Balanced Canavati type Fractional Opial Inequalities

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Here we present L_p, $p > 1$, fractional Opial type inequalities subject to high order boundary conditions. They involve the right and left Canavati type generalised fractional derivatives. These derivatives are mixed together into the balanced Canavati type generalised fractional derivative. This balanced fractional derivative is introduced and activated here for the first time.

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Key Words and Phrases: Opial inequality, fractional inequality, Canavati fractional derivative, boundary conditions.

1 Introduction

This article is inspired by the famous theorem of Z. Opial [10], 1960, which follows

Theorem 1 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

In (1), the constant $\frac{h}{4}$ is the best possible. Inequality (1) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases}$$

where $c > 0$ is an arbitrary constant.

To prove easier Theorem 1, Beesack [4] proved the following well-known Opial type inequality which is used very commonly.

This is another insiration to our work.

Theorem 2 *Let $x(t)$ be absolutely continuous in $[0, a]$, and $x(0) = 0$. Then*

$$\int_0^a |x(t) x'(t)| \, dt \leq \frac{a}{2} \int_0^a (x'(t))^2 \, dt. \tag{2}$$

Inequality (2) is sharp, it is attained by $x(t) = ct$, $c > 0$ is an arbitrary constant.

Opial type inequalities are used a lot in proving uniqueness of solutions to differential equations, also to give upper bounds to their solutions.

By themselves have made a great subject of intensive research and there exists a great literature about them.

Typical and great sources on them are the monographs [1], [2].

We define here the balanced Canavati type fractional derivative and we prove related Opial type inequalities subject to boundary conditions.

These have smaller constants than in other Opial inequalities when using traditional fractional derivatives.

2 Background

Let $\nu > 0$, $n := [\nu]$ (integral part of ν), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. The gamma function Γ is given by $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. Here $[a, b] \subseteq \mathbb{R}$, $x, x_0 \in [a, b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a, b])$ and define the left Riemann-Liouville integral

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x - t)^{\nu-1} f(t) \, dt, \tag{3}$$

$x_0 \leq x \leq b$. We define the subspace $C_{x_0}^\nu([a.b])$ of $C^n([a, b])$:

$$C_{x_0}^\nu([a, b]) := \left\{ f \in C^n([a, b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0, b]) \right\}. \tag{4}$$

For $f \in C_{x_0}^\nu([a, b])$, we define the left generalized ν-fractional derivative of f over $[x_0, b]$ as

$$D_{x_0}^\nu f := \left(J_{1-\alpha}^{x_0} f^{(n)}\right)', \tag{5}$$

see [2], p. 24, and Canavati derivative in [5].

Notice that $D_{x_0}^\nu f \in C([x_0, b])$.

We need the following generalization of Taylor's formula at the fractional level, see [2], pp. 8-10, and [5].

Theorem 3 *Let $f \in C_{x_0}^{\nu}([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $\nu \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + ... + f^{(n-1)}(x_0)\frac{(x - x_0)^{n-1}}{(n-1)!} \tag{6}$$
$$+ \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0.$$

(ii) If $0 < \nu < 1$ we get

$$f(x) = \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0 \tag{7}$$

We will use (6) and (7).

Furthermore we need:

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha - 1} f(J)\, dJ, \tag{8}$$

$x \in [a,b]$, see also [3], [6], [7], [8], [11]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f := (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)}\right)', \tag{10}$$

see [3]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [3], we need the following Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

1) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b_-)}{k!} (x - b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{11}$$

2) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \quad \forall\, x \in [a,b]. \tag{12}$$

We will use (11) and (12).

We introduce a new concept

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{\alpha+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

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3 Main Result

We give our main result

Theorem 6 *Let $f \in C([a,b])$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. Assume further that*

$$f^{(k)}(a) = f^{(k)}(b) = 0, \quad k = 0, 1, ..., m-1; \tag{14}$$

$$p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \quad \text{and} \quad \alpha > \frac{1}{q}.$$

(i) Case of $1 < q \leq 2$. Then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{15}$$

$$\frac{2^{-\left(\alpha + \frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q \, d\omega\right)^{\frac{2}{q}}.$$

(ii) Case of $q > 2$. Then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{16}$$

$$\frac{2^{-\left(\alpha + \frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q \, d\omega\right)^{\frac{2}{q}}.$$

(iii) When $p = q = 2$, $\alpha > \frac{1}{2}$, then

$$\int_a^b |f(\omega)| \, |D^{\alpha} f(\omega)| \, d\omega \leq \tag{17}$$

$$\frac{2^{-\left(\alpha + \frac{1}{2}\right)} (b-a)^{\alpha}}{\Gamma(\alpha) \left[\sqrt{2\alpha(2\alpha-1)}\right]} \left(\int_a^b |D^{\alpha} f(\omega)|^2 \, d\omega\right).$$

Remark 7 *Let us say that $\alpha = 1$, then by (17) we obtain*

$$\int_a^b |f(\omega)| \, |f'(\omega)| \, d\omega \leq \frac{(b-a)}{4} \left(\int_a^b (f'(\omega))^2 \, d\omega\right), \tag{18}$$

that is reproving and recovering Opial's inequality (1), see [10], see also Olech's result [9].

Proof. of Theorem 6. Let $x \in \left[a, \frac{a+b}{2}\right]$, we have by assumption $f^{(k)}(a) = 0,\ k = 0, 1, ..., m-1$ and Theorem 3 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_a^\alpha f(\tau)\, d\tau. \tag{19}$$

Let $x \in \left[\frac{a+b}{2}, b\right]$, we have by assumption $f^{(k)}(b) = 0,\ k = 0, 1, ..., m-1$ and Theorem 4 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} D_{b-}^\alpha f(\tau)\, d\tau. \tag{20}$$

Using Hölder's inequality on (19) we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} |D_a^\alpha f(\tau)|\, d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left((x-\tau)^{\alpha-1}\right)^p d\tau\right)^{\frac{1}{p}} \left(\int_a^x |D_a^\alpha f(\tau)|^q\, d\tau\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x |D_a^\alpha f(\tau)|^q\, d\tau\right)^{\frac{1}{q}}. \tag{21}$$

Set

$$z(x) := \int_a^x |D_a^\alpha f(\tau)|^q\, d\tau, \quad (z(a) = 0).$$

Then

$$z'(x) = |D_a^\alpha f(x)|^q,$$

and

$$|D_a^\alpha f(x)| = (z'(x))^{\frac{1}{q}}, \quad \text{all } a \leq x \leq \frac{a+b}{2}.$$

Therefore by (21) we have

$$|f(\omega)|\, |D_a^\alpha f(\omega)| \leq \frac{1}{\Gamma(\alpha)} \frac{(\omega-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} (z(\omega) z'(\omega))^{\frac{1}{q}}, \tag{22}$$

all $a \leq \omega \leq x \leq \frac{a+b}{2}$.

Next working similarly with (20) we obtain

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} \left|D_{b-}^\alpha f(\tau)\right| d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^b \left((\tau-x)^{\alpha-1}\right)^p d\tau\right)^{\frac{1}{p}} \left(\int_x^b \left|D_{b-}^\alpha f(\tau)\right|^q d\tau\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)}\frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_x^b\left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau\right)^{\frac{1}{q}}. \tag{23}$$

Set

$$\lambda(x):=\int_x^b\left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau=-\int_b^x\left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau,\quad (\lambda(b)=0).$$

Then

$$\lambda'(x)=-\left|D_{b-}^{\alpha}f(x)\right|^q$$

and

$$\left|D_{b-}^{\alpha}f(x)\right|=\left(-\lambda'(x)\right)^{\frac{1}{q}},\quad \text{all } \frac{a+b}{2}\leq x\leq b.$$

Therefore by (23) we have

$$|f(\omega)|\left|D_{b-}^{\alpha}f(\omega)\right|\leq\frac{1}{\Gamma(\alpha)}\frac{(b-\omega)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(-\lambda(\omega)\lambda'(\omega)\right)^{\frac{1}{q}}, \tag{24}$$

all $\frac{a+b}{2}\leq x\leq\omega\leq b$.

Next we integrate (22) over $[a,x]$ to obtain

$$\int_a^x|f(\omega)|\left|D_a^{\alpha}f(\omega)\right|d\omega\leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_a^x(\omega-a)^{\frac{p(\alpha-1)+1}{p}}\left(z(\omega)z'(\omega)\right)^{\frac{1}{q}}d\omega\leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_a^x(\omega-a)^{p(\alpha-1)+1}d\omega\right)^{\frac{1}{p}}\left(\int_a^x z(\omega)z'(\omega)d\omega\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\frac{(x-a)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}}\frac{z(x)^{\frac{2}{q}}}{2^{\frac{1}{q}}}=$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x\left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{25}$$

So we have proved

$$\int_a^x|f(\omega)|\left|D_a^{\alpha}f(\omega)\right|d\omega\leq$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x\left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{26}$$

for all $a\leq x\leq\frac{a+b}{2}$.

By (26) we get

$$\int_a^{\frac{a+b}{2}}|f(\omega)|\left|D_a^{\alpha}f(\omega)\right|d\omega\leq$$

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$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}}\, 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^{\frac{a+b}{2}}\left|D_a^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{27}$$

Similarly we integrate (24) over $[x,b]$ to obtain

$$\int_x^b |f(\omega)|\left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_x^b (b-\omega)^{\frac{p(\alpha-1)+1}{p}}\left(-\lambda(\omega)\lambda'(\omega)\right)^{\frac{1}{q}} d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_x^b (b-\omega)^{p(\alpha-1)+1} d\omega\right)^{\frac{1}{p}}\left(\int_x^b -\lambda(\omega)\lambda'(\omega)\, d\omega\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\frac{(b-x)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}}\frac{(\lambda(x))^{\frac{2}{q}}}{2^{\frac{1}{q}}}. \tag{28}$$

We have proved that

$$\int_x^b |f(\omega)|\left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{2^{-\frac{1}{q}}(b-x)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_x^b\left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{29}$$

for all $\frac{a+b}{2} \leq x \leq b$.

By (29) we get

$$\int_{\frac{a+b}{2}}^b |f(\omega)|\left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}}\, 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_{\frac{a+b}{2}}^b\left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{30}$$

Adding (27) and (30) we get

$$\int_a^b |f(\omega)|\left|D^\alpha f(\omega)\right| d\omega \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\cdot$$

$$\left[\left(\int_a^{\frac{a+b}{2}}\left|D_a^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}+\left(\int_{\frac{a+b}{2}}^b\left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}\right] =: (*.) \tag{31}$$

Assume $1 < q \leq 2$, then $\frac{2}{q} \geq 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)}(b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\cdot$$

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$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^{\alpha} f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} = \tag{32}$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[\left(p(\alpha-1)+1\right)\left(p(\alpha-1)+2\right)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{33}$$

So for $1 < q \leq 2$ we have proved (15).

Assume now $q > 2$, then $0 < \frac{2}{q} < 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)} 2^{1-\frac{2}{q}}}{\Gamma(\alpha)\left[\left(p(\alpha-1)+1\right)\left(p(\alpha-1)+2\right)\right]^{\frac{1}{p}}}.$$

$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^{\alpha} f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^{\alpha} f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} =$$

$$\frac{2^{-\left(\alpha+\frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[\left(p(\alpha-1)+1\right)\left(p(\alpha-1)+2\right)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^{\alpha} f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{34}$$

So when $q > 2$ we have established (16).

(iii) The case of $p = q = 2$, see (17), is obvious, it derives from (15) immediately. ■

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Canavati fractional Ostrowski type inequalities

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Abstract

Here we present Ostrowski type inequalities involving left and right Canavati type generalised fractional derivatives. Combining these we obtain fractional Ostrowski type inequalities of mixed form. Then we establish Ostrowski type inequalities for ordinary and fractional derivatives involving complex valued functions defined on the unit circle.

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Keywords and Phrases: deviation of a function, fractional derivative, Ostrowski inequality, function average.

1 Introduction

In 1938, A. Ostrowski [12] proved the following important inequality:

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \cdot (b-a)\,\|f'\|_\infty\,, \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we present various general Ostrowski type inequalities involving fractional derivatives of Canavati type.

At the end we give applications to complex valued functions defined on the unit circle.

2 Background

Let $\nu > 0$, $n := [\nu]$ (integral part of ν), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. The gamma function Γ is given by $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. Here $[a,b] \subseteq \mathbb{R}$, $x, x_0 \in [a,b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a,b])$ and define the left Riemann-Liouville integral

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} f(t)\, dt, \tag{2}$$

$x_0 \leq x \leq b$. We define the subspace $C_{x_0}^\nu([a.b])$ of $C^n([a,b])$:

$$C_{x_0}^\nu([a,b]) := \left\{ f \in C^n([a,b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0,b]) \right\}. \tag{3}$$

For $f \in C_{x_0}^\nu([a,b])$, we define the left generalized ν-fractional derivative of f over $[x_0,b]$ as

$$D_{x_0}^\nu f := \left(J_{1-\alpha}^{x_0} f^{(n)} \right)', \tag{4}$$

see [4], p. 24, and Canavati derivative in [6].

Notice that $D_{x_0}^\nu f \in C([x_0,b])$.

We need the following generalization of Taylor's formula at the fractional level, see [4], pp. 8-10, and [6].

Theorem 2 *Let $f \in C_{x_0}^\nu([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $\nu \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \ldots + f^{(n-1)}(x_0)\frac{(x-x_0)^{n-1}}{(n-1)!} + \left(J_\nu^{x_0} D_{x_0}^\nu f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0. \tag{5}$$

(ii) If $0 < \nu < 1$ we get

$$f(x) = \left(J_\nu^{x_0} D_{x_0}^\nu f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0 \tag{6}$$

We will use (5).

Furthermore we need:

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\, dJ, \tag{7}$$

$x \in [a,b]$, see also [5], [9], [10], [11], [13]. Define the subspace of functions

$$C_{b-}^\alpha([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{8}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f := (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{9}$$

see [5]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [5], we need the following Taylor fractional formula.

Theorem 3 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

1) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b_-)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \forall\, x \in [a,b]. \tag{10}$$

2) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \quad \forall\, x \in [a,b]. \tag{11}$$

We will use (10).

In [4], pp. 589-594, and [3], we proved the first fractional Ostrowski inequality.

Theorem 4 *Let $a \leq x_0 < b$, x_0 is fixed. Let $f \in C_{x_0}^{\nu}([a,b])$, $\nu \geq 1$, $n := [\nu]$. Assume $f^{(i)}(x_0) = 0$, $i = 1, ..., n-1$. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{\infty,[x_0,b]}}{\Gamma(\nu+2)} \cdot (b-x_0)^{\nu}. \tag{12}$$

Inequality (12) is sharp, namely it is attained by

$$f(x) := (x-x_0)^{\nu}, \ \ \nu \geq 1, \ \ x \in [a,b]. \tag{13}$$

When $1 \leq \nu < 2$ the assumption $f^{(i)}(x_0) = 0$, $i = 1, ..., n-1$ is void.

3 Main Results

We give

Theorem 5 *Same assumptions as in Theorem 4. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\, dy - f(x_0) \right| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_1([x_0,b])} (b-x_0)^{\nu-1}}{\Gamma(\nu+1)}. \tag{14}$$

Proof. By (5) we get

$$f(y)-f(x_0)=\frac{1}{\Gamma(\nu)}\int_{x_0}^{y}(y-w)^{\nu-1}\left(D_{x_0}^{\nu}f\right)(w)\,dw,\quad \forall\, y\geq x_0. \tag{15}$$

Hence

$$|f(y)-f(x_0)|\leq\frac{1}{\Gamma(\nu)}\int_{x_0}^{y}(y-w)^{\nu-1}\left|\left(D_{x_0}^{\nu}f\right)(w)\right|dw$$

$$\leq\frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)}\int_{x_0}^{y}\left|\left(D_{x_0}^{\nu}f\right)(w)\right|dw\leq\frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)}\int_{x_0}^{b}\left|\left(D_{x_0}^{\nu}f\right)(w)\right|dw$$

$$=\frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)}\left\|D_{x_0}^{\nu}f\right\|_{L_1([x_0,b])}.$$

I.e.

$$|f(y)-f(x_0)|\leq\frac{(y-x_0)^{\nu-1}}{\Gamma(\nu)}\left\|D_{x_0}^{\nu}f\right\|_{L_1([x_0,b])},\quad \forall\, y\in[x_0,b]. \tag{16}$$

Therefore we get

$$\left|\frac{1}{b-x_0}\int_{x_0}^{b}f(y)\,dy-f(x_0)\right|=\frac{1}{b-x_0}\left|\int_{x_0}^{b}(f(y)-f(x_0))\,dy\right|\leq$$

$$\frac{1}{b-x_0}\int_{x_0}^{b}|f(y)-f(x_0)|\,dy\overset{(16)}{\leq}$$

$$\frac{1}{b-x_0}\left(\int_{x_0}^{b}(y-x_0)^{\nu-1}dy\right)\frac{\left\|D_{x_0}^{\nu}f\right\|_{L_1([x_0,b])}}{\Gamma(\nu)}=\frac{(b-x_0)^{\nu-1}}{\nu}\frac{\left\|D_{x_0}^{\nu}f\right\|_{L_1([x_0,b])}}{\Gamma(\nu)}=$$

$$\frac{(b-x_0)^{\nu-1}}{\Gamma(\nu+1)}\left\|D_{x_0}^{\nu}f\right\|_{L_1([x_0,b])},$$

proving the claim. ■

We continue with

Theorem 6 *Same assumptions as in Theorem 4. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{1}{b-x_0}\int_{x_0}^{b}f(y)\,dy-f(x_0)\right|\leq\frac{\left\|D_{x_0}^{\nu}f\right\|_{L_q([x_0,b])}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-x_0)^{\nu-1+\frac{1}{p}}. \tag{17}$$

Proof. We notice that

$$|f(y)-f(x_0)|\leq\frac{1}{\Gamma(\nu)}\int_{x_0}^{y}(y-w)^{\nu-1}\left|\left(D_{x_0}^{\nu}f\right)(w)\right|dw$$

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$$\leq \frac{1}{\Gamma(\nu)} \left(\int_{x_0}^{y} (y-w)^{p(\nu-1)} \, dw \right)^{\frac{1}{p}} \left(\int_{x_0}^{y} \left| \left(D_{x_0}^{\nu} f \right)(w) \right|^q dw \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\nu)} \left(\frac{(y-x_0)^{p(\nu-1)+1}}{p(\nu-1)+1} \right)^{\frac{1}{p}} \left(\int_{x_0}^{b} \left| \left(D_{x_0}^{\nu} f \right)(w) \right|^q dw \right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\nu)} \frac{(y-x_0)^{(\nu-1)+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \left\| \left(D_{x_0}^{\nu} f \right) \right\|_{L_q([x_0,b])}.$$

That is

$$|f(y)-f(x_0)| \leq \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)} \frac{(y-x_0)^{(\nu-1)+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}, \quad \forall\, y \in [x_0,b]. \tag{18}$$

Consequently we obtain

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\,dy - f(x_0) \right| \leq \frac{1}{b-x_0} \int_{x_0}^{b} |f(y)-f(x_0)|\,dy \overset{(18)}{\leq}$$

$$\left(\frac{1}{b-x_0} \right) \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \int_{x_0}^{b} (y-x_0)^{(\nu-1)+\frac{1}{p}}\,dy =$$

$$\frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}} \frac{(b-x_0)^{\nu+\frac{1}{p}-1}}{\left(\nu+\frac{1}{p}\right)}, \tag{19}$$

proving the claim. ■

Combining Theorems 4-6 we derive

Proposition 7 *Let all as in Theorem 6. Then*

$$\left| \frac{1}{b-x_0} \int_{x_0}^{b} f(y)\,dy - f(x_0) \right| \leq \min \left\{ \frac{\left\| D_{x_0}^{\nu} f \right\|_{\infty,[x_0,b]}}{\Gamma(\nu+2)} (b-x_0)^{\nu}, \right. \tag{20}$$

$$\left. \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_1([x_0,b])} (b-x_0)^{\nu-1}}{\Gamma(\nu+1)}, \frac{\left\| D_{x_0}^{\nu} f \right\|_{L_q([x_0,b])} (b-x_0)^{\nu-1+\frac{1}{p}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu+\frac{1}{p}\right)} \right\}.$$

We continue with right Canavati fractional Ostrowski inequalities.

Theorem 8 *Let $\alpha \geq 1$, $m=[\alpha]$, $f \in C_{b-}^{\alpha}([a,b])$. Assume $f^{(k)}(b_-)=0$, $k=1,...,m-1$; which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_{a}^{b} f(x)\,dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}. \tag{21}$$

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Proof. Let $x \in [a,b]$. By (10) we get

$$f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha-1} D_{b-}^{\alpha} f(J)\, dJ. \tag{22}$$

Then, as before, we get

$$|f(x) - f(b)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}, \quad \forall\, x \in [a,b]. \tag{23}$$

Hence it holds

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, dx \overset{(23)}{\leq}$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)(b-a)} \int_a^b (b-x)^{\alpha}\, dx = \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(\Gamma(\alpha+1))(\alpha+1)} (b-a)^{\alpha}$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}, \tag{24}$$

proving the claim. ■

We continue with

Theorem 9 *Same assumptions as in Theorem 8. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])} (b-a)^{\alpha-1}}{\Gamma(\alpha+1)}. \tag{25}$$

Proof. We have again

$$|f(x) - f(b)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ \tag{26}$$

$$\leq \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}, \quad \forall\, x \in [a,b].$$

Hence

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, dx \overset{(26)}{\leq}$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(\Gamma(\alpha))(b-a)} \int_a^b (b-x)^{\alpha-1}\, dx = \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1}, \tag{27}$$

proving the claim. ■

We also have

Theorem 10 *Same assumptions as in Theorem 8. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} (b-a)^{\alpha-1+\frac{1}{p}}. \tag{28}$$

Proof. As before we obtain

$$|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} (b-x)^{\alpha-1+\frac{1}{p}}, \quad \forall\, x \in [a,b]. \tag{29}$$

Hence

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\,dx \overset{(29)}{\leq}$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} (b-a)} \left(\int_a^b (b-x)^{\alpha-1+\frac{1}{p}}\,dx \right) =$$

$$\frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \Gamma(\alpha)} \frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\left(\alpha + \frac{1}{p}\right)}, \tag{30}$$

proving the claim. ■

Combining Theorems 8-10 we derive

Proposition 11 *Here all as in Theorem 10. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(b) \right| \leq \min \left\{ \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}, \right. \tag{31}$$

$$\left. \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])} (b-a)^{\alpha-1}}{\Gamma(\alpha+1)}, \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])} (b-a)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \right\}.$$

We also give optimality of (21).

Proposition 12 *Inequality (21) is sharp, namely it is attained by*

$$f_*(J) = (b-J)^{\alpha}, \;\; \alpha \geq 1, \;\; J \in [a,b], \tag{32}$$

$m := [\alpha]$.

Proof. We have that

$$f_*^{(m)}(J) = (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)(b-J)^{\alpha-m}. \tag{33}$$

We also notice

$$\left(J_{b-}^{1-(\alpha-m)} f_*^{(m)}\right)(x) \overset{(7)}{=} \frac{1}{\Gamma(1-\alpha+m)} \int_x^b (J-x)^{-\alpha+m} f_*^{(m)}(J)\, dJ$$

$$= \frac{1}{\Gamma(1-\alpha+m)} \int_x^b (J-x)^{m-\alpha} (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)$$

$$\cdot (\alpha-m+1)(b-J)^{\alpha-m}\, dJ$$

$$= \frac{(-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)}{\Gamma(1-\alpha+m)} \cdot$$

$$\int_x^b (b-J)^{(\alpha-m+1)-1} (J-x)^{(1+m-\alpha)-1}\, dJ$$

$$= \frac{(-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)}{\Gamma(1-\alpha+m)} \cdot$$

$$\frac{\Gamma(\alpha-m+1)\Gamma(1+m-\alpha)}{\Gamma(2)}(b-x)$$

$$= (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)(\alpha-m+1)\Gamma(\alpha-m+1)(b-x)$$

$$= (-1)^m \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-m+2)\Gamma(\alpha-m+2)(b-x)$$

$$= \ldots = (-1)^m \Gamma(\alpha+1)(b-x). \tag{34}$$

That is

$$\left(J_{b-}^{1-\alpha+m} f_*^{(m)}\right)(x) = (-1)^m \Gamma(\alpha+1)(b-x). \tag{35}$$

Therefore it holds

$$\left(D_{b-}^{\alpha} f_*\right)(x) \overset{(9)}{=} (-1)^{m-1}(-1)^m \Gamma(\alpha+1)(-1) = \Gamma(\alpha+1). \tag{36}$$

So that

$$\left\| D_{b-}^{\alpha} f_* \right\|_{\infty,[a,b]} = \Gamma(\alpha+1). \tag{37}$$

We also notice that $f_*^{(k)}(b_-) = 0$, $k = 1, \ldots, m-1$, and $f_*(b) = 0$.

We observe further that

$$L.H.S.(21) = \frac{(b-a)^{\alpha}}{\alpha+1}, \tag{38}$$

and

$$R.H.S.(21) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}(b-a)^{\alpha} = \frac{(b-a)^{\alpha}}{\alpha+1}, \tag{39}$$

proving the claim. ■

Next we present mixed Canavati fractional Ostrowski type inequalities.

Theorem 13 *Let $\alpha \geq 1$, $m = [\alpha]$, $x \in [a,b]$ fixed, $f \in C([a,b])$ with $f \in C^{\alpha}_{x-}([a,x])$ and $f \in C^{\alpha}_{x}([x,b])$. Assume that $f^{(k)}(x) = 0$, $k = 1,...,m-1$, which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \tag{40}$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]} (x-a)^{\alpha+1} + \left\| D^{\alpha}_{x} f \right\|_{\infty,[x,b]} (b-x)^{\alpha+1} \right\} \leq$$

$$\left(\frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)} \right) \max \left\{ \left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]}, \left\| D^{\alpha}_{x} f \right\|_{\infty,[x,b]} \right\}. \tag{41}$$

Proof. Let $x \in [a,b]$. By (10) we get

$$f(y) - f(x) = \frac{1}{\Gamma(\alpha)} \int_y^x (J-y)^{\alpha-1} D^{\alpha}_{x-} f(J)\,dJ, \quad \forall\, y \in [a,x]. \tag{42}$$

Hence

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_y^x (J-y)^{\alpha-1} \left| D^{\alpha}_{x-} f(J) \right| dJ \leq \frac{(x-y)^{\alpha}}{\Gamma(\alpha+1)} \left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]}, \tag{43}$$

$\forall\, y \in [a,x]$.

Similarly, by (5) we get

$$f(y) - f(x) = \frac{1}{\Gamma(\alpha)} \int_x^y (y-w)^{\alpha-1} \left(D^{\alpha}_{x} f \right)(w)\,dw, \quad \forall\, y \in [x,b]. \tag{44}$$

Hence

$$|f(y) - f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^y (y-w)^{\alpha-1} \left| D^{\alpha}_{x} f(w) \right| dw \leq \left\| D^{\alpha}_{x} f \right\|_{\infty,[x,b]} \frac{(y-x)^{\alpha}}{\Gamma(\alpha+1)}, \tag{45}$$

$\forall\, y \in [x,b]$.

We observe that

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \frac{1}{b-a} \int_a^b |f(y) - f(x)|\,dy = \tag{46}$$

$$\frac{1}{b-a} \left\{ \int_a^x |f(y) - f(x)|\,dy + \int_x^b |f(y) - f(x)|\,dy \right\} \overset{\text{by } ((43),\,(45))}{\leq}$$

$$\frac{1}{b-a} \left\{ \frac{\left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \int_a^x (x-y)^{\alpha}\,dy + \frac{\left\| D^{\alpha}_{x} f \right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \int_x^b (y-x)^{\alpha}\,dy \right\} = \tag{47}$$

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$$\frac{1}{(b-a)\,\Gamma(\alpha+2)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha+1}+\left\|D_{x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha+1}\right\}, \tag{48}$$

proving the claim. ∎

We continue with the optimality of Theorem 13.

Proposition 14 *Inequalities (40), (41) are sharp, namely are attained by*

$$\overline{f}(J)=\begin{cases}(x-J)^{\alpha}, & J\in[a,x],\\ (J-x)^{\alpha}, & J\in[x,b],\end{cases} \tag{49}$$

where $\alpha\geq 1$, $x\in[a,b]$ is fixed.

See that $\overline{f}^{(k)}(x_-)=\overline{f}^{(k)}(x_+)=0$, $k=0,1,...,m-1$.

