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Tel.(+33)(0)3.20.43.42.18
Fax (+33)(0)3.20.43.43.02
Catalin.Badea@math.univ-lille1.fr
Approximation Theory, Functional
Analysis, Operator Theory.

4) Erik J.Balder
Mathematical Institute
Universiteit Utrecht
P.O.Box 80 010
3508 TA UTRECHT
The Netherlands
Tel.+31 30 2531458
Fax+31 30 2518394
balder@math.uu.nl
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5) Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site: http://www.unipg.it/~bardaro/
Functional Analysis and Approximation
Theory,
Signal Analysis, Measure Theory, Real
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6) Heinrich Begehr
Freie Universitaet Berlin
I. Mathematisches Institut, FU Berlin,
Arnimallee 3,D 14195 Berlin
Germany,
Tel. +49-30-83875436, office
   +49-30-83875374, Secretary
Fax +49-30-83875403
begehr@math.fu-berlin.de
Complex and Functional Analytic
Methods in PDEs, Complex Analysis,
History of Mathematics.

7) Fernando Bombal
Departamento de Analisis Matematico
Universidad Complutense
Plaza de Ciencias,3
28040 Madrid, SPAIN
Tel. +34 91 394 5020
Fax +34 91 394 4726
fernando_bombal@mat.ucm.es

28) Miroslav Krbec
Mathematical Institute
Academy of Sciences of Czech Republic
Zitna 25
CZ-115 67 Praha 1
Czech Republic
Tel +420 222 090 743
Fax +420 222 211 638
krbecm@matsrv.math.cas.cz
Function spaces,Real Analysis,Harmonic
Analysis,Interpolation and
Extrapolation Theory,Fourier Analysis.

29) Peter M.Maass
Center for Industrial Mathematics
Universitaet Bremen
Bibliotheksstr.1,
MZH 2250,
28359 Bremen
Germany
Tel +49 421 218 9497
Fax +49 421 218 9562
pmaass@math.uni-bremen.de
Inverse problems,Wavelet Analysis and
Operator Equations,Signal and Image
Processing.

30) Julian Musielak
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Ul.Umultowska 87
61-614 Poznan
Poland
Tel (48-61) 829 54 71
Fax (48-61) 829 53 15
Grzegorz.Musielak@put.poznan.pl
Functional Analysis, Function Spaces,
Approximation Theory,Nonlinear Operators.

31) Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel:: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
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Department of Mathematics
National Technical University of Athens

Operators on Banach spaces,
Tensor products of Banach spaces,
Polymeasures, Function spaces.

8) Michele Campiti
Department of Mathematics "E.De Giorgi"
University of Lecce
P.O. Box 193
Lecce,ITALY
Tel. +39 0832 297 432
Fax +39 0832 297 594
michele.campiti@unile.it
Approximation Theory,
Semigroup Theory, Evolution problems,
Differential Operators.

9)Domenico Candeloro
Dipartimento di Matematica e Informatica
Universita degli Studi di Perugia
Via Vanvitelli 1
06123 Perugia
ITALY
Tel +39(0)75 5855038
    +39(0)75 5853822,
    +39(0)744 492936
Fax +39(0)75 5855024
candelor@dipmat.unipg.it
Functional Analysis, Function spaces,
Measure and Integration Theory in
Riesz spaces.

10) Pietro Cerone
School of Computer Science and
Mathematics, Faculty of Science,
Engineering and Technology,
Victoria University
P.O.14428,MCMC
Melbourne,VIC 8001,AUSTRALIA
Tel +613 9688 4689
Fax +613 9688 4050
Pietro.cerone@vu.edu.au
Approximations, Inequalities,
Measure/Information Theory,
Numerical Analysis, Special Functions.

11)Michael Maurice Dodson
Department of Mathematics
University of York,
York YO10 5DD, UK
Tel +44 1904 433098
Fax +44 1904 433071
Mmd1@york.ac.uk
Harmonic Analysis and Applications to
Signal Theory,Number Theory and
Dynamical Systems.

Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability.

33) Pier Luigi Papini
Dipartimento di Matematica
Piazza di Porta S.Donato 5
40126 Bologna
ITALY
Fax +39(0)51 582528
papini@dm.unibo.it
Functional Analysis, Banach spaces,
Approximation Theory.

34) Svetlozar (Zari) Rachev, Professor of Finance,
College of Business,and
Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
Phone: +1-631-632-1998,
Email : svetlozar.rachev@stonybrook.edu

35) Paolo Emilio Ricci
Department of Mathematics
Rome University "La Sapienza"
P.le A.Moro,2-00185
Rome,ITALY
Tel ++3906-49913201 office
    ++3906-87136448 home
Fax ++3906-44701007
Paoloemilio.Ricci@uniroma1.it
riccip@uniroma1.it
Special Functions, Integral and Discrete
Transforms, Symbolic and Umbral Calculus,
ODE, PDE,Asymptotics, Quadrature,
Matrix Analysis.

36) Silvia Romanelli
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona 4
70125 Bari, ITALY.
Tel (INT 0039)-080-544-2668 office
         080-524-4476 home
         340-6644186 mobile
Fax -080-596-3612 Dept.
romans@dm.uniba.it
PDEs and Applications to Biology and
Finance, Semigroups of Operators.

12) Sever S.Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001,AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever@csm.vu.edu.au
Inequalities,Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

13) Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530, Ankara,
Turkey, oduman@etu.edu.tr
Classical Approximation Theory, Summability
Theory,
Statistical Convergence and its Applications

14) Paulo J.S.G.Ferreira
Department of Electronica e
Telecomunicacoes/IEETA
Universidade de Aveiro
3810-193 Aveiro
PORTUGAL
Tel +351-234-370-503
Fax +351-234-370-545
pjf@ieeta.pt
Sampling and Signal Theory,
Approximations, Applied Fourier Analysis,
Wavelet, Matrix Theory.

15) Gisele Ruiz Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA.
Tel 901-678-2513
Fax 901-678-2480
ggoldste@memphis.edu
PDEs, Mathematical Physics,
Mathematical Geophysics.

16) Jerome A.Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA
Tel 901-678-2484
Fax 901-678-2480
jgoldste@memphis.edu
PDEs,Semigroups of Operators,
Fluid Dynamics,Quantum Theory.

37) Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620,USA
Tel 813-974-9710
boris@math.usf.edu
Approximation Theory, Banach spaces,
Classical Analysis.

38) Rudolf Stens
Lehrstuhl A fur Mathematik
RWTH Aachen
52056 Aachen
Germany
Tel ++49 241 8094532
Fax ++49 241 8092212
stens@mathA.rwth-aachen.de
Approximation Theory, Fourier Analysis,
Harmonic Analysis, Sampling Theory.

39) Juan J.Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271.LaLaguna.Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-
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Fourier Transforms, Special functions,
Approximations,and Applications.

40) Tamaz Vashakmadze
I.Vekua Institute of Applied Mathematics
Tbilisi State University,
2 University St. , 380043,Tbilisi, 43,
GEORGIA.
Tel (+99532) 30 30 40 office
   (+99532) 30 47 84 office
   (+99532) 23 09 18 home
Vasha@viam.hepi.edu.ge
tamazvashakmadze@yahoo.com
Applied Functional Analysis, Numerical
Analysis, Splines, Solid Mechanics.

41) Ram Verma
International Publications
5066 Jamieson Drive, Suite B-9,
Toledo, Ohio 43613,USA.
Verma99@msn.com
rverma@internationalpubls.com
Applied Nonlinear Analysis, Numerical
Analysis, Variational Inequalities,
Optimization Theory, Computational
Mathematics, Operator Theory.

17) Heiner Gonska
Institute of Mathematics
University of Duisburg-Essen
Lotharstrasse 65
D-47048 Duisburg
Germany
Tel +49 203 379 3542
Fax +49 203 379 1845
gonska@math.uni-duisburg.de
Approximation and Interpolation Theory,
Computer Aided Geometric Design,
Algorithms.

18) Karlheinz Groechenig
Institute of Biomathematics and Biometry,
GSF-National Research Center
for Environment and Health
Ingolstaedter Landstrasse 1
D-85764 Neuherberg,Germany.
Tel 49-(0)-89-3187-2333
Fax 49-(0)-89-3187-3369
Karlheinz.groechenig@gsf.de
Time-Frequency Analysis, Sampling Theory,
Banach spaces and Applications,
Frame Theory.

19) Vijay Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka
New Delhi 110075, India
e-mail: vijay@nsit.ac.in;
vijaygupta2001@hotmail.com
Approximation Theory

20) Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method,
Numerical PDE, Variational inequalities,
Computational mechanics

21) Tian-Xiao He
Department of Mathematics and
Computer Science
P.O.Box 2900,Illinois Wesleyan University
Bloomington,IL 61702-2900,USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations,Wavelet, Integration Theory,
Numerical Analysis, Analytic Combinatorics.

42) Gianluca Vinti
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia
ITALY
Tel +39(0) 75 585 3822,
   +39(0) 75 585 5032
Fax +39 (0) 75 585 3822
mategian@unipg.it
Integral Operators, Function Spaces,
Approximation Theory, Signal Analysis.

43) Ursula Westphal
Institut Fuer Mathematik B
Universitaet Hannover
Welfengarten 1
30167 Hannover,GERMANY
Tel (+49) 511 762 3225
Fax (+49) 511 762 3518
westphal@math.uni-hannover.de
Semigroups and Groups of Operators,
Functional Calculus, Fractional Calculus,
Abstract and Classical Approximation
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44) Ronald R.Yager
Machine Intelligence Institute
Iona College
New Rochelle,NY 10801,USA
Tel (212) 249-2047
Fax(212) 249-1689
Yager@Panix.Com
ryager@iona.edu
Fuzzy Mathematics, Neural Networks,
Reasoning,
Artificial Intelligence, Computer Science.

45) Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University,AL 36849-5310
USA.
Tel 334-844-6557 office
       678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory,Chebychev Systems,
Wavelet Theory.

22) Don Hong
Department of Mathematical Sciences
Middle Tennessee State University
1301 East Main St.
Room 0269, Blgd KOM
Murfreesboro, TN 37132-0001
Tel (615) 904-8339
dhong@mtsu.edu
Approximation Theory,Splines,Wavelet,
Stochastics, Mathematical Biology Theory.

23) Hubertus Th. Jongen
Department of Mathematics
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
Tel +49 241 8094540
Fax +49 241 8092390
jongen@rwth-aachen.de
Parametric Optimization, Nonconvex
Optimization, Global Optimization.

NEW MEMBERS

46) Jianguo Huang
Department of Mathematics
Shanghai Jiao Tong University
Shanghai 200240
P.R. China
jghuang@sjtu.edu.cn
Numerical PDE's

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Journal of Applied Functional Analysis

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SOME RESULTS CONCERNING SYSTEM OF MODULATES FOR $L^2(\mathbb{R})$

A. Zothansanga[1] and Suman Panwar[2]

[1,2] Department of Mathematics,
University of Delhi, Delhi 110 007. INDIA
[1] e-mail:azothansanga26@yahoo.com
[2] e-mail: spanwar87@gmail.com

Abstract: Frames by integer translates were studied by Kim and Kwon [6]. Christensen et.al [3] studied Riesz sequences of translates and their generalized duals. In this paper we consider a system of modulates for $L^2(\mathbb{R})$. A necessary and sufficient condition for a system of modulates for $L^2(\mathbb{R})$ to be a frame sequence (Riesz sequence,orthonormal sequence) is given. We also prove a result related to the frame operator of a frame sequence $\{E_k\phi\}_{k\in\mathbb{Z}}$. Finally, it is proved that for $n \geq 1$, the system of modulates $\{E_k B_n\}_{k\in\mathbb{Z}}$ is a Riesz Fischer sequence.

AMS Subject Classification 2010: 42C15, 42C30

Key Words: Frame, system of modulates, Bessel sequence, Riesz-Fischer sequence.

1   Introduction

A basic approach to the decomposition of a signal in terms of elementary signals was given by Dennis Gabor [7] in 1946. While answering some deep problems in non-harmonic Fourier series, Duffin and Schaeffer [5] in 1952 abstracted Gabor's method and defined frames for Hilbert spaces. Frames are widely used in mathematics, science and engineering. In particular they are used in filter banks, wireless sensor networks, multiple-antenna code design, signal processing, image processing and many more. For a nice and comprehensive survey on various types of frames, one may refer to [1,2,4,8,10].

Coherent frames $\{f_k\}_{k\in I}$ are the frames for which the elements f_k appear by action of some operators (belonging to a special class) on a single element f in a Hilbert space. This feature is necessary for applications because it simplifies calculations and makes it simpler to collect information about the frame.

In this paper, we consider the case where the operators act by modulation. A necessary and sufficient condition for a system of modulates for $L^2(\mathbb{R})$ to be a frame sequence(Riesz sequence,orthonormal) is given. We also prove a result related to the frame operator of a frame sequence $\{E_k\phi\}_{k\in\mathbb{Z}}$. Finally, it is proved that for $n \geq 1$, the system of modulates $\{E_k B_n\}_{k\in\mathbb{Z}}$ is a Riesz Fischer sequence.

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2 Preliminaries

Definition 2.1. [4] Let $\mathbb{H}$ denote a Hilbert space, I denotes a countable index set and $[f_i]_{i\in I}$ denotes the closed linear span of the sequence $\{f_i\}_{i\in I}$. A family of vectors $\{f_i\}_{i\in I}$ is called a frame for $\mathbb{H}$ if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \text{for all } f \in \mathbb{H}. \tag{1}$$

The positive constants A and B are called lower frame bound and upper frame bound for the family $\{f_i\}_{i\in I}$, respectively. The inequality (1) is called the frame inequality. If in (1) only the upper inequality holds, then $\{f_i\}_{i\in I}$ is called a Bessel sequence.

If the family of vectors $\{f_i\}_{i\in I}$ is a frame for $[f_i]_{i\in I}$, then it is called a frame sequence.

Definition 2.2. [4]A family of vectors $\{f_i\}_{i\in I}$ is called a Riesz basis for $\mathbb{H}$ if it is complete in $\mathbb{H}$ and if there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\sum_{i\in I} |c_i|^2 \le \|\sum_{i\in I} c_i f_i\|^2 \le B\sum_{i\in I} |c_i|^2. \tag{2}$$

If the family of vectors $\{f_i\}_{i\in I}$ satisfies (2) , then it is called a Riesz sequence and if it satisfies the left hand inequality of (2), then it is called a Riesz-Fischer sequence.

For $a \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R})$, the modulation operator E_a on $L^2(\mathbb{R})$ is defined as $E_a\phi(x) = e^{2\pi i a x}\phi(x)$, $x \in \mathbb{R}$ and the dilation operator D_a, $a \neq 0$ on $L^2(\mathbb{R})$ is defined as $D_a\phi(x) = \dfrac{1}{|a|}\phi(\frac{x}{a})$, $x \in \mathbb{R}$.

Definition 2.3. Let $\phi \in L^2(\mathbb{R})$. The sequence $\{E_k\phi\}_{k\in\mathbb{Z}}$ is called a system of modulates for $L^2(\mathbb{R})$. Further,

(I) If $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then it is called a frame of modulates.

(II) If $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$, then it is called a Bessel sequence of modulates.

For a system of modulates $\{E_k\phi\}_{k\in\mathbb{Z}}$, the associated pre-frame operator is $T : l^2(\mathbb{Z}) \to L^2(\mathbb{R})$, given by $T\{\alpha_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} \alpha_k E_k\phi$. The adjoint operator of T or the analysis operator $T^* : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$, is given by $T^*\{f\} = \{\langle f, E_k\phi\rangle\}_{k\in\mathbb{Z}}$, $f \in L^2(\mathbb{R}$.

If the system of modulates $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a Bessel sequence, then by composing T and T^*, we obtain the frame operator $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as $Sf = \sum_{k\in\mathbb{Z}} \langle f, E_k\phi\rangle E_k\phi$, $f \in L^2(\mathbb{R})$.

If the system of modulates $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame, then the frame operator S is invertible.

Definition 2.4. For $n \in \mathbb{N}$, the B-splines B_n are defined inductively as $B_1 = \chi_{[-\frac{1}{2}, \frac{1}{2}]}$ and $B_{n+1} = B_n * B_1$, that is $B_{n+1}(x) = \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} B_n(x - t)dt$.

3   Main Results

We give necessary and sufficient condition for the system of modulates $\{E_k\phi\}_{k \in \mathbb{Z}}, \phi \in L^2(\mathbb{R})$, to be a frame sequence, Riesz sequence or an orhonormal sequence in terms of a function Φ associated with ϕ.

Theorem 3.1. *For $\phi \in L^2(\mathbb{R})$ consider the function Φ on $\mathbb{R}$ defined as $\Phi(x) = \sum\limits_{k \in \mathbb{Z}} |\phi(x + k)|^2$. Consider the set $N = \{x \in [0, 1], \Phi(x) = 0\}$. Then for $A, B > 0$, the following characterizations hold:*

(1) $\{E_k\phi\}_{k \in \mathbb{Z}}$ is a frame sequence with bounds A and B if and only if

$A \leq \Phi(x) \leq B, a.e\ x \in [0, 1] - N$.

(2) $\{E_k\phi\}_{k \in \mathbb{Z}}$ is a Riesz sequence with bounds A and B if and only if

$A \leq \Phi(x) \leq B, a.e\ x \in [0, 1]$.

(3) $\{E_k\phi\}_{k \in \mathbb{Z}}$ is an orthonormal sequence if and only if

$\Phi(x) = 1, a.e\ x \in [0, 1]$.

Proof. (1) Let $\{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$ and $\{E_k\phi\}_{k \in \mathbb{Z}}$ be a Bessel sequence with the pre-frame operator T. Using Theorem 3.2 in [9], we have

$$(3) \qquad \|T\{c_k\}_{k \in \mathbb{Z}}\|^2 = \int\limits_0^1 |\sum_{k \in \mathbb{Z}} c_k e^{2\pi ikx}|^2 \Phi(x)dx.$$

Let the Kernal of T be denoted as Ker(T). Then using Equation (3)

$$\text{Ker}(T) = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}), \sum_{k \in \mathbb{Z}} c_k e^{2\pi ikx} = 0, \text{for all } x \in [0, 1] - N. \right\}$$

Let $\{c_k\}_{k \in \mathbb{Z}} \in \text{Ker}(T)^\perp$. Then $\langle \{c_k\}_{k \in \mathbb{Z}}, \{d_k\}_{k \in \mathbb{Z}} \rangle = 0, \text{for all} \{d_k\}_{k \in \mathbb{Z}} \in \text{Ker}(T)$.

As $\{E_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(0, 1)$, $\{c_k\}_{k \in \mathbb{Z}} \in \text{Ker}(T)^\perp$ if and only if

$$\int\limits_0^1 \sum_{k \in \mathbb{Z}} c_k e^{2\pi ikx} \overline{\sum_{k \in \mathbb{Z}} d_k e^{2\pi ikx}} dx = 0.$$

Hence,

$$\text{Ker}(T)^\perp = \left\{ \{c_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}), \sum_{k \in \mathbb{Z}} c_k e^{2\pi ikx} = 0, \text{for all } x \in N. \right\}$$

Using the definition of $\text{Ker}(T)^\perp$, for all $\{c_k\}_{k \in \mathbb{Z}} \in \text{Ker}(T)^\perp$, we have

$$(4) \qquad \sum_{k \in \mathbb{Z}} |c_k|^2 = A \int\limits_{[0,1]-N} |\sum_{k \in \mathbb{Z}} c_k e^{2\pi ikx}|^2 dx$$

Now using equation (3) and (4)and Lemma 5.4.5 in [4], $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence with lower bound A if and only if

$$A\int\limits_{[0,1]-N}|\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx \leq \int\limits_{[0,1]-N}|\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2\phi(x)dx.$$

which is equivalent to

$$A \leq \Phi(x) \text{ a.e } x \in [0,1]-N.$$

Also, $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence with upper bound B if and only if

$$B\int\limits_{[0,1]-N}|\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx \geq \int\limits_{[0,1]-N}|\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2\phi(x)dx.$$

which is equivalent to

$$B \geq \Phi(x) \text{ a.e } x \in [0,1]-N.$$

Hence, we obtain

$$A \leq \Phi(x) \leq B \text{ a.e } x \in [0,1]-N.$$

(2) Let $\{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Then using Theorem 3.2 in [9] , we have

$$(5) \qquad \|T\{c_k\}_{k\in\mathbb{Z}}\|^2 = \int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2\Phi(x)dx \text{ and}$$

$$(6) \qquad \sum_{k\in\mathbb{Z}}|c_k|^2 = \int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2 \ dx.$$

Also, $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence with bounds A and B if and only if

$$(7) \qquad A\sum_{k\in\mathbb{Z}}|c_k|^2 \leq \|T\{c_k\}_{k\in\mathbb{Z}}\|^2 \leq B\sum_{k\in\mathbb{Z}}|c_k|^2.$$

Using (5) and (6) in (7) we get that $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a Riesz sequence with bounds A and B if and only if

$$A\int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx \leq \int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2\Phi(x)dx \leq B\int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx.$$

which implies that

$$A \leq \Phi(x) \leq B, \text{a.e } x \in [0,1].$$

(3) Let $\{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$. We know that $\{E_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal sequence if and only if

$$(8) \qquad \sum_{k\in\mathbb{Z}}|c_k|^2 \leq \|T\{c_k\}_{k\in\mathbb{Z}}\|^2 \leq \sum_{k\in\mathbb{Z}}|c_k|^2.$$

Thus, $\{E_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal sequence with bounds A and B if and only if

$$\int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx \leq \int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2\Phi(x)dx \leq \int_0^1 |\sum_{k\in\mathbb{Z}}c_ke^{2\pi ikx}|^2dx.$$

This proves (3). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

SOME RESULTS CONCERNING SYSTEM OF MODULATES FOR $L^2(\mathbb{R})$

From general frame theory we know that if for $\phi \in L^2(\mathbb{R})$, $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence. Then, for all $f \in [E_k\phi]_{k\in\mathbb{Z}}$, we have

$$f = \sum_{k\in\mathbb{Z}} \langle f, S^{-1}E_k\phi\rangle E_k\phi.$$

In order to apply the above frame decomposition, we should be able to calculate $S^{-1}E_k\phi$ for all $k \in \mathbb{Z}$. The following theorem helps to simplify this process by proving that instead of calculating $S^{-1}E_k\phi$ for all $k \in \mathbb{Z}$, it is sufficient to calculate $S^{-1}\phi$.

Theorem 3.2. *For $\phi \in L^2(\mathbb{R})$ assume that $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence. Then, for all $f \in [E_k\phi]_{k\in\mathbb{Z}}$,*

(1) $SE_kf = E_kSf$, for all $k \in \mathbb{Z}$.

(2) $S^{-1}E_kf = E_kS^{-1}f$, for all $k \in \mathbb{Z}$.

where S denotes the frame operator for $[E_k\phi]_{k\in\mathbb{Z}}$.

Proof. (1) Given $f \in [E_k\phi]_{k\in\mathbb{Z}}$ and $k \in \mathbb{Z}$, we have

$$SE_kf = \sum_{k'\in\mathbb{Z}} \langle E_kf, E_{k'}\phi\rangle E_{k'}\phi.$$
$$= \sum_{k'\in\mathbb{Z}} \langle f, E_{k'-k}\phi\rangle E_{k'}\phi.$$

letting $k' \longrightarrow k' + k$, we finally obtain

$$SE_kf = E_k\sum_{k'\in\mathbb{Z}} \langle f, E_{k'}\phi\rangle E_{k'}\phi$$
$$= E_kSf.$$

(2) Since $\{E_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence, it is a frame for $[E_k\phi]_{k\in\mathbb{Z}}$. Therefore, the operator S is invertible. Therefore, the proof of (2) follows from (1). $\square$

4 Applications

We now prove that the B-spline B_1 is a Riesz sequence with Riesz bound 1.

Theorem 4.1. *For $B_1 = \chi_{[\frac{-1}{2}, \frac{1}{2}]}$, $\{E_kB_1\}_{k\in\mathbb{Z}}$ is a Riesz sequence with Riesz bound 1.*

Proof. Note that for $k \in \mathbb{Z}$, $\|E_kB_1\|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |e^{2\pi ikx}|^2 dx$.

For $k \neq j$, $\langle E_kB_1, E_jB_1\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i(k-j)x} dx$.

Thus, $\{E_kB_1\}_{k\in\mathbb{Z}}$ is an orthonormal sequence and therefore in view of Theorem 3.1, the result follows.

$\square$

As an important outcome of Theorem 3.1, we now prove that the integer modulates of any B-spline form a Riesz-Fischer sequence.

Theorem 4.2. *For $n \geq 1$, $\{E_k B_n\}_{k \in \mathbb{Z}}$ is a Riesz Fischer sequence with lower bound 1.*

Proof. By Theorem 3.1, for $n \geq 1$, $\{E_k B_n\}_{k \in \mathbb{Z}}$ is a Riesz Fischer sequence with lower bound 1 if and only if

$$1 \leq \Phi(x), \quad x \in [0,1], \text{ where } \Phi(x) = \sum_{k \in \mathbb{Z}} |B_n(x+k)|^2.$$

By corollary 6.2.1 in [4], we have

$$\sum_{k \in \mathbb{Z}} B_n(x+k) = 1, \text{ for all } x \in \mathbb{R}.$$

Now,

$$(9) \qquad 1 = |\sum_{k \in \mathbb{Z}} B_n(x+k)|^2 \leq \sum_{k \in \mathbb{Z}} |B_n(x+k)|^2, \text{ for all } x \in \mathbb{R}.$$

Since $\{E_k B_1\}_{k \in \mathbb{Z}}$ is a Riesz sequence with Riesz bound 1, by Theorem 3.1 we have

$$(10) \qquad \sum_{k \in \mathbb{Z}} |B_1(x+k)|^2 = 1, \text{ a.e } x \in [0,1].$$

Using (9) and (10), we have

$$1 \leq \sum_{k \in \mathbb{Z}} |B_n(x+k)|^2, \text{ a.e } x \in [0,1].$$

Thus, for $n \geq 1$, $\{E_k B_n\}_{k \in \mathbb{Z}}$ is a Riesz Fischer sequence with lower bound 1. $\qquad \square$

Acknowledgement

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SOME DEFINITION OF A NEW INTEGRAL TRANSFORM BETWEEN ANALOGUE AND DISCRETE SYSTEMS

S.K.Q. AL-OMARI

Department of Applied Sciences, Faculty of Engineering Technology
Al-Balqa Applied University, Amman 11134, Jordan
s.k.q.alomari@fet.edu.jo

ABSTRACT

The $\mathcal{D}$ transform is proclaimed as a new integral transform which states an equivalence between analogue and discrete systems [16]. In this article, we establish a convolution theorem of the $\mathcal{D}$ transform and give its definition in the space of generalized functions. The definition of the $\mathcal{D}$ transform of a distribution is a one to one and linear mapping. Over and above, we establish that the $\mathcal{D}$ transform of a Boehmian is a distribution and retain its classical properties.

Keywords: $\mathcal{D}$ Transformation; Integral Transform; Analogue System; Discrete System; Distribution; Boehmian.

1   INTRODUCTION

The sampling techniques are based on the transformation $x_\alpha \to x$, where x_α is an analogue function and x is the sequence

$$x\left(n\right) = x_\alpha\left(n\mathcal{T}\right), \tag{1}$$

$\mathcal{T}$ being a sampling period.

Let x_α be an analogue signal which is casual; i.e. $t < 0 \Rightarrow x_\alpha\left(t\right) = 0$. Then the $\mathcal{D}$ transform of x_α is defined by [16]

$$\mathcal{D}\left(x_\alpha\right)\left(m\right) := \int\limits_0^\infty x_\alpha\left(t\right) e^{\frac{-t}{\mathcal{T}}} \frac{\left(\frac{t}{\mathcal{T}}\right)^m}{m!} dt, m \in \mathbb{N}. \tag{2}$$

Given an analogue system which is characterized by an impulse response h_α then a discrete system is described by an impulse response $\mathcal{D}\left(h_\alpha\right)$ whose input signal is $\mathcal{D}\left(x_\alpha\right)$, where x_α is an input signal of the analogue system with output $\mathcal{D}\left(y_\alpha\right)$ given by

$$\mathcal{D}\left(x_\alpha * y_\alpha\right) = \mathcal{D}\left(x_\alpha\right) * \mathcal{D}\left(x_\alpha\right), \tag{3}$$

where

$$x_\alpha * y_\alpha\left(t\right) = \int\limits_0^t x_\alpha\left(\tau\right) y_\alpha\left(t - \tau\right) d\tau. \tag{4}$$

It also clear that the cited transform extends possibilities concerning linear difference and differential equations as well. For, we recall the following example from the given citation.

Example [16] Given a system which is described by the linear differential equation

$$\left(\frac{\mathcal{L}}{\mathcal{R}}\right)\frac{\mathsf{d}y_\alpha}{\mathsf{d}t} + y_\alpha = x_\alpha,$$

where $\mathcal{L}$ and $\mathcal{R}$ are given as in Figure 2 of [16]. Then an equivalent discrete system under the $\mathcal{D}$ transform can be written as

$$\left(\frac{\mathcal{L}}{\mathcal{R}}\right)\Delta_1 y + y = x$$

where $\Delta_1 y\left(n\right) = \dfrac{y\left(n\right) - y\left(n-1\right)}{\mathcal{T}}.$

Hence, calculations then lead to the solution $x_\alpha\left(t\right) = \delta\left(t\right),$ where δ represents a Dirac delta distribution.

We enumerate certain properties of the $\mathcal{D}$ transform summarized as follows :

(1) Linearity : $\mathcal{D}\left(\lambda x_\alpha + \mu y_\alpha\right) = \lambda \mathcal{D}\left(x_\alpha\right) + \mu \mathcal{D}\left(y_\alpha\right).$

(2) Derivation : $\mathcal{D}\left(\dfrac{\mathsf{d}^n x_\alpha}{\mathsf{d}t^n}\right) = \Delta_n \mathcal{D}\left(x_\alpha\right),$ where

$$\Delta_n = \Delta_1\left(\Delta_{n-1} x\right)\ .$$

(3) Integration : For $f = \dfrac{\mathsf{d}\mathcal{F}}{\mathsf{d}t}$ we have

$$\mathcal{D}\left(\mathcal{F}\right)\left(m\right) = \mathcal{T}\sum_{k=0}^{m}\mathcal{D}\left(f\right)\left(k\right)t\mathcal{T}.\mathcal{F}\left(0\right).$$

(4) Sum and integral correspondence : $\displaystyle\int_{0}^{\infty} x_\alpha\left(t\right)dt = \sum_{k=0}^{m}\mathcal{D}\left(x_\alpha\right)\left(m\right).$

2   CONVOLUTION THEOREM OF $\mathcal{D}$ TRANSFORM

Denote by $\bullet$ the usual convolution product but extended in its upper limit. That is,

$$\left(f\bullet g\right)\left(t\right) = \int_{0}^{\infty} f\left(\tau\right)g\left(t-\tau\right)d\tau. \tag{5}$$

Then we establish the following theorem.

Theorem 1 (Convolution Theorem) Let x_α and y_α be casual analogue signals. Then we have

$$\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) = \sum_{k=0}^{m} \mathcal{D}\left(x_\alpha\right)(k)\,\mathcal{D}\left(y_\alpha\right)(m-k). \qquad (6)$$

Proof Let x_α and y_α be arbitrary signals; then, by (5) and (1), we get that

$$\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) = \int_0^\infty \left(\int_0^\infty x_\alpha\left(\tau\right) y_\alpha\left(t-\tau\right) d\tau\right) e^{\frac{-t}{T}} \frac{\left(\frac{t}{T}\right)^m}{m!} d\tau.$$

Fubinitz theorem also gives

$$\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) = \int_0^\infty x_\alpha\left(\tau\right) \int_0^\infty y_\alpha\left(t-\tau\right) e^{\frac{-t}{T}} \frac{\left(\frac{t}{T}\right)^m}{m!} dt d\tau.$$

The change of variables, $\xi = t - \tau$, implies that

$$\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) = \int_0^\infty x_\alpha\left(\tau\right) \int_0^\infty y_\alpha\left(\xi\right) e^{\frac{-\xi+\tau}{T}} \frac{\left(\frac{\xi+\tau}{T}\right)^m}{m!} d\xi d\tau.$$

Hence, the binomial theorem, $(\alpha + \beta)^m = \sum_{k=0}^m \binom{m}{k} \alpha^{m-k} \beta^k$, suggests

$$\begin{aligned}
\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) \quad &= \quad \sum_{k=0}^m \frac{\binom{m}{k}}{m!} k!\,(m-k)! \left(\int_0^\infty x_\alpha\left(\tau\right) e^{\frac{-\tau}{T}} \frac{\left(\frac{\tau}{T}\right)^k}{k!} d\tau\right) \\
&\quad \times \left(\int_0^\infty y_\alpha\left(\xi\right) e^{\frac{-\xi}{T}} \frac{\left(\frac{\xi}{T}\right)^{m-k}}{(m-k)!} d\xi\right). \qquad (7)
\end{aligned}$$

Using the formula $\binom{m}{k} = \dfrac{m!}{(m-k)!k!}$, Equation (7) can be simplified to mean

$$\mathcal{D}\left(x_\alpha \bullet y_\alpha\right)(m) = \sum_{k=0}^{m} \mathcal{D}\left(x_\alpha\right)(k)\,\mathcal{D}\left(y_\alpha\right)(m-k)$$

This completes the proof of the theorem.

3   GENERALIZED $\mathcal{D}$ TRANSFORM

In this section we discuss the $\mathcal{D}$ transform on a distribution space [3,10,13] of compact support and a quotient space of Boehmians [7]. To this end, reader is supposed to be aquainted with the concept of distribution spaces [17]. The minimal structure of Boehmians has been given in Section 3.2. of this note.

3.1 DISTRIBUTIONAL $\mathcal{D}$ TRANSFORM

Denote by $\mathtt{E}\left(\mathtt{R}^{+}\right)$ the space of smooth functions over $\mathtt{R}^{+}$ equipped with its usual topology [13,10] then for real values $t, t > 0$, we certainly get that

$$e^{\frac{-t}{\mathcal{T}}}\frac{\left(\frac{t}{\mathcal{T}}\right)^{m}}{m!} \in \mathtt{E}\left(\mathtt{R}^{+}\right), \tag{8}$$

$\mathcal{T}$ being the sampling period.

By (8), we define the distributionl $\mathcal{D}$ transform of a distribution in $\mathtt{E}'\left(\mathtt{R}^{+}\right)$, the space of distributions of compact support, as

$$\overset{\Rrightarrow}{\mathcal{D}}\left(x_{\alpha}\right)\left(m\right) = \left\langle x_{\alpha}\left(t\right), e^{\frac{-t}{\mathcal{T}}}\frac{\left(\frac{t}{\mathcal{T}}\right)^{m}}{m!}\right\rangle. \tag{9}$$

Reader can simply establish some properties of $\overset{\Rrightarrow}{\mathcal{D}}$ transform which are listed in the following remark.

Remark 1 Let $x_{\alpha} \in \mathtt{E}'\left(\mathtt{R}^{+}\right)$ be given; then the following are satisfied in the distributional sense :

(1) $\overset{\Rrightarrow}{\mathcal{D}}$ is linear.

(2) $\overset{\Rrightarrow}{\mathcal{D}}$ is one to one.

Proof is straightforward.

Moreover, the generalized convolution of (5) can also be described by the inner product

$$\left\langle \left(x_{\alpha} \bullet y_{\alpha}\right)\left(t\right), \varphi\left(t\right)\right\rangle = \left\langle x_{\alpha}\left(t\right), \left\langle y_{\alpha}\left(\tau\right), \varphi\left(t+\tau\right)\right\rangle\right\rangle,$$

where $\varphi\left(\tau\right) \in \mathtt{E}\left(\mathtt{R}^{+}\right)$ is arbitrary. Hence, the distributional $\overset{\Rrightarrow}{\mathcal{D}}$ transform therefore acts on the convolution $\bullet$ by the equation

$$\overset{\Rrightarrow}{\mathcal{D}}\left(x_{\alpha} \bullet y_{\alpha}\right)\left(m\right) = \sum_{k=0}^{m}\overset{\Rrightarrow}{\mathcal{D}}\left(x_{\alpha}\right)\left(k\right)\overset{\Rrightarrow}{\mathcal{D}}\left(y_{\alpha}\right)\left(m-k\right). \tag{10}$$

3.2 GENERAL BOEHMIAN

The space of Boehmians as the youngest space of generalized functions consists of the following elements : A linear space $\mathtt{Y}$ and a subspace $\mathtt{X}$ of $\mathtt{Y}$. To all pairs $\left(f, \phi\right), \left(g, \psi\right), f, g \in \mathtt{Y}, \phi, \psi \in \mathtt{X}$, is assigned the products $f * \phi, g * \psi$ such that the following holds :

(1) $\phi * \psi \in \mathtt{X}$ and $\phi * \psi = \psi * \phi$.

(2) $\left(f * \phi\right) * \psi = f * \left(\phi * \psi\right)$.

(3) $\left(f + g\right) * \phi = f * \phi + g * \phi$.

(4) $k\left(f * \phi\right) = \left(kf\right) * \phi = f * \left(k\phi\right), k \in \mathtt{R}$.

Let Δ be the collection of sequences from $\mathtt{X}$ satisfying :

(5)If $\left(\epsilon_{n}\right) \in \Delta$ and $f * \epsilon_{n} = g * \epsilon_{n}, n = 1, 2, ...$, then $f = g$;

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(6) $(\epsilon_n), (\tau_n) \in \Delta \Rightarrow (\epsilon_n * \tau_n) \in \Delta$;

Then elements of Δ are called delta sequences or approximating identities.

Let $\mathtt{A}$ be the class of pairs of sequences defined by

$$\mathtt{A} = \left\{ ((f_n), (\epsilon_n)) : (f_n) \subseteq \mathtt{Y}^{\mathbb{N}}, (\epsilon_n) \in \Delta \right\},$$

$\forall n \in \mathbb{N}$.

Then the pair $((f_n), (\epsilon_n)) \in \mathtt{A}$ is said to be quotient of sequences, $\dfrac{f_n}{\epsilon_n}$, when

$$f_n * \epsilon_m = f_m * \epsilon_n, \forall n, m \in \mathbb{N}.$$

Quotients of sequences $\dfrac{f_n}{\epsilon_n}$ and $\dfrac{g_n}{\tau_n}$ are equivalent, $\dfrac{f_n}{\epsilon_n} \sim \dfrac{g_n}{\tau_n}$, if

$$f_n * \epsilon_m = g_m * \tau_n, \forall n, m \in \mathbb{N}.$$

The operation $\sim$ is an equivalent relation on $\mathtt{A}$ and hence, splits $\mathtt{A}$ into equivalence classes.

The equivalence class containing $\dfrac{f_n}{\epsilon_n}$ is denoted by $\left[\dfrac{f_n}{\epsilon_n} \right]$. These equivalence classes are called *Boehmians* and the space of all Boehmians is denoted by $\mathtt{B}(\mathtt{Y}, \mathtt{X}, \Delta, *)$.

The sum and multiplication by a scalar of two Boehmians are naturally defined in the respective ways :

$$\left[\frac{f_n}{\epsilon_n} \right] + \left[\frac{g_n}{\tau_n} \right] = \left[\frac{f_n * \tau_n + g_n * \epsilon_n}{\epsilon_n * \tau_n} \right]$$

and

$$\kappa \left[\frac{f_n}{\epsilon_n} \right] = \left[\frac{\kappa f_n}{\epsilon_n} \right], \kappa \text{ being complex number.}$$

The operations $*$ and $\mathcal{D}^k$, the k-th derivative, are defined by $\left[\dfrac{f_n}{\epsilon_n} \right] * \left[\dfrac{g_n}{\tau_n} \right] = \left[\dfrac{f_n * g_n}{\epsilon_n * \tau_n} \right]$ and $\mathcal{D}^k \left[\dfrac{f_n}{\epsilon_n} \right] = \left[\dfrac{\mathcal{D}^k f_n}{\epsilon_n} \right]$.

Many a time, $\mathtt{Y}$ is equipped with a notion of convergence. The intrinsic relationship between the notion of convergence and the product $*$ are given by:

(1) If $f_n \to f$ as $n \to \infty$ in $\mathtt{Y}$ and, $\phi \in \mathtt{X}$ is any fixed element, then

$$f_n * \phi \to f * \phi \text{ in } \mathtt{Y} \text{ as } n \to \infty.$$

(2) If $f_n \to f$ as $n \to \infty$ in $\mathtt{Y}$ and $(\epsilon_n) \in \Delta$, then $f_n * \epsilon_n \to f$ in $\mathtt{Y}$ as $n \to \infty$.

The operation $*$ is extended to $\mathtt{B}(\mathtt{Y}, \mathtt{X}, \Delta, *) \times \mathtt{X}$ by the following definition.

More details of the construction of Boehmians are given in [1, 2, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15].

6   *S.K.Q. Al-Omari*

3.3 $\mathcal{D}$ TRANSFORM OF A BOEHMIAN

Denote by $\mathtt{U}$ the space of test functions of bounded support and $\mathtt{U}'$ its conjugate space of distributions defined on $\mathtt{R}^+$. By Δ we mean the subset of $\mathtt{U}$ of all sequences of analogue signals $(\theta_{\alpha i})_0^\infty$ in $\mathtt{U}$ such that :

$(i) \int\limits_0^\infty \theta_{\alpha i}(t)\, dt = 1;$

$(ii)\, \theta_{\alpha i}(t) > 0, \forall i \in \mathbb{N};$

$(iii)\, \mathrm{supp}\, \theta_{\alpha i}(t) \to 0$ as $i \to \infty.$

Then each sequence of such signals is called delta sequences which correspond to dirac delta function.

In the next, let us be concerned with the Boehmian space $\mathtt{B}(\mathtt{E}, \mathtt{U}, \Delta, \bullet)$ where $\mathtt{E}$ is the group, $\mathtt{U}$ is the subgroup of $\mathtt{E}$ and, $\bullet$ is the operation, then, with Δ, a typical element of $\mathtt{B}(\mathtt{E}, \mathtt{U}, \Delta, \bullet)$ is written in the form

$$\beta := \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right],$$

where $x_{\alpha i}$ is an analogue sequence from $\mathtt{E}(\mathtt{R}^+)$ and $\theta_{\alpha i} \in \Delta.$

Sum, derivation, multiplication by scalars and the operation $\bullet$ between Boehmians in $\mathtt{B}(\mathtt{E}, \mathtt{U}, \Delta, \bullet)$ are respectively defined in the natural way as :

$$\left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] + \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right] = \left[\frac{x_{\alpha i} \bullet \beta_{\alpha i} + y_{\alpha i} \bullet \theta_{\alpha i}}{\theta_{\alpha i} \bullet \beta_{\alpha i}} \right],$$

$$\frac{\mathtt{d}^k}{\mathtt{d}x^k} \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = \left[\frac{\dfrac{\mathtt{d}^k}{\mathtt{d}x^k} x_{\alpha i}}{\theta_{\alpha i}} \right],$$

$$\kappa \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = \left[\frac{\kappa x_{\alpha i}}{\theta_{\alpha i}} \right], \kappa \text{ being complex number,}$$

and

$$\left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] \bullet \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right] = \left[\frac{x_{\alpha i} \bullet y_{\alpha i}}{\theta_{\alpha i} \bullet \beta_{\alpha i}} \right]$$

Theorem 2 Let $(x_{\alpha i}) \in \mathtt{E}$, $(\theta_{\alpha i}) \in \Delta$ be a pair of analogue sequences be given such that $\beta = \left[\dfrac{x_{\alpha i}}{\theta_{\alpha i}} \right] \in \mathtt{B}(\mathtt{E}, \mathtt{U}, \Delta, \bullet)$. Then $\mathcal{D}x_{\alpha i}$ converges uniformly in $\mathtt{U}'(\mathtt{R}^+).$

Proof Let $\varphi \in \mathtt{U}(\mathtt{R}^+)$ be arbitrary. Choose $(\theta_{\alpha k_0}) \in \Delta$ such that $(\theta_{\alpha k_0})(m-k) > 0$ on the support of $\varphi, 1 \le k \le m$. Then the fact that $\dfrac{x_{\alpha i}}{\theta_{\alpha i}}$ represents a quotient means

$$x_{\alpha i} \bullet \theta_{\alpha j} = x_{\alpha j} \bullet \theta_{\alpha i}, i, j \in \mathbb{N}. \tag{11}$$

Applying $\mathcal{D}$ transform to (11) yields

$$\sum_{k=0}^m \mathcal{D}x_{\alpha i}(k)\, \mathcal{D}\theta_{\alpha j}(m-k) = \sum_{k=0}^m \mathcal{D}x_{\alpha j}(k)\, \mathcal{D}\theta_{\alpha i}(m-k), \tag{12}$$

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from which we get

$$\mathcal{D}x_{\alpha i}\left(k\right)\mathcal{D}\theta_{\alpha j}\left(m-k\right) = \mathcal{D}x_{\alpha j}\left(k\right)\mathcal{D}\theta_{\alpha i}\left(m-k\right). \tag{13}$$

Therefore,

$$
\begin{aligned}
\mathcal{D}x_{\alpha i}\left(k\right)\left(\varphi\right) &= \mathcal{D}x_{\alpha i}\left(k\right)\left(\frac{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}\varphi\right) \\
&= \mathcal{D}x_{\alpha i}\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)\left(\frac{\varphi}{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}\right).
\end{aligned}
$$

By (13) we write

$$
\begin{aligned}
\mathcal{D}x_{\alpha i}\left(k\right)\left(\varphi\right) &= \mathcal{D}x_{\alpha k_0}\left(k\right)\mathcal{D}\theta_{\alpha i}\left(m-k\right)\left(\frac{\varphi}{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}\right) \\
&= \mathcal{D}x_{\alpha k_0}\left(\frac{\mathcal{D}\theta_{\alpha i}\left(m-k\right)}{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}\right). \tag{14}
\end{aligned}
$$

But the fact that

$$\mathcal{D}\theta_{\alpha i}\left(m-k\right) \to e^{\frac{-t}{T}}\frac{\left(\frac{t}{T}\right)^k}{k!}, \tag{15}$$

as $i \to \infty$, when inserted in (14), yields

$$\mathcal{D}x_{\alpha i}\left(k\right)\left(\varphi\right) \to \Phi \text{ as } i \to \infty$$

where $\Phi := e^{\frac{-t}{T}}\dfrac{\left(\frac{t}{T}\right)^k}{k!}\dfrac{\varphi}{\mathcal{D}\theta_{\alpha k_0}\left(m-k\right)}$.

Once again, the fact that Φ is ofcourse smooth function with a property that

$$\operatorname{supp}\Phi \subseteq \operatorname{supp}\varphi$$

implies that $\mathcal{D}x_{\alpha i}\left(k\right)$ defines a distribution in $\mathtt{U}'\left(\mathtt{R}^+\right).$

This completes the proof of the theorem.

Theorem 3 Let $\begin{bmatrix}x_{\alpha i}\\\theta_{\alpha i}\end{bmatrix} = \begin{bmatrix}y_{\alpha i}\\\beta_{\alpha i}\end{bmatrix} \in \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right);$ then $\mathcal{D}x_{\alpha i}\left(k\right)$ and $\mathcal{D}y_{\alpha i}\left(k\right)$ converge to the same limit in $\mathtt{U}'\left(\mathtt{R}^+\right).$

Proof Let $\begin{bmatrix}x_{\alpha i}\\\theta_{\alpha i}\end{bmatrix} = \begin{bmatrix}y_{\alpha i}\\\beta_{\alpha i}\end{bmatrix} \in \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right);$ then defining a set $\mathtt{A}_{\alpha i}$ and $\mathtt{B}_{\alpha i}$ by

$$
\left(\mathtt{A}_\alpha\right)_i = \begin{cases} x_{\alpha\frac{i+1}{2}}\bullet\beta_{\alpha\frac{i+1}{2}}, & \text{if } i \text{ odd} \\ y_{\alpha\frac{i}{2}}\bullet\theta_{\alpha\frac{i}{2}}, & \text{if } i \text{ even} \end{cases}
$$

and

$$
\mathtt{B}_{\alpha i} = \begin{cases} \theta_{\alpha\frac{i+1}{2}}\bullet\beta_{\alpha\frac{i+1}{2}}, & \text{if } i \text{ odd} \\ \theta_{\alpha\frac{i}{2}}\bullet\beta_{\alpha\frac{i}{2}}, & \text{if } i \text{ even} \end{cases}
$$

implies $\begin{bmatrix}\mathtt{A}_{\alpha i}\\\mathtt{B}_{\alpha i}\end{bmatrix} = \begin{bmatrix}x_{\alpha i}\\\theta_{\alpha i}\end{bmatrix} = \begin{bmatrix}y_{\alpha i}\\\beta_{\alpha i}\end{bmatrix}.$

Therefore $\mathcal{D}\mathtt{A}_{\alpha i}$ converges in $\mathtt{U}'\left(\mathtt{R}^+\right)$ and

$$\lim_{i \to \infty} \mathcal{D}\mathtt{A}_{\alpha(2i-1)}\left(\varphi\right) = \lim_{i \to \infty} \mathcal{D}x_{\alpha i}\left(\varphi\right).$$

Thus, $\mathcal{D}\mathtt{A}_{\alpha i}$ and $\mathcal{D}x_{\alpha i}$ converge to same limit.

Similarly we proceed for $\mathcal{D}\mathtt{A}_{\alpha i}$ and $\mathcal{D}y_{\alpha i}$.

Hence the proof is completed.

Now, by virtue of above, we introduce the $\mathcal{D}$ transform of the equivalence class $\left[\dfrac{x_{\alpha i}}{\theta_{\alpha i}}\right] \in \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right)$ as

$$\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right] = \lim_{i \to \infty} \mathcal{D}x_{\alpha i} \tag{16}$$

on compact subsets of $\mathtt{R}^+$.

Theorem 4 The transform $\overset{\leftrightarrows}{\mathcal{D}} : \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right) \to \mathtt{U}'\left(\mathtt{R}^+\right)$ is well defined.

Proof Considering the equation $\left[\dfrac{x_{\alpha i}}{\theta_{\alpha i}}\right] = \left[\dfrac{y_{\alpha i}}{\beta_{\alpha i}}\right]$ we get, by the concept of quotients, that

$$x_{\alpha i} \bullet \beta_{\alpha j} = y_{\alpha i} \bullet \theta_{\alpha i}. \tag{17}$$

Applying $\mathcal{D}$ transform to (17) then using the idea of (12) and (13) we get

$$\mathcal{D}x_{\alpha i}\left(k\right)\mathcal{D}\beta_{\alpha j}\left(m-k\right) = \mathcal{D}y_{\alpha j}\left(k\right)\mathcal{D}\theta_{\alpha i}\left(m-k\right).$$

In particular, for $i = j$, we get

$$\mathcal{D}x_{\alpha i}\left(k\right)\mathcal{D}\beta_{\alpha i}\left(m-k\right) = \mathcal{D}y_{\alpha i}\left(k\right)\mathcal{D}\beta_{\alpha i}\left(m-k\right). \tag{18}$$

Considering limits as $i \to \infty$, from (18), we find

$$\lim_{i \to \infty} \mathcal{D}x_{\alpha i}\left(k\right) = \lim_{i \to \infty} \mathcal{D}y_{\alpha i}\left(k\right).$$

Hence the theorem.

Theorem 5 The transform $\overset{\leftrightarrows}{\mathcal{D}} : \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right) \to \mathtt{U}'\left(\mathtt{R}^+\right)$ is infinitely smooth.

Proof Let $\left[\dfrac{x_{\alpha i}}{\theta_{\alpha i}}\right] \in \mathtt{B}\left(\mathtt{E},\mathtt{U},\Delta,\bullet\right)$ and $\mathtt{K}$ be an open bounded set of $\mathtt{R}^+$ then, there is $j \in \mathbb{N}$ such that

$$\mathcal{D}\theta_{\alpha j}\left(m-k\right) > 0, 1 \leq k \leq m, \text{ on } \mathtt{K}.$$

Hence,

$$\begin{aligned} \overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right] &= \lim_{i \to \infty} \mathcal{D}x_{\alpha i}\left(k\right) \\ &= \lim_{i \to \infty} \mathcal{D}x_{\alpha i}\left(k\right)\frac{\mathcal{D}\theta_{\alpha j}\left(m-k\right)}{\mathcal{D}\theta_{\alpha j}\left(m-k\right)} \end{aligned}$$

By (13) we get

$$\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right] = \lim_{i\to\infty} \frac{\mathcal{D}x_{\alpha j}(k)\,\mathcal{D}\theta_{\alpha i}(m-k)}{\mathcal{D}\theta_{\alpha j}(m-k)}$$

$$\to \frac{e^{-\frac{t}{T}}\left(\frac{t}{T}\right)^{k}}{k!}\frac{\mathcal{D}x_{\alpha j}(k)}{\mathcal{D}\theta_{\alpha j}(m-k)} \text{ on } \mathtt{K}.$$

Last step follows from (15).

Since $\mathcal{D}x_{\alpha j}$ and $\mathcal{D}x_{\alpha j}$ are in $\mathtt{E}(\mathtt{R}^{+})$ our result follows.
This completes the proof of the theorem.

Theorem 6 Let $(x_{\alpha i}),(y_{\alpha i}) \in \mathtt{E},(\theta_{\alpha i}),(\beta_{\alpha i}) \in \Delta$ be given sequences of signals such that $\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right],\left[\frac{y_{\alpha i}}{\beta_{\alpha i}}\right] \in \mathtt{B}(\mathtt{E},\mathtt{U},\Delta,\bullet)$. Then

$$\overset{\leftrightarrows}{\mathcal{D}}\left(\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right]\bullet\left[\frac{y_{\alpha i}}{\beta_{\alpha i}}\right]\right) = \frac{e^{-\frac{t}{T}}}{k!}\left(\frac{t}{T}\right)^{k}\left(\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right]\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{y_{\alpha i}}{\beta_{\alpha i}}\right]\right).$$

Proof of this theorem is a result of (15) and (16).

Theorem 7 Let $\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right]$ define a Boehmian in $\mathtt{B}(\mathtt{E},\mathtt{U},\Delta,\bullet)$ then

$$\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right](\mathcal{D}\theta_{\alpha j})(m-k) = e^{-\frac{t}{T}}\frac{\left(\frac{t}{T}\right)^{k}}{k!}\mathcal{D}x_{\alpha j}(k), 1 \le k \le m.$$

Proof Let $\varphi \in \mathtt{U}(\mathtt{R}^{+})$ then (15) and (16) give

$$\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right](\mathcal{D}\theta_{\alpha j})(m-k)(\varphi) = \lim_{i\to\infty}\mathcal{D}x_{\alpha i}(k)(\mathcal{D}\theta_{\alpha i})(m-k)(\varphi)$$

The equation in (13) then yields

$$\overset{\leftrightarrows}{\mathcal{D}}\left[\frac{x_{\alpha i}}{\theta_{\alpha i}}\right](\mathcal{D}\theta_{\alpha j})(m-k)(\varphi) = \lim_{i\to\infty}\mathcal{D}x_{\alpha j}(k)(\mathcal{D}\theta_{\alpha i})(m-k)(\varphi)$$

$$= e^{-\frac{t}{T}}\frac{\left(\frac{t}{T}\right)^{k}}{k!}(\mathcal{D}x_{\alpha j}(k))(\varphi).$$

Hence the theorem.

Theorem 8 The transform $\overset{\leftrightarrows}{\mathcal{D}} : \mathtt{B}(\mathtt{E},\mathtt{U},\Delta,\bullet) \to \mathtt{U}'(\mathtt{R}^{+})$ is continuous with respect to δ - convergence.

Proof Let $(\beta_{i}) \in \mathtt{B}(\mathtt{E},\mathtt{U},\Delta,\bullet),\beta \in \mathtt{B}(\mathtt{E},\mathtt{U},\Delta,\bullet)$ be such that $\beta_{i} \overset{\delta}{\to} \beta$. There can exists $(x_{\alpha_{i,k}}),(x_{\alpha k}) \in \mathtt{E},(\theta_{\alpha i}) \in \Delta$, such that $\beta_{n} = \left[\frac{x_{\alpha_{i,k}}}{\theta_{\alpha_{i}}}\right],\beta = \left[\frac{x_{\alpha k}}{\theta_{\alpha_{i}}}\right]$ and $x_{\alpha_{i},k} \to x_{\alpha k}$ for every $k \in \mathbf{N}$ as $i \to \infty$ in $\mathtt{E}$. Continuity of $\mathcal{D}$ implies $\mathcal{D}(x_{\alpha i,k}) \to \mathcal{D}(x_{\alpha k})$ as $i \to \infty$ in $\mathtt{U}'(\mathtt{R}^{+})$.

Thus, $\overleftrightarrow{\mathcal{D}}\beta_i \overset{\delta}{\to} \overleftrightarrow{\mathcal{D}}\beta$ as $i \to \infty$.

This completes the proof of the theorem.

Theorem 9 The transform $\overleftrightarrow{\mathcal{D}} : \mathtt{B}\,(\mathtt{E},\mathtt{U},\Delta,\bullet) \to \mathtt{U}^{'}\,(\mathtt{R}^{+})$ is linear.

Proof Let $k_1, k_2 \in \mathtt{R}$ then we get

$$\overleftrightarrow{\mathcal{D}} \left(k_1 \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] + k_2 \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right] \right) = \left[\frac{k_1 x_{\alpha i}}{\theta_{\alpha i}} \right] + \left[\frac{k_2 y_{\alpha i}}{\beta_{\alpha i}} \right]$$

That is

$$\overleftrightarrow{\mathcal{D}} \left(k_1 \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] + k_2 \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right] \right) = k_1 \lim_{i \to \infty} \mathcal{D}x_{\alpha i} + k_2 \lim_{i \to \infty} \mathcal{D}y_{\alpha i}$$

That is

$$\overleftrightarrow{\mathcal{D}} \left(k_1 \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] + k_2 \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right] \right) = k_1 \overleftrightarrow{\mathcal{D}} \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] + k_2 \overleftrightarrow{\mathcal{D}} \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right]$$

This completes the proof of the theorem.

Theorem 10 The transform $\overleftrightarrow{\mathcal{D}} : \mathtt{B}\,(\mathtt{E},\mathtt{U},\Delta,\bullet) \to \mathtt{U}^{'}\,(\mathtt{R}^{+})$ is one - to - one.

Proof Let $\left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right]$; then $x_{\alpha i} \bullet \beta_{\alpha j} = y_{\alpha j} \bullet \theta_{\alpha i}$. Applying $\mathcal{D}$ transform then using (13) and considering limits for $i = j$, imply $\lim x_{\alpha i} = \lim y_{\alpha i}$ as $i \to \infty$. Thus,

$$\overleftrightarrow{\mathcal{D}} \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = \overleftrightarrow{\mathcal{D}} \left[\frac{y_{\alpha i}}{\beta_{\alpha i}} \right].$$

Hence the proof.

We state without proof the following theorems. Due simplicity of the proofs we prefer they be omitted.

Theorem 11 $\overleftrightarrow{\mathcal{D}} \left(\left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] \bullet \beta_{\alpha i}\,(m - k) \right) = \overleftrightarrow{\mathcal{D}} \left(\beta_{\alpha i} \bullet \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] \right), (\beta_{\alpha})_i \in \Delta.$

Theorem 12 If $\overleftrightarrow{\mathcal{D}} \left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = 0$, then $\left[\frac{x_{\alpha i}}{\theta_{\alpha i}} \right] = 0$ in the sense of Boehmians of $\mathtt{B}\,(\mathtt{E},\mathtt{U},\Delta,\bullet)$.

Theorem 13 If ρ_n is a sequence of Boehmians in $\mathtt{B}\,(\mathtt{E},\mathtt{U},\Delta,\bullet)$ such that $\rho_n \overset{\Delta}{\to} \rho$ as $n \to \infty$, then $\overleftrightarrow{\mathcal{D}}\rho_n \overset{\Delta}{\to} \overleftrightarrow{\mathcal{D}}\rho$ as $n \to \infty$ in $\mathtt{U}^{'}\,(\mathtt{R}^{+})$ on compact subsets.

Conflict of Interests The author declare that there is no conflict of interests regarding the publication of this paper.

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SOME FIXED POINT THEOREMS FOR ITERATED CONTRACTION MAPS

Santosh Kumar

Department of Mathematics

College of Natural and Applied Sciences

University of Dar es salaam

P.O.Box-35062

Tanzania.

E. Mail: drsengar2002@gmail.com

ABSTRACT

The study of iterated contraction was initiated by Rheinboldt in 1969. The concept of iterated contraction proves to be very useful in the study of certain iterative process and has wide applicability. In this paper a brief introduction of iterated contraction maps is given and some fixed point results are discussed.

1. INTRODUCTION

In 1969, Rheinboldt [4] gave unified convergence theory for a class of iterative processes. He initiated the process of finding solution of the nonlinear operator equation $Fx = 0$ using metric fixed point theory. Rheinboldt developed a general convergence theory for iterative processes of the form $x_{k+1} = Gx_k, k = 0, 1, ...$; and founded on certain nonlinear estimates for the iteration function G as well as on a so called concept of majorizing sequences, His new approach reduces the study of the iterative process to that of a second order nonlinear difference equation.

The purpose of this paper is to establish yet another fixed point theorem for iterated contractions maps in different settings. Our results extends several results contained in [1], [2], [6] etc.

2. PRELIMINARIES

If f is a contractive mapping, then under certain conditions the sequence $x_{n+1} = fx_n$ tends to the unique fixed-point of f. Ortega, J. M. and Rheinboldt, W. C. [3] proved this type results. The following definitions are useful to our discussion.

Definition 2.1 If $f : X \to X$ is map such that $d(fx, ffx) \leq k \ d(x, fx)$ for all $x \in X, 0 \leq k < 1$, then f is said to be an iterated contraction map.

In case $d(fx, ffx) < d(x, fx), x \neq fx$, then f is an iterated contractive map. See for further results [3,4].

Definition 2.2 A map $f : X \to X$, satisfying $d(fx, fy) \leq kd(x, y), 0 \leq k < 1$, for all $x, y \in X$, is called a contraction map.
A contraction map is continuous and is an iterated contraction.

For example, if $y = fx$, then $d(fx, ffx) \leq kd(x, fx)$ is satisfied. However, converse is not true.
If $f : [-1/2, 1/2] \to [-1/2, 1/2]$ is given by $fx = x^2$, then f is an iterated contraction but not a contraction map.
If $f : R \to R$, defined by $fx = 0$ for $x \in [0, 1/2)$ and $fx = 1$ for $x \in [1/2, 1]$, then f is not continuous at $x = 1/2$, and f is an iterated contraction.

Remark 2.1 A contraction map has a unique fixed point. However, an iterated contraction map may have more than one fixed point.
For example, the iterated contraction function $fx = x^2$ on $[0, 1]$ has $f0 = 0$ and $f1 = 1$, two fixed points.

Remark 2.2 If $f : R \to R$ is given by $fx = 2x+1$, then f is a continuous map but f is not an iterated contraction. It is easy to see that $d(fx, ffx) \leq kd(x, fx)$ is not satisfied for $0 \leq k < 1$.

Remark 2.3 A discontinuous function need not always be an iterated contraction.

Let $X = [0, 1]$. Define $f : [0, 1] \to [0, 1]$ by $fx = 1/2, x \in [0, 1/2)$, and

$fx = 0, x \in [1/2, 1]$. Then f is discontinuous.

If we take $x = 1/4$, then $d(fx, ffx) \leq kd(x, fx), 0 \leq k < 1$, is not satisfied and hence f is not an iterated contraction.

If $f : [0, 1] \to [0, 1]$ defined by $fx = 1/4, x \in [0, 1/2)$, and $fx = 3/4, x \in [1/2, 1]$,then f is discontinuous at $x = 1/2$, f has fixed points when $x = 1/4$ and $x = 3/4$.

If $f : R \to R$ defined by $fx = x/3 + 1/3$ for $x \leq 0$, and $fx = x/3$ for $x > 0$, then f is an iterated contraction and f has no fixed point.

If $f : [0, 1] \to [0, 1]$ is defined by $fx = 0$ for $x \in [0, 1/2)$, and $fx = 1/2$ for $x \in [1/2, 1]$, then f is discontinuous at $x = 1/2$ and f is an iterated contraction map and has a fixed point $f(1/2) = 1/2$.

The following is a fixed point theorem for iterated contraction map.

Theorem 2.1 If $f : X \to X$ is a continuous iterated contractive map and the sequence of iterates $\{x_n\}$ defined by $x_{n+1} = fx_n, n = 1, 2, ...$ for $x \in X$, has a subsequence converging to $y \in X$, then $y = fy$, that is,f has a fixed point.

Proof: Let $x_{n+1} = fx_n, n = 1, 2, ...$ Then the sequence $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence of reals. It is bounded below by 0, and therefore has a limit. Since the subsequence converges to y and f is continuous on X, so $f(x_{n_i})$ converges to fy and $ff(x_{n_i})$ converges to ffy.

Thus $d(y, fy) = lim\ d(x_{n_i}, x_{n_{i+1}}) = lim\ d(x_{n_{i+1}}, x_{n_{i+2}}) = d(fy, ffy)$.

If $y \neq fy$, then $d(ffy, fy) < d(fy, y)$, since f is an iterative contractive map. Consequently, $d(fy, y) = d(ffy, fy) < d(fy, y)$, a contradiction and hence $fy = y$.

We give the following example to show that if f is an iterated contraction that is not continuous, then f may not have a fixed point.

Let $f : R \to R$, given by $fx = x/5 + 1/5$ for $x \leq 0$, and $fx = x/5$ for $x > 0$. Then f does not have a fixed point. Here f is discontinuous at $x = 0$. If f is continuous but X is not complete, then f need not have a fixed point.

For example, if $f : (0, 1] \to (0, 1]$ given by $fx = x/2$, then f is an iterated contraction, but has no fixed point.

A continuous function f that is not an iterated contraction may not have a fixed point.

The following example illustrates the fact:

If $f : R \to R$ is a translation map $fx = x + 2$, then f has no fixed point but f is continuous.

Note 2.1 If f is not contraction but some powers of f is contraction, then f has a unique fixed point on a complete metric space.

For example, if $f : R \to R$ is defined by $fx = 1$ for x rational and $fx = 0$ for x irrational, then $fofx = f^2x = 1$ a constant map, is a contraction map, and f has a unique fixed point.

Note 2.2 Continuity of an iterated contraction is sufficient but not necessary.

Let $f : [0,1] \to [0,1]$ be defined by $fx = 0$ on $[0, 1/2)$, and $fx = 2/3$, for $x \in [1/2, 1]$. Then f is an iterated contraction and $f0 = 0, f2/3 = 2/3$, and f is not continuous.

As stated in Note 2.1 that if f is not contraction still f may have a unique fixed point when some powers of f is a contraction map. The same is true for iterated contraction map.

Theorem 2.2 Let $f : X \to X$ be an iterated contraction map on a complete metric space X. If for some power of f, say f^r is an iterated contraction, that is, $d(f^r x, f^r f^r x) \leq k \ d(x, f^r x)$ and f^r is continuous at y, where $y = lim(f^r)^n x$, for any arbitrary $x \in X$. Then f has a fixed point.

Proof: Since X is a complete metric space and f is an iterated contraction that is continuous at y, therefore $y = f^r y$, where $y = lim(f^r)^n x$.

It is easy to show that $d(y, fy) \leq k^r \ d(y, fy)$. Since $k^r \leq 1$, therefore $d(y, fy) = 0$ and hence f has a fixed point.

We give the following example to illustrate the theorem.

Example 2.1 If $f : [0,1] \to [0,1]$ is given by $fx = 1/4, x \in [0, 1/4], f(x) = 0, x \in (1/4, 1/2)$, and $fx = 1/2, x \in [1/2, 1]$, then f is not continuous at $x = 1/4$. However, if we take iteration, then $f^r(1/4) = 1/4$ on $[0, 1/2)$ and $f^r(1/4) = 1/2$ on $[1/2, 1]$ for $r = 2, 3, ...$ It is clear that

$f(1/4) = 1/4$.

Note 2.3 If f is not an iterated contraction in Theorem 2.2, but f^r is an iterated contraction with $f^r y = y$, then f has a fixed point.

The following example is worth mentioning.

Example 2.2 Let $f : [0,1] \rightarrow [0,1]$ defined by $fx = 1/4$ for $x \in [0,1/4], fx = 0$ for $x \in (1/4, 1/2), fx = 1/2$ for $x \in [1/2, 3/4]$ and $fx = 3/4$ for $x \in (3/4, 1]$.

Here f is not iterated contraction but f^r is iterated contraction for $r = 2, 3, ... f^2(x) = 1/4$ for $x \in [0, 1/2)$ and $f^2(x) = 1/2$ for $x \in [1/2, 1]$. In this example f has a fixed point at $x = 1/4$.

3. MAIN RESULTS

In this section, we will prove some fixed point theorems for iterated maps under different settings. Our results will corollarize many existing results. The first theorem is as follows:

Theorem 3.1 If $f : X \rightarrow X$ is an iterated contraction map, and X is a complete metric space, then the sequence of iterates x_n converges to $y \in X$. In case, f is continuous at y, then $y = fy$, that is,f has a fixed point.

Proof: Let $x_{n+1} = fx_n, n = 1, 2...; x_1 \in X$. It is easy to show that $\{x_n\}$ is a Cauchy sequence, since f is an iterated contraction. The Cauchy sequence $\{x_n\}$ converges to $y \in X$, since X is a complete metric space. Moreover, if f is continuous at y, then $x_{n+1} = fx_n$ converges to fy. It follows that $y = fy$.

Note 3.1 A continuous iterated contraction map on a complete metric space has a unique fixed point. If an iterated contraction map is not continuous, even then it may have a fixed point.

For example, if $f : [0,1] \rightarrow [0,1]$ is given by $fx = 0$ for $x \in [0, 1/2)$ and $fx = 2/3$ for $x \in [1/2, 1]$. In this example, $X = [0,1]$ is complete but f is not continuous at $x = 1/2$. However, f is continuous at $x = 0, f0 = 0$ and f is continuous at $x = 2/3, f2/3 = 2/3$.

Theorem 3.2 If $f : C \rightarrow C$ is an iterated contraction and is continuous, where C is closed subset of a metric space X, then f has a fixed point provided that $f(C)$ is compact.

Proof: We show that the sequence $\{x_n\}$ has a convergent subsequence. Using iterated contraction and continuity of f we get a fixed point.

Definition 3.1 Let X be a metric space and $f : X \to X$. Then f is said to be an iterated nonexpansive map if

$$d(fx, ffx) \leq d(x, fx)$$

for all $x \in X$.

The following is a fixed point theorem for the iterated nonexpansive map.

Theorem 3.3 Let X be a metric space and $f : X \to X$ an iterated nonexpensive map satisfying the following :

(i) if $x \neq fx$, then $d(ffx, fx) < d(fx, x)$,

(ii) if for some $x \in X$, the sequence of iterates $x_{n+1} = fx_n$ has a convergent subsequence converging to y say and f is continuous at y. Then f has a fixed point.

Proof: It is easy to show that the sequence $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence of positive reals bounded below by 0. The sequence has a limit.

Hence, $d(fy, y) = lim\ d(x_{n_i}, x_{n_{i+1}}) = lim\ d(x_{n_{i+1}}, x_{n_{i+2}}) = d(ffy, fy)$.

This is a contradiction to (i). Therefore f has a fixed point, that is, $fy = y$.

If C is a compact subset of a metric space X and $f : C \to C$ an iterated nonexpansive map, satisfying condition (i) of the above theorem, then f has a fixed point.

Note 3.2 If C is compact, then condition (ii) of Theorem 3.3 is satisfied, and hence the result.

If C is a closed subset of a metric space X and $f : C \to C$ an iterated contraction. If the sequence $\{x_{n_n}\}$ converges to y, where f is continuous at y, then $fy = y$, that is, f has a fixed point.

The following theorem deals with two metrics on X.

Theorem 3.4 Let $f : X \to X$ satisfy the following:

(i) X is complete with metric d and $d(x, fx) \leq \delta(x, fx)$ for all $x, fx \in X$,

(ii)f is iterated contraction with respect to δ,

Then for $x \in X$, the sequence of iterates $x_n = f^n x$ converges to $y \in X$. If f is continuous at y, then f has a fixed point, say $fy = y$.

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Proof: It is easy to show that $\{x_n\}$ is a Cauchy sequence with respect to δ. Since $d(x, fx) \leq \delta(x, fx)$, therefore $\{x_n\}$ is a Cauchy sequence with respect to d. The sequence $\{x_n\}$ converges to y in (X, d) since it is complete. The function f is continuous at y, therefore $fy = f\ lim\ x_n = lim\ fx_n = lim\ x_{n+1} = y$. Hence the result.

Kannan [2] considered the following map.

Let $f : X \to X$ satisfy $d(fx, fy) \leq k[d(x, fx) + d(y, fy)]$ for all $x, y \in X$ and $0 \leq k < 1/2$. Then f is said to be a Kannan map.

If $y = fx$, then we get $d(fx, ffx) \leq k[d(x, fx) + d(fx, ffx)]$. This gives $d(fx, ffx) \leq \frac{k}{(1-k)} d(x, fx)$, where $0 \leq \frac{k}{(1-k)} < 1$, that is, f is an iterated contraction. A Kannan map is an iterated contraction map.

The following result due to Kannan [2] is valid for iterated contraction map.

Theorem 3.5 Let $f : X \to X$ satisfy $d(fx, fy) \leq k[d(x, fx) + d(y, fy)]$ for all $x, y \in X, 0 \leq k < 1/2, f$ continuous on X and let the sequence of iterates $\{x_n\}$ have a subsequence $\{x_{n_i}\}$ converging to y. Then f has a fixed point.

The following result deals with two mappings.

Theorem 3.6 Let $f : X \to X$ and $g : X \to X$, where X is a complete metric space, satisfy $d(fx, gfx) \leq kd(x, fx)$ and $d(gx, fgx) \leq kd(x, gx)$ for some $k, 0 \leq k < 1$ and for each $x_1 \in X$. If f and g are continuous on X, then f and g have a common fixed point.

Proof: We consider the sequence of iterates as follows. Let $x_2 = fx_1, x_3 = gx_2, x_4 = fx_2$, and so on. In this case it is easy to show that the sequence $\{x_n\}$ is a Cauchy sequence and converges in X since X is a complete metric space.

Let $lim\ x_n = y$. Then $lim\ x_{2n} = y$ and $lim\ x_{2n-1} = y$. Since f and g are continuous on X, therefore,

$$fy = f\ lim\ x_{2n-1} = lim\ fx_{2n-1} = lim\ x_{2n} = y.$$

$$gy = g\ lim\ x_{2n} = lim\ gx_{2n} = lim\ x_{2n+1} = y.$$

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Hence $y = fy = gy$ and y is a common fixed point.

Results of this type for topological vector spaces are given in [5]. We are giving the following:

Theorem 3.7 Let $f : X \to X$ be a continuous iterated contraction map such that:

(i) if $x \neq fx$, then $d(fx, ffx) < d(x, fx)$, and

(ii) the sequence $x_{n+1} = f(x_n)$ has a convergent subsequence converging to y(say).

Then the sequence $\{x_n\}$ converges to a fixed point of f.

Proof: It is easy to see that the sequence $\{d(x_n, x_{n+1})\}$ is a nonincreasing and bounded below by 0. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging to y. Then,

$$d(y, fy) = \lim d(x_{n_i}, x_{n_{i+1}}) = d(x_{n_{i+1}}, x_{n_{i+2}}) = d(fy, ffy) < d(y, fy).$$

This is a contradiction so $y = fy$. Since $d(x_{n+1}, y) < d(x_n, y)$ for all n, so $\{x_n\}$ converges to $y = fy$.

The result due to Cheney and Goldstein [1] follows as a corollary.

Corollary 3.1 Let f be a map of a metric space X into itself such that

(i) f is a nonexpansive map on X, that is, $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$,

(ii) if $x \neq fx$, then $d(fx, ffx) < d(x, fx)$,

(iii) the sequence $x_{n+1} = f(x_n)$ has a convergent subsequence converging to y(say). Then the sequence $\{x_n\}$ converges to a fixed point of f.

The following is well known for contractive maps.

Note 3.3 If $f : X \to X$ is a contractive map and $f(X)$ compact, then f has a unique fixed point.

It is easy to see that the sequence of iterates $\{x_n\}$ converges to a unique fixed point of f. However, for nonexpansive map, a sequence of iterates need not converge to a fixed point of f. For example, if $fx = -x$, then the sequence of iterates $\{x_n\}$ does not converge to a fixed point of $f(f0 = 0)$.

Note 3.4 If $g = afx + (1 - a)x, 0 < a < 1$, then the fixed point of f is the same as of g.

Let $fy = y$. Then $gy = afy + (1 - a)y$, that is, $gy = y$ since $fy = y$.

Let $gy = y$. Then we show that $y = fy$. Here $gy = afy + (1-a)y = y$. Then $fy = y$, that is, f has a fixed point y [6]. In case the sequence $x_{n+1} = gx_n$ converges to y a fixed point of g, then $y = fy$.

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On Leave from:

Department of Applied Mathematics

Inderprastha Engineering College,

63, Site-IV, Surya Nagar Flyover Road, Sahibabad

Ghaziabad, U.P-201010.

India.

Nonlinear Regularized Nonconvex Random Variational Inequalities with Fuzzy Event in q-uniformly Smooth Banach Spaces

Salahuddin

Department of Mathematics

Jazan University, Jazan

Kingdom of Saudi Arabia

salahuddin12@mailcity.com

Abstract

The main purpose of this paper is to introduced and studied a new class of *non-linear regularized nonconvex random variational inequalities with fuzzy mappings* in q-uniformly smooth Banach spaces. An existence theorem of random solutions for this kind of *nonlinear regularized nonconvex random variational inequalities with fuzzy mappings* is established and a new iterative random algorithm with random errors is suggested and discussed. The results presented in this paper generalized, improved and unified some recent works in this fields.

Keywords: Nonlinear regularized nonconvex random variational inequalities, fuzzy mappings, uniformly r-prox regular sets, random iterative sequences, randomly relaxed (κ_t, ζ_t)-cocoercive mapping, randomly accretive mappings, convergence analysis, random mixed errors, q-uniformly smooth Banach spaces.

AMS Mathematics Subject Classification: 49J40, 47H06.

1 Historical Backbone

Variational inequalities are an important and generalization of classical variational inequalities which have wide applications in many fields for example mechanics, physics, optimization and control theory, nonlinear programming, economics and engineering sciences and in face, the problems for random variational inequalities are just so. It is known that accretivity of the underlying operator plays indispensable roles in the theory of variational inequalities and its generalizations. In 2001, Huang and Fang [16] were first to introduced generalized m-accretive mapping and gave the definition of the resolvent operators for generalized m-accretive mapping in Banach spaces.

The fuzzy set theory which was given by Zadeh [29] at university of California in 1965 has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of the fuzzy set theory can be found in many branches of regional, physical, mathematical and engineering sciences. In 1989, Chang and Zhu [8] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities, quasi variational inequalities and complementarity problems with fuzzy mappings were considered by Agarwal *et al.* [1], Anastassiou *et al.* [2], Chang and Huang [7], Ding *et al.* [11], Huang [15], Lee *et al.* [18, 19], Salahuddin [22, 23],and Zhang and Bi [30]. Note that most of results in this direction for variational inequalities have been done in the setting of Hilbert spaces.

On the other hand, random variational inequality problems and random quasi-variational inequality problems have been studied by Bharucha and Reid [4], Chang [5, 6], Chang and Huang [7], Huang [15], Khan and Salahuddin [17], Salahuddin [22], Tan [24] and Yuan [28], *etc.*.

The projection method is an important tools for finding an approximate solutions of various types of variational inequalities and quasi variational inequalities. The idea of this techniques is to established the equivalence between the variational inequalities and fixed point problems using the concepts of projection. The most of results regarding the existence and the iterative approximation of solutions to variational inequality problems have been investigated and considered so far the case where the underlying set is a convex set. Recently the concepts of convex set has been generalized in many directions which has pointed and important applications in various fields. It is well known that the uniformly prox regular set are nonconvex and included the convex sets as special cases . This class of uniformly prox regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions, *see* [9, 10, 26].

Inspired by the recent works going on this fields, *see* [3, 13, 15, 16, 17, 18, 20, 21, 25], in this paper, we introduced and considered a *nonlinear regularized nonconvex random variational inequalities with fuzzy mapping* involving randomly relaxed cocoercive and randomly accretive mappings in q-uniformly smooth Banach spaces. We suggested a random perturbed projection iterative algorithm with random mixed errors for finding a random solutions of aforementioned problems for fuzzy mappings. We also proved the convergence of the defined random iterative sequences under some suitable assumptions. The results presented in this paper generalized, improve and unify some recent results in this paper.

2   Conceptual Backbone

Throughout this paper, we suppose that $(\Omega, \mathbb{R}, \mu)$ is a complete σ-finite measurable space and $\mathcal{X}$ is a separable real Banach space endowed with dual space $\mathcal{X}^*$, the norm $\|\cdot\|$ and an dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{X}$ and $\mathcal{X}^*$. We denote by $\mathcal{B}(\mathcal{X})$ the class of Borel σ-fields in $\mathcal{X}; 2^{\mathcal{X}}$ and $CB(\mathcal{X})$ denote the family of all nonempty subsets of $\mathcal{X}$, the family of all nonempty closed bounded subsets of $\mathcal{X}$, respectively. The generalized duality mapping $J_q : \mathcal{X} \to 2^{\mathcal{X}^*}$ is defined by

$$J_q(x) = \{f^* \in \mathcal{X}^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\} \ \forall x \in \mathcal{X}$$

where $q > 1$ is a constant. In particular J_2 is a usual normalized duality mapping. It is known that in general $J_q(x) = \|x\|^{q-1} J_2(x)$ for all $x \neq 0$ and J_q is single valued if $\mathcal{X}^*$ is strictly convex. In the sequel, we always assume that $\mathcal{X}$ is a real Banach space such that J_q is a single valued. If $\mathcal{X}$ is a Hilbert space then J_q becomes the identity mapping on $\mathcal{X}$. The modulus of smoothness of $\mathcal{X}$ is the function $\pi_{\mathcal{X}} : [0, \infty) \to [0, \infty)$ is defined by

$$\pi_{\mathcal{X}}(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space $\mathcal{X}$ is called uniformly smooth if

$$\lim_{t \to 0} \frac{\pi_{\mathcal{X}}(t)}{t} = 0.$$

$\mathcal{X}$ is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\pi_{\mathcal{X}}(t) < ct^q, q > 1.$$

Note that J_q is a single valued if $\mathcal{X}$ is uniformly smooth. Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces. Xu [27] proved the following results.

Lemma 2.1 *The real Banach space $\mathcal{X}$ is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in \mathcal{X}$*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Let $\mathfrak{D}(\cdot, \cdot)$ represent the Hausdorff metric on $CB(\mathcal{X})$ defined by

$$\mathfrak{D}(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\right\} \quad \text{for all} \quad A, B \in CB(\mathcal{X}),$$

where

$$d(a, B) = d(B, a) = \inf_{b \in B} \|a - b\| \quad \text{for} \quad a \in A.$$

Definition 2.2 *A mapping $x : \Omega \to \mathcal{X}$ is said to be measurable if for any $\mathbf{B} \in \mathcal{B}(\mathcal{X}), \{t \in \Omega, x(t) \in \mathbf{B} \in \mathbb{R}\}$.*

Definition 2.3 *A mapping $T : \Omega \times \mathcal{X} \to \mathcal{X}$ is said to be a random mapping if for each fixed $x \in \mathcal{X}, T(t, x) = y(t)$ is measurable. A random mapping T_t is said to be continuous if for each fixed $t \in \Omega, f(t, \cdot) : \Omega \times \mathcal{X} \to \mathcal{X}$ is continuous.*

Definition 2.4 *A multivalued mapping $T : \Omega \to 2^{\mathcal{X}}$ is said to be measurable if for any $B \in \mathcal{B}(\mathcal{X}), T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.*

Definition 2.5 *A mapping $u : \Omega \to \mathcal{X}$ is said to be a measurable selection of a measurable mapping $T : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$, if u is measurable and for any $t \in \Omega, u(t) \in T_t(x(t))$.*

Definition 2.6 *A mapping $T : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is said to be a random multivalued mapping if for each fixed $x \in \mathcal{X}, T(\cdot, x) : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is a measurable multivalued mapping. A random multivalued mapping $T : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ is said to be $\mathfrak{D}$-continuous if for each fixed $t \in \Omega, T(t, \cdot) : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is a randomly continuous with respect to the Hausdorff metric on $\mathfrak{D}$.*

Definition 2.7 *A multivalued mapping $T : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is said to be a random multivalued mapping if for any $x \in \mathcal{X}, T(\cdot, x)$ is measurable (denoted by $T_{t,x}$ or T_t).*

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Let Ω be a set and $\mathfrak{F}(\mathcal{X})$ be a collection of fuzzy sets over $\mathcal{X}$. A mapping $\tilde{F} : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ is called a fuzzy mapping. For each $x \in \mathcal{X}$, $\tilde{F}(x)$ (denote it by $\tilde{T}_x$ in the sequel) is a fuzzy mapping on $\mathcal{X}$ and $\tilde{F}_x(y)$ is the membership-grade of y in $\tilde{F}_x$.

