network operators",

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{34}$$

where $0<\alpha<1$ and $x\in\mathbb{R}$, $n\in\mathbb{N}$. The terms in the ratio of sums (34) can be nonzero iff

$$\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right| \leq T, \text{ i.e. } \left|x-\frac{k}{n}\right| \leq \frac{T}{n^{1-\alpha}},$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{35}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{36}$$

and $card(k)$:=cardinality$(k) \geq 1$ over $[nx-Tn^{\alpha}, nx+Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right). \tag{37}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number, and by $\lceil\cdot\rceil$ the ceiling of the number.

Without loss of generality ([13]) we may assume that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{38}$$

So here we assume (37) and we have

$$(H_n(f))(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{39}$$

We need

Theorem 44 *([13]) We consider $f:\mathbb{R}\to\mathbb{R}$. Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in AC^N([a,b])$, $\forall\ [a,b]\subset\mathbb{R}$, with $f^{(N)}\in L_{\infty}(\mathbb{R})$. Let also $x\in\mathbb{R}$, $T>0$, $n\in\mathbb{N}: n\geq\max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D_{*x}^{\beta}f$, $D_{x-}^{\beta}f$ are uniformly continuous functions or continuous and bounded on $[x,+\infty)$, $(-\infty,x]$, respectively.*

Then

1)

$$|H_n(f)(x) - f(x)| \leq \left(\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right| T^j}{j! n^{(1-\alpha)j}}\right) + \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}}. \tag{40}$$

$$\left\{ \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\},$$

above $\sum_{j=1}^{0} \cdot = 0,$

2)

$$\left| H_n (f)(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \left(H_n \left((\cdot - x)^j \right) \right)(x) \right| \leq \frac{T^{\beta}}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{41}$$

$$\left\{ \omega_1 \left(D_{*x}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{[x,+\infty)} + \omega_1 \left(D_{x-}^{\beta} f, \frac{T}{n^{1-\alpha}} \right)_{(-\infty,x]} \right\} =: \lambda_n(x),$$

3) assume further that $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we get*

$$|H_n(f)(x) - f(x)| \leq \lambda_n(x), \tag{42}$$

4) in case of $N = 1$, *we obtain again*

$$|H_n(f)(x) - f(x)| \leq \lambda_n(x). \tag{43}$$

Here we get fractionally with rates the poitwise convergence of $(H_n(f))(x) \to f(x)$, *as* $n \to \infty$, $x \in \mathbb{R}$.

We will use (40).

4 Fractional convergence with rates of Fuzzy normalized neural network operators

Here b as Definition 42, $n \in \mathbb{N}$ as in (37), and $f \in C_{\mathcal{F}}(\mathbb{R})$.

Based on (34), (39) we define the corresponding fuzzy operators

$$\left(H_n^{\mathcal{F}}(f) \right)(x) := \frac{\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]*} f\left(\frac{k}{n}\right) \odot b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{44}$$

where $0 < \alpha < 1$, and $x \in \mathbb{R}$.

We notice that ($r \in [0,1]$)

$$\left[\left(H_n^{\mathcal{F}}(f) \right)(x) \right]^r = \sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left[f\left(\frac{k}{n}\right) \right]^r \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left[f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right) \right]^r \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} = \tag{45}$$

$$\left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f_{-}^{(r)}\left(\frac{k}{n}\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)},\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} f_{+}^{(r)}\left(\frac{k}{n}\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}\right] = \tag{46}$$

$$\left[\left(H_n\left(f_{-}^{(r)}\right)\right)(x), \left(H_n\left(f_{+}^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(H_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = H_n\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1], \tag{47}$$

respectively.

We present

Theorem 45 *Let $x \in \mathbb{R}$; then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{T}{n^{1-\alpha}}\right)_{\mathbb{R}}. \tag{48}$$

Notice that (48) gives $H_n^{\mathcal{F}}(f) \overset{D}{\to} f$ pointwise and uniformly, as $n \to \infty$, when $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$.

Proof. We observe that

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) = \sup_{r\in[0,1]} \max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_{-}^{(r)}(x) - f_{-}^{(r)}(x)\right|,\right.$$

$$\left.\left|\left(H_n^{\mathcal{F}}(f)\right)_{+}^{(r)}(x) - f_{+}^{(r)}(x)\right|\right\} \overset{(47)}{=}$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(H_n\left(f_{-}^{(r)}\right)\right)(x) - f_{-}^{(r)}(x)\right|, \left|\left(H_n\left(f_{+}^{(r)}\right)\right)(x) - f_{+}^{(r)}(x)\right|\right\} \overset{\text{(by [12])}}{\leq} \tag{49}$$

$$\sup_{r\in[0,1]} \max\left\{\omega_1\left(f_{-}^{(r)}, \frac{T}{n^{1-\alpha}}\right)_{\mathbb{R}}, \omega_1\left(f_{+}^{(r)}, \frac{T}{n^{1-\alpha}}\right)_{\mathbb{R}}\right\} \tag{50}$$

(by Proposition 7)

$$= \omega_1^{(\mathcal{F})}\left(f, \frac{T}{n^{1-\alpha}}\right)_{\mathbb{R}}.$$

■

We also give

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Theorem 46 *We consider $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C^N_{\mathcal{F}}(\mathbb{R})$, with $D\left(f^{(N)}(\cdot), \widetilde{o}\right) \in L_\infty(\mathbb{R})$. Let also $x \in \mathbb{R}$, $T > 0$, $n \in \mathbb{N}$: $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. We further assume that $D^{\beta\mathcal{F}}_{*x} f$, $D^{\beta\mathcal{F}}_{x-} f$ are fuzzy uniformly continuous functions or fuzzy continuous and bounded on $[x, +\infty)$, $(-\infty, x]$, respectively.*

Then

1)

$$D\left(\left(H^{\mathcal{F}}_n(f)\right)(x), f(x)\right) \leq \sum_{j=1}^{N-1} \frac{T^j}{j! n^{(1-\alpha)j}} D\left(f^{(j)}(x), \widetilde{o}\right) + \frac{T^\beta}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{51}$$

$$\left\{\omega^{(\mathcal{F})}_1\left(D^{\beta\mathcal{F}}_{*x} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega^{(\mathcal{F})}_1\left(D^{\beta\mathcal{F}}_{x-} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\} =: \lambda^{\mathcal{F}}_n(x),$$

above $\sum_{j=1}^{0} \cdot = 0$,

2) in case of $N = 1$ or $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, all $j = 1, ..., N-1$, $N \geq 2$, we get that

$$D\left(\left(H^{\mathcal{F}}_n(f)\right)(x), f(x)\right) \leq \frac{T^\beta}{\Gamma(\beta+1) n^{(1-\alpha)\beta}} \cdot \tag{52}$$

$$\left\{\omega^{(\mathcal{F})}_1\left(D^{\beta\mathcal{F}}_{*x} f, \frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)} + \omega^{(\mathcal{F})}_1\left(D^{\beta\mathcal{F}}_{x-} f, \frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}.$$

Here we get fractionally with high rates the fuzzy pointwise convergence of $\left(H^{\mathcal{F}}_n(f)\right)(x) \xrightarrow{D} f(x)$, as $n \to \infty$, $x \in \mathbb{R}$.

Proof. We have

$$D\left(f^{(N)}(x), \widetilde{o}\right) = \sup_{r \in [0,1]} \max\left\{\left|(f^{(r)}_-)^{(N)}(x)\right|, \left|(f^{(r)}_+)^{(N)}(x)\right|\right\},$$

so that

$$\left|(f^{(r)}_\pm)^{(N)}(x)\right| \leq D\left(f^{(N)}(x), \widetilde{o}\right), \ \forall\ r \in [0,1], \text{ any } x \in \mathbb{R}. \tag{53}$$

Thus

$$(f^{(r)}_\pm)^{(N)} \in L_\infty(\mathbb{R}), \ \ \forall\ r \in [0,1].$$

Also we have $f^{(r)}_\pm \in C^N(\mathbb{R})$, hence $f^{(r)}_\pm \in AC^N([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$; $\forall\ r \in [0,1]$.

By assumptions we get that $\left(D^{\beta\mathcal{F}}_{*x} f\right)^{(r)}_\pm \in C_U([x, +\infty))$ or in $C_b([x, +\infty))$ (bounded and continuous on $[x, +\infty)$ functions), also it holds $\left(D^{\beta\mathcal{F}}_{x-} f\right)^{(r)}_\pm \in C_U((-\infty, x])$ or in $C_b((-\infty, x])$, $\forall\ r \in [0,1]$.

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By (30) we have

$$\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}=D_{*x}^{\beta}\left(f_{\pm}^{(r)}\right),\quad \forall r\in[0,1]. \tag{54}$$

And by (33) we have that

$$\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}=D_{x-}^{\beta}\left(f_{\pm}^{(r)}\right),\quad \forall r\in[0,1]. \tag{55}$$

Therefore all asumptions of Theorem 44 are fulfilled by each of $f_{\pm}^{(r)}$, $\forall r\in[0,1]$; thus (40) is valid for these functions.

We observe that

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x),f(x)\right)=\sup_{r\in[0,1]}\max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_{-}^{(r)}(x)-f_{-}^{(r)}(x)\right|,\right.$$

$$\left.\left|\left(H_n^{\mathcal{F}}(f)\right)_{+}^{(r)}(x)-f_{+}^{(r)}(x)\right|\right\}\overset{(47)}{=}$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(H_n\left(f_{-}^{(r)}\right)\right)(x)-f_{-}^{(r)}(x)\right|,\left|\left(H_n\left(f_{+}^{(r)}\right)\right)(x)-f_{+}^{(r)}(x)\right|\right\}\overset{(40)}{\leq} \tag{56}$$

$$\sup_{r\in[0,1]}\max\left\{\sum_{j=1}^{N-1}\frac{\left|\left(f_{-}^{(r)}\right)^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}+\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot\right.$$

$$\left\{\omega_1\left(D_{*x}^{\beta}\left(f_{-}^{(r)}\right),\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}\left(f_{-}^{(r)}\right),\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

$$\sum_{j=1}^{N-1}\frac{\left|\left(f_{+}^{(r)}\right)^{(j)}(x)\right|T^j}{j!n^{(1-\alpha)j}}+\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot \tag{57}$$

$$\left.\left\{\omega_1\left(D_{*x}^{\beta}\left(f_{+}^{(r)}\right),\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(D_{x-}^{\beta}\left(f_{+}^{(r)}\right),\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}=$$

(by Remark 12, (54), (55))

$$\sup_{r\in[0,1]}\max\left\{\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}\right)_{-}^{(r)}(x)\right|T^j}{j!n^{(1-\alpha)j}}+\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot\right.$$

$$\left\{\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_{-}^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_{-}^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\},$$

$$\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}\right)_{+}^{(r)}(x)\right|T^j}{j!n^{(1-\alpha)j}}+\frac{T^{\beta}}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot \tag{58}$$

$$\left\{\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}\overset{(1)}{\leq}$$

$$\sum_{j=1}^{N-1}\frac{T^j}{j!n^{(1-\alpha)j}}\sup_{r\in[0,1]}\max\left\{\left|\left(f^{(j)}\right)_-^{(r)}(x)\right|,\left|\left(f^{(j)}\right)_+^{(r)}(x)\right|\right\}+ \tag{59}$$

$$\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_-^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\right.$$

$$\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_-^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]},\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+$$

$$\left.\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\overset{\text{(by Definition 1, (1))}}{\leq}$$

$$\sum_{j=1}^{N-1}\frac{T^j}{j!n^{(1-\alpha)j}}D\left(f^{(j)}(x),\widetilde{o}\right)+\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_-^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)},\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}\right\}+\right.$$

$$\left.\sup_{r\in[0,1]}\max\left\{\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_-^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]},\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_+^{(r)},\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}\right\}$$

$$\overset{\text{(by Proposition 7)}}{=}\sum_{j=1}^{N-1}\frac{T^j}{j!n^{(1-\alpha)j}}D\left(f^{(j)}(x),\widetilde{o}\right)+\frac{T^\beta}{\Gamma(\beta+1)n^{(1-\alpha)\beta}}\cdot$$

$$\left\{\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{T}{n^{1-\alpha}}\right)_{[x,+\infty)}+\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{T}{n^{1-\alpha}}\right)_{(-\infty,x]}\right\}, \tag{61}$$

proving the claim. ■

We need

Definition 47 *([14], [1]) Let the nonnegative function $S:\mathbb{R}\to\mathbb{R}$, S has compact support $[-T,T]$, $T>0$, and is nondecreasing there and it can be continuous only on either $(-\infty,T]$ or $[-T,T]$. S can have jump discontinuities. We call S the "squashing function".*

Remark 48 *If in (44) we replace b by S we derive the so called normalized fuzzy squashing operator $K_n^{\mathcal{F}}$. Then under the same terms and assumptions as in Theorems 45 and 46, we get*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x),f(x)\right)\leq\omega_1^{(\mathcal{F})}\left(f,\frac{T}{n^{1-\alpha}}\right)_{\mathbb{R}}, \tag{62}$$

(see (48)), and

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x),f(x)\right)\leq\lambda_n^{\mathcal{F}}(x), \tag{63}$$

(see (51)), respectively.

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Fuzzy Fractional neural network approximation by Fuzzy Quasi-interpolation operators

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Abstract

Here we consider the univariate fuzzy fractional quantitative approximation of fuzzy real valued functions on a compact interval by quasi-interpolation sigmoidal and hyperbolic tangent fuzzy neural network operators. These approximations are derived by establishing fuzzy Jackson type inequalities involving the fuzzy moduli of continuity of the right and left Caputo fuzzy fractional derivatives of the engaged function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy neural networks are with one hidden layer. Our fuzzy fractional approximation results into higher order converges better than the fuzzy ordinary ones.

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Keywords and Phrases: sigmoidal and hyperbolic tangent functions, neural network fuzzy fractional approximation, fuzzy quasi-interpolation operator, fuzzy modulus of continuity, fuzzy fractional derivative.

1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support.

The author motivated by [22], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [12], [14], [15], [16], [17], [18], by treating both the univariate and multivariate cases.

Continuation of the author's last work ([18]) is this article where fuzzy neural network approximation is taken at the fractional level resulting into higher rates of approximation. We involve the right and left Caputo fuzzy fractional derivatives of the fuzzy function under approximation and we establish tight fuzzy Jackson type inequalities. An extensive background is given on fuzzyness, fractional calculus and neural networks, all needed to expose our work.

Our fuzzy feed-forward neural networks (FFNNs) are with one hidden layer. About neural networks in general, see [28], [31], [32].

2 Fuzzy Mathematical Analysis Background

We need the following basic background

Definition 1 *(see [35]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e. $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where supp$(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([27]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [35]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall \ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \rightarrow \mathbb{R}_\mathcal{F}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ \ u, v \in \mathbb{R}_\mathcal{F}.$$

We have that D is a metric on $\mathbb{R}_\mathcal{F}$.

Then $(\mathbb{R}_\mathcal{F}, D)$ is a complete metric space, see [35], [36].

Here $\sum^{*}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_\mathcal{F}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_\mathcal{F}.$$

Denote

$$D^*(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \rightarrow \mathbb{R}_\mathcal{F}$.

We mention

Definition 2 *Let $f : X \subseteq \mathbb{R} \rightarrow \mathbb{R}_\mathcal{F}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y \in X, \ |x-y| \leq \delta} D(f(x), f(y)), \ \ \delta > 0.$$

When $g : X \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we define

$$\omega_1(g,\delta)_X = \sup_{x,y \in X, \ |x-y| \leq \delta} |g(x) - g(y)|.$$

We define by $C_\mathcal{F}^U(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \rightarrow \mathbb{R}_\mathcal{F}$, also $C_\mathcal{F}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_\mathcal{F})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 3 *([5]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_1^{\mathcal{F}}(f,\delta)_X < \infty$, for any $\delta > 0$.*

Proposition 4 *([5]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$.

Proposition 5 *([5]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 6 *It is clear by Propositions 4, 5, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$).*

Proposition 7 *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)},\delta\right)_X$, $\omega_1\left(f_+^{(r)},\delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)},\delta\right)_X, \omega_1\left(f_+^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [9]. ∎

We need

Remark 8 *([3]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. Suppose that*

$$\sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 9 *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 10 *([34]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$.

We mention

Theorem 11 *([29]) Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in \mathbb{R}$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall\, r \in [0,1].$$

Remark 12 *([4]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we obtain*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, ..., N$.

Note 13 *([4]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we have $f_{\pm}^{(r)} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.*

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$), see [35].

Definition 14 *([26], p. 644) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have*

$$D\left(\sum_{P}^{*} (v-u) \odot f(\xi), I \right) < \varepsilon.$$

We write

$$I := (FR) \int_a^b f(x)\, dx. \tag{3}$$

We mention

Theorem 15 *([27]) Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR) \int_a^b f(x)\, dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR) \int_a^b f(x)\, dx \right]^r = \left[\int_a^b (f)_-^{(r)}(x)\, dx, \int_a^b (f)_+^{(r)}(x)\, dx \right],$$

$\forall\ r \in [0, 1]$.

For the definition of general fuzzy integral we follow [30] next.

Definition 16 *Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0, 1]$ defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [30].

Theorem 17 *([30]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\},$$

the following are equivalent
(1) F is measurable,
(2) $\forall\ r \in [0, 1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [30], given that for each $r \in [0, 1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \le r \le 1 \right\} \tag{4}$$

is a fuzzy real number for each $A \in \Sigma$.
The last fact leads to

Definition 18 *([30]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \le r \le 1 \right\}$$

is integrable if for each $r \in [0, 1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\}.$$

By [30], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 19 *([30]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \leq \int_A D(F, G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu \right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu \right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall \ r \in [0, 1].$$

We need

Definition 20 *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a, b])$ (space of functions f with $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [23], pp. 49-52, [25], [33]) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n - \nu)} \int_a^x (x - t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall \ x \in [a, b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.

Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall \ x \in [a, b]$.

Lemma 21 *([8]) Let* $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ *and* $f^{(n)} \in L_\infty([a,b])$. *Then* $D_{*a}^{\nu} f(a) = 0$.

Definition 22 *(see also [6], [24], [25]) Let* $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *The right Caputo fractional derivative of order* $\alpha > 0$ *is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) \, d\zeta, \tag{7}$$

$\forall\, x \in [a,b]$. *We set* $D_{b-}^{0} f(x) = f(x)$. *Notice that* $D_{b-}^{\alpha} f \in L_1([a,b])$ *and* $D_{b-}^{\alpha} f$ *exists a.e. on* $[a,b]$.

Lemma 23 *([8]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. *Then* $D_{b-}^{\alpha} f(b) = 0$.

Convention 24 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \text{ for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \ \text{ for } x > x_0, \tag{9}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 25 *([8]) Let* $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a,b]$.

Also we have

Proposition 26 *([8]) Let* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(x)$ *is continuous in* $x \in [a,b]$.

We further mention

Proposition 27 *([8]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt, \tag{10}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 28 *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{11}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

We need

Proposition 29 *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)\, dt, \quad \text{for } x \geq x_0, \tag{12}$$

and $L(x, x_0) = 0$, for $x < x_0$.

Then L is jointly continuous in (x, x_0) on $[a,b]^2$.

We mention

Proposition 30 *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$K(x, x_0) = \int_{x_0}^{x} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \quad \text{for } x \leq x_0, \tag{13}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Propositions 29, 30 we derive

Corollary 31 *([8]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We need

Theorem 32 *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta)_{[x,b]}, \tag{14}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous in $x \in [a,b]$.

Also it holds

Theorem 33 *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta)_{[\alpha,x]}, \tag{15}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

So that for $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in [a,b]$, we have that $\omega_1 (D^{\alpha}_{*x} f, h)_{[x,b]}$, $\omega_1 \left(D^{\alpha}_{x-} f, h\right)_{[a,x]}$ are continuous functions in $x \in [a,b]$, $h > 0$ is fixed.

We make

Remark 34 *([8]) Let* $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_{\infty}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. *Then we have*

$$|D^{\nu}_{*a} f(x)| \leq \frac{\left\| f^{(n)} \right\|_{\infty}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{16}$$

Thus we observe

$$\omega_1 (D^{\nu}_{*a} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D^{\nu}_{*a} f(x) - D^{\nu}_{*a} f(y)| \tag{17}$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\| f^{(n)} \right\|_{\infty}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\| f^{(n)} \right\|_{\infty}}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2 \left\| f^{(n)} \right\|_{\infty}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{18}$$

Consequently

$$\omega_1 (D^{\nu}_{*a} f, \delta) \leq \frac{2 \left\| f^{(n)} \right\|_{\infty}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{19}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, *then*

$$\omega_1 \left(D^{\alpha}_{b-} f, \delta \right) \leq \frac{2 \left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{20}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, *we find*

$$s_1(\delta) := \sup_{x_0 \in [a,b]} \omega_1 \left(D^{\alpha}_{*x_0} f, \delta \right)_{[x_0,b]} \leq \frac{2 \left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{21}$$

and

$$s_2(\delta) := \sup_{x_0 \in [a,b]} \omega_1 \left(D^{\alpha}_{x_0-} f, \delta \right)_{[a,x_0]} \leq \frac{2 \left\| f^{(m)} \right\|_{\infty}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{22}$$

By Proposition 15.114, p. 388 of [7], we get here that $D^{\alpha}_{*x_0} f \in C([x_0,b])$, *and by [13] we obtain that* $D^{\alpha}_{x_0-} f \in C([a,x_0])$. *Clearly here we have that*

$$s_1(\delta), \ s_2(\delta) \to 0, \quad as \quad \delta \to 0. \tag{23}$$

We need

Definition 35 *([11]) Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.*

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)\,dt, \quad x \in [a,b], \tag{24}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\,dt, \quad x \in [a,b], \tag{25}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 36 *([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ for every finite, pairwise disjoint, family*

$$(c_k, d_k)_{k=1}^n \subseteq (a,b) \quad \textit{with} \quad \sum_{k=1}^n (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^n D(f(d_k), f(c_k)) < \epsilon. \tag{26}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 37 *([11]) $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 38 *([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\,dt$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\,dt, \right.\right.$$

$$\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)} \right)_+^{(r)} (t)\, dt \right) |0 \le r \le 1 \Big\} =$$

$$= \Big\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)} \right)^{(n)} (t)\, dt, \right.$$

$$\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)} \right)^{(n)} (t)\, dt \right) |0 \le r \le 1 \Big\}. \quad (27)$$

So, we get

$$\left[D_{*a}^{\nu \mathcal{F}} f(x) \right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)} \right)^{(n)} (t)\, dt, \right. \right.$$

$$\left. \left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)} \right)^{(n)} (t)\, dt \right) \right], \quad 0 \le r \le 1. \quad (28)$$

That is

$$\left(D_{*a}^{\nu \mathcal{F}} f(x) \right)_\pm^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_\pm^{(r)} \right)^{(n)} (t)\, dt = \left(D_{*a}^{\nu} \left(f_\pm^{(r)} \right) \right)(x),$$

see [7], [23].

