

Generalized Fractional Ostrowski and Grüss type inequalities involving several Banach algebra valued functions

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Abstract

Using generalized Caputo fractional left and right vectorial Taylor formulae we establish mixed fractional Ostrowski and Grüss type inequalities involving several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

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1 Introduction

The following results motivate our work.

Theorem 1 (1938, Ostrowski [10]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a, b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \quad (1)$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

Theorem 2 (1882, Čebyšev [6]) *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \quad (2)$$

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([4], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [2], Chapters 24-26 and [3], Chapters 2-6.

In this article we generalize [4], Ch. 8,9 for several Banach algebra valued functions. Now our left and right Caputo fractional derivatives are for Banach space valued functions and our integrals are of Bochner type. Several applications finish this article. Inspiration came also from [7], [8].

2 Vectorial background fractional calculus

Here all come from [5].

We need

Definition 3 ([5], p. 106) *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the left generalized g -fractional derivative X -valued of f of order α as follows:

$$(D_{a+;g}^\alpha f)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) (f \circ g^{-1})^{(n)}(g(t)) dt, \quad (3)$$

$\forall x \in [a, b]$, where Γ is the gamma function. The last integral is of Bochner type ([1], pp. 422-428; [9]).

If $\alpha \notin \mathbb{N}$, by Theorem 4.10 ([5], p. 98), we have that $(D_{a+;g}^\alpha f) \in C([a, b], X)$.

We set

$$D_{a+;g}^n f(x) := \left((f \circ g^{-1})^n \circ g \right)(x) \in C([a, b], X), \quad n \in \mathbb{N}, \quad (4)$$

$$D_{a+;g}^0 f(x) = f(x), \quad \forall x \in [a, b].$$

When $g = id$, then

$$D_{a+;g}^\alpha f = D_{a+;id}^\alpha f = D_{*a}^\alpha f, \quad (5)$$

the usual left X -valued Caputo fractional derivative, see [5], Ch. 1.

We also need

Definition 4 ([5], p. 107) Let $\alpha > 0$, $[\alpha] = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.

We define the right generalized g -fractional derivative X -valued of f of order α as follows:

$$(D_{b-;g}^\alpha f)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) (f \circ g^{-1})^{(n)}(g(t)) dt, \quad (6)$$

$\forall x \in [a, b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by Theorem 4.11 ([5], p. 101), we have that $(D_{b-;g}^\alpha f) \in C([a, b], X)$.

We set

$$D_{b-;g}^n f(x) := (-1)^n \left((f \circ g^{-1})^n \circ g \right)(x) \in C([a, b], X), \quad n \in \mathbb{N}, \quad (7)$$

$$D_{b-;g}^0 f(x) := f(x), \quad \forall x \in [a, b].$$

When $g = id$, then

$$D_{b-;g}^\alpha f(x) = D_{b-;id}^\alpha f(x) = D_{b-}^\alpha f, \quad (8)$$

the usual right X -valued Caputo fractional derivative, see [5], Ch. 2.

We mention the following generalized fractional Taylor formulae with integral remainders over Banach spaces.

Theorem 5 ([5], p. 107) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then

$$\begin{aligned} f(x) &= f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} (f \circ g^{-1})^{(i)}(g(a)) + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (D_{a+;g}^\alpha f)(t) dt = \\ &\quad f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} (f \circ g^{-1})^{(i)}(g(a)) + \\ &\quad \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{a+;g}^\alpha f) \circ g^{-1})(z) dz. \end{aligned} \quad (9)$$

We also mention

Theorem 6 ([5], p. 108) *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then*

$$\begin{aligned} f(x) &= f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} (f \circ g^{-1})^{(i)}(g(b)) + \\ &\frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) (D_{b-;g}^\alpha f)(t) dt = \\ &f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} (f \circ g^{-1})^{(i)}(g(b)) + \\ &\frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} ((D_{b-;g}^\alpha f) \circ g^{-1})(z) dz. \end{aligned} \quad (10)$$

If $0 < \alpha \leq 1$, then the sums in (9), (10) disappear.

Also in (9), (10), we have that $\left\| \left(D_{a+;g}^\alpha f \right) \circ g^{-1} \right\|_{\infty, [g(a), g(b)]} < \infty$,

$$\left\| \left(D_{b-;g}^\alpha f \right) \circ g^{-1} \right\|_{\infty, [g(a), g(b)]} < \infty.$$

3 Banach Algebras background

All here come from [11].