Let $A \in \mathfrak{F}(\mathcal{X})$, $\alpha \in (0, 1]$, then the set

$$A_\alpha = \{x \in \mathcal{X} : A(x) \geq \alpha\}$$

is called an α-cut of A.

Definition 2.8 *A fuzzy mapping* $\tilde{T} : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ *is called measurable, if for any* $\alpha \in (0, 1]$, $(\tilde{T}(\cdot))_\alpha : \Omega \to 2^{\mathcal{X}}$ *is a measurable multivalued mapping.*

Definition 2.9 *A fuzzy mapping* $\tilde{T} : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ *is a random fuzzy mapping if for any* $x \in \mathcal{X}$, $\tilde{T}(\cdot, x) : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ *is a measurable fuzzy mapping (denoted by* $\tilde{T}_{t,x}$ *short down* $\tilde{T}_t$).

Definition 2.10 *Let* $\mathcal{K}$ *be a nonempty closed subsets of q-uniformly smooth Banach space* $\mathcal{X}$. *The proximal normal cone of* $\mathcal{K}$ *at a point* $u \in \mathcal{X}$ *is given by*

$$N_{\mathcal{K}}^P(u) = \{\zeta \in \mathcal{X} : u \in P_{\mathcal{K}}(u + \alpha\zeta) \text{ for some } \alpha > 0\}$$

where $\alpha > 0$ *is a constant and*

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\}.$$

Here $d_{\mathcal{K}}(u)$ *is the usual distance function to the subset* $\mathcal{K}$ *that is*

$$d_{\mathcal{K}}(u) = \inf_{v \in \mathcal{K}} \|u - v\|.$$

Lemma 2.11 *Let* $\mathcal{K}$ *be a nonempty closed subset in q-uniformly smooth Banach space* $\mathcal{X}$. *Then* $\zeta \in N_{\mathcal{K}}^P(u)$ *if and only if there exists a constant* $\alpha = \alpha(\zeta, u) > 0$ *such that*

$$\langle \zeta, j_q(v - u) \rangle \leq \alpha \|v - u\|^2 \quad \forall v \in \mathcal{K}.$$

Lemma 2.12 *Let* $\mathcal{K}$ *be a nonempty closed and convex subset in q-uniformly smooth Banach space* $\mathcal{X}$. *Then* $\zeta \in N_{\mathcal{K}}^P(u)$ *if and only if*

$$\langle \zeta, j_q(v - u) \rangle \leq 0 \quad \forall v \in \mathcal{K}.$$

Lemma 2.13 *[5] Let* $G : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ *be a* $\mathcal{D}$-*continuous random multivalued mapping. Then for a measurable mapping* $u : \Omega \to \mathcal{X}$, *a multivalued mapping* $G(\cdot, u(\cdot)) : \Omega \to CB(\mathcal{X})$ *is measurable.*

Lemma 2.14 *[5] Let* $A, T : \Omega \to CB(X)$ *be measurable multivalued mappings and* $u : \Omega \to \mathcal{X}$ *be a measurable selection of* A. *Then there exists a measurable selection* $v : \Omega \to \mathcal{X}$ *of* T *such that for all* $t \in \Omega$ *and* $\epsilon > 0$,

$$\|u(t) - v(t)\| \leq (1 + \epsilon)\mathfrak{D}(A(t), T(t)).$$

Definition 2.15 *The tangent cone $T_{\mathcal{K}}(x)$ to $\mathcal{K}$ at a point $x \in \mathcal{K}$ is defined as follows*

$$T_{\mathcal{K}}(x) = \{v \in \mathcal{X} : d^o_{\mathcal{K}}(x; v) = 0\}.$$

The normal cone of $\mathcal{K}$ at x by polarity with $T_{\mathcal{K}}(x)$ defined as follows

$$N_{\mathcal{K}}(x) = \{\zeta : \langle \zeta, j_q(v) \rangle \leq 0 \;\; \forall v \in T_{\mathcal{K}}(x)\}.$$

The Clarke normal cone $N^C_{\mathcal{K}}(x)$ is given by

$$N^C_{\mathcal{K}}(x) = \overline{co}[N^P_{\mathcal{K}}(x)]$$

where $\overline{co}(S)$ mean the closure of the convex hull of S. Clearly $N^P_{\mathcal{K}}(x) \subseteq N^C_{\mathcal{K}}(x)$ but the converse is not true. Since $N^P_{\mathcal{K}}(x)$ is always closed and convex cone where $N^C_{\mathcal{K}}(x)$ is convex but may not be closed, *see* [9, 10, 21].

Definition 2.16 *For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of $\mathcal{X}$ is called the normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball that is $\forall u \in \mathcal{K}_r$ and $0 \neq \zeta \in N^P_{\mathcal{K}_r}(u), \|\zeta\| = 1$ one has*

$$\langle \zeta, j_q(v - u) \rangle \leq \frac{1}{2r} \|v - u\|^2 \;\; \forall v \in \mathcal{K}.$$

Proposition 2.17 *Let $r > 0$ and $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox regular subset of q-uniformly smooth Banach space $\mathcal{X}$. Set*

$$\mathcal{U}(r) = \{u \in \mathcal{X} : 0 \leq d_{\mathcal{K}_r}(u) < r\}.$$

Then the following statements are hold:

(a) for all $x \in \mathcal{U}(r), P_{\mathcal{K}_r}(x) \neq \emptyset$;

(b) for all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on

$$\mathcal{U}(r') = \{u \in \mathcal{X} : 0 \leq d_{\mathcal{K}_r}(u) < r'\};$$

(c) the proximal normal cone is closed as a set valued mapping.

Let $\tilde{T} : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ be a random fuzzy mapping satisfying the condition $(*)$: $(*)$: there exists a function $a : \mathcal{X} \to (0, 1]$ such that for all $(t, x) \in \Omega \times \mathcal{X}$, we have $(\tilde{T}_t)_{\alpha(x)} \in CB(\mathcal{X})$ where $CB(\mathcal{X})$ denotes the family of all nonempty bounded closed subsets of $\mathcal{X}$. By using the random fuzzy mapping $\tilde{T}_t$, we can define a random multivalued mapping $T_t : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ by $T_t = (\tilde{T}_t)_{\alpha(x)}$ for $x \in \mathcal{X}$. In the sequel T_t is called the multivalued mapping induced by the random fuzzy mapping $\tilde{T}_t$. Let $\alpha : \mathcal{X} \to (0, 1]$ be a function and $g, h : \Omega \times \mathcal{X} \to \mathcal{X}$ be the nonlinear random single valued mapping such that $\mathcal{K}_r \subset g_t(\mathcal{X})$. For a given measurable mapping $\eta : \Omega \to (0, 1)$ we finding measurable mappings $x, u : \Omega \to \mathcal{X}$ such that

$$\langle \eta_t u(t) + h_t(x(t)) - g_t(x(t)), g_t(y(t)) - h_t(x(t)) \rangle + \frac{1}{2r} \|g_t(y(t)) - h_t(x(t))\|^2 \; \forall \; g_t(y(t)) \in \mathcal{K}_r,$$

$$(2.1)$$

which is called a *nonlinear regularized nonconvex random variational inequalities with fuzzy mappings.*

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Lemma 2.18 *Let $\mathcal{K}_r$ be a uniformly r-prox regular set then the problem (2.1) is equivalent to finding $x(t) \in \mathcal{X}, u(t) \in T_t(x(t))$ such that $h_t(x(t)) \in \mathcal{K}_r$ and*

$$0 \in \eta_t u(t) + h_t(x(t)) - g_t(x(t)) + N_{\mathcal{K}_r}^P(h_t(x(t))), \tag{2.2}$$

where $N_{\mathcal{K}_r}^P(s)$ denotes the P-normal cone of $\mathcal{K}_r$ at s in the sense of nonconvex analysis.

Proof. Let $(x(t), u(t))$ with $x(t) \in \mathcal{X}, h_t(x(t)) \in \mathcal{K}_r$ and $u(t) \in T_t(x(t))$ be the random solution sets of the problem (2.2). If

$$\eta_t u(t) + h_t(x(t)) - g_t(x(t)) = 0$$

because the vector zero always belongs to any normal cone, then

$$0 \in \eta_t u(t) + h_t(x(t)) - g_t(x(t)) + N_{\mathcal{K}_r}^P(h_t(x(t))).$$

If

$$\eta_t u(t) + h_t(x(t)) - g_t(x(t)) \neq 0$$

then for all $x(t) \in \mathcal{X}$ with $g_t(x(t)) \in \mathcal{K}_r$ one has

$$\langle -(\eta_t u(t) + h_t(x(t)) - g_t(x(t))), g_t(y(t)) - h_t(x(t)) \rangle \leq \frac{1}{2r} \|g_t(y(t)) - h_t(x(t))\|^2. \tag{2.3}$$

From Lemma 2.11, we have

$$-(\eta_t u(t) + h_t(x(t)) - g_t(x(t))) \in N_{\mathcal{K}_r}^P(h_t(x(t)))$$

and

$$0 \in \eta_t u(t) + h_t(x(t)) - g_t(x(t)) + N_{\mathcal{K}_r}^P(h_t(x(t))). \tag{2.4}$$

Conversely if $(x(t), u(t))$ with $x(t) \in \mathcal{X}, h_t(x(t)) \in \mathcal{K}_r$ and $u(t) \in T_t(x(t))$ is a random solution sets of the problem (2.2) then from Definition 2.16, $x(t) \in \mathcal{X}$ and $u(t) \in T_t(x(t))$ with $h_t(x(t)) \in \mathcal{K}_r$ are random solution sets of the problem (2.1).
∎

Lemma 2.19 *Let $\tilde{T}_t : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ be a random fuzzy mapping induced by a multivalued mapping $T : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ and $g, h : \Omega \times \mathcal{X} \to \mathcal{X}$ be the random single valued mappings and $\eta : \Omega \to (0,1)$ be a measurable mapping, then $(x(t), u(t))$ with $(x(t)) \in \mathcal{X}, h_t(x(t)) \in \mathcal{K}_r, u(t) \in \tilde{T}_t(x(t))$ is a random solution sets of the problem (2.1) if and only if*

$$h_t(x(t)) = P_{\mathcal{K}_r}[g_t(x(t)) - \eta_t u(t)], \tag{2.5}$$

where $P_{\mathcal{K}_r}$ is the projection of $\mathcal{X}$ onto the uniformly r-prox regular set $\mathcal{K}_r$.

Proof. Let $(x(t), u(t))$ with $x(t) \in \mathcal{X}, h_t(x(t)) \in \mathcal{K}_r, u(t) \in \tilde{T}_t(x(t))$ be a random solution sets of the problem (2.1). Then from Lemma 2.18, we have

$$0 \in \eta_t u(t) + h_t(x(t)) - g_t(x(t)) + N_{\mathcal{K}_r}^P(h_t(x(t))) \tag{2.6}$$

$\Leftrightarrow$

$$g_t(x(t)) - \eta_t u(t) \in (I + N^P_{\mathcal{K}_r})(h_t(x(t))) \tag{2.7}$$

$\Leftrightarrow$

$$h_t(x(t)) = P_{\mathcal{K}_r}[g_t(x(t)) - \eta_t u(t)], \tag{2.8}$$

where I is an identity mapping and $P_{\mathcal{K}_r} = (I + N^P_{\mathcal{K}_r})^{-1}$. ∎

Remark 2.20 *The inequality* (2.5) *can be written as follows*

$$x(t) = x(t) - h_t(x(t)) + P_{\mathcal{K}_r}[g_t(x(t)) - \eta_t u(t)], \tag{2.9}$$

where $\eta : \Omega \to (0,1)$ *is a measurable mapping.*

Algorithm 2.21 *Let* $\tilde{T}_t : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ *be a random fuzzy mapping satisfying the condition* (∗) *and* $T : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ *be a multivalued random mapping induced by the random fuzzy mapping* T_t. *Let* $g, h : \Omega \times \mathcal{X} \to \mathcal{X}$ *be the random single valued mappings. For any given* $x_0 : \Omega \to \mathcal{X}$ *the random single valued mappings,* $T_t(x_0(t)) : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ *is measurable by Lemma 2.13. We know that for any* $x_0(t) \in \mathcal{X}, T_t(x_0(t))$ *is measurable and there exists a measurable selection* $u_0(t) \in T_t(x_0(t))$, *see [13, 14]. Set*

$$x_1(t) = (1 - \lambda_t)x_0(t) + \lambda_t[x_0(t) - h_t(x_0(t)) + P_{\mathcal{K}_r}[g_t(x_0(t)) - \eta_t u_0(t)] + \lambda_t e_0(t) + r_0(t)$$

where $\eta, \lambda : \Omega \to (0,1)$ *are measurable mappings and* $0 < \lambda_t < 1$ *is a random constant and* $e_0(t), r_0(t) : \Omega \to \mathcal{X}$ *is the random measurable mapping which is a random errors to take into account of a possible inexact computation for the proximal point. Then it is easy to see that* $x_1 : \Omega \to \mathcal{X}$ *is a random measurable mapping. Since* $u_0(t) \in T_t(x_0(t))$ *then there exists a random measurable selection* $u_1(t) \in T_t(x_1(t))$ *for all* $t \in \Omega$

$$\|u_0(t) - u_1(t)\| \leq (1 + (\frac{1}{1})\mathfrak{D}(T_t(x_0(t)), T_t(x_1(t))).$$

By induction, we can define the measurable random sequences $x_n(t), u_n(t) : \Omega \to \mathcal{X}$ *inductively satisfying*

$$x_{n+1}(t) = (1 - \lambda_t)x_n(t) + \lambda_t[x_n(t) - h_t(x_n(t)) + P_{\mathcal{K}_r}(g_t(x_n(t)) - \eta_t u_n(t)) + \lambda_n(t)e_n(t) + r_n(t), \tag{2.10}$$

$$u_n(t) \in T_t(x_n(t)); \|u_n(t) - u_{n+1}(t)\| \leq (1 + (1+n)^{-1})\mathfrak{D}(T_t(x_n(t)), T_t(x_{n+1}(t))) \tag{2.11}$$

where $0 < \lambda_t < 1$ *is a random constant and* $\{e_n(t)\}^\infty_{n=0}, \{r_n(t)\}^\infty_{n=0}$ *are random sequences in* $\mathcal{X}$ *to take into account of a possible inexact computation for the resolvent operator satisfying the following conditions:*

$$\lim_{n \to \infty} e_n(t) = \lim_{n \to \infty} r_n(t) = 0,$$

$$\sum_{n=1}^{\infty} \|e_n(t) - e_{n-1}(t)\| < \infty \quad and \quad \sum_{n=1}^{\infty} \|r_n(t) - r_{n-1}(t)\| < \infty. \tag{2.12}$$

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Definition 2.22 *Let $\mathcal{X}$ be a q-uniformly smooth Banach spaces. A random single valued mapping $g : \Omega \times \mathcal{X} \to \mathcal{X}$ is called*

(i) *randomly accretive if*

$$\langle g_t(x(t)) - g_t(y(t)), j_q(x(t) - y(t)) \rangle \geq 0 \quad \forall x(t), y(t) \in \mathcal{X}, t \in \Omega;$$

(ii) *randomly μ_t-strongly accretive if there exists a measurable mapping $\mu : \Omega \to (0,1)$ such that*

$$\langle g_t(x(t)) - g_t(y(t)), j_q(x(t) - y(t)) \rangle \geq \mu_t \|x(t) - y(t)\|^q \quad \forall x(t), y(t) \in \mathcal{X}, t \in \Omega;$$

(iii) *randomly relaxed μ_t-accretive mapping if there exists a measurable mapping $\mu : \Omega \to (0,1)$ such that*

$$\langle g_t(x(t)) - g_t(y(t)), j_q(x(t) - y(t)) \rangle \geq -\mu_t \|x(t) - y(t)\|^q \quad \forall x(t), y(t) \in \mathcal{X}, t \in \Omega;$$

(iv) *randomly α_t-Lipschitz continuous mapping if there exists a measurable mapping $\alpha : \Omega \to (0,1)$ such that*

$$\|g_t(x(t)) - g_t(y(t))\| \leq \alpha_t \|x(t) - y(t)\| \quad \forall x(t), y(t) \in \mathcal{X}, t \in \Omega.$$

Definition 2.23 *Let $g : \Omega \times \mathcal{X} \to \mathcal{X}$ be a random single valued mapping and $T : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ a random multivalued mapping. Then T_t is said to be*

(i) *randomly accretive if*

$$\langle u(t) - v(t), j_q(x(t) - y(t)) \rangle \geq 0, \quad \forall x(t), y(t) \in \mathcal{X}, u(t) \in T_t(x(t)), v(t) \in T_t(y(t)),$$

(ii) *randomly (κ_t, ζ_t)-relaxed cocoercive mapping with respect to g_t if there exist the measurable mappings $\kappa, \zeta : \Omega \to (0,1)$ such that for all $x(t), y(t) \in \mathcal{X}, t \in \Omega$*

$$\langle u(t) - v(t), j_q(g_t(x(t)) - g_t(y(t))) \rangle \geq -\kappa_t \|u(t) - v(t)\|^q + \zeta_t \|g_t(x(t)) - g_t(y(t))\|^q$$

$$\forall u(t) \in T_t(x(t)), v(t) \in T_t(y(t)),$$

(iii) *randomly $\alpha - \mathfrak{D}$-Lipschitz continuous mapping if there exists a measurable mapping $\alpha : \Omega \to (0,1)$ such that for all $x(t), y(t) \in \mathcal{X}, t \in \Omega$*

$$\mathfrak{D}(T_t(x(t)), T_t(y(t))) \leq \alpha_t \|x(t) - y(t)\|$$

where $\mathfrak{D}$ is the Hausdorff pseudo metric defined in $\mathcal{X}$.

3 Main Backbone

In this section, we prove the existence and the convergence of problem (2.1).

Theorem 3.1 *Let $\mathcal{X}$ be a q-uniformly smooth real Banach space. Let $\tilde{T}_t : \Omega \times \mathcal{X} \to \mathfrak{F}(\mathcal{X})$ be a random fuzzy mapping satisfying the condition $(*)$ and $T_t : \Omega \times \mathcal{X} \to CB(\mathcal{X})$ be a random multivalued mapping induced by random fuzzy mapping. Suppose that T_t is a randomly $\mathfrak{D}$-Lipschitz continuous with random measurable mapping $\alpha : \Omega \to (0,1)$. Let T_t be the randomly (κ_t, ζ_t)-relaxed cocoercive with respect to the random mapping g_t and $g : \Omega \times \mathcal{X} \to \mathcal{X}$ be the randomly Lipschitz continuous with a measurable mapping $\beta : \Omega \to (0,1)$. Let $h : \Omega \times \mathcal{X} \to \mathcal{X}$ be the randomly relaxed μ_t-accretive mapping with measurable mapping $\mu : \Omega \to (0,1)$ and randomly Lipschitz continuous with the measurable mapping $\gamma : \Omega \to (0,1)$. If for any $t \in \Omega, y(t), z(t) \in \mathcal{X}$, we have*

$$\|P_{\mathcal{K}_r}(y(t)) - P_{\mathcal{K}_r}(z(t)) \le \frac{r}{r - r'}\|y(t) - z(t)\|. \tag{3.1}$$

Assume that the measurable mapping $\eta : \Omega \to (0,1)$ satisfying the following assumptions

$$0 < \theta_t = 1 - \lambda_t(1 - p_t) < 1,$$

$$Q_t + \frac{r}{r - r'}\sqrt[q]{\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q + \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q} < 1,$$

$$Q_t = \sqrt[q]{1 + q\mu_t + c_q\gamma_t^q}$$

$$Q_t < 1, \qquad 0 < \lambda_t < 1, \qquad p_t < 1$$

and

$$\lim_{n\to\infty} \|e_n(t)\| = \lim_{n\to\infty} \|r_n(t)\| = 0,$$

$$\sum_{n=0}^{\infty} \|e_n(t) - e_{n-1}(t)\| < \infty, \quad \sum_{n=0}^{\infty} \|r_n(t) - r_{n-1}(t)\| < \infty, \tag{3.2}$$

where $r' \in (0,r)$, then the random mappings $x_n(t), u_n(t) : \Omega \times \mathcal{X} \to \mathcal{X}$ generated by random algorithm converge strongly to the random mappings $x^*(t), u^*(t) : \Omega \times \mathcal{X} \to \mathcal{X}$ and $x^*(t), u^*(t) \in \mathcal{X}$ with $h_t(x(t)) \in \mathcal{K}_r$ is a random solution sets of problem (2.1).

Proof. From Algorithm 2.21 and (3.1), $t \in \Omega$ and $0 < \lambda_t < 1$, we have

$$\|x_{n+1}(t)-x_n(t)\| \le (1-\lambda_t)\|x_n(t)-x_{n-1}(t)\|+\lambda_t(\|x_n(t)-x_{n-1}(t)-(h_t(x_n(t))-h_t(x_{n-1}(t)))\|$$

$$+\frac{r}{r-r'}\|g_t(x_n(t))-g_t(x_{n-1}(t))-\eta_t(u_n(t)-u_{n-1}(t))\|)+\lambda_t\|e_n(t)-e_{n-1}(t)\|+\|r_n(t)-r_{n-1}(t)\|. \tag{3.3}$$

Since h_t is randomly relaxed μ_t-accretive mapping and randomly Lipschitz continuous mapping, we have

$$\|x_n(t) - x_{n-1}(t) - (h_t(x_n(t)) - h_t(x_{n-1}(t)))\|^q = \|x_n(t) - x_{n-1}(t)\|^q$$

$$-q\langle h_t(x_n(t)) - h_t(x_{n-1}(t)), j_q(x_n(t) - x_{n-1}(t))\rangle + c_q\|h_t(x_n(t)) - h_t(x_{n-1}(t))\|^q$$

$$\leq \|x_n(t) - x_{n-1}(t)\|^q + q\mu_t\|x_n(t) - x_{n-1}(t)\|^q + \gamma_t^q c_q\|x_n(t) - x_{n-1}(t)\|^q$$

$$\leq (1 + q\mu_t + \gamma_t^q c_q)\|x_n(t) - x_{n-1}(t)\|^q$$

$$\Rightarrow \|x_n(t) - x_{n-1}(t) - (h_t(x_n(t)) - h_t(x_{n-1}(t)))\| \leq \sqrt[q]{1 + q\mu_t + c_q\gamma_t^q}\|x_n(t) - x_{n-1}(t)\|. \quad (3.4)$$

Since T_t is randomly $\alpha_t - \mathfrak{D}$-Lipschitz continuous mapping, then we have

$$\|u_n(t) - u_{n-1}(t)\| \leq (1 + \frac{1}{n})\mathfrak{D}(T_t(x_n(t)), T_t(x_{n-1}(t)))$$

$$\leq \alpha_t(1 + \frac{1}{n})\|x_n(t) - x_{n-1}(t)\|. \quad (3.5)$$

Again T_t is randomly (κ_t, ζ_t)-relaxed cocoercive mapping with respect to g_t and g_t is randomly Lipschitz continuous mapping with measurable mapping $\beta : \Omega \to (0,1)$, we have

$$\|g_t(x_n(t)) - g_t(x_{n-1}(t)) - \eta_t(u_n(t) - u_{n-1}(t))\|^q \leq \|g_t(x_n(t)) - g_t(x_{n-1}(t))\|^q$$

$$-q\eta_t\langle u_n(t) - u_{n-1}(t), j_q(g_t(x_n(t)) - g_t(x_{n-1}(t))))\rangle + c_q\eta_t^q\|u_n(t) - u_{n-1}(t)\|^q$$

$$\leq \|g_t(x_n(t)) - g_t(x_{n-1}(t))\|^q - q\eta_t(-\kappa_t\|u_n(t) - u_{n-1}(t)\|^q + \zeta_t\|g_t(x_n(t)) - g_t(x_{n-1}(t))\|^q)$$

$$+c_q\eta_t^q\|u_n(t) - u_{n-1}(t)\|^q$$

$$\leq \beta_t^q\|x_n(t) - x_{n-1}(t)\|^q + q\eta_t\kappa_t\alpha_t^q(1 + \frac{1}{n})^q\|x_n(t) - x_{n-1}(t)\|^q - q\eta_t\zeta_t\beta_t^q\|x_n(t) - x_{n-1}(t)\|^q$$

$$+c_q\eta_t^q\alpha_t^q(1 + \frac{1}{n})^q\|x_n(t) - x_{n-1}(t)\|^q$$

$$\leq (\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q(1 + \frac{1}{n})^q + \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q(1 + \frac{1}{n})^q)\|x_n(t) - x_{n-1}(t)\|^q$$

$$\Rightarrow \quad \|g_t(x_n(t)) - g_t(x_{n-1}(t)) - \eta_t(u_n(t) - u_{n-1}(t))\|$$

$$\leq \sqrt[q]{\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q(1 + \frac{1}{n})^q + \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q(1 + \frac{1}{n})^q}\|x_n(t) - x_{n-1}(t)\|. \quad (3.6)$$

From (3.3),(3.4) and (3.6), we get

$$\|x_{n+1}(t) - x_n(t)\| \leq (1 - \lambda_t)\|x_n(t) - x_{n-1}(t)\| + \lambda_t(\sqrt[q]{1 + q\mu_t + c_q\gamma_t^q}$$

$$+\frac{r}{r - r'}\sqrt[q]{\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q(1 + \frac{1}{n})^q - \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q(1 + \frac{1}{n})^q})\|x_n(t) - x_{n-1}(t)\|$$

$$+\lambda_t\|e_n(t) - e_{n-1}(t)\| + \|r_n(t) - r_{n-1}(t)\|$$

$$\leq (1 - \lambda_t + \lambda_t p_{n,t})\|x_n(t) - x_{n-1}(t)\| + \lambda_t\|e_n(t) - e_{n-1}(t)\| + \|r_n(t) - r_{n-1}(t)\|$$

$$\leq \theta_{n,t}\|x_n(t) - x_{n-1}(t)\| + \lambda_t\|e_n(t) - e_{n-1}(t)\| + \|r_n(t) - r_{n-1}(t)\| \quad (3.7)$$

where $\theta_{n,t} = 1 - \lambda_t + \lambda_t p_{n,t}$

$$p_{n,t} = Q_t + \frac{r}{r - r'} \sqrt[q]{\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q(1 + \frac{1}{n})^q + \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q(1 + \frac{1}{n})^q}$$

and $Q_t = \sqrt[q]{1 + q\mu_t + c_q\gamma_t^q}$.
Let $\theta_t = 1 - \lambda_t + \lambda_t p_t$ and

$$p_t = Q_t + \frac{r}{r - r'} \sqrt[q]{\beta_t^q - q\eta_t(-\kappa_t\alpha_t^q + \zeta_t\beta_t^q) + c_q\eta_t^q\alpha_t^q}.$$

We have $p_{n,t} \to p_t$ and $\theta_{n,t} \to \theta_t$ as $n \to \infty$. It follows from (3.2), $0 < \lambda_t < 1$ and $\theta_t^* \in (0,1)$ such that $\theta_{n,t} < \theta_t^*$ for all $n \geq N_0$. Therefore from (3.7), we have

$$\|x_{n+1}(t) - x_n(t)\| \leq \theta_t^*\|x_n(t) - x_{n-1}(t)\| + \lambda_t\|e_n(t) - e_{n-1}(t)\| + \|r_n(t) - r_{n-1}(t)\|, \forall n \geq 1$$

without loss of generality, we may assume that

$$\|x_{n+1}(t) - x_n(t)\| \leq \theta_t^*\|x_n(t) - x_{n-1}(t)\| + \lambda_t\|e_n(t) - e_{n-1}(t)\| + \|r_n(t) - r_{n-1}(t)\|, \forall n \geq 1.$$

Hence for any $m > n > 0$, we have

$$\|x_{n+1}(t) - x_n(t)\| \leq \sum_{i=n}^{m-1} \|x_{i+1}(t) - x_i(t)\| \leq \sum_{i=n}^{m-1} \theta_t^{*,i}\|x_1(t) - x_0(t)\|$$

$$+\lambda_t \sum_{i=n}^{m-1}\sum_{j=1}^{i} \theta_t^{*,i-j}\|e_j(t) - e_{j-1}(t)\| + \sum_{i=n}^{m-1}\sum_{j=1}^{i} \theta_t^{*,i-j}\|r_j(t) - r_{j-1}(t)\|, \forall n \geq 1. \qquad (3.8)$$

It follows from condition (3.2) that $\|x_m(t) - x_n(t)\| \to 0$ as $n \to \infty$ and so $\{x_n(t)\}$ is a Cauchy sequence in $\mathcal{X}$. Let $x_n(t) \to x^*(t)$ as $n \to \infty$. By the random $\mathfrak{D}$-Lipschitz continuity of $T_t(\cdot)$ we have

$$\|u_{n+1}(t) - u_n(t)\| \leq (1 + \frac{1}{1+n})\mathfrak{D}(T_t(x_{n+1}(t)), T_t(x_n(t)))$$

$$\leq (1 + \frac{1}{1+n}))\alpha_t\|x_{n+1}(t) - x_n(t)\| \to 0, \text{ as } n \to \infty. \qquad (3.9)$$

It follows that $\{u_n(t)\}$ is a Cauchy sequence in $\mathcal{X}$. We can assume that $u_n(t) \to u^*(t)$. Note that $u_n(t) \in T_t(x_n(t))$ we have

$$d(u^*(t), T_t(x^*(t))) \leq \|u^*(t) - u_n(t)\| + d(u_n(t), T_t(x^*(t)))$$

$$\leq \|u^*(t) - u_n(t)\| + (1 + \frac{1}{n})\mathfrak{D}(T_t(x_n(t)), T_t(x^*(t)))$$

$$\leq \|u^*(t) - u_n(t)\| + (1 + \frac{1}{n})\alpha_t\|x_n(t) - x^*(t)\| \to 0 \text{ as } n \to \infty. \quad (3.10)$$

Hence $d(u^*(t), T_t(x^*(t))) = 0$ and therefore $u^*(t) \in T_t(x^*(t)) \in \mathcal{X}$.
By the random $\mathfrak{D}$-Lipschitz continuity of $T_t(\cdot)$, Lemma 2.19 and condition (3.2) and

$$\lim_{n \to \infty} \|e_n(t)\| = \lim_{n \to \infty} \|r_n(t)\| = 0,$$

we have

$$x^*(t) = (1 - \lambda_t)x^*(t) + \lambda_t[x^*(t) - h_t(x^*(t)) + P_{\mathcal{K}_r}(g_t(x^*(t)) - \eta_t u^*(t)). \qquad (3.11)$$

By Lemma 2.19, we know that $(x^*(t), u^*(t))$ is a random solution sets of problem (2.1). This completes the proof. ∎

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New White Noise Functional Solutions For Wick-type Stochastic Coupled KdV Equations Using F-expansion Method

Hossam A. Ghany[1,2] and M. Zakarya[3]
[1]Department of Mathematics, Faculty of Science,
Taif University, Taif, Saudi Arabia.
[2]Department of Mathematics, Helwan University, Cairo, Egypt.
h.abdelghany@yahoo.com
[3]Department of Mathematics, Faculty of Science, Al-Azhar University,
Assiut 71524, Egypt.
mohammed_zakarya@rocketmail.com

Abstract

Wick-type stochastic coupled KdV equations are researched. By means of Hermite transformation, white noise theory and F-expansion method, three types of exact solutions for Wick-type stochastic coupled KdV equations are explicitly given. These solutions include the white noise functional solutions of Jacobi elliptic function (JEF) type, trigonometric type and hyperbolic type.

Keywords: Coupled KdV equations; F-expansion method; Hermite transform; Wick-type product; White noise theory.
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1 Introduction

In this paper, we shall explore exact solutions for the following variable coefficients coupled KdV equations.

$$\begin{cases} u_t + h_1(t)uu_x + h_2(t)vv_x + h_3(t)u_{xxx} = 0, \\ \\ v_x + h_4(t)uv_x + h_3(t)v_{xxx} = 0, \end{cases} , (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \qquad (1.1)$$

where $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random wave is an important subject of stochastic partial differential equations

(PDEs). Many authors have studied this subject. Wadati first introduced and studied the stochastic KdV equations and gave the diffusion of soliton of the KdV equation under Gaussian noise in [28, 30] and others [3–6, 23] also researched stochastic KdV-type equations. Xie first introduced Wick-type stochastic KdV equations on white noise space and showed the auto- Backlund transformation and the exact white noise functional solutions in [35]. Furthermore, Xie [36–39], Ghany et al. [12–14, 16–19] researched some Wick-type stochastic wave equations using white noise analysis.

In this paper we use F-expansion method for finding new periodic wave solutions of nonlinear evolution equations in mathematical physics, and we obtain some new periodic wave solution for coupled KdV equations. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear (PDEs) arising in mathematical physics. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fibers. Many effective methods have been presented, such as variational iteration method [7,8], tanh-function method [9,32,40], homotopy perturbation method [11, 27, 33], homotopy analysis method [1], tanh-coth method [29, 31, 32], exp-function method [21, 22, 34, 41, 42] , Jacobi elliptic function expansion method [10,24–26], the F-expansion method [43–46]. The main objective of this paper is using the F-expansion method to construct white noise functional solutions for wick-type stochastic coupled KdV equations via hermite transform, wick-type product, white noise theory. If equation (1.1) is considered in a random environment, we can get stochastic coupled KdV equations. In order to give the exact solutions of stochastic coupled KdV equations, we only consider this problem in white noise environment. We shall study the following Wick-type stochastic coupled KdV equations.

$$\begin{cases} U_t + H_1(t) \diamond U \diamond U_x + H_2(t) \diamond V \diamond V_x + H_3(t) \diamond U_{xxx} = 0, \\ V_x + H_4(t) \diamond U \diamond V_x + H_3(t) \diamond V_{xxx} = 0, \end{cases} \tag{1.2}$$

where "$\diamond$" is the Wick product on the Kondratiev distribution space (S_{-1}) which was defined in [20], $H_1(t), H_2(t), H_3(t)$ and $H_4(t)$ are (S_{-1})-valued functions.

2   Description of the F-expansion Method

In order to simultaneously obtain more periodic wave solutions expressed by various Jacobi elliptic functions to nonlinear wave equations, we introduce an F-expansion method which can be thought of as a succinctly over-all generalization of Jacobi elliptic function expansion. We briefly show what is F-expansion method and how to use it to obtain various periodic wave solutions to nonlinear wave equations. Suppose a nonlinear wave equation for u(t,x) is given by.

$$p(u, u_t, u_x, u_{xx}, u_{xxx}, ...) = 0, \tag{2.1}$$

2

where $u = u(t, x)$ is an unknown function, p is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a deformation F-expansion method.

Step 1. Look for traveling wave solution of Eq.(2.1)by taking

$$u(t, x) = u(\xi) \ , \xi(t, x) = kx + s \int_0^t \delta(\tau) d\tau + c, \tag{2.2}$$

Hence, under the transformation (2.2). Eq.(2.1) can be transformed into ordinary differential equation (ODE) as following.

$$O(u, s\delta u', ku', k^2 u'', k^3 u''', ...) = 0, \tag{2.3}$$

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$ of the form.

$$u(t, x) = u(\xi) = \sum_{i=1}^N a_i F^i(\xi), \tag{2.4}$$

where $a_0, a_1, ..., a_N$ are constants to be determined later, while $F'(\xi)$ in(2.4) satisfy

$$[F'(\xi)]^2 = PF^4(\xi) + QF^2(\xi) + R, \tag{2.5}$$

and hence holds for $F(\xi)$

$$\begin{cases} F'F'' = 2PF^3 F' + QFF', \\ F'' = 2PF^3 + QF, \\ F''' = 6PF^2 F' + QF', \\ \dots \end{cases} \tag{2.6}$$

where P, Q, and R are constants.

Step 3. The positive integer N can be determined by considering the homogeneous balance between the highest derivative term and the nonlinear terms appearing in (2.3). Therefore, we can get the value of N in (2.4).

Step 4. Substituting (2.4) into (2.3) with the condition (2.5), we obtain polynomial in $f^i(\xi)[f'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. Setting each coefficient of this polynomial to be zero yields a set of algebraic equations for $a_0, a_1, ..., a_N, s$ and δ.

Step 5. Solving the algebraic equations with the aid of Maple we have $a_0, a_1, ..., a_N, s$ and δ can be expressed by P, Q and R. Substituting these results into F-expansion (2.4), then a general form of traveling wave solution of Eq. (2.1) can be obtained.

Step 6. Since the general solutions of (2.4) have been well known for us Choose properly $(P, Q$ and R.) in ODE (2.5) such that the corresponding solution $F(\xi)$ of it is one of Jacobi elliptic functions. (See Appendix A, B and C.)

3 New Exact Traveling Wave Solutions of Eq. (1.2)

Taking the Hermite transform, white noise theory, and F-expansion method to explore new exact wave solutions for Eq.(1.2). Applying Hermite transform to Eq.(1.2), we get the deterministic equations:

$$
\begin{cases}
\tilde{U}_t(t,x,z) + \tilde{H}_1(t,z)\tilde{U}(t,x,z)\tilde{U}_x(t,x,z) + \tilde{H}_2(t,z)\tilde{V}(t,x,z)\tilde{V}_x(t,x,z) \\
+\tilde{H}_3(t,z)\tilde{U}_{xxx}(t,x,z) = 0, \\
\\
\tilde{V}_t(t,x,z) + \tilde{H}_4\tilde{U}(t,x,z)\tilde{V}_x(t,x,z) + \tilde{H}_3\tilde{V}_{xxx}(t,x,z) = 0,
\end{cases}
\tag{3.1}
$$

where $z = (z_1, z_2, ...) \in (\mathbb{C}^{\mathbb{N}})$ is a vector parameter. To look for the travelling wave solution of Eq.(3.1), we make the transformations $\tilde{H}_1(t,z) := h_1(t,z)$, $\tilde{H}_2(t,z) := h_2(t,z)$, $\tilde{H}_3(t,z) := h_3(t,z)$, $\tilde{H}_4(t,z) := h_4(t,z)$, $\tilde{U}(t,x,z) =: u(t,x,z) = u(\xi(t,x,z))$ and $\tilde{V}(t,x,z) =: v(t,x,z) = v(\xi(t,x,z))$ with.

$$
\xi(t,x,z) = kx + s \int_0^t \delta(\tau,z)d\tau + c,
$$

where k, s and c are arbitrary constants which satisfy $sk \neq 0$, $\delta(\tau)$ is a nonzero function of the indicated variables to be determined later. So, Eq.(3.1) can be transformed into the following (ODE).

$$
\begin{cases}
s\delta u' + kh_1 uu' + kh_2 vv' + k^3 h_3 u''' = 0, \\
s\delta v' + kh_4 uv' + k^3 h_3 v''' = 0,
\end{cases}
\tag{3.2}
$$

where the prime denote to the differential with respect to ξ. In view of F-expansion method, the solution of Eq. (3.1), can be expressed in the form.

$$
\begin{cases}
u(t,x,z) = u(\xi) = \Sigma_{i=1}^{N}(a_i F^i(\xi)), \\
v(t,x,z) = v(\xi) = \Sigma_{i=1}^{M}(b_i F^i(\xi)),
\end{cases}
\tag{3.3}
$$

where a_i and b_i are constants to be determined later. considering homogeneous balance between u''' and uu', vv' and the order of v''' and uv' in (3.2), then we can obtain $N = M = 2$, so (3.3) can be rewritten as following.

$$
\begin{cases}
u(t,x,z) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \\
v(t,x,z) = b_0 + b_1 F(\xi) + b_2 F^2(\xi),
\end{cases}
\tag{3.4}
$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are constants to be determined later. Substituting (3.4) with the conditions (2.5),(2.6) into (3.2) and collecting all terms with the same power

of $f^i(\xi)[f'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. as following.

$$
\begin{cases}
2k[12k^2 a_2 Ph_3 + b_2^2 h_2 + a_2^2 h_1]F^3 F' \\
+3k[2k^2 a_1 Ph_3 + a_1 a_2 h_1 + b_1 b_2 h_2]F^2 F' \\
+[2a_2 s\delta + kb_1^2 h_2 + 8k^3 a_2 Qh_3 + 2ka_0 a_2 h_1 + ka_1^2 h_1 + 2kb_0 b_2 h_2]FF' \\
+[sa_1 h_3 + ka_0 a_1 h_1 + kb_0 b_1 h_2 + k^3 a_1 Qh_3]F' = 0, \\
\\
2kb_2[12k^2 Ph_3 + a_2 h_4]F^3 F' \\
+k[6k^2 b_1 Ph_3 + 2a_1 b_2 h_4 + a_2 b_1 h_4]F^2 F' \\
+[2sb_2\delta + 2ka_0 b_2 h_4 + ka_1 b_1 h_4 + 8k^3 b_2 Qh_3]FF' \\
+b_1[s\delta + ka_0 h_4 + k^3 Qh_3]F' = 0.
\end{cases}
\tag{3.5}
$$

Setting each coefficient of $f^i(\xi)[f'(\xi)]^j$ to be zero, we get a system of algebraic equations which can be expressed by.

$$2k[12k^2 a_2 Ph_3 + b_2^2 h_2 + a_2^2 h_1] = 0,$$

$$3k[2k^2 a_1 Ph_3 + a_1 a_2 h_1 + b_1 b_2 h_2] = 0,$$

$$2a_2 s\delta + kb_1^2 h_2 + 8k^3 a_2 Qh_3 + 2ka_0 a_2 h_1 + ka_1^2 h_1 + 2kb_0 b_2 h_2 = 0,$$

$$sa_1 h_3 + ka_0 a_1 h_1 + kb_0 b_1 h_2 + k^3 a_1 Qh_3 = 0,$$

$$2kb_2[12k^2 Ph_3 + a_2 h_4] = 0,$$
$$\tag{3.6}$$

$$k[6k^2 b_1 Ph_3 + 2a_1 b_2 h_4 + a_2 b_1 h_4] = 0,$$

$$2sb_2\delta + 2ka_0 b_2 h_4 + ka_1 b_1 h_4 + 8k^3 b_2 Qh_3 = 0,$$

$$b_1[s\delta + ka_0 h_4 + k^3 Qh_3] = 0.$$

with solving by Maple to get the following coefficient.

$$
\begin{cases}
a_1 = b_1 = 0, a_0 = \text{arbitrary constant}, \\
a_2 = -\frac{12k^2 Ph_3(t,z)}{h_4(t,z)}, \\
b_2 = \pm i\frac{12k^2 Ph_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}} = \mp ia_2\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}}, \\
b_0 = \pm\frac{[3k^2 Qh_3(t,z)-a_0(h_1(t,z)-h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}, \\
\delta = \frac{-k[a_0 h_4(t,z)+k^2 Qh_3(t,z)]}{S}.
\end{cases}
\tag{3.7}
$$

Substituting by coefficient (3.7) into (3.4) yields general form solutions of eq. (1.2).

$$u(t, x, z) = a_0 - [\frac{12k^2 Ph_3(t, z)}{h_4(t, z)}]F^2(\xi(t, x, z)),
\tag{3.8}$$

$$v(t, x, z) = \pm \frac{[3k^2 Q h_3(t,z) + a_0(h_1(t,z) - h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z) - h_4(t,z)]}}$$

$$\pm \frac{12ik^2 P h_3(t,z)}{h_4(t,z)} \sqrt{\frac{h_1(t,z) - h_4(t,z)}{h_2(t,z)}} F^2(\xi(t, x, z)),$$

(3.9)

with

$$\xi(t, x, z) = k \left[x - \int_0^t \{ a_0 h_4(\tau, z) + k^2 Q h_3(\tau, z) \} d\tau \right].$$

From Appendix C, we give the special cases as following:

Case 1. if we take $P = 1, Q = (2 - m^2)$ and $R = (1 - m^2)$, we have $F(\xi) \to cs(\xi)$,

$$u_1(t, x, z) = a_0 - \left[\frac{12k^2 h_3(t, z)}{h_4(t, z)} \right] cs^2(\xi_1(t, x, z)),$$

(3.10)

$$v_1(t, x, z) = \pm \frac{[3k^2 h_3(t,z)(2 - m^2) + a_0(h_1(t,z) - h_4(t,z))]}{\sqrt{h_2(t,z)(h_1(t,z) - h_4(t,z))}}$$

$$\pm \frac{12ik^2 h_3(t,z)}{h_4(t,z)} \sqrt{\frac{h_1(t,z) - h_4(t,z)}{h_2(t,z)}} cs^2(\xi_1(t, x, z)),$$

(3.11)

with

$$\xi_1(t, x, z) = k \left[x - \int_0^t \{ a_0 h_4(\tau, z) + k^2(2 - m^2)h_3(\tau, z) \} d\tau \right].$$

In the limit case when $m \to o$ we have $cs(\xi) \to \cot(\xi)$, thus (3.10),(3.11) become.

$$u_2(t, x, z) = a_0 - \left[\frac{12k^2 h_3(t, z)}{h_4(t, z)} \right] \cot^2(\xi_2(t, x, z)),$$

(3.12)

$$v_2(t, x, z) = \pm \frac{[6k^2 h_3(t,z) + a_0(h_1(t,z) - h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z) - h_4(t,z)]}}$$

$$\pm \frac{12ik^2 h_3(t,z)}{h_4(t,z)} \sqrt{\frac{h_1(t,z) - h_4(t,z)}{h_2(t,z)}} \cot^2(\xi_2(t, x, z)),$$

(3.13)

with

$$\xi_2(t, x, z) = k \left[x - \int_0^t \{ a_0 h_4(\tau, z) + 2k^2 h_3(\tau, z) \} d\tau \right].$$

In the limit case when $m \to 1$ we have $cs(\xi) \to \operatorname{csch}(\xi)$, thus (3.10).(3.11) become.

$$u_3(t, x, z) = a_0 - \left[\frac{12k^2 h_3(t, z)}{h_4(t, z)} \right] \operatorname{csch}^2(\xi_3(t, x, z)),$$

(3.14)

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$$v_3(t, x, z) = \pm \frac{[3k^2h_3(t,z)+a_0(h_1(t,z)-h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{12ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}} \ \operatorname{csch}^2(\xi_3(t, x, z)),$$

(3.15)

with

$$\xi_3(t, x, z) = k\left[x - \int_0^t \left\{a_0h_4(\tau, z) + k^2h_3(\tau, z)\right\}d\tau\right].$$

Case 2. if we take $P = 1, Q = (2m^2 - 1)$ and $R = -m^2(1 - m^2)$, then $F(\xi) \to ds(\xi)$,

$$u_4(t, x, z) = a_0 - [\frac{12k^2h_3(t, z)}{h_4(t, z)}] \ ds^2(\xi_4(t, x, z)),$$

(3.16)

$$v_4(t, x, z) = \pm \frac{[3k^2h_3(t,z)(2m^2-1)+a_0(h_1(t,z)-h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{12ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}} \ ds^2(\xi_4(t, x, z)),$$

(3.17)

with

$$\xi_4(t, x, z) = k\left[x - \int_0^t \left\{a_0h_4(\tau, z) + k^2(2m^2 - 1)h_3(\tau, z)\right\}d\tau\right].$$

In the limit case when $m \to o$ we have $ds(\xi) \to \csc(\xi)$, thus (3.16),(3.17) become.

$$u_5(t, x, z) = a_0 - [\frac{12k^2h_3(t, z)}{h_4(t, z)}] \ \csc^2(\xi_5(t, x, z)),$$

(3.18)

$$v_5(t, x, z) = \pm \frac{[-3k^2h_3(t,z)+a_0(h_1(t,z)-h_4(t,z))]}{\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{12ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}} \ \csc^2(\xi_5(t, x, z)),$$

(3.19)

with

$$\xi_5(t, x, z) = k\left[x - \int_0^t \left\{a_0h_4(\tau, z) - k^2h_3(\tau, z)\right\}d\tau\right].$$

Remark that. In the limit case when $m \to 1$ we have $ds(\xi) = cs(\xi) \to \operatorname{csch}(\xi)$, thus (3.16),(3.17) become the same solutions in case 1.

Case 3. if we take $P = \frac{1}{4}, Q = \frac{1-2m^2}{2}$ and $R = \frac{1}{4}$, then $F(\xi) \to ns(\xi) \pm cs(\xi)$,

$$u_6(t, x, z) = a_0 - \frac{3k^2h_3(t, z)}{h_4(t, z)}\left[ns(\xi_6(t, x, z)) \pm cs(\xi_6(t, x, z))\right]^2,$$

(3.20)

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$$v_6(t, x, z) = \pm \frac{[3k^2h_3(t,z)(1-2m^2)+2a_0(h_1(t,z)-h_4(t,z))]}{2\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{3ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}}\left[ns(\xi_6(t, x, z)) \pm cs(\xi_6(t, x, z))\right]^2,$$

(3.21)

with

$$\xi_6(t, x, z) = k\left[x - \frac{1}{2}\int_0^t \left\{2a_0h_4(\tau, z) + k^2(1-2m^2)h_3(\tau, z)\right\}d\tau\right].$$

In the limit case when $m \to o$ we have $(ns(\xi) \pm cs(\xi)) \to (\csc(\xi) \pm \cot(\xi))$, thus (3.22),(3.23) become.

$$u_7(t, x, z) = a_0 - \frac{3k^2h_3(t, z)}{h_4(t, z)}\left[\csc(\xi_7(t, x, z)) \pm \cot(\xi_7(t, x, z))\right]^2,$$

(3.22)

$$v_7(t, x, z) = \pm \frac{[3k^2h_3(t,z)+2a_0(h_1(t,z)-h_4(t,z))]}{2\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{3ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}}\left[\csc(\xi_7(t, x, z)) \pm \cot(\xi_7(t, x, z))\right]^2,$$

(3.23)

with

$$\xi_7(t, x, z) = k\left[x - \frac{1}{2}\int_0^t \left\{2a_0h_4(\tau, z) + k^2h_3(\tau, z)\right\}d\tau\right].$$

In the limit case when $m \to 1$ we have $(ns(\xi) \pm cs(\xi)) \to (\coth(\xi) \pm csch(\xi))$, thus (3.22),(3.23) become.

$$u_8(t, x, z) = a_0 - \frac{3k^2h_3(t, z)}{h_4(t, z)}\left[\coth(\xi_8(t, x, z)) \pm \operatorname{csch}(\xi_8(t, x, z))\right]^2,$$

(3.24)

$$v_8(t, x, z) = \pm \frac{[-3k^2h_3(t,z)+2a_0(h_1(t,z)-h_4(t,z))]}{2\sqrt{h_2(t,z)[h_1(t,z)-h_4(t,z)]}}$$

$$\pm \frac{3ik^2h_3(t,z)}{h_4(t,z)}\sqrt{\frac{h_1(t,z)-h_4(t,z)}{h_2(t,z)}}\left[\coth(\xi_8(t, x, z)) \pm \operatorname{csch}(\xi_8(t, x, z))\right]^2,$$

(3.25)

with

$$\xi_8(t, x, z) = k\left[x - \frac{1}{2}\int_0^t \left\{2a_0h_4(\tau, z) - k^2h_3(\tau, z)\right\}d\tau\right].$$

Remark that: there are another solutions for Eq.(1.2). These solutions come from setting different values for the cofficients P, Q and R.(see Appendex C.). The above mentioned cases are just to clarify how far our technique is applicable.

4 White Noise Functional Solutions of Eq.(1.2)

In this section, we employ the results of the Section 3 by using Hermite transform to obtain exact white noise functional solutions for Wick-type stochastic coupled KdV equations (1.2). The properties of exponential and trigonometric functions yield that there exists a bounded open set $\mathbf{D} \subset \mathbb{R}_+ \times \mathbb{R}$, $\rho < \infty$, $\lambda > 0$ such that the solution $u(t, x, z)$ of Eq.(3.1) and all its partial derivatives which are involved in Eq. (3.1) are uniformly bounded for $(t, x, z) \in \mathbf{D} \times K_\rho(\lambda)$, continuous with respect to $(t, x) \in \mathbf{D}$ for all $z \in K_\rho(\lambda)$ and analytic with respect to $z \in K_\rho(\lambda)$, for all $(t, x) \in \mathbf{D}$. From Theorem 4.1.1 in [20], there exists $U(t, x, z) \in (\mathcal{S})_{-1}$ such that $u(t, x, z) = \tilde{U}(t, x)(z)$ for all $(t, x, z) \in \mathbf{D} \times K_\rho(\lambda)$ and $U(t, x)$ solves Eq.(1.2) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to the results of Section 3, we get New exact white noise functional solutions of Eq.(1.2) as follows:

- New Wick-type stochastic solutions of (JEF):

$$U_1(t, x) = a_0 - \left[\frac{12k^2 H_3(t)}{H_4(t)}\right] \diamond cs^{\diamond 2}(\Xi_1(t, x)), \tag{4.1}$$

$$V_1(t, x) = \pm \frac{[3k^2 H_3(t)(2-m^2)+a_0(H_1(t)-H_4(t))]}{\sqrt{H_2(t)\diamond[H_1(t)-H_4(t)]}}$$

$$\pm \frac{12ik^2 H_3(t)}{H_4(t)} \diamond \sqrt{\frac{H_1(t)-H_4(t)}{H_2(t)}} \diamond cs^{\diamond 2}(\Xi_1(t, x)), \tag{4.2}$$

$$U_2(t, x) = a_0 - \left[\frac{12k^2 H_3(t)}{H_4(t)}\right] \diamond ds^{\diamond 2}(\Xi_2(t, x)), \tag{4.3}$$

$$V_2(t, x) = \pm \frac{[3k^2 H_3(t)(2m^2-1)+a_0(H_1(t)-H_4(t))]}{\sqrt{H_2(t)\diamond[H_1(t)-H_4(t)]}}$$

$$\pm \frac{12ik^2 H_3(t)}{H_4(t)} \diamond \sqrt{\frac{H_1(t)-H_4(t)}{H_2(t)}} \diamond ds^{\diamond 2}(\Xi_2(t, x)), \tag{4.4}$$

$$U_3(t, x) = a_0 - \frac{3k^2 H_3(t)}{H_4(t)} \diamond \left[ns^\diamond(\Xi_3(t, x)) \pm cs^\diamond(\Xi_3(t, x))\right]^{\diamond 2}, \tag{4.5}$$

$$V_3(t, x) = \pm \frac{[3k^2 H_3(t)(1-2m^2)+2a_0(H_1(t)-H_4(t))]}{2\sqrt{H_2(t)\diamond[H_1(t)-H_4(t)]}}$$

$$\pm \frac{3ik^2 H_3(t)}{H_4(t)} \diamond \sqrt{\frac{H_1(t)-H_4(t)}{H_2(t)}} \diamond \left[ns^\diamond(\Xi_3(t, x)) \pm cs^\diamond(\Xi_3(t, x))\right]^{\diamond 2}, \tag{4.6}$$

with

$$\Xi_1(t, x) = k\left[x - \int_0^t \{a_0 H_4(\tau) + k^2(2 - m^2)H_3(\tau)\}d\tau\right],$$

$$\Xi_2(t, x) = k\left[x - \int_0^t \{a_0 H_4(\tau) + k^2(2m^2 - 1)H_3(\tau)\}d\tau\right],$$

$$\Xi_3(t, x) = k\left[x - \frac{1}{2}\int_0^t \{2a_0 H_4(\tau) + k^2(1 - 2m^2)H_3(\tau)\}d\tau\right].$$

• New Wick-type stochastic solutions of trigonometric functions:

$$U_4(t, x) = a_0 - \frac{12k^2 H_3(t)}{H_4(t)} \diamond \cot^{\diamond 2}(\Xi_4(t, x)), \qquad (4.7)$$

$$V_4(t, x) = \pm\frac{[6k^2 H_3(t) + a_0(H_1(t) - H_4(t))]}{\sqrt{H_2(t)\diamond(H_1(t) - H_4(t))}}$$

$$\pm\frac{12ik^2 H_3(t)}{H_4(t)} \diamond \sqrt{\frac{H_1(t) - H_4(t)}{H_2(t)}} \diamond \cot^{\diamond 2}(\Xi_4(t, x)), \qquad (4.8)$$

$$U_5(t, x) = a_0 - [\frac{12k^2 H_3(t)}{H_4(t)}] \diamond \csc^{\diamond 2}(\Xi_5(t, x)), \qquad (4.9)$$

$$V_5(t, x) = \pm\frac{[-3k^2 H_3(t) + a_0(H_1(t) - H_4(t))]}{\sqrt{H_2(t)\diamond[H_1(t) - H_4(t)]}} \pm \frac{12ik^2 H_3(t)}{H_4(t)}$$

$$\diamond\sqrt{\frac{H_1(t) - H_4(t)}{H_2(t)}} \diamond \csc^{\diamond 2}(\Xi_5(t, x)), \qquad (4.10)$$

$$U_6(t, x) = a_0 - \frac{3k^2 H_3(t)}{H_4(t)} \diamond \left[\csc^{\diamond}(\Xi_6(t, x)) \pm \cot^{\diamond}(\Xi_6(t, x))\right]^{\diamond 2}, \qquad (4.11)$$

$$V_6(t, x) = \pm\frac{[3k^2 H_3(t) + 2a_0(H_1(t) - H_4(t))]}{2\sqrt{H_2(t)\diamond[H_1(t) - H_4(t)]}} \pm \frac{3ik^2 H_3(t)}{H_4(t)}$$

$$\diamond\sqrt{\frac{H_1(t) - H_4(t)}{H_2(t)}} \diamond \left[\csc^{\diamond}(\Xi_6(t, x)) \pm \cot^{\diamond}(\Xi_6(t, x))\right]^{\diamond 2}, \qquad (4.12)$$

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with

$$\Xi_4(t,x) = k\left[x - \int_0^t \{a_0 H_4(\tau) + 2k^2 H_3(\tau)\}d\tau\right],$$

$$\Xi_5(t,x) = k\left[x - \int_0^t \{a_0 H_4(\tau) - k^2 H_3(\tau)\}d\tau\right],$$

$$\Xi_6(t,x) = k\left[x - \frac{1}{2}\int_0^t \{2a_0 H_4(\tau) + k^2 H_3(\tau)\}d\tau\right].$$

- New Wick-type stochastic solutions of hyperbolic functions:

$$U_7(t,x) = a_0 - [\frac{12k^2 H_3}{H_4}] \diamond \operatorname{csch}^{\diamond 2}(\Xi_7(t,x)), \tag{4.13}$$

$$V_7(t,x) = \pm\frac{[3k^2 H_3(t) + a_0(H_1(t) - H_4(t))]}{\sqrt{H_2(t)\diamond[H_1(t) - H_4(t)]}} \pm \frac{12ik^2 H_3(t)}{H_4(t)}$$

$$\diamond\sqrt{\frac{H_1(t) - H_4(t)}{H_2(t)}} \diamond \operatorname{csch}^{\diamond 2} \Xi_7(t,x), \tag{4.14}$$

$$U_8(t,x) = a_0 - \frac{3k^2 H_3(t)}{H_4(t)} \diamond \left[\coth^{\diamond}(\Xi_8(t,x)) \pm \operatorname{csch}^{\diamond}(\Xi_8(t,x))\right]^{\diamond 2}, \tag{4.15}$$

$$V_8(t,x) = \pm\frac{[-3k^2 H_3(t) + 2a_0(H_1(t) - H_4(t))]}{2\sqrt{H_2(t)\diamond[H_1(t) - H_4(t)]}} \pm \frac{3ik^2 H_3(t)}{H_4(t)}$$

$$\diamond\sqrt{\frac{H_1(t) - H_4(t)}{H_2(t)}} \diamond \left[\coth^{\diamond}(\Xi_8(t,x)) \pm \operatorname{csch}^{\diamond}(\Xi_8(t,x))\right]^{\diamond 2}, \tag{4.16}$$

with

$$\Xi_7(t,x) = k\left[x - \int_0^t \{a_0 H_4(\tau) + k^2 H_3(\tau)\}d\tau\right],$$

$$\Xi_8(t,x) = k\left[x - \frac{1}{2}\int_0^t \{2a_0 H_4(\tau) - k^2 H_3(\tau)\}d\tau\right].$$

We observe that. For different forms of H_1, H_2, H_3 and H_4, we can get different exact white noise functional solutions of Eq.(1.2) from Eqs.(4.1)-(4.16).

5 Example

It is well known that Wick version of function is usually difficult to evaluate. So, in this section, we give non-Wick version of solutions of Eq.(1.2). Let $W_t = \dot{B}_t$ be the Gaussian white noise, where B_t is the Brownian motion. We have the Hermite transform $\tilde{W}_t(z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)ds$ [20]. Since $\exp^{\diamond}(B_t) = \exp(B_t - \frac{t^2}{2})$, we have $\sin^{\diamond}(B_t) = \sin(B_t - \frac{t^2}{2})$, $\cos^{\diamond}(B_t) = \cos(B_t - \frac{t^2}{2})$, $\cot^{\diamond}(B_t) = \cot(B_t - \frac{t^2}{2})$, $\csc^{\diamond}(B_t) = \csc(B_t - \frac{t^2}{2})$, $\coth^{\diamond}(B_t) = \coth(B_t - \frac{t^2}{2})$ and $\operatorname{csch}^{\diamond}(B_t) = \operatorname{csch}(B_t - \frac{t^2}{2})$. Suppose that $H_1(t) = H_2(t) = \lambda_1 H_3(t), H_3(t) = \lambda_2 H_4(t)$ and $H_4(t) = \Gamma(t) + \lambda_3 W_t$ where λ_1, λ_2 and λ_3 are arbitrary constants and $\Gamma(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. Therefore, for $H_1(t)H_2(t)H_3(t)H_4(t) \neq 0$. thus exact white noise functional solutions of Eq.(1.2) are as follows:

$$U_9(t,x) = 3k^2 \left\{ \frac{a_0}{3k^2} - 4\lambda_2 \cot^2(\Phi_1(t,x)) \right\}, \tag{5.1}$$

$$V_9(t,x) = \pm 6k^2\lambda_2 \left\{ \frac{\left[1 + \frac{a_0}{6k^2\lambda_2(\lambda_1\lambda_2-1)} \right]}{\sqrt{\lambda_1\lambda_2(\lambda_1\lambda_2-1)}} + \right.$$
$$\left. 2i\sqrt{\frac{(\lambda_1\lambda_2-1)}{\lambda_1\lambda_2}} \cot^2(\Phi_1(t,x)) \right\}, \tag{5.2}$$

$$U_{10}(t,x) = 3k^2 \left\{ \frac{a_0}{3k^2} - 4\lambda_2 \csc^2(\Phi_2(t,x)) \right\}, \tag{5.3}$$

$$V_{10}(t,x) = \pm 3k^2\lambda_2 \left\{ \frac{\left[\frac{a_0}{3k^2\lambda_2(\lambda_1\lambda_2-1)} - 1 \right]}{\sqrt{\lambda_1\lambda_2(\lambda_1\lambda_2-1)}} + \right.$$
$$\left. 4i\sqrt{\frac{(\lambda_1\lambda_2-1)}{\lambda_1\lambda_2}} \csc^2(\Phi_2(t,x)) \right\}, \tag{5.4}$$

$$U_{11}(t,x) = a_0 - 3k^2\lambda_2 \left[\csc(\Xi(t,x)) \pm \cot(\Phi_3(t,x)) \right]^2, \tag{5.5}$$

$$V_{11}(t,x) = \pm 3k^2\lambda_2 \left\{ \frac{\left[1 + \frac{2a_0}{3k^2\lambda_2(\lambda_1\lambda_2-1)} \right]}{2\sqrt{\lambda_1\lambda_2(\lambda_1\lambda_2-1)}} + \right.$$
$$\left. i\sqrt{\frac{(\lambda_1\lambda_2-1)}{\lambda_1\lambda_2}} \left[\csc(\Phi(t,x)) \pm \cot(\Phi_3(t,x)) \right]^2 \right\}, \tag{5.6}$$

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$$U_{12}(t,x) = a_0 - 12k^2\lambda_2 \operatorname{csch}^2(\Phi_4(t,x)), \tag{5.7}$$

$$V_{12}(t,x) = \pm 3k^2\lambda_2 \left\{ \frac{\left[1 + \frac{a_0}{3k^2\lambda_2(\lambda_1\lambda_2-1)}\right]}{\sqrt{\lambda_1\lambda_2(\lambda_1\lambda_2-1)}} + \right.$$
$$\left. 4i\sqrt{\frac{(\lambda_1\lambda_2-1)}{\lambda_1\lambda_2}} \operatorname{csch}^2(\Phi_4(t,x)) \right\}, \tag{5.8}$$

$$U_{13}(t,x) = a_0 - 3k^2\lambda_2 \left[\coth(\Phi_5(t,x)) \pm \operatorname{csch}(\Phi_5(t,x))\right]^2, \tag{5.9}$$

$$V_{13}(t,x) = \pm 3k^2\lambda_2 \left\{ \frac{\left[\frac{2a_0}{3k^2\lambda_2(\lambda_1\lambda_2-1)} - 1\right]}{2\sqrt{\lambda_1\lambda_2(\lambda_1\lambda_2-1)}} + \right.$$
$$\left. 3i\sqrt{\frac{(\lambda_1\lambda_2-1)}{\lambda_1\lambda_2}} \left[\coth(\Phi_5(t,x)) \pm \operatorname{csch}(\Phi_5(t,x))\right]^{\diamond 2} \right\}, \tag{5.10}$$

with

$$\Phi_1(t,x) = k\left[x - (a_0 + 2k^2\lambda_2)\left\{\int_0^t \Gamma(\tau)d\tau + \lambda_3[B_t - \frac{t^2}{2}]\right\}\right],$$

$$\Phi_2(t,x) = k\left[x - (a_0 - k^2\lambda_2)\left\{\int_0^t \Gamma(\tau)d\tau + \lambda_3[B_t - \frac{t^2}{2}]\right\}\right],$$

$$\Phi_3(t,x) = k\left[x - \frac{(2a_0 + k^2\lambda_2)}{2}\left\{\int_0^t \Gamma(\tau)d\tau + \lambda_3[B_t - \frac{t^2}{2}]\right\}\right],$$

$$\Phi_4(t,x) = k\left[x - (a_0 + k^2\lambda_2)\left\{\int_0^t \Gamma(\tau)d\tau + \lambda_3[B_t - \frac{t^2}{2}]\right\}\right]$$

and

$$\Phi_5(t,x) = k\left[x - \frac{(2a_0 - k^2\lambda_2)}{2}\left\{\int_0^t \Gamma(\tau)d\tau + \lambda_3[B_t - \frac{t^2}{2}]\right\}\right].$$

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6   Conclusion

We have discussed the solutions of stochastic (PDEs) driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space. This connection was given by Benth and Gjerde [2]. We can see in the section 4.9 [20] clearly. Hence, by the aid of the connection, we can derive some stochastic exact soliton solutions if the coefficients , and are Poisson white noise functions in Eq. (1.2). In this paper, using Hermite transformation, white noise theory and F-expansion method, we study the white noise solutions of the Wick-type stochastic coupled KdV equations. This paper shows that the F-expansion method is sufficient to solve the stochastic nonlinear equations in mathematical physics. The method which we have proposed in this paper is powerful, direct and computerized method, which allows us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics such as modified Hirota-Satsuma coupled KdV, (2+1)-dimensional coupled KdV, KdV-Burgers, schamel KdV, modified KdV Burgers, Sawada-Kotera, Zhiber-Shabat equations and Benjamin-Bona-Mahony equations. Since the equation (1.2) has other solutions if select other values of P, Q and R (see Appendix A, B and C). So there are many other of exact solutions for wick-type stochastic coupled KdV equations.

Appendix A.
the jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$.

$$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1, sc\xi \to \tan\xi, sd\xi \to \sin\xi, cd\xi \to \cos\xi,$$
$$ns\xi \to \csc\xi, nc\xi \to \sec\xi, nd\xi \to 1, cs\xi \to \cot\xi, ds\xi \to \csc\xi, dc\xi \to \sec\xi.$$

Appendix B.
the jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$.

$$sn\xi \to \tan\xi, cn\xi \to \text{sech}\,\xi, dn\xi \to \text{sech}\,\xi, sc\xi \to \sinh\xi, sd\xi \to \sinh\xi, cd\xi \to 1,$$
$$ns\xi \to \coth\xi, nc\xi \to \cosh\xi, nd\xi \to \cosh, cs\xi \to \text{csch}\,\xi, ds\xi \to \text{csch}\,\xi, dc\xi \to 1.$$

Appendix C. The ODE and Jacobi Elliptic Functions
Relation between values of (P , Q , R) and corresponding $F(\xi)$ in ODE

$$(F')^2(\xi) = PF^4(\xi) + QF^2(\xi) + R,$$

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P	Q	R	$F(\xi)$
m^2	$-1-m^2$	1	$\text{sn}\xi,\ \text{cd}\xi = \frac{cn\xi}{dn\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\text{cn}\xi$
-1	$2-m^2$	m^2-1	$\text{dn}\xi$
1	$-1-m^2$	m^2	$\text{ns}\xi = \frac{1}{\text{sn}\xi},\ \text{dc}\xi = \frac{\text{dn}\xi}{\text{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\text{nc}\xi = \frac{1}{\text{cn}\xi}$
m^2-1	$2-m^2$	-1	$\text{nd}\xi = \frac{1}{\text{dn}\xi}$
$1-m^2$	$2-m^2$	1	$\text{sc}\xi = \frac{\text{sn}\xi}{\text{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\text{sd}\xi = \frac{\text{sn}\xi}{\text{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\text{cs}\xi = \frac{\text{cn}\xi}{\text{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\text{ds}\xi = \frac{\text{dn}\xi}{\text{sn}\xi}$
$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\text{sn}\xi}{1\pm\text{dn}\xi},\ \frac{\text{cn}\xi}{\sqrt{1-m^2}\pm\text{dn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{sn}\xi\pm i\text{cn}\xi,\ \frac{\text{dn}\xi}{i\sqrt{1-m^2}\text{sn}\xi\pm\text{cn}\xi},\ \frac{\text{m}\,\text{sn}\xi}{1\pm\text{dn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\text{ns}\xi\pm\text{cs}\xi,\ \frac{\text{cn}\xi}{\sqrt{1-m^2}\text{sn}\xi\pm\text{dn}\xi},\ \frac{\text{sn}\xi}{1\pm\text{cn}\xi},$
$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{\text{dn}\xi}{1\pm m\text{sn}\xi}$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\text{nc}\xi\pm i\text{sc}\xi\,\frac{\text{cn}\xi}{1\pm\text{sn}\xi}$
$\frac{-1}{4}$	$\frac{m^2+1}{2}$	$\frac{-(1-m^2)^2}{4}$	$m\text{cn}\xi\pm\text{dn}\xi$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\text{ns}\xi\pm\text{ds}\xi$

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A NOTE ON n-BANACH LATTICES

BİRSEN SAĞIR[1] AND NİHAN GÜNGÖR[2]

ABSTRACT. In this paper we describe n-Banach lattices by Riesz space equipped with n-norm. We consider spaces of reguler operators and order bounded operators between n-Banach lattices and Banach lattices, we generate norm on these space by the aid of n-norm. Then we investigate properties of operators on these space with generated norm.

1. INTRODUCTION

Riesz spaces, also called vector lattices, K-lineals which were first considered by F. Riesz, L. Kantorovic, and H. Freudenthal in the middle of nineteen thirties. L. Kantorovic and his school first recognized the importance of studying vector lattices in connection with Banach's theory of normed vector spaces; they investigated normed vector lattices as well as order-related linear operators between such vector lattices. In the middle seventies the research on this subject was essentially influenced by H.H. Schaefer. H. H. Schaefer present the theory of Banach lattices and positive operators as an inseparable part of the general Banach space and operator theory. In particular, deep results of the general theory and classical analysis were used to prove related properties in case of general Banach lattices. More recently other important study concerning this subject appeared CD. Aliprantis and O. Burkinshaw.

Let a real vector space X be an ordered vector space equipped with an order relation $\preceq$. A vector x in an ordered vector space X is called positive whenever $0 \preceq x$ holds. The set of all positive vectors denoted by X^+. An operator is a map between two vector space. An operator $T : X \to Y$ between two ordered vector space is called positive if $0 \preceq T(x)$ for all $x \in X$. A Riesz space is an ordered vector space X with property that for each pair vectors $x, y \in X$ the supremum and the infimum of the set $\{x, y\}$ both exists in X. For any vector x in a Riesz space X, define $x^+ := x \vee 0 = \sup\{x, 0\}$, $x^- := x \wedge 0 = \sup\{-x, 0\}$, $|x| := \sup\{x, -x\}$. If $T : X \to Y$ be an linear operator between two Riesz spaces such that $\sup\{|Ty| : |y| \preceq x\}$ exists in Y for each $x \in X^+$, then the modulus of T exists and $|T| x = \sup\{|Ty| : |y| \preceq x\}$ for each $x \in X^+$. Also $|T(x)| \preceq |T|(|x|)$ holds for all $x \in X$. A subset A of a Riesz space is said to be bounded above (bounded below) whenever there exists some x satisfying $y \preceq x$ $(x \preceq y)$ for all $y \in \dot{X}$. A subset in a Riesz space is called order bounded if it is bounded both above and below. A Riesz space is called Dedekind complete whenever every nonempty bounded above subset has supremum (whenever every nonempty bounded below subset has infimum). An linear operator $T : X \to Y$ between two Riesz spaces is

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said to be reguler if it can be written as a difference of two positive operators. The vector space of all reguler operators form X to Y denoted by $L_r(X, Y)$. Of course this is equivalent to saying there exists a positive operator $S : X \to Y$ satisfying $T \preceq S$. An linear operator $T : X \to Y$ between two Riesz spaces is said to be order bounded if it maps an order bounded subsets of X to order bounded subsets of Y. The vector space of all order bounded operators from X to Y denoted by $L_b(X, Y)$. Every positive operator is order bounded. Therefore every reguler operator is also order bounded. Hence $L_r(X, Y) \subseteq L_b(X, Y)$. A norm $\|.\|$ on a Riesz space is said to be a lattice norm whenever $|x| \preceq |y|$ implies $\|x\| \leq \|y\|$. For more detailed, it can be looked [1, 2, 6, 7].

2. Operators on n-Banach Lattices

Fistly, we will give the some required definitions. The notion of n-normed space as well as its properties was introduced by S. Gähler in [4] as a generalization of the normed space.

Let X be a real linear space. A function $\|., \ldots, .\| : X^n \to \mathbb{R}$ is called a n-norm on X^n if it satisfies following conditions, for all $\alpha \in \mathbb{R}$ and $x_1, x_2, ..., x_n, y \in X$;

i) $\|x_1, x_2, ..., x_n\| = 0 \Leftrightarrow x_1, x_2, ..., x_n$ are linearly dependent

ii) $\|x_1, x_2, ..., x_n\|$ is invariant under permutations of $x_1, x_2, ..., x_n$

iii) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$

iv) $\|x_1 + y, x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|y, x_2, ..., x_n\|$.

Then the pair $(X, \|., \ldots, .\|)$ is called n-normed space.

A sequence $\{x_k\}$ of n-normed space X is said to be convergent if there exists an element $x \in X$ such that

$$\lim_{k \to \infty} \|x_k - x, z_1, ..., z_{n-1}\| = 0$$

for all $z_1, ..., z_{n-1} \in X$ [5].

Let $(X, \|., \ldots, .\|)$ and $(Y, \|., \ldots, .\|)$ be n-normed spaces. For an operator $T : (X, \|., \ldots, .\|) \to (Y, \|., \ldots, .\|)$, define $[.]_n$ by

$$[T]_n := \sup \left\{ \frac{\|T(x_1), T(x_2), ..., T(x_n)\|}{\|x_1, x_2, ..., x_n\|} : \|x_1, x_2, ..., x_n\| \neq 0 \right\}.$$

But, this natural definition of $[.]_n$ is not a norm [8] .For this reason, we need to take Y as normed space.

Let X be n-normed space and Y be normed space. An operator $T : X^n \to Y$ be a n-linear operator on X if T is linear in each of variable. An n-linear operator is bounded if there exists a constant $k > 0$ such that ,

$$\|T(x_1, x_2, ..., x_n)\| \leq k \|x_1, x_2, ..., x_n\|$$

for all $(x_1, x_2, ..., x_n) \in X^n$. If T is a bounded n-linear operator, then the n-operator norm defined by

$$\|\|T\|\| := \sup \{\|T(x_1, x_2, ..., x_n)\| : \|x_1, x_2, ..., x_n\| \leq 1\}$$

and the space of all bounded n-linear operators from X^n to Y denoted by $L(X^n, Y)$. If Y is a Banach space, then $L(X^n, Y)$ is a Banach space with n-operator norm. Also it's known that when $T : X^n \to Y$ is a n-linear operator, T is bounded if and only if T is continuous [8].

Let X be a Riesz space. The cartesian product X^n is a Riesz space under the ordering

$$x = (x_1, x_2, ..., x_n) \preceq z = (z_1, z_2, ..., z_n) \iff x_i \preceq z_i$$

holds in X for each $i = 1, 2, .., n$. If $x = (x_1, x_2, ..., x_n)$ and $z = (z_1, z_2, ..., z_n)$ are vectors of X^n, then

$$x \vee y = (x_1 \vee z_1, x_2 \vee z_2, ..., x_n \vee z_n) \text{ and } x \wedge y = (x_1 \wedge z_1, x_2 \wedge z_2, ..., x_n \wedge z_n)$$

[2].Hence we can see

$$|x| = |(x_1, x_2, ..., x_n)| = (|x_1|, |x_2|, ..., |x_n|)$$

for all $x = (x_1, x_2, ..., x_n) \in X^n$. Also, it's clearly that

$$x^+ = (x_1, x_2, ..., x_n)^+ = \left(x_1^+, x_2^+, ..., x_n^+\right) \text{ and } x^- = (x_1, x_2, ..., x_n)^- = \left(x_1^-, x_2^-, ..., x_n^-\right).$$

We demonstrate the space of positive vectors $x^+ = (x_1, x_2, ..., x_n)^+$ by X^{n^+}. Through this paper X^n will be equipped with this ordering.

In [2, 10], the space of reguler operators and order bounded operators between Banach lattices is defined and equipped with reguler norm and bound norm, respectively. Also some properties of these operators are showed. In this study, we define n-Banach lattice with the aid of Riesz space equipped with n-norm and then we generalize some of these properties by using n-normed.

Definition 1. *Let X be a Riesz space. A n-norm $\|\cdot, \ldots, \cdot\|$ on X is said to be a n-lattices norm whenever $|x| \preccurlyeq |y|$ implies $\|x, z_1, ..., z_{n-1}\| \leq \|y, z_1, ..., z_{n-1}\|$ for all $z_1, ..., z_{n-1} \in X$. When X equipped with a n-lattice norm, it's defined as a n-normed Riesz space. If n-normed Riesz space X is complete, then it's called as n-Banach lattice.*

Since $|x| \preceq ||x||$ and $||x|| \preceq |x|$ for all x element of n-normed Riesz space X, we can see easily that the following equality

$$\|x, z_1, ..., z_{n-1}\| = \||x|, z_1, ..., z_{n-1}\|$$

for all $z_1, ..., z_{n-1} \in X$. Also $\|x^+ - y^+, z_1, ..., z_{n-1}\| \leq \|x - y, z_1, ..., z_{n-1}\|$ and $\||x| - |y|, z_1, ..., z_{n-1}\| \leq \|x - y, z_1, ..., z_{n-1}\|$ for all $z_1, ..., z_{n-1} \in X$.

It's known that the space l_p with $1 \leq p < \infty$ is n-normed space with

$$\|x_1, x_2, ..., x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det(x_{ij_k})|^p\right]^{\frac{1}{p}}$$

, $i = 1, ..., n$ [5].If l_p equipped with an order relation $\preceq$ such that $|x| \preceq |y| \iff |x_k| \preceq |y_k|$ whenever $x = (x_k)$, $y = (y_k)$ for all $k \in \mathbb{N}$, then l_p is a Riesz space according to this relation and $\|., .\|_p$ is a n-lattice norm. So $\left(l_p, \|., .\|_p\right)$ is an example of n-normed Riesz space.

The following lemma will used in the proof of the future theorem. We prove the theorem by using the method which is well-known as in [3, 9].

Lemma 1. *n-normed space X is a n-Banach space if and only if every convergent series is absolutely convergent.*

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Proof. Firstly, let X be a n-Banach space. Assume that $\sum_{k=1}^{\infty} \|x_k, z_1, ..., z_{n-1}\|$ is convergent for all $z_1, ..., z_{n-1} \in X$. From convergent principle of real series $\sum_{k=m+1}^{\infty} \|x_k, z_1, ..., z_{n-1}\| \to 0 \; (m \to \infty)$. If we take $S_m = \sum_{k=1}^{m} x_k$, then

$$
\begin{aligned}
\|S_l - S_m, z_1, ..., z_{n-1}\| &= \|x_{m+1} + x_{m+2} + \cdots + x_l, z_1, ..., z_{n-1}\| \\
&\leq \|x_{m+1}, z_1, ..., z_{n-1}\| + \|x_{m+2}, z_1, ..., z_{n-1}\| + \\
&\quad + \cdots + \|x_l, z_1, ..., z_{n-1}\| \\
&\leq \sum_{k=m+1}^{l} \|x_k, z_1, ..., z_{n-1}\|.
\end{aligned}
$$

So that $\|S_l - S_m, z_1, ..., z_{n-1}\| \to 0 \; (m, l \to \infty)$ and then we see that (S_m) is a Cauchy sequence. Since X is n-Banach space, $\sum_{k=1}^{\infty} x_k$ is convergent.

For the converse, every convergent series is absolutely convergent. Let $\{x_k\}$ be a Cauchy sequence of X, so $\|x_k - x_m, z_1, ..., z_{n-1}\| \to 0 \; (k, m \to \infty)$. Then there exists strictly increasing sequence $\{k_m\}$ of natural numbers such that

$$
\|x_{k_m+1} - x_{k_m}, z_1, ..., z_{n-1}\| < \frac{1}{2^m}
$$

for all $m \in \mathbb{N}$. Hence we obtain $\sum_{m=1}^{\infty} \|x_{k_m+1} - x_{k_m}, z_1, ..., z_{n-1}\| < \infty$, then $\sum_{k=1}^{\infty} (x_{k_m+1} - x_{k_m})$ is convergent from the hypothesis. If we take $T_v = \sum_{m=1}^{v} (x_{k_m+1} - x_{k_m})$, then $\lim_{v \to \infty} T_v = \lim_{v \to \infty} (x_{k_v+1} - x_{k_1})$ exists and so the subsequence $\{x_{k_v+1}\}$ is convergent. Therefore $\{x_k\}$ is convergent and hence the proof is completed. □

Theorem 1. *Let X be n-Banach lattice and Y be normed Riesz space. If $T : X^n \to Y$ is positive surjective n-linear operator, then it's continuous.*

Proof. It's known that if $T : X^n \to Y$ is n-linear operator, then boundedness and continuity are equivalent [8]. Assume that T is not bounded. Then there exists a sequence $\{x_k\}$ of X such that $\|x_k, z_1, ..., z_{n-1}\| = 1$ and $\|T(x_k, z_1, ..., z_{n-1})\| \geq k^3$ for all $k \in \mathbb{N}$ and $z_1, ..., z_{n-1} \in X$. Since T is positive operator, we can write

$$
|T(x_k, z_1, ..., z_{n-1})| \preceq T |(x_k, z_1, ..., z_{n-1})| = T(|x_k|, |z_1|, ..., |z_{n-1}|).
$$

Hence we find,

$$
\|T(|x_k|, |z_1|, ..., |z_{n-1}|)\| \geq \| |T(x_k, z_1, ..., z_{n-1})| \| = \|T(x_k, z_1, ..., z_{n-1})\| \geq k^3.
$$

If we consider that $\|x_k, z_1, ..., z_{n-1}\| = \| |x_k|, z_1, ..., z_{n-1}\|$ and $y_k = |x_k|$, then there is a sequence $\{y_k\}$ such that

(2.1) $$\|T(y_k, |z_1|, ..., |z_{n-1}|)\| \geq k^3$$

with $\|y_k, z_1, ..., z_{n-1}\| = 1$ and $0 \preceq y_k$ for every $k \in \mathbb{N}$.