I.e. we get that

$$\left(D_{*a}^{\nu \mathcal{F}} f(x) \right)_\pm^{(r)} = \left(D_{*a}^{\nu} \left(f_\pm^{(r)} \right) \right)(x), \quad (29)$$

$\forall\ x \in [a,b]$, *in short*

$$\left(D_{*a}^{\nu \mathcal{F}} f \right)_\pm^{(r)} = D_{*a}^{\nu} \left(f_\pm^{(r)} \right), \quad \forall\ r \in [0,1]. \quad (30)$$

We need

Lemma 39 *([11]) $D_{*a}^{\nu \mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 40 *([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$D_{b-}^{\nu \mathcal{F}} f(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt$$

$$= \Big\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)} \right)_-^{(r)} (t)\, dt, \right.$$

$$\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)} \right)_+^{(r)} (t)\, dt \right) |0 \le r \le 1 \Big\} \quad (31)$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)} (t)\, dt, \right. \right.$$

$$\left. \left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)} (t)\, dt \right) | 0 \leq r \leq 1 \right\}.$$

We get

$$\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)} (t)\, dt, \right. \right.$$

$$\left. \left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)} (t)\, dt \right) \right], \quad 0 \leq r \leq 1.$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)} (t)\, dt = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [6].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{32}$$

$\forall\ x \in [a,b]$, *in short*

$$\left(D_{b-}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right) \leq D_{b-}^{\nu}\left(f_{+}^{(r)}\right), \quad \forall\ r \in [0,1].$$

We need

Lemma 41 *([11]) $D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

3 Fractional neural network approximation by real quasi-interpolation operators ([18])

We consider here the sigmoidal function of logarithmic type

$$s(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x\to+\infty} s(x) = 1$ and $\lim_{x\to-\infty} s(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks.

As in [22], we consider

$$\Phi(x) := \frac{1}{2}(s(x+1) - s(x-1)), \quad x \in \mathbb{R}. \tag{34}$$

We notice the following properties:

i) $\Phi(x) > 0, \ \forall\, x \in \mathbb{R}$,

ii) $\sum_{k=-\infty}^{\infty} \Phi(x-k) = 1, \ \forall\, x \in \mathbb{R}$,

iii) $\sum_{k=-\infty}^{\infty} \Phi(nx-k) = 1, \ \forall\, x \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi(x)\,dx = 1$,

v) Φ is a density function,

vi) Φ is even: $\Phi(-x) = \Phi(x), \ x \geq 0$.

We see that ([22])

$$\Phi(x) = \left(\frac{e^2-1}{2e^2}\right)\frac{1}{(1+e^{x-1})(1+e^{-x-1})}. \tag{35}$$

vii) By [22] Φ is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

viii) By [17] for $n \in \mathbb{N}$, $0 < \beta < 1$, we get

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| > n^{1-\beta} \end{cases}}^{\infty} \Phi(nx-k) < \left(\frac{e^2-1}{2}\right) e^{-n^{(1-\beta)}} = 3.1992 e^{-n^{(1-\beta)}}. \tag{36}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

ix) By [17] it holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < \frac{1}{\Phi(1)} = 5.250312578, \ \forall\, x \in [a,b]. \tag{37}$$

x) By [17] it holds $\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1$, for at least some $x \in [a,b]$.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We consider further (see also [17]) the positive linear neural network operator

$$G_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \quad x \in [a,b]. \tag{38}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We present here at fractional level the pointwise and uniform convergence of $G_n(f,x)$ to $f(x)$ with rates.

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$. Furthermore it holds $\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} > 0$.

This function plays also the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{39}$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Here we follow [14]

Proposition 42 *$\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty} \Psi(x) = 0 = \lim_{x\to\infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is at zero, $\Psi(0) = 0.3809297$.

Theorem 43 *We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1$, $\forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Furthermore we get:

Since Ψ is even it holds $\sum_{i=-\infty}^{\infty} \Psi(i-x) = 1$, $\forall x \in \mathbb{R}$.

Hence $\sum_{i=-\infty}^{\infty} \Psi(i+x) = 1$, $\forall\, x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1$, $\forall\, x \in \mathbb{R}$.

Theorem 44 *It holds* $\int_{-\infty}^{\infty}\Psi(x)\,dx = 1$.

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 45 *Let* $0 < \beta < 1$ *and* $n \in \mathbb{N}$. *It holds*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Psi(nx-k) \leq e^{4} \cdot e^{-2n^{(1-\beta)}}. \tag{40}$$

Theorem 46 *Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < 4.1488766 = \frac{1}{\Psi(1)}. \tag{41}$$

Also by [14], we obtain

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

Definition 47 *Let* $f \in C([a,b])$ *and* $n \in \mathbb{N}$ *such that* $\lceil na \rceil \leq \lfloor nb \rfloor$.

We further treat, as in [14], the positive linear neural network operator

$$F_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}, \quad x \in [a,b]. \tag{42}$$

We consider F_n similarly to G_n.

Here $\|\cdot\|_{\infty}$ stands for the supremum norm.

From [18] we need

Theorem 48 *Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N([a,b])$, *with* $f^{(N)} \in L_{\infty}([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$. *Then*

i)

$$\left| G_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} G_n\left((\cdot - x)^j\right)(x) - f(x) \right| \leq \tag{43}$$

$$\frac{(5.250312578)}{\Gamma(\alpha+1)} \cdot \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n_{\beta}}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\},$$

ii) if $f^{(j)}(x)=0$*, for* $j=1,...,N-1$*, we have*

$$|G_n(f,x)-f(x)| \leq \frac{(5.250312578)}{\Gamma(\alpha+1)} \cdot \tag{44}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left. 3.1992e^{-n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\},$$

when $\alpha>1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

iii)

$$|G_n(f,x)-f(x)| \leq (5.250312578) \cdot \tag{45}$$

$$\left\{ \sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^{j}(3.1992)e^{-n^{(1-\beta)}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left. 3.1992e^{-n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\},$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|G_nf-f\|_{\infty} \leq (5.250312578) \cdot \tag{46}$$

$$\left\{ \sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^{j}(3.1992)e^{-n^{(1-\beta)}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{ \frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left. 3.1992e^{-n^{(1-\beta)}}(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}.$$

Above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $G_n\to I$ *the unit operator, as* $n\to\infty$.

Also from [18] we need

Theorem 49 *Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N([a,b])$, *with* $f^{(N)} \in L_\infty([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$. *Then*

i)

$$\left| F_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} F_n\left((\cdot - x)^j\right)(x) - f(x) \right| \leq$$

$$\frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n_\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (47)$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$|F_n(f,x) - f(x)| \leq \frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n_\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (48)$$

when $\alpha > 1$ *notice here the extremely high rate of convergence of* $n^{-(\alpha+1)\beta}$,

iii)

$$|F_n(f,x) - f(x)| \leq (4.1488766) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left| f^{(j)}(x) \right|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n_\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. e^4 e^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\} \right\}, \quad (49)$$

$\forall\, x \in [a,b]$,

and

iv)

$$\|F_n f - f\|_\infty \leq (4.1488766) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n_{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. e^4 e^{-2n^{(1-\beta)}} (b-a)^{\alpha} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\} \right\}. \quad (50)$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $F_n \to I$ the unit operator, as $n \to \infty$.

Also from [18] we mention for $\alpha = \frac{1}{2}$ the next

Corollary 50 *Let $0 < \beta < 1$, $f \in AC([a,b])$, $f' \in L_\infty([a,b])$, $n \in \mathbb{N}$. Then*

i)

$$\|G_n f - f\|_\infty \leq \frac{(10.50062516)}{\sqrt{\pi}} \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. 3.1992 e^{-n^{(1-\beta)}} \sqrt{b-a} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} \Bigg\}, \quad (51)$$

and

ii)

$$\|F_n f - f\|_\infty \leq \frac{(8.2977532)}{\sqrt{\pi}} \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. e^4 e^{-2n^{(1-\beta)}} \sqrt{b-a} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} \Bigg\}. \quad (52)$$

Denote $\omega_1^{(\mathcal{F})}(f,\delta)_{[a,b]} = \omega_1^{(\mathcal{F})}(f,\delta)$ and $\omega_1(f,\delta)_{[a,b]} = \omega_1(f,\delta)$.
From [12] we need

Theorem 51 *Let $f \in C([a,b])$, $0 < \beta < 1$, $n \in \mathbb{N}$, $x \in [a,b]$. Then*
i)

$$|G_n(f,x) - f(x)| \leq (5.250312578) \cdot \left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984)\|f\|_\infty e^{-n^{(1-\beta)}}\right\} =: \lambda, \tag{53}$$

ii)

$$\|G_n(f) - f\|_\infty \leq \lambda, \tag{54}$$

where $\|\cdot\|_\infty$ is the supremum norm.

Finally from [12] we need

Theorem 52 *Let $f \in C([a,b])$, $0 < \beta < 1$, $n \in \mathbb{N}$, $x \in [a,b]$. Then*
i)

$$|F_n(f,x) - f(x)| \leq (4.1488766) \cdot \left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4\|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda^*, \tag{55}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda^*. \tag{56}$$

4 Fractional Approximation by Fuzzy Quasi - Interpolation Neural Network Operators

Let $f \in C_{\mathcal{F}}([a,b])$, $n \in \mathbb{N}$. We define the Fuzzy Quasi-Interpolation Neural Network operators

$$G_n^{\mathcal{F}}(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \tag{57}$$

$\forall\ x \in [a,b]$, see also (38).
Similarly, we define

$$F_n^{\mathcal{F}}(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)}, \tag{58}$$

$\forall\ x \in [a,b]$, see also (42).

The fuzzy sums in (57) and (58) are finite.

Let $r \in [0,1]$, we observe that

$$\left[G_n^{\mathcal{F}}(f,x)\right]^r = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f\left(\frac{k}{n}\right)\right]^r \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right)\right] \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right) = \tag{59}$$

$$\left[\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_-^{(r)}\left(\frac{k}{n}\right) \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right), \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_+^{(r)}\left(\frac{k}{n}\right) \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right)\right]$$

$$= \left[G_n\left(f_-^{(r)}, x\right), G_n\left(f_+^{(r)}, x\right)\right]. \tag{60}$$

We have proved that

$$\left(G_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)} = G_n\left(f_{\pm}^{(r)}, x\right), \tag{61}$$

respectively, $\forall\, r \in [0,1]$, $\forall\, x \in [a,b]$.

Similarly, it holds

$$\left(F_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)} = F_n\left(f_{\pm}^{(r)}, x\right), \tag{62}$$

respectively, $\forall\, r \in [0,1]$, $\forall\, x \in [a,b]$.

Therefore we get

$$D\left(G_n^{\mathcal{F}}(f,x), f(x)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(G_n\left(f_-^{(r)}, x\right)\right) - f_-^{(r)}(x)\right|, \left|\left(G_n\left(f_+^{(r)}, x\right)\right) - f_+^{(r)}(x)\right|\right\}, \tag{63}$$

and

$$D\left(F_n^{\mathcal{F}}(f,x), f(x)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(F_n\left(f_-^{(r)}, x\right)\right) - f_-^{(r)}(x)\right|, \left|\left(F_n\left(f_+^{(r)}, x\right)\right) - f_+^{(r)}(x)\right|\right\}, \tag{64}$$

$\forall\, x \in [a,b]$.

We present

Theorem 53 *Let* $f \in C_{\mathcal{F}}([a,b])$, $0<\beta<1$, $n \in \mathbb{N}$, $x \in [a,b]$. *Then*

1)

$$D\left(G_n^{\mathcal{F}}(f,x), f(x)\right) \leq (5.250312578) \cdot \tag{65}$$

$$\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right) + (6.3984)\, D^*(f, \widetilde{o})\, e^{-n^{(1-\beta)}}\right] =: \lambda^{(\mathcal{F})},$$

2)

$$D^*\left(G_n^{\mathcal{F}}(f), f\right) \leq \lambda^{(\mathcal{F})}. \tag{66}$$

Proof. Since $f \in C_{\mathcal{F}}([a,b])$ we have that $f_{\pm}^{(r)} \in C([a,b])$, $\forall\ r \in [0,1]$. Hence by (53) we obtain

$$\left|G_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq (5.250312578) \cdot$$

$$\left[\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^\beta}\right) + (6.3984)\left\|f_{\pm}^{(r)}\right\|_\infty e^{-n^{(1-\beta)}}\right]$$

(by Proposition 7 and $\left\|f_{\pm}^{(r)}\right\|_\infty \leq D^*(f, \widetilde{o})$)

$$\leq (5.250312578)\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right) + (6.3984)\, D^*(f, \widetilde{o})\, e^{-n^{(1-\beta)}}\right].$$

Taking into account (63) the proof of the claim is completed. ■

We also give

Theorem 54 *Let* $f \in C_{\mathcal{F}}([a,b])$, $0<\beta<1$, $n \in \mathbb{N}$, $x \in [a,b]$. *Then*

1)

$$D\left(F_n^{\mathcal{F}}(f,x), f(x)\right) \leq (4.1488766) \cdot \tag{67}$$

$$\left[\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right) + 2e^4 D^*(f, \widetilde{o})\, e^{-2n^{(1-\beta)}}\right] =: \lambda_*^{(\mathcal{F})},$$

2)

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) \leq \lambda_*^{(\mathcal{F})}. \tag{68}$$

Proof. Similar to Theorem 53 based on Theorem 52. ■

Next we present

Theorem 55 *Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^N([a,b])$, $0<\beta<1$, $x \in [a,b]$, $n \in \mathbb{N}$. *Then*

i)

$$D\left(G_n^{\mathcal{F}}(f,x), f(x)\right) \leq (5.250312578) \cdot \tag{69}$$

$$\left\{\sum_{j=1}^{N-1} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j \cdot (3.1992)\, e^{-n^{(1-\beta)}}\right\} + \right.$$

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$$+\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}\right.$$

$$+(3.1992)\,e^{-n^{(1-\beta)}}\cdot$$

$$\left.\left(D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}(x-a)^\alpha+D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}(b-x)^\alpha\right)\right\}\right\},$$

ii) if $f^{(j)}(x_0)=0,\ j=1,...,N-1,$ *we have*

$$D\left(G_n^{\mathcal{F}}(f,x_0),f(x_0)\right)\leq\frac{(5.250312578)}{\Gamma(\alpha+1)}\cdot\tag{70}$$

$$\left\{\frac{\left(\omega_1^{(\mathcal{F})}\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x_0]}+\omega_1^{(\mathcal{F})}\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x_0,b]}\right)}{n^{\alpha\beta}}\right.$$

$$+(3.1992)\,e^{-n^{(1-\beta)}}\cdot$$

$$\left.\left(D^*\left(\left(D_{x_0-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x_0]}(x_0-a)^\alpha+D^*\left(\left(D_{*x_0}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x_0,b]}(b-x_0)^\alpha\right)\right\},$$

when $\alpha>1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$, *and*

iii)

$$D^*\left(G_n^{\mathcal{F}}(f),f\right)\leq(5.250312578)\cdot\tag{71}$$

$$\left\{\sum_{j=1}^{N-1}\frac{D^*\left(f^{(j)}(x),\widetilde{o}\right)}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right\}+\right.$$

$$+\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}}\right.$$

$$+(3.1992)\,e^{-n^{(1-\beta)}}(b-a)^\alpha\cdot$$

$$\left.\left.\left(\sup_{x\in[a,b]}D^*\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^*\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right)\right\}\right\},$$

above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0.$

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of $G_n^{\mathcal{F}}\to I$ *the unit operator, as* $n\to\infty$.

Proof. Here $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall\ r \in [0,1]$, and $D_{x-}^{\alpha\mathcal{F}} f$, $D_{*x}^{\alpha\mathcal{F}} f$ are fuzzy continuous over $[a,b]$, $\forall\ x \in [a,b]$, so that $\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)} \in C([a,b])$, $\forall\ r \in [0,1]$, $\forall\ x \in [a,b]$.

We observe by (45) that (respectively in $\pm$)

$$\left|G_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq (5.250312578) \cdot$$

$$\left\{\sum_{j=1}^{N-1} \frac{\left|\left(f_{\pm}^{(r)}\right)^{(j)}(x)\right|}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j \cdot (3.1992)\, e^{-n^{(1-\beta)}}\right\}\right. \tag{72}$$

$$+\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}\right.$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}} \cdot$$

$$\left.\left.\left(\left\|D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} =$$

$$(5.250312578)\left\{\sum_{j=1}^{N-1} \frac{\left|\left(f^{(j)}(x)\right)_{\pm}^{(r)}\right|}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j \cdot (3.1992)\, e^{-n^{(1-\beta)}}\right\}\right.$$

$$+\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}\right. \tag{73}$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}} \cdot$$

$$\left.\left.\left(\left\|\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} \leq$$

$$(5.250312578)\left\{\sum_{j=1}^{N-1} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j \cdot (3.1992)\, e^{-n^{(1-\beta)}}\right\}\right. +$$

$$+\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}\right. \tag{74}$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}} \cdot$$

$$\left.\left.\left(D^*\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]}(x-a)^{\alpha} + D^*\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right)\right\}\right\},$$

along with (63) proving all inequalities of theorem.

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Here we notice that

$$\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{(-1)^{N}}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds, \tag{75}$$

where $a\le t\le x$.

Hence

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\le\frac{1}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds\le \tag{76}$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(x-t)^{N-\alpha}\le\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}$$

$$\le\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

So we have

$$\left|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\le\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{77}$$

all $a\le t\le x$.

And it holds

$$\left\|\left(D_{x-}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}\le\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{78}$$

that is

$$D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\le\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}\le\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}<\infty. \tag{79}$$

Similarly we have

$$\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{1}{\Gamma(N-\alpha)}\int_{x}^{t}(t-s)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds, \tag{80}$$

where $x\le t\le b$.

Hence

$$\left|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\le\frac{1}{\Gamma(N-\alpha)}\int_{x}^{t}(t-s)^{N-\alpha-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\le \tag{81}$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},$$

$x\leq t\leq b$.

So we have

$$\left\|\left(D_{*x}^{\alpha\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},\tag{82}$$

that is

$$D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha},\tag{83}$$

and

$$\sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty.\tag{84}$$

Furthermore we notice

$$\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}=\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t)\right)\leq$$

$$\sup_{\substack{s,t\in[a,x]\\|s-t|\leq\frac{1}{n^{\beta}}}}\left\{D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(s),\widetilde{o}\right),D\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right)(t),\widetilde{o}\right)\right\}\leq2D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}$$

$$\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}.\tag{85}$$

Therefore it holds

$$\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty.\tag{86}$$

Similarly we observe

$$\omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}=\sup_{\substack{s,t\in[x,b]\\|s-t|\leq\frac{1}{n^{\beta}}}}D\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right)(s),\left(D_{*x}^{\alpha\mathcal{F}}f\right)(t)\right)\leq$$

$$2D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}.\tag{87}$$

Consequently it holds

$$\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\leq\frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\alpha+1\right)}\left(b-a\right)^{N-\alpha}<+\infty.\tag{88}$$

So everything in the statements of the theorem makes sense.

The proof of the theorem is now completed. ■

We also give

Theorem 56 *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^{N}([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$. Then*

i)

$$D\left(F_n^{\mathcal{F}}(f,x), f(x)\right) \leq (4.1488766) \cdot \tag{89}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \cdot e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$+ \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1^{(\mathcal{F})} \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1^{(\mathcal{F})} \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} \right.$$

$$\left. \left. + e^4 e^{-2n^{(1-\beta)}} \left(D^* \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[a,x]} (x-a)^\alpha + D^* \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x,b]} (b-x)^\alpha \right) \right\} \right\},$$

ii) if $f^{(j)}(x_0) = 0$, $j = 1, ..., N-1$, we have

$$D\left(F_n^{\mathcal{F}}(f, x_0), f(x_0)\right) \leq \frac{(4.1488766)}{\Gamma(\alpha+1)} \cdot \tag{90}$$

$$\left\{ \frac{\left(\omega_1^{(\mathcal{F})} \left(\left(D_{x_0-}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[a,x_0]} + \omega_1^{(\mathcal{F})} \left(\left(D_{*x_0}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[x_0,b]} \right)}{n^{\alpha\beta}} + e^4 e^{-2n^{(1-\beta)}} \cdot \right.$$

$$\left. \left(D^* \left(\left(D_{x_0-}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[a,x_0]} (x_0-a)^\alpha + D^* \left(\left(D_{*x_0}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x_0,b]} (b-x_0)^\alpha \right) \right\},$$

when $\alpha > 1$ notice here the extremely high rate of convergence at $n^{-(\alpha+1)\beta}$,

iii)

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) \leq (4.1488766) \cdot \tag{91}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{D^*\left(f^{(j)}, \widetilde{o}\right)}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \cdot e^4 e^{-2n^{(1-\beta)}} \right\} + \right.$$

$$+ \frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})} \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})} \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} \right.$$

$$+ e^4 e^{-2n^{(1-\beta)}} (b-a)^\alpha \cdot$$

$$\left. \left. \left(\sup_{x\in[a,b]} D^* \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[a,x]} + \sup_{x\in[a,b]} D^* \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x,b]} \right) \right\} \right\}.$$

Above, when $N = 1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of $F_n^{\mathcal{F}} \to I$, as $n \to \infty$.

Proof. Similar to Theorem 55, using Theorem 49. ■

We make

Remark 57 *Looking at (63) and (64), and applying the principle of iterated suprema we obtain*

$$D^*\left(G_n^{\mathcal{F}}(f), f\right) = \sup_{r\in[0,1]} \max\left\{\left\|G_n\left(f_-^{(r)}\right) - f_-^{(r)}\right\|_\infty, \left\|G_n\left(f_+^{(r)}\right) - f_+^{(r)}\right\|_\infty\right\}, \tag{92}$$

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) = \sup_{r\in[0,1]} \max\left\{\left\|F_n\left(f_-^{(r)}\right) - f_-^{(r)}\right\|_\infty, \left\|F_n\left(f_+^{(r)}\right) - f_+^{(r)}\right\|_\infty\right\}, \tag{93}$$

where $f \in C_{\mathcal{F}}([a,b])$, see also proof of Theorem 53.

We finish with

Corollary 58 *Let $0 < \beta < 1$, $f \in C_{\mathcal{F}}^1([a,b])$, $n \in \mathbb{N}$. Then*

(i)

$$D^*\left(G_n^{\mathcal{F}}(f), f\right) \leq \left(\frac{10.50062516}{\sqrt{\pi}}\right) \cdot \tag{94}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} \right.$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}} \sqrt{b-a} \cdot$$

$$\left.\left(\sup_{x\in[a,b]} D^*\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]} D^*\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}\right)\right\},$$

and

(ii)

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) \leq \left(\frac{8.2977532}{\sqrt{\pi}}\right) \cdot \tag{95}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} \right.$$

$$+ e^4 e^{-2n^{(1-\beta)}} \sqrt{b-a} \cdot$$

$$\left.\left(\sup_{x\in[a,b]} D^*\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]} D^*\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}\right)\right\}.$$

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Proof. By (51) we get

$$\left\| G_n^{\mathcal{F}}\left(f_{\pm}^{(r)}\right) - \left(f_{\pm}^{(r)}\right)\right\|_{\infty} \leq \left(\frac{10.50062516}{\sqrt{\pi}}\right) \cdot \tag{96}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} \right.$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}}\sqrt{b-a}\cdot$$

$$\left.\left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}\right)\right\} =$$

$$\left(\frac{10.50062516}{\sqrt{\pi}}\right) \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(\left(D_{x-}^{\frac{1}{2}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(\left(D_{*x}^{\frac{1}{2}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} \right. \tag{97}$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}}\sqrt{b-a}\cdot$$

$$\left.\left(\sup_{x\in[a,b]} \left\| \left(D_{x-}^{\frac{1}{2}} f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| \left(D_{*x}^{\frac{1}{2}} f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}\right)\right\} \leq$$

$$\left(\frac{10.50062516}{\sqrt{\pi}}\right) \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})} \left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})} \left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} \right. \tag{98}$$

$$+ (3.1992)\, e^{-n^{(1-\beta)}}\sqrt{b-a}\cdot$$

$$\left.\left(\sup_{x\in[a,b]} D^{*} \left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]} D^{*} \left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}\right)\right\},$$

proving the claim with the use of Remark 57.