We need

Definition 7 ([11], p. 245) *A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \quad (11)$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \quad (12)$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \quad (13)$$

for all x, y and z in A and for all scalars α .

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A, y \in A) \quad (14)$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \quad (15)$$

and

$$\|e\| = 1, \quad (16)$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 8 Commutativity of A will be explicited stated when needed.

There exists at most one $e \in A$ that satisfies (15).

Inequality (14) makes multiplication to be continuous, more precisely left and right continuous, see [11], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [11], p. 247-248, § 10.3.

We also make

Remark 9 Next we mention about integration of A -valued functions, see [11], p. 259, § 10.22:

If A is a Banach algebra and f is a continuous A -valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [11], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f d\mu = \int_Q x f(p) d\mu(p) \quad (17)$$

and

$$\left(\int_Q f d\mu \right) x = \int_Q f(p) x d\mu(p). \quad (18)$$

The Bochner integrals we will involve in our article follow (17) and (18). Also, let $f \in C([a, b], X)$, where $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4 Main Results

We start with mixed generalized fractional Ostrowski type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 10 Let $(A, \|\cdot\|)$ be a Banach algebra, $x_0 \in [a, b] \subset \mathbb{R}$, $\alpha > 0$, $n = \lceil \alpha \rceil$, $f_i \in C^n([a, b], A)$, $i = 1, \dots, r \in \mathbb{N} - \{1\}$; $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, with $(f_i \circ g^{-1})^{(k)}(g(x_0)) = 0$, $k = 1, \dots, n-1$; $i = 1, \dots, r$. Denote by

$$\theta(f_1, \dots, f_r)(x_0) := \sum_{i=1}^r \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0) \right]. \quad (19)$$

Then

$$\begin{aligned} \|\theta(f_1, \dots, f_r)(x_0)\| &\leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^r \left[\left\| (D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right\|_{\infty, [g(a), g(x_0)]} \right. \\ &\quad \left. (g(x_0) - g(a))^\alpha \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right] + \\ &\quad \left[\left\| (D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^\alpha \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right]. \end{aligned} \quad (20)$$

Proof. Since $(f_i \circ g^{-1})^{(k)}(g(x_0)) = 0$, $k = 1, \dots, n-1$; $i = 1, \dots, r$, we have by Theorem 5 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz, \quad (21)$$

$\forall x \in [x_0, b]$,

and by Theorem 6 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz, \quad (22)$$

$\forall x \in [a, x_0]$;

for all $i = 1, \dots, r$.

Left multiplying (21) and (22) with $\left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right)$ we get, respectively,

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x_0) = \quad (23)$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right)}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz,$$

$\forall x \in [x_0, b],$

and

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x_0) = \quad (24)$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right)}{\Gamma(\alpha)} \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz,$$

$\forall x \in [a, x_0];$

for all $i = 1, \dots, r$.

Adding (23) and (24) as separate groups, we obtain

$$\sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x) - \sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz, \quad (25)$$

$\forall x \in [x_0, b],$

and

$$\sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x) - \sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz, \quad (26)$$

$\forall x \in [a, x_0].$

Next we integrate (25) and (26) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^r \int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) f_i(x) dx - \sum_{i=1}^r \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x)\right) dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right], \quad (27)$$

and

$$\begin{aligned} & \sum_{i=1}^r \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx - \sum_{i=1}^r \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0) = \\ & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right]. \end{aligned} \quad (28)$$

Finally, adding (27) and (28) we obtain the useful identity

$$\begin{aligned} & \theta(f_1, \dots, f_r)(x_0) := \\ & \sum_{i=1}^r \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0) \right] = \\ & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right] \right. \\ & \left. + \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right] \right]. \end{aligned} \quad (29)$$

Therefore we get that

$$\begin{aligned} & \|\theta(f_1, \dots, f_r)(x_0)\| = \\ & \left\| \sum_{i=1}^r \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0) \right] \right\| \leq \frac{1}{\Gamma(\alpha)} \\ & \sum_{i=1}^r \left\| \left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right] \right\| \end{aligned} \quad (30)$$

$$\begin{aligned}
& + \left\| \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz \right) dx \right] \right\| \leq \\
& \frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\left[\int_a^{x_0} \left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) dz \right) \right\| dx \right] \right. \\
& + \left. \left[\int_{x_0}^b \left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) dz \right) \right\| dx \right] \right] \stackrel{(31)}{\leq} \\
& \frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \|((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z)\| dz \right) dx \right] \right. \\
& + \left. \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \|((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z)\| dz \right) dx \right] \right] =: (*). \\
& \hspace{15cm} (32)
\end{aligned}$$

