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PREFACE (JAFA – JCAAM)

These special issues are devoted to a part of proceedings of AMAT 2012 - International Conference on Applied Mathematics and Approximation Theory - which was held during May 17-20, 2012 in Ankara, Turkey, at TOBB University of Economics and Technology. This conference is dedicated to the distinguished mathematician George A. Anastassiou for his 60th birthday.

AMAT 2012 conference brought together researchers from all areas of Applied Mathematics and Approximation Theory, such as ODEs, PDEs, Difference Equations, Applied Analysis, Computational Analysis, Signal Theory, and included traditional subfields of Approximation Theory as well as under focused areas such as Positive Operators, Statistical Approximation, and Fuzzy Approximation. Other topics were also included in this conference, such as Fractional Analysis, Semigroups, Inequalities, Special Functions, and Summability. Previous conferences which had a similar approach to such diverse inclusiveness were held at the University of Memphis (1991, 1997, 2008), UC Santa Barbara (1993), the University of Central Florida at Orlando (2002).

Around 200 scientists coming from 30 different countries participated in the conference. There were 110 presentations with 3 parallel sessions. We are particularly indebted to our plenary speakers: George A. Anastassiou (*University of Memphis - USA*), Dumitru Baleanu (*Çankaya University - Turkey*), Martin Bohner (*Missouri University of Science & Technology - USA*), Jerry L. Bona (*University of Illinois at Chicago - USA*), Weimin Han (*University of Iowa - USA*), Margareta Heilmann (*University of Wuppertal - Germany*), Cihan Orhan (*Ankara University - Turkey*). It is our great pleasure to thank all the organizations that contributed to the conference, the Scientific Committee and any people who made this conference a big success.

Finally, we are grateful to "TOBB University of Economics and Technology", which was hosting this conference and provided all of its facilities, and also to "Central Bank of Turkey" and "The Scientific and Technological Research Council of Turkey" for financial support.

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POSITIVE PERIODIC SOLUTIONS FOR HIGHER-ORDER FUNCTIONAL q-DIFFERENCE EQUATIONS

MARTIN BOHNER AND ROTCHANA CHIEOCHAN

ABSTRACT. In this paper, using the recently introduced concept of periodic functions in quantum calculus, we study the existence of positive periodic solutions of a certain higher-order functional q-difference equation. Just as for the well-known continuous and discrete versions, we use a fixed point theorem in a cone in order to establish the existence of a positive periodic solution.

THIS PAPER IS DEDICATED TO PROFESSOR GEORGE A. ANASTASSIOU ON THE OCCASION OF HIS 60TH BIRTHDAY

1. INTRODUCTION

The existence of positive periodic solutions of functional difference equations has been studied by many authors such as Zhang and Cheng [2], Zhu and Li [5], and Wang and Luo [6]. Some well-known models which are first-order functional difference equations are, for example (see [6]),

(i) the discrete model of blood cell production:

$$\begin{aligned}\Delta x(n) &= -a(n)x(n) + b(n)\frac{1}{1+x^k(n-\tau(n))}, \quad k \in \mathbb{N},\\ \Delta x(n) &= -a(n)x(n) + b(n)\frac{x(n-\tau(n))}{1+x^k(n-\tau(n))}, \quad k \in \mathbb{N},\end{aligned}$$

(ii) the periodic Michaelis–Menton model:

$$\Delta x(n) = a(n)x(n)\left[1 - \sum_{j=1}^{k} \frac{a_j(n)x(n-\tau_j(n))}{1+c_j(n)x(n-\tau_j(n))}\right], \quad k \in \mathbb{N},$$

(iii) the single species discrete periodic population model:

$$\Delta x(n) = x(n)\left[a(n) - \sum_{j=1}^{k} b_j(n)x(n-\tau_j(n))\right], \quad k \in \mathbb{N}.$$

Key words and phrases. Functional difference equation, q-difference equation, periodic solutions.

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This paper studies the existence of periodic solutions of the m-order functional q-difference equations

$$x(q^m t) = a(t)x(t) + f(t, x(t/\tau(t))), \tag{1.1}$$
$$x(q^m t) = a(t)x(t) - f(t, x(t/\tau(t))), \tag{1.2}$$

where $a : q^{\mathbb{N}_0} \to [0, \infty)$ with $a(t) = a(q^\omega t)$, $f : q^{\mathbb{N}_0} \times \mathbb{R} \to [0, \infty)$ is continuous and ω-periodic, i.e., $f(t, u) = q^\omega f(q^\omega t, u)$, and $\tau : q^{\mathbb{N}_0} \to q^{\mathbb{N}_0}$ satisfies $t \geq \tau(t)$ for all $t \in q^{\mathbb{N}_0}$. A few examples of the function a are given by $a(t) = c$, where c is constant for any $t \in q^{\mathbb{N}_0}$, and $a(t) = d_t$, where d_t are constants assigned for each $t \in \{q^k : 0 \leq k \leq \omega - 1\}$. By applying the fixed point theorem (Theorem 1.2) in a cone, we will prove later that (1.1) and (1.2) have positive periodic solutions. The definition of periodic functions on the so-called q-time scale $q^{\mathbb{N}_0}$ has recently been given by the authors [1] as follows.

Definition 1.1 (Bohner and Chieochan [1])**.** A function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying

$$f(t) = q^\omega f(q^\omega t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}$$

is called ω-*periodic*.

Theorem 1.2 (Fixed point theorem in a cone [3, 4])**.** *Let X be a Banach space and P be a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that $\Phi : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that*

(i) *$\|\Phi u\| \leq \|u\|$ for all $u \in P \cap \partial\Omega_1$, and there exists $\psi \in P \setminus \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for all $u \in P \cap \partial\Omega_2$ and $\lambda > 0$, or*

(ii) *$\|\Phi u\| \leq \|u\|$ for all $u \in P \cap \partial\Omega_2$, and there exists $\psi \in P \setminus \{0\}$ such that $u \neq \Phi u + \lambda\psi$ for all $u \in P \cap \partial\Omega_1$ and $\lambda > 0$.*

Then Φ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Positive Periodic Solutions of (1.1)

In this section, we consider the existence of positive periodic solutions of (1.1). Let

$$X := \left\{x = \{x(t)\} : \ x(t) = q^\omega x(q^\omega t) \quad \text{for all} \quad t \in q^{\mathbb{N}_0}\right\}$$

and employ the maximum norm

$$\|x\| := \max_{t \in Q_\omega} |x(t)|, \quad \text{where} \quad Q_\omega := \left\{q^k : \ 0 \leq k \leq \omega - 1\right\}.$$

Then X is a Banach space. Throughout this section, we assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$, where $m \in \mathbb{N}$ is the order of (1.1). We define $l := \gcd(m, \omega)$ and $h = \omega/l$.

Lemma 2.1. *$x \in X$ is a solution of* (1.1) *if and only if*

$$x(t) = \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^{i} a(q^{jm}t)}. \tag{2.3}$$

Proof. From (1.1) and $x \in X$, we get

$$\begin{aligned}
\frac{x(q^m t)}{a(t)} - x(t) &= \frac{f(t, x(t/\tau(t)))}{a(t)}, \\
\frac{x(q^{2m} t)}{a(q^m t)a(t)} - \frac{x(q^m t)}{a(t)} &= \frac{f(q^m t, x(q^m t/\tau(q^m t)))}{a(q^m t)a(t)}, \\
\frac{x(q^{3m} t)}{a(q^{2m} t)a(q^m t)a(t)} - \frac{x(q^{2m} t)}{a(q^m t)a(t)} &= \frac{f(q^{2m} t, x(q^{2m} t/\tau(q^{2m} t)))}{a(q^{2m} t)a(q^m t)a(t)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{x(q^{hm} t)}{\prod_{i=0}^{h-1} a(q^{im} t)} - \frac{x(q^{(h-1)m} t)}{\prod_{i=0}^{h-2} a(q^{im} t)} &= \frac{f(q^{(h-1)m} t, x(q^{(h-1)m} t/\tau(q^{(h-1)m} t)))}{\prod_{i=0}^{h-1} a(q^{im} t))}.
\end{aligned}$$

By summing all equations above and since $x(t) = q^\omega x(q^\omega t)$ for all $t \in q^{\mathbb{N}_0}$, we arrive at (2.3). $\square$

In order to obtain a cone in the Banach space X, we define

$$M^* := \max\left\{ q^{hm} \prod_{i=0}^{h-1} a(q^{im} t) : \ t \in Q_\omega \right\},$$

$$M_* := \min\left\{ q^{hm} \prod_{i=0}^{h-1} a(q^{im} t) : \ t \in Q_\omega \right\},$$

and

$$\delta := \frac{M_*^2 (1 - M^*)}{M^* (1 - M_*)}.$$

Note $0 < \delta < 1$. Now we define the cone P and the mapping $T : X \to X$ by

$$P := \left\{ y \in X : \ y(t) \geq 0, \ y(t) \geq \delta \|y\|, \ t \in q^{\mathbb{N}_0} \right\},$$

$$(Tx)(t) := \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im} t)}{1 - q^{hm} \prod_{i=0}^{h-1} a(q^{im} t)} \sum_{i=0}^{h-1} \frac{f(q^{im} t, x(q^{im} t/\tau(q^{im} t)))}{\prod_{j=0}^{i} a(q^{jm} t)},$$

respectively. Since we have

$$\begin{aligned}
\frac{q^{hm} M_*}{1 - M_*} \sum_{i=0}^{h-1} f(q^{im} t, x(q^{im} t/\tau(q^{im} t))) &\leq (Tx)(t) \\
&\leq \frac{q^{hm} M^*}{M_*(1 - M^*)} \sum_{i=0}^{h-1} f(q^{im} t, x(q^{im} t/\tau(q^{im} t)))
\end{aligned}$$

for any $x \in P$, it follows that $T(P) \subset P$. Define

$$\varphi(s) := \max\left\{ \frac{q^m t f(t, u)}{1 - q^m a(t)} : \ t \in Q_\omega, \ \delta s \leq u \leq s \right\},$$

$$\psi(s) := \min\left\{ \frac{q^m \delta f(t, u(t))}{(1 - q^m a(t)) u(t)} : \ t \in Q_\omega, \ \delta s \leq u \leq s \right\}.$$

Then both functions φ and ψ are continuous on $\mathbb{R}$.

Theorem 2.2. *Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$, where m is the order of the functional q-difference (1.1). Suppose there exist two real numbers $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that $\varphi(\alpha) \leq \alpha$ and $\psi(\beta) \geq 1$. Then (1.1) has at least one positive solution $x \in X$ satisfying*

$$\min\{\alpha,\beta\} \leq \|x\| \leq \max\{\alpha,\beta\}.$$

Proof. Without loss of generality, we can assume $\alpha < \beta$. Let

$$\Omega_1 := \{x \in X : \ \|x\| < \alpha\} \quad \text{and} \quad \Omega_2 := \{x \in X : \ \|x\| < \beta\}.$$

First, we show

$$\|T(x)\| \leq \|x\| \quad \text{for all} \quad x \in P \cap \partial\Omega_1. \tag{2.4}$$

Let $x \in P \cap \partial\Omega_1$. Then $\|x\| = \alpha$ and $\delta\alpha \leq x(t) \leq \alpha$ for all $t \in q^{\mathbb{N}_0}$. Since

$$\frac{q^m t f(t,u)}{1 - q^m a(t)} \leq \varphi(\alpha) \leq \alpha$$

and

$$\frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{mi}t)}{q^{(i+1)m}\prod_{j=0}^{i} a(q^{jm}t)} = 1$$

for all $t \in q^{\mathbb{N}_0}$, we obtain

$$\begin{aligned}
(Tx)(t) \quad &= \quad \frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^{i} a(q^{jm}t)} \\
&\leq \quad \frac{\alpha}{t}\frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)}{1 - q^{hm}\prod_{i=0}^{h-1} a(q^{im}t)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{mi}t)}{q^{(i+1)m}\prod_{j=0}^{i} a(q^{jm}t)} \\
&\leq \quad \alpha = \|x\|
\end{aligned}$$

for all $t \in q^{\mathbb{N}_0}$. Hence (2.4) holds. Next, we show that

$$x \neq Tx + \lambda \quad \text{for all} \quad x \in P \cap \partial\Omega_2, \quad \text{for some} \quad \lambda > 0. \tag{2.5}$$

Suppose (2.5) does not hold, i.e., there exist $x^* \in P \cap \partial\Omega_2$ and λ_0 such that $x^* = Tx^* + \lambda_0$. Let

$$\chi := \min\{x^*(t) : \ t \in Q_\omega\}.$$

Since $x^* \in P \cap \partial\Omega_2$, $\|x^*\| = \beta$ and $\delta\beta \leq x^*(t) \leq \beta$ for all $t \in q^{\mathbb{N}_0}$. Thus we have $\chi = x^*(t_0)$ for some $t_0 \in Q_\omega$. Since

$$1 \leq \psi(\beta) \leq \frac{q^m \delta f(t_0,u)}{(1 - q^m a(t_0))u}$$

and

$$\frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)}{1 - q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{1 - q^m a(q^{im}t_0)}{q^{(1+i)m}\prod_{j=0}^{i} a(q^{jm}t_0)} = 1,$$

we obtain

$$\begin{aligned}
x^*(t_0) &= \lambda_0 + Tx^*(t_0) \\
&= \lambda_0 + \frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)}{1-q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{f(q^{im}t_0, x^*(q^{im}t_0/\tau(q^{im}t_0)))}{\prod_{j=0}^{i} a(q^{jm}t_0)} \\
&\geq \lambda_0 + \frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)}{1-q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{(1-q^m a(q^{im}t_0))x^*(q^{im}t_0/\tau(q^{im}t_0))}{\delta q^m \prod_{j=0}^{i} a(q^{jm}t_0)} \\
&\geq \lambda_0 + \beta \frac{q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)}{1-q^{hm}\prod_{i=0}^{h-1} a(q^{im}t_0)} \sum_{i=0}^{h-1} \frac{1-q^m a(q^{im}t_0)}{q^{(1+i)m}\prod_{j=0}^{i} a(q^{jm}t_0)} \\
&= \lambda_0 + \beta \\
&\geq \lambda_0 + \chi > \chi.
\end{aligned}$$

This gives a contradiction since $x^*(t_0) = \chi$ and hence (2.5) holds. Therefore, by applying Theorem 1.2, it follows that T has a fixed point $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This fixed point is a positive ω-periodic solution of (1.1). $\square$

Corollary 2.3. *Assume* $0 < a(t) < 1/q^m$ *for all* $t \in q^{\mathbb{N}_0}$. *Suppose that one of the following conditions holds:*

(i) $\lim_{s\to 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$ *and* $\lim_{s\to\infty} \psi(s) = \psi_\infty > 1$,

(ii) $\lim_{s\to\infty} \frac{\varphi(s)}{s} = \varphi_\infty < 1$ *and* $\lim_{s\to 0^+} \psi(s) = \psi_0 > 1$.

Then (1.1) *has at least one positive solution* $x \in X$ *with* $\|x\| > 0$.

Proof. It is sufficient to show only case (i). Since $\lim_{s\to 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$, there exists $\delta > 0$ such that for all $s \in (0, \delta)$,

$$\left|\frac{\varphi(s)}{s} - \varphi_0\right| < \frac{1-\varphi_0}{2}, \quad \text{i.e.,} \quad \frac{3\varphi_0 - 1}{2} < \frac{\varphi(s)}{s} < \frac{1+\varphi_0}{2} < 1.$$

Hence there exists $\alpha \in (0, \delta)$ such that $\varphi(\alpha) < \alpha$. Since $\lim_{s\to\infty} \psi(s) = \psi_\infty > 1$, there exists $\delta > 0$ such that for all $s \in (0, \delta)$,

$$|\psi(s) - \psi_\infty| < \frac{\psi_\infty - 1}{2}, \quad \text{i.e.,} \quad 1 < \frac{1+\psi_\infty}{2} < \psi(s) < \frac{3\psi_\infty - 1}{2}.$$

Hence there exists $\beta > 0$ such that $\psi(\beta) > 1$. Thus, by Theorem 2.2, (1.1) has at least one positive solution $x \in X$ with $\|x\| > 0$. $\square$

Theorem 2.4. *Assume* $0 < a(t) < 1/q^m$ *for all* $t \in q^{\mathbb{N}_0}$. *Suppose there exist* $N+1$ *positive constants* $p_1 < p_2 < \ldots < p_N < p_{N+1}$ *such that one of the following conditions is satisfied:*

(i) $\varphi(p_{2k-1}) < p_{2k-1}$, $k \in \{1, 2, \ldots, [(N+2)/2]\}$ *and*
$\psi(p_{2k}) > 1$, $k \in \{1, 2, \ldots, [(N+1)/2]\}$,

(ii) $\varphi(p_{2k}) < p_{2k}$, $k \in \{1, 2, \ldots, [(N+1)/2]\}$ *and*
$\psi(p_{2k-1}) > 1$, $k \in \{1, 2, \ldots, [(N+2)/2]\}$,

where $[d]$ *denotes the integer part of* d. *Then* (1.1) *has at least* N *positive solutions* $x_k \in X$ *with*

$$p_k < \|x_k\| < p_{k+1} \quad \text{for all} \quad k \in \{1, 2, \ldots, N\}.$$

Proof. It is sufficient to show only case (i). Since $\varphi, \psi : (0, \infty) \to [0, \infty)$ are continuous for each pair $\{p_k, p_{k+1}\}$ and each $k \in \{1, 2, \ldots N\}$, there exist $p_k < \alpha_k < \beta_k < p_{k+1}$ for all $k \in \{1, 2, \ldots N\}$ such that

$$\varphi(\alpha_{2k-1}) < \alpha_{2k-1}, \quad \psi(\beta_{2k-1}) > 1, \quad k \in \{1, 2, \ldots, [(N+2)/2]\},$$
$$\varphi(\alpha_{2k}) < \alpha_{2k}, \quad \psi(\beta_{2k}) > 1, \quad k \in \{1, 2, \ldots, [(N+1)/2]\}.$$

By Theorem 2.2, (1.1) has at least one positive periodic solution $x_k \in X$ for every pair of numbers $\{\alpha_k, \beta_k\}$ with $p_k < \alpha_k \le \|x\| \le \beta_k < p_{k+1}$. The proof is complete. □

By applying Theorem 2.2, we can easily prove the following two corollaries.

Corollary 2.5. *Assume* $0 < a(t) < 1/q^m$ *for all* $t \in q^{\mathbb{N}_0}$. *Suppose that the following conditions hold:*

(i) $\lim_{s \to 0^+} \frac{\varphi(s)}{s} = \varphi_0 < 1$ *and* $\lim_{s \to \infty} \frac{\varphi(s)}{s} = \varphi_\infty < 1$,
(ii) *there exists a constant* $\beta > 0$ *such that* $\psi(\beta) > 1$.

Then (1.1) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < \|x_1\| < \beta < \|x_2\| < \infty.$$

Corollary 2.6. *Assume* $0 < a(t) < 1/q^m$ *for all* $t \in q^{\mathbb{N}_0}$. *Suppose that the following conditions hold:*

(i) $\lim_{s \to 0^+} \psi(s) = \psi_0 > 1$ *and* $\lim_{s \to \infty} \psi(s) = \psi_\infty > 1$,
(ii) *there exists a constant* $\alpha > 0$ *such that* $\varphi(\alpha) < \alpha$.

Then (1.1) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < \|x_1\| < \alpha < \|x_2\| < \infty.$$

3. Positive Periodic Solutions of (1.2)

In this section, we discuss the existence of positive periodic solutions of (1.2). Throughout this section, we assume $a(t) > \frac{1}{q^m}$ for all $t \in q^{\mathbb{N}_0}$, where m is the order of the functional q-difference equation (1.2). The proofs of the following results are omitted as they can be done similarly to the proofs of the corresponding results in Section 2.

Lemma 3.1. $x \in X$ *is a solution of* (1.1) *if and only if*

$$x(t) = \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) - 1} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^{i} a(q^{jm}t)}$$

for all $t \in q^{\mathbb{N}_0}$.

We also define M^* and M_* as in Section 2 but we choose

$$\delta^* := \frac{M_* - 1}{M^*(M^* - 1)}.$$

Clearly, $\delta^* \in (0,1)$. Then we define the cone

$$P := \{y \in X : \; y(t) \geq 0, \; t \in q^{\mathbb{N}_0}, \; y(t) \geq \delta^* \|y\|\}$$

and the mapping $T : X \to X$ by

$$Tx(t) = \frac{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t)}{q^{hm} \prod_{i=0}^{h-1} a(q^{im}t) - 1} \sum_{i=0}^{h-1} \frac{f(q^{im}t, x(q^{im}t/\tau(q^{im}t)))}{\prod_{j=0}^{i} a(q^{jm}t)}.$$

Thus $Tx(t) = q^\omega Tx(q^\omega t)$ and also $T(P) \subset P$. Define

$$\widetilde{\varphi}(s) := \max\left\{ \frac{q^m t f(t,u)}{1 - q^m a(t)} : \; t \in Q_\omega, \; \delta^* s \leq u \leq s \right\},$$

$$\widetilde{\psi}(s) := \min\left\{ \frac{q^m \delta^* f(t,u(t))}{(1 - q^m a(t))u(t)} : \; t \in Q_\omega, \; \delta^* s \leq u \leq s \right\}.$$

Theorem 3.2. *Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose there exist two real numbers $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that $\widetilde{\varphi}(\alpha) \leq \alpha$ and $\widetilde{\psi}(\beta) \geq 1$. Then* (1.2) *has at least one positive solution $x \in X$ with*

$$\min\{\alpha, \beta\} \leq \|x\| \leq \max\{\alpha, \beta\}.$$

Corollary 3.3. *Assume $0 < a(t) < 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose that one of the following condition holds:*

(i) $\lim_{s \to 0^+} \frac{\widetilde{\varphi}(s)}{s} = \widetilde{\varphi}_0 < 1$ *and* $\lim_{s \to \infty} \widetilde{\psi}(s) = \widetilde{\psi}_\infty > 1$,

(ii) $\lim_{s \to \infty} \frac{\widetilde{\varphi}(s)}{s} = \widetilde{\varphi}_\infty < 1$ *and* $\lim_{s \to 0^+} \widetilde{\psi}(s) = \widetilde{\psi}_0 > 1$.

Then (1.2) *has at least one positive solution $x \in X$ with $\|x\| > 0$.*

Theorem 3.4. *Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose there exist $N+1$ positive constants $p_1 < p_2 < \ldots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:*

(i) $\widetilde{\varphi}(p_{2k-1}) < p_{2k-1}$, $k \in \{1, 2, \ldots, [(N+2)/2]\}$ *and*
$\widetilde{\psi}(p_{2k}) > 1$, $k \in \{1, 2, \ldots, [(N+1)/2]\}$,

(ii) $\widetilde{\varphi}(p_{2k}) < p_{2k}$, $k \in \{1, 2, \ldots, [(N+1)/2]\}$ *and*
$\widetilde{\psi}(p_{2k-1}) > 1$, $k \in \{1, 2, \ldots, [(N+2)/2]\}$,

where $[d]$ denotes the integer part of d. Then (1.2) *has at least N positive solutions $x_k \in X$, $k \in \{1, 2, \ldots, N\}$ with*

$$p_k < \|x_k\| < p_{k+1}.$$

Corollary 3.5. *Assume $a(t) > 1/q^m$ for all $t \in q^{\mathbb{N}_0}$. Suppose that the following conditions are satisfied:*

(i) $\lim_{s \to 0^+} \frac{\widetilde{\varphi}(s)}{s} = \widetilde{\varphi}_0 < 1$ *and* $\lim_{s \to \infty} \frac{\widetilde{\varphi}(s)}{s} = \widetilde{\varphi}_\infty < 1$,

(ii) *there exists a constant $\beta > 0$ such that $\widetilde{\psi}(\beta) > 1$.*

Then (1.2) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < \|x_1\| < \beta < \|x_2\| < \infty.$$

Corollary 3.6. *Assume* $a(t) > 1/q^m$ *for all* $t \in q^{\mathbb{N}_0}$. *Suppose the following conditions are satisfied:*

(i) $\lim_{s\to 0^+} \widetilde{\psi}(s) = \widetilde{\psi}_0 > 1$ *and* $\lim_{s\to\infty} \widetilde{\psi}(s) = \widetilde{\psi}_\infty > 1$,
(ii) *there exists a constant* $\alpha > 0$ *such that* $\widetilde{\varphi}(\alpha) < \alpha$.

Then (1.2) *has at least two positive solutions* $x_1, x_2 \in X$ *with*

$$0 < \|x_1\| < \alpha < \|x_2\| < \infty.$$

4. Some Examples

In this section, we show some examples of equations of the form (1.1) and (1.2) and apply the main results of the previous sections.

Example 4.1. Consider the q-difference equation

$$x(q^3 t) = a x(t) + \frac{1}{t x(q^2 t)}, \tag{4.6}$$

where a is a constant with $0 < a < 1/q^3$, $f(t,x) = 1/(tx)$, and $\tau(t) = 1/q^2$ for all $t \in q^{\mathbb{N}_0}$. We have

$$\lim_{s\to\infty} \frac{\varphi(s)}{s} = \varphi_\infty = 0 < 1 \quad \text{and} \quad \lim_{s\to 0^+} \psi(s) = \psi_0 = \infty > 1.$$

By Corollary 2.3 (ii), (4.6) has at least one positive ω-periodic solution.

Example 4.2. Let $q = 2$, $m = 4$, $\omega = 5$. Consider the q-difference equation

$$x(16t) = a x(t) + t^{99} x^{100}(4t) + \frac{1}{16000 t e^{t x(4t)}}, \tag{4.7}$$

where a is a constant with $0 < a < 1/20$, $f(t,x) = t^{99}x^{100} + 1/(16000 t e^{tx})$, and $\tau(t) = 1/4$ for all $t \in q^{\mathbb{N}_0}$. We have

$$\lim_{s\to\infty} \psi(s) = \psi_\infty = \infty > 1 \quad \text{and} \quad \lim_{s\to 0^+} \psi(s) = \psi_0 = \infty > 1.$$

Since there exists $\alpha = 1/100$ such that $\varphi(\alpha) < \alpha$, by Corollary 2.6, (4.7) has at least two positive ω-periodic solutions.

Example 4.3. Consider the q-difference equation

$$x(q^5 t) = a_t x(t) - t^2 x^3(qt), \tag{4.8}$$

where $a(t) = a_t$ are constants assigned for each $t \in Q_\omega$ and $a(t) = a(q^\omega t)$ for all $t \in q^{\mathbb{N}_0}$. We have $\tau(t) = 1/q$, $f(t,x) = t^2 x^3$,

$$\lim_{s\to 0^+} \frac{\widetilde{\varphi}(s)}{s} = \widetilde{\varphi}_0 = 0 < 1 \quad \text{and} \quad \lim_{s\to\infty} \widetilde{\psi}(s) = \widetilde{\psi}_\infty = \infty > 1.$$

By Corollary 3.3 (i), (4.8) has at least one positive ω-periodic solution.

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APPROXIMATE SOLUTION OF SOME JUSTIFYING MATHEMATICAL MODELS CORRESPONDING TO 2DIM REFINED THEORIES

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Abstract. In this paper, by using projective-variational discrete method, we solved approxiamately some BVPs for thin walled elastic structures corresponding to justifying mathematical models of Kirchhoff-von Kármán-Reissner-Midlin type refined theories.

1. To Justifying 2dim Mathematical Models of Kirchhoff-von Kármán-Reissner-Midlin type refined theories

At first, we consider the linear problems for elastic thin walled structures by using generalised Hellinger-Reissner's principle (see section 2.4 of [8]). For isotropic, homogeneous, static bending case we have

$$\begin{aligned}\frac{2h^3}{3}\left[\mu\Delta w_\alpha+(\lambda^*+\mu)graddivw_+\right]-\frac{\mu h}{(1+2\gamma)}(w_\alpha+v_{3,\alpha})=\\ \int_{-h}^{h}tf_\alpha dt-h(g_\alpha^++g_\alpha^-)-\frac{\lambda}{2(\lambda+2\mu)}\int_{-h}^{h}t\sigma_{33,\alpha}dt,\end{aligned}\tag{1.1}$$

$$\frac{\mu h}{(1+2\gamma)}\left[\Delta v_3+w_{\alpha,\alpha}\right]=\int_{h}^{-h}f_3dt-(g_3^+-g_3^-).\tag{1.2}$$

These expressions, which are constructed without simplifying hyphotheses, represent general form for all well known refined theories and also new ones, if we choose arbitrary control parameter γ correspondingly.

Now, if we take σ_3 vector as (see [8], p.60):

$$\sigma_3=\frac{(z-h^-)g^+}{2h}+\frac{(h^+-z)g^-}{2h}+\sum_{s=1}^{\infty}\sigma_3^s(x,y)(P_{s+1}(\frac{z-h^*}{h})-P_{s-1}(\frac{z-h^*}{h})),\tag{1.3}$$

where $h^*=0.5(h^++h^-)$, the form of expressions of main physical values for all RT and *Filon-Kirchhoff* (FK) type systems of DEs are invariants and the boundary conditions will be satisfied exactly for all models. In fact for shearing forces $Q_{\alpha3}$, bending and twisting moments $M_{\alpha\beta}$, and surface efforts $T_{\alpha\beta}$ we have:

$$Q_{\alpha3}=h(g_\alpha^++g_\alpha^-)-2h\sigma_{\alpha3}^1,\tag{1.4a}$$

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$$M_{\alpha\beta}=\frac{2h^2}{3}\sigma^1_{\alpha\beta}, \tag{1.4b}$$

$$T_{\alpha\beta}=2h\sigma^0_{\alpha\beta}, \tag{1.4c}$$

For (σ_{33},t) and ψ_α we have

$$\begin{aligned}M_{33}=\int_{-h}^{h}t\sigma_{33}dt=\frac{h^2(1+2\gamma)}{3}(g_3^+-g_3^-)+r_1\left[t\sigma_{33};\gamma\right]=\\ h^2\left(\frac{g_3^+-g_3^-}{2}-\frac{2}{3}\sigma^2_{33}\right),\end{aligned} \tag{1.4d}$$

$$\psi_\alpha=\frac{1}{2}\int_{-h}^{h}(h^2-t^2)\sigma_{\alpha3}dt=\frac{h^2(1+2\gamma)}{3}Q_{\alpha3}+r_2\left[t\int_0^t\sigma_{\alpha3}dt;\gamma\right]. \tag{1.4e}$$

For reminder members $r_1\left[\,;\right]$ and $r_2\left[\,;\right]$ see (2.15) and (2.16) of [8]. It is evident that if we find the solutions of any BVPs for RT and FK (generalised plane stress case) it's possible to define first and second coefficients of (1.3). Inversely, if we solve the BVPs corresponding to Vekua first kind system (6.13) for N=2, formulas (6.9)-(6.12) of [8] define the coefficients $\sigma^1_{\alpha3},\ \sigma^s_{33},\ s=1,2$. By inserting these coefficients into (1.4) we have the explicit form for solutions of BVPs of all RTs and FK. We remind that the conditions $\sigma_{33}|_{S^\pm}=g_3^\pm$ are satisfied among the refined theories in only Reissner's theory with an additional artificial assumption of $\sigma_{33,3}|_{S^\pm}=0$.

For completeness, we consider the BVPs for systems of partial differential equations when $N=2$ according to Vekua theory [11]. If we know the values $\sigma^s_{\alpha\beta}, s=0,1,2;\sigma^s_{i3}, s=1,2$ then the boundary conditions on $S^\pm$ satisfied $\forall N\le\infty$. We remark that for finding the solutions of Refined Theories in wide sense we must study the BVPs for the following partial differential equations:

$$\left\{\begin{aligned}l_2u_+^0+h^{-1}\lambda gradu_3^1=F_+^0,\\ l_2u_+^1+3h^{-1}grad\left(\lambda u_3^2-\mu u_3^0\right)-3\mu h^{-2}u_+^1=F_+^1,\\ l_2u_+^2+5h^{-1}grad(-\mu u_3^1)-15\mu h^{-2}u_+^2=F_+^2,\\ \mu\Delta u_3^0+h^{-1}\mu divu_+^1=F_1^0,\\ \mu\Delta u_3^1+3h^{-1}div(\mu u_+^2-\lambda u_+^0)-3\left(\lambda+2\mu\right)h^{-2}u_3^1=F_3^1,\\ \mu\Delta u_3^2+5h^{-1}(-\lambda divu_+^1)-15(\lambda+2\mu)h^{-2}u_3^2=F_3^2,\end{aligned}\right. \tag{1.5a,f}$$

where

$$u_i\approx u_i^0+P_1\left(z/h\right)u_i^1+P_2\left(z/h\right)u_i^2;$$

$$\sigma_{\alpha\beta}\approx\sigma^0_{\alpha\beta}+P_1\left(z/h\right)\sigma^1_{\alpha\beta}+P_2\left(z/h\right)\sigma^2_{\alpha\beta}+P_3(z/h)\sigma^3_{\alpha\beta},$$

$$\sigma_3=\frac{(z-h^-)g^+}{2h}+\frac{(h^+-z)\,g^-}{2h}+$$

$$\sum_{s=1}^{\infty}\sigma_3^s(x,y)\left(P_{s+1}\left(\frac{z-h^*}{h^+-h^-}\right)-P_{s-1}\left(\frac{z-h^*}{h^+-h^-}\right)\right),$$

$$\sigma_3=\left(\sigma_{13},\sigma_{23},\sigma_{33}\right)^T,$$

l_2 is the planar differential operator of theory of elasticity and Δ is 2Dim Laplacian operator.

After solving BVP (1.5) we immediately have:

$$\text{(1.6a,c)}\quad \begin{cases} \sigma_{12}^s = 2\mu(u_{1,2}^s + u_{2,1}^s), s = 0,1,2,\ \sigma_{12}^3 = 0; \\ \sigma_{\alpha 3}^s = 0.5(g_\alpha^+ - (-1)^s g_\alpha^-) - \mu(u_{3,\alpha}^{s-1} + (2s-1)h^{-1}u_\alpha^s), \\ \sigma_{33}^s = 0.5(g_3^+ - (-1)^s g_3^-) - \left(\lambda u_\alpha^{s-1} + (2s-1)h^{-1}(\lambda+2\mu)u_3^s\right), \\ s = 1,2. \end{cases}$$

$$\text{(1.6d,k)}\quad \begin{cases} u_\alpha^* = \dfrac{3}{2h^3}(u_\alpha, z) = u_\alpha^1, \bar{u}_i = \dfrac{1}{2h}(u_i, h) = u_i^0, \\ u_3^* = \dfrac{3}{4h^3}(u_3, h^2 - z^2) = u_3^0 - 0.2u_3^2, \\ Q_{\alpha 3} = h(g_\alpha^+ + g_\alpha^-) - 2h\sigma_{\alpha 3}^1, \\ M_{\alpha\beta} = \dfrac{2h^2}{3}\sigma_{\alpha\beta}^1, \\ T_{\alpha\beta} = 2h\sigma_{\alpha\beta}^0, \\ \displaystyle\int_{-h}^h t\sigma_{33}dt = h^2\left(\frac{g_3^+ - g_3^-}{2} - \frac{2}{3}\sigma_{33}^2\right), \\ \psi_\alpha = \dfrac{1}{2}\displaystyle\int_{-h}^h \left(h^2 - t^2\right)\sigma_{\alpha 3}dt = \frac{h^2(1+2\gamma)}{3}Q_{\alpha 3} + r_2\left[t\int_0^t \sigma_{\alpha 3}dt;\gamma\right]. \\ \displaystyle\int_{-h}^h \sigma_{33}dt = h\left(g_3^+ + g_3^-\right) - 2h\sigma_{33}^1. \end{cases}$$

We remark that for BVP of any refined theories it is not necessary to investigate the problems of existence and uniqness of classical or general solutions (when on ∂D displacements are zero or it is free) and there are true Korn type inequalities for any $N \leq \infty$ when $1 + 2\gamma \geq 0$(see details in chapter 2 of [8], inequalities (6.19) and (6.23)):

$$(-L_N U_N, U_N) \geq \mu\left(\kappa^2\left\|U_N^+\right\|_1^2\right) + \left\|U_N^3\right\|_2^2,$$

$$(-L_{v_1}U, U) \geq (4h\mu)\left(\kappa_1^2\left\|gradU^+\right\|_1^2 + \kappa_2^2\left\|U^3\right\|_2^2\right),$$

$$(u^m, v^n)_1 = \left(\sqrt{(2m+1)(2n+1)}\right)^{-1}(u^m, v^n),$$

$$(u^m, v^n)_2 = h^{-2}\sqrt{(2m+1)(2n+1)}\left(\sum_{i\geq m(2)} u^{i+1}, \sum_{i\geq m(2)} v^{i+1}\right).$$

One of the most principal objects in development of mechanics and mathematics is a system of nonlinear differential equations for elastic isotropic plate constructed by *von Kármán.* This system represents the most essential part of the main manuals in elasticity theory [1, 2]. In spite of this in 1978 *Truesdell* expressed an idea about neediness of "Physical Soundness" of *von Kármán* system. This circumstance generated the problem of justification of *von Kármán* system. Afterwards this problem is studied by many authors, but with most attention it was investigated by Ciarlet [3]. In particular, he wrote: "the *von Kármán* equations may be given a full justification by means of the leading term of a formal asymptotic expansion" ([3], p.368). This result obviously is not sufficient for a justification of "Physical Soundness" of *von Kármán* system as representations by asymptotic expansions is dissimilar: leading terms are only coefficients of power series without any physical meaning. Based on [8], the method of constructing such anisotropic inhomogeneous 2D nonlinear models of *von Kármán-Mindlin-Reissner(KMR)* type for elastic plates with variable thickness is given, by means of which terms take quite determined "Physical Soundness". The corresponding variables are quantities with certain physical meaning: averaged components of the displacement vector, bending and twisting moments, shearing forces, rotation of normals, surface efforts. In addition the corresponding equations are constructed taking into account the conditions of equality of the main vector and moment to zero. By choosing parameters in the isotropic case from *KMR* type system (having a continuum power) the *von Kármán* system as one of the possible models is obtained. The given method differs from the classical one by the fact that according to the classical method, one of the equations of *von Kármán* system represents one of *St-Venant's* compatibility conditions, i.e. it's obtained at the bases of geometry and not taking into account the equilibrium equations. This remark is essential for dynamical problems.

Using methodology of [8], from ch.1 (in the case when thin-walled structure is an elastic isotropic homogeneous plate with constant thickness) we have the following nonlinear systems of PDEs of *KMR* type:

$$\left\{\begin{aligned} D\Delta^2\,\bar{u}_3 &= \left(1-\frac{h^2\,(1+2\gamma)\,(2-\nu)}{3\,(1-\nu)}\Delta\right)\left(g_3^+-g_3^-\right)\\ &+2h\left(1-\frac{2h^2\,(1+2\gamma)}{3\,(1-\nu)}\Delta\right)[\bar{u}_3,\Phi^*]+h\left(g_{3,\alpha}^+-g_{3,\alpha}^-\right)\\ &-\int_{-h}^{h}\left(zf_{\alpha,\alpha}-(1-\frac{1}{1-\nu}\Delta\left(h^2-z^2\right)f_3)\right)dz+R_1[\bar{u}_3;\gamma], \end{aligned}\right. \tag{1.7}$$

$$\Delta^2\Phi^*=-\frac{\mathrm{E}}{2}[\bar{u}_3,\bar{u}_3]+\frac{\nu}{2}\Delta\left(g_3^++g_3^-\right)+\frac{1+\nu}{2h}\bar{f}_\alpha+R_2\left[\Phi^*\right], \tag{1.8}$$

$$\left\{\begin{aligned} &Q_{\alpha 3}-\frac{1+2\gamma}{3}h^2\Delta Q_{\alpha 3}=-D\Delta\,\bar{u}_{3,\alpha}\\ &+\frac{h^2\,(1+2\gamma)}{3\,(1-\nu)}\partial_\alpha\left(g_3^+-g_3^-+2h\,(1+\nu)\right)[\bar{u}_3,\Phi^*]+h\left(g_\alpha^+-g_\alpha^-\right)\\ &-\int_{-h}^{h}zf_\alpha dz+\frac{1+\nu}{2\,(1-\nu)}\int_{-h}^{h}\left(h^2-z^2\right)f_{3,\alpha}dz+R_{2+\alpha}\left[Q_{\alpha 3};\gamma\right]. \end{aligned}\right. \tag{1.9}$$

The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von KMR type systems as in the stationary, as well in nonstationary cases. We remind that even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that *von Kármán, Love, Timoshenko, Landau & Lifshits* and et al. considered the compatibility conditions of *St.Venant-Beltrami* as one of the equations of the corresponding system.

In the presented model we demonstrated a correct equation that is especially important for dynamic problems. Further for isotropic and generalized transversal elastic plates along the quantities describing the vertical directions and surface wave processes it is necessary to take into account the quantity $\Delta\partial_{tt}\Phi$, corresponding to wave processes in the horizontal directions, in the equilibrium equations. The equations have the following form [9]:

$$\left\{\begin{aligned}&\left(D\Delta^2+2h\rho\partial_{tt}-2DE^{-1}\left(1+\upsilon\right)\rho\partial_{tt}\Delta\right)w=\\&\left(1-\frac{h^2\left(1+2\gamma\right)\left(2-\upsilon\right)}{3\left(1-\upsilon\right)}\Delta\right)\left(g_3^+-g_3^-\right)\\&+2h\left(1-\frac{2h^2\left(1+2\gamma\right)}{3\left(1-\upsilon\right)}\Delta\right)\left[u_3^*,\Phi\right]+h\left(g_{\alpha,\alpha}^+-g_{\alpha,\alpha}^-\right),\end{aligned}\right. \tag{1.10}$$

$$\begin{gathered}\left(\Delta^2-\frac{1-\nu^2}{E}\rho_1\Delta\partial_{tt}\right)\Phi=\\-\frac{E}{2}\left[w,w\right]+\frac{\nu}{2}\left(\Delta-\frac{2\rho_1}{E}\partial_{tt}\right)\left(g_3^++g_3^-\right)+\frac{1+\nu}{2h}f_{\alpha,\alpha}.\end{gathered} \tag{1.11}$$

From (1.10)-(1.11) follows *von Kármán* equations if in (1.10) $\gamma=-0.5, g_\alpha^\pm=0$ and in (1.11) $f_\alpha=\rho_1=\Delta g_3^\pm=0$. In addition, an equation *corresponding to* (1.11) *by von Kármán, A. Föppl, Love, Lukasievicz, Tomoshenko, Donnel, Landau, Ciarlet, Antman et al.* were constructed by the condition $\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=-0.5\left[u_3,u_3\right]$ and Hooke's law (but without using the equilibrium equations!). As we prove in works [8, 9] the form (1.11) follows immediately for more general cases, when thin-walled elastic structures are anisotropic and if we use Hooke's law, equilibrium equations with and nonlinear relations between strain tensor and displacement vector:

$$\varepsilon_{\alpha\beta}=0.5\left(u_{\alpha,\beta}+u_{\beta,\alpha}+u_{3,\alpha}u_{3,\beta}\right).$$

Now we prove that (1.11) **equations in dynamical case has the following form** [10]:

$$\left(-\frac{1-\nu^2}{E}\rho_1\Delta\partial_{tt}\right)\Phi=\frac{\nu}{2}\left(\Delta-\frac{2\rho_1}{E}\partial_{tt}\right)\left(g_3^++g_3^-\right)+\frac{1+\nu}{2h}f_{\alpha,\alpha}. \tag{1.12}$$

Thus we must demonstrate that both way give the expression $\Delta^2\Phi-0.5E\left[w,w\right]$ In fact, we constructed (1.11) by using the following expression (see [9]) :

$$\left\{\begin{aligned}&\left(\lambda^*+2\mu\right)\Delta\left(\bar\varepsilon_{11}+\bar\varepsilon_{22}\right)=\\&\left(2\mu(3\lambda+2\mu)\right)^{-1}\left(\lambda+2\mu\right)\left(\lambda^*+2\mu\right)\Delta\left(\bar\sigma_{11}+\bar\sigma_{22}\right)+...=\\&\mu\left(\left(-1\right)^{\alpha+\beta}\partial_{3-\alpha}\partial_{3-\beta}\bar u_{3,\alpha}\bar u_{3,\beta}\right)+...,\end{aligned}\right. \tag{1.13}$$

where **dots** denote other different members from (1.11). Let us $\bar\sigma_{\alpha\beta}=\left(-1\right)^{\alpha+\beta}\partial_{3-\alpha}\partial_{3-\beta}\Phi$, then from preliminary equation follows (1.11) or: $\Delta^2\Phi=-0.5E\left[w,w\right]+...$ From St.Venant-Beltrami compatibility conditions it is evident that

$$\begin{gathered}\varepsilon_{11,22}-2\varepsilon_{12,12}+\varepsilon_{22,11}=\\\left(2\mu\left(3\lambda+2\mu\right)\right)^{-1}\left[2\left(\lambda+\mu\right)\Delta\bar\sigma_{\alpha\alpha}-\lambda\bar\sigma_{\alpha\alpha,\alpha\alpha}\right]-\left(\mu\right)^{-1}\bar\sigma_{12,12}=2E^{-1}\Delta^2\Phi,\end{gathered}$$

or

$$\Delta^2\Phi + 0.5E\,[w,w] \equiv 0.$$

The mathematical models considered in [8] , ch.I contain a new quantity, which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (*Babushka, Lukasievicz, Mazia, Saponjan*), but also is very important for example for process of generating cracks and holes (details see in [8], ch.1, par. 3.3). Further, let us note that in works [9] equations of (1.11) type are constructed with respect to certain components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (1.11) type equation, are equivalent to the system, where the order of each equation is not higher than two. For example, in the isotropic case, obviously, for coefficients we have [9]: $c_{\alpha\alpha} = \lambda^* + 2\mu$, $c_{66} = 2\mu$, $c_{12} = \lambda^*$, $c_{\alpha 6} = 0$, $\lambda^* = 2\lambda\mu\,(\lambda + 2\mu)^{-1}$, λ and μ are the Lamé constants. Then the system (1.7) of [9] is presented in the form:

$$(\lambda^* + 2\mu)\,\partial_1\tau + \mu\partial_2\omega = \frac{1}{2h}\,\bar{f}_1 + \mu(\partial_1(\bar{u}_{3,2}) - \partial_2(\bar{u}_{3,1}\,\bar{u}_{3,2})) - \frac{\lambda}{2h\,(\lambda + 2\mu)}\,(\sigma_{33,1}, 1)\,, \tag{1.14a}$$

$$\mu\partial_1\omega + (\lambda^* + 2\mu)\,\partial_2\tau = \frac{1}{2h}\,\bar{f}_2 + \mu(\partial_2(\bar{u}_{3,1}) - \partial_1(\bar{u}_{3,1}\,\bar{u}_{3,2})) - \frac{\lambda}{2h\,(\lambda + 2\mu)}\,(\sigma_{33,2}, 1)\,, \tag{1.14b}$$

where the functions: $\tau = \bar{\varepsilon}_{\alpha\alpha},$, $\omega = \bar{u}_{1,2} - \bar{u}_{2,1}$ correspond to plane expansion and rotation respectively.

Thus, in the dynamical case the KMR type systems are (1.10) and (1.11). In the statical case from (1.14) immediately follows such relations:

$$\frac{\nu}{2}\Delta\left(g_3^+ + g_3^-\right) + \frac{1+\nu}{2h} f_{\alpha,\alpha} = 0.$$

In general this relation is not true or if it is true then these expressions are consequences of compatability conditions (see p.204 of [4])

$$\iiint_{\Omega_h} f d\omega + \iint_{S+S^{\pm}} g ds = 0.$$

2. Variation-Discrete Method

For demonstration, we described shortly the **Variation-Discrete method** for a strongly elliptic system of PDEs which contains the special case ((6.13) of [8] for N=2):

$$A_1\Delta u_+ + B_1\mathrm{grad}(\mathrm{div} u_+) = f_+, \tag{2.1a}$$

$$A_2\Delta u_3 + B_2(\mathrm{div} u_*) = f_3, \tag{2.1b}$$

$$A_3\Delta u_* + B_3\mathrm{grad}(\mathrm{div} u_*) + C_3\mathrm{grad} u_3 + D_3 u_* = f_*, \tag{2.1c}$$

where the closure of domain $\overline{D} := [-1,1]^2$, $u_+ = (u_1(x,y), u_2(x,y))^{\mathsf{T}}$, $u_3 = u_3(x,y)$, $u_* = (u_4(x,y), u_5(x,y))^{\mathsf{T}}$; $f_+ = (f_1(x,y), f_2(x,y))^{\mathsf{T}}$, $f_3 = f_3(x,y)$,

$f_* = (f_4(x,y), f_5(x,y))^{\mathsf{T}}$, the coefficients $A_i, B_i (i = 1,2,3); C_3$ and D_3 are constants.

Let us denote system (2.1) as

$$\mathsf{L}(\partial_1, \partial_2)u(x,y) = f(x,y), \quad (x,y) \in D := (-1,1) \mathsf{X} (-1,1)\,, \tag{2.2a}$$

with Dirichlet type boundary conditions

$$u|_{\partial D} = g, \quad g = \begin{cases} g_1(y), & (x,y) \in \{1\} \mathsf{X} [-1,1]\,, \\ g_2(x), & (x,y) \in [-1,1] \mathsf{X} \{1\}, \\ g_3(y), & (x,y) \in \{-1\} \mathsf{X} [-1,1]\,, \\ g_4(x), & (x,y) \in [-1,1] \mathsf{X} \{-1\}. \end{cases} \tag{2.2b}$$

where $u\,(x,y) \in \mathrm{C}^2(D) \bigcap \mathrm{C}(\bar{D}),\ f\,(x,y) \in \mathrm{C}(D)$[1] and $\mathsf{L}(\partial_1, \partial_2)$ is a linear elliptic type operator.

Instead of $u(x,y)$ we take a series expansion having a homogeneous baoundary values and add a function $v(x,y)$ who satisfies the heterogenous boundary conditions (2.2b)

$$u(x,y) = \sum_{i,j=1}^{\infty} u^{ij} \varphi_{ij}(x,y) + v(x,y), \tag{2.3}$$

where, u^{ij} is coefficients of $u(x,y)$ in $\varphi_{ij}(x,y)$ basis or coordinate functions which is defined by the multiplication of Legendre polynomials differences (with respect to indices) in the following way

$$\varphi_{ij}(x,y) := \chi P_i(x) \chi P_j(y), \quad \chi P_i(x) := \frac{1}{\sqrt{2(2i+1)}} \left(P_{i+1}(x) - P_{i-1}(x)\right), \tag{2.4a,b}$$

$$\begin{cases} v(x,y) = & G_1(x,y) \mathsf{H}\,(y+1)\, \mathsf{H}\,(1-y) + G_2(x,y) \mathsf{H}\,(x+1)\, \mathsf{H}\,(1-x) + \\ & E_1 \delta(x-1)\delta(y-1) + E_2 \delta(x-1)\delta(y+1) + \\ & E_3 \delta(x+1)\delta(y-1) + E_4 \delta(x+1)\delta(y+1), \end{cases} \tag{2.4c}$$

where

$$\mathsf{H}\,(x-a) := \begin{cases} 1, & x > a, \\ 0, & x \le a; \end{cases} \qquad \delta(x-a) := \begin{cases} 1, & x = a, \\ 0, & x \ne a, \end{cases}$$

$$G_1(x,y) = \left[\frac{x+1}{2} g_1(y) + \frac{1-x}{2} g_3(y)\right],$$

$$G_2(x,y) = \left[\frac{y+1}{2} g_2(x) + \frac{1-y}{2} g_4(x)\right],$$

$$E_1 = g_1\,(1) = g_2\,(1)\,, \quad E_2 = g_1\,(-1) = g_4\,(1)\,,$$

$$E_3 = g_2\,(-1) = g_3\,(1)\,, \quad E_4 = g_3\,(-1) = g_4\,(-1)\,.$$

The difference in (2.4b) is taken in such a way that the homogeneous boundary condition is satisfied and the function $v(x,y)$ is proposed in such a way that the heterogenous boundary conditions given in (2.2b) are satisfied. The difference in (2.4b) is between either odd or even ordered polynomials and since $P_i(\pm 1) = (\pm 1)^i$ it is always true that $\chi P_i(\pm 1) = 0$. Coordinate functions $\varphi_{ij}(x,y)$ constitute a complete system. The coefficient in operator χ is selected so that after several operations it can be simplified by other coefficients which come out of the integration

[1]For simplicity f is taken from $\mathrm{C}(D)$. The only condition f to satisfy is that it is integrable in the general sense over D. Therefore f can be selected from a more general class.

operations given in (2.6). For the numerical realisation we take the first N terms of the series given in (2.3) and it becomes

$$\overset{N}{u}(x,y) = \sum_{m,n=1}^{N} u^{mn}\varphi_{mn} + v(x,y). \tag{2.5}$$

Then the method starts with inserting approximate value $\overset{N}{u}$ instead of the exact value of u in the differential Eq.(2.2a) and then continues by multiplying both sides by coordinate functions φ_{ij} and taking integration over the domain D. Finally we have the projected approximate equation

$$\iint_D \mathsf{L}(\partial_1,\partial_2)\,\overset{N}{u}(x,y)\varphi_{ij}dxdy = \iint_D f(x,y)\varphi_{ij}dxdy =: (f,\varphi_{ij})\,. \tag{2.6}$$

To find algebraic equivalent system for the BVP (2.1) we need corresponding templates for the identity, first, direct and mixed second order operators. Let us call the equivalent operators as $\mathbf{I}, \mathbf{I_1}, \mathbf{I_{11}}, \mathbf{I_{12}}$ respectively for the identity, first, direct and mixed second order operators. Application of (2.2)-(2.6) and the following properties of Legendre polynomials

$$\int\limits_{-1}^{1} P_m P_n dt = \frac{2\delta_{mn}}{m+n+1}, \qquad P'_{m+1} - P'_{m-1} = (2m+1)P_m, \tag{2.7a,b}$$

where prime sign in (2.7b) denotes derivative with respect to the relevant argument x or y, gives the required templates as below:

$$\mathbf{I_{11}} := \left(\partial_{11}\overset{N}{u},\varphi_{ij}\right) = \sum_{n=-1}^{1} u^{i,j+2n}\left[(|n|-1)\,c_j + |n|a_{j+n}\right] + (\partial_{11}v,\varphi_{ij})\,,$$

$$\mathbf{I_{12}} := \left(\partial_{12}\overset{N}{u},\varphi_{ij}\right) = \sum_{m,n=-1}^{1} -u^{i+m,j+n}|mn|(-1)^{\frac{|m+n|}{2}} b_{i+\frac{m+1}{2},j+\frac{n+1}{2}} + (\partial_{12}v,\varphi_{ij})\,,$$

$$\mathbf{I_1} := \left(\partial_1\overset{N}{u},\varphi_{ij}\right) =$$

$$\sum_{m,n=-1}^{1} u^{i+m,j+2n}|m|(-1)^{\frac{m+3}{2}+n} e_{i+\frac{m+1}{2}} S_{1j}a_{j+n}S_{2j}c_j + (\partial_1 v,\varphi_{ij})\,,$$

$$\mathbf{I} := \left(\overset{N}{u},\varphi_{ij}\right) = \sum_{m,n=-1}^{1} u^{i+2m,j+2n} R_1 c_i R_2 c_j R_3 a_{i+m} R_4 a_{j+n} + (v,\varphi_{ij})\,,$$

where

$$a_j = d_{j+1}\sqrt{d_j d_{j+2}}\,,\ \ b_{i,j} = \sqrt{d_i d_{i+1} d_j d_{j+1}},\ \ c_j = \frac{1}{2}(d_j - d_{j+2}),$$

$$e_i = \sqrt{d_i d_{i+1}},\ \ d_i = \frac{1}{2i-1},$$

$$R_1 = 1 + |m|\left(\frac{1}{c_i} - 1\right),\ \ R_2 = 1 + |n|\left(\frac{1}{c_j} - 1\right),$$

$$R_3 = |m| + \frac{1}{a_i}\left(|m| - 1\right), \quad R_4 = |n| + \frac{1}{a_j}\left(|n| - 1\right),$$

$$S_{1j} = \frac{1}{a_j}\left(1 - |n|\right) + |n|, \quad S_{2j} = 1 + |n|\left(\frac{1}{c_j} - 1\right),$$

$$(v, \varphi_{ij}) = (G_1 + G_2, \varphi_{ij}),$$

$$(\partial_1 v, \varphi_{ij}) = -\sqrt{\frac{2i+1}{2}}\left(v, P_i(x)\,\chi P_j(y)\right),$$

$$(\partial_{12} v, \varphi_{ij}) = \sqrt{\frac{2i+1}{2}}\sqrt{\frac{2j+1}{2}}\left(v, P_i(x)\,P_j(y)\right),$$

$$(\partial_{11} v, \varphi_{ij}) = \sqrt{\frac{2i+1}{2}}\left[\left(v, P_i^{'}(x)\,\chi P_j(y)\right) + \int_{-1}^{1}\left[(-1)^i g_3(y) - g_1(y)\right]\chi P_j(y)\,dy\right].$$

Variational-Discrete method applied here represents Ritz method (for the proof see p.146 of [8]). For projective methods, one of the crucial point is the problem of stability. For these coordinate systems φ_{ij}, corresponding Gram type matrix has the same structure with the matrix corresponding to the finite difference method for 2Dim Laplacian. Thus, this fact opens the new way of possibility for sufficent large class of BVPs to investigate Gram type functional matices by methods of numerical mathematics. In our case, Gram matrix is bounded from below by non-negative value when the order of the matrix tends to infinity. This implies that the process of finding u^{ij} and approximate solution $\overset{N}{u}$ is stable (see Ch.III, section 12.1 of [8]). For demostration of some properties of this method below we consider 3 well known classical BVPs .

Example 1. We have the Poisson equation with a unit source function

$$-\Delta u(x,y) = 1, \quad u|_{\partial D} = 0, \tag{2.8}$$

where $\overline{D} := [-1,1]^2$and Δ is the 2D Laplacian operator.

By noting that due to the homogeneous boundary conditions $v = 0$ and using the algebraic equivalent of Laplacian operator $\mathbf{I_\Delta} = \mathbf{I_{11}} + \mathbf{I_{22}}$, the projected approximate equation related to the BVP (2.8) becomes

$$u^{i,j}(c_i + c_j) - u^{i+2,j}a_{i+1} - u^{i-2,j}a_{i-1} - u^{i,j+2}a_{j+1} - u^{i,j-2}a_{j-1} = g^{ij}. \tag{2.9}$$

where $g^{ij} = (1, \varphi_{ij})$. By using the orthogonality property (2.7a) of Legendre polynomials, the integral in the expression of g^{ij} simply yields $g^{11} = 2/3;\ \ g^{ij} = 0,\ \ if\ \ i \neq 1 \neq j$.

The system obtained in (2.9) is in fact consists of **four independent subsystems**. Indices (i,j) can take either odd or even values between $\overline{1,N}$. Each

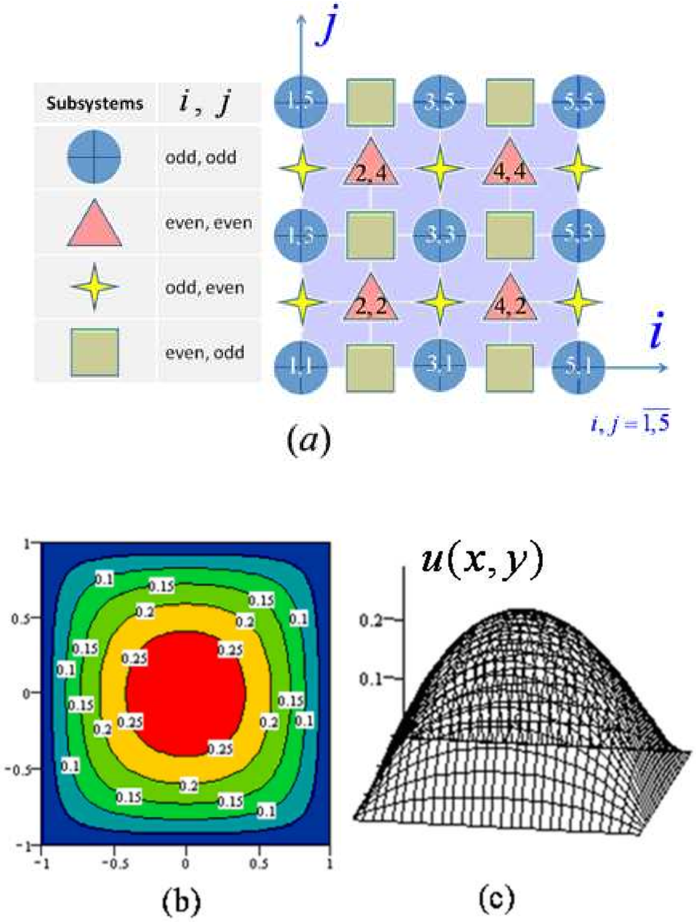

FIGURE 1. Template for Laplacian (a), solution of the Poisson equation: contour plot (b) and 3-D graph (c)

combination results in the same type of unknown coefficient indices, hence constitutes an independent subsystem (see Fig.1a). From the number of members' point of view the obtained scheme resembles the classical finite difference scheme.

The solution of the BVP is given in Fig.1b,c for $N = 3$ and the comparison of the results with [5, 6, 7] is shown in Table.1. The results given in terms of $T/(\mu\theta) = 4\iint_D u(x,y)dxdy$. This parameter is taken for computational convenience and at the same time it has rich physical meanings.

TABLE 1. Comparison of the results for the Poisson's equation

Methods	**Exact solution by series**	**Reduction Method to ODE**	**Variation-Discrete Method**
$T/(\mu\theta)$	2.249232 (for N=200, [5, 6]) (for N=200-500, [7])	2.234 (first order, [6])	2.222222 (for N=2) 2.249208 (N=5) 2.249232 (N=10)

The results shown in Table.1 which are given in 6 decimal point are new, the other with 3 decimal point is from the classical monograph [6]. We should also note that the exact result, which is misgiven/miswritten as 2.244 in [6], is corrected and refined here as 2.249232. This exact result is recalculated by using series expansions given in [5, 6] and [7] upto the first $N = 200$ and 500 terms respectively. From the table it is seen that even for $N = 10$ the Variation-Discrete method gives the same result with the exact solution by series upto the 6 decimal point.

Example 2. The general BVP given in (2.1a) corresponds to tension-compression problem of a 2D isotropic plate after inserting the corresponding material constants instead of A_1 and B_1. With homogeneous boundary conditions it can be formulated as below (see [8])

$$\mu\Delta u + (\lambda^* + \mu)\mathrm{grad}(\mathrm{div}u) = f, \quad u|_{\partial D} = 0, \tag{2.10}$$

where $\overline{D} := [-1, 1]^2$, the displacement vector $u = (u_1(x, y), u_2(x, y))^\mathsf{T}$,the generalized force function $f = (f_1(x, y), f_2(x, y))^\mathsf{T}$,$\lambda^* = 2\lambda\mu(\lambda + 2\mu)^{-1}$, λ and μ are Lamé constants.

(2.10) yields two coupled equations

$$(\lambda^* + 2\mu)\partial_{11}u_1 + \mu\partial_{22}u_1 + (\lambda^* + \mu)\partial_{12}u_2 = f_1, \tag{2.11a}$$

$$(\lambda^* + 2\mu)\partial_{22}u_2 + \mu\partial_{11}u_2 + (\lambda^* + \mu)\partial_{12}u_1 = f_2. \tag{2.11b}$$

Considering templates for $\mathbf{I_{11}}$ and $\mathbf{I_{12}}$ the approximate algebraic equations for (2.11a) and (2.11b) become respectively

$$\begin{cases} -\big((\lambda^* + 2\mu)c_j + \mu c_i\big)u_1^{i,j} + (\lambda^* + 2\mu)\big(u_1^{i,j+2}a_{j+1} + u_1^{i,j-2}a_{j-1}\big) \\ \qquad +\mu\big(u_1^{i+2,j}a_{i+1} + u_1^{i-2,j}a_{i-1}\big) + (\lambda^* + \mu)\big(u_2^{i+1,j+1}b_{i+1,j+1} \\ \qquad\quad +u_2^{i-1,j-1}b_{i,j} - u_2^{i-1,j+1}b_{i,j+1} - u_2^{i+1,j-1}b_{i+1,j}\big) = g_1^{ij}, \end{cases} \tag{2.12a}$$

$$\begin{cases} -\big((\lambda^* + 2\mu)c_i + \mu c_j\big)u_2^{i,j} + (\lambda^* + 2\mu)\big(u_2^{i+2,j}a_{i+1} + u_2^{i-2,j}a_{i-1}\big) \\ \qquad +\mu\big(u_2^{i,j+2}a_{j+1} + u_2^{i,j-2}a_{j-1}\big) + (\lambda^* + \mu)\big(u_1^{i+1,j+1}b_{i+1,j+1} \\ \qquad\quad +u_1^{i-1,j-1}b_{i,j} - u_1^{i-1,j+1}b_{i,j+1} - u_1^{i+1,j-1}b_{i+1,j}\big) = g_2^{ij}, \end{cases} \tag{2.12b}$$

where $g_k^{ij} = (f_k, \varphi_{ij})$, $k = 1, 2$.

To validate the correctness of the schema obtained in (2.12), displacements are taken to be $u_1(x, y) = \chi P_2(x)\chi P_1(y)$, and $u_2(x, y) = u_1(y, x)$. The material coefficients λ^*, μ are taken to be one[2]. Inserting these test functions into (2.11) we get the forces as $f_1(x, y) = x\sqrt{15}(-12 + 15y^2 + x^2)/4$, $f_2(x, y) = f_1(y, x)$. After

[2]Here and in Example 3 all coefficients are taken to be one for the computational simplicity.

inserting these force functions the algebraic system of equations (2.12) are solved and the results is exactly the same as the test functions.

Example 3. The general BVP given in (2.1b,c) corresponds to bending problem of a 2D isotropic plate and after inserting the corresponding material constants for homogeneous boundary conditions it becomes (see [8]):

$$\mu\Delta u_3 + \mu(\mathrm{div} u_*) = f_3, \tag{2.14a}$$

$$\frac{\mu h^2}{2}\Delta u_* + \frac{h^2}{2}(\lambda^* + \mu)\mathrm{grad}(\mathrm{div} u_*) - \mu\left(\mathrm{grad} u_3 + u_*\right) = f_*, \tag{2.14b}$$

where the closure of domain $\overline{D} := [-1,1]^2$, $u_3 = u_3(x,y)$, $u_* = (u_4(x,y), u_5(x,y))^{\mathsf{T}}$; $f_3 = f_3(x,y)$, $f_* = (f_4(x,y), f_5(x,y))^{\mathsf{T}}$.

(2.14) yields three coupled equations

$$\mu\left(\partial_{11}u_3 + \partial_{22}u_3\right) + \mu\left(\partial_1 u_4 + \partial_2 u_5\right) = f_3, \tag{2.15a}$$

$$\frac{\mu h^2}{2}\left(\partial_{11}u_4 + \partial_{22}u_4\right) + \frac{h^2}{2}(\lambda^* + \mu)\left(\partial_{11}u_4 + \partial_{12}u_5\right) - \mu\left(\partial_1 u_3 + u_4\right) = f_4, \tag{2.15b}$$

$$\frac{\mu h^2}{2}\left(\partial_{11}u_5 + \partial_{22}u_5\right) + \frac{h^2}{2}(\lambda^* + \mu)\left(\partial_{12}u_4 + \partial_{22}u_5\right) - \mu\left(\partial_2 u_3 + u_5\right) = f_5. \tag{2.15c}$$

Considering templates for $\mathbf{I_1}, \mathbf{I_{11}}, \mathbf{I_{12}}$ and $\mathbf{I}$ the approximate algebraic equations for (2.15a), (2.15b) and (2.15c) become respectively

$$\left\{\begin{aligned} &\mu\sum_{m=-1}^{1}\Big(u_3^{i+2m,j}\left[\left(|m|-1\right)c_i + |m|a_{i+m}\right] \\ &\qquad + u_3^{i,j+2m}\left[\left(|m|-1\right)c_j + |m|a_{j+m}\right]\Big) \\ &+\mu\sum_{m,n=-1}^{1}|m|(-1)^{\frac{m+3}{2}+n}\Big(u_4^{i+m,j+2n}e_{i+\frac{m+1}{2}}S_{1j}a_{j+n}S_{2j}c_j \\ &\qquad + u_5^{i+2n,j+m}e_{j+\frac{m+1}{2}}S_{1i}a_{i+n}S_{2i}c_i\Big) = g_3^{ij}, \end{aligned}\right. \tag{2.16a}$$

$$
\text{(2.16b)} \quad \begin{cases}
\dfrac{\mu h^2}{2}\displaystyle\sum_{m=-1}^{1} u_4^{i+2m,j}\left[\left(|m|-1\right)c_i+|m|a_{i+m}\right] \\
+\dfrac{h^2}{2}(\lambda^*+2\mu)\displaystyle\sum_{n=-1}^{1} u_4^{i,j+2n}\left[\left(|n|-1\right)c_j+|n|a_{j+n}\right] \\
-\dfrac{h^2}{2}(\lambda^*+\mu)\displaystyle\sum_{m,n=-1}^{1} u_5^{i+m,j+n}|mn|(-1)^{\frac{|m+n|}{2}}b_{i+\frac{m+1}{2},j+\frac{n+1}{2}} \\
-\mu\displaystyle\sum_{m,n=-1}^{1}|m|(-1)^{\frac{m+3}{2}+n}u_3^{i+m,j+2n}e_{i+\frac{m+1}{2}}S_{1j}a_{j+n}S_{2j}c_j \\
-\mu\displaystyle\sum_{m,n=-1}^{1} u_4^{i+2m,j+2n}R_1c_iR_2c_jR_3a_{i+m}R_4a_{j+n}=g_4^{ij}
\end{cases}
$$

$$
\text{(2.16c)} \quad \begin{cases}
\dfrac{\mu h^2}{2}\displaystyle\sum_{n=-1}^{1} u_5^{i,j+2n}\left[\left(|n|-1\right)c_j+|n|a_{j+n}\right] \\
+\dfrac{h^2}{2}(\lambda^*+2\mu)\displaystyle\sum_{m=-1}^{1} u_5^{i+2m,j}\left[\left(|m|-1\right)c_i+|m|a_{i+m}\right] \\
-\dfrac{h^2}{2}(\lambda^*+\mu)\displaystyle\sum_{m,n=-1}^{1} u_4^{i+m,j+n}|mn|(-1)^{\frac{|m+n|}{2}}b_{i+\frac{m+1}{2},j+\frac{n+1}{2}} \\
-\mu\displaystyle\sum_{m,n=-1}^{1}|m|(-1)^{\frac{m+3}{2}+n}u_3^{i+2n,j+m}e_{j+\frac{m+1}{2}}S_{1i}a_{i+n}S_{2i}c_i \\
-\mu\displaystyle\sum_{m,n=-1}^{1} u_5^{i+2m,j+2n}R_1c_iR_2c_jR_3a_{i+m}R_4a_{j+n}=g_5^{ij}
\end{cases}
$$

where $g_k^{ij}=\left(f_k,\varphi_{ij}\right)$, $k=3,4,5$.

To validate the correctness of the schema obtained in (2.16), the displacements are taken to be $u_3(x,y)=\chi P_2(x)\chi P_2(y)$, $u_4(x,y)=\chi P_2(x)\chi P_1(y)$ and $u_5(x,y)=u_4(y,x)$. Inserting these test functions into (2.15) we get the forces as

$$
f_3(x,y)=\frac{15}{4}\left(xy^3+x^3y-2xy\right)+\frac{\sqrt{15}}{4}\left(1+3x^2y^2-2y^2-2x^2\right),
$$
$$
f_4(x,y)=\frac{5}{8}\left(y^3+3yx^2-3x^2y^3-y\right)+\frac{\sqrt{15}}{8}\left(16y^2x+2x^3-x^3y^2-13x\right),
$$
$$
f_5(x,y)=f_4(y,x).
$$

After inserting these force functions the algebraic system of equations (2.16) are solved and the results is exactly the same as the test functions.

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TRIGONOMETRIC APPROXIMATION OF SIGNALS (FUNCTIONS) BELONGING TO WEIGHTED $(L^p, \xi(t))$-CLASS BY HAUSDORFF MEANS

UADAY SINGH AND SMITA SONKER

ABSTRACT. Rhoades [13] has obtained the degree of approximation of functions belonging to the weighted Lipschitz class $W(L^p, \xi(t))$ by Hausdorff means of their Fourier series, where $\xi(t)$ is an increasing function. The first result of Rhoades [13] generalizes the result of Lal [2]. In a very recent paper Rhoades *et al.* [14] have obtained the degree of approximation of functions belonging to the $Lip\alpha$ class by Hausdorff means of their Fourier series and generalized the result of Lal and Yadav [7]. The authors in [14] have made some important remarks, namely, increasing nature of $\xi(t)$ alone is not sufficient to prove the results of Lal [2], Lal and Singh [6], Qureshi [11] and Rhoades [13]; and the condition $1/\sin^{\beta}(t) = O(1/t^{\beta})$, $1/n \leq t \leq \pi$ used by all these authors is not valid since $\sin t \to 0$ as $t \to \pi$. They have also suggested a modification in the definition of weighted $(L^p, \xi(t))$ - class and leave an open question for determining a correct set of conditions to prove the results of Rhoades [13]. We note that the same types of errors can also be seen in the papers of Lal [3, 4], Nigam [8, 9] and Nigam and Sharma [10]. Being motivated by the remarks of Rhoades *et al.* [14], in this paper, we determine the degree of approximation of functions belonging to the weighted $(L^p, \xi(t))$ class by Hausdorff means of their Fourier series and rectify the above errors by using proper set of conditions. We also deduce some important corollaries from our result.

1. INTRODUCTION

For a given $2\pi-$periodic signal (function) $f \in L^p = L^p[0, 2\pi]$, $p \geq 1$, let

$$s_n(f) = s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \tag{1.1}$$

denote the partial sum, called trigonometric polynomial of degree (or order) n, of the first $(n + 1)$ terms of the Fourier series of f.

The L^p-norm of signal f is defined by
$\|f\|_p = \frac{1}{2\pi}\left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}$ $(1 \leq p < \infty)$, and $\|f\|_\infty = \sup_{x\in[0,2\pi]} |f(x)|$.

A signal (function) f is approximated by trigonometric polynomial T_n of order (or degree) n and the degree of approximation $E_n(f)$ is given by

$$E_n(f) = Min_n\|f(x) - T_n(x)\|_p.$$

This method of approximation is called trigonometric Fourier approximation.

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A function $f \in Lip\alpha$, if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1,$$

and $f \in Lip(\alpha, p)$, if

$$\|f(x+t) - f(x)\|_p = O(|t|^{\alpha}), 0 < \alpha \leq 1, p \geq 1.$$

For a positive increasing function $\xi(t)$ and $p \geq 1$, $f \in Lip(\xi(t), p)$, if

$$\|f(x+t) - f(x)\|_p = O(\xi(t)),$$

and $f \in W(L^p, \xi(t))$, if

$$\left\|[f(x+t) - f(x)] \sin^{\beta}(x/2)\right\|_p = O(\xi(t)),\ \beta \geq 0, p \geq 1. \tag{1.2}$$

If $\beta = 0$, $W(L^p, \xi(t)) \equiv Lip(\xi(t), p)$ and for $\xi(t) = t^{\alpha}(0 < \alpha \leq 1), Lip(\xi(t), p) \equiv Lip(\alpha, p)$. $Lip(\alpha, p) \to Lip\alpha$ as $p \to \infty$. Thus

$$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t)).$$

Hausdorff matrix $H \equiv (h_{n,k})$ is an infinite lower triangular matrix defined by

$$h_{n,k} = \begin{cases} \binom{n}{k} \triangle^{n-k} \mu_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

where $\triangle$ is the forward difference operator defined by $\triangle \mu_n = \mu_n - \mu_{n+1}$ and $\triangle^{k+1} \mu_n = \triangle^k (\triangle \mu_n)$. If H is regular, then $\{\mu_n\}$, known as moment sequence, has the representation

$$\mu_n = \int_0^1 u^n d\gamma(u),$$

where $\gamma(u)$ known as mass function, is continuous at $u = 0$ and belongs to $BV[0,1]$ such that $\gamma(0) = 0$, $\gamma(1) = 1$; and for $0 < u < 1$, $\gamma(u) = [\gamma(u+0) + \gamma(u-0)]/2$ [1]. The Hausdorff means of the Fourier series of f are defined by

$$H_n(f; x) = \sum_{k=0}^{n} h_{n,k} s_k(f; x), \quad n \geq 0. \tag{1.3}$$

For the mass function $\gamma(u)$ given by

$$\gamma(u) = \begin{cases} 0, & 0 \leq u \leq a, \\ 1, & a \leq u \leq 1, \end{cases}$$

where $a = 1/(1+q)$, $q > 0$, we can verify that $\mu_k = 1/(1+q)^k$ and

$$h_{n,k} = \begin{cases} \binom{n}{k} \frac{q^{n-k}}{(1+q)^n}, & 0 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Thus Hausdorff matrix $H \equiv (h_{n,k})$ reduces to Euler matrix (E, q) of order $q > 0$ and defines the corresponding (E, q) means by

$$E_n^q(f; x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(f; x). \tag{1.4}$$

One more example of Hausdorff matrix [$\gamma(u) = u$ for $0 \le u \le 1$] is the well known Cesáro matrix of order 1 $(C, 1)$ and defines the corresponding means by

$$\sigma_n(f;x) = \frac{1}{(n+1)} \sum_{k=0}^{n} s_k(f;x). \tag{1.5}$$

The details of Hausdorff matrices and their examples can be seen in [1, 12]. We shall denote by H_1, the class of all regular Hausdorff matrices with moment sequence $\{\mu_n\}$ associated with mass function $\gamma(u)$.

We use the notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

and

$$g(u,t) = Im\left[\sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} e^{i(k+1/2)t}\right].$$

2. Known Results

The degree of approximation of functions belonging to various function classes through their Fourier series has been studied by various investigators. In the sequel Lal [2-4], Lal and Kushwaha [5], Lal and Singh [6], Lal and Yadav [7], Nigam [8-9], Nigam and Sharma [10], Qureshi [11] Rhoades [13] and Rhoades *et al.* [14] have studied the degree of approximation of periodic functions in $Lip\alpha$, $Lip(\alpha, p)$, $Lip(\xi(t), p)$ and weighted $(L^p, \xi(t))$ classes through various summability means such as Nörlund, Hausdorff, $T \equiv (a_{n,k})$, $C^1.N_p$, $(C,1)(E,1)$ and $(C,1)(E,q)$, of the Fourier series associated with the functions. In this paper, we consider the result of Rhoades [13] in which the result of Lal [2] has been extended from $(C,1)(E,1)$ means to Hausdorff means by keeping other conditions unaltered. Rhoades [13] proved the following:

Theorem 2.1. *Let f be a $2\pi-$ periodic function belonging to the weighted $W(L^p, \xi(t))$ class, $H \in H_1$. Then its degree of approximation is given by*

$$\|H_n(f;x) - f(x)\|_p = O(n^{\beta+1/p}\xi(1/n)), \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{\int_0^{1/n} \left(\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right)^p dt\right\}^{1/p} = O\left(\frac{1}{n}\right), \tag{2.2}$$

and

$$\left\{\int_{1/n}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n^\delta), \tag{2.3}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$, $p^{-1}+q^{-1} = 1$, $1 \le p < \infty$ conditions (2.2) and (2.3) hold uniformly in x.

In second theorem, Rhoades [13, p. 313] has proved the same result for $H \equiv (E,q)$, $q > 0$.

Remark 2.1. *In the light of Rhoades et al.* [14], *we observe that in [13, pp. 310-311], the author has used* $1/\sin^{\beta} t = O(1/t^{\beta})$ *in the interval* $[1/n, \pi]$ *and considered* $\xi(1/y)$ *non-decreasing. Both the arguments are invalid since* $\sin t \to 0$ *as* $t \to \pi$ *and the increasing nature of* $\xi(t)$ *implies that* $\xi(1/y)$ *is non-increasing. We also observe that condition (2.2) of Theorem 2.1 leads to a divergent integral of the form* $\int_0^{1/n} t^{-(1+\beta)q} dt$ *for* $\beta \geq 0$ *[13, pp. 310, 313]. The same type of errors can also be seen in [2-4],* [6] *and [8-11].*

3. Main Results

As mentioned in the introduction of this paper, the $(C,1)$ and (E,q) are Hausdorff matrices, and product of two Hausdorff matrices is a Hausdorff matrix [1, 12, 13], all these matrices can be replaced by a regular Hausdorff matrix. This and the Remark 2.1 has motivated us to determine the degree of approximation of signals (functions) belonging to $W(L^p, \xi(t))-$ class by using Hausdorff means of their Fourier series with a proper set of conditions. In order to rectify the errors mentioned in Remark 2.1, we have defined $W(L^p, \xi(t))$ in (1.2) by replacing $\sin t$ with $\sin(t/2)$ in the definition given by the authors in [2-4], [8-11] and [13]. Further, we shall use increasing function $\xi(t)$ such that $\xi(t)/t$ is non-increasing and also modify the condition (2.2). More precisely, we prove the following:

Theorem 3.1. *Let* f *be a* $2\pi-$ *periodic function belonging to the weighted Lipschitz class* $W(L^p, \xi(t))$, *with* $0 \leq \beta < 1 - 1/p$. *Then its degree of approximation by Hausdorff means generated by* $H \in H_1$ *is given by*

$$\|H_n(f;x) - f(x)\|_p = O((n+1)^{\beta+1/p} \xi(1/n+1)), \tag{3.1}$$

provided positive increasing function $\xi(t)$ *satisfies the following conditions:*

$$\xi(t)/t \, is \, non-increasing, \tag{3.2}$$

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{|\phi(t)| \sin^{\beta}(t/2)}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left((n+1)^{-1/p} \right), \tag{3.3}$$

and

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta} |\phi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O((n+1)^{\delta}), \tag{3.4}$$

where δ *is an arbitrary number such that* $0 < \delta < \beta + 1/p$, $p^{-1} + q^{-1} = 1$ *and* $1 \leq p < \infty$. *The conditions (3.3) and (3.4) hold uniformly in* x.

Remark 3.1. *If we replace the Hausdorff matrix* H *by* (E,q) *in Theorem 2.1, we get Theorem 2 of Rhoades [13, p. 313].*

4. Lemma

For the proof of our Theorem 2.1, we need the following lemma.

Lemma 4.1. *Let* $g(u,t) = Im\left[\sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} e^{i(k+1/2)t}\right]$ *for* $0 \leq u \leq 1$*and* $0 \leq t \leq \pi$. *Then*

$$\left| \int_0^1 g(u,t) d\gamma(u) \right| = \begin{cases} O\left((n+1)t\right), & 0 \leq t \leq \pi/(n+1), \\ O\left(\frac{1}{(n+1)t}\right), & \pi/(n+1) \leq t \leq \pi. \end{cases}$$

Proof. We can write

$$\begin{aligned} g(u,t) &= Im\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}e^{i(k+1/2)t} \\ &= (1-u)^n Im\left\{e^{it/2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{ue^{it}}{1-u}\right)^k\right\} = Im\left\{e^{it/2}\left(1-u+ue^{it}\right)^n\right\}, \end{aligned}$$

which is continuous for $u \in [0,1]$.
Now for $0 < t \le \pi$,

$$\begin{aligned} \int_0^1 g(u,t)du &= \int_0^1 Im\left\{e^{it/2}\left(1-u+ue^{it}\right)^n\right\}du \\ &= Im\int_0^1 \frac{e^{it/2}(1-u+ue^{it})^n}{(-1+e^{it})}(-1+e^{it})du \\ &= Im\left\{\frac{(1-u+ue^{it})^{n+1}}{e^{-it/2}(n+1)(-1+e^{it})}\right\}_0^1 \\ &= Im\left\{\frac{e^{i(n+1)t}-1}{(n+1)(e^{it/2}-e^{-it/2})}\right\} \\ &= \frac{1-\cos(n+1)t}{2(n+1)\sin(t/2)} \\ &= \frac{\sin^2(n+1)t/2}{(n+1)\sin(t/2)} \ge 0. \end{aligned}$$

Therefore, if $M = \sup\limits_{0\le u\le 1}\{\gamma^{'}(u)\}$, then

$$\int_0^1 g(u,t)d\gamma(u) = \int_0^1 g(u,t)\frac{d\gamma}{du}du \le M\int_0^1 g(u,t)du = M\frac{sin^2(n+1)t/2}{(n+1)\sin(t/2)}.$$

Thus for $0 < t < \pi/(n+1)$, we have

$$\left|\int_0^1 g(u,t)d\gamma(u)\right| \le M\frac{\{(n+1)t/2\}^2}{n+1}(\pi/t) = O\{(n+1)t\}, \tag{4.1}$$

in view of $(\sin t)^{-1} \le \pi/2t$ for $0 < t \le \pi/2$ and $\sin t \le t$ for $t \ge 0$.
For $\pi/(n+1) \le t \le \pi$, we have

$$\left|\int_0^1 g(u,t)d\gamma(u)\right| \le M\frac{1}{n+1}(\pi/t) = O\left(\frac{1}{(n+1)t}\right), \tag{4.2}$$

in view of $(\sin t)^{-1} \le \pi/2t$ for $0 < t \le \pi/2$ and $|\sin t| \le 1$ for all t. Collecting (4.1) and (4.2), we get Lemma 4.1. □

Proof of Theorem 3.1. We have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \frac{\phi(t)}{\sin(t/2)}sin(n+1/2)tdt.$$

Therefore,

$$\begin{aligned}
H_n(f;x) - f(x) &= \sum_{k=0}^{n} h_{n,k}\{s_k(f;x) - f(x)\} \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \sum_{k=0}^{n} h_{n,k}\sin(k+1/2)tdt \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \sum_{k=0}^{n} \binom{n}{k} \triangle^{n-k}\mu_k \sin(k+1/2)tdt \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \sum_{k=0}^{n} \binom{n}{k} \int_0^1 u^k(1-u)^{n-k} d\gamma(u) Im e^{i(k+1/2)t} dt \\
&= \frac{1}{2\pi}\int_0^{\pi} \left(\frac{\phi(t)}{\sin(t/2)} \int_0^1 Im\left[\sum_{k=0}^{n}\binom{n}{k} u^k(1-u)^{n-k} e^{i(k+1/2)t}\right] d\gamma(u)\right) dt \\
&= \frac{1}{2\pi}\int_0^{\pi} \left(\frac{\phi(t)}{\sin(t/2)} \int_0^1 g(u,t) d\gamma(u)\right) dt.
\end{aligned}$$

Using $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, we have

$$\begin{aligned}
|H_n(f;x) - f(x)| &\leq \frac{1}{2\pi}\int_0^{\pi} \frac{|\phi(t)|}{t} \left|\int_0^1 g(u,t)d\gamma(u)\right| dt \\
&= \frac{1}{2\pi}\left(\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right) \frac{|\phi(t)|}{t} \left|\int_0^1 g(u,t)d\gamma(u)\right| dt \\
&= I_1 + I_2, \quad say, \qquad (4.3)
\end{aligned}$$

Now using Lemma 4.1 and Hölder inequality, we have

$$\begin{aligned}
I_1 &= O\left\{\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \frac{t^{-1}|\phi(t)|\sin^{\beta}(t/2)}{\xi(t)} \frac{(n+1)t\xi(t)}{\sin^{\beta}(t/2)} dt\right\} \\
&= O\left\{(n+1)\int_0^{\pi/(n+1)} \left(\frac{|\phi(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right\} \times \\
&\quad \left\{\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} \left(\frac{\xi(t)}{\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q} \\
&= O\left[(n+1)^{1-1/p}\xi(\pi/(n+1))\left(\lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/(n+1)} t^{-\beta q}dt\right)^{1/q}\right] \\
&= O\left[(n+1)^{1-1/p}\xi(\pi/(n+1))\left((n+1)^{\beta q-1}\right)^{1/q}\right] \\
&= O\left((n+1)^{\beta}\xi(\pi/(n+1))\right), \qquad (4.4)
\end{aligned}$$

in view of (3.3), mean value theorem for integrals, $1 - \beta q > 0$ and $p^{-1} + q^{-1} = 1$. Again using Lemma 4.1, Hölder inequality and $(\sin(t/2))^{-1} \leq \pi/t$ for $0 < t \leq \pi$, we have

$$
\begin{aligned}
I_2 &= O\left(\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)}{(n+1)\xi(t)} \frac{t^{-1}\xi(t)}{t^{-\delta}t\sin^{\beta}(t/2)}dt\right) \\
&= O\left\{\frac{1}{n+1}\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)}{\xi(t)}\right)^{p}dt\right\}^{1/p} \\
&\quad \left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-1}\xi(t)}{t^{-\delta+\beta+1}}\right)^{q}dt\right\} \\
&= O\left[(n+1)^{\delta-1}\xi(\pi/(n+1))\frac{n+1}{\pi}\left(\int_{\pi/(n+1)}^{\pi} t^{-(\beta-\delta+1)q}dt\right)^{1/q}\right] \\
&= O\left[(n+1)^{\delta}\xi(\pi/(n+1))(n+1)^{\beta-\delta+1-1/q}\right] \\
&= O\left[(n+1)^{\beta+1/p}\xi(\pi/(n+1))\right], \qquad (4.5)
\end{aligned}
$$

in view of (3.4), mean value theorem for integrals, $0 < \delta < \beta+1/p$ and $p^{-1}+q^{-1} = 1$. Finally collecting (4.3)-(4.5) and taking L_p-norm, we get (3.1).
Thus proof of Theorem 3.1 is complete.

□

5. Corollaries

The following corollaries can be derived from Theorem1.

Corollary 5.1. *If* $\beta = 0$, *then for* $f \in Lip(\xi(t), p)$,

$$\|H_n(f;x) - f(x)\|_p = O\left((n+1)^{1/p}\xi(\pi/(n+1))\right).$$

Corollary 5.2. *If* $\beta = 0$, $\xi(t) = t^{\alpha}$ $(0 < \alpha \leq 1)$, *then for* $f \in Lip(\alpha, p)$ $(\alpha > 1/p)$,

$$\|H_n(f;x) - f(x)\|_p = O\left((n+1)^{1/p-\alpha})\right).$$

Corollary 5.3. *If* $p \to \infty$ *in Corollary 5.2, then for* $f \in Lip\alpha$ $(0 < \alpha < 1)$,

$$\|H_n(f;x) - f(x)\|_{\infty} = O\left((n+1)^{-\alpha})\right).$$

which is a result due to Rhoades *et al.* [14] for $0 < \alpha < 1$.
Further, since the product of two Hausdorff matrices is a Hausdorff matrix [13], the results proved by Lal [2], Lal and Kushwaha [5], Lal and Singh [6], Lal and Yadav [7], Nigam [8, 9] and Nigam and Sharma [10] pertaining to the product of $(C, 1)$ and (E, q), $q > 0$, which are Huasdorff matrices, are also particular cases of our Theorem 3.1.

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SOME PROPERTIES OF q-BERNSTEIN SCHURER OPERATORS

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ABSTRACT. In this paper, we study some shape preserving properties of the q-Bernstein Schurer operators and compute the rate of convergence of these operators by means of Lipschitz class functions, the first and the second modulus of continuity. Furthermore, we give the order of convergence of the approximation process in terms of the first modulus of continuity of the derivative of the function.

1. INTRODUCTION

In 1962, Schurer [9] introduced and studied the Bernstein Schurer operators. Let $C[a,b]$ denotes the space of continuous functions on $[a,b]$. For all $n \in \mathbb{N}$ and $f \in C[0,p+1]$, the Bernstein Schurer operators are defined by

$$B_n^p(f;x) = \sum_{r=0}^{n+p} f\left(\frac{r}{n}\right) \binom{n+p}{r} x^r (1-x)^{n+p-r}, \qquad x \in [0,1].$$

Over two decades ago, in 1987 A. Lupaş [5] introduced the q-based Bernstein operators and initiated an intensive research in the intersection of q-calculus and Korovkin type approximation theory. In 1996, another q-based Bernstein operator was proposed by Phillips [8].