Part (ii) follows similarly. ■

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Higher Order Multivariate Fuzzy Approximation by basic Neural Network Operators

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Abstract

Here are studied in terms of multivariate fuzzy high approximation to the multivariate unit basic sequences of multivariate fuzzy neural network operators. These operators are multivariate fuzzy analogs of earlier studied multivariate real ones. The produced results generalize earlier real ones into the fuzzy setting. Here the high order multivariate fuzzy pointwise convergence with rates to the multivariate fuzzy unit operator is established through multivariate fuzzy inequalities involving the multivariate fuzzy moduli of continuity of the Nth order ($N \geq 1$) H-fuzzy partial derivatives, of the engaged multivariate fuzzy number valued function.

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1 Fuzzy real Analysis Background

We need the following background

Definition 1 *(see [14]) Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R};\ \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\},\ \forall\ x, y \in \mathbb{R},\ \forall\ \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$*, i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{\sup p(\mu)}$ *is compact in* $\mathbb{R}$*, (where* $\sup p(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$*.*

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0*.*

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \tag{1}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([11]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [14]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u,\ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}, \tag{2}$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [14], [15].

Let $f, g : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) = \sup_{x \in \mathbb{R}^m} D(f(x), g(x)).$$

Here Σ^* stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 2 *([5]). Here $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. Suppose that*

$$\sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max \left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)} \right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right). \tag{3}$$

Definition 3 *Let $f \in C(\mathbb{R}^m)$, $m \in \mathbb{N}$, which is bounded or uniformly continuous, we define $(h > 0)$*

$$\omega_1(f, h) := \sup_{all \ x_i, x_i' \in \mathbb{R}, \ \left| x_i - x_i' \right| \leq h, \ for \ i=1,...,m} \left| f(x_1, ..., x_m) - f(x_1', ..., x_m') \right|. \tag{4}$$

Definition 4 *Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{x, y \in \mathbb{R}, \ |x_i - y_i| \leq \delta, \ for \ i=1,...,m} D(f(x), f(y)), \ \ \delta > 0, \tag{5}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

For $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)} \right], \tag{6}$$

where $f_{\pm}^{(r)} : \mathbb{R}^m \to \mathbb{R}$, $\forall \ r \in [0,1]$.

We need

Proposition 5 *Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f, \delta)$, $\omega_1 \left(f_-^{(r)}, \delta \right)$, $\omega_1 \left(f_+^{(r)}, \delta \right)$ are finite for any $\delta > 0$, $r \in [0,1]$.*

Then

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0,1]} \max \left\{ \omega_1 \left(f_-^{(r)}, \delta \right), \omega_1 \left(f_+^{(r)}, \delta \right) \right\}. \tag{7}$$

Proof. By Proposition 1 of [8]. ■

We define by $C_{\mathcal{F}}^{U}(\mathbb{R}^m)$ the space of fuzzy uniformly continuous functions from $\mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R}^m)$ is the space of fuzzy continuous functions on $\mathbb{R}^m$, and $C_b(\mathbb{R}^m, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 6 *([7]) Let $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$. Then $\omega_1^{(\mathcal{F})}(f, \delta) < \infty$, for any $\delta > 0$.*

Proposition 7 *([7]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f, \delta) = \omega_1^{(\mathcal{F})}(f, 0) = 0, \tag{8}$$

iff $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$.

Proposition 8 *([7]) Let $f \in C_{\mathcal{F}}(\mathbb{R}^m)$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}^m$, respectively in $\pm$.*

Note 9 *It is clear by Ptopositions 5, 7, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R}^m)$ (uniformly continuous on $\mathbb{R}^m$).*

We need

Definition 10 *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 11 *([14]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{9}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Definition 12 *We denote by $C_{\mathcal{F}}^{N}(\mathbb{R}^m)$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from $\mathbb{R}^m$ into $\mathbb{R}_{\mathcal{F}}$.*

Here fuzzy partial derivatives are defined via Definition 11 in the obvious way as in the ordinary real case.

We mention

4

Theorem 13 *([12]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \le r \le 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_\pm^{(r)} = \left(f_\pm^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{10}$$

Remark 14 *(see also [6]) Let $f \in C^N(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, $N \ge 1$. Then by Theorem 13 we obtain $f_\pm^{(r)} \in C^N(\mathbb{R})$ and*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have

$$\left(f^{(i)} \right)_\pm^{(r)} = \left(f_\pm^{(r)} \right)^{(i)}, \tag{11}$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^N(\mathbb{R}^m)$, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, ..., \widetilde{\alpha_m})$, $\widetilde{\alpha}_i \in \mathbb{Z}^+$, $i = 1, ..., m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^m \widetilde{\alpha}_i \le N, \quad N > 1.$$

Then by Theorem 13 we get that

$$\left(f_\pm^{(r)} \right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_\pm^{(r)}, \ \forall\, r \in [0,1], \tag{12}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \le N$. Here $f_\pm^{(r)} \in C^N(\mathbb{R}^m)$.

For the definition of general fuzzy integral we follow [13] next.

Definition 15 *Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to \mathbb{R}_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \quad \text{all } w \in \Omega$$

is measurable, see [13].

Theorem 16 *([13]) For $F:\Omega\rightarrow\mathbb{R}_{\mathcal{F}}$,*

$$F(w)=\{(F_{-}^{(r)}(w),F_{+}^{(r)}(w))|0\leq r\leq 1\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\ r\in[0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are measurable.

Following [13], given that for each $r\in[0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_{-}^{(r)}d\mu,\int_A F_{+}^{(r)}d\mu\right)|0\leq r\leq 1\right\}$$

is a fuzzy real number for each $A\in\Sigma$.

The last fact leads to

Definition 17 *([13]) A measurable function $F:\Omega\rightarrow\mathbb{R}_{\mathcal{F}}$,*

$$F(w)=\{(F_{-}^{(r)}(w),F_{+}^{(r)}(w))|0\leq r\leq 1\}$$

is integrable if for each $r\in[0,1]$, $F_{\pm}^{(r)}$ are integrable, or equivalently, if $F_{\pm}^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A\in\Sigma$ is defined by

$$\int_A Fd\mu:=\left\{\left(\int_A F_{-}^{(r)}d\mu,\int_A F_{+}^{(r)}d\mu\right)|0\leq r\leq 1\right\}. \tag{13}$$

By [13] F is integrable iff $w\rightarrow\|F(w)\|_{\mathcal{F}}$ is real-valeud integrable. Here

$$\|u\|_{\mathcal{F}}:=D\left(u,\widetilde{0}\right),\ \ \forall\ u\in\mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 18 *([13]) Let $F,G:\Omega\rightarrow\mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a,b\in\mathbb{R}$, then $aF+bG$ is integrable and for each $A\in\Sigma$,

$$\int_A(aF+bG)\,d\mu=a\int_A Fd\mu+b\int_A Gd\mu;$$

(2) $D(F,G)$ is a real-valued integrable function and for each $A\in\Sigma$,

$$D\left(\int_A Fd\mu,\int_A Gd\mu\right)\leq\int_A D(F,G)\,d\mu. \tag{14}$$

In particular,

$$\left\|\int_A Fd\mu\right\|_{\mathcal{F}}\leq\int_A\|F\|_{\mathcal{F}}\,d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r := \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{15}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \tag{16}$$

$\forall\, r \in [0,1]$, respectively.

We use

Notation 19 *We denote*

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^2 f(\overrightarrow{x}) := \tag{17}$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right).$$

In general we denote ($j = 1, ..., N$)

$$\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x}) := \tag{18}$$

$$\sum_{(j_1,...,j_m)\in\mathbb{Z}_+^m : \sum_{i=1}^m j_i = j} \frac{j!}{j_1! j_2! ... j_m!} D\left(\frac{\partial^j f(x_1,...,x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_m^{j_m}}, \widetilde{0}\right).$$

2 Convergence with rates of real multivariate neural network operators

Here we follow [9].

We need the following (see [10]) definitions.

Definition 20 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.*

Definition 21 *(see [10]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,*

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 22 *(from [10]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function.*

Assumption 23 *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function or a uniformly continuous function.*

Here we mention the study ([9]) of poitwise convergence with rates over $\mathbb{R}^d$, to the unit operator I, of the "normalized bell" real multivariate neural network operators

$$M_n(f)(\overrightarrow{x}) := \tag{19}$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ... \frac{k_d}{n}\right) b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)},$$

where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly, M_n is a positive linear operator.

The terms in the ratio of multiple sums (19) can be nonzero iff simultaneously

$$\left| n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i, \text{ all } i = 1, ..., d,$$

i.e., $\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\alpha}}$, all $i = 1, ..., d$, iff

$$nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha, \text{ all } i = 1, .., d. \tag{20}$$

To have the order

$$-n^2 \leq nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha \leq n^2, \tag{21}$$

we need $n \geq T_i + |x_i|$, all $i = 1, .., d$. So (21) is true when we take

$$n \geq \max_{i \in \{1, ..., d\}} (T_i + |x_i|). \tag{22}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (21) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Consider

$$\widetilde{I}_i := [nx_i - T_i n^\alpha, nx_i + T_i n^\alpha], \ i = 1, ..., d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^\alpha$. By Proposition 1 of [1], we get that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card(k_i) \geq \max(2T_i n^\alpha - 1, 0)$, any $i \in \{1, ..., d\}$. In order to have $card(k_i) \geq 1$, we need $2T_i n^\alpha - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\alpha}}$, any $i \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (21) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, ..., d$ is that

$$n \geq \max_{i \in \{1,\ldots,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\alpha}} \right\}. \tag{23}$$

Clearly as $n \to +\infty$ we get that $card(k_i) \to +\infty$, all $i = 1, ..., d$. Also notice that $card(k_i)$ equals to the cardinality of integers in $[\lceil nx_i - T_i n^\alpha \rceil, [nx_i + T_i n^\alpha]]$ for all $i = 1, ..., d$. Here, $[\cdot]$ denotes the integral part of the number, while $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this article we will assume (23). Furthermore it holds

$$(M_n(f))(\overrightarrow{x}) := \frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} f\left(\frac{k_1}{n}, \ldots \frac{k_d}{n}\right)}{V(\overrightarrow{x})} \tag{24}$$

$$\cdot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)$$

all $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, where

$$V(\overrightarrow{x}) :=$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{25}$$

From [9], we need and mention

Theorem 24 *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{26}$$

Inequality (26) is attained by constant functions.

Inequalities (26) gives $M_n(f)(\overrightarrow{x}) \to f(\overrightarrow{x})$, pointwise with rates, as $n \to +\infty$, where $\overrightarrow{x} \in \mathbb{R}^d$, $d \geq 1$, provided that f is uniformly continuous on $\mathbb{R}^d$. In the last case it is clear that $M_n \to I$, uniformly.

From [9], we also need and mention

Theorem 25 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then*

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \tag{27}$$

$$\left\{\sum_{j=1}^{N}\frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f(\overrightarrow{x})\right)\right\}+\frac{(T^*)^N d^N}{N!n^{N(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (27) is attained by constant functions. Also, (27) gives us with rates the pointwise convergences of $M_n(f)\to f$ over $\mathbb{R}^d$, as $n\to+\infty$.

3 Main Results - Convergence with rates of fuzzy multivariate neural networks

Here b is as in Definition 21.

Assumption 26 *We suppose that $b(\overrightarrow{x})$ is of compact support $B:=\prod_{i=1}^{d}[-T_i,T_i]$, $T_i>0$, and it may have jump discontinuities there. We consider $f:\mathbb{R}^d\to\mathbb{R}_{\mathcal{F}}$ to be fuzzy continuous and fuzzy bounded function or fuzzy uniformly continuous function.*

In this section we study the D-metric pointwise convergence with rates over $\mathbb{R}^d$, to the fuzzy unit operator $I_{\mathcal{F}}$, of the fuzzy multivariate neural network operators ($0<\alpha<1$, $\overrightarrow{x}:=(x_1,...,x_d)\in\mathbb{R}^d$, $n\in\mathbb{N}$)

$$M_n^{\mathcal{F}}(f)(\overrightarrow{x}):= \tag{28}$$

$$\frac{\sum_{k_1=-n^2}^{n^2*}...\sum_{k_d=-n^2}^{n^2*}f\left(\frac{k_1}{n},...\frac{k_d}{n}\right)\odot b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2}...\sum_{k_d=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$=\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]*}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]*}f\left(\frac{k_1}{n},...\frac{k_d}{n}\right) \tag{29}$$

$$\odot\frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{V(\overrightarrow{x})},$$

where $V(\overrightarrow{x})$ as in (25) and under the assumption (23).

We notice for $r\in[0,1]$ that

$$\left[M_n^{\mathcal{F}}(f)(\overrightarrow{x})\right]^r=\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]}\left[f\left(\frac{k_1}{n},...\frac{k_d}{n}\right)\right]^r$$

$$\cdot\frac{b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}{V(\overrightarrow{x})} \tag{30}$$

$$=\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]}\left[f_-^{(r)}\left(\frac{k_1}{n},...\frac{k_d}{n}\right),f_+^{(r)}\left(\frac{k_1}{n},...\frac{k_d}{n}\right)\right]$$

$$\cdot \frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}$$

$$= \Bigg[\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} f_-^{(r)}\left(\frac{k_1}{n}, ... \frac{k_d}{n}\right)$$

$$\cdot \frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V\left(\overrightarrow{x}\right)},$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} f_+^{(r)}\left(\frac{k_1}{n}, ... \frac{k_d}{n}\right)$$

$$\cdot \frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} \Bigg] \tag{31}$$

$$= \left[\left(M_n\left(f_-^{(r)}\right)\right)\left(\overrightarrow{x}\right), \left(M_n\left(f_+^{(r)}\right)\right)\left(\overrightarrow{x}\right)\right].$$

We have proved that

$$\left(M_n^{\mathcal{F}}(f)\right)_\pm^{(r)} = M_n\left(f_\pm^{(r)}\right), \quad \forall\, r \in [0,1], \tag{32}$$

respectively.

We present

Theorem 27 *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)\left(\overrightarrow{x}\right), f\left(\overrightarrow{x}\right)\right) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{33}$$

Notice that (33) gives $M_n^{\mathcal{F}} \overset{D}{\to} I_{\mathcal{F}}$ pointwise and uniformly, as $n \to \infty$, when $f \in C_{\mathcal{F}}^U\left(\mathbb{R}^d\right)$.

Proof. We observe that

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)\left(\overrightarrow{x}\right), f\left(\overrightarrow{x}\right)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(M_n^{\mathcal{F}}(f)\right)_-^{(r)}\left(\overrightarrow{x}\right) - f_-^{(r)}\left(\overrightarrow{x}\right)\right|, \left|\left(M_n^{\mathcal{F}}(f)\right)_+^{(r)}\left(\overrightarrow{x}\right) - f_+^{(r)}\left(\overrightarrow{x}\right)\right|\right\} \overset{(32)}{=}$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(M_n\left(f_-^{(r)}\right)\right)\left(\overrightarrow{x}\right) - f_-^{(r)}\left(\overrightarrow{x}\right)\right|, \left|\left(M_n\left(f_+^{(r)}\right)\right)\left(\overrightarrow{x}\right) - f_+^{(r)}\left(\overrightarrow{x}\right)\right|\right\} \overset{(26)}{\leq}$$

$$\sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \frac{T^*}{n^{1-\alpha}}\right), \omega_1\left(f_+^{(r)}, \frac{T^*}{n^{1-\alpha}}\right)\right\} \overset{(7)}{=} \omega_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}}\right),$$

proving the claim. ■

We continue with

Theorem 28 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C_{\mathcal{F}}^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that all of its fuzzy partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are fuzzy uniformly continuous or fuzzy continuous and fuzzy bounded. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \leq \tag{34}$$

$$\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right]\right\}$$

$$+\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

As $n \to \infty$, we get $D\left(\left(M_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \to 0$ pointwise with rates.

Proof. As before we have

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \overset{(32)}{=}$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(M_n\left(f_-^{(r)}\right)\right)(\overrightarrow{x}) - f_-^{(r)}(\overrightarrow{x})\right|, \left|\left(M_n\left(f_+^{(r)}\right)\right)(\overrightarrow{x}) - f_+^{(r)}(\overrightarrow{x})\right|\right\} \overset{(27)}{\leq}$$

$$\sup_{r\in[0,1]} \max\left\{\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_-^{(r)}(\overrightarrow{x})\right]\right\}\right.$$

$$+\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_-^{(r)}\right)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right),$$

$$\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_+^{(r)}(\overrightarrow{x})\right]\right\}$$

$$\left.+\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_+^{(r)}\right)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right)\right\} \overset{(3)}{\leq} \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \cdot \tag{36}$$

$$\sup_{r\in[0,1]} \max\left\{\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_-^{(r)}(\overrightarrow{x})\right), \left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_+^{(r)}(\overrightarrow{x})\right)\right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f_-^{(r)}\right)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right), \omega_1\left(\left(f_+^{(r)}\right)_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right)\right\}$$

$$\overset{\text{(by (3), (7), (12), (18))}}{\leq} \left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right]\right\} + \tag{37}$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right),$$

proving the claim. ■

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4 Main Results - The fuzzy multivariate "normalized squashing type operators" and their fuzzy convergence to the fuzzy unit with rates

We give the following definition

Definition 29 *Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d} (-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [10]).*

Let $f : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$ be either fuzzy uniformly continuous or fuzzy continuous and fuzzy bounded function.

For $\overrightarrow{x} \in \mathbb{R}^d$, we define the fuzzy multivariate " normalized squashing type operator",

$$L_n^{\mathcal{F}}(f)(\overrightarrow{x}) := \tag{38}$$

$$\frac{\sum_{k_1=-n^2}^{n^2 *} \ldots \sum_{k_d=-n^2}^{n^2 *} f\left(\frac{k_1}{n}, \ldots \frac{k_d}{n}\right) \odot S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{W(\overrightarrow{x})},$$

where $0 < \alpha < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1,\ldots,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\alpha}} \right\}, \tag{39}$$

and

$$W(\overrightarrow{x}) := \sum_{k_1=-n^2}^{n^2 *} \ldots \sum_{k_d=-n^2}^{n^2 *} S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{40}$$

It is clear that

$$\left(L_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}) := \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}\right] *} \frac{f\left(\frac{\overrightarrow{k}}{n}\right) \odot S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\Phi(\overrightarrow{x})}, \tag{41}$$

where

$$\Phi(\overrightarrow{x}) := \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}\right]} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right). \tag{42}$$

Here, we study the D-metric pointwise convergence with rates of $\left(L_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}) \to f(\overrightarrow{x})$, as $n \to +\infty$, $\overrightarrow{x} \in \mathbb{R}^d$.

This is given first by the next result.

Theorem 30 *Under the above terms and asumptions, we find that*

$$D\left(\left(L_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{43}$$

Notice that (43) gives $L_n^{\mathcal{F}} \xrightarrow{D} I_{\mathcal{F}}$ *pointwise and uniformly, as* $n \to \infty$*, when* $f \in C_{\mathcal{F}}^U\left(\mathbb{R}^d\right)$.

Proof. Similar to (33). ∎

We also give

Theorem 31 *Let* $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C_{\mathcal{F}}^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, *such that all of its fuzzy partial derivatives* $f_{\widetilde{\alpha}}$ *of order* N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, *are fuzzy uniformly continuous or fuzzy continuous and fuzzy bounded. Then*

$$D\left(\left(L_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}), f(\overrightarrow{x})\right) \leq \tag{44}$$

$$\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(\overrightarrow{x})\right]\right\}$$

$$+ \frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (44) gives us with rates the poitwise convergence of $D(\left(L_n^{\mathcal{F}}(f)\right)(\overrightarrow{x}),$ $f(\overrightarrow{x})) \to 0$ *over* $\mathbb{R}^d$, *as* $n \to \infty$.

Proof. Similar to (34). ∎

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High degree Multivariate Fuzzy Approximation by Quasi-Interpolation Neural Network Operators

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Abstract

Here are considered in terms of multivariate fuzzy high approximation to the multivariate unit sequences of multivariate fuzzy quasi-interpolation neural network operators. These operators are multivariate fuzzy analogs of earlier considered multivariate real ones. The derived results generalize earlier real ones into the fuzzy setting. Here the high degree multivariate fuzzy pointwise and uniform convergences with rates to the multivariate fuzzy unit operator are given through multivariate fuzzy inequalities involving the multivariate fuzzy moduli of continuity of the Nth order ($N \geq 1$) H-fuzzy partial derivatives, of the involved multivariate fuzzy number valued function.

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Key words and phrases: multivariate fuzzy real analysis, multivariate fuzzy quasi-interpolation neural network operators, high degree multivariate fuzzy approximation, multivariate fuzzy modulus of continuity and multivariate Jackson type inequalities.

1 Fuzzy Real Analysis Background

We need the following background

Definition 1 *(see [23]) Let* $\mu : \mathbb{R} \to [0, 1]$ *with the following properties*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x+(1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x,y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$, *i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0)+\varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{\sup p(\mu)}$ *is compact in* $\mathbb{R}$, *(where* $\sup p(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, *for any* $x_0 \in \mathbb{R}$, *where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \tag{1}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ ([17]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [23]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}, \tag{2}$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \ \ u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [23], [24].

Let $f,g:W\subseteq\mathbb{R}^m\to\mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g)=\sup_{x\in W}D(f(x),g(x)).$$

Remark 2 *We try to determine better and use*

$$D^*(f,\widetilde{o})=\sup_{x\in W}D(f(x),\widetilde{o})=$$

$$\sup_{x\in W}\sup_{r\in[0,1]}\max\left\{\left|f_-^{(r)}(x)\right|,\left|f_+^{(r)}(x)\right|\right\}.$$

By the principle of iterated suprema we find that

$$D^*(f,\widetilde{o})=\sup_{r\in[0,1]}\max\left\{\left\|f_-^{(r)}\right\|_\infty,\left\|f_+^{(r)}\right\|_\infty\right\},\tag{3}$$

under the assumption $D^*(f,\widetilde{o})<\infty$, that is f is a fuzzy bounded function.

Above $\|\cdot\|_\infty$ is the supremum norm of the function over $W\subseteq\mathbb{R}^m$.

Another direct proof of (3) follows:

We easily see that

$$D^*(f,\widetilde{o})\leq\sup_{r\in[0,1]}\max\left\{\left\|f_-^{(r)}\right\|_\infty,\left\|f_+^{(r)}\right\|_\infty\right\}.$$

On the other hand we observe that $\forall\ x\in W$: each

$$\left|f_\pm^{(r)}(x)\right|\leq\max\left\{\left|f_\pm^{(r)}(x)\right|\right\}\leq\sup_{r\in[0,1]}\max\left\{\left|f_\pm^{(r)}(x)\right|\right\}\leq$$

$$\sup_{x\in W}\sup_{r\in[0,1]}\max\left\{\left|f_\pm^{(r)}(x)\right|\right\}=D^*(f,\widetilde{o}).$$

That is, each

$$\left|f_\pm^{(r)}(x)\right|\leq D^*(f,\widetilde{o}),\ \ \forall\ x\in W,$$

hence each

$$\left\|f_\pm^{(r)}\right\|_\infty\leq D^*(f,\widetilde{o}),$$

and

$$\max\left\{\left\|f_-^{(r)}\right\|_\infty,\left\|f_+^{(r)}\right\|_\infty\right\}\leq D^*(f,\widetilde{o}),$$

and

$$\sup_{r\in[0,1]}\max\left\{\left\|f_-^{(r)}\right\|_\infty,\left\|f_+^{(r)}\right\|_\infty\right\}\leq D^*(f,\widetilde{o}),$$

proving (3).