Hence it holds

$$\|\theta(f_1, \dots, f_r)(x_0)\| \leq (*). \quad (33)$$

We have that

$$\begin{aligned}
& (*) \leq \frac{1}{\Gamma(\alpha+1)} \\
& \sum_{i=1}^r \left[\left[\left\| \|((D_{x_0-;g}^\alpha f_i) \circ g^{-1})\| \right\|_{\infty, [g(a), g(x_0)]} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x_0) - g(x))^\alpha dx \right] \right. \\
& + \left. \left[\left\| \|((D_{x_0+;g}^\alpha f_i) \circ g^{-1})\| \right\|_{\infty, [g(x_0), g(b)]} \int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x) - g(x_0))^\alpha dx \right] \right] \leq \\
& \hspace{15cm} (34)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^r \left[\left[\left\| \|((D_{x_0-;g}^\alpha f_i) \circ g^{-1})\| \right\|_{\infty, [g(a), g(x_0)]} \right. \right. \\
& \left. \left. (g(x_0) - g(a))^\alpha \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right] \right] + \\
& \hspace{15cm} (35)
\end{aligned}$$

$$\left[\left\| \left\| \left((D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right) \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^\alpha \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right\| \right],$$

proving (20). ■

Next comes an L_1 estimate.

Theorem 11 *All as in Theorem 10, with $\alpha \geq 1$. Then*

$$\begin{aligned} \|\theta(f_1, \dots, f_r)(x_0)\| &\leq \frac{1}{\Gamma(\alpha)} \\ \sum_{i=1}^r &\left[\left\| \left\| (D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x_0) - g(x))^{\alpha-1} dx \right\| \right. \\ &\quad (36) \\ &\quad \left. + \left\| \left\| (D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x) - g(x_0))^{\alpha-1} dx \right\| \right]. \end{aligned}$$

Proof. Next we use that $\alpha \geq 1$. We observe that

$$\begin{aligned} (*) &\stackrel{(33)}{\leq} \frac{1}{\Gamma(\alpha)} \\ \sum_{i=1}^r &\left[\left\| \left\| (D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right\|_{L_1([g(a), g(x_0)])} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x_0) - g(x))^{\alpha-1} dx \right\| \right. \\ &\quad (37) \\ &\quad \left. + \left\| \left\| (D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right\|_{L_1([g(x_0), g(b)])} \int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) (g(x) - g(x_0))^{\alpha-1} dx \right\| \right], \end{aligned}$$

proving Theorem 11. ■

An L_p estimate follows.

Theorem 12 *All as in Theorem 10. Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

$$\begin{aligned} \|\theta(f_1, \dots, f_r)(x_0)\| &\leq \frac{1}{(p(\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(\alpha)} \\ \sum_{i=1}^r &\left[\left\| \left\| (D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right\|_{q, [g(a), g(x_0)]} \left(\int_a^{x_0} (g(x_0) - g(x))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right\| \right. \\ &\quad (38) \end{aligned}$$

$$+ \left\| \left\| (D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right\| \right\|_{q,[g(x_0),g(b)]} \\ \left(\int_{x_0}^b (g(x) - g(x_0))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \Bigg] =: \Phi(x_0).$$

Proof. We have that

$$\begin{aligned} (*) &\stackrel{(33)}{\leq} \frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \right. \right. \\ &\quad \left. \left(\int_{g(x)}^{g(x_0)} \left\| ((D_{x_0-;g}^\alpha f_i) \circ g^{-1})(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] + \\ &\quad \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \right. \\ &\quad \left. \left(\int_{g(x_0)}^{g(x)} \left\| ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] \Bigg] \leq \quad (39) \\ &\frac{1}{\Gamma(\alpha)} \sum_{i=1}^r \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \frac{(g(x_0) - g(x))^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left\| ((D_{x_0-;g}^\alpha f_i) \circ g^{-1}) \right\| \right\|_{q,[g(a),g(x_0)]} dx \right] \right. \\ &\quad \left. + \left[\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) \frac{(g(x) - g(x_0))^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left\| ((D_{x_0+;g}^\alpha f_i) \circ g^{-1})(z) \right\| \right\|_{q,[g(x_0),g(b)]} dx \right] \right] \\ &= \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)} \\ &\sum_{i=1}^r \left[\left\| \left\| (D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right\| \right\|_{q,[g(a),g(x_0)]} \left(\int_a^{x_0} (g(x_0) - g(x))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right. \\ &\quad \left. + \left\| \left\| (D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right\| \right\|_{q,[g(x_0),g(b)]} \left(\int_{x_0}^b (g(x) - g(x_0))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right], \quad (40) \end{aligned}$$