Recently Muraru [6] introduced and investigated the q-Bernstein Schurer operators. She obtained the Korovkin type approximation theory and the rate of convergence of the operators in terms of the first modulus of continuity. These operators were defined for fixed $p \in \mathbb{N}_0$ and for all $x \in [0,1]$, by

$$B_n^p(f;q;x) = \sum_{r=0}^{n+p} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x), \tag{1.1}$$

where, for any real number $q>0$ and $r>0$, the q-integer of the number r is defined by [3]

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1 \\ r, & q = 1, \end{cases}$$

q-factorial is defined by

$$[r]! = \begin{cases} [r][r-1]\dots[1], & r = 1,2,3,..., \\ 1, & r = 0 \end{cases}$$

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and q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]!\,[r]!}$$

for $n \geq 0$, $r \geq 0$.
Note that the case $p = 0$ reduces to the Phillips q-Bernstein operators.

We organize the paper as follows:
In section two, we study some shape preserving properties of the operators. In section three, we obtain the rate of convergence of the q-Bernstein Schurer operators by means of Lipschitz class functions and the first and the second modulus of continuity. Furthermore, we compute the degree of convergence of the approximation process in terms of the first modulus of continuity of the derivative of the function.

2. Shape Properties

In this section, we investigate the shape preserving properties of q-Bernstein Schurer operators defined by (1.1). First of all let us recall the first three moments of the q-Bernstein Schurer operators [6]:

Lemma 2.1. *Let $B_n^p(f;q;x)$ be given in (1.1). Then*
i) $B_n^p(1;q;x) = 1$.
ii) $B_n^p(t;q;x) = \dfrac{[n+p]}{[n]}x$.
iii) $B_n^p(t^2;q;x) = \dfrac{[n+p-1]\,[n+p]}{[n]^2}qx^2 + \dfrac{[n+p]}{[n]^2}x$.
Note that the proof of the above lemma has been given by Muraru [6].

Theorem 2.2. *If $f(x)$ is convex and non-decreasing on $[0,1]$, then*

$$B_n^p(f;q;x) \geq f(x), \qquad 0 \leq x \leq 1, \tag{2.1}$$

for all $n+p \geq 1$ and for $0 < q < 1$.

Proof. For each $x \in [0,1]$ and $q \in (0,1)$, let us define

$$x_r = \frac{[r]}{[n]} \text{ and } \lambda_r = \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r+1} (1-q^s x)\,,\ 0 \leq r \leq n+p.$$

So that x_r is the quotient of the q-integers $[r]$ and $[n]$, and $\begin{bmatrix} n+p \\ r \end{bmatrix}$ denotes the q-binomial coefficients.
We see that $\lambda_r \geq 0$ when $0 < q < 1$ and $x \in [0,1]$. Since

$$B_n^p(1;q;x) = 1,$$

then

$$\lambda_0 + \lambda_1 + \cdots + \lambda_{n+p} = 1.$$

Also, since $B_n^p(t;q;x) = \dfrac{[n+p]}{[n]}x$, then

$$\lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{n+p} x_{n+p} = \frac{[n+p]}{[n]}x.$$

Using the above informations and the fact that $f(x)$ is a convex and non-decreasing function, we have the inequality

$$B_n^p(f;q;x)=\sum_{r=0}^{n+p}\lambda_r f(x_r)\geq f\left(\sum_{r=0}^{n+p}\lambda_r x_r\right)=f\left(\frac{[n+p]}{[n]}x\right)\geq f(x).$$

□

Corollary 2.3. *If we choose $p=0$ in* (1.1)*, we get the $q-$Bernstein operators* [4]*. In this case, the condition that $f(x)$ is non-decreasing is revealed.*

3. Rate of Convergence

In this section we compute the rate of convergence of the operators in terms of the elements of Lipschitz classes and the first and the second modulus of continuity of the function. Furthermore, we calculate the order of convergence in terms of the first modulus of continuity of the derivative of the function.

The following lemma gives an estimate for second central moment:

Lemma 3.1. *For the second central moment we have the following inequality*

$$\left|B_n^p\left((t-x)^2;q;x\right)\right|\leq\frac{x^2}{[n]^2}[p]^2+\frac{[n+p]}{[n]^2}x.$$

Proof. We can write

$$B_n^p\left((t-x)^2;q;x\right)=\frac{[n+p-1][n+p]}{[n]^2}qx^2+\frac{[n+p]}{[n]^2}x\leq x^2\left(\frac{[n+p]}{[n]}-1\right)^2+\frac{[n+p]}{[n]^2}x$$

$$=\frac{x^2}{[n]^2}q^{2n}[p]^2+\frac{[n+p]}{[n]^2}x\leq\frac{x^2}{[n]^2}[p]^2+\frac{[n+p]}{[n]^2}x. \tag{3.1}$$

The proof is completed. □

Now, we will give the rate of convergence of the operators B_n^p in terms of the Lipschitz class $Lip_M(\alpha)$, for $0<\alpha\leq 1$. Note that a function $f\in C[0,p+1]$ belongs to $Lip_M(a)$ if

$$|f(t)-f(x)|\leq M|t-x|^{\alpha}\qquad (t,x\in[0,1])$$

satisfied.

Theorem 3.2. *Let $f\in Lip_M(\alpha)$, then*

$$|B_n^p(f;q;x)-f(x)|\leq M(\lambda_n(x))^{\alpha/2}$$

where $\lambda_n(x)=\dfrac{x^2}{[n]^2}[p]^2+\dfrac{[n+p]}{[n]^2}x.$

Proof. Considering the monotonicity and the linearity of the operators, and taking into account that $f \in Lip_M(\alpha)$ $(0 < \alpha \leq 1)$

$$\begin{aligned}
& |B_n^p(f;q;x) - f(x)| \\
= \; & |\sum_{r=0}^{n+p}(f(\frac{[r]}{[n]}) - f(x)\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\,| \\
\leq \; & \sum_{r=0}^{n+p}\left|f(\frac{[r]}{[n]}) - f(x)\right|\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) \\
\leq \; & M\sum_{r=0}^{n+p}|\frac{[r]}{[n]} - x|^{\alpha}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\,.
\end{aligned}$$

Using Hölder's inequality, with $p = \dfrac{2}{\alpha}$, $q = \dfrac{2}{2-\alpha}$, we get

$$\begin{aligned}
& |B_n^p(f;q;x) - f(x)| \\
& = M\sum_{r=0}^{n+p}[(\frac{[r]}{[n]} - x)^2\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)]^{\frac{\alpha}{2}}[\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)]^{\frac{2-\alpha}{2}} \\
& \leq M\left[\{\sum_{r=0}^{n+p}([(\frac{[r]}{[n]} - x)^2\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)])\}^{\frac{\alpha}{2}}\right. \\
& \left.\times\,\{\sum_{r=0}^{n+p}[\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)])\}^{\frac{2-\alpha}{\alpha}}\right] \\
& = M[B_n^p((t-x)^2;q;x)]^{\frac{\alpha}{2}} \\
& \leq M(\lambda_n(x))^{\frac{\alpha}{2}}.
\end{aligned}$$

Whence the result. □

It is clear that the norm of the operator $B_n^p(f;q;x)$ is given by

$$||B_n^p(f;q;\cdot)|| = 1, \tag{3.2}$$

since

$$||B_p^n(f;q;\cdot)|| = \sup_{||f||=1}||B_n^p(f;q;\cdot)|| = B_n^p(1;q;\cdot) = 1.$$

Now we will give the rate of convergence of the operators by means of the first and the second modulus of continuity. Recall that the first modulus of continuity of f on the interval I for $\delta > 0$ is given by

$$\omega(f;\delta) = \max_{\substack{|h|\leq\delta \\ t,x\in I}}|\Delta_h f(x)| = \max_{\substack{|h|\leq\delta \\ t,x\in I}}|\Delta_h f(x+h) - f(x)|$$

or equivalently,

$$\omega(f;\delta) = \max_{\substack{|t-x|\leq\delta \\ t,x\in I}}|f(t) - f(x)|.$$

On the other hand by denoting $C^2(I)$, the space of all functions $f \in C(I)$ such that $f', f'' \in C(I)$. Let $\|f\|$ denote the usual supremum norm of f. The classical Peetre's

K-functional and the second modulus of smoothness of the function $f \in C(I)$ are defined respectively by

$$K(f,\delta) := \inf_{g \in C^2(I)} \left[\|f-g\| + \delta \|g''\| \right]$$

and

$$\omega_2(f,\delta) := \sup_{\substack{0<h\leq\delta, \\ x, x+h \in I}} |f(x+2h) - 2f(x+h) + f(x)|$$

where $\delta > 0$. It is known that[2, p. 177], there exist a constant $A > 0$ such that

$$K(f,\delta) \leq A\omega_2\left(f, \sqrt{\delta}\right).$$

Theorem 3.3. *Let $q \in (0,1)$. Then, for every $n \in \mathbb{N}$, $x \in [0,1]$ and $f \in C[0,p+1]$, we have*

$$|B_n^p(f;q;x) - f(x)| \leq C\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega(f, x\alpha_n)$$

for some positive constant C, where

(3.3)

$$\delta_{n,q}(x) := \left(x^2 \left(\frac{[n+p]^2}{[n]^2} + \frac{[n+p-1][n+p]}{[n]^2} q - 4\frac{[n+p]}{[n]} + 2 \right) + \frac{[n+p]}{[n]} x \right)^{1/2}$$

and

$$\alpha_{n,q} := \frac{[n+p]}{[n]} - 1. \tag{3.4}$$

Proof. Define an auxiliary operator $B_{n,p}^*(f;q;x) : C[0,p+1] \to C[0,p+1]$ by

$$B_{n,p}^*(f;q;x) := B_n^p(f;q;x) - f\left(\frac{[n+p]}{[n]}x\right) + f(x). \tag{3.5}$$

Then, by Lemma 1, we get

$$\begin{aligned} B_{n,p}^*(1;q;x) &= 1 \\ B_{n,p}^*(\varphi;q;x) &= 0, \end{aligned} \tag{3.6}$$

where $\varphi = t - x$. From (3.2) we get

$$||B_{n,p}^*(f;q;\cdot)|| \leq 3.$$

Now, for a given $g \in C^2[0,p+1]$, it follows the Taylor formula that

$$g(y) - g(x) = (y-x)g'(x) + \int_x^y (y-u)g''(u)\,du, \quad y \in [0,p+1].$$

Taking into account (3.5) and using (3.6) we get, for every $x \in [0,1]$, that

$$\begin{aligned}\left|B_{n,p}^{*}\left(g;q;x\right)-g\left(x\right)\right| &= \left|B_{n,p}^{*}\left(g\left(y\right)-g\left(x\right);q;x\right)\right| \\ &= \left|g^{'}\left(x\right)B_{n,p}^{*}\left(\varphi;q;x\right)+B_{n,p}^{*}\int\limits_{x}^{y}\left(y-u\right)g^{''}\left(u\right)du;q;x\right| \\ &= \left|B_{n,p}^{*}\int\limits_{x}^{y}\left(y-u\right)g^{''}\left(u\right)du;q;x\right| \\ &= \left|B_{n}^{p}\int\limits_{x}^{y}\left(y-u\right)g^{''}\left(u\right)du;q;x-\int\limits_{x}^{\frac{[n+p]}{[n]}x}\left(\frac{[n+p]}{[n]}x-u\right)g^{''}\left(u\right)du\right|.\end{aligned}$$

Since

$$\left|B_{n}^{p}\int\limits_{x}^{y}\left(y-u\right)g^{''}\left(u\right)du;q;x\right|\leq\frac{\|g''\|}{2}B_{n}^{p}\left(\varphi^{2};q;x\right)$$

and

$$\left|\int\limits_{x}^{\frac{[n+p]}{[n]}x}\left(\frac{[n+p]}{[n]}x-u\right)g^{''}\left(u\right)du\right|\leq\frac{\|g''\|}{2}\left(\frac{[n+p]}{[n]}-1\right)^{2}x^{2}$$

we get

$$\left|B_{n,p}^{*}\left(g;q;x\right)-g\left(x\right)\right|\leq\frac{\|g''\|}{2}B_{n}^{p}\left(\varphi^{2};q;x\right)+\frac{\|g''\|}{2}\left(\frac{[n+p]}{[n]}-1\right)^{2}x^{2}.$$

Hence Lemma 1 implies that

$$\begin{aligned}&\left|B_{n,p}^{*}\left(g;q;x\right)-g\left(x\right)\right| \\ &\leq\frac{\|g''\|}{2}\left[\left(\frac{[n+p-1]\,[n+p]}{[n]^{2}}q-2\frac{[n+p]}{[n]}+1\right)x^{2}+\frac{[n+p]}{[n]}x+\left(\frac{[n+p]}{[n]}-1\right)^{2}x^{2}\right] \\ &\leq\frac{\|g''\|}{2}\left[x^{2}\left(\frac{[n+p]^{2}}{[n]^{2}}+\frac{[n+p-1]\,[n+p]}{[n]^{2}}-4\frac{[n+p]}{[n]}+2\right)+\frac{[n+p]}{[n]}x\right].\end{aligned} \tag{3.7}$$

Now, considering (3.3) and (3.4), if $f \in C[0,p+1]$ and $g \in C^{2}[0,p+1]$, we may write from (3.7) that

$$\begin{aligned}\left|B_{n}^{p}\left(f;q;x\right)-f\left(x\right)\right| &\leq\left|B_{n,p}^{*}\left(f-g;q;x\right)-\left(f-g\right)\left(x\right)\right| \\ &\quad+\left|B_{n,p}^{*}\left(g;q;x\right)-g\left(x\right)\right|+\left|f\left(\frac{[n+p]}{[n]}x\right)-f\left(x\right)\right| \\ &\leq 4\left\|f-g\right\|+\delta_{n,q}\left(x\right)\frac{\|g''\|}{2}+\left|f\left(\frac{[n+p]}{[n]}x\right)-f\left(x\right)\right| \\ &\leq 4\left(\left\|f-g\right\|+\delta_{n,q}\left(x\right)\|g''\|+\omega\left(f,x\alpha_{n,q}\right)\right)\end{aligned}$$

which yields that

$$\begin{aligned} \left|B_n^p(f;q;x)-f(x)\right| &\leq 2K\left(f,\delta_{n,q}(x)\right)+\omega\left(f,x\alpha_{n,q}\right) \\ &\leq C\omega_2\left(f,\sqrt{\delta_{n,q}(x)}\right)+\omega\left(f,x\alpha_{n,q}\right), \end{aligned}$$

where

$$\delta_{n,q}(x):=\left(x^2\left(\frac{[n+p]^2}{[n]^2}+\frac{[n+p-1]\,[n+p]}{[n]^2}q-4\frac{[n+p]}{[n]}+2\right)+\frac{[n+p]}{[n]}x\right)^{1/2}$$

and

$$\alpha_{n,q}:=\frac{[n+p]}{[n]}-1.$$

□

Now, we will compute the rate of convergence of the operators B_n^p in terms of the modulus of continuity of the derivative of the function.

Theorem 3.4. *If $f(x)$ have a continuous derivative $f'(x)$ and $\omega\left(f',\delta\right)$ is the modulus of continuity of $f'(x)$ in $[0,1]$, then*

$$\begin{aligned} &|f(x)-B_n^p(f;q;x)| \\ &\leq M\frac{[p]}{[n]}+2\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\omega\left(f',\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\right), \end{aligned}$$

where M is a positive constant such that $|f'(x)|\leq M$ $(0\leq x\leq 1)$.

Proof. Using the mean value theorem we have

$$\begin{aligned} f\left(\frac{[r]}{[n]}\right)-f(x)&=\left(\frac{[r]}{[n]}-x\right)f'(\xi) \\ &=\left(\frac{[r]}{[n]}-x\right)f'(x)+\left(\frac{[r]}{[n]}-x\right)\left(f'(\xi)-f(x)\right), \end{aligned}$$

where $x<\xi<\dfrac{[r]}{[n]}$. Hence, we have

$$\begin{aligned} &\left|B_n^p(f;q;x)-f(x)\right| \\ &=f'(x)\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx) \\ &+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx) \\ &\leq\left|f'(x)\right|B_n^p\left((t-x);q;x\right) \\ &+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx) \\ &\leq M\left(\frac{[n+p]}{[n]}-1\right) \end{aligned}$$

$$+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)\left(f^{'}(\xi)-f^{'}(x)\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)$$

$$\leq M\frac{[p]}{[n]}$$

$$+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)\left(f^{'}(\xi)-f^{'}(x)\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)$$

$$\leq M\frac{[p]}{[n]}$$

$$+\sum_{r=0}^{n+p}\omega(f',\delta)\left(\frac{\left|\frac{[r]}{[n]}-x\right|}{\delta}+1\right)\left(\frac{[r]}{[n]}-x\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x),$$

since

$$|\xi-x|\leq\left|\frac{[r]}{[n]}-x\right|.$$

Therefore we can write the following inequality,

$$|B_n^p(f;q;x)-f(x)|$$

$$\leq M\frac{[p]}{[n]}$$

$$+\sum_{r=0}^{n+p}\omega(f',\delta)\left(\frac{\left|\frac{[r]}{[n]}-x\right|}{\delta}+1\right)\left(\frac{[r]}{[n]}-x\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x).$$

From the Cauchy-Schwarz inequality for the first term we get

$$|B_n^p(f;q;x)-f(x)|$$

$$\leq M\frac{[p]}{[n]}$$

$$+\omega(f',\delta)\sum_{r=0}^{n+p}\left|\frac{[r]}{[n]}-x\right|\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)$$

$$+\frac{\omega(f',\delta)}{\delta}\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)^{2}\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)$$

$$\leq M\frac{[p]}{[n]}$$

$$+\omega(f',\delta)\left(\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)^{2}\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)\,d_qt\right)^{1//2}$$

$$+\frac{\omega(f',\delta)}{\delta}\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}-x\right)^{2}\begin{bmatrix}n+p\\r\end{bmatrix}x^{r}\prod_{s=0}^{n+p-r-1}(1-q^{s}x)$$

$$=M\frac{[p]}{[n]}+\omega(f',\delta)\sqrt{B_n^p\left((t-x)^2;q;x\right)}+\frac{\omega(f',\delta)}{\delta}B_n^p\left((t-x)^2;q;x\right).$$

Therefore using lemma 2 we see that

$$\sup_{0\leq x\leq 1} B_n^p\left((u-x)^2;q;x\right)\leq \frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}.$$

Thus

$$\begin{aligned}
&\left|B_n^p\left(f;q;x\right)-f(x)\right| \\
&\leq M\frac{[p]}{[n]} \\
&+\omega\left(f',\delta\right)\left\{\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}+\frac{1}{\delta}\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)\right\}
\end{aligned}$$

Choosing $\delta:=\delta_{n,q}(p)=\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}$,

$$\begin{aligned}
&\left|B_n^p\left(f;q;x\right)-f(x)\right| \\
&\leq M\frac{[p]}{[n]}+\omega\left(f',\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\right) \\
&\times\left\{\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}+\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\right\} \\
&= M\frac{[p]}{[n]}+2\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\omega\left(f',\left(\frac{[p]^2}{[n]^2}+\frac{[n+p]}{[n]^2}\right)^{1/2}\right).
\end{aligned}$$

□

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CLUSTER FLOW MODELS AND PROPERTIES OF APPROPRIATE DYNAMIC SYSTEMS

ALEXANDER P. BUSLAEV, ALEXANDER G. TATASHEV, AND MARINA V. YASHINA

Abstract. A model of a traffic flow on a highway is investigated. A straight line or a ring is divided into segments. The flow density and particles velocity is constant on each segment. The rectangles that have as supports these segments are called clusters. The height of such cluster is equal to the density. Neighboring clusters interact each other according some defined rules. This interaction means moving of the supports boundaries or (and) clusters heights. A system of ordinary differential equations is derived that describes the cluster interaction. The properties of solution of the derived systems are investigated.

1. Introduction

A model of a flow on a highway is considered. A straight line or a ring is divided into segments. The particles on each segment are distributed uniformly, i.e., the density is constant on the segment and the velocity of all the particles are the same on the segment, too. A function is defined that describes the dependence of particles batch velocity on the density. Let *the rectangles* that have as supports the segments with a constant density be called clusters. The height of such the rectangle is equal to the density. Neighboring clusters interact each other according the rules that are defined below. A system of differential equations is derived that describes the clusters interaction. This interaction means moving of the boundaries of supports or (and) clusters heights. The interaction depends on scenarios and proceeds so that the conservation law is true. The properties of solutions of the derived systems of ordinary differential equations are investigated. The relevance of this approach is due to the following fact. A flow density function appears when mathematical equations are used instead the local flow specification (*car-following model*). This function is characterized of some smoothness as an equation solution, on the one hand, and this function is distribution of no more than one and a half hundred particles per kilometer, on the other hand. Even the creators of the *hydrodynamic* approach noticed its limitation, [5]. The latter fact made be relevant methods of stochastic modeling of the particles. Models that are based on the system of differential equations, [2, 4, 5, 7], are used along with stochastic models, [1, 6]. In the present paper and in [3] an approach is offered that uses the concept of the flow density, on the one hand, and allows to make be discrete some processes, which accompany traffic flow processes, on the other hand. So the offered construction can be treated as an attempt to combine the continuum approach for modeling and the discrete approach.

2. Particles and clusters

Let the function $v = f(\rho)$ describe the dependence of flow velocity on density. This function is defined on the segment $[0, \rho_{max}]$ and decreases strictly on the segment $[\rho_{min}, \rho_{max}]$ from the value of $v_{\max}$ until 0. Here $v_{\max}$ is maximum permissible velocity, which corresponds to the density $\rho_{\min}$, $0 \le \rho \le \rho_{\min}$.

Let us introduce some concepts.

a) A *cluster* is particles batch that is characterized by a rectangle, the height of which corresponds to a constant density ρ of particles on the segment. Each cluster moves along the straight line with the velocity $v = f(\rho)$ (the *state function*), Fig. 1.

b) A *max-cluster* is a rectangle with the maximum possible height ρ_{max} and the moving velocity $v = 0$ (*jam*).

c) A *zero-cluster* is a rectangle with the height y, $0 \le y \le \rho_{min}$. This cluster moves with the maximum permissible velocity v_{max}.

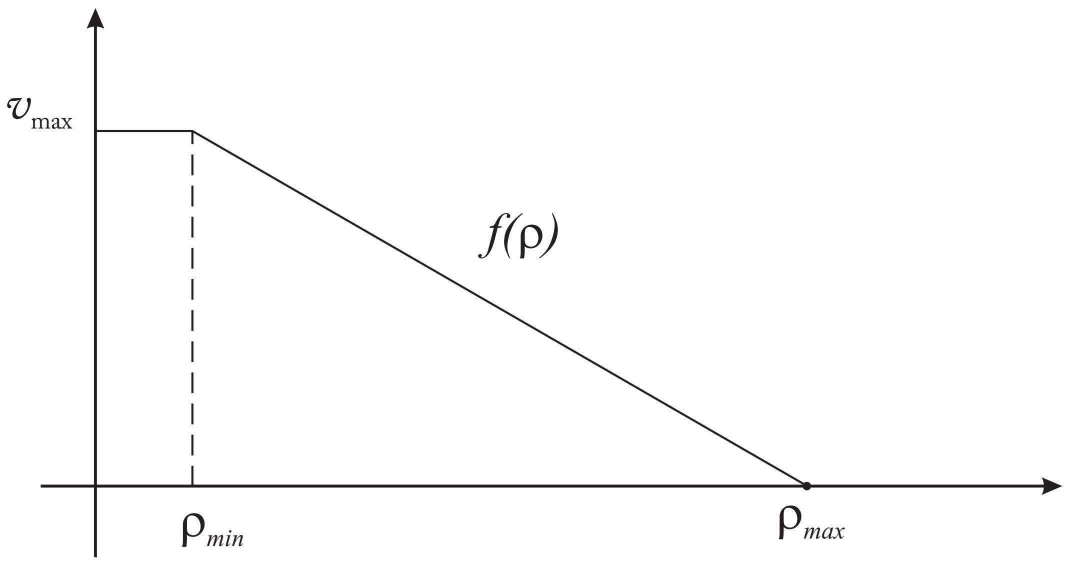

Figure 1. State function

An example of state function $v = f(x)$, which is linear and decreases from the value $v_{\max}$ for $\rho = \rho_{\min}$ until the value 0 for $\rho = \rho_{\max}$, is represented in Fig. 1. In the common case the state function $v = f(x)$ can be nonlinear. A natural requirement is imposed often for the function $v = f(x)$ that this function does not increase (monotonically decreases).

3. Principles of interaction

We consider a special case of movement: *totally-connected flow.* This strategy involves the adaptation of velocity mode of the outsider to the velocity mode of the leader. This adaptation prevents from flow separation into independent parts in the sense that is discussed below. Neighboring clusters, *leader and outsider,* interact together in accordance with the transfer of information within the outsider. If contact information is available only to the leading edge of the outsider cluster, then just this part begins to transform itself to adapt to the leader velocity. If contact information is made available to all particles of a outsider, then the adaptation of the velocity mode is synchronous. Some other possible scenarios are possible. However we focus on these two modes.

4. The interaction of two clusters with the local information

4.1. **The movement of the slow cluster behind the fast one (SF-pair).** As a leading cluster has greater velocity, the front part of the outsider enters into the tail part of the leader so the total mass of the particles is conserved. Let us derive differential equations that describe the interaction of two clusters. Suppose that at the time t the support of the left cluster (outsider) is the segment (x_1, x_2) and the height of the segment, i.e., the flow density on this segment, is equal to y_1, Fig. 2. The support of the leader is the segment (x_2, x_3), which has the height y_2, i.e., the flow density on the segment (x_2, x_3) equals y_2. The left boundary of the left cluster (outsider) moves with the velocity v_1 and therefore at the time $t + \Delta t$ the point $x_1 + v_1\Delta t$ corresponds to this boundary. The right boundary of the right cluster moves with the velocity v_2 and therefore at the time $t + \Delta t$ the point $x_3 + v_2\Delta t$ corresponds to this boundary. The heights of the clusters remain constant. The right boundary of the left cluster, which coincides with the left boundary of the right cluster, moves with the velocity that satisfies the condition that the sum of the rectangles squares remains constant. Let the rectangle square be called *mass* of the cluster. Let $x_2 + \Delta x_2$ be the coordinate of the point on the abscissa that corresponds to this boundary at the time $t + \Delta t$.

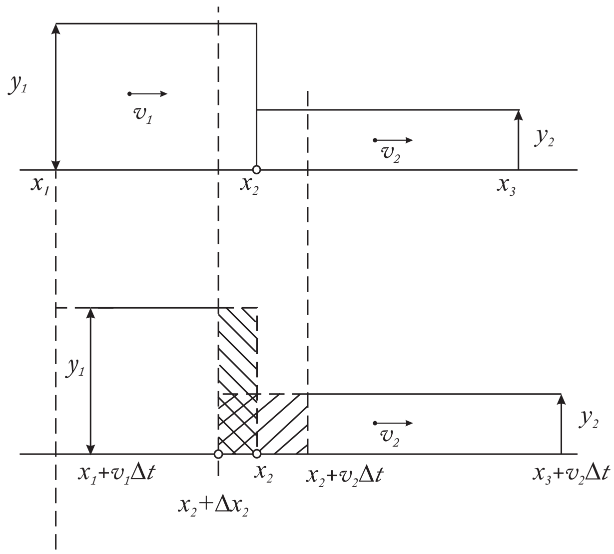

Figure 2. A slow cluster following the fast one

We have for case of the slow cluster following the fast one, Fig. 2,

$$(x_2 + v_2\Delta t - x_2 - \Delta x_2)\, y_2 = ((x_2 - x_1) - (x_2 + \Delta x_2 - x_1 - v_1\Delta t))y_1 \Longleftrightarrow$$

$$\iff (v_2\Delta t - \Delta x_2)y_2 = (-\Delta x_2 + v_1\Delta t)y_1 \iff$$
$$\iff (v_2y_2 - v_1y_1)\Delta t = \Delta x_2(y_2 - y_1) \iff$$
$$\iff \dot{x}_2 = \frac{v_2y_2 - v_1y_1}{y_2 - y_1} = \frac{q_2 - q_1}{y_2 - y_1},$$

where $q_i = y_iv_i$, $i = 1, 2$.

Hence,

$$\begin{cases} \dot{x_1} = v_1 = f(y_1), \\ \dot{x_2} = \frac{v_2y_2 - v_1y_1}{y_2 - y_1} = \frac{q_2 - q_1}{y_2 - y_1}, \\ \dot{x_3} = v_2 = f(y_2). \end{cases} \tag{1}$$

4.2. **The movement of a slow cluster ahead of a fast one (FS-pair).** Suppose a fast cluster follows slow one, Fig. 3. The front boundary of the outsider, which is fast, transforms into the stern part of the slow cluster and the clusters junction point changes according to the particles conservation law.

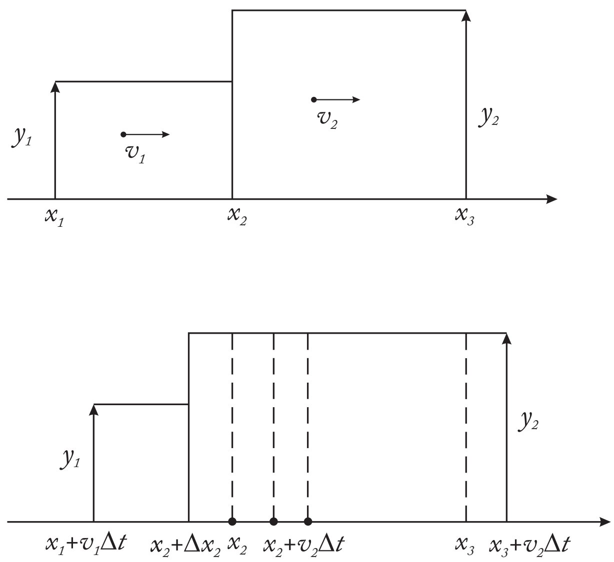

FIGURE 3. A fast cluster following slow one

As in the case of Section 4.1, we have

$$(x_2 + v_2\Delta t - x_2 - \Delta x_2)y_2 = ((x_2 - x_1) - (x_2 + \Delta x_2 - x_1 - v_1\Delta t))y_1 \iff$$

$$\iff (v_2\Delta t - \Delta x_2)y_2 = (-\Delta x_2 + v_1\Delta t)y_1 \iff$$

$$\iff (v_2y_2 - v_1y_1)\Delta t = (y_2 - y_1)\Delta x_2 \iff$$

$$\iff \dot{x}_2 = \frac{v_2y_2 - v_1y_1}{y_2 - y_1} = \frac{q_2 - q_1}{y_2 - y_1}.$$

As in the case of (1), we obtain

$$\begin{cases} \dot{x}_1 = v_1, \\ \dot{x}_2 = \frac{v_2 y_2 - v_1 y_1}{y_2 - y_1} = \frac{q_2 - q_1}{y_2 - y_1}, \\ \dot{x}_3 = v_2. \end{cases}$$

5. Support of an isolated clusters pair in the case of $t \to \infty$

Let us consider behavior of two clusters. Denote $\Delta_1(t) = x_2(t) - x_1(t)$, $\Delta_2(t) = x_3(t) - x_2(t)$.

Lemma 1. *Let $\Delta_1^0 = \Delta_1(0)$ and $\Delta_2^0 = \Delta_2(0)$ be the length of the leader and the outsider accordingly. Then, after the time time interval of duration $t^* = \Delta_1^0(y_2 - y_1)(y_2(v_1 - v_2))^{-1}$, the outsider vanishes, and $\Delta_2(t^*) = \Delta_2^0 + y_1\Delta_1^0 y_2{}^{-1}$. Besides, if the leader is the slow cluster, then $\Delta_2(t^*) < \Delta_1^0 + \Delta_2^0$, and, if the leader is the fast cluster, then $\Delta_2(t^*) > \Delta_1^0 + \Delta_2^0$.*

Proof. Let us find the difference of clusters edges velocities, i.e., the velocity of clusters lengths change. We have

$$\dot{x}_2 - \dot{x}_1 = \frac{v_2 y_2 - v_1 y_1}{y_2 - y_1} - v_1 = \frac{v_2 y_2 - v_1 y_2}{y_2 - y_1} = y_2 \cdot \frac{v_2 - v_1}{y_2 - y_1},$$

$$\dot{x}_3 - \dot{x}_2 = v_2 - \frac{v_2 y_2 - v_1 y_1}{y_2 - y_1} = \frac{v_1 y_1 - v_2 y_1}{y_2 - y_1} =$$

$$= y_1 \cdot \frac{v_1 - v_2}{y_2 - y_1} = -y_1 \cdot \frac{v_2 - v_1}{y_2 - y_1}.$$

If the fast cluster moves behind the slow one, then

$$y_2 > y_1,\ v_2 < v_1,\ \frac{v_2 - v_1}{y_2 - y_1} < 0.$$

If the slow cluster moves behind the fast one, then

$$y_2 < y_1,\ v_2 > v_1,\ \frac{v_2 - v_1}{y_2 - y_1} < 0.$$

Therefore we have in both the cases that the value of $\dot{x}_2 - \dot{x}_1$ is negative and the value of $\dot{x}_3 - \dot{x}_2$ is positive, i.e., the outsider support length decreases and the leader support length increases.

The velocity with that of the outsider support length decreases is constant, and equal to $y_2(v_1 - v_2)(y_2 - y_1)^{-1}$. Hence the outsider vanishes for the time segment of length $t^* = \Delta_1^0(y_2 - y_1)(y_2(v_1 - v_2))^{-1}$.

Since

$$\dot{x}_3 - \dot{x}_1 = (y_2 - y_1)\frac{v_2 - v_1}{y_2 - y_1} = v_2 - v_1,$$

it follows that $sgn(\dot{x}_3 - \dot{x}_1) = sgn(v_2 - v_1)$ and, therefore,

$$sgn(\Delta_1(t) + \Delta_2(t))' = sgn(\dot{x}_3 - \dot{x}_1) = sgn(v_2 - v_1).$$

Thus the statement of Lemma 1 about the clusters pair support is true. Lemma 1 has been proved. □

6. Tandems with zero-cluster

6.1. **Choleric-outsider.** If an arbitrary cluster follows a zero-cluster (Fig. 4), then we have, $0 \le y_0 \le \rho_{min}$,

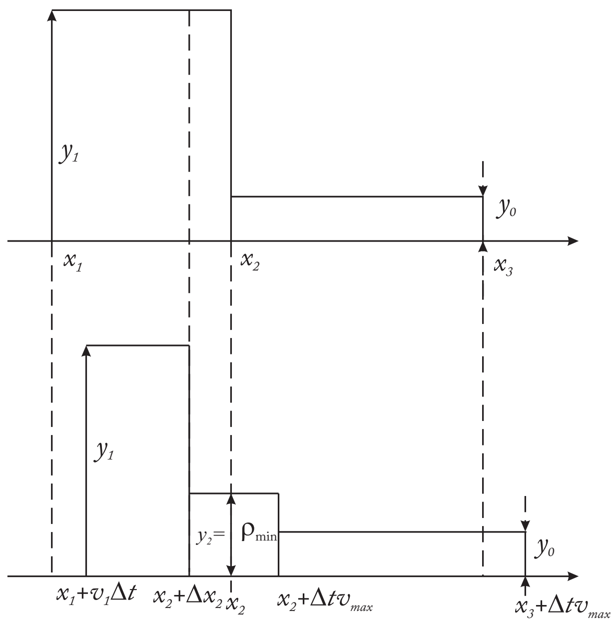

FIGURE 4. An arbitrary cluster follows a zero-cluster

$$(x_2 + v_{max}\Delta t - x_2 - \Delta x_2)\rho_{min} = (x_2 - x_2 - \Delta x_2)y_1 + v_1\Delta t y_1,$$

$$\Delta x_2(y_1 - \rho_0) = (v_1 y_1 - v_{max}\rho_{min})\Delta t,$$

$$\dot{x}_2 = \frac{v_{max}\rho_{min} - v_1 y_1}{\rho_{min} - y_1}.$$

Since

$$\dot{x}_2 - \dot{x}_1 = \dot{x}_2 - v_1 = \rho_{min}(v_{max} - v_1)(\rho_{min} - y_1)^{-1} < 0,$$

it follows that the time of transformation of the slow cluster into fast one is equal to $\Delta_1^0(y_1 - \rho_{min})(\rho_{min}(v_{max} - v_1))^{-1}$. If $\rho_{min} = 0$, then we get

$$\dot{x}_2 = \frac{-v_1 y_1}{0 - y_1} = v_1.$$

6.2. **Sanguine-outsider.** If $\rho_{min} = 0$, then we get, too,

$$\dot{x}_2 = \frac{-v_1 y_1}{0 - y_1} = v_1.$$

In this case outsider-cluster continues to move uniformly in accordance with the basic law (1), Fig. 5.

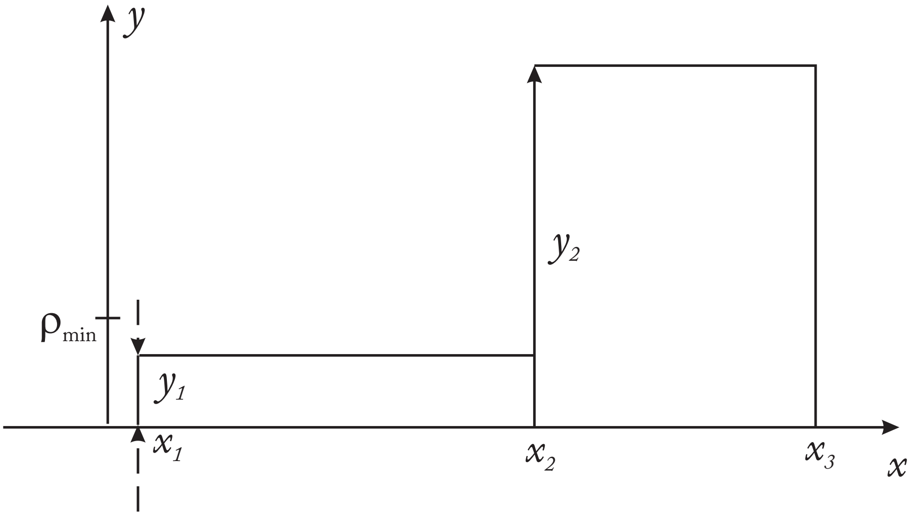

FIGURE 5. Outsidering zero-cluster

6.3. **Common case.** Let us suppose that y_2 is an arbitrary value, $0 < y_2 < \rho_{\max}$. Then we have

$$\dot{x}_2 = \frac{v_2 y_2 - v_1 y_1}{y_2 - y_1}.$$

Here y_2 is reaction of outsider cluster on zero-leader.

6.4. **Outsidering zero-cluster.** In the case when a zero-cluster follows an arbitrary cluster we have, Fig. 5, $0 \le y_1 \le \rho_{min}$,

$$(x_2 + v_2\Delta t - x_2 - \Delta x_2)y_2 = ((x_2 - x_1) - (x_2 + \Delta x_2 - x_1 - v_1\Delta t))y_1,$$

$$(v_2 y_2 - v_{\max} y_1)\Delta t = (y_2 - y_1)\Delta x_2,$$

$$\dot{x_2} = \frac{v_2 y_2 - v_{max} y_1}{y_2 - y_1},\ 0 < y_1 < \rho_{min}.$$

7. Connected chain of choleric-clusters with local interaction on the line

7.1. **Generalization of the problem to an arbitrary chain of clusters.** Let us generalize the problem to an arbitrary chain of clusters on the line.

Suppose n clusters follow each other on the segment $[x_1, x_{n+1}]$. Segments $[x_1, x_2]$, $[x_2, x_3], \ldots, [x_n, x_{n+1}]$ correspond to these clusters, $x_1 < \cdots < x_n$. Let $\Delta_i(t) = x_{i+1}(t) - x_i(t)$ be length of support of the i-th cluster at time t, $i = 1, \ldots, n$.

The height y_i, which is constant in time, corresponds to the i-th cluster, i.e., the cluster located on the segment $[x_i, x_{i+1}]$, $y_i \neq y_{i+1}$, $i = 1, \ldots, n$, and the velocity of cluster boundaries movement satisfies the system of equations

$$\begin{cases} \dot{x}_1 = v_1 = f(y_1), \\ \dot{x}_i = \frac{v_i y_i - v_{i-1} y_{i-1}}{y_i - y_{i-1}}, \ i = 2, \dots, n, \\ \dot{x}_{n+1} = v_n = f(y_n), \\ v_i = f(y_i), \ i = 1, \dots, n. \end{cases} \tag{2}$$

We suppose, if the length of some cluster becomes equal to zero at time t, i.e., $\Delta_i(t) = 0$ for some i, then the clusters are renumbered and, since time t, the movement of clusters is carried out in such a way as if their original number were equal to $n-1$ or less than $n-1$. The number of equations in system (2) decreases at least by one.

Let the product of the cluster length and its density be called the cluster *mass.* Let the sum of the clusters mass be called the flow mass.

Theorem 2. *Let the function $v = f(y)$ be decreasing strictly and $\Delta_i^0 = \Delta_i(0)$ be the initial length of the i-th cluster support, $i = 1, \dots, n$.*

Then the following statements are true:

(1) The length of the cluster $[x_1, x_2]$, which moves the latter, decreases over time.

(2) The length of the cluster $[x_n, x_{n+1}]$, which moves ahead, increases over time.

(3) Let u_i be the absolute value of change rate of the i-th cluster velocity, if the i-th cluster length decreases, and $u_i = 0$, if the i-th cluster length does not decrease, $i = 1, \dots, n$, (the velocity of change of cluster length is constant).

Then, after a time interval

$$t^* = \min_i \Delta_i^0 / u_i, \ i = 1, 2, \dots,$$

number of clusters decreases, where $\Delta_i^0 = \Delta_i(0)$ is the initial length of the i-th cluster.

(4) After a finite time interval, the chain of clusters is reduced to the front cluster.

(5) The flow mass does not change in time.

Proof. We calculate the difference of the velocities of the ends of the cluster that moves the latter, i.e., rate of change in the length of this cluster. Taking into account that the function $v = f(y)$ decreases, we see

$$\dot{x}_2 - \dot{x}_1 = \frac{v_2 y_2 - v_1 y_1}{y_2 - y_1} - v_1 = \frac{v_2 y_2 - v_1 y_2}{y_2 - y_1} = y_2 \cdot \frac{v_2 - v_1}{y_2 - y_1} < 0.$$

Hence the first statement of Theorem 1 is true.

We have for the change rate of the length of the cluster that moves ahead

$$\dot{x}_{n+1} - \dot{x}_n = v_n - \frac{v_n y_n - v_{n-1} y_{n-1}}{y_n - y_{n-1}} =$$

$$= \frac{v_{n-1} - v_n}{y_n - y_{n-1}} = -y_{n-1} \cdot \frac{v_{n-1} - v_n}{y_n - y_{n-1}} > 0.$$

Therefore the second statement of Theorem 1 is true.

The change rates of the lengths of cluster supports are constant. After the time interval of duration t^*, the length of support of one of cluster becomes equal to zero, and the cluster vanishes. This cluster cannot be the cluster that moved first. Hence the statements 3 and 4 of Theorem 1 are true.

Let m_i be mass of the cluster that is located on the segment (x_i, x_{i+1}). Let m be the flow mass.

We have

$$m = \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} y_i(x_{i+1} - x_i). \tag{3}$$

Using (2) and (3), we have for the derivative of the flow mass

$$\dot{m} = \sum_{i=1}^{n} \dot{m}_i = \sum_{i=1}^{n} y_i(\dot{x}_{i+1} - \dot{x}_i) =$$

$$= y_1\dot{x}_n + \sum_{i=1}^{n-1} \dot{x}_i(y_{i+1} - y_i) - y_n\dot{x}_n =$$

$$= y_1 f(y_1) + \sum_{i=1}^{n-1} \frac{y_i f(y_i) - y_{i+1} f(y_{i+1})}{y_{i+1} - y_i}(y_{i+1} - y_i) - y_n f(y_n) =$$

$$= y_1 f(y_1) + \sum_{i=1}^{n-1} (y_i f(y_i) - y_{i+1} f(y_{i+1})) - y_n x_n = 0.$$

Hence the last statement of Theorem 1 is true. Thus Theorem 1 has been proved. □

7.2. **Geometric interpretation.** Let us describe a geometric approach that represents the solutions of system (2). The solutions of this system can be represented by straight lines on the diagram with axes corresponding to the values t and x, Fig. 6. The slope of such the straight line $x_i(t)$ is equal to the slope of the segment $((y_i, q_i), (y_{i+1}, q_{i+1}))$ on the diagram of the function $v = q(y) = yf(y)$, Fig. 7. Each point of intersections of two straight lines from the set $x_i(t)$, $i = 1, \dots, n$, corresponds to a time of disappearance of a cluster.

8. Flow with local interaction on a circle. Choleric-clusters

Suppose a circle is divided into n parts

$$0 \le x_1^0 < x_2^0 < \dots < x_n^0 < 1,\ x_{n+1}^0 = x_1^0 + 1;$$

$$\Delta_i^0 = x_{i+1}^0 - x_i^0,\ 1 \le i \le n-1,\ \Delta_n^0 = 1 + x_1^0 - x_n^0;$$

$$\Delta_1^0 + \Delta_2^0 + \dots + \Delta_n^0 = 1.$$

The density y_i is defined on each segment $[x_i^0,\ x_{i+1}^0]$, $1 \le i \le n$, (Table 1). The flow velocity at the point is defined with the function $v = f(y)$, where v is the velocity; y is the density. The initial configuration of the points $x_1^0 \dots, x_n^0$ is defined.

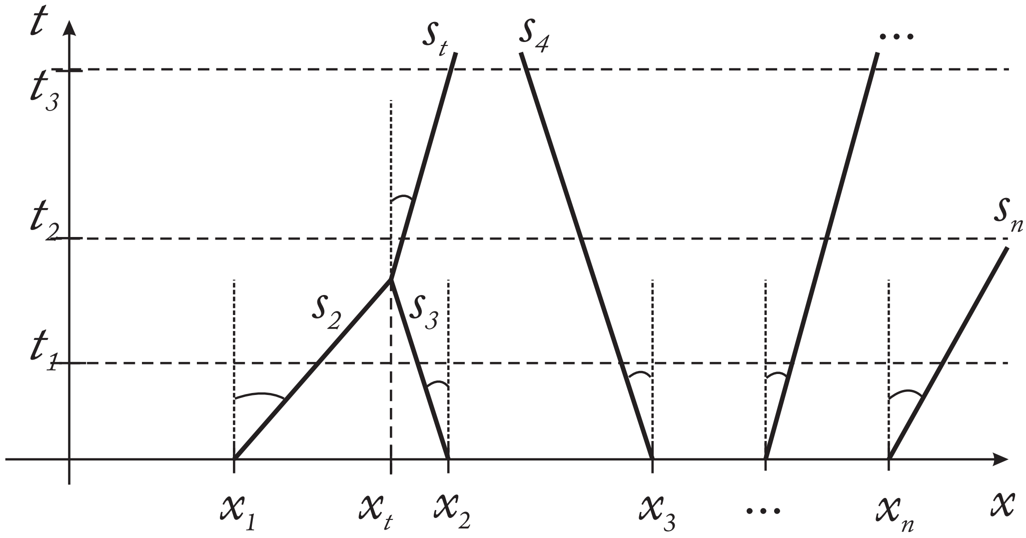

FIGURE 6. Geometrical interpretations. Solutions $s_i = \mathrm{tg}\,\varphi_i$ of system (2)

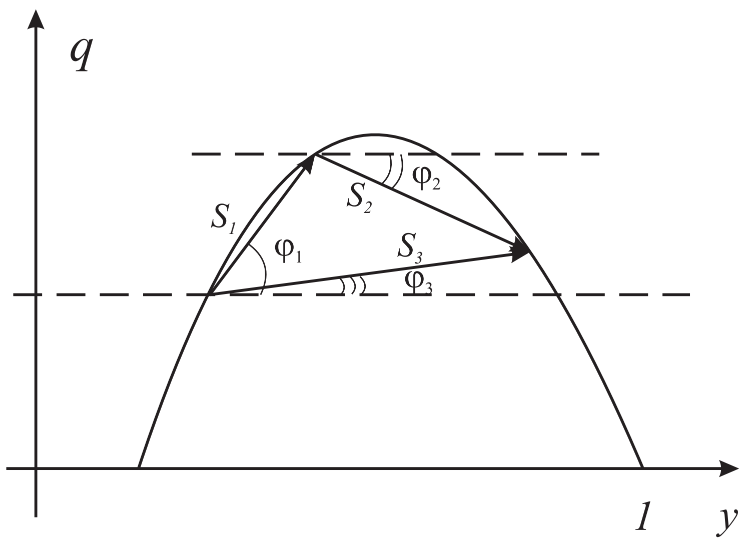

FIGURE 7. Geometrical interpretations. The slopes of the lines

Table 1. Initial requirements

$[x_1^0, x_2^0]$	$[x_2^0, x_3^0]$	......	$[x_n^0, x_{n+1}^0]$
y_1	y_2	......	y_n

The following system of equations defines dynamic of the points x_i

$$\dot{x}_{i+1} = \frac{q_{i+1} - q_i}{y_{i+1} - y_i} = \frac{v_{i+1}y_{i+1} - v_i y_i}{y_{i+1} - y_i},\ 1 \le i \le n-1, \tag{4}$$

where $v_i = f(y_i)$, $q_i = v_i y_i$.

It is clear that the densities values belong to the set $y_1, y_2, \dots, y_n$ at every time. The main question is the flow behavior, i.e., the behavior of the solutions of system (4) for the case of $t \to \infty$. Assume that

$$s_{ij} = s(i,j) = \frac{q(y_j) - q(y_i)}{y_j - y_i},\ 1 \le i, j \le n,\ s_i = s(i, i+1).$$

Then we can be rewrite (4) as

$$\dot{x}_{i+1} = \frac{q(y_{i+1}) - q(y_i)}{y_{i+1} - y_i} = s_i,\ 1 \le i \le n. \tag{5}$$

Assume that, if at some time for some i the length of the segment $[x_i^0,\ x_{i+1}^0]$ becomes equal to zero, i.e., the point x_i^0 coincides with the point x_{i+1}^0, then the further behavior of the model is so that at initial time the circle were divided into $n-1$ parts, and so on.

Suppose

$$t_i = \begin{cases} \frac{\Delta x_i}{|s_i|},\ s_i < 0, \\ \infty,\ s_i > 0, \end{cases} \tag{6}$$

$$t^* = \min(t_1 \dots, t_n).$$

Theorem 3. *Suppose*

$$\begin{gathered} y_i \ne y_j,\ i \ne j;\ s_{i,j} \ne 0,\ 1 \le i, j \le n, \\ s_{i_1,i_2} \ne s_{i_3,i_4},\ i_1 < i_2 \le i_3 < i_4,\ t_i \ne t_j,\ i \ne j. \end{gathered} \tag{7}$$

Then the following statements are true:

(1) Flow mass is constant in time.

(2) After the time t^* since beginning of the model functioning, the number of the segments, into which the circle is divided, decreases by one.

(3) The number of the segments, into which the circle is divided, decreases until this number becomes equal to two.

Proof. The proof of the two first statements of Theorem 2 is similar to the proof of Theorem 1. We take into account the rules of the model functioning and the assumptions made above. Since $s_{i_1,i_2} \ne s_{i_3,i_4}$ $(i_1 < i_2 \le i_3 < i_4)$, we have that the length of each cluster varies. Since $t_i \ne t_j,\ i \ne j$, we have that more one cluster cannot disappear simultaneously.

Let us prove the third statement. If $n = 2$, then

$$\dot{x}_1 = s_{1,2} = \frac{q(y_2) - q(y_1)}{y_2 - y_1} = \frac{q(y_1) - q(y_2)}{y_1 - y_2} = \dot{x}_2$$

and, therefore, the lengths of the segments, into which the circle is divided, are constant. Theorem 2 has been proved.

□

Remark 1. *Suppose requirement (7) can be not fulfilled. Then the number of clusters and the lengths of the segments can remain constant in time still, if the number of clusters is more than two.*

9. Movement in the presence of an obstacle

Suppose $\rho_{max} = 1$, $\rho_{min} = 0$, $f(y) = 1 - y$, $0 \leq y \leq 1$.

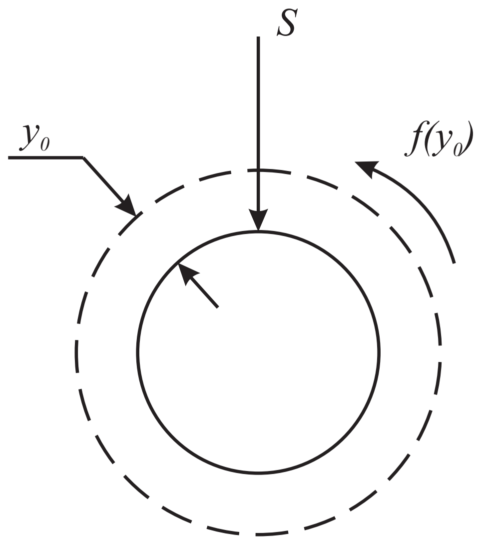

Figure 8. Full periodic cluster

9.1. **Movement in the presence of an obstacle: birth of clusters.** Assume that there is a single cluster, and its density is equal to y_0. The support of this cluster is the whole circle, Fig. 8. An obstacle comes into existence on the circle at the point x_1. This obstacle can be interpreted as the red traffic light. An obstacle appears in front of a cluster with a density $\rho_{max} = 1$, and the segment arises of zero density ahead of the obstacle in the direction of the movement. The length of the segment $[x_1, x_2]$, the density of which is zero, equals zero at the beginning of the existence of the obstacles, Fig. 9. The rear boundary of the segment is fixed at the point x_1 while the other moves forward, and its velocity equals $v_0 = f(y_0)$. The support of the cluster with the maximum density is the segment $[x_1, x_0]$. The coordinate of the point x_1 moves according to the law

$$\dot{x}_1 = -\frac{y_0 f(y_0)}{1 - y_0}$$

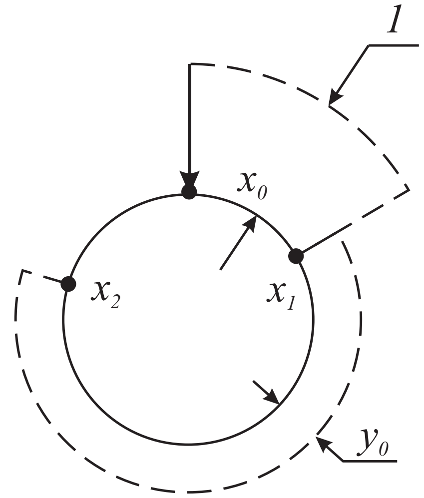

FIGURE 9. The flow is divided into three parts

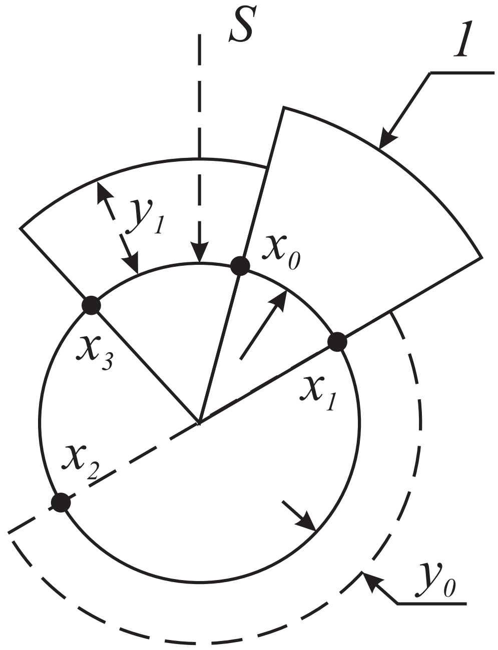

FIGURE 10. Movement for a green phase

such that the total mass of the resulting clusters does not change. An obstacle exists for some time T_r. After the disappearance of obstacles the cluster that has the density 1 is divided into a cluster of density 1 and a cluster of density y_1, $y_0 < y_1 < 1$, (a phase of the "green light" begins itself), Fig. 10. The points x_2 and x_3 have such velocity as the obstacle still existed. Density of the cluster that is located on the segment $[x_2, x_3]$ remains equal to y_0. Cluster of the density y_1 appears on the segment $[x_3, x_0]$. The cluster density on the segment $[x_3, x_2]$ is equal to 0. The cluster density on the segment $[x_1, x_0]$ is equal to 1. From this time the point x_1, which is the front boundary of this segment, has the velocity $v_1 = f(y_1)$. The point x_0, which is the rear boundary of the segment, moves such that its velocity ensures that the law of mass conservation is fulfilled:

$$\dot{x}_0 = -\frac{y_1 f(y_1)}{1 - y_1}.$$

After time interval T_g since beginning of the green light phase, a new red light phase can begin itself. The obstacle arises at the same point as previously (at the point of traffic-lights location). The red light phase begins only in the case if the given point is in a cluster that has density y_0. Otherwise, the green light phase repeats itself. At the red light phase the new cluster is formed with a density of 1 and, during the next phase of green light, the cluster is divided into clusters of densities y_1 and 1, etc.

Theorem 4. *Suppose l is the length of the circle, which is support of the cluster of density y_0. Then the following statements are true.*

(1) After time interval of duration not more than $\frac{l}{f(y_1) - f(y_0)}$, no cluster of density y_0 remains.

(2) After a finite time interval since beginning of model functioning, only clusters with densities y_1 and 0 remain.

Proof. After turning on red lights, clusters of density of 0 and 1 are born, and for the green light phase, clusters of densities y_1 $(0 < y_0 < y_1 < 1)$ are born also as described above.

The length of the cluster that has density 0 cannot decrease. From the mass conservation law, it follows that after the initial time there exist always at least one cluster of density y_1 or 1. Each cluster with y_0 is limited to the rear by a cluster of density 0. Hence there is no cluster of density y_0 the length of support of that is decreasing. When a cluster with density y_0 is divided into such two clusters (between which clusters of densities 1 and y_1 appear) the total length of the supports of the clusters of non-zero density does not increase. During the time intervals between such divisions the total length of the clusters of density y_0 decreases with a velocity, which is not less than $f(y_0) - f(y_1)$. Hence the first statement of Theorem 3 is true.

Let us prove the second statement.

A cluster of density y_1 arises in front of the cluster of density 1. Hence the length of the support of a cluster with density y_1 can only increase. Really, the front boundary of the cluster moves with velocity $f(y_1)$ in the direction of flow. The rear boundary of this cluster moves in the opposite direction. Therefore the clusters of density y_1 cannot disappear before the time when the clusters of density 1 disappear. At the time when the clusters of density y_0 disappears (in accordance with the first statement of the theorem such time will come) clusters of densities y_1, 0, and 1 or only clusters of densities

y_1 and 0. In the first case the total length of the clusters of density 1 decreases still, and the total length of the cluster support of density y_1 increases unless all the clusters of density 1 disappear. Thus in both the cases only the clusters of densities y_1 and 0 remain. Theorem 3 has been proved. □

10. Controlled clusters model

Suppose a full periodic cluster of density y_0 moves with velocity $f(y_0)$, Fig. 8, and the formula for f is

$$f(y) = 1 - y,\ 0 \leq y \leq 1. \tag{8}$$

At the pole S, prohibition of movement (traffic lights) is switched off since time $t = 0$ for the time interval T_r.

For this time interval, the flow is divided into three fragments, i.e., clusters (Fig. 9). The velocities of the boundaries are

$$\begin{cases} \dot{x}_2 = f(y_0), \\ x_0 \equiv f(y_0), \\ \dot{x}_1 = \frac{0 - y_0 f(y_0)}{1 - y_0}. \end{cases} \tag{9}$$

At the time $t = T_r$, the green light is switched at the point S allowing the movement that was banned previously. Let $y_1 \in (0, 1)$ be the density of the flow that goes out. Then four clusters are formed initially at $t > T_r$. The velocities of the boundaries are

$$\dot{x}_0 = \frac{y_1 f(y_1) - 0}{y_1 - 1},$$
$$\dot{x}_1 = \frac{0 - y_0 f(y_0)}{1 - y_0},$$
$$\dot{x}_2 = f(y_0),$$
$$\dot{x}_3 = f(y_1).$$

At the time $t = T_r + T_g$, Fig. 10, a red light phase begins itself and another boundary appears at the point S, $x_{-1}(T_r + T_g)) = 0$, i.e., since time $T_r + T_g$, the boundary is divided into two the boundaries $x{-}1$ and x_4, which velocities are

$$\dot{x_{-1}} = \frac{0 - y_1 f(y_1)}{1 - y_1},$$

$$\dot{x}_4 = \begin{cases} 0,\ T_r + T_g < t < 2T_r + T_g; \\ f(y_2), t > 2T_r + T_g \end{cases}$$

Therefore, at the general position, when the red light is switched on, at the point S two new clusters of densities 0 and 1 appear and, when the green light is switched, a cluster of density y_n appears too. Hence, for the interval

$T_r + T_g$, one cluster generates four new clusters (altogether there are five clusters) of densities 1, 0, y_n, y_{n-1}, i.e., there four boundary points.

The main objective is to study the limit state of the system, when control time management is large, and in the cases

a) periodic control;

b) adaptive control.

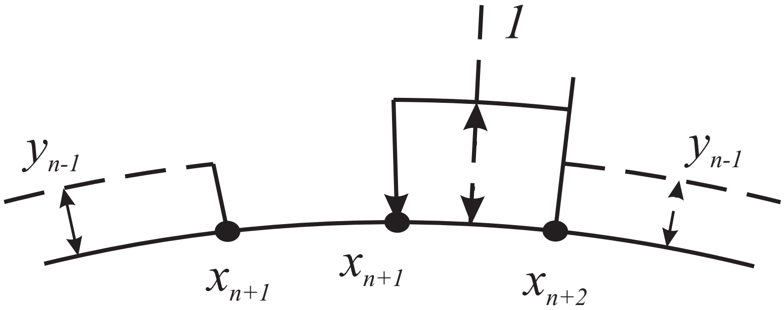

FIGURE 11. Movement in the neighborhood of S during the red time interval

Consider the processes in the neighborhood of the point S. According to Fig. 11, we have during the interval of red light phase, for $t = T_r^{(n)} = \Delta t$

$$\begin{cases} |x_{n-2} - x_{-n+1}| = \frac{y_{n-1} f(y_{n-1})}{1 - y_{n-1}} \delta t \\ |x_{n-1} - x_{-n+1}| = f(y_{n-1}) \Delta t. \end{cases} \tag{10}$$

For the interval of green light phase of duration δt, we have the situation represented in Fig. 12.

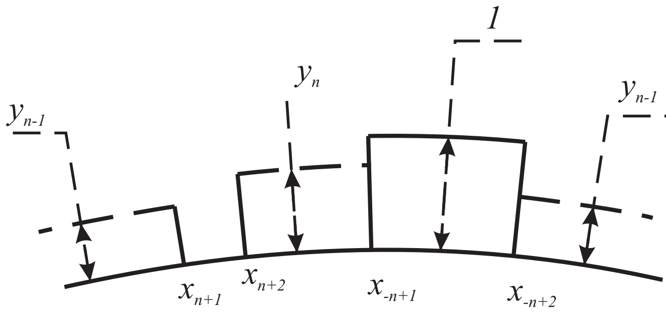

FIGURE 12. Movement in the neighborhood of S during the green time interval

$$\begin{cases} |x_{n+1} - x_{n+2}| = f(y_{n-1})(\Delta t + \delta t) - f(y_n)\delta t, \\ |x_{-n+1} - x_{-n+2}| = -\frac{y_n f(y_n)}{1 - y_n} \delta t + \frac{y_{n-1} f(y_{n-1}}{1 - y_{n-1}} (\delta t + \Delta t). \end{cases} \tag{11}$$

Remark 2. *The sum of the lengths of jam and zero cluster supports is a constant value.*

Proof. Since $f(y) = 1 - y$, we have $\frac{yf(y)}{1-y} = y$. We can rewrite equations (11) as

$$z_n = \begin{cases} x_{n+1} - x_{n+2} = (1 - y_{n-1})(\Delta t + \delta t) - (1 - y_n)\delta t = \Delta t - z_n, \\ x_{-n+1} + x_{-n+2} = -y_n \delta t + y_{n-1}(\delta t + \Delta t). \end{cases} \tag{12}$$

Thus the statement of Remark 2 is true. □

If the system is uncontrolled, then the number of clusters cannot increase. In the case of controlled system, the number of clusters can be also increase when a green or red phase begins itself. The behavior of the controlled system is to be studied.

11. Partially-connected movement of sanguine-clusters

Suppose that a circle is divided into n parts

$$0 \le x_1^0 < x_2^0 < \cdots < x_n^0 < 1, \; x_{n+1}^0 = x_1^0;$$
$$\Delta_i^0 = x_{i+1}^0 - x_i^0, \; 1 \le i \le n-1, \; \Delta_n^0 = 1 + x_1^0 - x_n^0;$$
$$\Delta_1^0 + \Delta_2^0 + \cdots + \Delta_n^0 = 1.$$

The value Δ_i^0 is equal to the length of segment $[x_i^0, \; x_{i+1}^0]$.

The density y_i^0 is defined on the segment $[x_i^0, \; x_{i+1}^0]$ $1 \le i \le n$. The flow velocity at the point is determined with the function $v = f(y)$, where v is velocity; y is the density. The initial configuration of the points $x_1^0 \ldots, x_n^0$ is defined.

If $y_i > 0$, then the segment $[x_i, x_{i+1}]$ corresponds to some cluster. If $y_i = 0$, then the $[x_i, x_{i+1}]$ corresponds to some gap between clusters.

Assume that at initial time all the clusters are divided by gaps. The dynamic of the points x_i is determined as follows.

If $y_{i-1} > 0$, $y_i = 0$, i.e., a cluster corresponds to the segment $[x_{i-1}, x_i]$, and a gap corresponds to $[x_i, x_{i+1}]$, then

$$\dot{x}_i = v_i = f(y_i). \tag{13}$$

If $y_{i-1} = 0$, $y_i > 0$, i.e., a gap corresponds to the segment $[x_{i-1}, x_i]$, and a cluster corresponds to the segment $[x_i, x_{i+1}]$, then also

$$\dot{x}_i = v_i = f(y_i).$$

If clusters correspond to segments $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, then

$$\dot{x}_i = \frac{v_i y_i - v_{i-1} y_{i-1}}{y_i - y_{i-1}}, \; 1 \le i \le n, \tag{14}$$

where $v_i = f(y_i)$.

Let several following one other clusters, non-divided by gaps, be called a batch.

Since it is assumed that at initial time all the clusters are divided by gaps and a slower cluster cannot reach a faster one, it follows that at initial time all the faster clusters move behind the slower ones. Then situation when a cluster moves after an other cluster and the interaction between clusters occurs in accordance with (14) arises only when the slower cluster is ahead more quickly one.

Density $y_i^0 \geq 0$, $1 \leq i \leq n$ is defined on the segment $[x_i^0,\ x_{i+1}^0]$. The function $v = f(y)$ is defined that can be interpreted as the dependence of velocity on the density. The initial configuration of points $x_1^0 \dots, x_n^0$ is defined.

If the density is not equal to zero on the segment $[x_i^0,\ x_{i+1}^0]$, then a rectangle corresponds to this segment, which is the support of the rectangle. The height of the rectangle is equal to y_i. This segment can be interpreted as a section on that the traffic flow is located with density y_i. Let the rectangle that corresponds to this segment be called a cluster, and the area of this rectangle be called the mass of the cluster.

Clusters that follow one after the other form clusters batches.

The number of groups can be reduced by merging clusters, which occurs because a faster cluster overtakes a slower group.

If the point x_i is the boundary of two clusters such that the greater density corresponds to the cluster moving ahead, then this boundary moves with the velocity that is determined by (14).

Consider some clusters batch, which is located on the segment $[x_1, x_k+1]$. Points $x_2 < \cdots < x_k$ are the boundaries of clusters that are contained in the batch. Let m_i, $i = 1, \dots, k$, be mass of the cluster that be located on the segment $[x_i,\ x_{i+1}]$. Denote by $m = m_1 + \cdots + m_k$ mass of the cluster batch.

Velocity of the point x_1 is determined by the equation

$$\dot{x}_1 = f(y_1).$$

The point x_n moves with velocity

$$\dot{x}_n = f(y_n).$$

Let $x_1 = x_{10}, \dots, x_n = x_{n0}$ be the distribution of the points on the straight line at the initial time $t = 0$. The point x_i moves with the velocity that is determined by (14). The considered phase ends at the time when some points merge. After this a similar phase begins with a less number of segments, if there exists yet more than one segment. If a single segment remains, then its edges move with the same velocity. It follows from proved below Theorem that the situation when a single segment remains is realized in a finite time and the flow mass does not change in time.

Denote

$$g_i = \frac{y_{i+1}f(y_{i+1}) - y_i f(y_i)}{y_{i+1} - y_i} - \frac{y_i f(y_i) - y_i f(y_{i-1})}{y_i - y_{i-1}},\ i = 1, \dots, k.$$

Theorem 5. *The following statements are true.*

(1) The clusters batch mass remains constant in time, if this batch does not merge with any other batch cluster.

(2) Suppose it is determined the initial distribution of the boundaries of the segments that are located within the group

$$x_1 = x_{10}, \dots, x_k = x_{k+1,0}.$$

Let i^* be the value of i at that the maximum of $g_i/(x_{i+1} - x_i)$ is attained. Then the number of clusters that are contained in the batch decreases after the period of duration $g_i/(x_{i+1} - x_i)$, when the points x_{i^*+1} and x_{i^*} merge.

(3) An only cluster remains after a finite period.

Proof. The first statement of Theorem 4 is proved similarly to Theorem 1. Let us prove the second statement. Since $f(x_0) < f(x_n)$, we have $\dot{x_0} - \dot{x_n} < 0$ and, therefore, length of the segment (x_i, x_{i+1}) decreases over time at least for one value of i. One of these value is the value i^*. The velocity of the segment (x_{i^*}, x_{i^*+1}) decreasing is constant and is equal to g_{i^*}. After a time interval of duration $\frac{x_{i^*+1} - x_{i^*}}{g_{i^*}}$, the points x_{i^*+1} and x_{i^*} merge and the phase for that there are n segments ends. The second statement of Theorem 4 has been proved. □

The total number of clusters cannot increase over time. In a finite time this number decreases. The total mass of clusters cannot change either when cluster merge or between merger time. Thus the last statement of Theorem 4 is also true.

12. Interaction of clusters with uniformly distributed information

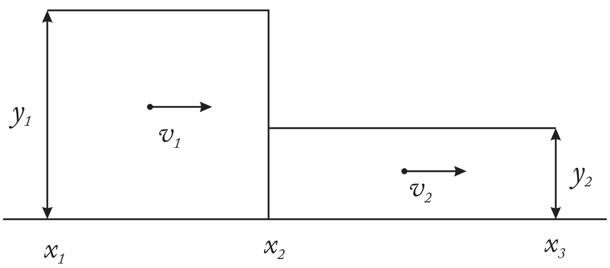

Figure 13. Interaction of two clusters with uniformly distributed information

Let us consider a model of interaction of two clusters that differs from the model of Section 4 in that the cluster height varies over time so that the area of the rectangle that corresponds to this cluster remains constant, Fig. 13. In physical terms it can be interpreted to mean that the next cluster adjusts to the leader, simultaneously changing its speed limits and keeping the same number of particles. Hence the information about the need to change speed limits delivers instantly to all the particles. Consider behavior of the cluster that is located on the segment $[x_i^0, x_{i+1}^0]$ (the i-th cluster), $i = 1, \dots, n$. The model is based on the fact that within a short time the density y_i changes in such a way as to compensate for the difference in velocity at the boundaries of the cluster.

We have up to an infinitesimal

$$x_i(t+\Delta t) = x_i(t) + \Delta x_i(t) \cong x_i(t) + v_i \Delta t,$$

$$x_{i+1}(t+\Delta t) = x_{i+1}(t) + \Delta x_{i+1}(t) \cong x_{i+1}(t) + v_{i+1} \Delta t.$$

From the conservation law it follows

$$(x_{i+1} - x_i) y_i = (x_{i+1} + \Delta x_{i+1} - x_i - \Delta x_i)(y_i + \Delta y_i) \cong$$

$$\cong (x_{i+1} + v_{i+1} t - x_i - v_i t)(y_i + \Delta y_i).$$

Hence,

$$0 = (x_{i+1} - x_i)\Delta y_i + (v_{i+1}\Delta t - v_i \Delta t) y_i$$

and

$$(x_{i+1} - x_i)\dot{y_i} + (v_{i+1} - v_i) y_i = 0.$$

Suppose

$$\dot{x_i} = v_i = f(y_i), \ i = 1, \dots, n.$$

Thus we have the system

$$\begin{cases} \dot{x_i} = f(y_i), \\ \dot{y_i} = y_i \frac{v_i - v_{i+1}}{x_{i+1} - x_i} = y_i \frac{f(y_i) - f(y_{i+1})}{x_{i+1} - x_i}, \ i = 1, \dots, n; \ x_{n+1} = 1 + x_1. \end{cases} \tag{15}$$

13. Qualitative properties of the flow with a uniformly distributed information

13.1. The behavior of solutions of the system on a circle in the case of two components. Consider the case $n = 2$. Suppose $y_1 < y_2$. Then we have for the solutions of system (15)

$$\dot{x}_1 = v_1 = f(y_1) > \dot{x}_2 = v_2 = f(y_2). \tag{16}$$

Theorem 6. *The following cases are possible, depending on the type of the function $f(y)$ and initial values.*

(1) Length of the segment $[x_1, x_2]$ becomes equal to zero at some time and the flow density becomes the same on all the circle;

(2) The value of becomes y_1 equal to y_2 at some time and, therefore, the flow density becomes the same on all the circle;

(3) The velocity of change of the segment $[x_1, x_2]$ length, which is equal to $[\dot{x_2} - \dot{x_1}]$, and the values $\dot{y_1}$ and $\dot{y_2}$ tends to zero as $t \to \infty$, although the length of this interval is non-zero, and the difference between $y_2 - y_1$ is positive. This case is possible only if the function $f(y)$ is defined appropriately.

Proof. From (16), it follows that the length of the segment $[x_2, x_1]$ increases by reducing the length of the segment $[x_1, x_2]$.

Hence the length of the segment $[x_2, x_1]$ increases for this solution, if the length of the segment $[x_1, x_2]$ decreases. Therefore the value of $\dot{y}_1$ is positive, and the value of $\dot{y}_2$ is negative. From this, the statements of Theorem 5 follow. $\square$

13.2. **The behavior of solutions for periodic distributed density.** Suppose that at initial time the considered circle is divided into n segments, which have the same length. Assume that the flow density on the segment $[x_i, x_{i+1}]$ is equal to h_1 for an odd i and this density is equal to $h_2 > h_1$ for an even i, $i = 1, \dots, n$. Then the solutions of system (15) are such that the length of the segment $[x_i, x_{i+1}]$ decreases for an odd i and this length increases for an even i. The flow density can become the same on the whole circle either because the length of the segments will decrease to zero, either because the flow densities become the same.

13.3. **The behavior of solutions in the common case.** Suppose $f(y)$ is strictly decreasing function on y. Then the derivative of the component y_i cannot become equal to 0 for $y_i \neq y_{i+1}$, $i, j = 1, \dots, n$, and, therefore, system (15) has no stationary points, for which the values of densities are different for any of the clusters.

Let $y_1, \dots, y_n$ correspond to a solution of system (15). Then the density y_i increases over time, if $y_i < y_{i+1}$, and this density decreases over time, if $y_i > y_{i+1}$, $i = 1, \dots, n$. The rectangle density that corresponds to the i-th cluster has to be conserved and the difference $x_{i+1} - x_i$, i.e., length of support of the i-th cluster decreases for $y_i < y_{i+1}$ and increases for $y_i > y_{i+1}$, $i = 1, \dots, n$.

The number of clusters decreases when the densities of the neighboring clusters become the same.

13.4. **The behavior of solutions on the circle in the case of two components.**

Theorem 7. *Suppose $n = 2$ and $y_1 < y_2$. The following cases are possible.*

(1) The length of the segment $[x_1, x_2]$ becomes equal to zero and the flow density becomes the same on the whole circle;

(2) At some time time the value y_1 becomes equal to the value y_2 and, therefore, the flow density becomes the same on the whole circle;

(3) The velocity of change of the segment length $[x_1, x_2]$, which is equal to $[\dot{x}_2 - \dot{x}_1]$, and the values $\dot{y}_1$ and $\dot{y}_2$ tend to zero as $t \to \infty$ although the length of this segment remains non-zero and the difference $y_2 - y_1$ remains positive. This case is possible only for the function $f(y)$ that is defined appropriately.

Proof. We have for the solutions of system (15)

$$\dot{x}_1 = v_1 = f(y_1) > \dot{x}_2 = v_2 = f(y_2).$$

Consequently, for this solution the length of the interval $[x_2, x_1]$ increases, if the length of the segment $[x_1, x_2]$ decreases. The value of $\dot{y}_1$ is positive and value of $\dot{y}_2$ is negative. From this, the statement of Theorem 6 follows. □

14. Qualitative properties of the flow with a uniformly distributed information. Clusters-sanguine

Let us consider a partial-connected model. In this model change the cluster height in accordance with (15) occurs only when the cluster of lower density (the fast cluster) follows the cluster of higher density (slow cluster). In this case the behavior of this cluster is similar to the behavior of cluster in the model described in Section 12.

If the slow cluster follows the faster cluster, then the fast cluster moves forward and its density does not change.

For a finite amount of time a group of clusters is formed in that fast clusters follow slow clusters. The subsequent behavior of the chain is carried out as in the model described in Section 12.

15. Conclusion

The mathematical model of the traffic flow, in which the highway is divided into segments with the flow density that is constant on each segment. We have derived systems of nonlinear ordinary differential equations according to that a change in the boundaries of these segments and their corresponding densities occur. We study the properties of the solutions of these systems.

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L_p- SATURATION THEOREM FOR AN ITERATIVE COMBINATION OF BERNSTEIN-DURRMEYER TYPE POLYNOMIALS

P. N. AGRAWAL, T. A. K. SINHA, AND K. K. SINGH

Abstract. Gupta and Maheshwari [5] introduced a new sequence of Durrmeyer type linear positive operators P_n to approximate p-th Lebesgue integrable functions on $[0, 1]$. It is observed that these operators are saturated with $O(n^{-1})$. In order to improve the rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $P_n(f;t)$. This technique was given by Micchelli [8] who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$.

In our paper [1] we obtained direct theorems in ordinary approximation in the L_p- norm by the operators $T_{n,k}$. Subsequently, we [10] proved a corresponding local inverse theorem over contracting intervals. The object of the present paper is to continue this work by proving the saturation theorem in a local set-up.

1. Introduction

For $f \in L_p[0, 1], 1 \le p < \infty$, the operators P_n can be expressed as

$$P_n(f;t) = \int_0^1 W_n(t,u)f(u)\,du,$$

where $$W_n(t,u) = n\sum_{\nu=1}^{n} p_{n,\nu}(t)p_{n-1,\nu-1}(u) + (1-t)^n\delta(u),$$

$$p_{n,\nu}(t) = \binom{n}{\nu} t^{\nu}(1-t)^{n-\nu},\ \ 0 \le t \le 1,$$

and $\delta(u)$ being the Dirac-delta function, is the kernel of the operators P_n.

For $f \in L_p[0, 1], 1 \leqslant p < \infty$, the iterative combination $T_{n,k}$ of the operators P_n is defined as

$$T_{n,k}(f;t) = \left(I - (I - P_n)^k\right)(f;t) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}P_n^r(f;t),\ \ k \in \mathbb{N},$$

where $P_n^0 \equiv I$ and $P_n^r \equiv P_n(P_n^{r-1})$ for $r \in \mathbb{N}$.

In what follows, we suppose that $0 < a < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < b < 1$. Also, $AC[a, b]$ and $BV[a, b]$ denote the classes of absolutely continuous functions and the functions of the bounded variation respectively in the interval $[a, b]$. Further, C denotes a constant not necessarily the same at each occurrence.

Key words and phrases. Linear positive operators, Bernstein-Durrmeyer type polynomials, iterative combination, inverse theorem, saturation theorem, Steklov mean.

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The aim of this paper is to establish a local saturation theorem for the operators $T_{n,k}(f,t)$ in the L_p-norm. The theorem shows that the sequence $T_{n,k}(.;t)$ is saturated with the order $O(n^{-k})$. The nature of saturation class depends on whether $p=1$ or $p>1$. The trivial class, however, remains the same for all p $(1 \le p < \infty)$.

We prove the following theorem (*saturation theorem*):

Theorem 1.1. *Let $f \in L_p[0,1], 1 \le p < \infty$. Then, in the following statements, the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and* (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) *hold:*

(1) *[(i)]*
(2) $\|T_{n,k}(f,.) - f\|_{L_p[a_1,b_1])} = O(n^{-k})$*;*
(3) *f coincides almost everywhere with a function F on $[a_2,b_2]$ having $2k$ derivatives such that:*
   (a) *when $p>1$, $F^{(2k-1)} \in AC[a_2,b_2]$ and $F^{(2k)} \in L_p[a_2,b_2]$,*
   (b) *when $p=1$, $F^{(2k-2)} \in AC[a_2,b_2]$ and $F^{(2k-1)} \in BV[a_2,b_2]$;*
(4) $\|T_{n,k}(f,.) - f\|_{L_p[a_3,b_3]} = O(n^{-k})$;
(5) $\|T_{n,k}(f,.) - f\|_{L_p[a_1,b_1]} = o(n^{-k})$;
(6) *f coincides almost everywhere with a function F on $[a_2,b_2]$, where F is $2k$ times continuously differentiable on $[a_2,b_2]$ and satisfies* $\sum_{\nu=1}^{2k} Q(\nu,k,t)F^{(\nu)}(t) = 0$, *where $Q(\nu,k,t)$ are the polynomials occurring in Theorem 2.8;*
(7) $\|T_{n,k}(f,.) - f\|_{L_p[a_3,b_3]} = o(n^{-k})$,
where $O(n^{-(k+1)})$ and $o(n^{-(k+1)})$ terms are with respect to n when $n \to \infty$.

Remark 1.1. *To prove the saturation theorem, we observe that without any loss of generality we may assume that $f(0)=0$. To prove this, let $f_1(u) = f(u) - f(0)$. By definition, $T_{n,k}(f_1,t) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}P_n^r(f_1;t)$. Further, using linearity,*
$P_n^r(f_1;t) = P_n^r(f;t) - f(0)P_n^r(1;t) = P_n^r(f;t) - f(0)$.
Since $T_{n,k}(f_1,t) = T_{n,k}(f,t) - f(0)$, it follows that
$T_{n,k}(f_1,t) - f_1(t) = T_{n,k}(f,t) - f(0) - (f(t) - f(0)) = T_{n,k}(f,t) - f(t)$, where $f_1(0) = 0$.

Since $f(0) = 0$ (in view of the above remark), it follows that $P_n f(0) = 0$. Consequently, $P_n^m f(0) = 0, \forall m \in \mathbb{N}$.

2. Preliminaries

In this section, we give some definitions and auxiliary results which are useful in establishing our main theorem.

Lemma 2.1. [10] *Let $r > 0$ and $V_n(x,t) =: n\sum_{\nu=1}^{n} p_{n,\nu}(x)p_{n-1,\nu-1}(t)$, then, for sufficiently large n*

$$\int_0^1 V_n(x,t)|x-t|^r\,dx = O(n^{-r/2}),$$

uniformly for all t in $[0,1]$.

For $m \in \mathbb{N}^0$ (the set of non-negative integers), the mth order moment for the operators P_n is defined as

$$\mu_{n,m}(t) = P_n\left((u-t)^m;t\right).$$

Lemma 2.2. [10]*For the function* $\mu_{n,m}(x)$*, we have* $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = \frac{(-x)}{(n+1)}$*, and for* $m \geq 1$ *there holds the recurrence relation*

$$(n+m+1)\mu_{n,m+1}(x) = x(1-x)\left\{\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)\right\} + (m(1-2x)-x)\mu_{n,m}(x).$$

Consequently,

(i) $\mu_{n,m}(x)$ *is a polynomial in* x *of degree* m*;*

(ii) *for every* $x \in [0,1], \mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$*, where* $[\beta]$ *is the integer part of* β*.*

Corollary 2.3. *For each* $r > 0$ *and for every* $x \in [0,1]$*, we have*

$$P_n(|t-x|^r, x) = O\left(n^{-r/2}\right), \text{ as } n \to \infty.$$

The mth order moment for the operator P_n^r is defined as $\mu^{[r]}_{n,m}(t) = P_n^r\left((u-t)^m; t\right)$, $r \in \mathbb{N}$. We denote $\mu^{[1]}_{n,m}(t)$ by $\mu_{n,m}(t)$.

Lemma 2.4. [2] *For* $r \in \mathbb{N}, m \in \mathbb{N}^0$ *and* $t \in [0,1]$ *we have*

$$\mu^{[r]}_{n,m}(t) = \mathcal{O}\left(n^{-[(m+1)/2]}\right).$$

Consequently, by Cauchy-Schwarz inequality, for every $t \in [0,1]$ one has

$$P_n^r\left(|u-t|^m; t\right) = O(n^{-m/2}).$$

Lemma 2.5. [2] *For* $k, l \in \mathbb{N}$ *and every* $t \in [0,1]$ *there holds*

$$T_{n,k}((u-t)^l; t) = \mathcal{O}(n^{-k}).$$

The next lemma gives a bound for the intermediate derivatives of f in terms of the highest order derivative and the function in L_p-norm.

Lemma 2.6. [4] *Let* $1 \leqslant p < \infty, f \in L_p[a,b]$*. Suppose* $f^{(k)} \in AC[a,b]$ *and* $f^{(k+1)} \in L_p[a,b]$*. Then*

$$\left\|f^{(j)}\right\|_{L_p[a,b]} \leqslant M_j\left(\left\|f^{(k+1)}\right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]}\right),\ j = 1,2,...,k,$$

where M_j *are certain constants independent of* f*.*

Let $f \in L_p[a,b]$, $1 \leqslant p < \infty$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\cdots\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\left(f(t) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i} f(t)\right)\prod_{i=1}^m dt_i, t \in [a_1, b_1],$$

where Δ_h^m is the mth order forward difference operator with step length h.

Lemma 2.7. *For the function* $f_{\eta,m}$*, we have*

(1) *[(a)]*
(2) $f_{\eta,m}$ *has derivatives up to order* m *over* $[a_1, b_1]$*;*
(3) $\|f^{(r)}_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_r\eta^{-r}\ \omega_r(f, \eta, [a,b]), r = 1,2,...,m;$
(4) $\|f - f_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+1}\ \omega_m(f, \eta, [a,b])$*;*
(5) $\|f_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+2}\ \|f\|_{L_p[a,b]}$*;*
(6) $\|f^{(m)}_{\eta,m}\|_{L_p[a_1,b_1]} \leqslant C_{m+3}\ \eta^{-m}\|f\|_{L_p[a,b]}$*,*

where $C_i's$ are certain constants that depend on i but are independent of f and η.

Following ([6], Theorem 18.17) or ([11], pp.163-165), the proof of the above lemma easily follows hence the details are omitted.

Theorem 2.8. [3] *Let $f \in L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$. If $f^{(2k)}$ exists at a point $t \in [0,1]$, then*

$$T_{n,k}(f;t) - f(t) = n^{-k} \sum_{\nu=1}^{2k} \frac{f^{(\nu)}(t)}{\nu!} Q(\nu,k,t) + o(n^{-k}), \text{ as } n \to \infty \tag{2.1}$$

and

$$[T_{n,k+1}(f;t) - f(t)] = o(n^{-k}), \text{ as } n \to \infty, \tag{2.2}$$

where $Q(\nu,k,t)$ are certain polynomials in t of degree ν. Further, the limits in (2.1) and (2.2) hold uniformly in $[0,1]$ if $f^{(2k)}(t)$ is continuous in $[0,1]$.

Theorem 2.9. *(Inverse theorem)* [10] *Let $f \in L_p[0,1], 1 \le p < \infty$, $0 < \alpha < 2k$ and $\|T_{n,k}(f,.) - f\|_{L_p[a_1,b_1]} = O(n^{-\alpha/2})$, as $n \to \infty$. Then, $\omega_{2k}(f,\tau,p,[a_2,b_2]) = O(\tau^\alpha)$, as $\tau \to 0$.*

Lemma 2.10. [9] *Let $1 \leqslant p < \infty$, $f \in L_p[a,b]$ and there holds*

$$\omega_m(f,\tau,p,[a,b]) = O(\tau^{r+\alpha}), (\tau \to 0),$$

where $m, r \in \mathbb{N}$ and $0 < \alpha < 1$. Then f coincides a.e. on $[c,d] \subset (a,b)$ with a function F possessing an absolutely continuous derivative $F^{(r-1)}$, the rth derivative $F^{(r)} \in L_p[c,d]$, and there holds $\omega(F^{(r)},\tau,p,[c,d]) = O(\tau^\alpha), (\tau \to 0)$.

Lemma 2.11. *Let $f \in L_p[0,1], 1 \le p < \infty$ and $\|T_{n,k}(f,.) - f\|_{L_p[a_1,b_1]} = O(n^{-k})$. Then for any function $g \in C_0^{2k}$ with $supp\, g \subset (a_1,b_1)$ there holds*

$$|\langle T_{n,k}(f,t) - f(t),\, g(t)\rangle| \leqslant \frac{C}{n^k}\left(\|f\|_{L_p[0,1]} + \|f^{(2k-1)}\|_{L_p[0,1]}\right),$$

where $\langle f,g\rangle = \int\limits_0^1 f(t)g(t)\,dt$.

Proof. By definition

$$\begin{aligned}
\left\langle M_n^r(f,t),\, g(t)\right\rangle &= \int_0^1 M_n^r(f,t)g(t)\,dt \\
&= \int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)f(u_r)\times \\
&\quad \left\{\sum_{i=0}^{2k-1}(t-u_r)^i g^{(i)}(u_r)+\frac{(t-u_r)^{(2k)}}{(2k)!}g^{(2k)}(\xi)\right\}du_r...du_1dt \\
&= \int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)f(u_r)g(u_r)du_r...du_1dt \\
&+ \int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)(t-u_r)h_1(u_r)du_r...du_1dt \\
&+ \int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)\frac{(t-u_r)^2}{2!}h_2(u_r)du_r...du_1dt \\
&+ ... \\
&+ \int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)\frac{(t-u_r)^{2k}}{(2k)!}f(u_r)g^{(2k)}(\xi)du_r...du_1dt \\
&= I_{0,r}+I_{1,r}+I_{2,r}+...+I_{2k,r}, \text{say,}
\end{aligned}$$

where $h_i(u)=f(u)g^{(i)}(u), i=1,2,...,2k-1$ and ξ lies between t and u_r.