We have that

$$\left\|D_{x-}^{\alpha}\overline{f}\right\|_{\infty,[a,x]}=\left\|D_{x}^{\alpha}\overline{f}\right\|_{\infty,[x,b]}=\Gamma(\alpha+1). \tag{50}$$

Furthermore we notice

$$L.H.S.\,(40)=\frac{1}{(\alpha+1)(b-a)}\left\{(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right\}, \tag{51}$$

and

$$R.H.S.\,(41)=\frac{\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right)}{(b-a)\,\Gamma(\alpha+2)}\Gamma(\alpha+1)$$
$$=\frac{\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right)}{(\alpha+1)(b-a)}, \tag{52}$$

proving the claim.

We continue with

Theorem 15 *All as in Theorem 13. Then ($x\in[a,b]$)*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy-f(x)\right|\leq \tag{53}$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+1)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}(x-a)^{\alpha}+\left\|D_{x}^{\alpha}f\right\|_{L_1([x,b])}(b-x)^{\alpha}\right\}\leq$$
$$\left(\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])},\left\|D_{x}^{\alpha}f\right\|_{L_1([x,b])}\right\}. \tag{54}$$

Proof. Let $x\in[a,b]$. From (42) we get ($y\in[a,x]$)

$$|f(y)-f(x)|\leq\frac{1}{\Gamma(\alpha)}\int_y^x(J-y)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)\right|dJ\leq \tag{55}$$

$$\frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}\int_y^x \left|D_{x-}^{\alpha}f(J)\right|dJ \leq \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)}\int_a^x \left|D_{x-}^{\alpha}f(J)\right|dJ. \tag{56}$$

That is

$$|f(y)-f(x)| \leq (x-y)^{\alpha-1}\frac{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}}{\Gamma(\alpha)}, \quad \forall\, y \in [a,x]. \tag{57}$$

Similarly from (44) we get

$$|f(y)-f(x)| \leq \frac{1}{\Gamma(\alpha)}\int_x^y (y-w)^{\alpha-1}\left|\left(D_x^{\alpha}f\right)(w)\right|dw \tag{58}$$

$$\leq \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha)}\left\|D_x^{\alpha}f\right\|_{L_1([x,b])}, \quad \forall\, y \in [x,b]. \tag{59}$$

From (46) we obtain

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \tag{60}$$

$$\frac{1}{b-a}\left\{\frac{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}dy + \frac{\left\|D_x^{\alpha}f\right\|_{L_1([x,b])}}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1}dy\right\} \tag{61}$$

$$= \frac{1}{(b-a)\Gamma(\alpha+1)}\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}(x-a)^{\alpha} + \left\|D_x^{\alpha}f\right\|_{L_1([x,b])}(b-x)^{\alpha}\right\} \leq$$

$$\left(\frac{(b-x)^{\alpha}+(x-a)^{\alpha}}{(b-a)\Gamma(\alpha+1)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_1([a,x])}, \left\|D_x^{\alpha}f\right\|_{L_1([x,b])}\right\}, \tag{62}$$

proving the claim. ∎

We also give

Theorem 16 *All as in Theorem 13. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then ($x\in[a,b]$)*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \tag{63}$$

$$\left\{(x-a)^{\alpha+\frac{1}{p}}\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])} + (b-x)^{\alpha+\frac{1}{p}}\left\|D_x^{\alpha}f\right\|_{L_q([x,b])}\right\} \leq$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}}+(x-a)^{\alpha+\frac{1}{p}}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right)\cdot$$

$$\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}, \left\|D_x^{\alpha}f\right\|_{L_q([x,b])}\right\}. \tag{64}$$

Proof. By (55) we get

$$|f(y)-f(x)| \leq \frac{1}{\Gamma(\alpha)}\left(\int_y^x (J-y)^{p(\alpha-1)}\,dJ\right)^{\frac{1}{p}} \left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}$$

$$= \frac{1}{\Gamma(\alpha)}\frac{(x-y)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}, \quad \forall\, y\in[a,x]. \tag{65}$$

Similarly from (58) we derive

$$|f(y)-f(x)| \leq \frac{1}{\Gamma(\alpha)}\left(\int_x^y (y-w)^{p(\alpha-1)}\,dw\right)^{\frac{1}{p}} \left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}$$

$$= \frac{1}{\Gamma(\alpha)}\frac{(y-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}, \quad \forall\, y\in[x,b]. \tag{66}$$

By (46) we derive

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\left\{\left(\int_a^x (x-y)^{\alpha-1+\frac{1}{p}}\,dy\right)\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])} + \right.$$

$$\left.\left(\int_x^b (y-x)^{\alpha-1+\frac{1}{p}}\,dy\right)\left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}\right\} \tag{67}$$

$$= \frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\cdot$$

$$\left\{(x-a)^{\alpha+\frac{1}{p}}\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])} + (b-x)^{\alpha+\frac{1}{p}}\left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}\right\} \leq \tag{68}$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}}+(x-a)^{\alpha+\frac{1}{p}}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_q([a,x])}, \left\|D_{x}^{\alpha}f\right\|_{L_q([x,b])}\right\}, \tag{69}$$

proving the claim. ■

Corollary 17 *All as in Theorem 13. Then*

$$\left|\frac{1}{b-a}\int_a^b f(y)\,dy - f(x)\right| \leq \tag{70}$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{2}}+(x-a)^{\alpha+\frac{1}{2}}}{(b-a)\Gamma(\alpha)\sqrt{2\alpha-1}\left(\alpha+\frac{1}{2}\right)}\right)\max\left\{\left\|D_{x-}^{\alpha}f\right\|_{L_2([a,x])}, \left\|D_{x}^{\alpha}f\right\|_{L_2([x,b])}\right\}.$$

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Proof. By Theorem 16. ■

Combining Theorems 13, 15, 16 we derive

Theorem 18 *Here all as in Theorem 13. Let any* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq$$

$$\min \left\{ \frac{1}{(b-a)\,\Gamma(\alpha+2)} \left\{ \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha+1} + \left\| D_{x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha+1} \right\}, \right.$$

$$\frac{1}{(b-a)\,\Gamma(\alpha+1)} \left\{ \left\| D_{x-}^{\alpha} f \right\|_{L_1([a,x])} (x-a)^{\alpha} + \left\| D_{x}^{\alpha} f \right\|_{L_1([x,b])} (b-x)^{\alpha} \right\}, \quad (71)$$

$$\frac{1}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \cdot$$

$$\left. \left\{ (x-a)^{\alpha+\frac{1}{p}} \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])} + (b-x)^{\alpha+\frac{1}{p}} \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])} \right\} \right\} \leq$$

$$\min \left\{ \left(\frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)} \right) \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}, \left\| D_{x}^{\alpha} f \right\|_{\infty,[x,b]} \right\}, \right.$$

$$\left(\frac{(b-x)^{\alpha} + (x-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)} \right) \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{L_1([a,x])}, \left\| D_{x}^{\alpha} f \right\|_{L_1([x,b])} \right\}, \quad (72)$$

$$\left(\frac{(b-x)^{\alpha+\frac{1}{p}} + (x-a)^{\alpha+\frac{1}{p}}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \right) \cdot$$

$$\left. \max \left\{ \left\| D_{x-}^{\alpha} f \right\|_{L_q([a,x])}, \left\| D_{x}^{\alpha} f \right\|_{L_q([x,b])} \right\} \right\}.$$

4 Applications

Inequalities for complex valued functions defined on the unit circle were studied extensively by S. Dragomir, see [7], [8].

We give here our version for these functions involved in Ostrowski type inequalities, by applying results of this article.

Let $t \in [a,b] \subseteq [0, 2\pi)$, the unit circle arc $A = \left\{ z \in \mathbb{C} : z = e^{it},\ t \in [a,b] \right\}$, and $f : A \to \mathbb{C}$ be a continuous function. Clearly here there exist functions $u, v : A \to \mathbb{R}$ continuous, the real and the complex part of f, respectively, such that

$$f\left(e^{it}\right) = u\left(e^{it}\right) + iv\left(e^{it}\right). \quad (73)$$

So that f is continuous, iff u, v are continuous.

Call $g(t) = f\left(e^{it}\right)$, $l_1(t) = u\left(e^{it}\right)$, $l_2(t) = v\left(e^{it}\right)$, $t \in [a,b]$; so that $g : [a,b] \to \mathbb{C}$ and $l_1, l_2 : [a,b] \to \mathbb{R}$ are continuous functions in t.

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If g has a derivative with respect to t, then l_1, l_2 have also derivatives with respect to t. In that case

$$f_t\left(e^{it}\right) = u_t\left(e^{it}\right) + iv_t\left(e^{it}\right), \tag{74}$$

(i.e. $g'(t) = l_1'(t) + il_2'(t)$), which means

$$f_t(\cos t + i\sin t) = u_t(\cos t + i\sin t) + iv_t(\cos t + i\sin t). \tag{75}$$

Let us call $x = \cos t$, $y = \sin t$. Then

$$u_t\left(e^{it}\right) = u_t(\cos t + i\sin t) = u_t(x + iy) = u_t(x, y) =$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u\left(e^{it}\right)}{\partial x}(-\sin t) + \frac{\partial u\left(e^{it}\right)}{\partial y}\cos t. \tag{76}$$

Similarly we find that

$$v_t\left(e^{it}\right) = \frac{\partial v\left(e^{it}\right)}{\partial x}(-\sin t) + \frac{\partial v\left(e^{it}\right)}{\partial y}\cos t. \tag{77}$$

So that

$$\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq \left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}, \tag{78}$$

and

$$\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq \left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}, \tag{79}$$

Consequently it holds

$$\left\|f_t\left(e^{it}\right)\right\|_{\infty,[a,b]} \leq$$

$$\left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]} + \left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]} + \left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}. \tag{80}$$

Since g is continuous on $[a,b]$, then $\int_a^b f\left(e^{it}\right) dt$ exists. Furthermore it holds

$$\int_a^b f\left(e^{it}\right) dt = \int_a^b u\left(e^{it}\right) dt + i\int_a^b v\left(e^{it}\right) dt. \tag{81}$$

Let now $t_0 \in [a,b]$. We observe that

$$\left|\frac{1}{b-a}\int_a^b f\left(e^{it}\right) dt - f\left(e^{it_0}\right)\right| =$$

$$\left|\frac{1}{b-a}\int_a^b u\left(e^{it}\right) dt + i\frac{1}{b-a}\int_a^b v\left(e^{it}\right) dt - u\left(e^{it_0}\right) - iv\left(e^{it_0}\right)\right| \leq \tag{82}$$

$$\left|\frac{1}{b-a}\int_a^b u\left(e^{it}\right)dt-u\left(e^{it_0}\right)\right|+\left|\frac{1}{b-a}\int_a^b v\left(e^{it}\right)dt-v\left(e^{it_0}\right)\right|\overset{\text{(by (1))}}{\leq}$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left[\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]}+\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]}\right]. \tag{83}$$

We have proved the following version of Ostrowski inequality for complex functions.

Theorem 19 *Let $f\in C(A,\mathbb{C})$ with its real and complex part $u\left(e^{it}\right),v\left(e^{it}\right)\in C^1([a,b])$ as functions of t, where $t_0\in[a,b]\subseteq[0,2\pi)$. Then*

$$\left|\frac{1}{b-a}\int_a^b f\left(e^{it}\right)dt-f\left(e^{it_0}\right)\right|\leq$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left[\left\|u_t\left(e^{it}\right)\right\|_{\infty,[a,b]}+\left\|v_t\left(e^{it}\right)\right\|_{\infty,[a,b]}\right]\leq \tag{84}$$

$$\left[\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\cdot \tag{85}$$

$$\left[\left\|\frac{\partial u\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]}+\left\|\frac{\partial v\left(e^{it}\right)}{\partial x}\right\|_{\infty,[a,b]}+\left\|\frac{\partial u\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}+\left\|\frac{\partial v\left(e^{it}\right)}{\partial y}\right\|_{\infty,[a,b]}\right].$$

Inequality (84) is sharp.

An explanation follows next.

For $z\in\mathbb{C}-\{0\}$ we call principal value of $\log(z)$ the complex valued function

$$Log(z):=\ln|z|+iArg(z), \tag{86}$$

where $0\leq Arg(z)<2\pi$.

For $t\in[0,2\pi)$ we have that

$$Log\left(e^{it}\right)=it. \tag{87}$$

Let here $a=0<b<2\pi$, and $t_0=0$. Here $l_1(t)=0$ and $l_2(t)=t$, with $l_2'(t)=1$.

Notice in general that

$$\left(\frac{1}{4}+\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)=\frac{(b-t_0)^2+(t_0-a)^2}{2(b-a)}, \tag{88}$$

for any $t_0\in[a,b]$, see [9], p. 498 and [1].

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Hence we have

$$L.H.S.\,(84) = \left| \frac{1}{b} \int_0^b Log\left(e^{it}\right) dt \right| = \left| \frac{1}{b} \int_0^b itdt \right| = \frac{1}{b} \int_0^b tdt = \frac{b}{2}. \tag{89}$$

Furthermore it holds

$$R.H.S.\,(84) = \frac{b}{2} \cdot 1 = \frac{b}{2}. \tag{90}$$

By (89) and (90) we conclude that inequality (84) is attained by Log at $t_0 = 0$ on $[0, b]$, that is a sharp inequality.

We now move at the fractional level.

Let $t_0 \in [a, b]$, we rewrite (82) as follows

$$\left| \frac{1}{b-a} \int_a^b f\left(e^{it}\right) dt - f\left(e^{it_0}\right) \right| \leq$$

$$\left| \frac{1}{b-a} \int_a^b l_1(t)\,dt - l_1(t_0) \right| + \left| \frac{1}{b-a} \int_a^b l_2(t)\,dt - l_2(t_0) \right|. \tag{91}$$

By applying Theorem 18 to each of the last two summands we derive the following complex fractional Ostrowski inequality.

Theorem 20 *Let $f \in C(A, \mathbb{C})$, $t, t_0 \in [a, b] \subseteq [0, 2\pi)$; any $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha \geq 1$, $m = [\alpha]$, with $l_1, l_2 \in C^{\alpha}_{t_0-}([a, t_0])$ and $l_1, l_2 \in C^{\alpha}_{t_0}([t_0, b])$. Assume that $l_1^{(k)}(t_0) = l_2^{(k)}(t_0) = 0$, $k = 1, ..., m-1$, which is void when $1 \leq \alpha < 2$. Then*

$$\left| \frac{1}{b-a} \int_a^b f\left(e^{it}\right) dt - f\left(e^{it_0}\right) \right| \leq$$

$$\min \left\{ \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{t_0-} l_1 \right\|_{\infty,[a,t_0]} (t_0-a)^{\alpha+1} + \left\| D^{\alpha}_{t_0} l_1 \right\|_{\infty,[t_0,b]} (b-t_0)^{\alpha+1} \right\}, \right.$$

$$\frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{t_0-} l_1 \right\|_{L_1([a,t_0])} (t_0-a)^{\alpha} + \left\| D^{\alpha}_{t_0} l_1 \right\|_{L_1([t_0,b])} (b-t_0)^{\alpha} \right\},$$

$$\frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\left. \left\{ (t_0-a)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{t_0-} l_1 \right\|_{L_q([a,t_0])} + (b-t_0)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{t_0} l_1 \right\|_{L_q([t_0,b])} \right\} \right\} +$$

$$\min \left\{ \frac{1}{(b-a)\Gamma(\alpha+2)} \left\{ \left\| D^{\alpha}_{t_0-} l_2 \right\|_{\infty,[a,t_0]} (t_0-a)^{\alpha+1} + \left\| D^{\alpha}_{t_0} l_2 \right\|_{\infty,[t_0,b]} (b-t_0)^{\alpha+1} \right\}, \right.$$

$$\frac{1}{(b-a)\Gamma(\alpha+1)} \left\{ \left\| D^{\alpha}_{t_0-} l_2 \right\|_{L_1([a,t_0])} (t_0-a)^{\alpha} + \left\| D^{\alpha}_{t_0} l_2 \right\|_{L_1([t_0,b])} (b-t_0)^{\alpha} \right\},$$

$$\frac{1}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\left\{ (t_0-a)^{\alpha+\frac{1}{p}} \left\| D_{t_0-}^{\alpha} l_2 \right\|_{L_q([a,t_0])} + (b-t_0)^{\alpha+\frac{1}{p}} \left\| D_{t_0}^{\alpha} l_2 \right\|_{L_q([t_0,b])} \right\} \Bigg\} \leq \tag{92}$$

$$\min \Bigg\{ \left(\frac{(b-t_0)^{\alpha+1} + (t_0-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)} \right) \max \left\{ \left\| D_{t_0-}^{\alpha} l_1 \right\|_{\infty,[a,t_0]}, \left\| D_{t_0}^{\alpha} l_1 \right\|_{\infty,[t_0,b]} \right\},$$

$$\left(\frac{(b-t_0)^{\alpha} + (t_0-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)} \right) \max \left\{ \left\| D_{t_0-}^{\alpha} l_1 \right\|_{L_1([a,t_0])}, \left\| D_{t_0}^{\alpha} l_1 \right\|_{L_1([t_0,b])} \right\},$$

$$\left(\frac{(b-t_0)^{\alpha+\frac{1}{p}} + (t_0-a)^{\alpha+\frac{1}{p}}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \right) \cdot$$

$$\max \left\{ \left\| D_{t_0-}^{\alpha} l_1 \right\|_{L_q([a,t_0])}, \left\| D_{t_0}^{\alpha} l_1 \right\|_{L_q([t_0,b])} \right\} \Bigg\} +$$

$$\min \Bigg\{ \left(\frac{(b-t_0)^{\alpha+1} + (t_0-a)^{\alpha+1}}{(b-a)\,\Gamma(\alpha+2)} \right) \max \left\{ \left\| D_{t_0-}^{\alpha} l_2 \right\|_{\infty,[a,t_0]}, \left\| D_{t_0}^{\alpha} l_2 \right\|_{\infty,[t_0,b]} \right\},$$

$$\left(\frac{(b-t_0)^{\alpha} + (t_0-a)^{\alpha}}{(b-a)\,\Gamma(\alpha+1)} \right) \max \left\{ \left\| D_{t_0-}^{\alpha} l_2 \right\|_{L_1([a,t_0])}, \left\| D_{t_0}^{\alpha} l_2 \right\|_{L_1([t_0,b])} \right\},$$

$$\left(\frac{(b-t_0)^{\alpha+\frac{1}{p}} + (t_0-a)^{\alpha+\frac{1}{p}}}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \right) \cdot$$

$$\max \left\{ \left\| D_{t_0-}^{\alpha} l_2 \right\|_{L_q([a,t_0])}, \left\| D_{t_0}^{\alpha} l_2 \right\|_{L_q([t_0,b])} \right\} \Bigg\}. \tag{93}$$

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Fractional Polya type integral inequality

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Abstract

Here we establish a fractional Polya type integral inequality with the help of generalised right and left fractional derivatives. The amazing fact here is that we do not need any boundary conditions as the classical Polya integral inequality requires.

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1 Introduction

We mention the following famous Polya's integral inequality, see [7], [8, p, 62], [9] and [10, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{1}$$

In [11], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup_{x\in[a,b]}|f'(x)|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ in (2) is the best constant.

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In this short note we present a fractional Polya type integral inequality, similar to (2), without the boundary conditions $f(a) = f(b) = 0$.

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [3] introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [3] the same over $[0,1]$ that appeared first.

Theorem 3 *Let $f \in C_{a+}^\alpha([a,b])$.*

(i) If $\alpha \geq 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b]. \tag{7}$$

We will use (7).

Notice that

$$\int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt = \int_a^x \left(D_{a+}^\alpha f\right)(t)\, d\left(\frac{(x-t)^\alpha}{-\alpha}\right)$$

$$= \left(D_{a+}^\alpha f\right)(\xi_x) \frac{(x-a)^\alpha}{\alpha}, \text{ where } \xi_x \in [a,x], \tag{8}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we get

$$f(x) = \left(D_{a+}^\alpha f\right)(\xi_x) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{9}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{10}$$

$x \in [a,b]$, see also [2], [4], [5], [6], [12]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{11}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{12}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \text{all } x \in [a,b]. \tag{13}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{14}$$

We will use (14).

Notice that

$$\int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt = \int_x^b \left(D_{b-}^{\alpha} f \right)(t)\, d\left(\frac{(t-x)^{\alpha}}{\alpha} \right)$$

$$= \left(D_{b-}^{\alpha} f \right)(\eta_x) \frac{(b-x)^{\alpha}}{\alpha}, \text{ where } \eta_x \in [x,b], \tag{15}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we obtain

$$f(x) = \left(D_{b-}^{\alpha} f \right)(\eta_x) \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{16}$$

2 Main Result

We present the following fractional Polya type integral inequality without any boundary conditions.

Theorem 5 *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M(f) = \max\left\{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}. \tag{17}$$

Then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx \le M(f)\,\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}. \tag{18}$$

Inequality (18) is sharp, namely it is attained by

$$f_*(x) = \left\{\begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array}\right\}, \quad 0 < \alpha < 1. \tag{19}$$

Clearly here non zero constant functions f are excluded.

Proof. By (9) we get

$$|f(x)| \le \left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[a, \frac{a+b}{2}\right]. \tag{20}$$

By (16) we derive

$$|f(x)| \le \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[\frac{a+b}{2}, b\right]. \tag{21}$$

Hence we get

$$\int_a^b |f(x)|\,dx = \int_a^{\frac{a+b}{2}} |f(x)|\,dx + \int_{\frac{a+b}{2}}^b |f(x)|\,dx$$

(by (20), (21))

$$\le \frac{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (x-a)^{\alpha}\,dx + \frac{\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b (b-x)^{\alpha}\,dx$$

$$= \frac{\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1} + \frac{\left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1} \tag{22}$$

$$= \frac{\left(\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2}\right)^{\alpha+1}. \tag{23}$$

4

So we have proved that

$$\int_a^b |f(x)|\,dx \le \max\left\{ \left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\} \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}, \quad (24)$$

proving (18).

Notice that

$$f_*\left(\left(\frac{a+b}{2}\right)_-\right) = f_*\left(\left(\frac{a+b}{2}\right)_+\right) = \left(\frac{b-a}{2}\right)^{\alpha},$$

so that $f_* \in C([a,b])$.

Here $m=0$. We see that

$$\left(J_{1-\beta}^{\alpha+}(\cdot - a)^{\alpha}\right)(x) = \left(J_{1-\alpha}^{a+}(\cdot - a)^{\alpha}\right)(x) =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} (t-a)^{\alpha}\, dt =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{(1-\alpha)-1} (t-a)^{(\alpha+1)-1}\, dt =$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\Gamma(\alpha+1)}{\Gamma(2)} (x-a) = \Gamma(\alpha+1)(x-a).$$

Hence

$$D_{a+}^{\alpha}(x-a)^{\alpha} = \Gamma(\alpha+1), \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right]. \quad (25)$$

Therefore

$$\left\|D_{a+}^{\alpha}(x-a)^{\alpha}\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} = \Gamma(a+1). \quad (26)$$

Furthermore we have

$$\left(J_{b-}^{1-\alpha}(b-\cdot)^{\alpha}\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} (b-t)^{\alpha}\, dt =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_x^b (b-t)^{(\alpha+1)-1} (t-x)^{(1-\alpha)-1}\, dt =$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(\alpha+1)\Gamma(1-\alpha)}{\Gamma(2)} (b-x) = \Gamma(\alpha+1)(b-x).$$

Therefore

$$D_{b-}^{\alpha}(b-x)^{\alpha} = \Gamma(\alpha+1), \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right], \quad (27)$$

and

$$\left\| D_{b-}^{\alpha}\left(b-x\right)^{\alpha}\right\|_{\infty,\left[\frac{a+b}{2},b\right]}=\Gamma\left(a+1\right). \tag{28}$$

Consequently we find that

$$M\left(f_{*}\right)=\Gamma\left(\alpha+1\right). \tag{29}$$

Applying f_* into (18) we obtain:

$$\text{R.H.S.(18) for } f_{*}\ =\Gamma\left(\alpha+1\right)\frac{\left(b-a\right)^{\alpha+1}}{\Gamma\left(\alpha+2\right)2^{\alpha}}=\frac{\left(b-a\right)^{\alpha+1}}{\left(\alpha+1\right)2^{\alpha}}, \tag{30}$$

while we get the same result from

$$\text{L.H.S. (18) for } f_{*}\ =\left|\int_{a}^{b}f_{*}\left(x\right)dx\right|=$$

$$\int_{a}^{\frac{a+b}{2}}\left(x-a\right)^{\alpha}dx+\int_{\frac{a+b}{2}}^{b}\left(b-x\right)^{\alpha}dx=\frac{\left(b-a\right)^{\alpha+1}}{\left(\alpha+1\right)2^{\alpha}}, \tag{31}$$

proving sharpness of (18). ■

We make

Remark 6 *When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements as in Theorem 5. If we set there $\alpha = 1$ we derive exactly Theorem 2. So we generalize Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

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Univariate Fractional Polya type integral inequalities

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Abstract

Here we establish a series of various fractional Polya type integral inequalities with the help of generalised right and left fractional derivatives. We give application to complex valued functions defined on the unit circle.

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1 Introduction

We mention the following famous Polya's integral inequality, see [12], [13, p, 62], [14] and [15, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\,dx. \tag{1}$$

In [16], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a) = f(b) = 0$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Then*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{(b-a)^2}{4} M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ in (2) is the best constant.

The above motivate the current paper.

In this article we present univariate fractional Polya type integral inequalities in various cases, similar to (2).

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \le x \le b$. We define the subspace $C_{a+}^{\alpha}([a.b])$ of $C^m([a,b])$:

$$C_{a+}^{\alpha}([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^{\alpha}([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^{\alpha} f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [5], introduced the above over $[0,1]$.

Notice that $D_{a+}^{\alpha} f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over $[0,1]$ that appeared first.

Theorem 3 *Let $f \in C_{a+}^{\alpha}([a,b])$.*

(i) If $\alpha \ge 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \dots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^{\alpha} f\right)(t)\, dt, \qquad \textit{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^{\alpha} f\right)(t)\, dt, \qquad \textit{all } x \in [a,b]. \tag{7}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{8}$$

$x \in [a,b]$, see also [2], [8], [9], [10], [17]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{10}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \text{all } x \in [a,b]. \tag{11}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{12}$$

We need from [3]

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [a]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

In [4] we proved the following fractional Polya type integral inequality without any boundary conditions.

Theorem 6 *Let $0 < \alpha < 1$, $f \in C([a,b])$. Assume $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. Set*

$$M_1(f) = \max\left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}. \tag{14}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \tag{15}$$

$$\frac{\left(\left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]} + \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2} \right)^{\alpha+1} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{16}$$

Inequalities (15), (16) are sharp, namely they are attained by

$$f_*(x) = \left\{ \begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \alpha < 1. \tag{17}$$

Clearly here non zero constant functions f are excluded.

The last result also motivates this work.

Remark 7 *(see [4]) When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements (15), (16), (17) as in Theorem 6. If we set there $\alpha = 1$ we derive exactly Theorem 2. So we have generalized Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

We continue here with other interesting univariate fractional Polya type inequalities.

2 Main Results

We present our first main result.

Theorem 8 *Let $\alpha \geq 1$, $m = [\alpha]$, $f \in C([a,b])$. Assume $f \in C^{\alpha}_{a+}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C^{\alpha}_{b-}\left(\left[\frac{a+b}{2}, b\right]\right)$, such that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$. Set*

$$M_2(f) = \max\left\{ \left\| D^{\alpha}_{a+} f \right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \left\| D^{\alpha}_{b-} f \right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \right\}. \tag{18}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \| D^{\alpha} f \|_{L_1([a,b])} \leq \frac{M_2(f)}{\Gamma(\alpha+1)\, 2^{\alpha-1}} (b-a)^{\alpha}. \tag{20}$$

Here f cannot be a non-trivial polynomial of degree m.

