Most General Fractional Representation formula for functions and implications

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present the most general fractional representation formulae for a function in terms of the most general fractional integral operators due to S. Kalla, [3], [4], [5]. The last include most of the well-known fractional integrals such as of Riemann-Liouville, Erdélyi-Kober and Saigo, etc. Based on these we derive very general fractional Ostrowski type inequalities.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Fractional Representation, Kalla Fractional integral, Ostrowski inequality.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [10]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P_1(x,t) f'(t)\, dt, \tag{1}$$

where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a,b]. \tag{4}$$

Properties of the above operator can be found in [9].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^\alpha f(b)}{b-a} - J_a^{\alpha-1} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f'(b)) \right\}. \tag{5}$$

When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).

Motivated by (5), here we establish a very general fractional representation formula based on the most general fractional integral due to S. Kalla, [3], [4], [5]. The last integral includes almost all other fractional integrals as special cases. We then establish a very general fractional Ostrowski type inequality.

We finish with applications.

2 Main Results

Here let $f : \mathbb{R}_+ \to \mathbb{R}$ differentiable with $f' : \mathbb{R}_+ \to \mathbb{R}$ be integrable. Let also $\Phi : [0,1] \to \mathbb{R}_+$ a general kernel function, which is differentiable with $\Phi' : [0,1] \to \mathbb{R}_+$ being integrable too. For z *in* $(0,1)$ we assume $\Phi(z) > 0$.

Let here the parameters γ, δ be such that $\gamma > -1$ and $\delta \in \mathbb{R}$. Set $\varepsilon := \delta - \gamma - 1$, that is $\delta = \varepsilon + \gamma + 1$.

The most general fractional integral operator was defined by S. Kalla ([3], [4], [5]), see also [7], as follows:

$$I_\Phi^{\gamma,\delta} f(x) := x^\delta \int_0^1 \Phi(\sigma) \sigma^\gamma f(x\sigma) \, d\sigma, \tag{6}$$

for any $x > 0$, with $I_\Phi^{\gamma,\delta} f(0) := 0$.

Here we consider $b > 0$ fixed, and $0 < x < b$. We operate on $[0,b]$.

By convenient change of variable we can rewrite $I_\Phi^{\gamma,\delta} f(x)$ as follows:

$$I_\Phi^{\gamma,\varepsilon} f(x) := x^\varepsilon \int_0^x \Phi\left(\frac{w}{x}\right) w^\gamma f(w) \, dw. \tag{7}$$

That is

$$I_\Phi^{\gamma,\varepsilon} f(x) = I_\Phi^{\gamma,\delta} f(x), \text{ for any } x > 0. \tag{8}$$

We take $\gamma > 0$ from now on.

We present the following most general fractional representation formula.

Theorem 2 *All as above described. Then*

$$f(x) = b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \left[\frac{1}{b} I_{\Phi}^{\gamma,\delta} f(b) + \gamma I_{\Phi}^{\gamma-1,\delta}\left(P_1(x,b) f(b)\right)\right.$$

$$\left. + \frac{1}{b} I_{\Phi'}^{\gamma,\delta}\left(P_1(x,b) f(b)\right) + I_{\Phi}^{\gamma,\delta}\left(P_1(x,b) f'(b)\right)\right]. \tag{9}$$

Proof. We observe that

$$I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b) f'(b)\right) = b^{\varepsilon} \int_0^b \Phi\left(\frac{w}{b}\right) w^{\gamma} P_1(x,w) f'(w)\, dw = \tag{10}$$

$$b^{\varepsilon}\left[\int_0^x \Phi\left(\frac{w}{b}\right) w^{\gamma} \frac{w}{b} f'(w)\, dw + \int_x^b \Phi\left(\frac{w}{b}\right) w^{\gamma}\left(\frac{w-b}{b}\right) f'(w)\, dw\right] = \tag{11}$$

$$b^{\varepsilon-1}\left[\int_0^x \Phi\left(\frac{w}{b}\right) w^{\gamma+1} f'(w)\, dw + \int_x^b \Phi\left(\frac{w}{b}\right)\left(w^{\gamma+1} - b w^{\gamma}\right) f'(w)\, dw\right] =$$

$$b^{\varepsilon-1}\left[\Phi\left(\frac{x}{b}\right) x^{\gamma+1} f(x) - \int_0^x f(w)\, d\left(\Phi\left(\frac{w}{b}\right) w^{\gamma+1}\right) - \right.$$

$$\left.\Phi\left(\frac{x}{b}\right)\left(x^{\gamma+1} - b x^{\gamma}\right) f(x) - \int_x^b f(w)\, d\left(\Phi\left(\frac{w}{b}\right)\left(w^{\gamma+1} - b w^{\gamma}\right)\right)\right] =$$

$$b^{\varepsilon-1}\left[b x^{\gamma} \Phi\left(\frac{x}{b}\right) f(x) - \int_0^x f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right) w^{\gamma+1} + (\gamma+1)\Phi\left(\frac{w}{b}\right) w^{\gamma}\right] dw - \right. \tag{12}$$

$$\left.\int_x^b f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1} - b w^{\gamma}\right) + \Phi\left(\frac{w}{b}\right)\left((\gamma+1) w^{\gamma} - b\gamma w^{\gamma-1}\right)\right] dw\right] =$$

$$b^{\varepsilon-1}\left[b x^{\gamma} \Phi\left(\frac{x}{b}\right) f(x) - \frac{1}{b}\int_0^x f(w)\, \Phi'\left(\frac{w}{b}\right) w^{\gamma+1} dw - \right.$$

$$(\gamma+1)\int_0^x f(w)\, \Phi\left(\frac{w}{b}\right) w^{\gamma} dw - \int_0^b f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1} - b w^{\gamma}\right) + \right.$$

$$\left.\Phi\left(\frac{w}{b}\right)\left((\gamma+1) w^{\gamma} - b\gamma w^{\gamma-1}\right)\right] dw + \int_0^x f(w)\left[\frac{1}{b}\Phi'\left(\frac{w}{b}\right)\left(w^{\gamma+1} - b w^{\gamma}\right) + \right. \tag{13}$$

$$\left.\left.\Phi\left(\frac{w}{b}\right)\left((\gamma+1) w^{\gamma} - b\gamma w^{\gamma-1}\right)\right] dw\right] =$$

$$b^{\varepsilon-1}\left[b x^{\gamma} \Phi\left(\frac{x}{b}\right) f(x) - \frac{1}{b}\int_0^b f(w)\, \Phi'\left(\frac{w}{b}\right) w^{\gamma+1} dw + \int_0^b f(w)\, \Phi'\left(\frac{w}{b}\right) w^{\gamma} dw - \right.$$

$$(\gamma+1)\int_0^b f(w)\, \Phi\left(\frac{w}{b}\right) w^{\gamma} dw + b\gamma \int_0^b f(w)\, \Phi\left(\frac{w}{b}\right) w^{\gamma-1} dw -$$

3

$$\left.\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw - b\gamma\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw\right] =: (\eta)\,. \tag{14}$$

We notice that

$$-\frac{1}{b}\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma+1}dw = -\left[\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)\frac{w}{b}w^{\gamma}dw+\right.$$

$$\left.\int_x^b f(w)\,\Phi'\left(\frac{w}{b}\right)\frac{(w-b)}{b}w^{\gamma}dw + \int_x^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw\right] = \tag{15}$$

$$-\int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma}dw - \int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw$$

$$+\int_0^x f(w)\,\Phi'\left(\frac{w}{b}\right)w^{\gamma}dw.$$

Furthermore we have

$$-\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw = -\gamma\left[b\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)\frac{w}{b}w^{\gamma-1}dw+\right.$$

$$\left.b\int_x^b f(w)\,\Phi\left(\frac{w}{b}\right)\frac{(w-b)}{b}w^{\gamma-1}dw + b\int_x^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw\right] = \tag{16}$$

$$-b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma-1}dw - b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw$$

$$+b\gamma\int_0^x f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma-1}dw.$$

Putting together (10), (14), (15), (16) we obtain

$$I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right) = (\eta) =$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x) - \int_0^b f(w)\,\Phi'\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma}dw-\right. \tag{17}$$

$$\left.\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)w^{\gamma}dw - b\gamma\int_0^b f(w)\,\Phi\left(\frac{w}{b}\right)P_1(x,w)\,w^{\gamma-1}dw\right] =$$

$$b^{\varepsilon-1}\left[bx^{\gamma}\Phi\left(\frac{x}{b}\right)f(x) - \frac{1}{b^{\varepsilon}}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right)\right.$$

$$\left.-\frac{1}{b^{\varepsilon}}I_{\Phi}^{\gamma,\varepsilon}f(b) - \gamma b^{1-\varepsilon}I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right)\right] = \tag{18}$$

$$b^{\varepsilon}x^{\gamma}\Phi\left(\frac{x}{b}\right)f(x) - \frac{1}{b}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right) - \frac{1}{b}I_{\Phi}^{\gamma,\varepsilon}f(b) - \gamma I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right).$$

That is

$$I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right) = b^{\varepsilon}x^{\gamma}\Phi\left(\frac{x}{b}\right)f(x) -$$

$$\frac{1}{b}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right) - \frac{1}{b}I_{\Phi}^{\gamma,\varepsilon}f(b) - \gamma I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right). \qquad (19)$$

Solving the last (19) for $f(x)$ we get

$$f(x) = b^{-\varepsilon}x^{-\gamma}\left(\Phi\left(\frac{x}{b}\right)\right)^{-1}\left[\frac{1}{b}I_{\Phi}^{\gamma,\varepsilon}f(b) + \gamma I_{\Phi}^{\gamma-1,\varepsilon}\left(P_1(x,b)\,f(b)\right) + \right.$$

$$\left.\frac{1}{b}I_{\Phi'}^{\gamma,\varepsilon}\left(P_1(x,b)\,f(b)\right) + I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right)\right], \qquad (20)$$

proving the claim. ■

Next we establish a very general fractional Ostrowski type inequality.

Theorem 3 *Here all as in Theorem 2. Then*

$$\left|f(x) - b^{\gamma+1-\delta}x^{-\gamma}\left(\Phi\left(\frac{x}{b}\right)\right)^{-1}\left[\frac{1}{b}I_{\Phi}^{\gamma,\delta}f(b) + \right.\right.$$

$$\left.\left.\gamma I_{\Phi}^{\gamma-1,\delta}\left(P_1(x,b)\,f(b)\right) + \frac{1}{b}I_{\Phi'}^{\gamma,\delta}\left(P_1(x,b)\,f(b)\right)\right]\right| \leq$$

$$b^{-1}x^{-\gamma}\left(\Phi\left(\frac{x}{b}\right)\right)^{-1}\|\Phi\|_{\infty,[0,1]}\|f'\|_{\infty,[0,b]}\left[\frac{\left(2x^{\gamma+2}-b^{\gamma+2}\right)}{\gamma+2} + \frac{b\left(b^{\gamma+1}-x^{\gamma+1}\right)}{\gamma+1}\right]. \qquad (21)$$

Proof. We observe that

$$\left|I_{\Phi}^{\gamma,\delta}\left(P_1(x,b)\,f'(b)\right)\right| = \left|I_{\Phi}^{\gamma,\varepsilon}\left(P_1(x,b)\,f'(b)\right)\right| = \qquad (22)$$

$$b^{\varepsilon}\left|\int_0^b \Phi\left(\frac{w}{b}\right)w^{\gamma}P_1(x,w)\,f'(w)\,dw\right| \leq b^{\varepsilon}\int_0^b \Phi\left(\frac{w}{b}\right)w^{\gamma}\left|P_1(x,w)\right|\left|f'(w)\right|dw \leq$$

$$b^{\varepsilon}\|\Phi\|_{\infty,[0,1]}\|f'\|_{\infty,[0,b]}\int_0^b w^{\gamma}\left|P_1(x,w)\right|dw = \qquad (23)$$

$$b^{\varepsilon}\|\Phi\|_{\infty,[0,1]}\|f'\|_{\infty,[0,b]}\left[\frac{1}{b}\int_0^x w^{\gamma+1}dw + \frac{1}{b}\int_x^b w^{\gamma}(b-w)\,dw\right] =$$

$$b^{\varepsilon-1}\|\Phi\|_{\infty,[0,1]}\|f'\|_{\infty,[0,b]}\left[\frac{2x^{\gamma+2}}{\gamma+2} + \frac{b}{\gamma+1}\left(b^{\gamma+1}-x^{\gamma+1}\right) - \frac{b^{\gamma+2}}{\gamma+2}\right]. \qquad (24)$$

That is we derived

$$\left|I_{\Phi}^{\gamma,\delta}\left(P_1(x,b)\,f'(b)\right)\right| \leq$$

$$b^{\delta-\gamma-2}\|\Phi\|_{\infty,[0,1]}\|f'\|_{\infty,[0,b]}\left[\frac{\left(2x^{\gamma+2}-b^{\gamma+2}\right)}{\gamma+2} + \frac{b\left(b^{\gamma+1}-x^{\gamma+1}\right)}{\gamma+1}\right]. \qquad (25)$$

The claim is proved. ■

5

3 Applications

We mention

Definition 4 *Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for $f \in C(\mathbb{R}_+)$ is defined by ([12], see also [6, p. 19], [11]):*

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)\, d\tau, \quad (26)$$

where the function ${}_2F_1$ in (26) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (27)$$

and $(a)_n$ is the Pochhammer symbol $(a)_n = a(a+1)\ldots(a+n-1)$, $(a)_0 = 1$; where $c \neq 0, -1, -2, \ldots$.

Note 5 *Given that $a+b<c$, ${}_2F_1$ converges on $[-1,1]$, see [2].*

Furthermore we have

$$\frac{d\ {}_2F_1(a,b;c;t)}{dt} = \left(\frac{ab}{c}\right)\ {}_2F_1(a+1, b+1; c+1; t), \quad (28)$$

which converges on $[-1,1]$ when $1+a+b<c$. So when $1+a+b<c$, then both (27) and (28) converge on $[-1,1]$. Therefore when $\eta > 1+\beta$ we get that both ${}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right)$ and its derivative with respect to τ : $\left(\frac{(\alpha+\beta)\eta}{t\alpha}\right)$ ${}_2F_1\left(\alpha+\beta+1, -\eta+1; \alpha+1; 1-\frac{\tau}{t}\right)$, converge on $[0,1]$; notice here $0 \leq 1-\frac{\tau}{t} \leq 1$, $t>0$.

Remark 6 *The integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by*

$$J_0^{\alpha}\{f(x)\} = I_{0,t}^{\alpha,-\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (\alpha>0), \quad (29)$$

and

$$I^{\alpha,\eta}\{f(t)\} = I_{0,t}^{\alpha,0,\eta}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\eta} f(\tau)\, d\tau \quad (\alpha>0,\ \eta \in \mathbb{R}). \quad (30)$$

Remark 7 *By a simple change of variable ($w = \frac{\tau}{t}$) we get*

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta}}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1}\ {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-w) f(tw)\, dw. \quad (31)$$

Similarly we find

$$J_0^{\alpha}\{f(t)\} = \frac{t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} f(tw)\, dw, \tag{32}$$

and

$$I^{\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-w)^{\alpha-1} w^{\eta} f(tw)\, dw. \tag{33}$$

Remark 8 *([8]) The above Saigo fractional integral (26) and its special cases of Riemann-Liouville and Erdélyi-Kober fractional integrals (29), (30), are all examples of the S. Kalla ([5]) generalized fractional integral in the reduced form*

$$K_{\Phi}^{\gamma} f(x) = x^{-\gamma-1} \int_0^x \Phi\left(\frac{w}{x}\right) w^{\gamma} f(w)\, dw = \int_0^1 \Phi(\sigma)\, \sigma^{\gamma} f(x\sigma)\, d\sigma, \tag{34}$$

where $x > 0$, $\gamma > -1$ and Φ continuous arbitrary Kernel function.

Notice that (by (6) and (34))

$$I_{\Phi}^{\gamma,\delta} f(x) = x^{\delta} K_{\Phi}^{\gamma} f(x), \tag{35}$$

for any $x > 0$, where $\gamma > -1$ and $\delta \in \mathbb{R}$.

So for $b > 0$ we get

$$I_{\Phi}^{\gamma,\delta} f(b) = b^{\delta} K_{\Phi}^{\gamma} f(b). \tag{36}$$

Next we restrict ourselves to $\gamma > 0$. By Theorem 2 and (36) we obtain the following general fractional representation formula

Theorem 9 *It holds*

$$f(x) = b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \Big[b^{\delta-1} K_{\Phi}^{\gamma} f(b) + \gamma b^{\delta} K_{\Phi}^{\gamma-1} \left(P_1(x,b) f(b)\right) +$$
$$b^{\delta-1} K_{\Phi'}^{\gamma} \left(P_1(x,b) f(b)\right) + b^{\delta} K_{\Phi}^{\gamma} \left(P_1(x,b) f'(b)\right) \Big]. \tag{37}$$

We finish the following very general fractional Ostrowski type inequality, a direct application of (21) and (36).

Theorem 10 *All as in Theorem 3. Then*

$$\Bigg| f(x) - b^{\gamma+1-\delta} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \Big[b^{\delta-1} K_{\Phi}^{\gamma} f(b) + \tag{38}$$
$$\gamma b^{\delta} K_{\Phi}^{\gamma-1} \left(P_1(x,b) f(b)\right) + b^{\delta-1} K_{\Phi'}^{\gamma} \left(P_1(x,b) f(b)\right) \Big] \Bigg| \leq$$
$$b^{-1} x^{-\gamma} \left(\Phi\left(\frac{x}{b}\right)\right)^{-1} \|\Phi\|_{\infty,[0,1]} \|f'\|_{\infty,[0,b]} \left[\frac{\left(2x^{\gamma+2} - b^{\gamma+2}\right)}{\gamma+2} + \frac{b\left(b^{\gamma+1} - x^{\gamma+1}\right)}{\gamma+1} \right].$$

Comment 11 *One can apply (37) and (38) for the Riemann-Liouville and Erdélyi-Kober fractional integrals, as well as many other fractional integrals. To keep article short we omit this task.*

7

References

[1] G. Anastassiou, M. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, *Montgomery Identities for Fractional Integrals and Related Fractional Inequalities*, Journal of Inequalities in Pure And Applied Mathematics, Vol. 10 (2009), No. 4, Article 97, 6 pages.

[2] E.A. Chistova (originator), *Hypergeometric function*, Encyclopedia of Mathematics, URL:http://www.encyclopediaofmath.org/index.php?title=Hypergeometric_function&oldid=12873, Encyclopedia of Mathematics - ISBN 1402006098.

[3] S.L. Kalla, *On operators of fractional integration*, I. Mat. Notae 22, 1970, 89-93.

[4] S.L. Kalla, *On operators of fractional integration*, II Mat. Notae 25, 1976, 29-35.

[5] S.L. Kalla, *Operators of fractional integration.* In: Proc. Conf. Analytic Functions, Kozubnik 1979, Lecture Notes of Math. 798, 1980, 258-280, Springer, Berlin.

[6] V.S. Kiryakova, *Generalized Fractional Calculus and Applications* (Pitman Res. Notes Math. Ser. 301), Longman Scientific & Technical, Harlow, 1994.

[7] V.S. Kiryakova, *All the special functions are fractional differintegrals of elementary functions*, J. Phys. A: Math. Gen. 30 (1997), 5085-5103.

[8] V.S. Kiryakova, *On two Saigo's fractional integral operators in the class of univalent functions*, Fractional Calculus & Applied Analysis, Vol. 9, No. 2 (2006), 159-176.

[9] S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993).

[10] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1994).

[11] R.K. Raina, *Solution of Abel-type integral equation involving the Appell hypergeometric function*, Integral Transforms Spec. Funct., 21 (7) (2010), 515-522.

[12] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. Kyushu Univ., 11 (1978), 135-143.

Fractional Representation Formulae under initial conditions and fractional Ostrowski type inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present very general fractional representation formulae for a function in terms of the fractional Riemann-Liouville integrals of different orders of the function and its ordinary derivatives under initial conditions. Based on these we derive general fractional Ostrowski type inequalities with respect to all basic norms.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Fractional integral, Representation formulae, Ostrowski inequality.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [2]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P_1(x,t)\, f'(t)\, dt, \tag{1}$$

where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a,b]. \tag{4}$$

Properties of the above operator can be found in [3].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^\alpha f(b)}{b-a} - J_a^{\alpha-1} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f'(b)) \right\}. \tag{5}$$

When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).

In this article we find higher order fractional representation for $f(x)$, similar to basic (5), and from there we derive interesting fractional Ostrowski type inequalities.

2 Main Results

Next we give higher order fractional representation of f subject to initial conditions.

Theorem 2 *Let $\alpha > 2$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ twice differentiable, with $f'' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{2-\alpha}}{\alpha - 1} \Big[-(b-a)^{\alpha-2} f(a) + \Gamma(\alpha) \Big\{ \frac{2}{b-a} J_a^{\alpha-1} f(b) - \tag{6}$$

$$J_a^{\alpha-2} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f''(b)) \Big\} \Big].$$

Proof. Let here $\alpha > 2$ and there exists $f'' : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$.

We have

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f''(b)) = \int_a^b (b-t)^{\alpha-1} P_1(x,t) f''(t)\, dt = \tag{7}$$

$$\int_a^x \left(\frac{t-a}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt + \int_x^b \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt =$$

$$\int_a^x \left(\frac{t-a}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt + \int_a^b \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt \tag{8}$$

$$- \int_a^x \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt =$$

2

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)\,dt.$$

That is

$$\Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f''(b)) =$$

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)\,dt =: (\xi_1)\,. \tag{9}$$

Next we use integration by parts, plus the assumption $f'(x)=0$. We have

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt = \int_a^x (b-t)^{\alpha-1}\,df'(t) =$$

$$-(b-a)^{\alpha-1} f'(a) - \int_a^x f'(t)\,d(b-t)^{\alpha-1} = -(b-a)^{\alpha-1} f'(a) \tag{10}$$

$$+(\alpha-1)\int_a^x (b-t)^{\alpha-2}\,df(t) = -(b-a)^{\alpha-1} f'(a)$$

$$+(\alpha-1)\left[(b-x)^{\alpha-2} f(x) - (b-a)^{\alpha-2} f(a) - \int_a^x f(t)\,d(b-t)^{\alpha-2}\right] =$$

$$-(b-a)^{\alpha-1} f'(a) + (\alpha-1)(b-x)^{\alpha-2} f(x) - (\alpha-1)(b-a)^{\alpha-2} f(a) + \tag{11}$$

$$(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt.$$

That is

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt = -(b-a)^{\alpha-1} f'(a) + (\alpha-1)(b-x)^{\alpha-2} f(x) -$$

$$(\alpha-1)(b-a)^{\alpha-2} f(a) + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt =: (\lambda_1)\,. \tag{12}$$

Next we observe

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)\,dt = -\frac{1}{(b-a)}\left[\int_a^b (b-t)^{\alpha}\,df'(t)\right] =$$

$$-\frac{1}{(b-a)}\left[-(b-a)^{\alpha} f'(a) + \alpha\int_a^b (b-t)^{\alpha-1}\,df(t)\right] = -\frac{1}{(b-a)}\cdot$$

$$\left[-(b-a)^{\alpha} f'(a) - \alpha(b-a)^{\alpha-1} f(a) + \alpha(\alpha-1)\int_a^b (b-t)^{\alpha-2} f(t)\,dt\right] = \tag{13}$$

$$(b-a)^{\alpha-1} f'(a) + \alpha(b-a)^{\alpha-2} f(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} f(t)\,dt.$$

That is

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)\,dt = (b-a)^{\alpha-1} f'(a) +$$

$$\alpha (b-a)^{\alpha-2} f(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} f(t)\,dt =: (\lambda_2). \qquad (14)$$

We have that

$$(\xi_1) = (\lambda_1) + (\lambda_2).$$

Thus

$$\Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f''(b)) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) + \qquad (15)$$

$$(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt - \frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt.$$

Notice that

$$-\alpha(\alpha-1) = -(\alpha-1)(\alpha-2) - 2(\alpha-1). \qquad (16)$$

We split

$$-\frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt = -\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt$$

$$-\frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt. \qquad (17)$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt = \qquad (18)$$

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)^{\alpha-2} f(t)\,dt + \int_x^b (b-t)^{\alpha-2} f(t)\,dt\right] =$$

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)(b-t)^{\alpha-3} f(t)\,dt + \int_x^b (b-t)(b-t)^{\alpha-3} f(t)\,dt\right] \qquad (19)$$

$$= -(\alpha-1)(\alpha-2)\cdot$$

$$\left[\int_a^x \left(1-\left(\frac{t-a}{b-a}\right)\right)(b-t)^{\alpha-3} f(t)\,dt - \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt\right] =$$

$$-(\alpha-1)(\alpha-2)\left[\int_a^x (b-t)^{\alpha-3} f(t)\,dt - \left[\int_a^x \left(\frac{t-a}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt\right.\right. \qquad (20)$$

$$\left.\left. + \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt\right]\right] =$$

4

$$-\left(\alpha-1\right)\left(\alpha-2\right)\int_a^x\left(b-t\right)^{\alpha-3}f\left(t\right)dt+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\int_a^b P_1\left(x,t\right)\left(b-t\right)^{\alpha-3}f\left(t\right)dt. \tag{21}$$

Therefore

$$-\frac{\alpha\left(\alpha-1\right)}{b-a}\int_a^b\left(b-t\right)^{\alpha-2}f\left(t\right)dt=-\frac{2\left(\alpha-1\right)}{b-a}\int_a^b\left(b-t\right)^{\alpha-2}f\left(t\right)dt+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\int_a^b P_1\left(x,t\right)\left(b-t\right)^{\alpha-3}f\left(t\right)dt \tag{22}$$

$$-\left(\alpha-1\right)\left(\alpha-2\right)\int_a^x\left(b-t\right)^{\alpha-3}f\left(t\right)dt.$$