Now,

$$\begin{aligned}
\left\langle T_{n,k}(f,t),\, g(t)\right\rangle &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left\langle M_n^r(f,t),\, g(t)\right\rangle \\
&= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}(I_{0,r}+I_{1,r}+I_{2,r}+...+I_{2k,r}). \qquad (2.3)
\end{aligned}$$

Since supp $g\subset(a_1,b_1)$, there follows

$$\int_0^1 W_n(t,u_1)\,dt = n\sum_{k=1}^{n}p_{n-1,k-1}(u)\int_0^1 p_{n,k}(t)\,dt=\frac{n}{n+1}. \qquad (2.4)$$

Using (2.4) and on interchanging integrals by Fubini's theorem, we have

$$
\begin{aligned}
I_{0,r} &= \int_0^1 W_n(u_{r-1},u_r)...\int_0^1 W_n(u_1,u_2)\left(\int_0^1 W_n(t,u_1)\,dt\right) f(u_r)g(u_r)du_1...du_r \\
&= \left(\frac{n}{n+1}\right)^r \left\{\int_0^1 f(u_r)g(u_r)\,du_r\right\} \\
&= \left\{1-\frac{r}{n}+\frac{r(r+1)}{2!n^2}+...\right\}\left(\int_0^1 f(t)g(t)\,dt\right). \qquad (2.5)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}I_{0,r} &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left\{1-\frac{r}{n}+\frac{r(r+1)}{2!n^2}+...\right\}\left(\int_0^1 f(t)g(t)\,dt\right) \\
&= \int_0^1 f(t)g(t)\,dt+0+0+...+O(n^{-k})\left(\int_0^1 f(t)g(t)\,dt\right) \\
&= \int_0^1 f(t)g(t)\,dt+O(n^{-k}).\|f\|_{L_p[0,1]}, \qquad (2.6)
\end{aligned}
$$

in view of the identities

$$
\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}r^m = \begin{cases} 0, & m=1,2,...,k-1 \\ (-1)^{k+1}(k!), & m=k. \end{cases}
$$

Next, in view of the hypothesis, inverse theorem 2.9 and Lemma 2.10, we have

$$
\begin{aligned}
I_{1,r} &= \int_0^1\int_0^1...\int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)(t-u_r)\times \\
&\quad \left\{h_1(t)+(u_r-t)h_1^{(1)}(t)+\frac{(u_r-t)^2}{2!}h_1^{(2)}(t)+...+\frac{(u_r-t)^{2k-2}}{(2k-2)!}h_1^{(2k-2)}(t)\right. \\
&+ \left.\frac{1}{(2k-2)!}\int_t^{u_r}(u_r-w)^{2k-2}h_1^{(2k-1)}(w)\,dw\right\}du_r...du_1dt \\
&= -\left(\int_0^1 h_1(t)\mu_{n,1}^{[r]}(t)dt+\frac{1}{2!}\int_0^1 h_1^{(1)}(t)\mu_{n,2}^{[r]}(t)dt+...+\frac{1}{(2k-2)!}\int_0^1 h_1^{(2k-2)}(t)\mu_{n,2k-1}^{[r]}(t)dt\right) \\
&+ \frac{1}{(2k-2)!}\int_0^1\int_0^1...\int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r)(t-u_r)\times \\
&\quad \left(\int_t^{u_r}(u_r-w)^{2k-2}h_1^{(2k-1)}(w)\,dw\right)du_r...du_1dt.
\end{aligned}
$$

Let

$$\Sigma_r := \frac{1}{(2k-2)!}\int_0^1\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-1},u_r) \times (t-u_r)\left(\int_t^{u_r}(u_r-w)^{2k-2}h_1^{(2k-1)}(w)\,dw\right) du_r...du_1dt.$$

Now, using Lemma 2.5 we get

$$\begin{aligned}\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}I_{1,r} &= -\Big(\int_0^1 h_1(t)T_{n,k}(u-t);t)dt \\ &+ \frac{1}{2!}\int_0^1 h_1^{(1)}(t)T_{n,k}(u-t)^2;t)dt + ...\Big) \\ &+ \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\{\Sigma_r\} \\ &= O(n^{-k})\left(\|h_1\|_{L_p[0,1]} + \|h_1^{(1)}\|_{L_p[0,1]} + ... + \|h_1^{(2k-1)}\|_{L_p[0,1]}\right) \\ &+ \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\{\Sigma_r\}. \end{aligned} \tag{2.7}$$

In order to estimate Σ_r, we break the interval of integration in u_r as follows: For each n there exists a non-negative integer $m(n)$ such that

$$\frac{m}{\sqrt{n}} \le \max\{b_1-a_2, b_2-a_1\} \le \frac{m+1}{\sqrt{n}}.$$

$$\Sigma_r \le \frac{1}{(2k-2)!}\int_0^1 ... \int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-2},u_{r-1})\times \left\{\int_0^1 W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\left|\int_t^{u_r}|h_1^{(2k-1)}(w)|dw\right| du_r\right\} du_{r-1}...du_1dt. \tag{2.8}$$

The expression inside the curly bracket in (2.8) is bounded by

$$\int_0^1 W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\left|\int_t^{u_r}|h_1^{(2k-1)}(w)|dw\right|du_r$$

$$\leq \sum_{l=0}^{m}\Bigg\{\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\int_t^{t+\frac{l+1}{\sqrt{n}}}|h_1^{(2k-1)}(w)|dw\,du_r$$

$$+\int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\int_{t-\frac{l+1}{\sqrt{n}}}^{t}|h_1^{(2k-1)}(w)|dw\,du_r\Bigg\}$$

$$\leq \sum_{l=1}^{m}\Bigg\{\frac{n^2}{l^4}\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k+3}\times \tag{2.9}$$

$$\int_t^{t+\frac{l+1}{\sqrt{n}}}|h_1^{(2k-1)}(w)|dw\,du_r$$

$$+\frac{n^2}{l^4}\int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k+3}\int_{t-\frac{l+1}{\sqrt{n}}}^{t}|h_1^{(2k-1)}(w)|dw\,du_r\Bigg\}$$

$$+\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}}|h_1^{(2k-1)}(w)|dw\,du_r.$$

On combining (2.8) and (2.9), we get

$$\Sigma_r \leq \frac{1}{(2k-2)!}\int_0^1 \ldots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\ldots W_n(u_{r-2},u_{r-1})\times$$

$$\Bigg\{\sum_{l=1}^{m}\Bigg(\frac{n^2}{l^4}\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k+3}\int_t^{t+\frac{l+1}{\sqrt{n}}}|h_1^{(2k-1)}(w)|dw\,du_r$$

$$+\frac{n^2}{l^4}\int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k+3}\int_{t-\frac{l+1}{\sqrt{n}}}^{t}|h_1^{(2k-1)}(w)|dw\,du_r\Bigg)$$

$$+\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}}|h_1^{(2k-1)}(w)|dw\,du_r\Bigg\}du_{r-1}\ldots du_1 dt.$$

$$= J_1 + J_2 + J_3, \text{ say.}$$

Now,

$$J_1 = \frac{1}{(2k-2)!} \int_0^1 \cdots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\ldots W_n(u_{r-2},u_{r-1}) \times$$
$$\left(\sum_{l=1}^{m} \frac{n^2}{l^4} \int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r - t|^{2k+3} \int_t^{t+\frac{l+1}{\sqrt{n}}} |h_1^{(2k-1)}(w)|dw\, du_r \right) du_{r-1}\ldots du_1 dt$$
$$\leq \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} \int_0^1 \cdots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\ldots W_n(u_{r-2},u_{r-1}) \times$$
$$\left(\int_0^1 \int_0^1 W_n(u_{r-1},u_r)|u_r - t|^{2k+3} \phi(w)|h_1^{(2k-1)}(w)|dw\, du_r \right) du_{r-1}\ldots du_1 dt$$
$$= \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} \int_0^1 \int_0^1 \Big\{ \int_0^1 \cdots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\ldots W_n(u_{r-1},u_r) \times$$
$$|u_r - t|^{2k+3}\, du_r du_{r-1}\ldots du_1 \Big\} \phi(w)|h_1^{(2k-1)}(w)|dw\, dt$$
$$= \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} \int_0^1 \int_0^1 P_n^r\left(|u_r - t|^{2k+3}; t\right) \phi(w)|h_1^{(2k-1)}(w)|dw\, dt,$$

where $\phi(w)$ denotes the characteristic function of the interval $[t, t+\frac{l+1}{\sqrt{n}}]$.

In view of Lemma 2.4 and interchanging integration in t and w by Fubini's theorem, we obtain

$$J_1 = \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} .O\left(\frac{1}{n^{(2k+3)/2}}\right) \int_0^1 \left(\int_0^1 \psi(w)dt \right) |h_1^{(2k-1)}(w)|dw$$
$$= \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} .O\left(\frac{1}{n^{(2k+3)/2}}\right) \int_0^1 \left(\int_{w-\frac{l+1}{\sqrt{n}}}^{w} \psi(w)dt \right) |h_1^{(2k-1)}(w)|dw$$
$$= \frac{1}{(2k-2)!} \sum_{l=1}^{m} \frac{n^2}{l^4} \left(\frac{l+1}{\sqrt{n}}\right) .O\left(\frac{1}{n^{(2k+3)/2}}\right) \int_0^1 |h_1^{(2k-1)}(w)|dw$$
$$= \left(\sum_{l=1}^{m} \frac{(l+1)}{l^4} \right) .O\left(\frac{1}{n^k}\right) \|h_1^{(2k-1)}\|_{L_p[0,1]}$$
$$= O(n^{-k}).\|h_1^{(2k-1)}\|_{L_p[0,1]}.$$

Treating J_2 in similar manner, we get $J_2 = O(n^{-k}).\|h_1^{(2k-1)}\|_{L_p[0,1]}$.

$$J_3 = \frac{1}{(2k-2)!}\int_0^1 \dots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\dots W_n(u_{r-2},u_{r-1})\times$$
$$\left\{\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}} W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\int_{t-\frac{1}{\sqrt{n}}}^{t+\frac{1}{\sqrt{n}}} |h_1^{(2k-1)}(w)|dw\,du_r\right\} du_{r-1}\dots du_1 dt$$
$$\leq \frac{1}{(2k-2)!}\int_0^1 \dots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\dots W_n(u_{r-2},u_{r-1})\times$$
$$\left\{\int_0^1\int_0^1 W_n(u_{r-1},u_r)|u_r-t|^{2k-1}\psi(w)\,|h_1^{(2k-1)}(w)|dw\,du_r\right\} du_{r-1}\dots du_1 dt$$
$$= \frac{1}{(2k-2)!}\int_0^1\int_0^1\left\{\int_0^1 \dots \int_0^1 W_n(t,u_1)W_n(u_1,u_2)\dots W_n(u_{r-1},u_r)\times\right.$$
$$\left.|u_r-t|^{2k-1}\,du_r du_{r-1}\dots du_1\right\}\psi(w)|h_1^{(2k-1)}(w)|dw\,dt$$
$$= \frac{1}{(2k-2)!}\int_0^1\int_0^1 P_n^r\left(|u_r-t|^{2k-1};t\right)\psi(w)|h_1^{(2k-1)}(w)|dw\,dt,$$

where $\psi(w)$ denotes the characteristic function of the interval $[t-\frac{1}{\sqrt{n}}, t+\frac{1}{\sqrt{n}}]$.

In view of Lemma 2.4 and interchanging integration in t and w by Fubini's theorem, we obtain

$$J_3 = \frac{1}{(2k-2)!}.O\left(\frac{1}{n^{(2k-1)/2}}\right).\int_0^1\left(\int_0^1 \psi(w)dt\right)|h_1^{(2k-1)}(w)|dw$$
$$= \frac{1}{(2k-2)!}.O\left(\frac{1}{n^{(2k-1)/2}}\right).\int_0^1\left(\int_{w-\frac{1}{\sqrt{n}}}^{w+\frac{1}{\sqrt{n}}} \psi(w)dt\right)|h_1^{(2k-1)}(w)|dw$$
$$= \frac{1}{(2k-2)!}\left(\frac{2}{\sqrt{n}}\right).O\left(\frac{1}{n^{(2k-1)/2}}\right)\int_0^1 |h_1^{(2k-1)}(w)|dw$$
$$= O(n^{-k}).\|h_1^{(2k-1)}\|_{L_p[0,1]}.$$

Hence, combining the estimates of $J_1 - J_3$, it follows that

$$|\Sigma_r| = O(n^{-k})\|h_1^{(2k-1)}\|_{L_p[0,1]}. \tag{2.10}$$

From (2.7) and (2.10), we have

$$\begin{aligned}\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}I_{1,r} &= O(n^{-k}).\left(\sum_{i=0}^{2k-1}\|h_1^{(i)}\|_{L_p[0,1]}\right)\\ &= O(n^{-k}).\left(\|f\|_{L_p[0,1]}+\|f^{(2k-1)}\|_{L_p[0,1]}\right),\end{aligned} \tag{2.11}$$

in view of Lemma 2.6.

Proceeding similarly, we can show that

$$\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}I_{j,r} = O(n^{-k}).\left(\|f\|_{L_p[0,1]}+\|f^{(2k-1)}\|_{L_p[0,1]}\right), \tag{2.12}$$

$j = 2, 3, ..., 2k-1$.

In order to estimate $I_{2k,r}$, we proceed as follows:

By multinomial theorem, we may write

$$(t-u_r)^{2k} = \sum_{m_1+m_2+...+m_r=2k}\binom{2k}{m_1, m_2, .., m_r}(t-u_1)^{m_1}(u_1-u_2)^{m_2}...(u_{r-1}-u_r)^{m_r}.$$

Hence, by Fubini's theorem

$$\begin{aligned}|I_{2k,r}| \leq{}& \frac{1}{(2k)!}\int_0^1\int_0^1...\int_0^1 W_n(t,u_1)W_n(u_1,u_2)...W_n(u_{r-2},u_{r-1})W_n(u_{r-1},u_r)\times\\ &\sum_{m_1+m_2+...+m_r=2k}\binom{2k}{m_1, m_2, .., m_r}|t-u_1|^{m_1}|u_1-u_2|^{m_2}...|u_{r-1}-u_r|^{m_r}\times\\ &|f(u_r)|\,|g^{(2k)}(\xi)|du_r\,du_{r-1}...du_1dt\\ \leq{}& \frac{\|g^{(2k)}\|}{(2k)!}\sum_{m_1+m_2+...+m_r=2k}\binom{2k}{m_1, m_2, .., m_r}\int_0^1...\int_0^1 W_n(u_1,u_2)...W_n(u_{r-2},u_{r-1})\times\\ &W_n(u_{r-1},u_r)\left(\int_0^1 W_n(t,u_1)|t-u_1|^{m_1}dt\right)|f(u_r)||u_1-u_2|^{m_2}...|u_{r-1}-u_r|^{m_r}\times\\ &du_1...du_{r-1}du_r.\end{aligned}$$

In view of the Remark 1.1 and Lemma 2.1, we have

$$\int_0^1 W_n(t,u_1)|t-u_1|^{m_1}\,dx = O(n^{-m_1/2}),$$

uniformly in $u_1 \in [a_1, b_1]$.

Next, we consider the integration in u_1. Again, applying Remark 1.1 and Lemma 2.1, we obtain

$$\int_0^1 W_n(u_1,u_2)|u_1-u_2|^{m_2}\,du_1 = O(n^{-m_2/2}),$$

uniformly in $u_2 \in [a_1, b_1]$.

Thus, with a repeated use of Remark 1.1 and Lemma 2.1 $r-$ times, we get

$$\begin{aligned}|I_{2k,r}| &\leq C\frac{\|g^{(2k)}\|}{(2k)!}\sum_{m_1+m_2+\ldots+m_r=2k}\binom{2k}{m_1,m_2,..,m_r}\frac{1}{n^{(m_1+m_2+\ldots+m_r)/2}}\times \\ &\quad \int_0^1 |f(u_r)|du_r \\ &\leq Cn^{-k}\|f\|_{L_1[0,1]}.\end{aligned}$$

Hence,

$$\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}I_{2k,r} = \|f\|_{L_p[0,1]}.\,O(n^{-k}). \tag{2.13}$$

From (2.3), (2.6) and (2.11)-(2.13), the required result follows.. □

3. Proof of Saturation Theorem

Proof. Assume (i). Then, it follows from Theorem 2.9 and Lemma 2.10 that for $a_1 < c < d < b_1, f$ coincides a.e. on $[c, d]$ with a function F possessing an absolutely continuous derivative $F^{(2k-2)}$ and a $(2k-1)$th derivative $F^{(2k-1)}$, which belongs to $L_p[c, d]$. Moreover, there holds for $0 < \beta < 1$

$$\omega\big(F^{(2k-1)}, \tau, p, [c, d]\big) = O(\tau^\beta), \quad (\tau \to 0).$$

We choose points $x_i, y_i, i = 1, 2$ such that $a_1 < x_1 < x_2 < a_2 < b_2 < y_2 < y_1 < b_1$. Let $q \in C_0^{2k}$ with $\operatorname{supp} q \subset (a_1, b_1)$ and $q(t) = 1$ for $t \in [x_1, y_1]$. Let us define a function $\mathcal{F}(u) = F(u)q(u)$, $u \in [0, 1]$. Then

$$\begin{aligned}\|T_{n,k}(\mathcal{F}, .) - \mathcal{F}\|_{L_p[x_2,y_2]} &\leqslant \|T_{n,k}(f, .) - f\|_{L_p[x_2,y_2]} \\ &+ \|T_{n,k}(\mathcal{F} - f, .)\|_{L_p[x_2,y_2]}.\end{aligned}$$

Since $\mathcal{F} = f$ on $[x_1, y_1]$, the contribution of the second term on the right hand side can be made arbitrarily small as $n \to \infty$. Hence, it follows that

$$\|T_{n,k}(\mathcal{F} - f, .)\|_{L_p[x_2,y_2]} = O(n^{-k}).$$

This alongwith the hypothesis that (i) holds, implies

$$\|T_{n,k}(\mathcal{F}, .) - \mathcal{F}\|_{L_p[x_2,y_2]} = O(n^{-k}).$$

Now, if $p > 1$, by Alaoglu's theorem there exists a function $H \in L_p[x_2, y_2]$, such that for some subsequence n_j and $g \in C_0^{2k}$ with $\operatorname{supp} g \subset (a_1, b_1)$, we have

$$\lim_{n_j\to\infty} n_j^k\big\langle T_{n_j,k}(\mathcal{F}, t) - \mathcal{F}(t),\, g(t)\big\rangle = \big\langle H(t),\, g(t)\big\rangle. \tag{3.1}$$

When $p = 1$, the functions ϕ_n defined by

$$\phi_n(u) = \int_{x_2}^{u} n^k\big\{T_{n,k}(\mathcal{F}, t) - \mathcal{F}(t)\big\}\,dt \tag{3.2}$$

are uniformly bounded and are of uniformly bounded variation. Making use of Alaoglu's theorem, it follows that there exists a function $\phi_0 \in BV[x_2, y_2]$ such that for some subsequence $\{n_j\}$ and for all $g \in C_0^{2k}$ with $\operatorname{supp} g \subset (x_2, y_2)$

$$\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t) - \phi_0(t)\right) \to 0, \quad (n_j \to \infty). \tag{3.3}$$

Now,

$$\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t) - \phi_0(t)\right) = \int_{x_2}^{y_2} g(t)d\phi_{n_j}(t) - \int_{x_2}^{y_2} g(t)d\phi_0(t).$$

From (3.2), Theorem 17.17 of [6] and the fact that $\operatorname{supp} g \subset (x_2, y_2)$, we get

$$\begin{aligned}\int_{x_2}^{y_2} g(t)d\left(\phi_{n_j}(t) - \phi_0(t)\right) &= n_j^k \int_{x_2}^{y_2} g(t)\left[T_{n_j,k}(\mathcal{F}, t) - \mathcal{F}(t)\right] dt \\ &+ \int_{x_2}^{y_2} g'(t)\phi_0(t)\, dt.\end{aligned}$$

This together with (3.3) implies that

$$\lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}; t) - \mathcal{F}(t),\, g\right\rangle = -\langle \phi_0(t),\, g'(t)\rangle. \tag{3.4}$$

As the Steklov means $\mathcal{F}_{\eta,2k}$ for $\mathcal{F}$ have continuous derivatives of order upto $2k$, using the property (c) of Lemma 2.7 for $i = 0, 1, ..., 2k-1$, there holds

$$\|\mathcal{F}_{\eta,2k}^{(i)} - \mathcal{F}^{(i)}\|_{L_p[a_1,b_1]} \to 0, \quad (\eta \to 0). \tag{3.5}$$

Now, by Theorem 2.8

$$T_{n_j,k}(\mathcal{F}_{\eta,2k}; t) - \mathcal{F}_{\eta,2k}(t) = \frac{1}{n_j^k}(P_{2k}D)\mathcal{F}_{\eta,2k}(t) + o\left(\frac{1}{n_j^k}\right), \tag{3.6}$$

where $P_{2k}D \equiv \sum_{i=1}^{2k} \frac{Q(i,k,t)}{i!} D^i$. Hence, if $P_{2k}^*(D)$ denotes the differential operator adjoint to $P_{2k}D$, by using (3.6), we have

$$\begin{aligned}\langle \mathcal{F}_{\eta,2k}(t), P_{2k}^*(D)g(t)\rangle &= \langle P_{2k}(D)\mathcal{F}_{\eta,2k}(t), g(t)\rangle \\ &= \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}_{\eta,2k}, t) - \mathcal{F}_{\eta,2k}(t),\, g(t)\right\rangle \\ &= \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}_{\eta,2k} - \mathcal{F}, t) - (\mathcal{F}_{\eta,2k}(t) - \mathcal{F}(t)),\, g(t)\right\rangle \\ &+ \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}, t) - \mathcal{F}(t),\, g(t)\right\rangle.\end{aligned}$$

i.e.

$$\begin{aligned}&\langle \mathcal{F}_{\eta,2k}(t), P_{2k}^*(D)g(t)\rangle - \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}, t) - \mathcal{F}(t),\, g(t)\right\rangle \\ &= \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}_{\eta,2k} - \mathcal{F}, t) - (\mathcal{F}_{\eta,2k}(t) - \mathcal{F}(t)),\, g(t)\right\rangle.\end{aligned}$$

Hence, by Lemma 2.2

$$\begin{aligned}&\langle \mathcal{F}_{\eta,2k}(t), P_{2k}^*(D)g(t)\rangle - \lim_{n_j \to \infty} n_j^k \left\langle T_{n_j,k}(\mathcal{F}, t) - \mathcal{F}(t),\, g(t)\right\rangle \\ &\leqslant C\, \|\mathcal{F}_{\eta,2k} - \mathcal{F}\|_{L_p[0,1]} + \|\mathcal{F}_{\eta,2k}^{(2k-1)} - \mathcal{F}^{(2k-1)}\|_{L_p[0,1]}.\end{aligned} \tag{3.7}$$

Taking limit as $\eta \to 0$ in (3.7) and using (3.5), we obtain

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \lim_{n_j\to\infty} n_j^k \big\langle T_{n_j,k}(\mathcal{F},t) - \mathcal{F}(t),\, g(t)\big\rangle. \tag{3.8}$$

Comparing (3.8) with (3.1) and (3.4), we have

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \begin{cases} \big\langle H(t),\, g(t)\big\rangle, & \text{if } p > 1; \\ -\langle \phi_0(t),\, g'(t)\rangle, & \text{if } p = 1. \end{cases}$$

Using integration by parts, it easily follows that

$$\langle \mathcal{F}(t), P^*_{2k}(D)g(t)\rangle = \langle Q(2k,t)\mathcal{F}(t) + \sum_{i=1}^{2k} I_i(b_iG)(t),\, g^{(2k)}(t)\rangle, \tag{3.9}$$

where $b_i(t)$ are certain polynomials in t and I_i denotes the ith iterated indefinite integral operator, namely

$$I_i = \overbrace{\int_0^t \dots \int_0^t}^{i\,\text{times}} dt \dots dt.$$

Similarly,

$$\big\langle H(t),\, g(t)\big\rangle = \big\langle I_{2k}H(t),\, g^{(2k)}(t)\big\rangle. \tag{3.10}$$

When $p > 1$, from (3.9) and (3.10) we have

$$\int_0^1 \big[Q(2k,t)\mathcal{F}(t) + \sum_{i=1}^{2k} I_i(b_iG)(t) - I_{2k}H(t)\big] g^{(2k)}(t)\, dt = 0.$$

It follows from Theorem 2.8 and Lemma 1.5.1 of [7] that $Q(2k,t) = C_k\big(t(1-t)\big)^k$, where C_k is a non-zero constant.

This implies by Lemma 1.1.1 [9] and the assumed smoothness for f that $\mathcal{F}^{(2k-1)} \in AC[x_2, y_2]$ and $\mathcal{F}^{(2k)} \in L_p[x_2, y_2]$. Since $\mathcal{F}(u) = F(u)$ for $u \in [x_1, y_1]$, we have $F^{(2k-1)} \in AC[a_2, b_2]$ and $F^{(2k)} \in L_p[a_2, b_2]$.

When $p = 1$, proceeding similarly, we obtain $F^{(2k-1)} \in BV[a_2, b_2]$. This completes the proof of the implication "$(i) \Rightarrow (ii)$".

The implication "$(ii) \Rightarrow (iii)$" follows from Theorem 3.1 of [1].

Assuming (iv) and proceeding as in the proof of the implication "$(i) \Rightarrow (ii)$", we first find that H and ϕ are zero functions. This does imply that F is $2k$ times continuously differentiable function and that $P_{2k}(D)F(t) = 0$.

Finally "$(v) \Rightarrow (vi)$" holds by Theorem 2.8.

This completes the proof. □

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A CAUCHY PROBLEM FOR SOME LOCAL FRACTIONAL ABSTRACT DIFFERENTIAL EQUATION WITH FRACTAL CONDITIONS

WEIPING ZHONG, XIAOJUN YANG, AND FENG GAO

Abstract. Fractional calculus is an important method for mathematics and engineering . In this paper, we review the existence and uniqueness of solutions to the Cauchy problem for the local fractional differential equation with fractal conditions

$$D^{\alpha}x(t) = f(t, x(t)), t \in [0, T], x(t_0) = x_0,$$

where $0 < \alpha \leq 1$ in a generalized Banach space. We use some new tools from Local Fractional Functional Analysis [25, 26] to obtain the results.

1. Introduction

In this paper, the some properties of the solution of the local fractional abstract differential equation

$$\begin{cases} \frac{d^{\alpha}x}{dt^{\alpha}} = f(t, x) \\ x(t_0) = x_0 \end{cases}, \tag{1.1}$$

where $\alpha \in (0\ ,\ 1], \frac{d^{\alpha}}{dt^{\alpha}}$ is the local fractional operator [25,26], $f(t, x)$ is a given function and both $f(t, x)$ and $x(t)$ are a non-differential function, have been the subject many investigation.

Local fractional calculus has revealed as one of useful tools in areas ranging from fundamental science to engineering [25-55]. It has gained importance and popularity during the past more than ten years, due to dealing with the fractal and continuously non-differentiable functions in the real world. The theory of local fractional integrals and derivatives was successfully applied in fractal elasticity [40-41], local fractional Fokker–Planck equation [34], local fractional transient heat conduction equation [42], local fractional diffusion equation [42], relaxation equation in fractal space [42], local fractional Laplace equation [45], fractal-time dynamical systems [31], local fractional partial differential equation [45], fractal signals [43,50], fractional Brownian motion in local fractional derivatives sense [39], fractal wave equation [53], Yang-Fourier transform [43,45,51,52], Yang-Laplace transform [45,47,51,53], discrete Yang-Fourier transform [46, 54], fast Yang-Fourier transform [48], local fractional Stieltjes transform in fractal space [44], local fractional Z transform in fractal space [51], local fractional short time transforms [25,26], local fractional wavelet transform [25, 26], and local fractional functional analysis [25,26,49].

Key words and phrases. Fractional analysis, local fractional differential equation, generalized Banach space, local fractional functional analysis.

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Based on the generalized Banach space [25, 26], the main aim of this paper is to show the existence and uniqueness of solutions to the Cauchy problem for the local fractional differential equation with fractal conditions.

The organization of this paper is as follows. In section 2, the preliminary results on the local fractional calculus and the generalized spaces are discussed. The existence and uniqueness of solutions to the Cauchy problem for the local fractional differential equation with fractal conditions is investigated in section 3. Conclusions are in section 4.

2. Preliminaries

2.1. Local fractional continuity of functions.

Definition 2.1. *If there exists [25,26,47,49,50]*

$$|f(x) - f(x_0)| < \varepsilon^{\alpha} \tag{2.1}$$

with $|x - x_0| < \delta$*,for* $\varepsilon, \delta > 0$ *and* $\varepsilon, \delta \in \mathrm{R}$*, now* $f(x)$ *is called local fractional continuous at* $x = x_0$*, denote by*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Then $f(x)$ *is called local fractional continuous on the interval* (a, b)*, denoted by*

$$f(x) \in C_{\alpha}(a, b). \tag{2.2}$$

2.2. Local fractional integrals.

Definition 2.2. *Let* $f(x) \in C_{\alpha}(a, b)$*. Local fractional integral of* $f(x)$ *of order* α *in the interval* $[a, b]$ *is given* [25, 26, 47, 49, 50]

$$\begin{aligned} & {}_aI_b^{(\alpha)} f(x) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)\,(dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\,(\Delta t_j)^{\alpha} \end{aligned}, \tag{2.3}$$

where $\Delta t_j = t_{j+1} - t_j$*,*$\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$ *and* $[t_j, t_{j+1}]$*,* $j = 0, ..., N-1$*,*$t_0 = a, t_N = b$*, is a partition of the interval* $[a, b]$*. For convenience, we assume that*

${}_aI_a^{(\alpha)} f(x) = 0$ *if* $a = b$ *and* ${}_aI_b^{(\alpha)} f(x) = -\,{}_bI_a^{(\alpha)} f(x)$ *if* $a < b$*. For any* $x \in (a, b)$*, we get*

$${}_aI_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a, b).$$

Remark 2.1. *If* $I_x^{(\alpha)}(a, b)$*, we have that*

$$f(x) \in C_{\alpha}(a, b).$$

Theorem 2.3. *(See* [25, 26]*) Suppose that* $f(x) \in C_{\alpha}[a, b]$*, then there is a function* $y(x) = {}_aI_x^{(\alpha)} f(x)$*, the function has its derivative with respect to* $(dx)^{\alpha}$*,*

$$\frac{d^{\alpha} y(x)}{dx^{\alpha}} = f(x), \quad a < x < b. \tag{2.4}$$

Theorem 2.4. *(Existence Theorem) Let $f(x,y)$ be local fractional continuous and bounded in the strip*

$$T = \{(x,y) : |x - x_0| \leq a, \left\| f(x,y) - f(x,y_0) \right\|_\alpha \leq L^\alpha \left\| y - y_0 \right\|_\alpha, L > 0\}.$$

Then the Cauchy value problem (1) has at least one solution in$|x - x_0| \leq a$.

2.3. **Local fractional derivative.**

Definition 2.5. *Let $f(x) \in C_\alpha(a,b)$. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is given [25,26,47,49,50]*

$$f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2.5}$$

where $\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$. For any $x \in (a,b)$, there exists

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x),$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a,b).$$

2.4. **Generalized Banach spaces.**

Definition 2.6. *(Generalized Banach space) (See [25, 26]) Let X be a generalized normed linear space. Since X is complete, the Cauchy sequence $\{x_n^\alpha\}_{n=1}^\infty$ is convergent; ie for each $\varepsilon > 0$ there exists a positive integer N such that*

$$\|x_n^\alpha - x_m^\alpha\|_\alpha < \varepsilon^\alpha \tag{2.6}$$

whenever $m,n \geq N$. This is equivalent to the requirement that

$$\lim_{m,n \to \infty} \|x_n^\alpha - x_m^\alpha\|_\alpha = 0. \tag{2.7}$$

A complete generalized normed linear space is called a generalized Banach space. There is an open ball in a generalized Banach space X:

$B_\alpha(x_0,r) = \{x \in X : \|x^\alpha - x_0^\alpha\|_\alpha < r^\alpha\}$ *with $r > 0$.*

Definition 2.7. *(Boundary of the fractal domain) (See [25, 26]) A set F in a generalized Banach space X is bounded if F is contained in some ball $B_\alpha(x_0,r)$ with $r > 0$.*

Definition 2.8. *(Local fractional continuity) (See [25, 26]) The function $f(x)$ with domain D is local fractional continuous at a if (i) the point a is in an open interval I contained in D, and (ii) for each positive number ε there is a positive number δ such that*

$|f(x) - f(x_0)| < \varepsilon^\alpha$ *whenever $|x - x_0| < \delta$ and $0 < \alpha \leq 1$.*

If a function $f(x)$ is said in the space $C_\alpha[a,b]$ if $f(x)$ is called local fractional continuous at $[a,b]$.

Definition 2.9. *(Local fractional uniform continuity) (See [25, 26]) A function $f(x)$ with domain D is said to be local fractional uniformly continuous on D if for each positive number ε there is a positive number δ such that*

$|f(x_1) - f(x_2)| < \varepsilon^\alpha$ *whenever $|x_1 - x_2| < \delta$, $x_1, x_2 \in D$ and $0 < \alpha \leq 1$.*

Definition 2.10. *(Convergence in fractal set) (See [25, 26]) A sequence $\{x_n^\alpha\}$ of fractal set$F$of fractal dimension α,$0 < \alpha \leq 1$, is said to converge to x^α, if given any neighborhood of x, there exists an integer m, such that $x_n^\alpha \in F$ whenever $n \geq m$.*

Definition 2.11. *(Cauchy sequence in fractal set) (See [25, 26]) A sequence $\{x_n^\alpha\}$ in a generalized Banach space X is a Cauchy sequence if for every $\varepsilon > 0$ there is a positive integer N such that $\|x_n^\alpha - x_m^\alpha\|_\alpha < \varepsilon^\alpha$ whenever $n, m > N$.*

2.5. **Generalized linear operators.** To begin with we give the definition of a generalized linear operator (See [25, 26]).

Definition 2.12. *(Generalized linear operator)(See [25, 26]) Let X and Y be generalized linear spaces over a field F and let $T : X \to Y$. If*

$$T\left(ax^\alpha + by^\alpha\right) = aT\left(x^\alpha\right) + bT\left(y^\alpha\right), \forall x^\alpha, y^\alpha \in X; \forall a, b \in F. \tag{2.8}$$

We say T is a generalized linear operator or a generalized linear transformation from X into Y.

Also, we write

$$T\left(X\right) = \left\{T\left(x^\alpha\right) : x^\alpha \in X\right\}.. \tag{2.9}$$

The local fractional differential operator D^α is a generalized linear operator [25, 26]:

$$D^\alpha f\left(x\right) = \lim_{x \to x_0} \frac{\Gamma\left(1+\alpha\right)\left[f\left(x\right) - f\left(x_0\right)\right]}{\left(x - x_0\right)^\alpha}. \tag{2.10}$$

The local fractional integral operator I^α is a generalized linear operator [25, 26]:

$$I^\alpha f\left(x\right) = \frac{1}{\Gamma\left(1+\alpha\right)} \int_a^x f\left(x\right)\left(dx\right)^\alpha. \tag{2.11}$$

2.6. **Contraction mapping on a generalized Banach space.**

Definition 2.13. *(Contraction mapping on a generalized Banach space) (See [25, 26]) Let X be a generalized Banach space, and let $T : X \to X$. If there exists a number $\beta \in (0, 1)$ such that*

$$\left\|T\left(x^\alpha\right) - T\left(y^\alpha\right)\right\|_\alpha \le \beta^\alpha \left\|x^\alpha - y^\alpha\right\|_\alpha \tag{2.12}$$

for all $x^\alpha, y^\alpha \in X$. We say that T is a contraction mapping on a generalized Banach space X.

It is remarked that the above definition is equal to [25,26], which is referred to fractional set theory [26,55].

Theorem 2.14. *(See [25, 26]) Let X be a generalized Banach space. A convergent sequence in X may have more than one limit in X.*

Theorem 2.15. *(Contraction Mapping Theorem in Generalized Banach Space) (See [25, 26]) A contraction mapping T defined on a complete generalized Banach space X has a unique fixed point.*

Theorem 2.16. *(Generalized Contraction Mapping Theorem in Generalized Banach Space) Suppose that $T : X \to X$ is a map on a generalized Banach space X such that for some $m \ge 1$, T^m is a contraction, ie., $\left\|T^m\left(y^\alpha\right) - T^m\left(x^\alpha\right)\right\|_\alpha \le \beta^\alpha \left\|x^\alpha - y^\alpha\right\|_\alpha$ for all $x^\alpha, y^\alpha \in X, \beta \in (0, 1)$. Then T has a unique fixed point.*

Proof. By Theorem 4, T^m has a unique fixed point x_0^α. Take into account

$$\begin{aligned} &\|Tx_0^\alpha - x_0^\alpha\|_\alpha \\ &= \left\|T^{m+1}x_0^\alpha - T^m x_0^\alpha\right\|_\alpha \\ &= \|T^m (Tx_0^\alpha) - T^m x_0^\alpha\|_\alpha \\ &\leq \beta^\alpha \|Tx_0^\alpha - x_0^\alpha\|_\alpha \end{aligned} \tag{2.13}$$

Hence $\|Tx_0^\alpha - x_0^\alpha\|_\alpha = 0$ and thus x_0^α is a fixed point of T. If $x_{0,0}^\alpha, x_{0,1}^\alpha$ are fixed points of T, they are fixed points of T^m and so $x_{0,0}^\alpha = x_{0,1}^\alpha$. □

3. Existence and uniqueness solution to the local fractional abstract differential equation

For the given equation

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = f(t,x) \\ x(t_0) = x_0 \end{cases} \tag{3.1}$$

form Theorem 1 and Theorem 2 we have that

$$x = x_0 + \frac{1}{\Gamma(1+\alpha)} \int_{t_0}^t f(t,x)(dt)^\alpha, \tag{3.2}$$

where $\|f(x_1,t) - f(x_0,t)\|_\alpha \leq k^\alpha \|x_1 - x_0\|_\alpha$.

Hence, by Theorem 2.4. we give the existence of solution to the local fractional abstract differential equation.

Furthermore, we suppose that the map $T : X \to X$ defined by

$$T(x(t)) = x_0 + \frac{1}{\Gamma(1+\alpha)} \int_{t_0}^t f(x,t)(dt)^\alpha \tag{3.3}$$

We claim that for all n,

$$\|T^n(x_1(t)) - T^n(x_0(t))\|_\alpha \leq k^{n\alpha} \frac{|t-t_0|^{n\alpha}}{\Gamma(1+n\alpha)} \|x_1 - x_0\|_\alpha. \tag{3.4}$$

The proof is by induction on n. The case $n = 0$ is trivial.

When $n = 1$, we have that

$$\|T(x_1(t)) - T(x_0(t))\|_\alpha \leq k^\alpha \frac{|t-t_0|^\alpha}{\Gamma(1+\alpha)} \|x_1 - x_0\|_\alpha. \tag{3.5}$$

The induction step is as follows:

$$\begin{aligned} &\left\|T^{n+1}(x_1(t)) - T^{n+1}(x_0(t))\right\|_\alpha \\ &= \left\| \tfrac{1}{\Gamma(1+\alpha)} \int_{t_0}^t f(t, T^n x_1(t)) - f(t, T^n x_0(t))(dt)^\alpha \right\|_\alpha \\ &\leq \tfrac{1}{\Gamma(1+\alpha)} \int_{t_0}^t k^\alpha \|f(t, T^n x_1(t)) - f(t, T^n x_0(t))\|_\alpha (dt)^\alpha \\ &\leq \tfrac{1}{\Gamma(1+\alpha)} \int_{t_0}^t \tfrac{k^{(n+1)\alpha}|t-t_0|^{n\alpha}}{\Gamma(1+n\alpha)} \|x_1 - x_0\|_\alpha (dt)^\alpha \\ &\leq \tfrac{1}{\Gamma(1+\alpha)} \int_{t_0}^t k^{(n+1)\alpha} \tfrac{|t-t_0|^{n\alpha}}{\Gamma(1+n\alpha)} \|x_1 - x_0\|_\alpha (dt)^\alpha \\ &\leq k^{(n+1)\alpha} \tfrac{|t-t_0|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \|x_1 - x_0\|_\alpha \end{aligned} \tag{3.6}$$

We have

$k^{(n+1)\alpha} \frac{|t-t_0|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \|x_1 - x_0\|_\alpha \to 0$ as $n \to 0$.

So far n sufficiently large,

$$0 < k^{(n+1)\alpha} \frac{|t-t_0|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} < 1 \tag{3.7}$$

and so T^n is a contraction on X.

Hence T has a unique fixed point in X, which gives a unique solution to the local fractional abstract differential equation.

4. Conclusions

Fractional calculus is an important method for mathematics and engineering. For more details, see [1-25]. In this paper we prove the generalized contraction mapping theorem in generalized Banach space. Finally, we show that the existence and uniqueness solution to the local fractional abstract differential equation for fractal condition by using some new tools from local fractional functional analysis to obtain the results, which are useful tools for dealing with local fractional operator.

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DIFFERENTIAL MAC MODELS IN CONTINUUM MECHANICS AND PHYSICS

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Abstract. The method of additional conditions or MAC was applied to create an integro-differential equation of the membrane problem [6]. This problem was presented at the Conference AMAT-2008. Another method can be used to create the differential MAC model of the same membrane problem. The obtained differential equation is much more easier to analyze and to obtain the exact solutions of the problem. Similar partial differential equation is considered in [9] but the exact solutions in our case are not given there.

The method to create the differential MAC models in mathematical physics is as follows. The classically stated problem is taken. Then the particular test problem is considered which solution could be compared with an experimental solution. For example we can take a circular elastic membrane with the fixed boundary condition at the contour and with the finite displacement in the center of membrane. The approximate experimental solution could be a cone. Substituting this solution into the classical membrane equation we will find the term which does not allow to satisfy the equation. We exclude this term from the equation and so the differential equation of the MAC model is created. We do not do anything except to correct mathematical model using an experiment.

It should be noted that mathematically similar test problems exist in the linear isotropic theory for cylinder and in the fluid mechanics for the Hagen-Poiseuille flow for a pipe. Then the differential MAC models for linear isotropic elasticity and for Navier-Stokes equations will be created.

The following differential MAC models are presented too: tension of an elastic rod, elastic string, beam, plate, heat conduction equation, Maxwell's equations, Schroedinger equations, Klein-Gordon equation.

1. Introduction

An elastic or fluid body with the given displacement of its one point create the infinite stresses acting near that point in the body [2], [3], [5], [4]. Then the elasticity or fluid mechanics theory should use the stress-strain or stress-rate of strain relations for infinite stresses. The experiment with the tension of a rod is an important tool to obtain the real stress-strain relations for an elastic body. And that experiments do not show the existance of such relations for infinite stresses. It means that we cannot apply the traditional elasticity theory to the case of point boundary conditions. For example if the force is applied to some point of the linear elastic body then the infinite displacements are at that point and the condition of finite displacement at that point could not be fulfilled.

We introduce and suggest to use the differential MAC models of elasticity to analyze the elastic problems not only with point boundary conditions but also in case

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of traditional distributed boundary conditions in form of displacements or stresses. The strength criteria could be used in the form which includes the strains but not stresses. The usual strength criteria involving stresses could be considered as a measure of strains.

The models of the membrane equation could be found in many problems of continuum mechanics. That equation and particular problem for them will be considered first of all and the differential MAC models for membrane will be introduced. Then these MAC models could be used to create the MAC models for other theories of continuum mechanics.

The membrane equation was considered in [6] where an integro-differential MAC model for membrane was introduced. The differential MAC models for membrane are considered in this paper.

2. Statement of the membrane problem

Let us consider an elastic membrane. The equation of motion of the membrane is given in [13] or in [10] or in [8]:

$$T_0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\rho\frac{\partial^2 u}{\partial t^2}+q(x,y,t), \tag{2.1}$$

where the membrane lies in the plane (x, y) in its natural state, T_0 is its tension per a unit of length, $u(x, y, t)$ is the transversal displacement of the point (x, y) of the initially plane membrane, ρ is the density of mass per unit area, t is time, $q(x, y, t)$ is the density of the transversal body forces per unit area. The tension T_0 is constant in this statement of the problem.

The nonlinear membrane equation was considered in [14], [15]. Unfortunately the experimental solutions taken in the present paper are not the solutions of the Zhilin's membrane equation [14] and the corresponding MAC model of membrane is not considered in this paper.The membrane equation in the paper [1] will not be satisfied with that experimental solutions and the corresponding MAC solution of that problem is not presented here.

We can write the equation (2.1) in the form

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}+p(x,y,t), \tag{2.2}$$

where

$$c^2=\frac{T_0}{\rho},\ p(x,y,t)=\frac{q(x,y,t)}{\rho}. \tag{2.3}$$

The correspondent initial and boundary conditions should be added to the equation (2.2) to obtain the unique solution of the problem.

Consider the steady state problem for the membrane without any given distributed forces $q = 0$. Then the function $u(x, y)$ does not depend on time t and the equation (2.2) becomes

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{2.4}$$

the membrane could be considered bounded or unbounded with Dirichlet's or Neumann's boundary conditions.

3. MAC model for membrane and conformal mapping

The MAC model based on conformal mapping was considered in [6]. The drawback of that MAC model is the constant transversal stiffness of membrane. That result is true also for some class of nonlinear distributions of the displacements. But we will not give the proof of these results in this paper. It could be mentioned that the given drawback for a mechanical membrane could be interesting if the similar equation will be applied to the other physical problems. We will not use the conformal mapping below to create the MAC models.

4. Differential MAC models for membrane

4.1. **Model 1.** Let us consider one particular problem for a circular elastic membrane with the fixed boundary conditions on the boundary of the circle and with the nonzero finite displacement at the center of the membrane. We know that the solution of that problem does not correspond to the results of the simple experiment with the real membrane [6]. Let us take the experimental solution and substitute it into the membrane equation (2.4). Then we will transform the classical equation of membrane to the form which includes the experimental function as a solution of the new equation.
Let us take the membrane equation (2.4) in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0, \tag{4.1}$$

where r, φ are the polar coordinates. Let the membrane occupies the circle $0 \le r \le R < \infty$, where R is the radius of a circle.
The boundary conditions are supposed to be

$$u(0) = u_0, u(R) = 0. \tag{4.2}$$

We accept the experimental solution as

$$u = u_0\left(1 - \frac{r}{R}\right). \tag{4.3}$$

The solution (4.3) is taken from the reality and it is just a function representing the experimental results obtained in experiments with the circular membranes.
Then substituting the function (4.3) into the equation (4.1) we obtain the nonzero term

$$\frac{1}{r}\frac{\partial u}{\partial r}, \tag{4.4}$$

which will be excluded from the equation (4.1). If we accept the equation (4.1) where the second term is excluded for all possible membrane solutions then we obtain the differential MAC model 1 for the steady state membrane problem in the following form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{4.5}$$

The equation (4.5) in Cartesian coordinates will take the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{x}{x^2+y^2}\frac{\partial u}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial u}{\partial y} = 0. \tag{4.6}$$

The MAC model 1 corresponding to the equation (2.2) has the equation

$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{x}{x^2+y^2}\frac{\partial u}{\partial x}-\frac{y}{x^2+y^2}\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial t^2}+p(x,y,t). \tag{4.7}$$

The equation (4.7) could be written in polar coordinates

$$c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right)=\frac{\partial^2 u}{\partial t^2}+\widetilde{p}(r,\varphi,t), \tag{4.8}$$

where $\widetilde{p}(r,\varphi,t)=p(x,y,t)$. The boundary and initial conditions should be added to the equation (4.7) or (4.8) to obtain an unique solution of the membrane problem. The methods to obtain the solutions of the presented equations could be taken for example in [9]. One remark to the obtained MAC equations should be given. We have excluded one term in the classical membrane equation and so we have changed the balance of forces acting on each small element of the membrane. That balance could be restored and the equation (4.8) will take the following form in case of symmetric problem

$$c^2\frac{\partial^2 u}{\partial r^2}=\frac{r}{R}\left(\frac{\partial^2 u}{\partial t^2}+\widetilde{p}(r,\varphi,t)\right), \tag{4.9}$$

where T_0 is a tension applied at the contour of membrane and the radial tension T is not constant but it is a function of r. We have in this case

$$T=\frac{T_0R}{r},\; c^2=\frac{T_0}{\rho}. \tag{4.10}$$

Then the equation (4.8) will be considered as the approximate MAC model of the membrane equation.
One of the methods to restore the balance of forces will be considered below. It can be mentioned that similar like the MAC model based on conformal mapping could be useful in another physical theories the MAC model (4.8) could find its applications. We will compare now these model (4.8) with the corresponding classical one.

4.2. **Comparison of classical and MAC solutions for circular membrane.**

4.2.1. *Problem 1.* Consider a circular membrane under constant pressure $-q$ in classical case. Then the stated problem is

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}=-\frac{q}{T_0},\; u(R)=0. \tag{4.11}$$

The solution of the problem (4.11) is

$$u(r)=\frac{q}{4T_0}(R^2-r^2). \tag{4.12}$$

The differential approximate MAC model 1 for membrane is

$$\frac{d^2u}{dr^2}=-\frac{q}{T_0},\; \frac{dU}{dr}(0)=0,\; u(R)=0, \tag{4.13}$$

where the equation (4.8) was used. The solution of the problem (4.13) is

$$u(r)=\frac{q}{2T_0}(R^2-r^2). \tag{4.14}$$

Then we see that the value $u(0) = \frac{qR^2}{2T_0}$ in an approximate MAC model is two times more as in the classical case.
If the equation of the MAC model (4.9) is taken then the solution will be

$$u(r) = \frac{q}{6RT_0}(R^3 - r^3) \tag{4.15}$$

and the MAC model gives the following value of the displacement in the center of membrane

$$u(0) = \frac{qR^2}{6T_0}. \tag{4.16}$$

4.2.2. *Problem 2.* Let us add the following condition to the above Problem 1:

$$u(0) = 0. \tag{4.17}$$

Then the solution in classical case does not exist at all. But the approximate MAC solution exists and is as follows

$$u(r) = \frac{qr}{2T_0}(R - r). \tag{4.18}$$

4.2.3. *Problem 3.* Let us consider now the free symmetric harmonic vibrations of a circular membrane. The stated problem in classical case is

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + \frac{\omega^2}{c^2}U = 0,\ \frac{dU}{dr}(0) = 0,\ U(R) = 0, \tag{4.19}$$

where $U(r)$ is the form of membrane corresponding to the eigenfrequency ω. The eigenfrequences of the problem (4.19) satisfy the equation

$$J_0\left(\frac{\omega R}{c}\right) = 0, \tag{4.20}$$

where $J_0(r)$ is the Bessel function of the first kind and of order zero.
The corresponding problem for approximate MAC model 1 is in this case:

$$\frac{d^2U}{dr^2} + \frac{\omega^2}{c^2}U = 0,\ \frac{dU}{dr}(0) = 0,\ U(R) = 0, \tag{4.21}$$

Solving the problem (4.21) we obtain the following eigenfrequences:

$$\omega_n = \frac{\pi c}{R}(0.5 + n),\ n = 0, 1, 2, \ldots. \tag{4.22}$$

4.2.4. *Problem 4.* Let us change the condition in the center of membrane in the problem 3 and apply

$$\frac{dU}{dr}(0) = \alpha \neq 0. \tag{4.23}$$

Then the classical case does not have any solution. The correspondent approximate MAC model 1 has the following solution

$$u = \frac{c\alpha}{\omega}\frac{\sin\frac{\omega(r-R)}{c}}{\cos\frac{\omega R}{c}}. \tag{4.24}$$

The resonance frequencies are

$$\omega_n = \frac{\pi c}{R}(0.5 + n),\ \ n = 1, 2\ldots. \tag{4.25}$$

4.2.5. *Problem 5.* Let us replace the condition in the center of the membrane in the classical and in the approximate MAC models of membrane in the Problem 3 through $U(0) = 0$. The eigenfrequences in classical model do not exist at all. And the approximate MAC model gives

$$\omega_n = \frac{\pi c}{R} n, \; n = 1, 2, \ldots . \tag{4.26}$$

4.3. **MAC solution for rectangular membrane.** The trigonometric series could be useful to consider the membrane problems for rectangular membrane like in classical case.

Consider the following problem for a rectangular membrane using the differential approximate MAC model 1:

$$c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{x}{x^2 + y^2} \frac{\partial u}{\partial x} - \frac{y}{x^2 + y^2} \frac{\partial u}{\partial y} \right) = p, \tag{4.27}$$

where p is a constant, $-a \leq x \leq a$, $-b \leq y \leq b$ and the boundary conditions are: $u(-a, y) = u(a, y) = u(x, -b) = u(x, b) = 0$.

Multiplying the equation (4.27) by $x^2 + y^2$ the solution of the problem could be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cos \frac{\pi x(2n - 1)}{2a} \cos \frac{\pi y(2n - 1)}{2b}, \tag{4.28}$$

where

(4.29)

$$a_{nm} = \frac{p}{c^2} \frac{(-1)^{n+m} 192 a^2 b^2 (a^2 + b^2)}{\pi^2 (2n - 1)(2m - 1)\{12 a^2 b^2 - \pi^2 (a^2 + b^2 [b^2 (2n - 1)^2 + a^2 (2m - 1)^2]\}}$$

for $n, m = 1, 2, \ldots$.

4.4. **Model 2.** The experimental solution of the real membrane test problem could be taken in more general form:

$$u(r) = u_0 \left(1 - \left(\frac{r}{R} \right)^{\alpha} \right), \tag{4.30}$$

where α is an experimental constant. If $\alpha = 1$ then we obtain the same experimental solution which was used in the MAC model 1 above. We may change the classical membrane equation for this symmetric problem to the following one:

$$\frac{d^2 u}{dr^2} + \frac{1 - \alpha}{r} \frac{du}{dr} = 0. \tag{4.31}$$

The solution (4.30) satisfies the equation (4.31) exactly. It is not an unique equation which includes the function (4.30) into its set of solutions. For example the following equations are satisfied using the solution (4.30):

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \alpha^2 \frac{1 - u}{r^2} = 0 \tag{4.32}$$

or

$$\frac{d^2 u}{dr^2} + \frac{\alpha^2 - \alpha}{r^2} (1 - u) = 0. \tag{4.33}$$

We take the equation (4.31) to create the approximate MAC model 2. Then the equation for the steady state membrane problem will be

$$\frac{\partial^2 u}{\partial r^2} + \frac{1-\alpha}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0. \tag{4.34}$$

The differential approximate MAC model 2 for membrane in polar coordinates therefore is

$$c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1-\alpha}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) = \frac{\partial^2 u}{\partial t^2} + p(r,\varphi,t). \tag{4.35}$$

The equations (4.34) and (4.35) in polar coordinates are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\alpha x}{x^2+y^2}\frac{\partial u}{\partial x} - \frac{\alpha y}{x^2+y^2}\frac{\partial u}{\partial y} = 0, \tag{4.36}$$

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\alpha x}{x^2+y^2}\frac{\partial u}{\partial x} - \frac{\alpha y}{x^2+y^2}\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial t^2} + p(x,y,t). \tag{4.37}$$

If the parameter $\alpha = 1$ then the approximate MAC model 2 coincides with the approximate MAC model 1.

4.5. **Model 3.** The MAC model 1 was created for a bounded membrane. If we consider the unbounded membrane then the experimental solution (4.3) will not satisfy both boundary conditions: at the origin and at the infinity. We can consider the following virtual experimental solution in this case

$$u = u_0 \exp(-\beta r), \tag{4.38}$$

where $\beta > 0$.
The function (4.38) may satisfy the following differential equation

$$\frac{d^2 u}{dr^2} + \beta\frac{du}{dr} = 0 \tag{4.39}$$

or

$$\frac{d^2 u}{dr^2} - \beta^2 u = 0. \tag{4.40}$$

The additional experiments with membrane should be used to choose the equation (4.39) or (4.40). If we choose the equation (4.39) then the corresponding membrane equation for the approximate MAC model 3 will take the form

$$c^2\left(\frac{\partial^2 u}{\partial r^2} + \beta\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right) = \frac{\partial^2 u}{\partial t^2} + p(r,\varphi,t). \tag{4.41}$$

We have considered some differential MAC models without changing the order of the partial differential equation of membrane. But it is possible to consider the MAC models introducing the differential equation of higher order as the classical one. It is not considered in this paper.

5. MAC model for membrane based on cones

The cones were used to create the MAC model for the linear thermoelasticity [7], where the balance of forces was satisfied. Similar approach is used in this section to consider the symmetric problems for a circular elastic membrane of the radius R. The origin is in the center of membrane and r is the distance of the origin. the transversal displacements of membrane are $u(r)$. The boundary condition is $u(R) = 0$. Let $Q(r)$ is an external transversal force per unit length applied at every point at the radius r. Suppose that the form of the displacements field could be the same as in the string which is obtained by two cuts along the diameter of the membrane [6].

If $u(a)$ is a given displacement at $r = a$ then the displacements field for $r \leq a$ is

$$u(r) = u(a), \tag{5.1}$$

and for $a \leq r \leq R$

$$u(r) = u(a)\frac{R-r}{R-a}. \tag{5.2}$$

The relation between $Q(a)$ and $u(a)$ follows from the balance of external forces applied to membrane.

$$Q(a) = \frac{u(a)RT_0}{a(R-a)}, \tag{5.3}$$

where T_0 is the tension applied at the boundary of the membrane. The formulas (5.1), (5.2), (5.3) allow to determine the displacements of membrane if the external forces are given.

5.1. **Example 1.** The constant pressure q is given. Then $Q(a) = qda$ and we obtain

$$u(r) = \int_0^r \frac{qa(R-r)}{T_0 R}da + \int_r^R \frac{qa(R-a)}{T_0 R}da = \frac{q}{6T_0 R}(R^3 - r^3). \tag{5.4}$$

We have $u(0) = \frac{qR^2}{6T_0}$ and this is 1.5 less then it is in classical case.

5.2. **Example 2.** If the center of membrane is fixed and the membrane is under a constant pressure q then we obtain the reaction at the origin from the equation (5.3)

$$S = 2\pi a Q(a)|_{a\to 0} = 2\pi u_S(0)T_0, \tag{5.5}$$

where the displacement under a force S should be equal -u(0) according to the equation (5.4). So we have got $u_S = -u(0)$ and then the reaction S is

$$S = -2\pi u(0)T_0 = 2\pi T_0 \cdot \frac{qR^2}{6T_0} = -\frac{\pi q R^2}{3}. \tag{5.6}$$

The displacements field is

$$u(r) = \frac{qr(R^2 - r^2)}{6T_0 R}. \tag{5.7}$$

5.3. **Example 3.** Consider the free symmetric vibrations of a circular membrane. Then $Q(a) = -\rho \frac{d^2u}{dt^2}(a)da$ and we obtain an integro-differential equation

$$u(r) = -\int_0^r \frac{a(R-r)}{T_0R}\rho \frac{\partial^2 u}{\partial t^2}(a)da - \int_r^R \frac{a(R-a)}{T_0R}\rho \frac{\partial^2 u}{\partial t^2}(a)da. \tag{5.8}$$

The boundary condition is $u(R) = 0$. The solution of the equation (5.8) is taken in the form $u(r,t) = U(r)\sin(\omega t)$, where ω is a constant. This form of solution and the equation (5.8) create the equation

$$U(r) = \frac{\rho\omega^2}{T_0R}\left[(R-r)\int_0^r aU(a)da + \int_r^R U(a)a(R-a)da\right]. \tag{5.9}$$

The boundary condition is transformed to $U(R) = 0$. Differentiating the equation (5.9) with respect to r we obtain

$$\frac{dU}{dr}(r) = -\frac{\rho\omega^2}{T_0R}\int_0^r aU(a)da. \tag{5.10}$$

The equation (5.10) gives the second condition at $r = 0$

$$\frac{dU}{dr}(0) = 0. \tag{5.11}$$

Differentiating the equation (87.9) with respect to r we get the equation

$$\frac{d^2U}{dr^2} + \frac{\rho\omega^2}{T_0R} rU(r) = 0. \tag{5.12}$$

Let us transform the equation (5.12) introducing the variable

$$\xi = -\sqrt[3]{\frac{\rho\omega^2}{T_0R}}r. \tag{5.13}$$

Then the equation(5.12) will take the following form

$$\frac{d^2U}{d\xi^2} - \xi U(\xi) = 0. \tag{5.14}$$

That is the Airy's equation [12]. The general solution of the equation (5.14) is

$$U(\xi) = C_1 Ai(\xi) + C_2 Bi(\xi), \tag{5.15}$$

where $Ai(\xi)$, $Bi(\xi)$ are the Airy functions, C_1, C_2 are arbitrary constants. Then the boundary condition and condition (5.11) could be satisfied and the frequency equation will be obtained

$$\sqrt{3}Ai\left(-\sqrt[3]{\frac{\rho\omega^2}{T_0R}}R\right) + Bi\left(-\sqrt[3]{\frac{\rho\omega^2}{T_0R}}R\right) = 0. \tag{5.16}$$

5.4. **Example 4.** Let us fix the center of membrane considered in Example 3. The integro-differential equation of this problem is

$$u(r) = -\int_0^r \frac{a(R-r)}{T_0 R}\rho\frac{\partial^2 u}{\partial t^2}(a)da - \int_r^0 \frac{a(R-a)}{T_0 R}\rho\frac{\partial^2 u}{\partial t^2}(a)da. \tag{5.17}$$

This equation (5.17) could be obtained if the value

$$u(0) = -\int_0^R \frac{a(R-a)}{T_0 R}\rho\frac{\partial^2 u}{\partial t^2}(a)da. \tag{5.18}$$

according to the equation (5.8) will be subtracted from the right side of the equation (5.8).
The boundary conditions are

$$u(0) = 0,\, u(R) = 0. \tag{5.19}$$

Substituting the function $u(r,t) = U(r)\sin(\omega t)$ into the equations (5.17), (5.19) yields the problem for the function $U(r)$:

$$U(r) = \frac{\rho\omega^2}{T_0 R}\left[(R-r)\int_0^r aU(a)da + \int_r^0 U(a)a(R-a)da\right], \tag{5.20}$$

$$U(0) = 0,\, U(R) = 0. \tag{5.21}$$

Differentiating two times the equation (5.20) with respect to r we find that the function $U(r)$ satisfies the same equation (5.12) which could be transformed to the Airy equation introducing the new variable (5.13. Then the frequency equation will be obtained if the general solution satisfies the boundary conditions. The frequency equation is in this case

$$\sqrt{3}Ai\left(-\sqrt[3]{\frac{\rho\omega^2}{T_0 R}}R\right) - Bi\left(-\sqrt[3]{\frac{\rho\omega^2}{T_0 R}}R\right) = 0. \tag{5.22}$$

The circular membrane on elastic support under constant pressure or its symmetric vibrations will have similar Airy's equations. We see that the Airy functions play an important role in solutions of MAC model for membrane. Both Airy's functions have not singularities on the whole plane. These property of the Airy functions differs them from the Bessel functions which are usually arising in the similar classical problems. One of two Bessel's functions has singularity at the origin.
These important property of nonsingularity of the fundamental functions of the corresponding differential MAC model conserves also in MAC model for an elastic plate. The MAC model equation will be Airy like equation but of the 4th order. And all their fundamental solutions have not singularities at the origin. But this MAC model for the plate will not be considered in this paper.

6. Partial differential equation for membrane MAC model

Let us differentiate the equation (5.8) two times with respect to r. Then the following partial differential equation of membrane will be obtained for symmetric vibrations

$$\frac{\partial^2 u}{\partial r^2} = \frac{r\rho}{T_0 R}\frac{\partial^2 u}{\partial t^2}. \tag{6.1}$$

The method of separation of variable could be applicable. For example the boundary conditions are

$$u(0,t)=0,\ u(R,t)=0 \tag{6.2}$$

and the initial conditions are taken as

$$u(r,0)=f(r), \tag{6.3}$$

where $f(r)$ is a given continuous function. The solution of the stated problem includes the Airy functions. The classical solutions of the similar problems for membrane should include the Bessels functions.

7. Differential MAC model for elasticity

Let us consider the following particular problem of the linear isotropic elasticity [11]. An elastic body occupies the unbounded cylinder $0 \le r \le R$, where R is the finite radius of the cylinder. Let the displacement field of the body is in cylindrical coordinates r, φ, z:

$$u_r = u_r(r,\varphi),\ u_\varphi = u_\varphi(r,\varphi),\ u_z = u_z(r). \tag{7.1}$$

The equations of the linear isotropic elasticity in cylindrical coordinates are

$$(\lambda+\mu)\frac{\partial e}{\partial r} + \mu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \varphi^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r}{r^2}\right) = 0, \tag{7.2}$$

$$\frac{(\lambda+\mu)}{r}\frac{\partial e}{\partial \varphi} + \mu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r^2}\right) = 0, \tag{7.3}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \varphi^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) = 0, \tag{7.4}$$

where r, φ, z are cylindrical coordinates, λ, μ are the Lame parameters, u_r, u_φ, u_z are components of the displacement vector in cylindrical coordinates,

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}. \tag{7.5}$$

Then the component u_z satisfies the equation

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = 0. \tag{7.6}$$

Let us apply the boundary conditions

$$u_z(0) = u_0 \neq 0,\ u_z(R) = 0. \tag{7.7}$$

We have

$$\tau_{rz} = \mu\frac{du_z}{dr}. \tag{7.8}$$

The equations (7.6), (7.7), (7.8) represent the same mathematical problem as for the membrane problem considered in the above sections. The parameter μ plays the same role as the tension T_0 in the membrane problem. The differential approximate and balanced MAC models of membrane could be applied in this elastic case. For example we may introduce the correspondent approximate MAC models for elasticity equations using the obtained approximate MAC models for membrane.

7.1. **MAC Model 1.** The differential approximate MAC model 1 equations for the linear isotropic elasticity in Cartesian coordinates could be given as

$$(\lambda+\mu)\frac{\partial e}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}-\frac{y}{y^2+z^2}\frac{\partial u_x}{\partial y}-\frac{z}{y^2+z^2}\frac{\partial u_x}{\partial z}\right)+\rho_0 B_x=\rho_0\frac{\partial^2 u_x}{\partial t^2}, \tag{7.9}$$

$$(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}-\frac{x}{x^2+z^2}\frac{\partial u_y}{\partial x}-\frac{z}{x^2+z^2}\frac{\partial u_y}{\partial z}\right)+\rho_0 B_y=\rho_0\frac{\partial^2 u_y}{\partial t^2}, \tag{7.10}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}-\frac{x}{x^2+y^2}\frac{\partial u_z}{\partial x}-\frac{y}{x^2+y^2}\frac{\partial u_z}{\partial y}\right)+\rho_0 B_z=\rho_0\frac{\partial^2 u_z}{\partial t^2}, \tag{7.11}$$

where

$$e=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}. \tag{7.12}$$

The initial and boundary conditions are taken as in classical theory of elasticity.

7.2. **MAC model 2.** The differential approximate MAC model 2 equations for the linear isotropic elasticity in Cartesian coordinates could be given as

$$(\lambda+\mu)\frac{\partial e}{\partial x}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}-\frac{\alpha y}{y^2+z^2}\frac{\partial u_x}{\partial y}-\frac{\alpha z}{y^2+z^2}\frac{\partial u_x}{\partial z}\right)+\rho_0 B_x=\rho_0\frac{\partial^2 u_x}{\partial t^2}, \tag{7.13}$$

$$(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}-\frac{\alpha x}{x^2+z^2}\frac{\partial u_y}{\partial x}-\frac{\alpha z}{x^2+z^2}\frac{\partial u_y}{\partial z}\right)+\rho_0 B_y=\rho_0\frac{\partial^2 u_y}{\partial t^2}, \tag{7.14}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z}+\mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}-\frac{\alpha x}{x^2+y^2}\frac{\partial u_z}{\partial x}-\frac{\alpha y}{x^2+y^2}\frac{\partial u_z}{\partial y}\right)+\rho_0 B_z=\rho_0\frac{\partial^2 u_z}{\partial t^2}, \tag{7.15}$$

The initial and boundary conditions could be taken as in the classical theory of elasticity. If $\alpha = 1$ then the approximate MAC model 1 for elasticity will be obtained.

8. MAC model for incompressible flow

Consider the fully developed laminar motion through a tube of radius a. Flow through a tube is frequently called a circular Poiseuille flow. We employ cylindrical coordinates (r, θ, x), with the $x-$ axis coinciding with the axis of the pipe. The only nonzero component of velocity is the axial velocity $u(r)$, and none of the flow variables depend on θ. The $x-$ momentum equation gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)=\frac{1}{\mu}\frac{dp}{dx}. \tag{8.1}$$

As the first term can only be a function of x, and the second term can only be a function of r, it follows that both terms must be constant. The pressure is therefore falls linearly along the length of pipe. The wall condition is $v = 0$ at $r = a$. The shear stress at any point is

$$\tau_{xr}=\mu\frac{dv}{dr}. \tag{8.2}$$

Let the boundary conditions are

$$v(0) = v_0 \neq 0,\ v(R) = 0. \tag{8.3}$$

The stated problem which is presented by the equations (8.1), (8.2), (8.3) is similar to the problem of membrane considered in the above sections. The parameter μ plays the role of the tension T_0 similar to the elasticity theory. The differential approximate and balanced MAC models of membrane could be applied in this case of fluid mechanics.

The classical solution of the steady state pipe problem is well known

$$v = \frac{r^2 - R^2}{4\mu}\frac{dp}{dx}. \tag{8.4}$$

The corresponding the balanced MAC model has the following solution

$$v = \frac{r^3 - R^3}{6\mu R}\frac{dp}{dx} \tag{8.5}$$

for the free flow on the axis of symmetry of a pipe. If that axis is fixed then the condition $v(0) = 0$ will be used. The MAC solution in this case is

$$v = \frac{r(r^2 - R^2)}{6\mu R}. \tag{8.6}$$

It should be mentioned that the classical solution in the last case does not exists. Then the differential approximate MAC model 2 of membrane will bring the following form in case of the Navier-Stokes equations

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = \tag{8.7}$$

$$= \rho B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} - \frac{\alpha y}{y^2+z^2}\frac{\partial v_x}{\partial y} - \frac{\alpha z}{y^2+z^2}\frac{\partial v_x}{\partial z}\right), \tag{8.8}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = \tag{8.9}$$

$$= \rho B_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} - \frac{\alpha z}{z^2+x^2}\frac{\partial v_y}{\partial z} - \frac{\alpha x}{z^2+x^2}\frac{\partial v_y}{\partial x}\right), \tag{8.10}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = \tag{8.11}$$

$$= \rho B_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} - \frac{\alpha x}{x^2+y^2}\frac{\partial v_z}{\partial x} - \frac{\alpha y}{x^2+y^2}\frac{\partial v_z}{\partial y}\right), \tag{8.12}$$

The fourth equation is supplied by the continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{8.13}$$

The initial and boundary conditions could be taken as in the classical theory of fluid mechanics. If $\alpha = 1$ then the approximate MAC model 1 for fluid mechanics will be obtained. Other MAC models could be easily obtained too.

8.1. **MAC model with integro-differential equation.**

8.1.1. *Statement of the problem.* Consider the following problem for incompressible flow. The Navier-Stokes equations for incompressible Newtonian fluids are taken in the form

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\nabla \mathbf{v})\mathbf{v}\right]=\rho\mathbf{B}-\nabla p+\mu\nabla^2\mathbf{v}, \tag{8.14}$$

where ρ is the mass density, $\rho\mathbf{B}$ is a body force per unit volume, p is the pressure, $\mathbf{v}$ is the velocity vector, μ is viscosity coefficient, ∇ is the gradient. The continuity equation should be added

$$\mathbf{div}\ \mathbf{v}=0. \tag{8.15}$$

the constitutive equations can be written in the form

$$T_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \tag{8.16}$$

where $i,j=1,2,3,$ v_i are the Cartesian components of the velocity vector and x_i are the components of the position vector. The variables $x=x_1, y=x_2, z=x_3$ are the Cartesian coordinates of a point belonging to the domain Ω:

$$x^2+y^2+z^2<R, x\neq x_0, y\neq y_0, z\neq z_0 \tag{8.17}$$

and the point $S(x_0,y_0,z_0$ is a given fixed point inside the sphere of radius R with the center of the sphere at the origin. There are four unknown functions in the four scalar equations (8.14), (8.15). We will consider the Dirichlet problem with the following boundary conditions consisting of two parts. The first part is

$$\mathbf{v}|_\Gamma=\mathbf{0}, \tag{8.18}$$

where Γ is a sphere $x^2+y^2+z^2=R^2$.
The second part of the boundary conditions is a given and nonzero value $\mathbf{v}_0$ of the function$\mathbf{v}(x,y,z)$ at the point $S(x_0,y_0,z_0)$:

$$\mathbf{v}(S)=\mathbf{v}_0. \tag{8.19}$$

8.1.2. *MAC Green's function.* Let us consider the MAC solution of the stated problem. We define the MAC solution as a union of the strait lines connecting the internal point $S(x_0,y_0,z_0)$ with each point of the boundary:

$$\mathbf{v}(x,y,z)=\mathbf{v}_0\sqrt{\frac{(x-x_\Gamma)^2+(y-y_\Gamma)^2+(z-z_\Gamma)^2}{(x_0-x_\Gamma)^2+(y_0-y_\Gamma)^2+(z_0-z_\Gamma)^2}}, \tag{8.20}$$

where the boundary point $(x_\Gamma,y_\Gamma,z_\Gamma)$ corresponds to the given point (x,y,z) of the domain Ω and satisfies the equations:

$$x_\Gamma^2+y_\Gamma^2+z_\Gamma^2=R^2, \tag{8.21}$$

$$\frac{x-x_\Gamma}{x_0-x_\Gamma}=\frac{y-y_\Gamma}{y_0-y_\Gamma}=\frac{z-z_\Gamma}{z_0-z_\Gamma}, \tag{8.22}$$

The force $\mathbf{Q}$ at the point $S(x_0,y_0,z_0)$ of the domain Ω could be found using its balance with viscous stresses applied to the external boundary of the sphere. Then

$$\mathbf{Q}=\int_\Gamma \mathbf{t}^\mu_\mathbf{n}d\Gamma, \tag{8.23}$$

where the viscous stress vector $\mathbf{t}^\mu_n$ is

$$\mathbf{t}^\mu_n=\mathbf{T}^\mu_n\mathbf{n}, \tag{8.24}$$

$\mathbf{n}$ is the outer normal to the sphere, the components of the viscous stress tensor $\mathbf{T}$ are

$$T_{ij}^{\mu} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{8.25}$$

The function $\mathbf{Q} = \mathbf{Q}(\mathbf{v_0}, x_0, y_0, z_0)$ in the equation (8.23) is obtained for the given function in (8.20) function $\mathbf{v}(x, y, z)$ and depends on the point $S(x_0, y_0, z_0)$ and the applied velocity vector $\mathbf{v_0}$. That function can be written in the form

$$\mathbf{Q} = \mathbf{S}\mathbf{v_0}, \tag{8.26}$$

where $\mathbf{S}$ is a stiffness matrix. Multiplying the equation (8.26) by the compliance matrix $\mathbf{C} = \mathbf{S^{-1}}$ we obtain

$$\mathbf{v_0} = \mathbf{C}\mathbf{Q}. \tag{8.27}$$

If we put $\mathbf{v_0}$ from the equation (8.27) into the equation (8.20) then we obtain

$$\mathbf{v}(x, y, z) = \mathbf{Q}\mathbf{C}\sqrt{\frac{(x - x_\Gamma)^2 + (y - y_\Gamma)^2 + (z - z_\Gamma)^2}{(x_0 - x_\Gamma)^2 + (y_0 - y_\Gamma)^2 + (z_0 - z_\Gamma)^2}}. \tag{8.28}$$

Introducing the MAC Green's matrix function of the ball domain Ω

$$\mathbf{M}(P, S) = \mathbf{C}\sqrt{\frac{(x - x_\Gamma)^2 + (y - y_\Gamma)^2 + (z - z_\Gamma)^2}{(x_0 - x_\Gamma)^2 + (y_0 - y_\Gamma)^2 + (z_0 - z_\Gamma)^2}}, \tag{8.29}$$

where $P(x, y, z), S(x_0, y_0, z_0)$ are any two points of the domain Ω and the components $x_\Gamma, y_\Gamma, z_\Gamma$ satisfy the equations (8.21), (8.22). Then the solution of the stated problem (8.28 is given in the form

$$\mathbf{v}(x, y, z) = \mathbf{Q}\mathbf{M}(P, S) = \mathbf{Q}\mathbf{M}(x, y, z, x_0, y_0, z_0). \tag{8.30}$$

8.1.3. *Integro-differential equation of MAC model for a ball.* The principle of superpositions allows to write the integro-differential equation of MAC model for a ball domain

$$\mathbf{v}(P, t) = \int_\Omega \mathbf{M}(P, S)\left[\rho(S)\left(\frac{\partial \mathbf{v}}{\partial t}(S) + (\nabla\mathbf{v})\mathbf{v}(S) - \mathbf{B}(S)\right) + \nabla p(S)\right] d\Omega, \tag{8.31}$$

where $\mathbf{M}(P, S)$ is the MAC Green's function of the ball domain Ω, $\mathbf{v}(x, y, z, t) = \mathbf{v}(P, t)$ is the velocity vector of the point P of the ball domain Ω, $\rho(S)$ is the mass-density per unit volume at a point S of the domain Ω, $\rho\mathbf{B}$ is the body force per unit volume, t is time.
The Navier-Stokes equations (8.14) are replaced by the equation (8.31) in the developed MAC model. The equation (8.15) remains in the MAC model. The boundary condition (8.18) remains also. The viscosity μ is taken just only at the boundary of considered domain.

8.1.4. *Diving method.* Let us consider an incompressible fluid flow in the domain $D \subset \Omega$.Consider the case when the velocity vector $\mathbf{v}$ is prescribed on the boundary surface ∂D:

$$\mathbf{v}|_{\partial D} = \mathbf{g}, \tag{8.32}$$

where $g(S)$ is a given vector function defined on the boundary ∂D, $S \in \partial D$. Then introducing the unknown density of the forces $\mathbf{q}dA$ on the boundary surface ∂D we obtain an integro-differential equation to find the density $\mathbf{q}$

$$\mathbf{g}(P_{\partial D}, t) = \int_{\partial D} \mathbf{M}(P_{\partial D}, S_{\partial D})\mathbf{q}(S_{\partial D}, t)dA+ \tag{8.33}$$

(8.34)

$$+\int_D \mathbf{M}(P_{\partial D}, S_D)\left[\rho(S_D)\left(\frac{\partial \mathbf{v}}{\partial t}(S_D,t) + (\nabla \mathbf{v})\mathbf{v}(S_D,t) - \mathbf{B}(S_D,t)\right) + \nabla p(S_D,t)\right] dD.$$

The second equation is to find the velocity vector $\mathbf{v}$

$$\mathbf{v}(P_D, t) = \int_{\partial D} \mathbf{M}(P_D, S_{\partial D})\mathbf{q}(S_{\partial D}, t)dA+ \tag{8.35}$$

(8.36)

$$+\int_D \mathbf{M}(P_D, S_D)\left[\rho(S_D)\left(\frac{\partial \mathbf{v}}{\partial t}(S_D,t) + (\nabla \mathbf{v})\mathbf{v}(S_D,t) - \mathbf{B}(S_D,t)\right) + \nabla p(S_D,t)\right] dD.$$

These two integro-differential equations should be added to the continuity equation (8.15). Then we obtain the MAC model using the diving method.
We don't consider the MAC models for ideal fluid in this paper. It can be also done using for example the velocity potential.

9. Differential MAC model for heat conduction equation

The heat conduction problem and the corresponding balanced MAC model was considered in [7] where an integro-differential equation was introduced. We will apply the developed differential MAC models from the above sections to the heat conduction problem.

9.1. **Statement of the problem.** Consider the following 3D heat conduction equation

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + q(x,y,z,t) = c_0\rho\frac{\partial u}{\partial t}, \tag{9.1}$$

where $u(x,y,z,t)$ is the temperature of the point $d(x,y,z)$ of the domain, $\rho(d)$ is the mass-density of the body per unit volume at a point d, t is time, c_0 is specific heat, k is the coefficient of thermal conduction, $q(x,y,z,t)$ is a rate of internal heat generation per unit volume produced in the body.
The equation (9.1) could be divided by $c_0\rho$ and then it will be written in the form

$$c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + p = \frac{\partial u}{\partial t}, \tag{9.2}$$

where

$$c^2 = \frac{k}{c_0\rho},\ p = \frac{q}{c_0\rho}. \tag{9.3}$$

The equation (9.3) is applied classically to the bounded and unbounded domains. The correspondent initial and boundary conditions are applied to obtain the unique solution of the problem.

The following steady state problem is considered very often. It consists of the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{9.4}$$

and the Dirichlet or Neumann boundary conditions.

9.2. **MAC model for 2D heat conduction based on cones.** The cones were used to create the MAC model for the heat conduction problem in [7], where the balance of heat fluxes was satisfied. Similar approach is used in this section to consider the symmetric problems for a circular cylinder of the radius R. We employ cylindrical coordinates (r, θ, x), with the $x-$ axis coinciding with the axis of the cylinder. Suppose that the nonzero temperature depends on r only. That is $u(r)$. The boundary condition is $u(R) = 0$. Let $Q(r)$ is an external heat flux per unit length applied at every point at the radius r. Suppose that the form of the temperature field could be the same as in the string which is obtained by two cuts along the diameter of the membrane [6].
If $u(a)$ is a given temperature at $r = a$ then the temperature field for $r \leq a$ is

$$u(r) = u(a), \tag{9.5}$$

and for $a \leq r \leq R$

$$u(r) = u(a)\frac{R-r}{R-a}. \tag{9.6}$$

The relation between $Q(a)$ and $u(a)$ follows from the balance of external heat fluxes applied to the cylinder.

$$Q(a) = \frac{u(a)Rk}{a(R-a)}, \tag{9.7}$$

where k is the coefficient of thermal conduction applied at the boundary of the cylinder. The formulas (9.5), (9.6), (9.7) allow to determine the temperature of the cylinder if the external heat fluxes are given.

9.2.1. *Example 1.* Consider the steady state problem. The constant heat source $q(r) = const = q$ is given. Then $Q(a) = qda$ and we obtain

$$u(r) = \int_0^r \frac{qa(R-r)}{kR}da + \int_r^R \frac{qa(R-a)}{kR}da = \frac{q}{6kR}(R^3 - r^3). \tag{9.8}$$

We have $u(0) = \frac{qR^2}{6k}$ and this is 1.5 less then it is in classical case.

9.2.2. *Example 2.* If the axis of the cylinder has a fixed zero temperature and the cylinder is under a constant heat flux q then we obtain the heat flux at the axis from the equation (9.7)

$$S = 2\pi a Q(a)|_{a\to 0} = 2\pi u_S(0)k, \tag{9.9}$$

where the temperature under a flux S should be equal -u(0) according to the equation (9.8). So we have got $u_S = -u(0)$ and then the flux S is

$$S = -2\pi u(0)k = 2\pi k \cdot \frac{qR^2}{6k} = -\frac{\pi q R^2}{3}. \tag{9.10}$$

The temperature field is

$$u(r) = \frac{qr(R^2 - r^2)}{6kR}. \tag{9.11}$$

9.2.3. *Example 3.* Consider the non stationary symmetric problem for a circular cylinder. Then $Q(a) = -c_0\rho\frac{\partial u}{\partial t}da$ and we obtain an integro-differential equation

$$u(r) = -\int_0^r \frac{a(R-r)}{kR} c_0\rho \frac{\partial u}{\partial t}(a)da - \int_r^R \frac{a(R-a)}{kR} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.12}$$

The boundary condition is $u(R) = 0$. The solution of the equation (9.12) is taken in the form $u(r,t) = U(r)\exp(-\omega t)$, where $\omega > 0$ is a constant. This form of solution and the equation (9.12) create the equation

$$U(r) = \frac{c_0\rho\omega}{kR}\left[(R-r)\int_0^r aU(a)da + \int_r^R U(a)a(R-a)da\right]. \tag{9.13}$$

The boundary condition is transformed to $U(R) = 0$. Differentiating the equation (9.13) with respect to r we obtain

$$\frac{dU}{dr}(r) = -\frac{c_0\rho\omega}{kR}\int_0^r aU(a)da. \tag{9.14}$$

The equation (9.14) gives the second condition at $r = 0$

$$\frac{dU}{dr}(0) = 0. \tag{9.15}$$

Differentiating the equation (116) with respect to r we get the equation

$$\frac{d^2U}{dr^2} + \frac{c_0\rho\omega}{kR} rU(r) = 0. \tag{9.16}$$

Let us transform the equation (9.16) introducing the variable

$$\xi = -\sqrt[3]{\frac{c_0\rho\omega}{kR}}\, r. \tag{9.17}$$

Then the equation(9.16) will take the following form

$$\frac{d^2U}{d\xi^2} - \xi U(\xi) = 0. \tag{9.18}$$

That is the Airy equation [12]. The general solution of the equation (9.18) is

$$U(\xi) = C_1 Ai(\xi) + C_2 Bi(\xi), \tag{9.19}$$

where $Ai(\xi)$, $Bi(\xi)$ are the Airy functions, C_1, C_2 are arbitrary constants. Then the boundary condition and condition (9.15) could be satisfied and the equation for ω will be obtained

$$\sqrt{3}Ai\left(-\sqrt[3]{\frac{c_0\rho\omega}{kR}}R\right) + Bi\left(-\sqrt[3]{\frac{c_0\rho\omega}{kR}}R\right) = 0. \tag{9.20}$$

9.2.4. *Example 4.* Let us fix the zero temperature on the axis of the cylinder considered in Example 3. The integro-differential equation of this problem is

$$u(r) = -\int_0^r \frac{a(R-r)}{kR} c_0\rho \frac{\partial u}{\partial t}(a)da - \int_r^0 \frac{a(R-a)}{kR} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.21}$$

This equation (9.21) could be obtained if the value

$$u(0) = -\int_0^R \frac{a(R-a)}{kR} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.22}$$

according to the equation (9.12) will be subtracted from the right side of the equation (9.12).

The boundary conditions are

$$u(0) = 0,\, u(R) = 0. \tag{9.23}$$

Substituting the function $u(r,t) = U(r)\exp(-\omega t)$ into the equations (9.22), (9.23) yields the problem for the function $U(r)$:

$$U(r) = \frac{c_0\rho\omega}{kR}\left[(R-r)\int_0^r aU(a)da + \int_r^0 U(a)a(R-a)da\right], \tag{9.24}$$

$$U(0) = 0,\, U(R) = 0. \tag{9.25}$$

Differentiating two times the equation (9.24) with respect to r we find that the function $U(r)$ satisfies the same equation (9.16) which could be transformed to the Airy equation introducing the new variable (9.17. Then the equation to find ω will be obtained if the general solution satisfies the boundary conditions. That equation is in this case

$$\sqrt{3}Ai\left(-\sqrt[3]{\frac{c_0\rho\omega}{kR}}R\right) - Bi\left(-\sqrt[3]{\frac{c_0\rho\omega}{kR}}R\right) = 0. \tag{9.26}$$

We see that the Airy functions play an important role in solutions of MAC model for heat conduction equation. Both Airy's functions have not singularities on the real axis. These property of the Airy functions differs them from the Bessel functions which are usually arising in the similar classical problems. One of two Bessel's functions has singularity at the origin.

These important property of nonsingularity of the fundamental functions of the corresponding differential MAC model conserves also in MAC model for 3D symmetric heat conduction problem. That will be described below.

9.2.5. *Partial differential equation for 2D heat conduction MAC model.* Let us differentiate the equation (9.12) two times with respect to r. Then the following partial differential equation for 2D heat conduction problem will be obtained for symmetric case

$$\frac{\partial^2 u}{\partial r^2} = \frac{c_0\rho}{kR} r \frac{\partial u}{\partial t}. \tag{9.27}$$

The method of separation of variables can be applied to the equation (9.27). For example the boundary conditions are

$$u(0,t) = 0,\, u(R,t) = 0 \tag{9.28}$$

and the initial conditions are taken as

$$u(r,0) = f(r), \tag{9.29}$$

where $f(r)$ is a given continuous function. The solution of the stated problem includes the Airy functions. The classical solutions of the similar problems for 2D heat conduction should include the Bessel functions.

9.3. **3D heat conduction MAC model.** The cones were used to create the MAC model for the heat conduction problem in above section in [7], where the balance of heat fluxes was satisfied. Similar approach is used in this section to consider the symmetric problems for a ball of the radius R. We employ spherical coordinates (r, θ, φ). Suppose that the nonzero temperature depends on r only. That is $u(r)$. The boundary condition is $u(R) = 0$. Let $Q(r)$ is an external heat flux per unit area applied at every point at the radius r. Suppose that the form of the temperature field could be the same as in the string which is obtained by cuts along the diameter of the ball.
If $u(a)$ is a given temperature at $r = a$ then the temperature field for $r \leq a$ is

$$u(r) = u(a), \tag{9.30}$$

and for $a \leq r \leq R$

$$u(r) = u(a)\frac{R-r}{R-a}. \tag{9.31}$$

The relation between $Q(a)$ and $u(a)$ follows from the balance of external heat fluxes applied to the ball.

$$Q(a) = \frac{u(a)R^2k}{a^2(R-a)}, \tag{9.32}$$

where k is the coefficient of thermal conduction applied at the boundary of the ball. The formulas (9.30), (9.31), (9.32) allow to determine the temperature of the cylinder if the external heat fluxes are given.

9.3.1. *Example 1.* Consider the steady state problem. The constant heat source $q(r) = const = q$ is given. Then $Q(a) = qda$ and we obtain

$$u(r) = \int_0^r \frac{qa^2(R-r)}{kR^2}da + \int_r^R \frac{qa^2(R-a)}{kR^2}da = \frac{q}{12kR^2}(R^4 - r^4). \tag{9.33}$$

We have $u(0) = \frac{qR^2}{12k}$.