proving Theorem 12. ■

We continue with generalized fractional Grüss-Cebysev type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 13 *Let $(A, \|\cdot\|)$ be a Banach algebra, $0 < \alpha \leq 1$, $f_i \in C^1([a, b], A)$, $i = 1, \dots, r \in \mathbb{N} - \{1\}$; $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$; $x_0 \in [a, b] \subset \mathbb{R}$ and $\theta(f_1, \dots, f_r)(x_0)$ as in (19). Assume that*

$$M(f_1, \dots, f_r) := \max_{i=1, \dots, r} \left\{ \sup_{x_0 \in [a, b]} \left\| (D_{x_0-}^\alpha f_i) \circ g^{-1} \right\|_{\infty, [g(a), g(x_0)]} \right. \quad (41)$$

$$\left. \sup_{x_0 \in [a, b]} \left\| (D_{x_0+}^\alpha f_i) \circ g^{-1} \right\|_{\infty, [g(x_0), g(b)]} \right\} < \infty.$$

Denote by

$$\Delta(f_1, \dots, f_r) := \int_a^b \theta(f_1, \dots, f_r)(x) dx =$$

$$\sum_{i=1}^r \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx \right) - \right.$$

$$\left. \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x) dx \right) \right]. \quad (42)$$

Then it holds

$$\|\Delta(f_1, \dots, f_r)\| =$$

$$\left\| \sum_{i=1}^r \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx \right) - \right. \right.$$

$$\left. \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x) dx \right) \right] \right\| \leq$$

$$\frac{M(f_1, \dots, f_r) (b-a)^2 (g(b) - g(a))^\alpha}{\Gamma(\alpha + 1)} \left(\sum_{i=1}^r \left\| \prod_{\substack{j=1 \\ j \neq i}}^r f_j \right\|_{\infty, [a, b]} \right). \quad (43)$$

Proof. We have that

$$\begin{aligned}
\|\Delta(f_1, \dots, f_r)\| &\leq \int_a^b \|\theta(f_1, \dots, f_r)(x_0)\| dx_0 \stackrel{(33)}{\leq} \\
&\frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^r \left[\left[\sup_{x_0 \in [a, b]} \left\| \left(D_{x_0-;g}^\alpha f_i \right) \circ g^{-1} \right\| \right]_{\infty, [g(a), g(x_0)]} \right. \\
&\left. \left\{ \int_a^b \left[(g(x_0) - g(a))^\alpha \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right] dx_0 \right\} \right] + \quad (44) \\
&\left[\sup_{x_0 \in [a, b]} \left\| \left(D_{x_0+;g}^\alpha f_i \right) \circ g^{-1} \right\| \right]_{\infty, [g(x_0), g(b)]} \\
&\left\{ \int_a^b \left[(g(b) - g(x_0))^\alpha \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right] dx_0 \right\} \right] \leq \\
&\frac{(g(b) - g(a))^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^r \left[\left[\sup_{x_0 \in [a, b]} \left\| \left(D_{x_0-;g}^\alpha f_i \right) \circ g^{-1} \right\| \right]_{\infty, [g(a), g(x_0)]} \right. \\
&\left. \left\{ \int_a^b \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) dx_0 \right\} \right] + \quad (45) \\
&\left[\sup_{x_0 \in [a, b]} \left\| \left(D_{x_0+;g}^\alpha f_i \right) \circ g^{-1} \right\| \right]_{\infty, [g(x_0), g(b)]} \\
&\left\{ \int_a^b \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) dx_0 \right\} \right] =: (**).
\end{aligned}$$