Since $\sum_{k=1}^{\infty} \frac{\| y_k, |z_1|, ..., |z_{n-1}| \|}{k^2} < \infty$ and X is a n-Banach space, then the series $\sum_{k=1}^{\infty} \frac{y_k}{k^2}$ is n-norm convergent in X. If we say that $y = \sum_{k=1}^{\infty} \frac{y_k}{k^2}$, then it's obvious that

$0 \preceq \frac{y_k}{k^2} \preceq y$ holds for every $k \in \mathbb{N}$. Hence, from the inequality (2.1) we obtain

$$k \leq \left\| T\left(\frac{y_k}{k^2}, |z_1|, ..., |z_{n-1}|\right) \right\| \leq \| T(y, |z_1|, ..., |z_{n-1}|) \| < \infty$$

for all $k \in \mathbb{N}$, but it's a contradiction. Therefore T must be bounded, so T is continuous. $\square$

When X and Y be normed Riesz space, the definition of reguler norm $\|.\|_r$ is given in [1, 2]. Now we will generate n-reguler norm by using n-norm.

Definition 2. *Let X be n-Banach lattice and Y be Banach lattice. If $T : X^n \to Y$ be a n-linear operator with modulus, then the n-reguler norm $|||T|||_r$ is defined by*

$$|||T|||_r = ||| |T| ||| := \sup \{\| |T(x_1, x_2, ..., x_n)| \| : \|x_1, x_2, ..., x_n\| \leq 1\}$$

for all $x_1, x_2, ..., x_n \in X$.
Also we can give equivalent definition of n-reguler norm as

$$|||T|||_r := \inf \{|||S||| : \pm T \preceq S\}.$$

Theorem 2. *If X be n-Banach lattice and Y be Banach lattice with Y Dedekind complete, then $L_r(X^n, Y)$ is a Banach lattice under the n-reguler norm. Also $|||T||| \leq |||T|||_r$ implies for all $T \in L_r(X^n, Y)$.*

Proof. From the definition of n-reguler norm, if T is a positive operator then $|||T|||_r = ||| |T| ||| = |||T|||$ holds. Let $T, S \in L_r(X^n, Y)$ satisfying $0 \preceq S \preceq T$. So, we can write

$$|S(x_1, x_2, ..., x_n)| \preceq S|(x_1, x_2, ..., x_n)| \preceq T|(x_1, x_2, ..., x_n)| = T(|x_1|, |x_2|, ..., |x_n|)$$

for all $x_1, x_2, ..., x_n \in X$ with $\|x_1, x_2, ..., x_n\| = \||x_1|, |x_2|, ..., |x_n|\| \leq 1$. Because of $0 \preceq S(x_1, x_2, ..., x_n)$, we find

$$\begin{aligned}
\|S(x_1, x_2, ..., x_n)\| &= \| |S(x_1, x_2, ..., x_n)| \| \leq \| |S|(x_1, x_2, ..., x_n)| \| \\
&\leq \| |T|(x_1, x_2, ..., x_n)| \| = \| T(|x_1|, |x_2|, ..., |x_n|) \| \\
&\leq |||T||| \| |x_1|, |x_2|, ..., |x_n| \| \leq |||T|||.
\end{aligned}$$

Therefore, we obtain

$$|||S||| = \sup \{\|S(x_1, x_2, ..., x_n)\| : \|x_1, x_2, ..., x_n\| \leq 1\} \leq |||T|||.$$

Let $T, S \in L_r(X^n, Y)$ satisfying $|S| \preceq |T|$. Since $|S|$ and $|T|$ are positive operators, from the preceding result we write

$$|||S|||_r = ||| |S| ||| \leq ||| |T| ||| = |||T|||_r.$$

Hence n-reguler norm is a lattice norm. Also, we get

$$\begin{aligned}
|||T||| &= \sup \{\|T(x_1, x_2, ..., x_n)\| : \|x_1, x_2, ..., x_n\| \leq 1\} \\
&= \sup \{\| |T(x_1, x_2, ..., x_n)| \| : \|x_1, x_2, ..., x_n\| \leq 1\} \\
&\leq \sup \{\| |T|(|x_1|, |x_2|, ..., |x_n|) \| : \| |x_1|, |x_2|, ..., |x_n| \| \leq 1\} \\
&\leq \sup \{\| |T|(z_1, z_2, ..., z_n) \| : \|z_1, z_2, ..., z_n\| \leq 1\} \\
&= |||T|||_r.
\end{aligned}$$

$\square$

Theorem 3. *If X be n-Banach lattice and Y be Banach lattice with Y Dedekind complete, then $L_b(X^n, Y)$ is a Dedekind complete Banach lattice under the n-reguler norm.*

Proof. It's known that $L_b(X^n, Y)$ is Dedekind complete whenever Y is Dedekind complete [1, 2]. Let $\{T_m\}$ be a Cauchy sequence of $L_b(X^n, Y)$ with respect to the n-reguler norm. Then, there is a subsequence such that

$$|||T_{m+1} - T_m|||_r = |||\,|T_{m+1} - T_m|\,||| < 2^{-m}$$

for all $m \in \mathbb{N}$. Since $|||T_{m+1} - T_m||| \leq |||T_{m+1} - T_m|||_r$ holds for all $m \in \mathbb{N}$ from Theorem 2, $\{T_m\}$ be a Cauchy sequence of $L(X^n, Y)$. So there exists a $T \in L(X^n, Y)$ with $|||T_m - T||| \to 0$. Now let any $x = (x_1, x_2, ..., x_n) \in X^{n^+}$. Since

$$(T - T_m)(z) = \sum_{i=m}^{\infty} (T_{i+1} - T_i)(z) \preceq \sum_{i=m}^{\infty} |T_{i+1} - T_i|(x)$$

for all $z = (z_1, z_2, ..., z_n) \in X^n$ with $|z| \preceq x$, the modulus of $T - T_m$ exists and

$$(2.2) \qquad |T - T_m|(x) = \sup\{(T - T_m)(z) : |z| \preceq x\} \preceq \sum_{i=m}^{\infty} |T_{i+1} - T_i|(x)$$

holds for each $x \in X^{n^+}$. T is a reguler operator i.e. $T \in L_b(X^n, Y)$, because we can write $T = (T - T_1) + T_1$. From (2.2), $|||T_m - T|||_r \leq \sum_{i=m}^{\infty} |||T_{i+1} - T_i|||_r \leq 2^{1-m}$ and hence $|||T_m - T|||_r \to 0$. $\qquad\square$

The order bound norm of an order bounded linear operators between Banach lattices is defined by [10]. We will introduce a new norm using n-norm on the space $L_b(X^n, Y)$ whenever X be n-Banach lattice and Y be Banach lattice. For this, we need the following proposition.

Proposition 1. *If X be n-Banach lattice and Y be Banach lattice and $T : X^n \to Y$ is an order bounded operator then there exists a real constant λ such that for all $x = (x_1, x_2, ..., x_n) \in X^{n^+}$ there is $y \in Y^+$ such that $|(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$ implies $|T(z_1, z_2, ..., z_n)| \preceq y$ with $\|y\| \leq \lambda \|x_1, x_2, ..., x_n\|$.*

Proof. Since T is an order bounded, it's obvious that $|(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$ implies $|T(z_1, z_2, ..., z_n)| \preceq y$ for any $(x_1, x_2, ..., x_n) \in X^{n^+}$. We will show n-norm condition, for this assume to the contrary that this condition. In this case, we can find a sequence $\{x_k\}$ of X^+ with $\|x_k, e_1, ..., e_{n-1}\| = 1$ for all $k \in \mathbb{N}$ such that for any $y_k \in Y^+$ with $|(z_1, z_2, ..., z_n)| \preceq (x_k, e_1, ..., e_{n-1})$ implies $|T(z_1, z_2, ..., z_n)| \preceq y_k$ it's must be

$$\|y_k\| \geq k 2^k.$$

Since X is n-Banach space the series $\sum_{k=1}^{\infty} 2^{-k} x_k$ convergent, so we write $x = \sum_{k=1}^{\infty} 2^{-k} x_k$. Now, let $y \in Y^+$ such that

$$|(z_1, z_2, ..., z_n)| \preceq (x, e_1, ..., e_{n-1}) \text{ implies } |T(z_1, z_2, ..., z_n)| \preceq y.$$

If $|(z_1, z_2, ..., z_n)| \preceq (x_k, e_1, ..., e_{n-1})$ then

$$|(2^{-k} z_1, z_2, ..., z_n)| \preceq (2^{-k} x_k, e_1, ..., e_{n-1}) \preceq (x, e_1, ..., e_{n-1})$$

and hence $|T(z_1, z_2, ..., z_n)| \preceq 2^k y$. Therefore we obtain $\|2^k y\| \geq k 2^k$ and so

$$\|y\| \geq k$$

for all $k \in \mathbb{N}$. It's a contradiction so the proof is completed. $\qquad\square$

Definition 3. *If X is n-Banach lattice, Y is Banach lattice and $T : X^n \to Y$ is an order bounded operator then we define the n-order bound norm of T,*
$|||T|||_b = \inf\{\lambda \in \mathbb{R} : there \ exists \ y \in Y^+ such \ that \ |(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$
implies $|T(z_1, z_2, ..., z_n)| \preceq y$ *with* $\|y\| \leq \lambda \|x_1, x_2, ..., x_n\|$ *for all* $(x_1, x_2, ..., x_n) \in X^{n^+}\}$.

Theorem 4. *If $T \in L_r(X^n, Y)$, then $|||T||| \leq |||T|||_b \leq |||T|||_r$. Moreover, if Y is a Dedekind complete, then $|||T|||_r = |||T|||_b$ holds for all $T \in L_r(X^n, Y)$.*

Proof. Let any $(x_1, x_2, ..., x_n) \in X^{n^+}$ with $\|x_1, x_2, ..., x_n\| \leq 1$. We can write $\|z_1, z_2, ..., z_n\| \leq \|x_1, x_2, ..., x_n\| \leq 1$ for all $(z_1, z_2, ..., z_n) \in X^n$ satisfying the condition $|(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$ in definition of $|||.|||_b$ norm. If we consider that $|T(z_1, z_2, ..., z_n)| \preceq y$ and $\|y\| \leq \lambda \|x_1, x_2, ..., x_n\|$, then we obtain

$$\|T(z_1, z_2, ..., z_n)\| \leq \|y\| \leq \lambda.$$

Therefore we find $|||T||| \leq \lambda$, then

$$|||T||| \leq |||T|||_b$$

holds for all $T \in L_r(X^n, Y)$.

Now, let $S \in L_r(X^n, Y)$ with $\pm T \preceq S$. If we take $y = S(x_1, x_2, ..., x_n)$, then $|(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$ implies $|T(z_1, z_2, ..., z_n)| \preceq |T||(z_1, z_2, ..., z_n)| \preceq S(x_1, x_2, ..., x_n) = y$. Hence we obtain

$$\|y\| = \|S(x_1, x_2, ..., x_n)\| \leq |||S||| \|x_1, x_2, ..., x_n\|.$$

So, $|||S||| \in \inf\{\lambda \in \mathbb{R} : there \ exists \ y \in Y^+ such \ that \ |(z_1, z_2, ..., z_n)| \preceq (x_1, x_2, ..., x_n)$ implies $|T(z_1, z_2, ..., z_n)| \preceq y$ with $\|y\| \leq \lambda \|x_1, x_2, ..., x_n\|$ for any $(x_1, x_2, ..., x_n) \in X^{n^+}\}$. Then we get

$$|||T|||_b \leq |||T|||_r.$$

$\square$

It's known that if $T : X \to Y$ is an order bounded disjointness preserving operator, then $|T|$ exists and $|T|(x) = |T(x)|$ for all $x \in X^+$ [2]. By the aid of this information, we will get the following proposition.

Proposition 2. *If X is n-Banach lattice and Y is Banach lattice and $T, S : X^n \to Y$ are order bounded disjointness preserving operators then*

$$||| |T| - |S| ||| \leq 2 |||T - S|||.$$

Proof. Let $(x_1, x_2, ..., x_n) \in X^{n^+}$. Then

$$| |T|(x_1, x_2, ..., x_n) - |S|(x_1, x_2, ..., x_n) | = | |T(x_1, x_2, ..., x_n)| - |S(x_1, x_2, ..., x_n)| |$$
$$\preceq |(T - S)(x_1, x_2, ..., x_n)|$$

so we find

$$\||T|(x_1, x_2, ..., x_n) - |S|(x_1, x_2, ..., x_n)\| \leq \|(T - S)(x_1, x_2, ..., x_n)\|$$
$$\leq |||T - S||| \|x_1, x_2, ..., x_n\|.$$

For $(x_1, x_2, ..., x_n) \in X^n$, we write

$$| |T|(x_1, x_2, ..., x_n) - |S|(x_1, x_2, ..., x_n)|$$
$$\leq \left| |T|\left((x_1, x_2, ..., x_n)^+\right) - |S|\left((x_1, x_2, ..., x_n)^+\right) \right|$$
$$+ \left| |T|\left((x_1, x_2, ..., x_n)^-\right) - |S|\left((x_1, x_2, ..., x_n)^-\right) \right|$$

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hence we obtain

$$\begin{aligned}
\|(|T| - |S|)(x_1, x_2, ..., x_n)\| &\leq \|T - S\| \left(\|x_1^+, x_2^+, ..., x_n^+\| + \|x_1^-, x_2^-, ..., x_n^-\| \right) \\
&\leq 2 \|T - S\| \|x_1, x_2, ..., x_n\|.
\end{aligned}$$

The proof is completed. $\square$

We will obtain the corollary by using the preceding proposition as the method in [10].

Corollary 1. *If X be n-Banach lattice and Y be Banach lattice, for each $k \in \mathbb{N}$ $T_k : X^n \to Y$ is an order bounded disjointness preserving operator and $T : X^n \to Y$ is a bounded operator with $T_n \to T$ for the n-operator norm then T is an order bounded disjointness preserving operator and $|T_n| \to |T|$ for the n-operator norm.*

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[1]ONDOKUZ MAYIS UNIVERSITY, FACULTY OF SCIENCES AND ARTS, DEPART-MENT OF MATHEMATICS, 55139 KURUPELİT SAMSUN / TURKEY
    E-mail address: bduyar@omu.edu.tr

Current address: [2]GÜMÜŞHANE UNIVERSITY, FACULTY OF ENGINEERING, DEPART-MENT OF MATHEMATICAL ENGINEERING, 29100 GÜMÜŞHANE / TURKEY
    E-mail address: nihangungor@gumushane.edu.tr

Differential subordinations using Ruscheweyh derivative and a multiplier transformation

Alb Lupaş Alina

Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro

Abstract

In this paper the author derives several interesting differential subordination results. These subordinations are established by means of a differential operator obtained using Ruscheweyh derivative $R^m f(z)$ and the multiplier transformations $I(m, \lambda, l) f(z)$ namely

$RI^\alpha_{m,\lambda,l}$ the operator given by $RI^\alpha_{m,\lambda,l} : \mathcal{A}_n \to \mathcal{A}_n$, $RI^\alpha_{m,\lambda,l} f(z) = (1-\alpha)R^m f(z) + \alpha I(m, \lambda, l) f(z)$, for $z \in U$, $m \in \mathbb{N}$, $\lambda, l \geq 0$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}$,.

A number of interesting consequences of some of these subordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

Keywords: differential subordination, convex function, best dominant, differential operator.
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1   Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}$ and $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots, z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in \mathcal{A}_n : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$, the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \qquad z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Definition 1.1 *(Ruscheweyh [13]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = zf'(z), \ldots$$
$$(m+1) R^{m+1} f(z) = z (R^m f(z))' + m R^m f(z), \ z \in U.$$

Remark 1.1 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+2}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=n+1}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

Definition 1.2 *([3]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$, the operator $I(m, \lambda, l) f(z)$ is defined by the following infinite series*

$$I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1) + l + 1}{l+1} \right)^m a_j z^j.$$

Remark 1.2 *It follows from the above definition that*

$$I(0, \lambda, l) f(z) = f(z),$$

$$(l+1) I(m+1, \lambda, l) f(z) = (l+1-\lambda) I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))', \quad z \in U.$$

Remark 1.3 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [11], which is reduced to the Sălăgean differential operator [14] for $\lambda = 1$.*

Definition 1.3 *([6]) Let $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$. Denote by $RI_{m,\lambda,l}^\alpha$ the operator given by $RI_{m,\lambda,l}^\alpha : \mathcal{A}_n \to \mathcal{A}_n$,*

$$RI_{m,\lambda,l}^\alpha f(z) = (1-\alpha) R^m f(z) + \alpha I(m, \lambda, l) f(z), \quad z \in U.$$

Remark 1.4 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then*
$RI_{m,\lambda,l}^\alpha f(z) = z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j$, *for $z \in U$.*

Remark 1.5 *For $\alpha = 0$, $RI_{m,\lambda,l}^0 f(z) = R^m f(z)$, where $z \in U$ and for $\alpha = 1$, $RI_{m,\lambda,l}^1 f(z) = I(m, \lambda, l) f(z)$, where $z \in U$, which was studied in [2], [8].*
    For $l = 0$, we obtain $RI_{m,\lambda,0}^\alpha f(z) = RD_{1,\alpha}^m f(z)$ which was studied in [4], [5], [10] and for $l = 0$ and $\lambda = 1$, we obtain $RI_{m,1,0}^\alpha f(z) = L_\alpha^m f(z)$ which was studied in [1], [7].
    For $m = 0$, $RI_{0,\lambda,l}^\alpha f(z) = (1-\alpha) R^0 f(z) + \alpha I(0, \lambda, l) f(z) = f(z) = R^0 f(z) = I(0, \lambda, l) f(z)$, where $z \in U$.

Lemma 1.1 *(Miller and Mocanu [12]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.*
    If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z), \quad z \in U,$$

and this result is sharp.

Lemma 1.2 *(Hallenbeck and Ruscheweyh [12, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\operatorname{Re} \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$.

2   Main results

Theorem 2.1 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{nz}{\delta} g'(z)$, $z \in U$. If $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$\left(\frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \right)^{\delta-1} \left(RI_{m,\lambda,l}^\alpha f(z) \right)' \prec h(z), \quad z \in U, \tag{2.1}$$

then

$$\left(\frac{RI_{m,\lambda,l}^\alpha f(z)}{z} \right)^\delta \prec g(z), \quad z \in U,$$

and this result is sharp.

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Proof. Consider $p(z) = \left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta = \left(\frac{z+\sum_{j=n+1}^\infty \left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z}\right)^\delta = 1 + p_{\delta n}z^\delta +$

$p_{\delta n+1}z^{\delta+1} + ..., \quad z \in U.$

We deduce that $p \in \mathcal{H}[1, \delta n]$.

Differentiating we obtain $\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^\alpha_{m,\lambda,l}f(z)\right)' = p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$.

Then (2.1) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z) = g(z) + \frac{nz}{\delta}g'(z), \quad z \in U.$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta \prec g(z), \quad z \in U.$$

∎

Theorem 2.2 *Let h be an holomorphic function which satisfies the inequality $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^\alpha_{m,\lambda,l}f(z)\right)' \prec h(z), \quad z \in U, \tag{2.2}$$

then

$$\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Let

$$
\begin{aligned}
p(z) &= \left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta = \left(\frac{z+\sum_{j=n+1}^\infty\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^j}{z}\right)^\delta \\
&= \left(1 + \sum_{j=n+1}^\infty\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j-1}\right)^\delta = 1 + \sum_{j=n\delta+1}^\infty p_j z^{j-1},
\end{aligned}
$$

for $z \in U$, $p \in \mathcal{H}[1, n\delta]$.

Differentiating, we obtain $\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^\alpha_{m,\lambda,l}f(z)\right)' = p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$, and (2.2) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.2, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta \prec q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt, \quad z \in U,$$

and q is the best dominant. ∎

Corollary 2.3 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \leq \beta < 1$. If $\alpha, \delta, l, \lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^{\delta-1}\left(RI^\alpha_{m,\lambda,l}f(z)\right)' \prec h(z), \quad z \in U, \tag{2.3}$$

then

$$\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta \prec q(z), \quad z \in U,$$

where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}} \int_0^z \frac{t^{\frac{\delta}{n}-1}}{1+t} dt$, $z \in U$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.2 and considering $p(z) = \left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta$, the differential subordination (2.3) becomes

$$p(z) + \frac{z}{\delta}p'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1+z}, \quad z \in U.$$

By using Lemma 1.2 for $\gamma = \delta$, we have $p(z) \prec q(z)$, i.e.

$$\left(\frac{RI^\alpha_{m,\lambda,l}f(z)}{z}\right)^\delta \prec q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z h(t)t^{\frac{\delta}{n}-1}dt =$$

$$\frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z t^{\frac{\delta}{n}-1}\frac{1 + (2\beta - 1)t}{1+t}dt = \frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z \left[(2\beta - 1)t^{\frac{\delta}{n}-1} + 2(1-\beta)\frac{t^{\frac{\delta}{n}-1}}{1+t}\right]dt$$

$$= (2\beta - 1) + \frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}} \int_0^z \frac{t^{\frac{\delta}{n}-1}}{1+t}dt, \quad z \in U.$$

∎

Remark 2.1 *For $m = 1$, $n = 1$, $l = 2$, $\lambda = 1$, $\alpha = \frac{1}{2}$, $\delta = 1$ we obtain the same example as in [9, Example 5.2.1, p. 179].*

Theorem 2.4 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{nz}{\delta}g'(z)$, $z \in U$. If $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination*

$$z\frac{\delta+1}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} - 2\frac{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)'}{RI^\alpha_{m+1,\lambda,l}f(z)}\right] \prec h(z), \quad z \in U$$

(2.4)

holds, then

$$z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2} \prec g(z), \quad z \in U,$$

and this result is sharp.

Proof. Consider $p(z) = z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}$ and we obtain

$$p(z) + \frac{z}{\delta}p'(z) = z\frac{\delta+1}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} - 2\frac{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)'}{RI^\alpha_{m+1,\lambda,l}f(z)}\right].$$

Relation (2.4) becomes

$$p(z) + \frac{z}{\delta}p'(z) \prec h(z) = g(z) + \frac{nz}{\delta}g'(z), \quad z \in U.$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2} \prec g(z), \quad z \in U.$$

∎

Theorem 2.5 *Let h be an holomorphic function which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, *and $h(0)=1$. If $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and satisfies the differential subordination*

$$z\frac{\delta+1}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}-2\frac{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)'}{RI^\alpha_{m+1,\lambda,l}f(z)}\right]\prec h(z),\quad z\in U,$$

(2.5)

then

$$z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\prec q(z),\quad z\in U,$$

where $q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Let $p(z)=z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}$, $z\in U$, $p\in\mathcal{H}[1,n]$.

Differentiating, we obtain

$$p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}-2\frac{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)'}{RI^\alpha_{m+1,\lambda,l}f(z)}\right],\ z\in U,\text{ and}$$

(2.5) becomes

$$p(z)+\frac{z}{\delta}p'(z)\prec h(z),\quad z\in U.$$

Using Lemma 1.2, we have

$$p(z)\prec q(z),\quad z\in U,\quad\text{i.e.}\quad z\frac{RI^\alpha_{m,\lambda,l}f(z)}{\left(RI^\alpha_{m+1,\lambda,l}f(z)\right)^2}\prec q(z)=\frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt,\quad z\in U,$$

and q is the best dominant. ∎

Theorem 2.6 *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+\frac{nz}{\delta}g'(z)$, $z\in U$. If $\alpha,\lambda,l,\delta\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and the differential subordination*

$$z^2\frac{\delta+2}{\delta}\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{RI^\alpha_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\right)^2\right]\prec h(z),\quad z\in U$$

(2.6)

holds, then

$$z^2\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\prec g(z),\quad z\in U.$$

This result is sharp.

Proof. Let $p(z)=z^2\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}$. We deduce that $p\in\mathcal{H}[0,n]$.

Differentiating, we obtain

$$p(z)+\frac{z}{\delta}p'(z)=z^2\frac{\delta+2}{\delta}\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}+\frac{z^3}{\delta}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{RI^\alpha_{m,\lambda,l}f(z)}-\left(\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\right)^2\right],\ z\in U.$$

Using the notation in (2.6), the differential subordination becomes

$$p(z)+\frac{1}{\delta}zp'(z)\prec h(z)=g(z)+\frac{nz}{\delta}g'(z).$$

By using Lemma 1.1, we have

$$p(z)\prec g(z),\quad z\in U,\quad\text{i.e.}\quad z^2\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\prec g(z),\quad z\in U,$$

and this result is sharp. ∎

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Theorem 2.7 *Let h be an holomorphic function which satisfies the inequality* $\mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$,
and $h(0) = 1$. *If* $\alpha, \lambda, l, \delta \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and satisfies the differential subordination*

$$z^2 \frac{\delta+2}{\delta} \frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{RI^\alpha_{m,\lambda,l}f(z)} - \left(\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\right)^2\right] \prec h(z), \quad z \in U, \tag{2.7}$$

then

$$z^2 \frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt$.

Proof. Let $p(z) = z^2 \frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}$, $z \in U$, $p \in \mathcal{H}[0,n]$.

Differentiating, we obtain $p(z) + \frac{z}{\delta}p'(z) = z^2 \frac{\delta+2}{\delta} \frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{RI^\alpha_{m,\lambda,l}f(z)} - \left(\frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)}\right)^2\right]$, $z \in$

U, and (2.7) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.2, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad z^2 \frac{\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{RI^\alpha_{m,\lambda,l}f(z)} \prec q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}}\int_0^z h(t)t^{\frac{\delta}{n}-1}dt, \quad z \in U,$$

and q is the best dominant. ∎

Theorem 2.8 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + nzg'(z)$,
$z \in U$. If $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination*

$$1 - \frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^\alpha_{m,\lambda,l}f(z)\right)'\right]^2} \prec h(z), \quad z \in U \tag{2.8}$$

holds, then

$$\frac{RI^\alpha_{m,\lambda,l}f(z)}{z\left(RI^\alpha_{m,\lambda,l}f(z)\right)'} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \frac{RI^\alpha_{m,\lambda,l}f(z)}{z\left(RI^\alpha_{m,\lambda,l}f(z)\right)'}$. We deduce that $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain $1 - \frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)''}{\left[\left(RI^\alpha_{m,\lambda,l}f(z)\right)'\right]^2} = p(z) + zp'(z)$, $z \in U$.

Using the notation in (2.8), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + nzg'(z).$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI^\alpha_{m,\lambda,l}f(z)}{z\left(RI^\alpha_{m,\lambda,l}f(z)\right)'} \prec g(z), \quad z \in U,$$

and this result is sharp. ∎

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Theorem 2.9 *Let h be an holomorphic function which satisfies the inequality* $\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$. *If* $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and satisfies the differential subordination*

$$1 - \frac{RI_{m,\lambda,l}^{\alpha}f(z) \cdot \left(RI_{m,\lambda,l}^{\alpha}f(z)\right)''}{\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2} \prec h(z), \quad z \in U, \tag{2.9}$$

then

$$\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$.

Proof. Let $p(z) = \frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}$, $z \in U$, $p \in \mathcal{H}[0,n]$.

Differentiating, we obtain $1 - \frac{RI_{m,\lambda,l}^{\alpha}f(z) \cdot \left(RI_{m,\lambda,l}^{\alpha}f(z)\right)''}{\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2} = p(z) + zp'(z)$, $z \in U$, and (2.9) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.2, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt, \quad z \in U,$$

and q is the best dominant. ∎

Corollary 2.10 *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0 \leq \beta < 1$. *If* $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and satisfies the differential subordination*

$$1 - \frac{RI_{m,\lambda,l}^{\alpha}f(z) \cdot \left(RI_{m,\lambda,l}^{\alpha}f(z)\right)''}{\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2} \prec h(z), \quad z \in U, \tag{2.10}$$

then

$$\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'} \prec q(z), \quad z \in U,$$

where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$. *The function q is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z) = \frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}$, the differential subordination (2.10) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

By using Lemma 1.2 for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.

$$\frac{RI_{m,\lambda,l}^{\alpha}f(z)}{z\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt =$$

$$\frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{1+(2\beta-1)t}{1+t}t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \left[(2\beta-1)t^{\frac{1}{n}-1} + 2(1-\beta)\frac{t^{\frac{1}{n}-1}}{1+t}\right]dt$$

$$= (2\beta-1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U.$$

∎

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Example 2.1 Let $h(z) = \frac{1-z}{1+z}$ a convex function in U with $h(0) = 1$ and $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, $\alpha = \frac{1}{2}$, we obtain $RI_{1,1,2}^{\frac{1}{2}}f(z) = \frac{1}{2}R^1f(z) + \frac{1}{2}I(1,1,2)f(z) = \frac{1}{3}f(z) + \frac{2}{3}zf'(z) = z + \frac{5}{3}z^2$, $z \in U$.

Then $\left(RI_{1,1,2}^{\frac{1}{2}}f(z)\right)' = 1 + \frac{10}{3}z$ and $\left(RI_{1,1,2}^{\frac{1}{2}}f(z)\right)'' = \frac{10}{3}$,

$$\frac{RI_{1,1,2}^{\frac{1}{2}}f(z)}{z\left(RI_{1,1,2}^{\frac{1}{2}}f(z)\right)'} = \frac{z + \frac{5}{3}z^2}{z\left(1 + \frac{10}{3}z\right)} = \frac{3 + 5z}{3 + 10z},$$

$$1 - \frac{RI_{1,1,2}^{\frac{1}{2}}f(z)\cdot\left(RI_{1,1,2}^{\frac{1}{2}}f(z)\right)''}{\left[\left(RI_{1,1,2}^{\frac{1}{2}}f(z)\right)'\right]^2} = 1 - \frac{\left(z + \frac{5}{3}z^2\right)\cdot\frac{10}{3}}{\left(1 + \frac{10}{3}z\right)^2} = \frac{50z^2 + 30z + 9}{(3 + 10z)^2}.$$

We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain

$$\frac{50z^2 + 30z + 9}{(3 + 10z)^2} \prec \frac{1 - z}{1 + z}, \quad z \in U,$$

induce

$$\frac{3 + 5z}{3 + 10z} \prec -1 + \frac{2\ln(1 + z)}{z}, \quad z \in U.$$

Theorem 2.11 *Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + nzg'(z)$, $z \in U$. If $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination*

$$\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2 + RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'' \prec h(z), \quad z \in U \tag{2.11}$$

holds, then

$$\frac{RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}{z} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \frac{RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}{z}$. We deduce that $p \in \mathcal{H}[0, n]$.

Differentiating, we obtain $\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2 + RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'' = p(z) + zp'(z)$, $z \in U$.

Using the notation in (2.11), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + nzg'(z).$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}{z} \prec g(z), \quad z \in U,$$

and this result is sharp. ∎

Theorem 2.12 *Let h be an holomorphic function which satisfies the inequality $Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 0$. If $\alpha, \lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$\left[\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'\right]^2 + RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'' \prec h(z), \quad z \in U, \tag{2.12}$$

then

$$\frac{RI_{m,\lambda,l}^{\alpha}f(z)\cdot\left(RI_{m,\lambda,l}^{\alpha}f(z)\right)'}{z} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$.

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Proof. Let $p(z) = \frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{z}$, $z \in U$, $p \in \mathcal{H}[0,n]$.

Differentiating, we obtain $\left[\left(RI^\alpha_{m,\lambda,l}f(z)\right)'\right]^2 + RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'' = p(z) + zp'(z)$, $z \in U$, and (2.12) becomes

$$p(z) + zp'(z) \prec h(z), \qquad z \in U.$$

Using Lemma 1.2, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{z} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt, \quad z \in U,$$

and q is the best dominant. ∎

Corollary 2.13 Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$. If $\alpha, \lambda, l \ge 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left[\left(RI^\alpha_{m,\lambda,l}f(z)\right)'\right]^2 + RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'' \prec h(z), \qquad z \in U, \tag{2.13}$$

then

$$\frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{z} \prec q(z), \qquad z \in U,$$

where q is given by $q(z) = (2\beta-1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.12 and considering $p(z) = \frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{z}$, the differential subordination (2.13) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \qquad z \in U.$$

By using Lemma 1.2 for $\gamma = 1$, we have $p(z) \prec q(z)$, i.e.

$$\frac{RI^\alpha_{m,\lambda,l}f(z) \cdot \left(RI^\alpha_{m,\lambda,l}f(z)\right)'}{z} \prec q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt =$$

$$\frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{1+(2\beta-1)t}{1+t}t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \left[(2\beta-1)t^{\frac{1}{n}-1} + 2(1-\beta)\frac{t^{\frac{1}{n}-1}}{1+t}\right]dt$$

$$= (2\beta-1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U.$$

∎

Example 2.2 Let $h(z) = \frac{1-z}{1+z}$ a convex function in U with $h(0) = 1$ and $\text{Re}\left(\frac{zh''(z)}{h'(z)}+1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, $\alpha = \frac{1}{2}$, we obtain $RI^{\frac{1}{2}}_{1,1,2}f(z) = \frac{1}{2}R^1f(z) + \frac{1}{2}I(1,1,2)f(z) = \frac{1}{3}f(z) + \frac{2}{3}zf'(z) = z + \frac{5}{3}z^2$, $z \in U$.

Then $\left(RI^{\frac{1}{2}}_{1,1,2}f(z)\right)' = 1 + \frac{10}{3}z$, $\left(RI^{\frac{1}{2}}_{1,1,2}f(z)\right)'' = \frac{10}{3}$,

$\frac{RI^{\frac{1}{2}}_{1,1,2}f(z) \cdot \left(RI^{\frac{1}{2}}_{1,1,2}f(z)\right)'}{z} = \frac{\left(z+\frac{5}{3}z^2\right)\left(1+\frac{10}{3}z\right)}{z} = \frac{50}{9}z^2 + 5z + 1$,

$\left[\left(RI^{\frac{1}{2}}_{1,1,2}f(z)\right)'\right]^2 + RI^{\frac{1}{2}}_{1,1,2}f(z) \cdot \left(RI^{\frac{1}{2}}_{1,1,2}f(z)\right)'' = \left(1+\frac{10}{3}z\right)^2 + \left(z+\frac{5}{3}z^2\right)\cdot\frac{10}{3} = \frac{50}{3}z^2 + 10z + 1$.

We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

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Using Theorem 2.12 we obtain

$$\frac{50}{3}z^2 + 10z + 1 \prec \frac{1-z}{1+z}, \quad z \in U,$$

induce

$$\frac{50}{9}z^2 + 5z + 1 \prec -1 + \frac{2\ln(1+z)}{z}, \quad z \in U.$$

Theorem 2.14 *Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + \frac{nz}{1-\delta}g'(z)$, $z \in U$. If $\alpha, \lambda, l \geq 0$, $\delta \in (0,1)$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination*

$$\left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{1-\delta}\left(\frac{\left(RI_{m+1,\lambda,l}^\alpha f(z)\right)'}{RI_{m+1,\lambda,l}^\alpha f(z)} - \delta\frac{\left(RI_{m,\lambda,l}^\alpha f(z)\right)'}{RI_{m,\lambda,l}^\alpha f(z)}\right) \prec h(z), \quad z \in U \qquad (2.14)$$

holds, then

$$\frac{RI_{m+1,\lambda,l}^\alpha f(z)}{z} \cdot \left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{z} \cdot \left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta$. We deduce that $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain $\left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{1-\delta}\left(\frac{\left(RI_{m+1,\lambda,l}^\alpha f(z)\right)'}{RI_{m+1,\lambda,l}^\alpha f(z)} - \delta\frac{\left(RI_{m,\lambda,l}^\alpha f(z)\right)'}{RI_{m,\lambda,l}^\alpha f(z)}\right) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$.

Using the notation in (2.14), the differential subordination becomes

$$p(z) + \frac{1}{1-\delta}zp'(z) \prec h(z) = g(z) + \frac{nz}{1-\delta}g'(z).$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{z} \cdot \left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \prec g(z), \quad z \in U,$$

and this result is sharp. ∎

Theorem 2.15 *Let h be an holomorphic function which satisfies the inequality $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha, \lambda, l \geq 0$, $\delta \in (0,1)$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination*

$$\left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{1-\delta}\left(\frac{\left(RI_{m+1,\lambda,l}^\alpha f(z)\right)'}{RI_{m+1,\lambda,l}^\alpha f(z)} - \delta\frac{\left(RI_{m,\lambda,l}^\alpha f(z)\right)'}{RI_{m,\lambda,l}^\alpha f(z)}\right) \prec h(z), \quad z \in U, \qquad (2.15)$$

then

$$\frac{RI_{m+1,\lambda,l}^\alpha f(z)}{z} \cdot \left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}}\int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt$.

Proof. Let $p(z) = \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{z} \cdot \left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta$, $z \in U$, $p \in \mathcal{H}[0,n]$.

Differentiating, we obtain $\left(\frac{z}{RI_{m,\lambda,l}^\alpha f(z)}\right)^\delta \frac{RI_{m+1,\lambda,l}^\alpha f(z)}{1-\delta}\left(\frac{\left(RI_{m+1,\lambda,l}^\alpha f(z)\right)'}{RI_{m+1,\lambda,l}^\alpha f(z)} - \delta\frac{\left(RI_{m,\lambda,l}^\alpha f(z)\right)'}{RI_{m,\lambda,l}^\alpha f(z)}\right) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$, and (2.15) becomes

$$p(z) + \frac{1}{1-\delta}zp'(z) \prec h(z), \quad z \in U.$$

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Using Lemma 1.2, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{RI^{\alpha}_{m+1,\lambda,l}f(z)}{z} \cdot \left(\frac{z}{RI^{\alpha}_{m,\lambda,l}f(z)} \right)^{\delta} \prec q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t)t^{\frac{1-\delta}{n}-1}dt, \quad z \in U,$$

and q is the best dominant. ∎

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On a certain subclass of analytic functions involving generalized Sălăgean operator and Ruscheweyh derivative

Alina Alb Lupaş[1] and Loriana Andrei[2]
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
[1]dalb@uoradea.ro, [2]lori_andrei@yahoo.com

Abstract

The main object of this paper is to study some properties of certain subclass of analytic functions in the open unit disc which is defined by the linear operator $RD_{\lambda,\alpha}^n : \mathcal{A} \to \mathcal{A}$, $RD_{\lambda,\alpha}^n f(z) = (1-\alpha)R^n f(z) + \alpha D_\lambda^n f(z)$, $z \in U$, where $R^n f(z)$ is the Ruscheweyh derivative, $D_\lambda^n f(z)$ the generalized Sălăgean operator and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. These properties include a coefficient inequality, distorsion theorem and extreme points of differential operator. We also discuss the boundedness properties associated with partial sums of functions in the class $\mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$.

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1   Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting the functions f of the form $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $z \in U$.

Definition 1.1 *(Al Oboudi [9]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
D_\lambda^0 f(z) &= f(z) \\
D_\lambda^1 f(z) &= (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \ \ldots \\
D_\lambda^{n+1} f(z) &= (1-\lambda) D_\lambda^n f(z) + \lambda z (D_\lambda^n f(z))' = D_\lambda (D_\lambda^n f(z)), \quad z \in U.
\end{aligned}
$$

Remark 1.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j$, $z \in U$.*
If $f \in \mathcal{T}$ and $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^n f(z) = z - \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j$, $z \in U$.

Remark 1.2 *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [14].*

Definition 1.2 *(Ruscheweyh [13]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
R^0 f(z) &= f(z) \\
R^1 f(z) &= z f'(z), \ \ldots \\
(n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U.
\end{aligned}
$$

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.*

If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z - \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.3 *[4] Let $\gamma, \alpha \geq 0$, $n \in \mathbb{N}$. Denote by $RD_{\lambda,\alpha}^n$ the operator given by $RD_{\lambda,\alpha}^n : \mathcal{A} \to \mathcal{A}$,*

$$
RD_{\lambda,\alpha}^n f(z) = (1-\alpha)R^n f(z) + \alpha D_\lambda^n f(z), \quad z \in U.
$$

Remark 1.4 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then*

$$RD_{\lambda,\alpha}^n f(z) = z + \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j, \ z \in U.$$

This operator was studied also in [5], [7], [8], [10], [11].
If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, *then*

$$RD_{\lambda,\alpha}^n f(z) = z - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j, \ z \in U.$$

Remark 1.5 *For* $\gamma = 0$, $RD_{\lambda,0}^n f(z) = R^n f(z)$, *where* $z \in U$ *and for* $\gamma = 1$, $RD_{\lambda,1}^n f(z) = D_\lambda^n f(z)$, *where* $z \in U$.
For $\lambda = 1$, *we obtain* $RD_{1,\alpha}^n f(z) = L_\alpha^n f(z)$ *which was studied in* [2], [3] *and* [6].

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.
We follow the works of A. Abubaker and M. Darus [1].

Definition 1.4 *Let* $0 < \beta \leq 1$, $\alpha, \lambda \geq 0$, $n \in \mathbb{N}$, $\gamma \in \mathbb{C} \backslash \{0\}$. *Then, the function* $f \in \mathcal{A}$ *is siad to be in the class* $\mathcal{S}_{\lambda,\alpha}^n (\beta, \gamma)$ *if*

$$\left| \frac{1}{\gamma} \left(\frac{z \left(RD_{\lambda,\alpha}^n f(z) \right)'}{RD_{\lambda,\alpha}^n f(z)} - 1 \right) \right| < \beta, \quad z \in U.$$

We define now the class $\mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$ *by*

$$\mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma) = \mathcal{S}_{\lambda,\alpha}^n (\beta, \gamma) \cap \mathcal{T}.$$

2 Coefficient Inequality

Theorem 2.1 *Let the function* $f \in \mathcal{T}$. *Then* $f(z)$ *is in the class* $\mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$ *if and only if*

$$\sum_{j=2}^{\infty} (j - 1 + \beta|\gamma|) \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \leq \beta|\gamma|, \tag{2.1}$$

where $\gamma \in \mathbb{C} - \{0\}$, $\lambda, \alpha \geq 0$, $n \in \mathbb{N}$, $z \in U$.

Proof. Let $f(z) \in \mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$. Then, we have

$$Re\{ \frac{z(RD_{\lambda,\gamma}^n f(z))'}{RD_{\lambda,\gamma}^n f(z)} - 1|\} > -\beta|\gamma|$$

$$\frac{z(RD_{\lambda,\gamma}^n f(z))'}{RD_{\lambda,\gamma}^n f(z)} = \frac{z - \sum_{j=2}^{\infty} j \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j}{z - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j}$$

$$Re \left\{ \frac{-\sum_{j=2}^{\infty} (j-1) \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j}{z - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j} \right\} > -\beta |\gamma|$$

By chossing the valkues of z on the real ais and letting $z \to 1^-$ through real values, the above inequality immediately yields the required condition (2.1).
Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we obtain

$$\left| \frac{z(RD_{\lambda,\gamma}^n f(z))'}{RD_{\lambda,\gamma}^n f(z)} - 1 \right| = \left| \frac{\sum_{j=2}^{\infty} (j-1) \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j}{z - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j z^j} \right|$$

$$\leq \frac{\beta |\gamma| \left(1 - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \right)}{1 - \sum_{j=2}^{\infty} \left\{ \alpha \left[1 + (j-1)\lambda\right]^n + (1-\alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j} \leq \beta|\gamma|.$$

2

Hence, by maximum modulus theorem, we have $f \in \mathcal{TS}^n_{\lambda,\alpha}(\beta, \gamma)$, which evidently completes the proof of theorem.

Finally, the result is sharp with the extremal functions f_j be in the class $\mathcal{TS}^n_{\lambda,\alpha}(\beta, \gamma)$ given by

$$f_j(z) = z - \frac{\beta \, |\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha\left[1+(j-1)\lambda\right]^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad \text{for } j \geq 2. \tag{2.2}$$

■

Corollary 2.2 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{TS}^n_{\lambda,\alpha}(\beta, \gamma)$. Then, we have*

$$a_j \leq \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha\left[1+(j-1)\lambda\right]^n + (1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad \text{for } j \geq 2. \tag{2.3}$$

The equality is attained for the functions f given by (2.2).

3   Growth and distortion theorems

Theorem 3.1 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{TS}^n_{\lambda,\alpha}(\beta, \gamma)$. Then for $|z| = r$, we have*

$$r - \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2 \leq |f(z)| \leq r + \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2.$$

with equality for

$$f(z) = z - \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} z^2.$$

Proof. In view of Theorem 2.1, we have

$$(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]\sum_{j=2}^{\infty} a_j \leq$$

$$\sum_{j=2}^{\infty}(j-1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right] a_j \leq \beta|\gamma|.$$

Hence

$$|f(z)| \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} a_j \leq r + \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2$$

and

$$|f(z)| \geq r - \sum_{j=2}^{\infty} a_j r^j \geq r - r^2 \sum_{j=2}^{\infty} a_j \geq r - \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2.$$

This completes the proof. ■

Theorem 3.2 *Let the function $f \in \mathcal{T}$ be in the class $\mathcal{TS}^n_{\lambda,\alpha}(\beta, \gamma)$. Then, for $|z| = r$ we have*

$$r - \frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2 \leq |f'(z)| \leq r + \frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2$$

with equality for

$$f(z) = z - \frac{\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} z^2.$$

Proof. We have

$$|f'(z)| \leq r + \sum_{j=2}^{\infty} j a_j r^j \leq r + 2r^2 \sum_{j=2}^{\infty} a_j \leq r + \frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2$$

and

$$|f'(z)| \geq r - \sum_{j=2}^{\infty} j a_j r^j \geq r - 2r^2 \sum_{j=2}^{\infty} a_j \geq r - \frac{2\beta|\gamma|}{(1+\beta|\gamma|)\left[\alpha\left(1+\lambda\right)^n + (1-\alpha)(n+1)\right]} r^2,$$

which completes the proof. ■

4   Extreme points

The extreme points of the class $\mathcal{TS}^n_{\lambda,\alpha}(\beta,\gamma)$ will be now determined.

Theorem 4.1 *Let* $f_1(z) = z$ *and* $f_j(z) = z - \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j$, $j \geq 2$.
Assume that f *is analytic in* U. *Then* $f \in \mathcal{TS}^n_{\lambda,\alpha}(\beta,\gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{j=1}^{\infty} \mu_j f_j(z),$$

where $\mu_j \geq 0$ *and* $\sum_{j=1}^{\infty} \mu_j = 1$.

Proof. Suppose that $f(z) = \sum_{j=1}^{\infty} \mu_j f_j(z)$ with $\mu_j \geq 0$ and $\sum_{j=1}^{\infty} \mu_j = 1$. Then

$$f(z) = \sum_{j=1}^{\infty} \mu_j f_j(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) =$$

$$\mu_1 z + \sum_{j=2}^{\infty} \mu_j \left(z - \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j \right) =$$

$$z - \sum_{j=2}^{\infty} \mu_j \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j.$$

Then

$$\sum_{j=2}^{\infty} \mu_j \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \frac{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|\gamma|} =$$

$$\sum_{j=2}^{\infty} \mu_j = \sum_{j=1}^{\infty} \mu_j - \mu_1 = 1 - \mu_1 \leq 1.$$

Thus $f \in \mathcal{TS}^n_{\lambda,\alpha}(\beta,\gamma)$ by Theorem 2.1.

Conversely, suppose that $f \in \mathcal{TS}^n_{\lambda,\alpha}(\beta,\gamma)$. By using (2.3) we may set and

$$\mu_j = \frac{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|\gamma|}a_j$$

for $j \geq 2$ and $\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j$.

Then

$$f(z) = z - \sum_{j=1}^{\infty} a_j z^j = z - \sum_{j=2}^{\infty} \mu_j \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j =$$

$$\mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) = \sum_{j=1}^{\infty} \mu_j f_j(z),$$

with $\mu_j \geq 0$ and $\sum_{j=1}^{\infty} \mu_j = 1$, which completes the proof. ∎

Corollary 4.2 *The extreme points of* $\mathcal{TS}^n_{\lambda,\alpha}(\beta,\gamma)$ *are the functions* $f_1(z) = z$ *and*

$$f_j(z) = \frac{\beta|\gamma|}{(j-1+\beta|\gamma|)\left\{\alpha[1+(j-1)\lambda]^n+(1-\alpha)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j, \ for \ j \geq 2.$$

4

5   Partial sums

We investigate in this section the partial sums of functions in the class $\mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$. We shall obtain sharp lower bounds for the real part of its ratios. We shall follow similar works done by Silverman [15] and Khairnar and Moreon [12] about the partial sums of analytic functions. In what follows, we will use the well known result that for an analytic function ω in U,

$$\text{Re} \left(\frac{1 + \omega(z)}{1 - \omega(z)} \right) > 0, \quad z \in U,$$

if and only if the inequality $|\omega(z)| < 1$ is satisfied.

Theorem 5.1 *Let* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in \mathcal{TS}_{\lambda,\alpha}^n (\beta, \gamma)$ *and define its partial sums by* $f_1(z) = z$ *and* $f_m(z) = z - \sum_{j=2}^{m} a_j z^j$, $m \geq 2$. *Then*

$$\text{Re} \left(\frac{f(z)}{f_m(z)} \right) \geq 1 - \frac{1}{c_{m+1}}, \quad z \in U, \, m \in \mathbb{N}, \tag{5.1}$$

and

$$\text{Re} \left(\frac{f_m(z)}{f(z)} \right) \geq \frac{c_{m+1}}{1 + c_{m+1}}, \quad z \in U, \, m \in \mathbb{N}, \tag{5.2}$$

where

$$c_j = \frac{(j + \beta|\gamma|) \left\{ \alpha \left[1 + (j-1)\lambda \right]^n + (1 - \alpha) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|\gamma|} \tag{5.3}$$

This estimates in (5.1) and (5.2) are sharp.

Proof. To prove (5.1), it suffices to show that

$$c_{m+1} \left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{c_{m+1}} \right) \right) \prec \frac{1 + z}{1 - z}, \quad z \in U.$$

By the subordination property, we can write

$$\frac{1 - \sum_{j=2}^{m} a_j z^{j-1} - c_{m+1} \sum_{j=m+1}^{\infty} a_j z^{j-1}}{1 - \sum_{j=2}^{m} a_j z^{j-1}} = \frac{1 + \omega(z)}{1 - \omega(z)}$$

for certain ω analytic in U with $|\omega(z)| \leq 1$. Notice that $\omega(0) = 0$ and

$$|\omega(z)| \leq \frac{c_{m+1} \sum_{j=m+1}^{\infty} a_j}{2 - 2\sum_{j=2}^{m} a_j - c_{m+1} \sum_{j=m+1}^{\infty} a_j}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{j=2}^{m} a_j + c_{m+1} \sum_{j=m+1}^{\infty} a_j \leq \sum_{j=2}^{\infty} c_j a_j \leq 1.$$

The above inequality holds because c_j is a non-decreasing sequence. This completes the proof of (5.1). Finally, it is observed that equality in (5.1) is attained for the function given by (2.3) when $z = re^{2\frac{\pi i}{m}}$ as $r \to 1^-$. Similarly, we take

$$(1 + c_{m+1}) \left(\frac{f_m(z)}{f(z)} - \frac{c_{m+1}}{1 + c_{m+1}} \right) = \frac{1 - \sum_{j=2}^{m} a_j z^{j-1} + c_{m+1} \sum_{j=m+1}^{\infty} a_j z^{j-1}}{1 - \sum_{j=2}^{m} a_j z^{j-1}} = \frac{1 + \omega(z)}{1 - \omega(z)},$$

where

$$|\omega(z)| \leq \frac{(1 + c_{m+1}) \sum_{j=m+1}^{\infty} a_j}{2 - 2\sum_{j=2}^{m} a_j + (1 - c_{m+1}) \sum_{j=m+1}^{\infty} a_j}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{j=2}^{m} a_j + c_{m+1} \sum_{j=m+1}^{\infty} a_j \leq \sum_{j=2}^{\infty} c_j a_j \leq 1.$$

This immediately leads to assertion (5.2) of Theorem 5.1. This completes the proof. ∎

Using a similar method, we can prove the following theorem.

Theorem 5.2 *If $f \in \mathcal{TS}_{\lambda,\alpha}^n(\beta,\gamma)$ and define the partial sums by $f_1(z) = z$ and $f_m(z) = z - \sum_{j=2}^{m} a_j z^j$. Then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1 - \frac{m+1}{c_{m+1}}, \quad z \in U,\ m \in \mathbb{N},$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f(z)}\right) \geq \frac{c_{m+1}}{m+1+c_{m+1}},$$

where c_j is given by (5.3). The result is sharp for every m, with extremal functions given by (2.2).

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ON THE $(r,s)−$CONVEXITY AND SOME HADAMARD-TYPE INEQUALITIES

▼M. EMIN ÖZDEMIR, ★ERHAN SET, AND ▼AHMET OCAK AKDEMIR

ABSTRACT. In this paper, we define a new class of convex functions which is called $(r,s)−$convex functions. We also prove some Hadamard's type inequalities related to this new class of functions.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then, the following double inequality is well known in literature as Hermite-Hadamard inequality holds for convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [1], Pearce *et al.* generalized this inequality to $r−$convex positive function f which is defined on an interval $[a, b]$, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$;

$$f(\lambda x + (1-\lambda)y) \leq \begin{cases} \left(\lambda [f(x)]^r + (1-\lambda)[f(y)]^r\right)^{\frac{1}{r}}, & if\ r \neq 0 \\ [f(x)]^\lambda [f(y)]^{1-\lambda}, & if\ r = 0 \end{cases}.$$

Obviously, one can see that $0−$convex functions are simply $\log −$convex functions and $1−$convex functions are ordinary convex functions. Another inequality which well known in the literature as Minkowski Inequality is given as following;

Let $p \geq 1$, $0 < \int_a^b f(x)^p dx < \infty$, and $0 < \int_a^b g(x)^p dx < \infty$. Then

$$(1.1) \qquad \left(\int_a^b (f(x)+g(x))^p\, dx\right)^{\frac{1}{p}} \leq \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} + \left(\int_a^b g(x)^p dx\right)^{\frac{1}{p}}.$$

Definition 1. *[See [4]] A function $f : I \to [0, \infty)$ is said to be $\log−$convex or multiplicatively convex if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:*

$$(1.2) \qquad f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

We note that a $\log −$convex function is convex, but the converse may not necessarily be true.

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In [6], Gill *et al.* proved following inequality for $r-$convex functions.

Theorem 1. *Suppose f is a positive $r-$convex function on $[a, b]$. Then*

$$(1.3) \qquad \frac{1}{b-a} \int_a^b f(t)dt \leq L_r\left(f(a), f(b)\right).$$

If f is a positive $r-$concave function, then the inequality is reversed.

For recent results on $r-$convexity see the papers [2], [5] and [6].

Definition 2. *[See [3]] Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s-$convex in the second sense if*

$$(1.4) \qquad f\left(tx + (1-t)y\right) \leq t^s f\left(x\right) + (1-t)^s f\left(y\right)$$

for all $x, y \in \mathbb{R}_+$ and $t \in [0, 1]$.

In [10], $s-$convexity introduced by Breckner as a generalization of convex functions. Also, Breckner proved the fact that the set valued map is $s-$convex only if the associated support function is $s-$convex function in [11]. Several properties of $s-$convexity in the first sense are discussed in the paper [3]. Obviously, $s-$convexity means just convexity when $s = 1$.

Theorem 2. *[See [8]] Suppose that $f : [0, \infty) \to [0, \infty)$ is an $s-$convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[0, 1]$, then the following inequalities hold:*

$$(1.5) \qquad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f\left(x\right) dx \leq \frac{f\left(a\right) + f\left(b\right)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.5). The above inequalities are sharp.

Some new Hermite-Hadamard type inequalities based on concave functions and $s-$convexity established by Kırmacı *et al.* in [9]. For related results see the papers [7], [8] and [9].

The main purpose of this paper is to give definition of $(r, s)-$convexity and to prove some Hadamard-type inequalities for $(r, s)-$convex functions.

2. MAIN RESULTS

Definition 3. *A positive function f is $(r, s)-$convex on $[a, b] \subset [0, \infty)$ if for all $x, y \in [a, b]$, $s \in (0, 1]$ and $\lambda \in [0, 1]$*

$$f(\lambda x + (1-\lambda)y) \leq \begin{cases} \left(\lambda^s f^r(x) + (1-\lambda)^s f^r(y)\right)^{\frac{1}{r}}, & if \ r \neq 0 \\ f^\lambda(x) f^{1-\lambda}(y) & , \ if \ r = 0 \end{cases}.$$

This definition of $(r, s)-$convexity naturally complements the concept of $(r, s)-$concave functions in which the inequality is reversed.

Remark 1. *We have that $(0, s)-$convex functions are simply $\log-$convex functions and $(1, 1)-$convex functions are ordinary convex functions.*

Remark 2. *We have that $(r, 1)-$convex functions are $r-$convex functions.*

Remark 3. *We have that $(1, s)-$convex functions are $s-$convex functions.*

Now, we will prove inequalities based on above definition.

Theorem 3. *Let* $f : [a, b] \subset [0, \infty) \rightarrow (0, \infty)$ *be* $(r, s)-$*convex function on* $[a, b]$ *with* $a < b$. *Then the following inequality holds;*

$$(2.1) \qquad \frac{1}{b-a} \int_a^b f(x)dx \leq \left(\frac{r}{r+s} \right) [f^r(a) + f^r(b)]^{\frac{1}{r}}$$

for $r, s \in (0, 1]$.

Proof. Since f is $(r, s)-$convex function and $r > 0$, we can write

$$f(ta + (1-t)b) \leq (t^s [f(a)]^r + (1-t)^s [f(b)]^r)^{\frac{1}{r}}$$

for all $t \in [0, 1]$ and $s \in (0, 1]$. It is easy to observe that

$$\frac{1}{b-a} \int_a^b f(x)dx \quad = \quad \int_0^1 f(ta + (1-t)b)dt$$

$$\leq \quad \int_0^1 (t^s [f(a)]^r + (1-t)^s [f(b)]^r)^{\frac{1}{r}} dt.$$

Using the inequality (1.1), we get

$$\int_0^1 (t^s [f(a)]^r + (1-t)^s [f(b)]^r)^{\frac{1}{r}} dt \quad \leq \quad \left[\left(\int_0^1 t^{\frac{s}{r}} f(a)dt \right)^r + \left(\int_0^1 (1-t)^{\frac{s}{r}} f(b)dt \right)^r \right]^{\frac{1}{r}}$$

$$= \quad \left[\left(\frac{r}{r+s} \right)^r (f^r(a) + f^r(b)) \right]^{\frac{1}{r}}$$

$$= \quad \left(\frac{r}{r+s} \right) [f^r(a) + f^r(b)]^{\frac{1}{r}}.$$

Thus

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \left(\frac{r}{r+s} \right) [f^r(a) + f^r(b)]^{\frac{1}{r}}$$

which completes the proof. $\qquad \square$

Corollary 1. *In Theorem 3, if we choose* $(1, s)-$*convex function on* $[a, b]$ *with* $a < b$. *Then, we have the following inequality;*

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \left(\frac{1}{s+1} \right) [f(a) + f(b)].$$

Corollary 2. *In Theorem 3, if we choose* $(r, 1)-$*convex function on* $[a, b]$ *with* $a < b$. *Then, we have the following inequality;*

$$\frac{1}{b-a} \int_a^b f(x)dx \leq \left(\frac{r}{r+1} \right) [f^r(a) + f^r(b)]^{\frac{1}{r}}.$$

Remark 4. *In Theorem 3, if we choose* $(1, 1)-$*convex function on* $[a, b] \subset [0, \infty)$ *with* $a < b$. *Then, we have the right hand side of Hadamard's inequality.*

Theorem 4. *Let $f, g : [a, b] \subset [0, \infty) \to (0, \infty)$ be $(r_1, s)-convex$ and $(r_2, s)-convex$ function on $[a, b]$ with $a < b$. Then the following inequality holds;*
(2.2)
$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \left(\frac{1}{s+1}\right) ([f(a)]^{r_1} + [f(b)]^{r_1})^{\frac{1}{r_1}} ([g(a)]^{r_2} + [g(b)]^{r_2})^{\frac{1}{r_2}}$$
for $r_1 > 1$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

Proof. Since f is $(r_1, s)-$convex function and g is $(r_2, s)-$convex function, we have
$$f(ta + (1-t)b) \leq (t^s [f(a)]^{r_1} + (1-t)^s [f(b)]^{r_1})^{\frac{1}{r_1}}$$
and
$$g(ta + (1-t)b) \leq (t^s [g(a)]^{r_2} + (1-t)^s [g(b)]^{r_2})^{\frac{1}{r_2}}$$
for all $t \in [0, 1]$ and $r_1, r_2, s \in (0, 1]$. Since f and g are non-negative functions, hence
$$f(ta + (1-t)b)g(ta + (1-t)b)$$
$$\leq (t^s [f(a)]^{r_1} + (1-t)^s [f(b)]^{r_1})^{\frac{1}{r_1}} (t^s [g(a)]^{r_2} + (1-t)^s [g(b)]^{r_2})^{\frac{1}{r_2}}.$$
Integrating both sides of the above inequality over $[0, 1]$ with respect to t, we obtain
$$\int_0^1 [f(ta + (1-t)b)g(ta + (1-t)b)] dt$$
$$\leq \int_0^1 \left[(t^s [f(a)]^{r_1} + (1-t)^s [f(b)]^{r_1})^{\frac{1}{r_1}} (t^s [g(a)]^{r_2} + (1-t)^s [g(b)]^{r_2})^{\frac{1}{r_2}}\right] dt.$$
By applying the Hölder's inequality, we have
$$\int_0^1 \left[(t^s [f(a)]^{r_1} + (1-t)^s [f(b)]^{r_1})^{\frac{1}{r_1}} (t^s [g(a)]^{r_2} + (1-t)^s [g(b)]^{r_2})^{\frac{1}{r_2}}\right] dt$$
$$\leq \left[\int_0^1 (t^s [f(a)]^{r_1} + (1-t)^s [f(b)]^{r_1} dt)\right]^{\frac{1}{r_1}} \left[\int_0^1 (t^s [g(a)]^{r_2} + (1-t)^s [g(b)]^{r_2} dt)\right]^{\frac{1}{r_2}}$$
$$= \left(\frac{1}{s+1}\right) ([f(a)]^{r_1} + [f(b)]^{r_1})^{\frac{1}{r_1}} ([g(a)]^{r_2} + [g(b)]^{r_2})^{\frac{1}{r_2}}.$$
By using the fact that
$$\frac{1}{b-a} \int_a^b f(x)g(x)dx = \int_0^1 [f(ta + (1-t)b)g(ta + (1-t)b)] dt.$$
We obtain the desired result. □

Corollary 3. *In Theorem 4, if we choose $s = 1$, $r_1 = r_2 = 2$ and $f(x) = g(x)$, we have the following inequality;*
$$\frac{1}{b-a} \int_a^b f(x)^2 dx \leq \frac{[f(a)]^2 + [f(b)]^2}{2}.$$

Corollary 4. *In Theorem 4, if we choose $s = 1$ and $r_1 = r_2 = 2$, we have the following inequality;*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \sqrt{\frac{[f(a)]^2 + [f(b)]^2}{2}} \sqrt{\frac{[g(a)]^2 + [g(b)]^2}{2}}.$$

Theorem 5. *Let $f : [a,b] \subset [0,\infty) \to (0,\infty)$ be a $(r,s)-$convex function on $[a,b]$ with $r, s \in (0,1]$. If $f \in L_1[a,b]$, then one has the following inequalities;*

$$(2.3) \qquad 2^{\frac{s}{r}-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f^r(x)dx \leq \frac{f^r(a) + f^r(b)}{s+1}.$$

Proof. By the $(r,s)-$convexity of f, we have that

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2^{\frac{s}{r}}} \left[f^r(x) + f^r(y)\right]$$

for all $x, y \in [a,b]$. If we take $x = ta + (1-t)b$ and $y = (1-t)a + tb$, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\frac{s}{r}}} \left[f^r(ta + (1-t)b) + f^r((1-t)a + tb)\right]$$

for all $t \in [0,1]$. Integrating the above inequality over $[0,1]$ with respect to t, we get

$$(2.4) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\frac{s}{r}}} \left[\int_0^1 f^r(ta + (1-t)b)dt + \int_0^1 f^r((1-t)a + tb)\,dt\right].$$

From the facts that

$$\int_0^1 f^r(ta + (1-t)b)dt = \frac{1}{b-a} \int_a^b f^r(x)dx$$

and

$$\int_0^1 f^r((1-t)a + tb)\,dt = \frac{1}{b-a} \int_a^b f^r(x)\,dx$$

we obtain the left hand side of the (2.3).

By the $(r,s)-$convexity of f, we also have that

$$(2.5) \qquad f(ta + (1-t)b) + f(tb + (1-t)a)$$
$$\leq \quad t^s f^r(a) + (1-t)^s f^r(b) + t^s f^r(b) + (1-t)^s f^r(a)$$

for all $t \in [0,1]$. Integrating the inequality (2.5) over $[0,1]$ with respect to t, we get

$$\frac{1}{b-a} \int_a^b f^r(x)dx \leq \frac{f^r(a) + f^r(b)}{s+1}.$$

which completes the proof. □

Remark 5. *In Theorem 5, if we choose $r = 1$, we have the inequality (1.5).*

Remark 6. *In Theorem 5, if we choose $s = r = 1$, we have the Hadamard's inequality.*

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▾Ataturk University, K. K. Education Faculty, Department of Mathematics, 25640, Kampus, Erzurum, Turkey
    E-mail address: emos@atauni.edu.tr

★Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu / Turkey
    E-mail address: erhanset@yahoo.com

▾Aǧri İbrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, 04100, Aǧri, Turkey
    E-mail address: ahmetakdemir@agri.edu.tr

COMPACT COMPOSITION OPERATORS ON WEIGHTED HILBERT SPACES

WALEED AL-RAWASHDEH

ABSTRACT. Let φ be an analytic self-map of open unit disk $\mathbb{D}$. A composition operator is defined as $(C_\varphi f)(z) = f(\varphi(z))$, for $z \in \mathbb{D}$ and f analytic on $\mathbb{D}$. Given an admissible weight ω, the weighted Hilbert space $\mathcal{H}_\omega$ consists of all analytic functions f such that $\|f\|^2_{\mathcal{H}_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 w(z) dA(z)$ is finite. In this paper, we study composition operators acting between weighted Bergman space A^2_α and the weighted Hilbert space $\mathcal{H}_\omega$. Using generalized Nevalinna counting functions associated with ω, we characterize the boundedness and compactness of these composition operators.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{D} : |z| < 1\}$ in the complex plane. Suppose φ is an analytic function maps $\mathbb{D}$ into itself and ψ is an analytic function on $\mathbb{D}$, the weighted composition operator $W_{\psi,\varphi}$ is defined on the space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ by

$$(W_{\psi,\varphi}f)(z) = \psi(z)C_\varphi f(z) = \psi(z)f(\varphi(z)),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The composition operator C_φ is a weighted composition operator with the weight function ψ identically equal to 1. It is well known that the composition operator $C_\varphi f = f \circ \varphi$ defines a linear operator C_φ which acts boundedly on various spaces of analytic or harmonic functions on $\mathbb{D}$. These operators have been studied on many spaces of analytic functions. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of $W_{\psi,\varphi}$ in terms of the function-theoretic properties of the induced maps φ and ψ. We refer to the monographs by Cowen and MacCluer [1], Duren and Schuster [2], Hedenmalm, Korenblum, and Zhu [3], Shapiro [6], and Zhu ([9], [10]) for the overview of the field as of the early 1990s.

For $\alpha > -1$, we define the measure dA_α on $\mathbb{D}$ by

$$dA_\alpha(z) = (-\log |z|)^\alpha dA(z),$$

where dA is the normalized Lebesgue measure on the unit disk $\mathbb{D}$. The weighted Bergman space A^2_α consists of all functions f analytic on $\mathbb{D}$ that

satisfy

$$\|f\|_\alpha^2 = \int_\mathbb{D} |f(z)|^2 dA_\alpha(z) < \infty.$$

In this definition, the measure dA_α can be replaced by the measure $(1 - |z|)^\alpha dA(z)$ and this results an equivalent norm; see [5] for example. Shapiro [8] showed that $f \in A_\alpha^2$ if and only if $f' \in A_{\alpha+2}^2$. In particular, he proved that

$$\|f\|_\alpha^2 = |f(0)|^2 + \|f'\|_{\alpha+2}^2,$$

this formula is just the Bergman space version of the Little-Paley identity.

The Hardy space $H^2(\mathbb{D})$ consists of functions f analytic on $\mathbb{D}$ that satisfy

$$\|f\|^2 = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where σ is the normalized Lebesgue measure on the boundary of the unit disk. In view of the Little-Payley identity, it is natural to define the Hardy space as A_{-1}^2.

Given a positive integrable function $\omega \in \mathrm{C}^2[0,1)$, we extend ω on $\mathbb{D}$ by setting $\omega(z) = w(|z|)$ for each $z \in \mathbb{D}$. Let ω be the weight function such that $\omega(z)dA(z)$ defines a finite measure on $\mathbb{D}$; that is, $\omega \in L^1(\mathbb{D}, dA)$. For such a weight ω, the weighted Hilbert space $\mathcal{H}_\omega$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{H}_\omega}^2 = |f(0)|^2 + \|f'\|_\omega^2,$$

where

$$\|f'\|_\omega^2 = \int_\mathbb{D} |f'(z)|^2 w(z)dA(z) < \infty.$$

For example, consider the weighted Hilbert space $\mathcal{H}_\alpha$ associated with the weight $\omega_\alpha(r) = (1-r^2)^\alpha$ where $\alpha > -1$. The weighted Hilbert space $\mathcal{H}_1$ is the Hardy space H^2. The Dirichlet space $\mathcal{D}_\alpha$ is $\mathcal{H}_\alpha$ for $0 \leq \alpha < 1$. The weight Bergman space A_α^2, where $\alpha > -1$, is the weighted Hilbert space $\mathcal{H}_{\alpha+2}$.

Recently, Kellay and Lefévre [4] characterized the compactness of composition operators acting on the weighted Hilbert space $\mathcal{H}_\omega$ with the weight function ω that satisfies the following conditions:

(1) ω is non-decreasing.
(2) $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$.
(3) $\lim_{r\to 1^-} \omega(r) = 0$.
(4) One of the two properties of convexity is fulfilled: ω is convex and $\lim_{r\to 1} \omega'(r) = 0$; or ω is concave.

In this paper, we consider the admissible weight ω that only satisfies two conditions; namely we say ω is admissible weight if it is convex and satisfies

$\lim\limits_{r \to 1^-} \omega(r) = \lim\limits_{r \to 1} \omega'(r) = 0$. We characterize boundedness and compactness of composition operators act between weighted bergman space A_α^2 and the weighted Hilbert space $\mathcal{H}_\omega$. The results we obtain about composition operators will be given in terms of a generalized Nevalinna counting functions, which we define next.

The Nevalinna counting functions play an essential role in study of composition operators on weighted Hilbert spaces. The classical Nevalinna function of the inducing map φ is defined by

$$N_\varphi(z) = \sum_{a \in \varphi^{-1}\{z\}} \log\left(\frac{1}{|a|}\right), \quad z \in \mathbb{D} \backslash \{\varphi(0)\},$$

where as usual $a \in \varphi^{-1}\{z\}$ is repeated according to the multiplicity of the zero of $\varphi - z$ at a. The Nevalinna counting function N_φ was first used by Shapiro in [8] to study composition operators on Hardy spaces H^2. In the same paper Shapiro also defined the generalized counting functions $N_{\varphi,\gamma}$ for $\gamma > 0$ by

$$N_{\varphi,\gamma}(z) = \sum_{a \in \varphi^{-1}\{z\}} \left(\log\left(\frac{1}{|a|}\right)\right)^\gamma, \quad z \in \mathbb{D} \backslash \{\varphi(0)\},$$

and used them to study composition operators from a weighted Bergman space to itself. For more information about the Nevalinna counting functions; see [8], [6], and [1], for example. Kellay and Lefèvre [4] defined the generalized Nevalinna function associated to the admissible weight ω by

$$N_{\varphi,\omega}(z) = \sum_{a \in \varphi^{-1}\{z\}} \omega(a), \quad z \in \mathbb{D} \backslash \{\varphi(0)\},$$

and used them to study composition operators from a weighted Hilbert space to itself. Note that if $\omega(r) = (\log(1/r))^\gamma$, then $N_{\varphi,\omega}$ is the generalized Nevalinna counting function $N_{\varphi,\gamma}$.

The next lemma explains how the generalized Nenalinna counting function $N_{\varphi,\omega}$ arise naturally in the study of composition operators on weighted Hilbert spaces, and its proof is a modification of that of (Theorem 2.32, [1]). This lemma is a generalization of the change of variables formula [8]. Shapiro's formula is a special case of Stanton's formula for integral means of analytic functions [7] that extends the Little-Paley identity. The utility of the change of variables formula in the study of composition operators on Hardy and Bergman spaces comes form Stanton's formula.

Lemma 1.1. *Let g and w be positive measurable functions, and φ is a nonconstant analytic self-map of $\mathbb{D}$. Then*

$$\int_{\mathbb{D}} g\left(\varphi(z)\right) |\varphi'(z)|^2 \omega(z) dA(z) = \int_{\varphi(\mathbb{D})} g(\lambda) N_{\varphi,\omega}(\lambda) dA(\lambda).$$

Proof. Suppose φ is a nonconstant analytic self-map of $\mathbb{D}$. Since $\varphi'(z)$ is not identically zero, φ is locally univalent on $\mathbb{D}$ except at the points at which the derivative equal to zero. Since the zeros of φ at most countably many, we can find a countable collection of disjoint open sets $\{R_n\}$ such that the area of $\mathbb{D} \backslash \cup R_n$ is zero, and such that φ is univalent on each R_n.

Let ψ_n be the inverse of $\varphi : R_n \to \varphi(R_n)$. Then the usual change of variables applied on R_n, with $z = \psi_n(\lambda)$ and $dA(\lambda) = |\varphi'(z)|^2 dA(z)$, gives

$$\int_{R_n} g(\varphi(z)) |\varphi'(z)|^2 \ \omega(z) dA(z) = \int_{\varphi(R_n)} g(\lambda) \ \omega(\psi_n(\lambda)) dA(\lambda).$$

Let χ_n denotes the characteristic function of the set $\varphi(R_n)$. Since the zeros of $\varphi'(z)$ is countable, we have

$$\int_{\mathbb{D}} g(\varphi(z)) |\varphi'(z)|^2 \ \omega(z) dA(z) = \sum_{n=1}^{\infty} \int_{\varphi(R_n)} g(\lambda) \ \omega(\psi_n(\lambda)) dA(\lambda)$$
$$= \int_{\varphi(\mathbb{D})} g(\lambda) \left(\sum_{n=1}^{\infty} \chi_n(\lambda) \ \omega(\psi_n(\lambda)) \right) dA(\lambda)$$
$$= \int_{\varphi(\mathbb{D})} g(\lambda) \left(\sum_{n=1}^{\infty} \omega(z_n(\lambda)) \right) dA(\lambda),$$

where $\{z_n(\lambda)\}$ denotes the sequence of zeros of $\varphi(z) - \lambda$, repeated according to multiplicity. This completes the proof, since the sum in the last equation is $N_{\varphi,\omega}(\lambda)$. $\qquad\square$

Recall that an operator is said to be compact if it takes bounded sets to sets with compact closure. The following lemma characterizes the compactness of a composition operator, and its proof is just a modification of that of (Proposition 3.11, [1]). The proof is a standard technique of Montel's Theorem and definition of a composition operator C_φ, so we omit the proof's details.

Lemma 1.2. *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : A_\alpha^2 \to \mathcal{H}_\omega$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in A_α^2 that converges to 0 uniformly on compact subsets of $\mathbb{D}$ then $\lim_{n\to\infty} \|C_\varphi f_n\|_{\mathcal{H}_\omega} = 0$.*

The next lemma shows that the generalized Nevalinna functions $N_{\varphi,\omega}$, while not subharmonic themselves, satisfy a subharmonic mean value property. This lemma is a modification of (Corollary 6.7, [8]) and its proof similar to that of (Corollary 4.6, [8]), so we omit the proof's details.

Lemma 1.3. *Let ω be an admissible weight. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi(0) \neq 0$. Then for every $0 < r < |\varphi(0)|$ we have*

$$N_{\varphi,\omega}(0) \leq \frac{1}{r^2} \int_{r\mathbb{D}} N_{\varphi,\omega}(\zeta) dA(\zeta).$$

2. COMPACTNESS OF A COMPOSITION OPERATOR

In this section we characterize the boundedness and compactness of a composition operator C_φ mapping weighted Bergman space A_α^2 into weighted Hilbert space $\mathcal{H}_\omega$ associated with the admissible weight ω. The notation $A \approx B$ means that there are two positive constants c_1 and c_2 independent of A and B such that $c_1 A \leq B \leq c_2 A$.

Theorem 2.1. *let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : A_\alpha^2 \to \mathcal{H}_\omega$ is bounded if and only if*

$$N_{\varphi,\omega}(\lambda) = O\left((-\log|\lambda|)^{2+\alpha}\right), \qquad as \ \ |\lambda| \to 1.$$

Proof. We first show that $N_{\varphi,\omega}$ satisfies the given growth condition. For $\lambda \in \mathbb{D}$, consider the test function

$$f_\lambda(z) = \frac{\left(1 - |\lambda|^2\right)^{(2+\alpha)/2}}{(1 - \bar{\lambda}z)^{2+\alpha}}.$$

Let $\psi_\lambda(z)$ be the Möbius self-map of $\mathbb{D}$ that interchanges 0 and λ, that is

$$\psi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}.$$

It is well-known that $\|f_\lambda\|_{A_\alpha^2}^2 \approx 1$. Then we have

$$\|C_\varphi(f_\lambda)\|_{\mathcal{H}_\omega}^2 = |f_\lambda(\varphi(0))|^2 + \int_{\mathbb{D}} |f_\lambda'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) dA(z)$$

$$= |f_\lambda(\varphi(0))|^2 + \int_{\varphi(\mathbb{D})} |f_\lambda'(z)|^2 N_{\varphi,\omega}(z) dA(z)$$

$$= |f_\lambda(\varphi(0))|^2 + \int_{\varphi(\mathbb{D})} |f_\lambda'(z)|^2 N_{\varphi,\omega}(z) dA(z)$$

$$= |f_\lambda(\varphi(0))|^2 + (2+\alpha)^2 |\lambda|^2 (1 - |\alpha|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\left|1 - \bar{\lambda}z\right|^{6+2\alpha}} dA(z)$$

$$= |f_\lambda(\varphi(0))|^2 + (2+\alpha)^2 |\lambda|^2 (1 - |\alpha|^2)^\alpha \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\left|1 - \bar{\lambda}z\right|^{2+2\alpha}} |\psi_\lambda'(z)|^2 dA(z)$$

$$= |f_\lambda(\varphi(0))|^2 + (2+\alpha)^2 |\lambda|^2 (1 - |\alpha|^2)^\alpha \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(\psi_\lambda(z))}{\left|1 - \bar{\lambda}\psi_\lambda(z)\right|^{2+2\alpha}} dA(z)$$

Now for any $|z| \leq \frac{1}{2}$, we get

$$\|C_\varphi(f_\lambda)\|_{\mathcal{H}_\omega}^2 \geq \frac{(2+\alpha)^2 |\lambda|^2}{4^{1+\alpha}(1 - |\lambda|^2)^{2+\alpha}} \int_{\frac{1}{2}\mathbb{D}} N_{\varphi,\omega}\left(\psi_\lambda(z)\right) dA(\lambda)$$

$$\geq \frac{(2+\alpha)^2 |\lambda|^2}{4^{2+\alpha}(1 - |\lambda|^2)^{2+\alpha}} N_{\varphi,\omega}(\lambda),$$

On the other hand, using similar argument as that in the proof of Theorem 2.1, there are positive constants C and C_1 such that

$$\|C_\varphi(f_n)\|_{\mathcal{H}_\omega}^2 \geq \int_{\mathbb{D}} |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) dA(z)$$

$$\geq C \frac{N_{\varphi,\omega}(\lambda_n)}{(1 - |\lambda_n|^2)^{2+\alpha}}$$

$$\geq CC_1 \frac{N_{\varphi,\omega}(\lambda_n)}{(-\log|\lambda_n|)^{2+\alpha}}$$

$$\geq CC_1 \beta.$$

Which contradicts the compactness of C_φ, because C and C_1 are constants independent of n.

For the converse, Let $\{f_n\}$ be a sequence in the unit ball of A_α^2 converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then Cauchy's estimate gives f_n' converges uniformly to 0 on compact subsets of $\mathbb{D}$. Let $\epsilon > 0$, by our hypothesis there exists $\delta \in (0,1)$ such that

$$N_{\varphi,\omega}(z) \leq \epsilon \left(-\log|z|\right)^{2+\alpha}, \qquad \text{for} \quad \delta < |z| < 1.$$

To end the proof, by Lemma 1.2, it is enough to show $\lim_{n\to\infty} \|C_\varphi(f_n)\|_{\mathcal{H}_\omega}^2 = 0$.

$$\|C_\varphi(f_n)\|_{\mathcal{H}_\omega}^2 = |f_n(\varphi(0))|^2 + \int_{\mathbb{D}} |f_n'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) dA(z)$$

$$= |f_n(\varphi(0))|^2 + \int_{\mathbb{D}} |f_n'(z)|^2 N_{\varphi,\omega}(z) dA(z)$$

$$= |f_n(\varphi(0))|^2 + \int_{|z|\leq\delta} |f_n'(z)|^2 N_{\varphi,\omega}(z) dA(z) + \int_{|z|>\delta} |f_n'(z)|^2 N_{\varphi,\omega}(z) dA(z)$$

$$= |f_n(\varphi(0))|^2 + \int_{|z|\leq\delta} |f_n'(z)|^2 N_{\varphi,\omega}(z) dA(z) + \epsilon \int_{|z|>\delta} |f_n'(z)|^2 (-\log|z|)^{2+\alpha} dA(z)$$

$$= |f_n(\varphi(0))|^2 + \int_{|z|\leq\delta} |f_n'(z)|^2 N_{\varphi,\omega}(z) dA(z) + \epsilon\|f_n\|_\alpha^2.$$

Since $\epsilon > 0$ is arbitrary, $\{f_n\}$ and $\{f_n'\}$ converge to 0 uniformly on compact subsets of $\mathbb{D}$, we can make the right hand side of the last inequality as small as we wish by choosing n sufficiently large. This completes the proof. $\square$

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(Waleed Al-Rawashdeh) DEPARTMENT OF MATHEMATICAL SCIENCES, MONTANA TECH OF THE UNIVERSITY OF MONTANA, BUTTE, MONTANA 59701, USA

E-mail address: `walrawashdeh@mtech.edu`

A NEW DOUBLE CESÀRO SEQUENCE SPACE DEFINED BY MODULUS FUNCTIONS

OĞUZ OĞUR

ABSTRACT. The object of this paper is to introduce a new double Cesàro sequence spaces $Ces^{(2)}(F, p)$ defined by a double sequence of modulus functions. We study some topologic properties of this space and obtain some inclusion relations.

1. INTRODUCTION

As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of positive integers, real numbers and nonnegative real numbers, respectively. A double sequence on a normed linear space X is a function x from $\mathbb{N} \times \mathbb{N}$ into X and briefly denoted by $x = (x(i, j))$. Throughout this work , w and w^2 denote the spaces of all single real sequences and double real sequences, respectively.

First of all, let us recall preliminary definitions and notations.

If for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\|x_{k,l} - a\|_X < \varepsilon$ whenever $k, l > n_\varepsilon$ then a double sequence $\{x_{k,l}\}$ is said to be converge (in terms of Pringsheim) to $a \in X$ [10].

A double sequence $\{x_{k,l}\}$ is called a Cauchy sequence if and only if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $|x_{k,l} - x_{p,q}| < \varepsilon$ for all $k, l, p, q \geq n_0$.

A double series is infinity sum $\sum_{k,l=1}^{\infty} x_{k,l}$ and its convergence implies the convergence by $\|.\|_X$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{k=1}^{n} \sum_{l=1}^{m} x_{k,l}$ (see [2],[3]).

If each double Cauchy sequence in X converges an element of X according to norm of X, then X is said to be a double complete space. A normed double complete space is said to be a double Banach space [2].

Let X be a linear space. A function $p : X \to \mathbb{R}$ is called *paranorm,* if

$i)$ $p(0) = 0$,

$ii)$ $p(x) \geq 0$ for all $x \in X$,

$iii)$ $p(-x) = p(x)$ for all $x \in X$,

$iv)$ $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,

$v)$ if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$ (continuity of multiplication by scalars).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called *total* [7].