9.3.2. *Example 2.* If the center of the ball has a fixed zero temperature and the ball is under a constant heat source q then we obtain the heat flux at the center from the equation (9.32)

$$S = 4\pi a^2 Q(a)|_{a\to 0} = 4\pi u_S(0)Rk, \tag{9.34}$$

where the temperature under a flux S should be equal -u(0) according to the equation (9.33). So we have got $u_S = -u(0)$ and then the flux S is

$$S = -4\pi u(0)Rk = -4\pi kR \cdot \frac{qR^2}{12k} = -\frac{\pi qR^3}{3}. \tag{9.35}$$

The temperature field is

$$u(r) = \frac{qr(R^3 - r^3)}{12kR^2}. \tag{9.36}$$

9.3.3. *Example 3.* Consider the nonstationary symmetric problem for a circular cylinder. Then $Q(a) = -c_0\rho\frac{\partial u}{\partial t}da$ and we obtain an integro-differential equation

$$u(r) = -\int_0^r \frac{a^2(R-r)}{kR^2} c_0\rho \frac{\partial u}{\partial t}(a)da - \int_r^R \frac{a^2(R-a)}{kR^2} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.37}$$

The boundary condition is $u(R) = 0$. The solution of the equation (9.37) is taken in the form $u(r,t) = U(r)\exp(-\omega t)$, where $\omega > 0$ is a constant. This form of solution and the equation (9.37) create the equation

$$U(r) = \frac{c_0\rho\omega}{kR^2}\left[(R-r)\int_0^r a^2U(a)da + \int_r^R U(a)a^2(R-a)da\right]. \tag{9.38}$$

The boundary condition is transformed to $U(R) = 0$. Differentiating the equation (9.38) with respect to r we obtain

$$\frac{dU}{dr}(r) = -\frac{c_0\rho\omega}{kR^2}\int_0^r a^2U(a)da. \tag{9.39}$$

The equation (9.39) gives the second condition at $r = 0$

$$\frac{dU}{dr}(0) = 0. \tag{9.40}$$

Differentiating the equation (9.40) with respect to r we get the equation

$$\frac{d^2U}{dr^2} + \frac{c_0\rho\omega}{kR^2} r^2 U(r) = 0. \tag{9.41}$$

Let us transform the equation (9.41) introducing the variable

$$\xi = \sqrt[4]{\frac{c_0\rho\omega}{kR^2}}\, r. \tag{9.42}$$

Then the equation(9.41) will take the following form

$$\frac{d^2U}{d\xi^2} + \xi^2 U(\xi) = 0. \tag{9.43}$$

The equation (9.43) is similar the Airy equation [12] in the sense that it has two fundamental solution without any finite point of singularity. The first independent fundamental solution of the equation (9.43) is

$$U_1(\xi) = \sum_{n=0}^{\infty} a_{4n}\xi^{4n}, \tag{9.44}$$

where

$$a_0 = 1,\ a_{4n} = -\frac{a_{4n-4}}{4n(4n-1)},\ n = 1, 2, 3, \ldots. \tag{9.45}$$

The second independent fundamental solution of the equation (9.43) is

$$U_2(\xi) = \sum_{n=0}^{\infty} a_{4n+1}\xi^{4n+1}, \tag{9.46}$$

where

$$a_1 = 1,\ a_{4n+1} = -\frac{a_{4n-3}}{4n(4n+1)},\ n = 1, 2, 3, \ldots. \tag{9.47}$$

The general solution of the equation (9.43) is

$$U(\xi) = C_1 U_1(\xi) + C_2 U_2(\xi), \tag{9.48}$$

where $U_1(\xi)$, $U_2(\xi)$ are the fundamental solutions (9.44), (9.46), C_1, C_2 are arbitrary constants. Then the boundary condition and condition (9.40) could be satisfied and the equation for ω will be obtained. We have

$$U_1\left(\sqrt[4]{\frac{c_0\rho\omega}{kR^2}}R\right) = 0. \tag{9.49}$$

9.3.4. *Example 4.* Let us fix the zero temperature at the center of the ball considered in Example 3. The integro-differential equation of this problem is

$$u(r) = -\int_0^r \frac{a^2(R-r)}{kR^2} c_0\rho \frac{\partial u}{\partial t}(a)da - \int_r^0 \frac{a^2(R-a)}{kR^2} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.50}$$

This equation (9.50) could be obtained if the value

$$u(0) = -\int_0^R \frac{a^2(R-a)}{kR^2} c_0\rho \frac{\partial u}{\partial t}(a)da. \tag{9.51}$$

according to the equation (9.37) will be subtracted from the right side of the equation (9.37).
The boundary conditions are

$$u(0) = 0,\ u(R) = 0. \tag{9.52}$$

Substituting the function $u(r,t) = U(r)\exp(-\omega t)$ into the equations (9.50), (9.50) yields the problem for the function $U(r)$:

$$U(r) = \frac{c_0\rho\omega}{kR^2}\left[(R-r)\int_0^r a^2U(a)da + \int_r^0 U(a)a^2(R-a)da\right], \tag{9.53}$$

$$U(0) = 0,\ U(R) = 0. \tag{9.54}$$

Differentiating two times the equation (9.53) with respect to r we find that the function $U(r)$ satisfies the same equation (9.41) which could be transformed to the Airy like equation introducing the new variable (9.43). Then the equation to find ω will be obtained if the general solution satisfies the boundary conditions. That equation is in this case

$$U_2\left(\sqrt[4]{\frac{c_0\rho\omega}{kR^2}}R\right) = 0. \tag{9.55}$$

We see that the Airy like functions play an important role in solutions of MAC model for 3D heat conduction equation. Both fundamental functions have not singularities on the real axis. These property of the Airy like functions differs them from the Bessel functions which are usually arising in the similar classical problems. One of two Bessel's functions has singularity at the origin.

9.3.5. *Partial differential equation for 3D heat conduction MAC model.* Let us differentiate the equation (9.37) two times with respect to r. Then the following partial differential equation for 3D heat conduction problem will be obtained for symmetric case

$$\frac{\partial^2 u}{\partial r^2} = \frac{c_0 \rho}{k R^2} r^2 \frac{\partial u}{\partial t}. \tag{9.56}$$

The method of separation of variables can be applied to the equation (9.56). For example the boundary conditions are

$$u(0,t) = 0,\ u(R,t) = 0 \tag{9.57}$$

and the initial conditions are taken as

$$u(r,0) = f(r), \tag{9.58}$$

where $f(r)$ is a given continuous function. The solution of the stated problem includes a set of Airy like functions. The classical solutions of the similar problems for 3D heat conduction should include the Bessel functions.

10. Tension of an elastic bar

10.1. **Statement of the problem.** Consider the simple tension of an elastic bar. The equation of one-dimensional motion of a bar is

$$\frac{\partial N}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} - q(x,t), \tag{10.1}$$

where $N-$ is the normal force applied to the cross-section of a bar, $x-$ is a Cartesian coordinate of a cross-section, $0 < x < L$, $L-$ is the length of a bar, $t-$ is time, $q(x,t)-$ is the density of the longitudinal body forces per unit length.
The Hook law is

$$N = EA\varepsilon, \tag{10.2}$$

where $E-$ is the Young modulus, $A-$ is the cross-sectional area, $\varepsilon-$ is the longitudinal strain which is supposed to be

$$\varepsilon = \frac{\partial u}{\partial x}. \tag{10.3}$$

Substituting the equations (10.2), (10.3) into the equation (10.1) we obtain the equation

$$EA\frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} - q(x,t) \tag{10.4}$$

or

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - p(x,t), \tag{10.5}$$

where

$$c^2 = \frac{EA}{\rho},\ p(x,t) = \frac{q(x,t)}{\rho}. \tag{10.6}$$

The equation (10.5) could be applied to the limited and also to the unlimited bar. The initial and boundary conditions should be applied to obtain the unique solution of the problem.
Consider the steady state problem for a bar as one particular problem. Let the

distributed forces are not given. Then the function u does not depend on time t and the equation (10.5) becomes

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{10.7}$$

Consider the boundary conditions

$$u(0) = u_0,\ u(L) = 0. \tag{10.8}$$

The general solution of the equation (10.7) is

$$u(x) = Ax + B, \tag{10.9}$$

where $A, B-$ are arbitrary constants. If the length of the bar is limited bar then the solution of problem (10.7), (10.8) is

$$u = u_0 \left(1 - \frac{x}{L}\right). \tag{10.10}$$

If the length of the bar is infinite then the solution of the stated problem could be obtained as a limit $L \to \infty$ in the solution (10.10) for the finite bar. The solution will take the form

$$u = u_0,\ 0 \leq x \leq \infty. \tag{10.11}$$

Another solution will be obtained if we take the general solution (10.9) and satisfy the second boundary condition (10.8) at infinity. Then we get

$$A = 0,\ B = 0 \tag{10.12}$$

and the solution is

$$u = u_0,\ x = 0, \tag{10.13}$$

$$u = 0,\ 0 < x < \infty. \tag{10.14}$$

The situation for unlimited bar is undetermined because we have two different solutions (10.12) and (10.13), (10.14). We can improve this situation introducing the MAC model which must have the unique determined solution for both limited and unlimited bars.

10.2. **Differential MAC model.** Let the linear term is introduced into the equation (10.7):

$$\frac{\partial^2 u}{\partial x^2} - au = 0,\ 0 < x < \infty, \tag{10.15}$$

where $a > 0$ is a parameter which should be determined from an experiment additionally. The Hook law corresponding to the equation (10.15) will take the following form

$$\frac{\partial N}{\partial x} = EA\frac{\partial \varepsilon}{\partial x} - EAau. \tag{10.16}$$

The general solution of the equation (10.15) is

$$u = A\exp(\sqrt{a}x) + B\exp(-\sqrt{a}x), \tag{10.17}$$

where $A, B-$ are arbitrary constants.
The finite bar with the boundary conditions (10.8) has the solution

$$u = u_0 \frac{\sinh[\sqrt{a}(L-x)]}{\sinh[\sqrt{a}L]}. \tag{10.18}$$

The solution (10.18) is suitable for the unlimited bar too.

11. Conclusion

The differential MAC models of many physical theories may be created in similar way replacing the Laplace operator through the given differential operators in MAC models for membrane. Examples of the theories which could give the differential MAC models are Navier-Stoke's equations, Maxwell's equations, Schroedinger equation, Klein- Gordon equation, heat conduction equation. The limited number of pages does not allow to consider all of them. But the idea and the presented methods should be enough to develop and apply the MAC theory in many cases of the real life situations.
The MAC model for a bar was given to show another way to introduce the MAC model, where was used a generalization of the Hook law.

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PAIRWISE LIKELIHOOD PROCEDURE FOR TWO-SAMPLE LOCATION PROBLEM

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ABSTRACT. This paper is about estimating shift parameter by using pairwise differences in the two-sample location problem, which assumes G(x)=F(x-Δ). The parameter Δ is called location *shift parameter* between populations of F(x) and G(x). Distribution and density functions of the pairwise differences can be found and used to construct a log likelihood function with respect to the shift parameter. An estimator of the shift parameter is found by Newton's one step algorithm from the log likelihood function. Asymptotic properties of the new estimator which is similar to a regular MLE estimator are shown under some regularity conditions. As an example, normal and Laplace Distribution model assumptions are investigated using the proposed approach. Moreover, a hypothesis testing procedure is developed and shown that pairwise difference approach is asymptotically equivalent to the Rao's score type likelihood test.

1. INTRODUCTION

Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be two independent i.i.d samples from continuous distribution functions F(x) and G(x), respectively. We assume a relationship of G(x)=F(x-Δ) where Δ is a location *shift parameter* between F(x) and G(x). Therefore, we will consider a location shift model and focus our attention to estimate of the shift parameter of Δ. A hypothesis testing for this model could be defined by,

$$H_0 : \Delta = \Delta_0 \text{ vs } H_a : \Delta \neq \Delta_0$$

If $\Delta_0 = 0$, the hypothesis test becomes:

$$H_0 : F(x) = G(x) \text{ vs } H_a : F(x) \neq G(x)$$

which is very common in two sample location problem.

The problem of estimating the shift parameter Δ_0 has been studied extensively in the past. It can be shown that the classic least squares method (minimizing the L_2 norm) leads to $\widehat{\Delta}_{LS} = \overline{Y} - \overline{X}$. It has been shown by Hettmansperger-McKean [3] that

$$\sqrt{n}(\widehat{\Delta}_{LS} - \Delta_0) \rightarrow \mathrm{N}(0, \sigma^2 \tfrac{1}{\lambda(1-\lambda)})$$

where σ^2 is the common variance of the population distributions, $G(x)$ and $F(x)$, and $n_1/n \rightarrow \lambda$ as $n \rightarrow \infty$. Hodges-Lehmann [4], showed that the shift parameter estimator based on Wilcoxon ranks is given by

$$\widehat{\Delta}_R = med_{i,j}\{Y_j - X_i\}$$

which is the median of the pairwise differences. Hodges-Lehmann [4] also showed that

$$\sqrt{n}(\widehat{\Delta}_R - \Delta_0) \rightarrow \mathrm{N}(0, \tau^2 \tfrac{1}{\lambda(1-\lambda)})$$

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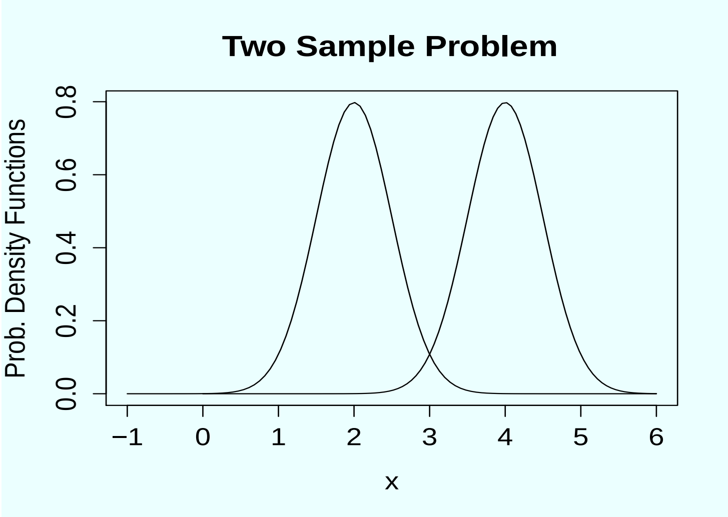

FIGURE 1. Illustration of Two Sample Location Problem

where the scale parameter $\tau = [\sqrt{12}\int f^2(x)dx]^{-1}$ and $n_1/n \to \lambda$ as $n \to \infty$. Anderson and Hettmansberger [1] showed that

$$\sqrt{n}(\widehat{\Delta}_G - \Delta_0) \to N\left(0, \frac{\delta^2 E[\tau^2(x)]}{[E\tau'(x)]^2}\frac{1}{\lambda(1-\lambda)}\right),$$

where δ is the scale parameter, $\tau(t) = \int \psi((t-u)/\delta)f(u)du$ and τ' is the derivative of τ. Tasdan-Sievers [6] proposed a smoothed Mann-Whitney-Wilcoxon approach to find an estimator for Δ. They showed that

$$\sqrt{n}(\widehat{\Delta}_s - \Delta_0) \to N(0, \frac{1}{c^2})$$

and the efficacy $c = \mu'(0)/\sigma(0)$, where

$$\mu'(0) = \int\int l(y-x)dF(x)dF(y) \text{ and } \sigma(0) = \sqrt{\frac{\sigma_1^2}{\lambda(1-\lambda)}}.$$

In Section 2, we will introduce the main idea of the study. It will be shown that by using pairwise differences, a likelihood function can be constructed and solved to estimate the shift parameter. In addition, a test procedure will be developed to test the hypothesis defined above. In Section 3, the properties of the estimator such as asymptotic normality will be shown. Another theorem proves that the proposed method is an equivalent of Rao's score type test. In Section 4, example of several models will be applied to the proposed solutions. The paper ends with a conclusion in Section 5.

2. PROPOSED PROCEDURE

The main idea behind the proposed procedure is to find the distribution function of the pairwise differences. First, consider that we F(x)=G(x), which assumes no shift model. Let $Z_{ij} = Y_j - X_i$ for all i and j differences and $H(z) = P(Y_j - X_i \le z)$.

We define

$$\begin{aligned}
P(Z_{ij} < z) &= P(Y_j - X_i < z) \\
&= \int P(Y_j - X_i \leq z | X_i = x) dF(x) \\
&= \int P(Y_j \leq z + x) dF(x)
\end{aligned}$$

$$H(z) = \int G(x+z) dF(x) \tag{2.1}$$

The resulting H(z) is the distribution function (CDF) of the $Z_{ij} = Y_j - X_i$ pairwise differences. Now consider that $F(x - \Delta) = G(x)$, which assumes a shift in the model. Then, we will have

$$H(z) = \int G(x+z) dF(x)$$

$$H_\Delta(z) = \int F(x + z - \Delta) dF(x) \tag{2.2}$$

Next, by assuming that it exists, the probability density function $h_\Delta(z)$, can be found by

$$h(z, \Delta) = \frac{dH(z)}{dz} = \int f(x + z - \Delta) f(x) dx \tag{2.3}$$

The result is like a convolution operation that convolutes two functions. Let $h_\Delta(z) = u(z - \Delta)$. Therefore, we can consider the problem as a location parameter problem. The log-likelihood function of the pairwise differences of the data by using $u(z - \Delta)$ is

$$L(\Delta) = \prod_i \prod_j u(y_j - x_i - \Delta)$$

$$l(\Delta) = log[L(\Delta)] = \sum_i \sum_j log[u(y_j - x_i - \Delta)]$$

$$l'(\Delta) = \frac{\partial}{\partial \Delta} log[L(\Delta)] = -\sum_i \sum_j \frac{u'(y_j - x_i - \Delta)}{u(y_j - x_i - \Delta)} \tag{2.4}$$

To estimate Δ parameter, $l'(\Delta)$ will be set to zero and solved for Δ. $l'(\Delta)$ can be considered a score function which determines the estimating equations for the MLE estimator of Δ. However, there might be no root or there might be more than one root. In that case, a maximizing value of the estimator should be taken as MLE estimator. Theorem 6.1.1 from Hogg-McKean-Craig [2] states that asymptotically the likelihood function is maximized at true value Δ_0 of the parameter. Therefore, it is appropriate to take the value that maximizes the likelihood function for more than one root cases. Still it could be difficult or impossible to find an explicit formula for some estimators but a solution can be found by a numerical approximation method. One of the iterative methods that could be used is the Newton's one-step estimator which requires that the initial value must be a consistent estimator. Newton's iteration starts with an initial estimate of Δ. Let $\widetilde{\Delta}$ be the initial value and a consistent estimator of Δ, then set

$$\widehat{\Delta} = \widetilde{\Delta} - \frac{l'(\widetilde{\Delta})}{l''(\widetilde{\Delta})} \tag{2.5}$$

The result is the one step estimator of Δ. An algorithm will be provided for the proposed estimator in the appendix section. In addition, R program has "uniroot" function available for this type of problem. The resulting estimator, we call $\widehat{\Delta}$, is the Maximum Likelihood Estimator (MLE) of the true shift parameter based on the pairwise difference. An example of the proposed solution will be given in section 4.

3. Properties of Proposed Solution

One of the advantages of using pairwise differences is that it can be treated as one sample location parameter problem. A score type likelihood test can be developed so that there is no need for an estimate of Δ. In the next two theorems, we show that under same regularity conditions, the proposed estimator is consistent and has an asymptotic distribution of $\widehat{\Delta}$. First, we will show that the proposed estimator is consistent by Theorem 3.1. Before that we need to make some assumptions (regularity conditions). These assumptions are similar to the regular maximum likelihood assumptions.

Assumptions(Regularity Conditions):
(A1) $h(z,\Delta)$ is a distinct pdf; i.e $\Delta \neq \Delta' \Rightarrow h(z,\Delta) \neq h(z,\Delta')$.
(A2) $h(z,\Delta)$ have common support for all $\Delta \in \Omega$.
(A3) The point Δ_0 is an interior point in Ω.
(A4) $h(z,\Delta)$ is three times differentiable as a function of Δ.
(A5) The integral $\int h(z,\Delta)dz$ can be differentiated twice under the integral sign a function of Δ.

Theorem 3.1. *Suppose that the regularity conditions A1-A2 hold and $h(z,\Delta)$ is differentiable with respect to Δ in Ω. Then, with probability approaching 1 as $n \to \infty$, there exist $\widehat{\Delta}$ such that $l'(\widehat{\Delta}) = 0$ and $\widehat{\Delta} \xrightarrow{P} \Delta_0$.*

Above theorem can be proven by Theorem 6.1.3 from Hogg-McKean-Craig [2]. Therefore, the proof will not be discussed here. By the following theorem, we show that the proposed estimator is asymptotically normal as $n \to \infty$.

Theorem 3.2. *Assume that the regularity conditions and Theorem 3.1 hold. Also assume that the Fisher information satisfies $0 < I(\Delta_0) < \infty$. Finally, assume that $l(\Delta)$ has three derivatives in a neighborhood of Δ_0 and $l'''(\delta)$ is uniformly bounded in this neighborhood. Then, we have*

$$\sqrt{n}(\widehat{\Delta} - \Delta_0) \xrightarrow{D} N(0, \tfrac{1}{I(\Delta_0)})$$

Proof. The proof is a typical MLE proof that can be found in Serfling [5] or Hogg-McKean-Craig [2]. By using second order Taylor expansion of $l'(\Delta)$ at Δ_0 and evaluating $l'(\Delta)$ at $\widehat{\Delta}$, we get

$$l'(\widehat{\Delta}) = l'(\Delta_0) + (\widehat{\Delta} - \Delta_0)l''(\Delta_0) + \tfrac{1}{2}(\widehat{\Delta} - \Delta_0)^2 l'''(\Delta^\star)$$

where $\Delta^\star$ is between $\widehat{\Delta}$ and Δ_0. Since $l'(\widehat{\Delta}) = 0$, we can rearrange the last equation as

$$\sqrt{n}(\widehat{\Delta} - \Delta_0) = \frac{\sqrt{n}l'(\Delta_0)}{-n^{-1}l''(\Delta_0) - (2n)^{-1}(\widehat{\Delta} - \Delta_0)l'''(\Delta^\star)}$$

By the Central Limit Theorem and Law of Large Numbers,

$$\frac{1}{\sqrt{n}} l'(\Delta_0) \xrightarrow{D} N[0, I(\Delta_0)]$$

and

$$-n^{-1} l''(\Delta_0) \xrightarrow{P} I(\Delta_0)$$

where $I(\Delta_0) = V[\frac{\partial}{\partial \Delta} \log u(Y - X - \Delta_0)]$. We will assume that the second term in the denominator of the expression goes to zero as $n \to \infty$ and $n^{-1} l'''(\Delta^\star)$ is bounded in probability. Therefore, the proof is complete. □

In the next theorem and definition, we show that the proposed pairwise likelihood method is equivalent to Rao's score type test.

Theorem 3.3. *Assume that the regularity conditions and Theorem 3.2 hold. Under the null hypothesis, $H_0 : \Delta = \Delta_0$,*

$$R_n^2 \xrightarrow{D} \chi^2(1)$$

where the test statistic $R_n^2 = (\frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}})^2$ and χ_1^2 is the Chi-Square random variable with degrees of freedom of 1.

Proof. By the central limit theorem and $I(\Delta) = Var\left(\frac{\partial}{\partial \Delta} log[u(Y - X - \Delta)]\right) < \infty$, we can write that

$$\frac{1}{\sqrt{n}} l'(\Delta_0) = \sqrt{n}\left(\frac{1}{n} \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} \frac{\partial}{\partial \Delta} log[u(y_j - x_i - \Delta)]\right) \xrightarrow{D} N[0, I(\Delta_0)]$$

where $n = n_1 n_2$. From the fundamental theorems of mathematical statistics, we know that the square of a standard normal random variable is a chi square with degrees of freedom of 1. Thus, we have

$$R_n = \frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}} \xrightarrow{D} N(0, 1) \tag{3.1}$$

and

$$R_n^2 = \left(\frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}}\right)^2 \xrightarrow{D} \chi^2(1) \tag{3.2}$$

Theorem 3.3 also proves that the pairwise likelihood approach is equivalent to the Rao's score type test at the asymptotic level. □

In the following definition, an asymptotic α level hypothesis test for the pairwise likelihood approach has been defined.

Definition 3.4. *Let Z_{ij} be the pairwise difference of $Y_j - X_i$ for all i and j. Z_{ij} are independent and identically distributed with distribution function $P(Z_{ij} \leq x) = H_\Delta(z - \Delta)$, where $h(z, \Delta) = H'_\Delta(z)$ exists. Also assume that $Var(Z_{ij}) = \sigma_z^2$. Then, an asymptotic α level test for , $H_0 : \Delta = \Delta_0$ vs $H_a : \Delta \neq \Delta_0$, is any test that rejects H_0 in favor of H_a when $|R_n| \geq z_{\alpha/2}$ where $R_n = \frac{(\overline{Z}_n - \Delta_0)}{\sigma_z/\sqrt{n}}$ and $z_{\alpha/2}$ is the critical value.*

It can be shown that likelihood ratio, Wald and Rao's score type tests are all asymptotically equivalent tests under H_o. Therefore, all three tests must reach the same decision with probability approaching 1 as $n \to \infty$.

4. Examples

Several examples will be provided in this section. Different population distributions result an estimator in different classes such as Normal distribution assumption results an estimator which is similar to the least square estimator, on the other hand, Laplace distribution assumption results Hodges-Lehmann type estimator.

4.1. **Example #1.** This example will demonstrate the proposed solution under the normality of the random samples assumption. Assume that $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ are two independent iid samples from $N(\mu_x, \sigma^2)$ and $N(\mu_y, \sigma^2)$ distributions, respectively. Define $H_0 : \Delta = \Delta_0$, where $\Delta = \mu_y - \mu_x$. Let $Z_{ij} = Y_j - X_i$ be the pairwise differences. By the equation (2.3), we have

$$h(z, \Delta) = \int f(x + z - \Delta) f(x) dx \tag{4.1}$$

By the normality assumption, $f(x) = \frac{1}{\sqrt{2\pi}} \exp^{-[(x)^2/2]}$, where we also assume that $\mu_x = 0$ and $\sigma^2 = 1$ to simplify the process. If we plug in $f(x)$ into $h(z, \Delta)$,

$$\begin{aligned}
h(z, \Delta) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp^{-[(x+z-\Delta)^2/2]} * \frac{1}{\sqrt{2\pi}} \exp^{-[(x)^2/2]} \, dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp^{-[(x+z-\Delta)^2 - x^2]/2} \, dx \\
&= \frac{1}{2\pi} \exp^{-(z-\Delta)^2/4} \int_{-\infty}^{+\infty} \exp^{-[x+(z-\Delta)/2]^2} \, dx \\
&= \frac{1}{2\pi} \exp^{-(z-\Delta)^2/4} \int_{-\infty}^{+\infty} \frac{\sqrt{\pi}}{\sqrt{\pi}} \exp^{-[x+(z-\Delta)/2]^2} \, dx \\
&= \frac{\sqrt{\pi}}{2\pi} \exp^{-(z-\Delta)^2/4} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} \exp^{-[x+(z-\Delta)/2]^2/(2*1/2)} \, dx. \qquad (4.2)
\end{aligned}$$

The integral inside the function is a normal pdf with $\mu = \frac{\Delta - z}{2}$ and $\sigma^2 = 1/2$. Therefore, by integrating it from $-\infty$ to $+\infty$, we get 1. The term in front of the integral is

$$h(z, \Delta) = \frac{1}{\sqrt{4\pi}} \exp^{-(z-\Delta)^2/4}, \qquad z \in (-\infty, +\infty). \tag{4.3}$$

which is a normal pdf with $\mu_z = \Delta$ and $\sigma_z^2 = 2$. If f(x) is normal pdf with $\mu_x = 0$ and σ^2, then $h(z, \Delta)$ will have a normal pdf with $\mu_z = \Delta$ and $\sigma_z^2 = 2\sigma^2$.

We set $h(z, \Delta) = u(z - \Delta)$ as defined by the equation (2.3) which assumes that we have a location model and the parameter is Δ. By the equation (2.4), we will have,

$$l'(\Delta) = -\sum_{i}^{n_1} \sum_{j}^{n_2} \left(\frac{z_{ij} - \Delta}{\sigma_z^2} \right)$$

We set $l'(\Delta) = 0$ and solve for Δ. It is not difficult to see that the estimator is $\widehat{\Delta} = \overline{Y} - \overline{Y}$. In fact, this is known as the least square estimator of shift parameter in the literature. Moreover, Rao's score type test can be developed by the result of the Theorem 3.3:

$$R_n^2 = \left(\frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}} \right)^2 \tag{4.4}$$

We first find the likelihood function, $L(\Delta)$, of the paired differences.

$$\begin{aligned} L(\Delta) &= \prod_i^{n_1} \prod_j^{n_2} u(z_{ij} - \Delta) \\ &= \prod_i^{n_1} \prod_j^{n_2} \left(\frac{1}{\sqrt{2\pi}\sigma_z} \right) e^{-\sum_i^{n_1} \sum_j^{n_2} \frac{(z_{ij}-\Delta)^2}{2\sigma_z^2}} \end{aligned}$$

By adding and subtracting $\overline{z}$ inside the exponential term, and working it out, we get,

$$\begin{aligned} &= (\frac{1}{2\pi\sigma_z^2})^{n_1 n_2/2} e^{-\sum_i^{n_1} \sum_j^{n_2} \frac{(z_i - \overline{z} + \overline{z} + \Delta)^2}{2\sigma_z^2}} \\ &= (\frac{1}{2\pi\sigma_z^2})^{n_1 n_2/2} e^{-\sum_i^{n_1} \sum_j^{n_2} \frac{(z_{ij} - \overline{z})^2}{\sigma_z^2}} e^{-\sum_i^{n_1} \sum_j^{n_2} \frac{(\overline{z} - \Delta)^2}{\sigma_z^2}} \end{aligned} \tag{4.5}$$

By setting $n = n_1 n_2$, taking the log of both sides and derivative with respect to Δ, we find that

$$l'(\Delta) = \frac{\partial}{\partial \Delta} log[L(\Delta)] = \frac{(\overline{z} - \Delta)}{\sigma_z^2 / n}$$

$$\frac{\sigma_z}{\sqrt{n}} l'(\Delta) = \frac{(\overline{z} - \Delta)}{\sigma_z / \sqrt{n}} \tag{4.6}$$

By taking the square of the above result, we have

$$\left[\frac{\sigma_z}{\sqrt{n}} l'(\Delta) \right]^2 = \left(\frac{\overline{z} - \Delta}{\sigma_z / \sqrt{n}} \right)^2 \tag{4.7}$$

We define a test statistic $R_n^2 = \left(\frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}} \right)^2$ where $I(\Delta_0) = \frac{1}{\sigma_z^2}$ which is the Fisher Information. The right side of the equation is z^2 and has a $\chi^2(1)$ under H_0. This is similar to Rao's score type test statistics and proven by the Theorem 3.3. If we use the equation (4.7) above, an α level test based on Normal Distribution model example can be developed as

To test hypothesis of $H_0 : \Delta = \Delta_0$ vs $H_1 : \Delta \neq \Delta_0$, we use the test statistics:

$$R_n^2 = \left(\frac{\overline{z} - \Delta_0}{\sigma_z / \sqrt{n}} \right)^2 \longrightarrow \chi^2(1)$$

A decision rule for a size α test is to reject H_0 if $R_n^2 \geq \chi_\alpha^2(1)$.

4.2. **Example #2.** In this example, we will demonstrate the proposed estimator under the assumption that F(x) is an exponential distribution with λ parameter. First, we assume that $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ are two independent iid samples from F(x) and G(x), respectively. Define $H_0 : \Delta = \Delta_0$, where $\Delta = \mu_y - \mu_x$. Let $Z_{ij} = Y_j - X_i$ be the pairwise differences. By the equation (2.3), we have

$$h(z, \Delta) = \int f(x + z - \Delta) f(x) dx \tag{4.8}$$

By the assumption that $F(x)$ has an exponential distribution with λ, we have $f(x) = \frac{1}{\lambda}\exp^{-x/\lambda}$ for $x > 0$. If we plug in $f(x)$ into $h(z, \Delta)$ and if $z - \Delta > 0$,

$$
\begin{aligned}
h(z,\Delta) &= \int_{-\infty}^{+\infty} \frac{1}{\lambda}\exp^{-(x+z-\Delta)/\lambda} * \frac{1}{\lambda}\exp^{-x/\lambda}\,dx \\
&= \frac{1}{\lambda^2}\int_{-\infty}^{+\infty} \exp^{-(x+z-\Delta)/\lambda - x/\lambda}\,dx \\
&= \frac{1}{\lambda^2}\int_{0}^{+\infty} \exp^{-(x+z-\Delta-x)/\lambda}\,dx \\
&= \frac{1}{\lambda^2}\exp^{-(z-\Delta)/\lambda}\int_{0}^{+\infty} \exp^{-(2x)/\lambda}\,dx \\
&= \frac{1}{\lambda^2}\exp^{-(z-\Delta)/\lambda}\int_{0}^{+\infty} \frac{\lambda/2}{\lambda/2}\exp^{-x/\lambda/2}\,dx \\
&= \frac{\lambda/2}{\lambda^2}\exp^{-(z-\Delta)/\lambda}\int_{0}^{+\infty} \frac{1}{\lambda/2}\exp^{-x/\lambda/2}\,dx \qquad (4.9)
\end{aligned}
$$

The integral part of the function inside is an exponential pdf with $\lambda/2$, therefore, the integrating it from 0 to $+\infty$ gives us 1. The term in front of the integral is

$$h(z,\Delta) = \frac{1}{2\lambda}\exp^{-(z-\Delta)/\lambda}, \qquad z - \Delta > 0. \tag{4.10}$$

If we assume $z - \Delta < 0$, and applying the similar approach as above, we get

$$h(z,\Delta) = \frac{1}{2\lambda}\exp^{-(\Delta-z)/\lambda}, \qquad z - \Delta < 0. \tag{4.11}$$

Therefore,

$$h_\Delta(z) = \frac{1}{2\lambda}e^{-|z-\Delta|/\lambda}, \qquad -\infty < z < \infty. \tag{4.12}$$

which is a Laplace distribution with $\mu_z = \Delta$ and $\sigma_z^2 = \lambda$. Define $H_0 : \Delta = \Delta_0$. The likelihood function is

$$L(\Delta) = (2\lambda)^{-n}e^{-\sum_i^{n_1}\sum_j^{n_2}|y_j - x_i - \Delta|/\lambda} \tag{4.13}$$

The score function is

$$l'(\Delta) = \frac{\partial}{\partial\Delta}log[L(\Delta)] = \sum_i^{n_1}\sum_j^{n_2} sign(y_j - x_i - \Delta)/\lambda \tag{4.14}$$

We set this result to "zero" and solve for Δ. We find that $\widehat{\Delta} = median\{y_j - x_i\}$ which is equivalent to the Hodges-Lehmann estimator of Δ.

An asymptotic α level test based on Laplace distribution model example can be developed as well. To test hypothesis of $H_0 : \Delta = \Delta_0$ vs $H_1 : \Delta \neq \Delta_0$, we use the test statistics:

$$R_n^2 = \left(\frac{l'(\Delta_0)}{\sqrt{nI(\Delta_0)}}\right)^2 = (S)^2/n \longrightarrow \chi_\alpha^2(1)$$

where Fisher information, $I(\Delta_0) = \lambda$ and $S = \sum_i^{n_1}\sum_j^{n_2} sign(y_j - x_i - \Delta_0)$

A decision rule for a size α test is to reject H_0 if $R_n^2 \geq \chi^2(1)$.

5. Conclusion

We showed that by using the pairwise differences of two random samples, an estimator of shift parameter, Δ, can be estimated. The proposed method uses $Z_{ij} = Y_j - X_i$ differences and assumes that Z_{ij} has a pdf of $h(z;\Delta)$, where Δ is the location parameter. The theory of the method is similar to the typical maximum likelihood theorems and conditions. An estimator of the shift, $\widehat{\Delta}$, can be found by Newton's one step estimator if there is no explicit result found for the estimator. In fact, R algorithm for this estimator is provided in the appendix. Asymptotic properties of the estimator are shown in section 3. It has been shown that an estimator from the pairwise differences has asymptotic normality under some regularity conditions. An asymptotic level score test (Rao's score test) is also developed for the estimator. Moreover, in section 4, two examples which are provided in the study show that under the normality of F(x), the resulting estimator is equal to the least squares estimator, $\widehat{\Delta} = \overline{Y} - \overline{X}$ and under the assumption of exponential distribution of F(x), the resulting estimator is equal to Hodges-Lehmann estimator, $\widehat{\Delta} = Median\{Y_j - X_i\}$. One of the main advantages of using pairwise differences is to estimate the shift parameter with only one known distribution function, F(x), instead of two. As a result, using pairwise differences of the two samples, a pdf of $h(z,\Delta)$ for the differences can be found. Also assuming Δ as a location parameter, two sample location problem can be treated as one sample location problem and $\widehat{\Delta}$ can be found by maximizing the log likelihood function of $h(z,\Delta)$.

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6. Appendix

This section contains R algorithms used in the estimation of shift parameter and pdf of $h(z,\Delta)$. These algorithms can also be reached from Tasdan [7].

```
plog<-function(t,x,y){
n1<-length(x)
n2<-length(y)
n<-n1*n2
sig<-spool(x,y)
a<-outer(y,x,"-")
da <- c(a[row(a) <= col(a)])
```

```
db <- c(a[row(a) > col(a)])
dab <- append(da, db)
b<-rep(0,n)
for(i in 1:n){
b[i]<-log(f1(dab[i],x,dab,t))
}
l<--sum(b)
l
}
```

Convolution function to find $h(z, \Delta)$:

```
f1<-function(z,dab,t){
    sig<-mad(dab) # robust estimate of deviation by MAD function
    m<-mean(dab)
    #a<-integrate(function(x) {dnorm((x+z-t),0,sig)*dnorm(x,0,sig)}, -Inf, Inf)$value
    if((z-t)>0){a<-integrate(function(x) {dexp((x+z-t),m)*dexp(x,m)}, 0, Inf)$value}
    else{a<-integrate(function(x) {dexp((x-(z-t)),m)*dexp(x,m)}, 0, Inf)$value}
    #a<-integrate(function(x) {dcauchy((x+z-t),0,sig)*dcauchy(x,0,sig)}, -Inf, Inf)$value
    #a<-integrate(function(x) {dlaplace((x+z-t),0,1/m)*dlaplace(x,0,1/m)}, 0, Inf)$value
    #a<-integrate(function(x) {dfun((x+z-t)/sig)*dfun(x/sig)}, 0, Inf)$value
    #a<-integrate(function(x) {dunif((x+z-t),0,1)*dunif(x,0,1)}, -Inf, Inf)$value
    a
}
```

Minimization of log likelihood to estimate the shift parameter. The function uses plog function from above:

```
finder<-function(x, y)
{
# Estimating Shift parameter by using nlm(nonlinear minimization) function in R
# function "plog" has to be used here
n1 <- length(x)
n2 <- length(y)
n <- n1 + n2
options(warn=-1)
d1<-nlm(plog,0,x=x,y=y)
d2 <- round(d1$estimate, 7)
d2
}
```

Newton's one step estimator:

```
finder1<-function(x,y,tol,dl,du){# finding shift parameter via
                              #Newton's one step...
n<-length(x)
m<-length(y)
change<-100
step<-0
dold<-du
while(change>tol&&step<50){
s1<-plog(x,y,dl)   #plog is required function
s2<-plog(x,y,du)
```

```
d<-dl-((s1*(du-dl))/(s2-s1))
change<-abs((d-dold)/d)
if(change<tol){
break
}
else{dold<-d
s3<-plog(x,y,d)
if((s1*s3)>0){
dl<-d
}
else{du<-d
}
}
step<-step+1
cat("step=",step,"Est=",round(d,4),"\n")
}
d
}
```

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A MODIFIED ADOMIAN APPROACH APPLIED TO NONLINEAR FREDHOLM INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we introduce the linearization method and the modified Adomian method applied to non linear Fredholm integral equations. To assess the applicability, simplicity and the accuracy of the modified Adomian technique, we applied the both methods on selected non-linear Fredholm integral equations. This study showed the applicability, simplicity, accuracy and the fast speed of convergent of the modified Adomian method, comparing with the linearization method, even when the accuracy of the linearization method improved by employing variable steps size.

1. LINEARIZATION METHOD FOR NONLINEAR FREDHOLM INTEGRAL EQUATIONS

The linearization method based on the piecewise linearization of the nonlinear integral equations, and the analytical solution of the resulting linear integral equation. Refs. [7, 2, 6, 9] applied this technique to find numerical solution for non-linear Volterra integral equation in the interval $[0, 1]$. In this section, we follow these studies and introduce the Linearization method for the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b k(x, t, u(t))\, dt, \tag{1.1}$$

where $u(x)$ is an unknown function, a and b are real constants and λ is a real (or complex) parameter. The kernel $K(x, t, u)$ and $f(x)$ are analytical functions on $\mathbb{R}^3$ and $\mathbb{R}$ respectively, where $K(x, t, u)$ is nonlinear function of u. Hence, equation (1.1) represents a nonlinear Fredholm integral equation of second kind.

Now, we are interested to find a numerical solution of (1.1) in the interval $[0, 1]$, so we consider the subintervals $[x_n, x_{n+1}]$, with $x_0 = 0$ and in each subinterval, we approximate $k(x, t, u)$ by the first three terms of its Taylor series expansion around (x_n, t_n, u_n). Hence, the three terms of this expansion are

$$\begin{aligned} k(x, t, u) = k(x_n, t_n, u_n) &+ (x - x_n)\frac{\partial k(x_n, t_n, u_n)}{\partial x} \\ &+ (t - t_n)\frac{\partial k(x_n, t_n, u_n)}{\partial t} + (u - u_n)\frac{\partial k(x_n, t_n, u_n)}{\partial u}. \end{aligned} \tag{1.2}$$

By substituting (1.2) into (1.1) we obtain for $x_n \le x \le x_{n+1}$

$$u(x) = f(x) + \lambda \int_a^b \big(K_n + (x - x_n)J_n + (t - t_n)Q_n + (u - u_n)Z_n\big)\, dt, \tag{1.3}$$

Key words and phrases. Adomian method, linearization method, non-linear integral equations.

1

where $u_n = u(x_n)$ and

$$K_n = k(x_n, t_n, u_n), \qquad J_n = \frac{\partial k(x_n, t_n, u_n)}{\partial x},$$
$$Q_n = \frac{\partial k(x_n, t_n, u_n)}{\partial t}, \quad Z_n = \frac{\partial k(x_n, t_n, u_n)}{\partial u}. \tag{1.4}$$

Since in the integration part of (1.3), t is an indipendent variable, u is a dependent variable and x is a parameter, therefore by integrating it with respect to t, we have

$$\begin{aligned} u(x) = f(x) + \lambda Z_n \int_a^b u(t)\, dt + \lambda\big(K_n + (x - x_n)J_n - u_n Z_n\big) \int_a^b dt \\ + \lambda Q_n \int_a^b (t - t_n)\, dt, \end{aligned} \tag{1.5}$$

which can be written in the form

$$\begin{aligned} u(x) = f(x) + \lambda(b - a)\big(K_n + (x - x_n)J_n - u_n Z_n\big) \\ + \frac{\lambda}{2}\big((b - t_n)^2 - (a - t_n)^2\big)Q_n + \lambda Z_n \int_a^b u(t)\, dt. \end{aligned} \tag{1.6}$$

Next, we differentiate (1.6) with respect to x, we obtain

$$u'(x) = f'(x) + \lambda(b - a)J_n. \tag{1.7}$$

Then by integrating the both sides of (1.7) with respect to x, from x_n to x_{n+1}, we obtain

$$\int_{x_n}^{x_{n+1}} u'(t)\, dt = \int_{x_n}^{x_{n+1}} f'(t)\, dt + \lambda(b - a)J_n \int_{x_n}^{x_{n+1}} dt, \tag{1.8}$$

this leads to the formula

$$u(x_{n+1}) = u(x_n) + f(x_{n+1}) - f(x_n) + \lambda J_n(b - a)(x_{n+1} - x_n). \tag{1.9}$$

At the end, the numerical solution of (1.1), with step size h and at the grid points: x_{n+1}; $(n = 0, 1, 2, \dots)$, can be obtained from the recurrent formula

$$\begin{aligned} u(x_0) &= u_0 \\ u_{n+1} &= u_n + (f_{n+1} - f_n) + \lambda J_n(b - a)h, \end{aligned} \tag{1.10}$$

where $h = x_{n+1} - x_n$, is the local step size, i.e. $x_n = x_0 + nh$; $(n = 0, 1, 2, \dots)$. Note that, the aim of [3], was to get the error function $e(x_r) \leq 10^{k_r}$, where k_r is any positive integer number. Hence, by assuming $\max(10^{-k_r}) = 10^{-k}$, the step size h can be decreasing as far as the inequality $e(x_r) \leq 10^k$ holds at each point x_r.

2. Modified Techniques of Adomian method

In this section, we introduce a modified technique of Adomian method for nonlinear Fredholm integral equations. To do that, let us first introduce the standard Adomian method [4, 5, 8, 3]. For simplicity, we assume the the kernel $K(x, t, u)$ can be split as $K(x, t, u) = \tilde{K}(x, t)F(u)$, where th kernel $\tilde{k}(x, t)$ is analytical function on $\mathbb{R}^2$ and F is nonlinear function of u. Now the nonlinear Fredholm integral equation (1.1) becomes

$$u(x) = f(x) + \lambda \int_a^b \tilde{k}(x, t)F(u)\, dt. \tag{2.1}$$

The first step of the standard Adomian method is to decompose u into $\sum_{n=0}^{\infty} u_n$ and assume that

$$u = \lim_{n\to\infty} \sum_{i=0}^{n} u_i. \tag{2.2}$$

Then we choose $u_0 = f(x)$ and set $F(u) = \sum_{n=0}^{\infty} A_n$, where $A_n; n \geq 0$ are special polynomials known as Adomian polynomials. Now equation (2.1) bocomes

$$\sum_{n=0}^{\infty} u_n = f(x) + \lambda \int_a^b \big(\tilde{k}(x,t) \sum_{n=0}^{\infty} A_n\big)\, dt. \tag{2.3}$$

This leads to the recursive formulas

$$u_0 = f(x), \quad u_{n+1} = \lambda \int_a^b \tilde{k}(x,t) A_n\, dt, \quad n = 0, 1, 2, \dots \tag{2.4}$$

In [1], close formulas of Adomian polynomials A_n for any analytic nonlinear function $F(u)$, introduced in the forms

$$\begin{aligned} &A_0 = F(u_0) \\ &A_n = \sum_{\nu=1}^{n} \Big(\frac{1}{\nu!} \sum_{i_1, i_2, \dots i_\nu = 1}^{n+1-\nu} \delta_{n, i_1+i_2+\cdots+i_\nu} y_{i_1} y_{i_2} \dots y_{i_\nu}\Big) \frac{d^\nu F(u_0)}{du_0^\nu}, \end{aligned} \tag{2.5}$$

where $n = 1, 2, \dots, n \geq \nu$ and $\delta_{n,m}$ is the Kronecker delta. In [2] we shown that the choice of the initial data u_0, plays an essential role on the speed of the convergence of Adomian method and we found the standard Adomian method encountered computational difficulties for certain types of non-homogeneous function $f(x)$. To reduce the computational difficulties and accelerate the convergence of standard method, we introduce a modified technique [10]. The modified technique assumed that the function $f(x)$ can be split as

$$f(x) = f_1(x) + f_2(x). \tag{2.6}$$

Based on this assumption, we can introduce a a slight change of the choice of the components u_0 and u_1 as following

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \int_a^b k(x,t) A_0(t)\, dt, \\ u_{n+1} &= \int_a^b k(x,t) A_n(t)\, dt, \quad n \geq 1. \end{aligned} \tag{2.7}$$

Note that, this choice of initial data u_0, as we will see in next section, reduces the computational difficulties work and accelerate the convergence of the Adomian decomposition method procedure.

3. Presentation of results

In order to asses both the applicability and the accuracy of the theoretical results of the pervious sections, we have applied these results to a variety of nonlinear Fredholm integral equations in the following examples:

x_n	h=0.1	h=0.01	h=0.001	h=0.0001
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.087500	0.087508	0.087509	0.087510
0.2	0.175038	0.175138	0.175152	0.175153
0.3	0.262845	0.263227	0.263273	0.263277
0.4	0.351381	0.352346	0.352457	0.352469
0.5	0.441350	0.443327	0.443551	0.443574
0.6	0.533720	0.537304	0.537703	0.537743
0.7	0.629766	0.635770	0.636433	0.636500
0.8	0.731147	0.740706	0.741763	0.741870
0.9	0.840030	0.854770	0.856407	0.856572
1.0	0.959284	0.981607	0.984106	0.984359

TABLE 1. Shows the Numerical solution presented by the Linearization method with h = 0.1, 0.01, 0.001, and 0.0001.

Example 3.1. *The integral equation*

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xtu^2(t)\,dt, \tag{3.1}$$

is a nonlinear Fredholm integral equation with a separable kernel. Using the direct computation method this integral equation has the solution $u(x) = x, 7x$.

To investigate both the applicability and the accuracy of the linearization method applied to nonlinear (3.1), we first reduced it to linear integral equation, then by using (1.10), a numerical solution of (2.1) at the grid points x_{n+1}, $(n = 0, 1, 2, \dots)$ can be found from the recurrent formula

$$\begin{aligned} u(x_0) &= u_0 = 0, \\ u_{n+1} &= u_n + h\left(\frac{7}{8} + \frac{1}{2}x_n u_n^2\right). \end{aligned} \tag{3.2}$$

By the help of Mathematica, numerical solutions with $h = 0.1$, $h = 0.01$, $h = 0.001$, and $h = 0.0001$, are presented in Table (1). Furthermore, figures (1) and (2), show the plotting of exact solution against the approximate solutions for for $h = 0.1$ and $h = 0.001$ respectively.

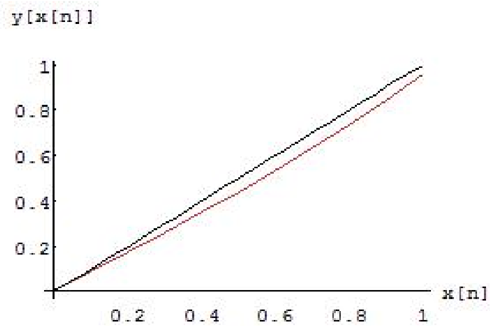

FIGURE 1. $h = 0.1$.

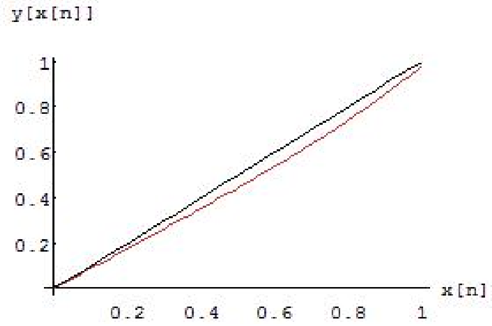

FIGURE 2. $h = 0.001$.

Next, we investigate both the applicability and the accuracy of the modified techniques of Adomian applied to nonlinear Fredholm integral equations (3.1). *First, we rewrite* (3.1) *in the form*

$$u(x) = x - \frac{1}{8}x + \frac{1}{2}\int_0^1 xtu^2(t)dt, \tag{3.3}$$

then we split the function $f(x)$ *as*

$$f_1(x) = x, \quad f_2(x) = -x/8. \tag{3.4}$$

Now we can use the modified recursive formula (2.7). *This gives*

$$\begin{aligned} u_0(x) &= x, \\ u_1(x) &= -\frac{1}{8}x + \frac{x}{2}\int_0^1 tu^2(t)dt, \\ u_{n+1}(x) &= \int_0^1 tA_n(t)dt, \quad n \geq 1, \end{aligned} \tag{3.5}$$

where

$$A_0 = u_0^2, \quad A_1 = 2u_0u_1, \quad A_2 = 2u_0u_2 + u_1^2, \quad A_3 = 2u_0u_3 + 2u_1u_2 \dots. \tag{3.6}$$

Now using (3.5) *and* (3.6) *we can calculate*

$$u_1(x) = -\frac{1}{8} + \frac{x}{2}\int_0^1 t^3 dt = 0, \quad u_{n+1}(x) = 0, \quad n \geq 1.$$

Hence this leads immediately to the exact solution $u(x) = x$.

Example 3.2. *The integral equation*

$$u(x) = \sec x - x + \int_0^1 x(u^2(t) - \tan^2 x)dt, \tag{3.7}$$

is a nonlinear Fredholm integral equation with a separable kernel and has the exact solutions $u(x) = \sec(x)$. *Hence by reducing it to linear integral equation and using* (1.10), *a numerical solution of* (3.7) *at the grid points* x_{n+1}, $(n = 0, 1, , 2, \dots)$ *can be found from the recurrent formula*

$$\begin{aligned} u(x_0) &= u_0 = 1, \\ u_{n+1} &= u_n + \sec(u_{n+1}) - \sec(u_n) + h(u_n^2 - \tan^2 u_n - 1). \end{aligned} \tag{3.8}$$

By the help of Mathematica, we found numerical solutions with $h = 0.1$, $h = 0.01$, $h = 0.001$, *and* $h = 0.0001$. *Figure* (3) *shows the plotting of the numerical solution for* $h = 0.1$ *against the exact solution.*

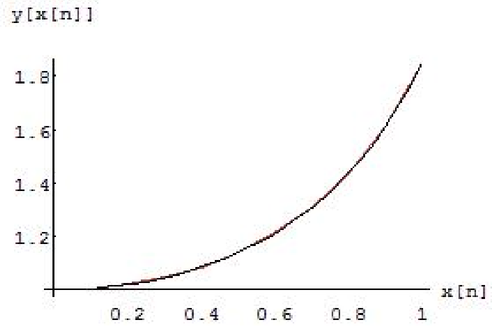

FIGURE 3. The numerical solution for $h = 0.1$.

Next, we investigate both the applicability and the accuracy of the modified techniques of Adomian applied to nonlinear Fredholm integral equations (3.7). *Now, the modified recursive formula* (2.7) *gives*

$$\begin{aligned} u_0(x) &= \sec(x) \\ u_1(x) &= -x\big(1 + \int_0^1 \tan^2(t)dt\big) + x\int_0^1 A_0(t)dt, \\ u_{n+1}(x) &= \int_0^1 A_n(t)dt, \quad n \geq 1. \end{aligned} \tag{3.9}$$

Now (3.6) *reduces* (3.9) *to*

$$\begin{aligned} u_0 &= \sec x, \\ u_1(x) &= -x\big(1 + \int_0^1 \tan^2(t)dt\big) + x\int_0^1 A_0(t)dt \\ &= -x\tan(1) + x\int_0^1 \sec^2(t)dt = 0, \\ u_{n+1}(x) &= 0, \quad n \geq 1, \end{aligned}$$

which leads to the exact solution $u(x) = \sec(x)$.

4. CONCLUSIONS

In this work we examined the accuracy, applicability and simplicity of both the modified Adomian technique and the linearization method applied to non linear Fredholm integral equations of the second kind. This study showed the accuracy and the applicability of both methods; however, this study showed the fast convergent modified Adomian technique, even when the accuracy of linearization method improved by employing variable steps. From this study, we conclude that using the right splitting of the non-homogeneous function $f(x)$, we can avoid the calculation difficulties of using the Adomian polynomials required for the non-linear terms, which minimize the number of iterations required for the standard Adomian method. Furthermore, we recommend using the modified technique, when the non-homogeneous function $f(x)$ is given in term of a polynomial or a combination of polynomial and trigonometric, or transcendental, functions. Furthermore, we recommend using the linearization method, for cases involving non separable kernels or when the right splitting is hard to find.

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PREFACE (JAFA – JCAAM)

These special issues are devoted to a part of proceedings of AMAT 2012 - International Conference on Applied Mathematics and Approximation Theory - which was held during May 17-20, 2012 in Ankara, Turkey, at TOBB University of Economics and Technology. This conference is dedicated to the distinguished mathematician George A. Anastassiou for his 60th birthday.

AMAT 2012 conference brought together researchers from all areas of Applied Mathematics and Approximation Theory, such as ODEs, PDEs, Difference Equations, Applied Analysis, Computational Analysis, Signal Theory, and included traditional subfields of Approximation Theory as well as under focused areas such as Positive Operators, Statistical Approximation, and Fuzzy Approximation. Other topics were also included in this conference, such as Fractional Analysis, Semigroups, Inequalities, Special Functions, and Summability. Previous conferences which had a similar approach to such diverse inclusiveness were held at the University of Memphis (1991, 1997, 2008), UC Santa Barbara (1993), the University of Central Florida at Orlando (2002).

Around 200 scientists coming from 30 different countries participated in the conference. There were 110 presentations with 3 parallel sessions. We are particularly indebted to our plenary speakers: George A. Anastassiou (*University of Memphis - USA*), Dumitru Baleanu (*Çankaya University - Turkey*), Martin Bohner (*Missouri University of Science & Technology - USA*), Jerry L. Bona (*University of Illinois at Chicago - USA*), Weimin Han (*University of Iowa - USA*), Margareta Heilmann (*University of Wuppertal - Germany*), Cihan Orhan (*Ankara University - Turkey*). It is our great pleasure to thank all the organizations that contributed to the conference, the Scientific Committee and any people who made this conference a big success.

Finally, we are grateful to “TOBB University of Economics and Technology”, which was hosting this conference and provided all of its facilities, and also to “Central Bank of Turkey” and “The Scientific and Technological Research Council of Turkey” for financial support.

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ON COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED PARTIAL METRIC SPACES

ERDAL KARAPINAR

Abstract. In this manuscript, we prove new coupled fixed point theorems in the context of partially ordered partial metric spaces. The main theorems of this paper extent by improving some earlier results in the literature. We also present applications of these new results through a number of examples.

1. Introduction and Preliminaries

In nonlinear phenomena, one of the crucial tools is known to be the fixed point theory. In addition to mathematics, fixed point theory has wide range of applications in many disciplines such as physics, biology, economics, computer sciences, and engineering. Banach contraction mapping principle [16], also referred to as Banach fixed point theorem, is the seminal and most important result of this topic. Banach showed not only the existence and uniqueness of a fixed point of a self-mapping but also how to determine this fixed point. This remarkable result of Banach has been the center of attention for many authors since its appearance. As a consequence, many different approaches toward a generalization of Banach fixed point theorem have been given in the literature.

In 1992, Matthews announced one of the interesting generalizations by defining a new notion, a partial metric space. The author proved the analog of Banach fixed point theorem in the context of partial metric space which is a generalization of a metric spaces. In brief, in a partial metric space self distance of some points may not be zero. This phenomena was discovered by Matthews [41] when he attempt to solve problems of applying metric space techniques in the subfield of computer science: semantics and domain theory (see e.g. [39, 40]). After this initial result of Mathews, a number of results have appeared on partial metric spaces (see e.g. [1]-[3],[5, 6, 7],[11]-[13],[15, 26],[30]-[35],[39, 40, 54, 58]).

Turinici [61] initiated a new trend in fixed point theory by introducing criteria that implies existence and uniqueness of a fixed point in partially ordered sets. In this paper, Turinici extended Banach contraction principle in partially ordered sets. Consequently, Ran and Reurings [52] applied Turinici's results to matrix equations. After these initial papers, a number of exceptionally good results have been published in this direction. (see e.g. [4, 5],[11]-[13],[15, 18, 19],[20]-[22],[24]-[28],[38],[42]-[50], [53]-[56],[58]). The concept of a coupled fixed point introduced by Gnana-Bhaskar and Lakshmikantham [17] in the class of partially ordered metric

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spaces. In this article, we prove the existence and uniqueness of coupled fixed points in ordered partial metric spaces.

We start with recalling basic definitions and crucial results in coupled fixed point theory from the view point of metric spaces. Throughout the manuscript, we always assume that $X \neq \emptyset$.

Definition 1.1. *(See [17]) Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The function F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and}$$
$$y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1), \ \text{for } y_1, y_2 \in X.$$

Definition 1.2. *(see [17]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$F(x, y) = x \ \text{and} \ \ F(y, x) = y.$$

The following two results were given by Bhaskar and Lakshmikantham in [17].

Theorem 1.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0, 1)$ with*

$$p(F(x, y), F(u, v)) \leq \frac{k}{2} \left[p(x, u) + p(y, v) \right], \ \text{for all } u \leq x, \ y \leq v. \tag{1.1}$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 1.4. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that X has the following properties:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, $\forall n$;
(i) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, $\forall n$.

Assume that there exists a $k \in [0, 1)$ with

$$p(F(x, y), F(u, v)) \leq \frac{k}{2} \left[p(x, u) + p(y, v) \right], \ \text{for all } u \leq x, \ y \leq v. \tag{1.2}$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

The following concept of a g-mixed monotone mapping was introduced by Lakshmikantham and Ćirić [42].

Definition 1.5. *Let $(X, \leq)$ be partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The function F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y, that is, for any $x, y \in X$,*

$$g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \quad \text{and} \tag{1.3}$$

$$g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1), \text{ for } y_1, y_2 \in X. \tag{1.4}$$

It is clear that Definition 1.5 reduces to Definition 1.1 when g is the identity.

Definition 1.6. *An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = g(x), \quad F(y, x) = g(y),$$

and is called a coupled common fixed of F and g, if

$$F(x, y) = g(x) = x, \quad F(y, x) = g(y) = y.$$

The mappings F and g are said to commute if

$$g(F(x, y)) = F(g(x), g(y)),$$

for all $x, y \in X$.

Definition 1.7. *Let $F : X \times X \to X$ and $g : X \to X$. The mappings F and g are said to commute if*

$$g(F(x, y)) = F(g(x), g(y)), \quad \text{for all } x, y \in X.$$

The main result of [42] is the following.

Theorem 1.8. *Let $(X, \leq)$ be partially ordered set and (X, d) be a complete metric space. Assume there exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and $\lim_{r \to r^+} \varphi(r) < t$ for each $t > 0$ and also suppose that $F : X \times X \to X$ and $g : X \to X$ where $X \neq \emptyset$. Suppose that F has the mixed g-monotone property and*

$$d(F(x, y), F(u, v)) \leq \varphi\left(\frac{[d(g(x), g(u)) + d(g(y), g(v))]}{2}\right) \tag{1.5}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(v) \leq g(y)$. Suppose $F(X \times X) \subset g(X)$, where g is sequentially continuous and commutes with F and also suppose either F is continuous or X has the following property:

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x, \text{ for all } n, \tag{1.6}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \leq y_n, \text{ for all } n. \tag{1.7}$$

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \leq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a couple coincidence.

After Gnana-Bhaskar and Lakshmikantham [17] and Lakshmikantham and Ćirić [42] many remarkable papers published in this direction (see e.g. [7]-[10],[15, 19], [20]-[22],[24]-[27],[35]-[38], [43]-[47],[49, 51],[53]-[56],[58, 59].)

Next we include necessary definitions and basic results on coupled fixed point theory in the context of partial metric spaces. A partial metric is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions

$(P1)$ If $p(x, x) = p(x, y) = p(y, y)$, then $x = y$,
$(P2)$ $p(x, y) = p(y, x)$,
$(P3)$ $p(x, x) \leq p(x, y)$,
$(P4)$ $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$,

for all $x, y, z \in X$. Then (X,p) is called a partial metric space. If p is a partial metric p on X, then the function $d_p : X \times X \to [0,\infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, a closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in X : p(x,y) \le p(x,x) + \varepsilon\}$. For more details see e.g. [5, 41].

Definition 1.9 (See e.g. [41, 5, 32])**.** *Let (X,p) be a partial metric space.*

- (i) *A sequence $\{x_n\}$ in X converges to $x \in X$ whenever $\lim\limits_{n\to\infty} p(x,x_n) = p(x,x)$,*
- (ii) *A sequence $\{x_n\}$ in X is called Cauchy whenever $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),*
- (iii) *(X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, that is, $\lim\limits_{n,m\to\infty} p(x_n,x_m) = p(x,x)$.*
- (iv) *A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.*

Lemma 1.10 (See e.g. [41, 5, 32, 1])**.** *Let (X,p) be a partial metric space.*

- (a) *A sequence $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X,d_p),*
- (b) *(X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) = p(x,x). \tag{1.8}$$

Lemma 1.11. *(See e.g.* [1]*) Let (X,p) be a partial metric space. Then*

- (A) *If $p(x,y) = 0$ then $x = y$.*
- (B) *If $x \neq y$, then $p(x,y) > 0$.*

Remark 1.1. *If $x = y$, $p(x,y)$ may not be* 0.

The triangle inequality $(P4)$ yields the following result.

Lemma 1.12. *(See* [1]*) Let $x_n \to z$ as $n \to \infty$ in a partial metric space (X,p) where $p(z,z) = 0$. Then $\lim\limits_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

Lemma 1.13. *(See e.g.* [34]*) Let $\lim_{n\to\infty} p(x_n,y) = p(y,y)$ and $\lim_{n\to\infty} p(x_n,z) = p(z,z)$. If $p(y,y) = p(z,z)$ then $y = z$.*

Remark 1.2. *Limit of a sequence $\{x_n\}$ in a partial metric space (X,p) is not unique.*

Example 1.14. *Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. Clearly, p is not a metric. Observe that the sequence $\{1 - \frac{1}{n+n^2}\}$ converges both for example to $x = 3$ and $y = 5$, so no uniqueness of the limit.*

Let (X,p) be a partial metric space. Note that the mappings $\rho_2 : X^2 \times X^2 \to [0,+\infty)$ defined by

$$\rho_2(\mathbf{x},\mathbf{y}) := \max\{p(x_1,y_1), p(x_2,y_2)\},$$

forms a partial metric on X^2 where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2) \in X^2$ where

$$X^2 = X \times X.$$

2. Existence of Coupled Fixed Points

We start this section with the following definition.

Definition 2.1. [29] *A function $\varphi : [0, \infty) \to [0, \infty)$ is called an alternating distance function if the following properties are satisfied:*

(*i*) *φ is monotone increasing and continuous,*
(*ii*) *$\varphi(t) = 0$ if and only if $t = 0$.*

The following theorem is our first main result.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered set and (X, p) be a complete partial metric space and ϕ, ψ are alternating distance functions. Let $F : X \times X \to X$ and $g : X \to X$ where $X \neq \emptyset$. Suppose that F has the mixed g-monotone property and*

$$\begin{aligned} \psi(\max\{p(F(x,y), F(u,v)), p(F(y,x), F(v,u))\}) \leq \; & \psi(\max\{p(g(x), g(u)), p(g(y), g(v))\}) \\ & -\phi(\max\{p(g(x), g(u)), p(g(y), g(v))\}) \end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ for which $g(x) \leq g(u)$ and $g(v) \leq g(y)$. Suppose $F(X \times X) \subset g(X)$, where g is continuous, and F and g are compatible mappings. Also suppose either

(a) *F is continuous or*
(b) *X has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x, \text{ for all } n \geq 0, \tag{2.2}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \leq y_n, \text{ for all } n \geq 0. \tag{2.3}$$

If there exist $x_0, y_0 \in X$ such that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \leq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, F and g have a couple coincidence.

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Since $F(X \times X) \subset g(X)$, then we can choose $x_1, y_1 \in X$ such that

$$gx_1 = F(x_0, y_0) \quad \text{and} \quad gy_1 = F(y_0, x_0). \tag{2.4}$$

Again, from $F(X \times X) \subset g(X)$, continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n). \tag{2.5}$$

We shall show that

$$gx_n \leq gx_{n+1}, \quad gy_{n+1} \leq gy_n. \tag{2.6}$$

We shall use the mathematical induction. Since, $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$ then by (2.4), we get

$$gx_0 \leq gx_1 \quad \text{and} \quad gy_1 \leq gy_0,$$

that is (2.6) holds for $n = 0$.
We presume that (2.6) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \leq gx_{n+1}$ and $gy_{n+1} \leq gy_n$, we obtain

$$\begin{aligned} gx_{n+1} = F(x_n, y_n) &\leq F(x_{n+1}, y_n) \\ &\leq F(x_{n+1}, y_n) \\ &\leq F(x_{n+1}, y_{n+1}) = gx_{n+2}, \end{aligned}$$

$$\begin{aligned} gy_{n+2} = F(y_{n+1}, x_{n+1}) &\leq F(y_{n+1}, x_n) \\ &\leq F(y_n, x_n) = gy_{n+1}, \end{aligned}$$

Thus, (2.6) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad \text{and} \quad gy_n = gy_{n+1},$$

then, by (2.5), (x_n, y_n) is a coupled coincidence point of F and g. From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \neq gx_{n+1} \quad \text{or} \quad gy_n \neq gy_{n+1}. \tag{2.7}$$

Due to (2.1),(2.5) and (2.6), we have $\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}))\} > 0$. Set $\delta_n = \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}))\}$. Then consider

$$\begin{aligned} \psi(p(gx_n, gx_{n+1})) &= \psi(p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \psi(\max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}) \\ &\quad -\phi(\max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \psi(p(gy_n, gy_{n+1})) &= \psi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq \psi(\max\{p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n)\}) \\ &\quad -\phi(\max\{p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n)\}), \end{aligned} \tag{2.9}$$

Using the monotone property (i) of ϕ together with (2.8) and (2.9), we obtain that

$$\begin{aligned} \psi(\max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}) &= \max\{\psi(p(gx_n, gx_{n+1})), \psi(p(gy_n, gy_{n+1}))\} \\ &\leq \psi(\max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}) \\ &\quad -\phi(\max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n)\}). \end{aligned} \tag{2.10}$$

So (2.10) turns into

$$\begin{aligned} \psi(\delta_n) &\leq \psi(\delta_{n-1}) - \phi(\delta_{n-1}) \\ &\leq \psi(\delta_{n-1}). \end{aligned} \tag{2.11}$$

By using the property of ϕ, for all $n \geq 0$ we have

$$\delta_n \leq \delta_{n-1}. \tag{2.12}$$

Thus, $\{\delta_n\}$ is a monotone decreasing sequence of non-negative real numbers. So, there exists a $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{2.13}$$

Suppose $\delta > 0$. Letting $n \to \infty$ in (2.10), then we get

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta)$$

which is a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = 0. \tag{2.14}$$

Hence, we have

$$\lim_{n\to\infty} p(gx_{n+1}, gx_n) = 0,$$

$$\lim_{n\to\infty} p(gy_{n+1}, gy_n) = 0.$$

By condition $(P3)$, we have

$$p(g(x_n), g(x_n)) \leq p(g(x_n), g(x_{n+1})),$$

so letting $n \to \infty$, we get

$$\lim_{n\to\infty} p(g(x_n), g(x_n)) = 0. \tag{2.15}$$

Analogously, we have

$$\lim_{n\to\infty} p(g(y_n), g(y_n)) = 0. \tag{2.16}$$

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not Cauchy. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$t_k = \max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\} \geq \varepsilon. \tag{2.17}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.17) and $n(k) > m(k) \geq k$. Thus,

$$\max\{p(gx_{n(k)-1}, gx_{m(k)}), p(gy_{n(k)-1}, gy_{m(k)})\} < \varepsilon. \tag{2.18}$$

By using the triangle inequality and having (2.17), (2.18) in mind

$$\begin{aligned} \varepsilon \quad &\leq t_k = \max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\} \\ &\leq \max\{p(gx_{n(k)}, gx_{n(k)-1}) + p(gx_{n(k)-1}, gx_{m(k)}), \\ &\qquad p(gy_{n(k)}, gy_{n(k)-1}) + p(gy_{n(k)-1}, gy_{m(k)})\} \\ &\leq \max\{p(gx_{n(k)}, gx_{n(k)-1}), p(gy_{n(k)}, gy_{n(k)-1})\} + \varepsilon \\ &\leq \delta_{n(k)-1} + \varepsilon. \end{aligned} \tag{2.19}$$

Letting $k \to \infty$ in (2.19) and using (2.14)

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} \max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\} = \varepsilon. \tag{2.20}$$

Set $t_{k+1} = \max\{p(gx_{n(k)+1}, gx_{m(k)+1}), p(gy_{n(k)+1}, gy_{m(k)+1})\}$. Again by the triangle inequality,

(2.21)

$$\begin{aligned} t_k \quad &= \max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\} \\ &\leq \max\{p(gx_{n(k)}, gx_{n(k)+1}) + p(gx_{n(k)+1}, gx_{m(k)+1}) + p(gx_{m(k)+1}, gx_{m(k)}) \\ &\qquad p(gy_{n(k)}, gy_{n(k)+1}) + p(gy_{n(k)+1}, gy_{m(k)+1}) + p(gy_{m(k)+1}, gy_{m(k)})\} \\ &\leq \max\{p(gx_{n(k)}, gx_{n(k)+1}), p(gy_{n(k)}, gy_{n(k)+1})\} \\ &\qquad + \max\{p(gx_{n(k)+1}, gx_{m(k)+1}), p(gy_{n(k)+1}, gy_{m(k)+1})\} \\ &\qquad + \max\{p(gx_{m(k)}, gx_{m(k)+1}), p(gy_{m(k)}, gy_{m(k)+1})\} \\ &\leq \delta_{n(k)+1} + t_{k+1} + \delta_{m(k)+1} \end{aligned}$$

analogously we have

$$t_{k+1} \leq \delta_{n(k)+1} + t_k + \delta_{m(k)+1}. \tag{2.22}$$

Letting $n \to \infty$ in (2.21) and (2.22), we get that

$$\begin{aligned} &\lim_{k\to\infty} t_{k+1} \\ &= \lim_{k\to\infty} \max\{p(gx_{n(k)+1}, gx_{m(k)+1}), p(gy_{n(k)+1}, gy_{m(k)+1})\} \\ &= \varepsilon. \end{aligned} \tag{2.23}$$

Since $n(k) > m(k)$, then

$$gx_{n(k)} \geq gx_{m(k)} \text{ and } gy_{n(k)} \leq gy_{m(k)}, \tag{2.24}$$

Hence using the property (i) of ϕ with (2.1), (2.5) and (2.24), we have

$$\begin{aligned} \psi(p(gx_{n(k)+1}, gx_{m(k)+1})) &= \psi(p(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}) \\ &\leq \psi\left(\max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\}\right) \\ &-\phi\left(\max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\}\right) \end{aligned} \tag{2.25}$$

$$\begin{aligned} \psi(p(gy_{n(k)+1}, gy_{m(k)+1})) &= \psi(p(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)})) \\ &\leq \psi\left(\max\{p(gy_{n(k)}, gy_{m(k)}), p(gx_{n(k)}, gx_{m(k)})\}\right) \\ &-\phi\left(\max\{p(gy_{n(k)}, gy_{m(k)}), p(gx_{n(k)}, gx_{m(k)})\}\right) \end{aligned} \tag{2.26}$$

From (2.25) and (2.26) and by using the monotone property of ψ, we get that

$$\begin{aligned} \psi(t_{k+1}) &= \psi(\max\{p(gx_{n(k)+1}, gx_{m(k)+1}), p(gy_{n(k)+1}, gy_{m(k)+1})\}) \\ &= \max\{\psi(p(gx_{n(k)+1}, gx_{m(k)+1})), \psi(p(gy_{n(k)+1}, gy_{m(k)+1}))\}) \\ &\leq \psi\left(\max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\}\right) \\ &- \phi\left(\max\{p(gx_{n(k)}, gx_{m(k)}), p(gy_{n(k)}, gy_{m(k)})\}\right) \\ &= \psi(t_k) - \phi(t_k). \end{aligned} \tag{2.27}$$

Letting $k \to \infty$ and having in mind (2.27) we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$$

which is a contradiction. This shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Thus, the sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy in $(g(X), p)$. By Lemma 1.10, $\{g(x_n)\}$ and $\{g(y_n)\}$ are also Cauchy in (X, d_p). Again by Lemma 1.10, $(X, d_p))$ is complete. Thus, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} d_p(x, g(x_n)) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, g(x_n)) = \lim_{n\to\infty} p(g(x_n), g(x_n)) = 0, \tag{2.28}$$

$$\lim_{n\to\infty} d_p(y, g(y_n)) = 0 \Leftrightarrow p(y, y) = \lim_{n\to\infty} p(y, g(y_n)) = \lim_{n\to\infty} p(g(y_n), g(y_n)) = 0. \tag{2.29}$$

Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \quad \lim_{n\to\infty} gy_n = y. \tag{2.30}$$

From (2.5), (2.30) and using the continuity of g, we have

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n)), \tag{2.31}$$

and

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n)). \tag{2.32}$$

Now we shall show that $gx = F(x, y)$ and $gy = F(y, x)$.

Since F and g are compatible, in addition with (2.30), we have

$$\lim_{n\to\infty} p(g(F(x_n,y_n)),F(g(x_n),g(y_n)))=0, \tag{2.33}$$

and

$$\lim_{n\to\infty} p(g(F(y_n,x_n)),F(g(y_n),g(x_n)))=0. \tag{2.34}$$

Suppose that F is continuous.

For all $n\geq 0$, we have,

$p(gx,F(gx_n,gy_n))\leq p(gx,g(F(x_n,y_n))+p(g(F(x_n,y_n)),F(gx_n,gy_n))$.

Taking the limit as $n\to\infty$, using (2.31), (2.33), (2.30) and the fact that F and g are continuous, we have $p(gx,F(x,y))=0$.

Similarly, by using (2.32), (2.34), (2.30) and also the fact that F and g are continuous, we have $p(gy,F(y,x))=0$ as $n\to\infty$.

Thus we have proved that F and g have a coupled coincidence point.

Suppose now the assumption (b) holds. Since $\{gx_n\}$ is non-decreasing and $gx_n\to x$ and also $\{gy_n\}$ is non-increasing with $gy_n\to y$, then by assumption (b) we have for all n

$$gx_n\geq x,\quad gy_n\leq y, \tag{2.35}$$

Now we have

$$p(gx,F(x,y))\leq p(gx,g(gx_{n+1}))+p(g(g(x_{n+1})),F(x,y)).$$

Taking the limit as $n\to\infty$ in the inequality above, using (2.31), (2.33) and (2.35) we have,

$$\begin{aligned} p(gx,F(x,y)) &\leq \lim_{n\to\infty} p(gx,g(gx_{n+1}))+\lim_{n\to\infty} p(g(F(x_n,y_n)),F(gx_n,gy_n))\\ &\quad+\lim_{n\to\infty} p(F(gx_n,gy_n),F(x,y))\\ &\leq \lim_{n\to\infty} p(F(gx_n,gy_n),F(x,y)). \end{aligned} \tag{2.36}$$

Analogously we get that

$$p(gy,F(y,x))\leq \lim_{n\to\infty} p(F(gy_n,gx_n),F(y,x)).$$

By using the properties of ψ function

$$\psi(\max\{p(gx,F(x,y)),p(gy,F(y,x)))\leq \lim_{n\to\infty}\psi(\max\{p(F(gx_n,gy_n),F(x,y)),p(F(gy_n,gx_n),F(y,x))\}).$$

In view of (2.1), for all $n\geq 0$ we have ,

$$\begin{aligned} &\psi(\max\{p(F(gx_n,gy_n),F(x,y)),p(F(gy_n,gx_n),F(y,x))\})\\ &\qquad\leq \lim_{n\to\infty}\psi\left(\max\{p(ggx_n,gx),p(ggy_n,gy)\}\right)\\ &\qquad\quad-\lim_{n\to\infty}\phi\left(\max\{p(ggx_n,gx),p(ggy_n,gy)\}\right)\\ &\qquad\leq \psi\left(\max\{\lim_{n\to\infty} p(ggx_n,gx),\lim_{n\to\infty} p(ggy_n,gy)\}\right)\\ &\qquad\quad-\phi\left(\max\{\lim_{n\to\infty} p(ggx_n,gx),\lim_{n\to\infty} p(ggy_n,gy)\}\right). \end{aligned}$$

By (2.31) and (2.32),

$$\psi(\max\{p(gx,F(x,y)),p(F(gy_n,gx_n),F(y,x))\})\leq\psi(0)-\phi(0).$$

Using the property of ψ, φ-function we obtain,

$$p(gx, F(x,y)) \leq 0 \text{ and } p(gy, F(y,x)) \leq 0$$

as $n \to \infty$. That is

$$gx = F(x,y).$$

Analogously, by using (2.31), (2.32), (2.33) and (2.34) we obtain

$$gy = F(y,x).$$

Thus, we proved that F and g have a coupled coincidence point in X.

$\square$

The following result is a consequence of Theorem 2.2.

Corollary 2.3. *Let $(X, \leq)$ be partially ordered set and (X, p) be a complete partial metric space and ϕ, ψ are alternating distance functions. Let $F : X \times X \to X$ be a mapping. Suppose that F has the mixed monotone property and*

(2.37)
$$\psi(\max\{p(F(x,y), F(u,v)), p(F(y,x), F(v,u))\}) \leq \begin{array}{l}\psi(\max\{p(x,u), p(y,v)\}) \\ -\phi(\max\{p(x,u), p(y,v)\})\end{array}$$

for all $x, y, u, v \in X$ for which $x \leq u$ and $v \leq y$. Also suppose either

(a) *F is continuous or*
(b) *X has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \leq x, \text{ for all } n \geq 0, \tag{2.38}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \leq y_n, \text{ for all } n \geq 0. \tag{2.39}$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point.

3. Uniqueness of Coupled Fixed Points

Let $(X, \leq)$ be a partially ordered set. We endow $X \times X$ with the following order $\leq_g$ where

$$(u,v) \leq_g (x,y) \Leftrightarrow g(u) < g(x),\ g(y) \leq g(v), \quad \text{for all} \quad (x,y),(u,v) \in X \times X. \tag{3.1}$$

Moreover, (u,v) and (x,y) are called g-comparable if either $(u,v) \leq_g (x,y)$ or $(u,v) \leq_g (x,y)$.In case $g = I_X$ we shortly say that (u,v) and (x,y) are comparable and denote by $(u,v) \leq (x,y)$. In this section, we shall prove the uniqueness of coupled fixed points.

Theorem 3.1. *In addition to the hypotheses of Theorem 2.2, assume that for all non g-comparable points (x,y), $(x^*, y^*) \in X^2$, there exists $(a,b) \in X^2$ such that $(F(a,b), F(b,a))$ is comparable to both $(g(x), g(y))$ and $(g(x^*), g(y^*))$. Then, F and g have a unique coupled common fixed point, that is, there exists $(u,v) \in X^2$ such that*

$$u = g(u) = F(u,v) \quad \text{and} \quad v = g(v) = F(v,u).$$

Proof. The set of coupled coincidence points of F and g is not empty due to Theorem 2.2. If (x,y) is the only coupled coincidence point of F and g, then commutativity of F and g implies that

$$g(g(x)) = g(F(x,y)) = F(g(x),g(y)) \text{ and } g(g(y)) = g(F(y,x)) = F(g(y),g(x)).$$

Hence, $(u,v) = (g(x),g(y))$ is a coupled coincidence point of F and g and by uniqueness we conclude that

$$F(x,y) = g(x) = x \text{ and } F(y,x) = g(y) = y.$$

Now suppose that $(x,y),(x^*,y^*) \in X^2$ are two coupled coincidence points of F and g. We show that $g(x) = g(x^*)$ and $g(y) = g(y^*)$. To this end we distinguish the following two cases.

First case: (x,y) is g-comparable to (x^*,y^*) with respect to the ordering in X^2, where

$$F(x,y) = g(x),\ F(y,x) = g(y),\ F(x^*,y^*) = g(x^*),\ F(y^*,x^*) = g(y^*).$$

If $p(g(x),g(x^*)) = 0 = p(g(y^*),g(y))$ then the theorem follows. Suppose that either $p(g(x),g(x^*)) \neq 0$ or $p(g(y^*),g(y)) \neq 0$. Without loss of the generality, we may assume that

$$g(x) = F(x,y) < F(x^*,y^*) = g(x^*), \qquad g(y) = F(y,x) \geq F(y^*,x^*) = g(y^*).$$

By definition of ρ_2 we have

$$\begin{aligned} 0 < \rho_2((g(x),g(y)),(g(x^*),g(y^*))) &= \max\{p(g(x),g(x^*)),p(g(y^*),g(y))\} \\ &= \max\{p(F(x,y),F(x^*,y^*)),p(F(y^*,x^*),F(y,x))\}. \end{aligned}$$

Due to 2.1, we have

$$\begin{aligned} \psi(\max\{p(g(x),g(x^*)),p(g(y^*),g(y))\}) &= \psi(\max\{p(F(x,y),F(x^*,y^*)),p(F(y^*,x^*),F(v,u))\}) \\ &\leq \psi(\max\{p(g(x),g(x^*)),p(g(y),g(y^*))\}) \\ &\qquad -\phi(\max\{p(g(x),g(x^*)),p(g(y),g(y^*))\}) \end{aligned}$$

This is a contradiction due to the property of ϕ and ψ. Therefore, we have $p(g(x),g(y)) = p(g(x^*),g(y^*)) = 0$. Hence

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*).$$

Second case: (x,y) is not g-comparable to (x^*,y^*).
By the assumption, there exists $(a,b) \in X^2$ such that $(F(a,b),F(b,a))$ is comparable to both $(g(x),g(y))$ and $(g(x^*),g(y^*))$. Then, we have

$$\begin{aligned} &g(x) = F(x,y) < F(a,b) \quad \text{and} \quad F(x^*,y^*) = g(x^*) < F(a,b), \\ &g(y) = F(y,x) \geq F(b,a) \quad \text{and} \quad F(y^*,x^*) = g(y^*) \geq F(b,a). \end{aligned} \tag{3.2}$$

Setting $x = x_0, y = y_0, a = a_0, b = b_0$, and $x^* = x_0^*, y^* = y_0^*$ as in the proof of Theorem 2.2, we get

$$g(x_{n+1}) = F(x_n,y_n) \ \text{ and } \ g(y_{n+1}) = F(y_n,x_n) \ \text{ for all } \ n = 0,1,2,\cdots, \tag{3.3}$$

$$g(a_{n+1}) = F(a_n,b_n) \ \text{ and } \ g(b_{n+1}) = F(b_n,a_n) \ \text{ for all } \ n = 0,1,2,\cdots \tag{3.4}$$

and

$$g(x^*_{n+1}) = F(x^*_n,y^*_n) \ \text{ and } \ g(y^*_{n+1}) = F(y^*_n,x^*_n) \ \text{ for all } \ n = 0,1,2,\cdots. \tag{3.5}$$

We have $g(x) \leq g(a_1)$ and $g(b_1) \leq g(y)$, since $(F(x,y), F(y,x)) = (g(x), g(y)) = (g(x_1), g(y_1))$ is comparable with $(F(a,b), F(b,a)) = (g(a_1), g(b_1))$. By using that F has the mixed $g-$ monotone property, we observe that $g(x) \leq g(a_n)$ and $g(b_n) \leq g(y)$ for all $n \geq 1$.

Thus, by 2.1, we get that

$$\begin{aligned} \psi(\max\{p(g(x), g(a_{n+1})), p(g(y), g(b_{n+1}))\}) &= \psi(\max\{p(F(x,y), F(a_n, b_n)), p(F(b_n, a_n), F(y,x))\}) \\ &\leq \psi(\max\{p(g(x), g(a_n)), p(g(y), g(b_n))\}) \\ &\quad -\phi(\max\{p(g(x), g(a_n)), p(g(y), g(b_n))\}). \end{aligned} \tag{3.6}$$

Letting $n \to \infty$ we conclude that

$$\lim_{n\to\infty} \max\{p(g(x), g(a_{n+1})), p(g(y), g(b_{n+1}))\} = 0.$$

Analogously, we get that

$$\lim_{n\to\infty} \max\{p(g(x^*), g(a_{n+1})), p(g(y^*), g(b_{n+1}))\} = 0.$$

By the triangle inequality, we have

$$\begin{aligned} p(g(x), g(x^*)) &\leq p(g(x), g(a_{n+1})) + p(g(x^*), g(a_{n+1})) - p(g(a_{n+1}), g(a_{n+1})) \\ &\leq p(g(x), g(a_{n+1})) + p(g(x^*), g(a_{n+1})) \to 0 \text{ as } n \to \infty, \end{aligned}$$

$$\begin{aligned} p(g(y), g(y^*)) &\leq p(g(y), g(b_{n+1})) + p(g(y^*), g(b_{n+1})) - p(g(b_{n+1}), g(b_{n+1})) \\ &\leq p(g(y), g(b_{n+1})) + p(g(y^*), g(b_{n+1})) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Combining all the observations above, we get that $p(g(x^*), g(x)) = 0$ and $p(g(y^*), g(y)) = 0$. Therefore,

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*). \tag{3.7}$$

In both of the cases above, we have shown that (3.7) holds. Now, let $g(x) = u$ and $g(y) = v$. By the commutativity of F and g and the fact that $g(x) = F(x,y)$ and $F(y,x) = g(y)$, we have

$$g(u) = g(g(x)) = g(F(x,y)) = F(g(x), g(y)) = F(u,v), \tag{3.8}$$

$$g(v) = g(g(y)) = g(F(y,x)) = F(g(y), g(x)) = (Fv, u). \tag{3.9}$$

Thus, (u,v) is a coupled coincidence point of F and g. Set $u = x^*$ and $v = y^*$ in (3.8), (3.9). Then, by (3.7) we have

$$u = g(x) = g(x^*) = g(u) \text{ and } v = g(y) = g(y^*) = g(v).$$

From (3.8), (3.9) we get that

$$u = g(u) = F(u,v) \text{ and } v = g(v) = F(v,u).$$

Hence, the pair (u,v) is a coupled common fixed point of F and g.

Finally, we prove the uniqueness of a coupled common fixed point of F. Actually, if (z,w) is another coupled common fixed point of F and g, then

$$u = g(u) = g(z) = z \text{ and } v = g(v) = g(w) = w$$

which follows from (3.7). □

Corollary 3.2. *In addition to the hypotheses of Theorem 2.2, assume that for all non comparable points (x, y), $(x^*, y^*) \in X^2$, there exists $(a, b) \in X^2$ such that $(F(a, b), F(b, a))$ is comparable to both (x, y) and (x^*, y^*). Then, F and g have a unique coupled common fixed point, that is, there exists $(u, v) \in X^2$ such that*

$$u = F(u, v) \quad \text{and} \quad v = F(v, u).$$

4. Examples

Example 4.1. *Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$. Set $g : X \to X$ and $F : X \times X \to X$ so that $g(x) = x^2$ and $F(x, y) = \frac{x^2 - y^2}{4}$, respectively.*

Then the operator F satisfies the mixed g-monotone property. Notice that

$$\max\{p(g(x), g(u)), p(g(y), g(v))\} = \max\{\max\{x^2, u^2\}, \max\{y^2, v^2\}\}.$$

On the other hand,

$$\begin{aligned} &\max\{p(F(x, y), F(u, v)), p(F(y, x), F(v, u))\} \\ &= \max\{\max\{\frac{x^2 - y^2}{8}, \frac{u^2 - v^2}{8}\}, \max\{\frac{y^2 - x^2}{8}, \frac{v^2 - u^2}{8}\}\} \end{aligned}$$

where $x \geq u$ and $y \leq v$. For $\psi(t) = t^2$ and $\phi(t) = \frac{t^2}{5}$ all conditions of Theorem 2.2 are satisfied. Therefore Theorem 2.2 yields a coupled coincidence point. In fact, $(0, 0)$ is the couple coincidence point of F and g.

Example 4.2. *Let X be a real line and $p(x, y) = \max\{x, y\}$. Suppose that $F : X \times X \to X$ is defined as $F(x, y) = \frac{2x - 2y}{7}$ for $x, y \in X$, respectively.*

Then the operator F satisfies mixed monotone property.

Let $x, y, u, v \in X$ with $x \geq u$, $y \leq v$ such that

$$\max\{p(x, u), p(y, v)\} = \max\{\max\{x, y\}, \max\{u, v\}\}. \tag{4.1}$$

On the other hand,

$$\max\{p(F(x, y), F(u, v)), p(F(y, x), F(v, u))\} \tag{4.2}$$

$$= \max\{\max\{\frac{2x - 2y}{7}, \frac{2u - 2v}{7}\}, \max\{\frac{2y - 2x}{7}, \frac{2v - 2u}{7}\}\} \tag{4.3}$$

For the alternating distance function $\psi(t) = t$ and $\phi(t) = \frac{t}{7}$, all conditions of Corollary 2.3 are satisfied. Consequently, Corollary 2.3 yields a coupled fixed point. Notice that $(0, 0)$ is the coupled fixed point of F.

5. Applications

We start this section with the following definition.

By Φ, we denote the class of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(a) ϕ is Lebesgue integrable function on each compact subset of $[0, \infty)$,

(b) $\displaystyle\int_0^{\varepsilon} \phi(s)ds > 0$ for any $\varepsilon > 0$.

Corollary 5.1. *Let $(X, \leq)$ be partially ordered set and (X, p) be a complete partial metric space. Assume that ϕ, ψ are alternating distance functions. Let $F : X \times X \to$*

X and $g : X \to X$ where $X \neq \emptyset$. Suppose that F has the mixed g-monotone property and

$$\int_0^{\max\{p(F(x,y),F(u,v)),p(F(y,x),F(v,u))\}} \psi(s)ds \leq \int_0^{\max\{p(g(x),g(u)),p(g(y),g(v))\}} \psi(s)ds - \int_0^{\max\{p(g(x),g(u)),p(g(y),g(v))\}} \psi(s)ds \tag{5.1}$$

where $\phi, \psi \in \Phi$. Suppose that there exist $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0), \quad gy_0 \geq F(y_0, x_0).$$

Assume that F is continuous. Then, F and g have a coupled coincidence point.

Proof. It is clear that the functions $t \to \int_0^t \phi(s)ds$ and $t \to \int_0^t \psi(s)ds$ are alternating functions. □

Finally we give the following corollary.