Hence it holds

$$\Gamma\left(\alpha\right)J_a^\alpha\left(P_1\left(x,b\right)f''\left(b\right)\right)=\left(\alpha-1\right)\left(b-x\right)^{\alpha-2}f\left(x\right)+\left(b-a\right)^{\alpha-2}f\left(a\right)-$$

$$\frac{2\left(\alpha-1\right)}{b-a}\int_a^b\left(b-t\right)^{\alpha-2}f\left(t\right)dt+\left(\alpha-1\right)\left(\alpha-2\right)\int_a^b P_1\left(x,t\right)\left(b-t\right)^{\alpha-3}f\left(t\right)dt= \tag{23}$$

$$\left(\alpha-1\right)\left(b-x\right)^{\alpha-2}f\left(x\right)+\left(b-a\right)^{\alpha-2}f\left(a\right)-\frac{2\left(\alpha-1\right)\Gamma\left(\alpha-1\right)}{b-a}J_a^{\alpha-1}f\left(b\right)+ \tag{24}$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\Gamma\left(\alpha-2\right)J_a^{\alpha-2}\left(P_1\left(x,b\right)f\left(b\right)\right)=$$

$$\left(\alpha-1\right)\left(b-x\right)^{\alpha-2}f\left(x\right)+\left(b-a\right)^{\alpha-2}f\left(a\right)-$$

$$\frac{2\Gamma\left(\alpha\right)}{b-a}J_a^{\alpha-1}f\left(b\right)+\Gamma\left(\alpha\right)J_a^{\alpha-2}\left(P_1\left(x,b\right)f\left(b\right)\right). \tag{25}$$

We have proved that

$$\left(\alpha-1\right)\left(b-x\right)^{\alpha-2}f\left(x\right)=-\left(b-a\right)^{\alpha-2}f\left(a\right)+\frac{2\Gamma\left(\alpha\right)}{b-a}J_a^{\alpha-1}f\left(b\right)-$$

$$\Gamma\left(\alpha\right)J_a^{\alpha-2}\left(P_1\left(x,b\right)f\left(b\right)\right)+\Gamma\left(\alpha\right)J_a^\alpha\left(P_1\left(x,b\right)f''\left(b\right)\right)= \tag{26}$$

$$-\left(b-a\right)^{\alpha-2}f\left(a\right)+$$

$$\Gamma\left(\alpha\right)\left\{\frac{2}{b-a}J_a^{\alpha-1}f\left(b\right)-J_a^{\alpha-2}\left(P_1\left(x,b\right)f\left(b\right)\right)+J_a^\alpha\left(P_1\left(x,b\right)f''\left(b\right)\right)\right\}. \tag{27}$$

We have produced (6). ■

We continue with

5

Theorem 3 *Let $\alpha > 3$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ three times differentiable, with $f''' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = f''(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{3-\alpha}}{(\alpha-1)(\alpha-2)} \left\{ -2(\alpha-1)(b-a)^{\alpha-3} f(a) - (b-a)^{\alpha-2} f'(a) + \right.$$

$$\left. \Gamma(\alpha) \left\{ \frac{3}{(b-a)} J_a^{\alpha-2} f(b) - J_a^{\alpha-3} \left(P_1(x,b) f(b)\right) + J_a^{\alpha} \left(P_1(x,b) f'''(b)\right) \right\} \right\}. \tag{28}$$

Proof. Let here $\alpha > 3$ and there exists $f''' : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. We have as before that

$$\Gamma(\alpha) J_a^{\alpha} \left(P_1(x,b) f'''(b)\right) =$$

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\, dt - \frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f'''(t)\, dt =: (\xi_2). \tag{29}$$

By assumption we have $f'(x) = f''(x) = 0$. We use repeatedly integration by parts next

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\, dt = \int_a^x (b-t)^{\alpha-1} df''(t) =$$

$$-(b-a)^{\alpha-1} f''(a) + (\alpha-1) \int_a^x (b-t)^{\alpha-2} f''(t)\, dt =$$

$$-(b-a)^{\alpha-1} f''(a) + (\alpha-1) \int_a^x (b-t)^{\alpha-2} df'(t) =$$

$$-(b-a)^{\alpha-1} f''(a) +$$

$$(\alpha-1) \left[-(b-a)^{\alpha-2} f'(a) + (\alpha-2) \int_a^x (b-t)^{\alpha-3} f'(t)\, dt \right] = \tag{30}$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) +$$

$$(\alpha-1)(\alpha-2) \int_a^x (b-t)^{\alpha-3} df(t) =$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2) \left[(b-x)^{\alpha-3} f(x) \right.$$

$$\left. -(b-a)^{\alpha-3} f(a) + (\alpha-3) \int_a^x (b-t)^{\alpha-4} f(t)\, dt \right] =$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) -$$

$$(\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f(a) + (\alpha-1)(\alpha-2)(\alpha-3) \int_a^x (b-t)^{\alpha-4} f(t)\, dt. \tag{31}$$

That is

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\,dt = -(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) +$$

$$(\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f(a) + \tag{32}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt =: (\omega_1).$$

Similarly we find

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f'''(t)\,dt = -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}\,df''(t) = \tag{33}$$

$$-\frac{1}{(b-a)}\left[-(b-a)^{\alpha} f''(a) + \alpha\int_a^b (b-t)^{\alpha-1} f''(t)\,dt\right] =$$

$$(b-a)^{\alpha-1} f''(a) - \frac{\alpha}{(b-a)}\int_a^b (b-t)^{\alpha-1}\,df'(t) =$$

$$(b-a)^{\alpha-1} f''(a) - \frac{\alpha}{(b-a)}\left[-(b-a)^{\alpha-1} f'(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2} f'(t)\,dt\right] \tag{34}$$

$$= (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2}\,df(t) =$$

$$(b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) - \tag{35}$$

$$\frac{\alpha(\alpha-1)}{(b-a)}\left[-(b-a)^{\alpha-2} f(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3} f(t)\,dt\right] =$$

$$(b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt. \tag{36}$$

That is we found

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f'''(t)\,dt = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =: (\omega_2). \tag{37}$$

Notice that

$$(\xi_2) = (\omega_1) + (\omega_2).$$

We have

$$\Gamma(\alpha)\, J_a^{\alpha}\left(P_1(x,b)\, f'''(b)\right) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) +$$

$$2(\alpha-1)(b-a)^{\alpha-3}f(a)+(a-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4}f(t)\,dt-$$
$$\frac{\alpha(a-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3}f(t)\,dt. \tag{38}$$

We notice that

$$-\alpha(\alpha-1)(\alpha-2)=-3(\alpha-1)(\alpha-2)-(\alpha-1)(\alpha-2)(\alpha-3). \tag{39}$$

Hence

$$\begin{gathered}-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3}f(t)\,dt=\\ -\frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3}f(t)\,dt-\\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3}f(t)\,dt.\end{gathered} \tag{40}$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3}f(t)\,dt=$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\left[\int_a^x (b-t)^{\alpha-3}f(t)\,dt+\int_x^b (b-t)^{\alpha-3}f(t)\,dt\right]= \tag{41}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$
$$\left[\int_a^x (b-t)(b-t)^{\alpha-4}f(t)\,dt+\int_x^b (b-t)(b-t)^{\alpha-4}f(t)\,dt\right]= \tag{42}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$
$$\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-4}f(t)\,dt-\int_x^b (t-b)(b-t)^{\alpha-4}f(t)\,dt\right]=$$
$$-(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x \left(1-\left(\frac{t-a}{b-a}\right)\right)(b-t)^{\alpha-4}f(t)\,dt-\right.$$
$$\left.\int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-4}f(t)\,dt\right]= \tag{43}$$
$$-(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x (b-t)^{\alpha-4}f(t)\,dt-\int_a^b P_1(x,t)(b-t)^{\alpha-4}f(t)\,dt\right].$$

8

We derived that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt = \tag{44}$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt.$$

Therefore we obtain

$$-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =$$

$$-\frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt - \tag{45}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt.$$

Combining (38) and (45) we find

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f'''(b)) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) +$$

$$2(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt + \tag{46}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt =$$

$$(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a) -$$

$$\frac{3(\alpha-1)(\alpha-2)\Gamma(\alpha-2)}{b-a} J_a^{\alpha-2} f(b) + \tag{47}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3) J_a^{\alpha-3}(P_1(x,b) f(b)) =$$

$$(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a) -$$

$$\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) + \Gamma(\alpha) J_a^{\alpha-3}(P_1(x,b) f(b)). \tag{48}$$

Consequently we get

$$(\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) = -(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$

$$+\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) - \Gamma(\alpha) J_a^{\alpha-3}(P_1(x,b) f(b)) + \Gamma(\alpha) J_a^\alpha (P_1(x,b) f'''(b)) = \tag{49}$$

9

$$-(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f(a) +$$

$$\Gamma(\alpha)\left\{\frac{3}{(b-a)} J_a^{\alpha-2} f(b) - J_a^{\alpha-3}(P_1(x,b) f(b)) + J_a^{\alpha}(P_1(x,b) f'''(b))\right\}, \quad (50)$$

proving the claim. ■

We continue with

Theorem 4 *Let $\alpha > 4$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ four times differentiable, with $f^{(4)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = f''(x) = f'''(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{4-\alpha}}{(\alpha-1)(\alpha-2)(\alpha-3)} \Big\{-3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(\alpha) -$$

$$2(\alpha-1)(b-a)^{\alpha-3} f'(a) - (b-a)^{\alpha-2} f^{(2)}(a) +$$

$$\Gamma(\alpha)\left\{\frac{4J_a^{\alpha-3}(f(b))}{(b-a)} - J_a^{\alpha-4}(P_1(x,b) f(b)) + J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right)\right\}\Big\}. \quad (51)$$

Proof. Let here $\alpha > 4$ and there exists $f^{(4)} : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. We have as before that

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right) = \int_a^x (b-t)^{\alpha-1} f^{(4)}(t)\,dt -$$

$$\frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f^{(4)}(t)\,dt =: (\xi_3).$$

By assumption we have $f'(x) = f''(x) = f'''(x) = 0$. We use repeatedly integration by parts next

$$\int_a^x (b-t)^{\alpha-1} f^{(4)}(t)\,dt = \int_a^x (b-t)^{\alpha-1} df^{(3)}(t) =$$

$$-(b-a)^{\alpha-1} f^{(3)}(a) + (\alpha-1)\int_a^x (b-t)^{\alpha-2} df^{(2)}(t) = -(b-a)^{\alpha-1} f^{(3)}(a) +$$

$$(52)$$

$$(\alpha-1)\left[-(b-a)^{\alpha-2} f^{(2)}(a) + (\alpha-2)\int_a^x (b-t)^{\alpha-3} df'(t)\right] =$$

$$-(b-a)^{\alpha-1} f^{(3)}(a) - (\alpha-1)(b-a)^{\alpha-2} f^{(2)}(a) +$$

$$(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} df'(t) =$$

$$-(b-a)^{\alpha-1} f^{(3)}(a) - (\alpha-1)(b-a)^{a-2} f^{(2)}(a) +$$

$$(\alpha-1)(\alpha-2)\left[-(b-a)^{\alpha-3} f'(a) + (\alpha-3)\int_a^x (b-t)^{\alpha-4} df(t)\right] = \quad (53)$$

$$-(b-a)^{\alpha-1}f^{(3)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a)-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f'(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x(b-t)^{\alpha-4}df(t)=$$
$$-(b-a)^{\alpha-1}f^{(3)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a)-(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f'(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4}f(x)-$$
$$(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f(a)+$$
$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x(b-t)^{\alpha-5}f(t)\,dt. \tag{54}$$

We find that

$$\int_a^x(b-t)^{\alpha-1}f^{(4)}(t)\,dt=-(b-a)^{\alpha-1}f^{(3)}(a)-(\alpha-1)(b-a)^{\alpha-2}f^{(2)}(a)-$$
$$(\alpha-1)(\alpha-2)(b-a)^{\alpha-3}f'(a)+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4}f(x)-$$
$$(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-4}f(a)+$$
$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x(b-t)^{\alpha-5}f(t)\,dt=:(\theta_1). \tag{55}$$

Next we observe that

$$-\frac{1}{(b-a)}\int_a^b(b-t)^{\alpha}f^{(4)}(t)\,dt=-\frac{1}{(b-a)}\int_a^b(b-t)^{\alpha}df^{(3)}(t)=$$
$$-\frac{1}{(b-a)}\left[-(b-a)^{\alpha}f^{(3)}(a)+\alpha\int_a^b(b-t)^{\alpha-1}df^{(2)}(t)\right]= \tag{56}$$
$$(b-a)^{\alpha-1}f^{(3)}(a)-\frac{\alpha}{b-a}\int_a^b(b-t)^{\alpha-1}df^{(2)}(t)=(b-a)^{\alpha-1}f^{(3)}(a)-$$
$$\frac{\alpha}{b-a}\left[-(b-a)^{\alpha-1}f^{(2)}(a)+(\alpha-1)\int_a^b(b-t)^{\alpha-2}df'(t)\right]=$$
$$(b-a)^{\alpha-1}f^{(3)}(a)+\alpha(b-a)^{\alpha-2}f^{(2)}(a)-\frac{\alpha(\alpha-1)}{(b-a)}\int_a^b(b-t)^{\alpha-2}df'(t)=$$
$$(b-a)^{\alpha-1}f^{(3)}(a)+\alpha(b-a)^{\alpha-2}f^{(2)}(a)-$$
$$\frac{\alpha(\alpha-1)}{(b-a)}\left[-(b-a)^{\alpha-2}f'(a)+(\alpha-2)\int_a^b(b-t)^{\alpha-3}df(t)\right]= \tag{57}$$
$$(b-a)^{\alpha-1}f^{(3)}(a)+\alpha(b-a)^{\alpha-2}f^{(2)}(a)+\alpha(\alpha-1)(b-a)^{\alpha-3}f'(a)-$$
$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b(b-t)^{\alpha-3}df(t)=$$

11

$$(b-a)^{\alpha-1} f^{(3)}(a) + \alpha (b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) - \tag{58}$$
$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)} \left[-(b-a)^{\alpha-3} f(a) + (\alpha-3) \int_a^b (b-t)^{\alpha-4} f(t)\,dt \right] =$$
$$(b-a)^{\alpha-1} f^{(3)}(a) + \alpha (b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$
$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt.$$

That is

$$-\frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f^{(4)}(t)\,dt = (b-a)^{\alpha-1} f^{(3)}(a) + \alpha(b-a)^{\alpha-2} f^{(2)}(a) +$$
$$\alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) -$$
$$\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt =: (\theta_2). \tag{59}$$

Notice that

$$(\xi_3) = (\theta_1) + (\theta_2). \tag{60}$$

We find that

$$\Gamma(\alpha) J_a^{\alpha} \left(P_1(x,b) f^{(4)}(b) \right) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$
$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) + \tag{61}$$
$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4) \int_a^x (b-t)^{\alpha-5} f(t)\,dt$$
$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt.$$

We have

$$-\alpha(\alpha-1)(\alpha-2)(\alpha-3) =$$
$$-4(\alpha-1)(\alpha-2)(\alpha-3) - (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4). \tag{62}$$

and

$$\frac{-\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt = \tag{63}$$
$$-\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt$$
$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\,dt.$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt =$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-5} f(t)\,dt\right.$$

$$\left.-\int_x^b (t-b)(b-t)^{\alpha-5} f(t)\,dt\right] =$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\left[\int_a^x (b-t)^{\alpha-5} f(t)\,dt\right.\tag{64}$$

$$\left.-\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt\right].$$

Therefore it holds

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt =$$

$$-\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x (b-t)^{\alpha-5} f(t)\,dt$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt.\tag{65}$$

Consequently we get

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$

$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) -$$

$$\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt+\tag{66}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt =$$

$$(b-a)^{\alpha-2} f^{(2)}(a)+2(\alpha-1)(b-a)^{\alpha-3} f'(a)+3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) -$$

$$\frac{4(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3)}{(b-a)} J_a^{\alpha-3}(f(b)) +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4) J_a^{\alpha-4}(P_1(x,b) f(b)) =\tag{67}$$

$$(b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) -$$
$$\frac{4\Gamma(\alpha)}{(b-a)} J_a^{\alpha-3}(f(b)) + \Gamma(\alpha) J_a^{\alpha-4}(P_1(x,b) f(b)). \tag{68}$$

That is

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$
$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) +$$
$$\Gamma(\alpha)\left\{-\frac{4J_a^{\alpha-3}(f(b))}{(b-a)} + J_a^{\alpha-4}(P_1(x,b) f(b))\right\}, \tag{69}$$

proving the claim. ■

We continue with

Theorem 5 *Let $\alpha > 5$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ five times differentiable, with $f^{(5)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1,2,3,4$. Then*

$$f(x) = \frac{(b-x)^{5-\alpha}}{\prod_{j=1}^{4}(\alpha-j)} \left\{ -4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5} f(a) - \right.$$
$$3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) - (b-a)^{\alpha-2} f^{(3)}(a) +$$
$$\left.\Gamma(\alpha)\left\{\frac{5}{(b-a)}\left(J_a^{\alpha-4}(f(b))\right) - J_a^{\alpha-5}(P_1(x,b) f(b)) + J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right)\right\}\right\}. \tag{70}$$

Proof. Let here $\alpha > 5$ and there exists $f^{(5)} : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$. We have as before that

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right) =$$
$$\int_a^x (b-t)^{\alpha-1} f^{(5)}(t)\,dt - \frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt =: (\xi_4). \tag{71}$$

By assumption we have $f^{(j)}(x) = 0$, $j = 1,2,3,4$. We use repeatedly integration by parts next

$$\int_a^x (b-t)^{\alpha-1} f^{(5)}(t)\,dt = \int_a^x (b-t)^{\alpha-1} df^{(4)}(t) =$$

$$
\begin{gathered}
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)+\left(\alpha-1\right)\int_{a}^{x}\left(b-t\right)^{\alpha-2}df^{(3)}\left(t\right)=-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)+\\
\left(\alpha-1\right)\left\{-\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)+\left(\alpha-2\right)\int_{a}^{x}\left(b-t\right)^{\alpha-3}df^{(2)}\left(t\right)\right\}=
\end{gathered}
\tag{72}
$$

$$
\begin{gathered}
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)+\\
\left(\alpha-1\right)\left(\alpha-2\right)\int_{a}^{x}\left(b-t\right)^{\alpha-3}df^{(2)}\left(t\right)=\\
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)+\\
\left(\alpha-1\right)\left(\alpha-2\right)\left\{-\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)+\left(\alpha-3\right)\int_{a}^{x}\left(b-t\right)^{\alpha-4}df'\left(t\right)\right\}=
\end{gathered}
\tag{73}
$$

$$
\begin{gathered}
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\int_{a}^{x}\left(b-t\right)^{\alpha-4}df'\left(t\right)\\
=-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)\\
-\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)+\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left\{-\left(b-a\right)^{\alpha-4}f'\left(a\right)+\left(\alpha-4\right)\int_{a}^{x}\left(b-t\right)^{\alpha-5}df\left(t\right)\right\}=\\
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4}f'\left(a\right)\\
+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\int_{a}^{x}\left(b-t\right)^{\alpha-5}df\left(t\right)=\\
-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4}f'\left(a\right)+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\cdot\\
\left\{\left(b-x\right)^{\alpha-5}f\left(x\right)-\left(b-a\right)^{\alpha-5}f\left(a\right)+\left(\alpha-5\right)\int_{a}^{x}\left(b-t\right)^{\alpha-6}f\left(t\right)dt\right\}.
\end{gathered}
\tag{74}
$$

That is

$$
\begin{gathered}
\int_{a}^{x}\left(b-t\right)^{\alpha-1}f^{(5)}\left(t\right)dt=-\left(b-a\right)^{\alpha-1}f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4}f'\left(a\right)+\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\left(b-x\right)^{\alpha-5}f\left(x\right)-\\
\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\left(b-a\right)^{\alpha-5}f\left(a\right)+
\end{gathered}
\tag{75}
$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x (b-t)^{\alpha-6} f(t)\,dt =: (\eta_1).$$

Next we observe that

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt = -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}\,df^{(4)}(t) =$$

$$-\frac{1}{(b-a)}\left\{-(b-a)^{\alpha} f^{(4)}(a) + \alpha\int_a^b (b-t)^{\alpha-1}\,df^{(3)}(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) - \frac{\alpha}{(b-a)}\int_a^b (b-t)^{\alpha-1}\,df^{(3)}(t) = \tag{76}$$

$$(b-a)^{\alpha-1} f^{(4)}(a) -$$

$$\frac{\alpha}{(b-a)}\left\{-(b-a)^{\alpha-1} f^{(3)}(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2}\,df^{(2)}(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2}\,df^{(2)}(t) =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) -$$

$$\frac{\alpha(\alpha-1)}{(b-a)}\left\{-(b-a)^{\alpha-2} f^{(2)}(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3}\,df'(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) -$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3}\,df'(t) =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) -$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\left\{-(b-a)^{\alpha-3} f'(a) + (\alpha-3)\int_a^b (b-t)^{\alpha-4}\,df(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + \tag{77}$$

$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4}\,df(t)$$

$$= (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) +$$

$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$

$$\left\{-(b-a)^{\alpha-4} f(a) + (\alpha-4)\int_a^b (b-t)^{\alpha-5} f(t)\,dt\right\}. \tag{78}$$

We proved that

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt = (b-a)^{\alpha-1} f^{(4)}(a) + \alpha (b-a)^{\alpha-2} f^{(3)}(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) +$$

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) - \tag{79}$$

$$\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt =: (\eta_2).$$

We have

$$(\xi_4) = (\eta_1) + (\eta_2).$$

Therefore it holds

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right) = (b-a)^{\alpha-2} f^{(3)}(a) + \tag{80}$$

$$2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x) +$$

$$4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x (b-t)^{\alpha-6} f(t)\,dt$$

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt.$$

We see that

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt =$$

$$-\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt \tag{81}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt.$$

We have

$$-\frac{\prod_{j=1}^{5}(\alpha-j)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt = -\frac{\prod_{j=1}^{5}(\alpha-j)}{(b-a)}\cdot$$

$$\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-6} f(t)\,dt - \int_x^b (t-b)(b-t)^{\alpha-6} f(t)\,dt\right] =$$

17

$$-\prod_{j=1}^{5}(\alpha-j)\left[\int_a^x (b-t)^{\alpha-6} f(t)\,dt - \int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)\,dt\right]. \quad (82)$$

Therefore it holds

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt =$$

$$-\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt -$$

$$\prod_{j=1}^{5}(\alpha-j)\int_a^x (b-t)^{\alpha-6} f(t)\,dt + \prod_{j=1}^{5}(\alpha-j)\int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)\,dt. \quad (83)$$

Consequently we get

$$\begin{gathered}\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right) = (b-a)^{\alpha-2} f^{(3)}(a) + \\ 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) \\ +(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x) + \\ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) - \\ \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt \\ +\prod_{j=1}^{5}(\alpha-j)\int_a^b P_1(x,t)(b-t)^{\alpha-6} f(t)\,dt.\end{gathered} \quad (84)$$

So that

$$\begin{gathered}\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right) = (b-a)^{\alpha-2} f^{(3)}(a) + \\ 2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) \\ +(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x) + \\ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) - \\ \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4)}{(b-a)}\left(J_a^{\alpha-4}(f(b))\right) \\ +\prod_{j=1}^{5}(\alpha-j)\Gamma(\alpha-5) J_a^{\alpha-5}\left(P_1(x,b) f(b)\right).\end{gathered} \quad (85)$$

And finally we derive

$$\prod_{j=1}^{4}(\alpha-j)(b-x)^{\alpha-5} f(x) = -4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) \quad (86)$$

$$-3\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-4}f'\left(a\right)$$

$$-2\left(\alpha-1\right)\left(b-a\right)^{\alpha-3}f^{(2)}\left(a\right)-\left(b-a\right)^{\alpha-2}f^{(3)}\left(a\right)+$$

$$\Gamma\left(\alpha\right)\left\{\frac{5}{\left(b-a\right)}\left(J_a^{\alpha-4}\left(f\left(b\right)\right)\right)-J_a^{\alpha-5}\left(P_1\left(x,b\right)f\left(b\right)\right)+J_a^{\alpha}\left(P_1\left(x,b\right)f^{(5)}\left(b\right)\right)\right\},$$

proving the claim. ■

In general holds the following fractional representation formula

Theorem 6 *Let $\alpha>n$, $n\in\mathbb{N}$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ n-times differentiable, with $f^{(n)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}\left(x\right)=0$, $j=1,...,n-1$. Then*

$$f\left(x\right)=\frac{\left(b-x\right)^{n-\alpha}}{\prod_{j=1}^{n-1}\left(\alpha-j\right)}\left\{-\left(n-1\right)\prod_{j=1}^{n-2}\left(\alpha-j\right)\left(b-a\right)^{\alpha-n}f\left(a\right)-\right. \tag{87}$$

$$\left(n-2\right)\prod_{j=1}^{n-3}\left(\alpha-j\right)\left(b-a\right)^{\alpha-n+1}f'\left(a\right)-\left(n-3\right)\prod_{j=1}^{n-4}\left(\alpha-j\right)\left(b-a\right)^{\alpha-n+2}f^{(2)}\left(a\right)$$

$$-\left(n-4\right)\prod_{j=1}^{n-5}\left(\alpha-j\right)\left(b-a\right)^{\alpha-n+3}f^{(3)}\left(a\right)-...$$

$$-\left(b-a\right)^{\alpha-2}f^{(n-2)}\left(a\right)+\Gamma\left(\alpha\right)\left\{\frac{n}{b-a}\left(J_a^{\alpha-n+1}\left(f\left(b\right)\right)\right)-J_a^{\alpha-n}\left(P_1\left(x,b\right)f\left(b\right)\right)\right.$$

$$\left.\left.+J_a^{\alpha}\left(P_1\left(x,b\right)f^{(n)}\left(b\right)\right)\right\}\right\}.$$

Above we assume that $\prod_{j=1}^{0}\left(\alpha-j\right)=1$, and $\prod_{j=1}^{k}\left(\alpha-j\right)=0$ if $k\in\{-1,-2,...\}$. Also set $f^{(-1)}\left(a\right):=0$.

Proof. Based on Theorems 1-5. ■

Theorems 1-5 are special cases of Theorem 6.

We give applications of Theorem 6 for $n=6,7$.