A double sequence space E is said to be solid (or normal) if $(y_{kl}) \in E$ whenever $|y_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$ and $(x_{kl}) \in E$.

A function $f : [0, \infty) \to [0, \infty)$ is said to be a modulus function if it satisfies the following:

1) $f(x) = 0$ if and only if $x = 0$;

2) $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$;

3) f is increasing;

4) f is continuous from right at 0.

It follows that f is continuous on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = x / (x + 1)$, then $f(x)$ is bounded. But, for $0 < p < 1$, $f(x) = x^p$ is not bounded [9].

By the condition 2), we have $f(nx) \leq n.f(x)$ for all $n \in \mathbb{N}$ and so $f(x) \leq f\left(nx\frac{1}{n}\right) \leq nf\left(\frac{x}{n}\right)$, hence

$$\frac{1}{n}f(x) \leq f\left(\frac{x}{n}\right) \text{ for all } n \in \mathbb{N}.$$

The FK-spaces $L(f)$, introduced by Ruckle in [11], is in the form

$$L(f) = \left\{ x \in w : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}$$

where f is a modulus function. This space is closely related to the space ℓ_1 which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$. Later on, this space was investigated by many authors (see $[1, 4, 5, 8, 13]$).

For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$Ces_p = \left\{ x \in w : \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p < \infty \right\},$$

equipped with norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p \right)^{\frac{1}{p}}.$$

This space was first introduced by Shiue [14]. It is very useful in the theory of matrix operators and others [6]. Sanhan and Suantai introduced and studied a generalized Cesàro sequence space $Ces(p)$, where $p = (p_j)$ is a bounded sequence of positive real numbers (see [12]).

Finally, Bala [1] introduced the space $Ces(f, p)$ using a modulus function f as follows;

$$Ces(f, p) = \left\{ x \in w : \sum_{j=1}^{\infty} \left[f\left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right) \right]^{p_j} < \infty \right\}$$

where $p = (p_j)$ is a bounded sequence of positive real numbers.

In this work, we introduce double sequence spaces $Ces^{(2)}(f, p)$ as follows;

Let $p = (p_{n,m})$ be a bounded double sequence of positive real numbers and $F = (f_{n,m})$ be a double sequence of modulus functions. The space $Ces^{(2)}(f, p)$ is defined by

$$Ces^{(2)}(F, p) = \left\{ x \in w : \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} < \infty \right\}.$$

If $f_{n,m}(x) = x$, $p_{n,m} = p$ $(1 \leq p < \infty)$ for all $n, m \in \mathbb{N}$, then the space $Ces^{(2)}(F, p)$ reduced to double Cesaro sequence space.

The following inequality will be used throughout this paper. Let $(p_{n,m})$ be a bounded double sequence of strictly positive real numbers and denote $H = \sup_{n,m} p_{n,m}$. For any complex $a_{n,m}$ and $b_{n,m}$ we have

$$|a_{n,m} + b_{n,m}|^{p_{n,m}} \leq D. \left(|a_{n,m}|^{p_{n,m}} + |b_{n,m}|^{p_{n,m}}\right)$$

where $D = \max\left(1, 2^{H-1}\right)$. Also, for any complex λ,

$$|\lambda|^{p_{n,m}} \leq \max\left(1, |\lambda|^{H}\right).$$

2. MAIN RESULTS

Theorem 1. *The double sequence space $Ces^{(2)}(F, p)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. Let $x, y \in Ces^{(2)}(F, p)$ and $\lambda, \beta \in \mathbb{C}$. Then there exist integers M_λ, N_β such that $|\lambda| \leq M_\lambda$ and $|\beta| \leq N_\beta$. Hence we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |\lambda x_{i,j} + \beta y_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |\lambda x_{i,j}| \right) + f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |\beta y_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[M_\lambda f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) + N_\beta f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |y_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq D. \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[M_\lambda f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$+ D. \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[N_\beta f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |y_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq D. \max\left(1, M_\lambda^{H-1}\right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[M_\lambda f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$+ D. \max\left(1, N_\beta^{H-1}\right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[N_\beta f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |y_{i,j}| \right) \right]^{p_{n,m}}$$

where $D = \max\left(1, 2^{H-1}\right)$. This shows that $\lambda x + \beta y \in Ces^{(2)}(F, p)$ and so $Ces^{(2)}(F, p)$ is a linear space. $\square$

Theorem 2. *The double sequence space $Ces^{(2)}(F, p)$ is paranormed space with the paranorm*

$$g(x) = \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m}\left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

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where $H = \sup_{n,m} p_{n,m} < \infty$ and $M = \max(1, H)$.

Proof. It is clear that $g(x) = g(-x)$ and $g(0) = 0$. For any $x, y \in Ces^{(2)}(F, p)$,

$$
\begin{aligned}
g(x + y) &= \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j} + y_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\leq \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) + f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |y_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\leq g(x) + g(y).
\end{aligned}
$$

For the continuity of scalar multiplication suppose that $\left\{ \lambda^{(s)} \right\}$ is sequence of scalar such that $\left| \lambda^{(s)} - \lambda \right| \to 0$ and $g(x^{(s)} - x) \to 0$ as $s \to \infty$. We shall show that $g(\lambda^{(s)} x^{(s)} - \lambda x) \to 0$ as $s \to \infty$.

$$
\begin{aligned}
g(\lambda^{(s)} x^{(s)} - \lambda x) &= \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| \lambda^{(s)} x_{i,j}^{(s)} - \lambda x_{i,j} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&= \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| \lambda^{(s)} x_{i,j}^{(s)} + \lambda^{(s)} x_{i,j} - \lambda^{(s)} x_{i,j} - \lambda x_{i,j} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\leq \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| \lambda^{(s)} x_{i,j}^{(s)} - \lambda^{(s)} x_{i,j} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\quad + \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| \lambda^{(s)} x_{i,j} - \lambda x_{i,j} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\leq (K)^{\frac{H}{M}} \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j}^{(s)} - x_{i,j} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \\
&\quad + \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{\left| \lambda^{(s)} - \lambda \right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}
\end{aligned}
$$

where $K = 1 + \max\left(1, \sup \lambda^{(s)}\right)$. We must show that

$$
\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{\left| \lambda^{(s)} - \lambda \right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \to 0
$$

as $s \to \infty$. For all $s \in \mathbb{N}$ there exists $T > 0$ such that $\left|\lambda^{(s)} - \lambda\right| \leq T$. Taking $\varepsilon > 0$. Since $x \in Ces^{(2)}(F, p)$, then there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} + \sum_{n=1}^{n_0} \sum_{m=m_0+1}^{\infty} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$+ \sum_{n=n_0+1}^{\infty} \sum_{m=1}^{m_0} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} \left[f_{n,m} \left(\frac{T}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} + \sum_{n=1}^{n_0} \sum_{m=m_0+1}^{\infty} \left[f_{n,m} \left(\frac{T}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$+ \sum_{n=n_0+1}^{\infty} \sum_{m=1}^{m_0} \left[f_{n,m} \left(\frac{T}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$< \frac{\varepsilon}{2}$$

for all $s \in \mathbb{N}$. Also, we have

$$\sum_{n=1}^{n_0} \sum_{m=1}^{m_0} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} < \frac{\varepsilon}{2}$$

as $s \to \infty$. Consequently,

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \to 0$$

as $s \to \infty$. Since

$$g(\lambda^{(s)} x^{(s)} - \lambda x) \leq (K)^{\frac{H}{M}} g(x^{(s)} - x) + \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{\left|\lambda^{(s)} - \lambda\right|}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

we get $g(\lambda^{(s)} x^{(s)} - \lambda x) \to 0$ as $s \to \infty$. $\square$

Theorem 3. *The double sequence spaces $Ces^{(2)}(F, p)$ is complete with respect to its paranorm.*

Proof. Let $\{x^{(s)}\}$ be a double Cauchy sequence in $Ces^{(2)}(F, p)$ such that $x^{(s)} = \left\{ x_{k,l}^{(s)} \right\}_{k,l=1}^{\infty}$ for all $s \in \mathbb{N}$. Then we have

$$(1) \qquad \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j}^{(s)} - x_{i,j}^{(t)} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}} \to 0$$

as $s, t \to \infty$. Hence for each fixed $i, j \in \mathbb{N}$, $\left| x_{i,j}^{(s)} - x_{i,j}^{(t)} \right| \to 0$ as $s, t \to \infty$ and so $\{x^{(s)}\}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $i, j \in \mathbb{N}$.

Then, there exists $x_{i,j} \in \mathbb{C}$ such that $x_{i,j}^{(s)} \to x_{i,j}$ as $s \to \infty$ and let define $x = (x_{i,j})$. From (1), we have that for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j}^{(s)} - x_{i,j}^{(t)} \right| \right) \right]^{p_{n,m}} < \varepsilon^M$$

for $s, t > N$ and so by taking $t \to \infty$ we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j}^{(s)} - x_{i,j} \right| \right) \right]^{p_{n,m}} < \varepsilon^M$$

for all $s \geq N$. Thus we get $g(x^{(s)} - x) \to 0$ for all $s \geq N$. Finally we must show that $x \in Ces^{(2)}(F, p)$. By definition of a modulus function, for all $s \geq N$ we get

$$\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

$$= \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j} - x_{i,j}^{(s)} + x_{i,j}^{(s)} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

$$\leq \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j} - x_{i,j}^{(s)} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

$$+ \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} \left| x_{i,j}^{(s)} \right| \right) \right]^{p_{n,m}} \right)^{\frac{1}{M}}$$

$$\leq \varepsilon + g(x^{(s)}),$$

which implies that $x \in Ces^{(2)}(F, p)$. Then $Ces^{(2)}(F, p)$ is complete. $\square$

Theorem 4. *Let $F = (f_{n,m})$ and $H = (h_{n,m})$ be two modulus functions. Then*

(i) $\limsup \frac{f_{n,m}(t)}{h_{n,m}(t)} < \infty$ *for all $n, m \in \mathbb{N}$ implies $Ces^{(2)}(H, p) \subset Ces^{(2)}(F, p)$,*

(ii) $Ces^{(2)}(H, p) \cap Ces^{(2)}(F, p) \subseteq Ces^{(2)}(F + H, p)$.

Proof. (i) By the hypothesis there exists $K > 0$ such that $f_{n,m}(t) \leq K.h_{n,m}(t)$ for all $n, m \in \mathbb{N}$. Let $x \in Ces^{(2)}(H, p)$, then we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[K.h_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$= C^H \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[h_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$< \infty,$$

where $C = \max(1, K)$. Hence we get $x \in Ces^{(2)}(F, p)$.

(ii) Let $x \in Ces^{(2)}(H,p) \cap Ces^{(2)}(F,p)$. Hence we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[(f_{n,m} + h_{n,m}) \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) + h_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$

$$\leq D. \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[h_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} \right\},$$

where $D = \max\left(1, 2^{H-1}\right)$. Therefore we get $x \in Ces^{(2)}(F+H,p)$ and this complete proof. $\square$

Corollary 1. *The space $Ces^{(2)}(F,p)$ is solid.*

Proof. Suppose that $x \in Ces^{(2)}(F,p)$ and $|y_{i,j}| \leq |x_{i,j}|$ for all $i, j \in \mathbb{N}$. Then, by the monotonity of the modulus function we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |y_{i,j}| \right) \right]^{p_{n,m}} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$
$$< \infty.$$

This implies that $y \in Ces^{(2)}(F,p)$ and so $Ces^{(2)}(F,p)$ is solid. $\square$

Theorem 5. *Let $(p_{n,m})$ and $(q_{n,m})$ be bounded double sequences of positive real numbers such that $0 < p_{n,m} \leq q_{n,m} < \infty$ for all $n, m \in \mathbb{N}$. Then for any modulus $F = (f_{n,m})$, $Ces^{(2)}(F,p) \subset Ces^{(2)}(F,q)$.*

Proof. Let $x \in Ces^{(2)}(F,p)$. Then we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}} < \infty.$$

Hence there exists $N \in \mathbb{N}$ such that $f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \leq 1$ for all $n, m \geq N$. Since $0 < p_{n,m} \leq q_{n,m}$, we get

$$\sum_{n=N}^{\infty} \sum_{m=N}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{q_{n,m}} \leq \sum_{n=N}^{\infty} \sum_{m=N}^{\infty} \left[f_{n,m} \left(\frac{1}{n.m} \sum_{i,j=1}^{n,m} |x_{i,j}| \right) \right]^{p_{n,m}}$$
$$< \infty$$

and so $x \in Ces^{(2)}(F,q)$. $\square$

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(O.Oğur) ONDOKUZ MAYIS UNIVERSITY, ART AND SCIENCE FACULTY, DEPARTMENT OF MATHEMATICS, KURUPELIT CAMPUS, SAMSUN, TURKEY

E-mail address: oguz.ogur@omu.edu.tr

Right Fractional Monotone Approximation

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Let $f \in C^p([-1,1])$, $p \geq 0$ and let L be a linear right fractional differential operator such that $L(f) \geq 0$ throughout $[-1,0]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L(Q_n) \geq 0$ over $[-1,0]$, furthermore f is approximated uniformly by Q_n. The degree of this restricted approximations is given by an inequalities using the modulus of continuity of $f^{(p)}$.

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1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h,k,p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j} \right] \tag{1}$$

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and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$*, there is a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C *is independent of* n *or* f.

In this article we extend Theorem 1 to the right fractional level. Now L is a linear right Caputo fractional differential operator. Here the monotonicity property is only true on the critical interval $[-1, 0]$. Quantitative unifom approximation remains true on all of $[-1, 1]$.

To the best of our knowledge this is the first time right fractional monotone Approximation is introduced.

We need and make

Definition 2 *([3]) Let* $\alpha > 0$ *and* $\lceil \alpha \rceil = m$, *(* $\lceil \cdot \rceil$ *ceiling of the number). Consider* $f \in C^m([-1, 1])$. *We define the right Caputo fractional derivative of* f *of order* α *as follows:*

$$\left(D_{1-}^{\alpha} f \right)(x) = \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^1 (t - x)^{m - \alpha - 1} f^{(m)}(t) \, dt, \tag{5}$$

for any $x \in [-1, 1]$, *where* Γ *is the gamma function.*

We set

$$D_{1-}^0 f(x) = f(x),$$

$$D_{1-}^m f(x) = (-1)^m f^{(m)}(x), \quad \forall \; x \in [-1, 1]. \tag{6}$$

2   Main Result

We present

Theorem 3 *Let* h, k, p *be integers,* h *is even,* $0 \leq h \leq k \leq p$ *and let* f *be a real function,* $f^{(p)}$ *continuous in* $[-1, 1]$ *with modulus of continuity* $\omega_1 \left(f^{(p)}, \delta \right)$, $\delta > 0$, *there. Let* $\alpha_j(x)$, $j = h, h+1, ..., k$ *be real functions, defined and bounded on* $[-1, 1]$ *and assume for* $x \in [-1, 0]$ *that* $\alpha_h(x)$ *is either* $\geq$ *some number* $\alpha > 0$ *or* $\leq$ *some number* $\beta < 0$. *Let the real numbers* $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < ... < \alpha_p < p$. *Here* $D_{1-}^{\alpha_j} f$ *stands for the right Caputo fractional derivative of*

f of order α_j anchored at 1. Consider the linear right fractional differential operator

$$L := \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j} \right] \tag{7}$$

and suppose, throughout $[-1, 0]$,

$$L(f) \geq 0. \tag{8}$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \quad throughout \quad [-1, 0], \tag{9}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n} \right), \tag{10}$$

where C is independent of n or f.

Proof. Let $n \in \mathbb{N}$. By a theorem of Trigub [6, 7], given a real function g, with $g^{(p)}$ continuous in $[-1, 1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| g^{(j)}(x) - q_n^{(j)}(x) \right| \leq R_p n^{j-p} \omega_1\left(g^{(p)}, \frac{1}{n} \right), \tag{11}$$

$j = 0, 1, ..., p$, where R_p is independent of n or g.

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$.

Let $\alpha_j > 0$, $j = 1, ..., p$, such that $0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < 3... < ... < \alpha_p < p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., p$.

We consider the right Caputo fractional derivatives

$$\left(D_{1-}^{\alpha_j} g \right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j - \alpha_j - 1} g^{(j)}(t) \, dt, \tag{12}$$

$$\left(D_{1-}^{j} g \right)(x) = (-1)^j g^{(j)}(x),$$

and

$$\left(D_{1-}^{\alpha_j} q_n \right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j - \alpha_j - 1} q_n^{(j)}(t) \, dt, \tag{13}$$

$$\left(D_{1-}^{j} q_n \right)(x) = (-1)^j q_n^{(j)}(x); \quad j = 1, ..., p,$$

where Γ is the gamma function

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt, \quad v > 0. \tag{14}$$

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We notice that
$$\left| \left(D_{1-}^{\alpha_j} g \right)(x) - \left(D_{1-}^{\alpha_j} q_n \right)(x) \right| =$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} g^{(j)}(t)\, dt - \int_x^1 (t-x)^{j-\alpha_j-1} q_n^{(j)}(t)\, dt \right| = \quad (15)$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} \left(g^{(j)}(t) - q_n^{(j)}(t) \right) dt \right| \leq$$

$$\frac{1}{\Gamma(j - \alpha_j)} \int_x^1 (t-x)^{j-\alpha_j-1} \left| g^{(j)}(t) - q_n^{(j)}(t) \right| dt \overset{(11)}{\leq} \quad (16)$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left(\int_x^1 (t-x)^{j-\alpha_j-1}\, dt \right) R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right) =$$

$$\frac{1}{\Gamma(j - \alpha_j)} \frac{(1-x)^{j-\alpha_j}}{(j - \alpha_j)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right) \leq \quad (17)$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right).$$

We proved that for any $x \in [-1, 1]$ we have

$$\left| \left(D_{1-}^{\alpha_j} g \right)(x) - \left(D_{1-}^{\alpha_j} q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right). \quad (18)$$

Hence it holds

$$\max_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} g \right)(x) - \left(D_{1-}^{\alpha_j} q_n \right)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right),$$
$$(19)$$

$j = 0, 1, ..., p$.

Above we set $D_{1-}^0 g(x) = g(x)$, $D_{1-}^0 q_n(x) = q_n(x)$, $\forall\ x \in [-1, 1]$, and $\alpha_0 = 0$, i.e. $\lceil \alpha_0 \rceil = 0$.

Put

$$s_j \equiv \sup_{-1 \leq x \leq 1} \left| \alpha_h^{-1}(x)\, \alpha_j(x) \right|, \quad j = h, ..., k, \quad (20)$$

and

$$\eta_n := R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \left(\sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-p} \right). \quad (21)$$

I. Suppose, throughout $[-1, 0]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j} \left(f(x) + \eta_n (h!)^{-1} x^h \right) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \overset{(19)}{\leq}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (22)$$

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$j = 0, 1, ..., p.$

In particular $(j = 0)$ holds

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) + \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (23)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) =$$

$$(h!)^{-1} R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \left(\sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-p} \right)$$

$$+ R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \leq \quad (24)$$

$$R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \right). \quad (25)$$

That is

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq$$

$$R_p \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \right) n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (26)$$

proving (10).

Here

$$L = \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j} \right],$$

and suppose, throughout $[-1, 0]$, $Lf \geq 0$.

So over $-1 \leq x \leq 0$, we get

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) + \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} + \quad (27)$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) - \frac{\eta_n}{h!} D_{1-}^{\alpha_j} x^h \right] \overset{(22)}{\geq}$$

$$\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} - \left(\sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-p} \right) R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) =$$

$$\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} - \eta_n = \eta_n \left[\frac{(1-x)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} - 1 \right] = \quad (28)$$

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$$\eta_n \left[\frac{(1-x)^{h-\alpha_h} - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right] \geq \eta_n \left[\frac{1 - \Gamma(h - \alpha_h + 1)}{\Gamma(h - \alpha_h + 1)} \right] \geq 0. \quad (29)$$

Explanation: We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus $\Gamma(h - \alpha_h + 1) \leq 1$ and

$$1 - \Gamma(h - \alpha_h + 1) \geq 0. \quad (30)$$

Hence

$$L(Q_n(x)) \geq 0, x \in [-1, 0]. \quad (31)$$

II. Suppose, throughout $[-1, 0]$, $\alpha_h(x) \leq \beta < 0$. In this case let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j} \left(f(x) - \eta_n (h!)^{-1} x^h \right) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \leq \quad (32)$$

$$\frac{2^{j - \alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right),$$

$j = 0, 1, ..., p.$

In particular holds $(j = 0)$

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) - \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (33)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \overset{(\text{as before})}{\leq}$$

$$R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \right). \quad (34)$$

That is

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq$$

$$R_p \left(1 + (h!)^{-1} \sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \right) n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (35)$$

reproving (10).

Again suppose, throughout $[-1, 0]$, $Lf \geq 0$.

Also if $-1 \leq x \leq 0$, then

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) - \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} + \quad (36)$$

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$$\sum_{j=h}^{k} \alpha_h^{-1}(x)\, \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) + \frac{\eta_n}{h!}\left(D_{1-}^{\alpha_j} x^h \right) \right] \overset{(32)}{\leq}$$

$$-\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \left(\sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p} \right) R_p \omega_1\left(f^{(p)}, \frac{1}{n} \right) =$$

$$\eta_n - \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} = \eta_n \left(1 - \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) =$$

$$\eta_n \left(\frac{\Gamma(h-\alpha_h+1) - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq \eta_n \left(\frac{1 - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq 0, \quad (37)$$

and hence again

$$L\left(Q_n(x) \right) \geq 0, \quad \forall\, x \in [-1,0]. \tag{38}$$

∎

Remark 4 *Based on [1], here we have that $D_{1-}^{\alpha_j} f$ are continuous functions, $j = 0, 1, ..., p$. Suppose that $\alpha_h(x), ..., \alpha_k(x)$ are continuous functions in $[-1,1]$, and $L(f) \geq 0$ on $[-1,0]$ is replaced by $L(f) > 0$ on $[-1,0]$. Disregard the assumption made in the Theorem 3 on $\alpha_h(x)$. For $n \in \mathbb{N}$, let $Q_n(x)$ be $q_n(x)$ of (19) for $g = f$. Then $Q_n(x)$ converges to f at the Jackson rate [4, p. 18, Theorem VIII] and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[-1,1]$, $L(Q_n) > 0$ on $[-1,0]$ for all n sufficiently large.*

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A differential sandwich-type result using a generalized Sălăgean operator and Ruscheweyh operator

Andrei Loriana

Department of Mathematics and Computer Science
University of Oradea
1 Universitatii street, 410087 Oradea, Romania
lori_andrei@yahoo.com

Abstract

The purpose of this paper is to introduce sufficient conditions for subordination and super-ordination involving the derivative operator $DR_\lambda^{m,n}$ and also to obtain sandwich-type result.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots, \; z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Denote by $K = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U \right\}$, the class of normalized convex functions in U.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p

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satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [17] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution product) of $f(z)$ and $g(z)$, written as $(f * g)(z)$, is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

Definition 1.1 *(Al Oboudi [7]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
D_\lambda^0 f(z) &= f(z) \\
D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\
&\quad \ldots \\
D_\lambda^m f(z) &= (1-\lambda) D_\lambda^{m-1} f(z) + \lambda z \left(D_\lambda^m f(z) \right)' = D_\lambda \left(D_\lambda^{m-1} f(z) \right), \text{ for } z \in U.
\end{aligned}
$$

Remark 1.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j z^j$, for $z \in U$.*

Remark 1.2 *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [20].*

Definition 1.2 *(Ruscheweyh [19]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
R^0 f(z) &= f(z) \\
R^1 f(z) &= z f'(z) \\
&\quad \ldots \\
(n+1) R^{n+1} f(z) &= z \left(R^n f(z) \right)' + n R^n f(z), \quad z \in U.
\end{aligned}
$$

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of M. Darus, K. Al-Shaqs [15], Shanmugam, Ramachandran, Darus and Sivasubramanian [21], R.W. Ibrahim, M. Darus [16].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.3 *[18] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1.1 *[18] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
   1. Q is starlike univalent in U and
   2. $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)} \right) > 0$ for $z \in U$.
   If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and q is the best dominant.

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Lemma 1.2 *[14] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

1. $\mathrm{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ *for $z \in U$ and*
2. $\psi(z) = zq'(z)\phi(q(z))$ *is starlike univalent in U.*

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2 Main results

Definition 2.1 *Let $\lambda \geq 0$ and $n, m \in \mathbb{N}$. Denote by $DR_\lambda^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the generalized Sălăgean operator D_λ^m and the Ruscheweyh operator R^n,*

$$DR_\lambda^{m,n} f(z) = (D_\lambda^m * R^n) f(z),$$

for any $z \in U$ and each nonnegative integers m, n.

Remark 2.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$DR_\lambda^{m,n} f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, *for $z \in U$.*
This operator was studied in [12].

Remark 2.2 *For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [1] of the Sălăgean operator S^n and Ruscheweyh derivative R^n, which was studied in [2], [3].*

Remark 2.3 *For $m = n$ we obtain the Hadamard product DR_λ^n [4] of the generalized Sălăgean operator D_λ^n and Ruscheweyh derivative R^n, which was studied in [5], [6], [8], [9], [10], [11].*

Using simple computation one obtains the next result.

Proposition 2.1 *[12] For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$z\left(DR_\lambda^{m,n} f(z)\right)' = (n+1) DR_\lambda^{m,n+1} f(z) - nDR_\lambda^{m,n} f(z). \tag{2.1}$$

We begin with the following

Theorem 2.2 *Let $\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}(U)$, $z \in U$, $f \in \mathcal{A}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0) = 1$. Assume that*

$$\mathrm{Re}\left(1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0, \quad z \in U, \tag{2.2}$$

for $\alpha, \beta \in \mathbb{C}, \beta \neq 0$, $z \in U$, and

$$\psi_\lambda^{m,n}(\alpha, \beta; z) := \tag{2.3}$$

$$\beta(n+1)\frac{\left(DR_\lambda^{m,n+1} f(z)\right)'}{\left(DR_\lambda^{m,n} f(z)\right)'} + (\alpha - .\beta)\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} - \beta n.$$

If q satisfies the following subordination

$$\psi_\lambda^{m,n}(\alpha, \beta, \mu; z) \prec \alpha q(z) + \frac{\beta zq'(z)}{q(z)}, \tag{2.4}$$

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for, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ *then*

$$\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \prec q(z), \quad z \in U, \tag{2.5}$$

and q is the best dominant.

Proof. Our aim is to apply Lemma 1.1. Let the function p be defined by $p(z) := \frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)}$,
$z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$

Differentiating this function, with respect to z, we get

$$zp'(z) = \frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)} + \frac{z^2(DR_\lambda^{m,n} f(z))''}{DR_\lambda^{m,n} f(z)} - \left[\frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)}\right]^2$$

By using the identity (2.1), we obtain

$$\frac{zp'(z)}{p(z)} = (n+1)\frac{\left(DR_\lambda^{m,n+1} f(z)\right)'}{\left(DR_\lambda^{m,n} f(z)\right)'} -$$

$$\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} - n \tag{2.6}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{\beta zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \frac{\beta zq'(z)}{q(z)}$, $z \in U$

If we derive the function Q, with respect to z, perform calculations, we have $\text{Re}\left(\frac{zh'(z)}{Q(z)}\right) =$
$\text{Re}\left(1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.6), we obtain

$$\alpha p(z) + \beta\frac{zp'(z)}{p(z)} = \alpha\frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)} + .\beta\left[(n+1)\frac{\left(DR_\lambda^{m,n+1} f(z)\right)'}{\left(DR_\lambda^{m,n} f(z)\right)'} - \frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)} - n\right] =$$

$$\beta(n+1)\frac{\left(DR_\lambda^{m,n+1} f(z)\right)'}{\left(DR_\lambda^{m,n} f(z)\right)'} + (\alpha - .\beta)\frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)} - \beta n.$$

By using (2.4), we have $\alpha p(z) + \beta\frac{zp'(z)}{p(z)} \prec \alpha q(z) + \beta\frac{zq'(z)}{q(z)}$.

Therefore, the conditions of Lemma 1.1 are met, so we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z(DR_\lambda^{m,n} f(z))'}{DR_\lambda^{m,n} f(z)}$
$\prec q(z)$, $z \in U$, and q is the best dominant. ∎

Corollary 2.3 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$. *Assume that (2.2) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(\alpha, \beta; z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.3), then*

$$\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.2 we get the corollary. ∎

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Corollary 2.4 *Let* $q(z) = \frac{1+z}{1-z}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$. *Assume that (2.2) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \frac{1+z}{1-z} + \beta \frac{2z}{1-z^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.3), then*

$$\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.2 for $q(z) = \frac{1+z}{1-z}$

Theorem 2.5 *Let* q *be convex and univalent in* U, *such that* $q(0) = 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that*

$$\operatorname{Re}\left(\frac{\alpha}{\beta} q(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \ \mu \neq 0, \ z \in U. \tag{2.7}$$

If $f \in \mathcal{A}$, $\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\psi_\lambda^{m,n}(\alpha, \beta, \mu; z)$ *is univalent in* U, *where* $\psi_\lambda^{m,n}(\alpha, \beta; z)$ *is as defined in (2.3), then*

$$\alpha q(z) + \beta \frac{zq'(z)}{p(z)} \prec \psi_\lambda^{m,n}(\alpha, \beta, \mu; z), \quad z \in U, \tag{2.8}$$

implies

$$q(z) \prec \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}, \quad z \in U, \tag{2.9}$$

and q *is the best subordinant.*

■

Proof. Let the function p be defined by $p(z) := \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since q is convex and univalent function, it follows that $\operatorname{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = \operatorname{Re}\left(\frac{\alpha}{\beta} q(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

By using (2.8) we obtain

$$\alpha q(z) + \beta \frac{zq'(z)}{q(z)} \prec \alpha p(z) + \beta \frac{zp'(z)}{p(z)}.$$

Using Lemma 1.2, we have

$$q(z) \prec p(z) = \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}, \quad z \in U,$$

and q is the best subordinant. ■

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Corollary 2.6 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $m, n \in \mathbb{N}$, $\lambda \ge 0$. *Assume that (2.7) holds. If* $f \in \mathcal{A}, \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha \frac{1 + Az}{1 + Bz} + \beta \frac{(A - B)z}{(1 + Az)(1 + Bz)} \prec \psi_\lambda^{m,n}(\alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, $-1 \le B < A \le 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.3), then*

$$\frac{1 + Az}{1 + Bz} \prec \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$ in Theorem 2.5 we get the corollary. ∎

Corollary 2.7 *Let* $q(z) = \frac{1+z}{1-z}$, $m, n \in \mathbb{N}$, $\lambda \ge 0$. *Assume that (2.7) holds. If* $f \in \mathcal{A}, \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}$ $\in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha \frac{1 + z}{1 - z} + \beta \frac{2z}{1 - z^2} \prec \psi_\lambda^{m,n}(\alpha, \beta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.3), then*

$$\frac{1 + z}{1 - z} \prec \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}$$

and $\frac{1+z}{1-z}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.5 for $q(z) = \frac{1+z}{1-z}$. ∎

Combining Theorem 2.2 and Theorem 2.5, we state the following sandwich theorem.

Theorem 2.8 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \ne 0$ *and* $q_2(z) \ne 0$, *for all* $z \in U$, *with* $zq_1'(z)$ *and* $zq_2'(z)$ *being starlike univalent. Suppose that* q_1 *satisfies (2.2) and* q_2 *satisfies (2.7). If* $f \in \mathcal{A}$, $\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\psi_\lambda^{m,n}(\alpha, \beta; z)$ *is as defined in (2.3) univalent in* U, *then*

$$\alpha q_1(z) + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_\lambda^{m,n}(\alpha, \beta; z) \prec \alpha q_2(z) + \frac{\beta z q_2'(z)}{q_2(z)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, *implies*

$$q_1(z) \prec \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \prec q_2(z),$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \le B_2 < B_1 < A_1 < A_2 \le 1$, we have the following corollary.

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Corollary 2.9 *Let $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.2) and (2.7) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ and*

$$\alpha \frac{1+A_1 z}{1+B_1 z} + \beta \frac{(A_1 - B_1) z}{(1+A_1 z)(1+B_1 z)} \prec \psi_\lambda^{m,n}(\alpha, \beta; z)$$

$$\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \beta \frac{(A_2 - B_2) z}{(1+A_2 z)(1+B_2 z)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.3), then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z},$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

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Predictor Corrector type method for solving certain non convex equilibrium problems

Wasim Ul-Haq[1,2]**, Manzoor Hussain**[2]

[1]Department of Mathematics, College of Science in Al-Zulfi, Majmaah University, Kingdom of Saudi Arabia

[2]Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan
Email: wasim474@hotmail.com, manzoor366@gmail.com

Abstract: In this paper, we propose and analyze a new predictor corrector type method for solving uniformly regular invex equilibrium problems by using the auxiliary principle technique. The convergence criteria of this new method under some mild restrictions is also studied. New and known methods for solving variational-like inequalities and equilibrium problems appear as special consequences of our work.

Key Words: Equilibrium problem, Invex function, Prox regular set.

MSC: 49J40, 91B50

1   Introduction

A variety of problems arising in optimization, economics, finance, networking, transportation, structural analysis and elasticity can be studied in the framework of Equilibrium problems. The theory of equilibrium problems provides productive and innovative techniques to investigate a wide range of such problems. Blum and Oettli [1] and Noor and Oettli [2] has shown that variational inequalities and mathematical programming problems can be seen as a special case of the abstract equilibrium problems.

There are a many numerical methods including the projection technique and its variant forms, Weiner-Hopf equations, the auxiliary principle technique and resolvent equations method for solving variational inequalities. However, it is known that projection, Weiner-Hopf equations, and proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving uniformly regular invex equilibrium problems and variational-like inequalities due to the presence of function $\eta\left(.,.\right)$. To overcome this shortcoming, one may use auxiliary principle technique. Several interesting generalizations and extensions of classical convexity have been studied and investigated in recent years. Hanson [3] introduced the invex functions as a generalization of convex functions. Later, subsequent works inspired from Hanson's result have greatly found the role

and applications of invexity in non linear optimization and other branches of pure and applied sciences. The basic properties and role of preinvex functions in optimization, equilibrium problems and variational inequalities was studied by Noor [2,4,5] and Wier and Mond [8]. Motivated from the above and recent work of Noor et al. [47], we use the auxiliary principle technique which is mainly due to Glowinski et al. [12] to propose and analyze a new predictor-corrector type method for solving the uniformly regular invex equilibrium problems. We also study the convergence of this new method under some suitable conditions.

2   Preliminaries

Let $\langle ., . \rangle$ and $\|.\|$ denote the inner product and norm respectively in a real Hilbert space H. In the sequel let $f \colon K \to H$ and $\eta(.,.) \colon K \times K \to H$ be a continuous function, where K is a nonempty closed convex set in H. Now, we recall some basic definitions and concepts of non-smooth analysis, for details see [48,49].

Definition 1. The proximal normal cone denoted by $N_K^P(u)$ of the set K at u is defined by

$$N_K^P(u) = \{\xi \in H : u \in P_K[u + \alpha\xi]\}$$

where $\alpha > 0$ is a constant and $P_K[.]$ is the projection of H onto the set K and defined as

$$P_K[u] = \{u^* \in K : d_K = \|v - u^*\|\}$$

Here $d_K(.)$ is the usual distance function to the subset K, and defined as

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone has the following characterization.

Lemma 1. Let K be a closed subset in H. Then $\xi \in N_K^P(u)$ if and only if there exists a constant such that

$$\langle \xi, v - u \rangle \leq \alpha \|v - u\|^2, \ \forall v \in K.$$

Definition 2. The Clarke normal cone, denoted by $N_K^C(u)$ is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)], \quad \forall u \in K$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^C(u)$ is always closed and convex, whereas $N_K^P(u)$ is convex but may not be closed, see [49].

Poliquin et al. [49] and Clarke et al. [48] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important role in many nonconvex

applications such as optimization, dynamical systems and differential inclusion. In particular, we have

Definition 3. For a given $r \in (0, \infty]$, a subset K_r is said to be uniformly $r-$proximal regular if and only if for each $u \in K_r$ and all proximal normal vectors $0 \neq \xi \in N_K^P(u)$,

$$\left\langle \frac{\xi}{\|\xi\|}, \eta(v, u) \right\rangle \leq \frac{1}{2r} \|\eta(v, u)\|^2, \quad \forall v \in K_r.$$

The class of uniformly prox-regular sets contains the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [48,49]. It is clearsimple to note that if $r = \infty$ then uniformly $r-$proximal regular set K reduces to convex set. This observation plays a key role in this work. It is known that if K is a uniformly $r-$prox-regular set, then the proximal normal cone $N_K^P(u)$ is closed as a set-valued mapping. Thus, we have $N_K^C(u) = N_K^P(u)$. From now onward the set K_r is considered to be uniformly proximal regular invex set. For a given continuous bifunction $F(.,.) : K_r \times K_r \to H$, consider the problem of finding $u \in K_r$ such that

$$F(u, v) + \left(\frac{k}{2r} \right) \|\eta(v, u)\|^2 \geq 0, \quad \forall v \in K_r,$$

where k is a positive constant. By letting $\lambda = \frac{k}{2r}$ such that $\lambda = 0$ for $r = \infty$, then the above problem becomes

$$F(u, v) + \lambda \|\eta(v, u)\|^2 \geq 0, \quad \forall v \in K_r. \tag{2.1}$$

This type of problem is called uniformly regular invex equilibrium problem. Note that for $\eta(v, u) = v - u$, then the uniformly regular invex equilibrium problem reduces to uniformly regular equilibrium problem introduced and studied by Noor [26, 3 − 5] and Noor and Noor [30] and for $r = \infty$ uniformly regular invex equilibrium problem reduces to classical equilibrium problem introduced and studied by Blum and Oettli [1] and Noor and Oettli [2]. If $F(u, v) = \langle Tu, \eta(v, u) \rangle$, where $T : H \to H$ is a nonlinear continuous operator, then problem (2.1) is equivalent to finding $u \in K_r$ such that

$$\left\langle Tu + \lambda \|\eta(v, u)\|^2, \eta(v, u) \right\rangle \geq 0, \quad \forall v \in K_r. \tag{2.2}$$

which is a variant form of varitainol-like inequality problem. Note that for $r = \infty$ problem (2.2) reduces to varitainol-like inequality problem studied by various authors in recent years, see Noor [19] and Yang and Chen [43]. If $\eta(v, u) = v - u$ and $r = \infty$ uniformly $r-$proximal regular set reduces to convex set K.and problem (2.2) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall \ v \in K. \tag{2.3}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [40]. For the recent applications, numerical methods and formulation of variational inequalities, variational-like inequalities and equilibrium problems, see [1-51] and the references therein.

Definition 4. The function $F(.,.) : K \times K \to H$ is said to be:

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(i). $\eta-$pseudomonotone

$$F(u, v) \geq 0 \Longrightarrow -F(v, u) \geq 0, \quad \forall u, v \in K.$$

(ii). partially relaxed strongly $\eta-$monotone, if there exist a constant $\alpha > 0$ such that

$$F(u, v) + F(v, z) \leq \alpha \left\| \eta(z, u) \right\|^2, \quad \forall u, v, z \in K,$$

Note that for $z = u$, $\eta(z, u) = 0$, $\forall\, u \in K$, thus partially relaxed strongly monotonicity reduces to

$$F(u, v) + F(v, u) \leq 0, \quad \forall u, v, \in K,$$

which is known as the $\eta-$monotonicity of $F\left(.,.\right)$. It is obvious that $\eta-$monotonicity implies $\eta-$pseudomonotonicity, but the converse is not true.

3 Iterative methods and convergence analysis

In this section, we introduce some iterative scheme for solving the uniformly regular invex equilibrium problem (2.1) by using the auxiliary principle technique, which is due to Glowinski et al. [12].

For a given $u \in K_r$, consider the auxiliary uniformly regular invex equilibrium problem of finding $w \in K_r$ such that

$$\rho F\left(u, v\right) + \rho \lambda \left\| \eta(v, u) \right\|^2 + \left\langle E'(w) - E'(w_n), \eta\left(v, u_{n+1}\right) \right\rangle \geq 0, \quad \forall\, v \in K_r, \quad (3.1)$$

where ρ is a positive constant and $E'(u)$ is the differential of a strongly preinvex function E at $u \in K$. From the strong preinvexity of the differentiable function $E(u)$, it follows that problem (3.1) has a unique solution. Note that if $w = u$, then w is a solution of uniformly regular invex equilibrium problem (2.1). This observation leads us to propose and analyze the following iterative scheme for solving the uniformly regular invex equilibrium problem (2.1).

Algorithm 1. Given $u_0 \in H$, calculate u_{n+1} from the following iterative scheme

$$\rho F\left(w_n, v\right) + \rho \lambda \left\| \eta(v, w_n) \right\|^2 + \left\langle E'(u_{n+1}) - E'(w_n), \eta\left(v, u_{n+1}\right) \right\rangle \geq 0, \quad \forall\, v \in K_r \quad (3.2)$$

$$\mu F\left(u_n, v\right) + \mu \lambda \left\| \eta(v, u_n) \right\|^2 + \left\langle E'(w_n) - E'(u_n), \eta\left(v, w_n\right) \right\rangle \geq 0, \quad \forall\, v \in K_r \quad (3.3)$$

where ρ, μ are both positive constants.

If $F\left(u, v\right) = \left\langle Tu, \eta(v, u) \right\rangle$, then we have

Algorithm 2. Given $u_0 \in H$, calculate u_{n+1} through iterative scheme

$$\left\langle \rho Tw_n + \rho \lambda \left\| \eta(v, w_n) \right\|^2 + E'(u_{n+1}) - E'(w_n), \eta\left(v, u_{n+1}\right) \right\rangle \geq 0, \quad \forall\, v \in K_r$$

$$\left\langle \mu T u_n + \mu \lambda \left\| \eta(v, u_n) \right\|^2 + E'(w_n) - E'(u_n), \eta(v, u_n) \right\rangle \geq 0, \quad \forall \, v \in K_r$$

where ρ, μ are both positive constants.

When $r = +\infty$, then $\lambda = 0$, so the uniformly regular invex equilibrium problem reduces to equilibrium problem. i.e.

$$F(u, v) \geq 0, \ \forall \, v \in K$$

and the uniformly prox-regular set K becomes convex set K. Thus, we have

Algorithm 3 [47]. For $u_0 \in H$ we calculate u_{n+1} through the following iterative scheme

$$\rho F(w_n, v) + \left\langle E'(u_{n+1}) - E'(w_n), v - u_{n+1} \right\rangle \geq 0, \quad \forall \, v \in K$$

$$\mu F(u_n, v) + \left\langle E'(w_n) - E'(u_n), v - w_n \right\rangle \geq 0, \quad \forall \, v \in K$$

where ρ, μ are both positive constants.

Now if, $F(u, v) = \langle Tu, v \rangle$, then we have

Algorithm 4[47]. For $u_0 \in H$ we calculate u_{n+1} through the following iterative scheme

$$\left\langle \rho T w_n + E'(u_{n+1}) - E'(w_n), v - u_{n+1} \right\rangle \geq 0, \quad \forall \, v \in K$$

$$\left\langle \mu T u_n + E'(w_n) - E'(u_n), v - w_n \right\rangle \geq 0, \quad \forall \, v \in K$$

where ρ, μ are both positive constants.

We now study the convergence criteria of Algorithm 1. For this purpose, we need the following condition.

Assumption 1[47]. $\forall \, u, v, z \in K$, the function $\eta(., .)$ satisfies the following condition

$$\eta(u, v) = \eta(u, z) + \eta(z, v). \tag{3.4}$$

Assumption has been used to study the existence of a solution of a variational-like inequalities. Note that $\eta(u, v) = 0$ if and only if $u = v$, $\forall u, v \in K$.

Theorem 1. Suppose the function $F(., .)$ is partially relaxed strongly $\eta-$monotone with constant $\alpha > 0$ and suppose $E(u)$ be a strongly preinvex function with $\beta > 0$. If $0 < \rho < \frac{\beta}{\alpha + \lambda}$, $0 < \mu < \frac{\beta}{\alpha + \lambda}$ and the above assumption holds, then the approximate solution obtained from Algorithm 1 converges to a solution $u \in K$ of the uniform regular invex equilibrium problem (2.1).

Proof Let $u \in K_r$ be a solution of (2.1), see [1,19]. Then

$$\rho F(u, v) + \rho \lambda \left\| \eta(v, u) \right\|^2 \geq 0, \ \forall v \in K_r \tag{3.5}$$

$$\mu F(u, v) + \mu \lambda \left\| \eta(v, u) \right\|^2 \geq 0, \ \forall v \in K_r, \tag{3.6}$$

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where ρ and μ are both positive constants.Taking $v = u_{n+1}$ in (3.5) and $v = u$ in (3.2), we have

$$\rho F(u, u_{n+1}) + \rho\lambda \|\eta (u_{n+1}, u)\|^2 \geq 0, \ \ \forall v \in K_r, \tag{3.7}$$

$$\rho F(w_n, u) + \rho\lambda \|\eta (u, w_n)\|^2 + \left\langle E^{'}(u_{n+1}) - E^{'}(w_n), \eta (u, u_{n+1}) \right\rangle \geq 0, \ \ \forall v \in K_r. \tag{3.8}$$

Now we consider the generalized Bergman function as

$$B(u, z) = E(u) - E(z) - \left\langle E^{'}(z), \eta(u, z) \right\rangle \geq \beta \|\eta (u, z)\|^2,$$

where we have used the fact that the function $E(u)$ is strongly preinvex.

Combining $(3.7) - (3.9)$ and (3.4), we have

$$B(u, w_n) - B(u, u_{n+1}) = E(u_{n+1}) - E(w_n) - \left\langle E^{'}(w_n), \eta(u, w_n) \right\rangle + \left\langle E^{'}(u_{n+1}), \eta(u, u_{n+1}) \right\rangle$$

$$= E(u_{n+1}) - E(w_n) - \left\langle E^{'}(w_n) - E^{'}(u_{n+1}), \eta(u, u_{n+1}) \right\rangle - \left\langle E^{'}(w_n), \eta(u_{n+1}, w_n) \right\rangle$$

$$\geq \beta \|\eta (u_{n+1}, w_n)\|^2 + \left\langle E^{'}(u_{n+1}) - E^{'}(w_n), \eta(u, u_{n+1}) \right\rangle$$

$$\geq \beta \|\eta (u_{n+1}, w_n)\|^2 - \rho\{F(u, u_{n+1}) + F(w_n, u)\} - \rho\lambda\{\|\eta (u_{n+1}, u)\|^2 + \|\eta (u, w_n)\|^2\}$$

$$\geq \{\beta - \rho(\alpha + \lambda)\} \|\eta (u_{n+1}, w_n)\|^2 .$$

since $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$.

In a similar way, we have

$$B(u, u_n) - B(u, w_n) \geq \beta \|\eta (u_n, w_n)\|^2 - \mu\{F(u, w_n) + F(u_n, u)\} - \mu\lambda\{\|\eta (u, u_n)\|^2 + \|\eta (w_n, u)\|^2\}$$

$$\geq \{\beta - \mu(\alpha + \lambda)\} \|\eta (u_n, w_n)\|^2 .$$

since $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$.

If $u_{n+1} = w_n = u_n$, then clearly u_n is a solution of (2.1). Otherwise, for $0 < \rho < \frac{\beta}{\alpha+\lambda}$ and $0 < \mu < \frac{\beta}{\alpha+\lambda}$, the sequences $B(u, w_n) - B(u, u_{n+1})$ and $B(u, u_n) - B(u, w_n)$ are nonnegative and we must have

$$\lim_{n\to\infty}(\|\eta (u_{n+1}, w_n)\|) = 0 \text{ and } \lim_{n\to\infty}(\|\eta (w_n, u_n)\|) = 0.$$

Thus

$$\lim_{n\to\infty} \|\eta (u_{n+1}, u_n)\| = \lim_{n\to\infty}(\|\eta (u_{n+1}, w_n)\|) + \lim_{n\to\infty}(\|\eta (w_n, u_n)\|) = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_j}\}$, and let $\{u_{n_j}\}$ be a subsequence converging towards $\bar{u}$. Now using the technique of Zhu and Marcotte [46], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the uniformly regular invex equilibrium problem (2.1).

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ON SOME ZWEIER I-CONVERGENT DIFFERENCE SEQUENCE SPACES

KULDIP RAJ AND SURUCHI PANDOH

ABSTRACT. In the present paper we introduce some Zweier ideal convergent difference sequence spaces defined by a sequence of Orlicz functions. We study some topological properties and inclusion relations between these spaces. We also make an effort to study these sequence spaces over n-normed spaces.

1. Introduction and Preliminaries

An ideal convergence is a generalization of statistical convergence. The ideal convergence plays a vital role not only in pure mathematics but also in other branches of science especially in computer science, information theory, biological science, dynamical systems, geographic information systems, and motion planning in robotics. Kostyrko et al. [14] introduced the notion of $I-$convergence based on the structure of admissible ideal I of subsets of natural number $\mathbb{N}$. For more details about ideal convergence see ([10], [15], [24], [27]).

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (Power set of X) is said to be an ideal if I is additive, that is, $A, B \in I \Rightarrow A \cup B \in I$ and $A \in I, B \subseteq A \Rightarrow B \in I$.

A non empty family of sets $\mathcal{L}(I) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \in \mathcal{L}(I)$, for $A, B \in \mathcal{L}(I)$ we have $A \cap B \in \mathcal{L}(I)$ and for each $A \in \mathcal{L}(I)$ and $A \subseteq B$ implies $B \in \mathcal{L}(I)$. An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal I is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

A subset A of $\mathbb{N}$ is said to have asymptotic density $\delta(A)$ if $\delta(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k)$ exists, where χ_A is the characteristic function of A.

If we take $I = I_f = \{A \subseteq \mathbb{N} : A$ is a finite subset$\}$, then I_f is a non trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence.

If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denotes the asymptotic density of the set A, then I_δ is a non trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

Recently, some new sequence spaces by mean of the matrix domain have been discussed by Malkowsky [17] and many others. Sengonul [25] defined the sequence $y = (y_i)$ which is frequently used as the Z^p-transformation of the sequence $x = (x_i)$, that is,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k); \\ 1-p, & (i-1 = k)(i, k \in \mathbb{N}); \\ 0, & \text{otherwise.} \end{cases}$$

Sengonul [25] introduced the Zweier sequence spaces Z and Z_0 as follows:

$$Z = \{x = (x_k) \in w : Z^p x \in c\}$$

and

$$Z_0 = \{x = (x_k) \in w : Z^p x \in c_0\},$$

where w denotes the space of all real or complex sequences $x = (x_k)$.

By l_∞, c, c_0 we denote the classes of all bounded, convergent and null sequence spaces. The notion of difference sequence spaces was introduced by Kızmaz [12] who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [3] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let n be a nonnegative integer, then for $\mathfrak{z} = c, c_0$ and l_∞, we have sequence spaces

$$\mathfrak{z}(\Delta^n) = \{x = (x_k) \in w : (\Delta^n x_k) \in \mathfrak{z}\},$$

where $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+v}.$$

Taking $n = 1$, we get the spaces studied by Et and Çolak [3]. For more details about sequence spaces see ([19], [21], [22]) and references therein.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

    (1) $p(x) \geq 0$ for all $x \in X$,
    (2) $p(-x) = p(x)$ for all $x \in X$,
    (3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
    (4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [28], Theorem 10.4.2, pp. 183).

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to define the following sequence space. Orlicz sequence space is defined as

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \ \text{for some } \rho > 0 \Big\}.$$

The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

It is shown by Lindenstrauss and Tzafriri [16] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$ which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leq p < \infty$. An Orlicz function M satisfies Δ_2−condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L) M(u)$ for all values of $u \geq 0$. Later on, Orlicz sequence spaces were investigated by Parashar and Chaudhary [20], Esi [4], Tripathy et al. [26], Bhardwaj and Singh [1], Et [2], Esi and Et [5] and many others. Also if M is an Orlicz function, then we may write $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Definition 1.1. A sequence $(x_k) \in w$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$. In this case we write $I - \lim x_k = L$.

Definition 1.2. A sequence $(x_k) \in w$ is said to be I-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Definition 1.3. A sequence $(x_k) \in w$ is said to be I-Cauchy if for every $\epsilon > 0$, there exist a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.4. A sequence $(x_k) \in w$ is said to be I-bounded if there exist $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 1.5. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| < 1$ for all $k \in \mathbb{N}$.

Definition 1.6. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of $\mathbb{N}$.

Definition 1.7. A sequence space E is said to be sequence algebra if $(x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.8. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Definition 1.9. Let $K = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ and let E be a sequence space. A $K-$step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (x_k) \in E\}$.

Definition 1.10. A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all the elements in λ_K^E, that is, y is in the canonical preimage of λ_K^E if and only if y is a canonical preimage of some $x \in \lambda_K^E$.

Definition 1.11. A sequence space E is said to be monotone if it contain the canonical preimages of its step spaces.

Throughout the article Z^I, Z_0^I, Z_∞^I, m_Z^I and $m_{Z_0}^I$ represents Zweier I-convergent, Zweier I-null, Zweier bounded I-convergent and Zweier bounded I-null sequence spaces, respectively.

Lemma 1.12. *The sequence space E is solid implies that E is monotone (See [13]).*

Let $\mathcal{M} = (M_n)$ be a sequence of Orlicz functions, $u = (u_n)$ be a sequence of strictly positive real numbers and $p = (p_n)$ be a bounded sequence of positive real numbers. In the present paper we define the following sequence spaces:

$Z^I(\mathcal{M}, \Delta^r, p, u) =$

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I \right\} \text{ for some } L \in \mathbb{C}.$$

$$Z_0^I(\mathcal{M}, \Delta^r, p, u) = \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I \right\}.$$

$Z_\infty^I(\mathcal{M}, \Delta^r, p, u) =$

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \exists\, K > 0, \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n} \geq K \right\} \in I \right\}.$$

Also, $m_Z^I(\mathcal{M}, \Delta^r, p, u) = Z^I(\mathcal{M}, \Delta^r, p, u) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u) = Z_0^I(\mathcal{M}, \Delta^r, p, u) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$.

If we take $(p_k) = 1$ for all k, then we have
$Z^I(\mathcal{M}, \Delta^r, p, u) = Z^I(\mathcal{M}, \Delta^r, u)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u) = Z_0^I(\mathcal{M}, \Delta^r, u)$, $Z_\infty^I(\mathcal{M}, \Delta^r, p, u) = Z_\infty^I(\mathcal{M}, \Delta^r, u)$, $m_Z^I(\mathcal{M}, \Delta^r, p, u) = Z^I(\mathcal{M}, \Delta^r, u) \cap Z_\infty^I(\mathcal{M}, \Delta^r, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u) = Z_0^I(\mathcal{M}, \Delta^r, u) \cap Z_\infty^I(\mathcal{M}, \Delta^r, u)$.

If we take $(u_k) = 1$ for all k, then we have
$Z^I(\mathcal{M}, \Delta^r, p, u) = Z^I(\mathcal{M}, \Delta^r, p)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u) = Z_0^I(\mathcal{M}, \Delta^r, p)$, $Z_\infty^I(\mathcal{M}, \Delta^r, p, u) = Z_\infty^I(\mathcal{M}, \Delta^r, p)$, $m_Z^I(\mathcal{M}, \Delta^r, p, u) = Z^I(\mathcal{M}, \Delta^r, p) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u) = Z_0^I(\mathcal{M}, \Delta^r, p) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p)$.

If we take $(M_n) = M$, $(p_n) = 1$, $(u_n) = 1$ for all $n \in \mathbb{N}$ and $r = 0$, then we get the sequence spaces defined by Hazarika et al. [11].

The main purpose of this paper is to introduce the sequence spaces $Z^I(\mathcal{M}, \Delta^r, p, u)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$. We study some topological properties and inclusion relations between these sequence spaces. An effort is made to study these sequence spaces over n-normed spaces in the section third of this paper.

2. **Main Results**

Theorem 2.1. *Let $\mathcal{M} = (M_n)$ be a sequence of Orlicz functions, $u = (u_n)$ be a sequence of strictly positive real numbers and $p = (p_n)$ be a bounded sequence of positive real numbers. Then the spaces $Z^I(\mathcal{M}, \Delta^r, p, u)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$ are linear over the complex field $\mathbb{C}$.*

Proof. We shall prove the result for the space $Z^I(\mathcal{M}, \Delta^r, p, u)$. Let $x = (x_k)$, $y = (y_k) \in Z^I(\mathcal{M}, \Delta^r, p, u)$ and α, β be scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L_1|}{\rho_1} \right]^{p_n} \geq \epsilon \right\} \in I \text{ for some } L_1 \in \mathbb{C}.$$

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n - L_2|}{\rho_2} \right]^{p_n} \geq \epsilon \right\} \in I \text{ for some } L_2 \in \mathbb{C}.$$

that is,

$$(2.1) \qquad A_1 = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L_1|}{\rho_1} \right]^{p_n} \geq \frac{\epsilon}{2} \right\} \in I.$$

$$(2.2) \qquad A_2 = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n - L_2|}{\rho_2} \right]^{p_n} \geq \frac{\epsilon}{2} \right\} \in I.$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_n)$ is a nondecreasing and convex function, we have

$$M_n \left[\frac{|(\alpha u_n(Z^p(\Delta^r x))_n + \beta u_n(Z^p(\Delta^r y))_n) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right]^{p_n}$$

$$\leq M_n \left[\frac{|\alpha||u_n(Z^p(\Delta^r x))_n - L_1|}{\rho_3} + \frac{|\beta||u_n(Z^p(\Delta^r y))_n - L_2|}{\rho_3} \right]^{p_n}$$

$$\leq M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L_1|}{\rho_1} \right]^{p_n} + M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n - L_2|}{\rho_2} \right]^{p_n}.$$

Now from equations (2.1) and (2.2), we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|(\alpha u_n(Z^p(\Delta^r x))_n + \beta u_n(Z^p(\Delta^r y))_n) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right]^{p_n} \geq \epsilon \right\}$$

$$\subset A_1 \cup A_2 \in I.$$

Therefore, $\alpha x + \beta y \in Z^I(\mathcal{M}, \Delta^r, p, u)$. Hence $Z^I(\mathcal{M}, \Delta^r, p, u)$ is a linear space. Similarly, we can prove that $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$ are linear spaces. $\qquad\square$

Theorem 2.2. *The spaces $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ are paranormed spaces, with the paranorm g defined by*

$$g(x) = \inf \left\{ \rho^{\frac{p_n}{H}} : \sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{\frac{p_n}{H}} \leq 1, \ \text{ for some } \rho > 0 \right\},$$

where $H = \max\{1, \sup_n p_n\}$.

Proof. Clearly $g(-x) = g(x)$ and $g(0) = 0$. Let $x = (x_k)$ and $y = (y_k) \in m_Z^I(\mathcal{M}, \Delta^r, p, u)$. Now for $\rho_1, \rho_2 > 0$ we denote

$$A_3 = \left\{ \rho_1 : \sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho_1} \right]^{p_n} \leq 1 \right\}$$

and

$$A_4 = \left\{ \rho_2 : \sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n|}{\rho_2} \right]^{p_n} \leq 1 \right\}.$$

If $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we obtain

$$M_n \left[\frac{|u_n(Z^p(\Delta^r (x+y)))_n|}{\rho} \right]^{p_n} \leq \frac{\rho_1}{\rho_1 + \rho_2} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho_1} \right]^{p_n}$$

$$+ \frac{\rho_2}{\rho_1 + \rho_2} M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n|}{\rho_2} \right]^{p_n}.$$

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Therefore,

$$\sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r(x+y)))_n|}{\rho} \right]^{p_n} \leq 1$$

and

$$
\begin{aligned}
g(x+y) &= \inf\left\{ (\rho_1+\rho_2)^{\frac{p_n}{H}} : M_n \left[\frac{|u_n(Z^p(\Delta^r(x+y)))_n|}{\rho} \right]^{\frac{p_n}{H}} \leq 1,\ \rho_1 \in A_3, \rho_2 \in A_4 \right\} \\
&\leq \inf\left\{ (\rho_1)^{\frac{p_n}{H}} : M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho_1} \right]^{\frac{p_n}{H}} \leq 1,\ \rho_1 \in A_3 \right\} \\
&+ \inf\left\{ (\rho_2)^{\frac{p_n}{H}} : M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n|}{\rho_2} \right]^{\frac{p_n}{H}} \leq 1,\ \rho_2 \in A_4 \right\} \\
&= g(x) + g(y).
\end{aligned}
$$

Let $t^m \to L$ and let $g(\Delta^r x^m - \Delta^r x) \to 0$ as $m \to \infty$. To prove that $g(t^m \Delta^r x^m - L\Delta^r x) \to 0$ as $m \to \infty$. We put

$$A_5 = \left\{ \rho_m > 0 : \sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r x^m))_n|}{\rho_m} \right]^{p_n} \leq 1 \right\}$$

and

$$A_6 = \left\{ \rho_s > 0 : \sup_n M_n \left[\frac{|u_n(Z^p(\Delta^r(x^m - x)))_n|}{\rho_s} \right]^{p_n} \leq 1 \right\}$$

by the continuity of $\mathcal{M} = (M_n)$, we observe that

$$
\begin{aligned}
M_n \left[\frac{|u_n(Z^p(t^m\Delta^r x^m - L\Delta^r x))_n|}{|t^m - L|\rho_m + |L|\rho_s} \right]^{p_n} &\leq M_n \left[\frac{|u_n(Z^p(t^m\Delta^r x^m - L\Delta^r x^m))_n|}{|t^m - L|\rho_m + |L|\rho_s} \right]^{p_n} \\
&+ M_n \left[\frac{|u_n(Z^p(L\Delta^r x^m - L\Delta^r x))_n|}{|t^m - L|\rho_m + |L|\rho_s} \right]^{p_n} \\
&\leq \frac{|t^m - L|\rho_m}{|t^m - L|\rho_m + |L|\rho_s} M_n \left[\frac{|u_n(Z^p(\Delta^r x^m)_n|}{\rho_m} \right]^{p_n} \\
&+ \frac{|L|\rho_s}{|t^m - L|\rho_m + |L|\rho_s} M_n \left[\frac{|u_n(Z^p(\Delta^r x^m - \Delta^r x))_n|}{\rho_s} \right]^{p_n}.
\end{aligned}
$$

From the above inequality it follows that

$$\sup_n M_n \left[\frac{|u_n(Z^p(t^m\Delta^r x^m - L\Delta^r x))_n|}{|t^m - L|\rho_m + |L|\rho_s} \right]^{p_n} \leq 1$$

and consequently

$$
\begin{aligned}
(2.3) \qquad g(t^m\Delta^r x^m - L\Delta^r x) &= \inf\{ [|t^m - L|\rho_m + |L|\rho_s]^{\frac{p_n}{H}} : \rho_m \in A_5, \rho_s \in A_6 \} \\
&\leq |t^m - L|^{\frac{p_n}{H}} \inf\{ (\rho_m)^{\frac{p_n}{H}} : \rho_m \in A_5 \} \\
&+ |L|^{\frac{p_n}{H}} \inf\{ (\rho_s)^{\frac{p_n}{H}} : \rho_s \in A_6 \} \\
&\leq \max\{1, |t^m - L|^{\frac{p_n}{H}}\} g(\Delta^r x^m) + \max\{1, |L|^{\frac{p_n}{H}}\} \\
&\quad g(\Delta^r x^m - \Delta^r x).
\end{aligned}
$$

It is obvious that $g(\Delta^r x^m) \leq g(\Delta^r x) + g(\Delta^r x^m - \Delta^r x)$, for all $m \in \mathbb{N}$. Clearly, the right hand side of the relation (2.3) tends to 0 as $m \to \infty$ and the result follows. This completes the proof. $\qquad\square$

Theorem 2.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Orlicz functions satisfying Δ_2-condition. Then*
(i) $W(\mathcal{M}_2, \Delta^r, p, u) \subseteq W(\mathcal{M}_1 o \mathcal{M}_2, \Delta^r, p, u)$.
(ii) $W(\mathcal{M}_1, \Delta^r, p, u) \cap W(\mathcal{M}_2, \Delta^r, p, u) = W(\mathcal{M}_1 + \mathcal{M}_2, \Delta^r, p, u)$ for $W = Z^I, Z_0^I, Z_\infty^I$.

Proof. (i) Let $x = (x_k) \in Z^I(\mathcal{M}_2, \Delta^r, p, u)$. Let $\epsilon > 0$, be given. For some $\rho > 0$, we have

$$(2.4) \qquad \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I \right\}.$$

Let $\epsilon > 0$ and choose $0 < \delta < 1$ such that $M_1(t) \leq \epsilon$ for $0 \leq t \leq \delta$. We define

$$y_n = M_2 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]$$

and consider $\lim\limits_{n \in \mathbb{N},\ 0 \leq y_n \leq \delta} [M_1(y_n)]^{p_n} = \lim\limits_{n \in \mathbb{N},\ y_n \leq \delta} [M_1(y_n)]^{p_n} + \lim\limits_{n \in \mathbb{N},\ y_n > \delta} [M_1(y_n)]^{p_n}.$
We have

$$(2.5) \qquad \lim_{n \in \mathbb{N},\ y_n \leq \delta} [M_1(y_n)]^{p_n} \leq [M_1(2)]^G + \lim_{n \in \mathbb{N},\ y_n \leq \delta} [(y_n)]^{p_n},\ G = \sup_n p_n.$$

For second summation (i.e. $y_n > \delta$), we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since M_1 is nondecreasing and convex, it follows that

$$M_1(y_n) < M_1 \left(1 + \frac{y_n}{\delta} \right) \leq \frac{1}{2} M_1(2) + \leq \frac{1}{2} M_1 \left(\frac{2 y_n}{\delta} \right).$$

Since M_1 satisfies the Δ_2-condition, we can write

$$M_1(y_n) < \frac{1}{2} K \frac{y_n}{\delta} M_1(2) + \frac{1}{2} K \frac{y_n}{\delta} M_1(2) = K \frac{y_n}{\delta} M_1(2).$$

We get the following estimates:

$$(2.6) \qquad \lim_{n \in \mathbb{N},\ y_n > \delta} [M_1(y_n)]^{p_n} \leq \max\{1, (K \delta^{-1} M_1(2))^H\} + \lim_{n \in \mathbb{N},\ y_n > \delta} [y_n]^{p_n}.$$

From equations (2.4), (2.5) and (2.6), it follows that

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left(M_2 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \right) \geq \epsilon \right\} \in I \right\}.$$

Hence $Z^I(\mathcal{M}_2, \Delta^r, p, u) \subseteq Z^I(\mathcal{M}_1 o \mathcal{M}_2, \Delta^r, p, u)$.

(ii) Let $x = (x_k) \in Z^I(\mathcal{M}_1, \Delta^r, p, u) \cap Z^I(\mathcal{M}_2, \Delta^r, p, u)$. Let $\epsilon > 0$, be given. Then there exists $\rho > 0$ such that

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I \right\}$$

and

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I \right\}.$$

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The rest of the proof follows from the following relations:

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} (M_1 + M_2) \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\}$$

$$\cup \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\}$$

Taking $M_2(x) = x$ and $M_1(x) = M(x)$ for all $x \in [0, \infty)$, we have the following result. □

Corollary 2.4. $W \subseteq W(\mathcal{M}, \Delta^r, p, u)$, where $W = Z^I, Z_0^I$.

Theorem 2.5. *The spaces $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ are solid and monotone.*

Proof. We shall prove the result for $Z_0^I(\mathcal{M}, \Delta^r, p, u)$. For $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$, the result can be proved similarly. Let $x = (x_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u)$, then there exists $\rho > 0$ such that

$$(2.7) \qquad \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I.$$

Let (α_k) be a sequence scalar with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from (2.7) and the following inequality

$$M_n \left[\frac{|\alpha_k u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n} \leq |\alpha_k| M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n}$$

$$\leq M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho} \right]^{p_n}$$

for all $n \in \mathbb{N}$. The space $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ is monotone follows from the Lemma (1.12). □

Theorem 2.6. *The spaces $Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ are neither monotone nor solid in general.*

Proof. The proof of this result follows from the following example. Let $I = I_f$, $M_n(x) = x$ for all $x \in [0, \infty)$, $p = (p_n) = 1$, $u = (u_n) = 1$, for all n and $r = 0$. Consider the K-step space T_k of T defined as follows.
Let $(x_k) \in T$ and $(y_k) \in T_k$ be such that

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is odd.} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined by $x_k = \frac{1}{2}$ for all $k \in \mathbb{N}$. Then $(x_k) \in Z^I(\mathcal{M}, \Delta^r, p, u)$ but its K- step space preimage does not belongs to $Z^I(\mathcal{M}, \Delta^r, p, u)$. Thus $Z^I(\mathcal{M}, \Delta^r, p, u)$ are not monotone. Hence, $Z^I(\mathcal{M}, \Delta^r, p, u)$ is not solid by Lemma (1.12). □

Theorem 2.7. *The spaces $Z^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ are sequence algebra.*

Proof. We prove that $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ is sequence algebra. For the space $Z^I(\mathcal{M}, \Delta^r, p, u)$, the result can be proved similarly. Let $x = (x_k), y = (y_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u)$. Then

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{\rho_1} \right]^{p_n} \geq \epsilon \right\} \in I, \ \text{ for some } \rho_1 > 0.$$

and

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r y))_n|}{\rho_2} \right]^{p_n} \geq \epsilon \right\} \in I, \text{ for some } \rho_2 > 0.$$

Let $\rho = \rho_1 \rho_2 > 0$. Then we can show that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n((Z^p(\Delta^r xy))_n)|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I.$$

Thus, $(x_k y_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u)$. Hence $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ is sequence algebra. $\square$

Theorem 2.8. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Then $Z_0^I(\mathcal{M}, \Delta^r, p, u) \subset Z^I(\mathcal{M}, \Delta^r, p, u) \subset Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$ and the inclusions are proper.*

Proof. Let $x = (x_k) \in Z^I(\mathcal{M}, \Delta^r, p, u)$. Then there exists $L \in \mathbb{C}$ and $\rho > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I.$$

We have

$$M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n|}{2\rho} \right]^{p_n} \leq \frac{1}{2} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} + M_n \frac{1}{2} \left[\frac{|L|}{\rho} \right]^{p_n}.$$

Taking supremum over n on both sides we get $x = (x_k) \in Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$. The inclusion $Z_0^I(\mathcal{M}, \Delta^r, p, u) \subset Z^I(\mathcal{M}, \Delta^r, p, u)$ is obvious. The inclusion is proper follows from the following example.
Let $I = I_d$, $M_n(x) = x^2$ for all $x \in [0, \infty)$, $u = (u_n) = 1$, $p = (p_n) = 1$ and $r = 0$ for all $n \in \mathbb{N}$.
(a) Consider the sequence (x_k) defined by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in Z^I(\mathcal{M}, \Delta^r, p, u)$ but $(x_k) \notin Z_0^I(\mathcal{M}, \Delta^r, p, u)$.
(b) Consider the sequence (y_k) defined as

$$y_k = \begin{cases} 2, & \text{if } k \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in Z_\infty^I(\mathcal{M}, \Delta^r, p, u)$ but $(y_k) \notin Z^I(\mathcal{M}, \Delta^r, p, u)$. $\square$

Theorem 2.9. *The spaces $Z^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_0^I(\mathcal{M}, \Delta^r, p, u)$ are not convergence free in general.*

Proof. The proof of this result follows from the following example. Let $I = I_f$, $M_n(x) = x^2$ for all $x \in [0, \infty)$, $u = (u_n) = 1$, $p = (p_n) = 1$ and $r = 0$ for all $n \in \mathbb{N}$.
Consider the sequence (x_k) and (y_k) defined by $x_k = \frac{1}{k^2}$ and $y_k = k^2$ for all $k \in \mathbb{N}$. Then (x_k) belongs to $Z^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_0^I(\mathcal{M}, \Delta^r, p, u)$, but (y_k) does not belongs to both $Z^I(\mathcal{M}, \Delta^r, p, u)$ and $Z_0^I(\mathcal{M}, \Delta^r, p, u)$. Hence, the spaces are not convergence free. $\square$

Theorem 2.10. *$m_Z^I(\mathcal{M}, \Delta^r, p, u)$ is a closed subspace of $l_\infty(\mathcal{M}, \Delta^r, p, u)$.*

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ such that $x^{(i)} \to x$. We show that $x \in m_Z^I(\mathcal{M}, \Delta^r, p, u)$. Since $(x_k^{(i)}) \in m_Z^I(\mathcal{M}, \Delta^r, p, u)$, then there exists a_i such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(i)}))_n - a_i|}{\rho} \right]^{p_n} \geq \epsilon \right\} \in I.$$

We need to show that
(i) (a_i) converges to a.

(ii) If $U = \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x))_n - L|}{\rho} \right]^{p_n} \geq \epsilon \right\} \right\}$, then $U^c \in I$.

(i) Since $(x_k^{(i)})$ be a Cauchy sequence in $m_Z^I(\mathcal{M}, \Delta^r, p, u)$, then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_n M_n \left[\frac{|u_n((Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n)|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \text{ for all } i, j \geq k_0.$$

For a given $\epsilon > 0$, we have

$$B_{ij} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n((Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n)|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$B_i = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(i)} - a_i))_n|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$B_j = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(j)} - a_j))_n|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

Then $B_{ij}, B_i, B_j \in I$. Let $B^c = B_{ji}^c \cup B_i^c \cup B_j^c$, where

$$B = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(a_i - a_j)|}{\rho} \right]^{p_n} < \epsilon \right\}$$

Then $B^c \in I$. We choose $n_0 \in B^c$, then for each $i, j \geq n_0$ we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(a_i - a_j)|}{\rho} \right]^{p_n} < \epsilon \right\}$$

$$\supseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(i)} - a_i))_n|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(j)} - a_j))_n|}{\rho} \right]^{p_n} < \frac{\epsilon}{3} \right\}.$$

Then (a_i) is a Cauchy sequence of scalars in $\mathbb{C}$ and so there exists a scalar $a \in \mathbb{C}$ such that $a_i \to a$ as $i \to \infty$.

(ii) Let $0 < \delta < 1$ be given. We show that if

$$U = \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x - a))_n|}{\rho} \right]^{p_n} < \delta \right\} \right\}, \text{ then } U^c \in I.$$

Since $x^{(i)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$(2.8) \quad P = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n((Z^p(\Delta^r x^{(q_0)}))_n - (Z^p(\Delta^r x))_n)|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}$$

which implies that $P^c \in I$. The number q_0 can be so chosen that together with (2.8),

we have

$$Q = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(a_{q_0} - a)|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Thus, we have $Q^c \in I$. Since

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n|}{\rho} \right]^{p_n} \geq \delta \right\} \in I.$$

Then we have a subset $S \in \mathbb{N}$ such that $S^c \in I$, where

$$S = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Let $U^c = P^c \cup Q^c \cup S^c$, where

$$U = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x - a))_n|}{\rho} \right]^{p_n} < \delta \right\}.$$

Therefore, for each $n \in U^c$, we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x - a))_n|}{\rho} \right]^{p_n} < \delta \right\}$$

$$\supseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n\big((Z^p(\Delta^r x^{(q_0)}))_n - (Z^p(\Delta^r x))_n\big)|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\frac{|u_n(a_{q_0} - a)|}{\rho} \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Hence, the result follows.

Note that the inclusion $m_Z^I(\mathcal{M}, \Delta^r, p, u) \subset l_\infty(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u) \subset l_\infty(\mathcal{M}, \Delta^r, p, u)$ are strict. So in view of Theorem (2.10). We have the following result.   □

Theorem 2.11. *The spaces $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ are nowhere dense in $l_\infty(\mathcal{M}, \Delta^r, p, u)$.*

Theorem 2.12. *The spaces $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ are not separable.*

Proof. We shall prove the result for the space $m_Z^I(\mathcal{M}, \Delta^r, p, u)$. Let A be an infinite subset of $\mathbb{N}$ of increasing natural numbers such that $A \in I$. Let

$$p_n = \begin{cases} 1, & \text{if } n \in A \\ 2, & \text{otherwise.} \end{cases}$$

$M_n(x) = x$, $u = (u_n) = 1$ and $r = 0$ for all n. Let $\{P_0 = (x_n) : x_n = 0 \text{ or } 1, n \in A, x_n = 0, \text{ otherwise}\}$. Since A is infinite, so P_0 is uncountable. Consider the class of open balls $B_1 = \{B(z, \frac{1}{z}) : z \in P_0\}$. Let C_1 be an open cover of $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ as well as $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$. Thus, $m_Z^I(\mathcal{M}, \Delta^r, p, u)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u)$ are not separable.   □

3. Sequence spaces over n-normed spaces

The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that for n-normed spaces one can see in Misiak [18]. Since then, many others have studied this concept and obtained various results, see Gunawan ([7], [8]) and Gunawan and Mashadi [9]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of reals of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{R}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. A complete n-normed space is said to be n-Banach space. For more details about $n-$ normed space see [23] and references therein.

Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space and $W(n - X)$ denotes the space of X-valued sequences. Let $\mathcal{M} = (M_n)$ be a sequence of Orlicz functions, $u = (u_n)$ be a sequence of strictly positive real numbers and $p = (p_n)$ be a bounded sequence of positive real numbers. For some $\rho > 0$, we define the following classes of sequences:

$$Z^I(\mathcal{M}, \Delta^r, p, u, ||\cdot, \cdots, \cdot||) =$$

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n (Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I, L \in \mathbb{R} \right\}$$

$$Z_0^I(\mathcal{M}, \Delta^r, p, u, ||\cdot, \cdots, \cdot||) =$$

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n (Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \right\}$$

$$Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) =$$

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \exists K > 0, \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq K \right\} \in I \right\}$$

Also, $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$
and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$.

If we take $(p_n) = 1$ for all n then we have
$Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$
$= Z_0^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|)$, $Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z_\infty^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|)$,
$m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|)$
and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z_0^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, u, \|\cdot, \cdots, \cdot\|)$.

If we take $(u_n) = 1$ for all n then we get
$Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$
$= Z_0^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|)$, $Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z_\infty^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|)$,
$m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|)$
and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = Z_0^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|) \cap Z_\infty^I(\mathcal{M}, \Delta^r, p, \|\cdot, \cdots, \cdot\|)$.

The main aim of this section is to study some topological properties and inclusion relations between the spaces $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$.

Theorem 3.1. *Let $\mathcal{M} = (M_n)$ be a sequence of Orlicz functions, $u = (u_n)$ be a sequence of strictly positive real numbers and $p = (p_n)$ be a bounded sequence of positive real numbers. Then the spaces $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are linear over the real field $\mathbb{R}$ of real numbers.*

Proof. We shall prove the result for the space $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Let $x = (x_k)$, $y = (y_k) \in Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and let α, β be scalars. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x))_n - L_1}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \text{ for some } L_1 \in \mathbb{R}.$$

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r y))_n - L_2}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \text{ for some } L_2 \in \mathbb{R}.$$

that is,

$$(3.1) \qquad D_1 = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x))_n - L_1}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \frac{\epsilon}{2} \right\} \in I.$$

$$(3.2) \qquad D_2 = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r y))_n - L_2}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \frac{\epsilon}{2} \right\} \in I.$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_n)$ is a nondecreasing and convex function, we have

$$M_n \left[\left\| \frac{(\alpha u_n(Z^p(\Delta^r x))_n + \beta u_n(Z^p(\Delta^r y))_n) - (\alpha L_1 + \beta L_2)}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$\leq M_n\left[|\alpha|\left\|\frac{u_n(Z^p(\Delta^r x))_n - L_1}{\rho_3}, z_1, \cdots, z_{n-1}\right\| + |\beta|\left\|\frac{u_n(Z^p(\Delta^r y))_n - L_2}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

$$\leq M_n\left[\left\|\frac{u_n(Z^p(\Delta^r x))_n - L_1}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} + M_n\left[\left\|\frac{u_n(Z^p(\Delta^r y))_n - L_2}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}.$$

Now from equations (3.1) and (3.2), we have

$$\left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n\left[\left\|\frac{(\alpha u_n(Z^p(\Delta^r x))_n + \beta u_n(Z^p(\Delta^r y))_n) - (\alpha L_1 + \beta L_2)}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}\right.$$

$$\left.\geq \epsilon\right\} \subset D_1 \cup D_2 \in I.$$

Therefore, $\alpha x + \beta y \in Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Hence $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is a linear space. Similarly, we can prove that $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are linear spaces. $\square$

Theorem 3.2. *The spaces $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are paranormed spaces, with the paranorm g defined by*

$$g(x) = \inf\left\{\rho^{\frac{p_n}{H}} : \sup_n M_n\left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{\frac{p_n}{H}} \leq 1, \ \text{for some } \rho > 0\right\},$$

where $H = \max\{1, \sup_n p_n\}$.

Proof. Clearly $g(-x) = g(x)$ and $g(0) = 0$. Let $x = (x_k)$ and $y = (y_k) \in m_Z^I(\mathcal{M}, \Delta^r, p, u)$. Now for $\rho_1, \rho_2 > 0$ we denote

$$D_3 = \left\{\rho_1 : \sup_n M_n\left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \leq 1\right\}$$

and

$$D_4 = \left\{\rho_2 : \sup_n M_n\left[\left\|\frac{u_n(Z^p(\Delta^r y))_n}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \leq 1\right\}.$$

If $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we obtain

$$M_n\left[\left\|\frac{u_n(Z^p(\Delta^r(x+y)))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} M_n\left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

$$+ \frac{\rho_2}{\rho_1 + \rho_2} M_n\left[\left\|\frac{u_n(Z^p(\Delta^r y))_n}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

which in terms give us, $\sup_n M_n\left[\left\|\dfrac{u_n(Z^p(\Delta^r(x+y)))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \leq 1$

and

$$g(x+y) = \inf\left\{(\rho_1 + \rho_2)^{\frac{p_n}{H}} : M_n\left[\left\|\frac{u_n(Z^p(\Delta^r(x+y)))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{\frac{p_n}{H}} \leq 1\right.$$

$$\rho_1 \in D_3, \rho_2 \in D_4 \Bigg\}$$

$$\leq \quad \inf\left\{ (\rho_1)^{\frac{p_n}{H}} : M_n\left[\left\| \frac{u_n(Z^p(\Delta^r x))_n}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right]^{\frac{p_n}{H}} \leq 1 \; \rho_1 \in D_3 \right\}$$

$$+ \quad \inf\left\{ (\rho_2)^{\frac{p_n}{H}} : M_n\left[\left\| \frac{u_n(Z^p(\Delta^r y))_n}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right]^{\frac{p_n}{H}} \leq 1 \; \rho_2 \in D_4 \right\}$$

$$= \quad g(x) + g(y).$$

Let $t^m \to L$ and let $g(\Delta^r x^m - \Delta^r x) \to 0$ as $m \to \infty$. To prove that $g(t^m \Delta^r x_k - L\Delta^r x) \to 0$ as $m \to \infty$. We put

$$D_5 = \left\{ \rho_m > 0 : \sup_n M_n\left[\left\| \frac{u_n(Z^p(\Delta^r x^m))_n}{\rho_m}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \leq 1 \right\}$$

and

$$D_6 = \left\{ \rho_s > 0 : \sup_n M_n\left[\left\| \frac{u_n(Z^p(\Delta^r(x^m - x)))_n}{\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \leq 1 \right\}$$

by the continuity of $\mathcal{M} = (M_n)$, we observe that

$$M_n\left[\left\| \frac{u_n(Z^p(t^m \Delta^r x^m - L\Delta^r x))_n}{|t^m - L|\rho_m + |L|\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$\leq M_n\left[\left\| \frac{u_n(Z^p(t^m \Delta^r x^m - L\Delta^r x^m))_n}{|t^m - L|\rho_m + |L|\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$+ M_n\left[\left\| \frac{u_n(Z^p(L\Delta^r x^m - L\Delta^r x))_n}{|t^m - L|\rho_m + |L|\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$\leq \frac{|t^m - L|\rho_m}{|t^m - L|\rho_m + |L|\rho_s} M_n\left[\left\| \frac{u_n(Z^p(\Delta^r x^m))_n}{\rho_m}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$+ \frac{|L|\rho_s}{|t^m - L|\rho_m + |L|\rho_s} M_n\left[\left\| \frac{u_n(Z^p(\Delta^r(x^m - x)))_n}{\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}.$$

From the above inequality it follows that

$$\sup_n M_n\left[\left\| \frac{u_n(Z^p(t^m \Delta^r x^m - L\Delta^r x))_n}{|t^m - L|\rho_m + |L|\rho_s}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \leq 1$$

and consequently

$$
\begin{aligned}
(3.3) \qquad g(t^m \Delta^r x^m - L\Delta^r x) &= \inf\{[|t^m - L|\rho_m + |L|\rho_s]^{\frac{p_n}{H}} : \rho_m \in D_5, \rho_s \in D_6\} \\
&\leq |t^m - L|^{\frac{p_n}{H}} \inf\{(\rho_m)^{\frac{p_n}{H}} : \rho_m \in D_5\} \\
&+ |L|^{\frac{p_n}{H}} \inf\{(\rho_s)^{\frac{p_n}{H}} : \rho_s \in D_6\} \\
&\leq \max\{1, |t^m - L|^{\frac{p_n}{H}}\} g(\Delta^r x^m) + \max\{1, |L|^{\frac{p_n}{H}}\} \\
& \quad g(\Delta^r x^m - \Delta^r x).
\end{aligned}
$$

It is obvious that $g(\Delta^r x^m) \leq g(\Delta^r x) + g(\Delta^r x^m - \Delta^r x)$, for all $m \in \mathbb{N}$. Clearly, the right hand side of the relation (3.3) tends to 0 as $m \to \infty$ and the result follows. This completes the proof. $\qquad\square$

Theorem 3.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be Orlicz functions that satisfies Δ_2-condition. Then*
(i) $W(\mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subseteq W(\mathcal{M}_1 o \mathcal{M}_2, , \Delta^r p, u, \|\cdot, \cdots, \cdot\|)$.
(ii) $W(\mathcal{M}_1, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \cap W(\mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) = W(\mathcal{M}_1 + \mathcal{M}_2, \Delta^r, p, u,$
$\|\cdot, \cdots, \cdot\|)$ for $W = Z^I, Z_0^I, Z_\infty^I$.