The assumption $D^*(f,\widetilde{o})<\infty$ implies $\left\|f_\pm^{(r)}\right\|_\infty<\infty$, $\forall\ r\in[0,1]$.

Here Σ^* stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 3 *([5]). Here* $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. *Suppose that*

$$\sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max \left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)} \right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right). \tag{4}$$

Definition 4 *Let* $f \in C(W)$, $W \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, *which is bounded or uniformly continuous, we define* $(h > 0)$

$$\omega_1(f, h) := \sup_{x, y \in W, \ \|x - y\|_\infty \leq h} |f(x) - f(y)|, \tag{5}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

Definition 5 *Let* $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, *we define the fuzzy modulus of continuity of* f *by*

$$\omega_1^{(\mathcal{F})}(f, h) = \sup_{x, y \in W, \ \|x - y\|_\infty \leq h} D(f(x), f(y)), \quad h > 0. \tag{6}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

For $f : W \to \mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^m$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)} \right], \tag{7}$$

where $f_\pm^{(r)} : W \to \mathbb{R}$, $\forall \ r \in [0,1]$.
We need

Proposition 6 *Let* $f : W \to \mathbb{R}_{\mathcal{F}}$. *Assume that* $\omega_1^{\mathcal{F}}(f, \delta)$, $\omega_1\left(f_-^{(r)}, \delta\right)$, $\omega_1\left(f_+^{(r)}, \delta\right)$ *are finite for any* $\delta > 0$, $r \in [0,1]$.
Then

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0,1]} \max \left\{ \omega_1\left(f_-^{(r)}, \delta\right), \omega_1\left(f_+^{(r)}, \delta\right) \right\}. \tag{8}$$

Proof. As in [5]. ■

We define by $C_{\mathcal{F}}^{U}(W)$ the space of fuzzy uniformly continuous functions from $W \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(W)$ is the space of fuzzy continuous functions on $W \subseteq \mathbb{R}^{m}$, and $C_{b}(W, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 7 *([7]) Let $f \in C_{\mathcal{F}}^{U}(W)$, where $W \subseteq \mathbb{R}^{m}$ is convex. Then $\omega_{1}^{(\mathcal{F})}(f, \delta) < \infty$, for any $\delta > 0$.*

Proposition 8 *([7]) It holds*

$$\lim_{\delta \to 0} \omega_{1}^{(\mathcal{F})}(f, \delta) = \omega_{1}^{(\mathcal{F})}(f, 0) = 0, \tag{9}$$

iff $f \in C_{\mathcal{F}}^{U}(W)$, $W \subseteq \mathbb{R}^{m}$.

Proposition 9 *([7]) Let $f \in C_{\mathcal{F}}(W)$, $W \subseteq \mathbb{R}^{m}$ open or compact. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over W, respectively in $\pm$.*

Note 10 *It is clear by Propositions 6, 8, that if $f \in C_{\mathcal{F}}^{U}(W)$, then $f_{\pm}^{(r)} \in C_{U}(W)$ (uniformly continuous on W).*

We need

Definition 11 *Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.*

Definition 12 *([23]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{10}$$

exist and are equal to $f'(x)$.

We call f' teh H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Definition 13 *We denote by $C_{\mathcal{F}}^{N}(W)$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from W into $\mathbb{R}_{\mathcal{F}}$, $W \subseteq \mathbb{R}^{m}$ open or compact which is convex.*

Here fuzzy partial derivatives are defined via Definition 12 in the obvious way as in the ordinary real case.

We mention

Theorem 14 *([19]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{11}$$

Remark 15 *(see also [6]) Let $f \in C^N([a,b], \mathbb{R}_{\mathcal{F}})$, $N \geq 1$. Then by Theorem 14 we obtain $f_{\pm}^{(r)} \in C^N([a,b])$ and*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_-^{(r)} \right)^{(i)}, \left(f(t)_+^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)}, \tag{12}$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^N(W)$, $W \subseteq \mathbb{R}^m$, open or compact, which is convex, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, ..., \widetilde{\alpha_m})$, $\widetilde{\alpha}_i \in \mathbb{Z}^+$, $i = 1, ..., m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{m} \widetilde{\alpha}_i \leq N, \quad N > 1.$$

Then by Theorem 14 we get that

$$\left(f_{\pm}^{(r)} \right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_{\pm}^{(r)}, \ \forall\, r \in [0,1], \tag{13}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \leq N$. Here $f_{\pm}^{(r)} \in C^N(W)$.

Notation 16 *We denote*

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^2 f(\overrightarrow{x}) := \tag{14}$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0} \right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0} \right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0} \right).$$

In general we denote $(j = 1, ..., N)$

$$\left(\sum_{i=1}^{m} D \left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^j f(\overrightarrow{x}) := \tag{15}$$

$$\sum_{(j_1,...,j_m) \in \mathbb{Z}_+^m : \sum_{i=1}^m j_i = j} \frac{j!}{j_1! j_2! ... j_m!} D \left(\frac{\partial^j f(x_1, ..., x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_m^{j_m}}, \widetilde{0} \right).$$

2 Basic on real Quasi-Interpolation Neural Network operators Approximation

I) Here all come from [16], [8].

We consider the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1 + e^{-x_i}}, \quad x_i \in \mathbb{R}, \, i = 1, ..., N; \; x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [16], we consider

$$\Phi_i(x_i) := \frac{1}{2} \left(s_i(x_i + 1) - s_i(x_i - 1) \right), \quad x_i \in \mathbb{R}, \, i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0$, $\forall \, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1$, $\forall \, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1$, $\forall \, x_i \in \mathbb{R}$; $n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i) \, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i)$, $x_i \geq 0$, for $i = 1, ..., N$.

We see that ([12])

$$\Phi_i(x_i) = \left(\frac{e^2 - 1}{2e^2} \right) \frac{1}{(1 + e^{x_i - 1})(1 + e^{-x_i - 1})}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Notice that $\Phi_i(x_i) = \Phi_i(0) = 0.231$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [12] we get

viii)

$$\sum_{\begin{cases} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{cases}}^{\infty} \Phi_i (nx_i - k_i) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

As in [12] we obtain

ix)

$$0 < \frac{1}{\sum_{k_i = \lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i (nx_i - k_i)} < \frac{1}{\Phi_i (1)} = 5.250312578,$$

$\forall \ x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [12], we see that

$$\lim_{n \to \infty} \sum_{k_i = \lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i (nx_i - k_i) \neq 1,$$

for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We use here

$$\Phi (x_1, ..., x_N) := \Phi (x) := \prod_{i=1}^{N} \Phi_i (x_i), \quad x \in \mathbb{R}^N. \tag{16}$$

It has the properties:

(i)' $\Phi (x) > 0$, $\forall \ x \in \mathbb{R}^N$,

(ii)'

$$\sum_{k=-\infty}^{\infty} \Phi (x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi (x_1 - k_1, ..., x_N - k_N) = 1, \tag{17}$$

$k := (k_1, ..., k_N)$, $\forall \ x \in \mathbb{R}^N$.

8

(iii)'

$$\sum_{k=-\infty}^{\infty} \Phi(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(nx_1-k_1, ..., nx_N-k_N) = 1, \quad (18)$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi(x)\, dx = 1,$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil),$$
$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor).$$

We also have

(v)'

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} \Phi(nx-k) \le 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

(vi)'

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < (5.250312578)^N,$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

(vii)'

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \Phi(nx-k) \le 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

(viii)'

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\neq 1 \tag{19}$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define ([11]) the multivariate positive linear neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$$G_n(f,x_1,...,x_N):=G_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)} \tag{20}$$

$$:=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi_i(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Phi_i(nx_i-k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

When $f\in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$) we define ([11])

$$\overline{G}_n(f,x):=\overline{G}_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)\Phi(nx-k) \tag{21}$$

$$:=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi_i(nx_i-k_i)\right),$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, $N\geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [11]:

Theorem 17 *Let* $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n,N\in\mathbb{N}$. *Then*

i)

$$|G_n(f,x)-f(x)|\leq(5.250312578)^N\cdot$$
$$\left\{\omega_1\left(f,\frac{1}{n^\beta}\right)+(6.3984)\|f\|_\infty e^{-n^{(1-\beta)}}\right\}=:\lambda_1, \tag{22}$$

ii)

$$\|G_n(f)-f\|_\infty\leq\lambda_1. \tag{23}$$

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Theorem 18 *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. Then*

i)

$$\left|\overline{G}_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + (6.3984)\|f\|_\infty e^{-n^{(1-\beta)}} =: \lambda_2, \tag{24}$$

ii)

$$\left\|\overline{G}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{25}$$

II) Here we follow [10], [8].

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{26}$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 19 *([9]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim\limits_{x\to-\infty} \Psi(x) = 0 = \lim\limits_{x\to\infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 20 *([9]) We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1$, $\forall\, x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 21 *([9]) It holds $\int_{-\infty}^{\infty} \Psi(x)\,dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 22 *([9]) Let $0 < \alpha < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 23 *([9]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [9] we get that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

We use (see [10])

$$\Theta(x_1,...,x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{27}$$

It has the properties:

(i)* $\Theta(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)*

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1,...,x_N-k_N) = 1, \tag{28}$$

where $k := (k_1,...,k_N)$, $\forall \ x \in \mathbb{R}^N$.

(iii)*

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1,...,nx_N-k_N) = 1, \tag{29}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)*
$$\int_{\mathbb{R}^N} \Theta(x)\, dx = 1,$$

that is Θ is a multivariate density function.

(v)*
$$\sum_{\substack{\left\{ \begin{array}{c} k=\lceil na\rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{array} \right.}}^{\lfloor nb \rfloor} \Theta(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}}, \tag{30}$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

(vi)*
$$0 < \frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb \rfloor} \Theta(nx-k)} < \frac{1}{(\Psi(1))^N} = (4.1488766)^N, \tag{31}$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $n \in \mathbb{N}$.

(vii)*
$$\sum_{\substack{\left\{ \begin{array}{c} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{array} \right.}}^{\infty} \Theta(nx-k) \leq e^4 \cdot e^{-2n^{(1-\beta)}}, \tag{32}$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

Also we get that
$$\lim_{n \to \infty} \sum_{k=\lceil na\rceil}^{\lfloor nb \rfloor} \Theta(nx-k) \neq 1, \tag{33}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i] \right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the multivariate positive linear neural network operator ([10]) ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i] \right)$)

$$F_n(f, x_1, ..., x_N) := F_n(f, x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb \rfloor} \Theta(nx-k)} \tag{34}$$

$$:= \frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^N \Psi(nx_i - k_i) \right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i) \right)}.$$

When $f \in C_B\left(\mathbb{R}^N\right)$ we define ([10]),

$$\overline{F}_n(f,x) := \overline{F}_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\Theta(nx-k) := \tag{35}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Psi(nx_i-k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [10]:

Theorem 24 *Let* $f \in C\left(\prod_{i=1}^N [a_i,b_i]\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^N [a_i,b_i]\right)$, $n, N \in \mathbb{N}$. *Then*

i)

$$|F_n(f,x) - f(x)| \leq (4.1488766)^N \cdot$$
$$\left\{\omega_1\left(f,\frac{1}{n^\beta}\right) + 2e^4\|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda_1, \tag{36}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda_1. \tag{37}$$

Theorem 25 *Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. *Then*

i)

$$\left|\overline{F}_n(f,x) - f(x)\right| \leq \omega_1\left(f,\frac{1}{n^\beta}\right) + 2e^4\|f\|_\infty e^{-2n^{(1-\beta)}} =: \lambda_2, \tag{38}$$

ii)

$$\left\|\overline{F}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{39}$$

Notation 26 *Let* $f \in C^m\left(\prod_{i=1}^N [a_i,b_i]\right)$, $m, N \in \mathbb{N}$. *Here* f_α *denotes a partial derivative of* f, $\alpha := (\alpha_1,...,\alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i=1,...,N$, *and* $|\alpha| := \sum_{i=1}^N \alpha_i = l$, *where* $l = 0,1,...,m$. *We write also* $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ *and we say it is of order* l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha,h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha,h). \tag{40}$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m}\left\{\|f_\alpha\|_\infty\right\}, \tag{41}$$

In the next we mention the high order of approximation by using the smoothness of f.

We give

Theorem 27 *([8]) Let* $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$. *Then*

i)

$$\left| G_n(f,x) - f(x) - \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) G_n\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \tag{42}$$

$$(5.250312578)^N \cdot \left\{ \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta} \right) + \right.$$

$$\left. \left(\frac{(6.3984) \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-n^{(1-\beta)}} \right\},$$

ii)

$$|G_n(f,x) - f(x)| \leq (5.250312578)^N \cdot \tag{43}$$

$$\left\{ \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \cdot (3.1992) e^{-n^{(1-\beta)}} \right] \right) + \right.$$

$$\left. \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{(6.3984) \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-n^{(1-\beta)}} \right\},$$

iii)

$$\|G_n(f) - f\|_\infty \leq (5.250312578)^N \cdot \tag{44}$$

$$\left\{ \sum_{j=1}^N \left(\sum_{|\alpha|=j} \left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) (3.1992) e^{-n^{(1-\beta)}} \right] \right) + \right.$$

$$\left. \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{(6.3984) \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-n^{(1-\beta)}} \right\},$$

iv) Assume $f_\alpha(x_0) = 0$, *for all* $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$. *Then*

$$|G_n(f, x_0) - f(x_0)| \leq (5.250312578)^N \cdot \tag{45}$$

$$\left\{ \frac{N^m}{m! n^{m\beta}} \omega_1^{\max}\left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{(6.3984) \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-n^{(1-\beta)}} \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

We also mention

Theorem 28 *([8]) Let* $f \in C^m \left(\prod_{i=1}^N [a_i, b_i] \right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$. *Then*

i)

$$\left| F_n(f,x) - f(x) - \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) F_n \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq \tag{46}$$

$$(4.1488766)^N \cdot \left\{ \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{2e^4 \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-2n^{(1-\beta)}} \right\},$$

ii)

$$|F_n(f,x) - f(x)| \leq (4.1488766)^N \cdot \tag{47}$$

$$\left\{ \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{|f_\alpha(x)|}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \cdot e^4 e^{-2n^{(1-\beta)}} \right] \right) + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{2e^4 \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-2n^{(1-\beta)}} \right\},$$

iii)

$$\|F_n(f) - f\|_\infty \leq (4.1488766)^N \cdot \tag{48}$$

$$\left\{ \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{\|f_\alpha\|_\infty}{\prod_{i=1}^N \alpha_i!} \right) \left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) e^4 e^{-2n^{(1-\beta)}} \right] \right) + \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{2e^4 \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-2n^{(1-\beta)}} \right\},$$

iv) Assume $f_\alpha(x_0) = 0$, *for all* $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^N [a_i, b_i] \right)$. *Then*

$$|F_n(f, x_0) - f(x_0)| \leq (4.1488766)^N \cdot \tag{49}$$

$$\left\{ \frac{N^m}{m! n^{m\beta}} \omega_1^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \left(\frac{2e^4 \|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) e^{-2n^{(1-\beta)}} \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

We need

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Notation 29 *([15]) Call* $L_n = G_n,\ \overline{G}_n,\ F_n,\ \overline{F}_n$.

Denote by

$$c_N = \begin{cases} (5.250312578)^N, & \text{if } L_n = G_n, \\ (4.1488766)^N, & \text{if } L_n = F_n, \\ 1, & \text{if } L_n = \overline{G}_n,\ \overline{F}_n, \end{cases} \tag{50}$$

$$\mu = \begin{cases} 6.3984, & \text{if } L_n = G_n,\ \overline{G}_n, \\ 2e^4, & \text{if } L_n = F_n, \overline{F}_n, \end{cases} \tag{51}$$

and

$$\gamma = \begin{cases} 1, & \text{when } L_n = G_n,\ \overline{G}_n, \\ 2 & \text{when } L_n = F_n, \overline{F}_n. \end{cases} \tag{52}$$

Based on the above notations Theorems 17, 18, 24, 25 can put in a unified way as follows.

Theorem 30 *([15]) Let* $f \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$ *or* $f \in C_B\left(\mathbb{R}^N\right)$*;* $n,\ N \in \mathbb{N}$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$ *or* $x \in \mathbb{R}^N$*. Then*

(i)

$$|L_n(f,x) - f(x)| \le c_N \left\{ \omega_1\left(f, \frac{1}{n^\beta}\right) + \mu \|f\|_\infty e^{-\gamma n^{(1-\beta)}} \right\} =: \rho_n, \tag{53}$$

(ii)

$$\|L_n(f) - f\|_\infty \le \rho_n. \tag{54}$$

For basic neural networks knowledge we refer to [21], [22].

3 Approximation by Fuzzy Quasi-Interpolation Neural Network Operators

Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $n \in \mathbb{N}$. We define the Fuzzy Quasi-Interpolation Neural Network operators

$$G_n^{\mathcal{F}}(f, x_1, ..., x_N) := G_n^{\mathcal{F}}(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \tag{55}$$

$\forall\ x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, see also (20).

Similarly we define

$$F_n^{\mathcal{F}}(f, x_1, ..., x_N) := F_n^{\mathcal{F}}(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\Theta(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx-k)}, \tag{56}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, see also (34).

Let $f \in C_b\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right)$. We define

$$\overline{G}_n^{\mathcal{F}}(f, x) := \overline{G}_n^{\mathcal{F}}(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty *} f\left(\frac{k}{n}\right) \odot \Phi(nx-k) \tag{57}$$

$$:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty *} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \odot \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i)\right),$$

and

$$\overline{F}_n^{\mathcal{F}}(f, x) := \sum_{k=-\infty}^{\infty *} f\left(\frac{k}{n}\right) \odot \Theta(nx-k), \tag{58}$$

$\forall\ x \in \mathbb{R}^N$, $N \geq 1$.

The sum in (55), (56) are finite.

Let $r \in [0,1]$, we observe that

$$\left[G_n^{\mathcal{F}}(f, x)\right]^r = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f\left(\frac{k}{n}\right)\right]^r \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right)\right] \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right) = \tag{59}$$

$$\left[\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_-^{(r)}\left(\frac{k}{n}\right) \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right), \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_+^{(r)}\left(\frac{k}{n}\right) \left(\frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}\right)\right]$$

$$= \left[G_n \left(f_-^{(r)}, x \right), G_n \left(f_+^{(r)}, x \right) \right]. \tag{60}$$

We have proved that

$$\left(G_n^{\mathcal{F}} (f, x) \right)_{\pm}^{(r)} = G_n \left(f_{\pm}^{(r)}, x \right), \tag{61}$$

respectively, $\forall\ r \in [0,1]$, $\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

Similarly, it holds

$$\left(F_n^{\mathcal{F}} (f, x) \right)_{\pm}^{(r)} = F_n \left(f_{\pm}^{(r)}, x \right), \tag{62}$$

respectively, $\forall\ r \in [0,1]$, $\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

We will prove also that

$$\left(\overline{G}_n^{\mathcal{F}} (f, x) \right)_{\pm}^{(r)} = \overline{G}_n \left(f_{\pm}^{(r)}, x \right), \tag{63}$$

$$\left(\overline{F}_n^{\mathcal{F}} (f, x) \right)_{\pm}^{(r)} = \overline{F}_n \left(f_{\pm}^{(r)}, x \right), \tag{64}$$

respectively, $\forall\ r \in [0,1]$, $\forall\ x \in \mathbb{R}^N$.

The sums in (63), (64) are doubly infinite and their proof is more complicated and follows.

We need

Remark 31 *(see also [20]) (1) Here* $k = (k_1, k_2) \in \mathbb{Z}^2$, $m = (m_1, m_2) \in \mathbb{Z}_-^2$, $n = (n_1, n_2) \in \mathbb{Z}_+^2$, $\infty = (\infty, \infty)$, $-\infty = (-\infty, -\infty)$, $\sum$ *is a double sum.*

Let $(u_k)_{k \in \mathbb{Z}^2} \in \mathbb{R}_{\mathcal{F}}$. *We denote the fuzzy double infinite series by* $\sum_{k=-\infty}^{\infty *} u_k$ *and we say that it converges to* $u \in \mathbb{R}_{\mathcal{F}}$ *iff* $\lim_{m \to \infty} \lim_{n \to \infty} D \left(\sum_{k=m}^{n*} u_k, u \right) = 0$. *We denote the last by* $\sum_{k=-\infty}^{\infty *} u_k = u$.

Let $(u_k)_{k \in \mathbb{Z}^2}$, $(v_k)_{k \in \mathbb{Z}^2}$, $u, v, \mathbb{R}_{\mathcal{F}}$ *such that* $\sum_{k=-\infty}^{\infty *} u_k = u$, $\sum_{k=-\infty}^{\infty *} v_k = v$. *Then*

$$\sum_{k=-\infty}^{\infty *} (u_k \oplus v_k) = u \oplus v = \sum_{k=-\infty}^{\infty *} u_k \oplus \sum_{k=-\infty}^{\infty *} v_k. \tag{65}$$

The last is true since

$$\lim_{m \to \infty} \lim_{n \to \infty} D \left(\sum_{k=m}^{n*} (u_k \oplus v_k), u \oplus v \right) =$$

$$\lim_{m\to\infty}\lim_{n\to\infty} D\left(\left(\sum_{k=m}^{n*} u_k\right) \oplus \left(\sum_{k=m}^{n*} v_k\right), u \oplus v\right)$$
$$\leq \lim_{m\to\infty}\lim_{n\to\infty}\left[D\left(\sum_{k=m}^{n*} u_k, u\right) + D\left(\sum_{k=m}^{n*} v_k, v\right)\right] = 0.$$

Let $\sum_{k=-\infty}^{\infty*} u_k = u \in \mathbb{R}_{\mathcal{F}}$, *then one has that*

$$\sum_{k=-\infty}^{\infty} (u_k)_-^{(r)} = u_-^{(r)} = \left(\sum_{k=-\infty}^{\infty*} u_k\right)_-^{(r)}, \tag{66}$$

and

$$\sum_{k=-\infty}^{\infty} (u_k)_+^{(r)} = u_+^{(r)} = \left(\sum_{k=-\infty}^{\infty*} u_k\right)_+^{(r)}, \tag{67}$$

$\forall\ r \in [0,1]$.

We prove the last claim:

We have that

$$0 = \lim_{m\to\infty}\lim_{n\to\infty} D\left(\sum_{k=m}^{n} u_k, u\right) =$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\sup_{r\in[0,1]}\max\left\{\left|\sum_{k=m}^{n} (u_k)_-^{(r)} - u_-^{(r)}\right|, \left|\sum_{k=m}^{n} (u_k)_+^{(r)} - u_+^{(r)}\right|\right\} \geq$$
$$\lim_{m\to\infty}\lim_{n\to\infty}\left\{\left|\sum_{k=m}^{n} (u_k)_-^{(r)} - u_-^{(r)}\right|, \left|\sum_{k=m}^{n} (u_k)_+^{(r)} - u_+^{(r)}\right|\right\},$$

$\forall\ r \in [0,1]$, *proving the claim.*

Also we need: let $(u_k)_{k\in\mathbb{Z}^2} \in \mathbb{R}_{\mathcal{F}}$ *with* $\sum_{k=-\infty}^{\infty*} u_k = u \in \mathbb{R}_{\mathcal{F}}$, *then clearly one has for any* $\lambda \in \mathbb{R}$ *that* $\sum_{k=-\infty}^{\infty*} \lambda u_k = \lambda u$.