Theorem 7 *Let $\alpha>6$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ six times differentiable, with $f^{(6)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}\left(x\right)=0$, $j=1,...,5$. Then*

$$f\left(x\right)=\frac{\left(b-x\right)^{6-\alpha}}{\prod_{j=1}^{5}\left(\alpha-j\right)}\left\{-5\prod_{j=1}^{4}\left(\alpha-j\right)\left(b-a\right)^{\alpha-6}f\left(a\right)-\right.$$

$$4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f'(a)-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(2)}(a)-$$
$$2(\alpha-1)(b-a)^{\alpha-3}f^{(3)}(a)-(b-a)^{\alpha-2}f^{(4)}(a)+ \tag{88}$$
$$\Gamma(\alpha)\left\{\frac{6}{b-a}\left(J_a^{\alpha-5}(f(b))\right)-J_a^{\alpha-6}(P_1(x,b)f(b))+J_a^{\alpha}\left(P_1(x,b)f^{(6)}(b)\right)\right\}\right\}.$$

Theorem 8 *Let $\alpha>7$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ seven times differentiable, with $f^{(7)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x)=0$, $j=1,...,6$. Then*

$$f(x)=\frac{(b-x)^{7-\alpha}}{\prod_{j=1}^{6}(\alpha-j)}\left\{-6\prod_{j=1}^{5}(\alpha-j)(b-a)^{\alpha-7}f(a)-\right. \tag{89}$$

$$5\prod_{j=1}^{4}(\alpha-j)(b-a)^{\alpha-6}f'(a)-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f''(a)$$
$$-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(3)}(a)-2(\alpha-1)(b-a)^{\alpha-3}f^{(4)}(a)$$
$$-(b-a)^{\alpha-2}f^{(5)}(a)+\Gamma(\alpha)\left\{\frac{7}{b-a}\left(J_a^{\alpha-6}(f(b))\right)-J_a^{\alpha-7}(P_1(x,b)f(b))\right.$$
$$\left.\left.+J_a^{\alpha}\left(P_1(x,b)f^{(7)}(b)\right)\right\}\right\}.$$

We make

Remark 9 *We rewrite (87) as follows:*

$$E_n(f,\alpha,x):=f(x)+\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\left\{(n-1)\prod_{j=1}^{n-2}(\alpha-j)(b-a)^{\alpha-n}f(a)+\right. \tag{90}$$

$$(n-2)\prod_{j=1}^{n-3}(\alpha-j)(b-a)^{\alpha-n+1}f'(a)+(n-3)\prod_{j=1}^{n-4}(\alpha-j)(b-a)^{\alpha-n+2}f^{(2)}(a)$$
$$+(n-4)\prod_{j=1}^{n-5}(\alpha-j)(b-a)^{\alpha-n+3}f^{(3)}(a)+...+(b-a)^{\alpha-2}f^{(n-2)}(a)+$$
$$\left.+\Gamma(\alpha)\left\{-\frac{n}{b-a}\left(J_a^{\alpha-n+1}(f(b))\right)+J_a^{\alpha-n}(P_1(x,b)f(b))\right\}\right\}=$$

20

$$\frac{(b-x)^{n-\alpha}\Gamma(\alpha)}{\prod_{j=1}^{n-1}(\alpha-j)}J_a^{\alpha}\left(P_1(x,b)\,f^{(n)}(b)\right)=$$

$$\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b(b-t)^{\alpha-1}P_1(x,t)\,f^{(n)}(t)\,dt. \tag{91}$$

We upper bound $E_n(f,\alpha,x)$, that is we upper bound the right hand side of (91).

Consequently we produce fractional Ostrowski type inequalities motivated by [1] done there for $n=1$.

Theorem 10 *Let $\alpha>n$, $n\in\mathbb{N}$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ n-times differentiable, with $f^{(n)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x)=0$, $j=1,...,n-1$, and $\left\|f^{(n)}\right\|_{\infty}<\infty$. Then*

$$|E_n(f,\alpha,x)|\leq\frac{\left\|f^{(n)}\right\|_{\infty}}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^{\alpha}}{\alpha(\alpha+1)}-\frac{(b-x)^{n}}{\alpha}+\frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right], \tag{92}$$

where $E_n(f,\alpha,x)$ as in (90).

Proof. We have that

$$|E_n(f,\alpha,x)|\leq\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b(b-t)^{\alpha-1}|P_1(x,t)|\left|f^{(n)}(t)\right|dt\leq$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_{\infty}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[\int_a^x(b-t)^{\alpha-1}(t-a)\,dt+\int_x^b(b-t)^{\alpha}\,dt\right]= \tag{93}$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_{\infty}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[\int_a^x(b-t)^{\alpha-1}((b-a)-(b-t))\,dt+\frac{(b-x)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|f^{(n)}\right\|_{\infty}(b-x)^{n-\alpha}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[(b-a)\int_a^x(b-t)^{\alpha-1}dt-\int_a^x(b-t)^{\alpha}dt+\frac{(b-x)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\left\|f^{(n)}\right\|_{\infty}(b-x)^{n-\alpha}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\cdot$$

$$\left[(b-a)\left(\frac{(b-a)^{\alpha}}{\alpha}-\frac{(b-x)^{\alpha}}{\alpha}\right)-\frac{(b-a)^{\alpha+1}}{\alpha+1}+\frac{2(b-x)^{\alpha+1}}{\alpha+1}\right]= \tag{94}$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_{\infty}}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-a)^{\alpha}}{\alpha(\alpha+1)}-\frac{(b-x)^{\alpha}}{\alpha}+\frac{2(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\right]= \tag{95}$$

$$\frac{\left\|f^{(n)}\right\|_{\infty}}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^{\alpha}}{\alpha(\alpha+1)}-\frac{(b-x)^{n}}{\alpha}+\frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right].$$

■

Theorem 11 *Let all as in Theorem 6. Then*

$$|E_n(f,\alpha,x)| \leq \left(\frac{(b-x)^{n-\alpha}(b-a)^{\alpha-2}}{2\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a+|a+b-2x|)\left\|f^{(n)}\right\|_{L_1([a,b])}. \tag{96}$$

Proof. We have that

$$|E_n(f,\alpha,x)| \leq \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b (b-t)^{\alpha-1}|P_1(x,t)|\left|f^{(n)}(t)\right|dt \leq \tag{97}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\max\{x-a,b-x\}\left\|f^{(n)}\right\|_{L_1([a,b])}= \tag{98}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\left(\frac{b-a+|a+b-2x|}{2}\right)\left\|f^{(n)}\right\|_{L_1([a,b])}.$$

■

22

Theorem 12 *Let $p, q, r > 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let all as in Theorem 6, but now $f^{(n)} \in L_r([a,b])$. Then*

$$|E_n(f,\alpha,x)| \leq$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1} (\alpha - j)} \right) \frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \frac{(b-x)^{q+1} + (x-a)^{q+1}}{(q+1)} \right\}^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])}. \tag{99}$$

Proof. We have

$$|E_n(f,\alpha,x)| \overset{(97)}{\leq} \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1} (\alpha - j)} \right) \left(\int_a^b (b-t)^{(\alpha-1)p} \, dt \right)^{\frac{1}{p}} \cdot$$

$$\left(\int_a^b |P_1(x,t)|^q \, dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} = \tag{100}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1} (\alpha - j)} \right) \frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left(\int_a^x (t-a)^q \, dt + \int_x^b (b-t)^q \, dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} =$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1} (\alpha - j)} \right) \frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\frac{(x-a)^{q+1}}{(q+1)} + \frac{(b-t)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} =$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1} (\alpha - j)} \right) \frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \frac{(b-x)^{q+1} + (x-a)^{q+1}}{(q+1)} \right\}^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])}, \tag{101}$$

proving the claim. ∎

References

[1] G. Anastassiou, M. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, *Montgomery Identities for Fractional Integrals and Related Fractional Inequalities*, Journal of Inequalities in Pure And Applied Mathematics, Vol. 10 (2009), No. 4, Article 97, 6 pages.

[2] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1994).

[3] S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993).

Multivariate Fractional Representation Formula and Ostrowski type inequality

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we derive a multivariate fractional representation formula involving ordinary partial derivatives of first order. Then we prove a related multivariate fractional Ostrowski type inequality with respect to uniform norm.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Multivariate Fractional integral, Representation formula, Multivariate Ostrowski inequality.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [3]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \int_a^b P_1(x,t)\,f'(t)\,dt, \tag{1}$$

where $P_1(x, t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a, b]. \tag{4}$$

1

Properties of the above operator can be found in [4].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^\alpha f(b)}{b-a} - J_a^{\alpha-1} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f'(b)) \right\}. \tag{5}$$

When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).

We may rewrite (5) as follows

$$f(x) = (b-x)^{1-\alpha} \left[\frac{1}{b-a} \int_a^b (b-t)^{\alpha-1} f(t)\, dt - \right.$$

$$\left. (\alpha-1) \int_a^b (b-t)^{\alpha-2} P_1(x,t) f(t)\, dt + \int_a^b (b-t)^{\alpha-1} P_1(x,t) f'(t)\, dt \right]. \tag{6}$$

In this article based on (5), we establish a multivariate fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i, b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate fractional Ostrowski type inequality.

2 Main Results

We make

Assumption 2 *Let $f \in C^1 \left(\prod_{i=1}^m [a_i, b_i] \right)$.*

Assumption 3 *Let $f : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1, ..., m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1, ..., t_i, x_{i+1}, ..., x_m)$ it is integrable on $\prod_{j=1}^i [a_j, b_j]$, for all $i = 1, ..., m$, for any $(x_{i+1}, ..., x_m) \in \prod_{j=i+1}^m [a_j, b_j]$.*

Convention 4 *We set*

$$\prod_{j=1}^0 \cdot = 1. \tag{7}$$

Notation 5 *Here $x = \overrightarrow{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1, ..., t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. We denote the kernel*

$$P_1(x_i, t_i) = \begin{cases} \frac{t_i - a_i}{b_i - a_i}, & a_i \leq t_i \leq x_i, \\ \frac{t_i - b_i}{b_i - a_i}, & x_i < t_i \leq b_i, \end{cases} \tag{8}$$

We need

Definition 6 *(see [2]) Let* $\prod_{i=1}^{m}[a_i,b_i]\subset\mathbb{R}^m$, $m\in\mathbb{N}-\{1\}$, $a_i<b_i$, $a_i,b_i\in\mathbb{R}$. *Let* $\alpha>0$, $f\in L_1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$. *We define the left mixed Riemann-Liouville fractional multiple integral of order* α:

$$\left(I_{a+}^{\alpha}f\right)(x):=\frac{1}{(\Gamma(\alpha))^m}\int_{a_1}^{x_1}\dots\int_{a_m}^{x_m}\left(\prod_{i=1}^{m}(x_i-t_i)\right)^{\alpha-1}f(t_1,\dots,t_m)\,dt_1\dots dt_m, \tag{9}$$

where $x_i\in[a_i,b_i]$, $i=1,\dots,m$, *and* $x=(x_1,\dots,x_m)$, $a=(a_1,\dots,a_m)$, $b=(b_1,\dots,b_m)$.

We present the following multivariate fractional representation formula

Theorem 7 *Let* f *as in Assumption 2 or Assumption 3,* $\alpha\geq1$, $x_i\in[a_i,b_i)$, $i=1,\dots,m$. *Then*

$$f(x_1,\dots,x_m)=\frac{\left(\prod_{i=1}^{m}(b_i-x_i)\right)^{1-\alpha}(\Gamma(\alpha))^m}{\prod_{i=1}^{m}(b_i-a_i)}\left(I_{a+}^{\alpha}f\right)(b)+$$

$$\sum_{i=1}^{m}A_i(x_1,\dots,x_m)+\sum_{i=1}^{m}B_i(x_1,\dots,x_m), \tag{10}$$

where for $i=1,\dots,m$:

$$A_i(x_1,\dots,x_m):=\frac{-(\alpha-1)\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j-a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i-1}(b_j-t_j)\right)^{\alpha-1}\cdot \tag{11}$$

$$(b_i-t_i)^{\alpha-2}P_1(x_i,t_i)\,f(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1\dots dt_i,$$

and

$$B_i(x_1,\dots,x_m):=\frac{\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j-a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i}(b_j-t_j)\right)^{\alpha-1}\cdot \tag{12}$$

$$P_1(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1dt_2\dots dt_i.$$

Proof. By (6) we have

$$f(x_1,\dots,x_m)=(b_1-x_1)^{1-\alpha}\left[\frac{1}{b_1-a_1}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}f(t_1,x_2,\dots,x_m)\,dt_1\right.$$

$$-(\alpha-1)\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)\,f(t_1,x_2,\dots,x_m)\,dt_1$$

3

$$+\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1\Bigg],\tag{13}$$

and

$$f(t_1,x_2,...,x_m)=(b_2-x_2)^{1-\alpha}\left[\frac{1}{b_2-a_2}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_2\right.$$

$$-(\alpha-1)\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-2}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$\left.+\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2\right].\tag{14}$$

We plug in (14) into (13). Hence

$$f(x_1,...,x_m)=(b_1-x_1)^{1-\alpha}\left[\frac{1}{b_1-a_1}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}(b_2-x_2)^{1-\alpha}\cdot\right.$$

$$\left[\frac{1}{b_2-a_2}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_2\right.$$

$$-(\alpha-1)\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-2}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2\tag{15}$$

$$\left.+\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2\right]dt_1$$

$$-(\alpha-1)\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1$$

$$\left.+\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1\right].$$

That is we have so far

$$f(x_1,...,x_m)=\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2-\tag{16}$$

$$\frac{(\alpha-1)(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2$$

4

$$+\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2$$

$$-(\alpha-1)(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1$$

$$+(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1.$$

Call

$$A_1(x_1,...,x_m):=-(\alpha-1)(b_1-x_1)^{1-\alpha}\cdot$$

$$\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1, \tag{17}$$

$$B_1(x_1,...,x_m):=(b_1-x_1)^{1-\alpha}\cdot$$

$$\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1, \tag{18}$$

$$A_2(x_1,x_2,...,x_m):=-\frac{(\alpha-1)(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot \tag{19}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2,$$

and

$$B_2(x_1,x_2,...,x_m):=\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot \tag{20}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2.$$

We rewrite (16) as follows

$$f(x_1,...,x_m)=\frac{((b_1-x_1)(b_2-x_2))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}((b_1-t_1)(b_2-t_2))^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2+ \tag{21}$$

$$A_2(x_1,...,x_m)+B_2(x_1,...,x_m)+A_1(x_1,...,x_m)+B_1(x_1,...,x_m).$$

We continue with

$$f(t_1,t_2,x_3,...,x_m)\overset{(6)}{=}\frac{(b_3-x_3)^{1-\alpha}}{b_3-a_3}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3$$

5

$$-\left(\alpha-1\right)\left(b_{3}-x_{3}\right)^{1-\alpha}\int_{a_{3}}^{b_{3}}\left(b_{3}-t_{3}\right)^{\alpha-2}P_{1}\left(x_{3},t_{3}\right)f\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{3}$$

$$+\left(b_{3}-x_{3}\right)^{1-\alpha}\int_{a_{3}}^{b_{3}}\left(b_{3}-t_{3}\right)^{\alpha-1}P_{1}\left(x_{3},t_{3}\right)\frac{\partial f}{\partial x_{3}}\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{3}. \tag{22}$$

Next plug (22) into (21). Hence it holds

$$f\left(x_{1},...,x_{m}\right)=\frac{\left(\left(b_{1}-x_{1}\right)\left(b_{2}-x_{2}\right)\left(b_{3}-x_{3}\right)\right)^{1-\alpha}}{\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)\left(b_{3}-a_{3}\right)}\cdot$$

$$\int_{a_{1}}^{b_{1}}\int_{a_{2}}^{b_{2}}\int_{a_{3}}^{b_{3}}\left(\left(b_{1}-t_{1}\right)\left(b_{2}-t_{2}\right)\left(b_{3}-t_{3}\right)\right)^{\alpha-1}f\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3}$$

$$-\frac{\left(\alpha-1\right)\left(\left(b_{1}-x_{1}\right)\left(b_{2}-x_{2}\right)\left(b_{3}-x_{3}\right)\right)^{1-\alpha}}{\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)}\cdot$$

$$\int_{a_{1}}^{b_{1}}\int_{a_{2}}^{b_{2}}\int_{a_{3}}^{b_{3}}\left(\left(b_{1}-t_{1}\right)\left(b_{2}-t_{2}\right)\right)^{\alpha-1}\left(b_{3}-t_{3}\right)^{\alpha-2}P_{1}\left(x_{3},t_{3}\right)\cdot$$

$$f\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3}$$

$$+\frac{\left(\left(b_{1}-x_{1}\right)\left(b_{2}-x_{2}\right)\left(b_{3}-x_{3}\right)\right)^{1-\alpha}}{\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)}\cdot \tag{23}$$

$$\int_{a_{1}}^{b_{1}}\int_{a_{2}}^{b_{2}}\int_{a_{3}}^{b_{3}}\left(\left(b_{1}-t_{1}\right)\left(b_{2}-t_{2}\right)\left(b_{3}-t_{3}\right)\right)^{\alpha-1}P_{1}\left(x_{3},t_{3}\right)\cdot$$

$$\frac{\partial f}{\partial x_{3}}\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3}$$

$$+A_{1}\left(x_{1},...,x_{m}\right)+A_{2}\left(x_{1},...,x_{m}\right)+B_{1}\left(x_{1},...,x_{m}\right)+B_{2}\left(x_{1},...,x_{m}\right).$$

Call

$$A_{3}\left(x_{1},...,x_{m}\right):=-\frac{\left(\alpha-1\right)\left(\left(b_{1}-x_{1}\right)\left(b_{2}-x_{2}\right)\left(b_{3}-x_{3}\right)\right)^{1-\alpha}}{\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)}\cdot \tag{24}$$

$$\int_{a_{1}}^{b_{1}}\int_{a_{2}}^{b_{2}}\int_{a_{3}}^{b_{3}}\left(\left(b_{1}-t_{1}\right)\left(b_{2}-t_{2}\right)\right)^{\alpha-1}\left(b_{3}-t_{3}\right)^{\alpha-2}P_{1}\left(x_{3},t_{3}\right)\cdot$$

$$f\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3},$$

and

$$B_{3}\left(x_{1},...,x_{m}\right):=\frac{\left(\left(b_{1}-x_{1}\right)\left(b_{2}-x_{2}\right)\left(b_{3}-x_{3}\right)\right)^{1-\alpha}}{\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)}\cdot \tag{25}$$

$$\int_{a_{1}}^{b_{1}}\int_{a_{2}}^{b_{2}}\int_{a_{3}}^{b_{3}}\left(\left(b_{1}-t_{1}\right)\left(b_{2}-t_{2}\right)\left(b_{3}-t_{3}\right)\right)^{\alpha-1}P_{1}\left(x_{3},t_{3}\right)\cdot$$

$$\frac{\partial f}{\partial x_{3}}\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3}.$$

Thus we have proved

$$f(x_1,...,x_m) = \frac{\left(\prod_{i=1}^{3}(b_i - x_i)\right)^{1-\alpha}}{\prod_{i=1}^{3}(b_i - a_i)} \cdot$$

$$\int_{\prod_{i=1}^{3}[a_i,b_i]} \left(\prod_{i=1}^{3}(b_i - t_i)\right)^{\alpha-1} f(t_1,t_2,t_3,x_4,...,x_m)\, dt_1 dt_2 dt_3 +$$

$$\sum_{i=1}^{3} A_i(x_1,...,x_m) + \sum_{i=1}^{3} B_i(x_1,...,x_m). \tag{26}$$

Working similarly we finally obtain the fractional representation formula

$$f(x_1,...,x_m) = \frac{\left(\prod_{i=1}^{m}(b_i - x_i)\right)^{1-\alpha}}{\prod_{i=1}^{m}(b_i - a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} \left(\prod_{i=1}^{m}(b_i - t_i)\right)^{\alpha-1} f\left(\overrightarrow{t}\right) d\overrightarrow{t}$$

$$+\sum_{i=1}^{m} A_i(x_1,...,x_m) + \sum_{i=1}^{m} B_i(x_1,...,x_m). \tag{27}$$

The proof of the theorem is now completed. ∎

We make

Remark 8 *Let* $f \in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i,b_i)$, $i = 1,...,m$. *Denote by*

$$\|f\|_{\infty}^{\sup} := \sup_{x \in \prod_{i=1}^{m}[a_i,b_i]} |f(x)|. \tag{28}$$

We observe that

$$|B_i(x_1,...,x_m)| \overset{(12)}{\leq} \frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j - a_j)} \left(\int_{\prod_{j=1}^{i}[a_j,b_j]} \left(\prod_{j=1}^{i}(b_j - t_j)\right)^{\alpha-1} \cdot\right. \tag{29}$$

$$\left.|P_1(x_i,t_i)|\, dt_1...dt_i\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup} =$$

$$\frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}}{\prod_{j=1}^{i-1}(b_j - a_j)} \left(\prod_{j=1}^{i-1}\int_{a_j}^{b_j}(b_j - t_j)^{\alpha-1} dt_j\right) \cdot$$

$$\left(\int_{a_i}^{b_i}(b_i - t_i)^{\alpha-1} |P_1(x_i,t_i)|\, dt_i\right) = \tag{30}$$

$$\frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}}{(b_i - a_i)\prod_{j=1}^{i-1}(b_j - a_j)} \frac{\left(\prod_{j=1}^{i-1}(b_j - a_j)\right)^{\alpha}}{\alpha^{i-1}} \cdot$$

$$\left[\int_{a_i}^{x_i}(b_i-t_i)^{\alpha-1}(t_i-a_i)\,dt_i+\int_{x_i}^{b_i}(b_i-t_i)^{\alpha-1}(b_i-t_i)\,dt_i\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}. \tag{31}$$

$$\left[\int_{a_i}^{x_i}(b_i-t_i)^{\alpha-1}[(b_i-a_i)-(b_i-t_i)]\,dt_i+\frac{(b_i-x_i)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}. \tag{32}$$

$$\left[(b_i-a_i)\left[\frac{(b_i-a_i)^{\alpha}}{\alpha}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]-\frac{(b_i-a_i)^{\alpha+1}}{\alpha+1}+\frac{2(b_i-x_i)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}.$$

$$\left[\frac{(b_i-a_i)^{\alpha+1}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{\alpha+1}-(b_i-a_i)\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]. \tag{33}$$

We have proved for $i=1,...,m$, that

$$|B_i(x_1,...,x_m)|\le\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}}.$$

$$\left[\frac{(b_i-a_i)^{\alpha}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{(\alpha+1)(b_i-a_i)}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]. \tag{34}$$

We have established the following multivariate fractional Ostrowski type inequality.

Theorem 9 *Let $f\in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha\ge1$, $x_i\in[a_i,b_i)$, $i=1,...,m$. Then*

$$\left|f(x_1,...,x_m)-\frac{\left(\prod_{i=1}^{m}(b_i-x_i)\right)^{1-\alpha}(\Gamma(\alpha))^m\left(I_{a+}^{\alpha}f\right)(b)}{\prod_{i=1}^{m}(b_i-a_i)}-\sum_{i=1}^{m}A_i(x_1,...,x_m)\right|$$

$$\le\sum_{i=1}^{m}\left\{\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}}.\right.$$

$$\left.\left[\frac{(b_i-a_i)^{\alpha}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{(\alpha+1)(b_i-a_i)}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]\right\}. \tag{35}$$

8

References

[1] G. Anastassiou, M. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, *Montgomery Identities for Fractional Integrals and Related Fractional Inequalities*, Journal of Inequalities in Pure And Applied Mathematics, Vol. 10 (2009), No. 4, Article 97, 6 pages.

[2] T. Mamatov, S. Samko, *Mixed fractional integration operators in mixed weighted Hölder spaces*, Fractional Calculus and Applied Analysis, Vol. 13, No. 3 (2010), 245-259.

[3] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1994).

[4] S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993).

Multivariate weighted Fractional Representation Formulae and Ostrowski type inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we derive multivariate weighted fractional representation formulae involving ordinary partial derivatives of first order. Then we present related multivariate weighted fractional Ostrowski type inequalities with respect to uniform norm.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Multivariate Fractional integral, weighted Representation formula, Multivariate weighted Ostrowski inequality.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Suppose now that $w : [a, b] \to [0, \infty)$ is some probability density function, i.e. it is a nonnegative integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a, b]$, $W(t) = 0$ for $t \leq a$ and $W(t) = 1$ for $t \geq b$. Then, the following identity (Pecarić, [5]) is the weighted generalization of the Montgomery identity ([4])

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x, t) f'(t)\,dt, \tag{1}$$

where the weighted Peano Kernel is

$$P_w(x, t) := \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases} \tag{2}$$

In [1] we proved

1

Theorem 1 *Let $w : [a,b] \to [0,\infty)$ be a probability density function, i.e. $\int_a^b w(t)\,dt = 1$, and set $W(t) = \int_a^t w(x)\,dx$ for $a \le t \le b$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$, $\alpha \ge 1$, and f is as in (1). Then the generalization of the weighted Montgomery identity for fractional integrals is the following*

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha) J_a^{\alpha}(w(b) f(b)) -$$

$$J_a^{\alpha-1}(Q_w(x,b) f(b)) + J_a^{\alpha}(Q_w(x,b) f'(b)), \tag{3}$$

where the weighted fractional Peano Kernel is

$$Q_w(x,t) := \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha) W(t), & a \le t \le x, \\ (b-x)^{1-\alpha}\Gamma(\alpha)(W(t)-1), & x < t \le b, \end{cases} \tag{4}$$

i.e. $Q_w(x,t) = (b-x)^{1-\alpha}\Gamma(\alpha) P_w(x,t)$.

When $\alpha = 1$ then the weighted generalization of the Montgomery identity for fractional integrals in (3) reduces to the weighted generalization of the Montgomery identity for integrals in (1).

So for $\alpha \ge 1$ and $x \in [a,b)$ we can rewrite (3) as follows

$$f(x) = (b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1} w(t) f(t)\,dt -$$

$$(b-x)^{1-\alpha}(\alpha-1)\int_a^b (b-t)^{\alpha-2} P_w(x,t) f(t)\,dt + \tag{5}$$

$$(b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1} P_w(x,t) f'(t)\,dt.$$

In this article based on (5), we establish a multivariate weighted general fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i,b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate weighted fractional Ostrowski type inequality. We finish with an application.