Proof. (i) Let $x = (x_k) \in Z^I(\mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Let $\epsilon > 0$, be given. For some $\rho > 0$, we have

$$(3.4) \quad \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\left\| \frac{u_n (Z^p (\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \right\}.$$

Let $\epsilon > 0$ and choose $0 < \delta < 1$ such that $M_1(t) \leq \epsilon$ for $0 \leq t \leq \delta$. We define

$$y_n = M_2 \left[\left\| \frac{u_n (Z^p (\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]$$

and consider $\lim_{n \in \mathbb{N}, \, 0 \leq y_n \leq \delta} [M_1(y_n)]^{p_n} = \lim_{n \in \mathbb{N}, \, y_n \leq \delta} [M_1(y_n)]^{p_n} + \lim_{n \in \mathbb{N}, \, y_n > \delta} [M_1(y_n)]^{p_n}$.
We have

$$(3.5) \qquad \lim_{n \in \mathbb{N}, \, y_n \leq \delta} [M_1(y_n)]^{p_n} \leq [M_1(2)]^G + \lim_{n \in \mathbb{N}, \, y_n \leq \delta} [(y_n)]^{p_n}, \; G = \sup_n p_n.$$

For second summation (i.e. $y_n > \delta$), we have $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$. Since M_1 is nondecreasing and convex, it follows that

$$M_1(y_n) < M_1 \left(1 + \frac{y_n}{\delta} \right) \leq \frac{1}{2} M_1(2) + \leq \frac{1}{2} M_1 \left(\frac{2 y_n}{\delta} \right).$$

Since M_1 satisfies the Δ_2-condition, we can write

$$M_1(y_n) < \frac{1}{2} K \frac{y_n}{\delta} M_1(2) + \frac{1}{2} K \frac{y_n}{\delta} M_1(2) = K \frac{y_n}{\delta} M_1(2).$$

We get the following estimates:

$$(3.6) \qquad \lim_{n \in \mathbb{N}, \, y_n > \delta} [M_1(y_n)]^{p_n} \leq \max\{1, (K \delta^{-1} M_1(2))^H\} + \lim_{n \in \mathbb{N}, \, y_n > \delta} [y_n]^{p_n}.$$

From equations (3.4), (3.5) and (3.6), it follows that

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left(M_2 \left[\left\| \frac{u_n (Z^p (\Delta^r x)) - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \right) \geq \epsilon \right\} \in I \right\}.$$

Hence $Z^I(\mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subseteq Z^I(\mathcal{M}_1 o \mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$.

(ii) Let $x = (x_k) \in Z^I(\mathcal{M}_1, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \cap Z^I(\mathcal{M}_2, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Let $\epsilon > 0$, be given. Then there exists $\rho > 0$ such that

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left[\left\| \frac{u_n (Z^p (\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \right\}$$

and

$$\left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\left\| \frac{u_n (Z^p (\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I \right\}.$$

The rest of the proof follows from the following relations:

$$\left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} (M_1 + M_2) \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_1 \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\}$$

$$\cup \left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_2 \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\}$$

Taking $M_2(x) = x$ and $M_1(x) = M(x)$ for all $x \in [0, \infty)$, we have the following result. $\square$

Corollary 3.4. $W \subseteq W(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, where $W = Z^I, Z_0^I$.

Theorem 3.5. *The spaces* $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *and* $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *are solid and monotone.*

Proof. We shall prove the result for $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. For $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, the result can be proved similarly. Let $x = (x_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, then there exists $\rho > 0$ such that

$$(3.7) \qquad \left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\} \in I.$$

Let (α_k) be a sequence scalar with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, we have

$$M_n \left[\left\|\frac{\alpha_k u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \leq |\alpha_k| M_n \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

$$\leq M_n \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n}$$

for all $n \in \mathbb{N}$. The space $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is monotone follows from the Lemma (1.12). $\square$

Theorem 3.6. *The spaces* $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *and* $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *are neither monotone nor solid in general.*

Proof. The proof of result follows from the Theorem (2.6). $\square$

Theorem 3.7. *The spaces* $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *and* $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *are sequence algebra.*

Proof. We prove that $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is sequence algebra. For the space $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, the result can be proved similarly. Let $x = (x_k), y = (y_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Then

$$\left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\|\frac{u_n(Z^p(\Delta^r x))_n}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\} \in I, \text{ for some } \rho_1 > 0.$$

and

$$\left\{n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\|\frac{u_n(Z^p(\Delta^r y))_n}{\rho_2}, z_1, \cdots, z_{n-1}\right\|\right]^{p_n} \geq \epsilon\right\} \in I, \text{ for some } \rho_2 > 0.$$

Let $\rho = \rho_1 \rho_2 > 0$. Then we can show that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n\big((Z^p(\Delta^r xy))_n\big)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I.$$

Thus, $(x_k y_k) \in Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Hence $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is sequence algebra. $\qquad\square$

Theorem 3.8. *Let $M = (M_n)$ be a sequence of Orlicz function. Then $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ $\subset Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subset Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and the inclusions are proper.*

Proof. Let $x = (x_k) \in Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Then there exists $L \in \mathbb{R}$ and $\rho > 0$ such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_k(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I.$$

We have

$$M_n \left[\left\| \frac{|u_n(Z^p(\Delta^r x))_n|}{2\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \leq \frac{1}{2} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}$$

$$+ \ M_n \frac{1}{2} \left[\left\| \frac{L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n}.$$

Taking supremum over n on both sides we get $x = (x_k) \in Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. The inclusion $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subset Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is obvious. The inclusion is proper follows from the following example.

Let $I = I_d$, $M_n(x) = x^2$ for all $x \in [0, \infty)$, $u = (u_n) = 1$, $p = (p_n) = 1$ and $r = 0$ for all $n \in \mathbb{N}$.

(a) Consider the sequence (x_k) defined by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ but $(x_k) \notin Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$.

(b) Consider the sequence (y_k) defined as

$$y_k = \begin{cases} 2, & \text{if } k \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in Z_\infty^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ but $(y_k) \notin Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. $\qquad\square$

Theorem 3.9. *The spaces $Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $Z_0^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are not convergence free in general.*

Proof. The proof follows from Theorem (2.9). $\qquad\square$

Theorem 3.10. *$m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ is a closed subspace of $l_\infty(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$.*

Proof. Let $(x_k^{(i)})$ be a Cauchy sequence in $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ such that $x^{(i)} \to x$. We show that $x \in m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$. Since $(x_k^{(i)}) \in m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, then there exists a_i such that

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(i)}))_n - a_i}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \in I.$$

We need to show that

(i) (a_i) converges to a.

(ii) If $\mathcal{V} = \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x))_n - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \epsilon \right\} \right\},$

then $\mathcal{V}^c \in I$.

(i) Since $(x_k^{(i)})$ be a Cauchy sequence in $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$, then for a given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_n M_n \left[\left\| \frac{u_n\big((Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n\big)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \text{ for all } i,j \geq k_0$$

For a given $\epsilon > 0$, we have

$$F_{ij} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n\big((Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n\big)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$F_i = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(i)} - a_i))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$F_j = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(j)} - a_j))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

Then $F_{ij}, F_i, F_j \in I$. Let $F^c = F_{ji}^c \cup F_i^c \cup F_j^c$, where

$$F = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(a_i - a_j)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \epsilon \right\}$$

Then $F^c \in I$. We choose $n_0 \in F^c$, then for each $i,j \geq n_0$ we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(a_i - a_j)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \epsilon \right\}$$

$$\supseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(i)} - a_i))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(i)}))_n - (Z^p(\Delta^r x^{(j)}))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(j)} - a_j))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \frac{\epsilon}{3} \right\}.$$

Then (a_i) is a Cauchy sequence of scalars in $\mathbb{R}$ and so there exists a scalar $a \in \mathbb{R}$ such that $a_i \to a$ as $i \to \infty$.

(ii) Let $0 < \delta < 1$ be given. We show that if

$$\mathcal{V} = \left\{ x = (x_k) : \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x - a))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \delta \right\} \right\}, \text{ then}$$

$\mathcal{V}^c \in I$.

Since $x^{(i)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$(3.8) \qquad \mathcal{G} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n\big((Z^p(\Delta^r x^{(q_0)}))_n - (Z^p(\Delta^r x))_n\big)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \right.$$

$$\left. \left(\frac{\delta}{3D} \right)^H \right\}$$

which implies that $\mathcal{G}^c \in I$. The number q_0 can be so choosen that together with (3.8), we have

$$\mathcal{H} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(a_{q_0} - a)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Thus, we have $\mathcal{H}^c \in I$. Since

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} \geq \delta \right\} \in I.$$

Then we have a subset $\mathcal{R} \in \mathbb{N}$ such that $\mathcal{R}^c \in I$, where

$$\mathcal{R} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Let $\mathcal{V}^c = \mathcal{G}^c \cup \mathcal{H}^c \cup \mathcal{R}^c$, where

$$\mathcal{V} = \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x - a))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \delta \right\}$$

Therefore, for each $n \in \mathcal{V}^c$, we have

$$\left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x - a))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \delta \right\}$$

$$\supseteq \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n((Z^p(\Delta^r x^{(q_0)}))_n - (Z^p(\Delta^r x))_n)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(Z^p(\Delta^r x^{(q_0)} - a_{q_0}))_n}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}$$

$$\cap \left\{ n \in \mathbb{N} : \sum_{n=1}^{\infty} M_n \left[\left\| \frac{u_n(a_{q_0} - a)}{\rho}, z_1, \cdots, z_{n-1} \right\| \right]^{p_n} < \left(\frac{\delta}{3D} \right)^H \right\}.$$

Hence, the result follows.

Note that the inclusion $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subset l_\infty(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|) \subset l_\infty(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are strict. So in view of Theorem (3.10). We have the following result. $\qquad \square$

Remark: The spaces $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ and $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ are nowhere dense in $l_\infty(\mathcal{M}, p, u, \|\cdot, \cdots, \cdot\|)$.

Theorem 3.11. *The spaces* $m_Z^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *and* $m_{Z_0}^I(\mathcal{M}, \Delta^r, p, u, \|\cdot, \cdots, \cdot\|)$ *are not separable.*

Proof. The proof follows from the Theorem (2.12). $\qquad \square$

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School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
E-mail address: kuldipraj68@gmail.com
E-mail address: suruchi.pandoh87@gmail.com

Composition operators on some general families of function spaces

R. A. Rashwan

Assiut University, Faculty of Science, Department of Mathematics Box 71515 Assiut - Egypt

A. El-Sayed Ahmed

Sohag University, Faculty of Science, Department of Mathematics, 82524 Sohag, Egypt

Current Address: Taif University, Faculty of Science, Mathematics Department

Box 888 El-Hawiyah, El-Taif 5700, Saudi Arabia e-mail: ahsayed80@hotmail.com

M. A. Bakhit

University of Alazhar, Assiut Branch, Faculty of Science Department of Mathematics Egypt

Abstract

In this paper, we give Carleson measure characterizations of the classes $F^{\#}(p,q,s)$, then we use these characterizations to study boundedness and compactness of the composition operator C_{ϕ} from the α-normal classes $\mathcal{N}^{\alpha}$ into $F^{\#}(p,q,s)$ classes. Moreover, necessary and sufficient conditions for C_{ϕ} from the spherical Dirichlet class $\mathcal{D}^{\#}$ to the classes $F^{\#}(p,q,s)$ to be compact are given in terms of the map ϕ.

1   Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ its boundary. Let $\mathcal{M}(\mathbb{D})$ be the class of all meromorphic functions and $\mathcal{H}(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. Also, $dA(z)$ be the normalized area measure, so that $A(\mathbb{D}) \equiv 1$. The Green's function is defined by

$$g(z,a) = \log\frac{1}{|\varphi_a(z)|},$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the automorphism of $\mathbb{D}$ interchanging the points zero and $a \in \mathbb{D}$. Note that $\varphi_a(\varphi_a(z)) = z$, and so $\varphi_a^{-1}(z) = \varphi_a(z)$. For all $z, a \in \mathbb{D}$, we know that

$$(1-|z|^2)|\varphi_a'(z)| = 1 - |\varphi_a(z)|^2 \leq 2g(z,a). \tag{1}$$

For a function $f \in \mathcal{M}(\mathbb{D})$, a natural analogue of $f'(z)$ is the spherical derivative

$$f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}.$$

For $\alpha \in (0,\infty)$ we say that the function $f \in \mathcal{M}(\mathbb{D})$ belongs to the α-normal class $\mathcal{N}^{\alpha} = \mathcal{N}^{\alpha}(\mathbb{D})$, if

$$\|f\|_{\mathcal{N}^{\alpha}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}f^{\#}(z) < \infty \quad (\text{see } [18]).$$

The little α-normal class $\mathcal{N}_0^{\alpha} = \mathcal{N}_0^{\alpha}(\mathbb{D})$ consists of all $f \in \mathcal{N}^{\alpha}$ such that

$$\lim_{|z|\to1}(1-|z|^2)^{\alpha}f^{\#}(z) = 0 \quad (\text{see } [18]).$$

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For some results on $\mathcal{N}^\alpha$ class, we refer to [1, 8, 11].

For some $r \in (0,1)$, we say that the function $f \in \mathcal{M}(\mathbb{D})$ belongs to the class of spherical Bloch functions $\mathcal{B}^\#$ (see [21]) if

$$\|f\|^2_{\mathcal{B}^\#} := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}(a,r)} (f^\#(z))^2 dA(z) < \infty,$$

where, $\mathbb{D}(a,r) = \{z : \rho(z,a) = |\varphi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| < r\}$ is the pseudo-hyperbolic disc with center a and radius r. We clearly have $\mathcal{N} \subset \mathcal{B}^\#$. In the analytic case, we know that the corresponding definition with $f^\#(z)$ replaced by $|f'(z)|$ both give the space of Bloch functions $\mathcal{B}$. In the meromorphic case, the situation is different.

A function $f \in \mathcal{M}(\mathbb{D})$ belongs to the spherical Dirichlet class $\mathcal{D}^\#$ (see [12]) if

$$\|f\|^2_{\mathcal{D}^\#} := \int_{\mathbb{D}} (f^\#(z))^2 dA(z) < \infty.$$

Let $f \in \mathcal{M}(\mathbb{D})$, $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If

$$\|f\|^p_{F^\#(p,q,s)} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^\#(z))^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

then $f \in F^\#(p,q,s)$ (see [18]). Moreover, if

$$\lim_{|a| \to 1} \int_{\mathbb{D}} (f^\#(z))^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F^\#_0(p,q,s)$. The classes $F^\#(p,q,s)$ were intensively studied by Rättyä in [18]. The classes $F^\#(p,q,s)$ and $F^\#_0(p,q,s)$ behave sometimes in a quite different way than their analytic counterparts. For example, the Green's function $g(z,a)$ cannot be replaced by the weight function $(1-|\varphi_a(z)|^2)$ in general (see [2] and [21]). Therefore the classes $M^\#(p,q,s)$ and $M^\#_0(p,q,s)$ defined as follows.

A function $f \in \mathcal{M}(\mathbb{D})$ is said to belong to the classes $M^\#(p,q,s)$ if

$$\|f\|^p_{p,q,s} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [f^\#(z)]^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) < \infty,$$

If

$$\lim_{|a| \to 1} \int_{\mathbb{D}} [f^\#(z)]^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) = 0,$$

then $f \in M^\#_0(p,q,s)$. For the inequality $1-|\varphi_a(z)|^2 \leq 2g(z,a)$, for all $z, a \in \mathbb{D}$, we see that $F^\#(p,q,s) \subset M^\#(p,q,s)$ and $F^\#_0(p,q,s) \subset M^\#_0(p,q,s)$, but, as mentioned before, the opposite inclusions do not hold in general.

If E is any set, the characteristic function χ_E of the set E is given by

$$\chi_E(z) = \begin{cases} 1, & \text{if } z \in E; \\ 0, & \text{if } z \notin E. \end{cases}$$

The function $\chi_E(z)$ is measurable if and only if E is measurable (see [19]).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $C > 0$ such that $\|T(f)\|_Y \leq C\|f\|_X$ for all maps $f \in X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y contained in $\mathcal{M}(\mathbb{D})$ or $\mathcal{H}(\mathbb{D})$, $T : X \to Y$ is compact if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to a function $f \in Y$.

Two quantities A and B are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$, written as $A \approx B$. Throughout this paper, the letter C denotes different positive constants which are not necessarily the same from line to line.

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The composition operator $C_\phi : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ is defined by $C_\phi = f \circ \phi$, where ϕ is a holomorphic self-map of the open unit disc in $\mathbb{C}$. There have been several attempts to study compactness and boundedness of composition operators in many function spaces (see e.g.[4, 5, 6, 7, 9, 10, 13, 17, 23] and others). Lappan and Xiao in [14] studied the boundedness of the composition operator C_ϕ in meromorphic case from the normal class $\mathcal{N}$ to their Möbius invariant subclasses.

In this paper, we show some clear differences between the analytic and the meromorphic cases of the classes $F(p, q, s)$. We give a Carleson measure characterization of the composition operator C_ϕ on $F^\#(p, q, s)$. Also, we characterize by means of s-Carleson measures and compact s-Carleson measures the boundedness and compactness of composition operator $C_\phi : \mathcal{M}(\mathbb{D}) \to \mathcal{M}(\mathbb{D})$ from α-normal classes $\mathcal{N}^\alpha$ to $F^\#(p, q, s)$ classes. Moreover, we investigate the compactness of composition operator C_ϕ from spherical Dirichlet class $\mathcal{D}^\#$ to $F^\#(p, q, s)$.

Now we quote several auxiliary results which will be used in the proofs of our main results. The following lemma can be found in [18], Corollary 3.2.3.

Lemma 1.1 Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$.

- If $M^\#(p, q, s) \subset \mathcal{N}^{\frac{q+2}{p}}$, then $F^\#(p, q, s) = M^\#(p, q, s)$.

- If $M_0^\#(p, q, s) \subset \mathcal{N}_0^{\frac{q+2}{p}}$, then $F_0^\#(p, q, s) = M_0^\#(p, q, s)$.

Taking the same technique used in lemma 1 of [14], we get the following lemma:

Lemma 1.2 For $\alpha > 0$, there are two functions $f_1, f_2 \in \mathcal{N}^\alpha$ such that

$$M_0 = \inf_{z \in \mathbb{D}} \left[(1 - |z|^2)^\alpha \left(f_1^\#(z) + f_2^\#(z) \right) \right] > 0,$$

where M_0 is a positive constant.

Using the standard arguments similar to those outlined in proposition 3.11 of [4], we have the following lemma:

Lemma 1.3 Let $\alpha \in (0, \infty), 0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map and $X = F^\#(p, q, s)$ or $B_p^\#$. Then $C_\phi : \mathcal{N}^\alpha \to X$ is compact if and only if for every bounded sequence $\{f_n\} \subset \mathcal{N}^\alpha$, which converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, $\lim_{n \to \infty} \|C_\phi f_n\|_X = 0$.

2 The classes $F^\#(p, q, s)$ and s-Carleson measures

In this section, we show some clear differences between the analytic and the meromorphic cases of the classes $F(p, q, s)$.

For $0 < s < \infty$, we say that a positive measure μ defined on $\mathbb{D}$ is a bounded s-Carleson measure, provided $\mu(S(I)) = O(|I|^s)$ for all sub-arcs I of $\partial \mathbb{D}$, where $|I|$ denotes the arc length of $I \subset \partial \mathbb{D}$ and $S(I)$ denotes the Carleson box based on I, that is

$$S(I) = \left\{ z \in \mathbb{D} : 1 - \frac{|I|}{2\pi} \leq |z| < 1, \ \frac{z}{|z|} \in I \right\}.$$

For $0 < s < \infty$, a positive Borel measure μ on $\mathbb{D}$ is a bounded s-Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi_a'(z)|^s d\mu < \infty; \tag{2}$$

and μ is a compact s-Carleson measure if and only if

$$\lim_{|a| \to 1} \int_{\mathbb{D}} |\varphi_a'(z)|^s d\mu = 0. \tag{3}$$

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Lemma 2.1 *Let* $0 < p < \infty, -2 < q < \infty$ *and* $f \in \mathcal{M}(\mathbb{D})$. *Then, for* $0 < s \le 1$, $f \in F^{\#}(p, q, s)$ *if and only if* $d\nu = (f^{\#}(z))^p(1 - |z|^2)^{q+s}dA(z)$ *is a bounded s-Carleson measure on* $\mathbb{D}$.

Proof: From (2) it follows that $d\nu = (f^{\#}(z))^p(1 - |z|^2)^{q+s}dA(z)$ is a bounded s-Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)|^s d\nu < \infty. \tag{4}$$

Using (1), we see that

$$
\begin{aligned}
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)|^s d\nu &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'_a(z)|^s (f^{\#}(z))^p(1 - |z|^2)^{q+s}dA(z) \\
&= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^p(1 - |z|^2)^q(1 - |\varphi_a(z)|^2)^s dA(z) \\
&\le C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^p(1 - |z|^2)^q g^s(z, a)dA(z) < \infty.
\end{aligned}
$$

If $d\nu$ is a bounded s-Carleson measure, then $f \in \mathcal{N}^{\frac{q+2}{p}}$, for $0 < s \le 1$ we have $f \in F^{\#}(p, q, s)$.

For $1 < s < \infty$ the meromorphic case differs from the analytic case (see Theorem 2.4 in [22]) as the following lemma shows.

Lemma 2.2 *Let* $0 < p < \infty, -2 < q < \infty$ *and* $f \in \mathcal{M}(\mathbb{D})$. *Then, for* $1 < s < \infty$, *the following statements are not equivalent:*

(i) $f \in F^{\#}(p, q, s)$;

(ii) $d\nu = (f^{\#}(z))^p(1 - |z|^2)^{q+s}dA(z)$ *is a bounded s-Carleson measure on* $\mathbb{D}$.

Proof: An easy calculation shows that $\mathcal{N}^{\alpha} \subset F^{\#}(p, \alpha p - 2, s)$ for $0 < p < \infty, 0 < \alpha < \infty$ and $1 < s < \infty$ (see theorems 3.2.1, 3.3.3 and also corollary 3.2.2 in [18]). In the case $\alpha = \frac{q+2}{p}$, we have that

$$\mathcal{N}^{\frac{q+2}{p}} \subset F^{\#}(p, q, s). \tag{5}$$

However, in this case it is possible that $F^{\#}(p, q, s) \subset \mathcal{N}^{\frac{q+2}{p}}$ as in the case of the class $Q_s^{\#}$ (see [2] and [3], Lemma 1).

Hence, $F^{\#}(p, q, s) = \mathcal{N}^{\frac{q+2}{p}}$ (for all $0 < p < \infty, -2 < q < \infty$ and $1 < s < \infty$), $d\nu$ being a bounded s-Carleson measure does not imply that $f \in F^{\#}(p, q, s)$ by Lemma 1.1. Thus the lemma is proved.

In case of compact s-Carleson measures we have a more complete description which is similar to the analytic case.

Lemma 2.3 *Let* $0 < p < \infty, -2 < q < \infty$ *and* $f \in \mathcal{M}(\mathbb{D})$. *Then, for* $0 < s < \infty$, *the following are equivalent:*

(i) $f \in F_0^{\#}(p, q, s)$;

(ii) $d\nu = (f^{\#}(z))^p(1 - |z|^2)^{q+s}dA(z)$ *is a compact s-Carleson measure on* $\mathbb{D}$.

Proof: (i)$\Rightarrow$ (ii). If $f \in F_0^{\#}(p, q, s)$, then, by (1) and (2), $d\nu$ is a compact s-Carleson measure on $\mathbb{D}$.
(ii)$\Rightarrow$ (i). For any $r \in (0, 1)$ and $\mathbb{D}(a, r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$, we have

$$
\begin{aligned}
I(a) &= \int_{\mathbb{D}} (f^{\#}(z))^p(1 - |z|^2)^q g^s(z, a)dA(z) \\
&= \left\{ \int_{\mathbb{D}(a,r)} + \int_{\mathbb{D} \setminus \mathbb{D}(a,r)} \right\} (f^{\#}(z))^p(1 - |z|^2)^q g^s(z, a)dA(z) \\
&= I_1(a) + I_2(a).
\end{aligned}
$$

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Suppose now that $d\nu$ is a compact s-Carleson measure on $\mathbb{D}$. Since

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|} \begin{cases} \geq \log 4 > 1, & |\varphi_a(z)| \leq \frac{1}{4}; \\ \leq 4(1 - |\varphi_a(z)|^2), & |\varphi_a(z)| > \frac{1}{4}, \end{cases}$$

we obtain for $s_0 = \max(s, 2)$ and $\mathbb{D}(a, \frac{1}{4}) = \{z \in \mathbb{D} : |\varphi_a(z)| < \frac{1}{4}\}$ that

$$\begin{aligned} I_1(a) &= \int_{\mathbb{D}(a,\frac{1}{4})} (f^\#(z))^p (1 - |z|^2)^q g^s(z,a) dA(z) \\ &\leq \int_{\mathbb{D}(a,\frac{1}{4})} (f^\#(z))^p (1 - |z|^2)^q g^{s_0}(z,a) dA(z) \end{aligned} \tag{6}$$

and

$$\begin{aligned} I_2(a) &= \int_{\mathbb{D}\setminus\mathbb{D}(a,\frac{1}{4})} (f^\#(z))^p (1 - |z|^2)^q g^s(z,a) dA(z) \\ &\leq 4^s \int_{\mathbb{D}\setminus\mathbb{D}(a,\frac{1}{4})} (f^\#(z))^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z). \end{aligned} \tag{7}$$

Hence, by (6) and (7), we get

$$\begin{aligned} I(a) &= \int_{\mathbb{D}} (f^\#(z))^p (1 - |z|^2)^q g^s(z,a) dA(z) \\ &\leq C \int_{\mathbb{D}} (f^\#(z))^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z). \end{aligned} \tag{8}$$

which means that $f \in F_0^\#(p,q,s)$. Hence the proof is completed.

3   Boundedness of composition operators from $\mathcal{N}^\alpha$ to $F^\#(p,q,s)$

We are going to work with composition operators acting on $\mathcal{N}^\alpha$ and $\mathcal{N}_0^\alpha$. For a function $\phi : \mathbb{D} \to \mathbb{D}$, we say that $C_\phi : \mathcal{N}^\alpha(\mathcal{N}_0^\alpha) \to F^\#(p,q,s)$ is bounded if

$$\|C_\phi f\|_{F^\#(p,q,s)} \leq C\|f\|_{\mathcal{N}^\alpha}, \quad f \in \mathcal{N}^\alpha(\mathcal{N}_0^\alpha).$$

For $0 < p < \infty, -2 < q < \infty$ and $0 < s < \infty$, we define the following notations:

$$d\mu_\phi(\alpha,p,q,s) = \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^{q+s} dA(z),$$

$$\Phi_\phi(\alpha,p,q,s,a) = \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z,a) dA(z),$$

$$\Psi_\phi(\alpha,p,q,s,a) = \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z).$$

Lemma 3.1 *If $\Psi_\phi(\alpha,p,q,s,a)$ is a finite at some point $a \in \mathbb{D}$, then, $\Psi_\phi(\alpha,p,q,s,a)$ is a continuous function of $a \in \mathbb{D}$.*

Proof: The proof is very similar to that of the lemma 2.3 in [20].

We proof the following result.

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Theorem 3.1 *Let* $2 \leq p < \infty, -2 < q < \infty$ *and* $0 < s < \infty$. *Then, the following statements are equivalent:*

(i) $C_\phi : \mathcal{N}^\alpha \to F^{\#}(p, q, s)$ *is bounded;*

(ii) *For* $p \geq 2$, $C_\phi : \mathcal{N}_0^\alpha \to F^{\#}(p, q, s)$ *is bounded;*

(iii) $\displaystyle\sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, q, s, a) < \infty;$

(iv) $d\mu_\phi(\alpha, p, q, s)$ *is a bounded s-Carleson measure.*

Proof: (i) $\Rightarrow$ (ii) is clear since $\mathcal{N}_0^\alpha \subset \mathcal{N}^\alpha$.

(ii) $\Rightarrow$ (iii). First, note that if $f \in \mathcal{N}^\alpha$ and $f_r(z) = f(rz)$ for $r \in (0,1)$, then $f_r \in \mathcal{N}_0^\alpha$ with $\|f_r\|_{\mathcal{N}^\alpha} \leq \|f\|_{\mathcal{N}^\alpha}$. Secondly, for the two functions f_1 and f_2 given by the Lemma 1.2, we have for $k = 1, 2$, and $a \in \mathbb{D}$,

$$\|C_\phi\|^p \|f_k\|_{\mathcal{N}^\alpha}^p \geq \|C_\phi\|^p \|(f_k)_r\|_{\mathcal{N}^\alpha}^p \geq \|C_\phi(f_k)_r\|_{F^{\#}(p,q,s)}^p$$

$$\geq \int_\mathbb{D} \left(f_k^{\#}(r\phi(z))\right)^p |r\phi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z).$$

Because of the Lemma 1.2,

$$(1 - |z|^2)^\alpha \left(f_1^{\#}(z) + f_2^{\#}(z)\right) \geq M_0 > 0, \quad \text{for all } z \in \mathbb{D},$$

we have that

$$\frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} \leq \frac{2^p}{M_0^p} \left((f_1 \circ \phi)^{\#}(z)\right)^p + \frac{2^p}{M_0^p} \left((f_2 \circ \phi)^{\#}(z)\right)^p.$$

Thus,

$$\sup_{a \in \mathbb{D}} \int_\mathbb{D} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \frac{2^p}{M_0^p} \sup_{a \in \mathbb{D}} \int_\mathbb{D} \left((f_1 \circ \phi)^{\#}(z)\right)^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$+ \frac{2^p}{M_0^p} \sup_{a \in \mathbb{D}} \int_\mathbb{D} \left((f_2 \circ \phi)^{\#}(z)\right)^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \frac{2^p}{M_0^p} \|C_\phi\|^p \left(\|f_1\|_{\mathcal{N}^\alpha}^p + \|f_2\|_{\mathcal{N}^\alpha}^p\right).$$

This inequality and Fatou's lemma imply (iii) holds. For all f with $\|f\|_{\mathcal{N}^\alpha} \leq 1$, it is clear that the boundedness of C_ϕ is identical to $\|C_\phi f\|_{F^{\#}(p,q,s)} < \infty$. Let $f \in \mathcal{N}^\alpha$ with $\|f\|_{\mathcal{N}^\alpha} \leq 1$, we have

$$\|C_\phi f\|_{F^{\#}(p,q,s)}^p = \sup_{a \in \mathbb{D}} \int_\mathbb{D} \left((f \circ \phi)^{\#}(z)\right)^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \|f\|_{\mathcal{N}^\alpha}^p \sup_{a \in \mathbb{D}} \int_\mathbb{D} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, q, s, a) < \infty.$$

Hence (iii) implies (i). So, it remains to check (iii) implies (iv), suppose that (iii) holds. By (2), we see that (iv) is equivalent to $\sup_{a \in \mathbb{D}} \Psi_\phi(\alpha, p, q, s, a) < \infty$. From (1), we get that

$$\sup_{a \in \mathbb{D}} \int_\mathbb{D} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z)$$

$$= \sup_{a \in \mathbb{D}} \int_\mathbb{D} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^{q+s} |\varphi_a'(z)|^s dA(z)$$

$$\leq \sup_{a \in \mathbb{D}} \int_\mathbb{D} |\varphi_a'(z)|^s d\mu_\phi(\alpha, p, q, s) < \infty,$$

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then (iv) holds. Now we suppose that (iv) is true. For $f \in \mathcal{N}^\alpha$, we get

$$
\begin{aligned}
\|C_\phi f\|_{F^\#(p,q,s)}^p &\leq C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} ((f \circ \phi)^\#(z))^p (1-|z|^2)^q (1-|\varphi_a(z)|^2)^s dA(z) \\
&\leq C\|f\|_{\mathcal{N}^\alpha}^p \sup_{a \in \mathbb{D}} \Psi_\phi(\alpha, p, q, s, a),
\end{aligned}
$$

and so $C_\phi : \mathcal{N}^\alpha \to F^\#(p,q,s)$ is bounded. Thus (iv) implies (i) and the proof is complete.

Theorem 3.2 *Let* $2 \leq p < \infty, -2 < q < \infty$ *and* $0 < s < \infty$. *Then, the following statements are equivalent:*

(i) $C_\phi : \mathcal{N}_0^\alpha \to F_0^\#(p,q,s)$ *is bounded;*

(ii) $\sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, q, s, a) < \infty$ *and* $\phi \in F_0^\#(p,q,s)$.

Proof: Suppose that (ii) holds, and let $f \in \mathcal{N}_0^\alpha$. Then $C_\phi : \mathcal{N}_0^\alpha \to F_0^\#(p,q,s)$ is bounded by Theorem 3.1. We only show that $C_\phi f \in F_0^\#(p,q,s)$. For any $\varepsilon > 0$, we can choose $r \in (0,1)$ such that

$$
(f^\#(w))^p (1-|w|^2)^{p\alpha} < \varepsilon, \quad \text{for } |w| > r.
$$

Thus

$$
\sup_{a \in \mathbb{D}} \int_{|\phi(z)|>r} ((f \circ \phi)^\#(z))^p (1-|z|^2)^q g^s(z,a) dA(z)
$$

$$
\leq \varepsilon \sup_{a \in \mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p (1-|z|^2)^q}{(1-|\phi(z)|^2)^{p\alpha}} g^s(z,a) dA(z)
$$

$$
\leq \varepsilon \sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, q, s, a).
$$

Since $\phi \in F_0^\#(p,q,s)$, for r as above, we have

$$
\lim_{|a| \to 1} \int_{|\phi(z)| \leq r} ((C_\phi f)^\#(z))^p (1-|z|^2)^q g^s(z,a) dA(z)
$$

$$
\leq \|f\|_{\mathcal{N}^\alpha}^p \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} \frac{|\phi'(z)|^p (1-|z|^2)^q}{(1-|\phi(z)|^2)^{p\alpha}} g^s(z,a) dA(z)
$$

$$
\leq \frac{\|f\|_{\mathcal{N}^\alpha}^p}{(1-r^2)^{p\alpha}} \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} |\phi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0.
$$

Combining above, we get that $C_\phi f \in F_0^\#(p,q,s)$.

Conversely, if $C_\phi : \mathcal{N}_0^\alpha \to F_0^\#(p,q,s)$ is bounded, then $C_\phi : \mathcal{N}_0^\alpha \to F^\#(p,q,s)$ is bounded. By Theorem 3.1, we get that $\sup_{a \in \mathbb{D}} \Phi_\phi(\alpha, p, q, s, a) < \infty$. Finally, by considering $f(z) = z$, the boundedness of C_ϕ implies $\phi \in F_0^\#(p,q,s)$. The proof is complete.

4 Compactness of composition operators from $\mathcal{N}^\alpha$ to $F^\#(p,q,s)$

Theorem 4.1 *Let* $\alpha \in (0, \infty), 0 < p, s < \infty, -2 < q < \infty$, *with* $q + s > -1$ *and let* $\phi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function. Then, for* $f \in \mathcal{M}(\mathbb{D})$ *the following are equivalent:*

(i) $C_\phi : \mathcal{N}^\alpha \to F^\#(p,q,s)$ *is compact;*

(ii) $C_\phi : \mathcal{N}_0^\alpha \to F^\#(p,q,s)$ *is compact;*

(iii) $\phi \in F^\#(p,q,s)$ *and*

$$
\lim_{r \to 1} \sup_{a \in \mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|^2)^q g^s(z,a) dA(z) = 0. \tag{9}
$$

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Proof: The proof follows from the proof of Theorem 1 in [23] by replacing the derivative f' with the spherical derivative $f^{\#}$.

Theorem 4.2 *Let* $\alpha \in (0, \infty), p \geq 1, -2 < q < \infty$, *and* $\phi : \mathbb{D} \to \mathbb{D}$. *Then, for* $f \in \mathcal{M}(\mathbb{D})$ *the following statements are equivalent:*

(i) $C_{\phi} : \mathcal{N}^{\alpha} \to F_0^{\#}(p, q, s)$ *is bounded;*

(ii) $C_{\phi} : \mathcal{N}^{\alpha} \to F_0^{\#}(p, q, s)$ *is compact;*

(iii) $\lim\limits_{|a| \to 1} \Phi_{\phi}(\alpha, p, q, s, a) = 0;$

(iv) $d\mu_{\phi}(\alpha, p, q, s)$ *is a compact s-Carleson measure;*

(v) $C_{\phi} : \mathcal{N}_0^{\alpha} \to F_0^{\#}(p, q, s)$ *is compact;*

(vi) $\phi \in F_0^{\#}(p, q, s)$ *and (9) holds.*

Proof: (ii)$\Rightarrow$ (i). This implication is obvious.
(i)$\Rightarrow$ (iii). Assume that (i) holds. Let $f_1, f_2 \in \mathcal{N}^{\alpha}$ be two functions from Lemma 1.2, such that

$$(1 - |z|^2)^{\alpha}\big(f_1^{\#}(z) + f_2^{\#}(z)\big) \geq M_0 > 0, \quad \text{for all } z \in \mathbb{D},$$

Since $C_{\phi} : \mathcal{N}^{\alpha} \to F_0^{\#}(p, q, s)$ is bounded, we get $C_{\phi}f_1 \in F_0^{\#}(p, q, s)$ and $C_{\phi}f_2 \in F_0^{\#}(p, q, s)$. Thus

$$
\begin{aligned}
\lim_{|a| \to 1} \Phi_{\phi}(\alpha, p, q, s, a) &= \lim_{|a| \to 1} \int_{\mathbb{D}} \frac{|\phi'(z)|^p}{\big(1 - |\phi(z)|^2\big)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) \\
&\leq \frac{2^p}{M_0^p} \lim_{|a| \to 1} \int_{\mathbb{D}} \Big[\big((f_1 \circ \phi)^{\#}(z)\big)^p + \big((f_2 \circ \phi)^{\#}(z)\big)^p\Big] (1 - |z|^2)^q g^s(z, a) dA(z) = 0.
\end{aligned}
$$

(iii)$\Rightarrow$ (iv). Suppose (iii) holds. By (2) (iv) is identical to $\lim\limits_{|a| \to 1} \Psi_{\phi}(\alpha, p, q, s, a) = 0$.
By (1) we have

$$\lim_{|a| \to 1} \Psi_{\phi}(\alpha, p, q, s, a) \leq \lim_{|a| \to 1} \Phi_{\phi}(\alpha, p, q, s, a) = 0.$$

(iv)$\Rightarrow$ (i). Suppose that (iv) holds, then for all $f \in \mathcal{N}^{\alpha}$, we get

$$
\begin{aligned}
\lim_{|a| \to 1} \int_{\mathbb{D}} \big((f \circ \phi)^{\#}(z)\big)^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) \\
\leq \|f\|_{\mathcal{N}^{\alpha}}^p \lim_{|a| \to 1} \Psi_{\phi}(\alpha, p, q, s, a) = 0.
\end{aligned}
$$

By Lemma 2.3, $C_{\phi}f \in F_0^{\#}(p, q, s)$, so $C_{\phi} : \mathcal{N}^{\alpha} \to F_0^{\#}(p, q, s)$ is bounded.
(iv)$\Rightarrow$ (ii). Let $\{f_n\} \in \mathcal{N}^{\alpha}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$ and let $\|f_n\|_{\mathcal{N}^{\alpha}} \leq 1$. Next, we will show that $\|C_{\phi}f_n\|_{F^{\#}(p,q,s)}$ converges to 0 as $n \to \infty$.
Since $\lim\limits_{|a| \to 1} \Psi_{\phi}(\alpha, p, q, s, a) = 0$, there is an $r > 0$ such that $\Psi_{\phi}(\alpha, p, q, s, a) < \varepsilon$ for every $\varepsilon > 0$ and $a \in G_r$, where $G_r = \{z \in \mathbb{D} : |z| > r\}$. So for any compact set $U \subset \mathbb{D}$, we have

$$I(a) = \int_{\mathbb{D}-U} \frac{|\phi'(z)|^p}{\big(1 - |\phi(z)|^2\big)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) < \epsilon, \quad \text{for } a \in G_r. \tag{10}$$

By Lemma 3.1, we see that $I(a)$ is also continuous. For any given a_i there is a compact set $U_i \subset \mathbb{D}$ such that (10) holds. The continuousness of $I(a)$ tells us that there is a neighborhood $U(a_i)$ of a_i such that (10) holds. By the compactness of $C(G_r) = \mathbb{D} - G_r$, we can choose finitely many $U(a_i)(i = 1, 2, \ldots, n)$ such that $C(G_r) = \bigcup\limits_{i=1}^n U_i$, then for all $a \in C(G_r)$, (8) holds for all $a \in \mathbb{D}$

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There exists a number $N > 0$, such that if $n \geq N$, $\sup_U [f_n^{\#}(\phi(z))]^p (1 - |\phi(z)|^2)^{\alpha p} < \varepsilon$. Then for $n \geq N$ and any $a \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} \left(f_n^{\#}(\phi(z)) \right)^p |\phi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \int_{\mathbb{D}} \left(f_n^{\#}(\phi(z)) \right)^p |\phi'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z)$$

$$\approx \left\{ \|f_n\|_{\mathcal{N}^\alpha}^p \int_U + \int_{\mathbb{D}-U} \right\} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dA(z) \leq \varepsilon.$$

Thus $\|C_\phi f_n\|_{F^{\#}(p,q,s)}$ converges to 0 as $n \to \infty$, i.e., $C_\phi : \mathcal{N}^\alpha \to F_0^{\#}(p, q, s)$ is compact.

(ii) $\Rightarrow$ (v). It is clear.

(v) $\Rightarrow$ (vi). Since the identical mapping belongs to $\mathcal{N}_0^\alpha$, $\phi \in F_0^{\#}(p, q, s)$. Since $C_\phi : \mathcal{N}_0^\alpha \to F_0^{\#}(p, q, s)$ is compact, $C_\phi : \mathcal{N}^\alpha \to F_0^{\#}(p, q, s)$ is compact. By Lemma 3.1, it is easy to see that (9) holds.

(vi) $\Rightarrow$ (i). Suppose (vi) is satisfied. For all $\varepsilon > 0$, there exists $\mathbb{D} : 0 < \mathbb{D} < 1$, such that if $\mathbb{D} < r < 1$, that for $a \in \mathbb{D}$

$$\int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \sup_{a \in \mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.$$

Therefore,

$$\int_{\Omega_r} \left((f_n \circ \phi)^{\#}(z) \right)^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \|f_n\|_{\mathcal{N}^\alpha}^p \int_{\Omega_r} \frac{|\phi'(z)|^p}{\left(1 - |\phi(z)|^2\right)^{\alpha p}} (1 - |z|^2)^q g^s(z, a) dA(z) < \varepsilon.$$

On the other hand,

$$\lim_{|a| \to 1} \int_{|\phi(z)| \leq r} \left((C_\phi f)^{\#}(z) \right)^p (1 - |z|^2)^q g^s(z, a) dA(z)$$

$$\leq \|f\|_{\mathcal{N}^\alpha}^p \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} \frac{|\phi'(z)|^p (1 - |z|^2)^q}{(1 - |\phi(z)|^2)^{p\alpha}} g^s(z, a) dA(z)$$

$$\leq \frac{\|f\|_{\mathcal{N}^\alpha}^p}{(1 - r^2)^{p\alpha}} \lim_{|a| \to 1} \int_{|\phi(z)| \leq r} |\phi'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0.$$

Combining above, we get that $C_\phi f \in F_0^{\#}(p, q, s)$, i.e., $C_\phi : \mathcal{N}^\alpha \to F_0^{\#}(p, q, s)$ is bounded. The proof is completed.

Now we consider the composition operators from the spherical Dirichlet class $\mathcal{D}^{\#}$ into $F^{\#}(p, q, s)$ classes. Our result is stated as follows:

Theorem 4.3 Let $\alpha \in (0, \infty), 2 \leq p < \infty$, $1 < s < \infty$ and $-2 < q < \infty$. If $\phi : \mathbb{D} \to \mathbb{D}$, then $C_\phi : \mathcal{D}^{\#} \to F^{\#}(p, q, s)$ is compact if and only if

$$\lim_{|a| \to 1} \|C_\phi(\varphi_a)\|_{F^{\#}(p,q,s)} = 0. \tag{11}$$

Proof: Assume that $C_\phi : \mathcal{D}^{\#} \to F^{\#}(p, q, s)$ is compact. Since $\{\varphi_a : a \in \mathbb{D}\}$ is a bounded set in $\mathcal{D}^{\#}$ and $(\varphi_a - a) \to 0$ uniformly on compact sets as $|a| \to 1$, the compactness of C_ϕ yields that

$$\|C_\phi(\varphi_a)\|_{F^{\#}(p,q,s)} \longrightarrow 0 \quad as \quad |a| \to 1.$$

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Conversely, let $\{f_n\} \subset \mathcal{D}^\#$ be a bounded sequence. Since $f_n \in \mathcal{D}^\# \subset \mathcal{N}$,

$$|f_n(z)| \le \sup_n \|f_n\|_{\mathcal{D}^\#} \left(1 + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\right), \quad \text{for } z \in \mathbb{D}.$$

Hence, $\{f_n\}$ is a normal family. Thus, there is a subsequence $\{f_{n_k}\}$, which converges to f analytic on $\mathbb{D}$ and both $f_{n_k} \to f$ and $f_{n_k}^\# \to f^\#$ uniformly on compact subsets of $\mathbb{D}$. It follows that $f \in \mathcal{D}^\#$ from the following estimates:

$$\begin{aligned}
\int_{\mathbb{D}} (f^\#(z))^2 dA(z) &= \int_{\mathbb{D}} \lim_{k\to\infty} \left(f_{n_k}^\#(z)\right)^2 dA(z) \\
&\le \liminf_{k\to\infty} \int_{\mathbb{D}} \left(f_{n_k}^\#(z)\right)^2 dA(z) \\
&\le \liminf_{k\to\infty} \|f_{n_k}^\#\|_{\mathcal{D}^\#}^2 \le C < \infty.
\end{aligned}$$

We replace f by $C_\phi f$, we remark that C_ϕ is compact by showing

$$\|C_\phi f_{n_k} - C_\phi f\|_{F^\#(p,q,s)} \longrightarrow 0 \quad as \ \ k \to \infty.$$

We write

$$\begin{aligned}
&\|C_\phi(\varphi_a)\|_{F^\#(p,q,s)}^p \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[(\varphi_a \circ \phi)^\#(z)\right]^P (1-|z|^2)^q g^s(z,a) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\varphi_a^\#(\phi(z))\right)^P |\phi'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\frac{|\varphi_a'(\phi(z))|}{1+|\varphi_a(\phi(z))|^2}\right]^P |\phi'(z)|^2 |\phi'(z)|^{p-2} (1-|z|^2)^q g^s(z,a) dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left[\frac{|\varphi_a'(w)|}{1+|\varphi_a(w)|^2}\right]^P N_{\phi,a}^{p,q,s}(w) dA(w) \\
&= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^p}{|1-\bar{a}w|^{2p}} \left[1 + \left|\frac{a-w}{1-\bar{a}w}\right|^2\right]^{-p} N_{\phi,a}^{p,q,s}(w) dA(w).
\end{aligned}$$

Here

$$N_{\phi,a}^{p,q,s}(w) = \sum_{z\in\phi^{-1}(w)} |\phi'(z)|^{p-2}(1-|z|^2)^q g^s(z,a),$$

is the counting function (see [7, 15]). Thus (11) is equivalent to

$$\lim_{|a|\to 1} \int_{\mathbb{D}} \frac{(1-|a|^2)^p}{|1-\bar{a}w|^{2p}} \left[1 + \left|\frac{a-w}{1-\bar{a}w}\right|^2\right]^{-p} N_{\phi,a}^{p,q,s}(w) dA(w) = 0.$$

Hence, for any $\varepsilon > 0$ there exists an δ, where $0 < \delta < 1$, such that for $|\lambda| > \delta$ and all $a \in \mathbb{D}$,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}(\lambda,1/2)} N_{\phi,a}^{p,q,s}(w) dA(w) < \varepsilon (1-|\lambda|)^p. \tag{12}$$

Thus, by Lemma 2.1 in [16],

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(f_{n_k}^\#(w) - f^\#(w)\right)^P N_{\phi,a}^{p,q,s}(w) dA(w) \\
&\le C \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \chi_{\mathbb{D}(w,1/2)}(u) \left(f_{n_k}^\#(u) - f^\#(u)\right)^P dA(u)\right) \frac{N_{\phi,a}^{p,q,s}(w)}{(1-|w|)^2} dA(w). \tag{13}
\end{aligned}$$

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Observe here that the characteristic function $\chi_{\mathbb{D}(w,1/2)}(u) = \chi_{\mathbb{D}(u,1/2)}(w)$.
Using (12) in (13), for sufficiently large k,

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(f_{n_k}^{\#}(w) - f^{\#}(w)\right)^p N_{\phi,a}^{p,q,s}(w) dA(w)$$

$$\leq \quad C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{\left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p}{(1 - |u|)^2} \int_{\mathbb{D}(u,1/2)} N_{\phi,a}^{p,q,s}(w) dA(w) dA(u)$$

$$= \quad C \sup_{a \in \mathbb{D}} \left(\int_{|u|>r} + \int_{|u| \leq r}\right) \frac{\left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p}{(1 - |u|)^2} \int_{\mathbb{D}(u,1/2)} N_{\phi,a}^{p,q,s}(w) dA(w) dA(u)$$

$$= \quad C \left(I_1(a) + I_2(a)\right).$$

For one hand, since $f_{n_k}, f \in \mathcal{D}^{\#} \subset \mathcal{N}$ and $2 \leq p < \infty$, we have

$$I_1(a) \quad = \quad \sup_{a \in \mathbb{D}} \int_{|u|>r} \frac{\left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p}{(1 - |u|)^2} \int_{\mathbb{D}(u,1/2)} N_{\phi,a}^{p,q,s}(w) dA(w) dA(u)$$

$$\leq \quad \varepsilon \sup_{a \in \mathbb{D}} \int_{|u|>r} \left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p (1 - |u|)^{p-2} dA(u)$$

$$\leq \quad \varepsilon \|f_{n_k} - f\|_{\mathcal{N}}^{p-2} \sup_{a \in \mathbb{D}} \int_{|u|>r} \left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^2 dA(u)$$

$$\leq \quad \varepsilon \|f_{n_k} - f\|_{\mathcal{N}}^{p-2} \|f_{n_k} - f\|_{\mathcal{D}^{\#}}^2$$

$$\leq \quad \varepsilon \|f_{n_k} - f\|_{\mathcal{D}^{\#}}^2.$$

On the other hand,

$$I_2(a) \quad = \quad \sup_{a \in \mathbb{D}} \int_{|u| \leq r} \frac{\left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p}{(1 - |u|)^2} \int_{\mathbb{D}(u,1/2)} N_{\phi,a}^{p,q,s}(w) dA(w) dA(u)$$

$$\leq \quad C \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} N_{\phi,a}^{p,q,s}(w) dA(w) \int_{|u| \leq r} \left(f_{n_k}^{\#}(u) - f^{\#}(u)\right)^p dA(u) \leq C \, \varepsilon,$$

Therefore, for sufficiently large k, the above discussion gives

$$\|C_{\phi} f_{n_k} - C_{\phi} f\|_{F^{\#}(p,q,s)}^p$$

$$= \quad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left((f_{n_k} \circ \phi)^{\#}(z) - (f \circ \phi)^{\#}(z)\right)^p (1 - |z|^2)^q g^s(z,a) dA(z)$$

$$= \quad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left(f_{n_k}^{\#}(w) - f^{\#}(w)\right)^p N_{\phi,a}^{p,q,s}(w) \, dA(w) < \, C \, \varepsilon.$$

It follows that C_{ϕ} is a compact operator. Therefore, the proof is completed.

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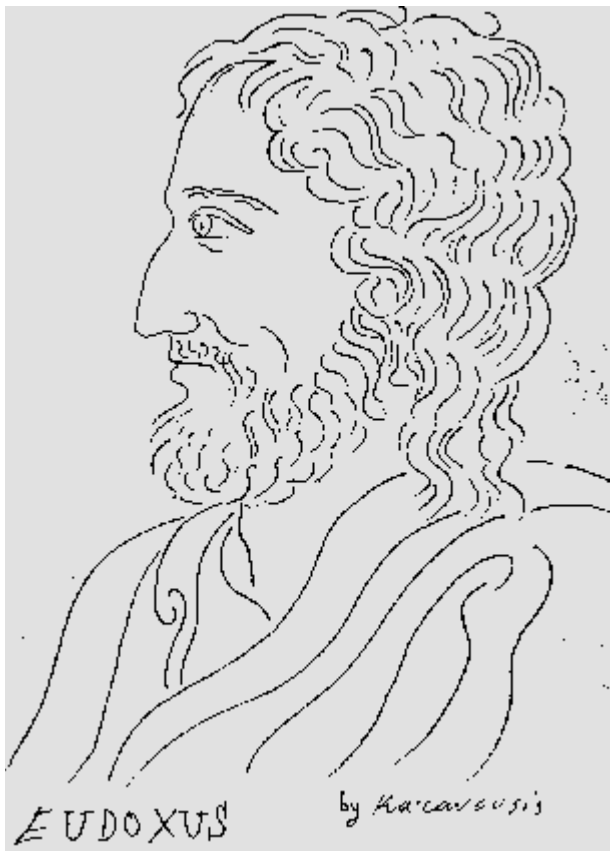# TABLE OF CONTENTS, JOURNAL OF APPLIED FUNCTIONAL ANALYSIS, VOL. 10, NO.'S 1-2, 2015

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences
The University of Memphis
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Tel.(+33)(0)3.20.43.42.18
Fax (+33)(0)3.20.43.43.02
Catalin.Badea@math.univ-lille1.fr
Approximation Theory, Functional
Analysis, Operator Theory.

4) Erik J.Balder
Mathematical Institute
Universiteit Utrecht
P.O.Box 80 010
3508 TA UTRECHT
The Netherlands
Tel.+31 30 2531458
Fax+31 30 2518394
balder@math.uu.nl
Control Theory, Optimization,
Convex Analysis, Measure Theory,
Applications to Mathematical
Economics and Decision Theory.

5) Carlo Bardaro
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia, ITALY
TEL+390755853822
+390755855034
FAX+390755855024
E-mail carlo.bardaro@unipg.it
Web site: http://www.unipg.it/~bardaro/
Functional Analysis and Approximation
Theory,
Signal Analysis, Measure Theory, Real
Analysis.

6) Heinrich Begehr
Freie Universitaet Berlin
I. Mathematisches Institut, FU Berlin,
Arnimallee 3,D 14195 Berlin
Germany,
Tel. +49-30-83875436, office
   +49-30-83875374, Secretary
Fax +49-30-83875403
begehr@math.fu-berlin.de
Complex and Functional Analytic
Methods in PDEs, Complex Analysis,
History of Mathematics.

7) Fernando Bombal
Departamento de Analisis Matematico
Universidad Complutense
Plaza de Ciencias,3
28040 Madrid, SPAIN
Tel. +34 91 394 5020
Fax +34 91 394 4726
fernando_bombal@mat.ucm.es

28) Miroslav Krbec
Mathematical Institute
Academy of Sciences of Czech Republic
Zitna 25
CZ-115 67 Praha 1
Czech Republic
Tel +420 222 090 743
Fax +420 222 211 638
krbecm@matsrv.math.cas.cz
Function spaces,Real Analysis,Harmonic
Analysis,Interpolation and
Extrapolation Theory,Fourier Analysis.

29) Peter M.Maass
Center for Industrial Mathematics
Universitaet Bremen
Bibliotheksstr.1,
MZH 2250,
28359 Bremen
Germany
Tel +49 421 218 9497
Fax +49 421 218 9562
pmaass@math.uni-bremen.de
Inverse problems,Wavelet Analysis and
Operator Equations,Signal and Image
Processing.

30) Julian Musielak
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Ul.Umultowska 87
61-614 Poznan
Poland
Tel (48-61) 829 54 71
Fax (48-61) 829 53 15
Grzegorz.Musielak@put.poznan.pl
Functional Analysis, Function Spaces,
Approximation Theory,Nonlinear Operators.

31) Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel:: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost Automorphy.

32) Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens

Operators on Banach spaces,
Tensor products of Banach spaces,
Polymeasures, Function spaces.

8) Michele Campiti
Department of Mathematics "E.De Giorgi"
University of Lecce
P.O. Box 193
Lecce,ITALY
Tel. +39 0832 297 432
Fax +39 0832 297 594
michele.campiti@unile.it
Approximation Theory,
Semigroup Theory, Evolution problems,
Differential Operators.

9)Domenico Candeloro
Dipartimento di Matematica e Informatica
Universita degli Studi di Perugia
Via Vanvitelli 1
06123 Perugia
ITALY
Tel +39(0)75 5855038
   +39(0)75 5853822,
   +39(0)744 492936
Fax +39(0)75 5855024
candelor@dipmat.unipg.it
Functional Analysis, Function spaces,
Measure and Integration Theory in
Riesz spaces.

10) Pietro Cerone
School of Computer Science and
Mathematics, Faculty of Science,
Engineering and Technology,
Victoria University
P.O.14428,MCMC
Melbourne,VIC 8001,AUSTRALIA
Tel +613 9688 4689
Fax +613 9688 4050
Pietro.cerone@vu.edu.au
Approximations, Inequalities,
Measure/Information Theory,
Numerical Analysis, Special Functions.

11)Michael Maurice Dodson
Department of Mathematics
University of York,
York YO10 5DD, UK
Tel   +44 1904 433098
Fax +44 1904 433071
Mmd1@york.ac.uk
Harmonic Analysis and Applications to
Signal Theory,Number Theory and
Dynamical Systems.

Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability.

33) Pier Luigi Papini
Dipartimento di Matematica
Piazza di Porta S.Donato 5
40126 Bologna
ITALY
Fax +39(0)51 582528
papini@dm.unibo.it
Functional Analysis, Banach spaces,
Approximation Theory.

34) Svetlozar (Zari) Rachev, Professor of Finance,
College of Business,and
Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
Phone: +1-631-632-1998,
Email : svetlozar.rachev@stonybrook.edu

35) Paolo Emilio Ricci
Department of Mathematics
Rome University "La Sapienza"
P.le A.Moro,2-00185
Rome,ITALY
Tel ++3906-49913201 office
   ++3906-87136448 home
Fax ++3906-44701007
Paoloemilio.Ricci@uniroma1.it
riccip@uniroma1.it
Special Functions, Integral and Discrete
Transforms, Symbolic and Umbral Calculus,
ODE, PDE,Asymptotics, Quadrature,
Matrix Analysis.

36) Silvia Romanelli
Dipartimento di Matematica
Universita' di Bari
Via E.Orabona 4
70125 Bari, ITALY.
Tel (INT 0039)-080-544-2668 office
        080-524-4476 home
        340-6644186 mobile
Fax -080-596-3612 Dept.
romans@dm.uniba.it
PDEs and Applications to Biology and
Finance, Semigroups of Operators.

12) Sever S.Dragomir
School of Computer Science and
Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001,AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever@csm.vu.edu.au
Inequalities,Functional Analysis,
Numerical Analysis, Approximations,
Information Theory, Stochastics.

13) Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530, Ankara,
Turkey, oduman@etu.edu.tr
Classical Approximation Theory, Summability
Theory,
Statistical Convergence and its Applications

14) Paulo J.S.G.Ferreira
Department of Electronica e
Telecomunicacoes/IEETA
Universidade de Aveiro
3810-193 Aveiro
PORTUGAL
Tel +351-234-370-503
Fax +351-234-370-545
pjf@ieeta.pt
Sampling and Signal Theory,
Approximations, Applied Fourier Analysis,
Wavelet, Matrix Theory.

15) Gisele Ruiz Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA.
Tel 901-678-2513
Fax 901-678-2480
ggoldste@memphis.edu
PDEs, Mathematical Physics,
Mathematical Geophysics.

16) Jerome A.Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA
Tel 901-678-2484
Fax 901-678-2480
jgoldste@memphis.edu
PDEs,Semigroups of Operators,
Fluid Dynamics,Quantum Theory.

37) Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620,USA
Tel 813-974-9710
boris@math.usf.edu
Approximation Theory, Banach spaces,
Classical Analysis.

38) Rudolf Stens
Lehrstuhl A fur Mathematik
RWTH Aachen
52056 Aachen
Germany
Tel ++49 241 8094532
Fax ++49 241 8092212
stens@mathA.rwth-aachen.de
Approximation Theory, Fourier Analysis,
Harmonic Analysis, Sampling Theory.

39) Juan J.Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271.LaLaguna.Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-
Operators-
Fourier Transforms, Special functions,
Approximations,and Applications.

40) Tamaz Vashakmadze
I.Vekua Institute of Applied Mathematics
Tbilisi State University,
2 University St. , 380043,Tbilisi, 43,
GEORGIA.
Tel (+99532) 30 30 40 office
    (+99532) 30 47 84 office
    (+99532) 23 09 18 home
Vasha@viam.hepi.edu.ge
tamazvashakmadze@yahoo.com
Applied Functional Analysis, Numerical
Analysis, Splines, Solid Mechanics.

41) Ram Verma
International Publications
5066 Jamieson Drive, Suite B-9,
Toledo, Ohio 43613,USA.
Verma99@msn.com
rverma@internationalpubls.com
Applied Nonlinear Analysis, Numerical
Analysis, Variational Inequalities,
Optimization Theory, Computational
Mathematics, Operator Theory.

17) Heiner Gonska
Institute of Mathematics
University of Duisburg-Essen
Lotharstrasse 65
D-47048 Duisburg
Germany
Tel   +49 203 379 3542
Fax   +49 203 379 1845
gonska@math.uni-duisburg.de
Approximation and Interpolation Theory,
Computer Aided Geometric Design,
Algorithms.

18) Karlheinz Groechenig
Institute of Biomathematics and Biometry,
GSF-National Research Center
for Environment and Health
Ingolstaedter Landstrasse 1
D-85764 Neuherberg,Germany.
Tel 49-(0)-89-3187-2333
Fax 49-(0)-89-3187-3369
Karlheinz.groechenig@gsf.de
Time-Frequency Analysis, Sampling Theory,
Banach spaces and Applications,
Frame Theory.

19) Vijay Gupta
School of Applied Sciences
Netaji Subhas Institute of Technology
Sector 3 Dwarka
New Delhi 110075, India
e-mail: vijay@nsit.ac.in;
vijaygupta2001@hotmail.com
Approximation Theory

20) Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method,
Numerical PDE, Variational inequalities,
Computational mechanics

21) Tian-Xiao He
Department of Mathematics and
Computer Science
P.O.Box 2900,Illinois Wesleyan University
Bloomington,IL 61702-2900,USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations,Wavelet, Integration Theory,
Numerical Analysis, Analytic Combinatorics.

42) Gianluca Vinti
Dipartimento di Matematica e Informatica
Universita di Perugia
Via Vanvitelli 1
06123 Perugia
ITALY
Tel +39(0) 75 585 3822,
    +39(0) 75 585 5032
Fax +39 (0) 75 585 3822
mategian@unipg.it
Integral Operators, Function Spaces,
Approximation Theory, Signal Analysis.

43) Ursula Westphal
Institut Fuer Mathematik B
Universitaet Hannover
Welfengarten 1
30167 Hannover,GERMANY
Tel (+49) 511 762 3225
Fax (+49) 511 762 3518
westphal@math.uni-hannover.de
Semigroups and Groups of Operators,
Functional Calculus, Fractional Calculus,
Abstract and Classical Approximation
Theory, Interpolation of Normed spaces.

44) Ronald R.Yager
Machine Intelligence Institute
Iona College
New Rochelle,NY 10801,USA
Tel (212) 249-2047
Fax(212) 249-1689
Yager@Panix.Com
ryager@iona.edu
Fuzzy Mathematics, Neural Networks,
Reasoning,
Artificial Intelligence, Computer Science.

45) Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University,AL 36849-5310
USA.
Tel 334-844-6557 office
       678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory,Chebychev Systems,
Wavelet Theory.

22) Don Hong
Department of Mathematical Sciences
Middle Tennessee State University
1301 East Main St.
Room 0269, Blgd KOM
Murfreesboro, TN 37132-0001
Tel (615) 904-8339
dhong@mtsu.edu
Approximation Theory,Splines,Wavelet,
Stochastics, Mathematical Biology Theory.

23) Hubertus Th. Jongen
Department of Mathematics
RWTH Aachen
Templergraben 55
52056 Aachen
Germany
Tel   +49 241 8094540
Fax   +49 241 8092390
jongen@rwth-aachen.de
Parametric Optimization, Nonconvex
Optimization, Global Optimization.

NEW MEMBERS

46) Jianguo Huang
Department of Mathematics
Shanghai Jiao Tong University
Shanghai 200240
P.R. China
jghuang@sjtu.edu.cn
Numerical PDE's

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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The University of Memphis
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Construction of Spline Type Orthogonal Scaling Functions and Wavelets

Tung Nguyen and Tian-Xiao He

Department of Mathematics

Illinois Wesleyan University

Bloomington, IL 61702-2900, USA

Abstract

In this paper, we present a method to construct orthogonal spline-type scaling functions by using B-spline functions. B-splines have many useful properties such as compactly supported and refinable properties. However, except for the case of order one, B-splines of order greater than one are not orthogonal. To induce the orthogonality while keeping the above properties of B-splines, we multiply a class of polynomial function factors to the masks of the B-splines so that they become the masks of a spline-type orthogonal compactly-supported and refinable scaling functions in L_2. In this paper we establish the existence of this class of polynomial factors and their construction. Hence, the corresponding spline-type wavelets and the decomposition and reconstruction formulas for their Multiresolution Analysis (MRA) are obtained accordingly.

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Key Words and Phrases: B-spline, wavelet, MRA.

1   Introduction

We start with the definitions of Multiresolution Analysis (MRA) and scaling functions.

Definition 1.1   *A Multiresolutional Analysis (MRA) generated by function ϕ consists of a sequence of closed subspaces $V_j, j \in \mathbb{Z}$, of $L_2(\mathbb{R})$ satisfying*

(i)   *(nested) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;*

(ii)   *(density) $\overline{\cup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R})$;*

(iii)   *(separation) $\cap_{j \in \mathbb{Z}} V_j = \{0\}$;*

(iv)   *(scaling) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;*

(v) *(Basis) There exists a function $\phi \in V_0$ such that $\{\phi(x - k) : k \in \mathbb{Z}\}$ is an orthonormal basis or a Riesz basis for V_0.*

The function whose existence asserted in (v) is called a scaling function of the MRA.

A scaling function ϕ must be a function in $L_2(\mathbb{R})$ with $\int \phi \neq 0$. Also, since $\phi \in V_0$ is also in V_1 and $\{\phi_{1,k} := 2^{j/2}\phi(2x - k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_1, there exists a unique sequence $\{p_k\}_{k=-\infty}^{\infty} \in l_2(\mathbb{Z})$ that describe the two-scale relation of the scaling function

$$\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k), \tag{1.1}$$

i.e., ϕ is of a two-scale refinable property. By taking a Fourier transformation on both sides of (1.1) and denoting the Fourier transformation of ϕ by $\hat{\phi}(\xi) := \int_{-\infty}^{\infty} \phi(x)e^{-i\xi x}dx$, we have

$$\hat{\phi}(\xi) = P(z)\hat{\phi}(\frac{\xi}{2}), \tag{1.2}$$

where

$$P(z) = \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k z^k \text{ and } z = e^{-i\xi/2} \tag{1.3}$$

Here, $P(z)$ is called the mask of the scaling function. Now, regarding the property that $\{\phi(x - k)\}$ must be either an orthonormal basis or a Riesz basis, we have the following characterization theorem (see, for example, Chs. 5 and 7 of [4])

Theorem 1.2 *Suppose the function ϕ satisfies the refinement relation $\phi(x) = \sum_{-\infty}^{\infty} p_k \phi(2x - k)$. Then we have the following statements*

(i) ϕ forms an orthonormal basis if and only if $|P(z)|^2 + |P(-z)|^2 = 1$ for $z \in \mathbb{C}$ with $|z| = 1$.
(ii) ϕ forms a Riesz basis if and only if $|P(z)|^2 + |P(-z)|^2 < 1$ for $z \in \mathbb{C}$ with $|z| = 1$.

Finally, from the scaling function ϕ, we can construct a corresponding wavelet function ψ by the following theorem (see, for example, [1, 7, 9])

Theorem 1.3 *Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA with scaling function ϕ, and ϕ satisfies refinement relation in (1.1). Then we can construct a corresponding wavelet function as*

$$\psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{p_{1-k}} \phi(2x - k) \tag{1.4}$$

and denote $W_j = span\{\psi(2^j x - k) : k \in \mathbb{Z}\}$.

Furthermore, $W_j \subset V_{j+1}$ is the orthogonal complement of V_j in V_{j+1}, and $\{\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_j.

In the next sections, we will examine the B-spline functions as a scaling functions and construct an orthogonal spline type scaling functions from them. The proof of the main result shown in Section 2 and some properties of the constructed spline type scaling functions are presented in Section 3. The corresponding spline type wavelets and their regularities as well as the decomposition and reconstruction formulas are also given. Finally, we illustrate our construction by using some examples in Section 4.

2   Construction of orthogonal scaling functions from B-splines

In this paper we are interested in a family of B-spline functions, $B_n(x)$, the uniform B-spline with integer knots 0, 1, ..., n+1 defined as follows (see [2, 3]).

Definition 2.1 *The cardinal B-splines with integer knots in $\mathbb{N}_0$, denoted by $B_n(x)$, is defined inductively by*

$$B_1(x) := \begin{cases} 1 & if \ x \in [0, 1] \\ 0 & otherwise, \end{cases}$$

$$and \ B_n(x) := (B_{n-1} * B_1)(x) = \int_{-\infty}^{\infty} B_{n-1}(x - t)B_0(t)dt \tag{2.1}$$

From the definition, it is easy to verify that $B_n(x)$ is compactly supported and in $L_2(\mathbb{R})$, which satisfies $\int B_n \neq 0$. By using Fourier transformation, we also have the refinement relation of $B_n(x)$

$$B_n(x) = \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} B_n(2x - j) \tag{2.2}$$

Next, we would like to see if $B_n(x)$ forms an orthornormal basis or not. We examine the mask $P_n(z)$ of $B_n(x)$. We have

$$P_n(z) = \frac{1}{2} \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} z^j = \frac{(1 + z)^n}{2^n} = \left(\frac{1 + z}{2}\right)^n \tag{2.3}$$

Thus considering theorem 1.2 we have

$$
\begin{aligned}
|P_n(z)|^2 + |P_n(-z)|^2 &= \left|\frac{1+z}{2}\right|^{2n} + \left|\frac{1-z}{2}\right|^{2n} \\
&= \left|\frac{1+cos(\xi/2) - isin(\xi/2)}{2}\right|^{2n} + \left|\frac{1-cos(\xi/2) + isin(\xi/2)}{2}\right|^{2n} \\
&= cos^{2n}(\xi/4) + sin^{2n}(\xi/4) \leq cos^2(\xi/4) + sin^2(\xi/4) = 1
\end{aligned}
$$

The equality happens only when n=1. Therefore, except for the case of order one (i.e., $n = 1$), $B_n(x)$ are generally not orthogonal (indeed they are Riesz basis). To induce orthogonality, we introduce a class of polynomial function factors $S(z)$. Hence, instead of $B_n(x)$, we consider a scaling function $\phi_n(x)$ with the mask $P_n(z)S_n(z)$, i.e

$$
\phi_n(\xi) = P_n(z)S_n(z)\phi_n(\xi/2) \tag{2.4}
$$

where $P_n(z)$ are defined as (2.3). We want to construct $S_n(z)$ such that the shift set of the new scaling function form an orthogonal basis. In other words, we need that $S_n(z)$ satisfy the following condition

$$
|P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 = 1 \tag{2.5}
$$

Now we consider $S_n(z)$ of the following type: $S_n(z) = a_1z + a_2z^2 + ... + a_nz^n$, $n \in \mathbb{N}$, and find their following expressions.

Lemma 2.2
Let $S_n(z)$ be defined as above. Then there holds

$$
\begin{aligned}
|S_n(z)|^2 \\
= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_i a_{i+2}cos(2\xi/2) + ... \\
+ 2a_1 a_n cos((n-1)\xi/2).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&|S_n(z)|^2 \\
&= |a_1(cos(\xi/2) - isin(\xi/2)) + ... + a_n(cos(n\xi/2) - isin(n\xi/2))|^2 \\
&= |(a_1cos(\xi/2) + ... + a_ncos(n\xi/2)) - i(a_1sin(\xi/2) + ... + a_nsin(n\xi/2))|^2 \\
&= (a_1cos(\xi/2) + ... + a_ncos(n\xi/2))^2 + (a_1sin(\xi/2) + ... + a_nsin(n\xi/2))^2 \\
&= a_1(cos^2(\xi/2) + sin^2(\xi/2)) + ... + a_n(cos^2(n\xi/2) + sin^2(n\xi/2)) \\
&\quad + \sum_{i \neq j} 2a_ia_j(cos(i\xi/2)cos(j\xi/2) + sin(i\xi/2)sin(j\xi/2)) \\
&= \sum_{i=1}^{n} a_i^2 + \sum_{i \neq j} 2a_ia_jcos((i-j)\xi/2) \\
&= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n-1} a_ia_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_ia_{i+2}cos(2\xi/2) + ... \\
&\quad + 2a_1a_ncos((n-1)\xi/2)
\end{aligned}
$$

A similar procedure can be applied to find $|S_n(-z)|^2$

$$
\begin{aligned}
&|S_n(-z)|^2 \\
&= \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n-1} a_ia_{i+1}cos(\xi/2) + 2\sum_{i=1}^{n-2} a_ia_{i+2}cos(2\xi/2) + ... \\
&\quad + (-1)^n 2a_1a_ncos((n-1)\xi/2).
\end{aligned}
$$

From Lemma 2.2, if we write each $cos(k\xi/2)$ as a polynomial of $cos(\xi/2)$, then $|S_n(z)|^2 = Q_n(x)$ where $x = cos(\xi/2)$. Obviously, $Q_n(x)$ has the degree of $n-1$. It is also easy to observe that $|S_n(-z)|^2 = Q_n(-x)$. Now equation (2.5) becomes

$$
\begin{aligned}
1 &= |P_n(z)S_n(z)|^2 + |P_n(-z)S_n(-z)|^2 \\
&= cos^{2n}(\xi/4)Q_n(x) + sin^{2n}(\xi/4)Q_n(-x) \\
&= \left(\frac{1 + cos(\xi/2)}{2}\right)^n Q_n(x) + \left(\frac{1 - cos(\xi/2)}{2}\right)^n Q_n(-x) \\
&= \left(\frac{1 + x}{2}\right)^n Q_n(x) + \left(\frac{1 - x}{2}\right)^n Q_n(-x)
\end{aligned}
$$

So finally we get

$$
\left(\frac{1 + x}{2}\right)^n Q_n(x) + \left(\frac{1 - x}{2}\right)^n Q_n(-x) = 1 \tag{2.6}
$$

Next, to show the existence of $Q(x)$ in the above equation, we make use of the Polynomial extended Euclidean algorithm (see [6, 16]).

Lemma 2.3 Polynomial extended Euclidean algorithm *If a and b are two nonzero polynomials, then the extended Euclidean algorithm produces the unique pair of polynomials (s, t) such that as+bt=gcd(a,b), where $deg(s) < deg(b)-deg(gcd(a,b))$ and $deg(t) < deg(a) - deg(gcd(a,b))$.*

We notice that $gcd((\frac{1+x}{2})^n, (\frac{1-x}{2})^n) = 1$, so by Lemma 2.3, there exists uniquely $Q(x)$ and $R(x)$ with degrees less than n such that $(\frac{1+x}{2})^n Q(x) + (\frac{1-x}{2})^n R(x) = 1$. If we replace x by $-x$ in the previous equation, we have $(\frac{1-x}{2})^n Q(-x) + (\frac{1+x}{2})^n R(-x) = 1$. Due to the uniqueness of the algorithm, we conclude that $R(x) = Q(-x)$. So we have showed the existence of a unique $Q(x) = Q_n(x)$ satisfying equation 2.6.

To construct $Q_n(x)$ explicitly, we use the Lorentz polynomials shown in [8, 15] and the following technique.

$$
\begin{aligned}
1 &= \left(\frac{1+x}{2} + \frac{1-x}{2}\right)^{2n-1} \\
&= \sum_{i=0}^{2n-1} \binom{2n-1}{i} \left(\frac{1+x}{2}\right)^{2n-1-i} \left(\frac{1-x}{2}\right)^i \\
&= \left(\frac{1+x}{2}\right)^n \left[\sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^i\right] \\
&+ \left(\frac{1-x}{2}\right)^n \left[\sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1-x}{2}\right)^{n-1-i} \left(\frac{1+x}{2}\right)^i\right],
\end{aligned}
$$

where the polynomials presenting in the brackets are the Lorentz polynomials. We notice that the degrees of the two polynomials in the brackets are $n-1$, and because $Q_n(x)$ in equation 2.6 is unique, we can conclude that

$$
Q_n(x) = \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+x}{2}\right)^{n-1-i} \left(\frac{1-x}{2}\right)^i \tag{2.7}
$$

With the construction of $Q_n(x)$, we take a step further by showing the existence of $\sum a_i^2, \sum a_i a_{i+1}, \dots$ in Lemma 2.2.

It is well-known that the set $\{1, cos(t), cos(2t), \dots, cos((n-1)t)\}$ is linearly independent. As a result, $\{1, cos(\xi/2), cos(2\xi/2), \dots, cos((n-1)\xi/2)\}$ forms a basis of the space $P_{n-1}(x) = \{P(x) : x = cos(\xi/2)$ and P is a polynomial of degree less than n. Based on this fact and the existence of $Q_n(x)$ in equation (2.6), it is obvious that the coefficients $\sum a_i^2, \sum a_i a_{i+1}, \dots$ in Lemma 2.2 must exist uniquely.

We now establish the main result of this paper.

Theorem 2.1 *Let $P_n(z)$ and $S_n(z)$ be defined as above. Then for $n = 1, 2, \ldots$, the spline type function $\phi_n(x)$ with the mask $P_n(z)S_n(z)$ is a scaling function that generates an orthogonal basis of V_0 in its MRA.*

From the construction of $\phi_n(x)$, we immediately know they are compactly supported and refinable. We will prove that they are in $L_2(\mathbb{R})$ in next section.

3   Properties of the constructed scaling function

3.1   The scaling function $\phi_n(x)$ is in $L_2(\mathbb{R})$

To show that the newly constructed scaling function $\phi_n(x)$ with mask $P_n(z)S_n(z)$ is in $L_2(\mathbb{Z})$, we make use of the following theorem

Theorem 3.1 [11, 12] *Let ϕ be a scaling function with mask $P(z)S(z)$ where $P(z) = (\frac{1+z}{2})^n$ and $S(z) = z^i \sum_{j=0}^{k} a_j z^j$. Then $\phi \in L_2(\mathbb{R})$ if*

$$(k+1) \sum_{j=0}^{k} a_j^2 < 2^{2n-1} \tag{3.1}$$

.

We also need the following lemma for the proof of Theorem 2.1.

Lemma 3.2 [14] *We have the following formula for binomial coefficients*

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n^k e^{-\frac{k^2}{2n} - \frac{k^3}{6n^2}}}{k!}(1 + o(1)) \tag{3.2}$$

Proof. (The proof of Theorem 2.1) In this case, with the $S_n(z)$ we use to construct $\phi_n(x)$, the condition (3.1) becomes

$$n \sum_{j=1}^{n} a_j^2 < 2^{2n-1} \tag{3.3}$$

Recall from Lemma 2.2 that

$$Q_n(x) = Q_n(cos(\xi/2))$$
$$= \sum_{i=1}^{n} a_i^2 + 2 \sum_{i=1}^{n-1} a_i a_{i+1} cos(\xi/2) + \ldots + 2a_1 a_n cos((n-1)\xi/2)$$

Taking the integration from 0 to 2π of both sides, we have

$$\int_0^{2\pi} Q_n(x)d\xi$$

$$= \int_0^{2\pi} \left(\sum_{i=1}^n a_i^2 + 2\sum_{i=1}^{n-1} a_i a_{i+1} cos(\xi/2) + ... + 2a_1 a_n cos((n-1)\xi/2) \right) d\xi$$

$$= 2\pi \sum_{i=1}^n a_i^2$$

On the other hands, from the expression of $Q_n(x)$ in (2.7) we have

$$\int_0^{2\pi} Q_n(x)d\xi = \int_0^{2\pi} \sum_{i=0}^{n-1} \left(\begin{array}{c} 2n-1 \\ i \end{array} \right) \left(\frac{1+x}{2} \right)^{n-1-i} \left(\frac{1-x}{2} \right)^i d\xi$$

$$= \int_0^{2\pi} \sum_{i=0}^{n-1} \left(\begin{array}{c} 2n-1 \\ i \end{array} \right) \left(\frac{1+cos(\xi/2)}{2} \right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2} \right)^i d\xi$$

Combining the two equations above we have

$$\sum_{i=1}^n a_i^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n-1} \left(\begin{array}{c} 2n-1 \\ i \end{array} \right) \left(\frac{1+cos(\xi/2)}{2} \right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2} \right)^i d\xi \quad (3.4)$$

Now we need to show that the expression on the right hand side of (3.3) is smaller than $\frac{2^{2n-1}}{n}$. As it is hard to obtain a simplification for the left hand side of (3.3), we will use estimation shown in Lemma 3.2 to get an upper bound for it. From lemma 3.2, we get

$$\left(\begin{array}{c} 2n-1 \\ i \end{array} \right) / \left(\begin{array}{c} n-1 \\ i \end{array} \right) \approx \frac{(2n-1)^i e^{-\frac{i^2}{2(2n-1)} - \frac{i^3}{6(2n-1)^2}}}{i!} \frac{i!}{(n-1)^i e^{-\frac{i^2}{2(n-1)} - \frac{i^3}{6(n-1)^2}}}$$

$$= \left(\frac{2n-1}{n-1} \right)^i e^{\frac{i^2}{2} \left(\frac{1}{n-1} - \frac{1}{2n-1} \right) + \frac{i^3}{6} \left(\frac{1}{(n-1)^2} - \frac{1}{(2n-1)^2} \right)}$$

Using the fact that $i \le n$, we can estimate the right hand side to approximate

$$\left(\frac{2n-1}{n-1} \right)^i e^{\frac{i^2}{2} \frac{1}{2i} + \frac{i^3}{6} \frac{1}{2i^2}} = \left(\frac{2n-1}{n-1} \right)^i e^{i/3} = \left(\frac{2n-1}{n-1} e^{1/3} \right)^i$$

Thus, from (3.3) we obtain

$$\sum_{i=1}^{n} a_i^2 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2}\right)^i d\xi$$

$$\approx \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{2n-1}{n-1}e^{1/3}\right)^i \left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2}\right)^i d\xi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1+cos(\xi/2)}{2} + \left(\frac{2n-1}{n-1}e^{1/3}\right)\left(\frac{1-cos(\xi/2)}{2}\right)\right)^n d\xi$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{2n-1}{n-1}e^{1/3}\right)^n d\xi$$

$$= \left(\frac{2n-1}{n-1}e^{1/3}\right)^n$$

Notice that the inequality above can be easily verified by checking that the maximum of the expression inside the integral is attained at $cos(\xi/2) = -1$.