Corollary 5.2. *Let $(X, \leq)$ be partially ordered set and (X, p) be a complete partial metric space. Assume that ϕ, ψ are alternating distance functions. Let $F : X \times X \to X$ where $X \neq \emptyset$. Suppose that F has the mixed monotone property and*

$$\int_0^{\max\{p(F(x,y),F(u,v)),p(F(y,x),F(v,u))\}} \psi(s)ds \leq \int_0^{\max\{p(x,u),p(y,v)\}} \psi(s)ds - \int_0^{\max\{p(x,u),p(y,v)\}} \psi(s)ds \tag{5.2}$$

where $\phi, \psi \in \Phi$. Suppose that there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0), \quad y_0 \geq F(y_0, x_0).$$

Assume that F is continuous. Then, F has a coupled fixed point.

Proof. It is clear that the functions $t \to \int_0^t \phi(s)ds$ and $t \to \int_0^t \psi(s)ds$ are alternating functions. □

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FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS IN ORDERED UNIFORM SPACE

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Abstract. In this work, we use the order relation on uniform spaces which is defined by [1] so we present some fixed point results for monotone operators in ordered uniform spaces using a weak generalized contraction-type assumption.

1. Introduction

There exists considerable literature of fixed point theory dealing with results on fixed or common fixed points in uniform space (e.g. [1,2,3,5,13,16,17,18]). But the majority of these results are proved for contractive or contractive type mapping (notice from the cited references). Recently, Aamri and El Moutawakil [1] have introduced the concept of E-distance function on uniform spaces and utilize it to improve some well known results of the existing literature involving both E-contractive or E- expansive mappings. Lately, I. Altun and M. Imdad [5] have introduced a partial ordering on uniform spaces utilizing E- distance function and have used the same to prove a fixed point theorem for single-valued non-decreasing mappings on ordered uniform spaces. The Banach contraction principle is the most celebrated fixed point theorem. Boyd and Wong [7] extended the Banach contraction principle to the case of nonlinear contraction mappings. Afterward many authors obtained important fixed point theorems (cf. [1-18]). Recently Bhaskar and Lakshmikantham [6], Nieto and Lopez [11,12]], Ran and Reurings [14] and Agarwal, El-Gebeily and O'Regan [4] presented some new results for contractions in partially ordered metric spaces.

In this work we use the order relation on uniform spaces which is defined by [5] so we present some fixed point results for monotone operators in ordered uniform spaces using a weak generalized contraction-type assumption.

Now, we mention some relevant definitions and properties from the foundation of uniform spaces. We call a pair (X, ϑ) to be a uniform space which consists of a non-empty set X together with an uniformity ϑ wherein the latter begins with a special kind of filter on $X \times X$ whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $m, n \geq N$. A uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

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A uniform space (X,ϑ) is said to be Hausdorff if and only if the intersection of all the $V\in\vartheta$ reduces to diagonal Δ of X i.e. $(x,y)\in V$ for $V\in\vartheta$ implies $x=y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V=V^{-1}=\{(y,x):(x,y)\in V\}$. Since each $V\in\vartheta$ contains a symmetrical $W\in\vartheta$ and if $(x,y)\in W$ then x and y are both W and V-close and then one may assume that each $V\in\vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X,ϑ), they are naturally interpreted with respect to the topological space $(X,\tau(\vartheta))$.

2. Preliminaries

We shall require the following definitions and lemmas in the sequel.

Definition 2.1 ([1])**.** *Let (X,ϑ) be an uniform space. A function $p:X\times X\to\mathbb{R}^+$ is said to be an E-distance if*

(p_1) *For any $V\in\vartheta$ there exists $\delta>0$ such that $p(z,x)\leq\delta$ and $p(z,y)\leq\delta$ for some $z\in X$, imply $(x,y)\in V$,*

(p_2) $p(x,y)\leq p(x,z)+p(z,y)$, $\forall x,y,z\in X$.

The following lemma embodies some useful properties of E-distance.

Lemma 2.2 ([1], [2])**.** *Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x,y,z\in X$, the following holds:*

(a) If $p(x_n,y)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $y=z$. In particular, if $p(x,y)=0$ and $p(x,z)=0$, then $y=z$.

(b) If $p(x_n,y_n)\leq\alpha_n$ and $p(x_n,z)\leq\beta_n$ for all $n\in\mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n,x_m)\leq\alpha_n$ for all $m>n$, then $\{x_n\}$ is a $p-$Cauchy sequence in (X,ϑ).

Let (X,ϑ) be an uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2.3 ([1], [2])**.** *Let (X,ϑ) be an uniform space and p be an E-distance on X. Then*

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x\in X$ with $\lim_{n\to\infty}p(x_n,x)=0$,

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x\in X$ with $\lim_{n\to\infty}x_n=x$ with respect to $\tau(\vartheta)$,

(iii) $f:X\to X$ is p-continuous if $\lim_{n\to\infty}p(x_n,x)=0$ implies

$$\lim_{n\to\infty}p(fx_n,fx)=0,$$

(iv) $f:X\to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty}x_n=x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty}fx_n=fx$ with respect to $\tau(\vartheta)$.

Remark 2.1 ([1])**.** *Let (X,ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x\in X$ such that*

$\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.

Lemma 2.4 ([4]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$. Define the relation " $\preceq$ " on X as follows;*

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x,y) \le \varphi(x) - \varphi(y).$$

Then " $\preceq$ " is a (partial) order on X induced by φ.

Definition 2.5. *Let (X, ϑ) be an uniform space, " $\preceq$ " is an order on X and $T : X \to X$. T is non-decreasing if $x, y \in X$, $x \preceq y$ implies $T(x) \preceq T(y)$.*

3. Main Results

Theorem 3.1. *Let (X, ϑ) be a uniform space, " $\preceq$ " is an order on X and suppose there is an $E-$distance p on X such that (X, p) is a $p-$Cauchy complete uniform space. Assume there is a non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ and also suppose T is a non-decreasing mapping with*

$$p(T(x), T(y)) \le \psi(p(x,y)) \text{ for all } x \succeq y.$$

Also suppose either

(i) *T is continuous*

or

(ii) *if $\{x_n\} \subseteq X$ is a non decreasing sequence with $x_n \to x$ in X then $x_n \preceq x$ for all n*

hold. If there exists an $x_0 \in X$ with $x_0 \preceq T(x_0)$ then T has a fixed point.

Proof. Since $\psi(t) < t$ for $t > 0$, ψ is non decreasing and suppose there exists $t_0 > 0$ with $t_0 \le \psi(t_0)$ then ψ is non decreasing as $t_0 \le \psi^n(t_0)$ for each $n \in \{1, 2, ...\}$. Also, $\psi(0) = 0$.

We take $T(x_0) = x_0$. In this case proof is completed. Therefore suppose $T(x_0) \ne x_0$. Since $x_0 \preceq T(x_0)$ and T is non-decreasing we have $x_0 \preceq T(x_0) \preceq T^2(x_0) \preceq ... \preceq T^n(x_0) \preceq T^{n+1}(x_0) \preceq ....$

As $x_0 \preceq T(x_0)$, we have $p(T^2(x_0), T(x_0)) \le \psi(p(T(x_0), x_0)$ and since $T(x_0) \preceq T^2(x_0)$ we have

$p(T^3(x_0), T^2(x_0)) \le \psi(p(T^2(x_0), T(x_0)) \le \psi^2(p(T(x_0), x_0))$.

Therefore, as use induction method,

$p(T^{n+1}(x_0), T^n(x_0)) \le \psi^n(p(T(x_0), x_0)$.

Now, let $\varepsilon > 0$ be fixed. Take $n \in \{1, 2, ...\}$ so that

$p(T^{n+1}(x_0), T^n(x_0)) < \varepsilon - \psi(\varepsilon)$.

As $T^n(x_0) \preceq T^{n+1}(x_0)$, then we have

$$\begin{aligned} p(T^{n+2}(x_0), T^n(x_0)) &\le p(T^{n+2}(x_0), T^{n+1}(x_0)) + (p(T^{n+1}(x_0), T^n(x_0)) \\ &\le \psi(p(T^{n+1}(x_0), T^n(x_0))) + [\varepsilon - \psi(\varepsilon)] \\ &\le \psi(\varepsilon - \psi(\varepsilon)) + [\varepsilon - \psi(\varepsilon)] \\ &\le \psi(\varepsilon) + [\varepsilon - \psi(\varepsilon)] \\ &= \varepsilon. \end{aligned}$$

Furthermore, since $T^n(x_0) \preceq T^{n+2}(x_0)$ we have

$$\begin{aligned} p(T^{n+3}(x_0), T^n(x_0)) &\le p(T^{n+3}(x_0), T^{n+1}(x_0)) + (p(T^{n+1}(x_0), T^n(x_0)) \\ &\le \psi(p(T^{n+2}(x_0), T^n(x_0))) + [\varepsilon - \psi(\varepsilon)] \\ &\le \psi(\varepsilon - \psi(\varepsilon)) + [\varepsilon - \psi(\varepsilon)] \end{aligned}$$

$= \varepsilon.$

Again, by use the induction method $p(T^{n+k}(x_0), T^n(x_0)) \leq \varepsilon$ for $k \in \{1,2,...\}$.

This inequality implies that $\{T^n(x_0)\}$ is a $p-$Cauchy sequence in X and also that $T^n(x_0) \preceq T^{n+1}(x_0)$ so there exists a $x \in X$ with $\lim_{n\to\infty} T^n(x_0) = x$.

If (i) holds then clearly $x = Tx$. Now suppose (ii) holds. Assume

$p(x, T(x)) = k < 0$. Therefore since $x = \lim_{n\to\infty} T^n(x_0)$ there exists $n^{\shortmid} \in \{1,2,...\}$ with $p(x, T^n(x_0)) < \frac{k}{2}$ for $n \geq n^{\shortmid}$. Since from (ii) that $T^n(x_0) \preceq x$, for $n \geq n^{\shortmid}$ we have

$$\begin{aligned} p(x, T(x)) &\leq p(x, T^{n+1}(x_0)) + (p(T(x), T^{n+1}(x_0)) \\ &< \tfrac{k}{2} + \psi(p(x, T^n(x_0))) < \tfrac{k}{2} + \psi(\tfrac{k}{2}) \leq k. \end{aligned}$$

This is a contradiction and then $T(x) = x$. □

Theorem 3.2. *Let (X, ϑ) be a uniform space, " $\preceq$ " is an order on X and suppose there is an $E-$distance p on X such that (X, p) is a $p-$Cauchy complete uniform space. Assume there is a non decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$ and also suppose T is a non-decreasing mapping with*

$$p(T(x), T(y)) \leq \psi\left(\max\{p(x,y), p(x,T(x)), p(y,T(y)), \frac{1}{2}[p(x,T(y)) + p(y,T(x))\}\right)$$

for all $x \succeq y$.

Also suppose either

(i) *T is continuous*

or

(ii) *if $\{x_n\} \subseteq X$ is a non decreasing sequence with $x_n \to x$ in X then $x_n \preceq x$ for all n*

hold. If there exists an $x_0 \in X$ with $x_0 \preceq T(x_0)$ then T has a fixed point.

Proof. Since $x_0 \preceq T(x_0)$ and T is non-decreasing we have

$x_0 \preceq T(x_0) \preceq T^2(x_0) \preceq ... \preceq T^n(x_0) \preceq T^{n+1}(x_0) \preceq ....$

Now, we claim that

$$p\left(T^{n+1}(x_0), T^n(x_0)\right) \leq \psi\left(p\left(T^n(x_0), T^{n-1}(x_0)\right)\right) \quad ...(1)$$

From (1) and $T^{n-1}(x_0) \preceq T^n(x_0)$

$$\begin{aligned} p(T^{n+1}(x_0), T^n(x_0)) &\leq \psi(\max\{p(T^n(x_0), T^{n-1}(x_0)), p(T^n(x_0), T^{n+1}(x_0)), \\ &\quad p(T^{n-1}(x_0), T^n(x_0)),\ \tfrac{1}{2}[p(T^n(x_0), T^n(x_0)) + p(T^{n-1}(x_0), T^{n+1}(x_0))]\}) \\ &\leq \psi(\alpha_n) \end{aligned}$$

Where

$$\begin{aligned} \alpha_n = \max\{&p\left(T^n(x_0), T^{n-1}(x_0)\right), p\left(T^n(x_0), T^{n+1}(x_0)\right), \\ &\tfrac{1}{2}[p(T^n(x_0), T^{n-1}(x_0)) + p(T^n(x_0), T^{n+1}(x_0))]\}. \end{aligned}$$

If $\alpha_n = p\left(T^n(x_0), T^{n-1}(x_0)\right)$ then (1) holds. If $\alpha_n = p\left(T^n(x_0), T^{n+1}(x_0)\right)$ then $p\left(T^n(x_0), T^{n+1}(x_0)\right) = 0$ since if not

$$p\left(T^n(x_0), T^{n+1}(x_0)\right) \leq \psi\left(p(T^n(x_0), T^{n+1}(x_0))\right) < p\left(T^n(x_0), T^{n+1}(x_0)\right).$$

Therefore this is a contradiction. Thus $p\left(T^n(x_0), T^{n+1}(x_0)\right) = 0$ and (1) is immediate. Lastly assume $\alpha_n = \frac{1}{2}[p(T^n(x_0), T^{n-1}(x_0)) + p(T^n(x_0), T^{n+1}(x_0))]$.

If $\alpha_n = 0$ then $p\left(T^n(x_0), T^{n+1}(x_0)\right) = 0$ and (1) is immediate.

If $\alpha_n \neq 0$ we have

$$\begin{aligned} p\left(T^n(x_0), T^{n+1}(x_0)\right) &\leq \psi(\tfrac{1}{2}[p(T^n(x_0), T^{n-1}(x_0)) + p(T^n(x_0), T^{n+1}(x_0))]) \\ &< \tfrac{1}{2}[\ p\left(T^n(x_0), T^{n-1}(x_0)\right) + p\left(T^n(x_0), T^{n+1}(x_0)\right)] \end{aligned}$$

and therefore
$\frac{1}{2}p\left(T^n(x_0), T^{n+1}(x_0)\right) < \frac{1}{2}p\left(T^n(x_0), T^{n-1}(x_0)\right).$
Then as a result
$$\begin{aligned}\alpha_n &= \tfrac{1}{2}[p(T^n(x_0), T^{n-1}(x_0)) + p(T^n(x_0), T^{n+1}(x_0))]\\ &< \tfrac{1}{2}p\left(T^n(x_0), T^{n-1}(x_0)\right) + \tfrac{1}{2}p\left(T^n(x_0), T^{n-1}(x_0)\right)\\ &= p\left(T^n(x_0), T^{n-1}(x_0)\right),\end{aligned}$$
this contradicts the definition of α_n. That is (1) is true in all cases. Thus
$p\left(T^{n+1}(x_0), T^n(x_0)\right) \leq \psi^n(p(T(x_0), x_0))$
and so $\lim_{n\to\infty} p(T^{n+1}(x_0), T^n(x_0)) = 0$. Let $\varepsilon > 0$ be fixed. Take $n \in \{1, 2, ...\}$ so that
$p\left(T^{n+1}(x_0), T^n(x_0)\right) < \varepsilon - \psi(\varepsilon).$
Finally, from theorem 3.1,
$p\left(T^{n+2}(x_0), T^n(x_0)\right) \leq p\left(T^{n+2}(x_0), T^{n+1}(x_0)\right) + [\varepsilon - \psi(\varepsilon)] \leq \varepsilon$ ...(2)
and
$p\left(T^{n+3}(x_0), T^n(x_0)\right) \leq p\left(T^{n+3}(x_0), T^{n+1}(x_0)\right) + [\varepsilon - \psi(\varepsilon)]$
also from (1) we have
$p\left(T^{n+2}(x_0), T^{n+1}(x_0)\right) \leq \psi(p\left(T^{n+1}(x_0), T^n(x_0)\right)) \leq \psi(\varepsilon)$ ...(3).
Since from (2) and (3)
$$\begin{aligned}p\left(T^{n+3}(x_0), T^n(x_0)\right) &\leq [\varepsilon - \psi(\varepsilon)] + \psi\max\{p\left(T^{n+2}(x_0), T^n(x_0)\right),\\ &p\left(T^{n+1}(x_0), T^n(x_0)\right), p\left(T^{n+3}(x_0), T^{n+2}(x_0)\right), \tfrac{1}{2}[p(T^{n+2}(x_0), T^{n+1}(x_0))+\\ &p(T^{n+3}(x_0), T^n(x_0))]\})\\ &\leq [\varepsilon - \psi(\varepsilon)] + \psi(\max\{\varepsilon, \varepsilon - \psi(\varepsilon), \psi^2(\varepsilon), \tfrac{1}{2}[\psi(\varepsilon) + p(T^{n+3}(x_0), T^n(x_0))]\})\\ &\leq [\varepsilon - \psi(\varepsilon)] + \psi(\beta_n)\end{aligned}$$
and thus from (1) and (3) we have
$p\left(T^{n+3}(x_0), T^{n+2}(x_0)\right) \leq \psi\left(p\left(T^{n+2}(x_0), T^{n+1}(x_0)\right)\right) \leq \psi^2(\varepsilon);$
where $\beta_n = \max\{\varepsilon, \frac{1}{2}[\psi(\varepsilon) + p\left(T^{n+3}(x_0), T^n(x_0)\right)]\}$.
If $\beta_n = \frac{1}{2}[\psi(\varepsilon) + p\left(T^{n+3}(x_0), T^n(x_0)\right)]$ (here $\beta_n > 0$), then
$p\left(T^{n+3}(x_0), T^n(x_0)\right) \leq [\varepsilon - \psi(\varepsilon)] + \frac{1}{2}[\psi(\varepsilon) + p\left(T^{n+3}(x_0), T^n(x_0)\right)]$
therefore
$\frac{1}{2}p\left(T^{n+3}(x_0), T^n(x_0)\right) < [\varepsilon - \psi(\varepsilon)] + \frac{1}{2}\psi(\varepsilon),$
and in conclusion
$\beta_n = \frac{1}{2}[\psi(\varepsilon) + p\left(T^{n+3}(x_0), T^n(x_0)\right)] < \frac{1}{2}\psi(\varepsilon) + \{[\varepsilon - \psi(\varepsilon)] + \frac{1}{2}\psi(\varepsilon)\} = \varepsilon.$
This contradicts with the definition of β_n. Consequently $\beta_n = \varepsilon$ and so
$p\left(T^{n+3}(x_0), T^n(x_0)\right) \leq [\varepsilon - \psi(\varepsilon)] + \psi(\varepsilon) = \varepsilon.$ ...(4).
Finally notice that
$p\left(T^{n+4}(x_0), T^n(x_0)\right) \leq p\left(T^{n+4}(x_0), T^{n+1}(x_0)\right) + [\varepsilon - \psi(\varepsilon)].$
Furthermore
$$\begin{aligned}p\left(T^{n+3}(x_0), T^{n+1}(x_0)\right) &\leq \psi(\max\{p\left(T^{n+2}(x_0), T^n(x_0)\right),\\ &p\left(T^{n+1}(x_0), T^n(x_0)\right), p\left(T^{n+3}(x_0), T^{n+2}(x_0)\right),\\ &\tfrac{1}{2}[p\left(T^{n+2}(x_0), T^{n+1}(x_0)\right) + p\left(T^{n+3}(x_0), T^n(x_0)\right)]\})\\ &\leq \psi(\max\{\varepsilon, \varepsilon - \psi(\varepsilon), \psi^2(\varepsilon), \tfrac{1}{2}[[\psi(\varepsilon) + \varepsilon]\})\end{aligned}$$
as from (1) we have
$p\left(T^{n+3}(x_0), T^{n+2}(x_0)\right) \leq \psi^2(p(T^{n+1}(x_0), T^n(x_0))) \leq \psi^2(\varepsilon).$
As a result $p\left(T^{n+3}(x_0), T^{n+1}(x_0)\right) \leq \psi(\varepsilon).$ ...(5).
So, (4) and (5)
$$\begin{aligned}p\left(T^{n+4}(x_0), T^n(x_0)\right) &\leq [\varepsilon - \psi(\varepsilon)] + p\left(T^{n+4}(x_0), T^{n+1}(x_0)\right)\\ &\leq [\varepsilon - \psi(\varepsilon)] + \psi(\max\{p\left(T^{n+3}(x_0), T^n(x_0)\right),\end{aligned}$$

$$p\left(T^{n+1}\left(x_0\right), T^n\left(x_0\right)\right), p\left(T^{n+4}\left(x_0\right), T^{n+3}\left(x_0\right)\right),$$
$$\tfrac{1}{2}[p\left(T^{n+3}\left(x_0\right), T^{n+1}\left(x_0\right)\right)+p\left(T^{n+4}\left(x_0\right), T^n\left(x_0\right)\right)]\})$$
$$\leq[\varepsilon-\psi(\varepsilon)]+\psi(\max\{\varepsilon, \varepsilon-\psi(\varepsilon), \psi^3(\varepsilon),\ \tfrac{1}{2}[\psi(\varepsilon)+p\left(T^{n+4}\left(x_0\right), T^n\left(x_0\right)\right)]\}).$$

Since from (1) we have

$p\left(T^{n+4}\left(x_0\right), T^{n+3}\left(x_0\right)\right) \leq \psi^3(p\left(T^{n+1}\left(x_0\right), T^n\left(x_0\right)\right)) \leq \psi^3(\varepsilon)$.

In conclusion

$p\left(T^{n+4}\left(x_0\right), T^n\left(x_0\right)\right) \leq[\varepsilon-\psi(\varepsilon)]+\psi\left(k_n\right)$,

$k_n=\max\{\varepsilon, \frac{1}{2}[\psi(\varepsilon)+p\left(T^{n+4}\left(x_0\right), T^n\left(x_0\right)\right)]\}$.

Thus, see that $k_n=\varepsilon$ and so,

$p\left(T^{n+4}\left(x_0\right), T^n\left(x_0\right)\right) \leq[\varepsilon-\psi(\varepsilon)]+\psi(\varepsilon)=\varepsilon$. ...(6).

Similarly for $k \epsilon\{1,2, \ldots\}$,

$p\left(T^{n+k-1}\left(x_0\right), T^{n+1}\left(x_0\right)\right) \leq \psi(\varepsilon)$ and $p\left(T^{n+k}\left(x_0\right), T^n\left(x_0\right)\right) \leq \varepsilon$ ...(7).

Therefore $\{T^n\left(x_0\right)\}$ is a $p-$Cauchy sequence in X, so there exists a $x \in X$ with $\lim\limits_{n \rightarrow \infty} T^n\left(x_0\right)=x$.

Since (i), $x=T(x)$. Assume (ii) holds and $p(x, T(x))=t>0$. Now since $x=\lim\limits_{n \rightarrow \infty} T^n\left(x_0\right)$ there exists $n_0 \epsilon\{1,2, \ldots\}$ with $p\left(x, T^n\left(x_0\right)\right)<\frac{t}{2}$ for $n \geq n_0$. Since from (ii) that $T^n\left(x_0\right) \preceq x$ then for $n \geq n_0$,

$$p(x, T(x)) \leq p\left(x, T^{n+1}\left(x_0\right)\right)+p\left(T(x), T^{n+1}\left(x_0\right)\right)$$
$$\leq p\left(x, T^{n+1}\left(x_0\right)\right)+\psi(\max\{p\left(x, T^n\left(x_0\right)\right),$$
$$p(x, T(x)), p\left(T^{n+1}\left(x_0\right), T^n\left(x_0\right)\right),$$
$$\tfrac{1}{2}[p\left(x, T^{n+1}\left(x_0\right)\right)+p\left(T(x), T^n\left(x_0\right)\right)]\}).$$

Furthermore $p\left(x, T^n\left(x_0\right)\right)<\frac{t}{2} \leq t=p(x, T(x))$,

$$p\left(T^{n+1}(x_0), T^n\left(x_0\right)\right) \leq p\left(x, T^n\left(x_0\right)\right)+p\left(x, T^{n+1}\left(x_0\right)\right)<\tfrac{t}{2}+\tfrac{t}{2}=t,$$

and also

$$\tfrac{1}{2}[p\left(x, T^{n+1}\left(x_0\right)\right)+p\left(T(x), T^n\left(x_0\right)\right)]\})< \tfrac{1}{2}[\tfrac{t}{2}+p(x, T(x))+p\left(x, T^n\left(x_0\right)\right)]$$
$$< \tfrac{1}{2}[\tfrac{t}{2}+t+\tfrac{t}{2}]=t.$$

Consequently we have $p(x, T(x)) \leq p\left(x, T^{n+1}\left(x_0\right)\right)+\psi(p(x, T(x)))$ for $n \geq n_0$, then letting $n \rightarrow \infty$ yields $p(x, T(x)) \leq \psi(p(x, T(x)))$ which is a contradiction. Thus $p(x, T(x))=0$. □

Theorem 3.3. *Let (X, ϑ) be a uniform space, " $\preceq$ " is an order on X and suppose there is an $E-$distance p on X such that (X, p) is a $p-$Cauchy complete uniform space. Assume there is a $\tau(\vartheta)-$continuous function $\psi:[0, \infty) \rightarrow[0, \infty)$ with $\lim\limits_{n \rightarrow \infty} \psi^n(t)=0$ for each $t>0$ and also suppose T is a non-decreasing mapping with*

$$p(T(x), T(y)) \leq \psi(\max\{p(x, y), p(x, T(x)), p(y, T(y))) \text{ for all } x \succeq y.$$

Also suppose either

(i) *T is continuous*

or

(ii) *if $\{x_n\} \subseteq X$ is a non decreasing sequence with $x_n \rightarrow x$ in X then $x_n \preceq x$ for all n*

hold. If there exists an $x_0 \in X$ with $x_0 \preceq T(x_0)$ then T has a fixed point.

Proof. Let $\gamma_n=p\left(T^{n+1}\left(x_0\right), T^n\left(x_0\right)\right)$. Notice since $T^n\left(x_0\right) \succeq T^{n-1}\left(x_0\right)$ that

$$\gamma_n \leq \psi(\max\{p\left(T^n\left(x_0\right), T^{n-1}\left(x_0\right)\right), p\left(T^n\left(x_0\right), T^{n+1}\left(x_0\right)\right),$$
$$p\left(T^{n-1}\left(x_0\right), T^n\left(x_0\right)\right)\})$$

$$= \psi\left(\max\{p\left(T^{n}\left(x_0\right), T^{n-1}\left(x_0\right)\right), p\left(T^{n+1}\left(x_0\right), T^{n}\left(x_0\right)\right)\}\right)$$
$$= \psi(\max\{\gamma_{n-1}, \gamma_n\}).$$

We now show

$$\gamma_n \leq \psi\left(\gamma_{n-1}\right). \ ...(8)$$

If $\max\{\gamma_{n-1}, \gamma_n\} = \gamma_{n-1}$ then above inequality is true, whereas if

$\max\{\gamma_{n-1}, \gamma_n\} = \gamma_n$ then $\gamma_n \leq \psi\left(\gamma_n\right)$ and so $\gamma_n = 0$, so (8) is immediate. Therefore (8) holds. Now since $\gamma_n \leq \psi\left(\gamma_{n-1}\right) \leq \gamma_{n-1}$ there exists $\gamma \geq 0$ with $\gamma_n \downarrow \gamma$. Now $\gamma_n \leq \psi\left(\gamma_{n-1}\right)$ together with the continuity of ψ implies $\gamma \leq \psi\left(\gamma\right)$ so $\gamma = 0$. As a result

$\gamma_n = p\left(T^{n+1}\left(x_0\right), T^{n}\left(x_0\right)\right) \to 0$ as $n \to \infty$. ...(9)

Thus $\{T^{n}\left(x_0\right)\}$ is a $p-$Cauchy sequence. ...(10)

Now, suppose (10) is false. Then we can find a $\delta > 0$ and two sequences of integers $\{m\left(k\right)\}$, $\{l\left(k\right)\}$, $m\left(k\right) > l\left(k\right) \geq k$ with

$r_k = p\left(T^{l(k)}\left(x_0\right), T^{m(k)}\left(x_0\right)\right) \geq \delta$ for $k \in \{1, 2, ...\}$. ...(11)

Also suppose

$p\left(T^{m(k)-1}(x_0)\right., T^{l(k)}(x_0)) < \delta$, ...(12)

by choosing $m\left(k\right)$ to be the smallest number exceeding $l(k)$ for which (11) holds. Now

$$\delta \leq r_k \leq p\left(T^{m(k)-1}(x_0), T^{l(k)}(x_0)\right) + p\left(T^{m(k)}\left(x_0\right), T^{m(k)-1}(x_0)\right) < \delta + \gamma_{m(k)-1},$$

so with this, (9) implies

$\lim\limits_{k\to\infty} r_k = \delta$. ...(2.14)

Furthermore note that $T^{m(k)}\left(x_0\right) \succeq T^{l(k)}\left(x_0\right)$ since $m\left(k\right) > l\left(k\right)$

$$\begin{aligned}
\delta \leq r_k &\leq p\left(T^{l(k)+1}\left(x_0\right), T^{l(k)}\left(x_0\right)\right) + p\left(T^{m(k)+1}\left(x_0\right), T^{m(k)}(x_0)\right) \\
&\quad + p\left(T^{m(k)+1}\left(x_0\right), T^{l(k)+1}(x_0)\right) \\
&= \gamma_{l(k)} + \gamma_{m(k)} + p\left(T^{m(k)+1}\left(x_0\right), T^{l(k)+1}(x_0)\right) \\
&\leq \gamma_{l(k)} + \gamma_{m(k)} + \psi(\max\{p\left(T^{m(k)}\left(x_0\right), T^{l(k)}(x_0)\right), \\
&\quad p\left(T^{m(k)}\left(x_0\right), T^{m(k)+1}(x_0)\right), p\left(T^{l(k)}\left(x_0\right), T^{l(k)+1}(x_0)\right)\} \\
&= \gamma_{l(k)} + \gamma_{m(k)} + \psi(r_k, \gamma_{l(k)}, \gamma_{m(k)}\})
\end{aligned}$$

and let $k \to \infty$, since (9), (13) and ψ are continuous then $\delta \leq \psi\left(\delta\right)$. Thus $\delta = 0$, which is a contradiction. As a result (10) holds, so there exists $x \in X$ with $\lim\limits_{n\to\infty} T^{n}\left(x_0\right) = x$.

If (i) holds then clearly $x = T\left(x\right)$. Now suppose (ii) holds. Since from (ii) that $T^{n}\left(x_0\right) \preceq x$ then

$$\begin{aligned}
p\left(x, T\left(x\right)\right) &\leq p\left(x, T^{n+1}\left(x_0\right)\right) + p\left(T(x), T^{n+1}\left(x_0\right)\right) \\
&\leq p\left(x, T^{n+1}\left(x_0\right)\right) + \psi(\max\{p\left(x, T^{n}\left(x_0\right)\right), \\
&\quad p\left(x, T\left(x\right)\right), p\left(T^{n+1}(x_0), T^{n}\left(x_0\right)\right)\}) \\
&\leq p\left(x, T^{n+1}\left(x_0\right)\right) + \psi(\max\{p\left(x, T^{n}\left(x_0\right)\right), p\left(x, T\left(x\right)\right), \gamma_n\})
\end{aligned}$$

and let $n \to \infty$ since ψ is continuous then obtain $p\left(x, T\left(x\right)\right) \leq \psi\left(p\left(x, T\left(x\right)\right)\right)$, so $p\left(x, T\left(x\right)\right) = 0$. □

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NONSTANDARD FINITE DIFFERENCE SCHEMES FOR FUZZY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a method for numerical approximation of fuzzy first order initial value problem is presented. We construct and develop nonstandard scheme for fuzzy differential equations. The scheme based on the nonstandard finite difference scheme is discussed. Examples are given, including nonlinear fuzzy first order differential equations.

1. INTRODUCTION

The theoretical framework of fuzzy differential equations (FDEs) has been an active research field over the last few years. Fuzzy differential equations are used in modelling problems in engineering and sciences. Namely in study of population models [15], quantum optic, gravity [12], medicine [3] and [5]. After introducing suffcient conditions for the existence of unique solutions of these equations, numerical methods for approximating these solutions were developed [1] and [19]. A comprehensive approach to FDEs has been the work of Seikkala [24], especially in its generalized form given by Buckley and Feuring [7]. Their work is important as it overcomes the existence of multiple definitions of the derivative of fuzzy functions, i.e.[11, 14, 19, 23, 24]. Moreover, in [7], a more general family of FDEs is faced from an analytical point of view. The results of [24] on a certain category of FDEs have inspired several authors who have applied numerical methods for the solution of these equations. Other methods were discussed by Puri and Ralescu [23] and Goetshchel and Voxman [14]. The use of fuzzy differential equations are natural way to model dynamical system under possibilistic uncertainty [25]. The concept of differential equations in a fuzzy environment was formulated by Kaleva [16]. The last few years, several authors have produced a wide range of results in both the theoretical and applied fields [6, 10, 16, 17, 24].

The most important contribution on these numerical methods is the Euler method provided by Ma [19]. Although this work is significant, it has the disadvantage that, when examining the convergence of their Euler method, the authors practically work on the convergence of the ODEs system that occurs when solving numerically. The authors of [2] develop runge kutta method for FDEs. However, their work shares the same problems as [19] and concentrates exclusively on this methods [8]. Following the results of, we apply nonstandard finite difference schemes for FDEs. The paper is organized as follows:

In Section 2, we give all the theoretical background we need and present, in sort terms, the theory of FDEs that is necessary for our goal. nonstandard finite

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difference schemes for solving FDEs are introduced in Section 3. The applications of the proposed numerical schemes are illustrated in Section 4. The conclusions are then given in the final part, Section 5.

2. SOME DEFINITONS AND THEOREM ABOUT FUZZY LOGIC

Firstly, we give some basic definitions and results. The solutions of FDEs are fuzzy functions, whose values are fuzzy numbers, for which we follow the definition of [4, 8, 20, 23].

Definition 2.1. *The membership function embodied the mathematical representation of membership in a set, and the notation used throughout this text for a fuzzy set is a set symbol with a tilde underscore, say A , where the functional mapping is given by;*

$$\mu_A : X \to [0,1]$$

$$x \in X \text{ and } \mu_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

and the symbol $\mu_A(x)$ is the degree of membership of element x in fuzzy set A. Therefore, $\mu_A(x)$ is a value on the unit interval that measures the degree to which element x belongs to fuzzy set A; equivalently, $\mu_A(x)$ is a degree to which $x \in A$ and fuzzy set A is given by;

$$A = \{(\mu_A(x), x) : x \in X\}$$

Definition 2.2. *A fuzzy number is a normalized fuzzy set A of $\mathbb{R}$, for which the following conditions hold:*
i) μ_A is upper semi continuous,
ii) A is convex
iii) Sets $\{x \in \mathbb{R}, \mu_A(x) = a\}$ are compact for $a \in (0,1]$.
We say that a fuzzy number is triangular if its membership function is a triangle (see Fig. 2.1). The membership function of a triangular fuzzy number C can be easily found if the interval $[C_1,C_3]$ of its basis and the summit $C_{2,}$ are known. For this reason, triangular fuzzy numbers are denoted by $(C_1,C_{2,},C_3)$. The set of fuzzy numbers is symbolized as $F(\mathbb{R})$. Before defining FDEs, we summarize a few things about them. And the other hand, we say that a fuzzy number is trapezoidal if its membership function is a trapezoidal (see Fig. 2.2). The membership function of a triangular fuzzy number C can be easily found if the interval $[C_1,C_4]$ of its basis and the summit $C_{2,}$ are known. For this reason, triangular fuzzy numbers are denoted

by $(C_1/C_{2,}, C_3/C_4)$. The set of fuzzy numbers are symbolized as $F(\mathbb{R})$.

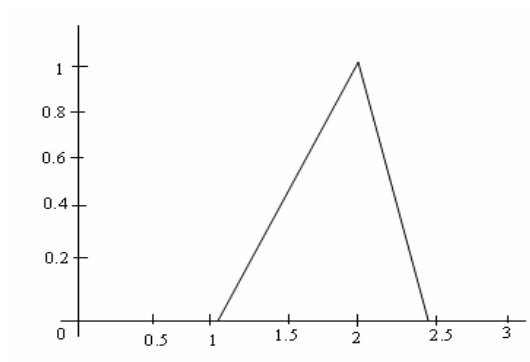

Figure 2.1. Triangular fuzzy numbers

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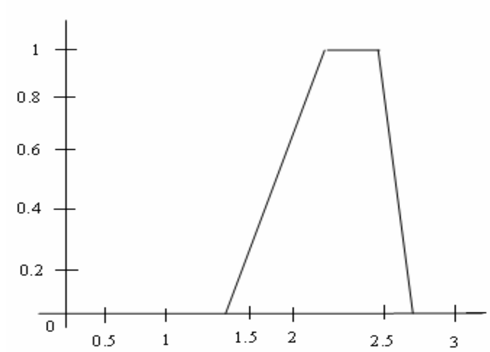

Figure 2.2. Trapezoidal fuzzy numbers

Definition 2.3. *We begin by considering a fuzzy set $A \in F(\mathbb{R})$, then define a $\alpha - cut$ set, A_α, where $0 \leq a \leq 1$. The set A_α is a crisp set called the $\alpha - cut$ (or lambda (λ)-cut) set of the fuzzy set A_α,where*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\} = [A_1^\alpha(x), A_2^\alpha(x)]$$

Note that the $\alpha-cut$ set A_α does not have a tilde underscore; it is a crisp set derived from its parent fuzzy set A_α. Any particular fuzzy set A_α can be transformed into an infinite number of $\alpha-cut$ sets, because there are an infinite number of values α on the interval $[0, 1]$. Any element $x \in A_\alpha$ belongs to A_α with a grade of membership that is greater than or equal to the value α.
Furthermore, we focus on fuzzy numbers with the property that for $A \in F(\mathbb{R})$ the set $\{x \in \mathbb{R} : \mu_A(x) > \alpha\}$ is bounded. This turns out to be a vital property when applying numerical methods. The following proposition gives arithmetic operations of fuzzy numbers in terms of their α-cuts.

Definition 2.4. *A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semi continuous membership function of bounded support. The class of fuzzy numbers will be denoted by $F(\mathbb{R})$. A fuzzy number u is completely determined by any pair $u(x;\alpha)$= $[u_1(x;\alpha), u_2(x;\alpha)]$ and $0 \leq a \leq 1$, which satisfy the three conditions:*
i) $u_1(x;\alpha)$ is a bounded left continuous monotonic increasing function $\alpha \in (0, 1]$,
ii) $u_2(x;\alpha)$ is a bounded left continuous monotonic decreasing function $\alpha \in (0, 1]$,
iii) $u_1(x;\alpha) \leq u_2(x;\alpha)$, $0 \leq a \leq 1$ [22].
A triangular fuzzy number U is defned by an ordered triple $U = (U_1, U_2, U_3) \in F(\mathbb{R})$ with $U_1 \leq U_2 \leq U_3$ where the graph of $U(x)$ is a triangular with base on the interval $[U_1, U_3]$ and vertex $x = U_2$. N is always a closed, bounded interval [18] and [22]. If $U = (U_1, U_2, U_3)$ then

$$U^\alpha = [U_1 + \alpha(U_2 - U_1), U_3 - \alpha(U_3 - U_2)]$$

for any $0 \leq a \leq 1$.

Proposition 2.5. *If $P, Q \in F(\mathbb{R})$ then for $\alpha \in (0, 1]$*

$$[P + Q]_\alpha = [P_1^\alpha + Q_1^\alpha, P_2^\alpha + Q_2^\alpha]$$

$$[P.Q]_\alpha = [\min\{P_1^\alpha.Q_1^\alpha, P_1^\alpha.Q_2^\alpha, P_2^\alpha.Q_1^\alpha, P_2^\alpha.Q_2^\alpha\}, \max\{P_1^\alpha.Q_1^\alpha, P_1^\alpha.Q_2^\alpha, P_2^\alpha.Q_1^\alpha, P_2^\alpha.Q_2^\alpha\}]$$

Let $P \in F(\mathbb{R})$. If there exists a fuzzy numbere R such that $P + R = Q$ then this number is unique and it is called Hukuhara differential of P, Q and is denoted by $Q - P$ [8, 23].
Let A; B two nonempty bounded subsets of $\mathbb{R}$. The Hausdorff distance between A and B is

$$d_H(A,B) = \max\left[\sup_{a\in A}\inf_{b\in B}|a-b|, \qquad \sup_{b\in B}\inf_{a\in A}|a-b|\right].$$

If $\tilde{P}, \tilde{Q} \in F(\mathbb{R})$ the distance D between $\tilde{P}$ and $\tilde{Q}$ is defined as

$$D(\tilde{P},\tilde{Q}) = \sup d_H([\tilde{P}]_a, [\tilde{Q}]_a)$$

Definition 2.6. *The supremum metric d_∞ on $F(\mathbb{R})$ is defined by*

$$d_\infty = \sup\{d_H([U]^\alpha, [V]^\alpha) : \alpha \in I\}$$

and $(F(\mathbb{R}), d_\infty)$ is a complete metric space.

Definition 2.7. *Let U be an open interval in $\mathbb{R}$. A fuzzy function $f : \mathbb{R} \to F(\mathbb{R})$ is called to be Hukuhara differentiable in $x_0 \in U$ if there exists $f'(x_0) \in F(\mathbb{R})$ such that*

$$\lim_{h\to 0^+} d_\infty\left(\frac{f(x_0+h)-f(x_0)}{h}, f'(x_0)\right) = 0$$

and

$$\lim_{h\to 0^+} d_\infty\left(\frac{f(x)-f(x_0-h)}{h}, f'(x_0)\right) = 0$$

both exist and they are equal to $f'(x_0)$ [8, 18, 23].
When this derivative exists, it is also written as

$$[f'(x)]_\alpha = [(f_1^\alpha)'(x), (f_2^\alpha)'(x)]$$

Let $(f_1^\alpha)'$,$(f_2^\alpha)'$ also be continuous functions with reference to both x and $\alpha \in (0,1]$. This property is called continuity condition. As we already mentioned in the introduction, in [4] *the following proposition is proved* [8].

Definition 2.8. *The fuzzy integral*

$$\int_a^b y(t)dt, \qquad 0 \le a \le b \le 1$$

is defined by

$$\left[\int_a^b y(t)dt\right]_\alpha = \left[\int_a^b y_1^\alpha(t)dt, \int_a^b y_2^\alpha(t)dt\right]$$

provided the Lebesgue integrals on the right exist [18].

Remark 2.1. *If $f : I \to F(\mathbb{R})$ is Hukuhara differentiable and its Hukuhara derivative f' is integrable over $[0,1]$, then*

$$f(t) = f(t_0) + \int_{t_0}^t f'(s)ds$$

for all values of t_0, t where $0 \le t_0 \le t \le 1$ [18].

Definition 2.9. *A mapping $y : I \to F(\mathbb{R})$ is called a fuzzy process. We denote*

$$[y(t)]_\alpha = [y_1(t), y_2(t)]$$

The Seikkala derivative $y'(t)$ of a fuzzy process y is defined by

$$[y'(t)]_\alpha = [y_1'(t), y_2'(t)]$$

provided the equation defines a fuzzy number $y'(t) \in F(\mathbb{R})$ [18].

Remark 2.2. *If $y : \mathbb{R} \to F(\mathbb{R})$ is Seikkala differentiable and its Seikkala derivative y' is integrable over* $[0, 1]$*, then*

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)ds$$

for all values of t_0, t where $t_0, t \in I$ [18].

Definition 2.10. *Consider the first-order fuzzy differential equation $y' = f(t, y)$, where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is Hukuhara or Seikkala fuzzy derivative of y. If an initial value $y(t_0) = y_0$ is given, a fuzzy cauchy problem of first-order will be obtained as follows:*

$$y'(t) = f(t, y(t)), \qquad t_0 \le t \le T, \qquad y(t_0) = y_0 \tag{2.1}$$

Sufficient conditions for the existence of a unique solution to Eq. (2.1) are:
i) Continuity of f,
ii) Lipschitz condition $d_\infty(f(t, x), f(t, y)) \le L\ d_\infty(f(t, x))$, $L > 0$.
By theorem 5.2 in [9] *we may replace Eq. (2.1) by equivalent system*

$$y'(t; \alpha) = f(t, y; \alpha) = (f_1(y, t), f_2(y, t)) = (F(t, y_1, y_2), G(t, y_1, y_2)) \tag{2.2}$$

$$y(t_0; \alpha) = (y_{1,0}, y_{2,0})$$

which possesses a unique solution (y_1, y_2) which is a fuzzy function, i.e. for each t, the pair $(y_1(t), y_2(t))$ is a fuzzy number.
In some cases the system given by Eq. (2.2) can be solved analytically [13]. *In most cases, however analytically solutions may not be found and a numerical approach must be considered. Some numerical methods such as the fuzzy Euler method, Adams–Bashforth, Adams–Moulton and predictor–corrector in FDE presented in* [4, 13, 18, 19].

3. NONSTANDART FINITE DIFFERENCE SCHEMES FOR FUZZY DIFFERENTIAL EQUATIONS

A fuzzy differential equation is

$$\frac{dy}{dt} = f(y, t, \lambda; \alpha), \tag{3.1}$$

where λ is n-parameter fuzzy vector. The simplest nonstandard finite difference schemes are constructed by making the replacements [21, 22].

$$t \to t_k = (\Delta t)k = hk, \qquad h = \Delta t$$

$$y(t; \alpha) = y(t_k; \alpha) = [y_k]_\alpha = [y_{1,k}, y_{2,k}]$$

$$\frac{dy}{dt} = \left[\frac{y_{1,k+1} - y_{1,k}}{\phi_1(h, \lambda_1)}, \frac{y_{2,k+1} - y_{2,k}}{\phi_2(h, \lambda_2)}\right] = [F(y_{1,k}, y_{1,k+1}, h, \lambda_1), G(y_{2,k}, y_{2,k+1}, h, \lambda_2)]$$

where $[\lambda]_\alpha = [\lambda_1, \lambda_2]$. The discrete derivate, on the left-side, is a generalization [22] , where the denominator fuzzy function $\phi(h, \lambda; \alpha) = [\phi_1(h, \lambda_1), \phi_2(h, \lambda_2)]$ has the property

$$\phi(h, \lambda; \alpha) = h + O(h^2).$$

Examples of fuzzy denominator functions $\phi(h, \lambda; \alpha)$ that satisfy this condition are

$$\phi(h, \lambda; \alpha) = \begin{cases} h \\ \sin(h) \\ e^h - 1 \\ 1 - e^{-h} \\ \frac{1-e^{-[\lambda]_\alpha h}}{[\lambda]_\alpha} \\ \vdots \end{cases}$$

4. NUMERICAL EXAMPLES

In this section, we show two examples. In example 4.2, the approximated solutions are obtained by nonstandard finite difference schemes and runge-kutta method are plotted in figures. While doing this, we use different nonlocal terms.

Example 4.1. *A fuzzy differential equation is*

$$y'(t) = \lambda y^2 + \beta y - 2, \qquad t \in [0, 1].$$

If we use nonlocal term following form

$$y(t; \alpha) \to [y_k]_\alpha$$
$$y^2(t; \alpha) \to [y_{k+1} y_k]_\alpha$$

we obtain

$$\frac{[y_{k+1}]_\alpha - [y_k]_\alpha}{h} = \lambda [y_{k+1} y_k]_\alpha + \beta [y_k]_\alpha - 2$$
$$[y_{k+1}]_\alpha = \frac{[y_k]_\alpha (1 + h\beta) - 2h}{1 - \lambda [y_k]_\alpha},$$

where denominator functions are given by

$$h = \phi(h, \lambda; \alpha)$$
$$1 + \beta h + O(\beta^2, h^2) = e^{\beta h}$$
$$h \to \frac{e^{\beta h} - 1}{\beta} = \phi(h, \beta; \alpha)$$

and we obtain,

$$[y_{k+1}]_\alpha = \frac{[y_k]_\alpha (1 + \phi(h, \lambda; \alpha)\beta) - 2\phi(h, \lambda; \alpha)}{1 - \lambda [y_k]_\alpha}.$$

If we choose different nonlocal terms:

$$y(t; \alpha) \to [y_k]_\alpha$$
$$y^2(t; \alpha) \to [y_k y_k]_\alpha$$

we obtain,

$$[y_{k+1}]_\alpha = \lambda h [y_k y_k]_\alpha + [y_k]_\alpha (1 + h\beta) - 2h$$
$$[y_{k+1}]_\alpha = \lambda \phi(h, \lambda; \alpha) [y_k y_k]_\alpha + [y_k]_\alpha (1 + \phi(h, \lambda; \alpha)\beta) - 2\phi(h, \lambda; \alpha).$$

A fuzzy differential equations system is

$$x'(t;\alpha) = -kyx - lx$$

$$y'(t;\alpha) = -kyx - x^2y + ly. \tag{4.1}$$

For $0 < \alpha \leq 1$, $[k]_\alpha = [k_1, k_2]$, $[l]_\alpha = [l_1, l_2]$, $[y_k]_\alpha = [y_{1,k}, y_{2,k}]$, $[x_k]_\alpha = [x_{1,k}, x_{2,k}]$ and $y(0;\alpha) = [0.1 + 0.1\alpha, 0.3 - 0.1\alpha]$, $x(0;\alpha) = [0.25 + 0.25\alpha, 1 - 0.5\alpha]$.

Case 1 : If we use these non-local terms in first equation of system (4.1):

$$x(t;\alpha) \to [x_k]_\alpha$$
$$y(t;\alpha) \to [y_{k+1}]_\alpha$$
$$(xy)(t;\alpha) \to [x_{k+1}y_k]_\alpha$$
$$x^2(t;\alpha) \to [x_{k+1}x_k]_\alpha$$
$$(x^2y)(t;\alpha) \to [x_{k+1}x_ky_k]_\alpha$$

we obtain,

$$\frac{[x_{k+1}]_\alpha - [x_k]_\alpha}{h} = -k[x_{k+1}y_k]_\alpha - l[x_k]_\alpha$$

and

$$[x_{k+1}]_\alpha = \frac{[x_k]_\alpha(1 - hl)}{(1 + kh[y_k]_\alpha)} \tag{4.2}$$

where denominator functions are given by

$$h \to \frac{1 - e^{-lh}}{l} = \phi_1(h, \lambda; \alpha).$$

We obtain

$$[x_{k+1}]_\alpha = \frac{[x_k]_\alpha(1 - \phi_1(h, \lambda; \alpha)l)}{(1 + k\phi_1(h, \lambda; \alpha)[y_k]_\alpha)}. \tag{4.3}$$

And if we use these non-local terms in systems (4.1),

$$\frac{[y_{k+1}]_\alpha - [y_k]_\alpha}{h} = -k[x_{k+1}y_k]_\alpha - [x_{k+1}x_ky_k]_\alpha + l[y_{k+1}]_\alpha$$

$$[y_{k+1}]_\alpha = \frac{[y_k]_\alpha - hk[x_{k+1}y_k]_\alpha - h[x_{k+1}x_ky_k]_\alpha}{1 - hl}$$

$$[y_{k+1}]_\alpha = \frac{[y_k]_\alpha - \phi_1(h, \lambda; \alpha)k[x_{k+1}y_k]_\alpha - \phi_1(h, \lambda; \alpha)[x_{k+1}x_ky_k]_\alpha}{1 - \phi_1(h, \lambda; \alpha)l} \tag{4.4}$$

For Case 1, nonstandard finite difference schemes(NFDS) solution and Runge Kutta(RK) solution are,in turn, given by Table1 and Table 2 at $t = 0.3$, $h = 0.1$, $k = (0.6/1/1.6)$, $l = (0.3/0.5/1)$.

α	$[x_1, x_2]$	$[y_1, y_2]$
0.0	.2244390804561129,.6576804535607970,	.1032047926283471,.2234385872420912
0.2	.2644511153616336,.6177753344071575,	.1226856257121523,.2222967264702617
0.4	.3026930617722152,.5731303484807911,	.1413768549298307,.2185949011368412
0.6	.3391322669787342,.5233153290117534,	.1591432957482870,.2122393855959653
0.8	.3737510387516800,.4679161611795227,	.1758703848701428,.2031877997687871
1.0	.4065461856588474,.4065461856588474	.1914641809109783,.1914641809109783

Table 1: NFDS solotions of system (4.1) for *Case* 1

α	$[x_1, x_2]$	$[y_1, y_2]$
0.0	.2343168288531087,.7787578518741462	.1028553497464426,.2007221062594857
0.2	.2775978932994965,.7164313414916264	.1220686035514132,.2050699613953046
0.4	.3194559435659985,.6513565295450350	.1403680147350854,.2061115756662631
0.6	.3598234574229595,.5831236878558108	.1575901611523887,.2036778273531992
0.8	.3986499888341699,.5113917939934632	.1735926541370104,.1977062976428317
1.0	.4359025487166450,.4359025487166450	.1882552928913357,.1882552928913357

Table 2: *RK* solutions of system (4.1)

Case 2 : Differently from Case 1, if we use these non-local terms and solve equation (4.1),

$$x(t;\alpha) \to [x_{k+1}]_\alpha$$
$$y(t;\alpha) \to [y_{k+1}]_\alpha$$
$$(xy)(t;\alpha) \to [x_{k+1}y_k]_\alpha$$
$$x^2(t;\alpha) \to [x_{k+1}x_k]_\alpha$$
$$(x^2y)(t;\alpha) \to [x_{k+1}x_k y_{k+1}]_\alpha$$

we obtain denominator functions;

$$h \to \frac{e^{lh}-1}{l} = \phi_2(h,\lambda;\alpha)$$

where we obtain solutions which are:

$$[x_{k+1}]_\alpha = \frac{[x_k]_\alpha}{1 + l\phi_2(h,\lambda;\alpha) + k\phi_2(h,\lambda;\alpha)[y_k]_\alpha} \tag{4.5}$$

$$[y_{k+1}]_\alpha = \frac{-\phi_1(h,\lambda;\alpha)k[x_{k+1}y_k]_\alpha + [y_k]_\alpha}{1 - \phi_1(h,\lambda;\alpha)l + \phi_1(h,\lambda;\alpha)[x_{k+1}x_k]_\alpha} \tag{4.6}$$

For Case 2, the nonstandard finite difference schemes solution is given by Table 3 (for $h = 0.1$).

α	$[x_1, x_2]$	$[y_1, y_2]$
0.0	.2244393158463085,.6569254938899954	.1031864918521038,.2284538214265367
0.2	.2644514586359461,.6173650478446476	.1226641534499551,.2255197656126929
0.4	.3026932468309790,.5729268990826093	.1413627023789611,.2205176325002552
0.6	.3391314883571007,.5232258689257032	.1591566020343204,.2132772481277070
0.8	.3737475157108241,.4678828617416123	.1759435222624581,.2036730727685865
1.	.4065365732029463,.4065365732029463	.1916440065194761,.1916440065194761

Table 3: *NFDS* solutions of system (4.1) for Case 2

Case 3 : Differently from Case 1 and Case 2, we use non-local terms:

$$x(t;\alpha) \to [x_{k+1}]_\alpha$$
$$y(t;\alpha) \to [y_k]_\alpha$$
$$(xy)(t;\alpha) \to [x_k y_k]_\alpha$$
$$x^2(t;\alpha) \to [x_k x_k]_\alpha$$
$$(x^2y)(t;\alpha) \to [x_k x_k y_k]_\alpha$$

where, we obtain solutions which are

$$[x_{k+1}]_\alpha = \frac{[x_k]_\alpha - \phi_2(h,\lambda;\alpha)k[x_k y_k]_\alpha}{1 + l\phi_2(h,\lambda;\alpha)} \tag{4.7}$$

$$[y_{k+1}]_\alpha = [y_k]_\alpha(1 + \phi_2(h,\lambda;\alpha)l - \phi_2(h,\lambda;\alpha)k[x_k]_\alpha - \phi_2(h,\lambda;\alpha)[x_k x_k]_\alpha) \tag{4.8}$$

For Case 3, the Nonstandard Finite Difference Schemes solutions are given by Table 4 (for $h = 0.1$). Table 5 and Table 6, shows absolute of error for $NFDS$ and RK solutions. Figure 4.1 and Figure 4.2 are shown graphics for solutions (for $h = 0.1$)

α	$[x_1, x_2]$	$[y_1, y_2]$
0.0	.2242950684129496,.6489023869377528	.1029836421924408,.2030236835539285
0.2	.2641841716892893,.6108002084916311	.1222641336629243,.2071325318795353
0.4	.3022404186434742,.5678566555488764	.1406445396599407,.2078133420041893
0.6	.3384154454005998,.5195484899654308	.1579576426725296,.2049707387763659
0.8	.3726760677253374,.4654021103117395	.1740551115790624,.1986019750715187
1.0	.4050046958176773,.4050046958176773	.1888087250626280,.1888087250626280

Table 4: $NFDS$ solutions of system (4.1) for Case 3

α	$Case$ 1	$Case$ 2	$Case$ 3
0.0	0.1309551469	0.1317098713	0.1398772257
0.2	0.1118027850	0.1122127284	0.1190448546
0.4	0.0949890628	0.0951923272	0.1007153990
0.6	0.0804995493	0.0805897880	0.0849832099
0.8	0.0683745828	0.0684114054	0.0719636048
1.0	0.0587127260	0.0587319510	0.0617957058

Table 5: For x absolute error $|NFDS-RK|$

α	$Case$ 1	$Case$ 2	$Case$ 3
0.0	0.0230659238	0.0280628573	0.0024298698
0.2	0.0178437872	0.0210453540	0.0022581006
0.4	0.0134921656	0.0154007445	0.0019782913
0.6	0.0101146927	0.0111658615	0.0016603929
0.8	0.0077592330	0.0083176434	0.0013581350
1.0	0.0064177760	0.0067774272	0.0011068644

Table 5: For y absolute error $|NFDS-RK|$

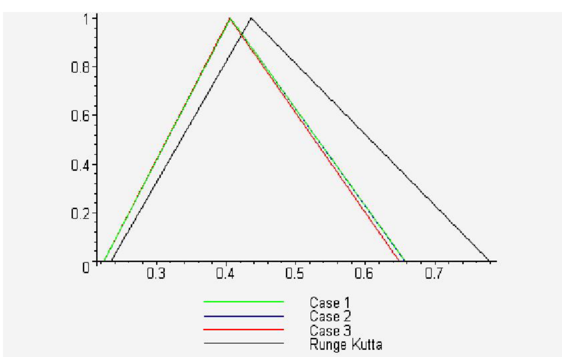

Figure 4.1: The results of x, for h=0.1 and t=0.3.

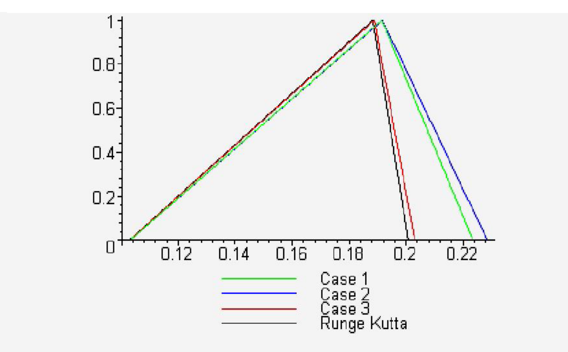

Figure 4.2: The results of y, for h=0.1 and t=0.3.

5. CONCLUSION

In this paper, a new method has been presented for solving fuzzy differential equations. NFDS used different non-local terms, which provides high accuracy compared to other methods.Two numerical methods based on fuzzy differential equations were compared: the Nonstandart Finite Difference Schemes and the

Runge Kutta method. We showed that our proposed Nonstandart Finite Difference Schemes, for different non-local terms, is more accurate and gives a better approximation than the method presented in.

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DYNAMICAL ANALYSIS OF A RATIO DEPENDENT HOLLING–TANNER TYPE PREDATOR–PREY MODEL WITH DELAY

CANAN ÇELIK

Abstract. In this paper, a ratio dependent delayed predator-prey model with Holling-Tanner type functional response is studied. The local stability of a positive equilibrium and the existence of Hopf bifurcations are established. By using the normal form theory and center manifold theorem, the explicit algorithm determining the stability, direction of the bifurcating periodic solutions are derived. Finally, numerical simulations for justifying the theoretical analysis are also presented.

1. Introduction

In recent years, the dynamics properties of the predator-prey models which have significant biological background have received much attention from many applied mathematicians and ecologists. In order to incorporate various realistic physical effects that may cause at least one of the physical variables to depend on the past history of the system, it is often necessary to introduce time-delays into these models. Many theoreticians and experimentalists concentrated on the stability of predator-prey systems and, more specifically they investigated the stability of such systems when time delays are incorporated into the models. Time delay may have very complicated impact on the dynamical behavior of the system such as the periodic structure, bifurcation, etc. For references see [1]-[8] and [10]-[38].

There have been many works which are devoted to the studies of dynamical behaviors for predator-prey systems with various functional responses. But, recently, many researchers found that when predators have to search for food and, therefore, have to share or compete for food, a more suitable general predator prey theory should be based on the so-called ratio-dependent theory, which can be roughly stated as that the per capita predator growth rate should be the so-called ratio dependent functional response. So our aim in this paper is to investigate the following delayed predator-prey system with Holling-Tanner type functional response

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)(1-N(t)) - \frac{N(t)P(t-\tau)}{N(t)+\alpha P(t-\tau)} \\ \frac{dP(t)}{dt} &= \beta P(t-\tau)(\delta - \frac{P(t-\tau)}{N(t)}) \end{aligned} \tag{1.1}$$

where α, β and δ are positive constants, and $N(t)$ and $P(t)$ can be interpreted as the densities of prey and predator populations at time t, respectively and $\tau \geq 0$

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denotes the time delay for the predator density. In this model, prey density is logistic with time delay and the carrying capacity proportional to predator density. In many of the studies related to stability of predator prey models, authors consider constant carrying capacity, however in this study, we focus on the carrying capacity proportional to prey density (ratio-dependent) which shows really interesting behavior in terms of dynamical structure.

The organization of this paper is as follows: In Section 2, we study the local stability of the equilibrium point of the corresponding characteristic equation. In Section 3, we illustrate the existence of Hopf bifurcation. The direction and stability of Hopf bifurcation are investigated in Section 4. Finally in Section 5, numerical simulations are performed to support our theoretical results.

2. Equilibrium and Local Stability Analysis

System (1.1) has a unique positive equilibrium point $E_0^* = (N_0^*, P_0^*)$ where $N_0^* = \frac{1+\alpha\delta-\delta}{1+\alpha\delta}, P_0^* = \delta(\frac{1+\alpha\delta-\delta}{1+\alpha\delta})$. To analyze the local stability of the positive equilibrium $E_0^* = (N_0^*, P_0^*)$,we first use the linear transformation $n(t) = N(t) - N_0^*$, and $p(t) = P(t) - P_0^*$ where $n \ll 1$ and $p \ll 1$ for which the system (1) turns out to be

$$\begin{aligned} \frac{dn}{dt} &= (n(t)+N_0^*)(1-n(t)-N_0^*) - \frac{(n(t)+N_0^*)(p(t-\tau)+P_0^*)}{n(t)+N_0^*+\alpha(p(t-\tau)+P_0^*)} \\ \frac{dP}{dt} &= \beta(p(t-\tau)+P_0^*)(\delta - \frac{p(t-\tau)+P_0^*}{n(t)+N_0^*}) \end{aligned} \tag{2.1}$$

and using relations $N_0^*(1-N_0^*) - \frac{N_0^* P_0^*}{N_0^*+\alpha P_0^*} = 0$ and $\beta P_0^*(\delta - \frac{P_0^*}{N_0^*}) = 0$, ignoring the higher order terms yield the following linear system

$$\begin{aligned} \frac{dn}{dt} &= (1 - 2N_0^* - \frac{P_0^*}{N_0^*+\alpha P_0^*} + \frac{P_0^* N_0^*}{(N_0^*+\alpha P_0^*)^2})n(t) \\ &\quad +(-\frac{N_0^*}{N_0^*+\alpha P_0^*} + \frac{\alpha P_0^* N_0^*}{(N_0^*+\alpha P_0^*)^2})p(t-\tau) \\ \frac{dp}{dt} &= (\beta\delta - 2\frac{\beta P_0^*}{N_0^*})p(t-\tau) + \frac{\beta(P_0^*)^2}{(N_0^*)^2}n(t) \end{aligned} \tag{2.2}$$

whose associated characteristic equation is given by the transcendental equation

$$\lambda^2 - A_1\lambda - A_4\lambda e^{-\lambda\tau} + (A_1A_4 - A_2A_3)e^{-\lambda\tau} = 0 \tag{2.3}$$

where $A_1 = 1 - 2N_0^* - \frac{P_0^*}{N_0^*+\alpha P_0^*} + \frac{P_0^* N_0^*}{(N_0^*+\alpha P_0^*)^2}$, $A_2 = -\frac{N_0^*}{N_0^*+\alpha P_0^*} + \frac{\alpha P_0^* N_0^*}{(N_0^*+\alpha P_0^*)^2}$, $A_3 = \frac{\beta(P_0^*)^2}{(N_0^*)^2}$ and $A_4 = \beta\delta - 2\frac{\beta P_0^*}{N_0^*}$. and

$$\lambda^2 - A_1\lambda - A_4\lambda e^{-\lambda\tau} + A_5 e^{-\lambda\tau} = 0 \tag{2.4}$$

where

$$A_5 = A_1A_4 - A_2A_3.$$

When there is no delay, i.e., $\tau = 0$, the corresponding characteristic equation (2.4) reduces to

$$\lambda^2 - (A_1 + A_4)\lambda + A_5 = 0 \tag{2.5}$$

Lemma 2.1. *Suppose the following conditions hold;*

$$i)\alpha\delta + 1 > \delta$$

$$ii)\delta(2 + \alpha\delta) < (1 + \delta\beta)(1 + \alpha\delta)$$

then the positive equilibrium E_0^* of the system (1.1) is locally asymptotically stable in the absence of τ.

Proof. In the absence of τ, the corresponding characteristic equation takes the form,

$$\lambda^2 - (trA)\lambda + \det A = 0$$

where $trA = (A_1 + A_4)$, i.e.,

$$trA = \frac{1}{(1 + \alpha\delta)^2}[\delta(2 + \alpha\delta) - (1 + \alpha\delta)^2(1 + \beta\delta)]$$

and

$$detA = (1 + \alpha\delta)^2(1 + \beta\delta) - \delta(2 + \alpha\delta).$$

Then it can be seen that under the conditions i) and ii), we obtain $trA < 0$ and $detA > 0$. Hence the equilibrium point E_0^* of the system (1.1), with $\tau = 0$, is locally asymptotically stable. ▮

Now we shall consider the distribution of the roots of the transcendental equation (2.4) since the stability of the point $(0, 0)$ of linear system (2.2) depends on the roots of the characteristic equation (2.4). By the continuous dependence of roots of $\lambda^2 - A_1\lambda - A_4\lambda e^{-\lambda\tau} + A_5e^{-\lambda\tau} = 0$ and the stability result for $\tau = 0$, $\exists\tau_0 > 0$ such that $Re\lambda(\tau) < 0$ for $\tau\epsilon\ [0, \tau_0)$. Since a loss of asymptotic stability of (N_0^*, P_0^*) will arise when $Re\lambda(\tau) = 0$, we shall examine whether there exists a $\tau^* > 0$ for which $Re\lambda(\tau^*) = 0$. i.e., we would like to know when equation (2.4) has purely imaginary roots. In this section we first obtain the local stability conditions of the equilibrium point.

Now suppose for $\tau = \tau^*$ and let $\lambda = iw$ be a root of (2.4) with w real and without loss of generality $w > 0$. Then w satisfies

$$(iw)^2 - A_1iw - A_4iwe^{-iw\tau} + A_5e^{-iw\tau} = 0$$

Separating real and imaginary parts, we obtain

$$\begin{aligned} A_5\cos(w\tau) - A_4w\sin(w\tau) &= w^2 \\ A_5\sin(w\tau) + A_4w\cos(w\tau) &= -A_1w \end{aligned} \tag{2.6}$$

that is equivalent to

$$w^4 + (A_1^2 - A_4^2)w^2 - A_5^2 = 0$$

Let $w^2 = z$, $p = A_1^2 - A_4^2$ and $q = -A_5^2$. Since $\lim_{z\to\infty} g(z) = \infty$ and $q < 0$, we conclude the following result

$$g(z) = z^2 + pz + q = 0 \tag{2.7}$$

Lemma 2.2. *Since $q < 0$, the polynomial equation (2.7) has at least one positive root.*

3. Existence of Hopf Bifurcation

By Lemma 2.2 and without loss of generality, we denote the positive root by z and $w = \sqrt{z}$. Solving the equations (2.6) for τ, we obtain

$$\cos(w\tau) = \frac{A_5w^2 - A_1A_4w^2}{A_5^2 + A_4^2w^2},$$

$$\sin(w\tau) = \frac{-A_4w^3 - A_1A_5w}{A_5^2 + A_4^2w^2},$$

and

$$\tan(w\tau) = (\frac{A_4w^2 + A_1A_5}{A_1A_4w - A_5w})$$

which leads to

$$\tau_k = \frac{1}{w}\{\arctan(\frac{A_4w^2 + A_1A_5}{A_1A_4w - A_5w}) + 2k\pi\} \tag{3.1}$$

for k=0,1,2,3,...

Let $\lambda(\tau) = \alpha(\tau)+iw(\tau)$ denote the root of (2.4) near $\tau = \tau_k$ satisfying $\alpha(\tau_k) = 0$ and $w(\tau_k) = w_1$, $k = 0, 1, 2....$ Then we have the following result.

Lemma 3.1. *Suppose $g^{'}(z_1) \neq 0$, then the following transversality condition is satisfied;*

$$\frac{d(Re\lambda(\tau_k))}{d\lambda} > 0, \qquad k = 0, 1, 2, 3, ...$$

and $g^{'}(z_1)$ and $\dfrac{dRe\lambda(\tau_k)}{d\tau}$ have the same sign.

Proof. Suppose that for $\tau = \tau_k$, let $\lambda = iw$ be a root of (2.4) with w real and without loss of generality $w > 0$. Differentiating the characteristic equation (2.4) with respect to τ, we get

$$2\lambda\frac{d\lambda}{d\tau} - A_1\frac{d\lambda}{d\tau} - [e^{-\lambda\tau}(-\frac{d\lambda}{d\tau}\tau - \lambda)](A_4\lambda - A_5) - e^{-\lambda\tau}A_4\frac{d\lambda}{d\tau} = 0,$$

that is

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{A_1 - 2\lambda}{\lambda(A_4\lambda - A_5)e^{-\lambda\tau}} - \frac{\tau}{\lambda} + \frac{A_4}{\lambda(A_4\lambda - A_5)}.$$

Then for $\lambda = iw$,

$$\begin{aligned}
\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=iw} &= \mathrm{Re}[\frac{A_1 - 2iw}{iw(A_4iw - A_5)e^{-i\lambda w}} - \frac{\tau}{iw} + \frac{A_4}{iw(A_4iw - A_5)}] \\
&= \mathrm{Re}[\frac{(A_1 - 2iw)(\cos(w\tau) + \sin(w\tau)) + A_4}{iw(A_4iw - A_5)}] \\
&= \mathrm{Re}[\frac{(2A_5w^2 - A_1A_4w^2)\cos(w\tau) - (A_1A_5w + 2A_4w^3)\sin(w\tau) - A_4^2w^2}{A_4^2w^4 + A_5^2w^2}]
\end{aligned}$$

and using the expressions for $\cos(w\tau)$ and $\sin(w\tau)$ above, we get

$$\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}|_{\lambda=iw} = A_4^2w^2(w^4+(A_1^2-A_4^2)w^2-A_5^2)+A_4^2w^6+(2A_5^2+A_1^2A_4^2)w^4+A_1^2A_5^2w^2$$

$$\text{Re}(\frac{d\lambda}{d\tau})^{-1} = A_4^2 w^6 + (2A_5^2 + A_1^2 A_4^2) w^4 + A_1^2 A_5^2 w^2,$$

$$\text{Re}(\frac{d\lambda}{d\tau})^{-1} \mid_{\lambda = iw} > 0.$$

Thus, lemma follows. ∎

Summarizing the above results, we have the following theorem on stability and Hopf bifurcation of the system (2.2).

Theorem 3.2. *For the system (2,2), the following results hold,*

i) If $\tau \epsilon [0, \tau_0)$, *then the equilibrium point* $(0,0)$ *of the system (2.2) is asymptotically stable,*

ii) If $g'(z_1) \neq 0$, *then the system (2.2) undergoes Hopf bifurcation at the equilibrium point* $(0,0)$ *when* $\tau = \tau_k, (k = 0, 1, 2...)$.

4. Direction and the stability of Hopf Bifurcation

In this section we shall determine the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions by applying the normal form theory and the center manifold theorem by Hassard et al. [9].

Throughout this section, we assume that the system (1.1) undergoes Hopf bifurcations at the positive equilibrium (N_0^*, P_0^*) at $\tau = \tau_k$ and iw_1 is the corresponding purely imaginary root of the characteristic equation at the positive equilibrium (N_0^*, P_0^*). For the sake of simplicity, we use the notation iw for iw_1.

We first consider the system (1.1) by the transformation

$$x_1 = N - N_0^*,\ x_2 = P - P_0^*,\ t = \frac{t}{\tau},\ \tau = \tau_k + \mu$$

which is equivalent to the following Functional Differential Equation(FDE) system in $C = C([-1, 0], R^2)$

$$\dot{x}(t) = L_\mu(x_t) + f(\mu, x_t) \tag{4.1}$$

where $x(t) = (x_1(t), x_2(t))^T \epsilon R^2$, and $L_\mu : C \to R^2$, $f : R \times C \to R^2$ are given respectively, by

$$\begin{aligned} L_\mu(x_t) &= (\tau_k + \mu) \begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(0) \\ \phi_2(0) \end{bmatrix} \\ &\quad + (\tau_k + \mu) \begin{bmatrix} 0 & A_2 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \end{bmatrix} \end{aligned}$$

and

$$f(\mu, \phi) = (\tau_k + \mu) \begin{bmatrix} f_{11} \\ f_{12} \end{bmatrix}$$

where

$$\begin{aligned} f_{11} &= -\phi_1^2(0) - \frac{\phi_2(-1)\phi_1(0)}{N^*+\alpha P^*} + \frac{P^*\phi_1^2(0) + N^*\phi_2(-1)\phi_1(0)}{(N^*+\alpha P^*)^2} \\ &\quad + \frac{\alpha P^*\phi_2(-1)\phi_1(0) + \alpha N^*\phi_2^2(-1)}{(N^*+\alpha P^*)^2} \\ &\quad - \frac{P^*N^*\phi_1^2(0) + 2\alpha P^*N^*\phi_2(-1)\phi_1(0) + \alpha^2 P^*N^*\phi_2^2(-1)}{(N^*+\alpha P^*)^3} \end{aligned}$$

and

$$f_{12} = -\frac{\beta\phi_2^2(-1)}{N^*} + \frac{2\beta P^*\phi_2(-1)\phi_1(0)}{(N^*)^2} - \frac{\beta(P^*)^2\phi_1^2(0)}{(N^*)^2}$$

where $\phi = (\phi_1, \phi_2)\ \epsilon C$.

By Riesz representation theorem, there exists a function $\eta(\theta,\mu)$ of bounded variation for $\theta\epsilon[-1,0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,0)\phi(\theta) \ \text{ for } \ \phi\epsilon C.$$

Indeed we may take

$$\begin{aligned} \eta(\theta,\mu) &= (\tau_k+\mu)\begin{bmatrix} A_1 & 0 \\ A_3 & 0 \end{bmatrix}\delta(\theta) \\ &\quad +(\tau_k+\mu)\begin{bmatrix} 0 & A_2 \\ 0 & A_4 \end{bmatrix}\delta(\theta+1) \end{aligned}$$

where δ is the Dirac delta function. For $\phi\epsilon C^1([-1,0],\mathbb{R}^2)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta\epsilon[-1,0) \\ \int_{-1}^{0} d\eta(\mu,s)\phi(s), & \theta = 0. \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, \theta\epsilon[-1,0) \\ f(\mu,\phi), \theta = 0. \end{cases}$$

Then the system (4.1) is equivalent to

$$x'(t) = A(\mu)x_t + R(\mu)x_t$$

where $x_t(\theta) = x(t+\theta)$ for $\theta\epsilon\,[-1,0)$ .For $\psi\epsilon C^1([-1,0],(\mathbb{R}^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s\epsilon(0,1] \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases}$$

and a bilinear inner product

$$\langle\psi(s),\phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{4.2}$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. Suppose that $q(\theta)$ and $q^*(s)$ are eigenvectors of A and A^* corresponding to $iw\tau_k$ and $-iw\tau_k$, respectively. Then suppose that $q(\theta) = (1,\alpha)^T e^{i\omega\tau_k\theta}$ is the eigenvector of $A(0)$

corresponding to $iw\tau_k$, then $A(0)q(\theta) = iw\tau_k q(\theta)$. It follows from the definition of $A(0)$, $L_\mu\phi$ and $\eta(\theta,\mu)$ that

$$\tau_k \begin{bmatrix} A_1 + iw & A_3 \\ A_2 e^{iw\tau_k} & A_4 e^{iw\tau_k} + iw \end{bmatrix} q(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then we can easily get

$$q(\theta) = (1,\alpha)^T e^{iw\tau_k\theta}. = q(0)e^{iw\tau_k\theta}$$

and similarly by definition of A^*,

$$\tau_k \begin{bmatrix} A_1 - iw & A_2 e^{-iw\tau_k} \\ A_3 & A_4 e^{-iw\tau_k} - iw \end{bmatrix} q^*(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and

$$q^*(\theta) = D(\alpha^*,1)^T e^{iw\tau_k\theta}. = q^*(0)e^{iw\tau_k\theta}.$$

To satisfy that $\langle q^*(s), q(\theta)\rangle = 1$, we evaluate the value of D. By the definition of the bilinear inner product

$$\begin{aligned}
\langle q^*(\theta), q(\theta)\rangle &= \overline{D}(\overline{\alpha}^*,1)(1,\alpha)^T - \int_{-1}^{0}\int_{\xi=0}^{\theta} \overline{D}(\overline{\alpha}^*,1)e^{iw\tau_k(\xi-\theta)}d\eta(\theta)(1,\alpha)^T e^{iw\tau_k\xi}d\xi \\
&= \overline{D}\left\{\alpha + \overline{\alpha}^* - \int_{-1}^{0}(\overline{\alpha}^*,1)e^{iw\tau_k\theta}\theta d\eta(\theta)(1,\alpha)^T\right\} \\
&= \overline{D}\left\{\alpha + \overline{\alpha}^* + \tau_k e^{-iw\tau_k}(A_4\overline{\alpha}^* + A_3)\right\}
\end{aligned}$$

Thus we can choose $\overline{D}$ as

$$\overline{D} = \frac{1}{\alpha + \overline{\alpha}^* + \tau_k e^{-iw\tau_k}(A_4\overline{\alpha}^* + A_3)}$$

such that $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \overline{q}(\theta)\rangle = 0$

In the following part, we use the theory by Hassard et al. [9] to compute the coordinates describing center manifold C_0 at $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t\rangle \ , \ W(t,\theta) = x_t - 2\,\mathrm{Re}\, z(t)q(\theta) \tag{4.3}$$

On the center manifold C_0 , we have

$$W(t,\theta) = W(z(t),\overline{z}(t),\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + ...$$

where z and $\overline{z}$ are local coordinates for centermanifold C_0 in the direction of q and $\overline{q}^*$. Note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \epsilon C_0$, since $\mu = 0$ and (4.1), we have

$$\begin{aligned} z' &= iw\tau_k z + \langle q^*(\theta), f(0, W(z,\overline{z},\theta) + 2\operatorname{Re} zq(\theta))\rangle \\ &= iw\tau_k z + q^*(0) f(0, W(z,\overline{z},0) + 2\operatorname{Re} zq(0)) \\ &\overset{def}{=} iw\tau_k z + q^*(0) f_0(z,\overline{z}) \\ &= iw\tau_k z + g(z,\overline{z}) \end{aligned}$$

where

$$g(z,\overline{z}) = q^*(\theta) f_0(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + ... \tag{4.4}$$

By using (4.3), we have $x_t(x_{1t}(\theta), x_{2t}(\theta)) = W(t,\theta) + zq(\theta) + \overline{zq}(\theta)$ and $q(\theta) = (1,\alpha)^T e^{iw\tau_k\theta}$, and then

$$\begin{aligned} x_{1t}(0) &= z + \overline{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3), \\ x_{2t}(0) &= z\alpha + \overline{z\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3), \\ x_{1t}(-1) &= ze^{-iw\tau_k\theta} + \overline{z}e^{iw\tau_k\theta} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3), \\ x_{2t}(-1) &= z\alpha e^{-iw\tau_k\theta} + \overline{z\alpha}e^{iw\tau_k\theta} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{2}(-1) z\overline{z} + W_{02}^{(2)}(-1)\frac{\overline{z}^2}{2} + O(|z,\overline{z}|^3), \end{aligned}$$

From the definition of $f(\mu, x_t)$,we have

$$g(z,\overline{z}) = \overline{q}^*(0) f_0(z,\overline{z}) = \overline{D}\tau_k(\overline{\alpha}^*, 1)\begin{bmatrix} f_{11}^0 \\ f_{12}^0 \end{bmatrix}$$

where

$$\begin{aligned} f_{11}^0 &= -x_{1t}^2(0) - \frac{x_{2t}(-1)x_{1t}(0)}{N^* + \alpha P^*} + \frac{P^* x_{1t}^2(0) + N^* x_{2t}(-1)x_{1t}(0)}{(N^* + \alpha P^*)^2} \\ &\quad + \frac{\alpha P^* x_{2t}(-1)x_{1t}(0) + \alpha N^* x_{2t}^2(-1)}{(N^* + \alpha P^*)^2} \\ &\quad - \frac{P^* N^* x_{1t}^2(0) + 2\alpha P^* N^* x_{2t}(-1)x_{1t}(0) + \alpha^2 P^* N^* x_{2t}^2(-1)}{(N^* + \alpha P^*)^3}, \end{aligned}$$

and

$$f_{12}^0 = -\frac{\beta x_{2t}^2(-1)}{N^*} + \frac{2\beta P^* x_{2t}(-1)x_{1t}(0)}{(N^*)^2} - \frac{\beta (P^*)^2 x_{1t}^2(0)}{(N^*)^2}.$$

Thus

$$\begin{aligned}
g(z,\overline{z}) &= \overline{D}\tau_k[\overline{\alpha}^*(-x_{1t}^2(0) - \frac{x_{2t}(-1)x_{1t}(0)}{(N^*+\alpha P^*)} \\
&+ \frac{P^*x_{1t}^2(0) + N^*x_{2t}(-1)x_{1t}(0)}{(N^*+\alpha P^*)^2} \\
&+ \frac{\alpha P^*x_{2t}(-1)x_{1t}(0) + \alpha N^*x_{2t}^2(-1)}{(N^*+\alpha P^*)^2} \\
&- \frac{P^*N^*x_{1t}^2(0) + 2\alpha P^*N^*x_{2t}(-1)x_{1t}(0)}{(N^*+\alpha P^*)^3} \\
&- \frac{\alpha^2P^*N^*x_{2t}^2(-1)}{(N^*+\alpha P^*)^3} - \frac{\beta x_{2t}^2(-1)}{N^*} + \frac{2\beta P^*N^*x_{2t}(-1)x_{1t}(0)}{(N^*)^2} \\
&- \frac{(P^*)^2x_{1t}^2(0)}{(N^*)^2}] + O(|(z,\overline{z})|^3)
\end{aligned}$$

By comparing the coefficients with (4.4), we get

$$\begin{aligned}
g_{20} &= 2\overline{D}\tau_k[-\overline{\alpha}^*e^{-2iw\tau_k\theta} - \frac{\overline{\alpha}^*\alpha e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\
&+ \frac{\overline{\alpha}^*P^*e^{-2iw\tau_k\theta} + \overline{\alpha}^*\alpha N^*e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\
&+ \frac{\overline{\alpha}^*\alpha^2P^*e^{-iw\tau_k\theta} + \overline{\alpha}^*\alpha^2N^*}{(N^*+\alpha P^*)^2} - \frac{\overline{\alpha}^*N^*P^*e^{-2iw\tau_k\theta}}{(N^*+\alpha P^*)^3} \\
&- \frac{2\overline{\alpha}^*\alpha^2N^*P^*e^{-iw\tau_k\theta}\overline{\alpha}^*\alpha^2N^*P^*}{(N^*+\alpha P^*)^3} \\
&- \frac{\beta\alpha^2}{N^*} + \frac{2\beta\alpha P^*e^{-iw\tau_k\theta} - 2\beta(P^*)^2e^{-2iw\tau_k\theta}}{(N^*)^2}]
\end{aligned}$$

$$\begin{aligned}
g_{11} &= \overline{D}\tau_k[-\overline{\alpha}^*2\alpha\overline{\alpha} - \frac{\overline{\alpha}^*\alpha e^{iw\tau_k\theta} + \overline{\alpha}^*\overline{\alpha}e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\
&+ \frac{2\overline{\alpha}^*P^* + \overline{\alpha}^*\alpha N^*e^{iw\tau_k\theta} + \overline{\alpha}^*\overline{\alpha}N^*e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\
&+ \frac{\overline{\alpha}^*\alpha^2P^*e^{iw\tau_k\theta} + \overline{\alpha}^*\alpha\overline{\alpha}P^*e^{-iw\tau_k\theta} + 2\overline{\alpha}^*\alpha^2\overline{\alpha}N^*}{(N^*+\alpha P^*)^2} \\
&- \frac{2\overline{\alpha}^*N^*P^* + 2\overline{\alpha}^*\alpha^2N^*P^*e^{iw\tau_k\theta}}{(N^*+\alpha P^*)^3} \\
&- \frac{2\overline{\alpha}^*\alpha\overline{\alpha}N^*P^*e^{-iw\tau_k\theta} + 2\overline{\alpha}^*\alpha^2\overline{\alpha}N^*P^*}{(N^*+\alpha P^*)^3} \\
&- \frac{2\beta\alpha\overline{\alpha}}{N^*} + \frac{2\beta\alpha P^*e^{iw\tau_k\theta} + 2\beta\overline{\alpha}P^*e^{-iw\tau_k\theta} - 2\beta(P^*)^2}{(N^*)^2}]
\end{aligned}$$

$$
\begin{aligned}
g_{02} \;=\;& 2\overline{D}\tau_k[-\overline{\alpha}^* e^{2iw\tau_k\theta} - \frac{\overline{\alpha}^*\overline{\alpha}e^{iw\tau_k\theta}}{(N^*+\alpha P^*)} \\
&+\frac{\overline{\alpha}^* P^* e^{2iw\tau_k\theta} + \overline{\alpha}^*\overline{\alpha}N^* e^{iw\tau_k\theta} + \overline{\alpha}^*\alpha\overline{\alpha}P^* e^{iw\tau_k\theta} + \overline{\alpha}^*\alpha\overline{\alpha}^2 N^*}{(N^*+\alpha P^*)^2} \\
&-\frac{\overline{\alpha}^* N^* P^* e^{2iw\tau_k\theta} + 2\overline{\alpha}^*\alpha\overline{\alpha}N^*P^* e^{iw\tau_k\theta} + \overline{\alpha}^*\alpha^2\overline{\alpha}^2 N^* P^*}{(N^*+\alpha P^*)^3} \\
&-\frac{\beta\overline{\alpha}^2}{N^*} + \frac{2\beta\overline{\alpha}P^* e^{iw\tau_k\theta} - \beta(P^*)^2 e^{2iw\tau_k\theta}}{(N^*)^2}]
\end{aligned}
$$

$$
\begin{aligned}
g_{21} \;=\;& 2\overline{D}\tau_k[-\overline{\alpha}^*(W_{20}^{(1)}(-1)e^{iw\tau_k\theta} + 2W_{11}^{(1)}(-1)e^{-iw\tau_k\theta}) \\
&-\frac{\overline{\alpha}^*(W_{11}^{(2)}(0)e^{-iw\tau_k\theta} + W_{11}^{(1)}(-1)\alpha + \frac{W_{20}^{(2)}(0)e^{iw\tau_k\theta}}{2} + \frac{W_{20}^{(1)}(-1)\overline{\alpha}}{2})}{(N^*+\alpha P^*)} \\
&+\frac{\overline{\alpha}^* P^*(W_{20}^{(1)}(-1)e^{iw\tau_k\theta} + 2W_{11}^{(1)}(-1)e^{-iw\tau_k\theta})+}{(N^*+\alpha P^*)^2} \\
&+\frac{\overline{\alpha}^* N^*(W_{11}^{(2)}(0)e^{-iw\tau_k\theta} + W_{11}^{(1)}(-1)\alpha + \frac{W_{20}^{(2)}(0)e^{iw\tau_k\theta}}{2} + \frac{W_{20}^{(1)}(-1)\overline{\alpha}}{2})}{(N^*+\alpha P^*)^2} \\
&+\frac{\overline{\alpha}^*\alpha P^*(W_{11}^{(2)}(0)e^{-iw\tau_k\theta} + W_{11}^{(1)}(-1)\alpha + \frac{W_{20}^{(2)}(0)e^{iw\tau_k\theta}}{2} + \frac{W_{20}^{(1)}(-1)\overline{\alpha}}{2})}{(N^*+\alpha P^*)^2} \\
&+\frac{\overline{\alpha}^*\alpha N^*(W_{20}^{(2)}(0)\overline{\alpha} + 2W_{11}^{(2)}(0)\alpha)}{(N^*+\alpha P^*)^2} \\
&-\frac{\overline{\alpha}^* N^* P^*(W_{20}^{(1)}(-1)e^{iw\tau_k\theta} + 2W_{11}^{(1)}(-1)e^{-iw\tau_k\theta})}{(N^*+\alpha P^*)^3} \\
&-\frac{2\overline{\alpha}^*\alpha N^* P^*(W_{11}^{(2)}(0)e^{-iw\tau_k\theta} + W_{11}^{(1)}(-1)\alpha + \frac{W_{20}^{(2)}(0)e^{iw\tau_k\theta}}{2} + \frac{W_{20}^{(1)}(-1)\overline{\alpha}}{2})}{(N^*+\alpha P^*)^3} \\
&-\frac{\overline{\alpha}^*\alpha^2 N^* P^*(W_{20}^{(2)}(0)\overline{\alpha} + 2W_{11}^{(2)}(0)\alpha)}{(N^*+\alpha P^*)^3} - \frac{\beta(W_{20}^{(2)}(0)\overline{\alpha} + 2W_{11}^{(2)}(0)\alpha)}{N^*} \\
&+\frac{2\beta\alpha P^*(W_{11}^{(2)}(0)e^{-iw\tau_k\theta} + W_{11}^{(1)}(-1)\alpha + \frac{W_{20}^{(2)}(0)e^{iw\tau_k\theta}}{2} + \frac{W_{20}^{(1)}(-1)\overline{\alpha}}{2})}{(N^*)^2} \\
&-\frac{\beta(P^*)^2(W_{20}^{(1)}(-1)e^{iw\tau_k\theta} + 2W_{11}^{(1)}(-1)e^{-iw\tau_k\theta})}{(N^*)^2}
\end{aligned}
$$

To determine g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. By (4.1) and (4.4), we have

$$
\begin{aligned}
W' &= x_t' - z'q - \overline{z'q} \\
&= \begin{cases} AW - 2\,\mathrm{Re}(\overline{q}^*(0) f_0 q(\theta)), & \theta\epsilon[-1,0) \\ AW - 2\,\mathrm{Re}(\overline{q}^*(0) f_0 q(\theta)) + f_0, & \theta = 0 \end{cases} \\
&\stackrel{def}{=} AW + H(z, \overline{z}, \theta).
\end{aligned} \tag{4.5}
$$

where

$$H(z, \overline{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + ... \tag{4.6}$$

Note that on the center manifold C_0 near to the origin,

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \bar{z}. \tag{4.7}$$

Thus we obtain,

$$(A - 2iw\tau_k)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11}(\theta) = -H_{11}(\theta). \tag{4.8}$$

By using (4.3), for $\theta\epsilon[-1,0)$,

$$H(z.\overline{z}, \theta) = \overline{q}^*(0) f_0 q(\theta) - q^*(0) f_0(0)\overline{q}(\theta) = -gq(\theta) - \overline{g}\overline{q}(\theta). \tag{4.9}$$

Comparing the coefficients with (4.6), we obtain the following

$$H_{20}(\theta) = -g_{20} q(\theta) - \overline{g}_{02}\overline{q}(\theta), \quad H_{11}(\theta) = -g_{11} q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{4.10}$$

From (4.8) and (4.10) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2iw\tau_k W_{20}(\theta) - g_{20} q(\theta) - \overline{g}_{02}\overline{q}(\theta).$$

Noticing $q(\theta) = q(0)e^{iw\tau_k\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\tau_k w} q(0) e^{iw\tau_k\theta} + \frac{i\overline{g}_{02}}{3\tau_k w}\overline{q}(0) e^{-iw\tau_k\theta} + E_1 e^{w_k\theta}, \tag{4.11}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})\epsilon R^2$ is a constant vector. Similarly, we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau_k w} q(0) e^{iw\tau_k\theta} + \frac{i\overline{g}_{11}}{\tau_k w}\overline{q}(0) e^{-iw\tau_k\theta} + E_2, \tag{4.12}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})\epsilon R^2$ is a constant vector. Now we will try to find E_1 and E_2. From the definition of A and (4.8), we obtain

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2iw\tau_k W_{20}(0) - H_{20}(0), \tag{4.13}$$

and

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{4.14}$$

where $d\eta(\theta) = \eta(\theta, 0)$.
By (4.8) and (4.9), we have

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) \\ &\quad + 2\tau_k \left[\begin{array}{c} -e^{2iw\tau_k\theta} - \frac{\alpha e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\ + \frac{P^* e^{-2iw\tau_k\theta} + \alpha N^* e^{-iw\tau_k\theta} + \alpha^2 P^* e^{-iw\tau_k\theta} + \alpha^2 N^*}{(N^*+\alpha P^*)^2} \\ - \frac{N^* P^* e^{-2iw\tau_k\theta} + 2\alpha^2 N^* P^* e^{-iw\tau_k\theta} \alpha^2 N^* P^*}{(N^*+\alpha P^*)^3} \\ \\ \frac{2\beta\alpha P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2 e^{-2iw\tau_k\theta}}{(N^*)^2} \end{array} \right] \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} H_{11}(\theta) &= -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) \\ &\quad + 2\tau_k \left[\begin{array}{c} -2\operatorname{Re}\alpha - \frac{\alpha e^{iw\tau_k\theta} + \overline{\alpha}^* \overline{\alpha} e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\ + \frac{2P^* + \alpha N^* e^{iw\tau_k\theta} + \overline{\alpha} N^* e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} + \frac{\alpha^2 P^* e^{iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\ + \frac{\alpha\overline{\alpha} P^* e^{-iw\tau_k\theta} + 2\operatorname{Re}\alpha N^*}{(N^*+\alpha P^*)^2} - \frac{2N^* P^* + 2\alpha^2 N^* P^* e^{iw\tau_k\theta}}{(N^*+\alpha P^*)^3} \\ - \frac{2\operatorname{Re} N^* P^* e^{-iw\tau_k\theta} + 2\operatorname{Re} N^* P^*}{(N^*+\alpha P^*)^3} \\ \\ - \frac{2\beta\alpha\overline{\alpha}}{N^*} + \frac{2\beta\alpha P^* e^{iw\tau_k\theta} + 2\beta\overline{\alpha} P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2}{(N^*)^2} \end{array} \right] \end{aligned} \tag{4.16}$$

Substituting (4.13) and (4.15) and noticing that

$$\begin{aligned} \left(iw\tau_k I - \int_{-1}^{0} e^{iw\tau_k\theta} d\eta(\theta) \right) q(0) &= 0 \\ \left(-iw\tau_k I - \int_{-1}^{0} e^{-iw\tau_k\theta} d\eta(\theta) \right) q(0) &= 0, \end{aligned}$$

we obtain

$$\left(2iw\tau_k I - \int_{-1}^{0} e^{2iw\tau_k\theta} d\eta(\theta)\right) E_1 = 2\tau_k \begin{bmatrix} -e^{2iw\tau_k\theta} - \frac{\alpha e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\ + \frac{P^* e^{-2iw\tau_k\theta} + \alpha N^* e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\ \frac{\alpha^2 P^* e^{-iw\tau_k\theta} + \alpha^2 N^*}{(N^*+\alpha P^*)^2} \\ - \frac{N^* P^* e^{-2iw\tau_k\theta} + 2\alpha^2 N^* P^* e^{-iw\tau_k\theta} \alpha^2 N^* P^*}{(N^*+\alpha P^*)^3} \\ \\ \frac{2\beta\alpha P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2 e^{-2iw\tau_k\theta}}{(N^*)^2} \end{bmatrix}$$

which is

$$\begin{bmatrix} 2iw - A_1 & -A_2 e^{-iw\tau_k} \\ A_3 & -A_4 e^{-iw\tau_k} + 2iw \end{bmatrix} E_1 = 2 \begin{bmatrix} -e^{2iw\tau_k\theta} - \frac{\alpha e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} \\ + \frac{P^* e^{-2iw\tau_k\theta} + \alpha N^* e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\ \frac{\alpha^2 P^* e^{-iw\tau_k\theta} + \alpha^2 N^*}{(N^*+\alpha P^*)^2} \\ - \frac{N^* P^* e^{-2iw\tau_k\theta} + 2\alpha^2 N^* P^* e^{-iw\tau_k\theta} \alpha^2 N^* P^*}{(N^*+\alpha P^*)^3} \\ \\ \frac{2\beta\alpha P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2 e^{-2iw\tau_k\theta}}{(N^*)^2} \end{bmatrix}$$

Now if we solve this system for E_1,we get

$$E_1^{(1)} = \frac{2}{B_1} \begin{vmatrix} E_{11}^{(1)} + E_{12}^{(1)} & -A_2 e^{-iw\tau_k} \\ \frac{2\beta\alpha P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2 e^{-2iw\tau_k\theta}}{(N^*)^2} & -A_4 e^{-iw\tau_k} + 2iw \end{vmatrix}$$

$$E_1^{(2)} = \frac{2}{B_1} \begin{vmatrix} 2iw - A_1 & E_{11}^{(1)} + E_{12}^{(1)} \\ A_3 & \frac{2\beta\alpha P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2 e^{-2iw\tau_k\theta}}{(N^*)^2} \end{vmatrix},$$

where

$$E_{11}^{(1)} = -e^{2iw\tau_k\theta} - \frac{\alpha e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} + \frac{P^* e^{-2iw\tau_k\theta} + \alpha N^* e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2}$$

$$E_{12}^{(1)} = \frac{\alpha^2 P^* e^{-iw\tau_k\theta} + \alpha^2 N^*}{(N^*+\alpha P^*)^2} - \frac{N^* P^* e^{-2iw\tau_k\theta} + 2\alpha^2 N^* P^* e^{-iw\tau_k\theta} \alpha^2 N^* P^*}{(N^*+\alpha P^*)^3}$$

$$B_1 = \begin{vmatrix} 2iw - A_1 & -A_2 e^{-iw\tau_k} \\ A_3 & -A_4 e^{-iw\tau_k} + 2iw \end{vmatrix}.$$

Similarly, substituting (4.12), (4.14) and (4.16), we obtain $\begin{bmatrix} -A_1 & A_2 \\ -A_3 & -A_4 \end{bmatrix} E_2 =$

$$2 \begin{bmatrix} -2\operatorname{Re}\alpha - \frac{\alpha e^{iw\tau_k\theta} + \overline{\alpha} e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)} + \frac{2P^* + \alpha N^* e^{iw\tau_k\theta} + \overline{\alpha} N^* e^{-iw\tau_k\theta}}{(N^*+\alpha P^*)^2} \\ + \frac{\alpha^2 P^* e^{iw\tau_k\theta}}{(N^*+\alpha P^*)^2} + \frac{\alpha\overline{\alpha}^2 P^* e^{-iw\tau_k\theta} + 2\operatorname{Re}\alpha N^*}{(N^*+\alpha P^*)^2} \\ - \frac{2N^* P^* + 2\alpha^2 N^* P^* e^{iw\tau_k\theta}}{(N^*+\alpha P^*)^3} - \frac{2\operatorname{Re} N^* P^* e^{-iw\tau_k\theta} + 2\operatorname{Re} N^* P^*}{(N^*+\alpha P^*)^3} \\ \\ - \frac{2\beta\alpha\overline{\alpha}}{N^*} + \frac{2\beta\alpha P^* e^{iw\tau_k\theta} + 2\beta\overline{\alpha} P^* e^{-iw\tau_k\theta} - 2\beta (P^*)^2}{(N^*)^2} \end{bmatrix},$$

which implies that

$$E_2^{(1)} = \frac{2}{B_2}\begin{vmatrix} E_{11}^{(2)} + E_{12}^{(2)} & A_2 \\ -\frac{2\beta\alpha\overline{\alpha}}{N^*} + \frac{2\beta\alpha P^* e^{iw\tau_k\theta} + 2\beta\overline{\alpha}P^* e^{-iw\tau_k\theta} - 2\beta(P^*)^2}{(N^*)^2} & -A_4 \end{vmatrix}$$

$$E_2^{(2)} = \frac{2}{B_2}\begin{vmatrix} A_1 & E_{11}^{(2)} + E_{12}^{(2)} \\ -A_3 & -\frac{2\beta\alpha\overline{\alpha}}{N^*} + \frac{2\beta\alpha P^* e^{iw\tau_k\theta} + 2\beta\overline{\alpha}P^* e^{-iw\tau_k\theta} - 2\beta(P^*)^2}{(N^*)^2} \end{vmatrix},$$

where

$$E_{11}^{(2)} = -2\operatorname{Re}\alpha - \frac{\alpha e^{iw\tau_k\theta} + \overline{\alpha}e^{-iw\tau_k\theta}}{(N^* + \alpha P^*)} + \frac{2P^* + \alpha N^* e^{iw\tau_k\theta} + \overline{\alpha}N^* e^{-iw\tau_k\theta}}{(N^* + \alpha P^*)^2}$$

$$E_{12}^{(2)} = \frac{\alpha^2 P^* e^{iw\tau_k\theta}}{(N^* + \alpha P^*)^2} + \frac{\alpha\overline{\alpha}^2 P^* e^{-iw\tau_k\theta} + 2\operatorname{Re}\alpha N^*}{(N^* + \alpha P^*)^2} - \frac{2N^*P^* + 2\alpha^2 N^* P^* e^{iw\tau_k\theta}}{(N^* + \alpha P^*)^3}$$

$$B_2 = \begin{vmatrix} -A_1 & A_2 \\ -A_3 & -A_4 \end{vmatrix}.$$

Thus we can compute $W_{20}(\theta)$ and $W_{11}(\theta)$ from (4.11) and (4.12) and determine the following values to investigate the qualities of bifurcating periodic solution in the center manifold at the critical value τ_k. For this purpose, we express $g_{ij}'s$ in terms of the parameters and delay. And then we can evaluate the following values;

$$c_1(0) = \frac{i}{2\omega\tau_k}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\lambda^{'}(\tau_k)\}},$$

$$\beta_2 = 2Re\{c_1(0)\},$$

$$T_2 = -\frac{Im\{c_1(0)\} + \mu_2 Im\{\lambda^{'}(\tau_k)\}}{\omega\tau_k}.$$

Theorem 4.1. *μ_2 determines the direction of Hopf bifurcation; if $\mu_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\tau > \tau_0$, if $\mu_2 < 0$, then the Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for $\tau < \tau_0$. β_2 determines the stability of the bifurcating periodic solutions; bifurcating periodic solutions are stable if $\beta_2 < 0$, unstable if $\beta_2 > 0$. T_2 determines the period of the bifurcating solution; the period increases if $T_2 > 0$, decreases if $T_2 < 0$.*

In the following section, we shall give a numerical example to verify the theoretical results.