2 Main Results

We make

Assumption 2 *Let $f \in C^1\left(\prod_{i=1}^m [a_i,b_i]\right)$.*

Assumption 3 *Let $f : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1,...,m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)$ it is integrable on $\prod_{j=1}^i [a_j,b_j]$, for all $i = 1,...,m$, for any $(x_{i+1},...,x_m) \in \prod_{j=i+1}^m [a_j,b_j]$.*

Convention 4 *We set*

$$\prod_{j=1}^{0} \cdot = 1. \tag{6}$$

Notation 5 *Here $x = \overrightarrow{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1, ..., t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. Here w_i, W_i correspond to $[a_i, b_i]$, $i = 1, ..., m$, and are as w, W of Theorem 1.*

We need

Definition 6 *(see [2] and [3]) Let $\prod_{i=1}^{m} [a_i, b_i] \subset \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$. Let $\alpha > 0$, $f \in L_1 \left(\prod_{i=1}^{m} [a_i, b_i]\right)$. We define the left mixed Riemann-Liouville fractional multiple integral of order α:*

$$\left(I_{a+}^{\alpha} f\right)(x) := \frac{1}{(\Gamma(\alpha))^m} \int_{a_1}^{x_1} ... \int_{a_m}^{x_m} \left(\prod_{i=1}^{m} (x_i - t_i)\right)^{\alpha - 1} f(t_1, ..., t_m)\, dt_1 ... dt_m, \tag{7}$$

where $x_i \in [a_i, b_i]$, $i = 1, ..., m$, and $x = (x_1, ..., x_m)$, $a = (a_1, ..., a_m)$, $b = (b_1, ..., b_m)$.

We present the following multivariate weighted fractional representation formula

Theorem 7 *Let f as in Assumption 2 or Assumption 3, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. Here P_{w_i} corresponds to $[a_i, b_i]$, $i = 1, ..., m$, and it is as in (2). The probability density function w_j is assumed to be bounded for all $j = 1, ..., m$. Then*

$$f(x_1, ..., x_m) = \left(\prod_{j=1}^{m} (b_j - x_j)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^{\alpha} \left(\prod_{j=1}^{m} w_j\right) f\right)(b) +$$

$$\sum_{i=1}^{m} A_i(x_1, ..., x_m) + \sum_{i=1}^{m} B_i(x_1, ..., x_m), \tag{8}$$

where for $i = 1, ..., m$:

$$A_i(x_1, ..., x_m) := -(\alpha - 1) \left(\prod_{j=1}^{i} (b_j - x_j)\right)^{1-\alpha} \int_{\prod_{j=1}^{i} [a_j, b_j]} \left(\prod_{j=1}^{i-1} (b_j - t_j)\right)^{\alpha - 1} \cdot \tag{9}$$

$$(b_i - t_i)^{\alpha - 2} \left(\prod_{j=1}^{i-1} w_j(t_j)\right) P_{w_i}(x_i, t_i)\, f(t_1, ..., t_i, x_{i+1}, ..., x_m)\, dt_1 ... dt_i,$$

and

$$B_i(x_1,...,x_m) := \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \int_{\prod_{j=1}^{i}[a_j,b_j]} \left(\prod_{j=1}^{i}(b_j - t_j)\right)^{\alpha-1} \cdot \tag{10}$$

$$\left(\prod_{j=1}^{i-1} w_j(t_j)\right) P_{w_i}(x_i,t_i) \frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)\, dt_1...dt_i.$$

Proof. We have that

$$f(x_1,x_2,...,x_m) \overset{(5)}{=} (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} w_1(t_1) f(t_1,x_2,...,x_m)\, dt_1 +$$

$$A_1(x_1,...,x_m) + B_1(x_1,...,x_m), \tag{11}$$

where

$$A_1(x_1,...,x_m) := -(\alpha-1)(b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-2} \cdot \tag{12}$$

$$P_{w_1}(x_1,t_1) f(t_1,x_2,...,x_m)\, dt_1,$$

and

$$B_1(x_1,...,x_m) := (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} \cdot \tag{13}$$

$$P_{w_1}(x_1,t_1) \frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\, dt_1.$$

Similarly it holds

$$f(t_1,x_2,...,x_m) \overset{(5)}{=} (b_2 - x_2)^{1-\alpha} \int_{a2}^{b_2} (b_2 - t_2)^{\alpha-1} w_2(t_2) f(t_1,t_2,x_3,...,x_m)\, dt_2$$

$$-(\alpha-1)(b_2 - x_2)^{1-\alpha} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-2} P_{w_2}(x_2,t_2) f(t_1,t_2,x_3,...,x_m)\, dt_2 +$$

$$(b_2 - x_2)^{1-\alpha} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} P_{w_2}(x_2,t_2) \frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\, dt_2. \tag{14}$$

Next we plug (14) into (11).

We get

$$f(x_1,...,x_m) = ((b_1 - x_1)(b_2 - x_2))^{1-\alpha} \cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} ((b_1 - t_1)(b_2 - t_2))^{\alpha-1} w_1(t_1) w_2(t_2) f(t_1,t_2,x_3,...,x_m)\, dt_1 dt_2 +$$

$$A_2(x_1,...,x_m) + B_2(x_1,...,x_m) + A_1(x_1,...,x_m) + B_1(x_1,...,x_m), \tag{15}$$

where

$$A_2(x_1,...,x_m) := -(\alpha-1)\left((b_1-x_1)(b_2-x_2)\right)^{1-\alpha}\cdot \tag{16}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-2}w_1(t_1)P_{w_2}(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2,$$

and

$$B_2(x_1,...,x_m) := \left((b_1-x_1)(b_2-x_2)\right)^{1-\alpha}\cdot \tag{17}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left((b_1-t_1)(b_2-t_2)\right)^{\alpha-1}w_1(t_1)P_{w_2}(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2.$$

We continue as above.

We also have

$$\begin{gathered}
f(t_1,t_2,x_3,...,x_m) \overset{(5)}{=} (b_3-x_3)^{1-\alpha}\cdot \\
\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}w_3(t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3 \\
-(\alpha-1)(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-2}P_{w_3}(x_3,t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3 \\
+(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}P_{w_3}(x_3,t_3)\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_3.
\end{gathered} \tag{18}$$

We plug (18) into (15). Therefore it holds

$$\begin{gathered}
f(x_1,...,x_m) = \left(\prod_{j=1}^{3}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{3}(b_j-t_j)\right)^{\alpha-1}\cdot \\
\left(\prod_{j=1}^{3}w_j(t_j)\right)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3+ \\
\sum_{j=1}^{3}A_j(x_1,...,x_m)+\sum_{j=1}^{3}B_j(x_1,...,x_m),
\end{gathered} \tag{19}$$

where

$$\begin{gathered}
A_3(x_1,...,x_m) := -(\alpha-1)\left(\prod_{j=1}^{3}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{2}(b_j-t_j)\right)^{\alpha-1}\cdot \\
(b_3-t_3)^{\alpha-2}\left(\prod_{j=1}^{2}w_j(t_j)\right)P_{w_3}(x_3,t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3,
\end{gathered} \tag{20}$$

5

and

$$B_3(x_1,...,x_m) := \left(\prod_{j=1}^{3}(b_j - x_j)\right)^{1-\alpha} \int_{\prod_{j=1}^{3}[a_j,b_j]} \left(\prod_{j=1}^{3}(b_j - t_j)\right)^{\alpha-1} \cdot \tag{21}$$

$$\left(\prod_{j=1}^{2} w_j(t_j)\right) P_{w_3}(x_3,t_3) \frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\, dt_1 dt_2 dt_3.$$

Continuing similarly we finally obtain

$$f(x_1,...,x_m) = \left(\prod_{j=1}^{m}(b_j - x_j)\right)^{1-\alpha} \cdot$$

$$\int_{\prod_{j=1}^{m}[a_j,b_j]} \left(\prod_{j=1}^{m}(b_j - t_j)\right)^{\alpha-1} \left(\prod_{j=1}^{m} w_j(t_j)\right) f\left(\overrightarrow{t}\right) d\overrightarrow{t}$$

$$+\sum_{i=1}^{m} A_i(x_1,...,x_m) + \sum_{i=1}^{m} B_i(x_1,...,x_m), \tag{22}$$

that is proving the claim. ■

We make

Remark 8 *Let* $f \in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i,b_i)$, $i = 1,...,m$. *Denote by*

$$\|f\|_{\infty}^{\sup} := \sup_{x \in \prod_{i=1}^{m}[a_i,b_i]} |f(x)|. \tag{23}$$

From (2) we get that

$$|P_w(x,t)| \leq \begin{Bmatrix} W(x), & a \leq t \leq x, \\ 1 - W(x), & x < t \leq b \end{Bmatrix}$$

$$\leq \max\{W(x), 1 - W(x)\} = \frac{1 + |2W(x) - 1|}{2}. \tag{24}$$

That is

$$|P_w(x,t)| \leq \frac{1 + |2W(x) - 1|}{2}, \tag{25}$$

for all $t \in [a,b]$, *where* $x \in [a,b]$ *is fixed.*

Consequently it holds

$$|P_{w_i}(x_i,t_i)| \leq \frac{1 + |2W_i(x_i) - 1|}{2}, \quad i = 1,...,m. \tag{26}$$

Assume here that

$$w_j(t_j) \leq K_j, \tag{27}$$

6

for all $t_j \in [a_j, b_j]$, *where* $K_j > 0$, $j = 1, ..., m$.

Therefore we derive

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\prod_{j=1}^{i-1} K_j\right) \cdot$$

$$\left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup} \prod_{j=1}^{i} \left(\int_{a_j}^{b_j} (b_j - t_j)^{\alpha - 1} dt_j\right). \tag{28}$$

That is

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^{i}(b_j - a_j)^{\alpha}}{\alpha^i}\right) \left(\prod_{j=1}^{i-1} K_j\right) \cdot$$

$$\left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}, \tag{29}$$

for all $i = 1, ..., m$.

Based on the above and Theorem 7 we have established the following multivariate weighted fractional Ostrowski type inequality.

Theorem 9 *Let* $f \in C^1\left(\prod_{i=1}^{m}[a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. *Here* P_{w_i} *corresponds to* $[a_i, b_i]$, $i = 1, ..., m$, *and it is as in (2). Assume that* $w_j(t_j) \leq K_j$, *for all* $t_j \in [a_j, b_j]$, *where* $K_j > 0$, $j = 1, ..., m$. *And* $A_i(x_1, ..., x_m)$ *is as in (9),* $i = 1, ..., m$. *Then*

$$\Bigg| f(x_1, ..., x_m) - \left(\prod_{j=1}^{m}(b_j - x_j)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^{\alpha}\left(\prod_{j=1}^{m} w_j\right) f\right)(b)$$

$$- \sum_{i=1}^{m} A_i(x_1, ..., x_m)\Bigg| \leq \sum_{i=1}^{m} \left\{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^{i}(b_j - a_j)^{\alpha}}{\alpha^i}\right) \cdot\right.$$

$$\left.\left(\prod_{j=1}^{i-1} K_j\right) \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\right\}. \tag{30}$$

3 Application

Here we operate on $[0, 1]^m$, $m \in \mathbb{N} - \{1\}$. We notice that

$$\int_0^1 \left(\frac{e^{-x}}{1 - e^{-1}}\right) dx = 1, \tag{31}$$

7

and
$$\frac{e^{-x}}{1-e^{-1}} \leq \frac{1}{1-e^{-1}}, \text{ for all } x \in [0,1]. \tag{32}$$
So here we choose as w_j the probability density function
$$w_j^*(t_j) := \frac{e^{-t_j}}{1-e^{-1}}, \tag{33}$$
$j = 1, ..., m$, $t_j \in [0,1]$.

So we have the corrsponding W_j as
$$W_j^*(t_j) = \frac{1-e^{-t_j}}{1-e^{-1}}, \quad t_j \in [0,1], \tag{34}$$
and the corresponding P_{w_j} as
$$P_{w_j}^*(x_j, t_j) = \begin{cases} \frac{1-e^{-t_j}}{1-e^{-1}}, & 0 \leq t_j \leq x_j, \\ \frac{e^{-1}-e^{-t_j}}{1-e^{-1}}, & x_j < t_j \leq 1, \end{cases} \tag{35}$$
$j = 1, ..., m$.

Set $\overrightarrow{0} = (0, ..., 0)$ and $\overrightarrow{1} = (1, ..., 1)$.

First we apply Theorem 7.

We have

Theorem 10 *Let $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. Then*
$$f(x_1, ..., x_m) = \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha} \left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m \left(I_{\overrightarrow{0}+}^{\alpha}\left(e^{-\sum_{j=1}^{m} t_j} f(\cdot)\right)\right)\left(\overrightarrow{1}\right)$$
$$+\sum_{i=1}^{m} A_i^*(x_1, ..., x_m) + \sum_{i=1}^{m} B_i^*(x_1, ..., x_m), \tag{36}$$
where for $i = 1, ..., m$:
$$A_i^*(x_1, ..., x_m) := \frac{-(\alpha-1)}{(1-e^{-1})^{i-1}} \left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha} \int_{[0,1]^i} \left(\prod_{j=1}^{i-1}(1-t_j)\right)^{\alpha-1} \cdot$$
$$(1-t_i)^{\alpha-2} e^{-\sum_{j=1}^{i-1} t_j} P_{w_i}^*(x_i, t_i) f(t_1, ..., t_i, x_{i+1}, ..., x_m)\, dt_1 ... dt_i, \tag{37}$$
and
$$B_i^*(x_1, ..., x_m) := \frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{(1-e^{-1})^{i-1}} \int_{[0,1]^i} \left(\prod_{j=1}^{i}(1-t_j)\right)^{\alpha-1} \cdot \tag{38}$$
$$e^{-\sum_{j=1}^{i-1} t_j} P_{w_i}^*(x_i, t_i) \frac{\partial f}{\partial x_i}(t_1, ..., t_i, x_{i+1}, ..., x_m)\, dt_1 ... dt_i.$$
Above we set $\sum_{i=1}^{0} \cdot = 0$.

8

Finally we apply Theorem 9.

Theorem 11 *Let* $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. *Here* $P^*_{w_i}$ *is as in (35) and* $A^*_i(x_1, ..., x_m)$ *as in (37),* $i = 1, ..., m$. *Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^{m} (1 - x_j) \right)^{1-\alpha} \left(\frac{\Gamma(\alpha)}{1 - e^{-1}} \right)^m \left(I^{\alpha}_{\overrightarrow{0}+} \left(e^{-\sum_{j=1}^{m} t_j} f(\cdot) \right) \right) \left(\overrightarrow{1} \right) \right.$$
$$\left. - \sum_{i=1}^{m} A^*_i(x_1, ..., x_m) \right| \leq \sum_{i=1}^{m} \left\{ \frac{\left(\prod_{j=1}^{i} (1 - x_j) \right)^{1-\alpha}}{\alpha^i (1 - e^{-1})^{i-1}} \cdot \right.$$
$$\left. \left(\frac{1 + |2W^*_i(x_i) - 1|}{2} \right) \left\| \frac{\partial f}{\partial x_i} \right\|^{\sup}_{\infty} \right\}. \tag{39}$$

References

[1] G. Anastassiou, M. Hooshmandasl, A. Ghasemi, F. Moftakharzadeh, *Montgomery Identities for Fractional Integrals and Related Fractional Inequalities*, Journal of Inequalities in Pure And Applied Mathematics, Vol. 10 (2009), No. 4, Article 97, 6 pages.

[2] T. Mamatov, S. Samko, *Mixed fractional integration operators in mixed weighted Hölder spaces*, Fractional Calculus and Applied Analysis, Vol. 13, No. 3 (2010), 245-259.

[3] S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, USA, (1993).

[4] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1994).

[5] J.E. Pečarić, *On the Čebyšev inequality*, Bul. Şti. Tehn. Inst. Politehn, "Traian Vuia" Timişoara, 25 (39) (1980), no. 1, pp. 5-9.

Basic and s-convexity Ostrowski and Grüss type inequalities involving several functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using the harmonic polynomials representation formula due to Dedic, Pečaric and Ujevic [5], we establish Ostrowski and Grüss type inequalities involving several functions. The estimates are with respect all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, and also take into account the s-convexity and s-concavity in the second sense of the involved functions.

2010 AMS Subject Classification : 26A51, 26D10, 26D15.
Keywords and Phrases: Grüss inequality, Ostrowski inequality, s-convexity and s-concavity in the second sense.

1 Introduction

The following results motivate our work.

Theorem 1 *(1938, Ostrowski [11]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Theorem 2 *(1935, Grüss [9]) Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a,b],$$

1

where $m, M, n, N \in \mathbb{R}$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \\ \leq \frac{1}{4} (M-m)(N-n). \tag{2}$$

Theorem 3 *(1998, Dragomir and Wang [7]) Let* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous function with* $f' \in L_p([a,b])$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in [a,b]$. *Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq$$

$$\frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p. \tag{3}$$

Ostrowski type inequalities are very useful in Numerical Analysis.

Theorem 4 *(1882, Čebyšev [3]) Let* $f, g : [a,b] \to \mathbb{R}$ *absolutely continuous functions with* $f', g' \in L_\infty([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \\ \leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{4}$$

Above is also assumed that the involved integrals exist.

Grüss type inequalities are very useful in Probability.

We are also biggly inspired by the great work of B.G. Pachpatte, see [12], [13], [14].

So here we produce Ostrowski and Grüss type inequalities for several functions, acting to all possible directions, including s-convexity and s-concavity in the second sense complete study. Our results are univariate.

2 Background

Let $(P_n)_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geq 1$, $P_0 = 1$. Furthermore, let $[a,b] \subset \mathbb{R}$, $a \neq b$, and $h : [a,b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some fixed $n \geq 1$. We use the notation

$$L_n[h(x)] = \frac{1}{n} \left[h(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) h^{(k)}(x) + \right.$$

$$\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b) \right] \Bigg],$$

$x \in [a,b]$, for convinience.

For $n = 1$ the above sums are defined to be zero, that is $L_1 [h(x)] = h(x)$.

Dedic, Pečaric and Ujevic, see [5], [4], established the following identity,

$$L_n [h(x)] - \frac{1}{b-a} \int_a^b h(t)\, dt = \frac{(-1)^{n+1}}{n(b-a)} \int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\, dt, \quad (6)$$

where

$$q(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b]. \quad (7)$$

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \geq 0$, the identity (6) reduces to the Fink identity in [8], (see also [5], p. 177).

3 Main Results

We present our first main result, a set of very general Ostrowski type inequalities involving several functions.

Theorem 5 *Let $n_j \in \mathbb{N}$, $j = 1, ..., r \in \mathbb{N} - \{1\}$, $n_1 \leq n_2 \leq ... \leq n_r$ and $f_j : [a,b] \to \mathbb{R}$ be such that $f_j^{(n_j-1)}$ is absolutely continuous function. Denote*

$$S_1(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right], \quad (8)$$

$$S_2(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} L_{n_i}[f_i(x)] \right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right], \quad (9)$$

$x \in [a,b]$. Then

1)

$$|S_1(f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\frac{1}{2} + \frac{|a+b-2x|}{2(b-a)} \right] \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \left\| f_j^{(n_j)} \right\|_{1,[a,b]} \right] \right], \quad (10)$$

and

$$|S_2(f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\frac{1}{2} + \frac{|a+b-2x|}{2(b-a)} \right] \cdot$$

3

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right], \tag{11}$$

2) let $p_{lj}>1:\sum_{lj=1}^{3}\frac{1}{p_{lj}}=1$, *with* $f_j^{(n_j)}\in L_{p_{3j}}\left([a,b]\right)$, $j=1,...,r$, *it holds*

$$\left|S_1\left(f_1,...,f_r\right)\right|\leq\frac{1}{n_1\left(b-a\right)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\left(\frac{\left(b-x\right)^{p_{2j}+1}+\left(x-a\right)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{12}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right|\leq\frac{1}{n_1\left(b-a\right)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\left(\frac{\left(b-x\right)^{p_{2j}+1}+\left(x-a\right)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{13}$$

3) assuming $f_j^{(n_j)}\in L_\infty\left([a,b]\right)$, $j=1,...,r$, *we get*

$$\left|S_1\left(f_1,...,f_r\right)\right|\leq\left(\frac{\left(b-x\right)^2+\left(x-a\right)^2}{2n_1\left(b-a\right)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right], \tag{14}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right|\leq\left(\frac{\left(b-x\right)^2+\left(x-a\right)^2}{2n_1\left(b-a\right)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right]. \tag{15}$$

Proof. For $j=1,...,r$, $r\in\mathbb{N}-\{1\}$, we have

$$L_{n_j}\left[f_j\left(x\right)\right]=\frac{1}{n_j}\left[f_j\left(x\right)+\sum_{k=1}^{n_j-1}\left(-1\right)^kP_k\left(x\right)f_j^{(k)}\left(x\right)+\right.$$

4

$$\sum_{k=1}^{n_j-1} \frac{(-1)^k (n_j-k)}{b-a} \left[P_k(a) f_j^{(k-1)}(a) - P_k(b) f_j^{(k-1)}(b) \right] \Bigg], \tag{16}$$

and

$$L_{n_j}[f_j(x)] - \frac{1}{b-a}\int_a^b f_j(t)\,dt \overset{(6)}{=} \frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dt. \tag{17}$$

Hence it holds

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a}\int_a^b f_j(t)\,dt \right] = \tag{18}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dt \right], \quad \text{for all } j=1,...,r.$$

Therefore by addition of (18), we derive the identity

$$S_1(f_1,...,f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a}\int_a^b f_j(t)\,dt \right] \right] \tag{19}$$

$$= \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dt \right] \right].$$

Similarly we produce the identity

$$S_2(f_1,...,f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} L_{n_i}[f_i(x)]\right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a}\int_a^b f_j(t)\,dt \right] \right] \tag{20}$$

$$= \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} L_{n_i}[f_i(x)]\right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dt \right] \right].$$

Consequently we have

$$|S_1(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} |f_i(x)|\right) \left[\frac{1}{n_j(b-a)} \int_a^b \left|P_{n_j-1}(t)\right| |q(x,t)| \left|f_j^{(n_j)}(t)\right| dt \right] \right], \tag{21}$$

5

and

$$|S_2(f_1,...,f_r)| \le$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{1}{n_j(b-a)}\int_a^b\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right]. \tag{22}$$

Furthermore it holds

$$|S_1(f_1,...,f_r)| \le$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right], \tag{23}$$

and

$$|S_2(f_1,...,f_r)| \le$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right]. \tag{24}$$

Since

$$|q(x,t)| \le \max(x-a,b-x) = \frac{(b-a)+|a+b-2x|}{2}, \tag{25}$$

we get

$$|S_1(f_1,...,f_r)| \le$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right], \tag{26}$$

and

$$|S_2(f_1,...,f_r)| \le$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]. \tag{27}$$

Let now $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$, with $f_j^{(n_j)} \in L_{p_{3j}}([a,b])$, $j = 1,...,r$.

Hence, by Hölder inequality for three functions, it holds

$$\int_a^b\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dt \le$$

$$\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\left(\int_a^b|q(x,t)|^{p_{2j}}dt\right)^{\frac{1}{p_{2j}}}\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} =$$

6

$$\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}. \tag{28}$$

Consequently we derive

$$|S_1(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{1}{n_j(b-a)}\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right. \tag{29}$$

$$\left.\left.\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\right]\right],$$

and

$$|S_2(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{1}{n_j(b-a)}\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right. \tag{30}$$

$$\left.\left.\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\right]\right].$$

Assuming that $f_j^{(n_j)} \in L_\infty([a,b])$, $j=1,...,r$, we find

$$|S_1(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{1}{n_j(b-a)}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\right.\right. \tag{31}$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\left(\frac{(b-x)^2+(x-a)^2}{2}\right)\right]\right],$$

and

$$|S_2(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{1}{n_j(b-a)}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\right.\right. \tag{32}$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\left(\frac{(b-x)^2+(x-a)^2}{2}\right)\right]\right].$$

The proof of the theorem is now complete. ■

We need

Definition 6 *([10]) A function $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y), \tag{33}$$

for all $x, y \in [0, \infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

When $s = 1$, s-convexity in the second sense reduces to ordinary convexity.
If "$\geq$" holds in (33), we talk about s-concavity in the second sense.
We also need

Definition 7 *(see also [1]) Let I be a subinterval of $\mathbb{R}_+$ and $f : I \to (0, \infty)$. We call f s-logarithmically convex (s-log-convex) in the second sense, iff $\log f(x)$ is s-convex in the second sense, iff*

$$f(\lambda x + (1-\lambda) y) \leq (f(x))^{\lambda^s} (f(y))^{(1-\lambda)^s}, \tag{34}$$

for all $x, y \in I$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

When $s = 1$, s-log-convexity in the second sense reduces to usual log-convexity.
If "$\geq$" holds in (34), we talk about s-log-concavity in the second sense.
We also need the s-convex Hadamard's inequality.

Theorem 8 *([6]) Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1([a,b])$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{35}$$

The constant $K = \frac{1}{s+1}$ is the best possible in the second inequality (35). The above inequalities are sharp.

Next we present general Ostrowski type inequalities for several functions under s-convexity and s-concavity in the second sense.

Theorem 9 *Same terms and assumptions as in Theorem 5. Assume that $a \geq 0$.*

1) Suppose $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$, $j = 1, ..., r$. Then

$$|S_1(f_1, ..., f_r)| \leq \frac{(b-a)}{n_1} \left[\frac{2}{s+2} \left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}} \right) - \right.$$

$$\left. \frac{1}{s+1} \left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}} \right) + \frac{2}{(s+1)(s+2)} \right]$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right], \tag{36}$$

and

$$|S_2(f_1,...,f_r)| \leq \frac{(b-a)}{n_1}\left[\frac{2}{s+2}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right)-\right.$$

$$\left.\frac{1}{s+1}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\right]$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right], \tag{37}$$

$x \in [a,b]$.

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$, *with* $f_j^{(n_j)} \in L_{p_{3j}}([a,b])$, $j = 1,...,r$.