Hence it holds

$$\begin{aligned}
(**) &\leq \frac{M(f_1, \dots, f_r)(g(b) - g(a))^\alpha (b-a)^2}{2\Gamma(\alpha+1)} \\
&\sum_{i=1}^r \left[\left[\sup_{x_0 \in [a, b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right\| \right]_{\infty, [a, x_0]} \right] +
\end{aligned}$$

$$\left[\sup_{x_0 \in [a, b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right\| \right]_{\infty, [x_0, b]} \leq \frac{M(f_1, \dots, f_r) (b-a)^2 (g(b) - g(a))^\alpha}{\Gamma(\alpha+1)} \left(\sum_{i=1}^r \left\| \prod_{\substack{j=1 \\ j \neq i}}^r \|f_j\| \right\|_{\infty, [a, b]} \right), \quad (46)$$

proving (43). ■

Next comes an L_p estimate

Theorem 14 *Let $(A, \|\cdot\|)$ be a Banach algebra, $0 < \alpha \leq 1$, $f_i \in C^1([a, b], A)$, $i = 1, \dots, r \in \mathbb{N} - \{1\}$; $g \in C^1([a, b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, $x_0 \in [a, b] \subset \mathbb{R}$. Assume that*

$$N(f_1, \dots, f_r) := \max_{i=1, \dots, r} \left\{ \sup_{x_0 \in [a, b]} \left\| (D_{x_0-}^\alpha f_i) \circ g^{-1} \right\|_{L_q([g(a), g(x_0)])} \right. \\ \left. \sup_{x_0 \in [a, b]} \left\| (D_{x_0+}^\alpha f_i) \circ g^{-1} \right\|_{L_q([g(x_0), g(b)])} \right\} < \infty, \quad (47)$$

and set

$$Z(f_1, \dots, f_r) := \max_{i=1, \dots, r} \left\{ \left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j\| \right) \right\|_{q, [a, b]} \right\}. \quad (48)$$

Then

$$\left\| \sum_{i=1}^r \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) f_i(x) dx \right) - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x) dx \right) \right] \right\| \leq \\ \frac{2rpN(f_1, \dots, f_r) Z(f_1, \dots, f_r)}{(p+1)(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)} (b-a)^{\frac{1}{p}+1} (g(b) - g(a))^{\alpha - \frac{1}{q}}. \quad (49)$$

Proof. Here $\Phi(x_0)$ is as in (38).

Clearly then it holds

$$\Phi(x_0) \leq \frac{N(f_1, \dots, f_r)}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^r \left[\left(\int_a^{x_0} (g(x_0) - g(x))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) + \right. \quad (50)$$

$$\left. \left(\int_{x_0}^b (g(x) - g(x_0))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) \right].$$

Therefore we obtain

$$\|\Delta(f_1, \dots, f_r)\| \leq \int_a^b \|\theta(f_1, \dots, f_r)(x_0)\| dx_0 \leq$$

$$\int_a^b \Phi(x_0) dx_0 \leq \frac{N(f_1, \dots, f_r)}{(p(\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^r \left[\left[\int_a^b \left(\int_a^{x_0} (g(x_0) - g(x))^{\left(\frac{p(\alpha-1)+1}{p}\right)} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) dx_0 \right] \right.$$

$$\left. + \left[\int_a^b \left(\int_{x_0}^b (g(x) - g(x_0))^{\left(\frac{p(\alpha-1)+1}{p}\right)} \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j(x)\| \right) dx \right) dx_0 \right] \right] \leq$$

$$\frac{N(f_1, \dots, f_r)}{(p(\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(\alpha)} \sum_{i=1}^r \left[\left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^r \|f_j\| \right) \right\|_{q, [a, b]} \right]$$

$$\left[\int_a^b \left(\int_a^{x_0} (g(x_0) - g(x))^{(p(\alpha-1)+1)} dx \right)^{\frac{1}{p}} dx_0 \right] + \quad (51)$$

$$\left[\int_a^b \left(\int_{x_0}^b (g(x) - g(x_0))^{(p(\alpha-1)+1)} dx \right)^{\frac{1}{p}} dx_0 \right] =: (\eta).$$