Proof. By assumption and Theorem 3 we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D^{\alpha}_{a+} f\right)(t)\, dt, \quad \text{all } x \in \left[a, \frac{a+b}{2}\right], \tag{21}$$

also it holds, by assumtion and Theorem 4, that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D^{\alpha}_{b-} f\right)(t)\, dt, \quad \text{all } x \in \left[\frac{a+b}{2}, b\right]. \tag{22}$$

By (21) we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left| \left(D^{\alpha}_{a+} f\right)(t) \right| dt$$

$$\leq \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left| \left(D^{\alpha}_{a+} f\right)(t) \right| dt, \quad \text{all } x \in \left[a, \frac{a+b}{2}\right]. \tag{23}$$

By (22) we derive

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left|\left(D_{b-}^{\alpha} f\right)(t)\right| dt$$

$$\leq \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left|\left(D_{b-}^{\alpha} f\right)(t)\right| dt, \quad \text{all } x \in \left[\frac{a+b}{2}, b\right]. \tag{24}$$

Consequently we have

$$\int_a^{\frac{a+b}{2}} |f(x)|\, dx \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1} dx\right) \left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)} \tag{25}$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)},$$

and

$$\int_{\frac{a+b}{2}}^b |f(x)|\, dx \leq \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1} dx\right) \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \tag{26}$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)}.$$

Therefore we obtain by adding (25) and (26) that

$$\int_a^b |f(x)|\, dx \leq \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left[\left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)} + \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)}\right]$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D^{\alpha} f\right\|_{L_1([a,b])} \leq$$

$$\max\left\{\left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)}, \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)}\right\} \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)\, 2^{\alpha-1}}, \tag{28}$$

proving the claim. ■

We continue with

Theorem 9 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = [\alpha]$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$, *such that* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$. *When* $\frac{1}{q} < \alpha < 1$, *the last boundary conditions are void.*

Set

$$M_3(f) = \max\left\{\left\|D_{a+}^{\alpha} f\right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \left\|D_{b-}^{\alpha} f\right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)}\right\}. \tag{29}$$

Then

$$\int_a^b |f(x)|\, dx \leq \frac{1}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}. \tag{30}$$

5

$$\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]\leq$$
$$\frac{M_3\left(f\right)}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}\left(b-a\right)^{\alpha+\frac{1}{p}}. \tag{31}$$

Again here f cannot be a non-trivial polynomial of degree m.

Proof. By Theorem 3 we have

$$\left|f\left(x\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_a^x\left(x-t\right)^{\alpha-1}\left|\left(D_{a+}^{\alpha}f\right)\left(t\right)\right|dt$$

$$\leq\frac{1}{\Gamma\left(\alpha\right)}\left(\int_a^x\left(x-t\right)^{p\left(\alpha-1\right)}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|\left(D_{a+}^{\alpha}f\right)\left(t\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(x-a\right)^{\frac{\left(p\left(\alpha-1\right)+1\right)}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\quad\text{for all } x\in\left[a,\frac{a+b}{2}\right]. \tag{32}$$

That is

$$\left|f\left(x\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(x-a\right)^{\alpha-1+\frac{1}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\quad\text{for all } x\in\left[a,\frac{a+b}{2}\right] \tag{33}$$

Similarly from Theorem 4 we get

$$\left|f\left(x\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_x^b\left(t-x\right)^{\alpha-1}\left|\left(D_{b-}^{\alpha}f\right)\left(t\right)\right|dt$$

$$\leq\frac{1}{\Gamma\left(\alpha\right)}\left(\int_x^b\left(t-x\right)^{p\left(\alpha-1\right)}dt\right)^{\frac{1}{p}}\left(\int_x^b\left|\left(D_{b-}^{\alpha}f\right)\left(t\right)\right|^q dt\right)^{\frac{1}{q}} \tag{34}$$

$$\leq\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(b-x\right)^{\left(\alpha-1\right)+\frac{1}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)},\quad\text{for all } x\in\left[\frac{a+b}{2},b\right]. \tag{35}$$

That is

$$\left|f\left(x\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\frac{\left(b-x\right)^{\alpha-1+\frac{1}{p}}}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)},\quad\text{for all } x\in\left[\frac{a+b}{2},b\right]. \tag{36}$$

Consequently we obtain by (33) that

$$\int_a^{\frac{a+b}{2}}\left|f\left(x\right)\right|dx\leq$$

$$\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\left(\int_a^{\frac{a+b}{2}}\left(x-a\right)^{\alpha-1+\frac{1}{p}}dx\right)\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}=$$

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$$\frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}. \quad (37)$$

Similarly it holds by (36) that

$$\int_{\frac{a+b}{2}}^{b}|f(x)|\,dx \leq$$

$$\frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}}\left(\int_{\frac{a+b}{2}}^{b}(b-x)^{\alpha-1+\frac{1}{p}}\,dx\right)\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)} =$$

$$\frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}. \quad (38)$$

Adding (37) and (38) we have

$$\int_{a}^{b}|f(x)|\,dx \leq \frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}} \quad (39)$$

$$\cdot\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right] \leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}, \quad (40)$$

proving the claim. ■

Combining Theorem 6, Remark 7, Theorem 8 and Theorem 9, we obtain

Theorem 10 *Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq 1$, $m=[\alpha]$, $f\in C([a,b])$. *Assume* $f\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ *and* $f\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, *such that* $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,...,m-1$.

Then

$$\int_{a}^{b}|f(x)|\,dx \leq$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)}\left(\frac{b-a}{2}\right)^{\alpha+1},\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}f\right\|_{L_1([a,b])}, \quad (41)$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\} \leq$$

$$\min\left\{ M_1(f)\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}}, \frac{M_2(f)}{\Gamma(\alpha+1)2^{\alpha-1}}(b-a)^{\alpha}, \right.$$

$$\left. \frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}} \right\}, \tag{42}$$

where $M_1(f)$ as in (14), $M_2(f)$ as in (18) and $M_3(f)$ as in (29).

Here f cannot be a non-trivial polynomial of degree m.

Corollary 11 *Here all as in Theorem 10. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{b-a}\int_a^b |f(x)|\,dx \leq$$

$$\min\left\{ \frac{\left(\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)2^{\alpha+1}}(b-a)^{\alpha}, \right. \tag{43}$$

$$\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(b-a)^{\alpha-1}\left\|D^{\alpha}f\right\|_{L_1([a,b])},$$

$$\left. \frac{\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)} + \left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha+\frac{1}{p}}}(b-a)^{\alpha+\frac{1}{p}-1} \right\} \leq$$

$$\min\left\{ M_1(f)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)2^{\alpha}}, \frac{M_2(f)}{\Gamma(\alpha+1)2^{\alpha-1}}(b-a)^{\alpha-1}, \right.$$

$$\left. \frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}-1} \right\}. \tag{44}$$

In 1938, A. Ostrowski [11] proved the following important inequality:

Theorem 12 *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_{\infty} := \sup_{t\in(a,b)}|f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\|f'\|_{\infty}, \tag{45}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

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In (45) for $x = \frac{a+b}{2}$ we get

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{b-a}{4}\right) \|f'\|_\infty . \tag{46}$$

We have proved the following

Theorem 13 *Let $f \in C^1([a,b])$, with $f\left(\frac{a+b}{2}\right) = 0$. Then*

$$\left| \int_a^b f(t)\, dt \right| \leq \frac{(b-a)^2}{4} \|f'\|_\infty , \tag{47}$$

where the constant $\frac{1}{4}$ is the best possible.

So we reproved (2) with only one initial condition.

3 Application

Inequalities for complex valued functions defined on the unit circle were studied extensively by S. Dragomir, see [6], [7].

We give here our version for these functions involved in a Polya type inequality, by applying a result of this article.

Let $t \in [a,b] \subseteq [0, 2\pi)$, the unit circle arc $A = \{z \in \mathbb{C} : z = e^{it},\ t \in [a,b]\}$, and $f : A \to \mathbb{C}$ be a continuous function. Clearly here there exist functions $u, v : A \to \mathbb{R}$ continuous, the real and the complex part of f, respectively, such that

$$f\left(e^{it}\right) = u\left(e^{it}\right) + iv\left(e^{it}\right). \tag{48}$$

So that f is continuous, iff u, v are continuous.

Call $g(t) = f\left(e^{it}\right)$, $l_1(t) = u\left(e^{it}\right)$, $l_2(t) = v\left(e^{it}\right)$, $t \in [a,b]$; so that $g : [a,b] \to \mathbb{C}$ and $l_1, l_2 : [a,b] \to \mathbb{R}$ are continuous functions in t.

If g has a derivative with respect to t, then l_1, l_2 have also derivatives with respect to t. In that case

$$f_t\left(e^{it}\right) = u_t\left(e^{it}\right) + iv_t\left(e^{it}\right), \tag{49}$$

(i.e. $g'(t) = l_1'(t) + il_2'(t)$), which means

$$f_t(\cos t + i \sin t) = u_t(\cos t + i \sin t) + iv_t(\cos t + i \sin t). \tag{50}$$

Let us call $x = \cos t$, $y = \sin t$. Then

$$u_t\left(e^{it}\right) = u_t(\cos t + i \sin t) = u_t(x + iy) = u_t(x, y) =$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u\left(e^{it}\right)}{\partial x}(-\sin t) + \frac{\partial u\left(e^{it}\right)}{\partial y}\cos t. \tag{51}$$

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Similarly we find that

$$v_t\left(e^{it}\right)=\frac{\partial v\left(e^{it}\right)}{\partial x}\left(-\sin t\right)+\frac{\partial v\left(e^{it}\right)}{\partial y}\cos t. \tag{52}$$

Since g is continuous on $[a,b]$, then $\int_a^b f\left(e^{it}\right)dt$ exists. Furthermore it holds

$$\int_a^b f\left(e^{it}\right)dt=\int_a^b u\left(e^{it}\right)dt+i\int_a^b v\left(e^{it}\right)dt. \tag{53}$$

We have here that

$$\left|\int_a^b f\left(e^{it}\right)dt\right|\leq\int_a^b\left|f\left(e^{it}\right)\right|dt\leq \tag{54}$$

$$\int_a^b\left|u\left(e^{it}\right)\right|dt+\int_a^b\left|v\left(e^{it}\right)\right|dt=\int_a^b\left|l_1\left(t\right)\right|dt+\int_a^b\left|l_2\left(t\right)\right|dt. \tag{55}$$

We give the following application of Theorem 10.

Theorem 14 *Let $f\in C\left(A,\mathbb{C}\right)$, $[a,b]\subseteq[0,2\pi)$; any $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq1$, $m=[\alpha]$. Assume $l_1,l_2\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ and $l_1,l_2\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, such that $l_1^{(k)}\left(a\right)=l_2^{(k)}\left(a\right)=l_1^{(k)}\left(b\right)=l_2^{(k)}\left(b\right)=0$, $k=0,1,...,m-1$. Then*

$$\left|\int_a^b f\left(e^{it}\right)dt\right|\leq\int_a^b\left|f\left(e^{it}\right)\right|dt\leq$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}l_1\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}l_1\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma\left(\alpha+2\right)}\left(\frac{b-a}{2}\right)^{\alpha+1},\right. \tag{56}$$

$$\frac{1}{\Gamma\left(\alpha+1\right)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}l_1\right\|_{L_1([a,b])},$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}l_1\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}l_1\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\}+$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}l_2\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}l_2\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma\left(\alpha+2\right)}\left(\frac{b-a}{2}\right)^{\alpha+1},\right.$$

$$\frac{1}{\Gamma\left(\alpha+1\right)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}l_2\right\|_{L_1([a,b])},$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}l_2\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}l_2\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\}\leq$$

$$\min\left\{M_1(l_1)\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}},\frac{M_2(l_1)}{\Gamma(\alpha+1)2^{\alpha-1}}(b-a)^{\alpha},\right.$$

$$\left.\frac{M_3(l_1)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}\right\}+ \tag{57}$$

$$\min\left\{M_1(l_2)\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}},\frac{M_2(l_2)}{\Gamma(\alpha+1)2^{\alpha-1}}(b-a)^{\alpha},\right.$$

$$\left.\frac{M_3(l_2)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}\right\},$$

where $M_1(l_i)$ *as in (14),* $M_2(l_i)$ *as in (18) and* $M_3(l_i)$ *as in (29),* $i=1,2$.

Here l_1, l_2 *cannot be non-trivial polynomials of degree* m.

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Multivariate Generalised Fractional Polya type integral inequalities

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Abstract

Here we present a set of multivariate generalised fractional Polya type integral inequalities on the ball and shell. We treat both the radial and non-radial cases in all possibilities. We give also estimates for the related averages.

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1 Introduction

We mention the following famous Polya's integral inequality, see [9], [10, p, 62], [11] and [12, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi\in[a,b]$ such that*

$$|f'(\xi)|>\frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{1}$$

In [13], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup\limits_{x\in[a,b]}|f'(x)|$. Then*

$$\left|\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{4}M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ *in (2) is the best constant.*

The above motivate the current paper.

In this article we present multivariate fractional Polya type integral inequalities in various cases, similar to (2).

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [5], introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over $[0,1]$ that appeared first.

Theorem 3 *Let* $f \in C_{a+}^\alpha([a,b])$.

(i) If $\alpha \geq 1$, *then*

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \ldots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \qquad \text{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, *we have*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \qquad \text{all } x \in [a,b]. \tag{7}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{8}$$

$x \in [a,b]$, see also [2], [6], [7], [8], [15]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{10}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad \textit{all } x \in [a,b]. \tag{11}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad \textit{all } x \in [a,b]. \tag{12}$$

We need from [3]

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \textit{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \textit{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

In [4] we proved the following fractional Polya type integral inequality without any boundary conditions.

Theorem 6 *Let $0 < \alpha < 1$, $f \in C([a,b])$. Assume $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. Set*

$$M_1(f) = \max \left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}. \tag{14}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \tag{15}$$

$$\frac{\left(\left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]} + \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2} \right)^{\alpha+1} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{16}$$

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Inequalities (15), (16) are sharp, namely they are attained by

$$f_*(x) = \begin{Bmatrix} (x-a)^\alpha, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^\alpha, & x \in \left[\frac{a+b}{2}, b\right] \end{Bmatrix}, \quad 0 < \alpha < 1. \tag{17}$$

Clearly here non zero constant functions f are excluded.

The last result also motivates this work.

Remark 7 *(see [4]) When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements (15), (16), (17) as in Theorem 6. If we set there $\alpha = 1$ we derive exactly Theorem 2. So we have generalized Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

We present Polya type integral inequalities on the ball and shell.

2 Main Results

We need

Remark 8 *We define the ball $B(0,R) = \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, and the sphere*

$$S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$. Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

Following [14, pp. 149-150, exercise 6] and [16, pp. 87-88, Theorem 5.2.2] we can write $F : \overline{B(0,R)} \to \mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega; \tag{18}$$

we use this formula a lot.

Initially the function $f : \overline{B(0,R)} \to \mathbb{R}$ is radial; that is, there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$, $\alpha > 0$, $m = [\alpha]$. Here we assume that $g \in C([0,R])$ with $g \in C^\alpha_{0+}\left(\left[0, \frac{R}{2}\right]\right)$ and

$g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$. In case of $0<\alpha<1$ then the last boundary conditions are void.

By assumption here and Theorem 3 we have

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\left(D_{0+}^{\alpha}g\right)(t)\,dt, \tag{19}$$

all $s\in\left[0,\frac{R}{2}\right]$,

also it holds, by assumption and Theorem 4, that

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_s^R (t-s)^{\alpha-1}\left(D_{R-}^{\alpha}g\right)(t)\,dt, \tag{20}$$

all $s\in\left[\frac{R}{2},R\right]$.

By (19) we get

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\left|\left(D_{0+}^{\alpha}g\right)(t)\right|dt$$

$$\leq\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}\frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}\,dt=\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{21}$$

for any $s\in\left[0,\frac{R}{2}\right]$.

That is

$$|g(s)|\leq\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{22}$$

for any $s\in\left[0,\frac{R}{2}\right]$.

Similarly we obtain

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_s^R (t-s)^{\alpha-1}\left|\left(D_{R-}^{\alpha}g\right)(t)\right|dt$$

$$\leq\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha)}\int_s^R (t-s)^{\alpha-1}\,dt=\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)}(R-s)^{\alpha}, \tag{23}$$

for any $s\in\left[\frac{R}{2},R\right]$.

I.e. it holds

$$|g(s)|\leq\frac{\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}}{\Gamma(\alpha+1)}(R-s)^{\alpha}, \tag{24}$$

for any $s\in\left[\frac{R}{2},R\right]$.

Next we observe that

$$\left|\int_{B(0,R)} f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\overset{(18)}{=}$$

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$$\int_{S^{N-1}}\left(\int_0^R |g(s)|\, s^{N-1}ds\right)d\omega=\left(\int_0^R |g(s)|\, s^{N-1}ds\right)\int_{S^{N-1}}d\omega=$$

$$\left(\int_0^R |g(s)|\, s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}= \tag{25}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_0^{\frac{R}{2}} |g(s)|\, s^{N-1}ds+\int_{\frac{R}{2}}^R |g(s)|\, s^{N-1}ds\right\}\overset{(by\ (22)\ and\ (24))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}\int_0^{\frac{R}{2}} s^{\alpha+N-1}ds+\right.$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\int_{\frac{R}{2}}^R (R-s)^{\alpha}\left(\left(s-\frac{R}{2}\right)+\frac{R}{2}\right)^{N-1}ds\right\}= \tag{26}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)}\left(\frac{R}{2}\right)^{\alpha+N}+\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\cdot\right.$$

$$\left.\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right]\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)}\left(\frac{R}{2}\right)^{\alpha+N}+\right. \tag{27}$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right]\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}. \tag{28}$$

We have proved that

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{29}$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

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Consider now

$$g_*(s) = \begin{cases} s^{\alpha}, & s \in \left[0, \frac{R}{2}\right], \\ (R-s)^{\alpha}, & s \in \left[\frac{R}{2}, R\right], \ \alpha > 0. \end{cases} \tag{30}$$

We have as in [4] that

$$D_{0+}^{\alpha} s^{\alpha} = \Gamma(\alpha+1), \quad \textit{all } s \in \left[0, \frac{R}{2}\right], \tag{31}$$

and

$$\left\| D_{0+}^{\alpha} s^{\alpha} \right\|_{\infty, \left[0, \frac{R}{2}\right]} = \Gamma(\alpha+1).$$

Similarly as in [] we get

$$D_{R-}^{\alpha} (R-s)^{\alpha} = \Gamma(\alpha+1), \quad \textit{all } s \in \left[\frac{R}{2}, R\right], \tag{32}$$

and

$$\left\| D_{R-}^{\alpha} (R-s)^{\alpha} \right\|_{\infty, \left[\frac{R}{2}, R\right]} = \Gamma(\alpha+1). \tag{33}$$

That is

$$\left\| D_{0+}^{\alpha} g_* \right\|_{\infty, \left[0, \frac{R}{2}\right]} = \left\| D_{R-}^{\alpha} g_* \right\|_{\infty, \left[\frac{R}{2}, R\right]} = \Gamma(\alpha+1). \tag{34}$$

Consequently we find that

$$\textit{R.H.S. (29)} = \frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1} \Gamma\left(\frac{N}{2}\right)} \left\{ \frac{1}{(\alpha+N)} + \right.$$

$$\left. (N-1)! \Gamma(\alpha+1) \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma(\alpha+N+1-k)} \right] \right\}. \tag{35}$$

Let $f_* : \overline{B(0,R)} \to \mathbb{R}$ *be radial such that* $f_*(x) = g_*(s)$, $s = |x|$, $s \in [0, R]$, $\forall$ $x \in \overline{B(0,R)}$.

Then we have

$$\textit{L.H.S. (29)} = \int_{B(0,R)} f_*(y)\, dy \overset{(18)}{=}$$

$$\left(\int_0^R g_*(s) s^{N-1} ds \right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} s^{\alpha+N-1} ds + \int_{\frac{R}{2}}^{R} (R-s)^{\alpha} s^{N-1} ds \right\} = \tag{36}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \left(\frac{R}{2}\right)^{\alpha+N} \frac{1}{(\alpha+N)} + \int_{\frac{R}{2}}^{R} (R-s)^{\alpha} \left(\left(s - \frac{R}{2}\right) + \frac{R}{2} \right)^{N-1} ds \right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)} + \right.$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\int_{\frac{R}{2}}^{R}(R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\Bigg\}= \tag{37}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N)}+\right.$$

$$\left.(N-1)!\Gamma(\alpha+1)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}\overset{(35)}{=}\text{ R.H.S. (29)}, \tag{38}$$

proving (29) sharp, infact it is attained.

We have proved the following main result.

Theorem 9 *Let $f:\overline{B(0,R)}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$, $\alpha>0$. Assume that $g\in C([0,R])$, with $g\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$, $m=[\alpha]$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{39}$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Inequalities (39) are sharp, namely they are attained by a radial function f_* such that $f_*(x)=g_*(s)$, all $s\in[0,R]$, where

$$g_*(s)=\begin{cases}s^{\alpha}, & s\in\left[0,\frac{R}{2}\right],\\ (R-s)^{\alpha}, & s\in\left[\frac{R}{2},R\right].\end{cases} \tag{40}$$

We continue with

Remark 10 *(Continuation of Remark 8) Here we assume that $\alpha \geq 1$. By (19) we get*

$$|g(s)| \leq \frac{s^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}, \tag{41}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Also, by (20), we obtain

$$|g(s)| \leq \frac{(R-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}, \tag{42}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence as in (25) we get

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_0^R |g(s)|\, s^{N-1} ds \right) = \tag{43}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^{R} |g(s)|\, s^{N-1} ds \right\} \overset{(by\ (41),\ (42))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left(\int_0^{\frac{R}{2}} s^{N+\alpha-2} ds \right) \left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{44}$$

$$\left. \left(\int_{\frac{R}{2}}^{R} (R-s)^{\alpha-1} s^{N-1} ds \right) \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \right\} =$$

(acting the same as before, see (26)-(28))

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\} \overset{(13)}{=} \tag{45}$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D^{\alpha} g \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{46}$$

We have proved

Theorem 11 *Here all terms and assumptions as in Theorem 9, however with $\alpha \geq 1$. Then*

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\|D_{R-}^{\alpha} g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{47}$$

We continue with

Remark 12 *(Also a continuation of Remark 8) Let here $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, with $\alpha > \frac{1}{q}$. By (19) we have*

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left|\left(D_{0+}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_0^s (s-t)^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} \left(\int_0^s \left|\left(D_{0+}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{s^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}, \tag{48}$$

all $s \in \left[0,\frac{R}{2}\right]$.

Similarly by (20) we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left|\left(D_{R-}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_s^R (t-s)^{p(\alpha-1)}\, dt\right)^{\frac{1}{p}} \left(\int_s^R \left|\left(D_{R-}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(R-s)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}, \tag{49}$$

all $s \in \left[\frac{R}{2},R\right]$.

Hence it holds

$$\int_{B(0,R)} |f(y)|\, dy \overset{(25)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{(\text{by } (48),\ (49))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\,\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left(\int_0^{\frac{R}{2}} s^{\alpha+N-2+\frac{1}{p}} ds \right) \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{50}$$

$$\left(\int_{\frac{R}{2}}^{R}(R-s)^{\alpha-1+\frac{1}{p}}\,s^{N-1}ds\right)\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{\left(\frac{R}{2}\right)^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right.$$

$$\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\left(\int_{\frac{R}{2}}^{R}(R-s)^{\left(\alpha+\frac{1}{p}-1\right)}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right)\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{51}$$

$$\left[\sum_{k=0}^{N-1}\frac{(N-1)!}{k!\,(N-k-1)!}\left(\frac{R}{2}\right)^{k}\frac{\Gamma\left(\alpha+\frac{1}{p}\right)\Gamma(N-k)}{\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\left(\frac{R}{2}\right)^{\alpha+\frac{1}{p}+N-k-1}\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{52}$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left(\frac{R^{\alpha+N-\frac{1}{q}}}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot$$

$$\left.\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}. \tag{53}$$

We have proved the following

Theorem 13 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. All other terms and assumptions as in Theorem 9. Then*

$$\int_{B(0,R)}|f(y)|\,dy\le$$

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$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\, 2^{\alpha+N-\frac{1}{q}-1}} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right.$$

$$\left. (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\}. \quad (54)$$

Combining Theorems 9, 11, 13 we derive

Theorem 14 *Let any* $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha \geq 1$. *And let* $f : \overline{B(0,R)} \to \mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x)=g(s)$, $s=|x|$, $s \in [0,R]$, $\forall\ x \in \overline{B(0,R)}$. *Assume that* $g \in C([0,R])$, *with* $g \in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ *and* $g \in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, *such that* $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$, $m=[\alpha]$. *When* $0<\alpha<1$ *the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \int_{B(0,R)} |f(y)|\,dy \leq$$

$$\min\left\{ \frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)} + \right.\right.$$

$$\left. (N-1)!\left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\},$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)!\left\|D_{R-}^{\alpha} g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)}\right] \right\},$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\, 2^{\alpha+N-\frac{1}{q}-1}} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right.$$

$$\left.\left. (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\}\right\}. \quad (55)$$

Note 15 *It holds*

$$Vol(B(0,R)) = \frac{2\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}\right) N}. \quad (56)$$

The corresponding estimate on the average follows

Corollary 16 *Let all terms and assumptions as in Theorem 14. Then*

$$\left|\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)} f\left(y\right)dy\right| \leq \frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)} \left|f\left(y\right)\right|dy \leq$$

$$\min\left\{\frac{NR^{\alpha}}{2^{\alpha+N}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\right.\right.$$

$$\left.\left(N-1\right)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\},$$

$$\frac{NR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-1\right)\Gamma\left(\alpha\right)}+\right.$$

$$\left.\left(N-1\right)!\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N-k\right)}\right]\right\},$$

$$\frac{NR^{\alpha-\frac{1}{q}}}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.\left.\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}\right\}. \tag{57}$$

We continue with Polya type inequalities on the ball for non-radial functions.

Theorem 17 *Let $f\in C\left(\overline{B\left(0,R\right)}\right)$ that is not necessarily radial, $0<\alpha<2$. Assume for any $\omega\in S^{N-1}$, that $f\left(\cdot\omega\right)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f\left(\cdot\omega\right)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f\left(0\right)=f\left(R\omega\right)=0$. When $0<\alpha<1$ the last boundary conditions are void. We further assume that*

$$\left\|\frac{\partial_{0+}^{\alpha}f\left(r\omega\right)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f\left(r\omega\right)}{\partial r^{\alpha}}\right\|_{\infty,\left(r\in\left[\frac{R}{2},R\right]\right)}\leq K, \tag{58}$$

for every $\omega\in S^{N-1}$, where $K>0$.

Then

(i)

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq\frac{K\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\cdot \tag{59}$$

$$\left\{\frac{1}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\left(N-1\right)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\},$$

and

(ii)

$$\left|\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}f\left(y\right)dy\right|\leq\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq \tag{60}$$

$$\frac{KNR^{\alpha}}{2^{\alpha+N}}\left\{\frac{1}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\left(N-1\right)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}.$$

Proof. In Remark 8, see (25), (26), (27), (28), we proved that

$$\int_0^R\left|g\left(s\right)\right|s^{N-1}ds\leq\left(\frac{R}{2}\right)^{\alpha+N}\cdot$$

$$\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left(N-1\right)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}. \tag{61}$$

In the above (61) we plug in $g\left(\cdot\right)=f\left(\cdot\omega\right)$, for $\omega\in S^{N-1}$ fixed, and we get

$$\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\overset{(58)}{\leq}K\left(\frac{R}{2}\right)^{\alpha+N}\cdot$$

$$\left\{\frac{1}{\left(\alpha+N\right)\Gamma\left(\alpha+1\right)}+\left(N-1\right)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+N+1-k\right)}\right]\right\}=:\lambda_1. \tag{62}$$

Consequently we obtain

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy=\int_{S^{N-1}}\left(\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\right)d\omega\leq$$

$$\lambda_1\int_{S^{N-1}}d\omega=\lambda_1\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{63}$$

proving the claims. ■

We continue with

Theorem 18 *Let* $f\in C\left(\overline{B\left(0,R\right)}\right)$ *that is not necessarily radial,* $1\leq\alpha<2$. *Assume for any* $\omega\in S^{N-1}$, *that* $f\left(\cdot\omega\right)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ *and* $f\left(\cdot\omega\right)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, *such that* $f\left(0\right)=f\left(R\omega\right)=0$. *We further assume*

$$\left\|\frac{\partial_{0+}^{\alpha}f\left(\cdot\omega\right)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f\left(\cdot\omega\right)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\leq M, \tag{64}$$

for every $\omega\in S^{N-1}$, *where* $M>0$.