2i) Assume again $\left|f_j^{(n_j)}\right|$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$|S_1(f_1,...,f_r)| \leq \frac{2}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{38}$$

and

$$|S_2(f_1,...,f_r)| \leq \frac{2}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right]. \tag{39}$$

2ii) Assume that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$|S_1(f_1,...,f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\frac{2^{\frac{1}{p_{3j}}} M_j (b-a)^{\frac{1}{p_{3j}}-1}}{(s+1)^{\frac{1}{p_{3j}}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \Bigg] \Bigg], \tag{40}$$

and

$$|S_2 (f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |L_{n_i} [f_i (x)]| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right.\right.$$

$$\left.\left. \frac{2^{\frac{1}{p_{3j}}} M_j (b-a)^{\frac{1}{p_{3j}}-1}}{(s+1)^{\frac{1}{p_{3j}}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right] \right]. \tag{41}$$

2iii) Assume that $\left| f_j^{(n_j)} \right|^{p_{3j}}$ *is s-concave in the second sense. Then*

$$|S_1 (f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right.\right.$$

$$\left.\left. 2^{\frac{s-1}{p_{3j}}} \left| f_j^{(n_j)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_{3j}}-1} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right] \right], \tag{42}$$

and

$$|S_2 (f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |L_{n_i} [f_i (x)]| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right.\right.$$

$$\left.\left. 2^{\frac{s-1}{p_{3j}}} \left| f_j^{(n_j)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_{3j}}-1} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right] \right]. \tag{43}$$

Proof. As in (23) and (24) we have that

$$|S_1 (f_1, ..., f_r)| \leq \frac{1}{n_1 (b-a)} \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) \left\| P_{n_j - 1} \right\|_{\infty, [a,b]} \int_a^b |q (x,t)| \left| f_j^{(n_j)} (t) \right| dt \right] \right], \tag{44}$$

and

$$|S_2 (f_1, ..., f_r)| \leq \frac{1}{n_1 (b-a)} \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt\right]\right]. \tag{45}$$

Set

$$p\left(x,t\right):=\begin{cases}t, & t\in\left[0,\frac{b-x}{b-a}\right),\\ t-1, & t\in\left[\frac{b-x}{b-a},1\right].\end{cases} \tag{46}$$

In [2], for $\lambda\in[0,1]$, we proved that

$$q\left(x,\lambda a+\left(1-\lambda\right)b\right)=\left(a-b\right)p\left(x,\lambda\right), \tag{47}$$

that is

$$\left|q\left(x,\lambda a+\left(1-\lambda\right)b\right)\right|=\left(b-a\right)\left|p\left(x,\lambda\right)\right|. \tag{48}$$

One can write

$$\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt= \tag{49}$$

$$\left(b-a\right)\int_0^1\left|q\left(x,\lambda a+\left(1-\lambda\right)b\right)\right|\left|f_j^{(n_j)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|d\lambda\overset{(48)}{=}$$

$$\left(b-a\right)^2\int_0^1\left|p\left(x,\lambda\right)\right|\left|f_j^{(n_j)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|d\lambda=:\left(*\right). \tag{50}$$

We notice under the assumption that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}\left(x\right)\right|\leq M_j$, $x\in[a,b]$, that

$$\left(*\right)\leq\left(b-a\right)^2\int_0^1\left|p\left(x,t\right)\right|\left(\lambda^s\left|f_j^{(n_j)}\left(a\right)\right|+\left(1-\lambda\right)^s\left|f_j^{(n_j)}\left(b\right)\right|\right)d\lambda\leq \tag{51}$$

$$M_j\left(b-a\right)^2\int_0^1\left|p\left(x,\lambda\right)\right|\left(\lambda^s+\left(1-\lambda\right)^s\right)d\lambda=$$

(as in [2])

$$M_j\left(b-a\right)^2\left[\frac{2}{s+2}\left(\frac{\left(b-x\right)^{s+2}+\left(x-a\right)^{s+2}}{\left(b-a\right)^{s+2}}\right)-\right.$$

$$\left.\frac{1}{s+1}\left(\frac{\left(b-x\right)^{s+1}+\left(x-a\right)^{s+1}}{\left(b-a\right)^{s+1}}\right)+\frac{2}{\left(s+1\right)\left(s+2\right)}\right]. \tag{52}$$

So we got that

$$\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt\leq M_j\left(b-a\right)^2\left[\frac{2}{s+2}\left(\frac{\left(b-x\right)^{s+2}+\left(x-a\right)^{s+2}}{\left(b-a\right)^{s+2}}\right)-\right.$$

$$\frac{1}{s+1}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\Bigg], \quad j=1,...,r. \tag{53}$$

Using (53) into (44) and (45) we derive (36) and (37).

Next we elaborate on (12) and (13).

Assume that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, acting as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq 2M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}, \tag{54}$$

$j=1,...,r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Next suppose that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ is s-convex in the second sense. As in [2] we get

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq \frac{2^{\frac{1}{p_{3j}}}M_j(b-a)^{\frac{1}{p_{3j}}}}{(s+1)^{\frac{1}{p_{3j}}}}, \tag{55}$$

$j=1,...,r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Finally assume that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ is s-concave in the second sense. Based on Theorem 8 and acting as in [2], we derive

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq 2^{\frac{s-1}{p_{3j}}}\left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right|(b-a)^{\frac{1}{p_{3j}}}. \tag{56}$$

The proof is completed. ■

Ostrowski type inequalities for several functions under s-log-convexity in the second sense follow.

Theorem 10 *Same terms and assumptions as in Theorem 5. Assume that $a \geq 0$. We further suppose that $\left|f_j^{(n_j)}\right| \neq 0$ is s-log-convex in the second sense, and $\left|f_j^{(n_j)}(a)\right|, \left|f_j^{(n_j)}(b)\right| \in (0,1]$, $j=1,...,r$. Call $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $s \in (0,1]$, $j=1,...,r$, and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z \in (0,\infty)-\{1\}, \\ 1, & \text{if } z=1. \end{cases} \tag{57}$$

1) It holds

$$|S_1(f_1,...,f_r)| \leq \left[\frac{(b-a)+|a+b-2x|}{2n_1}\right] \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^s\psi_s(A_j)\right]\right], \tag{58}$$

and

$$|S_2(f_1,...,f_r)| \leq \left[\frac{(b-a)+|a+b-2x|}{2n_1}\right] \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^s \psi_s(A_j)\right]\right]. \tag{59}$$

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$*, with* $f_j^{(n_j)} \in L_{p_{3j}}([a,b])$*, and set* $B_j := A_j^{p_{3j}}$*,* $j = 1,...,r$*. Then*

$$|S_1(f_1,...,f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.(b-a)^{\frac{1}{p_{3j}}-1}\left|f_j^{(n_j)}(b)\right|^s(\psi_s(B_j))^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{60}$$

and

$$|S_2(f_1,...,f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.(b-a)^{\frac{1}{p_{3j}}-1}\left|f_j^{(n_j)}(b)\right|^s(\psi_s(B_j))^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right]. \tag{61}$$

Proof. 1) See also [1]. Here we assume that $0 < \left|f_j^{(n_j)}(a)\right|, \left|f_j^{(n_j)}(b)\right| \leq 1$, set $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $\left|f_j^{(n_j)}\right|$ is s-logarithmically convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$, $j = 1,...,r$. From (26) we get

$$|S_1(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|f_i(x)|\right)\right.$$

$$\left.\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda\right]\right], \tag{62}$$

13

and from (27) we obtain

$$\left|S_2\left(f_1,...,f_r\right)\right| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\right.$$

$$\left.\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\int_0^1\left|f_j^{(n_j)}\left(\lambda a+(1-\lambda)b\right)\right|d\lambda\right]\right]. \tag{63}$$

We study separately the integral

$$\int_0^1\left|f_j^{(n_j)}\left(\lambda a+(1-\lambda)b\right)\right|d\lambda \overset{(34)}{\leq} \int_0^1\left|f_j^{(n_j)}(a)\right|^{\lambda^s}\left|f_j^{(n_j)}(b)\right|^{(1-\lambda)^s}d\lambda \leq \tag{64}$$

$$\int_0^1\left|f_j^{(n_j)}(a)\right|^{\lambda s}\left|f_j^{(n_j)}(b)\right|^{(1-\lambda)s}d\lambda = \left|f_j^{(n_j)}(b)\right|^s\int_0^1\left(\frac{\left|f_j^{(n_j)}(a)\right|}{\left|f_j^{(n_j)}(b)\right|}\right)^{\lambda s}d\lambda =$$

$$\left|f_j^{(n_j)}(b)\right|^s\int_0^1 A_j^{\lambda s}d\lambda =: (**)\,. \tag{65}$$

If $A_j = 1$, then

$$(**) = \left|f_j^{(n_j)}(b)\right|^s. \tag{66}$$

If $A_j \neq 1$, we get

$$(**) = \left|f_j^{(n_j)}(b)\right|^s\int_0^1 e^{s(\ln A_j)\lambda}d\lambda = \left|f_j^{(n_j)}(b)\right|^s\left(\frac{A_j^s-1}{s\ln A_j}\right). \tag{67}$$

Therefore we derive

$$\int_0^1\left|f_j^{(n_j)}\left(\lambda a+(1-\lambda)b\right)\right|d\lambda \leq \left|f_j^{(n_j)}(b)\right|^s\psi_s\left(A_j\right). \tag{68}$$

Hence by (68) used in (62) and (63) we derive (58) and (59).

2) As before and as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq (b-a)^{\frac{1}{p_{3j}}}\left|f_j^{(n_j)}(b)\right|^s\left(\psi_s\left(B_j\right)\right)^{\frac{1}{p_{3j}}}. \tag{69}$$

Using (69) into (12), (13) we derive (60), (61). ■

Next we give applications when $n_1 = n_2 = ... = n_r = 1$.

Corollary 11 *(to Theorem 5) Let $f_j : [a,b] \to \mathbb{R}$ is absolutely continuous function, $j = 1,...,r \in \mathbb{N}-\{1\}$. Denote*

$$S^*\left(f_1,...,f_r\right) := \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i\left(x\right)\right)\left[f_j\left(x\right)-\frac{1}{b-a}\int_a^b f_j\left(t\right)dt\right]\right], \tag{70}$$

14

$x \in [a,b]$.

Then

1)

$$|S^*(f_1,...,f_r)| \leq \left[\frac{1}{2} + \frac{|a+b-2x|}{2(b-a)}\right] \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|f_j'\right\|_{1,[a,b]}\right]\right], \tag{71}$$

2) let $p_{lj} > 1 : \sum_{lj=1}^{3} \frac{1}{p_{lj}} = 1$, *with* $f_j' \in L_{p_{3j}}([a,b])$, $j = 1,...,r$, *it holds*

$$|S^*(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)(b-a)^{\frac{1}{p_{1j}}-1}\left\|f_j'\right\|_{p_{3j},[a,b]}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right], \tag{72}$$

3) assuming $f_j' \in L_\infty([a,b])$, $j = 1,...,r$, *we get*

$$|S^*(f_1,...,f_r)| \leq \left(\frac{(b-x)^2+(x-a)^2}{2(b-a)}\right) \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left\|f_j'\right\|_{\infty,[a,b]}\right]\right]. \tag{73}$$

We continue with

Corollary 12 *(to Theorem 9) Same terms and assumptions as in Corollary 11. Assume that* $a \geq 0$.

1) Suppose $\left|f_j'\right|$ *is s-convex in the second sense and* $\left|f_j'(x)\right| \leq M_{1j}$, $x \in [a,b]$, $j = 1,...,r$. *Then*

$$|S^*(f_1,...,f_r)| \leq (b-a)\left[\frac{2}{s+2}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right) -\right.$$

$$\left.\frac{1}{s+1}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\right]$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)M_{1j}\right]\right], \tag{74}$$

15

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3} \frac{1}{p_{lj}} = 1$, *with* $f_j' \in L_{p_{3j}}([a,b])$, $j = 1, ..., r$.

2i) Assume again $\left|f_j'\right|$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$, $j = 1, ..., r$, $x \in [a,b]$. *Then*

$$|S^*(f_1, ..., f_r)| \leq 2 \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) (b-a)^{-\frac{1}{p_{2j}}} \right.\right.$$

$$\left.\left. M_{1j} (p_{3j} s + 1)^{-\frac{1}{p_{3j}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right] \right], \tag{75}$$

2ii) Assume that $\left|f_j'\right|^{p_{3j}}$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$, $j = 1, ..., r$, $x \in [a,b]$. *Then*

$$|S^*(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \right.$$

$$\left. \frac{2^{\frac{1}{p_{3j}}} M_{1j} (b-a)^{-\frac{1}{p_{2j}}}}{(s+1)^{\frac{1}{p_{3j}}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right], \tag{76}$$

2iii) Assume that $\left|f_j'\right|^{p_{3j}}$ *is s-concave in the second sense. Then*

$$|S^*(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \right.$$

$$\left. 2^{\frac{s-1}{p_{3j}}} \left| f_j' \left(\frac{a+b}{2} \right) \right| (b-a)^{-\frac{1}{p_{2j}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right]. \tag{77}$$

We also give

Corollary 13 *(to Theorem 10) Same terms and assumptions as in Corollary 11. Assume that* $a \geq 0$. *We further suppose that* $\left|f_j'\right| \neq 0$ *is* s-log-convex *in the second sense, and* $\left|f_j'(a)\right|, \left|f_j'(b)\right| \in (0,1]$, $j = 1, ..., r$. *Call* $A_{1j} := \left|\frac{f_j'(a)}{f_j'(b)}\right|$, $s \in (0,1]$, $j = 1, ..., r$, *and* ψ_s *as in (57).*

1) It holds

$$|S^*(f_1, ..., f_r)| \leq \left[\frac{(b-a) + |a+b-2x|}{2} \right].$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left| f_j'(b) \right|^s \psi_s(A_{1j}) \right] \right]. \tag{78}$$

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3} \frac{1}{p_{lj}} = 1$, *with* $f_j' \in L_{p_{3j}}([a,b])$, $B_{1j} := A_{1j}^{p_{3j}}$, $j = 1, ..., r$. *Then*

$$|S^*(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \right.$$

$$\left. (b-a)^{-\frac{1}{p_{2j}}} \left| f_j'(b) \right|^s (\psi_s(B_{1j}))^{\frac{1}{p_{3j}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \right]. \tag{79}$$

Next we present a set of very general Grüss type inequalities involving several functions.

Theorem 14 *Let* $n_j \in \mathbb{N}$, $j = 1, ..., r \in \mathbb{N} - \{1\}$, $n_1 \leq n_2 \leq ... \leq n_r$ *and* $f_j : [a,b] \to \mathbb{R}$ *be such that* $f_j^{(n_j-1)}$ *is absolutely continuous function. Denote*

$$\Delta(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) L_{n_j}[f_j(x)]\, dx \right) - \right.$$

$$\left. \frac{1}{b-a} \left(\int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) dx \right) \left(\int_a^b f_j(x)\, dx \right) \right]. \tag{80}$$

Then

1)

$$|\Delta(f_1, ..., f_r)| \leq \left(\frac{(b-a) + |a+b-2x|}{2n_1} \right) \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} \|f_i\|_{\infty,[a,b]} \right) \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \left\| f_j^{(n_j)} \right\|_{1,[a,b]} \right] \right], \tag{81}$$

2)

$$|\Delta(f_1, ..., f_r)| \leq (b-a) \left(\frac{(b-a) + |a+b-2x|}{2n_1} \right) \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} \|f_i\|_{\infty,[a,b]} \right) \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \left\| f_j^{(n_j)} \right\|_{\infty,[a,b]} \right] \right], \tag{82}$$

3) let $p_{i,j} > 1 : \sum_{i=1}^{r+2} \frac{1}{p_{i,j}} = 1$, *and* $f_j^{(n_j)} \in L_{(r+2),j}([a,b])$, $j = 1, ..., r$, *it holds*

$$|\Delta(f_1, ..., f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}}}{\left((p_{r+1,j}+1)(p_{r+1,j}+2)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left.\left.\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}\right]\right]. \tag{83}$$

Proof. From (19) we obtain

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)L_{n_j}[f_j(x)] - \frac{1}{b-a}\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)\int_a^b f_j(t)\,dt\right] =$$

$$= \sum_{j=1}^{r}\left[\frac{(-1)^{n_j+1}}{n_j(b-a)}\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)\int_a^b P_{n_j-1}(t)\,q(x,t)\,f_j^{(n_j)}(t)\,dt\right]. \tag{84}$$

Hence we get

$$\Delta(f_1, ..., f_r) := \sum_{j=1}^{r}\left[\left(\int_a^b\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)L_{n_j}[f_j(x)]\,dx\right) - \right.$$

$$\left.\frac{1}{b-a}\left(\int_a^b\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)dx\right)\left(\int_a^b f_j(t)\,dt\right)\right] =$$

$$= \sum_{j=1}^{r}\left[\frac{(-1)^{n_j+1}}{n_j(b-a)}\int_a^b\int_a^b\left(\prod_{\substack{i=1\\ i\neq j}}^{r} f_i(x)\right)P_{n_j-1}(t)\,q(x,t)\,f_j^{(n_j)}(t)\,dtdx\right]. \tag{85}$$

Therefore it holds

$$|\Delta(f_1, ..., f_r)| \leq$$

$$\frac{1}{n_1(b-a)}\sum_{j=1}^{r}\left[\int_a^b\int_a^b\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|f_i(x)|\right)\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dtdx\right]. \tag{86}$$

We first find

$$|\Delta(f_1, ..., f_r)| \leq \left(\frac{(b-a)+|a+b-2x|}{2n_1(b-a)}\right).$$

18

$$\left[\sum_{j=1}^{r}\left[\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}(b-a)\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]=$$

$$\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\left[\sum_{j=1}^{r}\left[\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]. \tag{87}$$

Also it holds

$$|\Delta(f_1,...,f_r)|\leq\left(\frac{(b-a)+|a+b-2x|}{2n_1(b-a)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}(b-a)^2\right]\right]=$$

$$(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right]. \tag{88}$$

Let $p_{i,j}>1:\sum_{i=1}^{r+2}\frac{1}{p_{i,j}}=1$, and $f_j^{(n_j)}\in L_{(r+2),j}([a,b])$, $j=1,...,r$. Hence, by Hölder's inequality we find

$$\int_a^b\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dtdx\leq$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left(\int_a^b\int_a^b|f_i(x)|^{p_{i,j}}dtdx\right)^{\frac{1}{p_{i,j}}}\right)\left(\int_a^b\int_a^b\left|P_{n_j-1}(t)\right|^{p_{j,j}}dtdx\right)^{\frac{1}{p_{j,j}}}\cdot \tag{89}$$

$$\left(\int_a^b\int_a^b|q(x,t)|^{p_{r+1,j}}dtdx\right)^{\frac{1}{p_{r+1,j}}}\left(\int_a^b\int_a^b\left|f_j^{(n_j)}(t)\right|^{p_{r+2,j}}dtdx\right)^{\frac{1}{p_{r+2,j}}}=$$

$$=\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}(b-a)^{\frac{1}{p_{i,j}}}\right)\left(\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}(b-a)^{\frac{1}{p_{j,j}}}\right)$$

$$\left(\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}(b-a)^{\frac{1}{p_{r+2,j}}}\right)\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{2}{p_{r+1,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}=$$

19

$$\frac{2^{\frac{1}{p_{r+1,j}}}\left(b-a\right)^{2+\frac{1}{p_{r+1,j}}}}{\left(\left(p_{r+1,j}+1\right)\left(p_{r+1,j}+2\right)\right)^{\frac{1}{p_{r+1,j}}}}\cdot$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left\|f_i\right\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}. \tag{90}$$

That is we found

$$\int_a^b\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left|P_{n_j-1}\left(t\right)\right|\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dtdx\leq$$

$$\frac{2^{\frac{1}{p_{r+1,j}}}\left(b-a\right)^{2+\frac{1}{p_{r+1,j}}}}{\left(\left(p_{r+1,j}+1\right)\left(p_{r+1,j}+2\right)\right)^{\frac{1}{p_{r+1,j}}}}\cdot$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left\|f_i\right\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}. \tag{91}$$

Using (91) into (86) we obtain (83).

The proof of the theorem now is complete. ■

Next we produce Grüss type inequalities for several functions under s-convexity and s-concavity in the second sense.

Theorem 15 *Here all as in Theorem 14, with $a\geq 0$.*

1) Suppose $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}\left(x\right)\right|\leq M_j$, $x\in[a,b]$, $j=1,...,r$. Then

$$\left|\Delta\left(f_1,...,f_r\right)\right|\leq\frac{4\left(b-a\right)^2}{\left(s+2\right)\left(s+3\right)n_1}\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left\|f_i\right\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right]. \tag{92}$$

2) Let $p_{i,j}>1:\sum_{i=1}^{r+2}\frac{1}{p_{i,j}}=1$, with $f_j^{(n_j)}\in L_{p_{r+2.j}}\left([a,b]\right)$, $j=1,...,r$.

2i) Assume again $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, and $\left|f_j^{(n_j)}\left(x\right)\right|\leq M_j$, $j=1,...,r$, $x\in[a,b]$. Then

$$\left|\Delta\left(f_1,...,f_r\right)\right|\leq\frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{1+\frac{1}{p_{r+1,j}}}\left(b-a\right)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{\left(\left(p_{r+1,j}+1\right)\left(p_{r+1,j}+2\right)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

20

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]} M_j\left(p_{r+2,j}s+1\right)^{-\frac{1}{p_{r+2,j}}}\Bigg]\Bigg]. \tag{93}$$

2ii) Assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right|\leq M_j,\ j=1,...,r,\ x\in[a,b]$*. Then*

$$|\Delta(f_1,...,f_r)|\leq\frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{\left((p_{r+1,j}+1)(p_{r+1,j}+2)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left.\left.\frac{M_j}{(s+1)^{\frac{1}{p_{r+2,j}}}}\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\right]\right]. \tag{94}$$

2iii) Assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ *is s-concave in the second sense. Then*

$$|\Delta(f_1,...,f_r)|\leq\frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{s-1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{\left((p_{r+1,j}+1)(p_{r+1,j}+2)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left.\left.\left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right|\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\right]\right]. \tag{95}$$

Proof. From (86) we get

$$|\Delta(f_1,...,f_r)|\leq\frac{1}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\right.\right. \tag{96}$$

$$\left.\left.\int_a^b\left(\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right)dx\right]\right].$$

Here $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}(x)\right|\leq M_j,\ x\in[a,b]$, $j=1,...,r.$

Using (53) we obtain

$$\int_a^b\left(\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right)dx\leq\frac{4M_j(b-a)^3}{(s+2)(s+3)}, \tag{97}$$

$j=1,...,r.$

Consequently by (97) and (96) we derive (92).

Next we elaborate on (83).

Assume that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, acting as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq 2M_j \left(\frac{b-a}{p_{r+2,j}s+1}\right)^{\frac{1}{p_{r+2,j}}}, \tag{98}$$

$j = 1, ..., r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Next suppose that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ is s-convex in the second sense. As in [2] we get

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq \frac{2^{\frac{1}{p_{r+2,j}}} M_j (b-a)^{\frac{1}{p_{r+2,j}}}}{(s+1)^{\frac{1}{p_{r+2,j}}}}, \tag{99}$$

$j = 1, ..., r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Finally assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ is s-concave in the second sense. Based on Theorem 8 and acting as in [2], we derive

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq 2^{\frac{s-1}{p_{r+2,j}}} \left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right| (b-a)^{\frac{1}{p_{r+2,j}}}. \tag{100}$$

The proof is done. ■

Grüss type inequalities for several functions under s-log-convexity in the second sense follow.

Theorem 16 *Same terms and assumptions as in Theorem 14, $a \geq 0$. We further suppose that $\left|f_j^{(n_j)}\right| \neq 0$ is* s-log-*convex in the second sense, and $\left|f_j^{(n_j)}(a)\right|$, $\left|f_j^{(n_j)}(b)\right| \in (0,1]$, $j = 1, ..., r$. Call $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $s \in (0,1]$, $j = 1, ..., r$, and $\psi_s(z)$ as in (57).*

1) It holds

$$|\Delta(f_1, ..., f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right) \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1 \\ i\neq j}}^{r} \|f_i\|_{\infty,[a,b]}\right) \left\|P_{n_j-1}\right\|_{\infty,[a,b]} \left|f_j^{(n_j)}(b)\right|^s \psi_s(A_j)\right]\right]. \tag{101}$$

2) Let $p_{i,j} > 1 : \sum_{i=1}^{r+2} \frac{1}{p_{i,j}} = 1$, and $f_j^{(n_j)} \in L_{(r+2),j}([a,b])$, $B_j^* := A_j^{p_{r+2,j}}$, $j = 1, ..., r$. Then

$$|\Delta(f_1, ..., f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\prod_{\substack{i=1\\i\neq j}}^{r}\left\|f_i\right\|_{p_{i,j},[a,b]}\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left|f_j^{(n_j)}(b)\right|^s\left(\psi_s\left(B_j^*\right)\right)^{\frac{1}{p_{r+2,j}}}\Bigg]\Bigg]. \tag{102}$$

Proof. 1) From (81) we get

$$|\Delta(f_1,...,f_r)|\leq(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda\right]\right]\overset{(\text{by }(68))}{\leq} \tag{103}$$

$$(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^s\psi_s(A_j)\right]\right]. \tag{104}$$

That is proving (101).

2) As in (69) we get

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}\leq(b-a)^{\frac{1}{p_{r+2,j}}}\left|f_j^{(n_j)}(b)\right|^s\left(\psi_s\left(B_j^*\right)\right)^{\frac{1}{p_{r+2,j}}}. \tag{105}$$

Using (105) into (83), we derive (102). ■

Finally we give applications to Grüss type inequalities for several functions when $n_1=n_2=...=n_r=1$.