For $n \geq 10$, we have

$$\left(\frac{2n-1}{n-1}e^{1/3}\right)^n = \left(\left(2+\frac{1}{n-1}\right)e^{1/3}\right)^n \leq \left(\left(2+\frac{1}{9}\right)e^{1/3}\right)^n \leq 2.95^n$$

We now prove that $2.95^n < \frac{2^{2n-1}}{n}$, or equivalently $2n < (\frac{4}{2.95})^n$. Consider the function $h(n) = (\frac{4}{2.95})^n - 2n$ for $n \geq 10$. We have

$$h'(n) = \left(\frac{4}{2.95}\right)^n \ln\left(\frac{4}{2.95}\right) - 2 \geq \left(\frac{4}{2.95}\right)^{10} \ln\left(\frac{4}{2.95}\right) - 2 > 0$$

Thus, $h(n) \geq h(10) > 0$. We have proven $\sum_{i=1}^{n} a_i^2 < 2^{2n-1}/n$ for all $n \geq 10$. Finally, we can easily check the inequality is also true for all $1 \leq n \leq 9$. The case of $n = 1$ is trivial since $S_1(z) = z$. For the cases of $2 \leq n \leq 9$, denote the difference of $\sum_{i=1}^{n} a_i^2 - 2^{2n-1}/n$ by α_n. Thus,

$$\alpha_n = \sum_{i=1}^{n} a_i^2 - \frac{2^{2n-1}}{n}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{n-1} \binom{2n-1}{i} \left(\frac{1+cos(\xi/2)}{2}\right)^{n-1-i} \left(\frac{1-cos(\xi/2)}{2}\right)^i d\xi - \frac{2^{2n-1}}{n}$$

Using Mathematica, we have the following table

n	2	3	4	5	6	7	8	9
α_n	-2	$-\frac{71}{12}$	-19	$-\frac{20223}{320}$	$-\frac{20687}{96}$	$-\frac{1343063}{1792}$	$-\frac{84763}{32}$	$-\frac{1398580967}{147456}$

As for n from 2 to 9, all the α_n are negative, inequality (3.3) is completely proven, which indicates that the scaling function $\phi_n(x)$ is in $L_2(\mathbb{R})$ and also completes the proof of Theorem 2.1

■

As examples, we consider the cases of $n = 1$ and 2. It is easy to find that $S_1(z) = z$ and $\phi_n(x)$ is the Haar function. For $n = 2$,

$$S_2(z) = \frac{1 + \sqrt{3}}{2}z + \frac{1 - \sqrt{3}}{2}z^2$$

and the corresponding $\phi_2(x)$ is the Daubechies scaling function.

3.2   Refinement relation and corresponding wavelets

Let $M_n(z) := P_n(z)S_n(z) = \frac{1}{2}\sum_{k=0}^{2n} c_k z^k$. Then $M_n(z)$ is the mask of $\phi_n(x)$ and thus we have the refinement equation

$$\phi_n(x) = \sum_{\forall k} c_k \phi_n(2x - k) \tag{3.5}$$

We also have the corresponding wavelets

$$\psi_n(x) = \sum_{\forall k} (-1)^k \overline{c_{1-k}} \phi_n(2x - k) \tag{3.6}$$

One natural question at this point is from the refinement equation to determine explicitly the scaling function $\phi_n(x)$ and thus the wavelet functions $\psi_n(x)$. We propose a method by the iterative procedure described in Theorem 5.23 of [1].

We start with $\phi_n^{(0)}(x) = B_1(x)$. From $\phi_n^{(0)}(x)$ we can construct $\phi_n^{(1)}(x)$ by using the refinement equation (3.5)

$$\phi_n^{(1)}(x) = \sum_{\forall k} c_k \phi_n^{(0)}(2x - k)$$

Now from $\phi_n^{(1)}(x)$ we can continue to construct $\phi_n^{(2)}(x)$ again by the refinement equation. Continuing the process for a certain number of times, we can obtain an approximation of $\phi_n(x)$.

Simple as it seems, this iteration method is not a very good way to construct ϕ as we do not know how many times should we repeat the process to find a good enough approximation for ϕ. For this reason, more direct method is developed and will be addressed in future paper.

From Theorem 2 of [11], we have the following result on the regularities of spline type wavelets $\psi_n(x)$.

Theorem 3.1 *Let $\phi_n(x)$ be constructed as the previous section with the mask shown in Theorem 2.1. Then the corresponding wavelets $\psi_n(x)$ established in (3.6) are in C^{β_n}, where β_n are greater than*

$$n - \frac{1}{2}\log_2\left(n\sum_{j=1}^{n} a_j^2\right).$$

3.3 Decomposition and reconstruction formula

Finally, upon obtaining the new scaling function and corresponding wavelet function, we may establish the corresponding decomposition and reconstruction formulas by using a routine argument. Roughly speaking consider a function $f \in L_2(R)$ with f_i being an approximation in V_j, a space generated by the jth order scaling function. Then

$$f_j = \sum_{k \in Z} < f, \phi_{jk} > \phi_{jk} \qquad (3.7)$$

Because we have the orthogonal direct sum decomposition $V_j = V_{j-1} + W_{j-1}$, we can express f_j using bases of V_{j-1} and bases of W_{j-1} as follows

$$f_j = f_{j-1} + g_{j-1} = \sum_{k \in Z} < f, \phi_{j-1,k} > \phi_{j-1,k} + \sum_{k \in Z} < f, \psi_{j-1,k} > \psi_{j-1,k} \qquad (3.8)$$

And now we have the following theorem regarding the relationship of these coefficients.

Let $\{V_j : j \in Z\}$ be an MRA with scaling function ϕ satisfying the scaling relation $\phi(x) = \sum_{k \in Z} p_k \phi(2x - k)$, and let W_j be the orthogonal complement of V_j in V_{j+1} with wavelet function ψ. Then the coefficients relative to the different bases in (3.7) and (3.8) satisfy the following decomposition formula

$$< f, \phi_{j-1,l} > = 2^{-1/2} \sum_{k \in Z} \overline{p_{k-2l}} < f, \phi_{jk} >$$

$$< f, \psi_{j-1,l} > = 2^{-1/2} \sum_{k \in Z} (-1)^k p_{1-k+2l} < f, \phi_{jk} >$$

and the reconstruction formula

$$< f, \phi_{jl} > = 2^{-1/2} \sum_{k \in Z} p_{l-2k} < f, \phi_{j-1,k} > + 2^{-1/2} \sum_{k \in Z} (-1)^k \overline{p_{1-l+2k}} < f, \phi_{j-1,k} >$$

4 Examples for $n = 3$ and $n = 4$

Besides the examples of the cases of $n = 1$ and 2 shown at the end of Section 3.1, in this section we give an examples on the cases of $n = 3$ and $n = 4$. According to the previous sections, we will construct an orthogonal scaling function $\phi_3(x)$ from the third order B-spline function $B_3(x)$.

In order to construct the function $\phi_3(x)$, we start with its mask $P_3(z)S_3(z)$, where $P_3(z) = (\frac{1+z}{2})^3$ is the mask of the third order B-spline. Let $S_3(z) = a_1 z + a_2 z^2 + a_3 z^3$,

then by Lemma 2.2, we have

$$
\begin{aligned}
Q_3(x) = |S_3(z)|^2 &= (a_1^2 + a_2^2 + a_3^2) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 2a_1a_3cos(\xi) \\
&= (a_1^2 + a_2^2 + a_3^2) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 2a_1a_3(2cos^2(\xi/2) - 1) \\
&= (a_1^2 + a_2^2 + a_3^2 - 2a_1a_3) + 2(a_1a_2 + a_2a_3)cos(\xi/2) + 4a_1a_3cos^2(\xi/2) \\
&= (a_1^2 + a_2^2 + a_3^2 - 2a_1a_3) + 2(a_1a_2 + a_2a_3)x + 4a_1a_3x^2
\end{aligned}
$$

where $x = cos(\xi/2)$ and $z = e^{-i\xi/2}$.

On the other hand, by equation (2.7) we have

$$
\begin{aligned}
Q_3(x) &= \sum_{i=0}^{2} \binom{5}{i} \left(\frac{1+x}{2}\right)^{2-i} \left(\frac{1-x}{2}\right)^{i} \\
&= \left(\frac{1+x}{2}\right)^2 + 5\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right) + 10\left(\frac{1-x}{2}\right)^2 \\
&= \frac{3}{2}x^2 - \frac{9}{2}x + 4
\end{aligned}
$$

Thus, from the above equations, we have the following system of equations

$$
\begin{cases}
a_1^2 + a_2^2 + a_3^2 - 2a_1a_3 = 4 \\
2(a_1a_2 + a_2a_3) = -\frac{9}{2} \\
4a_1a_3 = \frac{3}{2}
\end{cases}
\tag{4.1}
$$

Simplify (4.1) we get

$$
\begin{cases}
a_1^2 + a_2^2 + a_3^2 = \frac{19}{4} \\
a_1a_2 + a_2a_3 = -\frac{9}{4} \\
a_1a_3 = \frac{3}{8}
\end{cases}
\tag{4.2}
$$

From this system, we have $(a_1 + a_2 + a_3)^2 = a_1^2 + a_2^2 + a_3^2 + 2(a_1a_2 + a_2a_3 + a_1a_3) = 1$. Without loss of generality, consider the case $a_1 + a_2 + a_3 = 1$. Combining this with $a_1a_2 + a_2a_3 = -\frac{9}{4}$ and $a_1a_3 = \frac{3}{8}$ we have the following solution

$$
\begin{cases}
a_1 = \frac{1}{4}\left(1 + \sqrt{10} - \sqrt{5 + 2\sqrt{10}}\right) \\
a_2 = \frac{1}{2}\left(1 - \sqrt{10}\right) \\
a_3 = \frac{1}{4}\left(1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}\right)
\end{cases}
\tag{4.3}
$$

We verify the condition for ϕ_3 to be in $L_2(\mathbb{R})$

$$
a_1^2 + a_2^2 + a_3^2 = \frac{19}{4} < \frac{32}{3} = \frac{2^{2\cdot3-1}}{3}
$$

Thus ϕ_3 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_3 explicitly. It has the mask

$$
\begin{aligned}
P_3(z)S_3(z) &= \left(\frac{1+z}{2}\right)^3 (a_1z + a_2z^2 + a_3z^3) \\
&= 0.0249x - 0.0604x^2 - 0.095x^3 + 0.325x^4 + 0.571x^5 + 0.2352x^6
\end{aligned}
$$

By (1.1), (1.2) and (1.3) we have the refinement equation

$$\phi_3(x) = 0.0498\phi_3(2x - 1) - 0.121\phi_3(2x - 2) - 0.191\phi_3(2x - 3)$$
$$+ 0.650\phi_3(2x - 4) + 1.141\phi_3(2x - 5) + 0.4705\phi_3(2x - 6) \tag{4.4}$$

The corresponding spline type wavelet $\psi_3(x)$ can be expressed as

$$\psi_3(x) = 0.0498\phi_3(2x) + 0.121\phi_3(2x + 1) - 0.191\phi_3(2x + 2)$$
$$- 0.650\phi_3(2x + 3) + 1.141\phi_3(2x + 4) - 0.4705\phi_3(2x + 5),$$

which is in C^{β_3} and

$$\beta_3 > 4 - \frac{1}{2}\log_2(57).$$

We now consider the case $n = 4$. Again, we construct an orthogonal scaling function $\phi_4(x)$ from the forth order B-spline $B_4(x)$ with the mask $P_4(z) = (\frac{1+z}{2})^4$.

We examine the mask $P_4(z)S_4(z)$ of $\phi_4(x)$, where $S_4(z) = a_1z + a_2z^2 + a_3z^3 + a_4z^4$. By Lemma 2.2, we have

$$Q_4(x) = |S_4(z)|^2 = (a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)\cos(\xi/2)$$
$$+ 2(a_1a_3 + a_2a_4)\cos(\xi) + 2a_1a_4\cos(3\xi/2)$$
$$= (a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2(a_1a_2 + a_2a_3 + a_3a_4)\cos(\xi/2)$$
$$+ 2(a_1a_3 + a_2a_4)(2\cos^2(\xi/2) - 1) + 2a_1a_4(4\cos^3(\xi/2) - 3\cos(\xi/2))$$
$$= (a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4)$$
$$+ (2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4)\cos(\xi/2)$$
$$+ (4a_1a_3 + 4a_2a_4)\cos^2(\xi/2) + 8a_1a_4\cos^3(\xi/2)$$
$$= (a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4)$$
$$+ (2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4)x$$
$$+ (4a_1a_3 + 4a_2a_4)x^2 + 8a_1a_4x^3$$

where $x = \cos(\xi/2)$ and $z = e^{-i\xi/2}$.

Using equation (2.7) we get another expression for $Q_4(x)$

$$Q_4(x) = \sum_{i=0}^{3}\binom{7}{i}\left(\frac{1+x}{2}\right)^{3-i}\left(\frac{1-x}{2}\right)^i$$
$$= \left(\frac{1+x}{2}\right)^3 + 7\left(\frac{1+x}{2}\right)^2\left(\frac{1-x}{2}\right) + 21\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)^2 + 35\left(\frac{1-x}{2}\right)^3$$
$$= 8 - \frac{29}{2}x + 10x^2 - \frac{5}{2}x^3$$

Thus, from the above equations, we have the following system of equations

$$
\begin{cases}
a_1^2 + a_2^2 + a_3^2 + a_4^2 - 2a_1a_3 - 2a_2a_4 = 8 \\
2a_1a_2 + 2a_2a_3 + 2a_3a_4 - 6a_1a_4 = -\frac{29}{2} \\
4a_1a_3 + 4a_2a_4 = 10 \\
8a_1a_4 = -\frac{5}{2}
\end{cases}
\tag{4.5}
$$

Simplify (5.1) we have the following system

$$
\begin{cases}
a_1^2 + a_2^2 + a_3^2 + a_4^2 = 13 \\
a_1a_2 + a_2a_3 + a_3a_4 = -\frac{131}{16} \\
a_1a_3 + a_2a_4 = \frac{5}{2} \\
a_1a_4 = -\frac{5}{16}
\end{cases}
\tag{4.6}
$$

Solving for this system of equations yields 8 solutions. One of the numerical solutions is

$$
\begin{cases}
a_1 = 2.6064 \\
a_2 = -2.3381 \\
a_3 = 0.8516 \\
a_4 = -0.1199
\end{cases}
\tag{4.7}
$$

We verify the condition for ϕ_4 to be in $L_2(\mathbb{R})$

$$
a_1^2 + a_2^2 + a_3^2 + a_4^2 = 13 < 32 = \frac{2^{2\cdot 4 - 1}}{4}
$$

Thus ϕ_4 is indeed in $L_2(\mathbb{R})$. Now we will attempt to construct ϕ_4 explicitly. It has the mask

$$
\begin{aligned}
P_4(z)S_4(z) &= \left(\frac{1+z}{2}\right)^4 (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4) \\
&= 0.1629z + 0.5055z^2 + 0.4461z^3 - 0.0198z^4 - 0.1323z^5 + 0.0218z^6 \\
&\quad + 0.0233z^7 - 0.0075z^8
\end{aligned}
$$

By (1.1), (1.2) and (1.3) we have the refinement equation

$$
\begin{aligned}
\phi_4(x) = {}& 0.3258\phi_4(2x-1) + 1.011\phi_4(2x-2) + 0.8922\phi_4(2x-3) - 0.0396\phi_4(2x-4) \\
& - 0.2646\phi_4(2x-5) + 0.0436\phi_4(2x-6) + 0.0466\phi_4(2x-7) - 0.015\phi_4(2x-8)
\end{aligned}
\tag{4.8}
$$

The corresponding spline type wavelet $\psi_3(x)$ can be expressed as

$$
\begin{aligned}
\psi_4(x) = {}& 0.3258\phi_4(2x) - 1.011\phi_4(2x+1) + 0.8922\phi_4(2x+2) + 0.0396\phi_4(2x+3) \\
& - 0.2646\phi_4(2x+4) - 0.0436\phi_4(2x+5) + 0.0466\phi_4(2x+6) + 0.015\phi_4(2x+7)
\end{aligned}
$$

which is in C^{β_4} and

$$
\beta_4 > 4 - \frac{1}{2}\log_2(52).
$$

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Fixed point results for commuting mappings in modular metric spaces

Emine Kılınç[1], Cihangir Alaca[2,*]

[1] *Department of Mathematics, Institute of Natural and Applied Sciences, Celal Bayar University, 45140 Manisa, Turkey*

[2] *Department of Mathematics, Faculty of Science and Arts, Celal Bayar University, 45140 Manisa, Turkey*

Abstract: The purpose of this paper is to prove that two main fixed point theorems for commuting mappings in modular metric spaces. Our main results are more general than from those of Jungck [17] and Fisher [16].

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1   Introduction

The Banach Contraction Principle which published in 1922 [5] and then the beginning of fixed point theory in metric spaces is related to which in particular situation was already obtained by Liouville, Picard and Goursat. A number of authors have defined and studied contractive type mappings on a complate metric space X which are generalizations of well-known Banach contraction (see [18], [19],[26], [11]).

The notion of modular space was introduced by Nakano [24] and was intensively developed by Koshi, Shimogaki, Yamamuro (see [21, 27]) and others. A lot of mathematicians are interested fixed point of modular space. In 2008, Chistyakov introduced the notion of modular metric space generated by F-modular and developed the theory of this space [7], on the same idea was definedthe notion of a modular on an arbitrary set ad developed the theory of metric space generated by modular such that called the modular metric spaces in 2010 [8]. Afrah A. N. Abdou [1] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer et. al. [2] introduced the notion of modular G-metric spaces and proved some fixed point theorems of contractive and Azadifer et. al. [4] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in this

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*Corresponding author. Tel.: +90 236 2013209; fax: +90 236 2412158.

E-mail addresses: eklnc07@gmail.com (E. Kılınç), cihangiralaca@yahoo.com.tr (C. Alaca).

space. Kılınç and Alaca [20] defined $(\varepsilon, k)-$uniformly locally contractive mappings and η-chainable concept and proved a fixed point theorem for these concepts in a complete modular metric spaces. Recently, many authors [3, 6, 9, 10, 14, 23] studied on different fixed point results for modular metric spaces.

In this paper, we give modular metric versions of the fixed point theorem which was given by Jungck [17] and a generalization of this theorem which was given by Fisher [16].

2 Preliminaries

In this section, we will give some basic concepts and definitions about modular metric spaces.

Definition 2.1 [[8], Definition 2.1] Let X be a nonempty set, a function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on X if satisfying, for all $x, y, z \in X$ the following condition holds:

(i) $w_\lambda(x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;

(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$;

(iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ for all $\lambda, \mu > 0$.

If instead of (i), we have only the condition

$(\hat{i})$ $w_\lambda(x, x) = 0$ for all $\lambda > 0$, then w is said to be a (metric) pseudomodular on X.

The main property of a metric modular [[8]] w on a set X is the following: given $x, y \in X$, the function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), $(\hat{i})$ and (ii) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y).$$

It follows that at each point $\lambda > 0$ the right limit $w_{\lambda+0}(x, y) = \lim_{\mu \to \lambda+0} w_\mu(x, y)$ and the left limit $w_{\lambda-0}(x, y) = \lim_{\varepsilon \to +0} w_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y). \tag{2.1}$$

Definition 2.2 [[23]] Let X_w be a complete modular metric space. A self-mapping T on X_w is said to be a contraction if there exists $0 \leq k < 1$ such that

$$w_\lambda(Tx, Ty) \leq k w_\lambda(x, y) \tag{2.2}$$

for all $x, y \in X_w$ and $\lambda > 0$.

Theorem 2.1 [[23]] Let X_w be a complete modular metric space and T a contraction on X_w. Then, the sequence $(T^n x)_{n \in \mathbb{N}}$ converges to the unique fixed point of T in X_w for any initial $x \in X_w$.

Definition 2.3 [[23]], Definition 2.4] Let X_w be a modular metric space. Then following definitions exist:

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(1) The sequence $(x_n)_{n \in \mathbb{N}}$ in X_w is said to be convergent to $x \in X_w$ if $w_\lambda (x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.

(2) The sequence $(x_n)_{n \in \mathbb{N}}$ in X_w is said to be Cauchy if $w_\lambda (x_m, x_n) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.

(3) A subset C of X_w is said to be closed if the limit of a convergent sequence of C always belong to C.

(4) A subset C of X_w is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit is in C.

3   Main Results

In this section we will give two main theorems for commuting mappings in modular metric spaces, the fixed point theorem which was given by Jungck in 1976 [17] and a generalization of this theorem which was given by Fisher [16] in 1981 for metric spaces.

Theorem 3.1 Let I and T be commuting mappings of complete modular metric space X_w into itsef satisfiying the inequality

$$w_\lambda (Tx, Ty) \leq k.w_\lambda (Ix, Iy) \tag{3.1}$$

for all $x, y \in X_w$, where $0 < k < 1$.If the range of I contains the range of T ,I is continuous and $w_\lambda (x, Ix) < \infty$ then T and I have a common unique fixed point.

Proof.   Let $x_0 \in X_w$ be arbitrary.   Then Tx_0 and Ix_0 are well defined. Since $Tx_0 \in I(X_w)$, there is some $x_1 \in X_w$ such that $Ix_1 = Tx_0$. Then choose $x_2 \in X_w$ such that $Ix_2 = Tx_1$. In general if x_n is choosen, then we choose a point x_{n+1} in X_w such that $Ix_{n+1} = Tx_n$. We show that (x_n) is Cauchy sequence . From (3.1) we have

$$w_\lambda (Ix_n, Ix_{n+1}) = w_\lambda (Tx_{n-1}, Tx_n) \leq k.w_\lambda (Ix_{n-1}, Ix_n)$$

Then with induction we have

$$w_\lambda (Ix_n, Ix_{n+1}) \leq k.w_\lambda (Ix_{n-1}, Ix_n) \leq ... \leq k^n w_\lambda (Ix_0, Ix_1)$$

If we take the limit as $n \to \infty$, we get

$$w_\lambda (Ix_n, Ix_{n+1}) \to 0$$

Without loss of generality we can assume that for $\frac{\lambda}{m-n} > 0$ there exists $\frac{\varepsilon}{m-n}$ such that

$$w_{\frac{\lambda}{m-n}} (Ix_n, Ix_{n+1}) \leq \frac{\varepsilon}{m-n}$$

For $m, n \in \mathbb{N}$ and $m > n$ we have

$$
\begin{aligned}
w_\lambda (Ix_n, Ix_m) &\leq w_{\frac{\lambda}{m-n}} (Ix_n, Ix_{n+1}) + w_{\frac{\lambda}{m-n}} (Ix_{n+1}, Ix_{n+2}) + ... + w_{\frac{\lambda}{m-n}} (Ix_{m-1}, Ix_m) \\
&\leq \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + ... + \frac{\varepsilon}{m-n} \\
&\leq \varepsilon
\end{aligned}
$$

Thus (x_n) isa Cauchy sequence. Since X_w is complete, there is some $u \in X_w$ such that

$$\lim_{n \to \infty} I x_n = \lim_{n \to \infty} T x_{n-1} = u$$

Since I is continuous and T and I commute we get

$$
\begin{aligned}
Iu &= I\left(\lim_{n \to \infty} I x_n\right) = \lim_{n \to \infty} I^2 x_n \\
Iu &= I\left(\lim_{n \to \infty} T x_n\right) = \lim_{n \to \infty} I T x_n = \lim_{n \to \infty} T I x_n
\end{aligned}
$$

From (3.1) we get

$$w_\lambda\left(T I x_n, T u\right) \leq k.w_\lambda\left(I^2 x_n, I u\right)$$

Taking the limit as $n \to \infty$ we obtain

$$w_\lambda\left(I u, T u\right) \leq k.w_\lambda\left(I u, I u\right)$$

Hence we get $Iu = Tu$. Again from (3.1)

$$w_\lambda\left(T x_n, T u\right) \leq k.w_\lambda\left(I x_n, I u\right)$$

Letting n tend to ∞ we get

$$w_\lambda\left(u, T u\right) \leq k.w_\lambda\left(u, I u\right) = k.w_\lambda\left(u, T u\right)$$

Hence $Tu = u$. To show the uniqueness of u, assume that there exists $v \in X_w$ such that $Tv = Iv = v$, then

$$w_\lambda\left(u, v\right) = w_\lambda\left(I u, T v\right) = w_\lambda\left(T u, T v\right) \leq k.w_\lambda\left(I u, I v\right)$$

Then $u = v$. This completes the proof. ∎

Theorem 3.2 Let S and T be commuting mappings and I and J be commuting mappings of a complete modular metric space X_w into itself satisfying

$$w_\lambda\left(S x, T y\right) \leq k.w_\lambda\left(I x, J y\right)$$

for all $x, y \in X_w$, where $0 \leq k < 1$. If I and J are continuous and $w_\lambda\left(I x_0, J x_1\right) < \infty$, then all S, T, I and J have a unique common fixed point.

Proof. Let $x_0 \in X_w$ be arbitrary. Since $S x_0 \in J\left(X_w\right)$, let $x_1 \in X_w$ be such that $J x_1 = S x_0$ and also, as $T x_1 \in I\left(X_w\right)$, let $x_2 \in X_w$ such that $I x_2 = T x_1$. In general $x_{2n+1} \in X_w$ is choosen such that $J x_{2n+1} = S x_{2n}$ and $x_{2n+2} \in X_w$ suh that $I x_{2n+2} = T x_{2n+1}$, $n = 0, 1, ...$
Denote

$$
\begin{aligned}
y_{2n} &= J x_{2n+1} = S x_{2n} \\
y_{2n+1} &= I x_{2n+2} = T x_{2n+1}, \qquad n \geq 0
\end{aligned}
$$

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Then, it can be proved that (y_n) isa Cauchy sequence.

$$
\begin{aligned}
w_\lambda \left(Jx_{2n+1}, Ix_{2n+2} \right) &= w_\lambda \left(Sx_{2n}, Tx_{2n+1} \right) \leq k.w_\lambda \left(Ix_{2n}, Jx_{2n+1} \right) \\
&\leq k.w_\lambda \left(Tx_{2n-1}, Sx_{2n} \right) \leq k.k.w_\lambda \left(Jx_{2n-1}, Ix_{2n} \right) \\
&\leq k^2 w_\lambda \left(Jx_{2n-1}, Ix_{2n} \right) \\
&\leq k^3 w_\lambda \left(Jx_{2n-3}, Ix_{2n-2} \right) \\
&\qquad \dots \\
&\leq k^{2n} w_\lambda \left(Jx_3, Ix_2 \right) \\
&\leq k^{2n+2} w_\lambda \left(Jx_1, Ix_0 \right)
\end{aligned}
$$

Then taking $n \to \infty$ we have

$$
w_\lambda \left(Jx_{2n+1}, Ix_{2n+2} \right) \to 0
$$

Then there exists $\varepsilon > 0$ such that

$$
w_\lambda \left(Jx_{2n+1}, Ix_{2n+2} \right) \leq \varepsilon
$$

Without loss of generality we can assume that for $\frac{\lambda}{m-n} < 0$, there exists $\frac{\varepsilon}{m-n} > 0$ such that

$$
w_{\frac{\lambda}{m-n}} \left(Jx_{2n+1}, Ix_{2n+2} \right) \leq \frac{\varepsilon}{m-n}
$$

Hence choosing $m = 2k+1, n = 2l+2, k > l$ and using triangle inequality we get

$$
\begin{aligned}
w_\lambda \left(y_m, y_n \right) &= w_\lambda \left(y_{2k+1}, y_{2l+2} \right) = w_\lambda \left(Ix_{2k+2}, Jx_{2l+1} \right) \\
&\leq w_{\frac{\lambda}{m-n}} \left(Ix_{2k+2}, Jx_{2k+1} \right) + w_{\frac{\lambda}{m-n}} \left(Jx_{2k+1}, Ix_{2k} \right) + \dots + w_{\frac{\lambda}{m-n}} \left(Ix_{2l+2}, Jx_{2l+1} \right) \\
&\leq \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \dots + \frac{\varepsilon}{m-n} \\
&\leq \varepsilon
\end{aligned}
$$

Hence (y_n) is a Cauchy sequence. Let $u \in X_w$ be such that

$$
\lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Jx_{2n+1} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} Ix_{2n+2} = u
$$

Since I is continuous and I and S commute it follows that

$$
\lim_{n \to \infty} I^2 x_{2n+2} = Iu, \quad \lim_{n \to \infty} SIx_{2n} = \lim_{n \to \infty} ISx_{2n} = Iu = u
$$

From (3.2)

$$
w_\lambda \left(SIx_{2n}, Tx_{2n+1} \right) \leq k.w_\lambda \left(I^2 x_{2n}, Jx_{2n+1} \right)
$$

Taking the limit as $n \to \infty$ we get

$$
w_\lambda \left(Iu, u \right) \leq k.w_\lambda \left(Iu, u \right)
$$

Thence $Iu = u$.

Similarly we obtain $Ju = u$, as J is continuous and T and J are commute.

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From (3.2) we have, as $Iu = u$,

$$w_\lambda \left(Su, Tx_{2n+} \right) \leq k.w_\lambda \left(u, Jx_{2n+1} \right)$$

Taking the limit as $n \to \infty$ we get

$$w_\lambda \left(Su, u \right) = 0$$

Hence $Su = u$. Again from (3.2) considering the results above, we have

$$w_\lambda \left(Su, Tu \right) = 0$$

Hence $Tu = Su$. Thus we proved that

$$Su = Tu = Iu = Ju = u$$

Clearly u is the unique common fixed point of all S, T, I and J. This completes the proof. ■

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A general common fixed point theorem for weakly compatible mappings

Kristaq Kikina, Luljeta Kikina and Sofokli Vasili

Department of Mathematics and Computer Sciences,
University of Gjirokastra, Gjirokastra, Albania.
kristaqkikina@yahoo.com, gjonileta@yahoo.com

Abstract. The purpose of this paper is to complete and generalize many existing results in the literature for weakly compatible mappings and to perform the proof without the Hausdorff's condition.

1. Introduction and preliminaries

It is known that common fixed point theorems are generalizations of fixed point theorems. Over the past few decades, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems.

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions. Some of such generalizations are: the quasi-metric spaces, the generalized metric spaces and the generalized quasi-metric spaces.

The notion of quasi-metric, also known as b-metric, is in line of metric in which the triangular inequality $d(x, y) \leq d(x, z) + d(z, y)$ is replaced by quasi-triangular inequality $d(x, y) \leq k[d(x, z) + d(z, y)], k \geq 1$ (see [1], [6], [7], [8], [11], [12], [25], [30], [34], [35]).

In 2000 Branciari [8] introduced the concept of generalized metric space (**gms**), by replacing the triangle inequality with tetrahedral inequality. Starting with the paper of Branciari, many fixed point and common fixed point results have been established in this interesting space. For further information, the reader can refer to [2, 13-16, 20, 22-24, 28-29, 32-33, 36].

Recently L. Kikina and K. Kikina [26] introduced the concept of **generalized quasi-metric space (gqms)** by replacing the tetrahedral inequality $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$ with a more general inequality, namely quasi-tetrahedral inequality $d(x, y) \leq k[d(x, z) + d(z, w) + d(w, y)], k \geq 1$. The generalized metric space is a special case of generalized quasi-metric space (for $k = 1$). Also, every quasi-metric space is a **gqms**, while the converse is not true [26]. Further aspects may be found in ([21], [26], [27]). By using this generalized quasi-metric space successfully defined an "open" ball and hence a topology. On the other hand this topology fails to provide some useful topological properties: a "open" ball in generalized quasi-metric space needs not to be open set; the generalized quasi-metric needs not to be continuous; a convergent sequence in generalized quasi-metric space needs not to be

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Cauchy; the generalized metric space needs not to be Hausdorff and hence the uniqueness of limits cannot be guaranteed.

The above properties of generalized metric spaces (spesial case of gqms for $k = 1$) that do not hold for metric spaces were first observed by Das and Dey [13], [14] and also these facts were observed independently by Samet [32] and also by Sarma, Rao and Rao [33]. Initially these were considered to be true, implying non correct proofs of several theorems. This made some of the previous results to be reconsidered and to be corrected.

Under these circumstances, not every fixed point theorem from metric spaces can be extended in gms or in gqms. Even in the case this extension may be done, the proof of such theorem is technically more complicated than in the usual setting. Because of those difficulties, some authors have taken the Hausdorffity as additional condition in their theorems, but this is not always necessary. For example, the assertion in [33] that the space needs to be Hausdorff is superfluous, a fact first noted by Kikina and Kikina [23]. See also [36].

The aim of this paper is to present a general theorem, from which a lot of current results on common fixed points in different kind of spaces (metric, quasi metric, gms, gqms, etc.) are taken as corollaries.

The following definition was given by Kikina in 2012.

Definition 1.1 [26] Let X be a set. A function $d : X \times X \to R_+$ is called a *generalized quasi-metric* in X if and only if there exists a constant $k \geq 1$ such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from x and y the following conditions hold:

(i) $d(x, y) = 0 \Leftrightarrow x = y;$

(ii) $d(x, y) = d(y, x);$

(iii) $d(x, y) \leq k[d(x, z) + d(z, w) + d(w, y)]$ (*quasi-tetrahedral inequality*)

Inequality (3) is often called *quasi-tetrahedral inequality* and k is often called the *coefficient* of d. A pair (X, d) is called a *generalized quasi-metric space* (**gqms**) if X is a set and d *is* a *generalized quasi-metric* in *X*.

The set $B(a, r) = \{x \in X : d(x, a) < r\}$ is called "open" ball with center $a \in X$ and radius $r > 0$.

The family $\tau = \{Q \subset X : \forall a \in Q, \exists r > 0, B(a, r) \subset Q\}$ is a topology on X and it is called induced topology by the *generalized quasi-metric d* .

Definition 1.2 Let (X, d) be a **gqms**.

(1) *A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if and only if*

$\lim_{n \to \infty} d(x_n, x) = 0$. *We denote this by $x_n \to x$.*

(2) *A sequence $\{x_n\}$ is a Cauchy sequence if and only if for each $\varepsilon > 0$ there exists*

a natural number $n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n > n(\varepsilon)$.

(3) *(X, d) is called complete if every Cauchy sequence is convergent in* X.

Definition 1.3 *Let (X, d) be a gqms. The generalized quasi-metric d is said to be continuous if and only if for any sequences $\{x_n\}, \{y_n\} \subseteq X$, such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, we have*

$\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$.

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The following example illustrates the existence of the *generalized* quasi-*metric* space for an arbitrary coefficient $k \geq 1$:

Example 1.4 [26] Let $X = \left\{1 - \dfrac{1}{n} : n = 1, 2, \ldots\right\} \cup \{1, 2\}$, Define $d : X \times X \to R$ as follow:

$$d(x, y) = \begin{cases} 0 & \text{for } x = y \\ \dfrac{1}{n} & \text{for } x \in \{1, 2\} \text{ and } y = 1 - \dfrac{1}{n} \text{ or } y \in \{1, 2\} \text{ and } x = 1 - \dfrac{1}{n}, x \neq y \\ 3k & \text{for } x, y \in \{1, 2\}, x \neq y \\ 1 & \text{otherwise} \end{cases}$$

Then it is easy to see that (X, d) is a generalized quasi-metric space and is not a generalized metric space (for $k > 1$).

Note that the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ converges to both 1 and 2 and it is not a Cauchy sequence: $d(x_n, x_m) = d(1 - \frac{1}{n}, 1 - \frac{1}{m}) = 1, \forall n, m \in N$

The above example shows us some of the properties of metric spaces which do not hold in gqms (see [26]).

Kikina and Kikina in [27] extended the well-known Ciric's quasi-contraction principle [10] in a generalized quasi-metric space. We restate Theorem 2.6[27] as follows:

Theorem 1.5 *Let (X, d) be a complete gqms with the coefficient $k \geq 1$ and let $T : X \to X$ be a self-mapping satisfying:*

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (1)$$

for all $x, y \in X$, where $0 \leq h < \frac{1}{k}$. Then T has a unique fixed point in X,

By this Theorem, in special case $k = 1$, we obtain the Ciric's quasi-contraction principle in a generalized metric space which is presented in [15] and [22].

Let $\mathbb{F}$ be the set of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the condition $\xi(t) = 0$ if and only if $t = 0$. We denote by Ψ the set of functions $\psi \in \mathbb{F}$ such that ψ is continuous and non-decreasing. We reserve Φ for the set of functions $\alpha \in \mathbb{F}$ such that α is continuous. Finally, by Γ we denote the set of functions $\beta \in \mathbb{F}$ satisfying the following condition: β is lower semi-continuous.

Lakzian and Samet in [28] established the following fixed point theorem.

Theorem 1.6. ([28]). *Let (X, d) be a Hausdorff and complete gms and let $T : X \to X$ be a self-mapping satisfying $\psi(d(Tx, Ty) \leq \psi(d(x, y)) - \varphi(d(x, y))$. For all $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$. Then T has a unique fixed point in X.*

In [5], it is given a slightly improved version of Theorem 1.6, obtained by replacing the continuity condition of φ with a lower semi-continuity.

Definition 1.7 ([19]) *Let X be a non-empty set and $T, F : X \to X$. The mappings T and F are said to be weakly compatible if they commute at their coincidence points (i.e. $TFx = FTx$ whenever $Tx = Fx$). A point $y \in X$ is called point of coincidence of T and F if there exists a point $x \in X$ such that $y = Tx = Fx$.*

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Di Bari and Vetro [16] extended Theorem 1.6 of Lakzian and Samet to the following common fixed point theorem:

Theorem 1.8 ([16]) *Let* (X,d) *be a Hausdorff gms and let T and F be self-mappings on X such that* $TX \subseteq FX$. *Assume that* (FX,d) *is a complete gms and that the following condition holds:*

$$\psi(d(Tx,Ty) \leq \psi(d(Fx,Fy)) - \varphi(d(Fx,Fy))$$

for all $x,y \in X$, *where* $\psi \in \Psi$ *and* $\varphi \in \Gamma$. *Then T and F have a unique point of coincidence in X. Moreover, if T and F are weakly compatible, then T and F have a unique common fixed point.*

In [9], Choudury and Kundu established the (ψ,α,β)-weakly contraction principle in partially ordered metric spaces. Recently, Isik and Turkoglu [18], and Bilgili et al [5] extended the results in [9] to the set of generalized metric spaces:

Theorem 1.9 ([5]). Let (X,d) be a Hausdorff and complete gms, and let $T: X \to X$ be a self-mapping such that

$$\psi(d(Tx,Ty) \leq \alpha(d(x,y)) - \beta(d(x,y))$$

for all $x,y \in X$, *where* $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma$ *and these mappings satisfy condition*

$$\psi(t) - \alpha(t) + \beta(t) > 0, \textit{for all } t > 0 .$$

Then T has a unique fixed point in X.

Theorem 1.10 (Isik and Turkoglu [18]) *Let* (X,d) *be a Hausdorff and complete g.m.s. and let* $T,F: X \to X$ *be self-mappings such that* $TX \subseteq FX$, *and FX is a closed subspace of X, and that the following condition holds:*

$$\psi(d(Tx,Ty) \leq \alpha(d(Fx,Fy)) - \beta(d(Fx,Fy))$$

for all $x,y \in X$, *where* $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma$ *and these mappings satisfy condition*

$$\psi(t) - \alpha(t) + \beta(t) > 0, \textit{for all } t > 0$$

Then T and F have a unique coincidence point in X. Moreover, if T and F are weakly compatible, then T and F have a unique common fixed point.

The following lemma will play an important role in obtaining the main results of this paper.

Lemma 1.11 [17] *Let X be a nonempty set and f : X → X a function. Then there exists a subset* $E \subseteq X$ *such that f (E) = f (X) and f: E → X is one-to-one.*

2. Main results

In this section, we state and prove our main results. Firstly, we prove an important proposition that makes unnecessary the Hausdorffity condition for the most of well known theorems on gms and gqms.

Proposition 2.1 *If* $\{y_n\}$ *is a Cauchy sequence in a generalized quasi-metric space* (X,d) *with the coefficient* $k \geq 1$ *and* $\lim_{n \to \infty} y_n = u$, *then the limit u is unique.*

Proof. Assume that $u' \neq u$ is also $\lim_{n \to \infty} y_n$.

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If $\{y_n\}$ is a sequence of distinct points, $y_n \neq y_m$ for all $n \neq m$, there exists $n_0 \in N$ such that u and u' are different from y_n for all $n > n_0$. For $n > n_0$, by rectangular property of Definition 1.1 we obtain

$$d(u,u') \leq k[d(u,y_n) + d(y_n, y_{n+1}) + d(y_{n+1}, u')]$$

Letting n tend to infinity we get $d(u,u') = 0$ and so $u = u'$, a contradiction.

If $\{y_n\}$ is not a sequence of distinct points, then we have the following two cases:

a) The sequence $\{y_n\}$ contains a subsequence $\{y_{n_p}\}$ of distinct points, and in this case, we can use the same reasoning as above.

b) The sequence $\{y_n\}$ contains a constant subsequence $\{y_{n_p}\} : y_{n_p} = c; \forall p \in N$. Then

$c = \lim_{n \to \infty} y_n = u = u'$, a contradiction. This completes the proof of the Proposition.

Remark 2.2 From the above proposition it follows that even in the cases of gms and gqms, the including of the Hausdorffity condition in any theorem, in order to implies the uniqueness of limit of Cauchy sequence, *is unnecessary*

The following results are inspired by the techniques and ideas of [17].

The main Theorem

A natural way to unify and prove in a simple manner several fixed point and common fixed point theorems is by considering "functional" contractive condition like

$$F(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \leq 0, \ \forall x, y \in X$$

where $F : R_+^6 \to R$ is an **appropriate function**. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [31]; see also Turinici [37], Berinde and Vetro [3], Berinde [4], etc.

Let (X, d) be a **space** (metric, quasi-metric, generalized metric, generalized quasi-metric, etc.) in which the following theorem has been proved:

Theorem *: *If (X, d) is a complete **space** and $T : X \to X$ is a self-mapping satisfying*

$$F(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) \leq 0, \ \forall x, y \in X \qquad (4)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

We will prove here the main theorem:

Theorem 2.3 *Let (X, d) be a space on which the **Theorem *** holds and $T, f : X \to X$ be self-mappings such that $T(X) \subseteq f(X)$. Assume that (fX, d) is a complete space and that the following condition holds:*

$$F(d(Tx,Ty), d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)) \leq 0 \qquad (5)$$

for all $x, y \in X$. Then T and f have a unique coincidence point in X. Moreover if T and f are weakly compatible, then T and f have a unique common fixed point.

Proof. By Lemma 2.1, there exists $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one. Now, define a map $h : f(E) \to f(E)$ by $h(fx) = Tx$. Since f is one-to-one on E, h is well-defined. Note that,

$$F(d(h(fx), h(fy)), d(fx, fy), d(fx, h(fx)), d(fy, h(fy)), d(fx, h(fy)), d(fy, h(fx))) \leq 0$$

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for all $fx, fy \in f(E)$. Since $f(E) = f(X)$ is complete, by using **Theorem ***, there exists $x_0 \in X$ such that $h(fx_0) = fx_0$. Hence, T and f have a point of coincidence, which is also unique. It is clear that T and f have a unique common fixed point whenever T and f are weakly compatibles.

Corollary 2.4 (A generalization of the Ciric's quasi-contraction principle [10]) *Let (X, d) be a gqms with the coefficient $k \geq 1$ and let $T, f : X \rightarrow X$ be self-mappings such that $T(X) \subseteq f(X)$. Assume that (fX, d) is a complete gqms and that the following condition holds*:
$$d(Tx, Ty) \leq h \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \qquad (1)$$
for all $x, y \in X$, where $0 \leq h < \frac{1}{k}$. Then T and f have a unique coincidence point in X. Moreover if T and f are weakly compatible, then T and f have a unique common fixed point.
Proof. Corollary 2.4 *is taken from Theorem 2.3 in case when T***heorem* is the Theorem 1.5** and

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$

$$= d(Tx, Ty) - h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \text{ where } 0 \leq h < \frac{1}{k}$$

From this *Corollary*, in the special case $f = I_X$ we have the Theorem 1.5.

Corollary 2.5 *Theorem* 1.8 *is taken from Theorem* 2.3 *in case when* **T***heorem* is the* **Theorem 1.6** *and*
$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$
$$= \psi(d(Tx, Ty)) - \psi(d(x, y)) + \varphi(d(x, y))$$
Corollary 2.6 *Theorem* 1.10 *is taken from Theorem* 2.3 *in case when as* **Theorem* is the Theorem 1.9** *and*
$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))$$
$$= \psi(d(Tx, Ty)) - \alpha(d(x, y)) + \beta(d(x, y))$$
Corollary 2.7 *Let (X, d) be a gqms with the coefficient $k \geq 1$ and let $T, f : X \rightarrow X$ be self-mappings such that $T(X) \subseteq f(X)$. Assume that (fX, d) is a complete gqms and that the following condition holds*:
$$d(Tx, Ty) \leq h \max Uxy \qquad (2)$$
for all $x, y \in X$, where $0 \leq h < \frac{1}{k}$, and
$$U_{xy} \subseteq \{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \qquad (3)$$

Then T and f have a unique coincidence point in X. Moreover if T and f are weakly compatible, then T and f have a unique common fixed point.
Proof. By (2) and (3), we have:
$$d(Tx, Ty) \leq h \max Uxy \leq h \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$
Then it suffices to apply Corollary 2.4.
For different expression of U_{xy}, in the Corollary 2.2, we get different theorems. For example:
Corollary 2.3 *For $U_{xy} = \{d(fx, fy)\}$, we have an extension and generalization of the Banach contraction principle in a gqms.*

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Corollary 2.4 *For* $U_{xy} = \{d(fx, fy), d(fx, Tx), d(fy, Ty)\}$, *we have an extension and generalization of the Rhoades theorem* [31] *in a gqms*.

Remark 2.5 We can obtain many other similar results of Rhoades classification [31].

Conclusion 2.7 If in the future will be proved any new theorem of the type of Theorem* with a new appropriate form of the function F, then it holds also its corresponding Theorem 2.3.

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Integro-Differential equations of fractional order on unbounded domains

Xuhuan Wang[1,2], Yongfang Qi[1]

[1]Department of Mathematics, Pingxiang University, Pingxiang, Jiangxi 337055, P. R. China
[2]Department of Mathematics, Baoshan University, Baoshan, Yunnan 678000, P. R. China

Email: Xuhuan Wang - wangxuhuan84@163.com;

*Corresponding author

Xuhuan Wang

Abstract

In this paper, we discuss the nonlocal conditions for fractional order nonlinear integro-differential equations on an infinite interval, and prove the existence and uniqueness of bounded solutions. Our analysis relies on the some standard fixed point theorems.

Keywords: Fractional order; Integro-Differential equations; Existence; Fixed point theorem.

1 Introduction

It is well known that fractional differential equations are generalizations of classical differential equations to an arbitrary order. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order[3,4]. on the other hand, the integro-differential equations is an important branch of nonlinear analysis[5].

In 2008, Lakshmikanthama and Vatsala [1] studied basic theory of fractional differential equations

$$D^q u(t) = f(t, u(t)), \quad u(0) = u_0, \quad 0 < q < 1, \tag{1.1}$$

In 2012, Karthikeyan and Trujillo [2] consider existence and uniqueness results for fractional integrodif-

ferential equations with boundary value conditions by used fixed point theorem

$$\begin{cases} D^\alpha y(t) = f(t, y(t), Sy(t)), & 0 < \alpha \le 1, \quad t \in [0, T] \\ ay(0) + by(T) = c, \end{cases} \qquad (1.2)$$

Motivated by Lakshmikanthama et al [1] and Karthikeyan et al [2], in this paper, we consider the following

the nonlocal conditions for fractional order nonlinear integro-differential equations in E on an infinite interval:

$$D^\alpha u = f(t, u, Tu), \quad \forall t \in J, \quad u(0) + g(u) = u_0, \quad 0 < \alpha \le 1, \qquad (1.3)$$

where $J = [0, \infty)$ is an infinite interval, $u_0 \in E$, $f \in C(J \times E \times E, E)$, $g \in (J, E)$,

$$(Tu)(t) = \int_0^t k(t, s)u(s)ds, \forall t \in J, \qquad (1.4)$$

$k \in C[D, R]$, $D = \{(t, s) \in J \times J : t \ge s\}$.

Let $BC[J, E] = \{u \in C[J, E] : \sup_{t \in J} \|u(t)\| < \infty\}$ with norm $\|u\|_{BC} = \sup_{t \in J} \|u(t)\|$. Then $BC[J, E]$ is a

Banach space.

2   Preliminaries

Let us list some conditions.

(H_1) $k^* = \sup_{t \in J} \int_0^t |k(t, s)|ds < \infty$.

(H_2) $\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le \beta(t)[a\|u - \bar{u}\| + b\|v - \bar{v}\|]$, $\quad g(u) - g(v) \le L\|u - v\| \quad \forall \in J, u, \bar{u}, v, \bar{v} \in E$,

where constants $a \ge 0, b \ge 0, L \ge 0, \beta \in C[J, R_+]$ and

$$\beta^* = \int_0^\infty (t - s)^{\alpha - 1}\beta(s)ds < \infty, \quad \alpha^* = \int_0^\infty (t - s)^{\alpha - 1}\|f(s, \theta, \theta)\|ds < \infty,$$

here θ denotes the zero element of E.

(H_3) $c_0 = L + \dfrac{(a + bk^*)\beta^*}{\Gamma(\alpha)} < 1$.

3   Main Results

Theorem 3.1 If conditions $(H_1) - (H_3)$ are satisfied, then Eq (1.3) has a unique solution $u \in C^1[J, E] \cap$

$BC[J, E]$; moreover, for any $v_0 \in BC[J, E]$, the iterative sequence defined by

$$v_n(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha - 1}f(s, v_{n-1}(s), (Tv_{n-1})(s))ds \quad (n = 1, 2, 3, \cdots) \qquad (1.5)$$

converges to $u(t)$ uniformly in $t \in J$.

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Proof. It is clear that $u \in C^1[J, E] \cap BC[J, E]$ is a solution of Eq (1.3) if and only if $u \in BC[J, E]$ is a solution of the following integral equation:

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), (Tu)(s)) ds. \tag{1.6}$$

Define operator A by

$$(Au)(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), (Tu)(s)) ds. \tag{1.7}$$

By (H_2), we have

$$\|f(t, u, v)\| \le \|f(t, \theta, \theta)\| + \beta(t)[a\|u\| + b\|v\|], \quad \forall t \in J, u, v \in E. \tag{1.8}$$

It follows from (H_1), (H_2) and (1.7), (1.8) that, for $u \in BC[J, E]$,

$$\|(Au)(t)\| \le \|u_0\| + \|g(u)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\|f(s, \theta, \theta)\| + \beta(s)(a\|u(s)\| + b\|(Tu)(s)\|)] ds$$

$$\le \|u_0\| + \|g(u)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\|f(s, \theta, \theta)\| ds + \frac{(a + bk^*)\|u\|_{BC}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta(s) ds$$

$$\le \|u_0\| + \|g(u)\| + \frac{a^* + (a + bk^*)\|u\|_{BC}}{\Gamma(\alpha)}, \quad \forall t \in J$$

so $Au \in BC[J, E]$, i.e. $A : BC[J, E] \to BC[J, E]$. On the other hand, for $u, v \in BC[J, E]$, we have, by (1.7), (H_1) and (H_2),

$$\|(Au)(t) - (Av)(t)\| \le \|g(u) - g(v)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\beta(s)(a\|u(s) - v(s)\| + b\|(Tu)(s) - (Tv)(s)\|)] ds$$

$$\le \|g(u) - g(v)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\beta(s)(a\|u(s) - v(s)\| + bk^*\|u(s) - v(s)\|)] ds$$

$$\le L\|u - v\|_{BC} + \frac{(a + bk^*)\|u - v\|_{BC}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \beta(s) ds,$$

$$\le (L + \frac{(a + bk^*)\beta^*}{\Gamma(\alpha)})\|u - v\|_{BC} = c_0\|u - v\|_{BC}, \quad \forall t \in J \tag{1.9}$$

so,

$$\|Au - Av\|_{BC} \le c_0\|u - v\|_{BC}, \quad \forall u, v \in BC[J, E].$$

Since $c_0 = L + \frac{(a+bk^*)\beta^*}{\Gamma(\alpha)} < 1$ on account of (H_3), the Banach fixed point theorem implies that A has a unique fixed point u in $BC[J, E]$ and the sequence $\{v_n\}$ defined by (1.5) converges to u uniformly in $t \in J$. The theorem is proved.

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Theorem 3.2 Let conditions $(H_1) - (H_3)$ be satisfied and $u(t)$ be the unique solution in $C^1[J, E] \cap BC[J, E]$ of Eq (1.3). Then

$$\lim_{t \to \infty} u(t) = u(\infty) \tag{2.0}$$

exists, and

$$u(\infty) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^\infty (t - s)^{\alpha-1} f(s, u(s), (Tu)(s)) ds. \tag{2.1}$$

then problem (1.4) has a unique solution.

Proof. By the proof of Theorem 3.1, $u(t)$ satisfies Eq.(1.6). Let $t_2 > t_1 > 0$. Then, by a simple computation, we can get

$$\|u(t_2) - u(t_1)\| \le \|\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - s)^{\alpha-1} f(s, u(s), (Tu)(s)) ds\|$$

$$\le \|\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - s)^{\alpha-1} f(s, \theta, \theta) ds\| + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - s)^{\alpha-1} \beta(s)(a\|u(s)\| + b\|(Tu)(s)\|) ds$$

$$\le \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - s)^{\alpha-1} \|f(s, \theta, \theta)\| ds + \frac{(a + bk^*)\|u\|_{BC}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - s)^{\alpha-1} \beta(s) ds$$

which implies by virtue of (H_2) that limit (2.0) exists and (2.1) holds.

Theorem 3.3. Let conditions $(H_1) - (H_3)$ be satisfied. Denote by $u(t)$ and $u^*(t)$ the unique solutions in $C^1[J, E] \cap BC[J, E]$ of Eq. (1.3) and the following the nonlocal conditions

$$D^\alpha u = f(t, u, Tu), \quad \forall t \in J, \quad 0 < \alpha \le 1; \quad u(0) = u_0^*, \tag{2.2}$$

respectively. Then

$$\|u^* - u\|_{BC} \le \frac{1}{1 - c_0} \|u_0^* - u_0\|, \tag{2.3}$$

where c_0 is defined in (H_3).

proof. By the Theorem 3.1, we have

$$\|u_n - u\|_{BC} \to 0, \quad \|u_n^* - u^*\|_{BC} \to 0 \quad as \quad n \to \infty, \tag{2.4}$$

where $u_n(t)$ and $u_n^*(t)$ are defined by (1.5) and

$$v_n^*(t) = u_0^* + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, v_{n-1}^*(s), (Tv_{n-1}^*)(s)) ds \quad (n = 1, 2, 3, \cdots) \tag{2.5}$$

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$(v_0^* \in BC[J, E]$ is arbitrarily given). Similar to (1.9), we get from (1.5) and (2.5)

$$\|u_n^*(t) - v_n(t)\| \leq \|u_0^* - u_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\beta(s)(a\|v_{n-1}^*(s) - v_{n-1}(s)\| + b\|(Tv_{n-1}^*)(s) - (Tv_{n-1})(s)\|)]ds$$

$$\leq \|u_0^* - u_0\| + \frac{(a+bk^*)\|v_{n-1}^* - v_{n-1}\|_{BC}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\beta(s)ds$$

$$\leq \|u_0^* - u_0\| + \frac{(a+bk^*)\beta^*\|v_{n-1}^* - v_{n-1}\|_{BC}}{\Gamma(\alpha)}, \quad \forall t \in J$$

so

$$\|v_n^* - v_n\|_{BC} \leq \|u_0^* - u_0\| + c_0\|v_{n-1}^* - v_{n-1}\|_{BC} \quad (n = 1, 2, 3, \cdots),$$

and hence

$$\|v_n^* - v_n\|_{BC} \leq (1 + c_0 + \cdots + c_0^{n-1})\|u_0^* - u_0\| + c_0^n\|v_0^* - v_0\|_{BC}$$

$$= \frac{1 - c_0^n}{1 - c_0}\|u_0^* - u_0\| + c_0^n\|v_0^* - v_0\|_{BC} \quad (n = 1, 2, 3, \cdots). \tag{2.6}$$

Observing (2.4) and $0 \leq c_0 < 1$, and taking limits in (2.6), we obtain (2.3).

Remark. From (2.3) we know that $u^*(t) \to u(t)$ uniformly in $t \in J$ as $u_0^* \to u_0$. This means that, under conditions $(H_1) - (H_3)$, the unique solution of Eq.(1.3) is continuously dependent on the initial value u_0.

In the same way, we can discuss the nonlocal conditions for fractional order nonlinear integro-differential equation of mixed type:

$$D^\alpha u = f(t, u, Tu, Su), \quad \forall t \in J, \quad u(0) + g(u) = u_0, \quad 0 < \alpha \leq 1, \tag{2.7}$$

where $J = [0, \infty)$, $u_0 \in E$, $f \in C(J \times E \times E \times E, E)$, Tu, Su are defined by

$$(Tu)(t) = \int_0^t k(t, s)u(s)ds, \forall t \in J, \qquad (Su)(t) = \int_0^t h(t, s)u(s)ds, \forall t \in J, \tag{2.8}$$

$k, h \in C[D, R]$, here $D = \{(t, s) \in J \times J : t \geq s\}$. In this situation, conditions $(H_1) - (H_3)$ are replaced by the following conditions $(H_1') - (H_3')$:

(H_1') $k^* = \sup_{t \in J} \int_0^t |k(t,s)|ds < \infty$, $h^* = \sup_{t \in J} \int_0^\infty |h(t,s)|ds < \infty$ and
$\lim_{t' \to t} \int_0^\infty |h(t', s) - h(t, s)|ds = 0, \forall t \in J$

(H_2') $\|f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w})\| \leq \beta(t)[a\|u - \bar{u}\| + b\|v - \bar{v}\| + c\|w - \bar{w}\|]$, $g(u) - g(v) \leq L\|u - v\|$ $\forall \in$
$J, u, \bar{u}, v, \bar{v}, w, \bar{w} \in E$, where constants $a \geq 0, b \geq 0, c \geq 0, L \geq 0, \beta \in C[J, R_+]$ and

$$\beta^* = \int_0^\infty (t-s)^{\alpha-1}\beta(s)ds < \infty; \quad \alpha^* = \int_0^\infty (t-s)^{\alpha-1}\|f(s, \theta, \theta)\|ds < \infty,$$

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here θ denotes the zero element of E.

$(H_3^{'})$ $c_0 = L + \dfrac{(a + bk^* + ch^*)\beta^*}{\Gamma(\alpha)} < 1.$

We can prove similarly: if conditions $(H_1') - (H_3')$ are satisfied, then

(a) Eq.(2.7) has a unique solution $u \in C^1[J, E] \cap BC[J, E]$; moreover, for any $v_0 \in BC[J, E]$, the iterative sequence defined by

$$v_n(t) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, v_{n-1}(s), (Tv_{n-1})(s), (Su_{n-1})(s)) ds \qquad (n = 1, 2, 3, \cdots)$$

converges to $u(t)$ uniformly in $t \in J$.

(b) the limit (2.0) exists, and

$$u(\infty) = u_0 - g(u) + \frac{1}{\Gamma(\alpha)} \int_0^\infty (t - s)^{\alpha - 1} f(s, u(s), (Tu)(s), (Su)(s)) ds.$$

(c) inequality (2.3) hold, where c_0 is defined in (H_3') and $u^*(t)$ is the unique solutions in $C^1[J, E] \cap BC[J, E]$ of the following the nonlocal conditions

$$D^\alpha u = f(t, u, Tu, Su), \quad \forall t \in J, \qquad u(0) = u_0^*, \qquad 0 < \alpha \le 1.$$

4   Acknowledgements

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An iterative method for implicit general Wiener-Hopf equations and nonexpansive mappings

Manzoor Hussain, **Wasim Ul-Haq**

Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan

Email: manzoor366@gmail.com, wasim474@hotmail.com

Abstract: In this note, we suggest a new iterative method for finding the common element of the set of common fixed points of a finite family of psuedo-contractive mappings and set of solutions of extended general quasi variational inequality problem on the basis of Wiener-Hopf equations. The convergence criteria of this new method under some mild restrictions is also studied. New and known methods for solving various classes of Wiener-Hopf equations and variational inequalities also appear as a special consequences of our work.

Key Words: Variational inequality problem, Wiener-Hopf equations, nonexpansive mapping, psuedo-contractive mapping.

MSC: 49J40, 91B50

1. Introduction

A variety of problems arising in optimization, economics, finance, networking, transportation, structural analysis and elasticity can be addressed via the framework of variational inequalities. The theory of variational inequalities provides productive and innovative techniques to investigate a wide range of such problems. The origin of variational inequality problem can be traced back from that of Stampacchia's variational inequality problem [1] introduced in early 1964. It is a known fact that the variational inequality problem is equivalent to fixed point problem, which plays a central role in suggesting new and novel iterative schemes for solving variational inequality problem and its variant forms. The classical variationl inequality has been generalized in many directions by several researchers, see [3, 6, 7, 8, 9] and the references therein. Recently Noor et al. [3, 5] has intriduced and investigated a new class of variational inequality problem known as extended general quasi variational inequality problem. It has been shown by Noor et al. [3] that the extended general quasi variational inequalities are equivalent to the implicit fixed point problem.

On the other hand, several modifications and extentions hve been made to the classical projection iterative methods in many directions through a number of techniques It was Shi [2],who proved the equivalence between variational inequality problem and Weiner-Hopf equations by considering the problem of solving a system of nonlinear projections, known as Wiener-Hopf equations.

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Noor et al. [3] have extended the projection method and showed that a class
of variational inequalities known as extended general variational inequalities are
equivalent to implicit general Weiner-Hopf equations.

In 2008, Lu et al. [4] considered an iterative scheme for finding a common
element of set fixed points of a finite family of nonexpansive mappings and the
set of solution of the variational inequalities based on Wiener-Hopf equations.
Motivated from the above and recent work of Noor et al. [3], we suggest an
iterative algorithm to find a common element of set of.fixed points of a finite
family of psuedo-contractive mappings and the set of solutions of extended general
quasi variational inequalities on the basis of Wiener-Hopf equations. We also
study the convergence of this new method under some suitable conditions. The
results contained in this paper brings a significant improvement to already known
results and will continue to hold for them.

2. **Preliminaries**

Let $\langle ., . \rangle$ and $\|.\|$ denote the inner product and norm respectively in a real
Hilbert space H. Now, we recall some basic definitions and tools of a real Hilbert
space H.

Definition 1.1. A nonlinear mapping $T : H \to H$ is called

a). strongly monotone, if there exists some constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \ \forall y \in H,$$

b). $\alpha-$relaxed monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq (-\alpha) \|x - y\|^2, \ \forall y \in H.$$

Note that for $\alpha = 0$ the Definition 1.1(a) reduces to monotonicity of T.

c). Lipschitz continuous, if there exist a constant $\eta > 0$ such that

$$\|Tx - Ty\| \leq \eta \|x - y\|, \ \forall y \in H.$$

Definition 1.2. A mapping $S : K \to K$ is called $k-$strictly pseudo-
contractive in the sense of [13], if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \ \forall x, y \in K.$$

where I is the identity map. Observe that $k = 0$ implies that S is the usual
nonexpansive map.

Lemma 1.2 [4] Let the sequence $\{a_n\}$ of non-negative real numbers satis-
fying the following

$$a_{n+1} \leq (1 - \theta_n) a_n + b_n, \ \forall n \geq n_0$$

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where n_0 is some non-negative integer, $\{\theta_n\}$ is a sequence in $(0,1)$ such that $\sum \theta_n = \infty$, $b_n = o(\theta_n)$, then $\lim_{n \to \infty} a_n = 0$.

Lemma 1.3 [3, 5] Let $K(x)$ be a closed convex set in H. Then, for a given $z \in H$, $x \in K(x)$ satisfies the inequality

$$\langle x - z, y - x \rangle \geq 0, \ \forall y \in K(x),$$

if and only if

$$x = P_{K(x)} z,$$

where $P_{K(x)}$ is the projection of H onto the closed convex-valued set $K(x)$ in H.

Lemma 1.4 [3, 5] The function $x \in H : h(x) \in K(x)$ is a solution of the extended general quasi variational inequality if and only if $x \in H : h(x) \in K(x)$ satisfies the relation

$$h(x) = P_{K(x)}\{g(x) - \lambda Tx\},$$

where $P_{K(x)}$ is the projection mapping and $\lambda > 0$ is a positive constant.

Let $S : K \to K$ be a $k-$strict pseudo-contractive mapping with a fixed point. Let us denote by $\{S_k\}_{k=1}^n$, the finite family of nonexpansive mappings defined as:

$$S_k x = kx + (1-k)Sx, \ \forall x \in K. \tag{2.1}$$

with the set of fixed points $F(S_k)$ (i.e., $F(S_k) = \{x \in K : x = Tx\}$). Setting $F(S) = \cap_{n=1}^{\infty} F(S_k)$ then the following assertions hold:

Lemma 1.6 [11] Let K be a nonempty closed convex subset of a real Hilbert space H and let $S : K \to K$ be a $k-$strict pseudo-contraction with a fixed point. Then $F(S)$ is closed and convex. Define a mapping $S_k : K \to K$ by $S_k x = kx + (1-k)Sx, \ \forall x \in K$. Then S_k is nonexpansive such that $F(S_k) = F(S)$.

Lemma 1.7 [12] Let K be a nonempty closed convex subset of strictly convex Banach space X. Let $\{T_n : n \in N\}$ be a sequence of nonexpansive mappings on K. Suppose $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\delta_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \delta_n = 1$. Then a mapping S on K can be defined by

$$Sx = \sum_{n=1}^{\infty} \delta_n T_n x \tag{2.2}$$

for each $x \in K$ is well defined, nonexpansive and $F(S) = \cap_{n=1}^{\infty} F(T_n)$ holds.

3. **Iterative methods and convergence analysis**

Let $T, g, h \colon H \to H$ be certain nonlinear mappings and $K(x)$ is a nonempty closed convex-valued set in H. Consider the problem of finding an element $x \in H$,

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$h(x) \in K(x)$ such that

$$\langle Tx, g(y) - h(x) \rangle \geq 0, \ \forall y \in H, g(y) \in K(x) \tag{3.1}$$

such type of inequality known as extended general quasi variational inequalities introduced and studied by Noor et al. [3]. We denote by Λ the set of solutions of variational inequality (3.1).

Variational inequality (3.1) entails extended general variational inequality, general variational inequality, variational inequality and other numerous probelms as special cases as qouted in [3].

We also faces the probelm of solving the Wiener-Hopf equations which observe a close reltions to variational inequalities as proved by Shi [2]. A huge amount of interest has been shown by many reaschers in this regard, see e.g. [3, 4, 10]. To be more specific, let us denote by $Q_{K(x)} = I - S_k g h^{-1} P_{K(x)}$, where $P_{K(x)}$ is the implicit projection mapping, S_k is the finite family of nonexpansive mappings defined as by (2.1), I is identity map and assume that h^{-1} exist. Let us consider the problem of finding $z \in H$ such that

$$Th^{-1} S_k P_{K(x)} z + \lambda^{-1} Q_{K(x)} z = 0. \tag{3.2}$$

where $T, g, h : H \to H$ are given nonlinear mappings and λ is a positive constant. Equation (3.2) is known as implicit general Wiener-Hopf equations containing the finite family of nonexpansive mappings S_k.

Now we suggest the iterative method as follow:

Algorithm 1. For a given iterate $z_0 \in H$, compute z_{n+1} via the following scheme:

$$h(x_n) = (1 - \beta_n) P_{K(x_n)} z_n + \beta_n S_k P_{K(x_n)} z_n,$$
$$z^* = (1 - \gamma_n) z_n + \gamma_n [g(x_n) - \lambda T x_n],$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ and S_k is defined by (2.1).

If $1 - \beta_n = 0$, then Algorithm 1 becomes:

Algorithm 2. For a given iterate $z_0 \in H$, compute z_{n+1} via the following scheme:

$$h(x_n) = S_k P_{K(x_n)} z_n,$$
$$z^* = (1 - \gamma_n) z_n + \gamma_n [g(x_n) - \lambda T x_n],$$

where $\{\gamma_n\}$ is the sequence in $[0, 1]$ and S_k is defined by (2.1).

If $S_k = I$, then Algorithm 1 reduces to:

Algorithm 3. For a given iterate $z_0 \in H$, compute z_{n+1} via the following scheme:

$$h(x_n) = (1 - \beta_n) P_{K(x_n)} z_n + \beta_n P_{K(x_n)} z_n,$$
$$z^* = (1 - \gamma_n) z_n + \gamma_n [g(x_n) - \lambda T x_n],$$

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where $\{\gamma_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$.

If $S_k = I$ and $1 - \beta_n = 0$, then Algorithm 1 collapses with the same one as suggested by Noor et al. [3]. i.e.

Algorithm 4. For a given iterate $z_0 \in H$, compute z_{n+1} via the following scheme:

$$h(x_n) = P_{K(x_n)} z_n,$$
$$z^* = (1 - \gamma_n) z_n + \gamma_n [g(x_n) - \lambda T x_n],$$

where $\{\gamma_n\}$ is the sequence in $[0, 1]$.

In short, suitable and appropriate choices of mappings may leads one to obtain certain new and known algorithms for solving various classes of Wiener-Hopf equations and variational inequality problems.

Now we study the convergence of our proposed algorithm which is the main objective of this section. Since the implicit projection $P_{K(.)}$ is not nonexpansive but satisfy some Lipschitz type continuity as quoted by Noor et al. [3]. Also they pointed out that this mapping satisfy an inequality stated as below.

Assumption 1 [3] The implicit projection $P_{K(.)}$ mapping satisfies the following inequality

$$\left\| P_{K(x)} z - P_{K(y)} z \right\| \leq \tau \left\| x - y \right\|$$

where τ is a positive constant. This assumption plays an important part in studying the existence and convergence of iterative algorithms, see [3, 5].

We also need the following:

Lemma 1.5 [3, 5] The solution $x \in H : h(x) \in K(x)$ satisfies the extended general quasi variational inequality if and only if, $z \in H$ is a solution of the extended general implicit Wiener-Hopf equation, where

$$h(x) = S_k P_{K(x)} z$$
$$z = \{g(x) - \lambda T x\},$$

where $\lambda > 0$ is a positive constant.

Theorem 3.1. Let $T, g, h : H \rightarrow H$ be relaxed monotone mappings with constants $\alpha_1, \alpha_2,$ and α_3 and Lipschitz continuous with constants η_1, η_2 and η_3 respectively. Let S_k be defined by (2.1) such that $\Omega = F(S) \cap \Lambda \neq \varnothing$. Let $\{z_n\}$ and $\{x_n\}$ be two sequences generated by Algorithm 2.1. Let $\{\gamma_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ satisfying the following conditions:

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty.$

(ii) $\left| \lambda - \frac{\alpha_1}{\eta_1} \right| < \frac{\sqrt{\alpha_1^2 - \eta_1^2 \zeta(2 - \xi)}}{\eta_1^2}$

with $\alpha_1 > \eta_1 \sqrt{\zeta(2 - \xi)}, \xi < 1,$ where $\xi = \sqrt{1 + 2\alpha_2 + \eta_2^2} + \sqrt{1 + 2\alpha_3 + \eta_3^2} + \tau$

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If the Assumption 1 and conditions (**i**) and (**ii**) holds, then the sequence $\{z_n\}$ and $\{x_n\}$ converges strongly to $z^* \in \Delta = WHE(T, g, h)$ and $x^* \in \Omega$, respectively.

Proof. Let us assume that $x^* \in \Omega$ and $z^* \in \Delta$ be the solution. Then

$$\left.\begin{array}{l} h(x^*) = (1 - \beta_n) P_{K(x^*)} z^* + \beta_n S_k P_{K(x^*)} z^*, \\ z^* = (1 - \gamma_n) z^* + \gamma_n \left[g(x^*) - \lambda T x^* \right]. \end{array}\right\} \tag{3.3}$$

Consider

$$\begin{aligned} \|z_{n+1} - z^*\| &= \|(1 - \gamma_n) z_n + \gamma_n \left[g(x_n) - \lambda T x_n \right] - z^* \| \\ &= \|(1 - \gamma_n) z_n + \gamma_n \left[g(x_n) - \lambda T x_n \right] - \{(1 - \gamma_n) z^* + \gamma_n \left[g(x^*) - \lambda T x^* \right]\}\| \\ &\leq (1 - \gamma_n) \|z_n - z^*\| + \gamma_n \|g(x_n) - g(x^*) - \lambda(T x_n - T x^*)\| \\ &\leq (1 - \gamma_n) \|z_n - z^*\| + \gamma_n \{\|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \|x_n - x^* - \lambda(T x_n - T x^*)\|\}. \end{aligned} \tag{3.4}$$

Next, we estimate

$$\begin{aligned} \|x_n - x^* - \lambda(T x_n - T x^*)\|^2 &= \|x_n - x^*\|^2 - 2 \langle x_n - x^*, \lambda(T x_n - T x^*) \rangle + \|T x_n - T x^*\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\lambda \alpha_1 \|x_n - x^*\|^2 + \eta_1^2 \|x_n - x^*\|^2 \\ &\leq \left(1 + 2\lambda \alpha_1 + \eta_1^2\right) \|x_n - x^*\|^2, \end{aligned}$$

which yields

$$\|x_n - x^* - \lambda(T x_n - T x^*)\| \leq \sqrt{1 + 2\lambda \alpha_1 + \eta_1^2} \|x_n - x^*\|. \tag{3.5}$$

Similarly, we have

$$\begin{aligned} \|x_n - x^* - (g(x_n) - g(x^*))\|^2 &= \|x_n - x^*\|^2 - 2 \langle x_n - x^*, g(x_n) - g(x^*) \rangle \\ &\quad + \|(g(x_n) - g(x^*))\|^2 \\ &\leq \|x_n - x^*\|^2 + 2\alpha_2 \|x_n - x^*\|^2 + \eta_2^2 \|x_n - x^*\|^2 \\ &\leq \left(1 + 2\alpha_2 + \eta_2^2\right) \|x_n - x^*\|^2, \end{aligned}$$

which yields

$$\|x_n - x^* - (g(x_n) - g(x^*))\| \leq \sqrt{1 + 2\alpha_2 + \eta_2^2} \|x_n - x^*\|. \tag{3.6}$$

Exploiting (3.5) and (3.6) into (3.4), we obtain

$$\|z_{n+1} - z^*\| \leq (1 - \gamma_n) \|z_n - z^*\| + \gamma_n \left\{ \sqrt{1 + 2\lambda \alpha_1 + \eta_1^2} + \sqrt{1 + 2\alpha_2 + \eta_2^2} \right\} \|x_n - x^*\|. \tag{3.7}$$

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Now we estimate $\|x_n - x^*\|$. For this, consider

$$
\begin{aligned}
\|x_n - x^*\| &= \left\| x_n - x^* - h(x_n) + h(x^*) + P_{K(x_n)} - P_{K(x^*)} \right\| \\
&\leq \left\| x_n - x^* - (h(x_n) + h(x^*)) \right\| + \left\| P_{K(x_n)} z_n - P_{K(x^*)} z^* \right\| \\
&\leq \left\| x_n - x^* - (h(x_n) + h(x^*)) \right\| + \left\| P_{K(x_n)} z_n - P_{K(x^*)} z_n \right\| \\
&\quad + \left\| P_{K(x^*)} z_n - P_{K(x^*)} z^* \right\|.
\end{aligned}
$$

Invoking the Assumption 1 and Lipschitz type continuity of the implicit projection mapping, we have

$$
\|x_n - x^*\| \leq \left\| x_n - x^* - (h(x_n) + h(x^*)) \right\| + \tau \|x_n - x^*\| + \|z_n - z^*\|. \tag{3.8}
$$

Since

$$
\begin{aligned}
\left\| x_n - x^* - (h(x_n) + h(x^*)) \right\|^2 &= \|x_n - x^*\|^2 - 2 \langle x_n - x^*, h(x_n) - h(x^*) \rangle \\
&\quad + \|h(x_n) - h(x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\alpha_3 \|x_n - x^*\|^2 + \eta_3^2 \|x_n - x^*\|^2 \\
&\leq \left(1 + 2\alpha_3 + \eta_3^2 \right) \|x_n - x^*\|^2,
\end{aligned}
$$

which implies

$$
\left\| x_n - x^* - (h(x_n) + h(x^*)) \right\| \leq \sqrt{1 + 2\alpha_3 + \eta_3^2} \, \|x_n - x^*\|. \tag{3.9}
$$

Using (3.9) into (3.8) and sipmlifyng, we get

$$
\|x_n - x^*\| \leq \frac{1}{1 - \left(\tau + \sqrt{1 + 2\alpha_3 + \eta_3^2} \right)} \|z_n - z^*\|. \tag{3.10}
$$

On subtituting (3.10) into (3.7) we obtain

$$
\|z_{n+1} - z^*\| \leq (1 - \gamma_n) \|z_n - z^*\| + \gamma_n \left\{ \frac{\sqrt{1 + 2\lambda\alpha_1 + \eta_1^2} + \sqrt{1 + 2\alpha_2 + \eta_2^2}}{1 - \left(\tau + \sqrt{1 + 2\alpha_3 + \eta_3^2} \right)} \right\} \|z_n - z^*\|.
$$

Letting $\mu = \frac{\sqrt{1 + 2\lambda\alpha_1 + \eta_1^2} + \sqrt{1 + 2\alpha_2 + \eta_2^2}}{1 - (\tau + \sqrt{1 + 2\alpha_3 + \eta_3^2})}$, we get

$$
\begin{aligned}
\|z_{n+1} - z^*\| &\leq (1 - \gamma_n) \|z_n - z^*\| + \gamma_n \mu \|z_n - z^*\|. \\
&\leq \{1 - \gamma_n (1 - \mu)\} \|z_n - z^*\|.
\end{aligned}
$$

Then clearly

$$
\|z_{n+1} - z^*\| \leq \prod_{k=0}^{n} \{1 - \gamma_k (1 - \mu)\} \|z_0 - z^*\|.
$$

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Since $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $1 - \mu < 1$. Thus we have

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \{1 - \gamma_k(1-\mu)\} \right) = 0.$$

Consequently by virtue of Lemma 1.2, we obtain

$$\lim_{n\to\infty} \|z_n - z^*\| = 0,$$

implies the sequence $\{z_n\}$ strongly converges to $z^* \in \Delta$ satisfying (3.2). Similarly from (3.10) we get

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

implies that $\{x_n\}$ strongly converges to $x^* \in \Omega$ satisfying (3.1). This completes the proof.

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Parametric Duality Models for Multiobjective Fractional Programming Based on New Generation Hybrid Invexities

Ram U. Verma

Texas State University

Department of Mathematics

San Marcos, TX 78666, USA

verma99@msn.com

Abstract

In this paper we intend to introduce the new-generation second order hybrid $B - (b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-invexities, which generalizes most of the existing invexity concepts in the literature, and then we develop a wide range of higher order parametric duality models and results for multiobjective fractional programming problems. The obtained results generalize and unify, based on this new class for second order hybrid invexities, a wider spectrum of investigations in the literature on applications to other results on multiobjective fractional programming.

Keywords: New class of generalized invexities, Multiobjective fractional programming, efficient solutions, Parametric duality models

2000 MSC: 90C30, 90C32, 90C34

1 Introduction

In a series of publications, Zalmai [42] introduced Hanson-Antczak type invexities and applied to a class of second order parametric duality models for semiinfinite multiobjective fractional programming problems. Verma [26 - 28] investigated based on higher order univexities some results on parametric duality models to the context of fractional programming, while Mishra, Jaiswal and Verma [16] formulated some results on duality models for multiobjective fractional programming problems based on generalized invex functions. Verma [25] also introduced a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\epsilon-$efficient solutions.

Inspired by the accelerated recent advances on higher order invexities and other developments to the context of multiobjective fractional programming problems, we first introduce the new generation second order hybrid $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-invexities, which generalizes largely most of existing notions, including Antczak type $B - (\tilde{p}, \tilde{r})-$invexities, second we develop some duality models for two dual problems with relatively suitable constraints for proving weak and strong duality theorems under the new invexity conditions. The results established in this paper, not only generalize the results on exponential type hybrid invexities regarding

duality models for multiobjective fractional programming problems, but also generalize the higher order invexity results in general settings.

We consider under the general framework of the new-generation second order hybrid $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-invexities of functions, the following multiobjective fractional programming problem:

(P)

$$Minimize \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)} \right)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$, where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:

(Pλ)

$$Minimize \left(f_1(x) - \lambda_1 g_1(x), \cdots, f_p(x) - \lambda_p g_p(x) \right)$$

subject to $x \in Q$ with

$$\lambda = \left(\lambda_1, \lambda_2, \cdots, \lambda_p \right) = \left(\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)} \right),$$

where x^* is an efficient solution to (P).

Based on efficiency results, we develop some duality models and prove a number of appropriate duality theorems under several hybrid invexity constraints. We also recognize the significant role of second order hybrid invexiries in semiinfinite programming problems, especially in terms of applications to game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, decision and management sciences, optimal control problems, continuum mechanics, robotics, data envelopment analysis, and beyond. For more details, we refer the reader [1 - 43].

2   New Generation Second Order Invexities

The general invexity has been investigated in several directions. We generalize the notion of the first order Antczak type $B-(\tilde{p}, \tilde{r})$-invexiies to the case of the second order $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-invexities. These notions of the second order invexity encompass most of the existing notions in the literature. Let f be a twice continuously differentiable real-valued function defined on

X.