5. A numerical example.

In this section, we present some numerical simulations to verify the results in Lemma 2.1, Lemma 2.2, Theorem 3.1 and Theorem 4.1 by using MATLAB(7.6.0) programming. We simulate the predator-prey system (1.1) by choosing the parameters, $\alpha = 0.7$, $\beta = 0.9$ and $\delta = 0.6$, i.e., we consider the following system,

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)(1 - N(t)) - \frac{N(t)P(t-\tau)}{N(t) + 0.7P(t-\tau)} \\ \frac{dP(t)}{dt} &= 0.9P(t-\tau)(0.6 - \frac{P(t-\tau)}{N(t)}) \end{aligned} \tag{5.1}$$

which has only one positive equilibrium $E_0^* = (N_0^*, P_0^*) = (0.5775, 0.3465)$. By algorithms in the previous sections, we obtain $\tau_0 = 2.6124$, $w = 0.4670$. So by Theorem 3.1, the equilibrium point E^* is asymptotically stable when $\tau \epsilon [0, \tau_0) = [0, 2.6124)$ and unstable when $\tau > 2.6124$ and also Hopf bifurcation occurs at $\tau = \tau_0 = 2.6124$ as it is illustrated by computer simulations.

By the theory of Hassard et al. [9], as it is discussed in previous section, we also determine the direction of Hopf bifurcation and the other properties of bifurcating periodic solutions. From the formulaes in section 3 we evaluate the values of μ_2, β_2 and T_2 as

$$\mu_2 = -1.4654 < 0, \quad \beta_2 = 1.5368 > 0 \quad \text{and} \quad T_2 = 1.9723 > 0$$

from which we conclude that Hopf bifurcation of system (5.1) occurring at $\tau_0 = 2.6124$ is subcritical, the bifurcating periodic solution exists when τ crosses τ_0 to the left, and also the bifurcating periodic solution is unstable and the period increases.

In computer simulations, the initial conditions are taken as $(N_0, P_0) = (0.01, 0.01)$ and MATLAB DDE (Delay Differential Equations) solver is used to simulate the system (5.1). We first take $\tau = 1.8 < \tau_0$ and plot the density functions $N(t)$ and $P(t)$ in Fig-1,2 respectively which shows the positive equilibrium is asymptotically stable for $\tau < \tau_{0.}$.

However in Fig-3,4 below, we take $\tau = 2.3$ sufficiently close to τ_0 which illustrates the existence of bifurcating periodic solutions from the equilibrium point E_0^*.

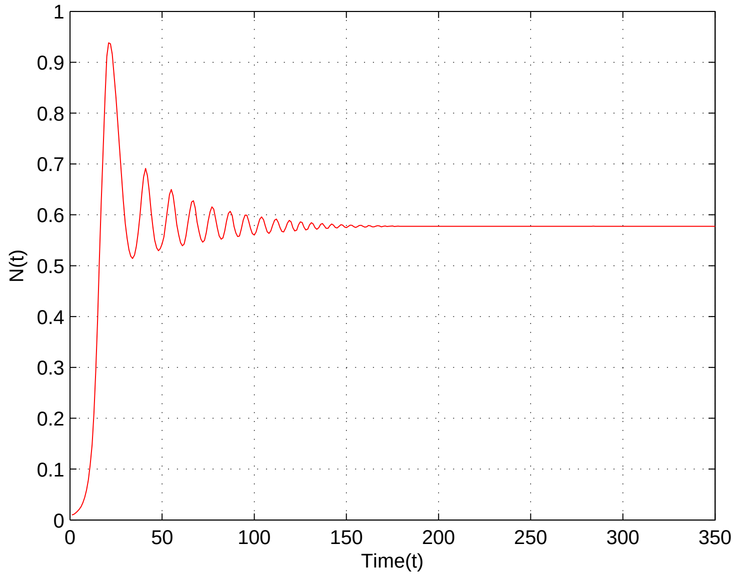

FIGURE 1. The trajectory of prey density versus time with the initial conditions $N_0 = 0.01$, $P_0 = 0.01$. When $\tau = 1.8 < \tau_0$ where the equilibrium point E^* is asymptotically stable.

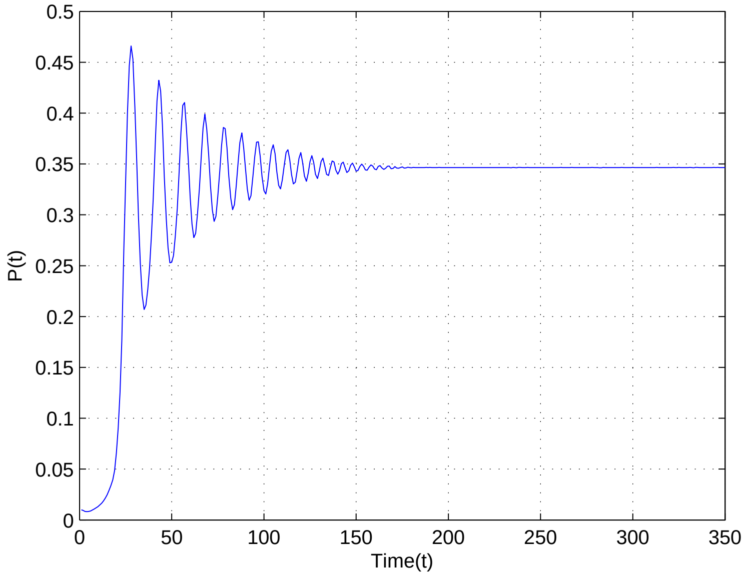

FIGURE 2. The trajectory of predator density versus time with the initial conditions $N_0 = 0.01$, $P_0 = 0.01$. When $\tau = 1.8 < \tau_0$ where the equilibrium point E^* is asymptotically stable.

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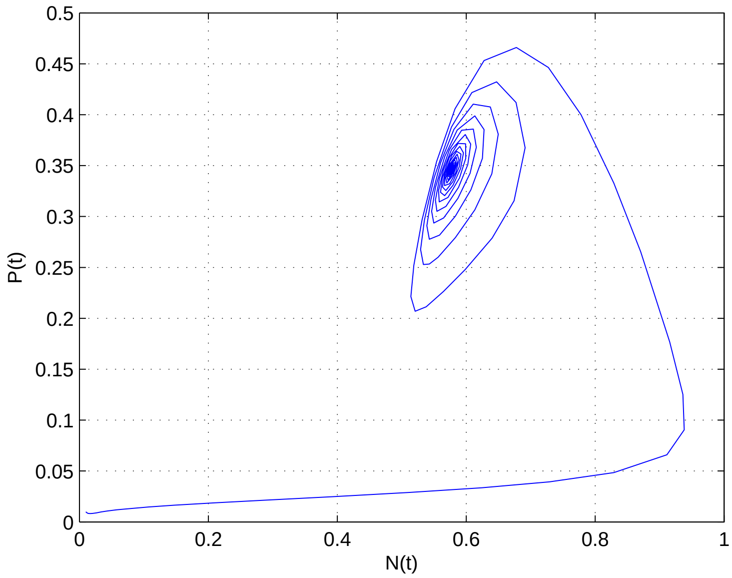

FIGURE 3. The phase portrait of Predator density versus Prey density for the same parameters as in Fig-1 when $\tau = 1.8 < \tau_0$.

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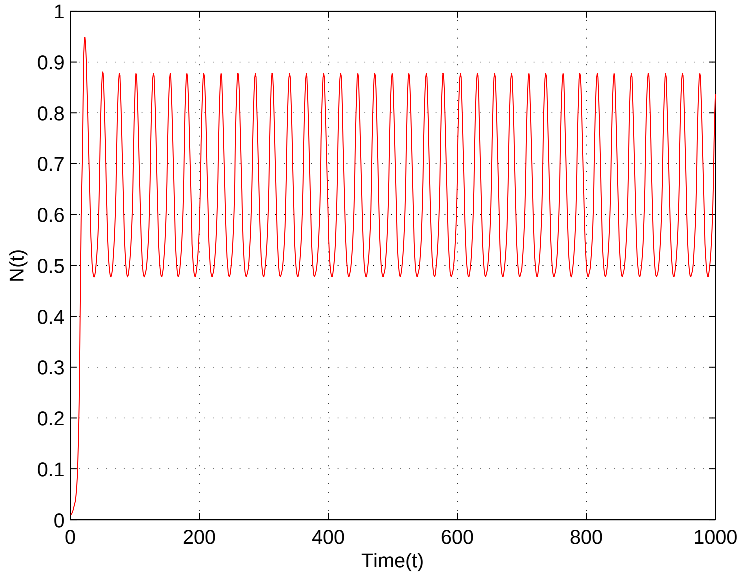

FIGURE 4. The trajectory of prey density versus time with the initial conditions $N_0 = 0.01$, $P_0 = 0.01$. When $\tau = 2.3$ the system periodic structure.

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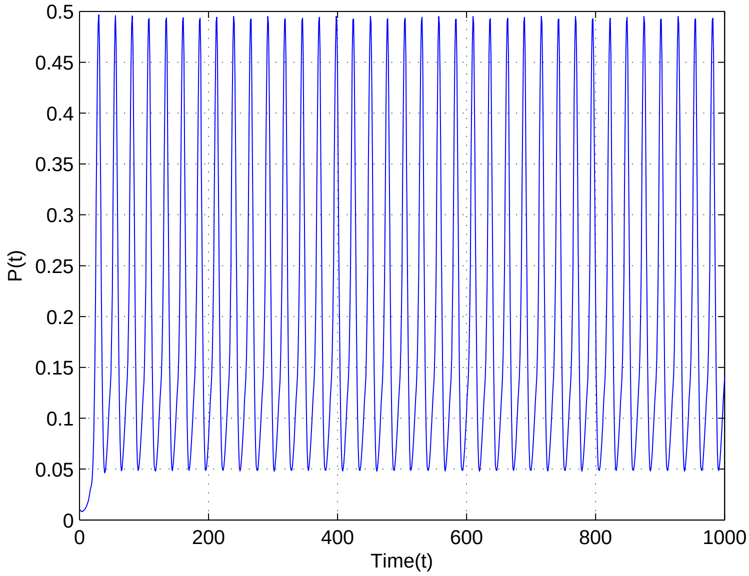

FIGURE 5. The trajectory of predator density versus time with the initial conditions $N_0 = 0.01$, $P_0 = 0.01$. When $\tau = 2.3$, the system shows periodic structure.

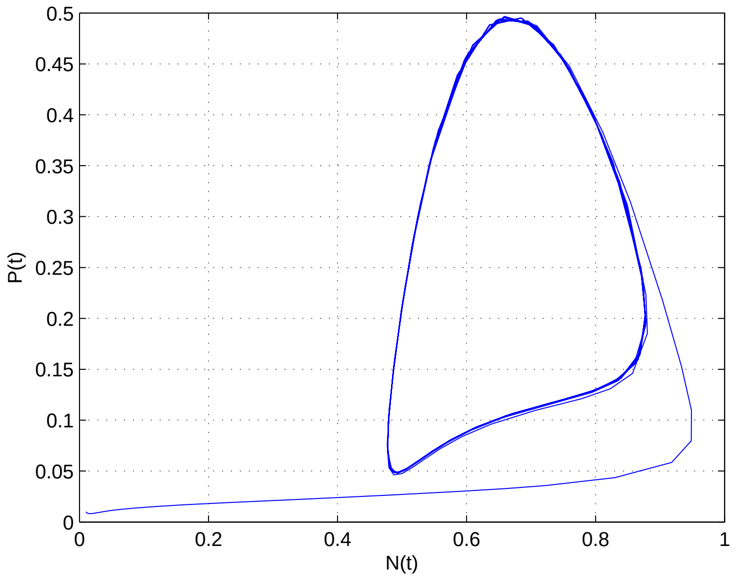

FIGURE 6. The phase portrait of Predator density versus Prey density for the same parameters as in Fig-1. When $\tau = 2.3$, the system shows the bifurcating periodic solutions from E^*.

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A DETERMINISTIC INVENTORY MODEL OF DETERIORATING ITEMS WITH STOCK AND TIME DEPENDENT DEMAND RATE

B. MUKHERJEE AND K. PRASAD

Abstract. In formulating inventory models, two fact of problem have been of growing interest, one being the deterioration of items, the other being the variation in demand rate. Time-varying demand patterns are usually used to reflect sales in deferent phases of the product life cycle in the market. The effect of deterioration of physical goods cannot be disregarded in many inventory systems. A deterministic inventory model for deteriorating item with inversely time dependent of two parameter weibull distribution to represent the deterioration rate has been studied in this paper. Time dependent and stock dependent demand rate separately has been studied by numerous authors while in this paper considering simultaneously both stock dependent and time dependent demand rate has been studied. The present-model has been solved analytically to minimize the cost. A numerical example has been carried out to illustrate the solution procedure.

1. Introduction

In the classical inventory model life time of an item is infinite while it is in storage. But effect of deterioration plays a vital role in the storage of some goods like vegetable, fruits, medicine etc. In such cases a certain part of these goods are either damaged or decayed and are not in a condition to satisfy the future demand of customer as a fresh unit. Mathematical model of inventory system has been developed by many of the researcher but most of them have consider the demand rate is constant also we know that now a day market is full of competitive environment as a result it is fluctuating day by day so in such a environment there are nothing fixed or constat. The inventory problem of deteriorating item was first researched by Within [17] who studies the problem of fashion goods at the end of inventory cycle. Sing et al. [16] they used constant rate of deterioration and linear rate of demand depending upon the current stock level. Ghare and Schrader [7]developed an inventory model with a constant rate of deterioration. An order level inventory model for items deteriorating at a constant rate was discussed by Shah and Jaiswal [15]. Aggarwal [1] reconsidered this model by rectifying the error in the work of Shah and Jaiswal [15] in calculating the average inventory holding cost. In all these models, the demand rate and the deterioration rate were constant, the replenishment rate was infinite and no shortage in inventory was allowed.

Key words and phrases. Inventory, Deterioration, Weibull distribution , Demand.
2010 *AMS Math. Subject Classification.* Primary 90B05; Secondary 65K10.

Researchers started to develop inventory systems allowing time variability in one or more than one parameters. Dave and Patel [5] discussed an inventory model for replenishment.This was followed by another model by Dave [4] with variable instantaneous demand, discrete opportunities for replenishment and shortages. Bahari-Kashani [2] discussed a heuristic model with time-proportional demand. An Economic Order Quantity (EOQ) model for deteriorating items with shortage and linear trend in demand was studied by Goswami and Chaudhuri [8]. On all these inventory systems, the deterioration rate is a constant.Another class of inventory models has been developed with time-dependent deterioration rate.

Covert and Philip [3] used a two-parameter Weibull distribution to represent the distribution of the time to deterioration. This model was further developed by Philip [12] taking three-parameter Weibull distribution for the time to deterioration. Mishra [10] analyzed an inventory model with a variable rate of deterioration, finite rate of replenishment and no shortage, but only a special case of the model was solved under very restrictive assumptions. Deb and Chaudhuri [6]studied a model with a finite rate of production and a time-proportional deterioration rate, allowing backlogging. Goswami and Chaudhuri [8] assumed that the demand rate, production rate and deterioration rate were all time dependent. Detailed information regarding inventory modeling for deteriorating items was given in the review articles of Nahmias [11] and Rafaat [13]. An order-level inventory model for deteriorating items without shortage has been developed by Jalan and Chaudhuri [9].

Here we have consider that demand is depending on time as well as current stock level of the system and the deterioration rate is considered as a two parameter weibull distribution which is function of time.

2. Notation and Assumptions

To develop an inventory model of deteriorating item the following notations and assumptions are used throughout the paper.

2.1. **Notations.** :

C_h holding cost per unit per unit time.
C_s shortage cost per unit per unit time.
C_d cost of a deteriorated unit.
C average cost of the system.
$q(t)$ inventory level at time t.
$\theta(t)$ the deterioration rate.
T duration of per cycle.
$D(q)$ demand function.
A replenishment cost.

2.2. **Assumptions.** :

(i) Shortages are allowed and backlogged.
(ii) T is the fixed duration of a cycle.
(iii) Lead time is zero and Replenishment is instantaneous.

(iv) The items considered in this model are deteriorating items with variable rate of deterioration $\theta(t)$.

(v) The deteriorating rate is defined as two parameter weibull distribution

$\theta(t) = \alpha\ \beta\ t^{\beta-1}$, Where $0 < \alpha < 1$, $0 < \beta \leq 1$ *and* $\beta = \frac{1}{\text{n}}$,where n is the natural number.

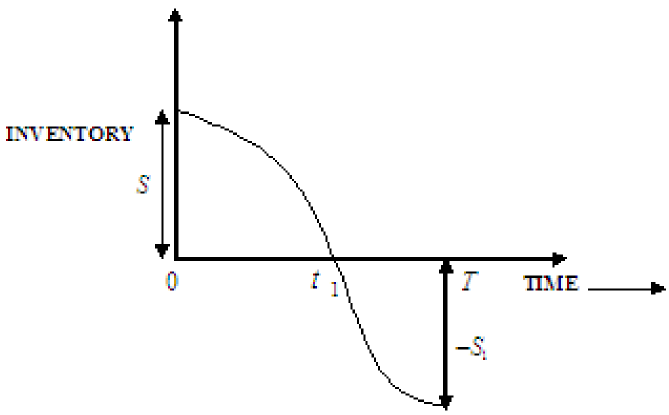
(vi) Demand rate is defined as the function of q(t) as $D(q(t)) = a + bt^{\beta-1}q(t)$.

3. Mathematical model and its analysis

On the basis of above mentioned assumptions, at the beginning that is at time $t = 0$,S units are hold for each cycle of the considered inventory system and the items are depleted gradually in the interval $[0, t_1]$ due to the combined effects of demand and deterioration.

At time $t = t_1$, the inventory level reaches zero and then inventory level depleted up to $-S_1$ due to demand only in the interval $[t_1, T]$ and the whole process is repeated.

Proposed model:



The variation of inventory level $q(t)$ with respect to time can be described by the following differential equation as follow:

$$\frac{dq}{dt} = -\theta q(t) - D(q(t)), \quad 0 \le t \le t_1. \tag{3.1}$$

$$\frac{dq}{dt} = -D(q(t)), \quad t_1 \le t \le T. \tag{3.2}$$

With the boundary conditions as

$$q(0) = S, \quad q(t_1) = 0, \quad q(T) = -S_1 \tag{3.3}$$

The solutions of above equations are given by

$$q(t) = \frac{a}{k_1\beta}\left(\frac{-1}{k_1}\right)^{\frac{1}{\beta}-1} e^{-k_1t^\beta}\left[I_{\frac{1}{\beta}} - (I_{\frac{1}{\beta}})_{t=t_1}\right], \quad 0 \le t \le t_1 \tag{3.4}$$

$$q(t) = \frac{a}{k\beta}\left(\frac{-1}{k}\right)^{\frac{1}{\beta}-1} e^{-kt^\beta}\left[I_{\frac{1}{\beta}} - (I_{\frac{1}{\beta}})_{t=t_1}\right], \quad t_1 \le t \le T \tag{3.5}$$

where $k_1 = \frac{\alpha\beta+b}{\beta}, k = \frac{b}{\beta}$ and

$$\begin{aligned} I_{\frac{1}{\beta}} &= \int e^{-z} z^{\frac{1}{\beta}-1} dz, \ where\ z = -k_1t^\beta \\ &= -(-k_1t^\beta)^{\frac{1}{\beta}-1}e^{k_1t^\beta} - \left(\frac{1}{\beta}-1\right)(-k_1t^\beta)^{\frac{1}{\beta}-2}e^{k_1t^\beta} \\ &- \left(\frac{1}{\beta}-1\right)\left(\frac{1}{\beta}-2\right)(-k_1t^\beta)^{\frac{1}{\beta}-3}e^{k_1t^\beta}... \\ &- \left(\frac{1}{\beta}-1\right)\left(\frac{1}{\beta}-2\right)...1\ e^{k_1t^\beta} \end{aligned} \tag{3.6}$$

Total number of deteriorating items in $(0,t_1)$

$$\begin{aligned} D_T &= S - \ Total\ demand\ in\ time\ (0, t_1) \\ &= S - \int_0^{t_1} [a + bt^{\beta-1}q(t)]dt. \end{aligned} \tag{3.7}$$

Case Study:

If $\beta = \frac{1}{2}$ the equation (3.7) in this case reduces to

$$\begin{aligned} D_T &= \frac{2a}{k_1^2}[1 - e^{k_1\sqrt{t_1}}(1 - k_1\sqrt{t_1})] - at_1 + \frac{2ab}{k_1^2}(k_1 t_1 - 2\sqrt{t_1}) \\ &+ \frac{4ab}{k_1^3}(1 - k_1\sqrt{t_1})(e^{k_1\sqrt{t_1}} - 1). \end{aligned} \tag{3.8}$$

Hence total Inventory during the time $(0, t_1)$

$$\begin{aligned} H_T &= \int_0^{t_1} q(t)dt \\ &= \int_0^{t_1} \frac{a}{k_1\beta}\left(\frac{-1}{k_1}\right)^{\frac{1}{\beta}-1} e^{-k_1 t^\beta}\left[I_{\frac{1}{\beta}} - \left(I_{\frac{1}{\beta}}\right)_{t=t_1}\right] dt \end{aligned} \tag{3.9}$$

which for present case reduces to

$$H_T = -\frac{4a}{3k_1}t_1^{\frac{3}{2}} + \frac{2at_1}{k_1^2} + \frac{4a}{k_1^4}\left(k_1\sqrt{t_1}e^{k_1\sqrt{t_1}} - e^{k_1\sqrt{t_1}} - k_1^2 t_1 + 1\right). \tag{3.10}$$

Similarly the total shortage during the time (t_1, T)

$$\begin{aligned} B_T &= -\int_{t_1}^{T} q(t)dt \\ &= -\int_{t_1}^{T} \frac{a}{k\beta}\left(\frac{-1}{k}\right)^{\frac{1}{\beta}-1} e^{-kt^\beta}\left[I_{\frac{1}{\beta}} - \left(I_{\frac{1}{\beta}}\right)_{t=t_1}\right] dt \end{aligned} \tag{3.11}$$

also for this case the equation (3.11) reduces to

$$\begin{aligned} B_T &= \frac{2a}{3b}\left(T^{\frac{3}{2}} - t_1^{\frac{3}{2}}\right) - \frac{a}{2b^2}(T - t_1) + \frac{a}{4b^4}\left[4b^2\sqrt{t_1 T}e^{2b(\sqrt{t_1}-\sqrt{T})} - 4b^2 t_1\right. \\ &+ \left. 2b\sqrt{t_1}e^{2b(\sqrt{t_1}-\sqrt{T})} - 2b\sqrt{T}e^{2b(\sqrt{t_1}-\sqrt{T})} - e^{2b(\sqrt{t_1}-\sqrt{T})} + 1\right]. \end{aligned} \tag{3.12}$$

Therefore average cost of the system

$$C(t_1, T) = \frac{A}{T} + C_d\frac{D_T}{T} + C_h\frac{H_T}{T} + C_s\frac{B_T}{T}. \tag{3.13}$$

Differentiating cost function with respect to t_1 and T using equations (3.8), (3.10) and (3.12), we have

$$\frac{\partial C}{\partial t_1} = \frac{C_d}{T}\frac{\partial D_T}{\partial t_1} + \frac{C_h}{T}\frac{\partial H_T}{\partial t_1} + \frac{C_s}{T}\frac{\partial B_T}{\partial t_1} \tag{3.14}$$

$$\frac{\partial C}{\partial T} = -\frac{A}{T^2} - C_d\frac{D_T}{T^2} - C_h\frac{H_T}{T^2} + \frac{C_s}{T}\frac{\partial B_T}{\partial T} - \frac{C_s}{T^2}B_T. \tag{3.15}$$

The optimal value of t_1 and T as t_1^*, T^* can be obtained by satisfying the necessary condition for minimization of the cost

$$\frac{\partial C}{\partial t_1} = 0, \quad \frac{\partial C}{\partial T} = 0.$$

provided they satisfy the sufficient conditions

$$\frac{\partial^2 C}{\partial t_1^2} > 0 \tag{3.16}$$

$$\left(\frac{\partial^2 C}{\partial t_1^2}\right)\left(\frac{\partial^2 C}{\partial T^2}\right) - \left(\frac{\partial^2 C}{\partial t_1 \partial T}\right)^2 > 0. \tag{3.17}$$

If the solution for t_1 and T do not satisfy the sufficient conditions (3.16) and (3.17) then no feasible solution will be optimal for the set of parameter value which has been used to solve the above equations. Such a situation will imply that the parameter values are inconsistent and there is some error in their estimation. A new parameter's value is required to analyse the situation further.

Numerical Example:

Numerical values of $t_1^*,\ T^*, C^*$ have been calculated with the help of C program for solution of system of non-linear equation using Newton Rapshon method by considering the parameters as $A = 8,\quad C_d = 1,\quad C_h = 2,\quad C_s = 5,\ a = 100,\ b = 0.30$

α	t_1^*	T^*	C^*
0.10	2.264520	3.240990	448.589996
0.20	1.751950	2.717921	440.370331
0.30	1.504149	2.553645	472.665436
0.40	1.362177	2.515763	513.252136
0.50	1.266038	2.522837	552.972595
0.60	1.192582	2.547586	590.161011
0.70	1.131593	2.579973	624.946655
0.80	1.078012	2.615901	657.758423
0.90	1.029138	2.653523	688.969910

It is observed that these result satisfy the sufficient conditions (3.16) and (3.17) for minimizing the cost of the system.

Conclusion:

Numerical calculation shows that as the value of the parameter alpha increases then optimal value t_1 is decreases continuously but the optimal value of T is decreases firstly up to $\alpha = 0.50$ and after then it starts to increase . The optimal cost is also decreases initially up to $\alpha = 0.20$ but after then it starts to increase. This shows that the time of duration of shortage is increases as deterioration is increases. From these result we may also conclude that as time passes the deterioration rate decreases which leads to reduction to the average cost of the system.

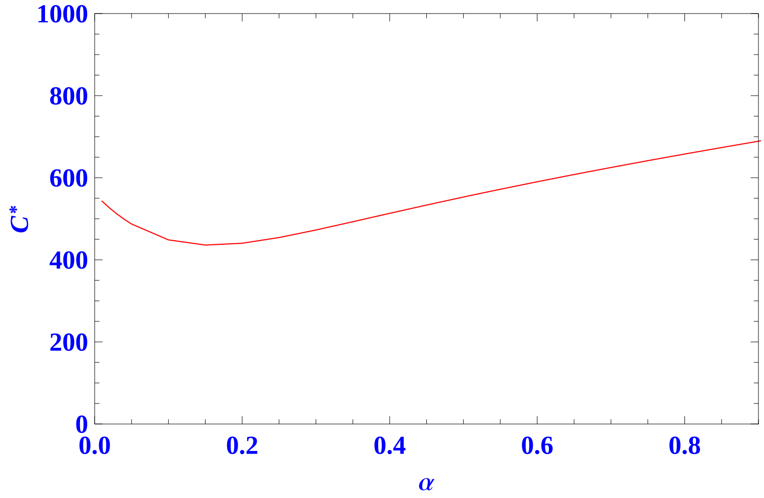


FIGURE 1. Deterioration parameter α verses optimal cost C^* of the system.

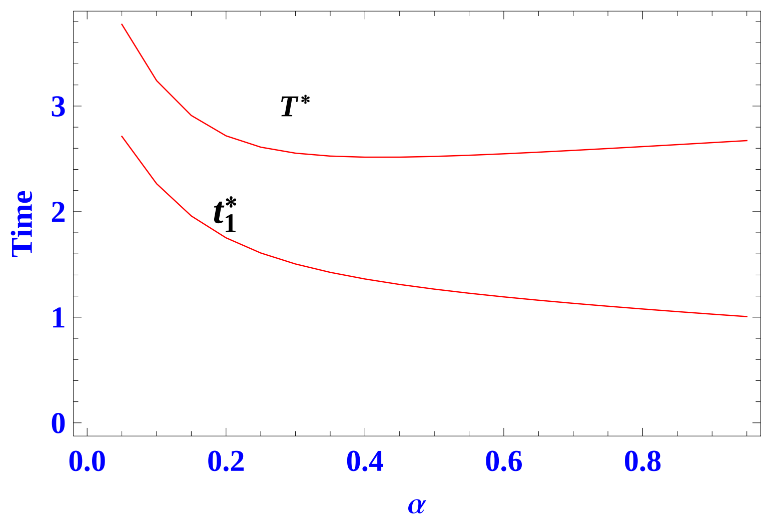


FIGURE 2. Comparative representation of T^* and t_1^* with deterioration parameter α

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OPEN PROBLEMS IN SEMI-LINEAR UNIFORM SPACES

ABDALLA TALLAFHA

ABSTRACT. Semi-linear uniform space is a new space defined by Tallafha, A and Khalil, R in [7]. The authors studied some cases of best approximation in such spaces, and gave some open problems in uniform spaces. Besides they defined a set valued map ρ on $X \times X$ and asked two questions about the properties of ρ. In 2011, Tallafha [8] defined another set valued map δ on $X \times X$, and give more properties of semi-linear uniform spaces using the maps ρ, δ and he answered the questions about ρ. The purpose of this paper is to introduce some open questions concerning this new spaces.

1. INTRODUCTION

Uniform spaces had been studied extensively through years. We refer the reader to [1],[2] , [3], [4] , [5] , [6] , [9] and[10] for the basic structure of uniform spaces.

Semi-linear uniform space is a new space defined by Tallafha, A and Khalil, R in [7], the authors define a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on uniform type spaces. They asked two question about the properties of ρ, besides they studied some cases of best approximation in such spaces, and gave some open problems in approximation theory in uniform spaces . In [8] , Tallafha, A. defined another set valued map δ on $X \times X$, and he gave more properties of semi-linear uniform spaces using ρ and δ. Besides he solved the two question about the properties of ρ.

Let X be a set and D_X be a collection of subsets of $X \times X$, such that each element V of D_X contains the diagonal

$$\Delta = \{(x, x) : x \in X\}$$

and

$$V = V^{-1} = \{(y, x) : (x, y) \in V\}$$

for all $V \in D_X$ (**symmetric**), D_X is called the family of all entourages of the diagonal. Let Γ be a sub collection of D_X, then the pair (X, Γ) is called a **uniform space** if

(i) V_1 and V_2 are in Γ then $V_1 \cap V_2 \in \Gamma$,
(ii) for every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$,
(iii) $\bigcap\limits_{V \in \Gamma} V = \Delta$,
(iv) if $V \in \Gamma$ and $V \subseteq W \in D_X$, then $W \in \Gamma$.

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2. Uniform type spaces

Let (X,Γ) be a uniform space. By a **chain** in $X\times X$ we mean a totally(or linearly) ordered collection of subsets of $X\times X$, where $V_1\le V_2$ means $V_1\subseteq V_2$.

Definition 2.1. [7] *We call* (X,Γ) *a* ***semi-linear uniform space*** *if it is a uniform space where* Γ *is a chain and condition* (vi) *is replaced by*

$$\bigcup_{V\in\Gamma} V = X\times X.$$

An example of a semi-linear uniform space is the following.

Example 2.2. *Let* $V_t=\{(x,y): y-t<x<y+t,\ \text{and}\ -\infty<y<\infty\}$. *Then* $(\mathbb{R},\Gamma)$, *with* $\Gamma=\{V_t: 0<t<\infty\}$ *is a semi-linear uniform space.*

One can generate semi-linear uniform spaces as follows. Let D_X be a chain in the power set of $X\times X$, such that, each element of D_X is symmetric, contains $\triangle$,

$$\bigcup_{U\,\in\, D_X} U = X\times X$$

and

$$\bigcap_{U\,\in\, D_X} U = \triangle.$$

Then one can easily see that (X,D_X) is a semi-linear uniform space.

We should remark that **the topology in metric and normed spaces can be generated by semi-linear uniformities**.

Throughout the rest of this paper, (X,Γ) will be assumed semi-linear uniform space. Let (X,Γ) be a semi-linear uniform space. For $x,y\in X$, let

$$C(x,y)=\cap\{V\in\Gamma:(x,y)\in V\},$$

and

$$\Sigma=\{C\ (x,y): x,y\in X\ \}\,.$$

Clearly $C\ (x,y)=\cap\{V^{-1}\in\Gamma:(x,y)\in V\}$.

Definition 2.3. [7] *Let* (X,Γ) *be a semi-linear uniform space. We define the set valued map:* $\rho: X\times X\ \to\Sigma$, $\rho(x,y)=C\ (x,y)$. *The map* ρ *will be called a* ***set metric*** *on* (X,Γ).

Proposition 2.4. [7] *For a semi-linear uniform space, we have the followings*:
 (i) $\rho(x,y)=\Delta$ *if and only if* $x=y$,
 (ii) $\rho(x,y)=\rho(y,x)$.

In [7] , the authors gave the following questions.

- **Question 1**. [7] Is $\rho(x,y)\subseteq\rho(x,z)\cap\rho(z,y)$?
- **Question 2.** [7] If $\rho(x,z)=\rho(x,w)$, for some $x\in X$, Must $w=z$?

The above questions is answered negatively by Tallafha, A. in [8] . Also Tallafha showed that the answer of question 1 still negative, if $\cap$ is replaced by $\cup$.

Definition 2.5. [7] *For* $x\in X$ *and* $E\subset X$, *we define*

$$\rho(x,E)=\underset{y\,\in\,E}{\cap}\rho(x,y).$$

Clearly, if $x\in E$, then $\rho(x,E)=\Delta$.

Definition 2.6. [7] *For $x \in X$ and $V \in \Gamma$, we define The open ball of center x and radius V to be*
$$B(x,V) = \{y : (x,y) \in V\}.$$
Equivalently
$$B(x,V) = \{y : \rho(x,y) \subseteq V\}.$$
Clearly if $y \in B(x,V)$, then there is a $W \in \Gamma$ such that $B(y,W) \subseteq B(x,V)$.

Definition 2.7. *$B \subseteq X$ is called bounded if $B \subseteq B(x,V)$, for some $V \in \Gamma, x \in X$.*

In [8] , the following concepts are defined, and the following results are proved.

Definition 2.8. *Let (X,Γ) be a semi-linear uniform space. then, the set valued map δ on $X \times X$ is defined by*
$$\delta(x,y) = \left\{ \begin{array}{ll} \cup\{V : V \in \Gamma^{c}_{(x,y)}\} & \text{if } x \neq y \\ \phi & \text{if } x = y \end{array} \right\}$$
where $\Gamma^{c}_{(x,y)}$ is the complement of $\Gamma_{(x,y)}$.

Clearly, if $x = y$ then $\Gamma^{c}_{(x,y)}$ is the empty set so we define $\delta(x,x)$ to be the empty set, and $\delta(x,y) = \delta(y,x)$. for all $(x,y) \in X \times X$. and $\Delta \subseteq \delta(x,y)$ for all $x \neq y$.

The first natural question that one should ask, is there a semi-linear uniform space which is not materialize?. The answer is yes as the following example shows.

Example 2.9. *Let $t \in (0,\infty)$,for $t \neq 1$ and*
$$V_t = \{(x,y) : x^2 + y^2 < t\} \cup \Delta, \Gamma = \{V_t : 0 < t < \infty\}.$$
Then $(\mathbb{R},\Gamma)$, is a semi-linear uniform space which is not materialize.

Proposition 2.10. *Let (X,Γ) be a semi-linear uniform space. Then,*

- (i) *If $V \in \Gamma^{c}_{(x,y)}$, then $V \subsetneqq \rho(x,y)$.*
- (ii) *$\delta(x,y) \subseteq \rho(x,y)$ for all $(x,y) \in X \times X$.*
- (iii) *If $V \in \Gamma_{(x,y)}$,then $\delta(x,y) \subseteq V$.*
- (iv) *If $(x,y) \in \rho(s,t)$ then $\rho(x,y) \subseteq \rho(s,t)$.*
- (v) *If $(x,y) \in \delta(s,t)$ then $\delta(x,y) \subseteq \delta(s,t)$.*
- (vi) *If $U \in \Gamma$ satisfies $U \subsetneqq \rho(x,y)$, then $U \subseteq \delta(x,y)$.*
- (vii) *If $U \in \Gamma$ satisfies $\delta(x,y) \subsetneqq U$, then $\rho(x,y) \subseteq U$.*
- (viii) *If $U \in \Gamma$ satisfies $\delta(x,y) \subseteq U \subseteq \rho(x,y)$, then $U = \delta(x,y)$ or $U = \rho(x,y)$.*
- (ix) *If $(s,t) \notin \delta(x,y)$ then $\delta(x,y) \subseteq \delta(s,t)$.*
- (x) *If $(s,t) \notin \rho(x,y)$ then $\rho(x,y) \subseteq \delta(s,t)$.*
- (xi) *If $\delta(x,y) \subsetneqq \delta(s,t)$, the there exist $U \in \Gamma$, such that $\delta(x,y) \subsetneqq U \subseteq \delta(s,t)$.*
- (xii) *If $\rho(x,y) \subsetneqq \rho(s,t)$, the there exist $U \in \Gamma$, such that $\rho(x,y) \subseteq U \subsetneqq \rho(s,t)$.*

Theorem 2.11. *Let (X,Γ) be a semi-linear uniform space. Then,*

- (i) *$\{\rho(x,y) : (x,y) \in X \times X\}$ is a chain.*
- (ii) *$\{\delta(x,y) : (x,y) \in X \times X,\ x \neq y\}$ is a chain.*

Theorem 2.12. *Let (X,Γ) be a semi-linear uniform space. Then, $\Theta = \boldsymbol{\rho} \cup \boldsymbol{\delta} \cup \Gamma$ is a chain.*

Theorem 2.13. *Let (X,Γ) be a semi-linear uniform space. Then,*

- (i) *$(X,\boldsymbol{\rho})$ is a semi-linear uniform space.*
- (ii) *$(X,\boldsymbol{\delta})$ is a semi-linear uniform space.*

Lemma 2.14. *Let (X,Γ) be a semi-linear uniform space. Then, $\rho(x,y) \subseteq \rho(s,t)$ if and only if $\delta(x,y) \subseteq \delta(s,t)$.*

Theorem 2.15. *Let (X,Γ) be a semi-linear uniform space. Then, $\rho(x,y) = \rho(s,t)$ if and only if $\delta(x,y) = \delta(s,t)$.*

In [7] , the authors defined the concepts of, **convergent, Cauchy** and they proved that (i) Every convergent sequence is Cauchy. (ii) Every Cauchy sequence is bounded. (iii) If (x_n) converges then the limit is unique. Also they gave the following open question.

- **Question 3**. If $\rho(x,E) = \Delta$, must $x \in E^{\ell}$?

Clearly the converse of the above question is true**.**

3. Proximinality in Semi-Linear Uniform Spaces

What is nice about semi-linear uniform spaces is that theory of best approximation can be studied in such spaces without tools that metric structure usually offers. In [7] the authors defined the following concepts and proved the following results.

Definition 3.1. *Let (X,Γ) be semi-linear uniform space, and $E \subset X$. The set E is called **proximinal** if for any $x \in X$, there exists some $e \in E$ such that $\rho(x,E) = \rho(x,e)$.*

Proposition 3.2. *If $E \subset X$ is proximinal, then E is closed.*

This question is given in [7] , is still open.

- **Question 4.** If E is compact, must E be proximinal?.

But the following partial answer is given.

Theorem 3.3. [7] *Let (X,Γ) be a semi-linear uniform space. Then every finite set is proximinal.*

Corollary 3.4. *If $E_1, E_2, ..., E_n$ are proximinal in (X,Γ), then $\bigcup\limits_{i=1}^{n} E_i$ is proximinal too.*

Also every sequence with it's limit is compact, so we have another partial answer to the question.

Theorem 3.5. *Let (X,Γ) be a semi-linear uniform space and (y_n) be a convergent sequence in X. Then $E = \{y, y_1, y_2, ...\}$ is proximinal, where $y = \lim y_n$.*

4. Fixed point in semi-linear uniform space

In [9], A.Tallafha defined **Lipschitz condition for functions and contractions functions** on semi-linear uniform spaces which enables us to study fixed points for such functions. Since **Lipschitz condition,** and **contractions** are usually discussed in metric and normed spaces, and never been studied in other weaker spaces. We believe that the structure of semi-linear uniform spaces is very rich, and all the known results on fixed point theory can be generalized.

Definition 4.1. [12] *Let $f : (X,\Gamma_X) \to (Y,\Gamma_Y)$. Then f is **uniformly continuous** if $\forall U \in \Gamma_Y$, $\exists V \in \Gamma_X$ such that if $(x,y) \in V$, then $(f(x), f(y)) \in U$.*

Clearly using our notation we have:

Proposition 4.2. *Let $f:(X,\Gamma_X)\to(Y,\Gamma_Y)$. Then f is* ***uniformly continuous*** *if and only if $\forall U\in\Gamma_Y$, $\exists V\in\Gamma_X$ such that, for all $x,y\in X$, if $\rho_X(x,y)\subseteq V$, then $\rho_Y(f(x),f(y))\subseteq U$.*

The following Proposition, shows that we may replace ρ by δ in Proposition 3.2.

Proposition 4.3. [9]. *Let $f:(X,\Gamma_X)\to(Y,\Gamma_Y)$. Then f is* ***uniformly continuous*** *if and only if $\forall U\in\Gamma_Y$, $\exists V\in\Gamma_X$, such that for all $x,y\in X$, if $\delta_X(x,y)\subseteq V$, then $\delta_Y(f(x),f(y))\subseteq U$.*

In [9], Tallafha gave the following.

Definition 4.4. *Let $f:(X,\Gamma)\longrightarrow(X,\Gamma)$, then f satisfied* ***Lipschitz condition*** *if there exist $m,n\in\mathbb{N}$ such that $m\delta(f(x),f(y))\subseteq n\delta(x,y)$. Moreover if $m>n$, then we call f a* ***contraction****.*

Remark 4.1. *One may use the set valued function ρ, instead of δ in the above definition.*

It is known that, every topological space (X,τ), whose topology induced by a metric or a norm on X, can be generated by a uniform space see[4], Also we now that if f is a contraction then it satisfies Lipschitz condition, if f satisfies Lipschitz condition, then it is uniformly continuous. In [9] Tallafha gave a similar results.

Theorem 4.5. [9]. *Every topological space whose topology induced by a metric or a norm on X, can be generated by a semi-linear uniform space. namely,*

$$\Gamma=\left\{V_\in,\in>0:V_\in=\bigcup_{x\in X}\{x\}\times B(x,\in)\right\}.$$

Theorem 4.6. [9]. *Let (X,Γ_X) be any semi-linear uniform space, and $f:(X,\Gamma)\longrightarrow(X,\Gamma)$, then.*

(1) *If f is a contraction then it satisfies Lipschitz condition.*
(2) *If f satisfies Lipschitz condition, then it is uniformly continuous.*

Definition 4.7. [7]. *A semi-linear uniform space (X,Γ) is called complete, if every Cauchy sequence is convergent.*

Fixed point theorems is one of the well known results in mathematics, and has a useful applications in many applied fields such as game theory, mathematical economics and the theory of quasi-variational inequalities. It states that every contraction from a complete metric space to it self has a unique fixed point. So the following question is natural.

- **Question3.8.** Let (X,Γ) be a complete semi-linear uniform space. And $f:(X,\Gamma)\to(X,\Gamma)$ be a contraction. Does f has a unique fixed point.

Remark 4.2. *All the results which was obtained using contraction on metric spaces can be consider as an open questions in semi-linear uniform space.*

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ALZER INEQUALITY FOR HILBERT SPACES OPERATORS

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ABSTRACT. In this paper, we give the Alzer inequality for Hilbert space operators as follows:

Let A, B be two selfadjoint operators on a Hilbert space $\mathcal{H}$ such that $0 < A, B \leq \frac{1}{2}I$, where I is identity operator on $\mathcal{H}$. Also, assume that $A\nabla_\lambda B := (1-\lambda)A + \lambda B$ and $A\sharp_\lambda B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\lambda A^{\frac{1}{2}}$ are arithmetic and geometric means of A, B, respectively, where $0 < \lambda < 1$. We show that if A and B are commuting, then

$$B' \nabla_\lambda A' - B' \sharp_\lambda A' \leq A \nabla_\lambda B - A \sharp_\lambda B\,,$$

where $A' := I - A$, $B' := I - B$ and $0 < \lambda \leq \frac{1}{2}$. Also, we state an open problem for an extension of Alzer inequality.

1. Introduction and preliminaries

Let $x_1, \cdots, x_n \in (0, \frac{1}{2}]$ and $\lambda_1, \cdots, \lambda_n > 0$ with $\sum_{j=1}^n \lambda_j = 1$. We denote by A_n and G_n, the arithmetic and geometric means of $x_1, \cdots, x_n$ respectively, i.e

$$A_n = \sum_{j=1}^n \lambda_j x_j, \qquad G_n = \prod_{j=1}^n x_j^{\lambda_j}\,,$$

and also by A'_n and G'_n, the arithmetic and geometric means of $1 - x_1, \cdots, 1 - x_n$ respectively, i.e.

$$A'_n = \sum_{j=1}^n \lambda_j (1 - x_j), \qquad G'_n = \prod_{j=1}^n (1 - x_j)^{\lambda_j}\,.$$

Alzer proved the following inequality and its refinement [1, 2]

$$A'_n - G'_n \leq A_n - G_n. \tag{1.1}$$

Throughout the paper, let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and I is the identity operator. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field and denote its identity by I_n. A selfadjoint operator $A \in \mathbb{B}(\mathcal{H})$ is called positive (strictly positive) if $\langle Ax, x\rangle \geq 0$ ($\langle Ax, x\rangle > 0$) holds for every $x \in \mathcal{H}$ and then we write $A \geq 0$ ($A > 0$) [6, 8]. For every selfadjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Let f be a continuous real valued function defined on an interval $[\alpha, \beta]$. The function

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f is called operator decreasing if $B \leq A$ implies $f(A) \leq f(B)$ for all A, B with spectra in $[\alpha, \beta]$. A function f is said to be operator concave on $[\alpha, \beta]$ if

$$\lambda f(A) + (1-\lambda) f(B) \leq f(\lambda A + (1-\lambda)B)$$

for any selfadjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in $[\alpha, \beta]$ and all $\lambda \in [0, 1]$.

The main result of this paper is the following theorem:

Theorem (Alzer Inequality). Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are commuting operators such that $0 < A \leq B \leq \frac{1}{2}I$, and let $A' := I - A$ and $B' = I - B$. If $0 < \lambda \leq \frac{1}{2}$, then

$$B' \nabla_\lambda A' - B' \sharp_\lambda A' \leq A \nabla_\lambda B - A \sharp_\lambda B\,.$$

2. Main results

In this section, we state an identity between arithmetic and geometric mean for positive operators and then we consequent the Alzer inequality.

We recall that, the *weighted arithmetic mean* ∇_λ and the *weighted geometric mean* (the λ-*power mean*) $\sharp_\lambda$ defined for $0 < \lambda < 1$:

$$\begin{aligned} A \nabla_\lambda B &:= (1-\lambda)A + \lambda B\,, \\ A \sharp_\lambda B &:= A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\lambda A^{\frac{1}{2}}\,. \end{aligned}$$

Also, we know that $A \sharp_\lambda B = B \sharp_{1-\lambda} A$ and if $AB = BA$ then $A \sharp_\lambda B = A^{1-\lambda} B^\lambda$.

Notice that if $\lambda = \frac{1}{2}$ in above definitions, we have the classic arithmetic and geometric means and denote its as follows:

$$\begin{aligned} \mathbf{A} &:= A \nabla B = A \nabla_{\frac{1}{2}} B = \frac{1}{2}A + \frac{1}{2}B\,, \\ \mathbf{G} &:= A \sharp B = A \sharp_{\frac{1}{2}} B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}\,. \end{aligned}$$

Also, we know that $\mathbf{A}' = A' \nabla B'$ and $\mathbf{G}' = A' \sharp B'$.

In the following theorem, we state distance between the arithmetic mean and the geometric mean as an infinite series.

Theorem 2.1. *Assume that A and B are two positive operators in $\mathbb{B}(\mathcal{H})$ such that $\|B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\| < 1$ and $\lambda \in (0, 1)$. Then we have*

$$A \nabla_\lambda B - A \sharp_\lambda B = \sum_{k=2}^{\infty} (-1)^{k-1} \binom{1-\lambda}{k} \left(AB^{-1} - I\right)^k B\,. \tag{2.1}$$

Proof. By using the binomial series, we have

$$\begin{aligned} \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\right)^{1-\lambda} &= \left(I + \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - I\right)\right)^{1-\lambda} \\ &= I + \sum_{k=1}^{\infty} \binom{1-\lambda}{k} \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - I\right)^k\,. \end{aligned} \tag{2.2}$$

Now, by multiplying each side (2.2) by $B^{\frac{1}{2}}$, we get

$$\begin{aligned}
&B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{1-\lambda}B^{\frac{1}{2}}\\
&\qquad = B+\sum_{k=1}^{\infty}\binom{1-\lambda}{k}B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}-I\right)^{k}B^{\frac{1}{2}}\\
&\qquad = B+\binom{1-\lambda}{1}(A-B)+\sum_{k=2}^{\infty}\binom{1-\lambda}{k}B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}-I\right)^{k}B^{\frac{1}{2}}\\
&\qquad = B+(1-\lambda)(-1)(B-A)+\sum_{k=2}^{\infty}\binom{1-\lambda}{k}B^{\frac{1}{2}}\left[B^{-\frac{1}{2}}(A-B)B^{-\frac{1}{2}}\right]^{k}B^{\frac{1}{2}}\\
&\qquad = (1-\lambda)A+\lambda B-\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}\left[(A-B)B^{-1}\right]^{k}B\,,
\end{aligned}$$

so, $B \sharp_{1-\lambda} A = A \nabla_\lambda B - \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}\left(AB^{-1}-I\right)^{k}B$, which completes the proof. □

We know that, if A and B are two commuting positive operators in $\mathbb{B}(\mathcal{H})$, then AB is positive operator and $(AB)^{\frac{1}{2}} = A^{\frac{1}{2}}B^{\frac{1}{2}}$. Furthermore, if B is invertible, then $AB^{-1} = B^{-1}A$. Also, we recall that if A and B are not commuting, then AB is not necessarily positive. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are positive but their product is not [10, p. 309].

Now, by using the above statements and Theorem 2.1, the following corollary is obvious.

Corollary 2.2. *With the assumptions in Theorem* 2.1*, if A and B are commuting, then*

$$A \nabla_\lambda B - A \sharp_\lambda B = \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}B^{\frac{1-k}{2}}(B-A)^{k}B^{\frac{1-k}{2}}\,.$$

In the following theorem we state the Alzer inequality for two commuting positive operator in $\mathbb{B}(\mathcal{H})$.

Theorem 2.3 (Alzer Inequality)**.** *Suppose that $A, B \in \mathbb{B}(\mathcal{H})$ are commuting operators such that $0 < A \le B \le \frac{1}{2}I$, and let $A' := I - A$ and $B' = I - B$. If $0 < \lambda \le \frac{1}{2}$, then*

$$B' \nabla_\lambda A' - B' \sharp_\lambda A' \le A \nabla_\lambda B - A \sharp_\lambda B\,. \tag{2.3}$$

Proof. It is clear that $0 < A \le B \le \frac{1}{2}I \le B' \le A' < I$ and also $A'B' = B'A'$. By using Corollary 2.2, we obtain

$$A \nabla_\lambda B - A \sharp_\lambda B = \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}B^{\frac{1-k}{2}}(B-A)^{k}B^{\frac{1-k}{2}}\,, \tag{2.4}$$

and

$$B' \nabla_\lambda A' - B' \sharp_\lambda A' = \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}A'^{\frac{1-k}{2}}(A'-B'^{k}A'^{\frac{1-k}{2}}\,. \tag{2.5}$$

Since $A' - B' = B - A$, $B \leq A'$ and $k \geq 2$ we have $A'^{\frac{1-k}{2}}(A' - B'^k A'^{\frac{1-k}{2}} \leq B^{\frac{1-k}{2}}(B-A)^k B^{\frac{1-k}{2}}$. On the other hand, since $0 < \lambda \leq \frac{1}{2}$ and $(-1)^{k-1}\binom{\alpha}{k} > 0$ for all $0 < \alpha < 1$ and $k \geq 2$, we get $(-1)^{k-1}\binom{1-\lambda}{k}A'^{\frac{1-k}{2}}(A' - B'^k A'^{\frac{1-k}{2}} \leq (-1)^{k-1}\binom{1-\lambda}{k}B^{\frac{1-k}{2}}(B-A)^k B^{\frac{1-k}{2}}$, which completes the proof. □

Corollary 2.4. *With the above notations, we have*

$$\mathbf{A}' - \mathbf{G}' \leq \mathbf{A} - \mathbf{G}.$$

Proof. Sufficient in the Theorem 2.3 we set $\lambda = \frac{1}{2}$ and use of this fact that $A\nabla B = B\nabla A$ and $A\sharp B = B\sharp A$. □

3. Open problem

In this section, we present an extension of Alzer inequality for Hilbert space operators as an open problem. For this purpose, first, we express some fundamental properties of the geometric mean. For to see many details c.f. [3, 4, 9, 11].

The geometric mean $\mathbf{G}_2 := \mathbf{G}_2(A,B)$ of two positive operators A and B was introduced as the solution of the matrix optimization problem, [3]

$$\mathbf{G}_2(A,B) := \max\left\{X \ : \ X^* = X, \quad \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}. \tag{3.1}$$

This operator mean can be also characterized as the strong limit of the arithmetic-harmonic sequence $\{\Phi_n(A,B)\}$ defined by [5, 7]

$$\begin{cases} \Phi_0(A,B) = \frac{1}{2}A + \frac{1}{2}B\,, \\ \Phi_{n+1}(A,B) = \frac{1}{2}\Phi_n(A,B) + \frac{1}{2}A(\Phi_n(A,B))^{-1}B \quad (n \geq 0)\,. \end{cases} \tag{3.2}$$

We know that, the explicit form of $\mathbf{G}_2(A,B)$ is given by

$$\mathbf{G}_2(A,B) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}\,. \tag{3.3}$$

M. Raïssouli, F. Leazizi and M. Chergui in [11] described an extended algorithm of (3.2) involving several positive operators. The main idea of such an extension comes from the fact that the arithmetic, harmonic and geometric means of m positive real numbers $a_1, a_2, \cdots, a_m$ can be written recursively as follows

$$\mathbf{A}_m(a_1, a_2, \cdots, a_m) := \frac{1}{m}\sum_{j=1}^m a_j = \frac{1}{m}a_1 + \frac{m-1}{m}\mathbf{A}_{m-1}(a_2, \cdots, a_m)\,, \tag{3.4}$$

$$\mathbf{H}_m(a_1, a_2, \cdots, a_m) := \left(\frac{1}{m}\sum_{j=1}^m a_j^{-1}\right)^{-1} = \left(\frac{1}{m}a_1^{-1} + \frac{m-1}{m}\mathbf{H}_{m-1}(a_2, \cdots, a_m)\right)^{-1}, \tag{3.5}$$

$$\mathbf{G}_m(a_1, a_2, \cdots, a_m) := \sqrt[m]{a_1 a_2 \cdots a_m} = a_1^{\frac{1}{m}}\left(\mathbf{G}_{m-1}(a_2, \cdots, a_m)\right)^{\frac{m-1}{m}}\,. \tag{3.6}$$

The extensions of (3.4) and (3.5) when the scalers variable $a_1, a_2, \cdots, a_m$ are positive operators can be immediately given, by setting $A^{-1} = \lim_{\epsilon \downarrow 0}(A + \epsilon I)^{-1}$. We know that the power geometric mean of two positive operators A and B defined by

$$\Phi_{\frac{1}{m}}(A,B) := B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{m}}B^{\frac{1}{2}}\,. \tag{3.7}$$

Assume that $A_1, \cdots, A_m \in \mathbb{B}(\mathcal{H}) \quad (m \geq 2)$ are m positive operators. In this section we introduce the geometric mean of $A_1, \cdots, A_m$. By using the algorithm (3.2), we define the recursive sequence $\{T_n\} := \{T_n(A, B)\}$, where $A, B \in \mathbb{B}(\mathcal{H})$ are two positive operators, as follows

$$\begin{cases} T_0 = \frac{1}{m}A + \frac{m-1}{m}B\,; \\ T_{n+1} = \frac{m-1}{m}T_n + \frac{1}{m}A(T_n^{-1}B)^{m-1} \quad (n \geq 0)\,. \end{cases} \tag{3.8}$$

In what follows, for simplicity we write $\{T_n\}$ instead of $\{T_n(A, B)\}$ and we set

$$T_n^{(-1)} = \left(T_n(A^{-1}, B^{-1})\right)^{-1}\,.$$

In the following theorem Raïssouli, Leazizi and Chergui [11] proved the convergence of the operator sequence $\{T_n\}$.

Theorem 3.1. *With the above assumptions, the sequence $\{T_n\} := \{T_n(A, B)\}$ converges decreasingly in $\mathbb{B}(\mathcal{H})$, with the limit*

$$\lim_{n\uparrow+\infty} T_n := \Phi_{\frac{1}{m}}(A, B) = B^{\frac{1}{2}}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{m}} B^{\frac{1}{2}}\,. \tag{3.9}$$

Further, the next estimation holds

$$0 \leq T_n - \Phi_{\frac{1}{m}}(A, B) \leq \left(1 - \frac{1}{m}\right)^n \left(T_0 - T_0^{(-1)}\right) \quad \forall n \geq 0\,. \tag{3.10}$$

Notice that $\Phi_{\frac{1}{m}}(A, B) = A^{\frac{1}{m}}B^{1-\frac{1}{m}}$ when A and B are two commuting positive operators and so, $\Phi_{\frac{1}{m}}(A, I) = A^{\frac{1}{m}}$, $\Phi_{\frac{1}{m}}(I, B) = B^{1-\frac{1}{m}}$ for all positive operators A and B. Also, the map $(A, B) \mapsto \Phi_{\frac{1}{m}}(A, B)$ satisfies the conjugate symmetry relation, i.e.

$$\Phi_{\frac{1}{m}}(A, B) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{m-1}{m}} A^{\frac{1}{2}} = \Phi_{\frac{m-1}{m}}(B, A)\,. \tag{3.11}$$

In the same paper, we see the definition of geometric operator mean of $A_1, \cdots, A_m$ as follows.

Definition 3.2. *Assume that $A_1, \cdots, A_m \in \mathbb{B}(\mathcal{H})$ are the positive operators. The geometric operator mean of $A_1, \cdots, A_m$ is defined by the relationship*

$$\mathbf{G}_m(A_1, A_2, \cdots, A_m) = \Phi_{\frac{1}{m}}\left(A_1, \mathbf{G}_{m-1}(A_2, \cdots, A_m)\right)\,. \tag{3.12}$$

It is easy to verify that, if $A_1, \cdots, A_m$ are commuting, then

$$\mathbf{G}_m(A_1, A_2, \cdots, A_m) = (A_1, A_2 \cdots A_m)^{\frac{1}{m}}\,.$$

In particular, for all positive operators $A \in \mathbb{B}(\mathcal{H})$ we have $\mathbf{G}_m(A, A, \cdots, A) = A$ and $\mathbf{G}_m(I, I, \cdots, A, I, \cdots, I) = A^{\frac{1}{m}}$. Also, we know that $(A, B) \mapsto \mathbf{G}_2(A, B)$ is symmetric, but $\mathbf{G}_m$ is not symmetric for $m \geq 3$, for more details see [11, Example 2.3].

The geometric operator mean $\mathbf{G}_m(A_1, A_2, \cdots, A_m)$ has nice properties that for seeing more details c.f. [11].

Open Problem. Let $A_1, \cdots, A_n$ be n selfadjoint operators on an Hilbert space $\mathcal{H}$ such that $0 < A_j \leq \frac{1}{2}I$, where I is identity operator on $\mathcal{H}$ [6, 8]. Also, let $\mathbf{A}_n := \mathbf{A}_n(A_1, \cdots, A_n)$ and $\mathbf{G}_n := \mathbf{G}_n(A_1, \cdots, A_n)$ be arithmetic and geometric means of $A_1, \cdots, A_n$ [11], and $\mathbf{A}'_n := \mathbf{A}_n(A'_1, \cdots, A'_n)$ and $\mathbf{G}'_n := \mathbf{G}_n(A'_1, \cdots, A'_n)$ be arithmetic and geometric means of $A'_1, \cdots, A'_n$ where $A'_j := I - A_j \quad (j = 1, \cdots, n)$, respectively. Then it seems that

$$\mathbf{A}'_n - \mathbf{G}'_n \leq \mathbf{A}_n - \mathbf{G}_n.$$

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DIRECT RESULTS ON THE q-MIXED SUMMATION INTEGRAL TYPE OPERATORS

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ABSTRACT. In this paper, we introduce a q-mixed summation integral type operators and investigate their approximation properties. We obtain a Voronovskaja type theorem and give direct results on degree of approximation for continuous functions.

1. INTRODUCTION

Let f be a locally integrable function on the interval $[0,\infty)$. the mixed summation integral type operators are defined as

$$S_n(f;x) = (n-1)\sum_{v=1}^{\infty} s_{n,v}(x)\int_0^{\infty} b_{n,v-1}(t)f(t)dt + e^{-nx}f(0) \tag{1.1}$$

where

$$s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!} \text{ and } b_{n,v}(t) = \left(\begin{array}{c} n+v-1 \\ v \end{array}\right) t^v(1+t)^{-n-v}.$$

are respectively Szász and Baskakov basis functions. This operators were studied in [6] and in [13]. Phillips [11] firstly studied Bernstein polynomials based on the $q-$integers. Gupta and Heping [7] studied the rate of convergence of $q-$Durrmeyer type operators. Aral and Gupta [1] introduced Durrmeyer type modification of the $q-$Baskakov type operators. Recently in [5], Gupta and Aral studied convergence of the $q-$ analogue of Szász-beta operators. Our aim is to obtain direct results on $q-$ mixed summation integral type operators. Before, we give some properties of $q-$ calculus. Throughout this paper we use following the notations and the formulas, which can be founded in [4], [8], [9] and [10] and [12]: For $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$, the $q-$integer and the $q-$ factorial are defined by

$$[n]_q = (1-q^n)/(1-q), \text{ for } 0<q<1; \ [n]_q = n, \text{ for } q=1 \tag{1.2}$$

and

$$[n]_q! = [1]_q[2]_q...[n]_q, \ n \in \mathbb{N}\backslash\{0\}; \ [0]_q! = 1. \tag{1.3}$$

The $q-$binomial coefficients are given by

$$\left[\begin{array}{c} n \\ v \end{array}\right]_q = \frac{[n]_q!}{[v]_q![n-v]_q!}, 0 \le v \le n. \tag{1.4}$$

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The $q-$derivative $D_q f$ of a function is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \text{ for } x \neq 0 \tag{1.5}$$

and $(D_q f)(0) = f'(0)$ provided that $f'(0)$ exists. The two $q-$analogues of the exponential function are defined by

(1.6)

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^\infty} \text{ and } E_q^x = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^\infty$$

where

$$(1+a)_q^\infty = \prod_{j=1}^{\infty} (1+q^{j-1}a).$$

The improper $q-$Jackson integral is defined as

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n\in\mathbb{Z}} f(\frac{q^n}{A}) \frac{q^n}{A}, \ A > 0. \tag{1.7}$$

The $q-$Gamma function and the $q-$Beta function are defined as

$$\Gamma_q(u) = K(A,u) \int_0^{\infty/A(1-q)} x^{u-1} e_q^{-qx} d_q x \tag{1.8}$$

and

$$B_q(u,v) = K(A,u) \int_0^{\infty/A} \frac{x^{u-1}}{(1+x)_q^{u+v}} d_q x = \frac{\Gamma_q(u)\Gamma_q(v)}{\Gamma_q(u+v)} \tag{1.9}$$

where

$$K(A,u) = \frac{A^u}{1+A} \left(1 + \frac{1}{A}\right)_q^u (1+A)_q^{1-u} \text{ and } (a+b)_q^n = \prod_{j=1}^{n} (a + q^{j-1} b).$$

In particular, for $u \in \mathbb{Z}$, $K(A,u) = q^{u(u-1)/2}$ and $K(A,0) = 1$.

2. Generalized $q-$mixed operators

Let $p, v \in \mathbb{N}$, $n \in \mathbb{N} \backslash \{0\}$, $A > 0$ and f be a real valued continuous function defined on the interval $[0, \infty)$. Using the formulas and the notations between (1.2) and (1.9), we introduce $q-$mixed summation integral type linear positive operators for $0 < q \leq 1$ as

(2.1)

$$S_{n,p,q}(f;x) = [n+p-1]_q \sum_{v=1}^{\infty} s_{n,p,v}(r(x);q) \int_0^{\infty/A} b_{n,p,v-1}(t;q) f(t) d_q t + e_q^{-[n+p]_q r(x)} f(0)$$

where

$$s_{n,p,v}(r(x);q) := \frac{([n+p]_q r(x))^v}{[v]_q!} e_q^{-[n+p]_q r(x)}, \ r(x) := \frac{q[n+p-2]_q}{[n+p]_q} x$$

and

$$b_{n,p,v}(t;q) := \begin{bmatrix} n+p+v-1 \\ v \end{bmatrix}_q \frac{q^{(v+1)v} t^v}{(1+t)_q^{n+p+v}}.$$

If we write $q=1$, $p=0$ and put x instead of $r(x)$ in (2.1), then the operators $S_{n,p,q}$ are reduced to mixed summation integral type operators given (1.1).

Now we give an auxiliary lemma for the Korovkin test functions.

Lemma 2.1. *Let $e_m(t)=t^m$, $m=0,1,2,3,4$. we have*

$$\begin{aligned}
(i)\ S_{n,p,q}(e_0;x) &= 1,\\
(ii)\ S_{n,p,q}(e_1;x) &= x,\\
(iii)\ S_{n,p,q}(e_2;x) &= \frac{[n+p-2]_q x^2}{q^2[n+p-3]_q} + \frac{[2]_q x}{q^2[n+p-3]_q},\\
(iv)\ S_{n,p,q}(e_3;x) &= \frac{[n+p-2]_q^2 x^3}{q^6[n+p-4]_q[n+p-3]_q} + \frac{([2]_q q+[4]_q)[n+p-2]_q x^2}{q^6[n+p-4]_q[n+p-3]_q}\\
&\quad + \frac{[2]_q[3]_q x}{q^5[n+p-4]_q[n+p-3]_q},\\
(v)\ S_{n,p,q}(e_4;x) &= \frac{[n+p-2]_q^3 x^4}{q^{12}[n+p-5]_q[n+p-4]_q[n+p-3]_q}\\
&\quad + \frac{\left([2]_q q^2+[4]_q q+[6]_q\right)[n+p-2]_q^2 x^3}{q^{12}[n+p-5]_q[n+p-4]_q[n+p-3]_q}\\
&\quad + \frac{([2]_q[3]_q q^2+[2]_q[5]_q q+[4]_q[5]_q)[n+p-2]_q x^2}{q^{11}[n+p-5]_q[n+p-4]_q[n+p-3]_q}\\
&\quad + \frac{[2]_q[3]_q[4]_q x}{q^9[n+p-5]_q[n+p-4]_q[n+p-3]_q}.
\end{aligned}$$

Proof. Using (1.8) and (1.9), we can obtain the estimate,

$$\begin{aligned}
\int_0^{\infty/A} \frac{b_{n,p,v}(t)t^m}{q^{(v+1)v}} d_q t &= \begin{bmatrix} n+p+v-1 \\ v \end{bmatrix}_q \int_0^{\infty/A} \frac{t^{v+m}}{(1+t)_q^{n+p+v}} d_q t\\
&= \frac{[n+p+v-1]_q!}{[v]_q![n+p-1]_q!} \frac{B_q(v+m+1, n+p-m-1)}{K(A, v+m+1)}\\
&= \frac{[v+m]_q![n+p-m-2]_q!}{[v]_q![n+p-1]_q! q^{(v+m+1)(v+m)/2}}. \qquad (2.2)
\end{aligned}$$

From (2.2) and (1.6), we get

$$\begin{aligned}
S_{n,p,q}(e_0;x) &= \sum_{v=1}^{\infty} q^{v(v-1)/2} s_{n,p,v}(r(x);q) + e_q^{-[n+p]_q r(x)}\\
&= e_q^{-[n+p]_q r(x)} \left(\sum_{v=1}^{\infty} q^{v(v-1)/2} \frac{([n+p]_q r(x))^v}{[v]_q!} + 1 \right)\\
&= e_q^{-[n+p]_q r(x)} E_q^{[n+p]_q r(x)}\\
&= 1,
\end{aligned}$$

which completes the proof of (i). By a direct computation

$$\begin{aligned} S_{n,p,q}(e_1;x) &= \sum_{v=1}^{\infty} q^{(v^2-3v)/2} \frac{[v]_q}{[n+p-2]_q} s_{n,p,v}(r(x);q) \\ &= qx \sum_{v=1}^{\infty} q^{(v^2-3v)/2} s_{n,p,v-1}(r(x);q), \end{aligned}$$

which gives proof of *(ii)*. Using the equality $[v+1]_q = [v-1]_q + [2]_q q^{v-1}$, we can write

$$\begin{aligned} S_{n,p,q}(e_2;x) &= \frac{[n+p-2]_q (qx)^2}{[n+p-3]_q} \sum_{v=2}^{\infty} q^{(v^2-5v-2)/2} s_{n,p,v-2}(r(x);q) \\ &+ \frac{[2]_q qx}{[n+p-3]_q} \sum_{v=1}^{\infty} q^{(v^2-3v-4)/2} s_{n,p,v-1}(r(x);q), \end{aligned}$$

which gives proof of *(iii)*. Using the equality

$$[v+1]_q[v+2]_q = [v-1]_q[v-2]_q + ([2]_q q + [4]_q) q^{v-2}[v-1]_q + [2]_q[3]_q q^{2v-2},$$

we can write

$$\begin{aligned} & S_{n,p,q}(e_3;x) \\ = & \frac{[n+p-2]_q^2 (qx)^3}{[n+p-4]_q[n+p-3]_q} \sum_{v=3}^{\infty} q^{(v^2-7v-6)/2} s_{n,p,v-3}(r(x);q) \\ & + \frac{([2]_q q + [4]_q)[n+p-2]_q (qx)^2}{[n+p-4]_q[n+p-3]_q} \sum_{v=2}^{\infty} q^{(k^2-5k-10)/2} s_{n,p,v-2}(r(x);q) \\ & + \frac{[2]_q[3]_q qx}{[n+p-4]_q[n+p-3]_q} \sum_{v=1}^{\infty} q^{(v^2-3v-10)/2} s_{n,p,v-1}(r(x);q), \end{aligned}$$

which gives the proof of (iv). For the proof of *(v)*, using the equality

$$\begin{aligned} & [v+1]_q[v+2]_q[v+3]_q \\ = & [v-1]_q[v-2]_q[v-3]_q + ([2]_q q^2 + [4]_q q + [6]_q) q^{v-3}[v-1]_q[v-2]_q \\ & + ([2]_q[3]_q q^2 + [2]_q[5]_q q + [4]_q[5]_q) q^{2v-4}[v-1]_q + [2]_q[3]_q[4]_q q^{3v-3}, \end{aligned}$$

we can write

$$
\begin{aligned}
& S_{n,p,q}(e_4;x) \\
= \quad & \frac{[n+p-2]_q^3(qx)^4}{[n+p-5]_q[n+p-4]_q[n+p-3]_q}\sum_{v=4}^{\infty} q^{(v^2-9v-12)/2} s_{n,p,v-4}(r(x);q) \\
& + \frac{\left([2]_q q^2+[4]_q q+[6]_q\right)[n+p-2]_q^2(qx)^3}{[n+p-5]_q[n+p-4]_q[n+p-3]_q}\sum_{v=3}^{\infty} q^{(v^2-7v-18)/2} s_{n,p,v-3}(r(x);q) \\
& + \frac{\left([2]_q[3]_q q^2+[2]_q[5]_q q+[4]_q[5]_q\right)[n+p-2]_q(qx)^2}{[n+p-5]_q[n+p-4]_q[n+p-3]_q} \\
& \times \sum_{v=2}^{\infty} q^{(v^2-5v-20)/2} s_{n,p,v-2}(r(x);q) \\
& + \frac{[2]_q[3]_q[4]_q qx}{[n+p-5]_q[n+p-4]_q[n+p-3]_q}\sum_{v=1}^{\infty} q^{(v^2-3v-18)/2} s_{n,p,v-1}(r(x);q).
\end{aligned}
$$

Thus, we get the desired result. □

Lemma 2.2. *Let $q \in (0,1)$, $n > 3$ and $p \in \mathbb{N}$. Then we have the following inequality*

$$S_{n,p,q}((t-x)^2;x) \leq \frac{4x(x+1)}{q^2[n+p-3]_q}.$$

Proof. From linearity of $S_{n,p,q}$ operators and Lemma 2.1, we can write the second moment as

$$S_{n,p,q}((t-x)^2;x) = \left(\frac{[n+p-2]_q}{q^2[n+p-3]_q}-1\right)x^2 + \frac{[2]_q}{q^2[n+p-3]_q}x.$$

Using the equality

$$[n+p-2]_q - q^2[n+p-3]_q = 1+q-q^{n+p-2},$$

we obtain

$$S_{n,p,q}((t-x)^2;x) = \left(\frac{1+q-q^{n+p-2}}{q^2[n+p-3]_q}\right)x^2 + \frac{[2]_q}{q^2[n+p-3]_q}x \tag{2.3}$$

then we reach the result of Lemma. □

Lemma 2.3. *Let $(q_n) \subset (0,1)$ a sequence such that $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. Then, for any $p \in \mathbb{N}$, we have the following limits*

$$
\begin{aligned}
(i)\quad & \lim_{n\to\infty}[n+p]_{q_n} S_{n,p,q_n}((t-x)^2;x) &=& \quad (2-a)x^2+2x \\
(ii)\quad & \lim_{n\to\infty}[n+p]_{q_n}^2 S_{n,p,q_n}((t-x)^4;x) &=& \quad \left(3a^2-12a+12\right)x^4+(3-12a)x^3+12x^2.
\end{aligned}
$$

Proof. (i). From (2.3), we obtain desired result

$$
\begin{aligned}
& \lim_{n\to\infty}[n+p]_{q_n} S_{n,p,q_n}((t-x)^2;x) \\
= \quad & \lim_{n\to\infty}\left(\left(\frac{\left(1+q_n-q_n^{n+p-2}\right)[n+p]_{q_n}}{q_n^2[n+p-3]_{q_n}}\right)x^2 + \frac{[2]_{q_n}[n+p]_{q_n}}{q_n^2[n+p-3]_{q_n}}x\right) \\
= \quad & (2-a)x^2+2x.
\end{aligned}
$$

(ii). From Lemma 2.1, using the linearity property of the S_{n,p,q_n} operators for $n > 5$, we can write

$$S_{n,p,q_n}((t-x)^4;x) = C_1(n,p,q_n)x^4 + C_2(n,p,q_n)x^3 + C_3(n,p,q_n)x^2 + C_4(n,p,q_n)x$$

where

$$\begin{aligned} C_1(n,p,q_n) &= \frac{[n+p-2]_{q_n}^3}{q_n^{12}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}} \\ &\quad - \frac{4[n+p-2]_{q_n}^2}{q_n^6[n+p-4]_{q_n}[n+p-3]_{q_n}} + \frac{6[n+p-2]_{q_n}}{q_n^2[n+p-3]_{q_n}} - 3, \end{aligned}$$

$$\begin{aligned} C_2(n,p,q_n) &= \frac{([2]_{q_n}q_n^2 + [4]_{q_n}q_n + [6]_{q_n})[n+p-2]_{q_n}^2}{q_n^{12}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}} \\ &\quad - \frac{4([2]_{q_n}q_n + [4]_{q_n})[n+p-2]_{q_n}}{q_n^6[n+p-4]_{q_n}[n+p-3]_{q_n}} + \frac{6[2]_{q_n}}{q_n^2[n+p-3]_{q_n}}, \end{aligned}$$

$$\begin{aligned} & C_3(n,p,q_n) \\ = \; & \frac{([2]_{q_n}[3]_{q_n}q_n^2 + [2]_{q_n}[5]_{q_n}q_n + [4]_{q_n}[5]_{q_n})[n+p-2]_{q_n}}{q_n^{11}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}} - \frac{4[2]_{q_n}[3]_{q_n}}{q_n^5[n+p-4]_{q_n}[n+p-3]_{q_n}}, \end{aligned}$$

and

$$C_4(n,p,q_n) = \frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}}{q_n^9[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}}.$$

It is obvious that

$$\lim_{n\to\infty}[n+p]_{q_n}^2 C_4(n,p,q_n) = 0. \tag{2.4}$$

Using the relations $[n+p-2]_{q_n} = [3]_{q_n} + q_n^3[n+p-5]_{q_n}$, $[n+p-3]_{q_n} = [2]_{q_n} + q_n^2[n+p-5]_{q_n}$ and $[n+p-4]_{q_n} = 1 + q_n[n+p-5]_{q_n}$, we will get following limits. Firstly,

$$\begin{aligned} & \lim_{n\to\infty}[n+p]_{q_n}^2 C_1(n,p,q_n) \\ = \; & \lim_{n\to\infty}\Bigg\{ \frac{[n+p-5]_{q_n}^2(1-q_n^{n+p-1})^2(-3q_n^4+3q_n^2+2q_n+1)}{q_n^3[n+p-4]_{q_n}[n+p-3]_{q_n}} \\ & + \frac{[n+p-5]_{q_n}[n+p]_{q_n}(1-q_n^{n+p-1})(6q_n^7-3q_n^6-9q_n^5-7q_n^4+q_n^3+9q_n^2+6q_n+3)}{q_n^6[n+p-4]_{q_n}[n+p-3]_{q_n}} \\ & + \frac{[n+p]_{q_n}^2\left(-3q_n^{10}+3q_n^9+6q_n^8+2q_n^7-8q_n^6-12q_n^5-5q_n^4+2q_n^3+9q_n^2+6q_n+3\right)}{q_n^9[n+p-4]_{q_n}[n+p-3]_{q_n}} \\ & + \frac{[n+p]_{q_n}^2(1+q_n+q_n^2)^3}{q_n^{12}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}} \Bigg\} \\ = \; & 3(1-a)^2 + 6(1-a) + 3. \end{aligned} \tag{2.5}$$

Secondly,

$$\begin{aligned}
&\lim_{n\to\infty}[n+p]_{q_n}^2 C_2(n,p,q_n)\\
=\ &\lim_{n\to\infty}\left\{\frac{[n+p-5]_{q_n}[n+p]_{q_n}(1-q_n^{n+p-1})\left(-2q_n^3+3q_n^2+q_n+1\right)(q_n+1)^2}{q_n^6[n+p-4]_{q_n}[n+p-3]_{q_n}}\right.\\
&+\frac{[n+p]_{q_n}^2(6q_n^{11}+6q_n^{10}-4q_n^9-8q_n^8-8q_n^7-4q_n^6+q_n^2+q_n+1)}{q_n^{12}[n+p-4]_{q_n}[n+p-3]_{q_n}}\\
&\left.+\frac{[n+p]_{q_n}^2(1+q_n+q_n^2)^2}{q_n^{12}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}}\right\}\\
=\ &12(1-a)-9. \qquad (2.6)
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lim_{n\to\infty}[n+p]_{q_n}^2 C_3(n,p,q_n)\\
=\ &\lim_{n\to\infty}\left\{\frac{[n+p]_{q_n}^2\left(q^7-q^6-2q^5+4q^3+6q^2+3q+1\right)}{q_n^8[n+p-4]_{q_n}[n+p-3]_{q_n}}\right.\\
&\left.+\frac{q^9+4q^8+10q^7+17q^6+22q^5+22q^4+17q^3+10q^2+4q+1}{q_n^{11}[n+p-5]_{q_n}[n+p-4]_{q_n}[n+p-3]_{q_n}}\right\}\\
=\ &12. \qquad (2.7)
\end{aligned}$$

Combining the limits between (2.4) and (2.7), we reach the desired result. □

3. Voronovskaja type theorem

Now we give a Voronovskaja type theorem for the S_{n,p,q_n} operators. $B[0,\infty)$ denotes the set of all bounded functions from $[0,\infty)$ to $\mathbb{R}$. $B[0,\infty)$ is a normed space with the norm $\|f\|_B=\sup\{|f(x)|:x\in[0,\infty)\}$. $C_B[0,\infty)$ denotes the subspace of all continuous functions in $B[0,\infty)$. The weighted Korovkin- type theorems were proved by Gadzhiev in [2] and [3]. We give the Gadzhiev's results in weighted spaces. Let $\rho(x)=1+\varphi^2(x)$, $\varphi(x)$ is a monotone increasing continuous function from $[0,\infty)$ to $\mathbb{R}$. $B_\rho[0,\infty)$ denotes the set of all functions f , from $[0,\infty)$ to $\mathbb{R}$, satisfying growth condition $|f(x)|\le M_f\ \rho(x)$,where M_f is a constant depending only on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho=\sup\left\{|f(x)|(\rho(x))^{-1}:x\in\mathbb{R}\right\}$. $C_\rho[0,\infty)$ denotes the subspace of all continuous functions in $B_\rho[0,\infty)$ and $C_\rho^*[0,\infty)$ denotes the subspace of all functions $f\in C_\rho[0,\infty)$ for which $\lim\limits_{|x|\to\infty}|f(x)|(\rho(x))^{-1}$ exists finitely.