Corollary 17 *(to Theorem 14) Let $f_j:[a,b]\to\mathbb{R}$ be absolutely continuous, $j=1,...,r\in\mathbb{N}-\{1\}$. Denote*

$$\Delta^*(f_1,...,f_r):=r\int_a^b\left(\prod_{i=1}^{r}f_i(x)\right)dx-$$

$$\frac{1}{b-a}\left[\sum_{j=1}^{r}\left[\left(\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i(x)\right)dx\right)\left(\int_a^b f_j(x)\,dx\right)\right]\right]. \tag{106}$$

Then

1)

$$|\Delta^*(f_1,...,f_r)|\leq\left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

23

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|f_j'\right\|_{1,[a,b]}\right]\right], \tag{107}$$

2)

$$|\Delta^*(f_1,...,f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|f_j'\right\|_{\infty,[a,b]}\right]\right], \tag{108}$$

3) let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$, *and* $f_j' \in L_{(r+2),j}([a,b])$, $j = 1,...,r$, *it holds*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|f_j'\right\|_{p_{r+2,j},[a,b]}\right]. \tag{109}$$

Corollary 18 *(to Theorem 15) Here all as in Corollary 17, with* $a \geq 0$.

1) Suppose $|f_j'|$ *is s-convex in the second sense and* $|f_j'(x)| \leq M_{1j}$, $x \in [a,b]$, $j = 1,...,r$. *Then*

$$|\Delta^*(f_1,...,f_r)| \leq \frac{4(b-a)^2}{(s+2)(s+3)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)M_{1j}\right]\right]. \tag{110}$$

2) Let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$, *with* $f_j' \in L_{p_{r+2.j}}([a,b])$, $j = 1,...,r$.

2i) Assume again $|f_j'|$ *is s-convex in the second sense, and* $|f_j'(x)| \leq M_{1j}$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{1+\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)M_{1j}(p_{r+2,j}s+1)^{-\frac{1}{p_{r+2,j}}}\right]. \tag{111}$$

2ii) Assume that $|f_j'|^{p_{r+2,j}}$ *is s-convex in the second sense, and* $|f_j'(x)| \leq M_{1j}$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\frac{M_{1j}}{(s+1)^{\frac{1}{p_{r+2,j}}}}\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\Bigg]. \tag{112}$$

2iii) Assume that $\left|f_j'\right|^{p_{r+2,j}}$ *is* s*-concave in the second sense. Then*

$$|\Delta^*(f_1,...,f_r)| \le \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{s-1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left|f_j'\left(\frac{a+b}{2}\right)\right|\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\right]. \tag{113}$$

Corollary 19 *(to Theorem 16) Here all as in Corollary 17, with* $a\ge 0$*. We further suppose that* $\left|f_j'\right|\neq 0$ *is* s-log-*convex in the second sense, and* $\left|f_j'(a)\right|,\left|f_j'(b)\right|$ $\in(0,1]$, $j=1,...,r$*. Call* $A_{1j}:=\left|\frac{f_j'(a)}{f_j'(b)}\right|$, $s\in(0,1]$, $j=1,...,r$, *and* $\psi_s(z)$ *as in (57).*

1) It holds

$$|\Delta^*(f_1,...,f_r)| \le (b-a)\left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left|f_j'(b)\right|^{s}\psi_s(A_{1j})\right]\right]. \tag{114}$$

2) Let $p_{i,j}>1:\sum_{i=1}^{r+2}\frac{1}{p_{i,j}}=1$, *and* $f_j'\in L_{(r+2),j}([a,b])$, $B_{1j}^{*}:=A_{1j}^{p_{r+2,j}}$, $j=1,...,r$*. Then*

$$|\Delta^*(f_1,...,f_r)| \le \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\left|f_j'(b)\right|^{s}\left(\psi_s\left(B_{1j}^{*}\right)\right)^{\frac{1}{p_{r+2,j}}}\right]. \tag{115}$$

Remark 20 *From (20) one can work out the analogous Grüss type inequalities general theory involving the functions* $L_{n_j}[f_j(x)]$, *for* $j=1,...,r\in\mathbb{N}-\{1\}$*. The results will be very similar to the results of Theorems 14-16, and when* $n_1=n_2=...=n_r=1$ *their applications will be identical to Corollaries 17-19. We choose to omit this study.*

References

[1] A.O.Akdemir, M. Tunc, *On some integral inequalities for s-logarithmically convex functions and their applications*, arXiv:1212.1584v1[math.FA]7Dec2012.

[2] G.A. Anastassiou, *General Grüss and Ostrowski type inequalities involving s-convexity*, submitted for publication 2013.

[3] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2(1882), 93-98.

[4] Lj. Dedic, M. Matic and J. Pečaric, *On some generalizations of Ostrowski' inequality for Lipschitz functions and functions of bounded variation*, Math. Inequal. Appl., 3(1)(2000), 1-14.

[5] Lj. Dedic, J. Pečaric and N. Ujevic, *On generalizations of Ostrowski inequality and some related results*, Czechoslovak Math. J., 53(128)(2003), 173-189.

[6] S.S. Dragomir and S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demostratio Math. 32(4)(1999), 687-696.

[7] S.S. Dragomir and S. Wang, *A new inequality of Ostrowski type in L_p norm*, Indian Jour. Math., 40(1998), 299-304.

[8] A.M. Fink, *Bounds of the deviation of a function from its averages*, Czechoslovak. Math. J., 42(117)(1992), 289-310.

[9] G. Grüss, *Über das maximum des absoluten Betraages von* $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx$, Math. Z., 39(1935), 215-226.

[10] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math., 48(1994), 100-111.

[11] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10(1938), 226-227.

[12] B.G. Pachpatte, *On Ostrowski-Grüss-Čebyšev type inequalities for functions whose modulus of derivatives are convex*, J. of Inequalities in Pure and Applied Mathematics, http://jipam.vu.edu.au/Vol.6, Issue 4, Article 128, 2005.

[13] B.G. Pachpatte, *New Ostrowski type inequalities involving the product of two functions*, J. of Inequalities in Pure and Applied Mathematics, http://jipam.vu.edu.au/Vol. 7, Issue 3, Article 104, 2006.

[14] B.G. Pachpatte, *On a new generalization of Ostrowski type inequality*, Tamkang Journal of Mathematics, Vol. 38, No. 4 (2007), 335-339.

Harmonic Multivariate Ostrowski and Grüss type Inequalities for several functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we derive very general multivariate Ostrowski and Grüss type inequalities for several functions by involving harmonic polynomials. Estimates are with respect to all basic norms. We give applications.

2010 AMS Subject Classification: 26B20, 26B40, 26D10, 26D15.
Keywords and Phrases: Multivariate Grüss inequality, Multivariate Ostrowski inequality, harmonic polynomials.

1 Introduction

The problem of estimating the difference of a value of a function from its average is a paramount one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions from the product of their averages is also a very important one. The answer to it are the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a vast literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very much interested to these inequalities due to their applications. So here we derive very general Ostrowski and Grüss type inequalities for several multivariate functions, acting to all possible directions.

We are motivated by the following results.

Theorem 1 *(1938, Ostrowski [7]) Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *whose derivative* $f' : (a, b) \to \mathbb{R}$ *is bounded on* (a, b)*, i.e.,*

$\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x\in[a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

Theorem 2 *(1935, Grüss [6]) Let* f,g *be integrable functions from* $[a,b]$ *into* $\mathbb{R}$, *that satisfy the conditions*

$$m \le f(x) \le M,\ n \le g(x) \le N,\ \ x\in[a,b],$$

where $m,M,n,N\in\mathbb{R}$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \qquad (2)$$

$$\le \frac{1}{4}(M-m)(N-n).$$

Theorem 3 *(1998, Dragomir and Wang [4]) Let* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous function with* $f'\in L_p([a,b])$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $x\in[a,b]$. *Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le$$

$$\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\|f'\|_p. \qquad (3)$$

Theorem 4 *(1882, Čebyšev [1]) Let* $f,g:[a,b]\to\mathbb{R}$ *absolutely continuous functions with* $f',g'\in L_\infty([a,b])$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \qquad (4)$$

$$\le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty.$$

Above is also assumed that the involved integrals exist.

2 Background

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geq 1$, $P_0 = 1$. Furthermore, let $[a,b] \subset \mathbb{R}$, $a \neq b$, and $h : [a,b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some fixed $n \geq 1$. We use the notation

$$L_n[h(x)] = \frac{1}{n}\Bigg[h(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b)\right]\Bigg], \tag{5}$$

$x \in [a,b]$, for convinience.

For $n = 1$ the above sums are defined to be zero, that is $L_1[h(x)] = h(x)$.

Dedic, Pečaric and Ujevic, see [2], [3], established the following identity,

$$L_n[h(x)] - \frac{1}{b-a}\int_a^b h(t)\,dt = \frac{(-1)^{n+1}}{n(b-a)}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{6}$$

where

$$q(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b]. \tag{7}$$

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \geq 0$, the identity (6) reduces to the Fink identity in [5], (see also [3], p. 177).

We rewrite (6) as follows:

$$h(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x) h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b)\right] = \frac{n}{b-a}\int_a^b h(t)\,dt + \frac{(-1)^{n+1}}{b-a}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{8}$$

$x \in [a,b]$.

That is the generalized Fink type representation formula.

$$h(x) = \sum_{k=1}^{n-1}(-1)^{k+1} P_k(x) h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(b) h^{(k-1)}(b) - P_k(a) h^{(k-1)}(a)\right] + \frac{n}{b-a}\int_a^b h(t)\,dt + \frac{(-1)^{n+1}}{b-a}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{9}$$

$x \in [a,b]$, $n \geq 1$, when $n = 1$ the above sums are zero.

3 Main Results

Here $\prod_{i=1}^{m}[a_i,b_i] \subseteq \mathbb{R}^m$, $m,n \in \mathbb{N}$.

We make

General Assumptions 5 *Let* $f : \prod_{i=1}^{m}[a_i,b_i] \to \mathbb{R}$. *We assume*

1) for $j = 1,...,m$ *we have that* $\frac{\partial^{n-1}f}{\partial x_j^{n-1}}(x_1,x_2,...,x_{j-1},s_j,x_{j+1},...,x_m)$ *is absolutely continuous in* s_j *on* $[a_j,b_j]$, *for every* $(x_1,x_2,...,x_{j-1},x_{j+1},...,x_m) \in \prod_{\substack{i=1\\ i\neq j}}^{m}[a_i,b_i]$,

2) for $j = 1,...,m$ *we have that* $\frac{\partial^{n}f(s_1,...,s_j,x_{j+1},...,x_m)}{\partial x_j^{n}}$ *is continuous on* $\prod_{i=1}^{j}[a_i,b_i]$, *for every* $(x_{j+1},...,x_m) \in \prod_{i=j+1}^{m}[a_i,b_i]$,

3) for each $j = 1,...,m$, *and for every* $l = 1,...,n-1$, *we have that* $\frac{\partial^{l}f}{\partial x_j^{l}}(s_1,s_2,...,s_{j-1},x_j,...,x_m)$ *is continuous on* $\prod_{i=1}^{j-1}[a_i,b_i]$, *for every* $(x_j,...,x_m) \in \prod_{i=j}^{m}[a_j,b_j]$.

4) f *is continuous on* $\prod_{i=1}^{m}[a_i,b_i]$.

Brief Assumptions 6 *Let* $f : \prod_{i=1}^{m}[a_i,b_i] \to \mathbb{R}$ *with* $\frac{\partial^{l}f}{\partial x_i^{l}}$ *for* $l = 0,1,...,n$; $i = 1,...,m$, *are continuous on* $\prod_{i=1}^{m}[a_i,b_i]$.

Definition 7 *We put*

$$q(x_i,s_i) = \begin{cases} s_i - a_i, & \text{if } s_i \in [a_i,x_i], \\ s_i - b_i, & \text{if } s_i \in (x_i,b_i], \end{cases} \quad x_i \in [a_i,b_i], \tag{10}$$

$i = 1,...,m$.

We present the following general representation result of Fink type.

Theorem 8 *Let* f *as in General Assumtions 5 or Brief Assumtions 6. Then*

$$f(x_1,...,x_n) = \frac{n^m}{\prod_{i=1}^{m}(b_i-a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} f(s_1,...,s_m)\,ds_1...ds_m \tag{11}$$

4

$$+\sum_{i=1}^{m} T_i\left(x_i, x_{i+1}, ..., x_m\right),$$

where

$$T_i\left(x_i, ..., x_m\right) := \frac{n^{i-1}}{\prod_{j=1}^{i-1}\left(b_j - a_j\right)} \cdot$$

$$\left[\sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_i\right) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f\left(s_1, ..., s_{i-1}, x_i, ..., x_m\right)}{\partial x_i^k} ds_1 ... ds_{i-1} + \right.$$

$$\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b_i - a_i} \cdot$$

$$\left[P_k\left(b_i\right) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f\left(s_1, ..., s_{i-1}, b_i, x_{i+1}, ..., x_m\right)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1} - \right.$$

$$\left. P_k\left(a_i\right) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f\left(s_1, ..., s_{i-1}, a_i, x_{i+1}, ..., x_m\right)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1}\right] + \tag{12}$$

$$\left. \frac{(-1)^{n+1}}{\left(b_i - a_i\right)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} P_{n-1}\left(s_i\right) q\left(x_i, s_i\right) \frac{\partial^n f\left(s_1, ..., s_i, x_{i+1}, ..., x_m\right)}{\partial x_i^n} ds_1 ... ds_i\right],$$

are continuous functions for all $i = 1, ..., m$.

Proof. We apply (9) repeatedly.

We have

$$f\left(x_1, ..., x_m\right) = \sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_1\right) \frac{\partial^k f\left(x_1, ..., x_m\right)}{\partial x_1^k} +$$

$$\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{\left(b_1 - a_1\right)}\left[P_k\left(b_1\right) \frac{\partial^{k-1} f\left(b_1, x_2, ..., x_m\right)}{\partial x_1^{k-1}} - P_k\left(a_1\right) \frac{\partial^{k-1} f\left(a_1, x_2, ..., x_m\right)}{\partial x_1^{k-1}}\right]$$

$$+\frac{n}{\left(b_1 - a_1\right)} \int_{a_1}^{b_1} f\left(s_1, x_2, ..., x_m\right) ds_1 +$$

$$\frac{(-1)^{n+1}}{\left(b_1 - a_1\right)} \int_{a_1}^{b_1} P_{n-1}\left(s_1\right) q\left(x_1, s_1\right) \frac{\partial^n f\left(s_1, x_2, ..., x_m\right)}{\partial x_1^n} ds_1, \tag{13}$$

any $x_1 \in [a_1, b_1]$,
under the assumption that $\frac{\partial^{n-1} f(\cdot, x_2, ..., x_m)}{\partial x_1^{n-1}} \in AC\left(\left[a_1, b_1\right]\right)$.

Call

$$T_1\left(x_1, ..., x_m\right) := \sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_1\right) \frac{\partial^k f\left(x_1, ..., x_m\right)}{\partial x_1^k} +$$

5

$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_1-a_1)}\left[P_k(b_1)\frac{\partial^{k-1}f(b_1,x_2,...,x_m)}{\partial x_1^{k-1}}-P_k(a_1)\frac{\partial^{k-1}f(a_1,x_2,...,x_m)}{\partial x_1^{k-1}}\right]$$

$$+\frac{(-1)^{n+1}}{(b_1-a_1)}\int_{a_1}^{b_1}P_{n-1}(s_1)\,q(x_1,s_1)\,\frac{\partial^n f(s_1,x_2...,x_m)}{\partial x_1^n}ds. \qquad (14)$$

Hence it holds

$$f(x_1,...,x_m)=\frac{n}{(b_1-a_1)}\int_{a_1}^{b_1}f(s_1,x_2,...,x_m)\,ds_1+T_1(x_1,...,x_m). \qquad (15)$$

Next similarly we get

$$f(s_1,x_2,...,x_m)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_2)\frac{\partial^k f(s_1,x_2,...,x_m)}{\partial x_2^k}+$$

$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_2-a_2)}\cdot$$

$$\left[P_k(b_2)\frac{\partial^{k-1}f(s_1,b_2,x_3,...,x_m)}{\partial x_2^{k-1}}-P_k(a_2)\frac{\partial^{k-1}f(s_1,a_2,x_3,...,x_m)}{\partial x_2^{k-1}}\right]+ \qquad (16)$$

$$\frac{n}{(b_2-a_2)}\int_{a_2}^{b_2}f(s_1,s_2,x_3,...,x_m)\,ds_2+$$

$$\frac{(-1)^{n+1}}{(b_2-a_2)}\int_{a_2}^{b_2}P_{n-1}(s_2)\,q(x_2,s_2)\,\frac{\partial^n f(s_1,s_2,x_3,...,x_m)}{\partial x_2^n}ds_2,$$

any $x_2\in[a_2,b_2]$.

Hence it holds

$$f(x_1,...,x_m)=\frac{n^2}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(s_1,s_2,x_3,...,x_m)\,ds_1ds_2+$$

$$T_1(x_1,...,x_m)+T_2(x_2,x_3,...,x_m), \qquad (17)$$

where

$$T_2(x_2,x_3,...,x_m):=\frac{n}{(b_1-a_1)}\left\{\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_2)\int_{a_1}^{b_1}\frac{\partial^k f(s_1,x_2,...,x_m)}{\partial x_2^k}ds_1\right.$$

$$+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_2-a_2)}\left[P_k(b_2)\int_{a_1}^{b_1}\frac{\partial^{k-1}f(s_1,b_2,x_3,...,x_m)}{\partial x_2^{k-1}}ds_1\right. \qquad (18)$$

$$\left.-P_k(a_2)\int_{a_1}^{b_1}\frac{\partial^{k-1}f(s_1,a_2,x_3,...,x_m)}{\partial x_2^{k-1}}ds_1\right]$$

$$+\frac{(-1)^{n+1}}{(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_{n-1}(s_2)\,q(x_2,s_2)\,\frac{\partial^n f(s_1,s_2,x_3,...,x_m)}{\partial x_2^n}ds_1ds_2\Bigg\}.$$

Next we see similarly that

$$f(s_1,s_2,x_3,...,x_m)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_3)\,\frac{\partial^k f(s_1,s_2,x_3,...,x_m)}{\partial x_3^k}+$$
$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_3-a_3)}\Bigg[P_k(b_3)\,\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,...,x_m)}{\partial x_3^{k-1}}-$$
$$P_k(a_3)\,\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,...,x_m)}{\partial x_3^{k-1}}\Bigg]+ \tag{19}$$
$$\frac{n}{(b_3-a_3)}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4,...,x_m)\,ds_3+$$
$$\frac{(-1)^{n+1}}{(b_3-a_3)}\int_{a_3}^{b_3}P_{n-1}(s_3)\,q(x_3,s_3)\,\frac{\partial^n f(s_1,s_2,s_3,x_4,...,x_m)}{\partial x_3^n}ds_3,$$

any $x_3\in[a_3,b_3]$.

So that we get

$$f(x_1,...,x_m)=\frac{n^3}{\prod_{j=1}^{3}(b_j-a_j)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4,...,x_m)\,ds_1ds_2ds_3+$$
$$T_1(x_1,...,x_m)+T_2(x_2,...,x_m)+T_3(x_3,...,x_m), \tag{20}$$

where

$$T_3(x_3,...,x_m):=\frac{n^2}{(b_1-a_1)(b_2-a_2)}\cdot$$
$$\Bigg[\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^k f(s_1,s_2,x_3,...,x_m)}{\partial x_3^k}ds_1ds_2+$$
$$\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_3-a_3)}\Bigg[P_k(b_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,...,x_m)}{\partial x_3^{k-1}}ds_1ds_2$$
$$-P_k(a_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,...,x_m)}{\partial x_3^{k-1}}ds_1ds_2\Bigg]+ \tag{21}$$
$$\frac{(-1)^{n+1}}{(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}P_{n-1}(s_3)\,q(x_3,s_3)\,\frac{\partial^n f(s_1,s_2,s_3,x_4,...,x_m)}{\partial x_3^n}ds_1ds_2ds_3\Bigg].$$

Furthermore we can write

$$f\left(s_1,s_2,s_3,x_4,...,x_m\right)=\sum_{k=1}^{n-1}\left(-1\right)^{k+1}P_k\left(x_4\right)\frac{\partial^k f\left(s_1,s_2,s_3,x_4...,x_m\right)}{\partial x_4^k}+$$
$$\sum_{k=1}^{n-1}\frac{\left(-1\right)^k\left(n-k\right)}{\left(b_4-a_4\right)}\left[P_k\left(b_4\right)\frac{\partial^{k-1}f\left(s_1,s_2,s_3,b_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}\right.$$
$$\left.-P_k\left(a_4\right)\frac{\partial^{k-1}f\left(s_1,s_2,s_3,a_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}\right]+ \tag{22}$$
$$\frac{n}{\left(b_4-a_4\right)}\int_{a_4}^{b_4}f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)ds_4+$$
$$\frac{\left(-1\right)^{n+1}}{\left(b_4-a_4\right)}\int_{a_4}^{b_4}P_{n-1}\left(s_4\right)q\left(x_4,s_4\right)\frac{\partial^n f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)}{\partial x_4^n}ds_4,$$

any $x_4\in\left[a_4,b_4\right]$.

Therefore it holds

$$f\left(x_1,...,x_m\right)=\frac{n^4}{\prod_{j=1}^{4}\left(b_j-a_j\right)}\cdot$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4}f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)ds_1ds_2ds_3ds_4+$$
$$T_1\left(x_1,...,x_m\right)+T_2\left(x_2,...,x_m\right)+T_3\left(x_3,...,x_m\right)+T_4\left(x_4,...,x_m\right), \tag{23}$$

where

$$T_4\left(x_4,...,x_m\right):=\frac{n^3}{\prod_{j=1}^{3}\left(b_j-a_j\right)}\left[\sum_{k=1}^{n-1}\left(-1\right)^{k+1}P_k\left(x_4\right)\cdot\right.$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^k f\left(s_1,s_2,s_3,x_4,...,x_m\right)}{\partial x_4^k}ds_1ds_2ds_3+$$
$$\sum_{k=1}^{n-1}\frac{\left(-1\right)^k\left(n-k\right)}{\left(b_4-a_4\right)}\left[P_k\left(b_4\right)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^{k-1}f\left(s_1,s_2,s_3,b_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}ds_1ds_2ds_3\right.$$
$$\left.-P_k\left(a_4\right)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^{k-1}f\left(s_1,s_2,s_3,a_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}ds_1ds_2ds_3\right] \tag{24}$$
$$+\frac{\left(-1\right)^{n+1}}{\left(b_4-a_4\right)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4}P_{n-1}\left(s_4\right)q\left(x_4,s_4\right)\cdot$$

8

$$\left.\frac{\partial^n f(s_1,s_2,s_3,s_4,x_5,...,x_m)}{\partial x_4^n} ds_1 ds_2 ds_3 ds_4\right],$$

etc.

The theorem is proved. ■

We make

Remark 9 *Let f_λ, $\lambda = 1,...,r \in \mathbb{N} - \{1\}$, as in Assumtions 5 or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ. Here $x = (x_1,...,x_m)$, $s = (s_1,...,s_m) \in \prod_{i=1}^{m} [a_i,b_i]$. Then*

$$f_\lambda(x) = \frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i - a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds + \sum_{i=1}^{m} T_{i\lambda}(x_i, x_{i+1},...,x_m). \tag{25}$$

Here we have

$$T_{i\lambda}(x_i,...,x_m) := \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1}(b_j - a_j)} \left[\sum_{k=1}^{n_\lambda - 1} (-1)^{k+1} P_k(x_i) \cdot\right.$$

$$\int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f_\lambda(s_1,...,s_{i-1},x_i,...,x_m)}{\partial x_i^k} ds_1...ds_{i-1} +$$

$$\sum_{k=1}^{n_\lambda - 1} \frac{(-1)^k (n_\lambda - k)}{b_i - a_i} \cdot$$

$$\left[P_k(b_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1,...,s_{i-1},b_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}} ds_1...ds_{i-1}\right.$$

$$\left.-P_k(a_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1,...,s_{i-1},a_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}} ds_1...ds_{i-1}\right] + \tag{26}$$

$$\left.\frac{(-1)^{n_\lambda + 1}}{(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} P_{n_\lambda - 1}(s_i)\, q(x_i,s_i) \frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}} ds_1...ds_i\right],$$

are continuous functions, $i = 1,...,m$; $\lambda = 1,...,r$.

Hence it holds

$$\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right) f_\lambda(x) = \left(\frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i - a_i)}\right) \left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right) \left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) + \tag{27}$$

$$\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\sum_{i=1}^{m} T_{i\lambda}(x_1,...,x_m)\right).$$

Therefore we derive

$$\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_{\rho}(x)\right) f_{\lambda}(x)\right)-$$

$$\frac{1}{\prod_{i=1}^{m}(b_i-a_i)}\left\{\sum_{\lambda=1}^{r}\left(n_{\lambda}^{m}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_{\lambda}(s)\,ds\right)\right)\right\} \tag{28}$$

$$=\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\sum_{i=1}^{m} T_{i\lambda}(x_i,...,x_m)\right)\right).$$

We notice that

$$T_{i\lambda}(x_i,...,x_m)=A_{i\lambda}(x_i,...,x_m)+B_{i\lambda}(x_i,...,x_m), \tag{29}$$

$i=1,...,m;$ *where*

$$A_{i\lambda}(x_i,...,x_m):=\frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i-1}(b_j-a_j)}\left[\sum_{k=1}^{n_{\lambda}-1}(-1)^{k+1}P_k(x_i)\cdot\right. \tag{30}$$

$$\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^k f_{\lambda}(s_1,...,s_{i-1},x_i,...,x_m)}{\partial x_i^k}ds_1...ds_{i-1}$$

$$+\sum_{k=1}^{n_{\lambda}-1}\frac{(-1)^k(n_{\lambda}-k)}{b_i-a_i}\cdot$$

$$\left[P_k(b_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1} f_{\lambda}(s_1,...,s_{i-1},b_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}-\right.$$

$$\left.\left.P_k(a_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1} f_{\lambda}(s_1,...,s_{i-1},a_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}\right]\right],$$

and

$$B_{i\lambda}(x_i,...,x_m):=\frac{n_{\lambda}^{i-1}(-1)^{n_{\lambda}+1}}{\prod_{j=1}^{i}(b_j-a_j)}\cdot \tag{31}$$

10

$$\left[\int_{a_1}^{b_1}...\int_{a_i}^{b_i} P_{n_\lambda-1}(s_i)\, q(x_i,s_i)\, \frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}} ds_1...ds_i\right],$$

for all $i=1,...,m;\ \lambda=1,...,r$.