Definition 2.1 *The function f is said to be second order $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-invex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$b(x, x^*)\left(\frac{1}{\tilde{r}}\left(e^{\tilde{r}[f(x)-f(x^*)]} - 1\right)\right) \geq \frac{1}{\tilde{p}}\left\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1} \right\rangle$$

$$- \frac{1}{2\tilde{p}}\left\langle \nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1} \right\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} \neq 0,$$

$$b(x, x^*)\left(\frac{1}{\tilde{r}}\left(e^{\tilde{r}[f(x)-f(x^*)]} - 1\right)\right) \geq \left\langle \nabla f(x^*) + \frac{1}{2}\nabla^2 f(x^*)z, \omega(x, x^*) \right\rangle$$

$$- \frac{1}{2}\left\langle \nabla^2 f(x^*)z, \xi(x, x^*) \right\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} \neq 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) \geq \frac{1}{\tilde{p}}\Big(\left\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1} \right\rangle\Big)$$

$$- \frac{1}{2}\left\langle \nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1} \right\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} = 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) \geq \left\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*) \right\rangle$$

$$- \frac{1}{2}\left\langle \nabla^2 f(x^*)z, \xi(x, x^*) \right\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} = 0.$$

Definition 2.2 *The function f is said to be second order $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-pseudoinvex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$\frac{1}{\tilde{p}}\left\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1} \right\rangle$$

$$- \frac{1}{\tilde{p}}\left\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1} \right\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$

$$\Rightarrow \ b(x, x^*)\left(\frac{1}{\tilde{r}}\left(e^{\tilde{r}[f(x)-f(x^*)]} - 1\right)\right) \geq 0 \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} \neq 0,$$

$$\Big(\left\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*) \right\rangle\Big)$$

$$- \left(\left\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x, x^*) \right\rangle\right) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0$$

$$\Rightarrow \ b(x, x^*)\left(\frac{1}{\tilde{r}}\left(e^{\tilde{r}[f(x)-f(x^*)]} - 1\right)\right) \geq 0 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} \neq 0,$$

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle\Big)$$

$$- \quad \frac{1}{\tilde{p}}\Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) \geq 0 \;\; for \; \tilde{p} \neq 0 \; and \; \tilde{r} = 0,$$

$$\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \quad \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) \geq 0 \;\; for \; \tilde{p} = 0 \; and \; \tilde{r} = 0.$$

Definition 2.3 *The function f is said to be second order strictly $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-pseudoinvex at $x^* \in X$ if there exist a function $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle\Big)$$

$$- \quad \frac{1}{\tilde{p}}\Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) > 0 \; for \; \tilde{p} \neq 0 \; and \; \tilde{r} \neq 0,$$

$$\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \quad \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) > 0 \;\; for \; \tilde{p} = 0 \; and \; \tilde{r} \neq 0,$$

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle\Big)$$

$$- \quad \frac{1}{\tilde{p}}\Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) > 0 \;\; for \; \tilde{p} \neq 0 \; and \; \tilde{r} = 0,$$

$$\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \quad \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \geq 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) > 0 \;\; for \; \tilde{p} = 0 \; and \; \tilde{r} = 0.$$

Definition 2.4 *The function f is said to be second order prestrictly $B - (b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-pseudoinvex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle\Big)$$

$$- \frac{1}{\tilde{p}}\Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0,$$

$$\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} \neq 0,$$

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle\Big)$$

$$- \frac{1}{\tilde{p}}\Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) \geq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} = 0,$$

$$\Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$

$$\Rightarrow \quad b(x,x^*)\Big([f(x) - f(x^*)]\Big) \geq 0 \text{ for } \tilde{p} = 0 \text{ and } \tilde{r} = 0.$$

Definition 2.5 *The function f is said to be second order $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-quasiinvex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0$$

$$\Rightarrow \quad \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle$$

$$- \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \le 0 \ \ for \ \tilde{p}=0 \ and \ \tilde{r} \ne 0$$

$$\Rightarrow \ \ \Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*)\big\rangle\Big)$$

$$- \ \ \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x, x^*)\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \le 0,$$

$$b(x, x^*)\Big([f(x)-f(x^*)]\Big) \le 0 \ \ for \ \tilde{p} \ne 0 \ and \ \tilde{r}=0$$

$$\Rightarrow \ \ \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x, x^*)} - \mathbf{1}\big\rangle$$

$$- \ \ \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x, x^*)} - \mathbf{1}\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \le 0,$$

$$b(x, x^*)\Big([f(x)-f(x^*)]\Big) \le 0 \ \ for \ \tilde{p}=0 \ and \ \tilde{r}=0$$

$$\Rightarrow \ \ \Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*)\big\rangle\Big)$$

$$- \ \ \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x, x^*)\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \le 0.$$

Definition 2.6 *The function f is said to be second order strictly $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-quasiinvex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \le 0 \ for \ \tilde{p} \ne 0 \ and \ \tilde{r} \ne 0$$

$$\Rightarrow \ \ \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x, x^*)} - \mathbf{1}\big\rangle$$

$$- \ \ \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x, x^*)} - \mathbf{1}\big\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0,$$

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big) \le 0 \ \ for \ \tilde{p}=0 \ and \ \tilde{r} \ne 0$$

$$\Rightarrow \ \ \Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*)\big\rangle\Big)$$

$$- \ \ \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x, x^*)\big\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) \leq 0 \ \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} = 0$$

$$\Rightarrow \quad \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle$$

$$- \quad \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 < 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) \leq 0 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} = 0$$

$$\Rightarrow \quad \Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \quad \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 < 0.$$

Definition 2.7 *The function f is said to be second order prestrictly $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-quasiinvex at $x^* \in X$ if there exist functions $\omega, \xi : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,*

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) < 0 \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} \neq 0$$

$$\Rightarrow \quad \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle$$

$$- \quad \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

$$b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) < 0 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} \neq 0$$

$$\Rightarrow \quad \Big(\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x,x^*)\big\rangle\Big)$$

$$- \quad \Big(\big\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x,x^*)\big\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) < 0 \ \ for \ \tilde{p} \neq 0 \ and \ \tilde{r} = 0$$

$$\Rightarrow \quad \frac{1}{\tilde{p}}\big\langle \nabla f(x^*) + \nabla^2 f(x^*)z, e^{\tilde{p}\omega(x,x^*)} - \mathbf{1}\big\rangle$$

$$- \quad \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 f(x^*)z, e^{\tilde{p}\xi(x,x^*)} - \mathbf{1}\big\rangle + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

$$b(x, x^*)\Big([f(x) - f(x^*)]\Big) < 0 \ \ for \ \tilde{p} = 0 \ and \ \tilde{r} = 0$$

$$\Rightarrow \ \Big(\langle \nabla f(x^*) + \nabla^2 f(x^*)z, \omega(x, x^*)\rangle\Big)$$

$$- \ \Big(\langle \frac{1}{2}\nabla^2 f(x^*)z, \xi(x, x^*)\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \leq 0.$$

Now we consider the efficiency solvability conditions for (P) and (Pλ) problems. We need to recall some auxiliary results crucial to the problem on hand.

Definition 2.8 *A point $x^* \in Q$ is an efficient solution to (P) if there exists no $x \in Q$ such that*

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)} \forall i = 1, \cdots, p.$$

Next to this context, we have the following auxiliary problem:

(P$\bar{\lambda}$)

$$minimize_{x \in Q} (f_1(x) - \bar{\lambda}_1 g_1(x), \cdots, f_p(x) - \bar{\lambda}_p g_p(x)),$$

subject to $x \in Q$,

where $\bar{\lambda}_i$ for $i \in \{1, \cdots, p\}$ are parameters, and $\bar{\lambda}_i = \frac{f_(x^*)}{g_i(x^*)}$.

Next, we introduce the solvability conditions for (P$\bar{\lambda}$) problem.

x^* is an efficient solution to (P) if there exists no $x \in Q$ such that

$$(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}) \leq (\frac{f_1(x^*)}{g_1(x^*)}, \frac{f_2(x^*)}{g_2(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}).$$

Lemma 2.1 [28] *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) *x^* is an efficient solution to (P).*

(ii) *x^* is an efficient solution to (P$\bar{\lambda}$), where*

$$\bar{\lambda} = (\frac{f_1(x^*)}{g_1(x^*)}, \cdots, \frac{f_p(x^*)}{g_p(x^*)}).$$

Lemma 2.2 [28] *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) *x^* is an efficient solution to (P).*

(ii) There exists $c = (c_1, \cdots, c_p) \in \Re_+^p \setminus \{0\}$ such that

$$\Sigma_{i=1}^p c_i [f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)}\Big) g_i(x)] \geq 0$$

$$= \quad \Sigma_{i=1}^p c_i [f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)}\Big) g_i(x^*)] \text{ for any } x \in Q.$$

Next, we recall the following result (Verma [28]) that is crucial to developing the results for the next section based on second order $(\phi, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-invexities.

Theorem 2.1 *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z,t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z,s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\{\Big(\nabla G_j(x^*,t), \langle y, \nabla^2 G_j(x^*,t)y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\}$$

$$+ \quad span\{\Big(\nabla H_k(x^*,s), \langle y, \nabla^2 H_k(x^*,s)y\rangle\Big) : s \in S_k, k \in \underline{r}\},$$

$$where \ \hat{T}_j(x^*) \equiv \{t \in T_j \ : \ G_j(x^*,t) = 0\},$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \leq \nu_0^* \leq \nu^* \leq n+1$, such that there exist ν_0^* indices j_m, with $1 \leq j_m \leq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*} \backslash \underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^p u_i^* [\nabla f_i(x^*) - \lambda^* (\nabla g_i(x^*))] + \sum_{m=1}^{\nu_0^*} v_m^* [\nabla G_{j_m}(x^*,t^m)$$

$$+ \quad \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*,s^m) = 0, \tag{2.1}$$

$$\langle y, \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*,t^m)$$

$$+ \quad \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*,s^m)\Big] y\rangle \geq 0, \tag{2.2}$$

where $\hat{T}_{j_m}(x^*) = \{t \in T_{j_m} : G_{j_m}(x^*,t) = 0\}$, $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^p u_i = 1\}$, and $\underline{\nu^*} \backslash \underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.

We next recall a set of second-order necessary optimality conditions for (P). This result will be needed for proving strong and strict converse duality theorems.

Theorem 2.2 [30] *Let x^* be an optimal solution of (P), let $\lambda^* = \phi(x^*) \equiv$ $\max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}$, and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are twice continuously differentiable at x^*, and that the second-order Guignard constraint qualification holds at x^*. Then for each $z^* \in C(x^*)$, there exist*

$$u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \ \sum_{i=1}^{p} u_i = 1\},$$

$v^* \in \mathbb{R}_+^q \equiv \{v \in \mathbb{R}^q : v \geq 0\}$, and $w^* \in \mathbb{R}^r$ such that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$

$$\left\langle z^*, \left\{ \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{j=1}^{q} v_j^* \nabla^2 G_j(x^*) + \sum_{k=1}^{r} w_k^* \nabla^2 H_k(x^*) \right\} z^* \right\rangle \geq 0,$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \ \ i \in \underline{p},$$

$$v_j^* G_j(x^*) = 0, \ \ j \in \underline{q},$$

where $C(x^*)$ is the set of all critical directions of (P) at x^*, that is,

$$C(x^*) = \{z \in \mathbb{R}^n : \langle \nabla f_i(x^*) - \lambda \nabla g_i(x^*), z \rangle = 0, \ i \in A(x^*), \ \langle \nabla H_j(x^*), z \rangle \leq 0, \ j \in B(x^*)\},$$

$$A(x^*) = \{j \in \underline{p} : f_j(x^*)/g_j(x^*) = \max_{1 \leq i \leq p} f_i(x^*)/g_i(x^*)\}, \ and \ B(x^*) = \{j \in \underline{m} : H_j(x^*) = 0\}.$$

We shall assume that the functions f_i, g_i, $i \in \underline{p}$, H_j, $j \in \underline{m}$ are twice continuously differentiable on the open set X. Moreover, we shall assume, without loss of generality, that $g_i(x) > 0$, $i \in \underline{p}$, and $\varphi(x) \geq 0$ for all $x \in X$.

3   Duality Model I

In this section, we consider two duality models with relatively standard constraint structures and prove weak and strong duality theorems. Consider the following two problems:

(DI) Maximize λ

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{m} v_m \nabla H_j(y) = 0, \tag{3.1}$$

$$\frac{1}{2}\left(\left\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{m} v_j \nabla^2 H_j(y) \right\} z \right) \geq 0, \tag{3.2}$$

$$\sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] \geq 0, \tag{3.3}$$

$$\sum_{j=1}^{m} v_j H_j(y) \geq 0, \tag{3.4}$$

$$y \in X, \ z \in \mathbb{R}^n, \ u \in U, \ v \in \mathbb{R}^q_+, \ \lambda \in \mathbb{R}_+.$$

$(\tilde{D}I)$ *Maximize* λ

subject to (3.2) - (3.4) and

$$\left\langle \sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{j=1}^{m} v_j \nabla H_j(y), \omega(x, y) \right\rangle \geq 0, \tag{3.5}$$

$$\left\langle \left\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{m} v_j \nabla^2 H_j(y) \right\} z, \omega(x, y) \right\rangle \geq 0 \ \text{ for all } x \in \mathbb{F}, \tag{3.6}$$

$$-\left\langle \frac{1}{2} \left\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^{m} v_j \nabla^2 H_j(y) \right\} z, \xi(x, y) \right\rangle \geq 0 \ \text{ for all } x \in \mathbb{F}, \tag{3.7}$$

where $\omega, \xi : X \times X \to \mathbb{R}^n$ are functions.

Comparing (DI) and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\tilde{D}I)$, but the converse is not necessarily true. Furthermore, we observe that (3.1) is a system of n equations, whereas (3.5) is a single inequality. We observe that from a computational point of view, (DI) is preferable to $(\tilde{D}I)$ because of the dependence of (3.5) - (3.7) on the feasible set of (P). We will just establish the duality theorems for the pair $(P) - (DI)$ since the duality results for the pair $(P) - (\tilde{D}I)$ are similar in nature.

We shall use the following list of symbols in the statements and proofs of our duality theorems:

$$\mathcal{C}(x, v) = \sum_{j=1}^{m} v_j H_j(x),$$

$$\mathcal{E}_i(x, \lambda) = f_i(x) - \lambda g_i(x), \quad i \in \underline{p},$$

$$\mathcal{E}(x, u, \lambda) = \sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)],$$

$$I_+(u) = \{i \in \underline{p} : u_i > 0\}, \quad J_+(v) = \{j \in \underline{m} : v_j \geq 0\}.$$

The next two theorems show that (DI) is a dual problem for (P).

Theorem 3.1 *(Weak Duality) Let x and $\mathcal{S} \equiv (y, z, u, v, \lambda)$ be arbitrary feasible solutions of (P) and (DI), respectively, and assume that either one of the following two sets of hypotheses is satisfied:*

(a)   (i) *for each $i \in I_+ \equiv I_+(u)$, $\mathcal{E}_i(\cdot, u, \lambda)$ is prestrictly $(b, \omega, \xi, \bar{\rho}_i, \theta, \tilde{p}, \tilde{r})$- quasiinvex at y,*

    (ii) *for each $j \in J_+ \equiv J_+(v)$, H_j is strictly $(b, \omega, \xi, \hat{\rho}_j, \theta, \tilde{p}, \tilde{r})$-quasiinvex at y;*

    (iii) $\sum_{i \in I_+} u_i \bar{\rho}_i(x, y) + \sum_{j \in J_+} v_j \hat{\rho}_j(x, y) \geq 0$;

(b) *the Lagrangian-type function*

$$\xi \to L(\xi, u, v, \lambda) = \sum_{i=1}^{p} u_i [f_i(\xi) - \lambda g_i(\xi)] + \sum_{j=1}^{m} v_j H_j(\xi)$$

    is $(b, \omega, \xi, \rho, \theta, \tilde{p}, \tilde{r})$-pseudoinvex at y, $\rho(x, y) \geq 0$.

Then $\varphi(x) \geq \lambda$ for $\varphi(x) = \left(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)} \right)$.

Proof: Based on the assumptions, we have

$$H_j(x) \leq 0 \leq H_j(y), \ j \in \underline{m}.$$

This implies

$$\frac{1}{\tilde{r}} (e^{\tilde{r} \sum_{j=1}^{m} [H_j(x) - H_j(y)]} - 1) \leq 0, \tag{3.8}$$

which further implies

$$b(x, y) \left(\frac{1}{\tilde{r}} (e^{\tilde{r} \sum_{j=1}^{m} [H_j(x) - H_j(y)]} - 1) \right) \leq 0. \tag{3.9}$$

Then by (ii), we have (for $j \in J_+$)

$$\frac{1}{\tilde{p}} \langle \nabla H_j(y) + \nabla^2 H_j(y) z, e^{\tilde{p} \omega(x,y)} - \mathbf{1} \rangle - \frac{1}{2\tilde{p}} \langle \nabla^2 H_j(y) z, e^{\tilde{p} \xi(x,y)} - \mathbf{1} \rangle$$
$$+ \quad \hat{\rho}_j(x, y) \|\theta(x, y)\|^2 < 0. \tag{3.10}$$

Using (3.1), (3.2), (3.10) and (iii), we have (for $i \in I_+$)

$$\frac{1}{\tilde{p}} \langle \nabla \mathcal{E}_i(y, u, \lambda) + \nabla^2 \mathcal{E}_i(y, u, \lambda) z, e^{\tilde{p} \omega(x,y)} - \mathbf{1} \rangle - \frac{1}{\tilde{p}} \langle \frac{1}{2} \nabla^2 \mathcal{E}_i(y, u, \lambda) z, e^{\tilde{p} \xi(x,y)} - \mathbf{1} \rangle$$
$$> -\bar{\rho}_i(x, y) \|\theta(x, y)\|^2, \tag{3.11}$$

which applying (i) implies

$$\frac{1}{\tilde{r}} \left(e^{\tilde{r} [\mathcal{E}_i(x, u, \lambda) - \mathcal{E}_i(y, u, \lambda)}} - 1) \right) \geq 0, \ i \in I_+. \tag{3.12}$$

Since the above inequality holds (for both cases of $\tilde{r}$), we have

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq \sum_{i=1}^{p} u_i[f_i(y) - \lambda g_i(y)] \geq 0.$$

It follows that

$$\varphi(x) \nleq \lambda.$$

(b): Using the dual feasibility of $\mathcal{S}$, nonnegativity of $\rho(x,y)$, (3.1), and (3.2), we obtain the following inequality:

$$\frac{1}{\tilde{p}}\langle \nabla L(y,u,v,\lambda) + \nabla^2 L(y,u,v,\lambda)z, e^{\tilde{p}\omega(x,y)} - \mathbf{1}\rangle - \frac{1}{\tilde{p}}\langle \frac{1}{2}\nabla^2 L(y,u,v,\lambda)z, e^{\tilde{p}\xi(x,y)} - \mathbf{1}\rangle$$
$$\geq -\rho(x,y)\|\theta(x,y)\|^2,$$

which in view of our $(b,\omega,\xi,\rho,\theta,\tilde{p},\tilde{r})$-pseudoinvexity assumption implies that

$$\frac{1}{\tilde{r}}b(x,y)\big(e^{\tilde{r}[L(x,u,v,\lambda)-L(y,u,v,\lambda)]} - 1\big) \geq 0,$$

which implies

$$L(x,u,v,\lambda) - L(y,u,v,\lambda) \geq 0.$$

Since $x \in \mathbb{F}$, $v \geq 0$, and (3.3) holds, we get

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0,$$

which is precisely (3.11). As seen in the proof of part (a), this inequality leads to the desired conclusion.

$\square$

Now we consider the second the theorem to show that $DI)$ is a dual of (P). We adopt the similar notations as for Theorem 3.1. and define real-valued functions $E_i(.,u)$ and $B_j(.,v)$ by

$$E_i(x,u,\lambda) = u_i[f_i(x) - \lambda g_i(x)], \ i \in \{1,\cdots,p\}$$

and

$$B_j(x,v) = v_j H_j(x), \ j = 1,\cdots,m.$$

Theorem 3.2 *(Weak Duality) Let x and $\mathcal{D} = (y,z,u,v,\lambda)$ be arbitrary feasible solutions of (P) and (DI), respectively. Let f_i, g_i for $i \in \{1,\cdots,p\}$ and H_j for $j \in \{1,\cdots,m\}$ be twice continuously differentiable at $y \in X$, and let there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \Sigma_{i=1}^p u_i = 1\}$ and $v \in \mathbb{R}_+^m$.*

Suppose, in addition, that any one of the following assumptions holds:

(i) $E_i(.\,;y,u^*,\lambda)$ $\forall\,i\in\{1,\cdots,p\}$ are second order $(b,\rho,\omega,\xi,\theta,\tilde{p},\tilde{r})$-pseudoinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$ and $z\in\mathbb{R}^n$, and $B_j(.\,,v^*)$ $\forall\,j\in\{1,\cdots,m\}$ are second order $(b,\rho,\omega,\xi,\theta,\tilde{p},\tilde{r})$-quasiinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, a function $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$, $z\in\mathbb{R}^n$, and $\rho(x,x^*)\geq 0$.

(ii) $E_i(.\,;y,u^*,\lambda)$ $\forall\,i\in\{1,\cdots,p\}$ are second order $(b,\rho_1,\omega,\xi,\theta,\tilde{p},\tilde{r})$-pseudoinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$ and $z\in\mathbb{R}^n$, and $B_j(.\,,v^*)$ $\forall\,j\in\{1,\cdots,m\}$ are second order $(b,\rho_2,\omega,\xi,\theta,\tilde{p},\tilde{r})$-quasiinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$, $z\in\mathbb{R}^n$, and $\rho_1(x,x^*),\rho_2(x,x^*)\geq 0$ with $\rho_2(x,x^*)\geq\rho_1(x,x^*)$.

(iii) $E_i(.\,;x^*,u^*,\lambda)$ $\forall\,i\in\{1,\cdots,p\}$ are second order prestrictly $(b,\rho_1,\omega,\xi,\theta,\tilde{p},\tilde{r})$-pseudoinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, a function $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$ and $z\in\mathbb{R}^n$, and $B_j(.\,,v^*)$ $\forall\,j\in\{1,\cdots,m\}$ are second order strictly $(b,\rho_2,\omega,\xi,\theta,\tilde{p},\tilde{r})$-quasiinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, a function $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$, $z\in\mathbb{R}^n$, and $\rho_1(x,x^*),\rho_2(x,x^*)\geq 0$ with $\rho_2(x,x^*)\geq\rho_1(x,x^*)$.

(iv) $E_i(.\,;y,u^*,\lambda)$ $\forall\,i\in\{1,\cdots,p\}$ are second order prestrictly $(b,\rho_1,\omega,\xi,\theta,\tilde{p},\tilde{r})$-quasiinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, a function $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$ and $z\in\mathbb{R}^n$, and $B_j(.\,,v^*)$ $\forall\,j\in\{1,\cdots,m\}$ are second order strictly $(b,\rho_2,\omega,\xi,\theta,\tilde{p},\tilde{r})$-pseudoinvex at $y\in X$ if there exist functions $\omega,\xi:X\times X\to\mathbb{R}^n$, a function $b:X\times X\to\mathbb{R}_+$, and real numbers $\tilde{r}$ and $\tilde{p}$ for all $x\in X$, $z\in\mathbb{R}^n$, and $\rho_1(x,x^*),\rho_2(x,x^*)\geq 0$ with $\rho_2(x,x^*)\geq\rho_1(x,x^*)$.

(v) For each $i\in\{1,\cdots,p\}$, f_i is second order $(b,\rho_1,\omega,\xi,\theta,\tilde{p},\tilde{r})-$invex and $-g_i$ is second order $B-(b,\rho_2,\omega,\xi,\theta,\tilde{p},\tilde{r})-$invex at x^*. $H_j(.\,,v^*)$ $\forall\,j\in\{1,\cdots,m\}$ is $(b,\rho_3,\omega,\xi,\theta,\tilde{p},\tilde{r})-$quasiinvex at y, and $\Sigma_{j=1}^m v_j^*\rho_3+\rho^*\geq 0$ for $\rho^*=\Sigma_{i=1}^p u_i^*(\rho_1+\phi(x^*)\rho_2)$ and for $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}$.

Then $\varphi(x)\geq\lambda$ for $\varphi(x)=\left(\frac{f_1(x)}{g_1(x)},\cdots,\frac{f_p(x)}{g_p(x)}\right)$.

Proof: If (i) holds, and since $x\in Q$ and $y\in\mathcal{D}$, it follows from (3.1) and (3.2) that

$$\frac{1}{\tilde{p}}\Big\langle\Sigma_{i=1}^p u_i^*[\nabla f_i(y)-\lambda\nabla g_i(y)]+\Sigma_{j=1}^m v_j^*\nabla H_j(y)$$

$$+\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(y)-\lambda\nabla^2 g_i(y)]+\Sigma_{j=1}^m v_j^*\nabla^2 H_j(y)\Big]z,e^{\tilde{p}\omega(x,y)}-\mathbf{1}\Big\rangle$$

$$-\frac{1}{2\tilde{p}}\Big\langle\Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(y)-\lambda\nabla^2 g_i(y)]+\sum_{j=1}^m v_j^*\nabla^2 H_j(y)\Big]z,e^{\tilde{p}\xi(x,y)}-\mathbf{1}\Big\rangle\geq 0.\qquad(3.13)$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 \leq \Sigma_{j=1}^m v_j^* H_j(y),$$

and so

$$b(x,y)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(y)]} - 1\big)\Big) \leq 0$$

since $\tilde{r} \neq 0$ and $b(x,y) \geq 0$ for all $x \in Q$. In light of the $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-quasiinvexity of $B_j(., v^*)$ at y, it follows that

$$\frac{1}{\tilde{p}}\big\langle \nabla H_j(y) + \nabla^2 H_j(y)z, e^{\tilde{p}\omega(x,y)} - \mathbf{1}\big\rangle - \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\nabla^2 H_j(y)z, e^{\tilde{p}\xi(x,y)} - \mathbf{1}\big\rangle + \rho(x,y)\|\theta(x,y)\|^2 \leq 0,$$

and hence,

$$\frac{1}{\tilde{p}}\big\langle \Sigma_{j=1}^m \nabla H_j(y) + \Sigma_{j=1}^m \nabla^2 H_j(y)z, e^{\tilde{p}\omega(x,y)} - \mathbf{1}\big\rangle$$

$$- \frac{1}{\tilde{p}}\big\langle \frac{1}{2}\Sigma_{j=1}^m \nabla^2 H_j(y)z, e^{\tilde{p}\xi(x,y)} - \mathbf{1}\big\rangle \leq -\rho(x,y)\|\theta(x,y)\|^2. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\frac{1}{\tilde{p}}\big\langle \Sigma_{i=1}^p u_i^*[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{i=1}^p u_i^*[\nabla^2 f_i(y)z - \lambda \nabla^2 g_i(y)z], e^{\tilde{p}\omega(x,y)} - \mathbf{1}\big\rangle$$

$$-\frac{1}{\tilde{p}}\big\langle \frac{1}{2}\sum_{i=1}^p u_i^*[\nabla^2 f_i(y)z - \lambda \nabla^2 g_i(y)z], e^{\tilde{p}\xi(x,y)} - \mathbf{1}\big\rangle \geq \rho(x,y)\|\theta(x,y)\|^2. \tag{3.15}$$

Since $\rho(x,y) \geq 0$, applying $(b, \rho, \omega, \xi, \theta, \tilde{p}, \tilde{r})-$pseudo-invexity at y to (3.15), we have

$$\frac{1}{\tilde{r}}b(x,y)\big(e^{\tilde{r}[E_i(\cdot,x,u^*)-E_i(\cdot,y,u^*)]} - 1\big) \geq 0. \tag{3.16}$$

Since $b(x,y) \geq 0$, (3.16) implies

$$\Sigma_{i=1}^p u_i^*[f_i(x) - \lambda g_i(x)]$$
$$\geq \quad \Sigma_{i=1}^p u_i^*[f_i(y) - \lambda g_i(y)])$$
$$\geq \quad 0.$$

Thus, we have

$$\Sigma_{i=1}^p u_i^*[f_i(x) - \lambda g_i(x)] \geq 0. \tag{3.17}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that

$$\frac{f_i(x)}{g_i(x)} - \lambda \geq 0 \; \forall \, i = 1, \cdots, p.$$

Hence, $\varphi(x) \geq \lambda$, i.e., (DI) is a dual for (P).

The proof for (ii) is similar to that of (i), but we include for the sake of the completeness. If (ii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$
\frac{1}{\tilde{p}} \langle \Sigma_{i=1}^p u_i^* [\nabla f_i(y) - \lambda \nabla g_i(y)] + \Sigma_{j=1}^m v_j^* \nabla H_j(y)
$$

$$
+ \ \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{j=1}^m v_j^* \nabla^2 H_j(y) \Big] z, e^{\tilde{p}\omega(x,y)} - \mathbf{1} \rangle
$$

$$
- \ \frac{1}{2\tilde{p}} \langle \Big[\sum_{i=1}^p u_i^* [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)]
$$

$$
+ \sum_{j=1}^m v_j^* \nabla^2 H_j(y) \Big] z, e^{\tilde{p}\xi(x,y)} - \mathbf{1} \rangle \geq 0 \tag{3.18}
$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$
\Sigma_{j=1}^m v_j^* H_j(x) \leq 0 = \Sigma_{j=1}^m v_j^* H_j(y),
$$

and so

$$
b(x,y) \Big(\frac{1}{\tilde{r}} \big(e^{\tilde{r}[H_j(x) - H_j(y)]} - 1 \big) \Big) \leq 0
$$

since $\tilde{r} \neq 0$ and $b(x,y) \geq 0$ for all $x \in Q$. In light of the $(b, \rho_2, \omega, \xi, \theta, \tilde{p}, \tilde{r})$-quasiinvexity of $B_j(., v^*)$ at y, it follows that

$$
\frac{1}{\tilde{p}} \langle \nabla H_j(y) + \nabla^2 H_j(y) z, e^{\tilde{p}\omega(x,y)} - \mathbf{1} \rangle
$$

$$
- \ \frac{1}{\tilde{p}} \langle \frac{1}{2} \nabla^2 H_j(y) z, e^{\tilde{p}\xi(x,y)} - \mathbf{1} \rangle + \rho_2(x,y) \|\theta(x,y)\|^2 \leq 0.
$$

Now it follows from (3.18) that

$$
\frac{1}{\tilde{p}} \langle \Sigma_{i=1}^p u_i^* [\nabla f_i(y) - \lambda \nabla g_i(y)]
$$

$$
+ \ \frac{1}{2} \sum_{i=1}^p u_i^* [\nabla^2 f_i(y) z - \lambda \nabla^2 g_i(y) z], e^{\tilde{p}\omega(x,y)} - 1 \rangle
$$

$$
- \ \frac{1}{\tilde{p}} \langle \Sigma_{i=1}^p u_i^* [\frac{1}{2} \sum_{i=1}^p u_i^* [\nabla^2 f_i(y) z - \lambda \nabla^2 g_i(y) z], e^{\tilde{p}\xi(x,y)} - 1 \rangle
$$

$$
\geq \ \rho_2(x,y) \|\theta(x,y)\|^2. \tag{3.19}
$$

Since $\rho_1(x,y), \rho_2(x,y) \geq 0$ with $\rho_2(x,y) \geq \rho_1(x,y)$, applying $B - (b, \rho_1, \omega, \xi, \theta, \tilde{p}, \tilde{r})-$pseudo-invexity at y to (3.19, we have

$$
\frac{1}{\tilde{r}} b(x,y) \big(e^{\tilde{r}[E_i(x,u^*,\lambda) - E_i(y,u^*,\lambda)]} - 1 \big) \geq 0. \tag{3.20}
$$

It follows from (3.20) that

$$\Sigma_{i=1}^{p} u_i^* [f_i(x) - \lambda g_i(x)]$$

$$\geq \quad \Sigma_{i=1}^{p} u_i^* [f_i(y) - \lambda g_i(y)])$$
$$\geq \quad 0.$$

Thus, we have

$$\Sigma_{i=1}^{p} u_i^* [f_i(x) - \lambda g_i(x)] \geq 0. \tag{3.21}$$

Next, the proofs for (iii)-(v) are similar to that of (i).

$\square$

Before we prove the theorem on strong duality, we recall that an efficient solution $x^* \in \mathbb{F}$ is referred to as a *normal* efficient solution of (P) if the generalized Guignard constraint qualification is satisfied at x^* and for each $i_0 \in \underline{p}$, the set $cone(\{\nabla G_j(x^*, t) : t \in \hat{T}_j(x^*), j \in \underline{q}\} \cup \{\nabla f_i(x^*) - \lambda_i^* \nabla g_i(x^*) : i \in \underline{p}, i \neq i_0\}) + span(\{\nabla H_k(x^*, s) : s \in S_k, k \in \underline{r}\})$ is closed, where $\lambda_i^* = f_i(x^*)/g_i(x^*)$, $i \in \underline{p}$.

Theorem 3.3 *(Strong Duality) Let x^* be a normal efficient solution of (P) and assume that for each feasible solution (y, z, u, v, λ) of (DI), any one of the four sets of conditions specified in Theorem 3.2 is satisfied. Then there exist u^*, v^*, λ^*, such that $(x^*, z^*, u^*, v^*, \lambda^*)$ is an efficient solution of (DI) and $\varphi(x^*) = \lambda^*$.*

Proof: The proof is similar to that of Theorem 3.2. $\square$

4   Concluding Remarks

We observe that the obtained results in this paper can be generalized to the case of multiobjective fractional subset programming with generalized invex functions, for instance based on the work of Mishra, Jaiswal and Verma [16], and Verma [29] to the case of the $\epsilon-$ efficiency and weak $\epsilon-$efficiency conditions to the context of minimax fractional programming problems involving n-set functions.

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On the Best Linear Prediction of Linear Processes

Mohammad Mohammadi[a,b] and Adel Mohammadpour[b]

[a]Department of Statistics, Faculty of Basic Science,

Khatam Alanbia University of Technology, Behbahan, Iran.

Email: mohammadi61@aut.ac.ir

[b]Department of Statistics, Faculty of Mathematics & Computer Science,

Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran, Iran.

Email: adel@aut.ac.ir

Abstract

Under mild conditions, we find the best linear prediction of linear processes with arbitrary innovations. The prediction method is based on the theory of Hilbert spaces. Using the presented method we can predict non-causal ARMA processes. Finally, we utilize the presented method for prediction of natural logarithms of volumes of Wal-Mart stock traded daily.

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Key words: ARMA process; Domain of attraction; Hilbert space; Linear process; Prediction.

1

1 Introduction

Infinite variance distributions have been used for modeling of data that arise in finance, telecommunications, insurance and medicine, see Kabašinskas et al. (2009), Nolan (2003), Tsay (2002), Rachev and Mittnik (2000) and Gallardo (2000), among many others. These applications justify the need to develop an effective methodology for the prediction of heavy tailed processes.

The set of all finite variance random variables constitutes a Hilbert space (denoted by H) with the inner product $< X, Y >_H = E(XY)$, for every X, Y in H. If values of a random process are in a Hilbert space then one can find the best linear prediction based on projections on the past values. When the second moment is infinite the classical method (prediction based on the Hilbert space H) for prediction fail. Several researchers have studied prediction of infinite variance processes. Cline and Brockwell (1985) used *minimum dispersion* criterion for the prediction of causal ARMA processes with i.i.d. innovations. They assumed that innovations have regularly varying tails. In the case of α-stable random processes, minimum dispersion criterion focuses tails of errors. This criterion is used in Kokoszka (1996) for predicting FARIMA processes with innovations in the domain of attraction of an α-stable distribution with $1 < \alpha \leq 2$. Generally, the predictor using minimum dispersion criterion is not unique and, except some cases, the calculations for finding predictors are not routine. We refer the reader interested in the prediction of infinite variance processes to Cambanis and Soltani (1984), Miamee and Pourahmadi (1988) and Soltani and Moeanaddin (1994).

In Mohammadi and Mohammadpour (2009), using integral representation of α-stable random variables a Hilbert space, is defined: the following Hilbert space

$$\left\{ \int_E f(x)\mathrm{d}M(x) \Big| f \in L^\alpha(E, \mathcal{E}, p) \right\},$$

where M is an α-stable, $0 < \alpha \leq 2$, random measure on $(E, \mathcal{E})$ with finite control measure p. The prediction method is based on the introduced Hilbert space with an inner product known as *stable covariation*. In Karchera et al. (2013) three methods for predicting α-stable random fields are investigated.

Non-causal ARMA processes with finite or infinite variance innovations are important process that the above described methods cannot be applied to. As we know, there is not any model based method for predicting such processes. It should be noted that there is a best linear predictor for non-causal ARMA processes with finite variance innovations, but, in practice we can not compute predictor.

In this article, our aim is to find the best linear prediction of the linear process $\{X_t | t \in \mathbb{Z}\}$ of the following form

$$X_t = \sum_{j=-\infty}^{+\infty} c_j W_{t-j}, \qquad t \in \mathbb{Z}, \tag{1}$$

where $\{W_t\}$ is an arbitrary sequence of random variables which satisfies many weak assumptions (see Theorem 2.1). Consider the process $\{X_t | t \in \mathbb{Z}\}$ which satisfies the following system

$$\Phi(B)X_t = \Theta(B)Z_t,$$

where $\{Z_t\}$ is an arbitrary sequence, $\Phi(B) = 1 - \phi_1 B^1 - \cdots - \phi_p B^p$, $\Theta(B) = 1 + \theta_1 B^1 + \cdots + \theta_q B^q$, $p, q \in \mathbb{N}$ and B is a backward shift operator. Obviously, two processes $\{\Phi(B)X_t\}$ and $\{\Theta(B)Z_t\}$ satisfy (1).

Section 2 is devoted to the construction of Hilbert spaces based on linear processes of the form (1). In Section 3, we explain the prediction method. Finally, in Section 4, we explain the application of the new prediction method for a financial time series which a non-causal ARMA$(2, 0)$ is an appropriate model for that.

2   Hilbert space generated by a linear process

In this section, under mild conditions we present a Hilbert space such that it contains the linear process of the form (1).

We can rewrite the linear process of the form (1) as

$$X_t = \sum_{j=-\infty}^{+\infty} c_{t+j} W_{-j}. \tag{2}$$

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In the sequel, we consider sequence of random variables $\{W_t\}$ such that for every sequence $\{a_j\} \subset \mathbb{C}$,

$$\sum_{j=-\infty}^{+\infty} a_{t+j}W_{-j} = 0 \text{ if and only if } a_j = 0, \text{ for every } j \in \mathbb{Z}. \tag{3}$$

Also, we consider linear processes of the form (2) such that

$$\sum_{j=-\infty}^{+\infty} |c_j|^2 < \infty \text{ and } \left| \sum_{j=-\infty}^{+\infty} c_{t+j}W_{-j} \right| < \infty, \text{ a.s.} \tag{4}$$

Theorem 2.1. *Let $\{W_t\}$ be a sequence of random variables satisfying (3). Let*

$$A = \left\{ \sum_{j=-\infty}^{+\infty} c_{t+j}W_{-j} + \mu \, \middle| \, t \in \mathbb{Z}, \mu \in \mathbb{C}, \text{ and} \{c_j\} \text{ satisfies } 4 \right\}.$$

For every two elements $X = \sum_{j=-\infty}^{+\infty} c_{t_1+j}W_{-j} + \mu_1$ and $Y = \sum_{j=-\infty}^{+\infty} d_{t_2+j}W_{-j} + \mu_2$ in A, we assign the following value to them,

$$< X, Y >_A = \sum_{j=-\infty}^{+\infty} c_{t_1+j}\bar{d}_{t_2+j} + \mu_1\bar{\mu}_2.$$

Then, there is a Hilbert space (denoted by $\bar{A}$) containing A such that

$$< X, Y >_A = < X, Y >_{\bar{A}}, \tag{5}$$

for every $X, Y \in A$.

Proof. It is obvious that A is a vector space with field $\mathbb{C}$. It is enough we show that A is an inner product space. Condition (3) ensures that the function $< .,. >_A$ is well defined and $< .,. >_A$ satisfies properties of the inner product. Therefore, A is an inner product space. Now, the completion of A is a Hilbert space satisfying (5). $\square$

Definition 2.2. *We call the space $\bar{A}$, the Hilbert space generated by the process $\{W_t\}$.*

Remark 2.3. *If $\{W_t\}$ is an i.i.d. sequence of random variables such that $Var(W_t) < +\infty$ and $E(W_t) = 0$, for every $t \in \mathbb{Z}$ then $\bar{A}$ is a subset of the Hilbert space H generated from all finite variance random variables. In this case, we have*

$$< X, Y >_H = Var(W_1) < X, Y >_A,$$

for every $X, Y \in A$.

2   HILBERT SPACE GENERATED BY A LINEAR PROCESS   5

Remark 2.4. *The inner product of the space A is not necessary based on distribution function. For example, even if the distributions W_1 and W_2 in A are different, it is possible to have $||W_1||_A = ||W_2||_A$, where $||.||_A = (<.,.>_A)^{1/2}$. Conversely, consider two non-zero random variables W_1 and aW_2 in A such that they have same distribution for $a \neq 1$. However, it is possible to have $||W_1||_A \neq ||aW_2||_A$.*

Example 2.5. *Assume that the process $\{X_t\}$ satisfies the system*

$$X_t = \sum_{j=-\infty}^{+\infty} c_j W_{t-j}, \ t \in \mathbb{Z},$$

where $\{c_j\}$ and $\{W_j\}$ satisfy (3) and (4). From Theorem (2.1), the process $\{W_j\}$ takes its values in the space A. On the other hand, by assuming that $\{X_t\}$ satisfies (3), the process $\{X_t\}$ takes its values in the following space

$$A_1 \ = \ \left\{ \sum_{j=-\infty}^{+\infty} d_{t+j} X_{-j} \Big| t \in \mathbb{Z}, \{d_j\} \subset \mathbb{C}, \ and \{d_j\} \ satisfies \ 4 \right\}.$$

Two functions $||.||_A$ and $||.||_{A_1}$ are not necessary equal. This result follows from the fact that the introduced inner product is not based on distribution function.

By a similar argument as in the proof of Theorem 2.1, we can generalize the space A to a larger space such that sum of linear processes takes its values in a Hilbert space.

Theorem 2.6. *Let the following two sets*

$$\left\{ \{W_t^\nu | t \in \mathbb{Z}\}, \nu = 1, \ldots, I \right\} \ and \left\{ \{c_t^\nu | t \in \mathbb{Z}\}, \nu = 1, \ldots, I \right\}$$

satisfy

$$\sum_{\nu=1}^{I} \sum_{j=-\infty}^{+\infty} c_{t_\nu+j}^\nu W_{-j}^\nu = 0 \ \ if \ and \ only \ if \ c_j^\nu = 0, \ for \ all \ j \in \mathbb{Z} \ and \ \nu = 1, \ldots, I, \tag{6}$$

where $I \in \mathbb{N}$. Moreover, assume that

$$\sum_{j=-\infty}^{+\infty} |c_j^\nu|^2 < \infty \ \ and \ \Big| \sum_{j=-\infty}^{+\infty} c_{t_\nu+j}^\nu W_{-j}^\nu \Big| < \infty, \ a.s. \tag{7}$$

for every $\nu = 1, \ldots, I$. Consider the space

$$B = \left\{ \sum_{\nu=1}^{I} \sum_{j=-\infty}^{+\infty} c_{t_\nu+j}^\nu W_{-j}^\nu + \mu \,\Big|\, t_\nu \in \mathbb{Z}, \mu \in \mathbb{C}, \text{ and } \{c_j^\nu\} \text{ satisfies } 6 \text{ and } 7 \right\}.$$

For every two elements

$$X = \sum_{\nu=1}^{I} \sum_{j=-\infty}^{+\infty} c_{t_\nu+j}^\nu W_{-j}^\nu + \mu_1 \ \text{ and } \ Y = \sum_{\nu=1}^{I} \sum_{j=-\infty}^{+\infty} d_{s_\nu+j}^\nu W_{-j}^\nu + \mu_2$$

in B assign the following value to them

$$<X,Y>_B \ = \ \sum_{\nu=1}^{I} \sum_{j=-\infty}^{+\infty} c_{t_\nu+j}^\nu \bar{d}_{s_\nu+j}^\nu + \mu_1 \bar{\mu}_2.$$

Then, there is a Hilbert space (denoted by $\bar{B}$) containing B such that

$$<X,Y>_B = <X,Y>_{\bar{B}},$$

for every $X, Y \in B$.

3 Prediction method

In the previous section, under conditions (3) and (4) we showed that every linear process $\{X_t\}$ in the form of (2) takes its values in a Hilbert space. Therefore, using the *projection theorem* (see Brockwell and Davis (1991) chapter 2), one can find the best linear prediction of X_{n+h} in terms of some past values $X_1, \ldots, X_n$. In other words, using the projection theorem, we can find the coefficient vector $\mathbf{a}_n = (a_{1n}, \ldots, a_{nn})'$ such that

$$\left\| \sum_{i=1}^{n} a_{in} X_{n+1-i} - X_{n+h} \right\|_{\bar{A}} = \inf_{K \in \overline{span}\{X_1,\ldots,X_n\}} \left\| K - X_{n+h} \right\|_{\bar{A}},$$

where $\bar{A}$ is the Hilbert space generated by the linear process $\{W_t\}$. The coefficient vector $\mathbf{a}_n$ satisfies the following equations

$$< \sum_{i=1}^{n} a_{in} X_{n+1-i} - X_{n+h}, X_{n+1-j} >_{\bar{A}} = 0, \quad j = 1, \ldots, n. \tag{8}$$

Set $\gamma(h) :=< X_0, X_h >_{\bar{A}}$, $h \in \mathbb{Z}$. Then, equations (8) can be rewritten as

$$\Gamma_n \mathbf{a}_n = \gamma_n, \tag{9}$$

where

$$\Gamma_n = [\gamma(i - j)]_{i,j=1,\ldots,n}$$

and $\gamma_n = (\gamma(h), \ldots, \gamma(n+h-1))'$. If Γ_n is non-singular then we have the unique solution,

$$\mathbf{a}_n = \Gamma_n^{-1} \gamma_n.$$

Remark 3.1. *Similarly to Proposition 5.1.1 in Brockwell and Davis (1991), it can be shown that if $\gamma(0) > 0$ and $\gamma(h) \to 0$, as $h \to +\infty$ then Γ_n is non-singular. For the linear process $\{X_t\}$ of the form (2) with conditions (3) and (4) we have $\gamma(h) \to 0$, as $h \to +\infty$. Therefore, Γ_n in (9) is non-singular.*

3.1   Prediction method for a class of linear processes with infinite variance

Consider the real valued process $\{X_t = \sum_{j=-\infty}^{+\infty} c_j W_{t-j} | t \in \mathbb{Z}\}$, with conditions (3) and (4) and $\{W_t\}$ is an i.i.d. sequence of random variables such that

$$P(|W_t| > x) = x^{-\alpha} L(x), \quad \frac{P(W_t > x)}{P(|W_t| > x)} \to p \text{ and } \frac{P(W_t < -x)}{P(|W_t| > x)} \to q, \tag{10}$$

as $x \to +\infty$, where $L(.)$ is a slowly varying function at infinity, $\alpha \in (0, 2)$, $p \in [0, 1]$ and $q = 1 - p$. In the sequel, for the process $\{X_t\}$ we describe a reasonable method for estimating the coefficient vector $\mathbf{a}_n$ in (9). Assume that

$$\rho(h) = \frac{\sum_{k=-\infty}^{+\infty} c_k c_{k+h}}{\sum_{k=-\infty}^{+\infty} c_k^2} \quad \text{and} \quad \hat{\rho}_n(h) = \frac{\sum_{t=1}^{n} X_t X_{t+h}}{\sum_{t=1}^{n} X_t^2}, \quad h \in \mathbb{Z}.$$

In Davis and Resnick (1985) it is shown that

$$\big(\hat{\rho}_n(1), \ldots, \hat{\rho}_n(H)\big) \to_P \big(\rho(1), \ldots, \rho(H)\big), \quad H > 0 \tag{11}$$

as $n \to +\infty$. The notation " $\to_P$ " denotes convergence in probability. By dividing two sides (9) on $\sum_{k=-\infty}^{+\infty} |c_k|^2$ we obtain

$$\Gamma_n^* \mathbf{a}_n = \gamma_n^*, \tag{12}$$

where

$$\Gamma_n^* = [\rho(i-j)]_{i,j=1,\dots,n} \text{ and } \gamma_n^* = (\rho(h), \dots, \rho(n+h-1))'.$$

Using (11), we can estimate Γ_n^* and γ_n^*. We summarize the obtained results in the following lemma. In this lemma a reasonable estimator for $\mathbf{a}_n$ is presented.

Lemma 3.2. *Let* $X_1, \dots, X_N$ *be a path of length* N *from the linear process* $\{X_t = \sum_{j=-\infty}^{+\infty} c_j W_{t-j}\}$ *with conditions (3) and (4) and* $\{W_t\}$ *is an i.i.d. sequence of random variables satisfying (10). Let*

$$\hat{\Gamma}_{n,N}^* = [\hat{\rho}_N(|i-j|)]_{i,j=1,\dots,n},$$

be non-singular and $\hat{\gamma}_{n,N}^* = (\hat{\rho}_N(h), \dots, \hat{\rho}_N(n+h-1))'$. *Let* $\hat{\mathbf{a}}_{n,N} = \hat{\Gamma}_{n,N}^{*-1} \hat{\gamma}_{n,N}^*$. *Then,* $\hat{\mathbf{a}}_{n,N} \to_P \mathbf{a}_n$, *as* $N \to +\infty$.

4   Application in finance

In this section, we utilize the new prediction method for predicting a set of financial data.

Andrews et al. (2009) showed that the natural logarithms of the volumes of Wal-Mart stock traded daily on the New York Stock Exchange between December 1, 2003 and December 31, 2004, can be fitted by the following non-causal ARMA(2,0) process

$$X_t + 2.0766 X_{t-1} - 2.0772 X_{t-2} = Z_t,$$

where $\{Z_t\}$ is an i.i.d. sequence of α-stable random variables with $\alpha = 1.8335$. As we know, there is not any model base method for predicting this process. For the values $\{x_t\}_{t=1}^{274}$ we found $\sum_{t=1}^{274} x_t / 274 = 16.08896$. Our aim is to predict X_{n+1} in terms of some past values. Firstly, we find the best linear prediction for the process $\{X_t^* = X_t - 16.08896 | t \in \mathbb{Z}\}$. Then, we consider $\hat{X}_{n+1}^* + 16.08896$ as a predictor of X_{n+1}.

4   APPLICATION IN FINANCE

We have $X_t^* + 2.0766X_{t-1}^* - 2.0772X_{t-2}^* = Z_t^*$, where $Z_t^* = Z_t - 16.07931$. We can rewrite X_t^* as $X_t^* = Z_t^* - 2.0766X_{t-1}^* + 2.0772X_{t-2}^*$. Using the introduced space in Theorem 2.6, the values of the process $\{Z_t^* - 2.0766X_{t-1}^* + 2.0772X_{t-2}^* | t \in \mathbb{Z}\}$ are in a Hilbert space. Therefore, for the process $\{X_t^*\}$ we have

$$
\gamma(h) = \begin{cases}
1 + 2.0766^2 + 2.0772^2 & h = 0 \\
-2.0766 \times 2.0772 & h = 1 \\
0 & \text{otherwise}
\end{cases}.
$$

Figure 1 shows the value of X_{n+1} and its prediction in terms of 20 past values, for $n = 20, \ldots, 273$. We calculated the mean square error of prediction and true values:

$$
\frac{1}{253} \sum_{n=21}^{273} (\hat{x}_{n+1} - x_{n+1})^2 = 0.2435914.
$$

The procedure described in this section can be used for predicting of causal and non-causal ARMA processes.

4   APPLICATION IN FINANCE
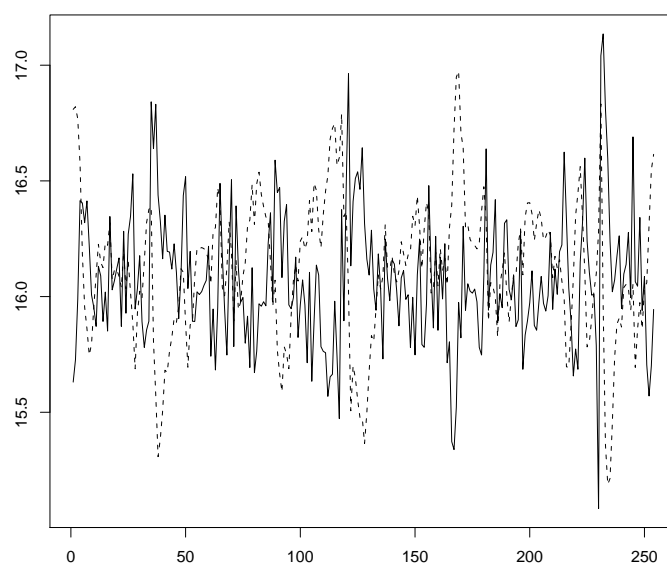
Figure 1: Dotted line displays the natural logarithms of the volumes of Wal-Mart stock traded daily on the New York Stock Exchange between December 21, 2003 and December 31, 2004. Smooth line displays the predicted values for X_{n+1} in terms of 20 past values, where $n = 20, \ldots, 273$.

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SOME ESTIMATE FOR THE EXTENDED FRESNEL TRANSFORM AND ITS PROPERTIES IN A CLASS OF BOEHMIANS

S.K.Q. Al-Omari

Department of Applied Sciences, Faculty of Engineering Technology
Al-Balqa Applied University, Amman 11134, Jordan
E-mail: s.k.q.alomari@fet.edu.jo

Abstract

The construction of Boehmians is similar to the construction of field of quotients and in some cases, it just gives the field of quotients. Strong Boehmians are introduced as a subclass of generalized functions to generalize distributions [4]. In this article, we continue the analysis obtained in [20] and [21]. We discuss the Fresnel transform on a certain space of strong Boehmians. The Fresnel transform and its inverse are extended to a context of Boehmians and are well-defined, linear, analytic and one-to-one mappings. New convergence is also defined.

Keywords: Fresnel Transform; Generalized Fresnel Transform; Tempered Distribution; Boehmian; Strong Boehmian.

1 INTRODUCTION

In wave optics theory, the generalized Fresnel transform is frequently used to describe beam propagation in paraxial approximation and Fresnel diffraction of light . The generalized Fresnel transform (the diffraction Fresnel transform) of an arbitrary function $f(x)$ is defined as [16, 9]

$$F_d f (\tau) = \int_{\mathbb{R}} E (\alpha_1, \gamma_1, \gamma_2, \alpha_2; x, \tau) f (x) \, dx. \tag{1}$$

where

$$E (\alpha_1, \gamma_1, \gamma_2, \alpha_2; x, \tau) := \frac{1}{\sqrt{2\pi i \gamma_1}} \exp \left(\frac{i}{2\gamma_1} \left(\alpha_1 x^2 - 2x\tau + \alpha_2 \tau^2 \right) \right)$$

is the transform kernel parameterized by a real matrix $\mathtt{M}$ being restricted by $\alpha_1 \alpha_2 - \gamma_1 \gamma_2 = 1$. Since diffraction Fresnel transforms are related to a wide class of optical transforms; their various properties and applications have brought great interests for physicists recently. In this article, we pay attention to a particular case of the diffraction Fresnel transform, namely, the fresnel transform $\alpha_1 = \alpha_2$. The reason we choose this simplified form of the diffraction Fresnel transform "Fresnel transform" lies in two aspects: the first is that the Fresnel transform can be analyzed more conveniently which provide a favourable performance; the second reason is related

to the complexity of finding a strong Boehmian space that can handle the diffraction Fresnel transform entangled with four parameters.

However, our results are new even for usual Boehmian spaces. We consider certain space of Boehmians named as "strong Boehmian space" for the distributional Fresnel transform. Enlightened by this space, the extended Fresnel transform is shown to constitute a usual Boehmian. Further properties are also discussed.

We spread the paper into five sections: The Fresnel transform is reviewed in Section 1. Section 2 presents a general construction of usual Boehmian spaces. The strong Boehmian space is obtained in Section 3. The image space of Boehmians is described in Section 4. Properties of the extended Fresnel transform and its inverse are discussed in Section 5.

By considering the case, $\alpha_1 = \alpha_2$, the reduction of Equ.(1) is the so-called Fresnel transform whose integral equation given by [8, p.p.207]

$$T f\left(\tau\right) := F\left(\tau\right) := \int_{\mathtt{R}} f\left(x\right) \exp i\left(\tau - x\right)^2 dx, \tag{2}$$

where $f\left(x\right)$ is a single valued function over $\mathtt{R}$. Let $E^{'}\left(\mathtt{R}\right)$ be the dual space of $E\left(\mathtt{R}\right)$ of all infinitely smooth complex-valued functions $\phi\left(x\right)$, over $\mathtt{R}$(the set of real numbers), such that

$$\gamma_k\left(\phi\right) = \sup_{x \in \mathtt{K}} \left| D_x^k \phi\left(x\right) \right| < \infty, \tag{3}$$

where $k \in \mathtt{N}\left(k = 0, 1, 2, ...\right)$ and $\mathtt{K}$ run through compact subsets of $\mathtt{R}$. Elements of $E^{'}\left(\mathtt{R}\right)$ are called distributions of compact support. The topology of $E\left(\mathtt{R}\right)$ is generated by the sequence $\left(\gamma_k\left(\phi\right)\right)_0^\infty$ of multinorms which makes $E\left(\mathtt{R}\right)$ locally convex Hausdörff topological vector space, $D\left(\mathtt{R}\right) \subset E\left(\mathtt{R}\right)$, $D\left(\mathtt{R}\right)$ is the test function space of compact support. The topology of $D\left(\mathtt{R}\right)$ is, therefore, stronger than that induced on $D\left(\mathtt{R}\right)$ by $E^{'}\left(\mathtt{R}\right)$, and the restriction of any element of $E\left(\mathtt{R}\right)$ to $D\left(\mathtt{R}\right)$ is in $D^{'}\left(\mathtt{R}\right)$; the dual space of Schwartz distributions.

The classical theory of the Fresnel transform is extended to distributions in $E^{'}\left(\mathtt{R}\right)$ by the formula

$$T f\left(\tau\right) := F\left(\tau\right) \overset{\Delta}{=} \left\langle f\left(x\right), \exp i\left(\tau - x\right)^2 \right\rangle, \tag{4}$$

where $\tau \in \mathtt{R}$ is arbitrary but fixed.

We state without proof the following theorem.

Theorem 1.1(Analyticity of T). Let $f \in E^{'}\left(\mathtt{R}\right)$, then T is differentiable and

$$D_\tau^k T f\left(\tau\right) = \left\langle f\left(x\right), D_\tau^k \exp i\left(\tau - x\right)^2 \right\rangle,$$

for every nonnegative integer k.

Theorem 1.2. The mapping T is linear and one-one .

Proof is straightforward.

Definition 1.3. Let f and $g \in E'(\mathtt{R})$. The generalized convolution product $\curlywedge$ of f and g is defined as

$$\langle (f \curlywedge g)(x), \phi(x) \rangle = \langle f(x), \langle g(\tau), \phi(x+\tau) \rangle \rangle,$$

for every $\phi \in E(\mathtt{R})$.

Theorem 1.4. For every $f \in E'(\mathtt{R})$, $\psi(x) = \langle f(\tau), \phi(x+\tau) \rangle$ is infinitely smooth and satisfies the relation

$$D_x^k \psi(x) = \left\langle f(\tau), D_x^k \phi(x+\tau) \right\rangle,$$

for all $k \in \mathtt{N}$.

Proof (see [16] for more details).

2   THE STRONG SPACE OF BOEHMIANS B_τ

Let $\mathtt{Y} = \mathtt{N} \times \mathtt{R}, \mathtt{R} = (-\infty, \infty)$, then $f(n, x) \in E(\mathtt{Y})$ if and only if $f(n, x)$ is infinitely smooth and

$$\gamma_k(f(n, x)) = \sup_{x \in \mathtt{K}} \left| D_x^k f(n, x) \right| < \infty, \tag{5}$$

for every $k \in \mathtt{N}$, $\mathtt{K}$ being compact subset of $\mathtt{R}$.

Let $f(n, x) \in E(\mathtt{Y})$ and $\omega \in D(\mathtt{R})$. The usual convolution product of $f(n, x)$ and ω with respect to x is given by

$$(f \curlywedge \omega)(n, x) = \int_{\mathtt{R}} f(n, y) \omega(x - y) \, dy, \tag{6}$$

$n \in \mathtt{N}$.

Let $E(\mathtt{R}) (\subset D(\mathtt{R}))$ be the set of all test functions ω such that

$$\int_{\mathtt{R}} \omega(x) \, dx = 1.$$

The pair (f, ω), or $\dfrac{f(n, x)}{\omega}$, is said to be a quotient of functions if and only if

$$f(n, x) \curlywedge d_m \omega(x) = f(m, x) \curlywedge d_n \omega(x),$$

$n, m \in \mathtt{N}$ and $d_r \omega(x) = r\omega(rx)$.

Two quotients of functions are equivalent, $\dfrac{f(n, x)}{\omega} \sim \dfrac{g(m, x)}{\psi}$, if and only if

$$f(n, x) \curlywedge d_m \psi(x) = g(m, x) \curlywedge d_n \omega(x). \tag{7}$$

Let $Q = \left\{ \dfrac{f(n, x)}{\omega} : f(n, x) \in E(\mathtt{Y}), \omega \in E(\mathtt{R}) \right\}$, then the equivalence class $\left[\dfrac{f(n, x)}{\omega} \right]$ containing $\dfrac{f(n, x)}{\omega}$ is said to be a strong Boehmian. The space of all such Boehmians is denoted by B_τ.

Addition, scalar multiplication, differentiation and convolution in B_τ are defined by

$$\left[\frac{f(n,x)}{\omega}\right] + \left[\frac{g(m,x)}{\psi}\right] = \left[\frac{f(n,x) \curlywedge \psi + g(m,x) \curlywedge \omega}{\omega \curlywedge \psi}\right];$$

$$k\left[\frac{f(n,x)}{\omega}\right] = \left[\frac{kf(n,x)}{\omega}\right]; D_x^k\left[\frac{f(n,x)}{\omega}\right] = \left[\frac{D_x^k f(n,x)}{\omega}\right]$$

and

$$\left[\frac{f(n,x)}{\omega}\right] \curlywedge \psi = \left[\frac{f(n,x) \curlywedge \psi}{\omega}\right],$$

respectively.

$\mathcal{E}$ **convergence** $\beta_\upsilon \xrightarrow{\mathcal{E}} \beta$ in B_τ if for some $f_\upsilon(n,x), f(n,x) \in E'(\mathtt{Y})$ such that $\beta_\upsilon = \left[\frac{f_\upsilon(n,x)}{\omega}\right], \beta = \left[\frac{f(n,x)}{\omega}\right]$ we have $f_\upsilon(n,x) \to f(n,x)$ as $\upsilon \to \infty$ in $E'(\mathtt{Y})$.

Theorem 2.1. The mapping $f(n,x) \to \left[\frac{f(n,x) \curlywedge d_n\omega}{\omega}\right]$ is continuous and imbedding from $E'(\mathtt{Y}) \to B_\tau$.

Proof see [7, Theorem 3.1] for detailed proof.

3   STRUCTURE OF USUAL BOEHMIANS

One of most youngest generalizations of functions, and more particularly of distributions, is the theory of Boehmians. The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [6]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose support is bounded from the left. In a concrete case, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions.

The construction of Boehmians is similar to the construction of the field of quotients and in some cases, it gives just the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as space $\mathcal{C}$ (the space of continous functions) with the operations of pointwise additions and convolution.

Let $\mathcal{G}$ be a linear space and $\mathcal{S}$ be a subspace of $\mathcal{G}$. We assume to each pair of elements $f \in \mathcal{G}$ and $\omega \in \mathcal{S}$, is assigned the product $f * g$ such that the following conditions are satisfied:

(1) If $\omega, \psi \in \mathcal{S}$, then $\omega \star \psi \in \mathcal{S}$ and $\omega \star \psi = \psi \star \omega$.
(2) If $f \in \mathcal{G}$ and $\omega, \psi \in \mathcal{S}$, then $(f * \omega) * \psi = f * (\omega \star \psi)$.
(3) If $f, g \in \mathcal{G}, \omega \in \mathcal{S}$ and $\lambda \in \mathtt{R}$, then

$$(f + g) * \omega = f * \omega + g * \omega \text{ and } \lambda(f * \omega) = (\lambda f) * \omega.$$

Let Δ be a family of sequences from $\mathcal{S}$, such that:

Δ_1 *If* $f, g \in \mathcal{G}, (\delta_n) \in \Delta$ *and* $f * \delta_n = g * \delta_n \, (n = 1, 2, ...)$, *then* $f = g, \forall n$.

Δ_2 *If* $(\omega_n), (\delta_n) \in \Delta$, *then* $(\omega_n \star \psi_n) \in \Delta$.

Elements of Δ *will be called delta sequences. Consider the class* $\mathcal{A}$ *of pair of sequences defined by*

$$\mathcal{A} = \left\{ ((f_n), (\omega_n)) : (f_n) \subseteq \mathcal{G}^{\mathbb{N}}, (\omega_n) \in \Delta \right\},$$

for each $n \in \mathbb{N}$.

An element $((f_n), (\omega_n)) \in \mathcal{A}$ *is called a quotient of sequences, denoted by,* $\left[\dfrac{f_v}{\omega_n} \right]$ *if* $f_n * \omega_m = f_m * \omega_n, \forall n, m \in \mathbb{N}$.

Two quotients of sequences $\dfrac{f_n}{\omega_n}$ *and* $\dfrac{g_n}{\psi_n}$ *are said to be equivalent,* $\dfrac{f_n}{\omega_n} \sim \dfrac{g_n}{\psi_n}$, *if* $f_n * \psi_m = g_m * \omega_n, \forall n, m \in \mathbb{N}$.

The relation $\sim$ *is an equivalent relation on* $\mathcal{A}$ *and hence, splits* $\mathcal{A}$ *into equivalence classes. The equivalence class containing* $\dfrac{f_n}{\omega_n}$ *is denoted by* $\left[\dfrac{f_n}{\omega_n} \right]$. *These equivalence classes are called* **Boehmians**; *or* **usual Boehmians**; *and the space of all Boehmians is denoted by* $\mathcal{B}$.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way $\left[\dfrac{f_n}{\omega_n} \right] + \left[\dfrac{g_n}{\psi_n} \right] = \left[\dfrac{f_n * \psi_n + g_n * \omega_n}{\omega_n * \psi_n} \right]$ *and* $\alpha \left[\dfrac{f_n}{\omega_n} \right] = \left[\alpha \dfrac{f_n}{\omega_n} \right] = \left[\dfrac{\alpha f_n}{\omega_n} \right], \alpha \in \mathbb{C}$, *space of complex numbers.*

The operation $*$ *and the differentiation are defined by* $\left[\dfrac{f_n}{\omega_n} \right] * \left[\dfrac{g_n}{\psi_n} \right] = \left[\dfrac{f_n * g_n}{\omega_n * \psi_n} \right]$ *and* $\mathcal{D}^{\alpha} \left[\dfrac{f_n}{\omega_n} \right] = \left[\dfrac{\mathcal{D}^{\alpha} f_n}{\omega_n} \right]$.

Many a time, $\mathcal{G}$ *is equipped with a notion of convergence. The relationship between the notion of convergence and* $*$ *are given by:*

(4) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\mathcal{G}$ *and,* $\omega \in \mathcal{S}$ *is any fixed element, then*

$$f_n * \omega \to f * \omega \text{ in } \mathcal{G} \, (as \, n \to \infty).$$

(5) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\mathcal{G}$ *and* $(\omega_n) \in \Delta$, *then*

$$f_n * \omega_n \to f \text{ in } \mathcal{G} \, (as \, n \to \infty).$$

The operation $*$ can be extended to $\mathcal{B} \times \mathcal{S}$ by : If $\left[\dfrac{f_n}{\omega_n} \right] \in \mathcal{B}$ and $\omega \in \mathcal{S}$, then

$$\left[\dfrac{f_n}{\omega_n} \right] * \omega = \left[\dfrac{f_n * \omega}{\omega_n} \right].$$

In $\mathcal{B}$, two types of convergence, δ convergence and Δ convergence, are defined as follows:

δ **convergence:** A sequence of Boehmians (β_n) in $\mathcal{B}$ is said to be δ convergent to a Boehmian β in $\mathcal{B}$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ω_n) such that

$$(\beta_n * \omega_n), (\beta * \omega_n) \in \mathcal{G}, \forall k, n \in \mathbb{N},$$

and

$$(\beta_n * \omega_k) \rightarrow (\beta * \omega_k) \text{ as } n \rightarrow \infty, \text{ in } \mathcal{G}, \text{ for every } k \in \mathbb{N}.$$

The following is equivalent for the statement of δ convergence:

Theorem 3.1. $\beta_n \xrightarrow{\delta} \beta \ (n \rightarrow \infty)$ in $\mathcal{B}$ if and only if there is $f_{n,k}, f_k \in \mathcal{G}$ and $(\omega_k) \in \Delta$ such that $\beta_n = \left[\dfrac{f_{n,k}}{\omega_k} \right], \beta = \left[\dfrac{f_k}{\omega_k} \right]$ and for each $k \in \mathbb{N}, f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ in $\mathcal{G}$.

Δ **convergence:** A sequence of Boehmians (β_n) in $\mathcal{B}$ is said to be Δ convergent to a Boehmian β in $\mathcal{B}$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\omega_n) \in \Delta$ such that $(\beta_n - \beta) * \omega_n \in \mathcal{G}, \forall n \in \mathbb{N}$, and $(\beta_n - \beta) * \omega_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{G}$. See; $[1 - 7, 15 - 17, 19]$, for further investigation.

4   THE IMAGE SPACE OF BOEHMIANS

The function $\ell(n, \tau)$ is a member in $\mathcal{L}(\mathtt{Y}), \mathtt{Y} = \mathbb{N} \times \mathbb{R}$, if there is $f(n, x) \in E'(\mathtt{Y})$ such that $\ell(n, \tau) := (Tf)(n, \tau) := F(n, \tau)$.

A sequence $(\ell_\upsilon(n, \tau))_{\upsilon=1}^\infty \rightarrow \ell(n, \tau)$ as $\upsilon \rightarrow \infty$, in $\mathcal{L}(\mathtt{Y})$, if and only if there are $f_\upsilon(n, x), f(n, x)$ in $E'(\mathtt{Y})$, such that

$$f_\upsilon(n, x) \rightarrow f(n, x) \text{ as } \upsilon \rightarrow \infty,$$

where, $\ell_\upsilon(n, \tau) = (Tf_\upsilon)(n, \tau), \ell(n, \tau) = (Tf)(n, \tau)$.

To $\mathcal{L}(\mathtt{Y})$ and $E(\mathbb{R})$, the usual convolution product $\curlyvee$ is interpreted to mean

$$(\ell \curlyvee \omega)(n, y) = \int_\mathbb{R} \ell(n, \tau - y) d_m \omega(y) dy, \ n, m \in \mathbb{N}, \tag{8}$$

where $\ell = Tf, \omega \in E(\mathbb{R})$.

Let

$$\tilde{E}(\mathbb{R}) = \{(d_n \omega) : \omega \in E(\mathbb{R}), \operatorname{supp} d_n \omega \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

then $\tilde{E}(\mathbb{R})$ is a family of delta sequences.

Following results constitute the space $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$ of general Boehmians.

Lemma 4.1. Let $f \in E'(\mathtt{Y})$ and $\omega \in E(\mathbb{R})$, then the following holds

$$T(f(n, x) \curlywedge d_m \omega(x))(n, \tau) = (\ell \curlyvee \omega)(n, \tau),$$

where $\ell = Tf$.

Proof. Let $f \in E'(\mathtt{Y})$ and $\omega \in E(\mathbb{R})$, then by using Equ.(4) we get

$$T(f(n, x) \curlywedge d_m \omega)(n, \tau) = \left\langle (f(n, x) \curlywedge d_m \omega)(x), \exp i(\tau - x)^2 \right\rangle.$$

Definition 1.3 implies

$$
\begin{aligned}
T\left(f\left(n,x\right)\curlywedge d_m\omega\right)\left(n,\tau\right) &= \left\langle f\left(n,x\right),\left\langle d_m\omega\left(y\right),\exp i\left(\tau-\left(x+y\right)\right)^2\right\rangle\right\rangle \\
&= \left\langle f\left(n,x\right),\left\langle d_m\omega\left(y\right),\exp i\left(\left(\tau-y\right)-x\right)^2\right\rangle\right\rangle.
\end{aligned}
$$

That is,

$$
T\left(f\left(n,x\right)\curlywedge d_m\omega\right)\left(n,\tau\right)=\int_{\mathbb{R}}\left\langle f\left(n,x\right),\exp i\left(\left(\tau-y\right)-x\right)^2\right\rangle d_m\omega\left(y\right)dy.
$$

Equ.(8) gives

$$
T\left(f\left(n,x\right)\curlywedge d_m\omega\right)\left(n,\tau\right)=\int_{\mathbb{R}}\ell\left(n,\tau-y\right)d_m\omega\left(y\right)dy=\left(\ell\left(n,\tau\right)\curlyvee\omega\right)\left(n,\tau\right),
$$

where $\ell=Tf$.

This completes the proof of the lemma.

Lemma 4.2. Let $\ell\in\mathcal{L}\left(\mathtt{Y}\right)$ and $\omega\in E\left(\mathbb{R}\right)$, then $\ell\curlyvee\omega\in\mathcal{L}\left(\mathtt{Y}\right)$.

Proof. Let $f\left(n,x\right)\in E'\left(\mathtt{Y}\right)$ be such that $\ell\left(n,\tau\right)=Tf\left(n,\tau\right)$. From Lemma 4.1. we see that
$$
\left(\ell\curlyvee\omega\right)\left(n,\tau\right)=T\left(f\left(n,x\right)\curlywedge d_m\omega\right)\left(n,\tau\right),
$$
$m,n\in\mathbb{N}$. The fact that $\omega\in E\left(\mathbb{R}\right)$ implies $\left(d_m\omega\right)\in E\left(\mathbb{R}\right)$, by [7, p.p.886,(2.4)]. Hence,
$$
f\left(n,x\right)\curlywedge d_m\omega\in E'\left(\mathtt{Y}\right).
$$
Thus, $\ell\curlyvee\omega\in\mathcal{L}\left(\mathtt{Y}\right)$.

This completes the proof of the lemma.

Lemma 4.3. Let $\ell\left(n,\tau\right)\in\mathcal{L}\left(\mathtt{Y}\right),\omega,\psi\in E\left(\mathbb{R}\right)$, then $\ell\curlyvee\left(\omega\curlywedge\psi\right)=\left(\ell\curlyvee\omega\right)\curlywedge\psi$.

Proof. The hypothesis of the lemma and Liebniz Theorem yield

$$
\begin{aligned}
\left(\ell\curlyvee\left(\omega\curlywedge\psi\right)\right)\left(n,\tau\right) &= \int_{\mathbb{R}}\ell\left(n,\tau-y\right)d_m\left(\omega\curlywedge\psi\right)\left(y\right)dy \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\ell\left(n,\tau-y\right)m\omega\left(my-t\right)dy\right)\psi\left(t\right)dt.
\end{aligned}
$$

The substitution $my-t=x$ implies

$$
\left(\ell\curlyvee\left(\omega\curlywedge\psi\right)\right)\left(n,\tau\right)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\ell\left(n,\tau-\frac{t+x}{m}\right)m\omega\left(x\right)dx\right)\psi\left(t\right)dt
$$

The substitution $x=mz$ implies

$$
\left(\ell\curlyvee\left(\omega\curlywedge\psi\right)\right)\left(n,\tau\right)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\ell\left(n,\tau-z-\frac{t}{m}\right)d_m\omega\left(z\right)dz\right)m\psi\left(t\right)dt
$$

Once again, the substitution $t = mr$ togather with Liebniz Theorem imply

$$(\ell \curlyvee (\omega \curlywedge \psi))(n, \tau) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \ell(n, \tau - z - r) \, d_m \omega(z) \, dz \right) d_m \psi(r) \, dr$$

Hence, by Equ.(8), we get that

$$\begin{aligned} (\ell \curlyvee (\omega \curlywedge \psi))(n, \tau) &= \int_{\mathbb{R}} (\ell \curlyvee \omega)(\tau - r) \, d_m \psi(r) \, dr \\ &= ((\ell \curlyvee \omega) \curlywedge \psi)(n, \tau) \end{aligned}$$

The lemma is therefore completely proved.

Lemma 4.4. Let $\ell_1, \ell_2 \in \mathcal{L}(\mathtt{Y}), \omega \in E(\mathbb{R})$, then

$(i)\,(\ell_1 + \ell_2) \curlyvee \omega = \ell_1 \curlyvee \omega + \ell_2 \curlyvee \omega;$
$(ii)\, k\,(\ell_1 \curlyvee \omega) = (k\ell_1) \curlyvee \omega = \ell_1 \curlyvee (k\omega)\,.$

Proof is straightforward from properties of integral operators.

Lemma. 4.5. Let $\ell_1, \ell_2 \in \mathcal{L}(\mathtt{Y}), (d_v \omega) \in \tilde{E}(\mathbb{R})$ and $\ell_1 \curlyvee \omega = \ell_2 \curlyvee \omega$, then $\ell_1 = \ell_2$ in $\mathcal{L}(\mathtt{Y})\,.$

Proof. For every $\ell_1(n, \tau), \ell_2(m, \tau) \in \mathcal{L}(\mathtt{Y})$, there are the corresponding distributions $f_1(n, x), f_2(m, x) \in E'(\mathtt{Y})$ such that $\ell_1 = Tf_1$ and $\ell_2 = Tf_2$. Hence, the hypothesis of the lemma and linearity of the distributional Fresnel transform, imply $T(f_1(n, x) - f_2(m, x)) \curlyvee \omega = 0, \forall n, m \in \mathtt{N}$. Therefore $T(f_1(n, x) - f_2(m, x)) = 0, n, m \in \mathtt{N}, x \in \mathbb{R}$, in $\mathcal{L}(\mathtt{Y})$. Thus $\ell_1(n, \tau) = \ell_2(m, \tau)$ in $\mathcal{L}(\mathtt{Y}), \forall n, m \in \mathtt{N}, \tau \in \mathbb{R}$.

Hence the lemma.

Lemma 4.6. Let $\ell_v \to \ell$ in $\mathcal{L}(\mathtt{Y})$ as $v \to \infty$ and $\omega \in E(\mathbb{R})$, then $\ell_v \curlyvee \omega \to \ell \curlyvee \omega$ in $\mathcal{L}(\mathtt{Y})$ as $v \to \infty$.

Proof is a straightforward consequence from definitions.

Lemma 4.7. Let $\ell_v \to \ell$ in $\mathcal{L}(\mathtt{Y})$ as $v \to \infty$ and $(d_v \omega) \in \tilde{E}(\mathbb{R})$, then $\ell_v \curlyvee \omega \to \ell$ as $v \to \infty$.

Proof. We have $\ell_v(n, \tau) \curlyvee \omega - \ell(n, \tau) = Tf_v(n, \tau) \curlyvee \omega - Tf(n, \tau)$, for some $f_v(n, x), f(n, x) \in E'(\mathbb{R})$. Hence, Lemma 4.1. implies

$$\ell_v(n, \tau) \curlyvee \omega - \ell(n, \tau) = T(f_v(n, x) \curlywedge d_r \omega)(n, \tau) - Tf(n, \tau), \text{ for some } r \in \mathtt{N}.$$

Once again, linearity of T and the fact that $(d_r \omega)$ is delta sequence yield

$$\begin{aligned} \ell_v(n, \tau) \curlyvee \omega - \ell(n, \tau) &= T(f_v(n, x) \curlywedge d_r \omega - f(n, x))(n, \tau) \\ &\to T(f_v(n, x) - f(n, x))(n, \tau) \text{ as } r \to \infty. \end{aligned}$$

Hence

$$\ell_v(n, \tau) \curlyvee \omega - \ell(n, \tau) \to 0 \text{ as } v \to \infty.$$

The proof is therefore completed.

The Boehmian space $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ is thus constructed.

A typical element in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ is denoted by

$$\left[\frac{\ell(n, \tau)}{d_n \omega}\right].$$

Two Boehmians $\left[\frac{\ell_1(n, \tau)}{d_n \omega}\right]$ and $\left[\frac{\ell_2(m, \tau)}{d_m \psi}\right]$ are said to be equal in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ if and only if

$$\ell_1(n, \tau) \Upsilon \psi = \ell_2(m, \tau) \Upsilon \omega. \tag{9}$$

In $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$, we define, addition, scalar multiplication, differentiation and convolution between Boehmians by

$$\left[\frac{\ell_1(n, \tau)}{d_n \omega}\right] + \left[\frac{\ell_2(m, \tau)}{d_m \psi}\right] = \left[\frac{\ell_1(n, \tau) \Upsilon \psi + \ell_2(m, \tau) \Upsilon \omega}{\omega \Upsilon \psi}\right];$$

$$k\left[\frac{\ell(n, \tau)}{d_n \omega}\right] = \left[\frac{k\ell(n, \tau)}{d_n \omega}\right]; D^k\left[\frac{\ell(n, \tau)}{d_n \omega}\right] = \left[\frac{D^k \ell(n, \tau)}{d_n \omega}\right]$$

and

$$\left[\frac{\ell(n, \tau)}{d_n \omega}\right] \Upsilon \psi = \left[\frac{\ell(n, \tau) \Upsilon \psi}{d_n \omega}\right],$$

respectively.

Definition 4.8. (δ convergence): $\beta_\upsilon \xrightarrow{\delta} \beta$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ is δ convergent if there exists a delta sequence $(d_\upsilon \omega) \in \tilde{E}(\mathtt{R})$ such that

$$(\beta_\upsilon \Upsilon \omega), (\beta \Upsilon \omega) \in \mathcal{L}, \forall \upsilon, n \in \mathtt{N},$$

and

$$(\beta_\upsilon \Upsilon \omega) \to (\beta \Upsilon \omega) \text{ as } \upsilon \to \infty, \text{ in } \mathcal{L}, \text{ for every } \upsilon \in \mathtt{N}.$$

Theorem 4.9. $\beta_\upsilon \xrightarrow{\delta} \beta$ $(\upsilon \to \infty)$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ if and only if there are $\ell_\upsilon(n, \tau), \ell(n, \tau) \in \mathcal{L}$ and $(d_n \omega) \in \tilde{E}(\mathtt{R})$ such that $\beta_\upsilon = \left[\frac{\ell_\upsilon(n, \tau)}{d_n \omega}\right], \beta = \left[\frac{\ell(n, \tau)}{d_n \omega}\right]$ and for each $\upsilon \in \mathtt{N}, \ell_\upsilon(n, \tau) \to \ell(n, \tau)$ as $\upsilon \to \infty$ in $\mathcal{L}$.

Definition 4.10.(Δ convergence): $(\beta_\upsilon) \xrightarrow{\Delta} \beta$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \Upsilon\right)$ is Δ convergent, if there exists a $(d_\upsilon \omega) \in \tilde{E}(\mathtt{R})$ such that $(\beta_\upsilon - \beta) \Upsilon \omega \in \mathcal{L}, \forall \upsilon \in \mathtt{N}$, and $(\beta_\upsilon - \beta) \Upsilon \omega \to 0$ as $\upsilon \to \infty$ in $\mathcal{L}$.

5   FRESNEL TRANSFORMS TO BOEHMIANS

This section is devoted for extending the Fresnel transform to Boehmians in B_τ.

Let $\left[\dfrac{f(n,x)}{\omega}\right] \in B_\tau$ then, the extended Fresnel transform of $\left[\dfrac{f(n,x)}{\omega}\right]$ is viewed as

$$\tilde{T}\left[\frac{f(n,x)}{\omega}\right] = \left[\frac{\ell(n,\tau)}{d_n\omega}\right] \in B\left(\mathcal{L},(E,\curlywedge),\tilde{E},\curlyvee\right), \qquad (10)$$

where $\ell = Tf$.

Theorem 5.1. The mapping $\tilde{T}: B_\tau \to B\left(\mathcal{L},(E,\curlywedge),\tilde{E},\curlyvee\right)$ is well-defined.

Proof. Let $\left[\dfrac{f(n,x)}{\omega}\right] = \left[\dfrac{g(m,x)}{\psi}\right] \in B_\tau$, then, using Equ.(7) yields

$$f(n,x) \curlywedge d_m\psi = g(m,x) \curlywedge d_n\omega. \qquad (11)$$

Applying the Fresnel transform on both sides of Equ.(11), and applying Lemma 4.1. imply $\ell_1(n,\tau) \curlyvee \psi = \ell_2(m,\tau) \curlyvee \omega$ where, $\ell_1 = Tf, \ell_2 = Tg, f,g \in E'(\mathbb{R})$. Hence, from Equ.(9), we obtain

$$\left[\frac{\ell_1(n,\tau)}{d_n\omega}\right] = \left[\frac{\ell_2(m,\tau)}{d_m\psi}\right]$$

in the space $B\left(\mathcal{L},(E,\curlywedge),\tilde{E},\curlyvee\right)$.

The proof is completed.

Theorem 5.2. The mapping $\tilde{T}: B_\tau \to B\left(\mathcal{L},(E,\curlywedge),\tilde{E},\curlyvee\right)$ is linear.

Proof Let $\left[\dfrac{f(n,x)}{\omega}\right], \left[\dfrac{g(m,x)}{\psi}\right] \in B_\tau$, then

$$\left[\frac{f(n,x)}{\omega}\right] + \left[\frac{g(m,x)}{\psi}\right] = \left[\frac{f(n,x) \curlywedge d_m\psi + g(m,x) \curlywedge d_n\omega}{\omega \curlywedge \psi}\right]. \qquad (12)$$

Employing the Fresnel transform for Equ.(12) yields

$$\begin{aligned}
\tilde{T}\left(\left[\frac{f(n,x)}{\omega}\right] + \left[\frac{g(m,x)}{\psi}\right]\right) &= \left[\frac{\ell_1(n,\tau) \curlyvee \psi + \ell_2(m,\tau) \curlyvee \omega}{\omega \curlyvee \psi}\right] \\
&= \left[\frac{\ell_1(n,\tau)}{d_n\omega}\right] + \left[\frac{\ell_2(m,\tau)}{d_m\psi}\right].
\end{aligned}$$

By using Lemma 4.1., we get

$$\tilde{T}\left(\left[\frac{f(n,x)}{\omega}\right] + \left[\frac{g(m,x)}{\psi}\right]\right) = \tilde{T}\left(\left[\frac{f(n,x)}{\omega}\right]\right) + \tilde{T}\left(\left[\frac{g(n,x)}{\psi}\right]\right),$$

where $\ell_1 = Tf, \ell_2 = Tg$.

Further, for each $k \in \mathbb{R}$, we have

$$k\tilde{T}\left(\left[\frac{f(n,x)}{\omega}\right]\right) = \tilde{T}\left(k\left[\frac{f(n,x)}{\omega}\right]\right) = \tilde{T}\left(\left[\frac{kf(n,x)}{\omega}\right]\right) = \left[\frac{k\ell_1(n,\tau)}{d_n\omega}\right],$$

$\ell_1 = Tf$.