Theorem 3.1. *Let $(q_n)\subset(0,1)$ a sequence such that $q_n\to1$ and $q_n^n\to a$ as $n\to\infty$. For any $f\in C[0,\infty)$ such that $f',f''\in C[0,\infty)$ we have the limit*

$$\lim_{n\to\infty}[n+p]_{q_n}\left(S_{n,p,q_n}(f;x)-f(x)\right)=\left(\frac{2-a}{2}x^2+x\right)f''(x).$$

Proof. By Taylor's expansion of f, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac12 f''(x)(t-x)^2+\varepsilon(t,x)(t-x)^2$$

where $\varepsilon(t,x)\to0$ as $t\to x$. Then, from Lemma 2.1, we obtain

$$S_{n,p,q_n}(f;x) = f(x) + \frac{1}{2}f''(x)S_{n,p,q_n}((t-x)^2;x) + S_{n,p,q_n}(\varepsilon(t,x)((t-x)^2;x).$$

For third term on the right side, using Cauchy-Schwarz inequality we write

$$S_{n,p,q_n}(\varepsilon(t,x)((t-x)^2;x) \leq \sqrt{S_{n,p,q_n}(\varepsilon^2(t,x);x)}\sqrt{S_{n,p,q_n}((t-x)^4;x)}.$$

Then

$$\begin{aligned} &\lim_{n\to\infty}[n+p]_{q_n}S_{n,p,q_n}(\varepsilon(t,x)((t-x)^2;x) \\ \leq\ &\sqrt{\lim_{n\to\infty}S_{n,p,q_n}(\varepsilon^2(t,x);x)}\sqrt{\lim_{n\to\infty}[n+p]_{q_n}^2S_{n,p,q_n}((t-x)^4;x)}. \end{aligned}$$

From Lemma 2.3 (ii), $\lim_{n\to\infty}[n+p]_{q_n}^2S_{n,p,q_n}((t-x)^4;x)$ is finite. Since $\lim_{n\to\infty}S_{n,p,q_n}(\varepsilon^2(t,x),x) = 0$, we have

$$\lim_{n\to\infty}[n+p]_{q_n}S_{n,p,q_n}(\varepsilon(t,x)((t-x)^2;x) = 0.$$

Thus, we obtain

$$\lim_{n\to\infty}[n+p]_{q_n}\left(S_{n,p,q_n}(f;x) - f(x)\right) = \frac{1}{2}f''(x)\lim_{n\to\infty}[n+p]_{q_n}S_{n,p,q_n}((t-x)^2;x).$$

Considering Lemma 2.3 (i), we get the desired result. □

4. Direct Results

In this section, we denote first modulus of continuity on finite interval $[0,b]$, $b>0$

$$\omega_{[0,b]}(f;\delta) = \sup_{0<h\leq\delta, x\in[0,b]}\left|f(x+h) - f(x)\right|. \tag{4.1}$$

The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf\left\{\|f-g\|_B + \delta\|g''\|_B : g\in W_\infty^2\right\},\ \ \delta>0$$

where $W_\infty^2 = \{g\in C_B[0,\infty) : g', g''\in C_B[0,\infty)\}$. By , p. 177, Theorem 2.4 in [14], there exists a positive constant M such that

$$K_2(f;\delta) \leq M\omega_2(f,\sqrt{\delta}) \tag{4.2}$$

where

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}\left|f(x+2h) - 2f(x+h) - f(x)\right|.$$

Theorem 4.1 ([2] and [3]). *(a) There exists a sequence of linear positive operators $L_n : C_\rho[0,\infty)\to B_\rho[0,\infty)$ such that*

$$\lim_{n\to\infty}\|L_n(\varphi^\upsilon) - \varphi^\nu\|_\rho = 0,\ \nu = 0,1,2, \tag{4.3}$$

and there exists a function $f^*\in C_\rho[0,\infty)\backslash C_\rho^*[0,\infty)$ with

$$\lim_{n\to\infty}\|L_n(f^*) - f^*\|_\rho \geq 1.$$

(b) If a sequence of linear positive operators $L_n : C_\rho[0,\infty)\to B_\rho[0,\infty)$ satisfies conditions (4.3), then

$$\lim_{n\to\infty}\|L_n(f) - f\|_\rho = 0,$$

for every $f\in C_\rho^*[0,\infty)$.

Throughout this paper we take growth condition as $\rho(x) = 1+x^2$.

Lemma 4.2. *Let $q \in (0,1)$, $n > 3$ and $p \in \mathbb{N}$. Then, for every $x \in [0,\infty)$ and $f'' \in C_B[0,\infty)$ we have the inequality*

$$|S_{n,p,q}(f;x) - f(x)| \leq \frac{2\|f''\|_B}{q^2[n+p-3]_q} x(x+1).$$

Proof. Using Taylor's expansion

$$f(t) = f(x) + (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

and from Lemma 2.1, we have

$$S_{n,p,q}(f;x) = S_{n,p,q}\left(\int_x^t (t-u)f''(u)du; x\right).$$

Then, using the inequality

$$\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_B \frac{(t-x)^2}{2}$$

we get

$$|S_{n,p,q}(f;x) - f(x)| \leq \|f''\| S_{n,p,q}\left(\frac{(t-x)^2}{2}; x\right) \leq \frac{2\|f''\|_B}{q^2[n+p-3]_q} x(x+1).$$

□

Theorem 4.3. *Let $(q_n) \subset (0,1)$ an sequence such that $q_n \to 1$ as $n \to \infty$. Then for every $n > 3$, $p \in \mathbb{N}$, $x \in [0,\infty)$ and $f \in C_B[0,\infty)$ we have the inequality*

$$|S_{n,p,q_n}(f;x) - f(x)| \leq 2M\omega_2\left(f; \sqrt{\frac{x(x+1)}{q_n^2[n+p-3]_{q_n}}}\right).$$

Proof. For any $g \in W_\infty^2$, we can write

$$|S_{n,p,q_n}(f;x) - f(x)| \leq |S_{n,p,q_n}(f-g,x) - (f-g)(x)| + |S_{n,p,q_n}(g,x) - g(x)|.$$

Then, from Lemma 4.2, we have

$$|S_{n,p,q_n}(f;x) - f(x)| \leq 2\,||f-g||_B + \frac{2x(x+1)}{q_n^2[n+p-3]_{q_n}} \|g''\|_B.$$

Now taking infimum over $g \in W_\infty^2$ on the right side of the above inequality and using the inequality (4.2), we get the desired result. □

Theorem 4.4. *Let $(q_n) \subset (0,1)$ an sequence such that $q_n \to 1$ as $n \to \infty$. Then, for every $p \in \mathbb{N}$ and $f \in C_\rho^*[0,\infty)$, we have*

$$\lim_{x\to\infty} \|S_{n,p,q_n}(f,x) - f(x)\|_\rho = 0.$$

Proof. From Lemma 1.1, it is obvious that $\|S_{n,p,q_n}(e_0,x)-1\|_\rho = 0$ and $\|S_{n,p,q_n}(e_1,x)-x\|_\rho = 0$. For every $n>3$ we write

$$\begin{aligned}
\|S_{n,p,q_n}(e_2;x)-x^2\|_\rho &= \sup_{x\in[0,\infty)} \frac{\left|\dfrac{[n+p-2]_{q_n}}{q_n^2[n+p-3]_{q_n}}x^2+\dfrac{[2]_{q_n}}{q_n^2[n+p-3]_{q_n}}x-x^2\right|}{1+x^2} \\
&\le \frac{4}{q_n^2[n+p-3]_{q_n}} \sup_{x\in[0,\infty)} \frac{x(x+1)}{1+x^2} \\
&= o(1).
\end{aligned}$$

Thus, from Theorem 4.1, we obtain desired result of Theorem. □

Theorem 4.5. *Let $f\in C_\rho[0,\infty), (q_n)\subset(0,1)$ a sequence such that $q_n\to 1$ as $n\to\infty$ and $\omega_{[0,b+1]}(f,\delta)$ be its modulus of continuity on the finite interval $[0,b+1]$, $b>0$. Then for every $n>3$ and $p\in\mathbb{N}$, there exists a constant $M>0$ such that the inequality holds*

$$\|S_{n,p,q_n}(f;x)-f(x)\|_{C[0,b]} \le M\left(\frac{b(1+b)^3}{q_n^2[n+p-3]_{q_n}}+\omega_{[0,b+1]}\left(f;\sqrt{\frac{4b(1+b)}{q_n^2[n+p-3]_{q_n}}}\right)\right).$$

Proof. Let $x\in[0,b]$ and $t>b+1$. Since $t-x>1$, we have

$$\begin{aligned}
|f(t)-f(x)| &\le M_f(2+(t-x+x)^2+x^2) \\
&\le 3M_f(1+b)^2(t-x)^2. \qquad (4.4)
\end{aligned}$$

Let $x\in[0,b]$, $t<b+1$ and $\delta>0$.Then, from (4.1), we have

$$|f(t)-f(x)| \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta). \qquad (4.5)$$

Due to(4.4) and (4.5), we can write

$$|f(t)-f(x)| \le 3M_f(1+b)^2(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).$$

Then, using Cauchy- Schwarz' s inequality and Lemma 2. 2, we get

$$\begin{aligned}
&|S_{n,p,q_n}(f;x)-f(x)| \\
\le\ & 3M_f(1+b)^2 S_{n,p,q_n}\left((t-x)^2;x\right)+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(S_{n,p,q_n}\left((t-x)^2;x\right)\right)^{1/2}\right] \\
\le\ & 12M_f(1+b)^2\frac{x(x+1)}{q_n^2[n+p-3]_{q_n}}+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(\frac{4x(x+1)}{q_n^2[n+p-3]_{q_n}}\right)^{1/2}\right].
\end{aligned}$$

Choosing,

$$\delta^2 := \frac{4b(1+b)}{q_n^2[n+p-3]_{q_n}}$$

and $M=\min\{12M_f,2\}$. We reach the proof of Theorem. □

Corollary 4.6. *Let $\alpha>0$, $(q_n)\subset(0,1)$ sequence such that $q_n\to 1$ as $n\to\infty$ and $f\in C_\rho^*[0,\infty)$. Then, we have*

$$\lim_{n\to\infty}\sup_{x\ge 0}\frac{|S_{n,p,q_n}(f;x)-f(x)|}{1+x^{2+\alpha}}=0.$$

Proof. For $\alpha > 0$, $f \in C^*_\rho[0,\infty)$ and $x_0 > 0$, Considering the inequality

$$\begin{aligned} & \sup_{x\geq 0} \frac{|S_{n,p,q_n}(f;x)-f(x)|}{1+x^{2+\alpha}} \\ \leq\ & \|S_{n,p,q_n}(f;x)-f(x)\|_{C[0,x_0]} + \sup_{x\geq x_0} \frac{|S_{n,p,q_n}(f;x)|}{1+x^{2+\alpha}} + \sup_{x\geq x_0} \frac{|f(x)|}{1+x^{2+\alpha}}, \end{aligned}$$

from Theorem 4.5 we get the desired result. $\square$

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NEW APPROACH FOR MULTIDIMENSIONAL SCALING WITH CATEGORICAL DATA

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ABSTRACT. Multidimensional scaling is the problem of representing n objects geometrically by n points, so that the interpoint distances correspond in some sense to experimental dissimilarities between objects. In this paper we consider a parametric family of multivariate multinomial distributions. We observe realizations w of W with

$$w = (h_{11}, ..., h_{k1}, h_{12}, ..., h_{kL}).$$

Here all frequencies h_{il} are nonnegative, $(h_{1l}, ..., h_{kl})$ is a realization of W_l with

$$\sum_{i=1}^{k} h_{il} = \tilde{n}_l,\ P(h_{1l}, ..., h_{kl}) = \frac{\tilde{n}_l}{h_{1l}! \cdot ... \cdot h_{kl}!} p_1(\mu, t_l)^{h_{1l}} \cdot ... \cdot p_k(\mu, t_l)^{h_{kl}}.$$

A categorical data is considered. We formulate a problem and find a scaling for these data. Using a stress function to fit our results we find a good configuration for the data.

1. INTRODUCTION

The traditional methods scaling need knowledge of the dimensions of the area being investigated [8]. The central motivating concepts of MDS is that the distances between the points representing the stimuli of interest should correspond in some sensible way to the observed proximities. With this in mind various authors have approached the problem by defining an objective function which measures the discrepancy between the observed proximities and the fitted distances [3]. In many situations, however, tables of counts resulting from the cross-classification of more than two categorical variables are of interest.

The analysis of three-dimensional tables poses entirely new conceptual problems as compared with the analysis of those of two dimensions. However, the extension from tables of three dimensions to those of four or more, whilst often increasing the complexity of both analysis and interpretation. Much work has been done on the analysis of multidimensional contingency tables [1]. Often data sets contain categorical data, e.g., levels of factors or names. There does not exist any ordering or any distance between these categories. At each level there are measured some metric or categorical values. We introduce a new method of scaling based on statistical decisions. For this we define empirical probabilities for the original observations and find a class of distributions in a metric space where these empirical probabilities can be found as approximations for equivalently defined probabilities. With this method we identify probabilities connected with the categorical data with probabilities in metric spaces. Here we get a mapping from the levels of factors or names into

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points of a metric space. This mapping yields the scale for the categorical data [6]. We use a stress function to compare the distances between the given data in any dimension and the results in R.

2. Measure of Similarity and Dissimilarity

Measures of similarity are often called similarity coefficients, and are some times, although not necessary, defined to lie in the range [0,1]. Often the measures of (dis)similarity are not observed directly but are obtained from a given $(n \times p)$ data matrix. Given observations on p variables for each of n individuals or objects, there are many ways of constructing an $(n \times n)$ matrix showing the similarity or dissimilarity of each pair of individuals. perhaps the most familiar measure of dissimilarity is Euclidean distance d_{rs}, such that [7]:

$$d_{rs} = \{\sum_{j=1}^{p}(x_{rj} - x_{sj})^2\}^{1/2} \tag{2.1}$$

3. Stress Function

We denote the dissimilarity between objects i and j by δ_{ij}, $1 \leq i, j \leq n$ and suppose that $\delta_{ij} = \delta_{ji}$ for all i, j.Representing points in R^k are collected in $n \times k$ matrix $X = (x_1, ..., x_n)' \in R^{n\times k}$, called a configuration matrix in what follows. $d_{ij}(X)$ denotes the distance between x_i and x_j w.r.t. the usual Euclidean distance in R^k. Fitting distances by least squares means minimizing stress, i.e.

$$f(X) = \sum_{1\leq i \leq j \leq n} (\delta_{ij} - d_{ij}(X))^2 \tag{3.1}$$

over all configurations $X \in R^{n\times k}$[5]. We observe realizations w of W with

$$w = (h_{11}, ..., h_{k1}, h_{12}, ..., h_{kL}). \tag{3.2}$$

Here all frequencies h_{il} are nonnegative, $(h_{1l}, ..., h_{kl})$ is a realization of W_l with

$$\sum_{i=1}^{k} h_{il} = \tilde{n}_l, \quad P(h_{1l}, ..., h_{kl}) = \frac{\tilde{n}_l}{h_{1l}! \cdot ... \cdot h_{kl}!} p_1(\mu, t_l)^{h_{1l}} \cdot ... \cdot p_k(\mu, t_l)^{h_{kl}}. \tag{3.3}$$

Such observation w can be represented as in Table (3.1).

Frequencies	h_{11}	h_{12}	h_{13}	$\cdots$	h_{1L}
	h_{21}	h_{22}	h_{23}	$\cdots$	h_{2L}
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	h_{k1}	h_{k2}	h_{k3}	$\cdots$	h_{kL}
Marginal sums	$h_{+1} = \tilde{n}_1$	$h_{+2} = \tilde{n}_2$	$h_{+3} = \tilde{n}_3$	$\cdots$	$h_{+L} = \tilde{n}_L$

Table 1. Structure of observations

The parameter μ is a common parameter for all variables $W_1, ..., W_L$ and t_l is a parameter only for W_l.

4. Most Separating Scales

Ahrens and Läuter [2] introduced a method for scaling which bases on a test statistic. This will be generalized for higher dimensional q-way classification tables. This was considered by Läuter [4] too. We will define scales for the factors on the basis of tests. This differs from the approach in the preceding chapter, but it is well motivated too. At first we denote the levels of the q factors in an arbitrary way by real numbers. The factor i has ν_i levels. Then we put τ_{ij} for the level j of the factor i, all levels are described by

$$\tau = (\tau_{11}, ..., \tau_{1\nu_1}, ..., \tau_{q\nu_q})^t$$

and altogether we have $\nu = \sum_i \nu_i$ levels.

Scale points are to be constructed on the basis of the observations. The observations are those which are given by the categories and the frequencies. In our understanding the categories are identified with points $t_1, ..., t_L \in \mathbb{R}^p$ and these points are to be determined in an optimal way. As in the preceding chapter a model can be formulated in spaces $\mathbb{R}^p$ for $1 \le p \le q$ depending on the specific background. The observations express the correspondence to some classes, denoted by $\{y_{11}, ..., y_{kn_k}\}$. Explicitly we have the observations

$$\{y_{11}, ..., y_{1n_1}\} = \{h_{11} \,\text{times}\, t_1, h_{12} \,\text{times}\, t_2, ..., h_{1L} \,\text{times}\, t_L\},$$

hence we have $n_1 = h_{1+}$. Or we write

$$y_{1j} = t_1, j = 1, ..., h_{11}; y_{1j} = t_2, j = h_{11}+1, ..., h_{11}+h_{12}; ...; y_{1j} = t_L, j = h_{1+}-h_{1L}, ..., h_{1+}.$$

In an analogous way we have for the other classes $i = 1, ..., k$

$$y_{ij} = t_1, j = 1, ..., h_{i1}; y_{ij} = t_2, j = h_{i1}+1, ..., h_{i1}+h_{i2}; ...; y_{ij} = t_L, j = h_{i+}-h_{iL}, ..., h_{i+}.$$

It holds $n_i = h_{i+}$. For statistical decisions one needs assumptions on the distributions. Depending on the meaning of the observations we can choose the distributions. Quite often binomial, normal or Poisson distributions are useful, but especially in reliability or survival analysis exponential or Weibull distributions are to be chosen. Now we derive the criterion for choosing the values τ_{ij}.

Assuming that we are given k distributions $P_{\vartheta_1}, ..., P_{\vartheta_k}$ and for each distribution P_{ϑ_i} with a density f_{ϑ_i} we have a random sample $Y_{i1}, ..., Y_{in_i}$. All random variables should be independent. For testing

$$H : P_{\theta_1} = ... = P_{\theta_k} \tag{4.1}$$

against K, that not all distributions are the same, we use the likelihood ratio test. The joint density for $Y = (Y_{11}, ..., Y_{kn_k})$ is denoted by $f_{\vartheta_1,...,\vartheta_k}$. As usually the LRT is given by

$$\varphi(y) = 1 \quad \text{if} \quad R_n(y) := \frac{\max_{\vartheta_1,...,\vartheta_k} f_{\vartheta_1,...,\vartheta_k}(y)}{\max_\vartheta f_{\vartheta,...,\vartheta}(y)} \ge c,$$

where c ensures the significance level. The aim is to find such a scale that the distributions or here classes can be discriminated as well as possible. Therefore we have to determine such a vector τ^* that maximizes the corresponding test statistic. Or we use an appropriate test statistic from an admissible test for H against K.

Definition 4.1. *If R denotes the test statistic where large values of R lead to the rejection of the hypothesis then τ^* with*

$$R(\tau^*) = \max_{\tau} R(\tau) \tag{4.2}$$

is called a most separating scale.

5. Model of Normal Distributions

We assume that

$$\begin{matrix} Y_{11} & \dots & Y_{1n_1} \\ \vdots & \ddots & \vdots \\ Y_{k1} & \dots & Y_{kn_k} \end{matrix}$$

are independent and normally distributed p-dimensional random variables, $Y_{ij} \sim N_p(\mu_i, \Sigma)$. Then we consider the test problem

$$H: \quad \mu_1 = \dots = \mu_k \quad \text{against} \quad K: \quad \text{not}\, H. \tag{5.1}$$

We denote the sample mean for the ith distribution by $y_{i\cdot}, \quad i = 1, ..., k$, the total mean by

$$y_{\cdot\cdot} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = \frac{1}{n}\sum_{j=1}^{k} n_j y_{j\cdot} \quad .$$

The unbiased estimator for the variance is

$$S = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - Y_{i\cdot})(Y_{ij} - Y_{i\cdot})^t.$$

Then

$$T_0^2(Y) = \frac{n-k-p+1}{(k-1)(n-k)p} \sum_{i=1}^{k} n_i (Y_{i\cdot} - Y_{\cdot\cdot})^t S^{-1} (Y_{i\cdot} - Y_{\cdot\cdot})$$

is approximately F-distributed. H. Ahrens and J. Läuter[2]proposed the approximation $T_0^2(Y) \approx F_{g_1,g_2}$ for

$$g_1 = \begin{cases} \frac{(k-1)(n-k-p)p}{n-(k-1)p-2} & \text{if} \quad n-(k-1)p-2 > 0 \\ \infty & \text{otherwise,} \end{cases}$$

$$g_2 = n - k - p + 1.$$

Then an admissible test is given by

$$\varphi(y) = \begin{cases} 1 & \text{if} \quad T_0^2(y) > F_{g_1,g_2;\alpha} \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

for the α-fractile of the F_{g_1,g_2}-distribution. Especially the normal model will be considered later. For testing H against K we use T_0^2 and therefore we use T_0^2 for determination of most separating scales

In section 4 the categories were identified by $t_1, ..., t_L$ and we defined the y_{ij}. For any t_l we find a $p \times \nu$ matrix C_l with $t_l = C_l \tau$. Every y_{ij} is one of the values $C_1\tau, ..., C_L\tau$. We assume

$$Y_{ij} \sim N_p(\mu_i, \Sigma), \quad i = 1, ..., k; j = 1, ..., n_i$$

We use

$$h_{t\cdot} = \frac{1}{L}\sum_{l=1}^{L} h_{tl}, \quad h_{\cdot l} = \frac{1}{k}\sum_{t=1}^{k} h_{tl}, \quad h_{\cdot\cdot} = \frac{1}{kL}\sum_{t=1}^{k}\sum_{l=1}^{L} h_{tl},$$

$$h_{t\cdot} \cdot L = \sum_{l=1}^{L} h_{tl} = n_t, \quad h_{\cdot\cdot} \cdot kL = n.$$

Then we calculate

$$y_{t\cdot} = \frac{1}{n_t}\sum_{s=1}^{n_t} y_{ts} = \frac{1}{n_t}\Big(h_{t1}C_1 + ... + h_{tL}C_L\Big)\tau, \quad y_{\cdot\cdot} = \frac{k}{n}\Big(h_{\cdot 1}C_1 + ... + h_{\cdot L}C_L\Big)\tau$$

$$y_{t\cdot} - y_{\cdot\cdot} = \Big((\frac{h_{t1}}{n_t} - \frac{kh_{\cdot 1}}{n})C_1 + ... + (\frac{h_{tL}}{n_t} - \frac{kh_{\cdot L}}{n})C_L\Big)\,\tau =: D_t\,\tau.$$

The test φ in (5.2) is an admissible test for H against K from (5.1) and so we can use T_0^2 for finding most separating scales. For calculating this statistic we use

$$H := \sum_{i=1}^{k} n_i\Big(y_{i\cdot} - y_{\cdot\cdot}\Big)\Big(y_{i\cdot} - y_{\cdot\cdot}\Big)^t = \sum_{i=1}^{k} n_i D_i\,\tau\,\tau^t D_i^t,$$

$$S := \frac{1}{n-k}\sum_{i=1}^{k}\sum_{s=1}^{n_i}\Big(y_{is} - y_{i\cdot}\Big)\Big(y_{is} - y_{i\cdot}\Big)^t = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{l=1}^{L} h_{il}F_{il}\,\tau\,\tau^t F_{il}^t$$

for

$$F_{il} = C_l - \frac{1}{n_i}\Big(h_{i1}C_1 + ... + h_{iL}C_L\Big)$$

and

$$T_0^2 = \frac{n-k-p+1}{(k-1)(n-k)p}\ \sum_{i=1}^{k} n_i(y_{i\cdot} - y_{\cdot\cdot})^t S^{-1}(y_{i\cdot} - y_{\cdot\cdot}) =$$

$$= \frac{n-k-p+1}{(k-1)(n-k)p}\ tr\Big(HS^{-1}\Big),$$

$$tr\Big(HS^{-1}\Big) \ = \ \tau^t\Big[\sum_{i=1}^{k} n_i D_i^t S^{-1} D_i\Big]\tau$$

so

$$T_0^2 = \frac{n-k-p+1}{(k-1)(n-k)p}\tau^t\Big[\sum_{i=1}^{k} n_i D_i^t S^{-1} D_i\Big]\tau. \tag{5.3}$$

with

$$S = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{l=1}^{L} h_{il}F_{il}\,\tau\,\tau^t F_{il}^t.$$

For a good decision in the analysis of variance it is necessary that the observed value of the test statistic is large. Then it is natural to look for such τ-values which maximize T_0^2.

The calculation of these τ^* is rather difficult. One has to use numerical methods. In special cases explicit solutions are given.

6. Calculation of Most Separating Scales

In general one has to use some optimization software for finding a maximal τ^*. We will consider in some detail the special case of normal distributions. In section 6.3 we considered the statistic T_0^2 is the statistic to be maximized. Up to a factor this coincides with

$$tr(HS^{-1}) = \tau^t \Big[\sum_{i=1}^{k} n_i D_i^t S^{-1} D_i\Big]\tau \tag{6.1}$$

with

$$S = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{l=1}^{L} h_{il} F_{il}\,\tau\,\tau^t F_{il}^t.$$

Now we consider q-way classification models and $p \leq q$. Then we have the $p \times \nu$ matrices C_l, D_i, F_{il} and with $H_\tau := H$, $S_\tau := S$ we have

$$tr(HS^{-1}) = tr(H_\tau S_\tau^{-1}) = \tau^t \Big[\sum_{i=1}^{k} n_i D_i^t S_\tau^{-1} D_i\Big]\tau \tag{6.2}$$

for

$$S_\tau = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{l=1}^{L} h_{il} F_{il}\tau\tau^t F_{il}^t. \tag{6.3}$$

Define

$$\psi(\tau, a) := a^t \Big[\sum_{i=1}^{k} n_i D_i^t S_\tau^{-1} D_i\Big] a \tag{6.4}$$

and then τ^* fulfills

$$\psi(\tau^*, \tau^*) = \max_\tau \psi(\tau, \tau). \tag{6.5}$$

We see that ψ does not change if τ is substituted by $\lambda\tau$ for any real λ.

Definition 6.1. *$\widetilde{\tau}$ is called a local extremum if*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi\Big((1-\lambda)\widetilde{\tau} + \lambda v, (1-\lambda)\widetilde{\tau} + \lambda v\Big)|_{\lambda=0} \leq 0 \quad \forall v \in \mathbb{R}^p.$$

We are interested in characterizing such a local extremum. This gives us the next theorem.

Theorem 6.2. *$\widetilde{\tau}$ is a local extremum if and only if $\alpha(\widetilde{\tau}) = 0$ with*

$$\alpha(\tau) := \sum_{i=1}^{k} n_i D_i^t S_\tau^{-1} D_i \tau - \frac{1}{n-k}\sum_{i=1}^{k} n_i \sum_{j=1}^{k}\sum_{l=1}^{L} h_{jl} F_{jl}^t S_\tau^{-1} D_i \tau\tau^t F_{jl}^t S_\tau^{-1} D_i \tau.$$

Proof. We put $\tau_\lambda = (1-\lambda)\widetilde{\tau} + \lambda v$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda = v - \tau_\lambda, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda\tau_\lambda^t|_{\lambda=0} = (v - \widetilde{\tau})\widetilde{\tau}^t + \widetilde{\tau}(v - \widetilde{\tau})^t,$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S_{\tau_\lambda}^{-1} = -S_{\tau_\lambda}^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\lambda} S_{\tau_\lambda}\Big) S_{\tau_\lambda}^{-1}$$

and consequently

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} S^{-1}_{\tau_\lambda}|_{\lambda=0} = -\frac{1}{n-k} S^{-1}_{\widetilde{\tau}} \sum_{j=1}^{k} \sum_{l=1}^{L} h_{jl} F_{jl} (v\widetilde{\tau}^t + \widetilde{\tau} v^t - 2\widetilde{\tau}\widetilde{\tau}^t) F^t_{jl} S^{-1}_{\widetilde{\tau}}.$$

Now we calculate in a direct way

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \psi(\tau_\lambda, \tau_\lambda)|_{\lambda=0} = 2v^t \alpha(\widetilde{\tau})$$

and so the theorem is proven. □

This theorem gives us a proposal for the calculation of a local extremum.
Step 1: Find dissimilarity matrix $d_{ij}(X)$ for X, where(X is given). Choose an initial point τ_0 then find $\delta_{ij}(\tau_0)$. If the stress function $f(X) \leq$ a tolerance STOP. Else go to step 2.
Step 2: Set $w := \frac{1}{|\alpha(\tau_0)|}\alpha(\tau_0)$ and $\widetilde{\tau}_\lambda = (1-\lambda)\tau_0 + \lambda w$ for euclidian norm $|\alpha(\tau_0)|$ of $\alpha(\tau_0)$.
Step 3: Determine such λ_1 that

$$\psi(\widetilde{\tau}_{\lambda_1}, \widetilde{\tau}_{\lambda_1}) = \max_{\lambda} \psi(\widetilde{\tau}_\lambda, \widetilde{\tau}_\lambda).$$

Step 4: Set $\tau_1 := \widetilde{\tau}_{\lambda_1}$ and calculate $\alpha(\tau_1)$. Check $f(X)$. In this way we get a sequence of q-vectors $\tau_0, \tau_1, \tau_2, ...$ and have

$$\psi(\tau_0, \tau_0) \leq \psi(\tau_1, \tau_1) \leq \psi(\tau_2, \tau_2) \leq ...$$

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Basic Fractional Integral Inequalities

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Abstract

Here we present basic Lp fractional integral inequalities for left and right Riemann-Liouville, generalized Riemann-Liouville, Hadamard, Erdelyi-Kober and multivariate Riemann-Liouville fractional integrals. Then we derive basic Lp fractional inequalities regarding the left Riemann-Liouville, the left and right Caputo and the left and right Canavati type fractional derivatives.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.

Key words and phrases: fractional integral, fractional derivative, Hardy type inequality, fractional inequality.

1 Introduction

We start with some facts about fractional integrals needed in the sequel, for more details see, for instance [1], [11].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann-Liouville fractional integrals, see [14]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (x > a), \tag{1}$$

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}f(t)(t-x)^{\alpha-1}dt,\quad (x<b), \tag{2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$, see also [16]. The first result yields that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1\leq p\leq\infty$, that is

$$\left\|I_{a+}^{\alpha}f\right\|_p\leq K\|f\|_p \quad , \quad \left\|I_{b-}^{\alpha}f\right\|_p\leq K\|f\|_p, \tag{3}$$

where

$$K=\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{4}$$

Inequality (3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [12].

In this article we prove basic Hardy type fractional integral inequalities and we are motivated by [12], [13], [6],[5].

2 Main Results

We present our first result.

Theorem 1. Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1;\alpha_i>0,i=1,...,m$. Let $f_i:(a,b)\to\mathbb{R}$,be Lebesgue measurable functions so that $\|f_i\|_q$ is finite, $i=1,...,m$. Then

$$\left\|\prod_{i=1}^{m}\left(I_{a+}^{\alpha_i}f_i\right)\right\|_p\leq\frac{(b-a)^{\sum\limits_{i=1}^{m}\alpha_i+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left[\left(p\sum\limits_{i=1}^{m}\alpha_i+m(1-p)+1\right)^{\frac{1}{p}}\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_q\right). \tag{5}$$

Proof. By (1) we have

$$\left(I_{a+}^{\alpha_i}f_i\right)(x)=\frac{1}{\Gamma(\alpha_i)}\int_{a}^{x}(x-t)^{\alpha_i-1}f_i(t)\,dt, \tag{6}$$

$x>a,i=1,...,m$.

We have that

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$$\left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right| \leq \frac{1}{\Gamma(\alpha_i)} \int_a^x (x-t)^{\alpha_i - 1} \left|f_i(t)\right| dt, \tag{7}$$

$x > a, i = 1, ..., m.$

By Hölder's inequality we get

$$\left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right| \leq \frac{1}{\Gamma(\alpha_i)} \left(\int_a^x (x-t)^{p(\alpha_i - 1)} dt\right)^{\frac{1}{p}} \left(\int_a^x \left|f_i(t)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \frac{(x-a)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b \left|f_i(t)\right|^q dt\right)^{\frac{1}{q}}, \tag{8}$$

$x > a, i = 1, ..., m.$

Therefore

$$\prod_{i=1}^m \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^p \leq \frac{1}{\left(\prod\limits_{i=1}^m \Gamma(\alpha_i)\right)^p} \frac{(x-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p)}}{\prod\limits_{i=1}^m (p(\alpha_i - 1) + 1)} \left(\prod_{i=1}^m \int_a^b \left|f_i(t)\right|^q dt\right)^{\frac{p}{q}}, \tag{9}$$

$x \in (a, b).$

Consequently we get

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^p\right) dx \leq \left(\frac{1}{\prod\limits_{i=1}^m \left(\Gamma(\alpha_i)^p \left(p(\alpha_i - 1) + 1\right)\right)}\right)$$

$$\cdot \left(\int_a^b (x-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p)} dx\right) \left(\prod_{i=1}^m \int_a^b \left|f_i(t)\right|^q dt\right)^{\frac{p}{q}} \tag{10}$$

$$= \frac{(b-a)^{p \sum\limits_{i=1}^m \alpha_i + m(1-p) + 1} \left(\prod\limits_{i=1}^m \int_a^b \left|f_i(t)\right|^q dt\right)^{\frac{p}{q}}}{\left[\left(p \sum\limits_{i=1}^m \alpha_i + m(1-p) + 1\right) \left(\prod\limits_{i=1}^m \left(\Gamma(\alpha_i)^p \left(p(\alpha_i - 1) + 1\right)\right)\right)\right]}, \tag{11}$$

proving the claim. ■

We give also the following general variant in

Theorem 2. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1, r > 0; \alpha_i > 0, i = 1, ..., m.$ Let $f_i : (a, b) \rightarrow \mathbb{R}$,be Lebesgue measurable functions so that $\|f_i\|_q$ is finite, $i = 1, ..., m$.

Then

$$\left\|\prod_{i=1}^{m}\left(I_{a+}^{\alpha_i}f_i\right)\right\|_r \leq \frac{(b-a)^{\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}+\frac{1}{r}}}{\left[\left(r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_q\right). \tag{12}$$

Proof. Using $r>0$ and (8) we get

$$\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r \leq \frac{1}{\Gamma(\alpha_i)^r}\frac{(x-a)^{r\left((\alpha_i-1)+\frac{1}{p}\right)}}{(p(\alpha_i-1)+1)^{\frac{r}{p}}}\left(\int_a^b|f_i(t)|^q\,dt\right)^{\frac{r}{q}}, \tag{13}$$

and

$$\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r \leq \frac{1}{\prod\limits_{i=1}^{m}\Gamma(\alpha_i)^r}\frac{(x-a)^{r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)}}{\left(\prod\limits_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}}\left(\prod_{i=1}^{m}\left(\int_a^b|f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r. \tag{14}$$

Consequently

$$\int_a^b\left(\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)(x)\right|^r\right)dx \leq \frac{\left(\int_a^b(x-a)^{r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)}dx\right)}{\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)^r\right)\left(\prod\limits_{i=1}^{m}(p(\alpha_i-1)+1)\right)^{\frac{r}{p}}}$$

$$\cdot\left(\prod_{i=1}^{m}\left(\int_a^b|f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r \tag{15}$$

$$=\frac{(b-a)^{r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1}}{\left(r\left(\sum\limits_{i=1}^{m}\alpha_i-m+\frac{m}{p}\right)+1\right)\left(\prod\limits_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)^r}, \tag{16}$$

$$\cdot\left(\prod_{i=1}^{m}\left(\int_a^b|f_i(t)|^q\,dt\right)^{\frac{1}{q}}\right)^r.$$

The claim is proved. ∎

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We continue with

Theorem 3. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m$. Let $f_i : (a, b) \to \mathbb{R}$,be Lebesgue measurable functions so that $\|f_i\|_q$ is finite, $i = 1, ..., m$. Then

$$\left\| \prod_{i=1}^{m} \left(I_{b-}^{\alpha_i} f_i \right) \right\|_p \leq \frac{(b-a)^{\sum_{i=1}^{m} \alpha_i + m\left(\frac{1}{p} - 1\right) + \frac{1}{p}}}{\left[\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right]}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_q \right). \tag{17}$$

Proof. By (2) we have

$$\left(I_{b-}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b (t-x)^{\alpha_i - 1} f_i(t)\, dt, \tag{18}$$

$x < b, i = 1, ..., m$.

We have that

$$\left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| \leq \frac{1}{\Gamma(\alpha_i)} \int_x^b (t-x)^{\alpha_i - 1} |f_i(t)|\, dt, \tag{19}$$

$x < b, i = 1, ..., m$.

By Hölder's inequality we get

$$\left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| \leq \frac{1}{\Gamma(\alpha_i)} \left(\int_x^b (t-x)^{p(\alpha_i - 1)} dt \right)^{\frac{1}{p}} \left(\int_x^b |f_i(t)|^q dt \right)^{\frac{1}{q}} \tag{20}$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \frac{(b-x)^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q dt \right)^{\frac{1}{q}}, \tag{21}$$

$x < b, i = 1, ..., m$.

Therefore

$$\prod_{i=1}^{m} \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^p \leq \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i) \right)^p} \frac{(b-x)^{p \sum_{i=1}^{m} \alpha_i + m(1-p)}}{\prod_{i=1}^{m} (p(\alpha_i - 1) + 1)} \left(\prod_{i=1}^{m} \int_a^b |f_i(t)|^q dt \right)^{\frac{p}{q}}, \tag{22}$$

$x \in (a, b)$.

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Consequently we get

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{b-}^{\alpha_i} f_i\right)(x)\right|^p\right) dx \leq \left(\frac{1}{\left(\prod_{i=1}^m \Gamma(\alpha_i)\right)^p \left(\prod_{i=1}^m (p(\alpha_i-1)+1)\right)}\right)$$

$$\cdot \left(\int_a^b (b-x)^{p\sum_{i=1}^m \alpha_i + m(1-p)} dx\right) \left(\prod_{i=1}^m \int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}} \tag{23}$$

$$= \frac{(b-a)^{p\sum_{i=1}^m \alpha_i + m(1-p)+1} \left(\prod_{i=1}^m \int_a^b |f_i(t)|^q dt\right)^{\frac{p}{q}}}{\left[\left(p\sum_{i=1}^m \alpha_i + m(1-p)+1\right)\left(\prod_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i-1)+1)\right)\right)\right]}, \tag{24}$$

proving the claim. ■

It follows

Theorem 4. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1, r > 0; \alpha_i > 0, i = 1, ..., m$. Let $f_i : (a,b) \to \mathbb{R}$, *be* Lebesgue measurable functions so that $\|f_i\|_q$ is finite, $i = 1, ..., m$.

Then

$$\left\|\prod_{i=1}^m \left(I_{b-}^{\alpha_i} f_i\right)\right\|_r \leq \frac{(b-a)^{\sum_{i=1}^m \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left[\left(r\left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p}\right)+1\right)^{\frac{1}{r}} \left(\prod_{i=1}^m \Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot \left(\prod_{i=1}^m \|f_i\|_q\right). \tag{25}$$

Proof. Using $r > 0$ and (21) we get

$$\left|\left(I_{b-}^{\alpha_i} f_i\right)(x)\right|^r \leq \frac{1}{\Gamma(\alpha_i)^r} \frac{(b-x)^{r\left((\alpha_i-1)+\frac{1}{p}\right)}}{(p(\alpha_i-1)+1)^{\frac{r}{p}}} \left(\int_a^b |f_i(t)|^q dt\right)^{\frac{r}{q}}, \tag{26}$$

and

$$\prod_{i=1}^m \left|\left(I_{b-}^{\alpha_i} f_i\right)(x)\right|^r \leq \frac{1}{\prod_{i=1}^m \Gamma(\alpha_i)^r} \frac{(b-x)^{r\left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p}\right)}}{\left(\prod_{i=1}^m (p(\alpha_i-1)+1)\right)^{\frac{r}{p}}} \left(\prod_{i=1}^m \left(\int_a^b |f_i(t)|^q dt\right)^{\frac{1}{q}}\right)^r. \tag{27}$$

Consequently it holds

$$\int_a^b \left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^r \right) dx \leq \frac{\left(\int_a^b (b-x)^{r\left(\sum\limits_{i=1}^m \alpha_i - m + \frac{m}{p} \right)} dx \right)}{\left(\prod\limits_{i=1}^m \Gamma(\alpha_i)^r \right) \left(\prod\limits_{i=1}^m (p(\alpha_i - 1) + 1) \right)^{\frac{r}{p}}}$$

$$\cdot \left(\prod_{i=1}^m \left(\int_a^b |f_i(t)|^q \, dt \right)^{\frac{1}{q}} \right)^r \tag{28}$$

$$= \frac{(b-a)^{r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1}}{\left(r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1\right)\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)^{r}}, \tag{29}$$

$$\cdot \left(\prod_{i=1}^{m}\left(\int_a^b |f_i(t)|^q\, dt\right)^{\frac{1}{q}}\right)^{r}.$$

The claim is proved. ■

We need

Definition 5. ([14, p.99])The fractional integrals of a function f with respect to given function g are defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is an increasing function on $[a, b]$ and $g \in C^1([a, b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a, b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\, f(t)\, dt}{(g(x)-g(t))^{1-\alpha}}, \quad x > a, \tag{30}$$

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\, f(t)\, dt}{(g(t)-g(x))^{1-\alpha}}, \quad x < b, \tag{31}$$

respectively.

We present

Theorem 6. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0, i = 1, ..., m$. Here $a, b \in \mathbb{R}$ and strictly increasing g with $I_{a+;g}^{\alpha_i}$ as in Definition 5, see (30). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions so that $\|f_i\|_{L_q(g)}$ is finite, $i = 1, ..., m$.

Then

$$\left\|\prod_{i=1}^{m}\left(I_{a+;g}^{\alpha_i} f_i\right)\right\|_{L_p(g)} \leq \frac{(g(b)-g(a))^{\sum_{i=1}^{m}\alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left[\left(p\sum_{i=1}^{m}\alpha_i + m(1-p)+1\right)^{\frac{1}{p}}\left(\prod_{i=1}^{m}\Gamma(\alpha_i)\left(p(\alpha_i-1)+1\right)^{\frac{1}{p}}\right)\right]}$$

$$\cdot\left(\prod_{i=1}^{m}\|f_i\|_{L_q(g)}\right). \tag{32}$$

Proof. By (30) we have

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)}\int_a^x \frac{g'(t) f_i(t)}{(g(x)-g(t))^{1-\alpha_i}} dt, \tag{33}$$

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$x > a, i = 1, ..., m$.

We have that

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| \leq \frac{1}{\Gamma(\alpha_i)} \int_a^x (g(x) - g(t))^{\alpha_i - 1} g'(t) |f_i(t)| \, dt$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_a^x (g(x) - g(t))^{\alpha_i - 1} |f_i(t)| \, dg(t), \tag{34}$$

$x > a, i = 1, ..., m$.

By Hölder's inequality we get

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| \leq \frac{1}{\Gamma(\alpha_i)} \left(\int_a^x (g(x) - g(t))^{p(\alpha_i - 1)} \, dg(t)\right)^{\frac{1}{p}} \left(\int_a^x |f_i(t)|^q \, dg(t)\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q \, dg(t)\right)^{\frac{1}{q}} \tag{35}$$

$$= \frac{1}{\Gamma(\alpha_i)} \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{36}$$

$x > a, i = 1, ..., m$.

So we got

$$\left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right| \leq \frac{(g(x) - g(a))^{\alpha_i - 1 + \frac{1}{p}}}{\Gamma(\alpha_i)(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{37}$$

$x > a, i = 1, ..., m$.

Hence

$$\prod_{i=1}^m \left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right|^p \leq \frac{(g(x) - g(a))^{p \sum_{i=1}^m \alpha_i + m(1-p)}}{\prod_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1)\right)} \prod_{i=1}^m \|f_i\|_{L_q(g)}^p, \tag{38}$$

$x \in (a, b)$.

Consequently, we obtain

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{a+;g}^{\alpha_i} f_i\right)(x)\right|^p\right) dg(x) \leq \frac{\prod_{i=1}^m \|f_i\|_{L_q(g)}^p \int_a^b (g(x) - g(a))^{p \sum_{i=1}^m \alpha_i + m(1-p)} dg(x)}{\prod_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1)\right)}$$

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$$= \prod_{i=1}^{m} \left[\frac{\|f_i\|_{L_q(g)}^p}{(\Gamma(\alpha_i)^p (p(\alpha_i - 1) + 1))} \right] \frac{(g(b) - g(a))^{p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1}}{\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)}, \tag{39}$$

proving the claim. ■

We also give

Theorem 7. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0, i = 1, ..., m$; $r > 0$. Here $a, b \in \mathbb{R}$ and strictly increasing g with $I_{a+;g}^{\alpha_i}$ as in Definition 5, see (30). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(g)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(I_{a+;g}^{\alpha_i} f_i \right) \right\|_{L_r(g)} \leq \frac{(g(b) - g(a))^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left[\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right) \right]}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right). \tag{40}$$

Proof. Using $r > 0$ and (37) we get

$$\left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r \leq \frac{(g(x) - g(a))^{r\left(\alpha_i - 1 + \frac{1}{p}\right)}}{\Gamma(\alpha_i)^r (p(\alpha_i - 1) + 1)^{\frac{r}{p}}} \|f_i\|_{L_q(g)}^r, \tag{41}$$

and

$$\prod_{i=1}^{m} \left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r \leq \frac{(g(x) - g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right)}}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right)^r} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r, \tag{42}$$

$x \in (a, b)$.

Consequently, it holds

$$\int_a^b \prod_{i=1}^{m} \left| \left(I_{a+;g}^{\alpha_i} f_i \right)(x) \right|^r dg(x) \leq \frac{\left(\int_a^b (g(x) - g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right)} dg(x) \right)}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right) \right)^r}$$

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$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r \tag{43}$$

$$= \frac{(g(b)-g(a))^{r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p}\right)+1} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r}{\left(r\left(\sum_{i=1}^{m}\alpha_i - m + \frac{m}{p} \right) + 1 \right) \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)^r}. \tag{44}$$

The claim is proved. ■

We continue with

Theorem 8. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m$. Here $a, b \in \mathbb{R}$ and strictly increasing g with $I_{b-;g}^{\alpha_i}$ as in Definition 5, see (31). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(g)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(I_{b-;g}^{\alpha_i} f_i \right) \right\|_{L_p(g)} \leq \frac{(g(b)-g(a))^{\sum_{i=1}^{m}\alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left[\left(p\sum_{i=1}^{m}\alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right]}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right). \tag{45}$$

Proof. By (31) we have

$$\left(I_{b-;g}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b \frac{g'(t) f_i(t)}{(g(t)-g(x))^{1-\alpha_i}} dt, \tag{46}$$

$x < b, i = 1, ..., m$.

We have that

$$\left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right| \leq \frac{1}{\Gamma(\alpha_i)} \int_x^b (g(t)-g(x))^{\alpha_i - 1} g'(t) |f_i(t)| \, dt$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_x^b (g(t)-g(x))^{\alpha_i - 1} |f_i(t)| \, dg(t), \tag{47}$$

$x < b, i = 1, ..., m$.

By Hölder's inequality we get

$$\left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right| \leq \frac{1}{\Gamma(\alpha_i)} \left(\int_x^b (g(t)-g(x))^{p(\alpha_i-1)}\, dg(t)\right)^{\frac{1}{p}} \left(\int_x^b |f_i(t)|^q\, dg(t)\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\Gamma(\alpha_i)} \frac{(g(b)-g(x))^{\alpha_i-1+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}} \left(\int_a^b |f_i(t)|^q\, dg(t)\right)^{\frac{1}{q}} \tag{48}$$

$$= \frac{1}{\Gamma(\alpha_i)} \frac{(g(b)-g(x))^{\alpha_i-1+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{49}$$

$x < b, i = 1, ..., m$.

So we got

$$\left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right| \leq \frac{(g(b)-g(x))^{\alpha_i-1+\frac{1}{p}}}{\Gamma(\alpha_i)(p(\alpha_i-1)+1)^{\frac{1}{p}}} \|f_i\|_{L_q(g)}, \tag{50}$$

$x < b, i = 1, ..., m$.

Hence

$$\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^p \leq \frac{(g(b)-g(x))^{p\sum_{i=1}^m \alpha_i+m(1-p)}}{\prod_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i-1)+1)\right)} \prod_{i=1}^m \|f_i\|_{L_q(g)}^p, \tag{51}$$

$x \in (a, b)$.

Consequently, we obtain

$$\int_a^b \left(\prod_{i=1}^m \left|\left(I_{b-;g}^{\alpha_i} f_i\right)(x)\right|^p\right) dg(x) \leq \frac{\prod_{i=1}^m \|f_i\|_{L_q(g)}^p \left(\int_a^b (g(b)-g(x))^{p\sum_{i=1}^m \alpha_i+m(1-p)}\, dg(x)\right)}{\prod_{i=1}^m \left(\Gamma(\alpha_i)^p (p(\alpha_i-1)+1)\right)}$$

$$= \prod_{i=1}^m \left[\frac{\|f_i\|_{L_q(g)}^p}{\left(\Gamma(\alpha_i)^p (p(\alpha_i-1)+1)\right)}\right] \frac{(g(b)-g(a))^{p\sum_{i=1}^m \alpha_i+m(1-p)+1}}{\left(p\sum_{i=1}^m \alpha_i+m(1-p)+1\right)}, \tag{52}$$

proving the claim. ■

We also give

Theorem 9. Let $p, q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m, r > 0$. Here $a, b \in \mathbb{R}$ and strictly increasing g with $I_{b-;g}^{\alpha_i}$ as in Definition 5, see (31).

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Let $f_i : (a,b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(g)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(I_{b-;g}^{\alpha_i} f_i \right) \right\|_{L_r(g)} \leq \frac{(g(b)-g(a))^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left[\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}} \left(\prod_{i=1}^{m} \Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right]} \cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right). \tag{53}$$

Proof. Using $r > 0$ and (50) we get

$$\left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right|^r \leq \frac{(g(b)-g(x))^{r\left(\alpha_i - 1 + \frac{1}{p}\right)}}{\Gamma(\alpha_i)^r \left(p(\alpha_i - 1) + 1 \right)^{\frac{r}{p}}} \|f_i\|_{L_q(g)}^r, \tag{54}$$

and

$$\prod_{i=1}^{m} \left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right|^r \leq \frac{(g(b)-g(x))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)}}{\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right)^r} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r, \tag{55}$$

$x \in (a,b)$.

Consequently, it holds

$$\int_a^b \prod_{i=1}^{m} \left| \left(I_{b-;g}^{\alpha_i} f_i \right)(x) \right|^r dg(x) \leq \frac{\left(\int_a^b (g(b)-g(x))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)} dg(x) \right)}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)^r} \cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r \tag{56}$$

$$= \frac{(g(b)-g(a))^{r\left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p}\right)+1} \left(\prod_{i=1}^{m} \|f_i\|_{L_q(g)} \right)^r}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right) \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)^r}. \tag{57}$$

The claim is proved. ■

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We need

Definition 10 ([13]). Let $0 < a < b < \infty$, $\alpha > 0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$\left(J_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{y}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x > a, \tag{58}$$

and

$$\left(J_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{y}{x}\right)^{\alpha-1} \frac{f(y)}{y} dy, \quad x < b, \tag{59}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g(x) = \ln x$ on $[a, b]$, $0 < a < b < \infty$.

Above f is a Lebesgue measurable function from (a, b) into $\mathbb{R}$, such that $\left(J_{a+}^{\alpha}(|f|)\right)(x)$ and/or $\left(J_{b-}^{\alpha}(|f|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a, b)$.

We present

Theorem 11. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m$. Here $0 < a < b < \infty$, and $J_{a+}^{\alpha_i}$ as in Definition 10, see (58). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(ln)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^m \left(J_{a+}^{\alpha_i} f_i\right) \right\|_{L_p(ln)} \leq \frac{\left(\ln(\frac{b}{a})\right)^{\sum_{i=1}^m \alpha_i + m(\frac{1}{p}-1) + \frac{1}{p}}}{\left(p \sum_{i=1}^m \alpha_i + m(1-p) + 1\right)^{\frac{1}{p}} \left(\prod_{i=1}^m \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}}\right)\right)}$$

$$\cdot \left(\prod_{i=1}^m \|f_i\|_{L_q(ln)}\right). \tag{60}$$

Proof. By Theorem 6, for $g(x) = \ln x$. ■

We also have

Theorem 12. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m; r > 0$. Here $0 < a < b < \infty$, and $J_{a+}^{\alpha_i}$ as in Definition 10, see (58). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(ln)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^m \left(J_{a+}^{\alpha_i} f_i\right) \right\|_{L_r(ln)} \leq \frac{\left(\ln(\frac{b}{a})\right)^{\sum_{i=1}^m \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p}\right) + 1\right)^{\frac{1}{r}} \left(\prod_{i=1}^m \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}}\right)\right)}$$

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$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(ln)} \right). \tag{61}$$

Proof. By Theorem 7, for $g(x) = \ln x$. ■

We continue with

Theorem 13. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m$. Here $0 < a < b < \infty$, and $J_{b-}^{\alpha_i}$ as in Definition 10, see (59). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(ln)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{L_p(ln)} \leq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m} \alpha_i + m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1\right)^{\frac{1}{p}} \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}} \right) \right)}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(ln)} \right). \tag{62}$$

Proof. By Theorem 8, for $g(x) = \ln x$. ■

We also have

Theorem 14. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m; r > 0$. Here $0 < a < b < \infty$, and $J_{b-}^{\alpha_i}$ as in Definition 10, see (59). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|f_i\|_{L_q(ln)}$ is finite, $i = 1, ..., m$. Then

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i \right) \right\|_{L_r(ln)} \leq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1\right)^{\frac{1}{r}} \left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}} \right) \right)}$$

$$\cdot \left(\prod_{i=1}^{m} \|f_i\|_{L_q(ln)} \right). \tag{63}$$

Proof. By Theorem 9, for $g(x) = \ln x$. ■

We need

Definition 15 ([16]). Let (a, b), $0 \leq a < b < \infty$; $\alpha, \sigma > 0$. We consider the left- and right-sided fractional integrals of order α as follows:

1) for $\eta > -1$, we define

$$\left(I_{a+;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)\, dt}{(x^\sigma - t^\sigma)^{1-\alpha}}, \tag{64}$$

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2) for $\eta > 0$, we define

$$\left(I_{b-;\sigma,\eta}^{\alpha} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\, dt}{(t^\sigma - x^\sigma)^{1-\alpha}}. \tag{65}$$

These are the Erdélyi-Kober type fractional integrals.

We present

Theorem 16. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i > 0, i = 1, ..., m$. Here $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > -1$, and $I_{a+;\sigma,\eta}^{\alpha_i}$ is as in Definition 15, see (64). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\|x^{\sigma\eta} f_i(x)\|_{L_q(x^\sigma)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i\right)(x) \right) \right\|_{L_p(x^\sigma)} \leq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^{m} \alpha_i + m(\frac{1}{p}-1)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1\right)^{\frac{1}{p}} \right) \right)} \left(\prod_{i=1}^{m} \|x^{\sigma\eta} f_i(x)\|_{L_q(x^\sigma)} \right). \tag{66}$$

Proof. By Definition 15, see (64), we have

$$\left(I_{a+;\sigma,\eta}^{\alpha_i} f_i\right)(x) = \frac{\sigma x^{-\sigma(\alpha_i+\eta)}}{\Gamma(\alpha_i)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f_i(t)\, dt}{(x^\sigma - t^\sigma)^{1-\alpha_i}}, \tag{67}$$

$x > a$. We rewrite (67) as follows:

$$L_1(f_i)(x) := x^{\sigma(\alpha_i+\eta)} \left(I_{a+;\sigma,\eta}^{\alpha_i} f_i\right)(x)$$

$$= \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} \left(t^{\sigma\eta} f_i(t)\right) dt^\sigma, \tag{68}$$

and by calling $F_{1i}(t) = t^{\sigma\eta} f_i(t)$, we have

$$L_1(f_i)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} F_{1i}(t) dt^\sigma, \tag{69}$$

$i = 1, ..., m$, $x > a$. Furthermore we notice that

$$|L_1(f_i)(x)| \leq \frac{1}{\Gamma(\alpha_i)} \int_a^x (x^\sigma - t^\sigma)^{\alpha_i - 1} |F_{1i}(t)|\, dt^\sigma, \tag{70}$$

$i = 1, ..., m$, $x > a$.

So that now we can act as in the proof of Theorem 6. ■

We continue with

Theorem 17. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0$, $i = 1, ..., m$, $r > 0$. Here $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > -1$, and $I^{\alpha_i}_{a+;\sigma,\eta}$ is as in Definition 15, see (64). Let $f_i : (a, b) \to \mathbb{R}$,be Lebesgue measurable functions and $\|x^{\sigma\eta} f_i(x)\|_{L_q(x^\sigma)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(x^{\sigma(\alpha_i+\eta)} \left(I^{\alpha_i}_{a+;\sigma,\eta} f_i \right)(x) \right) \right\|_{L_r(x^\sigma)} \leq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)} \left(\prod_{i=1}^{m} \|x^{\sigma\eta} f_i(x)\|_{L_q(x^\sigma)} \right). \tag{71}$$

Proof. Based on the proof of Theorem 16, and similarly acting as in the proof of Theorem 7. ■

We also have

Theorem 18. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0$, $i = 1, ..., m$. Here $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > 0$, and $I^{\alpha_i}_{b-;\sigma,\eta}$ is as in Definition 15, see (65). Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\left\| x^{-\sigma(\eta+\alpha_i)} f_i(x) \right\|_{L_q(x^\sigma)}$ is finite, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(x^{-\sigma\eta} \left(I^{\alpha_i}_{b-;\sigma,\eta} f_i \right)(x) \right) \right\|_{L_p(x^\sigma)} \leq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^{m} \alpha_i + m(\frac{1}{p} - 1) + \frac{1}{p}}}{\left(p \sum_{i=1}^{m} \alpha_i + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i) \left(p(\alpha_i - 1) + 1 \right)^{\frac{1}{p}} \right) \right)} \left(\prod_{i=1}^{m} \left\| x^{-\sigma(\eta+\alpha_i)} f_i(x) \right\|_{L_q(x^\sigma)} \right). \tag{72}$$

Proof. By Definition 15, see (65) we have

$$\left(I^{\alpha_i}_{b-;\sigma,\eta} f_i \right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha_i)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha_i)-1} f_i(t)\, dt}{(t^\sigma - x^\sigma)^{1-\alpha_i}}, \tag{73}$$

$x < b$. We rewrite (73) as follows:

$$L_2(f_i)(x) := x^{-\sigma\eta} \left(I^{\alpha_i}_{b-;\sigma,\eta} f_i \right)(x)$$

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$$= \frac{1}{\Gamma(\alpha_i)} \int_x^b (t^\sigma - x^\sigma)^{\alpha_i - 1} \left(t^{-\sigma(\eta+\alpha_i)} f_i(t) \right) dt^\sigma, \tag{74}$$

and by calling $F_{2i}(t) = t^{-\sigma(\eta+\alpha_i)} f_i(t)$, we have

$$L_2(f_i)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b (t^\sigma - x^\sigma)^{\alpha_i - 1} F_{2i}(t) dt^\sigma, \tag{75}$$

$i = 1, ..., m$, $x < b$. Furthermore we notice that

$$|L_2(f_i)(x)| \leq \frac{1}{\Gamma(\alpha_i)} \int_x^b (t^\sigma - x^\sigma)^{\alpha_i - 1} |F_{2i}(t)| dt^\sigma, \tag{76}$$

$i = 1, ..., m$, $x < b$.

So that now we can act as in the proof of Theorem 8. ∎

We continue with

Theorem 19. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\alpha_i > 0$, $i = 1, ..., m$, $r > 0$. Here $0 \leq a < b < \infty$, $\sigma > 0$, $\eta > 0$, and $I^{\alpha_i}_{b-;\sigma,\eta}$ is as in Definition 15, see (65) Let $f_i : (a, b) \to \mathbb{R}$, be Lebesgue measurable functions and $\left\| x^{-\sigma(\eta+\alpha_i)} f_i(x) \right\|_{L_q(x^\sigma)}$ is finite, $i = 1, ..., m$. Then

$$\left\| \prod_{i=1}^m \left(x^{-\sigma\eta} \left(I^{\alpha_i}_{b-;\sigma,\eta} f_i \right)(x) \right) \right\|_{L_r(x^\sigma)} \leq \frac{(b^\sigma - a^\sigma)^{\sum_{i=1}^m \alpha_i - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^m \alpha_i - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \left(\Gamma(\alpha_i) (p(\alpha_i - 1) + 1)^{\frac{1}{p}} \right) \right)} \left(\prod_{i=1}^m \left\| x^{-\sigma(\eta+\alpha_i)} f_i(x) \right\|_{L_q(x^\sigma)} \right). \tag{77}$$

Proof. Based on the proof of Theorem 18, and acting similarly as in the proof of Theorem 9. ∎

We make

Definition 20. Let $\prod_{i=1}^N (a_i, b_i) \subset \mathbb{R}^N$, $N > 1$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$. Let $\alpha_i > 0$, $i = 1, ..., N$; $f \in L_1 \left(\prod_{i=1}^N (a_i, b_i) \right)$, and set $a = (a_1, ..., a_N)$, $b = (b_1, ..., b_N)$, $\alpha = (\alpha_1, ..., \alpha_N)$, $x = (x_1, ..., x_N)$, $t = (t_1, ..., t_N)$.

We define the left mixed Riemann-Liouville fractional multiple integral of order α (see also [15]):

$$\left(I^\alpha_{a+} f \right)(x) := \frac{1}{\prod_{i=1}^N \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^N (x_i - t_i)^{\alpha_i - 1} f(t_1, ..., t_N) dt_1 ... dt_N, \tag{78}$$

with $x_i > a_i$, $i = 1, ..., N$.

We also define the right mixed Riemann-Liouville fractional multiple integral of order α (see also [13]):

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} ... \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f(t_1, ..., t_N)\, dt_1 ... dt_N, \tag{79}$$

with $x_i < b_i$, $i = 1, ..., N$.

Notice $I_{a+}^{\alpha}(|f|)$, $I_{b-}^{\alpha}(|f|)$ are finite if $f \in L_\infty\left(\prod_{i=1}^{N}(a_i, b_i)\right)$.

We present

Theorem 21. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Here all as in Definition 20, and (78) for I_{a+}^{α}. Let $f_j : \prod_{i=1}^{N}(a_i, b_i) \to \mathbb{R}$, $j = 1, ..., m$, such that $f_j \in L_q\left(\prod_{i=1}^{N}(a_i, b_i)\right)$.

Then it holds

$$\left\| \prod_{j=1}^{m} I_{a+}^{\alpha} f_j \right\|_{p, \prod_{i=1}^{N}(a_i, b_i)} \leq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i - 1) + \frac{1}{p}\right) + \frac{1}{p}\right)}}{(m(p(\alpha_i - 1) + 1) + 1)^{\frac{1}{p}} \left(\Gamma(\alpha_i)(p(\alpha_i - 1) + 1)^{\frac{1}{p}}\right)^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N}(a_i, b_i)} \right). \tag{80}$$

Proof. By Definition 20, see (78), we have

$$\left(I_{a+}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f_j(t_1, ..., t_N)\, dt_1 ... dt_N, \tag{81}$$

furthermore it holds

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} |f_j(t_1, ..., t_N)|\, dt_1 ... dt_N, \tag{82}$$

$j = 1, ..., m$, $x \in \prod_{i=1}^{N}(a_i, b_i)$.

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By Hölder's inequality we get

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \left(\int_{a_1}^{x_1} \ldots \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{p(\alpha_i - 1)} \, dt_1 \ldots dt_N\right)^{\frac{1}{p}}$$

$$\cdot \left(\int_{a_1}^{x_1} \ldots \int_{a_N}^{x_N} |f_j(t_1, \ldots, t_N)|^q \, dt_1 \ldots dt_N\right)^{\frac{1}{q}} \tag{83}$$

$$\leq \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \left(\prod_{i=1}^{N} \left(\int_{a_i}^{x_i} (x_i - t_i)^{p(\alpha_i - 1)} \, dt_i\right)^{\frac{1}{p}}\right) \left(\int_{\prod_{i=1}^{N}(a_i, b_i)} |f_j(t)|^q \, dt\right)^{\frac{1}{q}} \tag{84}$$

$$= \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \left(\prod_{i=1}^{N} \left(\frac{(x_i - a_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}}\right)\right) \left(\int_{\prod_{i=1}^{N}(a_i, b_i)} |f_j(t)|^q \, dt\right)^{\frac{1}{q}}. \tag{85}$$

Hence

$$\prod_{j=1}^{m} \left|\left(I_{a+}^{\alpha} f_j\right)(x)\right|^p \leq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mp}} \left(\prod_{i=1}^{N} \frac{(x_i - a_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}}\right)^{mp}$$

$$\cdot \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i, b_i)} |f_j(t)|^q \, dt\right)^{\frac{p}{q}}, \tag{86}$$

for $x \in \prod_{i=1}^{N} (a_i, b_i)$.

Consequently, we get

$$\int_{\prod_{i=1}^{N}(a_i, b_i)} \prod_{j=1}^{m} \left|\left(I_{a+}^{\alpha} f_j\right)(x)\right|^p dx \leq \frac{\left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N}(a_i, b_i)} |f_j(t)|^q \, dt\right)^{\frac{p}{q}}\right)}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mp} \left(\prod_{i=1}^{N} (p(\alpha_i - 1) + 1)^m\right)}$$

$$\cdot \left(\int_{\prod_{i=1}^{N}(a_i, b_i)} \prod_{i=1}^{N} (x_i - a_i)^{m(p(\alpha_i - 1) + 1)} \, dx_1 \ldots dx_N\right) \tag{87}$$

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$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{m(p(\alpha_i - 1)+1)+1}}{(m (p (\alpha_i - 1) + 1) + 1) (\Gamma (\alpha_i)^p (p (\alpha_i - 1) + 1))^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j (t)|^q \, dt \right)^{\frac{p}{q}} \right), \tag{88}$$

proving the claim. ■

We have

Theorem 22. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $r > 0$. Here all as in Definition 20, and (78) for I_{a+}^{α}. Let $f_j : \prod_{i=1}^{N} (a_i, b_i) \to \mathbb{R}$, $j = 1, ..., m$, such that $f_j \in L_q \left(\prod_{i=1}^{N} (a_i, b_i) \right)$.

Then

$$\left\| \prod_{j=1}^{m} I_{a+}^{\alpha} f_j \right\|_{r, \prod_{i=1}^{N} (a_i, b_i)} \leq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i - 1) + \frac{1}{p}\right) + \frac{1}{r}\right)}}{\left(mr \left((\alpha_i - 1) + \frac{1}{p}\right) + 1\right)^{\frac{1}{r}} \Gamma(\alpha_i)^m \, (p(\alpha_i - 1) + 1)^{\frac{m}{p}}} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N} (a_i, b_i)} \right). \tag{89}$$

Proof. We have

$$\left(I_{a+}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma (\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} f_j (t_1, ..., t_N) \, dt_1 ... dt_N, \tag{90}$$

furthermore it holds

$$\left|\left(I_{a+}^{\alpha} f_j\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{N} \Gamma (\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1} |f_j (t_1, ..., t_N)| \, dt_1 ... dt_N, \tag{91}$$

$j = 1, ..., m$, $x \in \prod_{i=1}^{N} (a_i, b_i)$.

By using (85) of the proof of Theorem 21 and $r>0$ we get

$$\prod_{j=1}^{m}\left|\left(I_{a+}^{\alpha}f_{j}\right)(x)\right|^{r}\leq\frac{1}{\left(\prod_{i=1}^{N}\Gamma\left(\alpha_{i}\right)\right)^{mr}}\left(\prod_{i=1}^{N}\left(\frac{\left(x_{i}-a_{i}\right)^{\left(\alpha_{i}-1\right)+\frac{1}{p}}}{\left(p\left(\alpha_{i}-1\right)+1\right)^{\frac{1}{p}}}\right)\right)^{mr}$$

$$\cdot\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\left|f_{j}\left(t\right)\right|^{q}dt\right)^{\frac{r}{q}},\tag{92}$$

for $x\in\prod_{i=1}^{N}\left(a_{i},b_{i}\right)$.

Consequently, we get

$$\int_{\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\prod_{j=1}^{m}\left|\left(I_{a+}^{\alpha}f_{j}\right)(x)\right|^{r}dx\leq\frac{1}{\left(\prod_{i=1}^{N}\Gamma\left(\alpha_{i}\right)\right)^{mr}}\frac{\left(\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\left|f_{j}\left(t\right)\right|^{q}dt\right)^{\frac{1}{q}}\right)^{r}}{\left(\prod_{i=1}^{N}\left(p\left(\alpha_{i}-1\right)+1\right)^{\frac{mr}{p}}\right)}$$

$$\cdot\left(\int_{\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\prod_{i=1}^{N}\left(x_{i}-a_{i}\right)^{mr\left(\left(\alpha_{i}-1\right)+\frac{1}{p}\right)}dx\right)\tag{93}$$

$$=\prod_{i=1}^{N}\left(\frac{\left(b_{i}-a_{i}\right)^{mr\left(\left(\alpha_{i}-1\right)+\frac{1}{p}\right)+1}}{\left(mr\left(\left(\alpha_{i}-1\right)+\frac{1}{p}\right)+1\right)\Gamma\left(\alpha_{i}\right)^{mr}\left(p\left(\alpha_{i}-1\right)+1\right)^{\frac{mr}{p}}}\right)$$

$$\cdot\left(\prod_{j=1}^{m}\left\|f_{j}\right\|_{q,\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\right)^{r},\tag{94}$$

proving the claim. ■

We also give

Theorem 23. Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Here all as in Definition 20, and (79) for I_{b-}^{α}. Let $f_{j}:\prod_{i=1}^{N}\left(a_{i},b_{i}\right)\rightarrow\mathbb{R}$, $j=1,...,m$, such that $f_{j}\in L_{q}\left(\prod_{i=1}^{N}\left(a_{i},b_{i}\right)\right)$.

Then it holds

$$\left\|\prod_{j=1}^{m}I_{b-}^{\alpha}f_{j}\right\|_{p,\prod_{i=1}^{N}\left(a_{i},b_{i}\right)}\leq\prod_{i=1}^{N}\left(\frac{\left(b_{i}-a_{i}\right)^{\left(m\left(\left(\alpha_{i}-1\right)+\frac{1}{p}\right)+\frac{1}{p}\right)}}{\left(m\left(p\left(\alpha_{i}-1\right)+1\right)+1\right)^{\frac{1}{p}}\left(\Gamma(\alpha_{i})\left(p(\alpha_{i}-1)+1\right)^{\frac{1}{p}}\right)^{m}}\right)$$

$$\cdot\left(\prod_{j=1}^{m}\|f_j\|_{q,\prod\limits_{i=1}^{N}(a_i,b_i)}\right). \tag{95}$$

Proof. By Definition 20, see (79), we have

$$\left(I_{b-}^{\alpha}f_j\right)(x)=\frac{1}{\prod\limits_{i=1}^{N}\Gamma(\alpha_i)}\int_{x_1}^{b_1}...\int_{x_N}^{b_N}\prod_{i=1}^{N}(t_i-x_i)^{\alpha_i-1}f_j(t_1,...,t_N)\,dt_1...dt_N, \tag{96}$$

furthermore it holds

$$\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|\leq\frac{1}{\prod\limits_{i=1}^{N}\Gamma(\alpha_i)}\int_{x_1}^{b_1}...\int_{x_N}^{b_N}\prod_{i=1}^{N}(t_i-x_i)^{\alpha_i-1}\left|f_j(t_1,...,t_N)\right|dt_1...dt_N, \tag{97}$$

$j=1,...,m$, $x\in\prod\limits_{i=1}^{N}(a_i,b_i)$.

By Hölder's inequality we get

$$\left|\left(I_{b-}^{\alpha}f_j\right)(x)\right|\leq\frac{1}{\prod\limits_{i=1}^{N}\Gamma(\alpha_i)}\left(\int_{x_1}^{b_1}...\int_{x_N}^{b_N}\prod_{i=1}^{N}(t_i-x_i)^{p(\alpha_i-1)}dt_1...dt_N\right)^{\frac{1}{p}}$$

$$\cdot\left(\int_{x_1}^{b_1}...\int_{x_N}^{b_N}\left|f_j(t_1,...,t_N)\right|^q dt_1...dt_N\right)^{\frac{1}{q}} \tag{98}$$

$$\leq\frac{1}{\prod\limits_{i=1}^{N}\Gamma(\alpha_i)}\left(\prod_{i=1}^{N}\left(\int_{x_i}^{b_i}(t_i-x_i)^{p(\alpha_i-1)}dt_i\right)^{\frac{1}{p}}\right)$$

$$\cdot\left(\int_{\prod\limits_{i=1}^{N}(a_i,b_i)}\left|f_j(t)\right|^q dt\right)^{\frac{1}{q}} \tag{99}$$

$$=\frac{1}{\prod\limits_{i=1}^{N}\Gamma(\alpha_i)}\left(\prod_{i=1}^{N}\left(\frac{(b_i-x_i)^{(\alpha_i-1)+\frac{1}{p}}}{(p(\alpha_i-1)+1)^{\frac{1}{p}}}\right)\right)$$

$$\cdot \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q \, dt \right)^{\frac{1}{q}}. \tag{100}$$

Hence

$$\prod_{j=1}^{m} \left| \left(I_{b-}^{\alpha} f_j \right)(x) \right|^p \leq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i) \right)^{mp}} \left(\prod_{i=1}^{N} \frac{(b_i - x_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \right)^{mp}$$

$$\cdot \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}}, \tag{101}$$

for $x \in \prod_{i=1}^{N} (a_i, b_i)$.

Consequently, we get

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} \prod_{j=1}^{m} \left| \left(I_{b-}^{\alpha} f_j \right)(x) \right|^p dx \leq \frac{\left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}} \right)}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i) \right)^{mp} \left(\prod_{i=1}^{N} (p(\alpha_i - 1) + 1)^m \right)}$$

$$\cdot \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} \prod_{i=1}^{N} (b_i - x_i)^{m(p(\alpha_i - 1) + 1)} \, dx_1 ... dx_N \right) \tag{102}$$

$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{m(p(\alpha_i - 1) + 1) + 1}}{(m(p(\alpha_i - 1) + 1) + 1)\left((\Gamma(\alpha_i))^p (p(\alpha_i - 1) + 1) \right)^m} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q \, dt \right)^{\frac{p}{q}} \right), \tag{103}$$

proving the claim. ∎

We have

Theorem 24. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $r > 0$. Here all as in Definition 20, and (79) for I_{b-}^{α}. Let $f_j : \prod_{i=1}^{N} (a_i, b_i) \to \mathbb{R}$, $j = 1, ..., m$, such that $f_j \in L_q \left(\prod_{i=1}^{N} (a_i, b_i) \right)$.

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Then

$$\left\| \prod_{j=1}^{m} I_{b-}^{\alpha} f_j \right\|_{r, \prod_{i=1}^{N} (a_i, b_i)} \leq \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{\left(m\left((\alpha_i - 1) + \frac{1}{p}\right) + \frac{1}{r}\right)}}{\left(mr\left((\alpha_i - 1) + \frac{1}{p}\right) + 1\right)^{\frac{1}{r}} \Gamma(\alpha_i)^m \left(p(\alpha_i - 1) + 1\right)^{\frac{m}{p}}} \right)$$

$$\cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N} (a_i, b_i)} \right). \tag{104}$$

Proof. We have

$$\left(I_{b-}^{\alpha} f_j\right)(x) = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} f_j(t_1, \ldots, t_N)\, dt_1 \ldots dt_N, \tag{105}$$

furthermore it holds

$$\left|\left(I_{b-}^{\alpha} f_j\right)(x)\right| \leq \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1} \left|f_j(t_1, \ldots, t_N)\right| dt_1 \ldots dt_N, \tag{106}$$

$j = 1, \ldots, m$, $x \in \prod_{i=1}^{N} (a_i, b_i)$.

By using (100) of the proof of Theorem 23 and $r > 0$ we get

$$\prod_{j=1}^{m} \left|\left(I_{b-}^{\alpha} f_j\right)(x)\right|^r \leq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mr}} \left(\prod_{i=1}^{N} \left(\frac{(b_i - x_i)^{(\alpha_i - 1) + \frac{1}{p}}}{(p(\alpha_i - 1) + 1)^{\frac{1}{p}}} \right) \right)^{mr}$$

$$\cdot \prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q\, dt \right)^{\frac{r}{q}}, \tag{107}$$

for $x \in \prod_{i=1}^{N} (a_i, b_i)$.

Consequently, we get

$$\int_{\prod_{i=1}^{N} (a_i, b_i)} \prod_{j=1}^{m} \left|\left(I_{b-}^{\alpha} f_j\right)(x)\right|^r dx \leq \frac{1}{\left(\prod_{i=1}^{N} \Gamma(\alpha_i)\right)^{mr}} \frac{\left(\prod_{j=1}^{m} \left(\int_{\prod_{i=1}^{N} (a_i, b_i)} |f_j(t)|^q\, dt \right)^{\frac{1}{q}} \right)^r}{\left(\prod_{i=1}^{N} (p(\alpha_i - 1) + 1)^{\frac{mr}{p}} \right)}$$

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$$\cdot \left(\int_{\prod_{i=1}^{N} (a_i,b_i)} \prod_{i=1}^{N} (b_i - x_i)^{mr\left((\alpha_i - 1)+\frac{1}{p}\right)} dx \right) \tag{108}$$

$$= \prod_{i=1}^{N} \left(\frac{(b_i - a_i)^{mr\left((\alpha_i-1)+\frac{1}{p}\right)+1}}{\left(mr\left((\alpha_i - 1) + \frac{1}{p}\right) + 1\right) \Gamma(\alpha_i)^{mr} (p(\alpha_i - 1) + 1)^{\frac{mr}{p}}} \right.$$

$$\left. \cdot \left(\prod_{j=1}^{m} \|f_j\|_{q, \prod_{i=1}^{N}(a_i,b_i)} \right) \right)^r, \tag{109}$$

proving the claim. ■

Definition 25 ([1], p. 448). The left generalized Riemann-Liouville fractional derivative of f of order $\beta > 0$ is given by

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\beta-1} f(y)\, dy, \tag{110}$$

where $n = [\beta] + 1$, $x \in [a, b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a, b)$ has an L_∞ fractional derivative $D_a^{\beta} f$ ($\beta > 0$) in $[a, b]$, if and only if

(1) $D_a^{\beta-k} f \in C([a, b])$, $k = 2, ..., n = [\beta] + 1$,

(2) $D_a^{\beta-1} f \in AC([a, b])$

(3) $D_a^{\beta} f \in L_\infty(a, b)$.

Above we define $D_a^0 f := f$ and $D_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \leq 1$.

From [1, p. 449] and [11] we mention and use

Lemma 26. Let $\beta > \alpha \geq 0$ and let $f \in L_1(a, b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a, b]$ and let $D_a^{\beta-k} f(a) = 0$, $k = 1, ..., [\beta] + 1$, then

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \tag{111}$$

for all $a \leq x \leq b$.

Here $D_a^{\alpha} f \in AC([a, b])$ for $\beta - \alpha \geq 1$, and $D_a^{\alpha} f \in C([a, b])$ for $\beta - \alpha \in (0, 1)$.

Notice here that

$$D_a^{\alpha} f(x) = \left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta} f\right)\right)(x), \quad a \leq x \leq b. \tag{112}$$

We present

Theorem 27. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\beta_i > \alpha_i \geq 0$, $i = 1, ..., m$. Let $f_i \in L_1(a, b)$ have an L_∞ fractional derivative $D_a^{\beta_i} f_i$ in $[a, b]$ and let $D_a^{\beta_i - k_i} f_i(a) = 0$, $k_i = 1, ..., [\beta_i] + 1$.

Then

$$\left\|\prod_{i=1}^{m}\left(D_a^{\alpha_i}f_i\right)\right\|_p \leq \frac{(b-a)^{\sum_{i=1}^{m}(\beta_i-\alpha_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^{m}(\beta_i-\alpha_i)+m(1-p)+1\right)^{\frac{1}{p}}}$$

$$\cdot\frac{1}{\left(\prod_{i=1}^{m}\Gamma(\beta_i-\alpha_i)\left(p(\beta_i-\alpha_i-1)+1\right)^{\frac{1}{p}}\right)}\left(\prod_{i=1}^{m}\left\|D_a^{\beta_i}f_i\right\|_q\right). \tag{113}$$

Proof. Using Theorem 1, see (5), and Lemma 26, see (112). ■

We also give

Theorem 28. Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$; $r>0$, $\beta_i>\alpha_i\geq 0$, $i=1,...,m$. Let $f_i\in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta_i}f_i$ in $[a,b]$ and let $D_a^{\beta_i-k_i}f_i(a)=0$, $k_i=1,...,[\beta_i]+1$.

Then

$$\left\|\prod_{i=1}^{m}\left(D_a^{\alpha_i}f_i\right)\right\|_r \leq \frac{(b-a)^{\sum_{i=1}^{m}(\beta_i-\alpha_i)-m+\frac{m}{p}+\frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m}(\beta_i-\alpha_i)-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}}$$

$$\cdot\frac{1}{\left(\prod_{i=1}^{m}\Gamma(\beta_i-\alpha_i)\left(p(\beta_i-\alpha_i-1)+1\right)^{\frac{1}{p}}\right)}\left(\prod_{i=1}^{m}\left\|D_a^{\beta_i}f_i\right\|_q\right). \tag{114}$$

Proof: Using Theorem 2, see (12), and Lemma 26, see (112). ■

We need

Definition 29 ([8], p. 50, [1], p. 449). Let $\nu\geq 0$, $n:=\lceil\nu\rceil$, $f\in AC^n([a,b])$. Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu}f(x)=\frac{1}{\Gamma(n-\nu)}\int_a^x(x-t)^{n-\nu-1}f^{(n)}(t)\,dt$$

$$=\left(I_{a+}^{n-\nu}f^{(n)}\right)(x), \tag{115}$$

and it exists almost everywhere for $x\in[a,b]$, in fact $D_{*a}^{\nu}f\in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n}f=f^{(n)}$, $n\in\mathbb{Z}_+$.

We also need

Theorem 30 ([4]). Let $\nu\geq\rho+1$, $\rho>0$, $\nu,\rho\notin\mathbb{N}$. Call $n:=\lceil\nu\rceil$, $m^*:=\lceil\rho\rceil$. Assume $f\in AC^n([a,b])$, such that $f^{(k)}(a)=0$, $k=m^*,m^*+1,...,n-1$, and

$D_{*a}^{\nu} f \in L_{\infty}(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)}\right)(x)$), and

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\,dt = \left(I_{a+}^{\nu-\rho}\left(D_{*a}^{\nu} f\right)\right)(x), \tag{116}$$

$\forall\, x \in [a,b]$.

We present

Theorem 31. Let $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$; and let $\nu_i \geq \rho_i + 1$, $\rho_i > 0$, $\nu_i, \rho_i \notin \mathbb{N}$, $i = 1,...,m$. Call $n_i := \lceil \nu_i \rceil$, $m_i^* := \lceil \rho_i \rceil$. Suppose $f_i \in AC^{n_i}([a,b])$, such that $f_i^{(k_i)}(a) = 0$, $k_i = m_i^*, m_i^*+1,...,n_i-1$, and $D_{*a}^{\nu_i} f_i \in L_{\infty}(a,b)$.

Then

$$\left\| \prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_i\right) \right\|_p \leq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i-\rho_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p\sum_{i=1}^{m}(\nu_i-\rho_i)+m(1-p)+1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m}\Gamma(\nu_i-\rho_i)\left(p(\nu_i-\rho_i-1)+1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^{m}\left\|D_{*a}^{\nu_i} f_i\right\|_q\right). \tag{117}$$

Proof. Using Theorem 1, see (5), and Theorem 30, see (116). ■

We also give

Theorem 32. Let $p,q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $r > 0$; and let $\nu_i \geq \rho_i + 1$, $\rho_i > 0$, $\nu_i, \rho_i \notin \mathbb{N}$, $i = 1,...,m$. Call $n_i := \lceil \nu_i \rceil$, $m_i^* := \lceil \rho_i \rceil$. Suppose $f_i \in AC^{n_i}([a,b])$, such that $f_i^{(k_i)}(a) = 0$, $k_i = m_i^*, m_i^*+1,...,n_i-1$, and $D_{*a}^{\nu_i} f_i \in L_{\infty}(a,b)$.

Then

$$\left\| \prod_{i=1}^{m} \left(D_{*a}^{\rho_i} f_i\right) \right\|_r \leq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i-\rho_i)-m+\frac{m}{p}+\frac{1}{r}}}{\left(r\left(\sum_{i=1}^{m}(\nu_i-\rho_i)-m+\frac{m}{p}\right)+1\right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m}\Gamma(\nu_i-\rho_i)\left(p(\nu_i-\rho_i-1)+1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^{m}\left\|D_{*a}^{\nu_i} f_i\right\|_q\right). \tag{118}$$

Proof. Using Theorem 2, see (12), and Theorem 30, see (116). ■

We need

Definition 33 ([2], [9], [10]). Let $\alpha \geq 0$, $n := \lceil \alpha \rceil$, $f \in AC^n([a,b])$. We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$\overline{D}_{b-}^{\alpha} f(x) := (-1)^n I_{b-}^{n-\alpha} f^{(n)}(x), \tag{119}$$

we set $\overline{D}_{-}^{0} f := f$, i.e.

$$\overline{D}_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (J-x)^{n-\alpha-1} f^{(n)}(J)\, dJ. \tag{120}$$

Notice that $\overline{D}_{b-}^{n} f = (-1)^n f^{(n)}$, $n \in \mathbb{N}$.

We need

Theorem 34 ([4]). Let $f \in AC^n([a,b])$, $\alpha > 0$, $n \in \mathbb{N}$, $n := \lceil \alpha \rceil$, $\alpha \geq \rho + 1$, $\rho > 0$, $r = \lceil \rho \rceil$, $\alpha, \rho \notin \mathbb{N}$. Assume $f^{(k)}(b) = 0$, $k = r, r+1, ..., n-1$, and $\overline{D}_{b-}^{\alpha} f \in L_\infty([a,b])$. Then

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho} \left(\overline{D}_{b-}^{\alpha} f \right) \right)(x) \in AC([a,b]), \tag{121}$$

that is

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f \right)(t)\, dt, \tag{122}$$

$\forall\, x \in [a,b]$.

We present

Theorem 35. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1; \alpha_i \geq \rho_i + 1, \rho_i > 0$, $i = 1, ..., m$. Suppose $f_i \in AC^{n_i}([a,b])$, $n_i \in \mathbb{N}$, $n_i := \lceil \alpha_i \rceil$, $r_i = \lceil \rho_i \rceil$, $\alpha_i, \rho_i \notin \mathbb{N}$, and $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1, ..., n_i - 1$, and $\overline{D}_{b-}^{\alpha_i} f_i \in L_\infty([a,b])$, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_p \leq \frac{(b-a)^{\sum_{i=1}^{m}(\alpha_i-\rho_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} (\alpha_i - \rho_i) + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\alpha_i - \rho_i) \left(p(\alpha_i - \rho_i - 1) + 1 \right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \overline{D}_{b-}^{\alpha_i} f_i \right\|_q \right). \tag{123}$$

Proof. Using Theorem 3, see (17), and Theorem 34, see (121). ∎

We also give

Theorem 36. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1, r > 0$; $\alpha_i \geq \rho_i + 1, \rho_i > 0$, $i = 1, ..., m$. Suppose$f_i \in AC^{n_i}([a,b])$, $n_i \in \mathbb{N}$, $n_i := \lceil \alpha_i \rceil$, $r_i = \lceil \rho_i \rceil$, $\alpha_i, \rho_i \notin \mathbb{N}$, and $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1, ..., n_i - 1$, and $\overline{D}_{b-}^{\alpha_i} f_i \in L_\infty([a,b])$, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_r \leq \frac{(b-a)^{\sum_{i=1}^{m} (\alpha_i - \rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} (\alpha_i - \rho_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma (\alpha_i - \rho_i) \left(p(\alpha_i - \rho_i - 1) + 1 \right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \overline{D}_{b-}^{\alpha_i} f_i \right\|_q \right). \tag{124}$$

Proof. Using Theorem 4, see (25), and Theorem 34, see (121). ■

We need

Definition 37. Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$ $(0 \leq \alpha < 1)$. Let $a, b \in \mathbb{R}$, $a \leq x \leq b$, $f \in C([a,b])$. We consider $C_a^{\nu}([a,b]) := \{f \in C^n([a,b]) : I_{a+}^{1-\alpha} f^{(n)} \in C^1([a,b])\}$. For $f \in C_a^{\nu}([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as

$$\Delta_a^{\nu} f := \left(I_{a+}^{1-\alpha} f^{(n)} \right)', \tag{125}$$

see [1], p. 24, and Canavati derivative in [7].

Notice here $\Delta_a^{\nu} f \in C([a,b])$.

So that

$$\left(\Delta_a^{\nu} f \right)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f^{(n)}(t)\, dt, \tag{126}$$

$\forall\, x \in [a,b]$.

Notice here that

$$\Delta_a^n f = f^{(n)}, \quad n \in \mathbb{Z}_+. \tag{127}$$

We need

Theorem 38([4]). Let $f \in C_a^{\nu}([a,b])$, $n = [\nu]$, such that $f^{(i)}(a) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, with $0 < \rho < \nu$. Then

$$\left(\Delta_a^{\rho} f \right)(x) = \frac{1}{\Gamma(\nu - \rho)} \int_a^x (x-t)^{\nu - \rho - 1} \left(\Delta_a^{\nu} f \right)(t)\, dt, \tag{128}$$

i.e.

$$\left(\Delta_a^{\rho} f \right) = I_{a+}^{\nu - \rho} \left(\Delta_a^{\nu} f \right) \in C([a,b]). \tag{129}$$

Thus $f \in C_a^{\rho}([a,b])$.