Hence we get

$$(\eta) \leq \frac{rN(f_1, \dots, f_r) Z(f_1, \dots, f_r)}{(p(\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\left[\left[\int_a^b \left(\int_a^{x_0} (g(x_0) - g(x))^{(p(\alpha-1)+1)} dx \right)^{\frac{1}{p}} dx_0 \right] + \right.$$

$$\left. \left[\int_a^b \left(\int_{x_0}^b (g(x) - g(x_0))^{(p(\alpha-1)+1)} dx \right)^{\frac{1}{p}} dx_0 \right] \right] \leq \quad (52)$$

$$\frac{2rpN(f_1, \dots, f_r) Z(f_1, \dots, f_r)}{(p+1)(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)} (g(b) - g(a))^{\alpha - \frac{1}{q}} (b-a)^{\frac{1}{p}+1},$$

proving (49). ■

5 Applications

We make

Remark 15 Assume from now on that $(A, \|\cdot\|)$ is a commutative Banach algebra. Then, we get that

$$\theta(f_1, \dots, f_r)(x_0) \stackrel{(19)}{=} r \int_a^b \left(\prod_{j=1}^r f_j(x) \right) dx - \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0), \quad (53)$$

$x_0 \in [a, b]$.

Furthermore, it holds ($0 < \alpha \leq 1$ case)

$$\begin{aligned} \Delta(f_1, \dots, f_r) &\stackrel{(42)}{=} r(b-a) \int_a^b \left(\prod_{j=1}^r f_j(x) \right) dx - \\ &\sum_{i=1}^r \left[\left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) \left(\int_a^b f_i(x) dx \right) \right]. \end{aligned} \quad (54)$$

When $r = 2$, we get that

$$\theta(f_1, f_2)(x_0) = 2 \int_a^b f_1(x) f_2(x) dx - f_1(x_0) \int_a^b f_2(x) dx - f_2(x_0) \int_a^b f_1(x) dx, \quad (55)$$

$x_0 \in [a, b]$,

and

$$\Delta(f_1, f_2) = 2 \left[(b-a) \int_a^b f_1(x) f_2(x) dx - \left(\int_a^b f_1(x) dx \right) \left(\int_a^b f_2(x) dx \right) \right], \quad (56)$$

$0 < \alpha \leq 1$.

We give

Corollary 16 *All as in Theorem 10, A is a commutative Banach algebra, $r = 2$. Then*

$$\begin{aligned} \|\theta(f_1, f_2)(x_0)\| &\leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^2 \left[\left\| \left((D_{x_0-;g}^\alpha f_i) \circ g^{-1} \right) \right\|_{\infty, [g(a), g(x_0)]} \right. \\ &\quad \left. (g(x_0) - g(a))^\alpha \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^2 \|f_j(x)\| \right) dx \right) \right] + \\ &\quad \left[\left\| \left((D_{x_0+;g}^\alpha f_i) \circ g^{-1} \right) \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^\alpha \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^2 \|f_j(x)\| \right) dx \right) \right]. \end{aligned} \quad (57)$$

Proof. By Theorem 10. ■

We continue with

Corollary 17 *All as in Theorem 13, A is a commutative Banach algebra, $r = 2$, $0 < \alpha \leq 1$, $M(f_1, f_2)$ as in (41). Then*

$$\|\Delta(f_1, f_2)\| \leq \frac{M(f_1, f_2)(b-a)^2 (g(b) - g(a))^\alpha}{\Gamma(\alpha+1)} \left[\|f_1\|_{\infty, [a, b]} + \|f_2\|_{\infty, [a, b]} \right]. \quad (58)$$

Proof. By Theorem 13. ■

Finally we derive:

Corollary 18 *All as in Corollary 16, for $g(t) = e^t$. Then*

$$\begin{aligned} \|\theta(f_1, f_2)(x_0)\| &\leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^2 \left[\left\| \left((D_{x_0-;e^t}^\alpha f_i) \circ \log \right) \right\|_{\infty, [e^a, e^{x_0}]} \right. \\ &\quad \left. (e^{x_0} - e^a)^\alpha \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^2 \|f_j(x)\| \right) dx \right) \right] + \\ &\quad \left[\left\| \left((D_{x_0+;e^t}^\alpha f_i) \circ \log \right) \right\|_{\infty, [e^{x_0}, e^b]} (e^b - e^{x_0})^\alpha \left(\int_{x_0}^b \left(\prod_{\substack{j=1 \\ j \neq i}}^2 \|f_j(x)\| \right) dx \right) \right]. \end{aligned} \quad (59)$$

Proof. By Corollary 16. ■

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