Clearly also here $\sum_{k=-\infty}^{\infty*} u_k = u \in \mathbb{R}_{\mathcal{F}}$, *here* $(u_k)_{k\in\mathbb{Z}^2} \in \mathbb{R}_{\mathcal{F}}$, *iff* $\sum_{k=-\infty}^{\infty} (u_k)_-^{(r)} = (u)_-^{(r)}$ *and* $\sum_{k=-\infty}^{\infty} (u_k)_+^{(r)} = (u)_+^{(r)}$, *uniformly in* $r \in [0,1]$, *see also [20].*

(2) By [20] we see: Let $\left(k \in \mathbb{Z}^2\right)$ $u_k := \left\{\left((u_k)_-^{(r)}, (u_k)_+^{(r)}\right) | 0 \leq r \leq 1\right\} \in \mathbb{R}_{\mathcal{F}}$ *such that* $\sum_{k=-\infty}^{\infty} (u_k)_-^{(r)} = (u)_-^{(r)}$ *and* $\sum_{k=-\infty}^{\infty} (u_k)_+^{(r)} = (u)_+^{(r)}$ *converge uniformly in* $r \in [0,1]$, *then* $u := \left\{\left((u)_-^{(r)}, (u)_+^{(r)}\right) | 0 \leq r \leq 1\right\} \in \mathbb{R}_{\mathcal{F}}$ *and* $u = \sum_{k=-\infty}^{\infty*} u_k$.

I.e. we have

$$\sum_{k=-\infty}^{\infty *}\left\{\left((u_k)_-^{(r)},(u_k)_+^{(r)}\right)|0\leq r\leq 1\right\}=$$

$$\left\{\left(\sum_{k=-\infty}^{\infty *}(u_k)_-^{(r)},\sum_{k=-\infty}^{\infty *}(u_k)_+^{(r)}\right)|0\leq r\leq 1\right\}. \tag{68}$$

All the content of Remark 31 goes through and is valid for $k\in\mathbb{Z}^N$, $N\geq 2$.

Proof. of (63) and (64).

The proof of (64) is totally similar to the proof of (63). So we prove only (63).

Let $f\in C_b\left(\mathbb{R}^N,\mathbb{R}_{\mathcal{F}}\right)$, then $f_{\pm}^{(r)}\in C_B\left(\mathbb{R}^N\right)$, $\forall\ r\in[0,1]$. We have

$$(f(x))_-^{(0)}\leq(f(x))_-^{(r)}\leq(f(x))_-^{(1)}\leq(f(x))_+^{(1)}$$

$$\leq(f(x))_+^{(r)}\leq(f(x))_+^{(0)},\quad\forall\ x\in\mathbb{R}^N. \tag{69}$$

We get that

$$\left|(f(x))_-^{(r)}\right|\leq\max\left\{\left|(f(x))_-^{(0)}\right|,\left|(f(x))_-^{(1)}\right|\right\}=$$

$$\frac{1}{2}\left\{\left|(f(x))_-^{(0)}\right|+\left|(f(x))_-^{(1)}\right|-\left|\left|(f(x))_-^{(0)}\right|-\left|(f(x))_-^{(1)}\right|\right|\right\}$$

$$=:A_-^{(0,1)}(x)\in C_B\left(\mathbb{R}^N\right). \tag{70}$$

Also it holds

$$\left|(f(x))_+^{(r)}\right|\leq\max\left\{\left|(f(x))_+^{(0)}\right|,\left|(f(x))_+^{(1)}\right|\right\}=$$

$$\frac{1}{2}\left\{\left|(f(x))_+^{(0)}\right|+\left|(f(x))_+^{(1)}\right|-\left|\left|(f(x))_+^{(0)}\right|-\left|(f(x))_+^{(1)}\right|\right|\right\}$$

$$=:A_+^{(0,1)}(x)\in C_B\left(\mathbb{R}^N\right). \tag{71}$$

I.e. we have obtained that

$$0\leq\left|(f(x))_-^{(r)}\right|\leq A_-^{(0,1)}(x), \tag{72}$$

$$0\leq\left|(f(x))_+^{(r)}\right|\leq A_+^{(0,1)}(x),\quad\forall\ r\in[0,1],\forall\ x\in\mathbb{R}^N.$$

Hence by positivity of the operators $\overline{G}_n$ we get

$$0\leq\left|\overline{G}_n\left(f_{\pm}^{(r)},x\right)\right|\leq\overline{G}_n\left(\left|f_{\pm}^{(r)}\right|,x\right)\leq\overline{G}_n\left(A_{\pm}^{(0,1)},x\right), \tag{73}$$

respectively in $\pm$, $\forall\ r\in[0,1]$, $\forall\ x\in\mathbb{R}^N$.

In detail one has

$$\left|\left(f\left(\frac{k}{n}\right)\right)_{\pm}^{(r)}\right|\Phi\left(nx-k\right)\leq A_{\pm}^{(0,1)}\left(\frac{k}{n}\right)\Phi\left(nx-k\right), \tag{74}$$

$\forall\ r\in[0,1]$, $k\in\mathbb{Z}^N$, $n\in\mathbb{N}$ fixed, $\forall\ x\in\mathbb{R}^N$ fixed, respectively in $\pm$.

We notice that

$$\sum_{k=-\infty}^{\infty}A_{\pm}^{(0,1)}\left(\frac{k}{n}\right)\Phi\left(nx-k\right)\leq\left\|A_{\pm}^{(0,1)}\right\|_{\infty}<\infty, \tag{75}$$

i.e. $\sum_{k=-\infty}^{\infty}A_{\pm}^{(0,1)}\left(\frac{k}{n}\right)\Phi\left(nx-k\right)$ converges, respectively in $\pm$.

Thus by Weierstrass M-test we obtain that $\overline{G}_n\left(\left|f_{\pm}^{(r)}\right|,x\right)$ as series converges uniformly in $r\in[0,1]$, respectively in $\pm$, $\forall\ n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$.

And by Cauchy criterion for series of uniform convergence we get that $\overline{G}_n\left(f_{\pm}^{(r)},x\right)$ as series converges uniformly in $r\in[0,1]$, $\forall\ n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, respectively in $\pm$.

Here for $k\in\mathbb{Z}^N$, $f\left(\frac{k}{n}\right)=\left\{\left(f\left(\frac{k}{n}\right)\right)_{-}^{(r)},\left(f\left(\frac{k}{n}\right)\right)_{+}^{(r)}|0\leq r\leq 1\right\}\in\mathbb{R}_{\mathcal{F}}$, and also

$$u_k:=\left\{\left(f\left(\frac{k}{n}\right)\right)_{-}^{(r)}\Phi\left(nx-k\right),\left(f\left(\frac{k}{n}\right)\right)_{+}^{(r)}\Phi\left(nx-k\right)|0\leq r\leq 1\right\} \tag{76}$$

$$=f\left(\frac{k}{n}\right)\odot\Phi\left(nx-k\right)\in\mathbb{R}_{\mathcal{F}}.$$

That is $\left(u_k\right)_{\pm}^{(r)}=\left(f\left(\frac{k}{n}\right)\right)_{\pm}^{(r)}\Phi\left(nx-k\right)$, respectively in $\pm$.

But we proved that

$$\sum_{k=-\infty}^{\infty}\left(u_k\right)_{\pm}^{(r)}=\overline{G}_n\left(f_{\pm}^{(r)},x\right)=:\left(u\right)_{\pm}^{(r)},$$

converges uniformly in $r\in[0,1]$, respectively in $\pm$.

Then by Remark 31 (2) we get

$$u:=\left\{\left(\left(u\right)_{-}^{(r)},\left(u\right)_{+}^{(r)}\right)|0\leq r\leq 1\right\}$$

$$=\left\{\left(\overline{G}_n\left(f_{-}^{(r)},x\right),\overline{G}_n\left(f_{+}^{(r)},x\right)\right)|0\leq r\leq 1\right\}\in\mathbb{R}_{\mathcal{F}} \tag{77}$$

and

$$u=\sum_{k=-\infty}^{\infty *}u_k=$$

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$$\sum_{k=-\infty}^{\infty *}\left\{\left(f\left(\frac{k}{n}\right)\right)_{-}^{(r)}\Phi\left(nx-k\right),\left(f\left(\frac{k}{n}\right)\right)_{+}^{(r)}\Phi\left(nx-k\right)|0\leq r\leq 1\right\} \quad (78)$$

$$=\sum_{k=-\infty}^{\infty *}\left\{\left(\left(f\left(\frac{k}{n}\right)\right)_{-}^{(r)},\left(f\left(\frac{k}{n}\right)\right)_{+}^{(r)}\right)|0\leq r\leq 1\right\}\odot\Phi\left(nx-k\right) \quad (79)$$

$$=\sum_{k=-\infty}^{\infty *}f\left(\frac{k}{n}\right)\odot\Phi\left(nx-k\right)=\overline{G}_{n}^{\mathcal{F}}\left(f,x\right).$$

So we have proved (63). ■

Conclusion 32 *Call $L_n^{\mathcal{F}}$ any of the operators $G_n^{\mathcal{F}}$, $F_n^{\mathcal{F}}$, $\overline{G}_n^{\mathcal{F}}$, $\overline{F}_n^{\mathcal{F}}$.*

Let also L_n be any of real operators G_n, F_n, $\overline{G}_n$, $\overline{F}_n$.
We have proved that

$$\left(L_n^{\mathcal{F}}\left(f,x\right)\right)_{\pm}^{(r)}=L_n\left(f_{\pm}^{(r)},x\right), \quad (80)$$

respectively in $\pm$ and operator matching couples, $\forall\, r\in[0,1]$, $\forall\, x\in\left(\prod_{i=1}^{N}\left([a_i,b_i]\right)\right)$ or $\forall\, x\in\mathbb{R}^N$, $N\in\mathbb{N}$.

We present our first main result

Theorem 33 *Let $f\in C_{\mathcal{F}}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ or $f\in C_b\left(\mathbb{R}^N,\mathbb{R}_{\mathcal{F}}\right)$; $n,N\in\mathbb{N}$, $0<\beta<1$, $x\in\prod_{i=1}^{N}[a_i,b_i]$ or $x\in\mathbb{R}^N$. Then*

(i)

$$D\left(L_n\left(f,x\right),f\left(x\right)\right)\leq c_N\left\{\omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)+\mu D^{*}\left(f,\widetilde{o}\right)e^{-\gamma n(1-\beta)}\right\}=:\rho_n^{\mathcal{F}}. \quad (81)$$

(ii)

$$D^{*}\left(L_n\left(f\right),f\right)\leq\rho_n^{\mathcal{F}}, \quad (82)$$

when c_N, μ, γ as in Notation 29.

Proof. We see that

$$D\left(L_n\left(f,x\right),f\left(x\right)\right)=$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(L_n\left(f,x\right)\right)_{-}^{(r)}-f_{-}^{(r)}\left(x\right)\right|,\left|\left(L_n\left(f,x\right)\right)_{+}^{(r)}-f_{+}^{(r)}\left(x\right)\right|\right\}$$

$$\overset{(80)}{=}\sup_{r\in[0,1]}\max\left\{\left|\left(L_n\left(f_{-}^{(r)},x\right)\right)-f_{-}^{(r)}\left(x\right)\right|,\left|\left(L_n\left(f_{+}^{(r)},x\right)\right)-f_{+}^{(r)}\left(x\right)\right|\right\}, \quad (83)$$

$\forall\ x \in \prod_{i=1}^{N} [a_i, b_i]$ or $\forall\ x \in \mathbb{R}^N$.

The assumption implies $f_{\pm}^{(r)} \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ or $f_{\pm}^{(r)} \in C_B\left(\mathbb{R}^N\right)$, respectively in $\pm$, $\forall\ r \in [0,1]$.

Hence by (53) we get

$$\left|\left(L_n\left(f_{\pm}^{(r)}, x\right)\right) - f_{\pm}^{(r)}(x)\right| \leq c_N \left\{\omega_1\left(f_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right) + \mu \left\|f_{\pm}^{(r)}\right\|_{\infty} e^{-\gamma n^{(1-\beta)}}\right\}$$

(by (8) and (3))

$$\leq c_N \left\{\omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^{\beta}}\right) + \mu D^*\left(f, \widetilde{o}\right) e^{-\gamma n^{(1-\beta)}}\right\}, \tag{84}$$

respectively in $\pm$.

Now combining (83) and (84) we prove (81). $\blacksquare$

Remark 34 *Let* $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *or* $f \in \left(C_b\left(\mathbb{R}^N, \mathbb{R}_{\mathcal{F}}\right) \cap C_{\mathcal{F}}^{U}\left(\mathbb{R}^N\right)\right)$. *Then* $\lim_{n\to\infty} D^*\left(L_n(f), f\right) = 0$, *quantitatively with rates from (82).*

Notation 35 *Let* $f \in C_{\mathcal{F}}^{m}\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $m, N \in \mathbb{N}$. *Here* f_{α} *denotes a fuzzy partial derivative as in Notation 26. We denote*

$$\omega_{1,m}^{(\mathcal{F})\max}\left(f_a, h\right) := \max_{\alpha:|\alpha|=m} \omega_1^{(\mathcal{F})}\left(f_a, h\right). \tag{85}$$

Call also

$$D_m^{*\max}\left(f_{\alpha}, \widetilde{o}\right) := \max_{\alpha:|\alpha|=m} \left\{D^*\left(f_{\alpha}, \widetilde{o}\right)\right\}. \tag{86}$$

We present

Theorem 36 *Let* $f \in C_{\mathcal{F}}^{m}\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $0 < \beta < 1$, $n, m, N \in \mathbb{N}$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$. *Then*

1)

$$D\left(G_n^{\mathcal{F}}(f, x), f(x)\right) \leq (5.250312578)^N \cdot \tag{87}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{D\left(f_{\alpha}(x), \widetilde{o}\right)}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^{N}\left(b_i - a_i\right)^{\alpha_i}\right)\cdot(3.1992)\, e^{-n^{(1-\beta)}}\right]\right)\right.$$

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$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$

$$\left(\frac{(6.3984)\|b-a\|_\infty^m D_m^{*\max}(f_\alpha,\widetilde{o})N^m}{m!}\right)e^{-n^{(1-\beta)}}\Bigg\},$$

2)

$$D^*\left(G_n^{\mathcal{F}}(f),f\right)\leq(5.250312578)^N\cdot \tag{88}$$

$$\left\{\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{D^*(f_\alpha,\widetilde{o})}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$

$$\left.\left(\frac{(6.3984)\|b-a\|_\infty^m D_m^{*\max}(f_\alpha,\widetilde{o})N^m}{m!}\right)e^{-n^{(1-\beta)}}\right\},$$

3) assume that $f_\alpha(x_0)=\widetilde{o}$, for all $\alpha:|\alpha|=1,...,m$; $x_0\in\left(\prod_{i=1}^N[a_i,b_i]\right)$, then

$$D\left(G_n^{\mathcal{F}}(f,x_0),f(x_0)\right)\leq(5.250312578)^N\cdot \tag{89}$$

$$\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{(6.3984)\|b-a\|_\infty^m D_m^{*\max}(f_\alpha,\widetilde{o})N^m}{m!}\right)e^{-n^{(1-\beta)}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Proof. We observe that

$$\left|G_n\left(f_\pm^{(r)},x\right)-f_\pm^{(r)}(x)\right|\overset{(43)}{\leq}(5.250312578)^N\cdot$$

$$\left\{\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{\left|\left(f_\pm^{(r)}\right)_\alpha(x)\right|}{\prod_{i=1}^N\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^N(b_i-a_i)^{\alpha_i}\right)\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right]\right)\right.$$

$$+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(\left(f_\pm^{(r)}\right)_\alpha,\frac{1}{n^\beta}\right)+ \tag{90}$$

$$\left.\left(\frac{(6.3984)\|b-a\|_\infty^m\left\|\left(f_\pm^{(r)}\right)_\alpha\right\|_{\infty,m}^{\max}N^m}{m!}\right)e^{-n^{(1-\beta)}}\right\}\overset{(13)}{=}(5.250312578)^N\cdot$$

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$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{\left|\left(f_{\alpha}\right)_{\pm}^{(r)}(x)\right|}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right]\right)\right.$$

$$+\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(\left(f_{\alpha}\right)_{\pm}^{(r)},\frac{1}{n^{\beta}}\right)+ \tag{91}$$

$$\left.\left(\frac{(6.3984)\,\|b-a\|_{\infty}^{m}\left\|\left(f_{\alpha}\right)_{\pm}^{(r)}\right\|_{\infty,m}^{\max}N^{m}}{m!}\right)e^{-n^{(1-\beta)}}\right\}\overset{((8),(3))}{\leq}(5.250312578)^{N}\cdot$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{D\left(f_{\alpha}(x),\widetilde{o}\right)}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right]\right)\right.$$

$$+\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+ \tag{92}$$

$$\left.\left(\frac{(6.3984)\,\|b-a\|_{\infty}^{m}D_{m}^{*\max}\left(f_{\alpha},\widetilde{o}\right)N^{m}}{m!}\right)e^{-n^{(1-\beta)}}\right\},$$

We have proved that

$$\left|G_{n}\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq(5.250312578)^{N}\cdot \tag{93}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{D\left(f_{\alpha}(x),\widetilde{o}\right)}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}\left(b_{i}-a_{i}\right)^{\alpha_{i}}\right)\cdot(3.1992)\,e^{-n^{(1-\beta)}}\right]\right)\right.$$

$$+\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+$$

$$\left.\left(\frac{(6.3984)\,\|b-a\|_{\infty}^{m}D_{m}^{*\max}\left(f_{\alpha},\widetilde{o}\right)N^{m}}{m!}\right)e^{-n^{(1-\beta)}}\right\},$$

respectively in $\pm$.

By (61) we get

$$D\left(G_{n}^{\mathcal{F}}(f,x),f(x)\right)=$$

$$\sup_{r\in[0,1]}\max\left\{\left|G_{n}\left(f_{-}^{(r)},x\right)-f_{-}^{(r)}(x)\right|,\left|G_{n}\left(f_{+}^{(r)},x\right)-f_{+}^{(r)}(x)\right|\right\}. \tag{94}$$

Combining (94) and (93) we have established claim. ■

Using (47), (62), (13), (8), (3) and acting similarly as in the proof of Theorem 36 we derive the last result of this article, coming next.

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Theorem 37 *All assumptions and terminology as in Theorem 36 and Notation 35. Then*

1)

$$D\left(F_n^{\mathcal{F}}(f,x), f(x)\right) \leq (4.1488766)^N \cdot \tag{95}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{D\left(f_\alpha(x), \widetilde{o}\right)}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\cdot e^4 e^{-2n^{(1-\beta)}}\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha, \frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha, \widetilde{o}\right)N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

2)

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) \leq (4.1488766)^N \cdot \tag{96}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{D^*\left(f_\alpha, \widetilde{o}\right)}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\cdot e^4 e^{-2n^{(1-\beta)}}\right]\right)+\right.$$

$$\left.\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha, \frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha, \widetilde{o}\right)N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

3) assume that $f_\alpha(x_0)=\widetilde{o}$, for all $\alpha : |\alpha| = 1, ..., m$; $x_0 \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$. Then

$$D\left(F_n^{\mathcal{F}}(f,x_0), f(x_0)\right) \leq (4.1488766)^N \cdot \tag{97}$$

$$\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{(\mathcal{F})\max}\left(f_\alpha, \frac{1}{n^\beta}\right)+\left(\frac{2e^4\|b-a\|_\infty^m D_m^{*\max}\left(f_\alpha, \widetilde{o}\right)N^m}{m!}\right)e^{-2n^{(1-\beta)}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Conclusion 38 *In Theorems 36, 37 we studied quantitatively with rates the high speed approximation of*

$$D\left(G_n^{\mathcal{F}}(f,x), f(x)\right) \to 0, \tag{98}$$

$$D\left(F_n^{\mathcal{F}}(f,x), f(x)\right) \to 0,$$

as $n \to \infty$.

Also we proved with rates that

$$D^*\left(G_n^{\mathcal{F}}(f), f\right) \to 0, \tag{99}$$

$$D^*\left(F_n^{\mathcal{F}}(f), f\right) \to 0,$$

as $n \to \infty$, involving smoothness of f.

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Multivariate Fuzzy-Random normalized Neural Network Approximation Operators

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Abstract

In this article we study the rate of multivariate pointwise convergence in the q-mean to the Fuzzy-Random unit operator or its perturbation of very precise multivariate normalized Fuzzy-Random neural network operators of Cardaliaguet-Euvrard and "Squashing" types. These multivariate Fuzzy-Ranfom operators arise in a natural and common way among multivariate Fuzzy-Random neural network. These rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function or its Fuzzy partial derivatives. Also several interesting results in multivariate Fuzzy-Random Analysis are given of independent merit, which are used then in the proof of the main results of the paper.

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1 Introduction

Let $(X, \mathcal{B}, P)$ be a probability space. Consider the set of all fuzzy-random variables $\mathcal{L}_F(X, \mathcal{B}, P)$. Let $f : \mathbb{R}^d \to \mathcal{L}_F(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a multivariate fuzzy-random function or fuzzy-stochastic process. Here for $\overrightarrow{t} \in \mathbb{R}^d$, $s \in X$ we denote $\left(f\left(\overrightarrow{t}\right)\right)(s) = f\left(\overrightarrow{t}, s\right)$ and actually we have $f : \mathbb{R}^d \times X \to \mathbb{R}_{\mathcal{F}}$, where $\mathbb{R}_{\mathcal{F}}$ is the set of fuzzy real numbers. Let $1 \leq q < +\infty$. Here we consider only multivariate fuzzy-random functions f which are (q-mean) uniformly continuous over $\mathbb{R}^d$. For each $n \in \mathbb{N}$, the multivariate fuzzy-random neural network we deal with has the following structure:

It is a three-layer feed forward network with one hidden layer. It has one input unit and one output unit. The hidden layer has $\left(2n^2+1\right)$ processing units. To each pair of connecting units (input to each processing unit) we assign the same weight $n^{1-\alpha}$, $0 < \alpha < 1$. The threshold values $\frac{\overrightarrow{k}}{n^{\alpha}}$ are one for each processing unit $\overrightarrow{k}$, $\overrightarrow{k} \in \mathbb{Z}^d$. The activation function b (or S) is the same for each processing unit. The Fuzzy-Random weights associated with the output unit are $f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{1}{\sum_{\overrightarrow{k}=-\overrightarrow{n}^2}^{\overrightarrow{n}^2} b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}$, one for each processing unit $\overrightarrow{k}$, $\odot$ denotes the scalar Fuzzy multiplication.

The above precisely described multivariate Fuzzy-Random neural networks induce some completely described multivariate Fuzzy-Random neural network operators of normalized Cardaliaguet-Euvrard and "Squashing "types.

We study here throughly the multivariate Fuzzy-Random pointwise convergence (in q-mean) of these operators to the unit operator and its perturbation. See Theorem 35, 38 and Comment 41. This is done with rates through multivariate Probabilistic-Jackson type inequalities involving Fuzzy-Random moduli of continuity of the engaged Fuzzy-Random function and its Fuzzy pertial derivatives.

On the way to establish these main results we produce some new independent and interersting results for multivariate Fuzzy-Random Analysis. The real ordinary theory of the above mentioned operators was presented earlier in [1], [2] and [9]. And the fuzzy case was treated in [3]. The fuzzy random case was studied first in [5]. Of course this article is strongly motivated from there and is a continuation.

The monumental revolutionizing work of L. Zadeh [13] is the foundation of this work, as well as another strong motivation. Fuzzyness in Computer Science and Engineering seems one of the main trends today. Also Fuzzyness has penetrated many areas of Mathematics and Statistics. These are other strong reasons for this work.