We present

Theorem 39. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\nu_i > \rho_i > 0$, $i = 1, ..., m$. Let $f_i \in C_a^{\nu_i}([a,b])$, $n_i = [\nu_i]$, such that $f_i^{(k_i)}(a) = 0$, $k_i = r_i, r_i + 1, ..., n_i - 1$, where $r_i := [\rho_i]$, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(\Delta_a^{\rho_i} f_i \right) \right\|_p \leq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i-\rho_i)+m\left(\frac{1}{p}-1\right)+\frac{1}{p}}}{\left(p \sum_{i=1}^{m} (\nu_i - \rho_i) + m(1-p) + 1 \right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) \left(p(\nu_i - \rho_i - 1) + 1\right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \Delta_a^{\nu_i} f_i \right\|_q \right). \quad (130)$$

Proof. Using Theorem 1, see (5), and Theorem 38, see (129). ∎

We also give

Theorem 40. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$; $\nu_i > \rho_i > 0$, $i = 1, ..., m$. Let $f_i \in C_a^{\nu_i}([a,b])$, $n_i = [\nu_i]$, such that $f_i^{(k_i)}(a) = 0$, $k_i = r_i, r_i + 1, ..., n_i - 1$, where $r_i := [\rho_i]$, $i = 1, ..., m$.

Then

$$\left\| \prod_{i=1}^{m} \left(\Delta_a^{\rho_i} f_i \right) \right\|_r \leq \frac{(b-a)^{\sum_{i=1}^{m}(\nu_i-\rho_i)-m+\frac{m}{p}+\frac{1}{r}}}{\left(r \left(\sum_{i=1}^{m} (\nu_i - \rho_i) - m + \frac{m}{p} \right) + 1 \right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^{m} \Gamma(\nu_i - \rho_i) \left(p(\nu_i - \rho_i - 1) + 1\right)^{\frac{1}{p}} \right)} \left(\prod_{i=1}^{m} \left\| \Delta_a^{\nu_i} f_i \right\|_q \right). \quad (131)$$

Proof. Using Theorem 2, see (12), and Theorem 38, see (129). ∎

We need

Definition 41 ([2]). Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C([a,b])$. Consider

$$C_{b-}^{\nu}([a,b]) := \{ f \in C^n([a,b]) : I_{b-}^{1-\alpha} f^{(n)} \in C^1([a,b]) \}. \quad (132)$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1} \left(I_{b-}^{1-\alpha} f^{(n)} \right)'. \quad (133)$$

We set $\Delta_{b-}^{0} f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f \right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (J - x)^{-\alpha} f^{(n)}(J)\, dJ, \quad (134)$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 42 ([4]). Let $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. Assume $f^{(i)}(b) = 0$, $i = r, r+1, ..., n-1$, where $r := [\rho]$, $n := [\nu]$. Then

$$\Delta_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_x^b (J-x)^{\nu-\rho-1} \left(\Delta_{b-}^{\nu} f\right)(J)\, dJ, \tag{135}$$

$\forall\ x \in [a,b]$, i.e.

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu-\rho}\left(\Delta_{b-}^{\nu} f\right) \in C([a,b]), \tag{136}$$

and $f \in C_{b-}^{\rho}([a,b])$.

We present

Theorem 43. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$; $\nu_i > \rho_i > 0$, $i = 1, ..., m$. Let $f_i \in C_{b-}^{\nu_i}([a,b])$ such that $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1, ..., n_i-1$, where $r_i := [\rho_i]$, $n_i := [\nu_i]$, $i = 1, ..., , m$.

Then

$$\left\| \prod_{i=1}^m \left(\Delta_{b-}^{\rho_i} f_i\right) \right\|_p \leq \frac{(b-a)^{\sum_{i=1}^m (\nu_i-\rho_i) + m\left(\frac{1}{p}-1\right) + \frac{1}{p}}}{\left(p \sum_{i=1}^m (\nu_i-\rho_i) + m(1-p) + 1\right)^{\frac{1}{p}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\nu_i-\rho_i)\left(p(\nu_i-\rho_i-1)+1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^m \left\|\Delta_{b-}^{\nu_i} f_i\right\|_q\right). \tag{137}$$

Proof. Using Theorem 3, see (17), and Theorem 42, see (136). ■

We also give

Theorem 44. Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $r > 0$; $\nu_i > \rho_i > 0$, $i = 1, ..., m$. Let $f_i \in C_{b-}^{\nu_i}([a,b])$ such that $f_i^{(k_i)}(b) = 0$, $k_i = r_i, r_i+1, ..., n_i-1$, where $r_i := [\rho_i]$, $n_i := [\nu_i]$, $i = 1, ..., , m$.

Then

$$\left\| \prod_{i=1}^m \left(\Delta_{b-}^{\rho_i} f_i\right) \right\|_r \leq \frac{(b-a)^{\sum_{i=1}^m (\nu_i-\rho_i) - m + \frac{m}{p} + \frac{1}{r}}}{\left(r\left(\sum_{i=1}^m (\nu_i-\rho_i) - m + \frac{m}{p}\right) + 1\right)^{\frac{1}{r}}}$$

$$\cdot \frac{1}{\left(\prod_{i=1}^m \Gamma(\nu_i-\rho_i)\left(p(\nu_i-\rho_i-1)+1\right)^{\frac{1}{p}}\right)} \left(\prod_{i=1}^m \left\|\Delta_{b-}^{\nu_i} f_i\right\|_q\right). \tag{138}$$

Proof. Using Theorem 4, see (25), and Theorem 42, see (136). ■

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The R-Transform of a Real-Valued Function and some of Its Applications

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Abstract

The role of the Difference Calculus, with all its applications to various branches of Applied Mathematics, is well established. One of the main applications of the Calculus of Differences is to provide methods for obtaining solutions to Difference Equations. However, while the published research on obtaining approximate solutions to various types of Differential Equations is quite extensive, the corresponding research for finding approximate solution of Difference Equations is rather limited. In this paper we present a method for obtaining approximate solutions of the Difference Equation $y(x+1) - y(x) = f(x)$, $a < x < \infty$, by means of an appropriate transformation, for a broad class of functions. Using the same transformation, it is possible to express in closed form sums of the form $\sum_{\lambda=0}^{K} f(x+\lambda)$, $K \leq \infty$. As a characteristic example, the Hurwitz Zeta Function will be considered.

Key words and phrases: Complete monotonicity; difference equation; approximate solution; Gamma function; Hurwitz zeta function.

1 Introduction

We begin by introducing the R-transform.

Definition 1. Let $f(x)$ be a real valued function of the real variable x, defined on the interval $[a, \infty)$. The function $f(x)$ is assumed to be continuous over its interval of definition. Given $f(x)$, we define a new function $R_1(x)$, named the R-transform of $f(x)$, by means of the formula

$$R\{f(x)\} := R_1(x) := \frac{1}{3}\{f(x) + 4f(x+1) + f(x+2)\} - \int_x^{x+2} f(t)dt \quad (1.1)$$

Starting with (1.1), we may define a family of functions $R_2(x) := R\{R_1(x)\}$, $R_3(x) := R\{R_2(x)\}$ and, in general

$$R_{k+1}(x) := R\{R_k(x)\}, \qquad k \in \mathbb{N} = \{0, 1, 2, ...\}, \tag{1.2}$$

where $R_0(x) := f(x)$.

Next, we list some basic properties of the R-Transform.

1. The R-transform of a function $f(x)$ is a linear transform, i.e.

$$R\left\{\sum_{k=1}^{n} c_k f_k(x)\right\} = \sum_{k=1}^{n} c_k R\left\{f_k(x)\right\}, \tag{1.3}$$

where $c_1, c_2, \ldots, c_n$, are constants.

2. Assuming that $f(x)$ is λ times differentiable on $[a, \infty)$, then

$$R_k\left\{\frac{d^\lambda f(x)}{dx^\lambda}\right\} = \frac{d^\lambda}{dx^\lambda} R_k\{f(x)\}, \qquad k, \lambda = 1, 2, \ldots . \tag{1.4}$$

3. If $R_1(x) = R\{f(x)\}$ and b is an arbitrary constant, then (assuming $x + b$ belongs to the domain of f)

$$R_1(x + b) = R\{f(x + b)\}. \tag{1.5}$$

The proofs of (1.3), (1.4), and (1.5) stem directly from Definition 1.

4.

$$R\left\{\int_x^{x+b} f(t)dt\right\} = \int_x^{x+b} R\{f(t)\}dt. \tag{1.6}$$

Proof. Let $F(t)$ be an antiderivative of $f(t)$. Then, by (1.3), (1.4), and (1.5)

$$R\left\{\int_x^{x+b} f(t)dt\right\} = R\{F(x+b)\} - R\{F(x)\} = \int_x^{x+b} \frac{d\{R(F(t)\}}{dt} dt = \int_x^{x+b} R\{f(t)\}dt.$$

■

5. If $f(x)$ is monotone on $[a, \infty)$, then

$$|R\{f(x)\}| = |R_1(x)| < \frac{2}{3} |f(x + 2) - f(x)| . \tag{1.7}$$

Proof. Let us assume that $f(x)$ is increasing on $[a, \infty)$. Then,

$$f(x) < \int_x^{x+1} f(t)dt < f(x + 1)$$

and

$$f(x + 1) < \int_{x+1}^{x+2} f(t)dt < f(x + 2),$$

from which, by addition, one obtains

$$f(x)+f(x+1)<\int_x^{x+2} f(t)dt<f(x+1)+f(x+2),$$

or equivalently,

$$-\frac{2}{3}f(x)+\frac{1}{3}f(x+1)+\frac{1}{3}f(x+2)>R\{f(x)\}>\frac{1}{3}f(x)+\frac{1}{3}f(x+1)-\frac{2}{3}f(x+2),$$

or even,

$$\frac{2}{3}\left[f(x+2)-f(x)\right]>R\{f(x)\}>-\frac{2}{3}\left[f(x+2)-f(x)\right],$$

since $f(x)$ was assumed to be increasing on $[a,\infty)$. It has thus been proved that

$$|R\{f(x)\}|=|R_1(x)|<\frac{2}{3}\left|f(x+2)-f(x)\right|.$$

In case where $f(x)$ is decreasing on $[a,\infty)$, $-f(x)$ will be increasing over the same interval and (1.7) is readily obtained. ■

6. If $f(x)$ is positive and decreasing, or is negative and increasing on $[a,\infty)$, then

$$|R\{f(x)\}|=|R_1(x)|<\frac{2}{3}\left|f(x)\right|. \tag{1.8}$$

Proof. Assuming that $f(x)$ is possitive and decreasing, then according to (1.7)

$$|R_1(x)|<\frac{2}{3}\left|f(x+2)-f(x)\right|=\frac{2}{3}(f(x)-f(x+2))<\frac{2}{3}f(x),$$

since $f(x+2)>0$, i.e.

$$|R_1(x)|<\frac{2}{3}\left|f(x)\right|.$$

In the case where $f(x)$ is negative and increasing, $-f(x)$ will be positive and decreasing, and (1.8) is obtained easily. ■

7. If $f(x)$ is positive and decreasing or is negative and increasing on $[a,\infty)$, and if

$$\lim_{x\to+\infty} f(x)=0,$$

then

$$\lim_{x\to+\infty} R_1(x)=0. \tag{1.9}$$

Proof. From equation (1.8) it follows that

$$0\leq|R_1(x)|<\frac{2}{3}\left|f(x)\right|,$$

and since, by assumption

$$\lim_{x\to+\infty} f(x) = 0,$$

it follows

$$\lim_{x\to+\infty} R_1(x) = 0.$$

■

8. If $f(x)$ is positive and decreasing, or is negative and increasing on $[a,\infty)$, then

$$\left|\sum_{\lambda=0}^{k} R_1(x+\lambda)\right| < \frac{2}{3}\left|f(x)+f(x+1)\right|, \qquad k=1,2,3,\dots. \tag{1.10}$$

Proof. Let us first assume that $f(x)$ is positive and decreasing on $[a,\infty)$. Then, making use of formula (1.7), one obtains

$$\left|\sum_{\lambda=0}^{k} R_1(x+\lambda)\right| \le \sum_{\lambda=0}^{k} \left|R_1(x+\lambda)\right| < \frac{2}{3}\sum_{\lambda=0}^{k}\left|f(x+\lambda+2)-f(x+\lambda)\right|,$$

i.e.

$$\left|\sum_{\lambda=0}^{k} R_1(x+\lambda)\right| \le \sum_{\lambda=0}^{k} \left|R_1(x+\lambda)\right| < \frac{2}{3}\sum_{\lambda=0}^{k}\left[f(x+\lambda)-f(x+\lambda+2)\right],$$

but

$$\frac{2}{3}\sum_{\lambda=0}^{k}\left[f(x+\lambda)-f(x+\lambda+2)\right] = \frac{2}{3}\left[f(x)+f(x+1)-f(x+k+1)-f(x+k+2)\right],$$

thus

$$\sum_{\lambda=0}^{k}\left|R_1(x+\lambda)\right| < \frac{2}{3}\left[f(x)+f(x+1)\right],$$

since, by assumption $f(x+k+1)$ and $f(x+k+2)$ are positive quantities. In case where $f(x)$ is negative and increasing, on $[a,\infty)$, the function $-f(x)$ will be positive and decreasing over the same interval, and thus the formula (1.10) is easily obtained. ■

9. On the assumption that the fourth order derivative of $f(x)$ exists on $[a,\infty)$, one has

$$R_1(x) = R\{f(x)\} = \frac{1}{90}f^{(4)}(\xi), \quad x<\xi<x+2. \tag{1.11}$$

Proof. Since $R\{f(x)\}$ is actually the error in evaluating the area under a given curve $f(t)$, from $t=x$ up to $t=x+2$, by means of the Simpson's rule, expression (1.11) for the error is well known, see for example [1]. ■

2 Completely Monotonic Functions

A function $f(x)$ is said to be *completely monotonic* (*c.m.*) on $[a, \infty)$, if
(i) $f(x)$ possesses derivatives of all orders and
(ii) $(-1)^k f^{(k)}(x) > 0$ or $(-1)^k f^{(k)}(x) < 0$, for $k = 0, 1, 2, \ldots$.

Functions of complete monotonicity have attracted special attention by various researchers, see for example [7], [14], [8], [9], [10], [3], and [4].

Definition 2. A (smooth) function defined on some interval $[a, \infty)$ is said to belong to the class $\mathcal{M}_4$, if its fourth derivative is completely monotonic, i.e.

$$f \in \mathcal{M}_4 \Leftrightarrow \{(-1)^k f^{(k)}(x) > 0 \quad \text{or} \quad (-1)^k f^{(k)}(x) < 0, \quad \text{for} \quad k = 4, 5, 6, \ldots\}$$

Typical functions belonging to the class $\mathcal{M}_4$, are the following:

- $x^{-p}, \quad p > 0,\ x > 0;$
- $x^{\frac{1}{q}}, \quad q > 1,\ x > 0;$
- $\ln x, \quad x > 0;$
- $e^{-x}, \quad x > 0;$
- The Laplace transform $f(x)$ of a positive function $F(t)$, $0 < t < \infty$, i.e. $f(x) = \int_0^\infty F(t) e^{-tx} dt$, see [2], [15], [16], and [5].

The R-Transform, when applied to functions of $\mathcal{M}_4$, leads to some quite interesting results, which are to be developed in the sequel.

Theorem 1. Let $f \in \mathcal{M}_4$. Then, the functions $R_k(x)$, $k = 1, 2, \ldots$, where $R_1(x) = R\{f(x)\}$ and $R_{k+1}(x) = R\{R_k(x)\}$, are completely monotonic on $[a, \infty)$ and have the sign of $f^{(4)}(x)$.

Proof. Let us assume without loss of generality that $(-1)^k f^{(k)}(x) > 0$, $k = 4, 5, \ldots$. Notice that, by virtue of (1.11), for $m = 0, 1, \ldots$, if $D = d/dx$, we have

$$(-1)^m D^m R_1(x) = (-1)^m D^m R\{f(x)\} = (-1)^m R\left\{D^m(f(x))\right\} = \frac{(-1)^m}{90} f^{(m+4)}(\xi_m),$$

where $x < \xi_m < x + 2$. Since $f \in \mathcal{M}_4$ and $(-1)^{m+4} = (-1)^m$ we see that $R_1(x)$ is completely monotonic and has the sign of $f^{(4)}(x)$. In a similar fashion we can show that the statement is true for $R_2(x)$. Indeed,

$$(-1)^m D^m R_2(x) = (-1)^m R\left\{D^m(R_1(x))\right\} = (-1)^m \frac{1}{90} R_1^{(m+4)}(\eta_m) > 0,$$

where $x < \eta_m < x+2$, $m = 0, 1, 2, \ldots$, since $R_1(x)$ is c.m. and hence in $\mathcal{M}_4$. Also, $R_2(x) > 0$, i.e. $R_2(x)$ and $f^{(4)}(x)$ have the same sign.

Proceeding in a similar way, we prove step by step, that all $R_k(x)$ are c.m. and positive. ■

Remark 1. Clearly, if f is c.m. on $[a,\infty)$, so are its derivatives of all orders. Likewise, if $f \in \mathcal{M}_4$, then $f^{(m)} \in \mathcal{M}_4$ for $m = 1, 2, \ldots$. Hence, by Theorem 1, if $f \in \mathcal{M}_4$, we have that $D^m R_k(x)$ is c.m. for all $m, k = 1, 2, \ldots$.

Theorem 2. On the assumption that $f \in \mathcal{M}_4$, $a \le x < \infty$, all functions $D^m R_k(x)$, $m, k = 1, 2, ...$, will be absolutely decreasing on the interval $[a,\infty)$, i.e. will be either positive and decreasing, or will be negative and increasing on $[a,\infty)$.

Proof. Assuming that $(-1)^k f^{(k)}(x) > 0$, $k = 4, 5, 6, \ldots$, all the $R_k(x)$'s are positive and decreasing (since $R_k(x) > 0$ and $DR_k(x) < 0$, $k = 1, 2, \ldots$), while the functions $DR_k(x)$, $k = 4, 5, 6, \ldots$, are negative and increasing (since $DR_k(x) < 0$ and $D^2R_k(x) > 0$, $k = 1, 2, \ldots$).

Likewise, step by step, we prove that the functions $D^2R_k(x)$, $k = 4, 5, 6, \ldots$, are positive and decreasing, $D^3R_k(x)$, $k = 4, 5, 6, \ldots$, are negative and increasing, etc.

The case $(-1)^k f^k(x) < 0$, $k = 4, 5, 6, \ldots$ is treated in a similar way. ■

Theorem 3. If $f \in \mathcal{M}_4$, then

$$|R_n(x)| < \left(\frac{2}{3}\right)^n |f(x)|. \tag{2.1}$$

Proof. Since $f \in \mathcal{M}_4$, by virtue of Theorem 2 the functions $R_k(x)$, $k = 1, 2, \ldots$ will be absolutely decreasing on $[a,\infty)$. Then by means of (1.8)

$$|R_k(x)| < \frac{2}{3}\,|R_{k-1}(x)|, \quad k = 1, 2, \ldots, n.$$

Multiplying together the inequalities above, from $k = 1$ up to $k = n$, (2.1) is easily obtained. ■

Corollary 1. Under the hypothesis of Theorem 3,

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for every} \quad x \in (a,\infty). \tag{2.2}$$

Corollary 2. Under the hypothesis of Theorem 3, and on the additional assumption that $\lim_{x\to\infty} f(x) = 0$ we have

$$\lim_{x\to\infty} R_n(x) = 0, \quad \text{for every} \quad n \in \mathbb{N}. \tag{2.3}$$

Theorem 4. If f belongs to $\mathcal{M}_4$, $x \in [a,\infty)$, then

$$\left|\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda)\right| < \frac{2}{3}\,|R_n(x) + R_n(x+1)|. \tag{2.4}$$

Proof. Since f is in $\mathcal{M}_4$, by Theorem 2 the functions $R_k(x)$, $k = 1, 2, 3, \dots$ will be absolutely decreasing on $[a, \infty)$. Making use of (1.10) one obtains

$$\left|\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda)\right| < \frac{2}{3}\left|R_n(x) + R_n(x+1)\right|.$$

■

Corollary 3. Under the hypothesis of Theorem 4,

$$\left|\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda)\right| < \left(\frac{2}{3}\right)^{n+1}\big(|f(x)| + |f(x+1)|\big). \tag{2.5}$$

Theorem 5. If $f \in \mathcal{M}_4$, then

$$|R_n(x)| < \frac{1}{90^n}\left|f^{(4n)}(x)\right|, \quad n = 1, 2, 3, \dots. \tag{2.6}$$

Proof. We will prove (2.6) using mathematical induction.

For $n = 1$ (2.6) is true because $|R_1(x)| = \frac{1}{90}\left|f^{(4)}(\xi)\right|$, $x < \xi < x+2$, and since $\left|f^{(4)}(x)\right|$ is absolutely decreasing on $[a, \infty)$

$$|R_1(x)| = \frac{1}{90}\left|f^{(4)}(\xi)\right| < \frac{1}{90}\left|f^{(4)}(x)\right|.$$

Assuming that (2.6) is true for $n = k$, we will show that (2.6) will also be true for $n = k+1$. Making use of (1.11)

$$R_{k+1}(x) = R\{R_k(x)\} = \frac{1}{90}R_k^{(4)}(\eta) \quad \text{where} \quad x < \eta < x+2.$$

Therefore,

$$|R_{k+1}(x)| = \frac{1}{90}\left|R_k^{(4)}(\eta)\right| < \frac{1}{90}\left|R_k^{(4)}(x)\right|$$

(by virtue of Theorem 2), i.e.

$$|R_{k+1}(x)| < \frac{1}{90}\left|R_k^{(4)}(x)\right| = \frac{1}{90}\left|R_k\left(f^{(4)}(x)\right)\right|$$

(from (1.4)), i.e.

$$|R_{k+1}(x)| < \frac{1}{90}\left|R_k\left(f^{(4)}(x)\right)\right| < \frac{1}{90}\frac{1}{90^k}\left|\left(f^{(4)}(x)\right)^{(4k)}\right|,$$

because of our assumption about $R_k(x)$, and finally,

$$|R_{k+1}(x)| < \frac{1}{90^{k+1}}\left|f^{(4k+4)}(x)\right|,$$

so by means of the principle of mathematical induction (2.6) is true for all $n = 1, 2, 3, \ldots$. ■

Theorem 6. If $f(x)$ is any continuous, real-valued function, defined on $[a, \infty)$ and k is any positive integer, then

$$\sum_{\lambda=0}^{k} f(x+\lambda) =$$

$$= \frac{5}{6} f(x) + \frac{1}{6} f(x+1) - \frac{5}{6} f(x+k+1) - \frac{1}{6} f(x+k+2)$$

$$+\frac{1}{2}\int_{x}^{x+k+1} f(t)dt + \frac{1}{2}\int_{x+1}^{x+k+2} f(t)dt + \frac{1}{2}\sum_{\lambda=0}^{k} R_1(x+\lambda). \tag{2.7}$$

Proof. The proof is straightforward. Starting with the definition of $R_1(t)$, applying it for $t = x, t = x+1, t = x+2, \ldots, t = x+k$ and adding the resulting equations, formula (2.7) is obtained. ■

Theorem 7. If $f(x)$ is any continuous, real-valued function defined on $[a, \infty)$ and k is any positive integer, then

$$\sum_{\lambda=0}^{k} f(x+\lambda)$$

$$= \frac{5}{6} s_n(x) + \frac{1}{6} s_n(x+1) - \frac{5}{6} s_n(x+k+1) - \frac{1}{6} s_n(x+k+2)$$

$$+\frac{1}{2}\int_{x}^{x+k+1} s_n(t)dt + \frac{1}{2}\int_{x+1}^{x+k+2} s_n(t)dt + \frac{1}{2^{n+1}}\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda), \tag{2.8}$$

where

$$s_n(x) := f(x) + \frac{1}{2}R_1(x) + \frac{1}{2^2}R_2(x) + \ldots + \frac{1}{2^n}R_n(x). \tag{2.9}$$

Proof. Starting with formula (2.7) and taking the R-transform of both sides m times successively, $m = 0, 1, ..., n$, one obtains

$$\sum_{\lambda=0}^{k} R_m(x+\lambda)$$

$$= \frac{5}{6} R_m(x) + \frac{1}{6} R_m(x+1) - \frac{5}{6} R_m(x+k+1) - \frac{1}{6} R_m(x+k+2)$$

$$+\frac{1}{2}\int_{x}^{x+k+1} R_m(t)dt + \frac{1}{2}\int_{x+1}^{x+k+2} R_m(t)dt + \frac{1}{2}\sum_{\lambda=0}^{k} R_{m+1}(x+\lambda).$$

Multiplying both sides by $(1/2)^m$, and adding the resulting equations from $m = 0$ up to $m = n$, one obtains (2.8), where the quantity $s_n(x)$ is defined by formula (2.9). ■

Theorem 8. If $f(x)$ is any continuous, real valued function, defined on the interval $[a, \infty)$, then

$$f(x) = \frac{5}{6}s_n(x) - \frac{4}{6}s_n(x+1) - \frac{1}{6}s_n(x+2) + \frac{1}{2}\int_x^{x+2} s_n(t)dt + \frac{1}{2^{n+1}}R_{n+1}(x), \tag{2.10}$$

where $s_n(x)$ is given by (2.9).

Proof. From formula (2.8), one obtains

$$\sum_{\lambda=0}^{k-1} f(x+1+\lambda)$$

$$= \frac{5}{6}s_n(x+1) + \frac{1}{6}s_n(x+2) - \frac{5}{6}s_n(x+k+1) - \frac{1}{6}s_n(x+k+2)$$

$$+\frac{1}{2}\int_{x+1}^{x+k+1} s_n(t)dt + \frac{1}{2}\int_{x+2}^{x+k+2} s_n(t)dt + \frac{1}{2^{n+1}}\sum_{\lambda=0}^{k-1} R_{n+1}(x+1+\lambda).$$

Subtracting this equation from (2.8), equation (2.10) is obtained. ■

3 An Approximate Solution of the Difference Equation $y(x+1) - y(x) = f(x)$

The Difference Equation $\Delta y(x) := y(x+1) - y(x) = f(x)$, $f(x)$ given, was first studied by Krull, in his pioneer work [11] and subsequently by other researchers [12], [6], and [13]. In the present work we derive, by means of the R-transform, an approximate solution to this equation for various functions $f(x)$, $x \geq a$.

Theorem 9. Consider the equation $\Delta y(x) = f(x)$, $a \leq x < \infty$, where $f(x)$ is given (the solution to this equation is determined up to an arbitrary periodic function $p(x)$ of period 1). Let us also define

$$r(n, x) := \frac{1}{2^{n+1}}\frac{R_{n+1}(x)}{f(x)}. \tag{3.1}$$

Then, the function

$$y_n(x) = -\frac{1}{6}s_n(x+1) - \frac{5}{6}s_n(x) + \frac{1}{2}\left[S_n(x+1) + S_n(x)\right], \tag{3.2}$$

where

$$S_n(x) = \int_c^x s_n(t)dt, \quad c = \text{constant},$$

($s_n(x)$ is given by (2.9)) will satisfy the Difference Equation

$$\Delta y_n(x) = y_n(x+1) - y_n(x) = [1 - r(n,x)]\, f(x),$$

i.e. $y_n(x)$ as defined in (3.2), is an approximate solution of $\Delta y(x) = f(x)$, provided that, for $x \geq a$ we have $r(n,x) \to 0$ as $n \to \infty$.

Proof. We have

$$\Delta y_n(x) = y_n(x+1) - y_n(x) = -\frac{1}{6}s_n(x+2) - \frac{4}{6}s_n(x+1) + \frac{5}{6}s_n(x) + \frac{1}{2}\int_x^{x+2} s_n(t)dt,$$

and taking into account (2.10) and (3.1),

$$\Delta y_n(x) = f(x) - \frac{1}{2^{n+1}}R_{n+1}(x) = [1 - r(n,x)]\, f(x).$$

■

Remark 2. The power of the method lies in the fact that $r(n,x)$ is negligible as compared to 1, even for $n = 1$ (first order approximation), over some interval $[a, \infty)$, for some family of functions, for example for the functions belonging to the class $\mathcal{M}_4$.

Example 1. The Difference Equation $\Delta y(x) = \ln x$, $x \in [3, \infty)$. For $n = 1$ Theorem 9 gives the approximate solution

$$y_1(x) = -\frac{1}{6}s_1(x+1) - \frac{5}{6}s_1(x) + \frac{1}{2}\left[S_1(x+1) + S_1(x)\right],$$

where by (2.9)

$$s_1(x) = f(x) + \frac{1}{2}R_1(x) \quad \text{and} \quad S_1(x) = \int_c^x s_1(t)dt.$$

Tedious but straightforward calculations yield

$$y_1(x) = \left[-\frac{(x+3)^2}{8} + \frac{x+3}{6} - \frac{1}{36}\right]\ln(x+3) + \left[-\frac{(x+2)^2}{8} + \frac{5(x+2)}{6} - \frac{9}{36}\right]\ln(x+2)$$
$$+\left[\frac{(x+1)^2}{8} + \frac{5(x+1)}{6} - \frac{27}{36}\right]\ln(x+1) + \left[\frac{x^2}{8} + \frac{x}{6} - \frac{35}{36}\right]\ln x - \frac{x}{2}. \quad (3.3)$$

The error term, over the interval $[3, \infty)$ is

$$|r(1,x)| = \frac{1}{2^2}\left|\frac{R_2(x)}{f(x)}\right|$$

and since $f(x) = \ln x$ belongs to $\mathcal{M}_4$,

$$|R_2(x)| < |R_2(3)| < \frac{1}{90^2}\left|\ln^{(8)}(x)\right|_{x=3}$$

(from Theorems 2 and 5), while $|\ln x| > |\ln 3|$ for $x \geq 3$, we have

$$|r(1,x)| < \frac{1}{2^2} \cdot \frac{1}{\ln 3} \cdot \frac{1}{90^2} \cdot \frac{7!}{3^8} \approx 2.158 \cdot 10^{-5}, \quad x \geq 3.$$

Now recall that the logarithm of the Gamma function $\ln \Gamma(x)$ also satisfies $\Delta y(x) = \ln x$. Hence, we expect that

$$\ln \Gamma(x) \approx y_1(x) + p(x), \qquad \text{where} \qquad p(x+1) = p(x). \tag{3.4}$$

To determine $p(x)$ we look at the asymptotic behavior of $\ln \Gamma(x)$ and $y_1(x)$ as $x \to \infty$. Stirling's formula gives

$$\ln \Gamma(x) = (1/2)\ln(2\pi) - x + (x - 1/2)\ln x + O\left(1/x\right),$$

while (3.3) yields

$$y_1(x) = 3/4 - x + (x - 1/2)\ln x + o(1).$$

Comparison of the above two formulas suggests that $p(x)$ of (3.4) is the constant $(1/2)\ln(2\pi) - 3/4$. Therefore,

$$\ln \Gamma(x) \approx y_1(x) + (1/2)\ln(2\pi) - 3/4. \tag{3.5}$$

The accuracy (3.5) is illustrated by the following list.

x	$y_1(x) + (1/2)\ln(2\pi) - 3/4$	$\ln \Gamma(x)$
3	0.693146	0.693147
3.45	1.14623	1.14623
4	1.791759	1.791759

In fact, the accuracy gets better as x increases.

Example 2. Solve the difference Equation $\Delta y(x) = x^{\frac{1}{q}}$, $q > 1$, on the interval $[5, \infty)$.
Taking $n = 1$, and proceeding as in Example 1 we find the approximate solution to be

$$y_1(x) = \left[-\frac{35}{36} + \frac{q}{6(q+1)}x + \frac{q^2}{4(q+1)(2q+1)}x^2\right] x^{\frac{1}{q}}$$

$$+ \left[-\frac{27}{36} + \frac{5q}{6(q+1)}(x+1) + \frac{q^2}{4(q+1)(2q+1)}(x+1)^2\right](x+1)^{\frac{1}{q}}$$

$$+ \left[-\frac{9}{36} + \frac{5q}{6(q+1)}(x+2) - \frac{q^2}{4(q+1)(2q+1)}(x+2)^2\right](x+2)^{\frac{1}{q}}$$

$$+ \left[-\frac{1}{36} + \frac{q}{6(q+1)}(x+3) - \frac{q^2}{4(q+1)(2q+1)}(x+3)^2\right](x+3)^{\frac{1}{q}}.$$

The error term $|r(1,x)|$, over the interval $[5,\infty)$ is $|r(1,x)| = \frac{1}{2^2}\left|\frac{R_2(x)}{f(x)}\right|$, and since $f(x) = x^{\frac{1}{q}} \in \mathcal{M}_4$ and $f(x)$ is increasing, $|r(1,x)| < |r(1,5)|$ for $x \geq 5$, we have

$$|r(1,x)| = \frac{1}{2^2}\left|\frac{R_2(x)}{f(x)}\right| < \frac{1}{2^2}\cdot\frac{1}{90^2}\left|\frac{f^{(8)}(x)}{f(x)}\right|_{x=5} = \frac{1}{2^2}\cdot\frac{1}{90^2}\left|\prod_{k=1}^{7}(1-kq)\right|\frac{1}{(5q)^8}.$$

For example, if $q=2$, in which case the expression for $y_1(x)$, gives the approximate solution of $\Delta y(x) = \sqrt{x}$, we get $|r(1,x)| < 4,17\cdot 10^{-8}$, for all $x \geq 5$.

4 Evaluating Finite/Infinite Sums

Finite sums of series can be computed with the aid of the R-transform. The main result is summarized in the following theorem.

Theorem 10. (**a**) For any function $f(x)$, continuous on the interval $[a,\infty)$, we have

$$\begin{aligned}\sum_{\lambda=0}^{k} f(x+\lambda) &= f(x)+f(x+1)+f(x+2)+\ldots+f(x+k) = \\ &= \frac{5}{6}s_n(x)+\frac{1}{6}s_n(x+1)-\frac{5}{6}s_n(x+k+1)-\frac{1}{6}s_n(x+k+2) \\ &\quad +\frac{1}{2}\int_{x}^{x+k+1}s_n(t)dt+\frac{1}{2}\int_{x+1}^{x+k+2}s_n(t)dt+e(n,x), \qquad (4.1)\end{aligned}$$

where $s_n(x)$ is given by (2.9) and the error term $e(n,x)$ is

$$e(n,x) = \frac{1}{2^{n+1}}\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda) \qquad (4.2)$$

(**b**) If we further assume that $f \in \mathcal{M}_4$, then

$$0 < |e(n,x)| < \frac{1}{3\cdot 180^n}\left[\left|f^{(4n)}(x)\right| + \left|f^{(4n)}(x+1)\right|\right]. \qquad (4.3)$$

It should be noted that the error $e(n,x)$ depends only on x and n and not on k. As a matter of fact, for a given n, the error, in absolute value, decreases as x increases, since $f \in \mathcal{M}_4$.

Proof. (**a**) The first part of the Theorem 10 follows directly from Theorem 7, equation (2.8).

(**b**) By virtue of (4.2), (2.4), and Theorem 4,

$$|e(n,x)| = \frac{1}{2^{n+1}}\left|\sum_{\lambda=0}^{k} R_{n+1}(x+\lambda)\right| < \frac{1}{3\cdot 2^n}\left|R_n(x)+R_n(x+1)\right|.$$

Now, making use of Theorem 5 yields

$$|e(n,x)| < \frac{1}{3 \cdot 2^n \cdot 90^n} \left(\left| f^{(4n)}(x) \right| + \left| f^{(4n)}(x+1) \right| \right).$$

■

Example 3. Evaluate the sum of the harmonic series

$$\sum_{\lambda=0}^{k} \frac{1}{x+\lambda},$$

on the interval $[5, \infty)$ (here k is an integer ≥ 1).

Applying (4.1) with $n = 1$, one obtains

$$\sum_{\lambda=0}^{k} \frac{1}{x+\lambda} = \Phi(x+k+1) - \Phi(x) + e(1,x),$$

where

$$\Phi(x) = \left(\frac{1}{6} - \frac{x+3}{4} \right) \ln(x+3) + \left(\frac{5}{6} - \frac{x+2}{4} \right) \ln(x+2)$$

$$+ \left(\frac{5}{6} + \frac{x+1}{4} \right) \ln(x+1) + \left(\frac{1}{6} + \frac{x}{4} \right) \ln x - \frac{1}{36} \left(\frac{1}{x+3} + \frac{9}{x+2} + \frac{27}{x+1} + \frac{35}{x} \right).$$

For the error term on $[5, \infty)$ we have

$$|e(1,x)| < |e(1,5)| = \frac{1 \cdot 4!}{3 \cdot 180} \left[\left(\frac{1}{5} \right)^5 + \left(\frac{1}{6} \right)^5 \right]$$

(since $\left| f^{(4)}(x) \right| = |(1/x)^{(4)}| = 4! |x|^{-5}$), i.e. $|e(1,x)| < 1.993 \cdot 10^{-5}, \quad x \geq 5$.

Theorem 11. (**a**) Assuming that $f \in \mathcal{M}_4$ and the series

$$\sum_{\lambda=0}^{\infty} f(x+\lambda)$$

converges, we have

$$\sum_{\lambda=0}^{\infty} f(x+\lambda) = \frac{5}{6} s_n(x) + \frac{1}{6} s_n(x+1) + \frac{1}{2} \int_x^{\infty} s_n(t) dt + \frac{1}{2} \int_{x+1}^{\infty} s_n(t) dt + e(n,x) \tag{4.4}$$

where $s_n(x)$ is given by (2.9) and

$$e(n,x) = \frac{1}{2^{n+1}} \sum_{\lambda=0}^{\infty} R_{n+1}(x+\lambda) \tag{4.5}$$

(**b**) With the same assumptions as in Part (**a**), we have

$$0 < |e(n,x)| < \frac{1}{3 \cdot 2^n} \left| R_n(x) + R_n(x+1) \right|. \tag{4.6}$$

Proof. (**a**) From (2.9)

$$s_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} R_k(x),$$

i.e.

$$0 < |s_n(x)| = \left| \sum_{k=0}^{n} \frac{1}{2^k} R_k(x) \right| \leq \sum_{k=0}^{n} \frac{1}{2^k} \left| R_k(x) \right|,$$

and making use of (2.1),

$$0 < |s_n(x)| \leq \sum_{k=0}^{n} \frac{1}{2^k} \left(\frac{2}{3} \right)^k |f(x)| = \frac{3}{2} \left[1 - \left(\frac{1}{3} \right)^{n+1} \right] |f(x)|. \tag{4.7}$$

Since the series

$$\sum_{\lambda=0}^{\infty} f(x+\lambda)$$

is assumed to be convergent, we must have

$$\lim_{x \to +\infty} f(x) = 0,$$

so from (4.7)

$$\lim_{x \to +\infty} s_n(x) = 0. \tag{4.8}$$

Equation (4.4) is obtained immediately from (4.1), if we pass to the limit as $k \to +\infty$, and make use of (4.8).
(**b**) Regarding the error term $e(n,x)$, we know that

$$0 < |e(n,x)| = \frac{1}{2^{n+1}} \left| \sum_{\lambda=0}^{k} R_{n+1}(x+\lambda) \right| < \frac{1}{3 \cdot 2^n} \left| R_n(x) + R_n(x+1) \right|$$

(from (2.4)) and if we pass to the limit as $k \to +\infty$, (4.6) is obtained. ■

Example 4 (The Hurwitz Zeta Function). The following series, is known as the Hurwitz Zeta Function

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s},$$

where here q and s are assumed to be real, with $q > 0$ and $s > 1$, so that the infinite series converges. Making use of Theorem 11, with $n = 1$, $\zeta(s,q)$ can be expressed as

$$\zeta(s,q) = \frac{35}{36} \frac{1}{q^s} + \frac{27}{36} \frac{1}{(q+1)^s} + \frac{9}{36} \frac{1}{(q+2)^s} + \frac{1}{36} \frac{1}{(q+3)^s}$$

$$+\frac{1}{s-1}\left[\frac{1}{6}\frac{1}{q^{s-1}}+\frac{5}{6}\frac{1}{(q+1)^{s-1}}+\frac{5}{6}\frac{1}{(q+2)^{s-1}}+\frac{1}{6}\frac{1}{(q+3)^{s-1}}\right]$$

$$+\frac{1}{4(s-1)(s-2)}\left[-\frac{1}{q^{s-2}}-\frac{1}{(q+1)^{s-2}}+\frac{1}{(q+2)^{s-2}}+\frac{1}{(q+3)^{s-2}}\right]+e(1,q,s), \tag{4.9}$$

where

$$0<|e(1,q,s)|<\frac{1}{6}\left|R_1(q,s)+R_1(q+1,s)\right|,$$

i.e.

$$0<|e(1,q,s)|<\frac{1}{6}\left|\frac{1}{3}\left(\frac{1}{q^s}+\frac{5}{(q+1)^s}+\frac{5}{(q+2)^s}+\frac{1}{(q+3)^s}\right)+\right.$$

$$\left.\frac{1}{s-1}\left(-\frac{1}{q^{s-1}}-\frac{1}{(q+1)^{s-1}}+\frac{1}{(q+2)^{s-1}}+\frac{1}{(q+3)^{s-1}}\right)\right|. \tag{4.10}$$

The function $R_1(q,s)$ considered as a function of q (s fixed) belongs to $\mathcal{M}_4$. The same function considered as a function of s ($q>1$ fixed) also belongs to $\mathcal{M}_4$, as can be easily shown. Indeed,

$$(-1)^n\frac{\partial^n}{\partial s^n}(q^{-s})=(\ln q)^n q^{-s}>0. \qquad q>1.$$

In the region $q\ge q_0>1$ and $s\ge s_0>1$, the error term satisfies

$$0<|e(1,q,s)|<|e(1,q_0,s_0)|,$$

since $R_1(q,s)$ and $R_1(q+1,s)$ belong to $\mathcal{M}_4$, therefore $R_1(q,s)+R_1(q+1,s)\in\mathcal{M}_4$, i.e. the function $f(s):=R_1(q,s)+R_1(q+1,s)$ is absolutely decreasing with respect to both variables q and s in the region $q\ge q_0>1$ and $s\ge s_0>1$. For example, in the region $q\ge 3$ and $s\ge 4$, the error term satisfies

$$0<|e(1,q,s)|<|e(1,3,4)|\approx 3.422\cdot 10^{-5}.$$

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COMMUTANTS OF A TOEPLITZ OPERATOR WITH A CERTAIN HARMONIC SYMBOL

ABDELRAHMAN YOUSEF

Abstract. In this paper we show, under some conditions, that only polynomials of $T_{z+\bar{z}}$ can commute with $T_{z+\bar{z}}$.

1. Introduction

Let $dA = \frac{1}{\pi} r dr d\theta$, where (r, θ) are the polar coordinates in the complex plane $\mathbb{C}$, denote the normalized Lebesgue area measure on the unit disk $\mathbb{D}$, so that the measure of $\mathbb{D}$ equals 1.

The Bergman space $L^2_a(\mathbb{D})$ is the Hilbert space consisting of all analytic functions in $L^2(\mathbb{D}, dA)$, the space of all square integrable functions on $\mathbb{D}$ with respect to the area measure dA. It is well known that $L^2_a(\mathbb{D})$ is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$, and has the set $\{\sqrt{n+1} z^n \mid n \geq 0\}$ as an orthonormal basis. Let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$.

For a function $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $L^2_a(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$, for $f \in L^2_a(\mathbb{D})$.

In [3], Čučković proved that if S is an operator in the closed norm subalgebra, generated by Toeplitz operators, such that S commutes with T_{z^n}, then $S = T_\psi$ where ψ is a bounded analytic function on $\mathbb{D}$. Later in [2], Axler, Čučković and Rao proved that if two Toeplitz operators on a Bergman space commute and the symbol of one of them is analytic and nonconstant, then the other one is also analytic. Also, they asked the following question:

Suppose ϕ is a bounded harmonic function on the disk that is neither analytic nor conjugate analytic. If ψ is a bounded measurable function on the disk such that T_ϕ and T_ψ commute, must ψ be of the form $a\phi + b$ for some constants a, b?

The only work in the literature that has been done regarding this question can be found in [9]. The authors there obtained a positive answer under some restrictions. In fact, they proved if $f \in L^1(\mathbb{D}, dA)$ is of the form $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ such that T_f is bounded, and T_f commutes with $T_{z+\bar{z}}$, then T_f must be a polynomial of $T_{z+\bar{z}}$ of degree at most 3. Using the same technique in their result, one can see that if $f \in L^\infty(\mathbb{D})$, then $T_f = aT_{z+\bar{z}} + bI$ for some constants a, b, which answers the question above partially. Moreover, and in a more general setting, the

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third author in [9] showed in [11] that if T_f commutes with $T_{z+\overline{g(z)}}$, where g is a bounded analytic function on $\mathbb{D}$ and $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ is bounded, then $T_f = aT_{z+\overline{g(z)}} + bI$ for some constants a, b.

Now, a related question to the above question and its partial answer is the following: what are the commutants of $T_{z+\bar{z}}$?, or in other words, are polynomials of $T_{z+\bar{z}}$ the only commutants of $T_{z+\bar{z}}$?. In section 3 of this paper, we shall give a partial answer to this question.

2. Preliminaries

A function f is said to be quasihomogeneous of degree p, where p is an integer, if it is of the form $e^{ip\theta}\phi$, where ϕ is a radial function. In this case the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz operator of degree p. Those Toeplitz operators were studied in [4] and [6]. The reason that we study such family of symbols is that any function f in $L^2(\mathbb{D}, dA)$ has the following polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], rdr)$.

Now, we need to introduce the Mellin transform that has been a very useful tool in obtaining many results. The Mellin transform $\widehat{f}$ of a radial function f in $L^1([0,1], rdr)$ is defined by

$$\widehat{f}(z) = \int_0^1 f(r) r^{z-1}\, dr.$$

It is well known that, for these functions, the Mellin transform is well defined on the right half-plane $\{z : \Re z \geq 2\}$ and it is analytic on $\{z : \Re z > 2\}$.

The following lemma, in [9], is helpful to avoid tedious calculations.

Lemma 1. *Let $k, p \in \mathbb{N}$ and let f be an integrable radial function. Then*

$$T_{e^{ip\theta}f}(z^k) = 2(k+p+1)\widehat{f}(2k+p+2)z^{k+p}$$

and

$$T_{e^{-ip\theta}f}(z^k) = \begin{cases} 0 & \text{if } 0 \leq k \leq p-1 \\ 2(k-p+1)\widehat{f}(2k-p+2)z^{k-p} & \text{if } k \geq p. \end{cases}$$

For convenience, we would like to remind the reader by the following property of the Bergman projection, that will be used to eliminate the calculations in the next lemma. Let n and m be a nonnegative integers. Then

$$P(z^n \bar{z}^m) = \begin{cases} \dfrac{n-m+1}{n+1} z^{n-m}, & if \quad n \geq m \\ 0, & if \quad n < m \end{cases} \tag{1}$$

For more about Bergman projection, one can see [5] or [12]. Now, using the above lemma and the property of the Bergman projection one can observe the following:
For $n = 1$, $T^n_{z+\bar{z}}(1) = z$, so for $n = 2$ we have, $T^2_{z+\bar{z}}(1) = T_{z+\bar{z}}(z) = z^2 + \frac{1}{2}$.

Now by induction, Suppose that $T_{z+\bar{z}}^{k-1}(1) = z^{k-1} + b_{k-2}z^{k-2} + \cdots + b_1 z + b_0$. This implies, using (1) , that

$$\begin{aligned} T_{z+\bar{z}}^{k}(1) &= T_{z+\bar{z}}(z^{k-1} + b_{k-2}z^{k-2} + \cdots + b_1 z + b_0) \\ &= z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 \end{aligned}$$

Hence, we can say that:

Remark 1. For every $k \in \mathbb{N}$, $T_{z+\bar{z}}^{k}(1) = q(z)$, where $q(z)$ is a monic polynomial of degree k.

3. Main Results

The main tool in [9] was the Mellin transform, and their idea was based on comparing the coefficients of the terms of the same degree on both sides, starting from the highest degree, which allowed them to compute the degree and find the symbol of each term. We state their result, [9, Theorem 2, P. 886], as:

Let $f(re^{i\theta}) = \sum_{k=-\infty}^{N} e^{ik\theta} f_k(r)$ be a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator T_f is bounded. If T_f commutes with $T_{z+\bar{z}}$, then $T_f = Q(T_{z+\bar{z}})$ where Q is a polynomial of degree at most 3.

In the following theorem, we will denote the commutator of two operators T and S by $[T, S] = TS - ST$. So, T commutes with S iff $[T, S] = 0$.

Theorem 1. *Let $f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r)$ be a function in $L^1(\mathbb{D}, dA)$ such that the Toeplitz operator T_f is bounded. Assume that there exist two positive integers N and M such that $f_{2N}(r) = cr^{2N}$ and $f_{2M+1}(r) = c'r^{2M+1}$, where c and c' are constants. If $[T_f, T_{z+\bar{z}}] = 0$, then $T_f = Q(T_{z+\bar{z}})$ where Q is a polynomial of degree at most 3.*

Proof. For simplicity let $h_k \equiv h_k(r, \theta) = f_k(r)e^{ik\theta}$. Since $[T_f, T_{z+\bar{z}}] = 0$, then for each $n \in \mathbb{N}$, we have $[T_f, T_{z+\bar{z}}](z^n) = 0$.

Now, the term in z of degree $n+2N+1$ is $([T_{h_{2N}}, T_z] + [T_{h_{2N+2}}, T_{\bar{z}}])(z^n) = 0$. But $[T_{h_{2N}}, T_z] = [T_{cz^{2N}}, T_z] = 0$, since Toeplitz operators with analytic symbols commute. Hence, $[T_{h_{2N+2}}, T_{\bar{z}}](z^n) = 0$. Which means $h_{2N+2}(r, \theta)$ is conjugate analytic, but this can only happen if $f_{2N+2}(r) \equiv 0$. Also, the coefficient of $z^{n+2N+(2m+1)}$ is zero, so we can obtain by induction, that $f_{2N+2m}(r) = 0$ for all $m \geq 1$.

Similarly, since the coefficient of $z^{n+2M+2m}$ equals 0, then again we can prove by induction that $f_{2M+2m+1}(r) = 0$ for all $m \geq 1$. Now, let $L = \max\{N, M\}$, then the symbol f becomes $f(re^{i\theta}) = \sum_{k=-\infty}^{L} e^{ik\theta} f_k(r)$. Hence, using [9, Theorem 2, P. 886] finishes the proof.

□

In the following lemma, which we will use in the next theorem, we show that any element in the orthogonal basis of the Bergman space can be written as a polynomial of $T_{z+\bar{z}}$ evaluated at 1.

Lemma 2. *For every $k \in \mathbb{N}$, $z^k = Q_k(T_{z+\bar{z}})(1)$ where Q_k is a monic polynomial of degree k.*

Proof. By remark 1, $T^k_{z+\bar{z}}(1) = z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0$. This implies,

$$z^k = T^k_{z+\bar{z}}(1) - a_{k-1}z^{k-1} - \cdots - a_1 z - a_0 \tag{2}$$

Similarly,

$$z^{k-1} = T^{k-1}_{z+\bar{z}}(1) - b_{k-2}z^{k-2} - \cdots - b_1 z - b_0 \tag{3}$$

Now, plug (3) into (2) to obtain:

$$z^k = T^k_{z+\bar{z}}(1) - a_{k-1}(T^{k-1}_{z+\bar{z}}(1) - b_{k-2}z^{k-2} - \cdots - b_1 z - b_0) - \cdots - a_1 z - a_0$$

Continuing the above process, gives us:

$$z^k = T^k_{z+\bar{z}}(1) + c_{k-1}T^{k-1}_{z+\bar{z}}(1) + c_{k-2}T^{k-2}_{z+\bar{z}}(1) + \cdots + c_1 T_{z+\bar{z}}(1) + c_0 I$$

which finishes the proof. □

The technique, used in [9], does not work if we replace T_f, by a finite sum of finite product of such operator. The idea used in the proof of next theorem is very simple, and one can use it to obtain the same result in [9] without using Mellin transform.

Theorem 2. *Let* $T = \displaystyle\sum_{l=1}^{n}\prod_{j=1}^{m_l} T_{f_{(l,j)}}$, *where* $f_{(l,j)}(re^{i\theta}) = \displaystyle\sum_{k=-\infty}^{N_{(l,j)}} e^{ik\theta}(f_{(l,j)})_k(r) \in L^\infty(\mathbb{D})$ *for every* $l = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m_l$. *If* $TT_{z+\bar{z}} = T_{z+\bar{z}}T$, *then there exists a polynomial* P_N *of degree* $N = \displaystyle\max_{1\le l\le n}\sum_{j=1}^{m_l} N_{(l,j)}$ *such that* $T = P_N(T_{z+\bar{z}})$.

Proof. Using lemma (1), one can see that if $f(re^{i\theta}) = \displaystyle\sum_{k=-\infty}^{M} e^{ik\theta} f_k(r)$, then

$$\begin{aligned} T_f(z^n) &= \sum_{k=-n}^{M} T_{e^{ik\theta}f_k(r)}(z^n) \\ &= \sum_{k=-n}^{M} 2(n+k+1)\widehat{f}(2n+k+2)z^{n+k} \end{aligned}$$

which is a polynomial of degree M. This implies that, for some $1 \le l \le n$ the product $(\prod_{j=1}^{m_l} T_{f_{(l,j)}})(1)$ is a polynomial of degree $\sum_{j=1}^{m_l} N_{(l,j)}$. But T is a finite sum of such products, so $T(1)$ is a sum of polynomials each of degree $\sum_{j=1}^{m_l} N_{(l,j)}$ for $l = 1, 2, \ldots, n$. Now, let $N = \max_{1\le l\le n}\sum_{j=1}^{m_l} N_{(l,j)}$ to obtain that $T(1) = a_N z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0$. By lemma (2), $z^k = Q_k(T_{z+\bar{z}})(1)$ for every $k \in \mathbb{N}$. This implies that, $T(1) = a_N Q_N(T_{z+\bar{z}})(1) + \cdots + a_1 Q_1(T_{z+\bar{z}})(1) + a_0 I$.

So, $T(1) = P_N(T_{z+\bar{z}})(1)$, where P_N is a polynomial of degree N. Now, using lemma (2) again, we have for every $k \in \mathbb{N}$, $T(z^k) = T(Q_k(T_{z+\bar{z}})(1))$. But T

commutes with $T_{z+\bar{z}}$, it follows that

$$\begin{aligned} T(z^k) &= T(Q_k(T_{z+\bar{z}})(1)) \\ &= Q_k(T_{z+\bar{z}})(T(1)) \\ &= Q_k(T_{z+\bar{z}})(P_N(T_{z+\bar{z}})(1)) \\ &= P_N(T_{z+\bar{z}})(Q_k(T_{z+\bar{z}})(1)) \\ &= P_N(T_{z+\bar{z}})(z^k) \end{aligned}$$

Hence, $T = P_N(T_{z+\bar{z}})$.

Now, consider the weakest topology on the algebra of all bounded linear operators acting on $L_a^2(\mathbb{D})$, in which the map $T \to T(p)$ is continuous for all polynomials p. It is easy to see that the commutant of $T_{z+\bar{z}}$ in the algebra of all bounded linear operators on $L_a^2(\mathbb{D})$ is the closure of the set $\{P_N(T_{z+\bar{z}}) : P_N \textit{ is polynomial}\}$. $\square$

Remark 2. Here are some remarks related to Theorem 2:

(1) It is shown, in [7, Corollary 6.5 , P. 533], that $T^n_{z+\bar{z}}$ is not a Toeplitz operator whenever $n \geq 4$. The authors in [9] obtained a polynomial of $T_{z+\bar{z}}$ of degree at most 3, because they were looking for commutants of $T_{z+\bar{z}}$ among Toeplitz operators.

(2) The question, "when the product of two Toeplitz operators with bounded symbols is a Toeplitz operator?", is still unsolved. So, it is worth mentioning here that the operator T in Theorem 2 above is not necessarily a Toeplitz operator. Also, we can't say that if the degree N of the polynomial $T = P_N(T_{z+\bar{z}})$ is greater than 4, implies that T is not a Toeplitz operator, because we have no control on the coefficients of such a polynomial, and the terms, in the expansion of T, that are not Toeplitz operators might cancel each other.

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APPLICATION OF ESTIMATES OF ALPHA-STABLE DISTRIBUTION TO DISTRESS FORECAST

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Abstract. This article proposes a method to evaluate the change in distress value of companies belonging to the US health care industry. The method involves the use of stable distribution parameters (as also the key financial ratios) in the formation of neural network committees. For each sector of the health care industry, we use genetic algorithms to select the most important parameters of alpha-stable distributions. As a result, the committee prediction error is significantly reduced. The proposed method is compared with neural network committees without stable distribution parameters (using key financial ratios only). The results show that the committees formed using genetic algorithms with stable distribution parameters (for each sector separately) are significantly better than the committees formed only with the key financial indicators.

1. Introduction

Adya and Collopy [1] showed that most authors compare the performance of neural networks with these traditional methods: discriminant analysis, logistic regression, regression models, decision trees, ID3, NEWQ, Probit, Logit, Five Qualitative response models, Five Software reliability models, k Nearest Neighbour, Experts, Leading indicators, and Factor-Logistic. Neural networks are chosen to predict the financial situation because they show significantly better results concurrently to the above mentioned methods.

The neural networks are connected to committees. Distress value direction (increase or decrease) for the coming year is forecasted from current year's data by two different committees of neural networks. Each neural network is trained to predict next year's distress direction (better, worse or unchanged value of distress) for each sector. Thus we have a total of nine neural networks corresponding to each of the following sectors of the health care industry: Medical Instruments and Supply (M_I_a_S), Medical Appliances and Equipment (M_A_E), Long Term Care (L_T_C), Home Health Care (H_H_C), Health Care Plans (H_C_P), Drug Manufactures Major (D_M_M), Drug Manufactures Other (D_M_O), Diagnostic Substances (D_S), and Biotechnology (B). In this way, each committee member reflects the specifics of the economic sector, and at the same time, the committee reflects the entire health care industry.

The available data covers the 2006–2010 period, during which the financial crisis produced strong shocks in the financial markets that resulted in the failure of businesses and financial institutions.

Neural network committees are formed by two different methods:

Key words and phrases. Genetic algorithms, neural networks, distress forecasting, alpha-stable distribution.

(1) Using the key financial indicators that best describe the company's distress level (Current Ratio, Total Asset Turnover, Gross Margin), and other supplementary financial indicators, which are listed in the Data section.
(2) Using the stable distribution's parameters (Alpha, Beta, Sigma, Mu) to describe a company's situation in the context of other sectors in addition to key financial ratios (Current Ratio, Total Asset Turnover, Gross Margin), with a genetic algorithm selecting the most important parameters.

In both cases, the committees combine nine neural networks, each of them trained to predict the direction of a single sector's distress. Neural networks are different only in the sets of features that have been used in training.

Real-world financial time series are often characterized by skewness, kurtosis, heavy tails [8, 3], self-similarity and multifractality [2]. One distribution supported by empirical evidence, first observed more than 45 years ago by Mandelbrot [4], is the stable distribution. Its advantages for modeling financial risk factors are now well documented (see, for example [6, 8, 3]). In our statistical analysis of TAT, CR, GM and OM factors, we also fit data series to the α-stable distribution.

2. Methodology

First neural networks are used for distress value prediction by using two different data sets for learning and testing purposes. So we have:

(1) 9 neural networks trained and tested using Current Ratio, Total Asset Turnover, Gross Margin, and the supplementary indicators that help companies evaluate distress more accurately,
(2) 9 neural networks trained and tested using stable distribution parameters in addition to the features in (1).

To train each of the above networks, we use data from all sectors, and to test the network, we use each sector's data separately. Thus, neural network weights hold information about the details of each sector's identity.

We form average and weighted committees of neural networks trained using the same set of attributes ((1) or (2)) for comparison.

After neural network ensembles are formed from neural networks trained and tested using stable distribution parameters, they are used as fitness functions for the genetic algorithm. In this way we select the stable distribution parameters that are important for each sector separately.

2.1. **Neural networks.**

A feedforward multi-layer perceptron (MLP) is used for forecasting of distress value direction, with one hidden layer and one output node. The training phase is the most important, as this determines the network's weights. During the training phase, the differences between the MLP output values and the known target values are minimized. Bayesian regularization is used for updating the weight and bias values according to Levenberg-Marquardt optimization.

Let x_1,... x_{29} be a vector of financial ratios, y be the distress value, w_1 be the matrix of linking weights from input to hidden layer, and w_2 the weights from hidden to output layer. The MLP with one hidden layer is a model:

$$y = f_2(w_2 f_1(w_1 x)). \tag{1}$$

During the training phase, mean squared errors (MSE) are minimized to estimate weight matrices.

The function for hidden nodes is the hyperbolic tangent sigmoid transfer function. The function for output node is linear transfer function. The MSE is computed by taking the differences between the target and the actual neural network output, squaring them, and averaging over all data vectors,

$$MSE = \frac{1}{N}\sum_{j=1}^{N}(a_j - y_j)^2 \tag{2}$$

where a_j represents the target value, y_j the network output for the jth training pattern, and N the number of training patterns.

2.2. **Genetic Algorithms.** A GA is chosen for searching of the local optima of data vectors consisting of financial ratios and considering many points in the search space simultaneously by probabilistic rules. GAs perform these stages:

- Initialization and Selection. A population of chromosomes (the combination of financial ratios) is selected as the starting point of the search. The MSE fitness function then maps each chromosome's performance to a scalar value.
- Using crossover, only the high scoring members are chosen for the new solution. The crossover occurs only at the one-point crossover rate.
- The data vector randomly changes during mutation process. The algorithm stops when the minimum of the MSE function is found.

The genetic optimization problem is defined by:

- The parameters that have to be coded for the problem. This is a 31 digit vector (a population of data vectors) generated by the GA (1010...0011) define which financial ratios used (1 used, 0 not used).
- Compute the fitness function to evaluate the performance of each data vector. Find a combination of financial ratios with minimum MSE of neural network. A solution of the GA population is used to construct input data set for neural network, which is then trained using a training set and tested with a test set. MSE is used to determine its fitness. The output of GA is a string, which defines proposed combination of financial ratios.

This process is repeated for each solution in the GA population. This allows exploring all possible combinations of 31 financial ratios and tends to favor the most likely solutions.

2.3. **Statistical analysis.** We start from TAT, CR, GM and OM empirical data analysis. We first estimate mean, variance, skewness, and asymmetry, then fit the normal and α-stable distributions to the data series.

Following the well-known definition (see [7, 9]) a random variable X has the α-stable distribution, denoted $X \stackrel{d}{=} S_\alpha(\sigma, \beta, \mu)$, if it has a characteristic function of the form:

$$\phi(t) = \begin{cases} \exp\left\{-\sigma^\alpha \cdot |t|^\alpha \cdot \left(1 - i\beta sgn(t)\tan(\frac{\pi\alpha}{2})\right) + i\mu t\right\}, \text{if} \alpha \neq 1 \\ \exp\left\{-\sigma \cdot |t| \cdot \left(1 + i\beta sgn(t)\frac{2}{\pi} \cdot \log|t|\right) + i\mu t\right\}, \text{if} \alpha = 1 \end{cases} . \tag{3}$$

Each stable distribution is described by four parameters, the first and most important being the stability index $\alpha \in (0, 2]$, which is essential for characterizing financial data. The others are a a skewness parameter $\beta \in [-1, 1]$, location parameter $\mu \in \mathbf{R}$, and scale parameter $\sigma > 0$.

The probability density function of an α-stable distribution is

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(t) \cdot \exp(-ixt) dt. \tag{4}$$

In the general case, this function (4) cannot be expressed in closed form. Infinite polynomial expressions of the density function are well known, but it is not very useful for maximum likelihood estimation (MLE) because of issues such as error estimation in the tails and difficulties with truncating the infinite series. We instead use an integral expression of the PDF in the standard parameterization with a Zolotarev-type formula ([3], section 2.1). The pth moment $E|X|^p = \int_0^\infty P(|X|^p > y)dy$ of an α-stable random variable X exists and is finite only if $0 < p < \alpha$. Thus for $\alpha < 2$, the variance does not exists, and for $\alpha < 1$, we cannot use mean as a positional characteristic.

3. Data

Each committee member reflects a cluster of companies grouped by activity (economic sector). The data vectors describing companies from D_S, D_M_M, D_M_O, H_C_P, H_H_C, L_T_C, M_I_a_S, M_A_E, and B sectors are chosen as input data for neural networks for the following reasons:

(1) It is difficult to separate companies into the sectors, as each company produces several kinds of products which can be attributed to a number of different sectors. Therefore, it is difficult to describe the dynamics of an industry according to company distributions within these sectors. That's why neural networks were trained with data from all sectors together and tested each sector separately and then combined into the committees.

(2) The same companies can be found in different sectors classified by Standard Industrial Classification system from the US Bureau of Census because of limitations in describing companies' activities. The companies can specialize in these biotech related applications: research, production processes or products. Almost all companies analyzed sell somewhat different products and rely on different technologies and usages. That's why their production processes diffuse into different industries.

(3) A more general concept of industry can be obtained by using knowledge-based perspective in addition to the classification into sectors including products, technologies and users perspectives. For example, the health care industry can be defined as a biomedical industry according to knowledge-based perspective. It overlaps medical technology, instruments, and medical supplies and is influenced by knowledge, techniques, tools. This has different impacts to various sectors over time.

(4) All segments are regulated by significant government supervision, taking into account President Obama's commitments to healthcare reform. For example, the Patient Protection and Affordable Care Act was passed by

Congress in 2010. It describes conditions in the healthcare industry. Because of healthcare reform, the amount of uninsured people is reduced. The high cost of developing new drugs and medical devices will tend to offset some of the cost savings because of higher taxes. Different segments of each industry are influenced differently by advantages and disadvantages of the healthcare reform process.

(5) Healthcare delivery and financing in the United States faces problems such as high cost and quality. All of them have influenced companies and citizens. For example total healthcare expenditures increased and the cost is rising faster than inflation.

The data for both committees are formed on the same principle. Each company is described by the data vector consisting of input data and the target value. Target values for both committees are the same (see the Output Data section 3.1), while the input data is different for both committees (see sections 3.2 and 3.3). Two different data sets are formed from financial ratios. Four data vectors describe each company. Each of them describes the situation of the company in different year from the selected period. Two different types of data vectors are formed:

(1) Data vectors are formed from the 31 financial ratios as input parameters to describe current year financial situation plus changes of distress value between the current year and one year ahead.
(2) Data vectors are formed from the data of 21 financial ratios as input parameters to describe current year financial situation plus changes of distress value between the current year and one year ahead.

3.1. **Output Data.** As already mentioned, the committees differ in the input data, while the choice of target values for both teams is identical, according to the table below (Table 1). A company's financial situation is measured by distress value (DV). There are three key financial indicators that can determine distress value: Total Asset Turnover (TAT), Current Ratio (CR), and Gross Margin (GM). Depending on the indicator values assigned to each company, the distress value varies from 1 to 8.

Table 1. Outputs of distress value

DV	CR	TAT	GM
1	< 1	< 50	Dec
2	< 1	≥ 50	Dec
3	< 1	< 50	Inc
4	< 1	≥ 50	Inc
5	≥ 1	< 50	Dec
6	≥ 1	≥ 50	Dec
7	≥ 1	< 50	Inc
8	≥ 1	≥ 50	Inc

Note: Distress value (DV): 1 - The situation is very tense; 2 - The situation is poor; 3 - Satisfactory situation; 4 - The average situation; 5 - On average, a good situation; 6 - A good situation; 7 – Stable situation; 8 - The situation is really good.

When a company is in a risky situation, the distress value is equal to 1, and when the company's financial situation is very good, the value is equal to 8. The neural network committees predict the DV change after one year. Thus the neural network output is (+1) if the situation will be improved, (-1) if it will get worse, and (0) if it will remain unchanged.

3.2. **Input data committee, trained by using key financial indicators.** Selected indicators can be divided into the following perspectives: liquidity, financial leverage, profitability, efficiency, productivity, and effectiveness. List of financial ratios are given below.

Liquidity ratios

(1) Total Current Assets / Total Current Liabilities
(2) (Total Current Assets- Inventories) / Total Current Liabilities
(3) Shareholder's Equity/Total Liabilities

Financial leverage ratios

(4) Total Liabilities*100/Total Assets
(5) Total Long Term Liabilities*100 / Total Assets
(6) Total Current Liabilities*100/Total Assets
(7) (Total Current Assets-Total Current Liabilities)/ Total Assets
(8) Shareholder's Equity*100/Revenue
(9) Total Long Term Liabilities*100/Revenue
(10) Total Current Liabilities*100/Revenue

Profitability ratios

(11) Gross Profit*100/Revenue
(12) Net Income*100/Revenue
(13) Net Income*100/Total Assets
(14) Total Operating Expenses*100/Cost of Revenue
(15) Net Income*100/Shareholder's Equity
(16) Operating Income*100/Revenue

Efficiency ratios

(17) Cost of Revenue*100/ Total Operating Expenses
(18) Receivables*360/Revenue
(19) Revenue/Total Assets
(20) Revenue/Total Long Term Assets
(21) Revenue/Shareholder's Equity

Productivity ratios

(22) Goodwill & Intangibles/Revenue
(23) Net Property, Plant & Equipment*100/Revenue
(24) Total Current Assets*100/Revenue
(25) (Total Current Assets-Inventory-Receivables)*100/Cash& Short Term Investments

Effectiveness ratios

(26) R& D as a percentage of sales
(27) R& D as a percentage of General and administrative + Selling and marketing
(28) Sales as a percentage of total operating cost
(29) Sales as a percentage of General and administrative + Selling and marketing

3.3. **Input data committee, trained using stable distribution parameters.** List of financial ratios (Gross Margin (GM), Operating Margin (OM), Current Ratio (CR) and Total Asset Turnover (TAT)) and stable distribution parameters (alpha, beta, sigma, mu) are provided below:

(1) CR,
(2) CR_alpha,
(3) CR_beta,
(4) CR_sigma,
(5) CR_mu,
(6) TAT,
(7) TAT_alpha,
(8) TAT_beta,
(9) TAT_sigma,
(10) TAT_mu,
(11) OM,
(12) OM_alpha,
(13) OM_beta,
(14) OM_sigma,
(15) OM_mu,
(16) GM,
(17) GM_alpha,
(18) GM_beta,
(19) GM_sigma,
(20) GM_mu
(21) Distress_Value of the current year.

All these mentioned parameters are calculated for particular year.

3.4. **Distributional analysis.** Empirical data analysis of all ratios and separately of Total Asset Turnover (TAT), Current Ratio (CR), and Gross Margin (GM) has shown that:

- all financial ratios are non-normally and non-stable distributed;
- the distributions of TAT, CR, and GM ratios differs for different periods and different industries, but usually they are stable distributed.

These results imply that given financial ratios cannot be used in linear regression. The non-linear regression or neural networks should be applied to forecast distress value.

Secondly TAT, CR, and GM data series are usually either normal or α-stable distributed. The Gaussian distribution is a special case of the α-stable law when α=2. This means that we may use estimates of α-stable distribution parameters to forecast the distress value changes. When these estimates are normally [5] distributed, linear regression (if necessary) may be used. However, we will use them to train neural networks with our genetic algorithm approach.

4. Experiments

The first experiment compares two committees of neural networks, which will be used to predict change of distress in next year. The above groups are used for training:

(1) the standard accounting ratios, the key indicators, and the current year distress value
(2) key indicators (TAT, GM, OM, and CR), stable distribution parameters of these indicators, and the current year distress value.

The second experiment aims to determine the most important parameters that are used for better prediction results showing committee training. This is expected to improve the forecasting performance and identify each sector.

The experiments are organized as follows:

(1) We calculate Total Asset Turnover, Current Ratio, and Gross Margin for each sector. We classify each company, using Table 1. Form the target values for each company as the change between current year class and one year ahead class. Therefore, these research strategies are selected:
   (a) The idea is that company performance dynamics fully reflect changes in the economic environment. The same financial ratios are used to evaluate performance for company distress as also to describe the development of economic environment. These financial indicators can be classified as liquidity, financial leverage, profitability, efficiency, productivity, and effectiveness ratios.
   (b) Using the parameters of the alpha-stable distribution (for Total Asset Turnover, Current Ratio, and Gross Margin) and indicators thmeself to describe the development of economic environment.

(2) Divide the data into learning and testing groups. The learning group consists of data from all sectors in the 2006, 2007, 2008, 2010 years. Testing data contains only 2009 for each sector separately (which is not used for learning process).

(3) Neural networks that are trained using the same set of features are connected to the mean and the weighting committees.

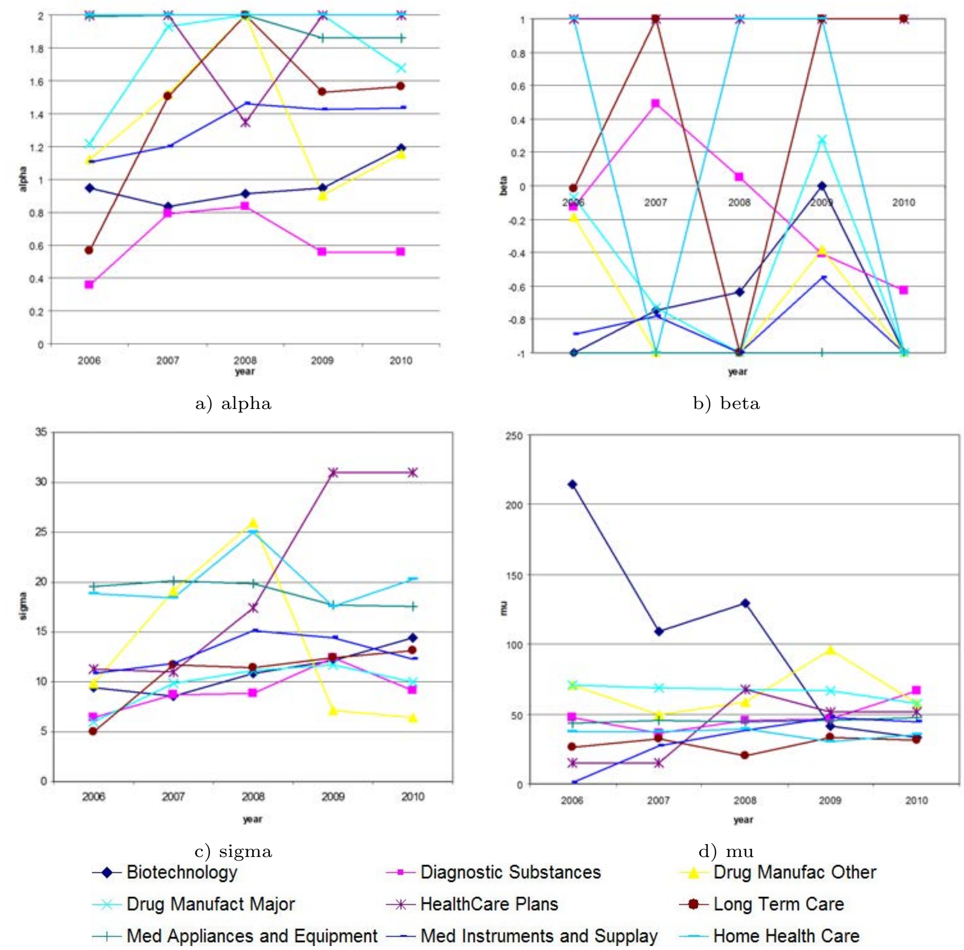
(4) The data sets that are used for committees, with better prediction results, for each sector separately are selected the most important collections. In this way it is expected to improve the forecasting results.

5. Results

This section presents all the results obtained during experiments mentioned in Section 4. We first give alpha-stable distribution parameter estimates of the input data. Secondly, we select the neural network with the lowest forecast error of distress value. Finally, we select the most important features for each health care sector using a genetic algorithm.

5.1. **Alpha-stable distribution parameter estimates of input data.** We estimate alpha-stable distribution parameters of Gross Margin (GM), Operating Margin (OM), Current Ratio (CR) and Total Asset Turnover (TAT). Section 3.3 gives a complete list of parameters estimated. Figure 5.1 shows the dynamics of alpha-stable parameters over our observation period. The Operating Margin case is given as an example. The complete list of alpha-stable distribution estimates is given in Table 2.

From Figure 5.1a, one may see that parameter alpha varies in the interval [0.36, 2] for the operating margin feature in all sectors. For Diagnostic Substances and Biotechnology sectors, parameter alpha is less that 1 almost all the time. Concurrently, OM for Home Health Care sector is always equal to 2, while in the Long Term Care and Drug Manufac Other sectors, OM oscillates. Higher values of alpha indicate smaller possible deviations in sector OM, while smaller α suggest extreme



a) alpha b) beta

c) sigma d) mu

FIGURE 1. Dynamics of Operating Margin stable parameter estimates over period 2006–2010: a) alpha, b) beta, c) sigma, and d) mu parameter

or even chaotic deviations. If α is more than 1, then forecast of operating margin may be made, i.e. expected future value may be found.

The asymmetry parameter beta has high variability in the health care industry, taking on values in the entire interval [-1,1]. Positive values indicate that operating margin has a higher probability to be bigger than "average" OM of that sector, and negative beta indicates that OM has higher probability to be lower than "average" OM of that sector. However, if parameter alpha is close to 2, beta is not an important parameter and may be treated as equal to 0, like in case of Home Health Care sector and etc.

TABLE 2. Estimates of alpha-stable distribution parameters

Sector	year	operating margin				current ratio				gross margin				total asset turnover			
		alpha	beta	sigma	mu	alpha	beta	sigma	mu	alpha	beta	sigma	mu	alpha	beta	sigma	mu
	2006	0.95	-1.00	9.45	214.9	0.23	1.00	4.01	2.62	0.37	-1.00	34.58	27.47	1.59	-1.00	0.22	0.41
	2007	0.83	-0.74	8.63	109.4	0.18	0.38	10.51	3.57	0.34	-1.00	38.27	33.88	1.54	-1.00	0.18	0.43
Biotechnology	2008	0.91	-0.64	10.83	129.5	0.17	0.24	8.90	3.19	0.26	-1.00	27.70	25.78	1.69	-1.00	0.26	0.31
	2009	0.95	0.00	12.17	41.29	0.80	0.73	0.89	0.42	0.30	-0.80	17.54	10.12	1.67	-1.00	0.28	0.26
	2010	1.20	-1.00	14.36	33.90	0.24	1.00	3.53	1.99	0.51	-0.79	27.34	26.00	1.91	-1.00	0.27	0.42
	2006	0.36	-0.12	6.47	47.26	0.67	0.64	0.91	2.18	0.16	-1.00	0.42	29.95	1.83	-1.00	0.26	0.49
	2007	0.79	0.49	8.76	35.95	0.73	0.31	1.62	4.42	0.18	-1.00	5.31	23.15	0.47	-1.00	0.05	0.93
Diagnostic	2008	0.83	0.05	8.93	45.49	1.29	0.07	1.57	5.53	0.32	-0.91	4.98	15.16	1.45	-1.00	0.10	0.66
Substances	2009	0.56	-0.40	12.50	46.21	0.92	-0.33	1.22	7.90	0.33	-0.57	7.63	14.38	0.76	-0.82	0.08	0.99
	2010	0.55	-0.63	9.19	66.57	1.64	1.00	1.63	5.41	0.18	-1.00	24.14	47.61	0.56	-0.23	0.05	0.70
	2006	1.13	-0.19	9.84	71.29	0.39	1.00	0.74	0.82	0.21	-1.00	12.76	33.06	1.60	-1.00	0.18	0.37
Drug	2007	1.53	-1.00	19.19	49.29	0.50	1.00	0.61	1.23	0.18	-1.00	5.93	6.45	2.00	-1.00	0.20	0.50
Manufacturer	2008	2.00	-1.00	25.97	58.61	0.56	0.40	0.87	2.26	0.03	-1.00	4.21	2.26	1.15	-0.28	0.10	0.45
Other	2009	0.91	-0.38	7.12	96.24	0.55	0.36	0.90	2.25	0.24	-0.39	7.20	6.60	1.57	-1.00	0.19	0.45
	2010	1.15	-1.00	6.50	57.92	0.48	1.00	0.66	0.65	0.49	-0.17	7.12	10.78	1.82	-1.00	0.27	0.33
	2006	1.21	-0.07	5.97	70.59	1.04	0.09	0.24	1.71	1.08	0.12	4.34	21.84	1.66	-1.00	0.10	0.44
Drug	2007	1.93	-0.73	9.89	69.09	0.47	0.67	0.13	1.24	0.41	0.90	2.12	14.80	1.30	-0.33	0.13	0.38
Manufacturer	2008	2.00	-1.00	11.11	68.20	0.77	-0.35	0.25	1.85	0.90	-0.06	6.18	18.70	1.79	0.63	0.14	0.43
Major	2009	2.00	0.28	11.70	67.30	0.38	-0.36	0.11	1.84	0.21	0.32	1.31	19.30	1.35	-0.35	0.11	0.39
	2010	1.68	-1.00	9.94	57.95	0.24	-0.42	0.15	2.03	0.58	-0.65	4.17	23.25	1.49	-0.43	0.16	0.43
	2006	2.00	1.00	11.35	15.45	1.99	1.00	0.51	1.16	2.00	1.00	2.05	2.05	2.00	1.00	0.18	0.35
	2007	2.00	1.00	10.97	15.28	1.93	1.00	0.50	1.29	0.04	1.00	0.00	-0.48	2.00	1.00	0.15	0.22
HealthCare	2008	1.35	1.00	17.45	67.58	0.91	-0.03	0.34	1.38	1.33	1.00	1.09	2.66	1.76	1.00	0.15	0.27
Plans	2009	2.00	1.00	31.00	51.34	0.57	0.12	0.26	1.11	1.98	1.00	1.86	2.32	2.00	1.00	0.17	0.29
	2010	2.00	1.00	31.05	51.42	0.41	0.81	0.13	0.79	2.00	1.00	1.95	2.58	1.27	-0.08	0.12	0.37
	2006	2.00	1.00	18.83	37.15	1.26	0.56	0.36	1.81	0.23	0.82	0.42	2.14	2.00	1.00	0.12	0.58
Home	2007	2.00	-1.00	18.41	37.45	1.25	0.05	0.32	1.69	1.11	0.57	2.51	11.66	1.66	-1.00	0.14	0.46
Health	2008	2.00	1.00	25.03	39.91	1.88	-1.00	0.43	1.35	1.58	1.00	1.77	3.28	1.71	-1.00	0.17	0.43
Care	2009	2.00	1.00	17.51	30.82	1.96	-1.00	0.35	1.94	2.00	1.00	2.93	1.93	2.00	-1.00	0.26	0.45
	2010	2.00	-1.00	20.34	35.32	1.96	1.00	0.45	2.16	2.00	1.00	2.44	3.30	2.00	-1.00	0.24	0.46
	2006	0.56	-0.02	5.04	25.93	2.00	-1.00	0.28	1.13	0.62	-0.81	1.45	3.69	1.76	0.34	0.14	0.24
Long	2007	1.50	1.00	11.75	32.88	2.00	1.00	0.41	1.23	1.21	-0.26	1.53	0.60	2.00	-1.00	0.14	0.28
Term	2008	2.00	-1.00	11.47	20.57	2.00	0.26	0.32	1.07	0.61	-0.62	1.17	4.04	2.00	1.00	0.15	0.19
Care	2009	1.53	1.00	12.46	33.42	2.00	-1.00	0.35	1.14	0.34	-0.26	0.28	-1.42	1.91	-0.39	0.12	0.29
	2010	1.56	1.00	13.15	31.10	2.00	1.00	0.37	1.25	0.15	-1.00	0.01	0.01	2.00	1.00	0.12	0.30
	2006	1.99	-1.00	19.57	43.74	0.18	0.65	5.93	2.88	0.24	0.39	2.65	-0.81	1.18	-0.69	0.09	0.55
Med	2007	2.00	-1.00	20.13	45.43	0.94	0.28	1.25	0.18	0.42	-0.60	4.21	9.15	1.26	-1.00	0.15	0.23
Appliances	2008	2.00	-1.00	19.85	44.12	1.13	0.66	93.98	446.1	0.53	-0.53	5.56	5.03	1.40	-1.00	0.14	0.45
and	2009	1.86	-1.00	17.76	46.01	0.16	0.08	23.89	3.82	0.53	-0.69	5.46	6.98	1.49	-1.00	0.14	0.53
Equipment	2010	1.86	-1.00	17.51	47.78	0.12	0.03	79.38	4.22	0.50	-0.10	4.55	2.47	1.60	-1.00	0.14	0.57
	2006	1.11	-0.89	10.80	0.68	0.24	1.00	2.98	1.34	2.00	-1.00	472.35	-203.5	1.58	-0.84	0.11	0.66
Med	2007	1.20	-0.78	11.83	27.13	1.14	0.50	1.63	7.23	0.28	-0.02	4.57	0.32	1.54	-1.00	0.13	0.55
Instruments	2008	1.46	-1.00	15.11	38.22	1.14	1.00	1.51	13.43	0.45	-0.20	4.95	3.17	1.46	-1.00	0.12	0.51
and Supplay	2009	1.43	-0.55	14.37	47.33	0.48	0.85	0.87	1.78	0.63	-0.68	8.30	12.74	1.46	-1.00	0.15	0.48
	2010	1.44	-1.00	12.30	44.12	0.70	0.85	1.08	1.10	0.10	-0.30	1319.8	15.00	1.38	-1.00	0.13	0.47

Parameter sigma is an identifier of scale/volatility. One may see that "volatility" in 2008 has increased for almost all sectors. However, sigma for the HealthCare Plans sector increses over all periods considered.

Position parameter mu indicates "average" OM in a particular sector. Figure 5.1d shows that mu decreases for almost all sectors, with the exception of Med Appliances and Equipment.

5.2. **Distress value prediction.** We compare committees of combined neural networks trained for distress value prediction within each sector separately. Training groups use different features sets described in sections 3.2 and 3.3. In Table 3 they are called the "31 ratios" (described in more detail in section 3.2) and "21 ratios" (described in more detail refer to section 3.3). Comparable results of both methods are given in Table 3.

Table 3. Error of forecast when financial ratios are used as input features

Sector	31 ratios	21 ratios
D_S	26.30	0.00
D_M_M	77.00	18.18
D_M_O	31.50	21.73
H_C_P	78.70	0.00
H_H_C	40.00	20.00
L_T_C	28.30	16.66
M_I_a_S	27.10	13.46
M_A_E	66.60	75.80
B	40.60	20.83
Average	46.23	20.74
Weighting	4.30	0.00

We can see that the neural network committee consisting of neural networks and stable distribution parameters have smaller errors in both case, with the exception of the M_A_E case.

5.3. **Features selection.** Using genetic algorithms we aim to select data sets of the most important features for each sector consisting of the 21 ratios (see section 3.3).

Table 4. Error of forecast when stable parameters are used as input features from genetic algorithm

Sector	NN	GA+NN
D_S	0.00	0.00
D_M_M	18.18	18.18
D_M_O	21.73	17.39
H_C_P	0.00	2.22
H_H_C	20.00	20.00
L_T_C	16.66	20.00
M_I_a_S	13.46	13.46
M_A_E	75.80	22.09
B	20.83	14.58
Average	20.74	14.21
Weighting	0.00	0.00

After selection of the most important features using genetic algorithms, forecast results improve for sectors D_M_O, M_A_E, and B, while they remain unchanged for sectors D_S, D_M_M, H_H_C, and M_I_a_S. Prediction results are worse for sectors H_C_P and L_T_C.

The results of weighting method has not changed when the networks are trained with all the features and selected most important features. Meanwhile, the average method group had better outcomes when the neural network is trained using the selected features group. Knowing which features are the most important for each sector, we can see how sectors differ from each other.

Lower alpha indicates a larger number of unprofitable companies in each sector. Beta describes the trend of companies. The company is susceptible to losses if beta<0 or profits if beta>0.

Mu describes the financial situation of the "average" company in each sector. Sigma (scale parameter) defines the differences between companies in each sector. The goal is to discover the most important features in each sector.

When comparing sectors by the selected features obtained using genetic algorithms, we identify similarities and differences:

(1) Similar sectors are: a) D_M_O and H_C_P (the most important indicators are CR and CR_sigma in these sectors) and b) L_T_C and B (the most important indicator is CR_sigma in these sectors).
(2) Similar sectors are: D_S and M_I_a_S (the most important indicators are TAT_alpha, TAT_beta, TAT_sigma, and TAT_mu in these sectors). Totally different sectors are: D_M_O (the most important indicators are (OM_alpha, OM_beta, OM_mu) in this sector and H_H_C (the most important indicators are (OM, OM_sigma) in this sector.
(3) Similar sectors are: a) D_S and M_I_a_S (the most important indicator are (OM_beta, OM_sigma) in these sectors. b) D_M_M and D_M_O (the most important indicators are (OM_alpha, OM_beta, OM_sigma) in these sectors. Totally different sectors are: H_C_P (the most important indicators are (OM_alpha, OM_beta, OM_mu) in this sector and H_H_C (the most important indicators are (OM, OM_sigma) in this sector.
(4) Similar sectors are: H_C_P and M_A_E. There are no important indicators from this group in these sectors.

It is also reflected in the Table 5 presented below, where 1 means that the feature is selected as important, and zero means that particular feature is not important for a particular sector.

Sector identity may be revealed by comparing key features set of each sector to the other sectors features set.

TABLE 5. Feature selection results from 21 attributes

Index	D_S	D_M_M	D_M_O	H_C_P	H_H_C	L_T_C	M_I_a_S	M_A_E	B
CR	0	0	1	1	0	0	0	1	0
CR_alpha	0	0	0	0	1	0	0	1	0
CR_beta	1	0	0	0	0	0	1	1	0
CR_sigma	1	1	1	1	0	1	1	1	1
CR_mu	0	1	0	0	0	0	1	0	0
TAT	0	0	1	0	0	1	0	0	1
TAT_alpha	1	1	1	0	0	0	1	0	0
TAT_beta	1	1	0	1	1	0	1	0	1
TAT_sigma	1	0	1	1	0	0	1	1	1
TAT_mu	1	0	0	0	1	1	1	1	1
OM	0	0	0	0	1	1	0	0	0
OM_alpha	0	1	1	1	0	1	0	1	1
OM_beta	1	1	1	1	0	1	1	1	0
OM_sigma	1	1	1	0	1	0	1	0	0
OM_mu	0	0	0	1	0	0	0	0	0
GM	0	0	1	0	0	1	1	0	0
GM_alpha	0	0	0	0	1	0	1	0	0
GM_beta	1	1	1	0	0	0	1	0	1
GM_sigma	0	1	0	0	0	1	0	0	0
GM_mu	0	0	1	0	0	1	0	0	1
DV, current	1	1	1	1	1	1	1	0	1

6. Conclusions

The neural network committee that used α-stable distribution parameters for training showed better forecasting results (average error 20.74%) than the committee which used basic standard financial ratios with no stable distribution parameters (average error 46.23%).

Using a genetic algorithm, we selected the most important features of each of the committee members whose training used stable distribution parameters. Feature selection for weighted committee had no significant impact on the outcome because in both cases (with and without feature selection), the weighted committee forecasting error is equal to zero. Meanwhile, the average error with feature selection reduced the forecasting error from 20.74 to 14.21 percent.

Finally, our results show that distress may be forecasted without direct information about the main financial ratios (TAT, GM, OM, and CR must be used). Meanwhile, if we have complete information about Gross Margin, Operating Margin, Current Ratio, Total Asset Turnover (as also, their historical alpha-stable parameter estimates) and Distress Value of the current year, we may forecast with much smaller error.

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