Then
(i)

$$\int_{B(0,R)}|f(y)|\,dy \leq \frac{M\pi^{\frac{N}{2}}R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)}\cdot \tag{65}$$

$$\left\{\frac{1}{(\alpha+N-1)\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\},$$

and
(ii)

$$\frac{1}{Vol\,(B(0,R))}\int_{B(0,R)}|f(y)|\,dy \leq \tag{66}$$

$$\frac{MNR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N-1)\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}.$$

Proof. In Remark 10, see (43), (44), (45), we proved that

$$\int_0^R |g(s)|\,s^{N-1}ds \leq \left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)}+\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}. \tag{67}$$

In the above (67) we plug in $g(\cdot)=f(\cdot\omega)$, for $\omega\in S^{N-1}$ fixed, and we derive

$$\int_0^R |f(s\omega)|\,s^{N-1}ds \overset{(64)}{\leq} M\left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{1}{(\alpha+N-1)\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}=:\lambda_2. \tag{68}$$

Hence

$$\int_{B(0,R)}|f(y)|\,dy=\int_{S^{N-1}}\left(\int_0^R|f(s\omega)|\,s^{N-1}ds\right)d\omega\leq$$

$$\lambda_2\int_{S^{N-1}}d\omega=\lambda_2\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{69}$$

proving the claims. ∎

We further have

Theorem 19 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{q}<\alpha<2$. Let $f\in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial. Assume for any $\omega\in S^{N-1}$, that $f(\cdot\omega)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$*

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and $f(\cdot\omega) \in C^{\alpha}_{R-}\left(\left[\frac{R}{2}, R\right]\right)$, *such that* $f(0) = f(R\omega) = 0$. *When* $\frac{1}{q} < \alpha < 1$ *the last boundary condition is void. We further assume*

$$\left\| \frac{\partial^{\alpha}_{0+} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}, \quad \left\| \frac{\partial^{\alpha}_{R-} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \leq \Phi, \tag{70}$$

for every $\omega \in S^{N-1}$, *where* $\Phi > 0$.

Then

(i)

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\Phi \pi^{\frac{N}{2}} R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\, 2^{\alpha+N-\frac{1}{q}-1}} \cdot \tag{71}$$

$$\left\{ \frac{1}{\left(\alpha+N-\frac{1}{q}\right)} + (N-1)!\,\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\,\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \right\},$$

and

(ii)

$$\frac{1}{Vol\,(B(0,R))} \int_{B(0,R)} |f(y)|\, dy \leq \frac{\Phi N R^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\, 2^{\alpha+N-\frac{1}{q}}} \cdot \tag{72}$$

$$\left\{ \frac{1}{\left(\alpha+N-\frac{1}{q}\right)} + (N-1)!\,\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\,\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \right\}.$$

Proof. In Remark 12, see (50), (51), (52), (53), we proved that

$$\int_0^R |g(s)|\, s^{N-1} ds \leq$$

$$\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}} \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \left\{ \frac{\left\|D^{\alpha}_{0+} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right. \tag{73}$$

$$\left. (N-1)!\,\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1} \frac{1}{k!\,\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right] \left\|D^{\alpha}_{R-} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\}.$$

In the above (73) we plug in $g(\cdot) = f(\cdot\omega)$, for $\omega \in S^{N-1}$ fixed, and we find

$$\int_0^R |f(s\omega)|\, s^{N-1} ds \overset{(70)}{\leq} \Phi \left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}} \frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

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$$\left\{ \frac{1}{\left(\alpha + N - \frac{1}{q}\right)} + (N-1)!\Gamma\left(\alpha + \frac{1}{p}\right) \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma\left(\alpha + \frac{1}{p} + N - k\right)} \right] \right\} =: \lambda_3. \tag{74}$$

Thus

$$\int_{B(0,R)} |f(y)|\, dy = \int_{S^{N-1}} \left(\int_0^R |f(s\omega)|\, s^{N-1} ds \right) d\omega \leq$$

$$\lambda_3 \int_{S^{N-1}} d\omega = \lambda_3 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{75}$$

proving the claims. ■

We make

Remark 20 *Let the spherical shell* $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. *Consider first that* $f : \overline{A} \to \mathbb{R}$ *is radial; that is, there exists* g *such that* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\, x \in \overline{A}$. *Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see ([14], p. 149-150 and [1], p. 421), furthermore for general* $F : \overline{A} \to \mathbb{R}$ *Lebesgue integrable function we have that*

$$\int_A F(x)\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega. \tag{76}$$

Let $d\omega$ *be the element of surface measure on* S^{N-1}, *then*

$$\omega_N := \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{77}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{2\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{N\Gamma\left(\frac{N}{2}\right)}. \tag{78}$$

We assume that $g \in C([R_1, R_2])$, *and* $\alpha > 0$, $m = [\alpha]$, *such that* $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ *and* $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. *When* $0 < \alpha < 1$ *the last boundary conditions are void.*

By assumption here and Theorem 3 we have

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{R_1}^{s} (s-t)^{\alpha-1} \left(D^{\alpha}_{R_1+} g\right)(t)\, dt, \tag{79}$$

all $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$,

also it holds, by assumption and Theorem 4, that

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{s}^{R_2} (t-s)^{\alpha-1} \left(D^{\alpha}_{R_2-} g\right)(t)\, dt, \tag{80}$$

all $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

By (79) we get

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_{R_1}^{s} (s-t)^{\alpha-1} \left|\left(D_{R_1+}^{\alpha} g\right)(t)\right| dt \tag{81}$$

$$\leq \left\|D_{R_1+}^{\alpha} g\right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]} \frac{(s-R_1)^{\alpha}}{\Gamma(\alpha+1)}, \tag{82}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly we obtain by (80) that

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_{s}^{R_2} (t-s)^{\alpha-1} \left|\left(D_{R_2-}^{\alpha} g\right)(t)\right| dt \tag{83}$$

$$\leq \left\|D_{R_2-}^{\alpha} g\right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \frac{(R_2-s)^{\alpha}}{\Gamma(\alpha+1)}, \tag{84}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Next we observe that

$$\left|\int_A f(y)\, dy\right| \leq \int_A |f(y)|\, dy \overset{(76)}{=} \tag{85}$$

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds\right) d\omega = \left(\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds\right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \tag{86}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{\int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds\right\} \overset{(by\ (82)\ and\ (84))}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{\left\|D_{R_1+}^{\alpha} g\right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]} \int_{R_1}^{\frac{R_1+R_2}{2}} (s-R_1)^{\alpha} s^{N-1} ds\right.$$

$$\left. + \left\|D_{R_2-}^{\alpha} g\right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \int_{\frac{R_1+R_2}{2}}^{R_2} (R_2-s)^{\alpha} s^{N-1} ds\right\} = \tag{87}$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \left\{\left\|D_{R_1+}^{\alpha} g\right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]}\right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right) +$$

$$\left.\left\|D_{R_2-}^{\alpha} g\right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \left[\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right]\right\}. \tag{88}$$

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We have proved that

$$\left|\int_A f(y)\,dy\right| \le \int_A |f(y)|\,dy \le \frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right.$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right]\right\}. \tag{89}$$

Consider now $f_*:\overline{A}\to\mathbb{R}$ *be radial such that* $f_*(x)=g_*(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, *where*

$$g_*(s)=\begin{cases}(s-R_1)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ (R_2-s)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right],\ \alpha>0.\end{cases} \tag{90}$$

We have, as in [4], that

$$\left\|D^{\alpha}_{R_1+}g_*\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}=\Gamma(\alpha+1),\quad \textit{and}\quad \left\|D^{\alpha}_{R_2-}g_*\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}=\Gamma(\alpha+1). \tag{91}$$

Hence

$$\textit{R.H.S. (89) (applied on } g_*)=\frac{\Gamma(\alpha+1)\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\sum_{k=0}^{N-1}\left(1+(-1)^{N+k-1}\right)\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right\}. \tag{92}$$

Furthermore we find

$$\textit{L.H.S. (89) (applied on } f_*)=\int_A f_*(y)\,dy=$$

$$\left(\int_{R_1}^{R_2}g_*(s)\,s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}(s-R_1)^{\alpha}s^{N-1}ds+\int_{\frac{R_1+R_2}{2}}^{R_2}(R_2-s)^{\alpha}s^{N-1}ds\right\}= \tag{93}$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right. \tag{94}$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}=$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\sum_{k=0}^{N-1}\left((-1)^{N+k-1}+1\right)\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right\}. \tag{95}$$

So that we find

$$\text{R.H.S. (89) (applied on } g_*) = \text{L.H.S. (89) (applied on } f_*), \tag{96}$$

proving sharpness of (89).

We have proved the following

Theorem 21 *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, $\alpha>0$, $m=[\alpha]$. We assume that $g\in C([R_1,R_2])$, such that $g\in C^{\alpha}_{R_1+}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $g\in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$,with $g^{(k)}(R_1)=g^{(k)}(R_2)=0$, $k=0,1,...,m-1$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_A f(y)\,dy\right|\leq\int_A|f(y)|\,dy\leq\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right. \tag{97}$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}.$$

Inequalities (97) are sharp, namely they are attained by the radial function $f_*:\overline{A}\to\mathbb{R}$ such that $f_*(x)=g_*(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, where

$$g_*(s)=\begin{cases}(s-R_1)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ (R_2-s)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right].\end{cases} \tag{98}$$

We continue with

Remark 22 *Here $\alpha\geq1$. By (81) we get*

$$|g(s)|\leq\frac{(s-R_1)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_1+}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}, \tag{99}$$

for any $s\in\left[R_1,\frac{R_1+R_2}{2}\right]$.

And by (83) we derive

$$|g(s)|\leq\frac{(R_2-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_2-}g\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}, \tag{100}$$

for any $s\in\left[\frac{R_1+R_2}{2},R_2\right]$.

Hence

$$\int_A |f(y)|\, dy \overset{(86)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (99) and (100))}}{\leq}$$

$$(101)$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s-R_1)^{\alpha-1} s^{N-1} ds \right) + \right.$$

$$(102)$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2-s)^{\alpha-1} s^{N-1} ds \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \tag{103}$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\}.$$

We have proved that

Theorem 23 *All terms and assumptions here as in Theorem 21, but with $\alpha \geq 1$. Then*

$$\int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\}. \tag{104}$$

We continue with

Remark 24 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \frac{1}{q}$. By (81) we get*

$$|g(s)| \leq \frac{(s-R_1)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \tag{105}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly by (83) we derive

$$|g(s)| \leq \frac{(R_2 - s)^{\alpha - 1 + \frac{1}{p}}}{\Gamma(\alpha)(p(\alpha - 1) + 1)^{\frac{1}{p}}} \left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1 + R_2}{2}, R_2\right]\right)}, \tag{106}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Hence

$$\int_A |f(y)|\, dy =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s - R_1)^{\alpha - 1 + \frac{1}{p}} s^{N-1} ds \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2 - s)^{\alpha - 1 + \frac{1}{p}} s^{N-1} ds \right) \right\} = \tag{107}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\frac{(N-1)!\,\Gamma\left(\alpha + \frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}} \right) \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha-\frac{1}{q}}}{k!\,\Gamma\left(N + \alpha + \frac{1}{p} - k\right)} \right) + \tag{108}$$

$$\left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\frac{(N-1)!\,\Gamma\left(\alpha + \frac{1}{p}\right)}{2^{\alpha+N-\frac{1}{q}}} \right) \cdot$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k-\frac{1}{q}}}{k!\,\Gamma\left(\alpha + \frac{1}{p} + N - k\right)} \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!\,\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\, 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\,\Gamma\left(N + \alpha + \frac{1}{p} - k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\,\Gamma\left(\alpha + N + \frac{1}{p} - k\right)} \right) \right\}. \tag{109}$$

We have proved

Theorem 25 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. All terms and assumptions as in Theorem 21. Then*

$$\int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)! \Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k! \Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}. \quad (110)$$

Combining Theorems 21, 23, 25 we derive

Theorem 26 *Let any $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. And let $f : \overline{A} \to \mathbb{R}$ be radial; that is, there exists a function g such that $f(x) = g(s)$, $s = |x|$, $s \in [R_1, R_2]$, $\forall\, x \in \overline{A}$; $\alpha \geq 1$, $m = [\alpha]$. We assume that $g \in C([R_1, R_2])$, such that $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. Then*

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \min \left\{ \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot \right.$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k! \Gamma(N-k+\alpha+1)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{\infty, \left[\frac{R_1+R_2}{2}, R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k! \Gamma(\alpha+1+N-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) +$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) \right\},$$

$$\frac{\pi^{\frac{N}{2}} (N-1)! \Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right) \Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-}g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\} \Bigg\}. \quad (111)$$

The corresponding estimate on the average follows

Corollary 27 *Let all terms and assumptions as in Theorem 26. Then*

$$\left| \frac{1}{Vol(A)} \int_A f(y)\,dy \right| \le \frac{1}{Vol(A)} \int_A |f(y)|\,dy \le \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot$$

$$\min \left\{ \left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right) \right. \right.$$

$$\left. + \left\| D^{\alpha}_{R_2-}g \right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \right\},$$

$$2 \left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right.$$

$$\left. + \left\| D^{\alpha}_{R_2-}g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\},$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right) 2^{\frac{1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+}g \right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-}g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\} \Bigg\}. \quad (112)$$

We need

Definition 28 *(see [1], p. 287) Let $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, and A is a spherical shell. Assume that there exists $\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} \in C\left(\overline{A}\right)$, given by*

$$\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} := \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \quad (113)$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in [R_1, R_2]$, and $\omega \in S^{N-1}$.

We call $\frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}}$ the left radial generalised fractional derivative of f of order α.

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We also need to introduce

Definition 29 *Let $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, and A is a spherical shell. Assume that there exists $\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} \in C\left(\overline{A}\right)$, given by*

$$\frac{\partial^\alpha_{R_2-} f(x)}{\partial r^\alpha} := (-1)^{m-1} \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_r^{R_2} (t-r)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \tag{114}$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in [R_1, R_2]$, and $\omega \in S^{N-1}$.

We call $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha}$ the right radial generalised fractional derivative of f of order α.

We present

Theorem 30 *Let the sperical shells $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$; $A_1 := B\left(0, \frac{R_1+R_2}{2}\right) - \overline{B(0, R_1)}$, $A_2 := B(0, R_2) - \overline{B\left(0, \frac{R_1+R_2}{2}\right)}$. Let $f \in C\left(\overline{A}\right)$, not necessarily radial, $\alpha > 0$, $m = [\alpha]$. Assume that $\frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \in C\left(\overline{A_1}\right)$, $\frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \in C\left(\overline{A_2}\right)$. For each $\omega \in S^{N-1}$, we assume further that $f(\cdot\omega) \in C^\alpha_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in C^\alpha_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. When $0 < \alpha < 1$ the last boundary conditions are void. Then*

(i)

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}, \tag{115}$$

and

(ii)

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \leq \frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot \tag{116}$$

$$\left\{ \left\| \frac{\partial^\alpha_{R_1+} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^\alpha_{R_2-} f}{\partial r^\alpha} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}.$$

Proof. By (86)-(88) we get

$$\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right) \left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{117}$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]} \left[\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right] \right\}.$$

For fixed $\omega \in S^{N-1}$, $f(\cdot\omega)$ sets like a radial function on $\overline{A}$. Thus plugging $f(\cdot\omega)$ into (117), we get

$$\int_{R_1}^{R_2} |f(s\omega)|\, s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right) \left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{118}$$

$$\left\{ \left\| \frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\} =: \gamma_1.$$

Therefore by (76) and (118) we derive

$$\int_A |f(y)|\, dy = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} |f(s\omega)|\, s^{N-1} ds \right) d\omega \leq$$

$$\gamma_1 \int_{S^{N-1}} d\omega = \gamma_1 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}}\right) \cdot \tag{119}$$

$$\left\{ \left\| \frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}} \right\|_{\infty,\overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\},$$

proving the claims of the theorem. ∎

We give also

Theorem 31 *Let $f \in C\left(\overline{A}\right)$, not necessarily radial, $\alpha \geq 1$, $m = [\alpha]$. For each $\omega \in S^{N-1}$, we assume that $f(\cdot\omega) \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in$*

$C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. *We further assume*

$$\left\| \frac{\partial^{\alpha}_{R_1+} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \left\| \frac{\partial^{\alpha}_{R_2-} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \leq \Psi_1, \tag{120}$$

for every $\omega \in S^{N-1}$, *where* $\Psi_1 > 0$.

Then

(i)

$$\int_A |f(y)|\, dy \leq \frac{\Psi_1 \pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \cdot \tag{121}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\},$$

and

(ii)

$$\frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \frac{\Psi_1 N!}{2^{\alpha+N-1}\left(R_2^N - R_1^N\right)} \cdot \tag{122}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) + \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \right\}.$$

Proof. Similar to Theorem 30, using (101)-(103). ■

We finish with

Theorem 32 *Let* $f \in C\left(\overline{A}\right)$, *not necessarily radial*, $\alpha > \frac{1}{q}$, *where* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $m = [\alpha]$. *For each* $\omega \in S^{N-1}$, *we assume that* $f(\cdot\omega) \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ *and* $f(\cdot\omega) \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, *with* $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. *When* $\frac{1}{q} < \alpha < 1$ *the last boundary conditions is void. We further assume*

$$\left\| \frac{\partial^{\alpha}_{R_1+} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \left\| \frac{\partial^{\alpha}_{R_2-} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \leq \Psi_2, \tag{123}$$

for every $\omega \in S^{N-1}$, *where* $\Psi_2 > 0$.

Then

(i)

$$\int_A |f(y)|\,dy \leq \frac{\Psi_2 \pi^{\frac{N}{2}} (N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}} \cdot \tag{124}$$

$$\left\{\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+\right.$$

$$\left.\left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\right\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A |f(y)|\,dy \leq \frac{N!\Gamma\left(\alpha+\frac{1}{p}\right)\Psi_2}{2^{\alpha+N-\frac{1}{q}}\left(R_2^N-R_1^N\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \tag{125}$$

$$\left\{\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+\right.$$

$$\left.\left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\right\}.$$

Proof. Similar to Theorem 30, using (107)-(109). ■

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The Reduction Method in Fractional Calculus and Fractional Ostrowski type inequalities

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Abstract

Here we study generalised fractional integrals and fractional derivatives. We present the reduction method of Fractional Calculus and we reduce them to basic fractional integrals and fractional derivatives. We give a series of generalised Ostrowski type fractional inequalities involving s–convexity. We apply all of the above to Hadamard and Erdélyi-Kober fractional integrals and fractional derivatives. We produce also important generalised fractional Taylor formulae.

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1 The Reduction Method in Fractional Calculus

We use a lot here the following generalised fractional integrals.

Definition 1 *(see also [8, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to given function g are defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a, b])$. We set

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt, \quad x \geq a, \tag{1}$$

clearly $\left(I_{a+;g}^{\alpha} f\right)(a) = 0$, *and*

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)f(t)\,dt,\quad x\leq b, \tag{2}$$

clearly $\left(I_{b-;g}^{\alpha}f\right)(b)=0.$

When g is the identity function id, we get that $I_{a+;id}^{\alpha}=I_{a+}^{\alpha}$ and $I_{b-;id}^{\alpha}=I_{b-}^{\alpha}$ the ordinary left and right Riemann-Liouville fractional integrals, where

$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt,\quad x\geq a, \tag{3}$$

$\left(I_{a+}^{\alpha}f\right)(a)=0,$ *and*

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt,\quad x\leq b, \tag{4}$$

$\left(I_{b-}^{\alpha}f\right)(b)=0.$

We need

Lemma 2 *Let $g\in AC([a,b])$ which is strictly increasing and $f\in L_{\infty}([a,b])$. Then*

$$\|f\|_{\infty,[a,b]}\geq\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}, \tag{5}$$

i.e. $\left(f\circ g^{-1}\right)\in L_{\infty}([g(a),g(b)])$.

If additionally $g^{-1}\in AC([g(a),g(b)])$ then

$$\|f\|_{\infty,[a,b]}=\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}. \tag{6}$$

Proof. Here m stands for the Lebesgue measure. By definition we have

$$\|f\|_{\infty,[a,b]}=ess\sup|f(t)|$$

$$=\inf\left\{M:m\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=m\{t:|f(t)|>M\}=0\right\}. \tag{7}$$

Furthermore we have

$$\left\|f\circ g^{-1}\right\|_{\infty,[g(a),g(b)]}=\inf\left\{L:m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>L\right\}=0\right\}. \tag{8}$$

Because g is absolutely continuous and strictly increasing function on $[a,b]$, by [9, p. 108] exercise 14, we get that

$$m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=m\left(g\left(\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}\right)\right)=0, \tag{9}$$

given that $m\left\{t:\left|\left(f\circ g^{-1}\right)(g(t))\right|>M\right\}=0.$

Therefore each M of (7) fulfills $M\in\left\{L:m\left\{g(t):\left|\left(f\circ g^{-1}\right)(g(t))\right|>L\right\}=0\right\}.$

The last implies (5). Similarly arguing reverse we derive (6). ■

We use (5) in the next

Remark 3 *We observe that*

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(g(x)-g(t))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt=$$

(by change of variable for Lebesgue integrals)

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad x\geq a, \tag{11}$$

equivalently $g(x)\geq g(a)$.

That is in the terms and assumtions of Definition 1 we get

$$\left(I_{a+;g}^{\alpha}f\right)(x)=\left(I_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad \text{for } x\geq a. \tag{12}$$

Similarly we observe that

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt$$

$$=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)), \tag{13}$$

for $x\leq b$.

That is

$$\left(I_{b-;g}^{\alpha}f\right)(x)=\left(I_{g(b)-}^{\alpha}\left(f\circ g^{-1}\right)\right)(g(x)),\quad \text{for } x\leq b. \tag{14}$$

So by (12) and (14) we have reduced the general fractional integrals to the ordinary left and right Riemann-Liouville fractional integrals.

We need

Definition 4 *([7]) Let* $0<a<b<\infty$, $\alpha>0$. *The left and right Hadamard fractional integrals of order* α *are given by*

$$\left(J_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left(\ln\frac{x}{y}\right)^{\alpha-1}\frac{f(y)}{y}dy,\quad x\geq a, \tag{15}$$

and

$$\left(J_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left(\ln\frac{y}{x}\right)^{\alpha-1}\frac{f(y)}{y}dy,\quad x\leq b, \tag{16}$$

respectively.

Here we take $f\in L_{\infty}([a,b])$.

Comparing to Definition 1 we have $g(x) = \ln x$ on $[a,b]$, $0 < a < b < \infty$. Comparing to (12) and (14) we get

$$\left(J_{a+}^{\alpha} f\right)(x) = \left(I_{(\ln a)+}^{\alpha} (f \circ \exp)\right)(\ln x), \quad \text{for } x \geq a, \tag{17}$$

and

$$\left(J_{b-}^{\alpha} f\right)(x) = \left(I_{(\ln b)-}^{\alpha} (f \circ \exp)\right)(\ln x), \quad \text{for } x \leq b. \tag{18}$$

We also consider

Definition 5 *Let $0 < a < b < \infty$; $\alpha, \sigma > 0$ and $\eta > -1$. Let $f \in L_\infty([a,b])$. We mention here the left and right Erdélyi-Kober type fractional integrals, respectively: as in [10] we define*

$$\left(I_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \quad x \geq a, \tag{19}$$

and similarly we also define

$$\left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_x^b (t^\sigma - x^\sigma)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \quad x \leq b. \tag{20}$$

Remark 6 *(following Definition 5) The above give rise to the following generalised weighted left and right fractional integrals.*

We set

$$\left(K_{a+;\sigma,\eta}^{\alpha} f\right)(x) = x^{\sigma(\alpha+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \tag{21}$$

$$\frac{\sigma}{\Gamma(\alpha)} \int_a^x (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (x^\sigma - t^\sigma)^{\alpha-1} \left(t^{\sigma\eta} f(t)\right) \sigma t^{\sigma-1} dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (x^\sigma - t^\sigma)^{\alpha-1} \left(t^{\sigma\eta} f(t)\right) dt^\sigma =$$

(setting $z = t^\sigma$)

$$\frac{1}{\Gamma(\alpha)} \int_{a^\sigma}^{x^\sigma} (x^\sigma - z)^{\alpha-1} \left(z^\eta f\left(z^{\frac{1}{\sigma}}\right)\right) dz = \left(I_{a^\sigma+}^{\alpha} \left(z^\eta f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^\sigma), \quad x \geq a, \tag{22}$$

that is

$$\left(K_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \left(I_{a^\sigma+}^{\alpha} \left(z^\eta f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^\sigma), \quad x \geq a. \tag{23}$$

Similarly we put

$$\left(K_{b-;\sigma,\eta}^{\alpha} f\right)(x) = x^{\sigma(\alpha+\eta)} \left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) =$$

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$$\frac{1}{\Gamma(\alpha)}\int_{x^{\sigma}}^{b^{\sigma}}(z-x^{\sigma})^{\alpha-1}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)dz=\left(I_{b^{\sigma}-}^{\alpha}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}),\quad x\leq b, \tag{24}$$

that is

$$\left(K_{b-;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{b^{\sigma}-}^{\alpha}\left(z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}),\qquad x\leq b. \tag{25}$$

Comparing to Definition 1 here, we have that $g(x)=x^{\sigma}\in C^{1}([a,b])$, *thus* $x^{\sigma}\in AC([a,b])$ *and it is strictly increasing. Clearly* $g^{-1}(z)=z^{\frac{1}{\sigma}}$, $z\in[a^{\sigma},b^{\sigma}]$. *We set* $F(t)=t^{\sigma\eta}f(t)$, $t\in[a,b]$. *Clearly we have* $F\in L_{\infty}([a,b])$. *Notice that* $F\circ g^{-1}=F\circ(id)^{\frac{1}{\sigma}}$, *and* $F(t)=\left(F\circ g^{-1}\right)(g(t))=\left(F\circ g^{-1}\right)(z)=z^{\eta}f\left(z^{\frac{1}{\sigma}}\right)$. *Thus a formal description of (23) and (25) follows.*

We have

$$\left(K_{a+;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{a^{\sigma}+}^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}),\qquad x\geq a, \tag{26}$$

and

$$\left(K_{b-;\sigma,\eta}^{\alpha}f\right)(x)=\left(I_{b^{\sigma}-}^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}),\qquad x\leq b, \tag{27}$$

where $F(x)=x^{\sigma\eta}f(x)$, $x\in[a,b]$.

We introduce

Definition 7 *Let* $\alpha>0$, $m=[\alpha]$, $\beta=\alpha-m$, $0<\beta<1$, $f\in C([a,b])$, $[a,b]\subset\mathbb{R}$; $g\in AC([a,b])$, *g is strictly increasing. We define the subspace* $C_{a+;g}^{\alpha}([a,b])$ *of* $C^{m}([a,b])$:

$$C_{a+;g}^{\alpha}([a,b])=\left\{f\in C^{m}([a,b]):\left(I_{a+;g}^{1-\beta}f^{(m)}\right)\in C^{1}([a,b])\right\}. \tag{28}$$

Denote $C_{a+}^{\alpha}=C_{a+;id}^{\alpha}$.

For $f\in C_{a+;g}^{\alpha}([a,b])$, *we define the left g-generalised* α*-fractional derivative of f over* $[a,b]$ *as*

$$D_{a+;g}^{\alpha}(f)=\left(I_{a+;g}^{1-\beta}f^{(m)}\right)'. \tag{29}$$

When $g=id$, *we denote*

$$D_{a+}^{\alpha}f=\left(I_{a+}^{1-\beta}f^{(m)}\right)', \tag{30}$$

called the left generalized α*-fractional derivative of f over* $[a,b]$, *see [4], [2], p. 24.*

We also introduce

Definition 8 *Let* $\alpha>0$, $m=[\alpha]$, $\beta=\alpha-m$, $f\in C([a,b])$, $[a,b]\subset\mathbb{R}$; $g\in AC([a,b])$, *g is strictly increasing. We define the subspace* $C_{b-;g}^{\alpha}([a,b])$ *of* $C^{m}([a,b])$:

$$C_{b-;g}^{\alpha}([a,b])=\left\{f\in C^{m}([a,b]):\left(I_{b-;g}^{1-\beta}f^{(m)}\right)\in C^{1}([a,b])\right\}. \tag{31}$$

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Denote $C^{\alpha}_{b-} = C^{\alpha}_{b-;id}$.

For $f \in C^{\alpha}_{b-;g}([a,b])$, *we define the right* g*-generalised* α*-fractional derivative of* f *over* $[a,b]$ *as*

$$D^{\alpha}_{b-;g}(f) = (-1)^{m-1}\left(I^{1-\beta}_{b-;g} f^{(m)}\right)'. \tag{32}$$

When $g = id$, *we denote*

$$D^{\alpha}_{b-} f = (-1)^{m-1}\left(I^{1-\beta}_{b-} f^{(m)}\right)', \tag{33}$$

called the right generalized α*-fractional derivative of* f *over* $[a,b]$, *see [3].*

Regarding fractional derivatives in this article from now on we consider only $0 < \alpha < 1$, i.e. $m = 0$ and $\beta = \alpha$.

So in this case we get

$$\left(D^{\alpha}_{a+;g} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (g(x) - g(t))^{-\alpha} g'(t) f(t)\, dt, \tag{34}$$

$$\left(D^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t)\, dt, \tag{35}$$

and

$$\left(D^{\alpha}_{b-;g} f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (g(t) - g(x))^{-\alpha} g'(t) f(t)\, dt, \tag{36}$$

$$\left(D^{\alpha}_{b-} f\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} f(t)\, dt, \tag{37}$$

for any $x \in [a,b]$.