Our approach is quantitative and recent on the topic, started in [1], [2] and continued in [3], [7], [8]. It determines precisely the rates of convergence through natural very tight inequalities using the measurement of smoothness of the engaged multivariate Fuzzy-Random functions.

2 Background

We begin with

Definition 1 *(see [12]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [12]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u, v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [12], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u, v), \quad \forall\, u, v, w \in \mathbb{R}_{\mathcal{F}}, \qquad (1) \\ D(k \odot u, k \odot v) &= |k|\, D(u, v), \quad \forall\, u, v \in \mathbb{R}_{\mathcal{F}}, \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u, w) + D(v, e), \quad \forall\, u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f, g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_{-}^{(r)} \leq v_{-}^{(r)}$ and $u_{+}^{(r)} \leq v_{+}^{(r)}$, $\forall\, r \in [0,1]$.

We need

Lemma 2 *([4]) For any $a, b \in \mathbb{R} : a \cdot b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leq |a - b| \cdot D(u, \widetilde{o}), \tag{2}$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{o} := \chi_{\{0\}}$.

Lemma 3 *([4])*

(i) If we denote $\widetilde{o} := \chi_{\{0\}}$, then $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \widetilde{o} = \widetilde{o} \oplus u = u$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$.

(ii) With respect to $\widetilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \widetilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.

(iii) Let $a, b \in \mathbb{R} : a \cdot b \geq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a + b) \odot u = a \odot u \oplus b \odot u$. For general $a, b \in \mathbb{R}$, the above property is false.

(iv) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) For any $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) If we denote $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$, then $\|\cdot\|_{\mathcal{F}}$ has the properties of a usual norm on $\mathbb{R}_{\mathcal{F}}$, i.e.,

$$\begin{gathered} \|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o}, \quad \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}}, \\ \|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \quad \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v). \end{gathered} \tag{3}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is not a linear space over $\mathbb{R}$; and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space.

As in Remark 4.4 ([4]) one can show easily that a sequence of operators of the form

$$L_n(f)(x) := \sum_{k=0}^{n*} f(x_{k_n}) \odot w_{n,k}(x), \quad n \in \mathbb{N}, \tag{4}$$

($\sum^{*}$ denotes the fuzzy summation) where $f : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$, $x_{k_n} \in \mathbb{R}^d$, $d \in \mathbb{N}$, $w_{n,k}(x)$ real valued weights, are linear over $\mathbb{R}^d$, i.e.,

$$L_n(\lambda \odot f \oplus \mu \odot g)(x) = \lambda \odot L_n(f)(x) \oplus \mu \odot L_n(g)(x), \tag{5}$$

$\forall \lambda, \mu \in \mathbb{R}$, any $x \in \mathbb{R}^d$; $f, g : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$. (Proof based on Lemma 3 (iv).)

We need

Definition 4 *(see [12]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then we call z the H-difference of x and y, denoted by $z := x - y$.*

Definition 5 *(see [12]) Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$*, with* $\beta > 0$*. A function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *is differentiable at* $x \in T$ *if there exists a* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$*. We call* f' *the derivative of* f *or* H*-derivative of* f *at* x*. If* f *is differentiable at any* $x \in T$*, we call* f *differentiable or* H*-differentiable and it has derivative over* T *the function* f'*.*

We need also a particular case of the Fuzzy Henstock integral ($\delta(x) = \frac{\delta}{2}$) introduced in [12], Definition 2.1 there.

That is,

Definition 6 *(see [11], p. 644) Let* $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$*. We say that* f *is Fuzzy-Riemann integrable to* $I \in \mathbb{R}_{\mathcal{F}}$ *if for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any division* $P = \{[u, v]; \xi\}$ *of* $[a, b]$ *with the norms* $\Delta(P) < \delta$*, we have*

$$D\left(\sum_{P}^{*} (v - u) \odot f(\xi), I\right) < \varepsilon. \tag{6}$$

We choose to write

$$I := (FR) \int_a^b f(x)\, dx. \tag{7}$$

We also call an f *as above* (FR)*-integrable.*

We mention the following fundamental theorem of Fuzzy Calculus:

Corollary 7 *([3]) If* $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ *has a fuzzy continuous derivative* f' *on* $[a, b]$*, then* $f'(x)$ *is* (FR)*-integrable over* $[a, b]$ *and*

$$f(s) = f(t) \oplus (FR) \int_t^s f'(x)\, dx, \quad \text{for any } s \geq t,\ s, t \in [a, b]. \tag{8}$$

Note. In Corollary 7 when $s < t$ the formula is invalid! Since fuzzy real numbers correspond to closed intervals etc.

We need also

Lemma 8 *([3]) If* $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *are fuzzy continuous (with respect to metric* D*), then the function* $F : [a, b] \to \mathbb{R}_+ \cup \{0\}$ *defined by* $F(x) := D(f(x), g(x))$ *is continuous on* $[a, b]$*, and*

$$D\left((FR) \int_a^b f(u)\, du, (FR) \int_a^b g(u)\, du\right) \leq \int_a^b D(f(x), g(x))\, dx. \tag{9}$$

Lemma 9 *([3]) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ fuzzy continuous (with respect to metric D), then $D(f(x),\widetilde{o})\leq M$, $\forall\, x\in[a,b]$, $M>0$, that is f is fuzzy bounded.*

We mention

Lemma 10 *([5]) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ have an existing fuzzy derivative f' at $c\in[a,b]$. Then f is fuzzy continuous at c.*

Note. Higher order fuzzy derivatives and all fuzzy partial derivatives are defined the obvious and analogous way to the real derivatives, all based on Definitions 4, 5, here.

We need the fuzzy multivariate Taylor formula.

Theorem 11 *([7], p.54) Let U be an open convex subset on $\mathbb{R}^n$, $n\in\mathbb{N}$ and $f:U\to\mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m\in\mathbb{N}$ exist and are fuzzy continuous. Let $z:=(z_1,...,z_n)$, $x_0:=(x_{01},...,x_{0n})\in U$ such that $z_i\geq x_{0i}$, $i=1,...,n$. Let $0\leq t\leq 1$, we define $x_i:=x_{0i}+t(z_i-z_{0i})$, $i=1,2,...,n$ and $g_z(t):=f(x_0+t(z-x_0))$. (Clearly $x_0+t(z-x_0)\in U$). Then for $N=1,...,m$ we obtain*

$$g_z^{(N)}(t)=\left[\left(\sum_{i=1}^{n*}(z_i-x_{0i})\odot\frac{\partial}{\partial x_i}\right)^N f\right](x_1,x_2,...,x_n). \tag{10}$$

Furthermore it holds the following fuzzy multivariate Taylor formula

$$f(z)=f(x_0)\oplus\sum_{N=1}^{m-1*}\frac{g_z^{(N)}(0)}{N!}\oplus\mathcal{R}_m(0,1), \tag{11}$$

where

$$\mathcal{R}_m(0,1):=\frac{1}{(m-1)!}\odot(FR)\int_0^1(1-s)^{m-1}\odot g_z^{(m)}(s)\,ds. \tag{12}$$

Comment 12 *(explaining formula (10)) When $N=n=2$ we have ($z_i\geq x_{0i}$, $i=1,2$)*

$$g_z(t)=f(x_{01}+t(z_1-x_{01}),x_{02}+t(z_2-x_{02})),\quad 0\leq t\leq 1.$$

We apply Theorems 2.18, 2.19, 2.21 of [7] repeatedly, etc. Thus we find

$$g_z'(t)=(z_1-x_{01})\odot\frac{\partial f}{\partial x_1}(x_1,x_2)\oplus(z_2-x_{02})\odot\frac{\partial f}{\partial x_2}(x_1,x_2).$$

Furthermore it holds

$$g_z''(t)=(z_1-x_{01})^2\odot\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2)\oplus 2(z_1-x_{01})(z_2-x_{02})\odot\frac{\partial^2 f(x_1,x_2)}{\partial x_1\partial x_2} \tag{13}$$

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$$\oplus\left(z_{2}-x_{02}\right)^{2}\odot\frac{\partial^{2}f}{\partial x_{2}^{2}}\left(x_{1},x_{2}\right).$$

When $n=2$ and $N=3$ we obtain

$$g_{z}^{\prime\prime\prime}\left(t\right)=\left(z_{1}-x_{01}\right)^{3}\odot\frac{\partial^{3}f}{\partial x_{1}^{3}}\left(x_{1},x_{2}\right)\oplus3\left(z_{1}-x_{01}\right)^{2}\left(z_{2}-x_{02}\right)\odot\frac{\partial^{3}f\left(x_{1},x_{2}\right)}{\partial x_{1}^{2}\partial x_{2}}$$

$$\oplus3\left(z_{1}-x_{01}\right)\left(z_{2}-x_{02}\right)^{2}\odot\frac{\partial^{3}f\left(x_{1},x_{2}\right)}{\partial x_{1}\partial x_{2}^{2}}\oplus\left(z_{2}-x_{02}\right)^{3}\odot\frac{\partial^{3}f}{\partial x_{2}^{3}}\left(x_{1},x_{2}\right). \quad (14)$$

When $n=3$ and $N=2$ we get ($z_i\geq x_{0i},\ i=1,2,3$)

$$g_{z}^{\prime\prime}\left(t\right)=\left(z_{1}-x_{01}\right)^{2}\odot\frac{\partial^{2}f}{\partial x_{1}^{2}}\left(x_{1},x_{2},x_{3}\right)\oplus\left(z_{2}-x_{02}\right)^{2}\odot\frac{\partial^{2}f}{\partial x_{2}^{2}}\left(x_{1},x_{2},x_{3}\right)$$

$$\oplus\left(z_{3}-x_{03}\right)^{2}\odot\frac{\partial^{2}f}{\partial x_{3}^{2}}\left(x_{1},x_{2},x_{3}\right)\oplus2\left(z_{1}-x_{01}\right)\left(z_{2}-x_{02}\right) \quad (15)$$

$$\odot\frac{\partial^{2}f\left(x_{1},x_{2},x_{3}\right)}{\partial x_{1}\partial x_{2}}\oplus2\left(z_{2}-x_{02}\right)\left(z_{3}-x_{03}\right)\odot\frac{\partial^{2}f\left(x_{1},x_{2},x_{3}\right)}{\partial x_{2}\partial x_{3}}$$

$$\oplus2\left(z_{3}-x_{03}\right)\left(z_{1}-x_{01}\right)\odot\frac{\partial^{2}f\left(x_{1},x_{2},x_{3}\right)}{\partial x_{3}\partial x_{1}},$$

etc.

3 Basic properties

We need

Definition 13 *(see also [11], Definition 13.16, p. 654) Let $(X,\mathcal{B},P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g:X\rightarrow\mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U\subseteq\mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1}\left(U\right)=\left\{s\in X;g\left(s\right)\in U\right\}\in\mathcal{B}). \quad (16)$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}\left(X,\mathcal{B},P\right)$. Let $g_n,g\in\mathcal{L}_{\mathcal{F}}\left(X,\mathcal{B},P\right)$, $n\in\mathbb{N}$ and $0<q<+\infty$. We say $g_{n}\left(s\right)\overset{\text{"}q\text{-}mean\text{"}}{\underset{n\rightarrow+\infty}{\rightarrow}}g\left(s\right)$ if

$$\lim_{n\rightarrow+\infty}\int_{X}D\left(g_{n}\left(s\right),g\left(s\right)\right)^{q}P\left(ds\right)=0. \quad (17)$$

Remark 14 *(see [11], p. 654) If $f,g\in\mathcal{L}_{\mathcal{F}}\left(X,\mathcal{B},P\right)$, let us denote $F:X\rightarrow\mathbb{R}_{+}\cup\{0\}$ by $F\left(s\right)=D\left(f\left(s\right),g\left(s\right)\right)$, $s\in X$. Here, F is $\mathcal{B}$-measurable, because $F=G\circ H$, where $G\left(u,v\right)=D\left(u,v\right)$ is continuous on $\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}$, and $H:X\rightarrow\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}$, $H\left(s\right)=\left(f\left(s\right),g\left(s\right)\right)$, $s\in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.*

Definition 15 *(see [11], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.*

Remark 16 *(see [11], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\ t \in T$, $s \in X$.*

Remark 17 *(see [11], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.*

Remark 18 *(see [11], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,*

$$\begin{aligned} (f \oplus g)(t, s) &= f(t, s) \oplus g(t, s), \\ (k \odot f)(t, s) &= k \odot f(t, s), \quad t \in T, s \in X. \end{aligned}$$

Definition 19 *(see also Definition 13.18, pp. 655-656, [11]) For a fuzzy-random function $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$

$$\sup \left\{ \left(\int_X D^q (f(x, s), f(y, s)) P(ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d, \ \|x - y\|_{l_1} \leq \delta \right\},$$

$0 < \delta$, $1 \leq q < \infty$.

Definition 20 *Here $1 \leq q < +\infty$. Let $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over $\mathbb{R}^d$, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:whenever $\|x - y\|_{l_1} \leq \delta$, $x, y \in \mathbb{R}^d$, implies that*

$$\int_X (D(f(x, s), f(y, s)))^q P(ds) \leq \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Proposition 21 *Let $f \in C_{FR}^{U_q}(\mathbb{R}^d)$. Then $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$, any $\delta > 0$.*

Proof. Let $\varepsilon_0 > 0$ be arbitrary but fixed. Then there exists $\delta_0 > 0$: $\|x - y\|_{l_1} \leq \delta_0$ implies

$$\int_X (D(f(x, s), f(y, s)))^q P(ds) \leq \varepsilon_0 < \infty.$$

That is $\Omega_1^{(\mathcal{F})}(f,\delta_0)_{L^q} \leq \varepsilon_0^{\frac{1}{q}} < \infty$. Let now $\delta > 0$ arbitrary, $x, y \in \mathbb{R}^d$ such that $\|x-y\|_{l_1} \leq \delta$. Choose $n \in \mathbb{N} : n\delta_0 \geq \delta$ and set $x_i := x + \frac{i}{n}(y-x)$, $0 \leq i \leq n$. Then

$$D(f(x,s), f(y,s)) \leq D(f(x,s), f(x_1,s)) + D(f(x_1,s), f(x_2,s))$$
$$+ ... + D(f(x_{n-1},s), f(y,s)).$$

Consequently

$$\left(\int_X (D(f(x,s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \leq \left(\int_X (D(f(x,s), f(x_1,s)))^q P(ds)\right)^{\frac{1}{q}}$$

$$+ ... + \left(\int_X (D(f(x_{n-1},s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \leq n\Omega_1^{(\mathcal{F})}(f,\delta_0)_{L^q} \leq n\varepsilon_0^{\frac{1}{q}} < \infty,$$

since $\|x_i - x_{i+1}\|_{l_1} = \frac{1}{n}\|x-y\|_{l_1} \leq \frac{1}{n}\delta \leq \delta_0$, $0 \leq i \leq n$.

Therefore $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \leq n\varepsilon_0^{\frac{1}{q}} < \infty$. ■

Proposition 22 *Let $f, g : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f,0)_{L^q} = 0$, iff $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

(iii) $\Omega_1^{(\mathcal{F})}(f,\delta_1+\delta_2)_{L^q} \leq \Omega_1^{(\mathcal{F})}(f,\delta_1)_{L^q} + \Omega_1^{(\mathcal{F})}(f,\delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1^{(\mathcal{F})}(f,n\delta)_{L^q} \leq n\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1^{(\mathcal{F})}(f,\lambda\delta)_{L^q} \leq \lceil\lambda\rceil \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \leq (\lambda+1)\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, where $\lceil\cdot\rceil$ is the ceilling of the number.

(vi) $\Omega_1^{(\mathcal{F})}(f \oplus g,\delta)_{L^q} \leq \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} + \Omega_1^{(\mathcal{F})}(g,\delta)_{L^q}$, $\delta > 0$. Here $f \oplus g$ is a fuzzy random function.

(vii) $\Omega_1^{(\mathcal{F})}(f,\cdot)_{L^q}$ is continuous on $\mathbb{R}_+$, for $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Proof. (i) is obvious.

(ii) $\Omega_1(f,0)_{L^q} = 0$.

($\Rightarrow$) Let $\lim_{\delta \downarrow 0} \Omega_1(f,\delta)_{L^q} = 0$. Then $\forall\, \varepsilon > 0$, $\varepsilon^{\frac{1}{q}} > 0$ and $\exists\, \delta > 0$, $\Omega_1(f,\delta)_{L^q} \leq \varepsilon^{\frac{1}{q}}$. I.e. for any $x, y \in \mathbb{R}^d : \|x-y\|_{l_1} \leq \delta$ we get

$$\int_X D^q(f(x,s), f(y,s)) P(ds) \leq \varepsilon.$$

That is $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

($\Leftarrow$) Let $f \in C_{FR}^{U_q}(\mathbb{R}^d)$. Then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: whenever $\|x-y\|_{l_1} \leq \delta$, $x, y \in \mathbb{R}^d$, it implies

$$\int_X D^q(f(x,s), f(y,s)) P(ds) \leq \varepsilon.$$

I.e. $\forall\ \varepsilon > 0\ \exists\ \delta > 0 : \Omega_1 (f, \delta)_{L^q} \leq \varepsilon^{\frac{1}{q}}$. That is $\Omega_1 (f, \delta)_{L^q} \to 0$ as $\delta \downarrow 0$.

(iii) Let $x_1, x_2 \in \mathbb{R}^d : \|x_1 - x_2\|_{l_1} \leq \delta_1 + \delta_2$. Set $x = \frac{\delta_2}{\delta_1+\delta_2} x_1 + \frac{\delta_1}{\delta_1+\delta_2} x_2$, so that $x \in \overline{x_1 x_2}$. Hence $\|x - x_1\|_{l_1} \leq \delta_1$ and $\|x_2 - x\|_{l_1} \leq \delta_2$. We have

$$\left(\int_X D^q \left(f (x_1, s), f (x_2, s) \right) P (ds) \right)^{\frac{1}{q}} \leq$$

$$\left(\int_X D^q \left(f (x_1, s), f (x, s) \right) P (ds) \right)^{\frac{1}{q}} + \left(\int_X D^q \left(f (x, s), f (x_2, s) \right) P (ds) \right)^{\frac{1}{q}} \leq$$

$$\Omega_1 \left(f, \|x_1 - x\|_{l_1} \right)_{L^q} + \Omega_1 \left(f, \|x_2 - x\|_{l_1} \right)_{L^q} \leq$$

$$\Omega_1 (f, \delta_1)_{L^q} + \Omega_2 (f, \delta_2)_{L^q} .$$

Therefore (iii) is true.

(iv) and (v) are obvious.

(vi) Notice that

$$\left(\int_X D^q \left((f \oplus g) (x, s), (f \oplus g) (y, s) \right) P (ds) \right)^{\frac{1}{q}} \leq$$

$$\left(\int_X D^q \left(f (x, s), f (y, s) \right) P (ds) \right)^{\frac{1}{q}} + \left(\int_X D^q \left(g (x, s), g (y, s) \right) P (ds) \right)^{\frac{1}{q}} .$$

That is (vi) is now clear.

(vii) By (iii) we get

$$\left| \Omega_1^{(\mathcal{F})} (f, \delta_1 + \delta_2)_{L^q} - \Omega_1^{(\mathcal{F})} (f, \delta_1)_{L^q} \right| \leq \Omega_1^{(\mathcal{F})} (f, \delta_2)_{L^q} .$$

Let now $f \in C_{FR}^{U_q} \left(\mathbb{R}^d \right)$, then by (ii) $\lim_{\delta_2 \downarrow 0} \Omega_1^{(\mathcal{F})} (f, \delta_2)_{L^q} = 0$. That is proving the continuity of $\Omega_1^{(\mathcal{F})} (f, \cdot)_{L^q}$ on $\mathbb{R}_+$. ■

We give

Definition 23 *([6]) Let $f (t, s)$ be a stochastic process from $\mathbb{R}^d \times (X, \mathcal{B}, P)$ into $\mathbb{R}$, $d \in \mathbb{N}$, where $(X, \mathcal{B}, P)$ is a probability space. We define the q-mean multivariate first moduli of continuity of f by*

$$\Omega_1 (f, \delta)_{L^q} :=$$

$$\sup \left\{ \left(\int_X |f (x, s) - f (y, s)|^q P (ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d,\ \ \|x - y\|_{l_1} \leq \delta \right\}, \quad (18)$$

$\delta > 0$, $1 \leq q < \infty$.

For more see [6].

We also give

Proposition 24 *Assume that $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ is finite, $\delta>0$, $1\leq q<\infty$. Then*

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \geq \sup_{r\in[0,1]} \max\left\{\Omega_1\left(f_-^{(r)},\delta\right)_{L^q}, \Omega_1\left(f_+^{(r)},\delta\right)_{L^q}\right\}. \tag{19}$$

The reverse direction "$\leq$" "is not possible.

Proof. We observe that

$$D(f(x,s), f(y,s)) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|f_-^{(r)}(x,s)-f_-^{(r)}(y,s)\right|, \left|f_+^{(r)}(x,s)-f_+^{(r)}(y,s)\right|\right\}$$

$$\geq \left|f_\pm^{(r)}(x,s)-f_\pm^{(r)}(y,s)\right|,$$

respectively in $+,-$.

Hence

$$\left(\int_X D^q(f(x,s), f(y,s))\,P(ds)\right)^{\frac{1}{q}} \geq \left(\int_X \left|f_\pm^{(r)}(x,s)-f_\pm^{(r)}(y,s)\right|^q P(ds)\right)^{\frac{1}{q}},$$

respectively in $+,-$.

Therefore it holds

$$\sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}} \left(\int_X D^q(f(x,s), f(y,s))\,P(ds)\right)^{\frac{1}{q}} \geq$$

$$\sup_{r\in[0,1]} \max_{\{+,-\}} \left\{ \sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}} \left(\int_X \left|f_\pm^{(r)}(x,s)-f_\pm^{(r)}(y,s)\right|^q P(ds)\right)^{\frac{1}{q}} \right\},$$

proving the claim. ■

Remark 25 *For each $s\in X$ we define the usual first modulus of continuity of $f(\cdot,s)$ by*

$$\omega_1^{(\mathcal{F})}(f(\cdot,s),\delta) := \sup_{\substack{x,y\in\mathbb{R}^d \\ \|x-y\|_{l_1}\leq\delta}} D(f(x,s), f(y,s)), \quad \delta>0. \tag{20}$$

Therefore

$$D^q(f(x,s), f(y,s)) \leq \left(\omega_1^{(\mathcal{F})}(f(\cdot,s),\delta)\right)^q,$$

$\forall\ s\in X$ and $x,y\in\mathbb{R}^d : \|x-y\|_{l_1}\leq\delta$, $\delta>0$.

Hence it holds

$$\left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}} \leq \left(\int_X \left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}},$$

$\forall\ x,y \in \mathbb{R}^d : \|x-y\|_{l_1} \leq \delta.$

We have that

$$\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q} \leq \left(\int_X \left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}, \tag{21}$$

under the assumption that the right hand side of (21) is finite.

The reverse "$\geq$" of the last (21) is not true.