We call and have the identity

$$S(f_1,...,f_r)(x) :=$$

$$\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left[f_\lambda(x) - \frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i-a_i)}\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) - \right.\right.$$

$$\left.\left.\sum_{i=1}^{m} A_{i\lambda}(x_i,...,x_m)\right]\right\} = \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left(\sum_{i=1}^{m} B_{i\lambda}(x_i,...,x_m)\right)\right), \quad (32)$$

true for any fixed $x\in\prod_{i=1}^{m}[a_i,b_i]$.

Then we have

$$|S(f_1,...,f_r)(x)| \leq \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} |f_\rho(x)|\right)\left(\sum_{i=1}^{m}|B_{i\lambda}(x_i,...,x_m)|\right)\right). \quad (33)$$

We estimate the right hand side of (33).

We also make

Remark 10 *We observe that*

$$|B_{i\lambda}(x_i,...,x_m)| \leq \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\left[\int_{a_1}^{b_1}...\int_{a_i}^{b_i}|P_{n_\lambda-1}(s_i)|\,|q(x_i,s_i)|\cdot\right.$$

$$\left.\left|\frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right| ds_1...ds_i\right] =: (\xi), \quad (34)$$

for all $i=1,...,m;\ \lambda=1,...,r$.

We know that

$$|q(x_i,s_i)| \leq \max(x_i-a_i, b_i-x_i) = \frac{(b_i-a_i)+|a_i+b_i-2x_i|}{2}. \quad (35)$$

We have

$$(\xi) \overset{(35)}{\leq} \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \left[\|P_{n_\lambda - 1}\|_{\infty,[a_i,b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \cdot \right.$$

$$\left. \left\| \frac{\partial^{n_\lambda} f_\lambda(..., x_{i+1}, ..., x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right]. \tag{36}$$

Thus

$$|S(f_1, ..., f_r)(x)| \leq \sum_{\lambda=1}^{r} \left(\left(\prod_{\substack{\rho=1 \\ \rho\neq\lambda}}^{r} |f_\rho(x)| \right) \left(\sum_{i=1}^{m} \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \cdot \right. \right.$$

$$\left[\|P_{n_\lambda - 1}\|_{\infty,[a_i,b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \cdot \right.$$

$$\left. \left. \left. \left\| \frac{\partial^{n_\lambda} f_\lambda(..., x_{i+1}, ..., x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right] \right) \right) =: \theta_1(x). \tag{37}$$

Next let $p_{li} > 1 : \sum_{li=1}^{3} \frac{1}{p_{li}} = 1$. *Then*

$$|B_{i\lambda}(x_i, ..., x_m)| \overset{(34)}{\leq} (\xi) \leq \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \cdot$$

$$\left[\left(\int_{a_1}^{b_1} ... \int_{a_i}^{b_i} |P_{n_\lambda - 1}(s_i)|^{p_{1i}} ds_1 ... ds_i \right)^{\frac{1}{p_{1i}}} \left(\int_{a_1}^{b_1} ... \int_{a_i}^{b_i} |q(x_i, s_i)|^{p_{2i}} ds_1 ... ds_i \right)^{\frac{1}{p_{2i}}} \right. \tag{38}$$

$$\left. \cdot \left\| \frac{\partial^{n_\lambda} f_\lambda(..., x_{i+1}, ..., x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right] =$$

$$\frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i,b_i])}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\cdot\right.$$

$$\left.\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]. \tag{39}$$

Therefore we get

$$|S(f_1,...,f_r)(x)|\overset{(39)}{\leq}\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\frac{n_\lambda^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\cdot\right.\right.$$

$$\left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i,b_i])}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\cdot\right.$$

$$\left.\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right])) \tag{40}$$

$$=:\theta_2(x).$$

We also have

$$|B_{i\lambda}(x_i,...,x_m)|\leq(\xi)\leq n_\lambda^{i-1}\left[\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\cdot\right.$$

$$\left.\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{i}[a_j,b_j]}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\right], \tag{41}$$

$i=1,...,m;\ \lambda=1,...,r.$

Consequently we find

$$|S(f_1,...,f_r)(x)|\leq\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\left[n_\lambda^{i-1}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\cdot\right.\right.\right.$$

$$\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda} f_\lambda\left(...,x_{i+1},...,x_m\right)}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod\limits_{j=1}^{i}[a_j,b_j]}\Bigg]))=:\theta_3\left(x\right). \tag{42}$$

Finally we derive that

$$\left|S\left(f_1,...,f_r\right)(x)\right|\leq\min\left\{\theta_1\left(x\right),\theta_2\left(x\right),\theta_3\left(x\right)\right\}. \tag{43}$$

We have proved the following general multivariate Ostrowski type inequality for several functions.

Theorem 11 *Let f_λ, $\lambda=1,...,r\in\mathbb{N}-\{1\}$, as in Assumptions 5 or Brief Assumptions 6; $n_\lambda\in\mathbb{N}$ associated with f_λ, $x=(x_1,...,x_m)$, $s=(s_1,...,s_m)\in\prod\limits_{i=1}^{m}[a_i,b_i]$. Here $A_{i\lambda}(x_i,...,x_m)$ as in (30), $i=1,...,m$. We put*

$$S\left(f_1,...,f_r\right)(x):=\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)\left[f_\lambda\left(x\right)-\frac{n_\lambda^m}{\prod\limits_{i=1}^{m}\left(b_i-a_i\right)}\cdot\right.\right.$$

$$\left.\left.\left(\int_{\prod\limits_{i=1}^{m}[a_i,b_i]}f_\lambda\left(s\right)ds\right)-\sum_{i=1}^{m}A_{i\lambda}\left(x_i,...,x_m\right)\right]\right\}. \tag{44}$$

Here $\theta_1(x)$ is as in (37). Let $p_{li}>1:\sum\limits_{li=1}^{3}\frac{1}{p_{li}}=1$, $i=1,...,m$, and $\theta_2(x)$ as in (40). And $\theta_3(x)$ as in (42). Then

$$\left|S\left(f_1,...,f_r\right)(x)\right|\leq\min\left\{\theta_1\left(x\right),\theta_2\left(x\right),\theta_3\left(x\right)\right\}. \tag{45}$$

We continue with

Remark 12 *Additionally assume that $\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}$ are continuous on $\prod\limits_{j=1}^{m}[a_j,b_j]$ for all $i=1,...,m$; $\lambda=1,...,r$.*

We define and observe

$$W:=\int_{\prod\limits_{j=1}^{m}[a_j,b_j]}S\left(f_1,...,f_r\right)(x)\,dx=r\int_{\prod\limits_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)dx- \tag{46}$$

14

$$\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\sum_{\lambda=1}^{r}n_{\lambda}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)dx\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]}f_{\lambda}(s)\,ds\right)-$$

$$\sum_{\lambda=1}^{r}\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)\left(\sum_{i=1}^{m}A_{i\lambda}(x_i,...,x_m)\right)\right)dx=$$

$$\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)\left(\sum_{i=1}^{m}B_{i\lambda}(x_i,...,x_m)\right)\right)dx\right\}. \tag{47}$$

Clearly here $B_{i\lambda}(x_i,...,x_m)$ is a continuous function for all $i=1,...,m$; $\lambda=1,...,r$.

Hence

$$|W|\overset{(47)}{\leq}\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_{\rho}(x)|\right)\left(\sum_{i=1}^{m}|B_{i\lambda}(x_i,...,x_m)|\right)\right\}dx\right\}=:(\omega_1). \tag{48}$$

That is

$$|W|\leq\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_{\rho}\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\left(\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx\right)\right)\right\}. \tag{49}$$

From (41) we obtain

$$|B_{i\lambda}(x_i,...,x_m)|\leq$$

$$n_{\lambda}^{i-1}\|P_{n_{\lambda}-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_{\lambda}}f_{\lambda}}{\partial x_i^{n_{\lambda}}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right), \tag{50}$$

all $i=1,...,m$; $\lambda=1,...,r$.

Then

$$\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx\leq$$

$$\frac{n_{\lambda}^{i-1}\|P_{n_{\lambda}-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_{\lambda}}f_{\lambda}}{\partial x_i^{n_{\lambda}}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}}{2(b_i-a_i)}.$$

15

$$\left(\prod_{\substack{j=1\\j\neq i}}^{m}(b_j-a_j)\right)\left(\int_{a_i}^{b_i}\left((b_i-x_i)^2+(x_i-a_i)^2\right)dx_i\right), \tag{51}$$

and consequently it holds,

$$\int_{\prod_{j=1}^{m}[a_j,b_j]}\left|B_{i\lambda}(x_i,...,x_m)\right|dx \leq$$

$$\frac{(b_i-a_i)\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}n_\lambda^{i-1}\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}, \tag{52}$$

for all $i=1,...,m$; $\lambda=1,...,r$.

Hence

$$\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left|B_{i\lambda}(x_i,...,x_m)\right|dx\right) \leq \left(\frac{\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}\right)\cdot$$

$$\left(\sum_{i=1}^{m}\left[(b_i-a_i)\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}n_\lambda^{i-1}\right]\right), \tag{53}$$

for $\lambda=1,...,r$.

Using (49) and (53) we obtain

$$|W| \leq \left(\frac{\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}\right)\left[\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\\rho\neq\lambda}}^{r}\left\|f_\rho\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\cdot\right.\right.$$

$$\left.\left.\left(\sum_{i=1}^{m}\left[(b_i-a_i)n_\lambda^{i-1}\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right)\right\}\right]=:A_1. \tag{54}$$

Notice next that

$$(\omega_1) = \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)|B_{i\lambda}(x_i,...,x_m)|\,dx\right) =: (\omega_2)\,. \quad (55)$$

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then

$$(\omega_2) \leq \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}\|_{L_q\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}}. \quad (56)$$

We have also proved that

$$|W| \leq \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}\|_{L_q\left(\prod_{j=i}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}} =: A_2. \quad (57)$$

From (50) we get

$$|B_{i\lambda}(x_i,...,x_m)| \leq n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\left(\frac{b_i-a_i}{2}\right), \quad (58)$$

all $i=1,...,m$; $\lambda=1,...,r$.

Thus it holds

$$\sum_{i=1}^{m}|B_{i\lambda}(x_i,...,x_m)| \leq$$

$$\frac{1}{2}\left\{\sum_{i=1}^{m}\left[(b_i-a_i)\,n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right\}, \quad (59)$$

$\lambda=1,...,r$.

Using (48) and (59) we finally derive

$$|W| \leq \frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}[\sum_{i=1}^{m}\left[(b_i-a_i)\,n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\cdot\right.\right.\right.$$

17

$$\left\| \frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^{m} [a_j, b_j]}] \Bigg] \Bigg\} \Bigg\} =: A_3. \tag{60}$$

We have proved the following general multivariate Grüss inequality.

Theorem 13 *Let f_λ, $\lambda = 1, ..., r \in \mathbb{N} - \{1\}$, as in Assumotions 5 plus $\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}$ are continuous on $\prod_{j=1}^{m} [a_j, b_j]$ for all $i = 1, ..., m$; $\lambda = 1, ..., r$, or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ. Here $A_{i\lambda}(x_i, ..., x_m)$ as in (30), and $B_{i\lambda}(x_i, ..., x_m)$ as in (31), $i = 1, ..., m$. We set*

$$W := r \int_{\prod_{j=1}^{m} [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x) \right) dx -$$

$$\frac{1}{\prod_{j=1}^{m} (b_j - a_j)} \sum_{\lambda=1}^{r} n_\lambda^m \left(\int_{\prod_{j=1}^{m} [a_j, b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x) \right) dx \right) \left(\int_{\prod_{i=1}^{m} [a_i, b_i]} f_\lambda(s) \, ds \right) -$$

$$\sum_{\lambda=1}^{r} \int_{\prod_{j=1}^{m} [a_j, b_j]} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x) \right) \left(\sum_{i=1}^{m} A_{i\lambda}(x_i, ..., x_m) \right) \right) dx. \tag{61}$$

Let A_1 as in (54); $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, A_2 as in (57), and A_3 as in (60).
Then

$$|W| \leq \min\{A_1, A_2, A_3\}. \tag{62}$$

4 Applications

We apply Theorems 8, 11 and 13 for the case of $n_1 = n_2 = ... = n_r = 1$.

We simplify General Assumptions 5 and Brief Assumptions 6, respectively, as follows:

General Assumptions 14 *Let $f : \prod_{j=1}^{m} [a_j, b_j] \to \mathbb{R}$ satisfying:*

1) for $j = 1, ..., m$ we have that $f(x_1, x_2, ..., x_{j-1}, s_j, x_{j+1}, ..., x_m)$ is absolutely continuous in $s_j \in [a_j, b_j]$, for every $(x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_m) \in \prod_{\substack{i=1 \\ i \neq j}}^{m} [a_i, b_i]$, *2) for $j=1,...,m$ we have that $\frac{\partial f(s_1,...,s_j,x_{j+1},...,x_m)}{\partial x_j}$ is continuous on $\prod_{i=1}^{j}[a_i,b_i]$, for every $(x_{j+1},...,x_m)\in\prod_{i=j+1}^{m}[a_i,b_i]$,*

3) f is continuous on $\prod_{i=1}^{m}[a_i,b_i]$.

Brief Assumptions 15 *Let $f:\prod_{j=1}^{m}[a_j,b_j]\to\mathbb{R}$ with $\frac{\partial^l f}{\partial x_i^l}$ for $l=0,1;\ j=1,...,m,$ are continuous on $\prod_{j=1}^{m}[a_j,b_j]$.*

We give the following multivariate representation result which is an application of Theorem 8

Theorem 16 *Let f as in General Assumtions 14 or Brief Assumtions 15. Then*

$$f(x_1,...,x_n)=\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\int_{\prod_{j=1}^{m}[a_j,b_j]}f(s_1,...,s_m)\,ds_1...ds_m+\sum_{i=1}^{m}T_i^*(x_i,...,x_m), \tag{63}$$

where

$$T_i^*(x_i,...,x_m)=\frac{1}{\prod_{j=1}^{i}(b_j-a_j)}\cdot$$

$$\int_{a_1}^{b_1}...\int_{a_i}^{b_i}q(x_i,s_i)\frac{\partial f(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i}ds_1...ds_i, \tag{64}$$

are continuous functions for all $i=1,...,m$.

Next we make

Remark 17 *Let f_λ as in Assumptions 14 or 15, $\lambda=1,...,r$. Then*

$$f_\lambda(x)=\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\int_{\prod_{j=1}^{m}[a_j,b_j]}f_\lambda(s)\,ds+\sum_{i=1}^{m}T_{i\lambda}^*(x_i,...,x_m). \tag{65}$$

Here the corresponding

$$A_{i\lambda}^*(x_i,...,x_m)=0,$$

and

$$B^*_{i\lambda}(x_i, ..., x_m) = \frac{1}{\prod_{j=1}^{i}(b_j - a_j)} \cdot$$

$$\int_{a_1}^{b_1} ... \int_{a_i}^{b_i} q(x_i, s_i) \frac{\partial f_\lambda(s_1, ..., s_i, x_{i+1}, ..., x_m)}{\partial x_i} ds_1...ds_i, \tag{66}$$

for all $i = 1, ..., m;\ \lambda = 1, ..., r.$

That is

$$T^*_{i\lambda}(x_i, ..., x_m) = B^*_{i\lambda}(x_i, ..., x_m), \tag{67}$$

$i = 1, ..., m;\ \lambda = 1, ..., r.$

We call and have the identity

$$S^*(f_1, ..., f_r)(x) :=$$

$$\sum_{\lambda=1}^{r} \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x) \right) \left[f_\lambda(x) - \frac{1}{\prod_{j=1}^{m}(b_j - a_j)} \left(\int_{\prod_{j=1}^{m}[a_j, b_j]} f_\lambda(s)\, ds \right) \right] \right\}$$

$$= \sum_{\lambda=1}^{r} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x) \right) \left(\sum_{i=1}^{m} B^*_{i\lambda}(x_i, ..., x_m) \right) \right), \tag{68}$$

true for any fixed $x \in \prod_{j=1}^{m}[a_j, b_j].$

Hence it holds

$$|S^*(f_1, ..., f_r)(x)| \leq \sum_{\lambda=1}^{r} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} |f_\rho(x)| \right) \left(\sum_{i=1}^{m} |B^*_{i\lambda}(x_i, ..., x_m)| \right) \right). \tag{69}$$

We obtain:

1) From (37) we derive

$$|S^*(f_1, ..., f_r)(x)| \leq \sum_{\lambda=1}^{r} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} |f_\rho(x)| \right) \left(\sum_{i=1}^{m} \frac{1}{\prod_{j=1}^{i}(b_j - a_j)} \cdot \right.\right.$$

$$\left[\left(\frac{(b_i-a_i)+|a_i+b_i-2x_i|}{2}\right)\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]\Bigg)\Bigg)\Bigg) \tag{70}$$

$$=: \theta_1^*(x).$$

2) Let $p_{li}>1:\sum_{li=1}^{3}\frac{1}{p_{li}}=1$. *Then by (40) we derive*

$$|S^*(f_1,...,f_r)(x)| \leq$$

$$\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\frac{1}{\prod_{j=1}^{i}(b_j-a_j)}\left[(b_i-a_i)^{\frac{1}{p_{1i}}}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\right.\right.\right.\cdot$$

$$\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\cdot$$

$$\left.\left.\left.\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]\right)\right) =: \theta_2^*(x). \tag{71}$$

3) We get by (42) that

$$|S^*(f_1,...,f_r)(x)| \leq \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\cdot\right.\right.$$

$$\left.\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{i}[a_j,b_j]}\right] =: \theta_3^*(x). \tag{72}$$

So as an applications of Theorem 11 we give the following multivariate Ostrowski type inequality.

Theorem 18 *All as in Remark 17. Then*

$$|S^*(f_1,...,f_r)(x)| \leq \min\{\theta_1^*(x),\theta_2^*(x),\theta_3^*(x)\}. \tag{73}$$

We also make

21

Remark 19 *Here Assumptions 14 hold and* $\frac{\partial f_\lambda}{\partial x_i}$ *are continuous on* $\prod_{j=1}^{m}[a_j,b_j]$ *for all* $i=1,...,m;\ \lambda=1,...,r,$ *or Assumptions 15 are valid.*

We set

$$W^* = \int_{\prod_{j=1}^{m}[a_j,b_j]} S^*(f_1,...,f_r)(x)\,dx = r\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\rho=1}^{r} f_\rho(x)\right)dx- \tag{74}$$

$$\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\sum_{\lambda=1}^{r}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)dx\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right)=$$

$$\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left(\sum_{i=1}^{m}B_{i\lambda}^*(x_i,...,x_m)\right)\right)dx\right\}. \tag{75}$$

Here $B_{i\lambda}^*$ is a continuous for any $i=1,...,m;\ \lambda=1,...,r.$

Then

1) following (49) we find

$$|W^*| \leq \sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\left(\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left|B_{i\lambda}^*(x_i,...,x_m)\right|dx\right)\right)\right\}. \tag{76}$$

We also get that

$$|W^*| \overset{(54)}{\leq} \left(\frac{\prod_{j=1}^{m}(b_j-a_j)}{3}\right)\left[\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\right.\right.$$

$$\left.\left.\left(\sum_{i=1}^{m}(b_i-a_i)\left\|\frac{\partial f_\lambda}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\right\}\right] =: A_1^*. \tag{77}$$

2) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1.$ Then

$$|W^*| \overset{(57)}{\leq} \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}^*\|_{L_q\left(\prod_{j=1}^{m}[a_j,b_j]\right)}. \tag{78}$$

22

$$\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}} =: A_2^*.$$

3) From (60) we obtain

$$|W^*| \le \frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left[\sum_{i=1}^{m}\left[(b_i-a_i)\left\|\frac{\partial f_\lambda}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right]\right\}\right\} \tag{79}$$

$$=: A_3^*.$$

We have proved the following multivariate Grüss type inequality as an application of Theorem 13.

Theorem 20 *Here all as in Remark 19. We derive*

$$|W^*| \le \min\{A_1^*, A_2^*, A_3^*\}. \tag{80}$$

References

[1] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2 (1882), 93-98.

[2] Lj. Dedic, M. Matic and J. Pečaric, *On some generalizations of Ostrowski's inequality for Lipschitz functions and functions of bounded variation*, Math. Inequal. Appl., 3(1)(2000), 1-14.

[3] Lj. Dedic, J. Pečaric and N. Ujevic, *On generalizations of Ostrowski inequality and some related results*, Czechoslovak Math. J., 53(128)(2003), 173-189.

[4] S.S. Dragomir and S. Wang, *A new inequality of Ostrowski type in L_p norm*, Indian Jour. Math., 40(1998), 299-304.

[5] A.M. Fink, *Bounds of the deviation of a function from its averages*, Czechoslovak. Math. J., 42(117)(1992), 289-310.

[6] G. Grüss, *Über das Maximum des absoluten Betrages von* $\left[\left(\frac{1}{b-a}\right)\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right)\right]$, Math. Z. 39 (1935), 215-226.

[7] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

Fractional Ostrowski and Grüss type inequalities involving several functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using Caputo fractional left and right Taylor formulae we establish mixed fractional Ostrowski and Grüss type inequalities involving several functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

2010 Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: fractional derivative, fractional inequalities, Ostrowski inequality, Grüss inequality.

1 Introduction

The following results motivate initially our work.

Theorem 1 *(1938, Ostrowski [10]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integration approximations in Numerical Analysis.

Theorem 2 *(1882, Čebyšev [6]) Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$
$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \quad (2)$$

The above integrals are assumed to exist.

Grüss type inequalities have applications to Probability.

We are also biggly inspired by the beautiful work of Deepak B. Pachpatte [11]. His article contains great ideas, but it is marred by many minor errors.

So here we present mixed fractional Ostrowski and Grüss type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [1], Chapters 24-26 and [3], Chapters 2-6.

2 Background

Let $\nu \geq 0$; the operator I_{a+}^ν, defined for $f \in L_1[(a, b)]$ is given by

$$I_{a+}^\nu f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \quad (3)$$

for $a \leq x \leq b$, is called the left Riemann-Liouville fractional integral operator of order ν. For $\nu = 0$, we set $I_{a+}^0 := I$, the identity operator, see [1], p. 392, also [7].

Let $\nu \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^n([a, b])$ (it means $f^{(n-1)} \in AC([a, b])$, absolutely continuous functions).

Then the left Caputo fractional derivative is given by

$$D_{*a}^\nu f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) dt = \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \quad (4)$$

and it exists almost everywhere for $x \in [a, b]$.

See Corollary 16.8, p. 394 of [1], and [7], pp. 49-50.

We need also the left Caputo fractional Taylor formula, see [1], p. 395 and [7], p. 54.

Theorem 3 *Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a, b])$. Then*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + I_{a+}^\nu D_{*a}^\nu f(x), \quad \forall\, x \in [a, b]. \quad (5)$$

Here $I_{a+}^\nu D_{*a}^\nu f \in AC^n([a, b])$.

Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator ([2], [8], [9]) of order α is denoted by

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \quad \forall\, x \in [a,b]. \tag{6}$$

We set $I_{b-}^{0} := I$, the identity operator.

Let now $f \in AC^m([a,b])$, $m \in \mathbb{N}$, with $m := \lceil \alpha \rceil$.

We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \tag{7}$$

we set $D_{b-}^{0} f := f$, that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{8}$$

We need the right Caputo fractional Taylor formula.

Theorem 4 *([2]) Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := \lceil a \rceil$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^z (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz. \tag{9}$$

We need

Proposition 5 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$; $x, x_0 \in [a,b] : x \geq x_0$. Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.*

Proposition 6 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$; $x, x_0 \in [a,b] : x \leq x_0$. Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.*

We also mention

Theorem 7 *([4], p. 362) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

Convention 8 *([4], p. 360) We suppose that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{10}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{11}$$

for all $x, x_0 \in [a,b]$.

Finally we are motivated by the following mixed Caputo fractional Ostrowski type inequalities.

Theorem 9 *([3], p. 44) Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$, *and* $\left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}$, $\left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} < \infty$, $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+2)} \cdot$$

$$\left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} (x_0-a)^{\alpha+1} + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} (b-x_0)^{\alpha+1} \right\} \leq \tag{12}$$

$$\frac{1}{\Gamma(\alpha+2)} \max\left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}, \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha}. \tag{13}$$

Inequality (12) is sharp, infact it is attained.

Theorem 10 *([3], p. 45) Le* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$ *and* $D^{\alpha}_{x_0-} f \in L_1([a,x_0])$, $D^{\alpha}_{*x_0} f \in L_1([x_0,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+1)} \cdot \tag{14}$$

$$\left\{ (x_0-a)^{\alpha} \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])} + (b-x_0)^{\alpha} \left\| D^{\alpha}_{*x_0} f \right\|_{L_1([x_0,b])} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \max\left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_1([x_0,b])} \right\} (b-a)^{\alpha-1}. \tag{15}$$

Theorem 11 *([3], p. 47) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$. *Assume* $D^{\alpha}_{x_0-} f \in L_q([a,x_0])$, *and* $D^{\alpha}_{*x_0} f \in L_q([x_0,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \cdot \tag{16}$$

$$\left\{ (x_0-a)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])} + (b-x_0)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\max\left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} (b-a)^{\alpha-\frac{1}{q}}. \tag{17}$$

In this article we generalize Theorems 9-11 for several functions. We also produce Caputo fractional Grüss type inequalities for several functions.

3 Main Results

We start with Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 12 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$, with $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $i = 1,...,r$. Assume that $\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,...,r$. Denote by*

$$\theta(f_1,...,f_r)(x_0) := r\int_a^b \left(\prod_{k=1}^r f_k(x) \right) dx - \sum_{i=1}^r \left[f_i(x_0) \int_a^b \left(\prod_{\substack{j=1 \\ j\neq i}}^r f_j(x) \right) dx \right]. \tag{18}$$

Then

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^r \left[\left[\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1} \left(\prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right. \tag{19}$$

$$\left. + \left[\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1} \left(\prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right].$$

Inequality (19) is sharp, infact it is attained.