We mention the following fractional Taylor formulae.

Theorem 9 *1) (see [2], pp. 8-10, [4]) Let* $f \in C^{\alpha}_{a+}([a,b])$, $0 < \alpha < 1$. *Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D^{\alpha}_{a+} f\right)(t)\, dt = \left(I^{\alpha}_{a+}\left(D^{\alpha}_{a+} f\right)\right)(x), \quad x \in [a,b]. \tag{38}$$

2) (see [3]) Let $f \in C^{\alpha}_{b-}([a,b])$, $0 < \alpha < 1$. *Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D^{\alpha}_{b-} f\right)(t)\, dt = \left(I^{\alpha}_{b-}\left(D^{\alpha}_{b-} f\right)\right)(x), \quad x \in [a,b]. \tag{39}$$

We make

Remark 10 *Here $0 < \alpha < 1$ and $g \in C^1([a,b])$, g is strictly increasing. Furthermore we assume that $\left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))$ exists. By (12) we have*

$$\left(I^{1-\alpha}_{a+;g} f\right)(x) = \left(I^{1-\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x)), \quad x \in [a,b]. \tag{40}$$

Hence there exists

$$\begin{aligned}\left(D^{\alpha}_{a+;g}(f)\right)(x) &= \left(I^{1-\alpha}_{a+;g} f\right)'(x) \overset{(40)}{=} \left(I^{1-\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)'(g(x))\, g'(x) \\ &= \left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))\, g'(x), \quad x \in [a,b]. \end{aligned} \tag{41}$$

We have established that there exists

$$\left(D^{\alpha}_{a+;g}(f)\right)(x) = \left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))\, g'(x), \quad x \in [a,b], \ f \in C([a,b]). \tag{42}$$

Next we assume that there exists $\left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))$. By (14) we get

$$\left(I^{1-\alpha}_{b-;g} f\right)(x) = \left(I^{1-\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x)), \quad x \in [a,b]. \tag{43}$$

Hence there exists

$$\begin{aligned}\left(D^{\alpha}_{b-;g}(f)\right)(x) &= -\left(I^{1-\alpha}_{b-;g} f\right)'(x) \overset{(43)}{=} -\left(I^{1-\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)'(g(x))\, g'(x) \\ &= \left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))\, g'(x), \quad x \in [a,b]. \end{aligned} \tag{44}$$

We have proved that there exists

$$\left(D^{\alpha}_{b-;g}(f)\right)(x) = \left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))\, g'(x), \quad x \in [a,b], \ f \in C([a,b]). \tag{45}$$

Next we apply (42) and (45).

We make

Remark 11 *(all as in Definition 4) We introduce the following Hadamard type fractional derivatives, see (46), (47). Here $f \in C([a,b])$. Let $0 < \alpha < 1$, and that $\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x)$ exists for $x \in [a,b]$, $a > 0$.*

Then by (42), we get

$$\left(D^{\alpha}_{a+;\ln}(f)\right)(x) = \frac{\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x)}{x}, \quad x \in [a,b]. \tag{46}$$

Assume next that $\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)(\ln x)$ exists for $x \in [a,b]$.

Then by (45), we find

$$\left(D^{\alpha}_{b-;\ln}(f)\right)(x) = \frac{\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)(\ln x)}{x}, \quad x \in [a,b]. \tag{47}$$

We make

Remark 12 *(refer to Definition 5, Remark 6) Let* $0 < \alpha < 1$*. By (26) we get*

$$\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x) = \left(I^{1-\alpha}_{a^{\sigma}+}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}), \quad x \in [a,b]. \tag{48}$$

And by (27)

$$\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x) = \left(I^{1-\alpha}_{b^{\sigma}-}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma}), \quad x \in [a,b]. \tag{49}$$

Above $F(x) = x^{\sigma\eta} f(x)$, $x \in [a,b]$.

Assume that

$$\frac{d\left(I^{1-\alpha}_{a^{\sigma}+}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})}{dx^{\sigma}} \tag{50}$$

and

$$\frac{d\left(I^{1-\alpha}_{b^{\sigma}-}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})}{dx^{\sigma}} \tag{51}$$

exist and are continuous in $x^{\sigma} \in [a^{\sigma}, b^{\sigma}]$, $f \in C([a,b])$.

Then

$$\frac{d\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x)}{dx} = \frac{d\left(I^{1-\alpha}_{a^{\sigma}+}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})}{dx^{\sigma}} \sigma x^{\sigma-1}, \tag{52}$$

and

$$\frac{d\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x)}{dx} = \frac{d\left(I^{1-\alpha}_{b^{\sigma}-}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})}{dx^{\sigma}} \sigma x^{\sigma-1}, \tag{53}$$

exist and are continuous in $x \in [a,b]$.

So we introduce the modified Erdélyi-Kober type left and right fractional derivatives of $f \in C([a,b])$*, as follows:*

$$\left(D^{\alpha}_{a+:\sigma,\eta} f\right)(x) = \frac{d\left(K^{1-\alpha}_{a+;\sigma,\eta} f\right)(x)}{dx}, \tag{54}$$

and

$$\left(D^{\alpha}_{b-:\sigma,\eta} f\right)(x) = -\frac{d\left(K^{1-\alpha}_{b-;\sigma,\eta} f\right)(x)}{dx}, \tag{55}$$

$x \in [a,b]$, $0 < \alpha < 1$.

That is, it holds

$$\left(D^{\alpha}_{a+:\sigma,\eta} f\right)(x) = \left(D^{\alpha}_{a^{\sigma}+}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})\, \sigma x^{\sigma-1}, \tag{56}$$

and

$$\left(D^{\alpha}_{b-:\sigma,\eta} f\right)(x) = \left(D^{\alpha}_{b^{\sigma}-}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})\, \sigma x^{\sigma-1}, \tag{57}$$

$x \in [a,b]$, $0 < \alpha < 1$, $a > 0$.

We make

Remark 13 *(continuation of Remark 11) Hence $f \in C([a,b])$. By (46) we get*

$$\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x) = x\left(D^{\alpha}_{a+;\ln}(f)\right)(x) = e^{\ln x}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{\ln x}\right), \tag{58}$$

$x \in [a,b]$.

Hence by (38) we obtain

$$f(x) = f\left(e^{\ln x}\right) = \left(I^{\alpha}_{(\ln a)+}\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)\right)(\ln x) = \tag{59}$$

$$\left(I^{\alpha}_{(\ln a)+}\left(e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln x}(\ln x - t)^{\alpha-1} e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)dt, \tag{60}$$

$x \in [a,b]$.

By (47) we have

$$\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)(\ln x) = x\left(D^{\alpha}_{b-;\ln}(f)\right)(x) = e^{\ln x}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{\ln x}\right), \tag{61}$$

$x \in [a,b]$.

Hence by (39) we obtain

$$f(x) = f\left(e^{\ln x}\right) = \left(I^{\alpha}_{(\ln b)-}\left(D^{\alpha}_{(\ln b)-}(f \circ \exp)\right)\right)(\ln x) =$$

$$\left(I^{\alpha}_{(\ln b)-}\left(e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln x}^{\ln b}(t - \ln x)^{\alpha-1} e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)dt, \tag{62}$$

$x \in [a,b]$.

We have proved the following Taylor Hadamard type fractional formulae.

Theorem 14 *Let $0 < \alpha < 1$, and all as in Definition 4, $f \in C([a,b])$, $a > 0$.*

1) Assume that $\left(D^{\alpha}_{(\ln a)+}(f \circ \exp)\right)(\ln x)$ exists and it is continuous, $x \in [a,b]$. Then

$$f(x) = \left(I^{\alpha}_{(\ln a)+}\left(e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x) =$$

$$\frac{1}{\Gamma(\alpha)}\int_{\ln a}^{\ln x}(\ln x - t)^{\alpha-1} e^{t}\left(D^{\alpha}_{a+;\ln}(f)\right)\left(e^{t}\right)dt, \tag{63}$$

$x \in [a,b]$.

2) Assume that $\left(D^{\alpha}_{(\ln b)-}(f\circ\exp)\right)(\ln x)$ *exists and it is continuous,* $x\in[a,b]$*. Then*

$$f(x)=\left(I^{\alpha}_{(\ln b)-}\left(e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)\right)\right)(\ln x)$$

$$=\frac{1}{\Gamma(\alpha)}\int_{\ln x}^{\ln b}(t-\ln x)^{\alpha-1}e^{t}\left(D^{\alpha}_{b-;\ln}(f)\right)\left(e^{t}\right)dt, \tag{64}$$

$x\in[a,b]$.

We make

Remark 15 *(continuation of Remark 12) By (56) and (57) we get*

$$\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})=\frac{x^{1-\sigma}}{\sigma}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)(x)$$

$$=\frac{(x^{\sigma})^{\left(\frac{1}{\sigma}-1\right)}}{\sigma}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left((x^{\sigma})^{\frac{1}{\sigma}}\right), \tag{65}$$

and

$$\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})=\frac{x^{1-\sigma}}{\sigma}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)(x)$$

$$=\frac{(x^{\sigma})^{\left(\frac{1}{\sigma}-1\right)}}{\sigma}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left((x^{\sigma})^{\frac{1}{\sigma}}\right), \tag{66}$$

$x\in[a,b]$, $0<\alpha<1$, $f\in C([a,b])$*. Above assume* $\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$, $\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)(x^{\sigma})$ *exist and are continuous in* $x^{\sigma}\in[a^{\sigma},b^{\sigma}]$.

Hence, by (38) it holds

$$x^{\sigma\eta}f(x)=\left(F\circ(id)^{\frac{1}{\sigma}}\right)(x^{\sigma})=\left(I^{\alpha}_{a^{\sigma}+}\left(D^{\alpha}_{a^{\sigma}+}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$=\frac{1}{\sigma}\left(I^{\alpha}_{a^{\sigma}+}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}) \tag{67}$$

$$=\frac{1}{\sigma\Gamma(\alpha)}\int_{a^{\sigma}}^{x^{\sigma}}(x^{\sigma}-t)^{\alpha-1}t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{a+;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)dt,\quad x\in[a,b]. \tag{68}$$

Similarly, by (39) we derive

$$x^{\sigma\eta}f(x)=\left(F\circ(id)^{\frac{1}{\sigma}}\right)(x^{\sigma})=\left(I^{\alpha}_{b^{\sigma}-}\left(D^{\alpha}_{b^{\sigma}-}\left(F\circ(id)^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma}) \tag{69}$$

$$=\frac{1}{\sigma}\left(I^{\alpha}_{b^{\sigma}-}\left(t^{\left(\frac{1}{\alpha}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^{\sigma})$$

$$=\frac{1}{\sigma\Gamma(\alpha)}\int_{x^{\sigma}}^{b^{\sigma}}(t-x^{\sigma})^{\alpha-1}t^{\left(\frac{1}{\sigma}-1\right)}\left(D^{\alpha}_{b-;\sigma,\eta}f\right)\left(t^{\frac{1}{\sigma}}\right)dt,\quad x\in[a,b]. \tag{70}$$

We give the following Taylor Erdélyi-Kober type fractional formulae.

Theorem 16 *Let $0 < \alpha < 1$, all as in Definition 5, (21), (24), (54), (55), $f \in C([a,b])$, $a > 0$; $F(x) = x^{\sigma\eta} f(x)$, $x \in [a,b]$.*

1) Assume that $\left(D_{a^\sigma+}^{\alpha}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)$ exists and it is continuous in $x^\sigma \in [a^\sigma, b^\sigma]$. Then

$$f(x) = \frac{x^{-\sigma\eta}}{\sigma}\left(I_{a^\sigma+}^{\alpha}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D_{a+;\sigma,\eta}^{\alpha} f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^\sigma)$$

$$= \frac{x^{-\sigma\eta}}{\sigma\Gamma(\alpha)}\int_{a^\sigma}^{x^\sigma}(x^\sigma - t)^{\alpha-1} t^{\left(\frac{1}{\sigma}-1\right)}\left(D_{a+;\sigma,\eta}^{\alpha} f\right)\left(t^{\frac{1}{\sigma}}\right)dt, \quad x \in [a,b]. \tag{71}$$

2) Assume that $\left(D_{b^\sigma-}^{\alpha}\left(F \circ (id)^{\frac{1}{\sigma}}\right)\right)(x^\sigma)$ exists and it is continuous in $x^\sigma \in [a^\sigma, b^\sigma]$. Then

$$f(x) = \frac{x^{-\sigma\eta}}{\sigma}\left(I_{b^\sigma-}^{\alpha}\left(t^{\left(\frac{1}{\sigma}-1\right)}\left(D_{b-;\sigma,\eta}^{\alpha} f\right)\left(t^{\frac{1}{\sigma}}\right)\right)\right)(x^\sigma)$$

$$= \frac{x^{-\sigma\eta}}{\sigma\Gamma(\alpha)}\int_{x^\sigma}^{b^\sigma}(t - x^\sigma)^{\alpha-1} t^{\left(\frac{1}{\sigma}-1\right)}\left(D_{b-;\sigma,\eta}^{\alpha} f\right)\left(t^{\frac{1}{\sigma}}\right)dt, \quad x \in [a,b]. \tag{72}$$

2 Fractional Ostrowski type Inequalities

We need the following

Lemma 17 *([11]) Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L_1([a,b])$, then for all $x \in [a,b]$ and $\alpha > 0$ we have:*

$$\left(\frac{(x-a)^\alpha + (b-x)^\alpha}{b-a}\right)f(x) - \frac{\Gamma(\alpha+1)}{(b-a)}\left[I_{x-}^{\alpha} f(a) + I_{x+}^{\alpha} f(b)\right] =$$

$$\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx + (1-t)a)\,dt - \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 t^\alpha f'(tx + (1-t)b)\,dt. \tag{73}$$

By (73), (12), (14), we obtain

Lemma 18 *Let $f \in C([a,b])$, $g \in C^1([a,b])$, g strictly increasing on $[a,b]$, $f \circ g^{-1}$ differentiable on $(g(a), g(b))$ with $\left(f \circ g^{-1}\right)' \in L_1([g(a), g(b)])$, $x \in [a,b]$, $a < b$, $a, b \in \mathbb{R}$, $\alpha > 0$. Then*

$$\left(\frac{(g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha}{g(b) - g(a)}\right)f(x) -$$

$$\frac{\Gamma(\alpha+1)}{(g(b) - g(a))}\left[\left(I_{x-;g}^{\alpha} f\right)(a) + \left(I_{x+;g}^{\alpha} f\right)(b)\right] =$$

$$\frac{(g(x)-g(a))^{\alpha+1}}{(g(b)-g(a))}\int_0^1 t^{\alpha}\left(f\circ g^{-1}\right)'(tg(x)+(1-t)g(a))\,dt \tag{74}$$
$$-\frac{(g(b)-g(x))^{\alpha+1}}{(g(b)-g(a))}\int_0^1 t^{\alpha}\left(f\circ g^{-1}\right)'(tg(x)+(1-t)g(b))\,dt.$$

We apply (74) for $g(x)=\ln x$, $x\in[a,b]$.

Lemma 19 *Let $0<a<b<\infty$, $\alpha>0$. Let $f\in C([a,b])$, $(f\circ\exp)$ is differentiable on $(\ln a,\ln b)$ with $(f\circ\exp)'\in L_1([\ln a,\ln b])$, $x\in[a,b]$. Then*

$$\left(\frac{\left(\ln\frac{x}{a}\right)^{\alpha}+\left(\ln\frac{b}{x}\right)^{\alpha}}{\ln\frac{b}{a}}\right)f(x)-\frac{\Gamma(\alpha+1)}{\ln\left(\frac{b}{a}\right)}\left[\left(J_{x-}^{\alpha}f\right)(a)+\left(J_{x+}^{\alpha}f\right)(b)\right]=$$

$$\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}}{\ln\frac{b}{a}}\int_0^1 t^{\alpha}(f\circ\exp)'(t\ln x+(1-t)\ln a)\,dt$$
$$-\frac{\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\int_0^1 t^{\alpha}(f\circ\exp)'(t\ln x+(1-t)\ln b)\,dt, \tag{75}$$

where $J_{x\pm}^{\alpha}f$ are the left and right Hadamard fractional integrals of order α anchored at $x\in[a,b]$, see (15), (16).

We apply (74) for $g(x)=x^{\sigma}$, $\sigma>0$, $x\in[a,b]$.

Lemma 20 *Let $0<a<b<\infty$, $\alpha>0$, $f\in C([a,b])$. Assume $\left(F\circ(id)^{\frac{1}{\sigma}}\right)$ is differentiable on (a^{σ},b^{σ}) with $\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\in L_1([a^{\sigma},b^{\sigma}])$, $x\in[a,b]$. Here $F(x)=x^{\sigma\eta}f(x)$, $x\in[a,b]$, $\eta>-1$. Then*

$$\left(\frac{(x^{\sigma}-a^{\sigma})^{\alpha}+(b^{\sigma}-x^{\sigma})^{\alpha}}{b^{\sigma}-a^{\sigma}}\right)x^{\sigma\eta}f(x)-$$

$$\frac{\Gamma(\alpha+1)}{(b^{\sigma}-a^{\sigma})}\left[\left(K_{x-;\sigma,\eta}^{\alpha}f\right)(a)+\left(K_{x+;\sigma,\eta}^{\alpha}f\right)(b)\right]=$$

$$\frac{(x^{\sigma}-a^{\sigma})^{\alpha+1}}{(b^{\sigma}-a^{\sigma})}\int_0^1 t^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(tx^{\sigma}+(1-t)a^{\sigma})\,dt$$
$$-\frac{(b^{\sigma}-x^{\sigma})^{\alpha+1}}{(b^{\sigma}-a^{\sigma})}\int_0^1 t^{\alpha}\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(tx^{\sigma}+(1-t)b^{\sigma})\,dt, \tag{76}$$

where $\left(K_{x\pm;\sigma,\eta}^{\alpha}f\right)$ as in (26), (27).

We need

Definition 21 *([6]) A function $f:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense if*

$$f\left(\lambda x+\left(1-\lambda\right)y\right)\leq\lambda^{s}f\left(x\right)+\left(1-\lambda\right)^{s}f\left(y\right), \tag{77}$$

for all $x,y\in[0,\infty)$, $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

This class of s-convex functions is denoted by K_s^2.

When $s=1$, s-convexity reduces to ordinary convexity.

If "$\geq$" holds in (77), we talk about s-concavity in the second sense.

In our proofs it is used a lot and it is built in the following

Theorem 22 *([5]) Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f'\in L_1([a,b])$, then it holds*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{s+1}, \tag{78}$$

where the constant $\frac{1}{s+1}$ is the best possible in the second inequality.

We are also motivated by the following Ostrowski type inequality in

Theorem 23 *([1]) Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I^0 such that $f'\in L_1([a,b])$, where $a,b\in I$, $a<b$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, for all $x\in[a,b]$, then*

$$\left|f\left(x\right)-\frac{1}{b-a}\int_a^b f\left(t\right)dt\right|\leq\frac{M}{b-a}\left[\frac{\left(x-a\right)^2+\left(b-x\right)^2}{s+1}\right], \tag{79}$$

for each $x\in[a,b]$.

We need

Theorem 24 *([11]) Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$, be a differentiable mapping on (a,b) with $a<b$, such that $f'\in L_1([a,b])$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$ and $|f'(x)|\leq M$, $x\in[a,b]$, $\alpha>0$, then*

$$\Delta_x\left(f\right):=\left|\left(\frac{\left(x-a\right)^{\alpha}+\left(b-x\right)^{\alpha}}{b-a}\right)f\left(x\right)-\frac{\Gamma\left(\alpha+1\right)}{\left(b-a\right)}\left[I_{x-}^{\alpha}f\left(a\right)+I_{x+}^{\alpha}f\left(b\right)\right]\right|$$

$$\leq\frac{M}{b-a}\left(1+\frac{\Gamma\left(\alpha+1\right)\Gamma\left(s+1\right)}{\Gamma\left(\alpha+s+1\right)}\right)\left[\frac{\left(x-a\right)^{\alpha+1}+\left(b-x\right)^{\alpha+1}}{\alpha+s+1}\right]. \tag{80}$$

We give the following general fractional Ostrowski type inequality. The proof comes by Lemma 18 and Theorem 24.

Theorem 25 *Let $f \in C([a,b])$, $g \in C^1([a,b])$, g strictly increasing on $[a,b]$, $f \circ g^{-1}$ differentiable on $(g(a), g(b))$ with $(f \circ g^{-1})' \in L_1([g(a), g(b)])$, $x \in [a,b]$, $a < b$, $a, b \in \mathbb{R}$, $\alpha > 0$. Assume $\left|(f \circ g^{-1})'\right|$ is s-convex in the second sense on $[g(a), g(b)] \subset [0, \infty)$ for some fixed $s \in (0,1]$ and $\left|(f \circ g^{-1})'(g(x))\right| \leq M$, $x \in [a,b]$. Then*

$$\Delta_{g(x)}(f) := \Bigg| \left(\frac{(g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}}{g(b) - g(a)} \right) f(x) -$$
$$\frac{\Gamma(\alpha+1)}{(g(b) - g(a))} \left[\left(I^{\alpha}_{x-;g} f\right)(a) + \left(I^{\alpha}_{x+;g} f\right)(b) \right] \Bigg|$$
$$\leq \frac{M}{(g(b) - g(a))} \left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right) \cdot$$
$$\left[\frac{(g(x) - g(a))^{\alpha+1} + (g(b) - g(x))^{\alpha+1}}{\alpha+s+1} \right]. \tag{81}$$

We need

Theorem 26 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Delta_x(f) \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right]. \tag{82}$$

We apply Theorem 26 and Lemma 18. We give the following fractional Ostrowski type inequality.

Theorem 27 *All as in Theorem 25, however here $\left|(f \circ g^{-1})'\right|^q$ is s-convex in the second sense on $[g(a), g(b)] \subset [0, \infty)$ for some fixed $s \in (0,1]$, $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\Delta_{g(x)}(f) \leq \frac{M}{(1+p\alpha)^{\frac{1}{p}}} \left(\frac{2}{s+1} \right)^{\frac{1}{q}} \left[\frac{(g(x) - g(a))^{\alpha+1} + (g(b) - g(x))^{\alpha+1}}{g(b) - g(a)} \right]. \tag{83}$$

We need

Theorem 28 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, with $q \geq 1$. Then*

$$\Delta_x(f) \leq M \left(\frac{1}{1+\alpha} \right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+s+1} \right)^{\frac{1}{q}} \cdot$$
$$\left(1 + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \right)^{\frac{1}{q}} \left[\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a} \right]. \tag{84}$$

We give with the use of (84) the following

Theorem 29 *Here all as in Theorem 27, however $q \geq 1$, p is not related. Then*

$$\Delta_{g(x)}(f) \leq M \left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}} \cdot$$

$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}} \left[\frac{(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}}{g(b)-g(a)}\right]. \quad (85)$$

We need

Theorem 30 *([11]) All as in Theorem 24, but here $|f'|^q$ is s-concave in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Delta_x(f) \leq \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(b-a)} \cdot$$

$$\left[(x-a)^{\alpha+1}\left|f'\left(\frac{x+a}{2}\right)\right| + (b-x)^{\alpha+1}\left|f'\left(\frac{b+x}{2}\right)\right|\right]. \quad (86)$$

Using (86) we get

Theorem 31 *All as in Theorem 25, but here $\left|\left(f\circ g^{-1}\right)'\right|^q$ is s-concave in the second sense on $[g(a),g(b)] \subset [0,\infty)$ for some fixed $s \in (0,1]$, $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Delta_{g(x)}(f) \leq \frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(g(b)-g(a))} \cdot$$

$$\left[(g(x)-g(a))^{\alpha+1}\left|\left(f\circ g^{-1}\right)'\left(\frac{g(x)+g(a)}{2}\right)\right| + \right. \quad (87)$$

$$\left.(g(b)-g(x))^{\alpha+1}\left|\left(f\circ g^{-1}\right)'\left(\frac{g(b)+g(x)}{2}\right)\right|\right].$$

We make

Remark 32 *Let $0<a<b<\infty$, $\alpha>0$. We have that*

$$\Delta_{\ln x}(f) = \left|\left(\frac{\left(\ln\frac{x}{a}\right)^{\alpha}+\left(\ln\frac{b}{x}\right)^{\alpha}}{\ln\frac{b}{a}}\right) f(x)\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\ln\frac{b}{a}}\left[\left(J_{x-}^{\alpha}f\right)(a)+\left(J_{x+}^{\alpha}f\right)(b)\right]\right|, \quad (88)$$

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where $J_{x\pm}^{\alpha}f$ *are the Hadamard fractional integrals, see (15), (16), and*

$$\Delta_{x^{\sigma}}(f)=\left|\left(\frac{(x^{\sigma}-a^{\sigma})^{\alpha}+(b^{\sigma}-x^{\sigma})^{\alpha}}{b^{\sigma}-a^{\sigma}}\right)x^{\sigma\eta}f(x)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{(b^{\sigma}-a^{\sigma})}\left[\left(K_{x-;\sigma,\eta}^{\alpha}f\right)(a)+\left(K_{x+;\sigma,\eta}^{\alpha}f\right)(b)\right]\right|, \tag{89}$$

where $K_{x\pm;\sigma,\eta}^{\alpha}(f)$ *as in (26), (27), the modified Erdélyi-Kober type fractional integrals, see also (19), (20), (21), and (24), where* $\sigma>0$, $\eta>-1$.

Using Theorem 25 we get

Theorem 33 *Let* $0<a<b<\infty$, $\alpha>0$. *Let* $f\in C([a,b])$, $(f\circ\exp)$ *is differentiable on* $(\ln a,\ln b)$ *with* $(f\circ\exp)'\in L_1([\ln a,\ln b])$, $x\in[a,b]$. *Assume* $\left|(f\circ\exp)'\right|$ *is s-convex in the second sense on* $[\ln a,\ln b]\subset[0,\infty)$ *for some fixed* $s\in(0,1]$ *and* $\left|(f\circ\exp)'(\ln x)\right|\leq M$, $x\in[a,b]$. *Then*

$$\Delta_{\ln x}(f)\leq\frac{M}{\ln\frac{b}{a}}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\cdot$$
$$\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\alpha+s+1}\right]. \tag{90}$$

Using Theorem 27 we derive

Theorem 34 *All as in Theorem 33, but here* $\left|(f\circ\exp)'\right|^{q}$ *is s-convex in the second sense on* $[\ln a,\ln b]\subset[0,\infty)$ *for some fixed* $s\in(0,1]$, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\Delta_{\ln x}(f)\leq\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\right]. \tag{91}$$

Using Theorem 29 we derive

Theorem 35 *All as in Theorem 34, however* $q\geq1$, p *is not related. Then*

$$\Delta_{\ln x}(f)\leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\cdot$$
$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{\left(\ln\frac{x}{a}\right)^{\alpha+1}+\left(\ln\frac{b}{x}\right)^{\alpha+1}}{\ln\frac{b}{a}}\right]. \tag{92}$$

Based on Theorem 31 we produce

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Theorem 36 *All as in Theorem 33, however here* $\left|(f\circ\exp)'\right|^q$ *is s-concave in the second sense on* $[\ln a,\ln b]\subset[0,\infty)$ *for some fixed* $s\in(0,1]$, $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\Delta_{\ln x}(f)\leq\frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}\left(\ln\frac{b}{a}\right)}\cdot$$

$$\left[\left(\ln\frac{x}{a}\right)^{\alpha+1}\left|(f\circ\exp)'\left(\frac{\ln(xa)}{2}\right)\right|+\left(\ln\frac{b}{x}\right)^{\alpha+1}\left|(f\circ\exp)'\left(\frac{\ln(bx)}{2}\right)\right|\right].\quad(93)$$

Based on Theorem 25 we give

Theorem 37 *Let* $0<a<b<\infty$, $f\in C([a,b])$, $\alpha,\sigma>0$, $\eta>-1$. *Assume* $\left(F\circ(id)^{\frac{1}{\sigma}}\right)$ *is differentiable on* (a^σ,b^σ) *with* $\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\in L_1([a^\sigma,b^\sigma])$, $x\in[a,b]$. *Here* $F(x)=x^{\sigma\eta}f(x)$, $x\in[a,b]$. *Assume* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\right|$ *is s-convex in the second sense on* $[a^\sigma,b^\sigma]$ *for some fixed* $s\in(0,1]$ *and* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'(x^\sigma)\right|\leq M$, $x\in[a,b]$. *Then*

$$\Delta_{x^\sigma}(f)\leq\frac{M}{(b^\sigma-a^\sigma)}\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)\cdot$$

$$\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{\alpha+s+1}\right].\quad(94)$$

By Theorem 27 we get

Theorem 38 *All as in Theorem 37, however here* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\right|^q$ *is s-convex in the second sense on* $[a^\sigma,b^\sigma]$ *for some fixed* $s\in(0,1]$, $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\Delta_{x^\sigma}(f)\leq\frac{M}{(1+p\alpha)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{b^\sigma-a^\sigma}\right].\quad(95)$$

Using Theorem 29 we get

Theorem 39 *Here all as in Theorem 38, however* $q\geq1$, *p is not related. Then*

$$\Delta_{x^\sigma}(f)\leq M\left(\frac{1}{1+\alpha}\right)^{1-\frac{1}{q}}\left(\frac{1}{\alpha+s+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(1+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right)^{\frac{1}{q}}\left[\frac{(x^\sigma-a^\sigma)^{\alpha+1}+(b^\sigma-x^\sigma)^{\alpha+1}}{b^\sigma-a^\sigma}\right].\quad(96)$$

Using Theorem 31 we obtain

Theorem 40 *All as in Theorem 37, however here* $\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\right|^q$ *is s-concave in the second sense on* $[a^\sigma, b^\sigma]$ *for some fixed* $s\in(0,1]$, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\Delta_{x^\sigma}(f)\leq\frac{2^{\frac{(s-1)}{q}}}{(1+p\alpha)^{\frac{1}{p}}(b^\sigma-a^\sigma)}\cdot$$
$$\left[(x^\sigma-a^\sigma)^{\alpha+1}\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\left(\frac{x^\sigma+a^\sigma}{2}\right)\right|+\right.$$
$$\left.(b^\sigma-x^\sigma)^{\alpha+1}\left|\left(F\circ(id)^{\frac{1}{\sigma}}\right)'\left(\frac{b^\sigma+x^\sigma}{2}\right)\right|\right].\tag{97}$$

3 Addendum

We make

Remark 41 *Let* $0<\alpha<1$, $f\in C([a,b])$, $g\in C^1([a,b])$, *g strictly increasing;* $\left(D^\alpha_{g(a)+}\left(f\circ g^{-1}\right)\right)(g(x))$, $\left(D^\alpha_{g(b)-}\left(f\circ g^{-1}\right)\right)(g(x))$ *exist and are continuous on* $[g(a),g(b)]$. *Also assume* $g'(x)\neq0$, *almost all* $x\in[a,b]$.