Also we have

Proposition 26 *([5]) (i) Let $Y(t,\omega)$ be a real valued stochastic process such that Y is continuous in $t \in [a,b]$. Then Y is jointly measurable in (t,ω).*

(ii) Further assume that the expectation $(E|Y|)(t) \in C([a,b])$, or more generally $\int_a^b (E|Y|)(t)\,dt$ makes sense and is finite. Then

$$E\left(\int_a^b Y\left(t,\omega\right)dt\right) = \int_a^b \left(EY\right)\left(t\right)dt. \tag{22}$$

According to [10], p. 94 we have the following

Definition 27 *Let $(Y,\mathcal{T})$ be a topological space, with its σ-algebra of Borel sets $\mathcal{B} := \mathcal{B}(Y,\mathcal{T})$ generated by $\mathcal{T}$. If $(X,\mathcal{S})$ is a measurable space, a function $f : X \to Y$ is called measurable iff $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.*

By Theorem 4.1.6 of [10], p. 89 f as above is measurable iff

$$f^{-1}\left(C\right) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We would need

Theorem 28 *(see [10], p. 95) Let $(X,\mathcal{S})$ be a measurable space and (Y,d) be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \to f(x)$ in Y. Then f is measurable. I.e., $\lim\limits_{n\to\infty} f_n = f$ is measurable.*

We need also

Proposition 29 *Let f,g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then*

(i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.

(ii) $f \oplus g$ is a fuzzy random variable.

Finally we need

Proposition 30 *([5]) Let $f : [a,b] \to L_{\mathcal{F}}(X,\mathcal{B},P)$, $a,b \in \mathbb{R}$, be a fuzzy-random function. We assume that $f(t,s)$ is fuzzy continuous in $t \in [a,b]$, $s \in X$. Then $(FR)\int_a^b f(t,s)\,dt$ exists and is a fuzzy-random variable.*

4 Main Results

We need the following (see [9]) definitions.

Definition 31 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L_1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.*

Definition 32 *(see [9]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,*

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 33 *(from [9]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function, e.g. b could be the characteristic function or the hat function on $[-1, 1]$.*

Assumption 34 *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B}^* := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Here we consider functions $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.*

In this article we study among others in q-mean $(1 \leq q < \infty)$ the pointwise convergence with rates over $\mathbb{R}^d$, to the fuzzy-random unit operator or a perturbation of it, of the following fuzzy-random multivariate neural network operators, $(0 < \alpha < 1$, $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $s \in X$, $(X, \mathcal{B}, P)$ a probability space, $n \in \mathbb{N})$

$$(M_n(f))(\overrightarrow{x}, s) =$$

$$\frac{\sum_{k_1=-n^2}^{n^2 *} \cdots \sum_{k_d=-n^2}^{n^2 *} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}, s\right) \odot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{23}$$

In short, we can write

$$(M_n(f))(\overrightarrow{x}, s) = \frac{\sum_{\overrightarrow{k}=-\overrightarrow{n^2}}^{\overrightarrow{n^2} *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=-\overrightarrow{n^2}}^{\overrightarrow{n^2}} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{24}$$

In this article we assume that

$$n \geq \max_{i \in \{1,...,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\alpha}} \right\}, \tag{25}$$

see also [2], p. 91.

So, by (25) we can rewrite ($[\cdot]$ is the integral part of a number, while $\lceil \cdot \rceil$ is the ceiling of a number)

$$(M_n(f))(\overrightarrow{x}, s) =$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]*} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]*} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}, s\right) \odot b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{26}$$

In short we can write

$$(M_n(f))(\overrightarrow{x}, s) = \frac{\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{27}$$

Denoting

$$V(\overrightarrow{x}) := \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{28}$$

we will write and use from now on that

$$(M_n(f))(\overrightarrow{x}, s) = \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\alpha]*} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}. \tag{29}$$

The above M_n are linear operators over $\mathbb{R}^d \times X$.

Related works were done in [7] and [8].

We present

Theorem 35 *Here all as above. Then*

$$\left(\int_X D^q\left(\left(M_n(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) P(ds)\right)^{\frac{1}{q}} \leq \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \quad (30)$$

As $n \to \infty$, we get that

$$(M_n f)(\overrightarrow{x},s) \overset{\text{"}q\text{-}mean\text{"}}{\rightarrow} f(\overrightarrow{x},s)$$

with rates.

Proof. We observe that

$$D\left(\left(M_n(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, f(\overrightarrow{x},s)\odot 1\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, f(\overrightarrow{x},s)\odot\frac{V(\overrightarrow{x})}{V(\overrightarrow{x})}\right) =$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})},\right.$$

$$\left.\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*} f(\overrightarrow{x},s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right) \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} D\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right).$$

That is it holds

$$D\left(\left(M_n(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) \leq \quad (31)$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} D\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right).$$

Hence

$$\left(\int_X D^q\left(\left(M_n(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lfloor n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rfloor} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lfloor n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rfloor} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\Omega_1^{(\mathcal{F})}\left(f, \left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_{l_1}\right)_{L^q} \leq \tag{32}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lfloor n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rfloor} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q} =$$

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \tag{33}$$

Condition (25) implies that

$$\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\alpha}}, \quad \text{all } i = 1, ..., d. \tag{34}$$

The proof of (30) is now finished. ■

Remark 36 *Consider the fuzzy-random perturbed unit operator*

$$\left(T_n(f)\right)(\overrightarrow{x},s) := f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right),$$

$\forall\ (\overrightarrow{x},s) \in \mathbb{R}^d \times X$; $\overrightarrow{T} := (T_1, ..., T_d)$, $n \in \mathbb{N}$, $1 \leq q < \infty$.

We observe that

$$\int_X D^q\left(\left(T_n(f)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} =$$

$$\left(\int_X D^q\left(f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} \leq \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^q}. \tag{35}$$

Given that $f \in C_{FR}^{U_q}\left(\mathbb{R}^d\right)$, *we get that* $\left(T_n(f)\right)(\overrightarrow{x}, s) \overset{\text{"q-mean"}}{\underset{n\to\infty}{\rightarrow}} f(\overrightarrow{x}, s)$, $\forall$ $(\overrightarrow{x}, s) \in \mathbb{R}^d \times X$.

Next we estimate in high order with rates the 1*-mean difference*

$$\int_X D\left(\left(M_n(f)\right)(\overrightarrow{x}, s), \left(T_n(f)\right)(\overrightarrow{x}, s)\right) P(ds), \quad n \in \mathbb{N}.$$

We make

Assumption 37 *Let* $\overrightarrow{x} \in \mathbb{R}^d$, $d \in \mathbb{N}$, $s \in X$; *where* $(X, \mathcal{B}, P)$ *is a probability space,* n *as in (25),* b *of compact support* $\mathcal{B}^*$, $0 < \alpha < 1$, M_n *as in (29).*

Let $f : \mathbb{R}^d \to L_{\mathcal{F}}(X, \mathcal{B}, P)$ *be a fuzzy continuous in* $\overrightarrow{x} \in \mathbb{R}^d$ *random function.*

We assume that all H*-fuzzy partial derivatives of* f *up to order* $N \in \mathbb{N}$ *exist and are fuzzy continuous in* $\overrightarrow{x} \in \mathbb{R}^d$ *and all belong to* $C_{FR}^{U_1}\left(\mathbb{R}^d\right)$.

Furthermore we assume that

$$\int_X D\left(f_\alpha(\overrightarrow{x}, s), \widetilde{o}\right) P(ds) < \infty, \tag{36}$$

for all $\alpha : |\alpha| = j$, $j = 1, ..., N$.

Call

$$\Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max}, \delta\right)_{L^1} := \max_{|\alpha|=N} \Omega_1^{(\mathcal{F})}\left(f_\alpha, \delta\right)_{L^1}, \quad \delta > 0. \tag{37}$$

We give

Theorem 38 *All here as in Assumption 37. Then*

$$\int_X D\left(\left(M_n(f)\right)(\overrightarrow{x}, s), \left(T_n(f)\right)(\overrightarrow{x}, s)\right) P(ds) \leq$$

$$\sum_{j=1}^{N}\left[\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\left.\left\{\int_X D\left(f_\alpha(\overrightarrow{x}, s), \widetilde{o}\right) P(ds) + \Omega_1^{(\mathcal{F})}\left(f_\alpha, \frac{\sum_{i=1}^{d} T_i}{n^{1-\alpha}}\right)_{L^1}\right\}\right]$$

$$+\left[\sum_{|\alpha|=N}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right]\Omega_{1,N}^{(\mathcal{F})}\left(f_\alpha^{\max}, \frac{2\left(\sum_{i=1}^{d} T_i\right)}{n^{1-\alpha}}\right)_{L^1}. \tag{38}$$

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By (38), as $n \to \infty$, we get

$$\int_X D\left(\left(M_n(f)\right)(\overrightarrow{x}, s), \left(T_n(f)\right)(\overrightarrow{x}, s)\right) P(ds) \to 0,$$

with rates in high order.

Proof. By (25) we get that

$$-\frac{T_i}{n^{1-\alpha}} \leq x_i - \frac{k_i}{n} \leq \frac{T_i}{n^{1-\alpha}}, \quad \text{all } i = 1, ..., d, \; x_i \in \mathbb{R}. \tag{39}$$

We consider the case

$$x_i - \frac{T_i}{n^{1-\alpha}} \leq \frac{k_i}{n}, \quad i = 1, ..., d. \tag{40}$$

Set

$$g_{\frac{\overrightarrow{k}}{n}}(t, s) := f\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}} + t\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} + \frac{\overrightarrow{T}}{n^{1-\alpha}}\right), s\right), \tag{41}$$

$0 \leq t \leq 1$, $\forall \; s \in X$.

We apply Theorem 11, and we have by the fuzzy multivariate Taylor formula that

$$f\left(\frac{\overrightarrow{k}}{n}, s\right) = f\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right) \oplus \sum_{j=1}^{N-1*} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0, s)}{j!} \oplus \mathcal{R}_N(0, 1, s), \tag{42}$$

where

$$\mathcal{R}_N(0, 1, s) := \frac{1}{(N-1)!} \odot (FR) \int_0^1 (1-\theta)^{N-1} \odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta, s) \, d\theta, \tag{43}$$

$\forall \; s \in X$.

Here for $j = 1, ..., N$, we obtain

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(\theta, s) = \left[\left(\sum_{i=1}^{d*}\left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right) \odot \frac{\partial}{\partial x_i}\right)^j f\right](x_1, x_2, ..., x_d, s), \tag{44}$$

$0 \leq \theta \leq 1$, $\forall \; s \in X$.

More precisely we have for $j = 1, ..., N$, that

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0, s) = \sum^{*}_{\substack{\alpha := (\alpha_1, ..., \alpha_d), \; \alpha_i \in \mathbb{Z}^+ \\ i = 1, ..., d, \; |\alpha| := \sum_{i=1}^{d} \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^{d} \alpha_i!}\right) \left(\prod_{i=1}^{d}\left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\odot f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right), \tag{45}$$

$\forall\ s\in X$, and

$$g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta,s)=\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i\in\mathbb{Z}^+\\ i=1,\ldots,d,\ |\alpha|:=\sum_{i=1}^{d}\alpha_i=N}}^{*}\left(\frac{N!}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{k_i}{n}-x_i+\frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$
$$\odot f_{\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}+\frac{\overrightarrow{T}}{n^{1-\alpha}}\right),s\right), \tag{46}$$

$0\leq\theta\leq 1$, $\forall\ s\in X$.

Multiplying (42) by $\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}$ and applying $\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil *}$ to both sides, we obtain

$$\left(M_n(f)\right)\left(\overrightarrow{x},s\right)=f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil *}\sum_{j=1}^{N-1 *}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!} \tag{47}$$
$$\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil *}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)},$$

$\forall\ s\in X$.

Next we observe

$$D\left(\left(M_n(f)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right)\right)\overset{((47),(1))}{=}$$
$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil *}\sum_{j=1}^{N-1 *}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\oplus\right.$$
$$\left.\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil *}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)},\widetilde{o}\right)\overset{(1)}{=} \tag{48}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\sum_{j=1}^{N-1*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s)}{j!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right.$$

$$\oplus\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\mathcal{R}_N(0,1,s)\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})},$$

$$\left.\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]*}\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\odot\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right)\overset{(2.1)}{\leq}$$

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})j!}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s),\widetilde{o}\right)+ \tag{49}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}D\left(\mathcal{R}_N(0,1,s),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\right).$$

So we obtain

$$D\left((M_n(f))(\overrightarrow{x},s),(T_n(f))(\overrightarrow{x},s)\right)\leq$$

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})j!}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0,s),\widetilde{o}\right)+ \tag{50}$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}D\left(\mathcal{R}_N(0,1,s),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\right),$$

$\forall\ s\in X$.

Notice that

$$\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}=\frac{1}{(N-1)!}\odot(FR)\int_0^1(1-\theta)^{N-1}\odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)\,d\theta. \tag{51}$$

We estimate

$$D\left(\mathcal{R}_N(0,1,s),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\right)\overset{(51)}{=}$$

$$D\left(\frac{1}{(N-1)!}\odot(FR)\int_0^1(1-\theta)^{N-1}\odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta,s)\,d\theta,\right.$$

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$$\frac{1}{(N-1)!} \odot (FR) \int_0^1 (1-\theta)^{N-1} \odot g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)\, d\theta \Big) \overset{(9)}{\leq} \tag{52}$$

$$\frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} D\left(g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta,s), g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)\right) d\theta \overset{(46)}{\leq}$$

$$\frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} \left[\sum_{|\alpha|=N} \left(\frac{N!}{\prod_{i=1}^d \alpha_i!}\right) \left(\prod_{i=1}^d \left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\left. D\left(f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}} + \theta\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} + \frac{\overrightarrow{T}}{n^{1-\alpha}}\right), s\right), f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right)\right)\right] d\theta \leq$$

$$\frac{1}{(N-1)!} \left[\sum_{|\alpha|=N} \left(\frac{N!}{\prod_{i=1}^d \alpha_i!}\right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right) \int_0^1 (1-\theta)^{N-1}\right.$$

$$\left. D\left(f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}} + \theta\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} + \frac{\overrightarrow{T}}{n^{1-\alpha}}\right), s\right), f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right)\right) d\theta\right]. \tag{53}$$

Therefore it holds

$$\int_X D\left(\mathcal{R}_N(0,1,s), \frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0,s)}{N!}\right) P(ds) \overset{\text{(by (53) and Tonelli's ([10]) theorem)}}{\leq}$$

$$\frac{1}{(N-1)!} \left[\sum_{|\alpha|=N} \left(\frac{N!}{\prod_{i=1}^d \alpha_i!}\right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right) \int_0^1 (1-\theta)^{N-1}\right. \tag{54}$$

$$\left(\int_X D\left(f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}} + \theta\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} + \frac{\overrightarrow{T}}{n^{1-\alpha}}\right), s\right),\right.\right.$$

$$\left.\left.\left. f_\alpha\left(\overrightarrow{x} - \frac{\overrightarrow{T}}{n^{1-\alpha}}, s\right)\right) P(ds)\right) d\theta\right] \leq$$

$$\frac{1}{(N-1)!} \left[\sum_{|\alpha|=N} \left(\frac{N!}{\prod_{i=1}^d \alpha_i!}\right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\int_0^1 (1-\theta)^{N-1} \Omega_1^{(\mathcal{F})} \left(f_\alpha, \theta \left(\sum_{i=1}^d \left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}} \right) \right) \right)_{L^1} d\theta \Bigg] \leq$$

$$\sum_{|\alpha|=N} \left(\frac{1}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \Omega_1^{(\mathcal{F})} \left(f_\alpha, \frac{2\left(\sum_{i=1}^d T_i \right)}{n^{1-\alpha}} \right)_{L^1} \leq \tag{55}$$

$$\left[\sum_{|\alpha|=N} \left(\frac{1}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \right] \Omega_{1,N}^{(\mathcal{F})} \left(f_\alpha^{\max}, \frac{2\left(\sum_{i=1}^d T_i \right)}{n^{1-\alpha}} \right)_{L^1} .$$

We have proved that

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^\alpha \rceil}^{[n\vec{x}+\vec{T}n^\alpha]} \frac{b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right)}{V(\vec{x})} \left(\int_X D\left(\mathcal{R}_N(0,1,s), \frac{g_{\frac{\vec{k}}{n}}^{(N)}(0,s)}{N!} \right) P(ds) \right)$$

$$\leq \left[\sum_{|\alpha|=N} \left(\frac{1}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{2T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \right] \Omega_{1,N}^{(\mathcal{F})} \left(f_\alpha^{\max}, \frac{2\left(\sum_{i=1}^d T_i \right)}{n^{1-\alpha}} \right)_{L^1} . \tag{56}$$

Next we extimate

$$\frac{1}{j!} D\left(g_{\frac{\vec{k}}{n}}^{(j)}(0,s), \widetilde{o} \right) =$$

$$\frac{1}{j!} D\left(\sum_{|\alpha|=j}^{*} \left(\frac{j!}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) \odot f_\alpha \left(\vec{x} - \frac{\vec{T}}{n^{1-\alpha}}, s \right), \widetilde{o} \right) \leq \tag{57}$$

$$\sum_{|\alpha|=j} \left(\frac{1}{\prod_{i=1}^d \alpha_i!} \right) \left(\prod_{i=1}^d \left(\frac{k_i}{n} - x_i + \frac{T_i}{n^{1-\alpha}} \right)^{\alpha_i} \right) D\left(f_\alpha \left(\vec{x} - \frac{\vec{T}}{n^{1-\alpha}}, s \right), \widetilde{o} \right) \leq$$

$$\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\left[D\left(f_\alpha\left(\overrightarrow{x},s\right),\widetilde{o}\right)+D\left(f_\alpha\left(\overrightarrow{x}-\frac{\overrightarrow{T}}{n^{1-\alpha}},s\right),f_\alpha\left(\overrightarrow{x},s\right)\right)\right]. \tag{58}$$

Consequently we have

$$\frac{1}{j!}\int_X D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right)P\left(ds\right)\leq\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)$$

$$\left\{\int_X D\left(f_\alpha\left(\overrightarrow{x},s\right),\widetilde{o}\right)P\left(ds\right)+\Omega_1^{(\mathcal{F})}\left(f_\alpha,\frac{\sum_{i=1}^{d}T_i}{n^{1-\alpha}}\right)_{L^1}\right\}. \tag{59}$$

Furthermore it holds

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)j!}\int_X D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right)P\left(ds\right)\leq$$

$$\sum_{j=1}^{N}\left[\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_i}{n^{1-\alpha}}\right)^{\alpha_i}\right)\right.$$

$$\left.\left\{\int_X D\left(f_\alpha\left(\overrightarrow{x},s\right),\widetilde{o}\right)P\left(ds\right)+\Omega_1^{(\mathcal{F})}\left(f_\alpha,\frac{\sum_{i=1}^{d}T_i}{n^{1-\alpha}}\right)_{L^1}\right\}\right]. \tag{60}$$

By (50) we get

$$\int_X D\left(\left(M_n\left(f\right)\right)\left(\overrightarrow{x},s\right),\left(T_n\left(f\right)\right)\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\leq$$

$$\sum_{j=1}^{N}\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)j!}\int_{X}D\left(g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0,s\right),\widetilde{o}\right)P\left(ds\right)+ \tag{61}$$

$$\sum_{\overrightarrow{k}=\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\right]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\left(\int_{X}D\left(\mathcal{R}_{N}\left(0,1,s\right),\frac{g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0,s\right)}{N!}\right)P\left(ds\right)\right)$$

$$\overset{((56),\ (60))}{\leq}\sum_{j=1}^{N}\left[\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_{i}!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_{i}}{n^{1-\alpha}}\right)^{\alpha_{i}}\right)\right.$$

$$\left.\left\{\int_{X}D\left(f_{\alpha}\left(\overrightarrow{x},s\right),\widetilde{o}\right)P\left(ds\right)+\Omega_{1}^{(\mathcal{F})}\left(f_{\alpha},\frac{\sum_{i=1}^{d}T_{i}}{n^{1-\alpha}}\right)_{L^{1}}\right\}\right]+ \tag{62}$$

$$\left[\sum_{|\alpha|=j}\left(\frac{1}{\prod_{i=1}^{d}\alpha_{i}!}\right)\left(\prod_{i=1}^{d}\left(\frac{2T_{i}}{n^{1-\alpha}}\right)^{\alpha_{i}}\right)\right]\Omega_{1,N}^{(\mathcal{F})}\left(f_{\alpha}^{\max},\frac{2\left(\sum_{i=1}^{d}T_{i}\right)}{n^{1-\alpha}}\right)_{L^{1}},$$

proving (38). ■

Remark 39 *Inequality (38) reveals that the operators M_n behave in good approximation like the simple operators T_n. So T_n is a good simplification of M_n.*

We give the following definition

Definition 40 *Let the nonnegative function $S:\mathbb{R}^d\to\mathbb{R}$, $d\geq 1$, S has compact support $\mathcal{B}^{*}:=\prod_{i=1}^{d}[-T_i,T_i]$, $T_i>0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d}(-\infty,T_i]$ or $\mathcal{B}^{*}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [9]).*

Comment 41 *If the operators M_n, see (23), replace b by S, we derive the normalized "squashing "operators K_n. Then Theorems 35, 38 remain valid for K_n, just replace M_n by K_n there.*

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Multivariate Fuzzy-Random Quasi-Interpolation Neural Network Approximation Operators

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Abstract

In this paper we study the rate of multivariate pointwise and uniform convergence in the q-mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation neural network operators. These multivariate Fuzzy-Random operators arise in a natural and common way among multivariate Fuzzy-Random neural networks. These rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function. The plain stochastic extreme analog of this theory is also presented.

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Key words and phrases: Fuzzy-Random analysis, Fuzzy-Random neural networks and operators, Fuzzy-Random modulus of continuity, Fuzzy-Random functions, Jackson type and probabilistic inequalities.

1 Fuzzy-Random functions theory Background

We begin with

Definition 1 *(see [22]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [22]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [22], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall\, u,v,w \in \mathbb{R}_{\mathcal{F}}, \qquad (1) \\ D(k \odot u, k \odot v) &= |k|\, D(u,v), \quad \forall\, u,v \in \mathbb{R}_{\mathcal{F}}, \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall\, u,v,w,e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let (M,d) metric space and $f, g : M \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in M} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0,1]$.

$\sum^{*}$ denotes the fuzzy summation, $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [23], [24].

We need

Definition 2 *(see also [18], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{2}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \overset{\text{"q-mean"}}{\underset{n\to+\infty}{\to}} g(s)$ if

$$\lim_{n\to+\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{3}$$

Remark 3 *(see [18], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.*

Definition 4 *(see [18], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.*

Remark 5 *(see [18], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\, t \in T$, $s \in X$.*

Remark 6 *(see [18], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.*

Remark 7 *(see [18], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,*

$$(f \oplus g)(t, s) = f(t, s) \oplus g(t, s),$$
$$(k \odot f)(t, s) = k \odot f(t, s), \quad t \in T, s \in X.$$

Definition 8 *(see also Definition 13.18, pp. 655-656, [18]) For a fuzzy-random function $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$

$$\sup\left\{\left(\int_X D^q(f(x, s), f(y, s)) P(ds)\right)^{\frac{1}{q}} : x, y \in W, \ \|x - y\|_\infty \le \delta\right\},$$

$0 < \delta$, $1 \le q < \infty$.

3

Definition 9 *Here $1 \leq q < +\infty$. Let $f : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over W, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:whenever $\|x - y\|_{\infty} \leq \delta$, $x, y \in W$, implies that*

$$\int_X (D(f(x,s), f(y,s)))^q P(ds) \leq \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}(W)$.

Proposition 10 *Let $f \in C_{FR}^{U_q}(W)$, where $W \subseteq \mathbb{R}^N$ is convex.*
Then $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$, any $\delta > 0$.