Theorem 13 *Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$; $x_0 \in [a,b]$ and $D_{x_0-}^{\alpha} f_i \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f_i \in L_1([x_0,b])$, for all $i = 1,...,r$. Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^r \left[\left[\left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a,x_0])} I_{a+}^{\alpha} \left(\prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right. \tag{20}$$

$$\left. + \left[\left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0,b])} I_{b-}^{\alpha} \left(\prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right].$$

Theorem 14 *Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $x_0 \in [a,b]$; $i = 1,...,r$. Assume $D_{x_0-}^{\alpha} f_i \in L_q([a,x_0])$, and $D_{*x_0}^{\alpha} f_i \in L_q([x_0,b])$, $i = 1,...,r$. Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

5

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\alpha\right)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{21}$$
$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Proof. of Theorems 12-14. Here $x_0 \in [a,b]$. Since $f_i^{(k)}(x_0)=0$, $k=1,...,m-1$; $i=1,...,r$, we have by Theorem 3 that

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-z\right)^{\alpha-1}D_{*x_0}^{\alpha}f_i\left(z\right)dz, \quad \forall\ x\in\left[x_0,b\right], \tag{22}$$

and by Theorem 4 that

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{x_0}\left(z-x\right)^{\alpha-1}D_{x_0-}^{\alpha}f_i\left(z\right)dz, \quad \forall\ x\in\left[a,x_0\right]; \tag{23}$$

for all $i=1,...,r$.

Multiplying (22) and (23) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)$ we get, respectively,

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$
$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-z\right)^{\alpha-1}D_{*x_0}^{\alpha}f_i\left(z\right)dz, \quad \forall\ x\in\left[x_0,b\right], \tag{24}$$

and

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$
$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)}{\Gamma\left(\alpha\right)}\int_{x}^{x_0}\left(z-x\right)^{\alpha-1}D_{x_0-}^{\alpha}f_i\left(z\right)dz, \quad \forall\ x\in\left[a,x_0\right]; \tag{25}$$

for all $i=1,...,r$.

Adding (24) and (25), separately, we obtain

$$r\left(\prod_{k=1}^{r}f_k\left(x\right)\right)-\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)\right]=$$

6

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right],\quad \forall\, x\in[x_0,b], \quad (26)$$

and

$$r\left(\prod_{k=1}^{r}f_k(x)\right)-\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x_0)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right],\quad \forall\, x\in[a,x_0]. \quad (27)$$

Next we integrate (26) and (27) with respect to $x\in[a,b]$. We have

$$r\int_{x_0}^{b}\left(\prod_{k=1}^{r}f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right], \quad (28)$$

and

$$r\int_{a}^{x_0}\left(\prod_{k=1}^{r}f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]. \quad (29)$$

Finally adding (28) and (29) we obtain the useful identity

$$\theta(f_1,...,f_r)(x_0):=$$

$$r\int_{a}^{b}\left(\prod_{k=1}^{r}f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1} D_{*x_0}^{\alpha} f_i(z)\,dz\right)dx\right]\Bigg]. \tag{30}$$

Hence it holds

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}\left|D_{x_0-}^{\alpha} f_i(z)\right|dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left|D_{*x_0}^{\alpha} f_i(z)\right|dz\right)dx\right]\right] =: (*). \tag{31}$$

We observe that

$$(*) \leq \frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha} f_i\right\|_{\infty,[a,x_0]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(x_0-x)^{\alpha}\,dx\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha} f_i\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(x-x_0)^{\alpha}\,dx\right]\right] = \tag{32}$$

$$\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha} f_i\right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha} f_i\right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \tag{33}$$

Based on Proposition 15.114, p. 388, [1] we get that $I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right) \in AC([a,b])$, so at x_0 is finite.

Also, based on [5] we get that $I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right) \in AC([a,b])$, so at x_0 is finite.

Next we prove that (19) is sharp, namely it is attained.

Set

$$\overline{f_i}(x) := \begin{cases}(x-x_0)^{\alpha}, & x \in [x_0,b],\\ (x_0-x)^{\alpha}, & x \in [a,x_0],\end{cases}$$

8

$\alpha > 0$, $a \leq x_0 \leq b$, $i = 1, ..., r$.

Observe that $\overline{f_i} \in AC^m([x_0, b])$, and in $AC^m([a, x_0])$. See that $\overline{f_{i-}}^{(k)}(x_0) = \overline{f_{i+}}^{(k)}(x_0) = 0$, $k = 0, 1, ..., m-1$. Hence there exists $\overline{f_i}^{(m-1)}$ at x_0, also $\overline{f_i}^{(m-1)} \in AC([a, b])$. That is $\overline{f_i} \in AC^m([a, b])$, $i = 1, ..., r$.

We find that

$$\left\| D_{x_0-}^{\alpha} \overline{f_i} \right\|_{\infty, [a, x_0]} = \Gamma(\alpha + 1),$$

and

$$\left\| D_{*x_0}^{\alpha} \overline{f_i} \right\|_{\infty, [x_0, b]} = \Gamma(\alpha + 1).$$

Consequently it holds

$$L.H.S.(19) = r\left[\frac{(b - x_0)^{\alpha r + 1} + (x_0 - a)^{\alpha r + 1}}{\alpha r + 1}\right] = R.H.S.(19),$$

proving optimality of (19).

Hence proving Theorem 12.

Next we notice, for $\alpha \geq 1$, that

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a, x_0])} \int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (x_0 - x)^{\alpha - 1} dx \right] + \right.$$

$$\left.\left[\left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0, b])} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (x - x_0)^{\alpha - 1} dx \right]\right] = \tag{34}$$

$$\sum_{i=1}^{r} \left[\left[\left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a, x_0])} I_{a+}^{\alpha} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right)\right] + \right.$$

$$\left.\left[\left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0, b])} I_{b-}^{\alpha} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right)\right]\right], \tag{35}$$

proving Theorem 13.

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right)\right.\right.$$

$$\left(\int_x^{x_0} (z-x)^{p(\alpha-1)}\,dz\right)^{\frac{1}{p}} \left(\int_x^{x_0} \left|D_{x_0-}^{\alpha} f_i(z)\right|^q dz\right)^{\frac{1}{q}} dx\Bigg] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x)|\right) \left(\int_{x_0}^{x} (x-z)^{p(\alpha-1)}\,dz\right)^{\frac{1}{p}} \left(\int_{x_0}^{x} \left|D_{*x_0}^{\alpha} f_i(z)\right|^q dz\right)^{\frac{1}{q}} dx\right]\Bigg] \leq \tag{36}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x)|\right) \frac{(x_0-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{x_0-}^{\alpha} f_i\right\|_{q,[a,x_0]} dx\right] +\right.$$

$$\left.\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x)|\right) \frac{(x-x_0)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{*x_0}^{\alpha} f_i\right\|_{q,[x_0,b]} dx\right]\right] = \tag{37}$$

$$\frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \cdot$$

$$\sum_{i=1}^{r} \left[\left[\left\|D_{x_0-}^{\alpha} f_i\right\|_{q,[a,x_0]} \int_a^{x_0} (x_0-x)^{\alpha+\frac{1}{p}-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x)|\right) dx\right] +\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha} f_i\right\|_{q,[x_0,b]} \int_{x_0}^{b} (x-x_0)^{\alpha+\frac{1}{p}-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x)|\right) dx\right]\right] = \tag{38}$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\left\|D_{x_0-}^{\alpha} f_i\right\|_{q,[a,x_0]} I_{a+}^{\alpha+\frac{1}{p}} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x_0)|\right)\right] + \right. \tag{39}$$

$$\left.\left[\left\|D_{*x_0}^{\alpha} f_i\right\|_{q,[x_0,b]} I_{b-}^{\alpha+\frac{1}{p}} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x_0)|\right)\right]\right],$$

that is proving Theorem 14. ■

Next follow Caputo fractional Grüss type inequalities for several functions.

Theorem 15 *Let* $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_i \in AC([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$. *Assume that* $\sup_{x_0 \in [a,b]} \left\|D_{x_0-}^{\alpha} f_i\right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\|D_{*x_0}^{\alpha} f_i\right\|_{\infty,[x_0,b]} < \infty$, $i = 1, ..., r$. *Denote by*

$$\Delta(f_1, ..., f_r)(x_0) := r(b-a) \int_a^b \left(\prod_{k=1}^{r} f_k(x)\right) dx- \tag{40}$$

10

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right)\right].$$

Then

$$|\Delta(f_1,...,f_r)| \leq (b-a)\cdot$$

$$\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]} I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]} I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \tag{41}$$

Theorem 16 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{q}<\alpha\leq 1$, and $f_i\in AC([a,b])$, $i=1,...,r\in\mathbb{N}-\{1\}$, $x_0\in[a,b]$. Assume that $\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}$, and $\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}<\infty$, $i=1,...,r$. Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{(b-a)\,\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\cdot$$

$$\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}\sup_{x_0\in[a,b]} I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}\sup_{x_0\in[a,b]} I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \tag{42}$$

Proof. of Theorems 15, 16. Here $0<\alpha\leq 1$, i.e. $m=1$, and $f_i\in AC([a,b])$, $i=1,...,r$. Now we are not tight up to any initial conditions, i.e. (22) and (23) are valid without initial conditions. Clearly here $\theta(f_1,...,f_r)(x_0)\in AC([a,b])$. Integrating (30) over $[a,b]$ with respect to $x_0\in[a,b]$ we derive

$$\Delta(f_1,...f_r) := \int_a^b \theta(f_1,...,f_r)(x_0)\,dx_0 = \tag{43}$$

$$r(b-a)\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)dx-\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\int_a^b\left\{\sum_{i=1}^r\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_x^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.\right.$$
$$\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_{x_0}^x(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right]\right]\right\}dx_0. \tag{44}$$

Hence it holds

$$|\Delta(f_1,...,f_r)|\leq$$
$$\frac{1}{\Gamma(\alpha)}\int_a^b\left|\sum_{i=1}^r\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_x^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.\right.$$
$$\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_{x_0}^x(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right]\right]\right|dx_0=:(**). \tag{45}$$

Using (19) we have

$$(**)\leq(b-a)\sum_{i=1}^r\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]\right.$$
$$\left.+\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]\right], \tag{46}$$

proving Theorem 15.

When $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $\frac{1}{q}<\alpha\leq1$, by (21) we get

$$(**)\leq\frac{(b-a)\,\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\cdot$$
$$\sum_{i=1}^r\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{q,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]+\right.$$
$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{q,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]\right], \tag{47}$$

proving Theorem 16. ■

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] G.A. Anastassiou, *On Right fractional calculus*, Chaos, Solitons and Fractals, 42 (2009), 365-376.

[3] G.A. Anastassiou, *Advances on Fractional Inequalities*, Research Monograph, Springer, New York, 2011.

[4] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Research Monograph, Springer, Berlin, Heidelberg, 2011.

[5] G.A. Anastassiou, *Fractional Representation Formulae and right fractional inequalities*, Mathematical and Computer Modelling, 54 (2011), (11-12), 3098-3115.

[6] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2(1882), 93-98.

[7] Kai Diethelm, T*he Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1 st edition, Springer, New York, Heidelberg, 2010.

[8] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, Vol. 3, No. 12 (2006), 81-95.

[9] R. Gorenflo and F. Mainardi, *Essentials of Fractional Calculus*, 2000, Maphysto Center, http://www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k0a.ps.

[10] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[11] D.B. Pachpatte, *On Right Caputo fractional Ostrowski Inequalities involving three functions*, International Journal of Analysis, Vol. 2013, Article ID 127061, 5 pages, http://dx.doi.org/10.1155/2013/127061.

Further interpretation of some fractional Ostrowski and Grüss type inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

We further interpret and simplify earlier produced fractional Ostrowski and Grüss type inequalities involving several functions.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.
Keywords and Phrases: fractional derivative, fractional inequalities, Ostrowski inequality, Grüss inequality.

1 Background

Let $\nu \geq 0$; the operator I_{a+}^{ν}, defined for $f \in L_1[(a,b)]$ is given by

$$I_{a+}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\, dt, \tag{1}$$

for $a \leq x \leq b$, is called the left Riemann-Liouville fractional integral operator of order ν. For $\nu = 0$, we set $I_{a+}^0 := I$, the identity operator, see [1], p. 392, also [7].

Let $\nu \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^n([a,b])$ (it means $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions).

Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt = \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{2}$$

and it exists almost everywhere for $x \in [a,b]$.

Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator ([2], [8], [9]) of order α is denoted by

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \quad \forall x \in [a,b]. \tag{3}$$

We set $I_{b-}^{0} := I$, the identity operator.

Let now $f \in AC^{m}([a,b])$, $m \in \mathbb{N}$, with $m := \lceil \alpha \rceil$.

We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^{m} I_{b-}^{m-\alpha} f^{(m)}(x), \tag{4}$$

we set $D_{b-}^{0} f := f$, that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)} \int_{x}^{b} (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{5}$$

We need

Proposition 1 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \geq x_0$. Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.*

Proposition 2 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \leq x_0$. Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.*

We also mention

Theorem 3 *([4], p. 362) Let $f \in C^{m}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

Convention 4 *([4], p. 360) We suppose that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{6}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{7}$$

for all $x, x_0 \in [a,b]$.

2 Motivation

We mention some Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 5 *([6]) Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_i \in AC^{m}([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$, with $f_i^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $i = 1, ..., r$. Assume that $\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1, ..., r$. Denote by*

$$\theta(f_1, ..., f_r)(x_0) := r \int_{a}^{b} \left(\prod_{k=1}^{r} f_k(x) \right) dx - \sum_{i=1}^{r} \left[f_i(x_0) \int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right]. \tag{8}$$

2

Then

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{9}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Inequality (9) is sharp, infact it is attained.

Theorem 6 *([6]) Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. *Suppose that* $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$; $x_0 \in [a,b]$ *and* $D_{x_0-}^{\alpha}f_i \in L_1([a,x_0])$, $D_{*x_0}^{\alpha}f_i \in L_1([x_0,b])$, *for all* $i = 1,...,r$. *Then*

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{10}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Theorem 7 *([6]) Let* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. *Suppose that* $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $x_0 \in [a,b]$; $i = 1,...,r$. *Assume* $D_{x_0-}^{\alpha}f_i \in L_q([a,x_0])$, *and* $D_{*x_0}^{\alpha}f_i \in L_q([x_0,b])$, $i = 1,...,r$. *Then*

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\alpha\right)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])} I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{11}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])} I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Next we mention some Caputo fractional Grüss type inequalities for several functions.

3

Theorem 8 *([6]) Let $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_i \in AC([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Assume that $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,...,r$. Denote by*

$$\Delta(f_1,...,f_r) := r(b-a) \int_a^b \left(\prod_{k=1}^{r} f_k(x) \right) dx - \tag{12}$$

$$\sum_{i=1}^{r} \left[\left(\int_a^b f_i(x)\, dx \right) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) \right].$$

Then

$$|\Delta(f_1,...,f_r)| \leq (b-a) \cdot$$

$$\sum_{i=1}^{r} \left[\left[\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]} \sup_{x_0 \in [a,b]} I_{a+}^{\alpha+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} \sup_{x_0 \in [a,b]} I_{b-}^{\alpha+1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right]. \tag{13}$$

Theorem 9 *([6]) Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, and $f_i \in AC([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$, $x_0 \in [a,b]$. Assume that $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_q([a,x_0])}$, and $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_q([x_0,b])} < \infty$, $i = 1,...,r$. Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{(b-a)\,\Gamma\left(\alpha + \frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \cdot$$

$$\sum_{i=1}^{r} \left[\left[\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_q([a,x_0])} \sup_{x_0 \in [a,b]} I_{a+}^{\alpha+\frac{1}{p}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_q([x_0,b])} \sup_{x_0 \in [a,b]} I_{b-}^{\alpha+\frac{1}{p}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right]. \tag{14}$$

3 Main Results

We make

Remark 10 *Let $g \in C([a,b])$, $\alpha > 0$, $x_0 \in [a,b] \subset \mathbb{R}$. Notice that*

$$I_{a+}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} g(z)\, dz. \tag{15}$$

Hence

$$\left|I_{a+}^{\alpha+1}(g)(x_0)\right| \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} |g(z)|\, dz \leq$$

$$\frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha}\, dz = \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \frac{(x_0 - a)^{\alpha+1}}{(\alpha+1)}$$

$$= \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{16}$$

That is

$$\left|I_{a+}^{\alpha+1}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{17}$$

Similarly we have

$$I_{b-}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha} g(z)\, dz, \tag{18}$$

and

$$\left|I_{b-}^{\alpha+1}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha}\, dz$$

$$= \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \frac{(b - x_0)^{\alpha+1}}{(\alpha+1)} = \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{19}$$

That is

$$\left|I_{b-}^{\alpha+1}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{20}$$

Consequently we derive

$$I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} |f_j(x_0)|\right) \overset{(17)}{\leq} \frac{\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}, \tag{21}$$

5

and

$$I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\overset{(20)}{\leq}\frac{\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}}{\Gamma\left(\alpha+2\right)}\left(b-x_0\right)^{\alpha+1}. \tag{22}$$

Therefore it holds

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right|\overset{(9)}{\leq}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right]\overset{((21),(22))}{\leq}$$

$$\frac{1}{\Gamma\left(\alpha+2\right)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right]\left(x_0-a\right)^{\alpha+1}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right]\left(b-x_0\right)^{\alpha+1}\right]=:\left(\xi_1\right). \tag{24}$$

Call

$$M_1\left(f_1,...,f_r\right)\left(x_0\right):=\max_{i=1,...,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]},\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\right\}. \tag{25}$$

Then

$$\left(\xi_1\right)\leq\frac{M_1\left(f_1,...,f_r\right)\left(x_0\right)}{\Gamma\left(\alpha+2\right)}\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\left(x_0-a\right)^{\alpha+1}+\right.$$

$$\left.\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\left(b-x_0\right)^{\alpha+1}\right]=:\left(\xi_2\right). \tag{26}$$

Call

$$\psi_1\left(f_1,...,f_r\right)\left(x_0\right):=\max\left\{\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]},\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \tag{27}$$

So that

$$(\xi_2) \leq \frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq \tag{28}$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}.$$

We have proved simpler interpretations of Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 11 *Here all as in Theorem 5, $M_1(f_1,...,f_r)(x_0)$ as in (25) and $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq \tag{29}$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{30}$$

We make

Remark 12 *Let $g \in C([a,b])$, $\alpha \geq 1$, $x_0 \in [a,b] \subset \mathbb{R}$. We have that*

$$\left|I_{a+}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{31}$$

and

$$\left|I_{b-}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{32}$$

Consequently we derive

$$I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right) \overset{(31)}{\leq} \frac{\left\|\prod_{\substack{j=1\\j\neq i}}^{r} f_j\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{33}$$

$$I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right) \overset{(32)}{\leq} \frac{\left\|\prod_{\substack{j=1\\j\neq i}}^{r} f_j\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{34}$$

Therefore it holds

$$|\theta(f_1,...,f_r)(x_0)| \overset{(10)}{\leq} \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right] \overset{((33),(34))}{\leq}$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right](x_0-a)^{\alpha}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right](b-x_0)^{\alpha}\right] =: (\eta). \quad (35)$$

Call

$$M_2(f_1,...,f_r)(x_0) := \max_{i=1,...,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])}, \left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\right\}. \quad (36)$$

Then

$$(\eta) \leq \frac{M_2(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\cdot$$

$$\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}(x_0-a)^{\alpha}+\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}(b-x_0)^{\alpha}\right] \quad (37)$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\left[(b-x_0)^{\alpha}+(x_0-a)^{\alpha}\right] \quad (38)$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \quad (39)$$

We have proved

Theorem 13 *Let all as in Theorem 6,* $M_2(f_1,...,f_r)(x_0)$ *as in (36) and* $\psi_1(f_1,...,f_r)(x_0)$ *as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\left[(b-x_0)^{\alpha}+(x_0-a)^{\alpha}\right] \quad (40)$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \quad (41)$$

8

Similarly we obtain

Theorem 14 *Let all as in Theorem 7. Call*

$$M_3(f_1,...,f_r)(x_0) := \max_{i=1,...,r}\left\{\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\}. \quad (42)$$

Here $\psi_1(f_1,...,f_r)(x_0)$ *as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\left[(b-x_0)^{\alpha+\frac{1}{p}}+(x_0-a)^{\alpha+\frac{1}{p}}\right] \leq \quad (43)$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}}. \quad (44)$$

Finally we give a simpler interpretation of Caputo fractional Grüss type inequalities (13), (14).

Theorem 15 *All as in Theorem 8. We define*

$$M_4(f_1,...,f_r) := \max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{\infty,[a,x_0]}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{\infty,[x_0,b]}\right\} \quad (45)$$

and

$$\psi_2(f_1,...,f_r)(x_0) :=$$

$$\max\left\{\sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}, \sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \quad (46)$$

Then

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_4(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\Gamma(\alpha+2)}(b-a)^{\alpha+2}. \quad (47)$$

Theorem 16 *All as in Theorem 9. We define*

$$M_5(f_1,...,f_r) := \max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\} \quad (48)$$

Here ψ_2 *is as in (46). Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_5(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}+1}. \quad (49)$$

We finish with applications.

4 Applications

We apply above theory for $r=2$. In that case

$$\theta(f_1,f_2)(x_0)=2\int_a^b f_1(x)\,f_2(x)\,dx-f_1(x_0)\int_a^b f_2(x)\,dx-f_2(x_0)\int_a^b f_1(x)\,dx, \tag{50}$$

$x_0\in[a,b]$,

$$M_1(f_1,f_2)(x_0)=$$
$$\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{\infty,[a,x_0]},\left\|D_{x_0-}^{\alpha}f_2\right\|_{\infty,[a,x_0]},\left\|D_{*x_0}^{\alpha}f_1\right\|_{\infty,[x_0,b]},\left\|D_{*x_0}^{\alpha}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{51}$$

$$\psi_1(f_1,f_2)(x_0)=\max\left\{\|f_1\|_{\infty,[a,x_0]}+\|f_2\|_{\infty,[a,x_0]},\|f_1\|_{\infty,[x_0,b]}+\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{52}$$

$$M_2(f_1,f_2)(x_0)=\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_1([a,x_0])},\right.$$
$$\left.\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_1([a,x_0])},\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_1([x_0,b])},\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_1([x_0,b])}\right\}, \tag{53}$$

$$M_3(f_1,f_2)(x_0):=\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_q([a,x_0])},\right.$$
$$\left.\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_q([a,x_0])},\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_q([x_0,b])},\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{54}$$

$$\Delta(f_1,f_2)=2\left[(b-a)\int_a^b f_1(x)\,f_2(x)\,dx-\left(\int_a^b f_1(x)\,dx\right)\left(\int_a^b f_2(x)\,dx\right)\right], \tag{55}$$

$$M_4(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_1\right\|_{\infty,[a,x_0]},\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_2\right\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_1\right\|_{\infty,[x_0,b]},\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{56}$$

$$\psi_2(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[x_0,b]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{57}$$

and

$$M_5(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_q([a,x_0])},\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_q([a,x_0])},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_q([x_0,b])},\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{58}$$

above $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.

Proposition 17 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_1, f_2 \in AC^m([a,b])$, with $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Assume that $\left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} \left[(b-x_0)^{\alpha+1} + (x_0-a)^{\alpha+1}\right] \tag{59}$$

$$\leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} (b-a)^{\alpha+1}. \tag{60}$$

Proof. By Theorem 11. ∎

Proposition 18 *Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f_1, f_2 \in AC^m([a,b])$. Suppose that $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$; $x_0 \in [a,b]$ and $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_1([x_0,b])$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} \left[(b-x_0)^{\alpha} + (x_0-a)^{\alpha}\right] \tag{61}$$

$$\leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} (b-a)^{\alpha}. \tag{62}$$

Proof. By Theorem 13. ∎

Proposition 19 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $f_1, f_2 \in AC^m([a,b])$. Suppose that $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$. Assume $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_q([a,x_0])$, and $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_q([x_0,b])$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \left[(b-x_0)^{\alpha+\frac{1}{p}} + (x_0-a)^{\alpha+\frac{1}{p}}\right] \tag{63}$$

$$\leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} (b-a)^{\alpha+\frac{1}{p}}. \tag{64}$$

Proof. By Theorem 14. ∎

Proposition 20 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_1, f_2 \in AC([a,b])$. Assume that $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. Then*

$$|\Delta(f_1,f_2)| \leq \frac{2M_4(f_1,f_2)\,\psi_2(f_1,f_2)}{\Gamma(\alpha+2)} (b-a)^{\alpha+2}. \tag{65}$$

Proof. By Theorem 15. ∎

Proposition 21 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, and $f_1, f_2 \in AC([a,b])$, $x_0 \in [a,b]$. Assume that $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_q([a,x_0])}, \sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_q([x_0,b])} < \infty$, $i = 1, 2$. Then*

$$|\Delta(f_1, f_2)| \leq \frac{2M_5(f_1, f_2)\,\psi_2(f_1, f_2)}{\left(\alpha + \frac{1}{p}\right)(p(\alpha - 1) + 1)^{\frac{1}{p}}\,\Gamma(\alpha)}(b-a)^{\alpha + \frac{1}{p} + 1}. \tag{66}$$

Proof. By Theorem 16. ■

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] G.A. Anastassiou, *On Right fractional calculus*, Chaos, Solitons and Fractals, 42 (2009), 365-376.

[3] G.A. Anastassiou, *Advances on Fractional Inequalities*, Research Monograph, Springer, New York, 2011.

[4] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Research Monograph, Springer, Berlin, Heidelberg, 2011.

[5] G.A. Anastassiou, *Fractional Representation Formulae and right fractional inequalities*, Mathematical and Computer Modelling, 54 (2011), (11-12), 3098-3115.

[6] G.A. Anastassiou, *Fractional Ostrowski and Grüss Type Inequalities Involving Several Functions*, submitted 2013.

[7] Kai Diethelm, T*he Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1 st edition, Springer, New York, Heidelberg, 2010.

[8] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, Vol. 3, No. 12 (2006), 81-95.

[9] R. Gorenflo and F. Mainardi, *Essentials of Fractional Calculus*, 2000, Maphysto Center, http://www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k0a.ps.