Then by (42) we get

$$\left(D^\alpha_{g(a)+}\left(f\circ g^{-1}\right)\right)(g(x))=(g'(x))^{-1}\left(D^\alpha_{a+;g}(f)\right)(x),\tag{98}$$

almost all $x\in[a,b]$.

Also by (45) we get

$$\left(D^\alpha_{g(b)-}\left(f\circ g^{-1}\right)\right)(g(x))=(g'(x))^{-1}\left(D^\alpha_{b-;g}(f)\right)(x),\tag{99}$$

almost all $x\in[a,b]$.

Then by (38) and (39) we obtain

$$f(x)=\left(f\circ g^{-1}\right)(g(x))=I^\alpha_{g(a)+}\left(D^\alpha_{g(a)+}\left(f\circ g^{-1}\right)\right)(g(x))\overset{(98)}{=}$$
$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-t)^{\alpha-1}(g'(t))^{-1}\left(D^\alpha_{a+;g}(f)\right)(t)\,dt,\tag{100}$$

and

$$f(x)=\left(f\circ g^{-1}\right)(g(x))=I^\alpha_{g(b)-}\left(D^\alpha_{g(b)-}\left(f\circ g^{-1}\right)\right)(g(x))\overset{(99)}{=}$$
$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(t-g(x))^{\alpha-1}(g'(t))^{-1}\left(D^\alpha_{b-;g}(f)\right)(t)\,dt,\tag{101}$$

for any $x\in[a,b]$.

We have proved the following generalized fractional Taylor formulae.

Theorem 42 *Let* $0 < \alpha < 1$, $f \in C([a,b])$, $g \in C^1([a,b])$, *g strictly increasing; each of* $\left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x))$, $\left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x))$ *exists and it is continuous on* $[g(a), g(b)]$. *Assume that* $g'(x) \neq 0$, *for almost all* $x \in [a,b]$. *Then*

1)

$$f(x) = I^{\alpha}_{g(a)+}\left(D^{\alpha}_{g(a)+}\left(f \circ g^{-1}\right)\right)(g(x)) =$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - t)^{\alpha-1} \left(g'(t)\right)^{-1} \left(D^{\alpha}_{a+;g}(f)\right)(t)\, dt, \tag{102}$$

and

2)

$$f(x) = I^{\alpha}_{g(b)-}\left(D^{\alpha}_{g(b)-}\left(f \circ g^{-1}\right)\right)(g(x)) =$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (t - g(x))^{\alpha-1} \left(g'(t)\right)^{-1} \left(D^{\alpha}_{b-;g}(f)\right)(t)\, dt, \tag{103}$$

for any $x \in [a,b]$.

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General Grüss and Ostrowski type inequalities involving s-convexity

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Abstract

Using the well known reoresentation formula for functions due to Fink [7], we establish a series of general Grüss and Ostrowski type inequalities involving s-convexity and s-concavity in the second sense, acting to all possible directions.

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1 Background

We are motivated by the following famous inequality due to G. Grüss of 1935.

Theorem 1 *([8]) Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a,b],$$

where $m, M, n, N \in \mathbb{R}$. Then

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left(\frac{1}{b-a} \int_a^b f(x) \, dx \right) \left(\frac{1}{b-a} \int_a^b g(x) \, dx \right) \right| \tag{1}$$
$$\leq \frac{1}{4} (M-m)(N-n).$$

We are also motivated by [3], Chapter 25, pp. 305-317.

Another motivation is [2].

We are strongly motivated in our method by

Proposition 2 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f' \in L_1([a,b])$, then*

$$f(x) = \frac{1}{b-a}\int_a^b f(u)\,du + (a-b)\int_0^1 p(x,t)\,f'(ta+(1-t)b)\,dt, \tag{2}$$

for all $x \in [a,b]$, where

$$p(x,t) = \begin{cases} t, & t \in \left[0, \frac{b-x}{b-a}\right), \\ t-1, & t \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \tag{3}$$

Proof. We use Montgomery identity ([5])

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b q(x,t)\,f'(t)\,dt, \tag{4}$$

for all $x \in [a,b]$, where

$$q(x,t) = \begin{cases} t-a, & t \in [a,x], \\ t-b, & t \in (x,b]. \end{cases} \tag{5}$$

We observe by change of variable that

$$\frac{1}{b-a}\int_a^b q(x,u)\,f'(u)\,du = \int_0^1 q(x,\lambda a+(1-\lambda)b)\,f'(\lambda a+(1-\lambda)b)\,d\lambda. \tag{6}$$

We notice that

$$q(x,\lambda a+(1-\lambda)b) = \begin{cases} (1-\lambda)(b-a), & \lambda a+(1-\lambda)b \in [a,x], \\ -\lambda(b-a), & \lambda a+(1-\lambda)b \in (x,b] \end{cases} \tag{7}$$

$$= \begin{cases} (1-\lambda)(b-a), & \lambda \in \left[\frac{b-x}{b-a}, 1\right], \\ -\lambda(b-a), & \lambda \in \left[0, \frac{b-x}{b-a}\right) \end{cases}$$

$$= (a-b)\begin{cases} \lambda, & \lambda \in \left[0, \frac{b-x}{b-a}\right), \\ \lambda-1, & \lambda \in \left[\frac{b-x}{b-a}, 1\right] \end{cases} = (a-b)\,p(x,\lambda). \tag{8}$$

Therefore it holds

$$\int_0^1 q(x,\lambda a+(1-\lambda)b)\,f'(\lambda a+(1-\lambda)b)\,d\lambda =$$

$$(a-b)\int_0^1 p(x,\lambda)\,f'(\lambda a+(1-\lambda)b)\,d\lambda, \tag{9}$$

that is

$$\frac{1}{b-a}\int_a^b q(x,u)\,f'(u)\,du = (a-b)\int_0^1 p(x,\lambda)\,f'(\lambda a+(1-\lambda)b)\,d\lambda, \tag{10}$$

proving the claim. ■

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Note 3 *From (3) we notice that*

$$|p(x,t)| \leq 1, \text{ all } x \in [a,b] \text{ and } t \in [0,1]. \tag{11}$$

Assume now that $|f'|$ is convex on $[a,b]$ with $f'(a), f'(b) \in \mathbb{R}$. Then $g(t) = |f'(ta + (1-t)b)| \geq 0$ is also a convex function in $t \in [0,1]$, as a composition of a convex and an affine function.

It is clear now by comparing the areas under g and the related trapezium that

$$\int_0^1 |f'(ta + (1-t)b)| \, dt \leq \frac{|f'(a)| + |f'(b)|}{2}. \tag{12}$$

Hence when $|f'|$ is convex with $f'(a), f'(b) \in \mathbb{R}$, we get

$$\int_0^1 |p(x,t)| \, |f'(ta + (1-t)b)| \, dt \leq \frac{|f'(a)| + |f'(b)|}{2}. \tag{13}$$

We need

Theorem 4 *([7], Fink) Let $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. Then*

$$f(x) = \frac{n}{b-a} \int_a^b f(t) \, dt \tag{14}$$

$$-\sum_{k=1}^{n-1} \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right)$$

$$+\frac{1}{(n-1)!(b-a)} \int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t) \, dt,$$

where

$$q(x,t) = \begin{cases} t-a, & t \in [a,x], \\ t-b, & t \in (x,b]. \end{cases} \tag{15}$$

When $n = 1$ the sum $\sum_{k=1}^{n-1}$ is zero.

We make

Remark 5 *(on Theorem 4) Here we transform the remainder of (14) as follows*

$$\frac{1}{(n-1)!(b-a)} \int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t) \, dt = \tag{16}$$

$$\frac{1}{(n-1)!} \int_0^1 (x - \lambda a - (1-\lambda) b)^{n-1} q(x, \lambda a + (1-\lambda) b) f^{(n)}(\lambda a + (1-\lambda) b) \, d\lambda$$

$$\overset{(8)}{=} \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x,\lambda) f^{(n)}(\lambda a + (1-\lambda) b) \, d\lambda. \tag{17}$$

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We have established the following result which is our main tool here

Theorem 6 *Let $a, b \in \mathbb{R}$, $a < b$, $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is absolutely continuous on $[a, b]$, $x \in [a, b]$. Then*

$$f(x) = \frac{n}{b-a} \int_a^b f(t)\, dt - \sum_{k=1}^{n-1} \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a} \right)$$

$$+ \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x, \lambda) f^{(n)}(\lambda a + (1-\lambda) b)\, d\lambda, \quad (18)$$

where

$$p(x, \lambda) = \begin{cases} \lambda, & \lambda \in \left[0, \frac{b-x}{b-a}\right), \\ \lambda - 1, & \lambda \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \quad (19)$$

When $n = 1$ the sum $\sum_{k=1}^{n-1}$ is zero.

Note 7 *When $n = 1$ formula (18) collapses to (2).*

We call the remainder of (18) as

$$Rem(18) = \frac{a-b}{(n-1)!} \int_0^1 ((x-b) + \lambda(b-a))^{n-1} p(x, \lambda) f^{(n)}(\lambda a + (1-\lambda) b)\, d\lambda. \quad (20)$$

We have that

$$|Rem(18)| \leq \frac{(b-a)^n}{(n-1)!} \int_0^1 \left| f^{(n)}(\lambda a + (1-\lambda) b) \right| d\lambda. \quad (21)$$

Assume now that $\left| f^{(n)} \right|$ is convex on $[a, b]$ with $f^{(n)}(a), f^{(n)}(b) \in \mathbb{R}$, then

$$\int_0^1 \left| f^{(n)}(\lambda a + (1-\lambda) b) \right| d\lambda \leq \frac{\left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right|}{2}. \quad (22)$$

So given that $\left| f^{(n)} \right|$ is convex on $[a, b]$ with $f^{(n)}(a), f^{(n)}(b) \in \mathbb{R}$, we derive

$$|Rem(18)| \leq \frac{(b-a)^n}{(n-1)!} \frac{\left(\left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right| \right)}{2}. \quad (23)$$

We need

Definition 8 *([9]) A function $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y), \quad (24)$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

When $s = 1$, s-convexity in the second sense reduces to ordinary convexity.

If "$\geq$" holds in (24), we talk about s-concavity in the second sense.

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We also need

Definition 9 *(see also [1])Let I be a subinterval of $\mathbb{R}_+$ and $f : I \to (0, \infty)$. We call f s-logarithmically convex (s-log-convex) in the second sense, iff $\log f(x)$ is s-convex in the second sense, iff*

$$f(\lambda x + (1-\lambda) y) \leq (f(x))^{\lambda^s} (f(y))^{(1-\lambda)^s}, \tag{25}$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

When $s = 1$, s-log-convexity in the second sense reduces to usual log-convexity.

If "$\geq$" holds in (25), we talk about s-log-concavity in the second sense.

We also need the s-convex Hadamard's inequality

Theorem 10 *([6]) Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1([a,b])$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{26}$$

The constant $K = \frac{1}{s+1}$ is the best possible in the second inequality (26). The above inequalities are sharp.

We mention also the s-convex Ostrowski type inequality

Theorem 11 *([2]) Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$ and $|f'(x)| \leq M$, $x \in [a,b]$, then it holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \frac{M}{b-a} \left(\frac{(x-a)^2 + (b-x)^2}{s+1} \right), \tag{27}$$

for each $x \in [a,b]$.

We are also motivated by

Theorem 12 *([4],Čebyšev 1882) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty([a,b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{28}$$

Next we derive general inequalities similar to (28) and (27).

2 Main Results

We present our first general main result that is about Grüss type inequalities and involves s-convexity and s-concavity in the second sense.

Theorem 13 *Let $a,b \in \mathbb{R}$, $a<b$, $f,g:[a,b]\to\mathbb{R}$, $n\in\mathbb{N}$; $f^{(n-1)}$, $g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x\in[a,b]$. Whenever we consider s-convexity or s-concavity of functions we take $a\geq 0$.*

Denote by

$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^{k}-f^{(k-1)}(a)(x-a)^{k}}{b-a}\right), \tag{29}$$

with $F_0^f(x):=0$, and

$$\Delta_{(f,g)} := \int_a^b f(x)\,g(x)\,dx - \frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$
$$-\frac{1}{2}\left[\int_a^b \left(g(x)\,F_{n-1}^{f}(x)+f(x)\,F_{n-1}^{g}(x)\right)dx\right]. \tag{30}$$

We distinguish the cases:

1) Here $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and

$$\left|f^{(n)}(x)\right|\leq M,\quad \left|g^{(n)}(x)\right|\leq K,\quad x\in[a,b]. \tag{31}$$

Then

$$\left|\Delta_{(f,g)}\right| \leq \frac{2(b-a)^{n+1}}{(n-1)!\,(s+2)(s+3)}\left(K\|f\|_{\infty}+M\|g\|_{\infty}\right). \tag{32}$$

2) Let $p_i>1:\sum_{i=1}^{4}\frac{1}{p_i}=1$, and $f^{(n)}$, $g^{(n)}\in L_{p_4}([a,b])$. Then

$$\left|\Delta_{(f,g)}\right|\leq$$
$$\frac{(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\left[\|f\|_{p_1}\left\|g^{(n)}\right\|_{p_4}+\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{33}$$

3) Let $p_i>1:\sum_{i=1}^{4}\frac{1}{p_i}=1$, with $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and $\left|f^{(n)}(x)\right|\leq M$, $\left|g^{(n)}(x)\right|\leq K$, $x\in[a,b]$. Then

$$\left|\Delta_{(f,g)}\right|\leq$$
$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}K+\|g\|_{p_1}M\right]}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(p_4 s+1)^{\frac{1}{p_4}}}. \tag{34}$$

4) Let $p_i > 1 : \sum_{i=1}^{4} \frac{1}{p_i} = 1$, with $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ are s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $\left|g^{(n)}(x)\right| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1} K + \|g\|_{p_1} M\right]}{2^{\frac{1}{p_1}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(s+1)^{\frac{1}{p_4}}}. \tag{35}$$

5) Here $p_i > 1 : \sum_{i=1}^{4} \frac{1}{p_i} = 1$. Let $f^{(n)}, g^{(n)} \in L_{p_4}([a,b])$, with $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ being s-concave in the second sense. Then

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g^{(n)}\left(\frac{a+b}{2}\right)\right| + \|g\|_{p_1}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right]}{2^{\left(\frac{1}{p_1}+\frac{2-s}{p_4}\right)}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{36}$$

Proof. Since $f, g : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, with $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$, by Theorem 6 we obtain

$$f(x) = \frac{n}{b-a}\int_a^b f(t)\,dt + F_{n-1}^{f}(x) + R_n^{f}(x), \tag{37}$$

where

$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right), \tag{38}$$

and

$$R_n^{f}(x) := \frac{a-b}{(n-1)!}\int_0^1 ((x-b)+\lambda(b-a))^{n-1} p(x,\lambda) f^{(n)}(\lambda a + (1-\lambda) b)\,d\lambda$$

$$= \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t)\,dt. \tag{39}$$

Similarly we have

$$g(x) = \frac{n}{b-a}\int_a^b g(t)\,dt + F_{n-1}^{g}(x) + R_n^{g}(x). \tag{40}$$

Then

$$f(x)g(x) = \frac{n}{b-a} g(x)\int_a^b f(t)\,dt + g(x)F_{n-1}^{f}(x) + g(x)R_n^{f}(x), \tag{41}$$

$$f(x)\,g(x)=\frac{n}{b-a}f(x)\int_a^b g(t)\,dt+f(x)\,F_{n-1}^{g}(x)+f(x)\,R_n^{g}(x)\,.$$

Then by integrating we obtain

$$\int_a^b f(x)\,g(x)\,dx=\frac{n}{b-a}\left(\int_a^b g(x)\,dx\right)\left(\int_a^b f(x)\,dx\right) \tag{42}$$

$$+\int_a^b g(x)\,F_{n-1}^{f}(x)\,dx+\int_a^b g(x)\,R_n^{f}(x)\,dx,$$

$$\int_a^b f(x)\,g(x)\,dx=\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$+\int_a^b f(x)\,F_{n-1}^{g}(x)\,dx+\int_a^b f(x)\,R_n^{g}(x)\,dx.$$

That is

$$\int_a^b f(x)\,g(x)\,dx-\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)=$$

$$\int_a^b g(x)\,F_{n-1}^{f}(x)\,dx+\int_a^b g(x)\,R_n^{f}(x)\,dx= \tag{43}$$

$$\int_a^b f(x)\,F_{n-1}^{g}(x)\,dx+\int_a^b f(x)\,R_n^{g}(x)\,dx.$$

Adding the last we derive

$$\Delta_{(f,g)}:=\int_a^b f(x)\,g(x)\,dx-\frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$-\frac{1}{2}\left[\int_a^b\left(g(x)\,F_{n-1}^{f}(x)+f(x)\,F_{n-1}^{g}(x)\right)dx\right]$$

$$=\frac{1}{2}\left[\int_a^b\left(f(x)\,R_n^{g}(x)+g(x)\,R_n^{f}(x)\right)dx\right]. \tag{44}$$

Next we estimate $\Delta_{(f,g)}$.

1) Estimate with respect to $\|\cdot\|_\infty$ and s-convexity in the second sense. We have that

$$\left|\Delta_{(f,g)}\right|\leq\frac{1}{2}\left[\|f\|_\infty\int_a^b\left|R_n^{g}(x)\right|dx+\|g\|_\infty\int_a^b\left|R_n^{f}(x)\right|dx\right]. \tag{45}$$

We notice under the assumption $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, that

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| \left|f^{(n)}(\lambda a + (1-\lambda) b)\right| d\lambda \tag{46}$$

$$\leq \frac{(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| \left(\lambda^s \left|f^{(n)}(a)\right| + (1-\lambda)^s \left|f^{(n)}(b)\right|\right) d\lambda$$

$$\leq \frac{M(b-a)^n}{(n-1)!} \int_0^1 |p(x,\lambda)| \left(\lambda^s + (1-\lambda)^s\right) d\lambda \tag{47}$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\int_0^{\frac{b-x}{b-a}} \lambda \left(\lambda^s + (1-\lambda)^s\right) d\lambda + \int_{\frac{b-x}{b-a}}^1 (1-\lambda)\left(\lambda^s + (1-\lambda)^s\right) d\lambda\right]$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\int_0^{\frac{b-x}{b-a}} \lambda^{s+1} d\lambda + \int_0^{\frac{b-x}{b-a}} \lambda (1-\lambda)^s d\lambda + \right. \tag{48}$$

$$\left. \int_{\frac{b-x}{b-a}}^1 \left(\lambda^s - \lambda^{s+1}\right) d\lambda + \int_{\frac{b-x}{b-a}}^1 (1-\lambda)^{s+1} d\lambda\right]$$

$$= \frac{M(b-a)^n}{(n-1)!} \left[\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2}\right) + \right.$$

$$\left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)}\right) +$$

$$\left(\frac{1}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{1}{(s+1)(s+2)}\right) +$$

$$\left.\left(\frac{1}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\right)\right] =$$

$$\frac{M(b-a)^n}{(n-1)!} \left[\frac{2}{s+2}\left(\frac{b-x}{b-a}\right)^{s+2} + \frac{2}{s+2}\left(\frac{x-a}{b-a}\right)^{s+2}\right.$$

$$\left. -\frac{1}{s+1}\left(\frac{x-a}{b-a}\right)^{s+1} - \frac{1}{s+1}\left(\frac{b-x}{b-a}\right)^{s+1} + \frac{2}{(s+1)(s+2)}\right] = \tag{49}$$

$$\frac{M(b-a)^n}{(n-1)!} \left[\frac{2}{s+2}\left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}}\right)\right.$$

$$\left. -\frac{1}{s+1}\left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}}\right) + \frac{2}{(s+1)(s+2)}\right] =: \frac{M(b-a)^n}{(n-1)!} \varphi(x,s). \tag{50}$$

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We have proved that

$$\left|R_n^f(x)\right| \le \frac{M(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{51}$$

Given that $\left|g^{(n)}\right|$ is s-convex in the second sense and $\left|g^{(n)}(x)\right| \le K$, $x \in [a,b]$, we similarly get

$$\left|R_n^g(x)\right| \le \frac{K(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{52}$$

We notice that

$$\int_a^b \varphi(x,s)\,dx = \frac{4(b-a)}{(s+2)(s+3)}. \tag{53}$$

Hence it holds

$$\int_a^b \left|R_n^f(x)\right| dx \le \frac{4M(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}, \tag{54}$$

and similarly

$$\int_a^b \left|R_n^g(x)\right| dx \le \frac{4K(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}. \tag{55}$$

Therefore we proved that

$$\left|\Delta_{(f,g)}\right| \le \frac{1}{2}\left[\|f\|_\infty \frac{4K(b-a)^{n+1}}{(n-1)!(s+2)(s+3)} + \|g\|_\infty \frac{4M(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}\right]. \tag{56}$$

Hence we derive

$$\left|\Delta_{(f,g)}\right| \le \frac{2(b-a)^{n+1}}{(n-1)!(s+2)(s+3)}\left(K\|f\|_\infty + M\|g\|_\infty\right). \tag{57}$$

2) Estimate with respect to $\|\cdot\|_{p_i}$ norms, $i = 1,2,3,4$; $p_i > 1$, $\sum_{i=1}^4 \frac{1}{p_i} = 1$.

We use Hölder's inequality for four functions, and we use the old remainder form, see (14).

So here

$$R_n^f(x) = \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} q(x,t) f^{(n)}(t)\,dt, \tag{58}$$

where $q(x,t)$ as in (15).

We have that

$$\left|\int_a^b g(x) R_n^f(x)\,dx\right| \le \int_a^b |g(x)| \left|R_n^f(x)\right| dx \le \tag{59}$$

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$$\frac{1}{(n-1)!\,(b-a)}\int_a^b\int_a^b |g(x)|\,|x-t|^{n-1}\,|q(x,t)|\,\left|f^{(n)}(t)\right| dtdx \leq$$

$$\frac{1}{(n-1)!\,(b-a)}\left(\int_a^b\int_a^b |g(x)|^{p_1}\,dtdx\right)^{\frac{1}{p_1}}\left(\int_a^b\int_a^b |x-t|^{(n-1)p_2}\,dtdx\right)^{\frac{1}{p_2}}\cdot$$

$$\left(\int_a^b\int_a^b |q(x,t)|^{p_3}\,dtdx\right)^{\frac{1}{p_3}}\left(\int_a^b\int_a^b \left|f^{(n)}(t)\right|^{p_4} dtdx\right)^{\frac{1}{p_4}} = \tag{60}$$

$$\frac{1}{(n-1)!\,(b-a)^{\frac{1}{p_2}+\frac{1}{p_3}}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}\left(\int_a^b\left(\int_a^b |x-t|^{(n-1)p_2}\,dt\right)dx\right)^{\frac{1}{p_2}}\cdot$$

$$\left(\int_a^b\left(\int_a^b |q(x,t)|^{p_3}\,dt\right)dx\right)^{\frac{1}{p_3}} =$$

$$\frac{\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\,(b-a)^{\frac{1}{p_2}+\frac{1}{p_3}}}\cdot\frac{2^{\frac{1}{p_2}}\,(b-a)^{n-1+\frac{2}{p_2}}}{\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}}\cdot \tag{61}$$

$$\frac{2^{\frac{1}{p_3}}\,(b-a)^{1+\frac{2}{p_3}}}{\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}} =$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{62}$$

We have established that

$$\left|\int_a^b g(x)\,R_n^f(x)\,dx\right| \leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{63}$$

Similarly we obtain

$$\left|\int_a^b f(x)\,R_n^g(x)\,dx\right| \leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}\,(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\,\|f\|_{p_1}\left\|g^{(n)}\right\|_{p_4}}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{64}$$

Consequently by (44), (63), (64), we get

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+\frac{1}{p_2}+\frac{1}{p_3}}\left[\|f\|_{p_1}\left\|g^{(n)}\right\|_{p_4}+\|g\|_{p_1}\left\|f^{(n)}\right\|_{p_4}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{65}$$

3) We notice also that if $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense, then

$$\left\|g^{(n)}\right\|_{p_4}^{p_4}=\int_a^b\left|g^{(n)}(t)\right|^{p_4}dt=$$

$$(b-a)\int_0^1\left|g^{(n)}(\lambda a+(1-\lambda)b)\right|^{p_4}d\lambda\leq \tag{66}$$

$$(b-a)\int_0^1\left(\lambda^s\left|g^{(n)}(a)\right|+(1-\lambda)^s\left|g^{(n)}(b)\right|\right)^{p_4}d\lambda\leq$$

$$K^{p_4}(b-a)\int_0^1\left(\lambda^s+(1-\lambda)^s\right)^{p_4}d\lambda\leq$$

$$2^{p_4-1}K^{p_4}(b-a)\left[\int_0^1\left(\lambda^{p_4s}+(1-\lambda)^{p_4s}\right)d\lambda\right]=\frac{(2K)^{p_4}(b-a)}{(p_4s+1)}. \tag{67}$$

Hence it holds

$$\left\|g^{(n)}\right\|_{p_4}\leq 2K\left(\frac{b-a}{p_4s+1}\right)^{\frac{1}{p_4}}. \tag{68}$$

Similarly we find

$$\left\|f^{(n)}\right\|_{p_4}\leq 2M\left(\frac{b-a}{p_4s+1}\right)^{\frac{1}{p_4}}. \tag{69}$$

So if $\left|f^{(n)}\right|$, $\left|g^{(n)}\right|$ are s-convex in the second sense and $\left|f^{(n)}(x)\right|\leq M$, $\left|g^{(n)}(x)\right|\leq K$, $x\in[a,b]$, then by (65), (68), (69), we get

$$\left|\Delta_{(f,g)}\right|\leq$$

$$\frac{2^{\frac{1}{p_2}+\frac{1}{p_3}}(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}K+\|g\|_{p_1}M\right]}{(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(p_4s+1)^{\frac{1}{p_4}}}. \tag{70}$$

4) Next assuming that $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ are s-convex in the second sense, then

$$\left\|g^{(n)}\right\|_{p_4}^{p_4}=(b-a)\int_0^1\left|g^{(n)}(\lambda a+(1-\lambda)b)\right|^{p_4}d\lambda\leq$$

$$(b-a)\int_0^1\left(\lambda^s\left|g^{(n)}(a)\right|^{p_4}+(1-\lambda)^s\left|g^{(n)}(b)\right|^{p_4}\right)d\lambda\leq \tag{71}$$

$$K^{p_4}(b-a)\int_0^1\left(\lambda^s+(1-\lambda)^s\right)d\lambda=\frac{2K^{p_4}(b-a)}{s+1}. \tag{72}$$

That is

$$\left\|g^{(n)}\right\|_{p_4} \leq \frac{2^{\frac{1}{p_4}} K (b-a)^{\frac{1}{p_4}}}{(s+1)^{\frac{1}{p_4}}}. \tag{73}$$

Similarly we get

$$\left\|f^{(n)}\right\|_{p_4} \leq \frac{2^{\frac{1}{p_4}} M (b-a)^{\frac{1}{p_4}}}{(s+1)^{\frac{1}{p_4}}}. \tag{74}$$

So if $\left|f^{(n)}\right|^{p_4}$, $\left|g^{(n)}\right|^{p_4}$ are s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $\left|g^{(n)}(x)\right| \leq K$, $x \in [a,b]$, then by (65), (73), (74), we derive

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} K + \|g\|_{p_1} M\right]}{2^{\frac{1}{p_1}} (n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}} (s+1)^{\frac{1}{p_4}}}. \tag{75}$$

5) Assume finally that $\left|g^{(n)}\right|^{p_4}$, $\left|f^{(n)}\right|^{p_4}$ are s-concave in the second sense, then by (26) we get

$$\int_a^b \left|g^{(n)}(t)\right|^{p_4} dt \leq 2^{s-1} \left|g^{(n)}\left(\frac{a+b}{2}\right)\right|^{p_4} (b-a) \tag{76}$$

and

$$\int_a^b \left|f^{(n)}(t)\right|^{p_4} dt \leq 2^{s-1} \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^{p_4} (b-a). \tag{77}$$

Therefore

$$\left\|g^{(n)}\right\|_{p_4} \leq 2^{\frac{s-1}{p_4}} \left|g^{(n)}\left(\frac{a+b}{2}\right)\right| (b-a)^{\frac{1}{p_4}}, \tag{78}$$

and

$$\left\|f^{(n)}\right\|_{p_4} \leq 2^{\frac{s-1}{p_4}} \left|f^{(n)}\left(\frac{a+b}{2}\right)\right| (b-a)^{\frac{1}{p_4}}. \tag{79}$$

Then by using (65), (78), (79), we get

$$\left|\Delta_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} \left|g^{(n)}\left(\frac{a+b}{2}\right)\right| + \|g\|_{p_1} \left|f^{(n)}\left(\frac{a+b}{2}\right)\right|\right]}{2^{\frac{1}{p_1}+\frac{2-s}{p_4}} (n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{80}$$

The proof of the theorem is complete. ■

We apply Theorem 13 for $n = 1$.