Proof. Let $\varepsilon_0 > 0$ be arbitrary but fixed. Then there exists $\delta_0 > 0$: $\|x - y\|_{\infty} \leq \delta_0$ implies

$$\int_X (D(f(x,s), f(y,s)))^q P(ds) \leq \varepsilon_0 < \infty.$$

That is $\Omega_1^{(\mathcal{F})}(f, \delta_0)_{L^q} \leq \varepsilon_0^{\frac{1}{q}} < \infty$. Let now $\delta > 0$ arbitrary, $x, y \in W$ such that $\|x - y\|_{\infty} \leq \delta$. Choose $n \in \mathbb{N} : n\delta_0 \geq \delta$ and set $x_i := x + \frac{i}{n}(y - x)$, $0 \leq i \leq n$. Then

$$D(f(x,s), f(y,s)) \leq D(f(x,s), f(x_1,s)) + D(f(x_1,s), f(x_2,s))$$
$$+ ... + D(f(x_{n-1},s), f(y,s)).$$

Consequently

$$\left(\int_X (D(f(x,s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \leq \left(\int_X (D(f(x,s), f(x_1,s)))^q P(ds)\right)^{\frac{1}{q}}$$

$$+ ... + \left(\int_X (D(f(x_{n-1},s), f(y,s)))^q P(ds)\right)^{\frac{1}{q}} \leq n\Omega_1^{(\mathcal{F})}(f, \delta_0)_{L^q} \leq n\varepsilon_0^{\frac{1}{q}} < \infty,$$

since $\|x_i - x_{i+1}\|_{\infty} = \frac{1}{n}\|x - y\|_{\infty} \leq \frac{1}{n}\delta \leq \delta_0$, $0 \leq i \leq n$.

Therefore $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} \leq n\varepsilon_0^{\frac{1}{q}} < \infty$. ■

Proposition 11 *Let $f, g : W \subseteq \mathbb{R}^N \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, be fuzzy random functions. It holds*
(i) $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.
(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f, 0)_{L^q} = 0$, iff $f \in C_{FR}^{U_q}(W)$.

Proof. (i) is obvious.
(ii) $\Omega_1(f, 0)_{L^q} = 0$.

($\Rightarrow$) Let $\lim_{\delta \downarrow 0} \Omega_1 (f, \delta)_{L^q} = 0$. Then $\forall\, \varepsilon > 0$, $\varepsilon^{\frac{1}{q}} > 0$ and $\exists\, \delta > 0$, $\Omega_1 (f, \delta)_{L^q} \leq \varepsilon^{\frac{1}{q}}$. I.e. for any $x, y \in W : \|x - y\|_\infty \leq \delta$ we get

$$\int_X D^q (f(x, s), f(y, s)) P(ds) \leq \varepsilon.$$

That is $f \in C_{FR}^{U_q} (W)$.

($\Leftarrow$) Let $f \in C_{FR}^{U_q} (W)$. Then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: whenever $\|x - y\|_\infty \leq \delta$, $x, y \in W$, it implies

$$\int_X D^q (f(x, s), f(y, s)) P(ds) \leq \varepsilon.$$

I.e. $\forall\, \varepsilon > 0\ \exists\, \delta > 0 : \Omega_1 (f, \delta)_{L^q} \leq \varepsilon^{\frac{1}{q}}$. That is $\Omega_1 (f, \delta)_{L^q} \to 0$ as $\delta \downarrow 0$. ■

We give

Definition 12 *(see also [6]) Let $f(t, s)$ be a random function (stochastic process) from $W \times (X, \mathcal{B}, P)$ into $\mathbb{R}$, $W \subseteq \mathbb{R}^N$, where $(X, \mathcal{B}, P)$ is a probability space. We define the q-mean multivariate first modulus of continuity of f by*

$$\Omega_1 (f, \delta)_{L^q} :=$$

$$\sup \left\{ \left(\int_X |f(x, s) - f(y, s)|^q P(ds) \right)^{\frac{1}{q}} : x, y \in W,\ \ \|x - y\|_\infty \leq \delta \right\}, \qquad (4)$$

$\delta > 0$, $1 \leq q < \infty$.

The concept of f being (q-mean) uniformly continuous random real function is defined the same way as in Definition 9, just replace D by $|\cdot|$, etc. We denote it as $f \in C_{\mathbb{R}}^{U_q} (W)$.

Similar properties as in Propositions 10, 11 are valid for $\Omega_1 (f, \delta)_{L^q}$.

Also we have

Proposition 13 *([3]) Let $Y(t, \omega)$ be a real valued stochastic process such that Y is continuous in $t \in [a, b]$. Then Y is jointly measurable in (t, ω).*

According to [17], p. 94 we have the following

Definition 14 *Let $(Y, \mathcal{T})$ be a topological space, with its σ-algebra of Borel sets $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ generated by $\mathcal{T}$. If $(X, \mathcal{S})$ is a measurable space, a function $f : X \to Y$ is called measurable iff $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.*

By Theorem 4.1.6 of [17], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We mention

Theorem 15 *(see [17], p. 95) Let* $(X, \mathcal{S})$ *be a measurable space and* (Y, d) *be a metric space. Let* f_n *be measurable functions from* X *into* Y *such that for all* $x \in X$, $f_n(x) \to f(x)$ *in* Y*. Then* f *is measurable. I.e.,* $\lim_{n\to\infty} f_n = f$ *is measurable.*

We need also

Proposition 16 *Let* f, g *be fuzzy random variables from* $\mathcal{S}$ *into* $\mathbb{R}_{\mathcal{F}}$*. Then*
(i) Let $c \in \mathbb{R}$*, then* $c \odot f$ *is a fuzzy random variable.*
(ii) $f \oplus g$ *is a fuzzy random variable.*

2 Basics on Neural Network Operators

I) Here all come from [16], [8].

We consider the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R},\ i = 1, ..., N;\ x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks.

As in [16], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R},\ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0, \ \forall\, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i)$, $x_i \geq 0$, for $i = 1, ..., N$.

We see that ([13])

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{(1+e^{x_i-1})(1+e^{-x_i-1})}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Notice that $\Phi_i(x_i) = \Phi_i(0) = 0,231$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [13] we get

viii)

$$\sum_{\begin{cases} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{cases}}^{\infty} \Phi_i(nx_i - k_i) \leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

Denote by $\lceil \cdot \rceil$ the ceiling of a number, and by $\lfloor \cdot \rfloor$ the integral part of a number. Consider here $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right) \subset \mathbb{R}^N$, $N \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$; $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

As in [13] we obtain

ix)

$$0 < \frac{1}{\sum_{k_i = \lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i)} < \frac{1}{\Phi_i(1)} = 5.250312578,$$

$\forall\, x_i \in [a_i, b_i]$, $i = 1, ..., N$.

x) As in [13], we see that

$$\lim_{n \to \infty} \sum_{k_i = \lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi_i(nx_i - k_i) \neq 1,$$

for at least some $x_i \in [a_i, b_i]$, $i = 1, ..., N$.

We will use here

$$\Phi(x_1, ..., x_N) := \Phi(x) := \prod_{i=1}^{N} \Phi_i(x_i), \quad x \in \mathbb{R}^N. \tag{5}$$

It has the properties:

(i)' $\Phi(x) > 0$, $\forall\, x \in \mathbb{R}^N$,

(ii)'

$$\sum_{k=-\infty}^{\infty} \Phi(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1 - k_1, ..., x_N - k_N) = 1, \tag{6}$$

$k := (k_1, ..., k_N)$, $\forall\, x \in \mathbb{R}^N$.

(iii)'

$$\sum_{k=-\infty}^{\infty} \Phi\left(nx-k\right) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \Phi\left(nx_1 - k_1, \dots, nx_N - k_N\right) = 1, \tag{7}$$

$\forall\ x \in \mathbb{R}^N;\ n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi\left(x\right) dx = 1,$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\left\{|x_1|, \dots, |x_N|\right\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, \dots, \infty)$, $-\infty := (-\infty, \dots, -\infty)$ upon the multivariate context, and

$$\lceil na \rceil := \left(\lceil na_1 \rceil, \dots, \lceil na_N \rceil\right),$$
$$\lfloor nb \rfloor := \left(\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor\right).$$

We also have

(v)'

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \Phi\left(nx-k\right) \leq 3.1992 e^{-n^{(1-\beta)}}, \tag{8}$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \left(\prod_{i=1}^N \left[a_i, b_i\right]\right)$.

(vi)'

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\left(nx-k\right)} < \left(5.250312578\right)^N, \tag{9}$$

$\forall\ x \in \left(\prod_{i=1}^N \left[a_i, b_i\right]\right),\ \ n \in \mathbb{N}$.

(vii)'

$$\sum_{\substack{k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \Phi\left(nx-k\right) \leq 3.1992 e^{-n^{(1-\beta)}},$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

(viii)'

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\neq 1$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

In general $\|.\|_\infty$ $s\tan ds\ for\ the\ \sup remum\ norm.$

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define ([11]) the multivariate positive linear neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$$G_n(f,x_1,...,x_N):=G_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}\tag{10}$$

$$:=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi_i(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\Phi_i(nx_i-k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We need, for $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ the first multivariate modulus of continuity

$$\omega_1(f,h):=\sup_{\substack{x,y\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)\\ \|x-y\|_\infty\leq h}}|f(x)-f(y)|,\quad h>0.\tag{11}$$

Similarly it is defined for $f\in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$). We have that $\lim_{h\to 0}\omega_1(f,h)=0$ when f is uniformly continuous.

When $f\in C_B\left(\mathbb{R}^N\right)$ we define ([11])

$$\overline{G}_n(f,x):=\overline{G}_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)\Phi(nx-k)\tag{12}$$

$$:=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi_i(nx_i-k_i)\right),$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, $N\geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [11]:

Theorem 17 *Let* $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n,N\in\mathbb{N}$. *Then* *i)*

$$\left|G_n(f,x)-f(x)\right| \leq (5.250312578)^N \cdot$$
$$\left\{\omega_1\left(f,\frac{1}{n^\beta}\right)+(6.3984)\|f\|_\infty e^{-n^{(1-\beta)}}\right\}=:\lambda_1, \tag{13}$$

ii)

$$\|G_n(f)-f\|_\infty \leq \lambda_1. \tag{14}$$

Theorem 18 *Let $f \in C_B(\mathbb{R}^N)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. Then*

i)

$$\left|\overline{G}_n(f,x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n^\beta}\right)+(6.3984)\|f\|_\infty e^{-n^{(1-\beta)}}=:\lambda_2, \tag{15}$$

ii)

$$\left\|\overline{G}_n(f)-f\right\|_\infty \leq \lambda_2. \tag{16}$$

II) Here we follow [8], [10].

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

It has the properties $\tanh 0 = 0$, $-1<\tanh x<1$, $\forall\ x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider

$$\Psi(x) := \frac{1}{4}\left(\tanh(x+1)-\tanh(x-1)\right)>0, \quad \forall\ x \in \mathbb{R}. \tag{26}$$

We easily see that $\Psi(-x)=\Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 19 *([9]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x\to-\infty}\Psi(x) = 0 = \lim_{x\to\infty}\Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0)=0.3809297$.

Theorem 20 *([9]) We have that $\sum_{i=-\infty}^{\infty}\Psi(x-i)=1$, $\forall\ x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i)=1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x+i)=1, \quad \forall x \in \mathbb{R}.$$

Theorem 21 *([9]) It holds* $\int_{-\infty}^{\infty} \Psi(x)\,dx = 1$.

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 22 *([9]) Let* $0<\alpha<1$ *and* $n \in \mathbb{N}$*. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

Theorem 23 *([9]) Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$*. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k)} < \frac{1}{\Psi(1)} = 4.1488766.$$

Also by [9] we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Psi(nx-k) \neq 1,$$

for at least some $x \in [a,b]$.

In this article we use (see [10])

$$\Theta(x_1,...,x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x=(x_1,...,x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \quad (17)$$

It has the properties:

(i) $\Theta(x) > 0,\ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1,...,x_N-k_N) = 1, \quad (18)$$

where $k := (k_1,...,k_N)$, $\forall\, x \in \mathbb{R}^N$.

(iii)

$$\sum_{k=-\infty}^{\infty}\Theta(nx-k):=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\dots\sum_{k_N=-\infty}^{\infty}\Theta(nx_1-k_1,\dots,nx_N-k_N)=1, \tag{19}$$

$\forall\ x\in\mathbb{R}^N;\ n\in\mathbb{N}.$

(iv)

$$\int_{\mathbb{R}^N}\Theta(x)\,dx=1,$$

that is Θ is a multivariate density function.

(v)

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\lfloor nb\rfloor}\Theta(nx-k)\leq e^4\cdot e^{-2n^{(1-\beta)}}, \tag{20}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$.

(vi)

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)}<\frac{1}{(\Psi(1))^N}=(4.1488766)^N, \tag{21}$$

$\forall\ x\in\left(\prod_{i=1}^N[a_i,b_i]\right),\ \ n\in\mathbb{N}.$

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\infty}\Theta(nx-k)\leq e^4\cdot e^{-2n^{(1-\beta)}}, \tag{22}$$

$0<\beta<1$, $n\in\mathbb{N}$, $x\in\mathbb{R}^N$.

Also we get that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Theta(nx-k)\neq1,$$

for at least some $x\in\left(\prod_{i=1}^N[a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^N[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,\dots,N$.

We introduce and define the multivariate positive linear neural network operator ([10]) $(x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i]\right))$

$$F_n(f, x_1, ..., x_N) := F_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Theta(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Theta(nx - k)} \tag{23}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Psi(nx_i - k_i)\right)}.$$

When $f \in C_B(\mathbb{R}^N)$ we define ([10]),

$$\overline{F}_n(f, x) := \overline{F}_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Theta(nx - k) := \tag{24}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Psi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \geq 1$, the multivariate quasi-interpolation neural network operator.

We mention from [10]:

Theorem 24 *Let* $f \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n, N \in \mathbb{N}$. *Then*

i)

$$|F_n(f, x) - f(x)| \leq (4.1488766)^N \cdot$$
$$\left\{\omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}}\right\} =: \lambda_1, \tag{25}$$

ii)

$$\|F_n(f) - f\|_\infty \leq \lambda_1. \tag{26}$$

Theorem 25 *Let* $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. *Then*

i)

$$\left|\overline{F}_n(f, x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2e^4 \|f\|_\infty e^{-2n^{(1-\beta)}} =: \lambda_2, \tag{27}$$

ii)

$$\left\|\overline{F}_n(f) - f\right\|_\infty \leq \lambda_2. \tag{28}$$

We are also motivated by [1], [2], [4], [5], [6], [7], [12], [14], [15]. For general knowledge on neural networks we recommend [19], [20], [21].

3 Main Results

I) q-mean Approximation by Fuzzy-Random Quasi-Interpolation Neural Network Operators

All terms and assumptions here as in Sections 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $1 \leq q < +\infty$, $n, N \in \mathbb{N}$, $0 < \beta < 1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define the following multivariate fuzzy random quasi-interpolation linear neural network operators

$$\left(G_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \tag{29}$$

(see also (10), and

$$\left(F_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{\overrightarrow{k}}{n}, s\right) \odot \frac{\theta(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \theta(nx-k)}, \tag{30}$$

(see also (23)).

We present

Theorem 26 *Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n, N \in \mathbb{N}$, $1 \leq q < +\infty$. Assume that $\int_X (D^*(f(\cdot,s), \widetilde{o}))^q P(ds) < \infty$. Then*

1)

$$\left(\int_X D^q\left(\left(G_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) P(ds)\right)^{\frac{1}{q}} \leq (5.250312578)^N \cdot \tag{31}$$

$$\left\{\Omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right)_{L^q} + (6.3984)\left(\int_X (D^*(f(\cdot,s), \widetilde{o}))^q P(ds)\right)^{\frac{1}{q}} e^{-n^{(1-\beta)}}\right\} =: \lambda_1^{(\mathcal{FR})},$$

2)

$$\left\|\left(\int_X D^q\left(\left(G_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x}, s), f(\overrightarrow{x}, s)\right) P(ds)\right)^{\frac{1}{q}}\right\|_{\infty, \left(\prod_{i=1}^{N}[a_i,b_i]\right)} \leq \lambda_1^{(\mathcal{FR})}. \tag{32}$$

(see also Theorem 17).

Proof. We notice that

$$D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)\leq D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),\widetilde{o}\right)+D\left(f\left(\overrightarrow{x},s\right),\widetilde{o}\right) \tag{33}$$

$$\leq 2D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right).$$

Hence

$$D^{q}\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)\leq 2^{q}D^{*q}\left(f\left(\cdot,s\right),\widetilde{o}\right), \tag{34}$$

and

$$\left(\int_{X}D^{q}\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq 2\left(\int_{X}\left(D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^{q}P\left(ds\right)\right)^{\frac{1}{q}}. \tag{35}$$

We observe that

$$D\left(\left(G_{n}^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)= \tag{36}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)},f\left(\overrightarrow{x},s\right)\odot 1\right)=$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)},f\left(\overrightarrow{x},s\right)\odot\frac{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}\right)= \tag{37}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)},\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\overrightarrow{x},s\right)\odot\frac{\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}\right)$$

$$\leq\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{\Phi\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right). \tag{38}$$

So that

$$D\left(\left(G_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) \leq$$

$$\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\frac{\Phi(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x},s)\right) = \tag{39}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\frac{\Phi(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x},s)\right) +$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\frac{\Phi(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) D\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x},s)\right).$$

Hence it holds

$$\left(\int_X D^q\left(\left(G_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right) P(ds)\right)^{\frac{1}{q}} \leq \tag{40}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\frac{\Phi(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) \left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x},s)\right) P(ds)\right)^{\frac{1}{q}} +$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na \rceil \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\frac{\Phi(nx-k)}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) \left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n}, s\right), f(\overrightarrow{x},s)\right) P(ds)\right)^{\frac{1}{q}} \leq$$

$$\left(\frac{1}{\sum_{\overrightarrow{k}=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} \right) \cdot \left\{ \Omega_1^{(\mathcal{F})}\left(f, \frac{1}{n^\beta}\right)_{L^q} + \right. \tag{41}$$

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$$2\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\left(\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\Bigg\}\leq$$

$$(5.250312578)^N\cdot$$

$$\left\{\Omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^\beta}\right)_{L^q}+2\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}(3.1992)\,e^{-n^{(1-\beta)}}\right\}. \tag{42}$$

We have proved claim. ■

Similarly we give

Theorem 27 *All assumptions as in Theorem 26. Then*

1)

$$\left(\int_X D^q\left(\left(F_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\leq(4.1488766)^N\cdot \tag{43}$$

$$\left\{\Omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^\beta}\right)_{L^q}+2e^4\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}e^{-2n^{(1-\beta)}}\right\}=:\lambda_2^{(\mathcal{FR})},$$

2)

$$\left\|\left(\int_X D^q\left(\left(F_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s),f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\right\|_{\infty,\left(\prod_{i=1}^N[a_i,b_i]\right)}\leq\lambda_2^{(\mathcal{FR})}. \tag{44}$$

(see also Theorem 24).

Proof. Similar to the proof of Theorem 26. ■

Conclusion 28 *By Theorems 26, 27, we obtain the pointwise and uniform convergences with rates in the q-mean and D-metric of the operators $G_n^{\mathcal{FR}}$, $F_n^{\mathcal{FR}}$ to the unit operator for $f\in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^N[a_i,b_i]\right)$.*

II) q-mean Approximation by Random Quasi-Interpolation Neural Network Operators

Let $g\in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0<\beta<1$, $\overrightarrow{x}\in\mathbb{R}^N$, $n,N\in\mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X}<\infty$, $(X,\mathcal{B},P)$ probability space, $s\in X$.

We define

$$\overline{G}_n^{(\mathcal{R})}(f,\overrightarrow{x}):=\sum_{\overrightarrow{k}=-\infty}^{\infty}f\left(\frac{\overrightarrow{k}}{n},s\right)\Phi(nx-k), \tag{45}$$

(see also (12), also we define

$$\overline{F}_n^{(\mathcal{R})}(f, \overrightarrow{x}) := \sum_{\overrightarrow{k}=-\infty}^{\infty} f\left(\frac{\overrightarrow{k}}{n}, s\right) \Theta(nx - k), \tag{46}$$

(see also (24)).

We give

Theorem 29 *Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$. Then*

1)

$$\int_X \left|\left(\overline{G}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s)\right| P(ds) \leq \tag{47}$$

$$\left\{\Omega_1\left(g, \frac{1}{n^\beta}\right)_{L^1} + (6.3984)\|g\|_{\infty,\mathbb{R}^N,X} e^{-n^{(1-\beta)}}\right\} =: \mu_1^{(\mathcal{R})},$$

2)

$$\left\|\int_X \left|\left(\overline{G}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s)\right| P(ds)\right\|_{\infty,\mathbb{R}^N} \leq \mu_1^{(\mathcal{R})}. \tag{48}$$

(see also Theorem 18).

Proof. Since $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, then

$$\left|g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s)\right| \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{49}$$

Hence

$$\int_X \left|g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s)\right| P(ds) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{50}$$

We observe that

$$\left(\overline{G}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) \Phi(nx - k) - g(\overrightarrow{x}, s) \sum_{\overrightarrow{k}=-\infty}^{\infty} \Phi(nx - k) = \tag{51}$$

$$\left(\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s)\right) \Phi(nx - k).$$

So that

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left|g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s)\right| \Phi(nx - k) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{52}$$

Hence

$$\left|\left(\overline{G}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x},s)-g(\overrightarrow{x},s)\right|\leq \tag{53}$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g(\overrightarrow{x},s)\right|\Phi(nx-k)=$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g(\overrightarrow{x},s)\right|\Phi(nx-k)+$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g(\overrightarrow{x},s)\right|\Phi(nx-k).$$

Furthermore it holds

$$\left(\int_X\left|\left(\overline{G}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x},s)-g(\overrightarrow{x},s)\right|P(ds)\right)\leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left(\int_X\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g(\overrightarrow{x},s)\right|P(ds)\right)\Phi(nx-k)+ \tag{55}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\left(\int_X\left|g\left(\frac{\overrightarrow{k}}{n},s\right)-g(\overrightarrow{x},s)\right|P(ds)\right)\Phi(nx-k)\leq$$

$$\Omega_1\left(g,\frac{1}{n^\beta}\right)_{L^1}+2\|g\|_{\infty,\mathbb{R}^N,X}\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\Phi(nx-k)\leq$$

$$\Omega_1\left(g,\frac{1}{n^\beta}\right)_{L^1}+\|g\|_{\infty,\mathbb{R}^N,X}(6.3984)e^{-n^{(1-\beta)}},$$

proving the claim. ■

We finish with

Theorem 30 *All as in Theorem 29. Then*

1)

$$\int_X\left|\left(\overline{F}_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x},s)-g(\overrightarrow{x},s)\right|P(ds)\leq \tag{57}$$

$$\left\{\Omega_1\left(g,\frac{1}{n^\beta}\right)_{L^1}+2e^4\|g\|_{\infty,\mathbb{R}^N,X}e^{-2n^{(1-\beta)}}\right\}=:\mu_2^{(\mathcal{R})},$$

2)
$$\left\| \int_X \left| \left(\overline{F}_n^{(\mathcal{R})} (g) \right) (\overrightarrow{x}, s) - g (\overrightarrow{x}, s) \right| P (ds) \right\|_{\infty, \mathbb{R}^N} \leq \mu_2^{(\mathcal{R})}. \tag{58}$$

(see also Theorem 25).

Proof. As similar to Theorem 29 is omitted. ■

Conclusion 31 *By Theorems 29, 30, we obtain pointwise and uniform convergences with rates in the q-mean of random operators* $\overline{G}_n^{(\mathcal{R})}$, $\overline{F}_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$.

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