Proposition 14 *Let $a, b \in \mathbb{R}$, $a < b$, $f, g : [a,b] \to \mathbb{R}$; f, g are absolutely continuous on $[a,b]$, $x \in [a,b]$. Whenever we consider s-convexity or s-concavity of functions we take $a \geq 0$.*

Denote by

$$\Delta^*_{(f,g)} := \int_a^b f(x)\, g(x)\, dx - \frac{1}{b-a}\left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right). \qquad (81)$$

We distinguish the cases:

1) Here $|f'|$, $|g'|$ are s-convex in the second sense and

$$|f'(x)| \leq M, \quad |g'(x)| \leq K, \quad x \in [a,b]. \qquad (82)$$

Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{2(b-a)^2}{(s+2)(s+3)}\left(K\|f\|_\infty + M\|g\|_\infty\right). \qquad (83)$$

2) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, and f', $g' \in L_{p_4}([a,b])$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{1+\frac{1}{p_2}+\frac{1}{p_3}}\left[\|f\|_{p_1}\|g'\|_{p_4} + \|g\|_{p_1}\|f'\|_{p_4}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_4}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \qquad (84)$$

3) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, with $|f'|$, $|g'|$ are s-convex in the second sense and $|f'(x)| \leq M$, $|g'(x)| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{2^{\frac{1}{p_3}}(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}K + \|g\|_{p_1}M\right]}{\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(p_4 s+1)^{\frac{1}{p_4}}}. \qquad (85)$$

4) Let $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$, with $|f'|^{p_4}$, $|g'|^{p_4}$ are s-convex in the second sense and $|f'(x)| \leq M$, $|g'(x)| \leq K$, $x \in [a,b]$. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}K + \|g\|_{p_1}M\right]}{2^{\frac{1}{p_1}+\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}(s+1)^{\frac{1}{p_4}}}. \qquad (86)$$

5) Here $p_i > 1 : \sum_{i=1}^4 \frac{1}{p_i} = 1$. Let $f', g' \in L_{p_4}([a,b])$, with $|f'|^{p_4}$, $|g'|^{p_4}$ being s-concave in the second sense. Then

$$\left|\Delta^*_{(f,g)}\right| \leq \frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g'\left(\frac{a+b}{2}\right)\right| + \|g\|_{p_1}\left|f'\left(\frac{a+b}{2}\right)\right|\right]}{2^{\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{2-s}{p_4}\right)}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \qquad (87)$$

We continue with

Theorem 15 *Let $0 \leq a < b$, $f, g : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}, g^{(n-1)}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$. We assume that $\left|f^{(n)}\right|, \left|g^{(n)}\right|$ are s-logarithmically convex in the second sense, and $\left|f^{(n)}(a)\right|$, $\left|f^{(n)}(b)\right|$, $\left|g^{(n)}(a)\right|$, $\left|g^{(n)}(b)\right| \in (0,1]$. Call $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $B := \frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}$, $s \in (0,1]$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z \in (0,\infty) - \{1\}, \\ 1, & \text{if } z = 1. \end{cases} \qquad (88)$$

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Denote by

$$F_{n-1}^{f}(x) := \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^{k}-f^{(k-1)}(a)(x-a)^{k}}{b-a}\right), \quad (89)$$

with $F_0^f(x) := 0$*, and*

$$\Delta_{(f,g)} := \int_a^b f(x)g(x)\,dx - \frac{n}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$
$$-\frac{1}{2}\left[\int_a^b \left(g(x)F_{n-1}^{f}(x)+f(x)F_{n-1}^{g}(x)\right)dx\right]. \quad (90)$$

We distinguish the cases:

1) Estimate with respect to $\|\cdot\|_\infty$*. It holds*

$$\left|\Delta_{(f,g)}\right| \leq \frac{(b-a)^{n+1}}{2(n-1)!}\left[\|f\|_\infty \left|g^{(n)}(b)\right|^{s}\psi_s(B)+\|g\|_\infty\left|f^{(n)}(b)\right|^{s}\psi_s(A)\right]. \quad (91)$$

2) Let $p_i > 1 : \sum_{i=1}^{4}\frac{1}{p_i}=1$*. Then*

$$\left|\Delta_{(f,g)}\right| \leq$$
$$\frac{(b-a)^{n+1-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g^{(n)}(b)\right|^{s}\left(\psi_s\left(B^{p_4}\right)\right)^{\frac{1}{p_4}}+\|g\|_{p_1}\left|f^{(n)}(b)\right|^{s}\left(\psi_s\left(A^{p_4}\right)\right)^{\frac{1}{p_4}}\right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}}(n-1)!\left((p_2(n-1)+1)(p_2(n-1)+2)\right)^{\frac{1}{p_2}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \quad (92)$$

Proof. 1) See also [1]. Here we assume that $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1$, set $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $\left|f^{(n)}\right|$ is s-logarithmically convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$.

We observe that

$$\left|R_n^f(x)\right| \overset{\text{(by (46), (11))}}{\leq} \frac{(b-a)^n}{(n-1)!}\int_0^1\left|f^{(n)}(\lambda a+(1-\lambda)b)\right|d\lambda \quad (93)$$

$$\leq \frac{(b-a)^n}{(n-1)!}\int_0^1\left|f^{(n)}(a)\right|^{\lambda^s}\left|f^{(n)}(b)\right|^{(1-\lambda)^s}d\lambda$$

$$\leq \frac{(b-a)^n}{(n-1)!}\int_0^1\left|f^{(n)}(a)\right|^{\lambda s}\left|f^{(n)}(b)\right|^{(1-\lambda)s}d\lambda$$

$$= \frac{(b-a)^n\left|f^{(n)}(b)\right|^{s}}{(n-1)!}\int_0^1\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{\lambda s}d\lambda \quad (94)$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \int_0^1 A^{\lambda s} d\lambda =: (*) .$$

If $A = 1$, then

$$(*) = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!}. \tag{95}$$

If $A \neq 1$ we get

$$(*) = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \int_0^1 e^{s(\ln A)\lambda} d\lambda = \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \frac{1}{s \ln A} \left(A^{s\lambda}|_0^1\right)$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \left(\frac{A^s - 1}{s \ln A}\right). \tag{96}$$

Therefore we derive that

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A) . \tag{97}$$

We further assume $0 < \left|g^{(n)}(a)\right|, \left|g^{(n)}(b)\right| \leq 1$, set $B := \frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}$, $\left|g^{(n)}\right|$ is s-log-convex in the second sense. Then, similarly, we obtain that

$$\left|R_n^g(x)\right| \leq \frac{(b-a)^n \left|g^{(n)}(b)\right|^s}{(n-1)!} \psi_s(B) . \tag{98}$$

Notice that

$$\int_a^b \left|R_n^f(x)\right| dx \leq \frac{(b-a)^{n+1} \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A) \tag{99}$$

and

$$\int_a^b \left|R_n^g(x)\right| dx \leq \frac{(b-a)^{n+1} \left|g^{(n)}(b)\right|^s}{(n-1)!} \psi_s(B) . \tag{100}$$

It is clear now

$$\left|\Delta_{(f,g)}\right| \leq \frac{1}{2} \left[\|f\|_\infty \left|g^{(n)}(b)\right|^s \psi_s(B) + \|g\|_\infty \left|f^{(n)}(b)\right|^s \psi_s(A)\right] \frac{(b-a)^{n+1}}{(n-1)!}. \tag{101}$$

2) Here we assume that $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \left|g^{(n)}(a)\right|, \left|g^{(n)}(b)\right| \leq 1$, set $C := \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}\right)^{p_4}$ and $D := \left(\frac{\left|g^{(n)}(a)\right|}{\left|g^{(n)}(b)\right|}\right)^{p_4}$; $\left|f^{(n)}\right|, \left|g^{(n)}\right|$ are s-log-convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$.

We have as in (66) that

$$\left\|g^{(n)}\right\|_{p_4}^{p_4} = (b-a) \int_0^1 \left|g^{(n)}(\lambda a + (1-\lambda) b)\right|^{p_4} d\lambda \tag{102}$$

$$\leq (b-a) \int_0^1 \left(\left|g^{(n)}(a)\right|^{\lambda^s}\right)^{p_4} \left(\left|g^{(n)}(b)\right|^{(1-\lambda)^s}\right)^{p_4} d\lambda$$

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$$= (b-a) \int_0^1 \left(\left| g^{(n)}(a) \right|^{p_4} \right)^{\lambda^s} \left(\left| g^{(n)}(b) \right|^{p_4} \right)^{(1-\lambda)^s} d\lambda$$

$$\leq (b-a) \int_0^1 \left(\left| g^{(n)}(a) \right|^{p_4} \right)^{\lambda s} \left(\left| g^{(n)}(b) \right|^{p_4} \right)^{(1-\lambda)s} d\lambda$$

$$= (b-a) \left| g^{(n)}(b) \right|^{p_4 s} \int_0^1 \left(\left(\frac{\left| g^{(n)}(a) \right|}{\left| g^{(n)}(b) \right|} \right)^{p_4} \right)^{s\lambda} d\lambda$$

$$= (b-a) \left| g^{(n)}(b) \right|^{p_4 s} \int_0^1 D^{s\lambda} d\lambda. \tag{103}$$

Hence

$$\left\| g^{(n)} \right\|_{p_4}^{p_4} \leq (b-a) \left| g^{(n)}(b) \right|^{p_4 s} \psi_s(D), \tag{104}$$

and

$$\left\| g^{(n)} \right\|_{p_4} \leq (b-a)^{\frac{1}{p_4}} \left| g^{(n)}(b) \right|^{s} (\psi_s(D))^{\frac{1}{p_4}}. \tag{105}$$

Similarly we obtain

$$\left\| f^{(n)} \right\|_{p_4} \leq (b-a)^{\frac{1}{p_4}} \left| f^{(n)}(b) \right|^{s} (\psi_s(C))^{\frac{1}{p_4}}. \tag{106}$$

Finally using (33) we find

$$\left| \Delta_{(f,g)} \right| \leq$$

$$\frac{(b-a)^{n+1-\frac{1}{p_1}} \left[\|f\|_{p_1} \left| g^{(n)}(b) \right|^s (\psi_s(D))^{\frac{1}{p_4}} + \|g\|_{p_1} \left| f^{(n)}(b) \right|^s (\psi_s(C))^{\frac{1}{p_4}} \right]}{2^{\frac{1}{p_1}+\frac{1}{p_4}} (n-1)! \left((p_2(n-1)+1)(p_2(n-1)+2) \right)^{\frac{1}{p_2}} \left((p_3+1)(p_3+2) \right)^{\frac{1}{p_3}}}. \tag{107}$$

The proof of the theorem now is complete. ■

We apply Theorem 15 for $n = 1$.

Proposition 16 *Let $0 \leq a < b$, $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$, $x \in [a,b]$. Assume that $|f'|, |g'|$ are s-log-convex in the second sense, and $|f'(a)|, |f'(b)|, |g'(a)|, |g'(b)| \in (0,1]$. Call $A^* := \frac{|f'(a)|}{|f'(b)|}$, $B^* := \frac{|g'(a)|}{|g'(b)|}$, $s \in (0,1]$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s - 1}{s \ln z}, & \text{if } z \in (0,\infty) - \{1\}, \\ 1, & \text{if } z = 1. \end{cases} \tag{108}$$

Denote by

$$\Delta^*_{(f,g)} := \int_a^b f(x) g(x) dx - \frac{1}{b-a} \left(\int_a^b f(x) dx \right) \left(\int_a^b g(x) dx \right). \tag{109}$$

We distinguish the cases:

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1) It holds

$$\left|\Delta^{*}_{(f,g)}\right| \leq \frac{(b-a)^{2}}{2}\left[\|f\|_{\infty}\left|g'(b)\right|^{s}\psi_{s}(B^{*})+\|g\|_{\infty}\left|f'(b)\right|^{s}\psi_{s}(A^{*})\right]. \tag{110}$$

2) Let $p_i > 1 : \sum_{i=1}^{4}\frac{1}{p_i}=1$. *Then*

$$\left|\Delta^{*}_{(f,g)}\right| \leq$$

$$\frac{(b-a)^{2-\frac{1}{p_1}}\left[\|f\|_{p_1}\left|g'(b)\right|^{s}\left(\psi_{s}(B^{*p_4})\right)^{\frac{1}{p_4}}+\|g\|_{p_1}\left|f'(b)\right|^{s}\left(\psi_{s}(A^{*p_4})\right)^{\frac{1}{p_4}}\right]}{2^{1-\frac{1}{p_3}}\left((p_3+1)(p_3+2)\right)^{\frac{1}{p_3}}}. \tag{111}$$

Next we present s-convexity general Ostrowski type inequalities.

Theorem 17 *Let* $a,b\in\mathbb{R}$, $a<b$, $f:[a,b]\to\mathbb{R}$, $n\in\mathbb{N}$; $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$, $x\in[a,b]$. *When we consider* s*-convexity or* s*-concavity of functions we take* $a\geq 0$.

Denote by

$$F^{f}_{n-1}(x):=\sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^{k}-f^{(k-1)}(a)(x-a)^{k}}{b-a}\right), \tag{112}$$

with $F^{f}_{0}(x):=0$, *and*

$$R^{f}_{n}(x):=f(x)-\frac{n}{b-a}\int_{a}^{b}f(t)\,dt-F^{f}_{n-1}(x). \tag{113}$$

We distinguish the cases:

1) Here $\left|f^{(n)}\right|$ *is* s*-convex in the second sense and* $\left|f^{(n)}(x)\right|\leq M$, $x\in[a,b]$. *Then*

$$\left|R^{f}_{n}(x)\right|\leq\frac{M(b-a)^{n}}{(n-1)!}\left[\frac{2}{(s+2)}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right)\right. \tag{114}$$

$$\left.-\frac{1}{(s+1)}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\right].$$

2) Let $p_i>1:\sum_{i=1}^{3}\frac{1}{p_i}=1$, *and* $f^{(n)}\in L_{p_3}([a,b])$. *Then*

$$\left|R^{f}_{n}(x)\right|\leq\frac{\left\|f^{(n)}\right\|_{p_3}}{(n-1)!(b-a)}\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\cdot$$

$$\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{115}$$

3) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$*. If* $\left|f^{(n)}\right|$ *is s-convex in the second sense and* $\left|f^{(n)}(x)\right| \leq M$*,* $x \in [a,b]$*, then*

$$\left|R_n^f(x)\right| \leq \frac{2M(b-a)^{\frac{1}{p_3}-1}}{(p_3 s+1)^{\frac{1}{p_3}}(n-1)!} \cdot \tag{116}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

4) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$*. If* $\left|f^{(n)}\right|^{p_3}$ *is s-convex in the second sense and* $\left|f^{(n)}(x)\right| \leq M$*,* $x \in [a,b]$*, then*

$$\left|R_n^f(x)\right| \leq \frac{2^{\frac{1}{p_3}}M(b-a)^{\frac{1}{p_3}-1}}{(s+1)^{\frac{1}{p_3}}(n-1)!} \cdot \tag{117}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

5) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$*,* $f^{(n)} \in L_{p_3}([a,b])$*, with* $\left|f^{(n)}\right|^{p_3}$ *being s-concave in the second sense. Then*

$$\left|R_n^f(x)\right| \leq \frac{2^{\frac{s-1}{p_3}}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|(b-a)^{\frac{1}{p_3}-1}}{(n-1)!} \cdot \tag{118}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}} \left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

Proof. 1) As in (50), (51), we get

$$\left|R_n^f(x)\right| \leq \frac{M(b-a)^n}{(n-1)!}\varphi(x,s), \quad x \in [a,b]. \tag{119}$$

2) Let $p_i > 1, \sum_{i=1}^{3} \frac{1}{p_i} = 1$. Then

$$\left|R_n^f(x)\right| \leq \frac{1}{(n-1)!(b-a)} \int_a^b |x-t|^{n-1} |q(x,t)| \left|f^{(n)}(t)\right| dt$$

$$\leq \frac{1}{(n-1)!(b-a)} \left(\int_a^b |x-t|^{p_1(n-1)} dt\right)^{\frac{1}{p_1}} \left(\int_a^b |q(x,t)|^{p_2} dt\right)^{\frac{1}{p_2}} \left\|f^{(n)}\right\|_{p_3} \tag{120}$$

$$= \frac{\left\|f^{(n)}\right\|_{p_3}}{(n-1)!(b-a)} \left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}.$$

$$\left(\frac{(b-x)^{p_2+1} + (x-a)^{p_2+1}}{p_2+1} \right)^{\frac{1}{p_2}}. \tag{121}$$

3) If $\left|f^{(n)}\right|$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, we get

$$\left\|f^{(n)}\right\|_{p_3} \leq 2M \left(\frac{b-a}{p_3 s+1} \right)^{\frac{1}{p_3}}, \tag{122}$$

see also (69).

4) If $\left|f^{(n)}\right|^{p_3}$ is s-convex in the second sense and $\left|f^{(n)}(x)\right| \leq M$, $x \in [a,b]$, we get

$$\left\|f^{(n)}\right\|_{p_3} \leq \frac{2^{\frac{1}{p_3}} M (b-a)^{\frac{1}{p_3}}}{(s+1)^{\frac{1}{p_3}}}, \tag{123}$$

see also (74).

5) If $\left|f^{(n)}\right|^{p_3}$ is s-concave in the second sense, then by (26) we find

$$\left\|f^{(n)}\right\|_{p_3} \leq 2^{\frac{s-1}{p_3}} \left|f^{(n)}\left(\frac{a+b}{2}\right)\right| (b-a)^{\frac{1}{p_3}}, \tag{124}$$

see also (79). ∎

We apply Theorem 17 for $n=1$.

Proposition 18 *Let $a,b \in \mathbb{R}$, $a<b$, $f:[a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, $x \in [a,b]$. In case of s-convexity or s-concavity we take $a \geq 0$.*

Denote by

$$R_1^f(x) = f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt. \tag{125}$$

We distinguish the cases:

1) Here $|f'|$ is s-convex in the second sense and $|f'(x)| \leq M$, $x \in [a,b]$. Then

$$\left|R_1^f(x)\right| \leq M(b-a) \left[\frac{2}{(s+2)} \left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}} \right) \right. \tag{126}$$
$$\left. - \frac{1}{(s+1)} \left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}} \right) + \frac{2}{(s+1)(s+2)} \right].$$

2) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$, and $f' \in L_{p_3}([a,b])$. Then

$$\left|R_1^f(x)\right| \leq \|f'\|_{p_3} (b-a)^{\frac{1}{p_1}-1} \left(\frac{(b-x)^{p_2+1} + (x-a)^{p_2+1}}{p_2+1} \right)^{\frac{1}{p_2}}. \tag{127}$$

3) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. *If* $|f'|$ *is s-convex in the second sense and* $|f'(x)| \leq M$, $x \in [a,b]$, *then*

$$\left|R_1^f(x)\right| \leq \frac{2M}{(p_3 s+1)^{\frac{1}{p_3}} (b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1} + (x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{128}$$

4) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. *If* $|f'|^{p_3}$ *is s-convex in the second sense and* $|f'(x)| \leq M$, $x \in [a,b]$, *then*

$$\left|R_1^f(x)\right| \leq \frac{2^{\frac{1}{p_3}} M}{(s+1)^{\frac{1}{p_3}} (b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1} + (x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{129}$$

5) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$, $f' \in L_{p_3}([a,b])$, *with* $|f'|^{p_3}$ *being s-concave in the second sense. Then*

$$\left|R_1^f(x)\right| \leq \frac{2^{\frac{s-1}{p_3}} \left|f'\left(\frac{a+b}{2}\right)\right|}{(b-a)^{\frac{1}{p_2}}} \left(\frac{(b-x)^{p_2+1} + (x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{130}$$

We continue with

Theorem 19 *Let* $0 \leq a < b$, $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$; $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$, $x \in [a,b]$. *We assume that* $\left|f^{(n)}\right|$ *is s-log-convex in the second sense, and* $\left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \in (0,1]$. *Call* $A := \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$, $s \in (0,1]$, *and*

$$\psi_s(z) := \begin{cases} \frac{z^s - 1}{s \ln z}, & \text{if } z \in (0,\infty) - \{1\}, \\ 1, & \text{if } z = 1. \end{cases} \tag{131}$$

Denote by

$$F_{n-1}^f(x) := \sum_{k=1}^{n-1} \left(\frac{n-k}{k!}\right) \left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right), \tag{132}$$

with $F_0^f(x) := 0$, *and*

$$R_n^f(x) := f(x) - \frac{n}{b-a} \int_a^b f(t)\,dt - F_{n-1}^f(x). \tag{133}$$

We distinguish the cases:

1) It holds generally

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!} \psi_s(A). \tag{134}$$

2) Let $p_i > 1 : \sum_{i=1}^{3} \frac{1}{p_i} = 1$. *Then*

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^{\frac{1}{p_3}-1}}{(n-1)!} \left|f^{(n)}(b)\right|^s \left(\psi_s\left(A^{p_3}\right)\right)^{\frac{1}{p_3}}. \tag{135}$$

$$\left(\frac{(b-x)^{p_1(n-1)+1}+(x-a)^{p_1(n-1)+1}}{p_1(n-1)+1}\right)^{\frac{1}{p_1}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}.$$

Proof. 1) As in (97) we get

$$\left|R_n^f(x)\right| \leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^s}{(n-1)!}\psi_s(A). \tag{136}$$

2) As in (106) we obtain

$$\left\|f^{(n)}\right\|_{p_3} \leq (b-a)^{\frac{1}{p_3}}\left|f^{(n)}(b)\right|^s \left(\psi_s\left(A^{p_3}\right)\right)^{\frac{1}{p_3}}. \tag{137}$$

Then we use (115). ■

At last we apply Theorem 19 for $n=1$.

Proposition 20 *Let $0 \leq a < b$, $f:[a,b] \to \mathbb{R}$ absolutely continuous on $[a,b]$, $x \in [a,b]$. We assume that $|f'|$ is s-*log*-convex in the second sense, and $|f'(a)|, |f'(b)| \in (0,1]$. Call $A^* := \frac{|f'(a)|}{|f'(b)|}$, $s \in (0,1]$, and $\psi_s(z)$ as in (131).*

Denote by

$$R_1^f(x) := f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{138}$$

We distinguish the cases:

1) It holds

$$\left|R_1^f(x)\right| \leq (b-a)\left|f'(b)\right|^s \psi_s(A^*). \tag{139}$$

2) Let $p_i > 1 : \sum_{i=1}^3 \frac{1}{p_i} = 1$. Then

$$\left|R_1^f(x)\right| \leq (b-a)^{-\frac{1}{p_2}}\left|f'(b)\right|^s \left(\psi_s\left(A^{*p_3}\right)\right)^{\frac{1}{p_3}}\left(\frac{(b-x)^{p_2+1}+(x-a)^{p_2+1}}{p_2+1}\right)^{\frac{1}{p_2}}. \tag{140}$$

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Generalised fractional Hermite-Hadamard Inequalities involving m-convexity and (s, m)-convexity

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Abstract

Here we present generalised fractional Hermite-Hadamard type inequalities involving m-convexity and (s, m)-convexity. These inequalities are with respect to generalised Riemann-Liouville fractional integrals. Our work is motivated by and expands [7] to the greatest generality and all possible directions.

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1 Background

We use a lot here the following generalised fractional integrals.

Definition 1 *(see also [3, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to given function g are defined as follows:*

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ (absolutely continuous functions) and is strictly increasing, $f \in L_\infty([a, b])$. We set

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t)\, dt, \quad x \geq a, \tag{1}$$

clearly $\left(I_{a+;g}^{\alpha} f\right)(a) = 0$,

and

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t)\, dt, \quad x \leq b, \tag{2}$$

clearly $\left(I^{\alpha}_{b-;g}f\right)(b)=0.$

When g *is the identity function id, we get that* $I^{\alpha}_{a+;id}=I^{\alpha}_{a+}$ *and* $I^{\alpha}_{b-;id}=I^{\alpha}_{b-}$ *the ordinary left and right Riemann-Liouville fractional integrals, where*

$$\left(I^{\alpha}_{a+}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x\geq a, \tag{3}$$

$\left(I^{\alpha}_{a+}f\right)(a)=0,$ *and*

$$\left(I^{\alpha}_{b-}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x\leq b, \tag{4}$$

$\left(I^{\alpha}_{b-}f\right)(b)=0.$

Remark 2 *(see also [1]) We observe that*

$$\left(I^{\alpha}_{a+;g}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt=$$

(by change of variable for Lebesgue integrals)

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)} (g(x)-z)^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I^{\alpha}_{g(a)+}\left(f\circ g^{-1}\right)\right)(g(x)), \quad x\geq a, \tag{5}$$

equivalently $g(x)\geq g(a)$.

That is in the terms and assumptions of Definition 1 we get

$$\left(I^{\alpha}_{a+;g}f\right)(x)=\left(I^{\alpha}_{g(a)+}\left(f\circ g^{-1}\right)\right)(g(x)), \quad \text{for } x\geq a. \tag{6}$$

Similarly we observe that

$$\left(I^{\alpha}_{b-;g}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(g(t))\,g'(t)\,dt$$

$$=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)} (z-g(x))^{\alpha-1}\left(f\circ g^{-1}\right)(z)\,dz=\left(I^{\alpha}_{g(b)-}\left(f\circ g^{-1}\right)\right)(g(x)), \tag{7}$$

for $x\leq b$.

That is

$$\left(I^{\alpha}_{b-;g}f\right)(x)=\left(I^{\alpha}_{g(b)-}\left(f\circ g^{-1}\right)\right)(g(x)), \quad \text{for } x\leq b. \tag{8}$$

So by (6) and (8) we have reduced the general fractional integrals to the ordinary left and right Riemann-Liouville fractional integrals.

When $g(x)=e^x$, $x\in[a,b]$ we have the application

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Definition 3 *The left and right fractional exponential integrals are defined as follows: Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$. We set*

$$\left(I^{\alpha}_{a+;e^x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(e^x - e^t\right)^{\alpha-1} e^t f(t)\, dt, \quad x \geq a, \tag{9}$$

and

$$\left(I^{\alpha}_{b-;e^x} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(e^t - e^x\right)^{\alpha-1} e^t f(t)\, dt, \quad x \leq b. \tag{10}$$

Note 4 *We see that*

$$\left(I^{\alpha}_{a+;e^x} f\right)(x) = \left(I^{\alpha}_{e^a+} (f \circ \ln)\right)(e^x), \quad x \geq a, \tag{11}$$

and

$$\left(I^{\alpha}_{b-;e^x} f\right)(x) = \left(I^{\alpha}_{e^b-} (f \circ \ln)\right)(e^x), \quad x \leq b. \tag{12}$$

Another example follows:

Definition 5 *Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$, $A > 1$. We introduce the fractional integrals:*

$$\left(I^{\alpha}_{a+;A^x} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_a^x \left(A^x - A^t\right)^{\alpha-1} A^t f(t)\, dt, \quad x \geq a, \tag{13}$$

and

$$\left(I^{\alpha}_{b-;A^x} f\right)(x) = \frac{\ln A}{\Gamma(\alpha)} \int_x^b \left(A^t - A^x\right)^{\alpha-1} A^t f(t)\, dt, \quad x \leq b. \tag{14}$$

We are motivated by

Theorem 6 *(1881, Hermite-Hadamard inequality, [4]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers, and $a, b \in I$, with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\, dt \leq \frac{f(a) + f(b)}{2}. \tag{15}$$

Additionally to the classical convex functions, Toader [6], Hudzik and Maligranda [2] and Pinheiro [5] generalized the concepts of classical convex functions to the concepts of m-convex function and (s,m)-convex function.

Definition 7 *The function $f : [0, b^*] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$ and $b^* > 0$ if for every $x, y \in [0, b^*]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t) f(y). \tag{16}$$

Definition 8 *The function $f : [0, b^*] \to \mathbb{R}$ is said to be (s,m)-convex, where $(s,m) \in [0,1]^2$ and $b^* > 0$, if for every $x, y \in [0, b^*]$ and $t \in [0,1]$, we have*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t^s) f(y). \tag{17}$$

We need the following list of Lemmas and Theorems from [7].

Lemma 9 *Let $\alpha > 0$, $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L_1([a,b])$, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)=$$

$$\frac{(b-a)^2}{2}\int_0^1 m(t)\,f''(ta+(1-t)b)\,dt, \tag{18}$$

where

$$m(t)=\begin{cases} t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in\left[0,\frac{1}{2}\right), \\ 1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in\left[\frac{1}{2},1\right). \end{cases} \tag{19}$$

Lemma 10 *Let $\alpha > 0$, $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L_1([a,b])$, $r > 0$, then*

$$\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]=$$

$$(b-a)^2\int_0^1 k(t)\,f''(ta+(1-t)b)\,dt, \tag{20}$$

where

$$k(t)=\begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1}, & t\in\left[0,\frac{1}{2}\right), \\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}, & t\in\left[\frac{1}{2},1\right). \end{cases} \tag{21}$$

Lemma 11 *Let $\alpha > 0$, $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $a < mb \le b$. If $f'' \in L_1([a,b])$, $r > 0$, then*

$$\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[I_{a+}^{\alpha}f(mb)+I_{mb-}^{\alpha}f(a)\right]=$$

$$(mb-a)^2\int_0^1 k(t)\,f''(ta+m(1-t)b)\,dt, \tag{22}$$

where $k(t)$ is defined in (21).

The following fractional m-convex Hermite-Hadamard type inequalities also come from [7].

Theorem 12 *Let $f : [0,b^*] \to \mathbb{R}$ be a twice differentiable mapping with $b^* > 0$, $\alpha > 0$. If $|f''|^q$ is measurable and m-convex on $\left[a, \frac{b}{m}\right]$ for some fixed $q \ge 1$, $0 \le a < b$ and $m \in (0,1]$ with $\frac{b}{m} \le b^*$, $r > 0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[I_{a+}^{\alpha}f(b)+I_{b-}^{\alpha}f(a)\right]\right|$$

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$$\le (b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)\cdot$$

$$\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_1^m(f). \tag{23}$$

Theorem 13 *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $|f''|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\frac{(b-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_2^m(f), \tag{24}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 14 *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $|f''|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\frac{(b-a)^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_3^m(f). \tag{25}$$

Theorem 15 *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $|f''|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\cdot$$

$$\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_4^m(f), \tag{26}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 16 *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $|f''|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$H^m(f):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[I_{a+}^\alpha f(b)+I_{b-}^\alpha f(a)\right]\right|$$

$$\le\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\cdot$$

$$\left(\frac{|f''(a)|^q+m\left|f''\left(\frac{b}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}=:R_5^m(f). \tag{27}$$

The following fractional (s,m)-convex Hermite-Hadamard type inequalities also come from [7].

Theorem 17 *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\le a<mb\le b$, $\alpha>0$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q\ge1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f):=$$

$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$

$$\le(mb-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}\cdot \tag{28}$$

$$\left[|f''(a)|^q I+m|f''(b)|^q\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}}=:R_{1s}^m(f),$$

where

$$I=\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$
$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right).$$

Theorem 18 *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\le a<mb\le b$, $\alpha>0$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$ $r>0$, then*

$$H_s^m(f):=$$

$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(mb-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}|f''(a)|^q+\frac{ms}{s+1}|f''(b)|^q\right)^{\frac{1}{q}} \tag{29}$$
$$=: R_{2s}^m(f),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 19 *Let* $f:[0,b]\to\mathbb{R}$ *be a twice differentiable mapping with* $0\leq a<mb\leq b$, $\alpha>0$. *If* $|f''|^q$ *is measurable and* (s,m)*-convex on* $[a,b]$ *for some fixed* $q>1$ *and* $(s,m)\in(0,1]^2$, $r>0$, *then*

$$H_s^m(f):=$$
$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{r(\alpha+1)}\left[|f''(a)|^q\left(\frac{1}{s+1}-\frac{1}{q(s+1)+s+1}-B(s+1,q(\alpha+1)+1)\right)\right. \tag{30}$$
$$+m|f''(b)|^q\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}+\frac{1}{q(\alpha+1)+s+1}\right.$$
$$\left.\left.+B(s+1,q(\alpha+1)+1)\right)\right]=: R_{3s}^m(f).$$

Theorem 20 *Let* $f:[0,b]\to\mathbb{R}$ *be a twice differentiable mapping with* $0\leq a<mb\leq b$, $\alpha>0$. *If* $|f''|^q$ *is measurable and* (s,m)*-convex on* $[a,b]$ *for some fixed* $q>1$ *and* $(s,m)\in(0,1]^2$, $r>0$, *then*

$$H_s^m(f):=$$
$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$
$$\leq \frac{(mb-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}.$$
$$\left(\frac{1}{s+1}|f''(a)|^q+\frac{ms}{s+1}|f''(b)|^q\right)^{\frac{1}{q}}=: R_{4s}^m(f), \tag{31}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 21 *Let* $f:[0,b]\to\mathbb{R}$ *be a twice differentiable mapping with* $0\leq a<mb\leq b$, $\alpha>0$. *If* $|f''|^q$ *is measurable and* (s,m)*-convex on* $[a,b]$ *for some fixed* $q>1$ *and* $(s,m)\in(0,1]^2$, $r>0$, *then*

$$H_s^m(f):=$$
$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[I_{a+}^\alpha f(mb)+I_{mb-}^\alpha f(a)\right]\right|$$

$$\leq \frac{(mb-a)^2}{r+1}\left[|f''(a)|^q H + m|f''(b)|^q\left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right.\right.$$

$$\left.\left.-\frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}-H\right)\right] =: R_{5s}^m(f), \tag{32}$$

where

$$H=\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^q t^s dt+\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^q t^s dt. \tag{33}$$

The aim of this article is to extend the results of [7] to generalized fractional integrals (1) and (2), in particular to fractional exponential integrals (9), (10) and to fractional trigonometric integrals (60), (61). That is to produce very general fractional m-convex and (s,m)-convex Hermite-Hadamard type inequalities.

2 Main Results

Combining Theorems 12-16 we get the following m-convex Hermite-Hadamard type inequality.

Theorem 22 *Let $f:[0,b^*]\to\mathbb{R}$ be a twice differentiable mapping with $b^*>0$, $\alpha>0$. If $|f''|^q$ is measurable and m-convex on $\left[a,\frac{b}{m}\right]$ for some fixed $q>1$, $0\leq a<b$ and $m\in(0,1]$ with $\frac{b}{m}\leq b^*$, $r>0$, then*

$$H^m(f)\leq\min\left\{R_1^m(f),R_2^m(f),R_3^m(f),R_4^m(f),R_5^m(f)\right\}. \tag{34}$$

Combining Theorems 17-21 we obtain the following (s,m)-convex Hermite-Hadamard type inequality.

Theorem 23 *Let $f:[0,b]\to\mathbb{R}$ be a twice differentiable mapping with $0\leq a<mb\leq b$, $\alpha>0$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$H_s^m(f)\leq\min\left\{R_{1s}^m(f),R_{2s}^m(f),R_{3s}^m(f),R_{4s}^m(f),R_{5s}^m(f)\right\}. \tag{35}$$

Next we generalize Lemmas 9-11.

Lemma 24 *Let $\alpha>0$, $a<b$, $f\in C([a,b])$, $g\in C^1([a,b])$, g strictly increasing on $[a,b]$, $\left(f\circ g^{-1}\right)$ is twice differentiable function on $(g(a),g(b))$ with $\left(f\circ g^{-1}\right)''\in L_1([g(a),g(b)])$. Then*

$$\frac{\Gamma(\alpha+1)}{2(g(b)-g(a))^\alpha}\left[I_{a+;g}^\alpha f(b)+I_{b-;g}^\alpha f(a)\right]-\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right)=$$

$$\frac{(g(b)-g(a))^2}{2}\int_0^1 m(t)\left(f\circ g^{-1}\right)''(tg(a)+(1-t)g(b))\,dt, \tag{36}$$

where $m(t)$ as in (19).

Lemma 25 *Let all as in Lemma 24, $r>0$. Then*

$$\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right)$$
$$-\frac{\Gamma(\alpha+1)}{r(g(b)-g(a))^\alpha}\left[I^\alpha_{a+;g}f(b)+I^\alpha_{b-;g}f(a)\right]$$
$$=(g(b)-g(a))^2\int_0^1 k(t)\left(f\circ g^{-1}\right)''(tg(a)+(1-t)g(b))\,dt, \tag{37}$$

where $k(t)$ as in (21).

Lemma 26 *Let all as Lemma 25, with $g(a)<mg(b)\le g(b)$. Then*

$$\frac{f(a)+\left(f\circ g^{-1}\right)(mg(b))}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+mg(b)}{2}\right)$$
$$-\frac{\Gamma(\alpha+1)}{r(mg(b)-g(a))^\alpha}\left[I^\alpha_{g(a)+}\left(f\circ g^{-1}\right)(mg(b))+I^\alpha_{mg(b)-}\left(f\circ g^{-1}\right)(g(a))\right]$$
$$=(mg(b)-g(a))^2\int_0^1 k(t)\left(f\circ g^{-1}\right)''(tg(a)+m(1-t)g(b))\,dt, \tag{38}$$

where $k(t)$ as in (21).

We apply Lemmas 24-26 to $g(x)=e^x$.

Lemma 27 *Let $\alpha>0$, $a<b$, $f\in C([a,b])$, $(f\circ\ln)$ is twice differentiable function on $\left(e^a,e^b\right)$ with $(f\circ\ln)''\in L_1\left(\left[e^a,e^b\right]\right)$. Then*

$$\frac{\Gamma(\alpha+1)}{2\left(e^b-e^a\right)^\alpha}\left[I^\alpha_{a+;e^x}f(b)+I^\alpha_{b-;e^x}f(a)\right]-(f\circ\ln)\left(\frac{e^a+e^b}{2}\right)=$$
$$\frac{\left(e^b-e^a\right)^2}{2}\int_0^1 m(t)(f\circ\ln)''\left(te^a+(1-t)e^b\right)dt, \tag{39}$$

where $m(t)$ as in (19).

Lemma 28 *Let all as in Lemma 27, $r>0$. Then*

$$\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}(f\circ\ln)\left(\frac{e^a+e^b}{2}\right)-\frac{\Gamma(\alpha+1)}{r\left(e^b-e^a\right)^\alpha}\left[I^\alpha_{a+;e^x}f(b)+I^\alpha_{b-;e^x}f(a)\right]$$
$$=\left(e^b-e^a\right)^2\int_0^1 k(t)(f\circ\ln)''\left(te^a+(1-t)e^b\right)dt, \tag{40}$$

where $k(t)$ as in (21).

Lemma 29 *Let all as in Lemma 28, with $e^a < me^b \leq e^b$. Then*

$$\frac{f(a)+(f\circ\ln)\left(me^b\right)}{r(r+1)}+\frac{2}{r+1}(f\circ\ln)\left(\frac{e^a+me^b}{2}\right)$$

$$-\frac{\Gamma(\alpha+1)}{r\left(me^b-e^a\right)^\alpha}\left[I^\alpha_{e^a+}(f\circ\ln)\left(me^b\right)+I^\alpha_{me^b-}(f\circ\ln)\left(e^a\right)\right]$$

$$=\left(me^b-e^a\right)^2\int_0^1 k(t)(f\circ\ln)''\left(te^a+(1-t)e^b\right)dt, \tag{41}$$

where $k(t)$ as in (21).

We need

Notation 30 *We denote by*

$$H^m(f,g):=\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+g(b)}{2}\right)-\right.$$

$$\left.\frac{\Gamma(\alpha+1)}{r(g(b)-g(a))^\alpha}\left[I^\alpha_{a+;g}f(b)+I^\alpha_{b-;g}f(a)\right]\right|, \tag{42}$$

$$R_1^m(f,g):=(g(b)-g(a))^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{43}$$

$$R_2^m(f,g):=\frac{(g(b)-g(a))^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{44}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_3^m(f,g):=\frac{(g(b)-g(a))^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}, \tag{45}$$

$$R_4^m(f,g):=\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(g(b)-g(a))^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\right.$$

$$-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\Bigg]^{\frac{1}{p}}\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^{q}+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{46}$$

where $\frac{1}{p}+\frac{1}{q}=1$,
and

$$R_5^m(f,g):=\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(g(b)-g(a))^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\right.$$

$$\left.\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\left(\frac{\left|\left(f\circ g^{-1}\right)''(g(a))\right|^{q}+m\left|\left(f\circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{47}$$

We present the following fractional generalised m-convex Hermite-Hadamard type inequality.

Theorem 31 *Let all as in Notation 30. Here $\alpha>0$, $b^*>0$, $f\in C([0,b^*])$, $g\in C^1([0,b^*])$, g is strictly increasing on $[0,b^*]$ with $g(0)=0$. Assume that $f\circ g^{-1}:[0,g(b^*)]\to\mathbb{R}$ is twice differentiable mapping. If $\left|\left(f\circ g^{-1}\right)''\right|^{q}$ is measurable and m-convex on $\left[g(a),\frac{g(b)}{m}\right]$ for some fixed $q>1$, $0\leq a<b\leq b^*$ and $m\in(0,1]$ with $\frac{g(b)}{m}\leq g(b^*)$, $r>0$, then*

$$H^m(f,g)\leq\min\left\{R_1^m(f,g),\ R_2^m(f,g),\ R_3^m(f,g),\ R_4^m(f,g),\ R_5^m(f,g)\right\}. \tag{48}$$

Proof. By Theorem 22. ■

We need

Notation 32 *We denote by*

$$H_s^m(f,g):=\left|\frac{f(a)+\left(f\circ g^{-1}\right)(mg(b))}{r(r+1)}+\frac{2}{r+1}\left(f\circ g^{-1}\right)\left(\frac{g(a)+mg(b)}{2}\right)-\right.$$

$$\left.\frac{\Gamma(\alpha+1)}{r(mg(b)-g(a))^{\alpha}}\left[I_{g(a)+}^{\alpha}\left(f\circ g^{-1}\right)(mg(b))+I_{mg(b)-}^{\alpha}\left(f\circ g^{-1}\right)(g(a))\right]\right|, \tag{49}$$

$$R_{1s}^m(f,g):=(mg(b)-g(a))^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}\cdot$$

$$\left[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^{q}I+m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^{q}\cdot\right. \tag{50}$$

$$\left.\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}},$$

where

$$I := \frac{1}{r(s+1)(s+\alpha+2)} - \frac{1}{r(\alpha+1)} B(s+1,\alpha+2) + \frac{1}{(r+1)(s+1)(s+2)} \left(1 - \left(\frac{1}{2}\right)^{s+1}\right), \tag{51}$$

$$R_{2s}^{m}(f,g) := \frac{(mg(b)-g(a))^2}{r(\alpha+1)} \left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \cdot \left(\frac{1}{s+1}\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + \frac{ms}{s+1}\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\right)^{\frac{1}{q}}, \tag{52}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_{3s}^{m}(f,g) := \frac{(mg(b)-g(a))^2}{r(\alpha+1)} \left[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q \left(\frac{1}{s+1} - \frac{1}{q(\alpha+1)+s+1} - B(s+1,q(\alpha+1)+1)\right) + m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q \left(\frac{s}{s+1} - \frac{2}{q(\alpha+1)+1} + \frac{1}{q(\alpha+1)+s+1} + B(s+1,q(\alpha+1)+1)\right)\right], \tag{53}$$

$$R_{4s}^{m}(f,g) := \frac{(mg(b)-g(a))^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}} \left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}} \left(\frac{1}{s+1}\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q + \frac{ms}{s+1}\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q\right)^{\frac{1}{q}}, \tag{54}$$

where $\frac{1}{p}+\frac{1}{q}=1$,
and

$$R_{5s}^{m}(f,g) := \frac{(mg(b)-g(a))^2}{r+1} \left[\left|\left(f\circ g^{-1}\right)''(g(a))\right|^q H + m\left|\left(f\circ g^{-1}\right)''(g(b))\right|^q \cdot \left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1} - \frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1} - H\right)\right], \tag{55}$$

where

$$H = \int_0^{\frac{1}{2}} \left(\frac{r+1}{r(\alpha+1)}+t\right)^q t^s dt + \int_{\frac{1}{2}}^1 \left(\frac{r+1}{r(\alpha+1)}+1-t\right)^q t^s dt. \tag{56}$$

Next we present a fractional generalised (s,m)-convex Hermite-Hadamard type inequality.

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Theorem 33 *Here all as in Notation 32. Let* $\alpha > 0$, $b > 0$, $f \in C([0,b])$, $g \in C^1([0,b])$, *g is strictly increasing on* $[0,b]$ *with* $g(0) = 0$. *Assume that* $f \circ g^{-1} : [0, g(b)] \to \mathbb{R}$ *is twice differentiable mapping, with* $0 \leq g(a) < mg(b) \leq g(b)$, $a \in [0,b]$. *If* $\left|\left(f \circ g^{-1}\right)''\right|^q$ *is measurable and* (s,m)*-convex on* $[g(a), g(b)]$ *for some fixed* $q > 1$ *and* $(s,m) \in (0,1]^2$, $r > 0$, *then*

$$H_s^m(f,g) \leq \min\left\{R_{1s}^m(f,g),\ R_{2s}^m(f,g),\ R_{3s}^m(f,g),\ R_{4s}^m(f,g),\ R_{5s}^m(f,g)\right\}. \tag{57}$$

Proof. By Theorem 23. ■

The case $q = 1$ is met separately.

Proposition 34 *Here* $H^m(f,g)$ *as in (42) of Notation 30. The rest of the assumptions as in Theorem 31 with* $q = 1$. *Then*

$$H^m(f,g) \leq (g(b) - g(a))^2 \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right) \cdot \left(\frac{\left|\left(f \circ g^{-1}\right)''(g(a))\right| + m\left|\left(f \circ g^{-1}\right)''\left(\frac{g(b)}{m}\right)\right|}{2}\right). \tag{58}$$

Proof. By Theorem 12. ■

Proposition 35 *Here* $H_s^m(f,g)$ *as in (49) of Notation 32. The rest of the assumptions as in Theorem 33 with* $q = 1$. *Then*

$$H_s^m(f,g) \leq (mg(b) - g(a))^2 \left[\left|\left(f \circ g^{-1}\right)''(g(a))\right| I + m\left|\left(f \circ g^{-1}\right)''(g(b))\right| \cdot \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)} - I\right)\right], \tag{59}$$

where I *as in (51).*

Proof. By Theorem 17. ■

We need

Definition 36 *Let* $a, b \in \left[0, \frac{\pi}{2}\right]$, $a < b$, $\alpha > 0$, $f \in L_\infty([a,b])$. *We consider the left and right fractional trigonometric integrals of* f *with respect to sine function denoted by sin :*

$$\left(I_{a+;\sin}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\sin x - \sin t)^{\alpha-1} \cos t f(t)\, dt, \quad x \geq a, \tag{60}$$

and

$$\left(I_{b-;\sin}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\sin t - \sin x)^{\alpha-1} \cos t f(t)\, dt, \quad x \leq b. \tag{61}$$

We need

Notation 37 *We denote by*

$$H_{*}^{m}(f,\sin) := \left| \frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}\left(f\circ\sin^{-1}\right)\left(\frac{\sin(a)+\sin(b)}{2}\right) - \right.$$

$$\left. \frac{\Gamma(\alpha+1)}{r(\sin(b)-\sin(a))^{\alpha}}\left[I_{a+;\sin}^{\alpha}f(b)+I_{b-;\sin}^{\alpha}f(a)\right]\right|, \tag{62}$$

$$R_{1*}^{m}(f,\sin) := (\sin(b)-\sin(a))^{2}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)\cdot$$

$$\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{63}$$

$$R_{2*}^{m}(f,\sin) := \frac{(\sin(b)-\sin(a))^{2}}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\cdot$$

$$\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{64}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_{3*}^{m}(f,\sin) := \frac{(\sin(b)-\sin(a))^{2}}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\cdot$$

$$\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{65}$$

$$R_{4*}^{m}(f,\sin) := \left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(\sin(b)-\sin(a))^{2}}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\right.$$

$$\left.\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{66}$$

where $\frac{1}{p}+\frac{1}{q}=1$,
and

$$R_{5*}^{m}(f,\sin) := \left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(\sin(b)-\sin(a))^{2}}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\right.$$

$$\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\Bigg]^{\frac{1}{q}}\left(\frac{\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}+m\left|\left(f\circ\sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}. \tag{67}$$

We present the following fractional generalised m-convex Hermite-Hadamard type inequality for sin function. So here $g(x)=\sin(x)$, $x\in\left[0,\frac{\pi}{2}\right]$.

Theorem 38 *Let all as in Notation 37. Here $\alpha>0$, $f\in C\left(\left[0,\frac{\pi}{2}\right]\right)$. Assume that $f\circ\sin^{-1}:[0,1]\to\mathbb{R}$ is twice differentiable mapping. If $\left|\left(f\circ\sin^{-1}\right)''\right|^{q}$ is measurable and m-convex on $\left[\sin(a),\frac{\sin(b)}{m}\right]$ for some fixed $q>1$, $0\leq a<b\leq\frac{\pi}{2}$ and $m\in(0,1]$ with $\sin(b)\leq m$, $r>0$, then*

$$H_{*}^{m}(f,\sin)\leq$$

$$\min\left\{R_{1*}^{m}(f,\sin),\ R_{2*}^{m}(f,\sin),\ R_{3*}^{m}(f,\sin),\ R_{4*}^{m}(f,\sin),\ R_{5*}^{m}(f,\sin)\right\}. \tag{68}$$

Proof. By Theorem 31. ■

We need

Notation 39 *We denote by*

$$H_{s*}^{m}(f,\sin):=\Bigg|\frac{f(a)+\left(f\circ\sin^{-1}\right)(m\sin(b))}{r(r+1)}+$$

$$\frac{2}{r+1}\left(f\circ\sin^{-1}\right)\left(\frac{\sin(a)+m\sin(b)}{2}\right)-\frac{\Gamma(\alpha+1)}{r(m\sin(b)-\sin(a))^{\alpha}}\cdot$$

$$\left[I_{\sin(a)+}^{\alpha}\left(f\circ\sin^{-1}\right)(m\sin(b))+I_{m\sin(b)-}^{\alpha}\left(f\circ\sin^{-1}\right)(\sin(a))\right]\Bigg|, \tag{69}$$

$$R_{1s*}^{m}(f,\sin):=(m\sin(b)-\sin(a))^{2}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}\cdot$$

$$\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^{q}I+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^{q}\cdot\right. \tag{70}$$

$$\left.\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}},$$

where

$$I:=\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$

$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right), \tag{71}$$

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$$R_{2s*}^{m}(f,\sin) := \frac{(m\sin(b)-\sin(a))^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \cdot$$

$$\left(\frac{1}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^q+\frac{ms}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^q\right)^{\frac{1}{q}}, \tag{72}$$

where $\frac{1}{p}+\frac{1}{q}=1$,

$$R_{3s*}^{m}(f,\sin) :=$$

$$\frac{(m\sin(b)-\sin(a))^2}{r(\alpha+1)}\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^q\left(\frac{1}{s+1}-\frac{1}{q(\alpha+1)+s+1}\right.\right.$$

$$-B(s+1,q(\alpha+1)+1))+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^q\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}\right.$$

$$\left.\left.+\frac{1}{q(\alpha+1)+s+1}+B(s+1,q(\alpha+1)+1)\right)\right], \tag{73}$$

$$R_{4s*}^{m}(f,\sin) := \frac{(m\sin(b)-\sin(a))^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\right.$$

$$\left.-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{1}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^q+\right.$$

$$\left.\frac{ms}{s+1}\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^q\right)^{\frac{1}{q}}, \tag{74}$$

where $\frac{1}{p}+\frac{1}{q}=1$,
and

$$R_{5s*}^{m}(f,\sin) :=$$

$$\frac{(m\sin(b)-\sin(a))^2}{r+1}\left[\left|\left(f\circ\sin^{-1}\right)''(\sin(a))\right|^q H+m\left|\left(f\circ\sin^{-1}\right)''(\sin(b))\right|^q\cdot\right.$$

$$\left.\left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}-H\right)\right], \tag{75}$$

where

$$H=\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^q t^s dt+\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^q t^s dt. \tag{76}$$

Next we present a fractional generalised (s,m)-convex Hermite-Hadamard type inequality involving $g(x)=\sin x$, $x\in\left[0,\frac{\pi}{2}\right]$.

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Theorem 40 *Here all as in Notation 39. Let* $\alpha > 0$, $a, b \in \left[0, \frac{\pi}{2}\right]$, $a < b$, $f \in C([0, b])$. *Assume that* $f \circ \sin^{-1} : [0, \sin(b)] \to \mathbb{R}$ *is twice differentiable mapping, with* $0 \leq \sin(a) < m \sin(b) \leq \sin(b)$. *If* $\left|\left(f \circ \sin^{-1}\right)''\right|^q$ *is measurable and* (s, m)*-convex on* $[\sin(a), \sin(b)]$ *for some fixed* $q > 1$ *and* $(s, m) \in (0, 1]^2$, $r > 0$, *then*

$$H_{s*}^m(f, \sin) \leq$$

$$\min\left\{R_{1s*}^m(f, \sin),\ R_{2s*}^m(f, \sin),\ R_{3s*}^m(f, \sin),\ R_{4s*}^m(f, \sin),\ R_{5s*}^m(f, \sin)\right\}. \tag{77}$$

Proof. By Theorem 33. ■

Finally we treat the case of $q = 1$ when $g(x) = \sin x$, $x \in \left[0, \frac{\pi}{2}\right]$.

Proposition 41 *Here* $H_*^m(f, \sin)$ *as in (62) of Notation 37. The rest of the assumptions as in Theorem 38 with* $q = 1$. *Then*

$$H_*^m(f, \sin) \leq (\sin(b) - \sin(a))^2 \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right) \cdot$$

$$\left(\frac{\left|\left(f \circ \sin^{-1}\right)''(\sin(a))\right| + m\left|\left(f \circ \sin^{-1}\right)''\left(\frac{\sin(b)}{m}\right)\right|}{2}\right). \tag{78}$$

Proof. By Proposition 34. ■

Proposition 42 *Here* $H_{s*}^m(f, \sin)$ *as in (69) of Notation 39. The rest of the assumptions as in Theorem 40 with* $q = 1$. *Then*

$$H_{s*}^m(f, \sin) \leq (m \sin(b) - \sin(a))^2 \left[\left|\left(f \circ \sin^{-1}\right)''(\sin(a))\right| I + \right.$$

$$\left. m\left|\left(f \circ \sin^{-1}\right)''(\sin(b))\right| \left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)} - I\right)\right], \tag{79}$$

where I *as in (51).*

Proof. By Proposition 35. ■

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