Hence the theorem is proved.

Theorem 5.3. $\tilde{T} : B_\tau \to B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$ is one-one mapping.

Proof Let $\left[\dfrac{f(n,x)}{\omega}\right], \left[\dfrac{g(m,x)}{\psi}\right] \in B_\tau$ be such that

$$\tilde{T}\left[\frac{f(n,x)}{\omega}\right] = \tilde{T}\left[\frac{g(m,x)}{\psi}\right] \text{ in } B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right).$$

Equ.(10) then yields

$$\left[\frac{\ell_1(n,\tau)}{d_n\omega}\right] = \left[\frac{\ell_2(m,\tau)}{d_m\psi}\right],$$

where $\ell_1(n,\tau) = Tf(n,x), \ell_2(m,\tau) = Tg(m,x)$.

From Equ.(9) we get, $\ell_1(n,\tau) \curlyvee \psi = \ell_2(m,\tau) \curlyvee \omega$. Applying Lemma 4.1. leads to

$$f(n,x) \curlywedge d_m\psi = g(m,x) \curlywedge d_n\omega.$$

Hence, upon considering Equ.(7), proof of our theorem is completed.

Definition 5.4. Let $\left[\dfrac{\ell(n,\tau)}{d_n\omega}\right] \in B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$, then we define its inverse Fresnel transform by

$$\tilde{T}^{-1}\left(\left[\frac{\ell(n,\tau)}{d_n\omega}\right]\right) = \left[\frac{f(n,x)}{\omega}\right],$$

where $\ell = Tf$.

Theorem 5.5. The mapping $\tilde{T}^{-1} : B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right) \to B_\tau$ is well defined.

Proof. Let $\left[\dfrac{\ell_1(n,\tau)}{d_n\omega}\right] = \left[\dfrac{\ell_2(m,\tau)}{d_m\psi}\right] \in B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$, then $\ell_1(n,\tau) \curlyvee \psi = \ell_2(m,\tau) \curlyvee \omega$. By virtue of Lemma 4.1. we get

$$T(f(n,x) \curlywedge d_m\psi) = T(g(m,x) \curlywedge d_n\omega),$$

for some f and g in $E^{'}(\mathbb{R})$, where $\ell_1 = Tf, \ell_2 = Tg$.

The property that T is one-one results in

$$f(n,x) \curlywedge d_m\psi = g(m,x) \curlywedge d_n\omega.$$

Therefore,

$$\left[\frac{f(n,x)}{\omega}\right] = \left[\frac{g(m,x)}{\psi}\right]$$

in B_τ.

The proof is therefore completed.

Theorem 5.6. The mapping $\tilde{T}^{-1} : B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right) \to B_\tau$ is linear.

Proof of this theorem is analogous to that of Theorem 5.2. Details are, thus, avoided.

Theorem 5.7. $\tilde{T}^{-1} : B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right) \to B_\tau$ is continuous with respect to δ convergence.

Proof   Let $\beta_v \to \beta$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$. Then, using Theorem 3.1, there are $\ell_v(n,\tau), \ell(n,\tau) \in B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$ and $(d_v\omega) \in \tilde{E}(\mathtt{R})$ such that $\beta_v = \left[\frac{\ell_v(n,\tau)}{d_n\omega}\right], \beta = \left[\frac{\ell(n,\tau)}{d_n\omega}\right]$ and $\ell_v(n,\tau) \to \ell(n,\tau)$ as $v \to \infty$ in $\mathcal{L}$. Hence, for some $f_v(n,x), f(n,x) \in E'(\mathtt{Y})$, where $\ell_v(n,\tau) = Tf_v(n,\tau), \ell(n,\tau) = Tf(n,\tau)$, we have $f_v(n,x) \to f(n,x)$ as $v \to \infty$ in $E'(\mathtt{Y})$. This implies

$$\left[\frac{f_v(n,x)}{\omega}\right] \to \left[\frac{f(n,x)}{\omega}\right]$$

in B_τ.

That is, $\tilde{T}^{-1}\beta_v \to \tilde{T}^{-1}\beta$ as $v \to \infty$.

This completes the proof of the theorem.

Theorem 5.8. $\tilde{T}^{-1} : B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right) \to B_\tau$ is continuous with respect to Δ convergence.

Proof   Let $\beta_v \to \beta$ as $v \to \infty$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$, then we find $\ell_v(n,\tau) \in B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$, $(d_v\omega) \in \tilde{E}(\mathtt{R})$ such that $(\beta_v - \beta) \curlyvee \omega = \left[\frac{\ell_v(n,\tau) \curlyvee \omega}{\omega}\right]$ and $\ell_v(n,\tau) \to 0$ as $v \to \infty$, $\ell_v(n,\tau) = Tf_v(n,\tau)$, for some $f_v(n,x) \in E'(\mathtt{Y})$. By aid of Lemma 4.1. we get

$$(\beta_v - \beta) \curlyvee \omega = \left[\frac{T(f(n,x) \curlywedge d_m\omega)}{\omega}\right].$$

Employing $\tilde{T}^{-1}$ yields

$$\tilde{T}^{-1}((\beta_v - \beta) \curlyvee \omega) = \left[\frac{f_v(n,x) \curlywedge d_m\omega}{\omega}\right] := f_v(n,x) \to 0,$$

as $v \to \infty$, by [7, Theorem 3.1]. Thus $\tilde{T}^{-1}\beta_v \to \tilde{T}^{-1}\beta$ in B_τ.

The theorem is completely proved.

Theorem 5.9. $\tilde{T} : B_\tau \to B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$ is continuous with respect to $\mathcal{E}$ convergence.

Proof. Let $\beta_\upsilon \xrightarrow{\mathcal{E}} \beta$ in B_τ, then for some $f_\upsilon(n, x), f(n, x) \in E'(\mathtt{Y})$ we have $\beta_\upsilon = \left[\dfrac{f_\upsilon(n, x)}{\omega}\right], \beta = \left[\dfrac{f(n, x)}{\omega}\right]$ and $f_\upsilon(n, x) \to f(n, x)$ as $\upsilon \to \infty$. Hence, applying the Fresnel transform implies $\ell_\upsilon(n, \tau) \to \ell(n, \tau)$ as $\upsilon \to \infty$. Therefore $\ell_\upsilon(n, \tau) \curlyvee \omega \to \ell(n, \tau) \curlyvee \omega$ as $\upsilon \to \infty$, or equivalently,

$$\left[\frac{\ell_\upsilon(n, \tau)}{d_n\omega}\right] \to \left[\frac{\ell(n, \tau)}{d_n\omega}\right]$$

as $\upsilon \to \infty$. Thus $\tilde{T}(\beta_\upsilon) \xrightarrow{\mathcal{E}} \tilde{T}(\beta)$ as $\upsilon \to \infty$. The proof is, therefore, completed.

It can further be observed that :

Theorem 5.10. $\tilde{T} : B_\tau \to B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$ is continuous with respect to Δ convergence.

Proof Let $\beta_\upsilon \xrightarrow{\Delta} \beta$ in B_τ as $\upsilon \to \infty$. Then, we find $f_\upsilon(n, x) \in E'(\mathtt{Y}), \omega \in E(\mathtt{R})$ such that $(\beta_\upsilon - \beta) \curlywedge d_m\omega = \left[\dfrac{f_\upsilon(n, x) \curlywedge d_m\omega}{\omega}\right]$ and $f_\upsilon(n, x) \to 0$ as $n \to \infty$. Hence, we have $\tilde{T}((\beta_\upsilon - \beta) \curlywedge d_m\omega) = \left[\dfrac{T(f_\upsilon(n, x) \curlywedge d_m\omega)}{\omega}\right] = \left[\dfrac{\ell_\upsilon(n, \tau) \curlyvee \omega}{\omega}\right] := \ell_\upsilon(n, \tau) \to 0$ as $\upsilon \to \infty$ in $\mathcal{E}'(\mathtt{Y})$. Therefore

$$\tilde{T}((\beta_\upsilon - \beta) \curlywedge d_m\omega) = \left(\tilde{T}\beta_\upsilon - \tilde{T}\beta\right) \curlyvee \omega \to 0 \text{ as } \upsilon \to \infty.$$

Hence, $\tilde{T}\beta_\upsilon \xrightarrow{\Delta} \tilde{T}\beta$ as $\upsilon \to \infty$ in $B\left(\mathcal{L}, (E, \curlywedge), \tilde{E}, \curlyvee\right)$.

The theorem is proved.

Conflict of Interests The author declares there is no conflict of interests regarding the publication of the paper.

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Resonance Problems for Nonlinear Elliptic Equations with Nonlinear Boundary Conditions

N. Mavinga

Department of Mathematics & Statistics, Swarthmore College,
Swarthmore, PA 19081-1390

E-mail: mavinga@swarthmore.edu

and

M. N. Nkashama

Department of Mathematics, University of Alabama at Birmingham,
Birmingham, AL 35294-1170

E-mail: nkashama@math.uab.edu

Abstract

We study the solvability of nonlinear second order elliptic partial differential equations with nonlinear boundary conditions where we impose asymptotic conditions on both nonlinearities in the differential equation and on the boundary in such a way that resonance occurs at a generalized eigenvalue; which is an eigenvalue of the linear problem in which the spectral parameter is both in the differential equation and on the boundary. The proofs are based on some variational techniques and topological degree arguments.

Keywords: nonlinear elliptic equations, nonlinear boundary conditions, weighted Robin-Neumann-Steklov eigenproblem, resonance conditions.

1   Introduction

In this paper we prove the existence of (weak) solutions to nonlinear second order elliptic partial differential equations with (possibly) nonlinear boundary conditions

$$\begin{cases} -\Delta u + c(x)u = f(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = g(x, u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the nonlinear reaction-function $f(x, u)$ and the nonlinear boundary function $g(x, u)$ interact, in some sense, with the generalized spectrum of the following linear problem (with possibly singular (m, ρ)-weights)

$$\begin{cases} -\Delta u + c(x)u = \mu m(x)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu \rho(x)u & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Notice that the eigenproblem (2) includes as special cases the weighted Steklov eigenproblem (when $m \equiv 0$ and $\rho \not\equiv 0$) which was considered in [3, 4, 6, 17, 24] as well as the weighted Robin-Neumann eigenproblem (when $\rho \equiv 0$ and $m \not\equiv 0$); the latter is also referred to in the literature as Neumann or regular oblique derivative boundary condition (see e.g. [1, 16] and references therein). When $m \not\equiv 0$ and $\rho \not\equiv 0$, we have then the eigenparameter μ both in the differential equation and on the boundary condition, we refer for instance to [5, 7, 8, 18].

Unlike previous results in the literature, what sets our results apart is that we compare both the reaction nonlinearity f in the differential and the boundary nonlinearity g in Eq.(1) with *higher* eigenvalues of the spectrum of problem (2), where the spectral parameter is both in the differential equation and on the boundary (with weights).

The nonlinear problem (1) has received much attention in recent years. Such problem (and its parabolic analog) has been studied in [9, 22] as a model for heat conduction in a body where cooling and heating appear inside and at the boundary at a rate proportional to a power of u. Problem (1) has also been considerably studied by many authors in the framework of sub and super-solutions method. We refer e.g. to [1, 2, 21], and references therein. Since it is based on (the so-called) comparison techniques, the (ordered) sub-super solutions method does not apply when the nonlinearities are compared with higher eigenvalues.

After Landesman-Lazer [14], much work has been devoted to the study of the solvability of elliptic boundary value problems (with linear homogeneous boundary conditions) where the reaction nonlinearity in the differential equation interacts with the eigenvalues of the corresponding linear differential equation with linear homogeneous boundary conditions (resonance and nonresonance problems). For some recent results in this direction we refer e.g. to [11, 12, 13, 19, 20, 23], and references therein.

A few results on a disk ($n = 2$) were obtained in the case of linear elliptic equations with nonlinear boundary conditions, where the nonlinearity on the boundary was compared with the first Steklov eigenvalue (that is, $m \equiv 0$ in Eq.(2)). We refer to Cushing [10] and Klingelhöfer [15] (the results in [15] were significantly generalized to higher dimensions in [1] in the framework of sub and super-solutions method as aforementioned). In [3, 17] the resonance problem for elliptic equations with nonlinear boundary conditions was analyzed using bifurcation theory (see Remark 3.6 herein). More recently, the authors in [19] proved nonresonance results for problem (1) in which the nonlinearities interact, in some sense, only with either the Steklov or the Neumann spectrum. In a very recent paper of one of the authors [18], *nonresonance* results for problem (1) were proved in which both nonlinearities in the differential equation and on the boundary interact, in some sense, with the generalized spectrum of problem (2).

It is our purpose in this paper to establish existence results for problem (1) by imposing asymptotic conditions on both nonlinearities in the differential equation and on the boundary in such a way that *resonance* occurs at a generalized eigenvalue of problem (2). Our results generalize earlier ones in [1, 2, 12, 23], and in some instances some of those in [3, 11, 17] which were obtained in a different setting.

The content of this paper is organized as follows. In Section 2, we state and prove some preliminary results that we shall need in the sequel. Section 3 is devoted to the statement and proof of existence results for Eq.(1) at *resonance*. The proof of the main result is based on variational and topological degree arguments. We conclude the paper with some remarks which show (among other) how our result can be extended to problems with variable

coefficients.

Throughout this paper we assume that Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with boundary $\partial\Omega$ of class $C^{0,1}$, $\partial/\partial\nu$ is the (unit) outer normal derivative. By a weak solution of Eq.(1) we mean a function $u \in H^1(\Omega)$ such that

$$\int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv = \int f(x,u)v + \oint g(x,u)v \quad \text{for all } v \in H^1(\Omega), \quad (3)$$

where $\int$ denotes the (volume) integral on Ω and $\oint$ denotes the (surface) integral on $\partial\Omega$.

Let us mention that $H^1(\Omega)$ denotes the usual real Sobolev space of functions on Ω endowed with the (c,σ)-inner product defined by

$$(u,v)_{(c,\sigma)} = \int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv \qquad (4)$$

with the associated norm denoted by $\|u\|_{(c,\sigma)}$. (The conditions on c and σ which imply that (4) is an inner product are given below.) This norm is equivalent to the standard $H^1(\Omega)$-norm. Besides the Sobolev spaces, we shall make use, in what follows, of the real Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$, and the compactness of the trace operator $\Gamma : H^1(\Omega) \to L^q(\partial\Omega)$ for $1 \leq q < \dfrac{2(n-1)}{n-2}$. (Sometimes we will just use u in place of Γu when considering the trace of a function on $\partial\Omega$.)

The functions $c : \Omega \to \mathbb{R}$, $\sigma : \partial\Omega \to \mathbb{R}$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions.

(C1) $c \in L^p(\Omega)$ with $p \geq n/2$ when $n \geq 3$ ($p > 1$ when $n = 2$) and $\sigma \in L^q(\partial\Omega)$ with $q \geq n-1$ when $n \geq 3$ ($q > 1$ when $n = 2$) with $(c,\sigma) > 0$; that is, $c(x) \geq 0$ a.e. on Ω and $\sigma(x) \geq 0$ a.e. on $\partial\Omega$ such that

$$\int c(x)\,dx + \oint \sigma(x)\,dx \neq 0.$$

(C2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (i.e., measurable in x for each u, and continuous in u for a.e. x).

(C3) There exist constants a_1, $a_2 > 0$ such that for a.e. $x \in \partial\Omega$ and all $u \in \mathbb{R}$,

$$|g(x,u)| \leq a_1 + a_2|u|^s \ \text{ with } 0 \leq s < \frac{n}{n-2}.$$

(C3') There exist constants b_1, $b_2 > 0$ such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$,

$$|f(x,u)| \leq b_1 + b_2|u|^s \ \text{ with } 0 \leq s < \frac{n+2}{n-2}.$$

2 Generalized Eigenproblems and Weighted Nonresonance

To put our results into context, we have collected in this short section some relevant results on generalized linear eigenproblems and *nonresonance* for nonlinear elliptic problem (1) needed for our purposes. We refer to a paper of one of the authors [18] for the proofs of these results.

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Consider the linear problem

$$
\begin{cases}
-\Delta u + c(x)u = \mu m(x)u & \text{in } \Omega, \\
\dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu \rho(x)u & \text{on } \partial\Omega,
\end{cases}
\tag{5}
$$

where $(m, \rho) \in L^p(\Omega) \times L^q(\partial\Omega)$ with p and q as in Section 1, and $(m, \rho) > 0$; that is,

$$
m(x) \geq 0 \text{ a.e. on } \Omega \text{ and } \rho(x) \geq 0 \text{ a.e. on } \partial\Omega \text{ with } \int m(x)\,dx + \oint \rho(x)\,dx \neq 0.
\tag{6}
$$

(We stress the fact that the weight-functions m and ρ may vanish on subsets of positive measure.)

The (generalized) eigenproblem is to find a pair $(\mu, \varphi) \in \mathbb{R} \times H^1(\Omega)$ with $\varphi \not\equiv 0$ such that

$$
\int \nabla\varphi\nabla v + \int c(x)\varphi v + \oint \sigma(x)\varphi v = \mu \left(\int m(x)\varphi v + \oint \rho(x)\varphi v \right) \text{ for all } v \in H^1(\Omega).
\tag{7}
$$

Picking $v = \varphi$ it follows that, if there is such an eigenpair, then one has that $\mu > 0$ and $\int m(x)\varphi^2 + \oint \rho(x)\varphi^2 > 0$. Therefore, one can split the Hilbert space $H^1(\Omega)$ as a direct (c, σ)-orthogonal sum in the following way,

$$
H^1(\Omega) = V_{(m,\rho)}(\Omega) \oplus H^1_{(m,\rho)}(\Omega),
\tag{8}
$$

where $V_{(m,\rho)}(\Omega) := \left\{ u \in H^1(\Omega) : \int m(x)u^2 + \oint \rho(x)u^2 = 0 \right\}$ and $H^1_{(m,\rho)}(\Omega) = \left[V_{(m,\rho)}(\Omega) \right]^\perp$. On $H^1_{(m,\rho)}(\Omega) \subset H^1(\Omega)$, we will also consider the inner product defined by

$$
(u, v)_{(m,\rho)} := \int m(x)uv + \oint \rho(x)uv,
\tag{9}
$$

with corresponding norm denoted by $\| \cdot \|_{(m,\rho)}$ (see e.g. [18] for details).

Assuming that the above assumptions are satisfied, one of the authors [18] proved that, for $n \geq 2$, the eigenproblem (5) has a sequence of real eigenvalues

$$
0 < \mu_1 < \mu_2 \leq \ldots \leq \mu_j \leq \ldots \to \infty, \text{ as } j \to \infty,
$$

each eigenvalue has a finite-dimensional eigenspace. The eigenfunctions φ_j corresponding to these eigenvalues form a complete orthonormal family in the (proper) subspace $H^1_{(m,\rho)}(\Omega)$. Moreover, the first eigenvalue μ_1 is simple, and its associated eigenfunction φ_1 is strictly positive (or strictly negative) in Ω and the following inequalities hold.

(i) For all $u \in H^1(\Omega)$,

$$
\mu_1 \left(\int m(x)u^2 + \oint \rho(x)u^2 \right) \leq \int |\nabla u|^2 + \int c(x)u^2 + \oint \sigma(x)u^2,
\tag{10}
$$

where $\mu_1 > 0$ is the least eigenvalue for Eq.(5). If equality holds in (10), then u is a multiple of an eigenfunction of Eq.(5) corresponding to μ_1.

(ii) For every $v \in \oplus_{i \leq j} E(\mu_i)$, and $w \in \oplus_{i \geq j+1} E(\mu_i)$, we have that

$$||v||^2_{(c,\sigma)} \leq \mu_j ||v||^2_{(m,\rho)} \qquad and \qquad ||w||^2_{(c,\sigma)} \geq \mu_{j+1}||w||^2_{(m,\rho)}, \qquad (11)$$

where $E(\mu_i)$ is the μ_i-eigenspace, $\oplus_{i \leq j} E(\mu_i)$ is the span of eigenfunctions associated with eigenvalues below and up to μ_j, and $|| \cdot ||_{(m,\rho)}$ is the norm induced by (9).

The next theorem concerns an existence result for the nonlinear problem (1) in the case of *weighted nonresonance*. We refer to [18] for the proof of this result.

Theorem 2.1 (Weighted nonresonance between consecutive generalized eigenvalues)
Suppose that the assumptions (C1)-(C3') and (6) are met, and that the following conditions hold.
(C4) There exist constants a, b, α, $\beta \in \mathbb{R}$ such that

$$\alpha m(x) \leq \liminf_{|u| \to \infty} \frac{f(x,u)}{u} \leq \limsup_{|u| \to \infty} \frac{f(x,u)}{u} \leq \beta m(x)$$

and

$$a\rho(x) \leq \liminf_{|u| \to \infty} \frac{g(x,u)}{u} \leq \limsup_{|u| \to \infty} \frac{g(x,u)}{u} \leq b\rho(x),$$

uniformly for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, where

$$\mu_j < \min(a,\alpha) \leq \max(b,\beta) < \mu_{j+1}. \qquad (12)$$

Then, Eq.(1) has at least one (weak) solution $u \in H^1(\Omega)$.

Remark 2.2 The result in Theorem 2.1 remains valid if one replaces the functions $f(x,u)$ and $g(x,u)$ with $f(x,u) + A(x)$ and $g(x,u) + B(x)$ respectively, where $A \in L^2(\Omega)$ and $B \in L^2(\partial\Omega)$.

3   Main Result

In this section, we prove an existence result for problem (1) at *resonance* which includes both the Steklov as well as Neumann and Robin problems with appropriate choices of the weights m and ρ as aforementioned. Notice that the nonlinearity in the boundary condition is at resonance as well. We require that the nonlinearities satisfy some sign conditions and a Landesman-Lazer type condition (possibly at a generalized higher eigenvalue).
Consider the following nonlinear problem

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j \rho(x)u + g(x,u) & \text{on } \partial\Omega, \end{cases} \qquad (13)$$

where μ_j is a generalized eigenvalue of the (weighted) problem (5).

Theorem 3.1 *(Resonance at any generalized eigenvalue)*
Suppose that the assumptions (C1)–(C3') and (6) are met, and that the following conditions hold.

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(C5) *There exists a constant β such that*

$$0 \le \liminf_{|u|\to\infty} \frac{f(x,u)}{u} \le \limsup_{|u|\to\infty} \frac{f(x,u)}{u} \le \beta m(x)$$

and

$$0 \le \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \le \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \le \beta\rho(x),$$

uniformly for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$, *where* $\beta < (\mu_{j+1} - \mu_j)$.

(C6) *Sign conditions: There exist functions* $a, A \in L^2(\Omega)$, $b, B \in L^2(\partial\Omega)$, *and constants* $r < 0 < R$ *such that*

$$f(x,u) \ge A(x) \quad and \quad g(x,u) \ge B(x)$$

for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$ *and all* $u \ge R$,

$$f(x,u) \le a(x) \qquad and \qquad g(x,u) \le b(x)$$

for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$ *and all* $u \le r$.

Then, Eq. (13) has at least one (weak) solution $u \in H^1(\Omega)$ *provided that the following Landesman-Lazer type condition holds:*

$$\int_{\varphi>0} f_+\varphi + \int_{\varphi<0} f_-\varphi + \oint_{\varphi>0} g_+\varphi + \oint_{\varphi<0} g_-\varphi > 0 \quad for\ all\ \varphi \in E(\mu_j) \setminus \{0\}, \qquad (14)$$

where $E(\mu_j)$ *is the* μ_j-*eigenspace,* $f_+(x) := \liminf_{u\to\infty} f(x,u)$, $g_+(x) := \liminf_{u\to\infty} g(x,u)$,

$f_-(x) := \limsup_{u\to-\infty} f(x,u)$, $g_-(x) := \limsup_{u\to-\infty} g(x,u)$ *and,* $\int_{\varphi>0}$ *and* $\oint_{\varphi>0}$ *denote the integrals on the sets* $\{x \in \Omega : \varphi(x) > 0\}$ *and* $\{x \in \partial\Omega : \varphi(x) > 0\}$ *respectively.*

Unlike previous results in the literature, what sets our results apart is that we compare both the reaction nonlinearity f in the differential equation and the boundary nonlinearity g with *higher* eigenvalues of the spectrum of problem (2), where the spectral parameter is both in the differential equation and on the boundary (with possibly singular weights). It should also be noted that the presence of the boundary nonlinearity extends the range of allowable 'forcing' terms in the condition (14). Our results generalize earlier ones in [1, 2, 3, 12, 17, 23] (see Remark 3.6 for details).

We will use variational and topological degree techniques combined with some duality arguments. Before giving a proof of our main result, we first prove several lemmas that are relevant in order to obtain a priori estimates. (For some of these lemmas, we borrow some techniques of proof from [12, 13].)

For $u \in H^1(\Omega)$, we shall write

$$u = u^0 + \bar{u} + \tilde{u} + v,$$

where $u^0 \in E^0 := E(\mu_j)$, $\bar{u} \in \oplus_{i \le j-1} E(\mu_i)$, $\tilde{u} \in \oplus_{i \ge j+1} E(\mu_i)$, and $v \in V_{(m,\rho)}$. Moreover, we shall set

$$w := \tilde{u} + v - \bar{u} - u^0 \quad and \quad u^\perp := \tilde{u} + v + \bar{u}.$$

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Lemma 3.2 *Let $\beta > 0$ be as in Theorem 3.1. Then there exists $\delta = \delta(\beta) > 0$ such that for all $u \in H^1(\Omega)$*

$$\int \nabla u \nabla w + \int c(x) uw + \oint \sigma(x) uw - (\mu_j + \beta) \left(\int m(x) uw + \oint \rho(x) uw \right) \geq \delta \left\| u^\perp \right\|_{(c,\sigma)}^2 .$$

Proof. Let $u \in H^1(\Omega)$, define $D_\beta(u)$ by

$$D_\beta(u) := \int \nabla u \nabla w + \int c(x) uw + \oint \sigma(x) uw - (\mu_j + \beta) \left(\int m(x) uw + \oint \rho(x) uw \right).$$

Taking into account the (c, σ)-orthogonality of $\tilde{u}$, v, $\bar{u}$ and u^0 in $H^1(\Omega)$ and the fact that $v \in V_{(m,\rho)}$ and $u^0 \in E^0$, one has that

$$D_\beta(u) = \|\tilde{u}\|_{(c,\sigma)}^2 + \|v\|_{(c,\sigma)}^2 - \|\bar{u}\|_{(c,\sigma)}^2 - (\mu_j + \beta) \|\tilde{u}\|_{(m,\rho)}^2 + (\mu_j + \beta) \|\bar{u}\|_{(m,\rho)}^2 + \beta \|u^0\|_{(m,\rho)}^2$$

$$\geq \left(\|\tilde{u}\|_{(c,\sigma)}^2 - \frac{(\mu_j + \beta)}{\mu_{j+1}} \|\tilde{u}\|_{(c,\sigma)}^2 \right) + \|v\|_{(c,\sigma)}^2 + \left(\frac{\mu_j}{\mu_{j-1}} \|\bar{u}\|_{(c,\sigma)}^2 - \|\bar{u}\|_{(c,\sigma)}^2 \right)$$

$$\geq \delta \left(\|\tilde{u}\|_{(c,\sigma)}^2 + \|v\|_{(c,\sigma)}^2 + \|\bar{u}\|_{(c,\sigma)}^2 \right) = \delta \left\| u^\perp \right\|_{(c,\sigma)}^2 ,$$

where $\delta = \min \left\{ 1, 1 - \dfrac{\mu_j + \beta}{\mu_{j+1}}, \dfrac{\mu_j}{\mu_{j-1}} - 1 \right\}$. The proof is complete. $\square$

Lemma 3.3 *Let $\beta > 0$ be as in Theorem 3.1, $\delta > 0$ be associated with β by Lemma 3.2, and $\epsilon > 0$. Then for all $\bar{\tau} \in L^p(\Omega)$ and $\tilde{\tau} \in L^q(\partial\Omega)$ satisfying $\bar{\tau}(x) \leq \beta m(x) + \epsilon$, $\tilde{\tau}(x) \leq \beta \rho(x) + \epsilon$ respectively, and all $u \in H^1(\Omega,)$ one has*

$$(u, w)_{(c,\sigma)} - \mu_j \left(\int m(x) uw + \oint \rho(x) uw \right) - \int \bar{\tau}(x) uw + \oint \tilde{\tau}(x) uw \geq C_\delta \left\| u^\perp \right\|_{(c,\sigma)}^2 ,$$

where $C_\delta > 0$ is a constant depending on δ and ϵ, provided that $\epsilon > 0$ is sufficiently small.

Proof. Let $D_\tau(u) := (u, w)_{(c,\sigma)} - \mu_j \left(\int m(x) uw + \oint \rho(x) uw \right) - \int \bar{\tau}(x) uw + \oint \tilde{\tau}(x) uw.$
Using the computations in the proof of Lemma 3.2 we obtain that

$$D_\tau(u) \geq \delta \left\| u^\perp \right\|_{(c,\sigma)}^2 - \epsilon \tilde{K} \left\| u^\perp \right\|_{(c,\sigma)}^2 = (\delta - \epsilon \tilde{K}) \left\| u^\perp \right\|_{(c,\sigma)}^2 = C_\delta \left\| u^\perp \right\|_{(c,\sigma)}^2 ,$$

where $\tilde{K}$ is a constant. If ϵ is sufficiently small then we get that $C_\delta > 0$. The proof is complete $\square$

Lemma 3.4 *Assume (C1)–(C3') and (6) are met, and that (in addition) f and g satisfy the sign-condition (C6). Then, for each real number $K > 0$ there are decompositions*

$$f(x, u) = p_K(x, u) + f_K(x, u) \quad and \quad g(x, u) = q_K(x, u) + g_K(x, u) \tag{15}$$

of f and g such that

$$0 \leq u \, p_K(x, u) \quad and \quad 0 \leq u \, q_K(x, u) \tag{16}$$

for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$, *and all* $u \in \mathbb{R}$. *Moreover, there exist functions* $\bar{\omega} \in L^2(\Omega)$ *and* $\tilde{\omega} \in L^2(\partial\Omega)$ *depending on* a, A, f *and* b, B, g *respectively such that*

$$|f_K(x,u)| \le \bar{\omega}(x) \quad and \quad |g_K(x,u)| \le \tilde{\omega}(x) \tag{17}$$

for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$, *and all* $u \in \mathbb{R}$.

Proof. Given $K > 0$, define $\hat{g}_K(x,u) := \begin{cases} \inf\{g(x,u), K\} & \text{if } u \ge 1, \\ \sup\{g(x,u), K\} & \text{if } u \le -1, \end{cases}$
and $\hat{q}_K(x,u) := g(x,u) - \hat{g}_K(x,u)$ for $x \in \overline{\Omega}$ and $|u| \ge 1$.

Set $q_K(x,u) := \begin{cases} \hat{q}_K(x,u) & \text{if } |u| \ge 1, \\ u\,\hat{q}_K(x, u/|u|) & \text{if } 0 < |u| \le 1, \\ 0 & \text{if } u = 0. \end{cases}$

Finally, define $g_K := g - q_K$. By an easy calculation, one can check that all the conditions of the lemma are satisfied with $\tilde{\omega} = \tilde{C} + \max\{|b(x)|, |B(x)|, K\}$, where the constant $\tilde{C} > 0$ depends on R, $-r$, 1, b_1, b_2. Similar arguments are used in the case of the function f. The proof is complete. $\square$

Lemma 3.5 *Assume (C1)–(C3') and (6) are met, and that in (addition)* f *and* g *satisfy (C5) and (C6). Then, for each real number* $K > 0$, *the functions* p_K *and* q_K *provided by Lemma 3.4 satisfy the following additional conditions*

$$|p_K(x,u)| \le (\beta\,m(x) + \epsilon)|u| - K \quad and \quad |q_K(x,u)| \le (\beta\,\rho(x) + \epsilon)|u| - K \tag{18}$$

for a.e. $x \in \Omega$, *respectively for a.e.* $x \in \partial\Omega$, *and all* $u \in \mathbb{R}$ *with* $|u| \ge \max\{1, \epsilon\}$.

Proof. It follows from (C5) that for all $\epsilon > 0$ there exists $\kappa = \kappa(\epsilon) > 0$ such that

$$|f(x,u)| \le (\beta\,m(x) + \epsilon)|u| \quad and \quad |g(x,u)| \le (\beta\,\rho(x) + \epsilon)|u| \tag{19}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and $u \in \mathbb{R}$ with $|u| \ge \kappa$.

Let $u \in \mathbb{R}$ with $|u| \ge 1$. Then $\hat{g}_K(x,u) := \begin{cases} g(x,u) & \text{if } u \ge 1 \text{ and } g(x,u) \le K, \\ K & \text{if } u \ge 1 \text{ and } g(x,u) \ge K, \\ g(x,u) & \text{if } u \le -1 \text{ and } g(x,u) \ge -K, \\ K & \text{if } u \le -1 \text{ and } g(x,u) \le -K. \end{cases}$

It follows that $q_K(x,u) = \begin{cases} 0 & \text{if } u \ge 1 \quad \text{and} \quad g(x,u) \le K, \\ g(x,u) - K & \text{if } u \ge 1 \quad \text{and} \quad g(x,u) \ge K, \\ 0 & \text{if } u \le -1 \text{ and } g(x,u) \ge -K, \\ g(x,u) + K & \text{if } u \le -1 \text{ and } g(x,u) \le -K. \end{cases}$

By (19) we get that

$$0 \le q_K(x,u) \le (\beta\,\rho(x) + \epsilon)\,u - K \quad \text{if } u \ge \max\{1, \kappa\}$$

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and

$$0 \geq q_K(x, u) \geq (\beta \rho(x) + \epsilon) u + K \quad \text{if } u \leq -\max\{1, \kappa\}.$$

Therefore

$$|q_K(x, u)| \leq (\beta \rho(x) + \epsilon) |u| - K$$

for a.e. $x \in \partial\Omega$ and all $u \in \mathbb{R}$ with $|u| \geq \max\{1, \kappa\}$. Similar arguments are used in the case of f. The proof is complete. $\square$

Proof of Theorem 3.1. The proof is divided into four steps.

Step 1. Let δ be associated to the constant β by Lemma 3.2. Then, by assumption (C5), there exists $\kappa \equiv \kappa_\delta > 0$ such that

$$|f(x, u)| \leq (\beta m(x) + \tilde{d})|u| \qquad \text{and} \qquad |g(x, u)| \leq (\beta \rho(x) + \tilde{d})|u| \tag{20}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \kappa$, where $\tilde{d}$ is a sufficiently small constant such that $0 < \tilde{d} << \delta$. By using Lemma 3.4 with $K = 1$, Eq.(13) is equivalent to

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + p_1(x, u) + f_1(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j \rho(x)u + q_1(x, u) + g_1(x, u) & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where p_1, f_1, q_1 and g_1 are defined in Lemma 3.4 and satisfy conditions (16) and (17). Moreover, since f and g verify the inequalities (20), by Lemma 3.5 we get that

$$|p_1(x, u)| \leq (\beta m(x) + \tilde{d})|u| + 1 \qquad \text{and} \qquad |q_1(x, u)| \leq (\beta \rho(x) + \tilde{d})|u| + 1 \tag{22}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \max\{1, \kappa\}$ (see the construction of p_1 and q_1 in Lemma 3.4). Let us choose $\bar{\kappa} \geq \max\{1, \kappa\}$ so that $(1/|u|) < \tilde{d}$. It follows that

$$0 \leq \frac{p_1(x, u)}{u} \leq \beta m(x) + d \qquad \text{and} \qquad 0 \leq \frac{q_1(x, u)}{u} \leq \beta \rho(x) + d, \tag{23}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$ with $|u| \geq \bar{\kappa}$, where $d = 2\tilde{d} << \delta$.

Step 2. Let us define $\bar{\gamma}, \tilde{\gamma} : \bar{\Omega} \times \mathbb{R}$ by

$$\bar{\gamma}(x, u) = \begin{cases} \dfrac{p_1(x, u)}{u} & \text{for } |u| \geq \bar{\kappa} \\ \dfrac{p_1(x, \bar{\kappa}) + p_1(x, -\bar{\kappa})}{2\bar{\kappa}^2} u + \dfrac{p_1(x, \bar{\kappa}) - p_1(x, -\bar{\kappa})}{2\bar{\kappa}} & \text{for } |u| < \bar{\kappa}. \end{cases}$$

$$\tilde{\gamma}(x, u) = \begin{cases} \dfrac{q_1(x, u)}{u} & \text{for } |u| \geq \bar{\kappa} \\ \dfrac{q_1(x, \bar{\kappa}) + q_1(x, -\bar{\kappa})}{2\bar{\kappa}^2} u + \dfrac{q_1(x, \bar{\kappa}) - q_1(x, -\bar{\kappa})}{2\bar{\kappa}} & \text{for } |u| < \bar{\kappa}. \end{cases}$$

The functions $\bar{\gamma}$ and $\tilde{\gamma}$ are Carathéodory in $\overline{\Omega} \times \mathbb{R}$ since p_1 and q_1 are. Moreover, by (23) one has

$$0 \leq \bar{\gamma}(x, u) \leq \beta m(x) + d \qquad \text{and} \qquad 0 \leq \tilde{\gamma}(x, u) \leq \beta \rho(x) + d, \tag{24}$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$.
Define $\bar{h}, \tilde{h} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\bar{h}(x, u) = f_1(x, u) + p_1(x, u) - \bar{\gamma}(x, u)\, u \qquad \text{and} \qquad \tilde{h}(x, u) = g_1(x, u) + q_1(x, u) - \tilde{\gamma}(x, u)\, u,$$

then it follows from (17) that for some $\bar{\zeta}(x) \in L^2(\Omega)$ and $\tilde{\zeta}(x) \in L^2(\partial\Omega)$,

$$|\bar{h}(x, u)| \leq \bar{\zeta}(x) \quad and \quad |\tilde{h}(x, u)| \leq \tilde{\zeta}(x)$$

for a.e. $x \in \Omega$, respectively for a.e. $x \in \partial\Omega$, and all $u \in \mathbb{R}$, where $\bar{\zeta}, \tilde{\zeta}$ depend on β, m, ρ, $\bar{\kappa}$ and the bounds of f_1 and g_1.
Finally, Eq.(21) is equivalent to

$$\begin{cases} -\Delta u + c(x)u = \mu_j m(x)u + \bar{\gamma}(x, u)\, u + \bar{h}(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u = \mu_j \rho(x)u + \tilde{\gamma}(x, u)\, u + \tilde{h}(x, u) & \text{on } \partial\Omega. \end{cases} \tag{25}$$

We will use the Leray-Schauder Fixed Point Theorem to prove that Eq.(25) has at least one (weak) solution. In order to apply this theorem, we need to show the existence of an *a priori* bound for all possible (weak) solutions of the family of equations

$$\begin{cases} -\Delta u + c(x)u - \mu_j\, m(x)u - (1 - \lambda)d\, m(x))u - \lambda\left[\bar{\gamma}(x, u) + \bar{h}(x, u)\right] = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u - \mu_j\, \rho(x)u - (1 - \lambda)d\, \rho(x)u - \lambda\left[\tilde{\gamma}(x, u) + \tilde{h}(x, u)\right] = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

where $\lambda \in [0, 1]$.
It is clear that for $\lambda = 0$, Eq.(26) has only the trivial weak solution. Now, if u is a (weak) solution of (26) for some $\lambda \in (0, 1]$, it follows from inequalities (24) that

$$(1 - \lambda)d\, m(x) + \lambda\bar{\gamma}(x, u) \leq (\beta + d)m(x) + d \quad \text{and} \quad (1 - \lambda)d\, \rho(x) + \lambda\tilde{\gamma}(x, u) \leq (\beta + d)\rho(x) + d.$$

Therefore, using Lemma 3.3, Hölder inequality, the Sobolev Embedding Theorem, and the continuity of the trace operator, one gets that

$$\begin{aligned} 0 = (u, w)_{(c,\sigma)} &- \mu_j \left(\int m(x)uw + \oint \rho(x)uw \right) - \int \bar{\tau}(x, u)w + \oint \tilde{\tau}(x, u)w \\ &- \lambda \int \bar{h}(x, u)w - \lambda \oint \tilde{h}(x, u)w \\ &\geq (\delta - \tilde{k}\, d) \left\| u^{\perp} \right\|^2_{(c,\sigma)} - \overline{k} \left(\left\| u^{\perp} \right\|_{(c,\sigma)} + \left\| u^0 \right\|_{(c,\sigma)} \right) \\ &\geq C_\delta \left\| u^{\perp} \right\|^2_{(c,\sigma)} - \overline{k} \left(\left\| u^{\perp} \right\|_{(c,\sigma)} + \left\| u^0 \right\|_{(c,\sigma)} \right), \end{aligned}$$

where $w = \tilde{u} + v - \bar{u} - u^0$, $u^{\perp} = \tilde{u} + v + \bar{u}$, $\bar{\tau}(x, u) = (1 - \lambda)d\, m(x) + \lambda\bar{\gamma}(x, u)$ and $\tilde{\tau}(x, u) = (1 - \lambda)d\, \rho(x) + \lambda\tilde{\gamma}(x, u)$. For d sufficiently small, it follows that $C_\delta > 0$. Taking $a = \dfrac{\overline{k}}{2\overline{C_\delta}}$ we get that

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$$\left\|u^{\perp}\right\|_{(c,\sigma)} \le a + \left(a^2 + 2a\left\|u^0\right\|_{(c,\sigma)}\right)^{1/2}. \tag{27}$$

Step 3. We claim that there exists a constant $C > 0$ such that

$$\|u\|_{H^1} < C \tag{28}$$

for any (possible) weak solution $u \in H^1(\Omega)$ of (26) (C is independent of u and λ). If we assume that the claim does not hold, then there exist sequences (λ_n) in the interval $(0, 1]$ and (u_n) in $H^1(\Omega)$ with $\|u_n\|_{H^1} \to \infty$ such that u_n is a (weak) solution of the following problem

$$\begin{cases} -\Delta u + c(x)u - \mu_j\, m(x)u - (1-\lambda_n)d\, m(x))u - \lambda_n\, f(x,u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \sigma(x)u - \mu_j\, \rho(x)u - (1-\lambda_n)d\, \rho(x)u - \lambda_n\, g(x,u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

That is,

$$0 = (u_n, v)_{(c,\sigma)} - \mu_j(u_n, v)_{(m,\rho)} - (1-\lambda_n)\, d\, (u_n, v)_{(m,\rho)} - \lambda_n \int f(x, u_n)v - \lambda_n \oint g(x, u_n)v \tag{30}$$

for every $v \in H^1(\Omega)$. From (27), it follows that

$$\left\|u_n^0\right\|_{(c,\sigma)} \to \infty \quad \text{and} \quad \frac{\left\|u_n^{\perp}\right\|_{(c,\sigma)}}{\left\|u_n^0\right\|_{(c,\sigma)}} \to 0. \tag{31}$$

Therefore, $\dfrac{u_n}{\left\|u_n^0\right\|_{(c,\sigma)}}$ is bounded in $H^1(\Omega)$. By the reflexivity of $H^1(\Omega)$, the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ and the compactness of the trace operator, one can assume (taking a subsequence if it is necessary) that

$$\frac{u_n}{\left\|u_n^0\right\|_{(c,\sigma)}} \rightharpoonup w \text{ in } H^1(\Omega); \quad \frac{u_n}{\left\|u_n^0\right\|_{(c,\sigma)}} \to w \text{ in } L^2(\Omega) \text{ (also in } L^2(\partial\Omega));$$

$$\frac{u_n^0}{\left\|u_n^0\right\|_{(c,\sigma)}} \to w \text{ in } L^2(\Omega) \text{ (also in } L^2(\partial\Omega)).$$

Set $v_n = \dfrac{u_n^0}{\left\|u_n^0\right\|_{(c,\sigma)}}$. Substituting v in (30) by (v_n/λ_n), and taking into account the orthogonality and the fact that $v_n \in E^0$, we get

$$0 \le (1-\lambda_n)\lambda_n^{-1}d \left\|u_n^0\right\|_{(c,\sigma)}^{-1} \left\|u_n^0\right\|_{(m,\rho)}^2 = -\int f(x, u_n)v_n - \oint g(x, u_n)v_n \tag{32}$$

By taking the liminf as $n \to \infty$, we have that

$$\liminf_{n\to\infty} \left(\int f(x, u_n)v_n + \oint g(x, u_n)v_n\right) \le 0. \tag{33}$$

Let $I^+ = \{x \in \Omega : w(x) > 0\}$ and $I^- = \{x \in \Omega : w(x) < 0\}$. Then for a.e. $x \in I^+$ there exists $\nu(x) \in \mathbb{N}$ such that for all $n \geq \nu(x)$, one has (passing to a subsequence if necessary),

$$\left|u_n^\perp(x)\right| \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} < \frac{1}{4} w(x)$$

and

$$\left|u_n^0(x) \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} - w(x)\right| < \frac{1}{4} w(x).$$

Therefore, for all $n \geq \nu(x)$ one has

$$u_n(x) \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} = (u_n^0(x) + u_n^\perp(x))(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \geq (u_n^0(x) - |u_n^\perp(x)|)(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \geq \frac{1}{2} w(x).$$

It follows that for a.e. $x \in I^+$, there exists an integer $\nu(x) \in \mathbb{N}$ such that for all $n \geq \nu(x)$,

$$v_n(x) > 0 \quad \text{and} \quad u_n(x) \geq \frac{1}{2} w(x)(\left\|u_n^0\right\|_{(c,\sigma)}) \to \infty \quad (\text{since } \left\|u_n^0\right\|_{(c,\sigma)}) \to \infty).$$

On the other hand, for a.e. $x \in I^-$ there exists $\vartheta(x) \in \mathbb{N}$ such that for all $n \geq \vartheta(x)$,

$$u_n(x) \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} = (u_n^0(x) + u_n^\perp(x))(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \leq (u_n^0(x) + |u_n^\perp(x)|)(\left\|u_n^0\right\|_{(c,\sigma)})^{-1} \leq \frac{1}{2} w(x).$$

Therefore, for $n \geq \vartheta(x)$, $u_n(x) \leq \frac{1}{2} w(x)(\left\|u_n^0\right\|_{(c,\sigma)}) \to -\infty$.

In order to apply Fatou's Lemma we need to find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$f(x, u_n) v_n \geq \bar{l}(x) \text{ a.e.} \quad \text{and} \quad g(x, u_n) v_n \geq \tilde{l}(x) \text{ a.e.}, \tag{34}$$

for some $\bar{l} \in L^1(\Omega)$ and $\tilde{l} \in L^1(\partial\Omega)$. Indeed, from (27) one gets

$$\left\|u_n^\perp\right\|_{(c,\sigma)}^2 \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \leq 2a \left\|u_n^\perp\right\|_{(c,\sigma)} \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} + 2a.$$

Therefore, by (31) one has that for $n \geq n_0$, $\left\|u_n^\perp\right\|_{(c,\sigma)}^2 \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \leq C$, where C is a constant independent of n. Since $\bar{\gamma}(x, u_n(x)) \geq 0$, one has that for $n \geq n_0$,

$$\bar{\gamma}(x, u_n(x))u_n(x)v_n(x) = \bar{\gamma}(x, u_n(x))u_n(x)u_n^0(x) \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1}$$

$$= \frac{1}{2}\bar{\gamma}(x, u_n(x)) \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} [(u_n(x))^2 + (u_n^0(x))^2 - (u_n(x) - u_n^0(x))^2]$$

$$\geq -\frac{1}{2}\bar{\gamma}(x, u_n(x))(u_n^\perp(x))^2 \left(\left\|u_n^0\right\|_{(c,\sigma)}\right)^{-1} \geq -\bar{C}\bar{\gamma}(x, u_n(x)) \, l_1(x),$$

where $l_1 \in L^1(\Omega)$, and $\bar{C} > 0$ are independent of n. Therefore, for $n \geq n_0$,

$$\bar{\gamma}(x, u_n(x))u_n(x)v_n(x) \geq -\bar{C}(\beta \, m(x) + d) \, l_1(x).$$

Now, using the decomposition of f, one has that for $n \geq n_0$,

$$f(x, u_n(x))v_n(x) = \bar{\gamma}(x, u_n(x))u_n(x)v_n(x) + \bar{h}(x, u_n(x))v_n(x)$$
$$\geq -\bar{C}(\beta \, m(x) + d) \, l_1(x) - K_1 \, l_2(x) = \bar{l}(x),$$

where $l_2 \in L^1(\Omega)$. We use similar arguments to obtain the function $\tilde{l}$ in (34). Notice that it follows from (32) and (34) that $\sup \int f(x, u_n)v_n < \infty$ and $\sup \oint g(x, u_n)v_n < \infty$. Therefore, by Fatou's Lemma and the properties of lim inf, one has

$$\int_{w>0} f_+ w \le \liminf_{n\to\infty} \int_{w>0} f(x, u_n)v_n; \quad \oint_{w>0} g_+ w \le \liminf_{n\to\infty} \oint_{w>0} g(x, u_n)v_n$$

$$\int_{w<0} f_- w \le \liminf_{n\to\infty} \int_{w<0} f(x, u_n)v_n; \quad \oint_{w<0} g_- w \le \liminf_{n\to\infty} \oint_{w<0} g(x, u_n)v_n,$$

and that

$$\int_{w>0} f_+ w + \oint_{w>0} g_+ w + \int_{w<0} f_- w + \oint_{w<0} g_- w \le 0,$$

which contradicts the assumption (14). Thus the claim holds.

Step 4. We use the Leray-Schauder Fixed Point Theorem combined with some duality arguments.

Define $T : H^1(\Omega) \to (H^1(\Omega))^*$ by

$$T(u)v = \int \nabla u \nabla v + \int c(x)uv + \oint \sigma(x)uv - (\mu_j + d)\left(\int m(x)uw + \oint \rho(x)uw\right).$$

It follows from Theorem 2.1 and Remark 2.2 that T is linear, continuous and bijective. Therefore, by the Open Mapping Theorem we have that T^{-1} is continuous. From (26) one sees that

$$0 = T(u)v - \lambda\left(\int (f(x, u) + d\, m(x)u)v + \oint (g(x, u) + d\, \rho(x)u)v\right)$$

where $\lambda \in [0, 1]$. Applying T^{-1} we get $0 = u - \lambda[T^{-1}J'_f(u) - T^{-1}J'_g(u)]$ where $J'_f(u)v = \int (f(x, u) + d\, m(x)u)v$, $J'_g(u)v = \oint (g(x, u) + d\, \rho(x)u)v$. Now, let M be defined by

$$Mu := T^{-1}J'_f(u) - T^{-1}J'_g(u).$$

Notice that from the continuity of T^{-1} and the compactness of J'_f and J'_g (see [19]) we have that M is a compact operator from $H^1(\Omega)$ to itself. Therefore, one sees that $u - \lambda Mu = 0$. It follows from the a priori estimate (28) and the Leray-Schauder Fixed Point Theorem that M has a fixed point. Thus, Problem (13) has a (weak) solution. The proof is complete. $\square$

Remark 3.6 We (briefly) indicate how some of our results extend previous ones in the literature.

(i) In [3] no reaction term f is considered and the nonlinear perturbation g is sublinear at infinity.

(ii) In [17] the p-Laplacian is considered and the nonlinear perturbations f and g are bounded.

Here, we consider the case $p = 2$ and the nonlinear perturbations f and g may be unbounded, with at most linear growth asymptotically.

Remark 3.7 If the boundary nonlinearity g is Lipschitz in u, uniformly in x and the functions $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in C^1(\bar{\Omega})$, one can show with a slight modification of the proof that the solution obtained in Theorem 3.1 is actually in $H^2(\Omega)$.

Remark 3.8 Our resonance results remain valid if one considers nonlinear equations with a more general linear part (in divergence form) with variable coefficients:

$$
\begin{cases}
-\sum_{i,j=1}^n \dfrac{\partial}{\partial x_j}\left(a_{ij}(x)\dfrac{\partial u}{\partial x_i}\right) + c(x)u = f(x, u) & \text{in } \Omega, \\
\dfrac{\partial u}{\partial \nu} + \sigma(x)u = g(x, u) & \text{on } \partial\Omega,
\end{cases}
\tag{35}
$$

where $\sigma \in L^\infty(\partial\Omega)$ with $\sigma(x) \geq 0$ a.e. on $\partial\Omega$, and $\partial/\partial\nu := \nu \cdot A\nabla$ is the (unit) outer conormal derivative. The matrix $A(x) := \big(a_{ij}(x)\big)$ is symmetric with $a_{ij} \in L^\infty(\Omega)$ such that there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$
\langle A(x)\xi, \xi \rangle \geq \gamma |\xi|^2 \quad \text{a.e. on } \Omega.
$$

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Approximation by Complex Generalized Discrete Singular Operators

George A. Anastassiou & Merve Kester

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A.
`ganastss@memphis.edu`
`mkester@memphis.edu`

Abstract. In this article, we work on the general complex-valued discrete singular operators over the real line regarding their convergence to the unit operator with rates in the L_p norm for $1 \leq p \leq \infty$. The related established inequalities contain the higher order L_p modulus of smoothness of the engaged function or its higher order derivative. Also we study the complex-valued fractional generalized discrete singular operators on the real line, regarding their convergence to the unit operator with rates in the uniform norm. The related established inequalities involve the higher order moduli of smoothness of the associated right and left Caputo fractional derivatives of the related function.

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Key Words and Phrases: general singular integral, fractional singular integral, discrete singular operator, modulus of smoothness, fractional derivative, complex valued function.

1 Background

In [5], Chapter 19, the authors studied the approximation propeties of the general complex- valued singular integral operators $\Theta_{r,\xi}(f;x)$ defined as follows:

They considered the complex valued Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + i f_2$, $i := \sqrt{-1}$. Here $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are implied to be real valued Borel measurable functions.

Let $\xi > 0$ and μ_ξ be a Borel probability measure on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ they put

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-n}, & j = 0. \end{cases} \tag{1}$$

Then, they defined the general complex-valued singular integral operators as

$$\Theta_{r,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) d\mu_\xi(t), \forall \xi > 0. \tag{2}$$

Clearly by the definition of R.H.S.(2) they had

$$\Theta_{r,\xi}(f;x) = \Theta_{r,\xi}(f_1;x) + i\Theta_{r,\xi}(f_2;x). \tag{3}$$

They supposed that $\Theta_{r,\xi}(f_j;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $j = 1, 2$.

In [5], Chapter 19, the authors defined the modulus of smoothness finite as follows:

Let $f_1, f_2 \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ with the rth modulus of smoothness finite, that is

$$\omega_r(f_{\widetilde{j}}^{(n)}, h) := \sup_{|t| \le h} \|\Delta_t^r f_{\widetilde{j}}^{(n)}(x)\|_{\infty,x} < \infty, \tag{4}$$

$h > 0$, where

$$\Delta_t^r f_{\widetilde{j}}^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f_{\widetilde{j}}^{(n)}(x + jt), \tag{5}$$

$\widetilde{j} = 1, 2$.

In [5], Chapter 19, The authors also defined the rth L_p-modulus of smoothness for f_1, $f_2 \in C^n(\mathbb{R})$ with $f_1^{(n)}, f_2^{(n)} \in L_p(\mathbb{R})$, $1 \le p < \infty$ as

$$\omega_r(f_{\widetilde{j}}^{(n)}, h)_p := \sup_{|t| \le h} \|\Delta_t^r f_{\widetilde{j}}^{(n)}(x)\|_{p,x}, \quad h > 0, \text{ with } \widetilde{j} = 1, 2. \tag{6}$$

There they supposed that $\omega_r(f_{\widetilde{j}}^{(n)}, h)_p < \infty$, $h > 0$, $\widetilde{j} = 1, 2$.

They denoted

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}.$$

They assumed that the integrals

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$$

are finite, $k = 1, \ldots, n$.

They noticed the inequalities

$$|\Theta_{r,\xi}(f;x) - f(x)| \le |\Theta_{r,\xi}(f_1;x) - f_1(x)| + |\Theta_{r,\xi}(f_2;x) - f_2(x)|, \tag{7}$$

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$$\|\Theta_{r,\xi}(f;x) - f(x)\|_{\infty,x} \le \|\Theta_{r,\xi}(f_1;x) - f_1(x)\|_{\infty,x} + \|\Theta_{r,\xi}(f_2;x) - f_2(x)\|_{\infty,x},$$
(8)

and

$$\|\Theta_{r,\xi}(f;x) - f(x)\|_{p,x} \le \|\Theta_{r,\xi}(f_1;x) - f_1(x)\|_{p,x} + \|\Theta_{r,\xi}(f_2;x) - f_2(x)\|_{p,x}, p \ge 1.$$
(9)

Furthermore, they stated that the equality

$$f^{(k)}(x) = f_1^{(k)}(x) + i f_2^{(k)}(x),$$
(10)

holds for all $k = 1, \dots, n$.

The authors stated

Theorem 1 ([5] p.365) *Let* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$.
Let $f_j \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$. *Assume that*

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Suppose also that $\omega_r(f_j^{(n)}, h) < \infty, \forall h > 0$. *Then*

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x}$$
(11)

$$\le \frac{1}{n!} \left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \left(\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi) \right).$$

When $n = 0$ *the sum in L.H.S.* (11) *collapses.*

They gave their results for the $n = 0$ case as follows.

Corollary 2 ([5] p.366) *Let* $f : \mathbb{R} \to \mathbb{C} : f = f_1 + if_2$. *Here* $j = 1, 2$. *Let*
$f_j \in C(\mathbb{R})$. *Assume*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Suppose also $\omega_r(f_j, h) < \infty, \forall h > 0$. *Then*

$$\|\Theta_{r,\xi}(f) - f\|_\infty \le \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \left(\omega_r(f_1, \xi) + \omega_r(f_2, \xi) \right).$$
(12)

Next they stated

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Theorem 3 ([5] p.366) *Let $g \in C^{n-1}(\mathbb{R})$, such that $g^{(n)}$ exists, $n, r \in \mathbb{N}$. Furthermore assume that for each $x \in \mathbb{R}$ the function $g^{(\widetilde{j})}(x + jt) \in L_1(\mathbb{R}, \mu_\xi)$ as a function of t, for all $\widetilde{j} = 0, 1, \ldots, n-1$; $j = 1, \ldots, r$. Suppose that there exist $\lambda_{\widetilde{j}, j} \geq 0$, $\widetilde{j} = 1, \ldots, n; j = 1, \ldots, r$, with $\lambda_{\widetilde{j}, j} \in L_1(\mathbb{R}, \mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$|g^{(\widetilde{j})}(x + jt)| \leq \lambda_{\widetilde{j}, j}(t), \tag{13}$$

for μ_ξ−almost all $t \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, 2, \ldots, r$. Then $g^{(\widetilde{j})}(x + jt)$ defines a μ_ξ−integrable function with respect to t for each $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, and

$$(\Theta_{r,\xi}(g; x))^{(\widetilde{j})} = \Theta_{r,\xi}\left(g^{(\widetilde{j})}; x\right), \tag{14}$$

for all $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$.

They presented the following approximation result

Theorem 4 ([5] p.366) *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^{n+\rho}(\mathbb{R})$, $n, \rho \in \mathbb{Z}^+$, and $\omega_r(f_j^{(n+\widetilde{j})}, h) < \infty, \forall h > 0$, for $\widetilde{j} = 0, 1, \ldots, \rho$. Assume*

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

We consider the assumptions of Theorem 3 valid regarding f_1, f_2 for $n = \rho$. Then

$$\left\|(\Theta_{r,\xi}(f; x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(k+\widetilde{j})}(x)}{k!} \delta_k c_{k,\xi}\right\|_{\infty, x} \tag{15}$$

$$\leq \frac{1}{n!}\left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t)\right]\left(\omega_r(f_1^{(n+\widetilde{j})}, \xi) + \omega_r(f_2^{(n+\widetilde{j})}, \xi)\right),$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$. When $n = 0$ the sum in L.H.S.(15) collapses.

They started to present their L_p results with the following theorem

Theorem 5 ([5] p.367) *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$\int_{-\infty}^{\infty}\left[\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right]|t|^{np-1} d\mu_\xi(t) < \infty,$$

and $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. Then

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{p,x} \tag{16}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_{\xi}(t) \right]^{\frac{1}{p}}$$
$$\cdot \xi^{\frac{1}{p}} \left(\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p \right).$$

For the case of $p = 1$, they obtained

Theorem 6 ([5] p.368)*Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$. Suppose that*

$$\int_{-\infty}^{\infty} \left[\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right] |t|^{n-1} d\mu_{\xi}(t) < \infty,$$

and $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. Then

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{1,x} \tag{17}$$

$$\leq \frac{1}{(n-1)!(r+1)} \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_{\xi}(t) \right]$$
$$\cdot \xi \left(\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1 \right).$$

For $n = 0$, they showed

Proposition 7 ([5] p.368) *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^{rp} d\mu_{\xi}(t) < \infty.$$

Then

$$\| \Theta_{r,\xi}(f) - f \|_p \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^{rp} d\mu_{\xi}(t) \right]^{\frac{1}{p}} (\omega_r(f_1, \xi)_p + \omega_r(f_2, \xi)_p). \tag{18}$$

Finally, they gave the case of $n = 0$, $p = 1$ as

Proposition 8 ([5] p.368) *Let* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + i f_2$, *where* $f_1, f_2 \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$. *Suppose*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_1 \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t)\right] (\omega_r(f_1, \xi)_1 + \omega_r(f_2, \xi)_1). \quad (19)$$

Next, the authors demonstrated their simultenous L_p approximation results. They started with

Theorem 9 ([5] p.369) *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + i f_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}, \rho \in \mathbb{Z}^+$, *with* $f_j^{(n+\widetilde{j})} \in L_p(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots, \rho$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Suppose that* $\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right) |t|^{np-1} d\mu_\xi(t) < \infty$, *and* $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. *We consider the assumptions of Theorem 3 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\|(\Theta_{r,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(k+\widetilde{j})}(x)}{k!} \delta_k c_{k,\xi}\right\|_{p,x} \leq \frac{1}{((n-1)!)} \quad (20)$$

$$\frac{1}{(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_p + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_p\right)$$

$$\cdot \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right) |t|^{np-1} d\mu_\xi(t)\right]^{\frac{1}{p}} \xi^{\frac{1}{p}},$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.

They had

Proposition 10 ([5] p.369) *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + i f_2$. *Let* $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Suppose that*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) < \infty.$$

We consider the assumptions of Theorem 3 valid regarding f_1, f_2 *for* $n = \rho$. *Then*

$$\left\|(\Theta_{r,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})}\right\|_p \quad (21)$$

$$\leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t)\right]^{\frac{1}{p}} \left(\omega_r(f_1^{(\widetilde{j})}, \xi)_p + \omega_r(f_2^{(\widetilde{j})}, \xi)_p\right),$$

$\widetilde{j} = 0, 1, \ldots, \rho$.

Next they gave the related L_1 result as

Theorem 11 ([5] p.369) *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + i f_2$, *and* $j = 1, 2$.
Let $f_1, f_2 \in C^{n+\rho}(\mathbb{R})$, *with* $f_j^{(n+\widetilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$.
Suppose that

$$\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) < \infty,$$

and $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. *We consider the assumptions of Theorem 3 valid
regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\| (\Theta_{r,\xi}(f; x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(k+\widetilde{j})}(x)}{k!} \delta_k c_{k,\xi} \right\|_{1,x} \tag{22}$$

$$\leq \frac{1}{(n-1)!(r+1)} \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) \right] \xi$$

$$\cdot \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_1 \right),$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.

Their last simultaneous approximation result follows

Proposition 12 ([5] p.370) *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + i f_2$. *Here*
$f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *Suppose*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^{r} d\mu_\xi(t) < \infty.$$

We consider the assumptions of Theorem 3 valid regarding f_1, f_2 *for* $n = \rho$.
Then

$$\left\| (\Theta_{r,\xi}(f))^{(j)} - f^{(j)} \right\|_1 \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^{r} d\mu_\xi(t) \right] \left(\omega_r(f_1^{(j)}, \xi)_1 + \omega_r(f_2^{(j)}, \xi)_1 \right),$$

$$\tag{23}$$

for all $j = 0, 1, \ldots, \rho$.

Next in [5], Chapter 19, the authors defined the generalized complex frac-
tional integral operators as follows:
Let $\xi > 0$, $x, x_0 \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$ Borel measurable, such that $f = f_1 + i f_2$. Sup-
pose $f_1, f_2 \in C^m(\mathbb{R})$, with $\left\| f_1^{(m)} \right\|_\infty < \infty$, $\left\| f_2^{(m)} \right\|_\infty < \infty$. Let μ_ξ probability

Borel measure on $\mathbb{R}$, $\forall \xi > 0$. Then

$$\overline{\Theta}_{r,\xi}(f;x) \quad : \quad = \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f(x+jt) \right) d\mu_\xi(t) \tag{24}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_1(x+jt) \right) d\mu_\xi(t) + i \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_2(x+jt) \right) d\mu_\xi(t)$$

$$= \overline{\Theta}_{r,\xi}(f_1;x) + i\overline{\Theta}_{r,\xi}(f_2;x),$$

where

$$\overline{\alpha}_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-\gamma}, & j = 1, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-\gamma}, & j = 0. \end{cases} \tag{25}$$

for $r \in \mathbb{N}$ and $\gamma > 0$.

Additionally, they denoted

$$\overline{\delta}_k = \sum_{j=1}^{r} \alpha_j j^k, k = 1, \ldots, m-1, \tag{26}$$

where $m = \lceil \gamma \rceil$, $\lceil \cdot \rceil$ is the ceiling of a number.

They supposed here that $\overline{\Theta}_{r,\xi}(f_1,x), \overline{\Theta}_{r,\xi}(f_2,x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.
   The authors also assumed existence of $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, m-1$,
and the existence of $\int_{-\infty}^{\infty} |t|^{\gamma+k} d\mu_\xi(t)$, $k = 0, 1, \ldots, r$.

   Finally, they gave their fractional result as

Theorem 13 ([5] p.371) *It holds*

$$\left\| \overline{\Theta}_{r,\xi}(f,\cdot) - f(\cdot) - \sum_{k=1}^{m-1} \frac{f^{(k)}(\cdot)}{k!} \overline{\delta}_k c_{k,\xi} \right\|_\infty \tag{27}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k} \int_{-\infty}^{\infty} |t|^{\gamma+k} d\mu_\xi(t) \right]$$

$$\cdot \left(\sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_1, \xi \right), \omega_r \left(D_{*x}^\gamma f_1, \xi \right) \right] \right\} \right.$$

$$\left. + \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_2, \xi \right), \omega_r \left(D_{*x}^\gamma f_2, \xi \right) \right] \right\} \right).$$

If $m = 1$ the sum disappears in L.H.S.(27).

   Let $j = 1, 2$. Above $D_{x_0-}^\gamma f_j$ is the right Caputo fractional derivative of order $\gamma > 0$ is given by

$$D_{x_0-}^\gamma f_j(x) := \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^{x_0} (\zeta-x)^{m-\gamma-1} f_j^{(m)}(\zeta)d\zeta, \tag{28}$$

$\forall x \leq x_0 \in \mathbb{R}$ *fixed*.

They supposed $D_{x_0-}^{\gamma} f(x) = 0, \forall x > x_0$.

Also $D_{*x_0}^{\gamma} f_j$ is the left Caputo fractional derivative of order $\gamma > 0$ is given by

$$D_{*x_0}^{\gamma} f_j(x) := \frac{1}{\Gamma(m-\gamma)} \int_{x_0}^{x} (x-t)^{m-\gamma-1} f_j^{(m)}(t)dt, \qquad (29)$$

$\forall x \geq x_0 \in \mathbb{R}$ fixed, where $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt, \nu > 0$.

They assumed $D_{*x_0}^{\gamma} f_j(x) = 0$, for $x < x_0$.

On the other hand, in [1], the authors defined important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ as follows :

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \qquad (30)$$

they defined the generalized discrete Picard operators as

$$P_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \qquad (31)$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \qquad (32)$$

they defined the generalized discrete Gauss-Weierstrass operators as

$$W_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \qquad (33)$$

iii) Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}, \qquad (34)$$

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they defined the generalized discrete Poisson-Cauchy operators as

$$
Q_{r,\xi}^{*}\left(f;x\right) := \frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty}\left(\nu^{2\alpha}+\xi^{2\alpha}\right)^{-\beta}}. \tag{35}
$$

They observed that for c constant they have

$$
P_{r,\xi}^{*}\left(c;x\right) = W_{r,\xi}^{*}\left(c;x\right) = Q_{r,\xi}^{*}\left(c;x\right) = c. \tag{36}
$$

They assumed that the operators $P_{r,\xi}^{*}\left(f;x\right)$, $W_{r,\xi}^{*}\left(f;x\right)$, and $Q_{r,\xi}^{*}\left(f;x\right) \in \mathbb{R}$, for $x \in \mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}} < \infty$.

iv) When

$$
\mu_{\xi}(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}, \tag{37}
$$

they defined the generalized discrete non-unitary Picard operators as

$$
P_{r,\xi}\left(f;x\right) := \frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}. \tag{38}
$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$
m_{\xi,P} := \frac{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}}. \tag{39}
$$

They observed that

$$
\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{\frac{-|\nu|}{\xi}}}, \tag{40}
$$

which is the probability measure (30) defining the operators $P_{r,\xi}^{*}$.

v) When

$$
\mu_{\xi}(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^{2}}{\xi}}}{\sqrt{\pi\xi}\left(1-\mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}, \tag{41}
$$

with $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^{2}}dt$, $\mathrm{erf}(\infty) = 1$, they defined the generalized discrete non-unitary Gauss-Weierstrass operators as

$$
W_{r,\xi}\left(f;x\right) := \frac{\sum\limits_{\nu=-\infty}^{\infty}\left(\sum\limits_{j=0}^{r}\alpha_{j}f(x+j\nu)\right)e^{\frac{-\nu^{2}}{\xi}}}{\sqrt{\pi\xi}\left(1-\mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1}. \tag{42}
$$

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Here $\mu_{\xi,W}(\nu)$ has mass

$$m_{\xi,W} := \frac{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{43}$$

They observed that

$$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{44}$$

which is the probability measure (32) defining the operators $W_{r,\xi}^*$.
The authors observed that $P_{r,\xi}(f;x)$, $W_{r,\xi}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$.

In [1], for $k = 1, ..., n$, the authors defined the ratios of sums

$$c_{k,\xi}^* := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{45}$$

$$p_{k,\xi}^* := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{46}$$

and for $\alpha \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, they introduced

$$q_{k,\xi}^* := \frac{\sum_{\nu=-\infty}^{\infty} \nu^k \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{47}$$

Furthermore, they proved that these ratios of sums $c_{k,\xi}^*$, $p_{k,\xi}^*$, and $q_{k,\xi}^*$ are finite
for all $\xi \in (0, 1]$.

In [1], the authors also proved

$$m_{\xi,P} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \to 1 \text{ as } \xi \to 0^+ \tag{48}$$

and

$$m_{\xi,W} = \frac{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}}{1 + \sqrt{\pi\xi}\left(1 - \text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)} \to 1 \text{ as } \xi \to 0^+. \tag{49}$$

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The authors introduced also

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{50}$$

Additionally, in [1], the authors defined the following error quantities:

$$E_{0,P}(f,x) \quad : \quad = P_{r,\xi}(f;x) - f(x) \tag{51}$$

$$= \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} - f(x),$$

$$E_{0,W}(f,x) \quad : \quad = W_{r,\xi}(f;x) - f(x) \tag{52}$$

$$= \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} - f(x).$$

Furthermore, they introduced the errors $(n \in \mathbb{N})$:

$$E_{n,P}(f,x) := P_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \tag{53}$$

and

$$E_{n,W}(f,x) := W_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1}. \tag{54}$$

Next, they obtained the inequalities

$$|E_{0,P}(f,x)| \leq m_{\xi,P} \left| P_{r,\xi}^{*}(f;x) - f(x) \right| + |f(x)| \left| m_{\xi,P} - 1 \right|, \tag{55}$$

$$|E_{0,W}(f,x)| \leq m_{\xi,W} \left| W_{r,\xi}^{*}(f;x) - f(x) \right| + |f(x)| \left| m_{\xi,W} - 1 \right|, \tag{56}$$

and

$$|E_{n,P}(f,x)| \tag{57}$$

$$\leq \quad m_{\xi,P} \left| P_{r,\xi}^{*}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^{*} \right| + |f(x)| \left| m_{\xi,P} - 1 \right|,$$

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with

$$|E_{n,W}(f,x)| \tag{58}$$

$$\leq m_{\xi,W} \left| W_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^* \right| + |f(x)| \, |m_{\xi,W} - 1| \, .$$

In [1], they first gave the following simultaneous approximation results for unitary operators. They showed

Theorem 14 *Let* $f \in C^n(\mathbb{R})$ *with* $f^{(n)} \in C_u(\mathbb{R})$ *(uniformly continuous functions on* $\mathbb{R}$*).*
i) *For* $n \in \mathbb{N}$,

$$\left\| P_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_{\infty,x} \tag{59}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right),$$

and

$$\left\| W_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_{\infty,x} \tag{60}$$

$$\leq \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right).$$

ii) *For* $n = 0$,

$$\left\| P_{r,\xi}^*(f;x) - f(x) \right\|_{\infty,x} \leq \omega_r(f,\xi) \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right), \tag{61}$$

and

$$\left\| W_{r,\xi}^*(f;x) - f(x) \right\|_{\infty,x} \leq \omega_r(f,\xi) \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right). \tag{62}$$

In the above inequalities (59) *-* (62)*, the ratios of sums in their right hand sides* $(R.H.S.)$ *are uniformly bounded with respect to* $\xi \in (0,1]$*.*

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In [1], they had also

Theorem 15 *Let* $f \in C^n(\mathbb{R})$ *with* $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, *and* $\beta > \frac{n+r+1}{2\alpha}$.
i) *For* $n \in \mathbb{N}$,

$$\left\| Q_{r,\xi}^* (f;x) - f(x) - \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_{\infty,x} \tag{63}$$

$$\leq \quad \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \right).$$

ii) *For* $n = 0$,

$$\left\| Q_{r,\xi}^* (f;x) - f(x) \right\|_{\infty,x} \leq \omega_r(f,\xi) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \right). \tag{64}$$

In the above inequalities (63) - (64), *the ratios of sums in their R.H.S. are uniformly bounded with respect to* $\xi \in (0,1]$.

Next, they stated their results in [1] for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$. They had

Corollary 16 *Let* $f \in C_u(\mathbb{R})$. *Then*

i)

$$|E_{0,P}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|)e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) + |f(x)| \, |m_{\xi,P} - 1|, \tag{65}$$

ii)

$$|E_{0,W}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \omega_r(f,|\nu|)e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right) + |f(x)| \, |m_{\xi,W} - 1|. \tag{66}$$

In [1], for $E_{n,P}$ and $E_{n,W}$, the authors presented

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Theorem 17 *Let $f \in C^n(\mathbb{R})$ with $f^{(n)} \in C_u(\mathbb{R})$, $n \in \mathbb{N}$, and $\|f\|_{\infty,\mathbb{R}} < \infty$. Then*

i)

$$\|E_{n,P}(f,x)\|_{\infty,x} \tag{67}$$

$$\leq \quad \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right) + \|f\|_{\infty,\mathbb{R}} |m_{\xi,P}-1|,$$

ii)

$$\|E_{n,W}(f,x)\|_{\infty,x} \tag{68}$$

$$\leq \quad \frac{\omega_r(f^{(n)},\xi)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1+\frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \right) + \|f\|_{\infty,\mathbb{R}} |m_{\xi,W}-1|.$$

In the above inequalities (67) - (68), the ratios of sums in their R.H.S. are uniformly bounded with respect to $\xi \in (0,1]$.

In [2], the authors represented simultaneous L_p approximation results. They started with

Theorem 18 i) *Let $f \in C^n(\mathbb{R})$, with $f^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p,q > 1 : \frac{1}{p}+\frac{1}{q}=1$, and rest as above in this section. Then*

$$\left\| P_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_p \tag{69}$$

$$\leq \quad \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(M_{p,\xi}^*\right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p$$

where

$$M_{p,\xi}^* := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1\right) |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{70}$$

which is uniformly bounded for all $\xi \in (0,1]$.
Additionally, as $\xi \to 0^+$ we obtain that R.H.S. of (69) goes to zero.
ii) *When $p = 1$, let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| P_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_1 \tag{71}$$

$$\leq \quad \frac{1}{(n-1)!(r+1)} M_{1,\xi}^* \xi \omega_r(f^{(n)},\xi)_1$$

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holds where $M^*_{1,\xi}$ is defined as in (70). Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (71) goes to zero.

iii) When $n = 0$, let $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above in this section. Then

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_p \leq \left(\bar{M}^*_{p,\xi} \right)^{1/p} \omega_r(f, \xi)_p \tag{72}$$

where

$$\bar{M}^*_{p,\xi} := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} e^{\frac{-|\nu|}{\xi}}}{\displaystyle\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{73}$$

which is uniformly bounded for all $\xi \in (0, 1]$.

Hence, as $\xi \to 0^+$, we obtain that $P^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

iv) When $n = 0$ and $p = 1$, let $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$ and the rest as above in this section. Then the inequality

$$\left\| P^*_{r,\xi}(f;x) - f(x) \right\|_1 \leq \bar{M}^*_{1,\xi} \omega_r(f, \xi)_1 \tag{74}$$

holds where $\bar{M}^*_{1,\xi}$ is defined as in (73). Furthermore, we get $P^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

Next, the authors presented their quantitative results for the Gauss-Weierstrass operators, see [2]. They started with

Theorem 19 i) Let $f \in C^n(\mathbb{R})$, with $f^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and the rest as above in this section. Then

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right\|_p \tag{75}$$

$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \left(N^*_{p,\xi} \right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p$$

where

$$N^*_{p,\xi} := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi} \right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}}}{\displaystyle\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \tag{76}$$

which is uniformly bounded for all $\xi \in (0, 1]$.

Additionally, as $\xi \to 0^+$ we obtain that R.H.S. of (75) goes to zero.

ii) *For $p = 1$, let $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p^*_{k,\xi} \right\|_1 \tag{77}$$

$$\leq \quad \frac{1}{(n-1)!\,(r+1)} N^*_{1,\xi} \xi \omega_r(f^{(n)}, \xi)_1$$

holds where $N^*_{1,\xi}$ is defined as in (76). Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (77) goes to zero.

iii) *For $n = 0$, let $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and the rest as above in this section. Then*

$$\left\| W^*_{r,\xi}(f;x) - f(x) \right\|_p \leq \left(\bar{N}^*_{p,\xi} \right)^{1/p} \omega_r(f, \xi)_p \tag{78}$$

where

$$\bar{N}^*_{p,\xi} := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} e^{\frac{-\nu^2}{\xi}}}{\displaystyle\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \tag{79}$$

which is uniformly bounded for all $\xi \in (0, 1]$.
Hence, as $\xi \to 0^+$, we obtain that $W^*_{r,\xi} \to$ unit operator I in the L_p norm for $p > 1$.

iv) *For $n = 0$ and $p = 1$, let $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$ and the rest as above in this section. Then the inequality*

$$\left\| W^*_{r,\xi}(f;x) - f(x) \right\|_1 \leq \bar{N}^*_{1,\xi} \omega_r(f, \xi)_1 \tag{80}$$

holds where $\bar{N}^*_{1,\xi}$ is defined as in (79). Furthermore, we get $W^*_{r,\xi} \to I$ in the L_1 norm as $\xi \to 0^+$.

For the Possion-Cauchy operators, in [2], the authors showed

Theorem 20 i) *Let $f \in C^n(\mathbb{R})$, with $f^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+n)+1}{2\alpha}$, $\alpha \in \mathbb{N}$, and the rest as above in this section. Then*

$$\left\| Q^*_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q^*_{k,\xi} \right\|_p \tag{81}$$

$$\leq \quad \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(S^*_{p,\xi} \right)^{\frac{1}{p}} \xi^{\frac{1}{p}} \omega_r(f^{(n)}, \xi)_p$$

where

$$S^*_{p,\xi} := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi} \right)^{rp+1} - 1 \right) |\nu|^{np-1} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}} \tag{82}$$

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is uniformly bounded for all $\xi \in (0, 1]$.

Additionally, as $\xi \to 0^+$, *we obtain that R.H.S. of* (81) *goes to zero.*

ii) *When* $p = 1$, *let* $f \in C^n(\mathbb{R})$, $f^{(n)} \in L_1(\mathbb{R})$, $\beta > \frac{r+n+1}{2\alpha}$, *and* $n \in \mathbb{N} - \{1\}$. *Then*

$$\left\| Q_{r,\xi}^* (f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_1 \tag{83}$$

$$\leq \frac{1}{(n-1)!(r+1)} S_{1,\xi}^* \xi \omega_r(f^{(n)}, \xi)_1$$

holds where $S_{1,\xi}^*$ *is defined as in* (82). *Hence, as* $\xi \to 0^+$, *we obtain that R.H.S. of* (83) *goes to zero.*

iii) *When* $n = 0$, *let* $f \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+2)+1}{2\alpha}$, *and the rest as above in this section. Then*

$$\left\| Q_{r,\xi}^* (f;x) - f(x) \right\|_p \leq \left(\bar{S}_{p,\xi}^* \right)^{1/p} \omega_r(f, \xi)_p \tag{84}$$

where

$$\bar{S}_{p,\xi}^* := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}} \tag{85}$$

which is uniformly bounded for all $\xi \in (0, 1]$.

Hence, as $\xi \to 0^+$, *we obtain that* $Q_{r,\xi}^* \to$ *unit operator* I *in the* L_p *norm for* $p > 1$.

iv) *When* $n = 0$ *and* $p = 1$, *let* $f \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $\beta > \frac{r+3}{2\alpha}$ *and the rest as above in this section. The inequality*

$$\left\| Q_{r,\xi}^* (f;x) - f(x) \right\|_1 \leq \bar{S}_{1,\xi}^* \omega_r(f, \xi)_1 \tag{86}$$

holds where $\bar{S}_{1,\xi}^*$ *is defined as in* (85). *Furthermore, we get* $Q_{r,\xi}^* \to I$ *in the* L_1 *norm as* $\xi \to 0^+$.

Next in [2], they stated their results for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$ as follows

Theorem 21 i) *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ *such that* $np \neq 1$,

$f \in L_p(\mathbb{R})$, *and the rest as above in this section. Then*

$$\|E_{n,P}(f,x)\|_p \tag{87}$$

$$\leq \frac{\xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{p}}}{1+2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \|f(x)\|_p |m_{\xi,P}-1|$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (87) goes to zero.
ii) *When $p=1$, let $f \in C^n(\mathbb{R}), f \in L_1(\mathbb{R}), f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$.*
Then

$$\|E_{n,P}(f,x)\|_1 \tag{88}$$

$$\leq \frac{\xi \omega_r(f^{(n)},\xi)_1}{(n-1)!\,(r+1)}$$

$$\times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1+\frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \|f(x)\|_1 |m_{\xi,P}-1|$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (88) goes to zero.
iii) *When $n=0$, let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$\|E_{0,P}(f,x)\|_p \leq \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}} \tag{89}$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1+\frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \right)^{1/p}}{1+2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \|f(x)\|_p |m_{\xi,P}-1|$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (89) goes to zero.

iv) *When $n = 0$ and $p = 1$, the inequality*

$$
\|E_{0,P}(f,x)\|_1 \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) \omega_r(f,\xi)_1 \qquad (90)
$$

$$
+ \|f(x)\|_1 |m_{\xi,P} - 1|
$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (90) goes to zero.

Next in [2], the authors gave quantitative results for $E_{n,W}(f,x)$

Theorem 22 i) *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ such that $np \neq 1$, $f \in L_p(\mathbb{R})$, and the rest as above in this section. Then*

$$
\|E_{n,W}(f,x)\|_p \qquad (91)
$$

$$
\leq \quad \frac{\xi^{\frac{1}{p}} \omega_r(f^{(n)},\xi)_p \left(\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}}
$$

$$
\times \left[\frac{\left(\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]
$$

$$
+ \|f(x)\|_p |m_{\xi,W} - 1|
$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (91) goes to zero.
ii) *For $p = 1$, let $f \in C^n(\mathbb{R}), f \in L_1(\mathbb{R}), f^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then*

$$
\|E_{n,W}(f,x)\|_1 \qquad (92)
$$

$$
\leq \quad \frac{\xi \omega_r(f^{(n)},\xi)_1}{(n-1)! (r+1)}
$$

$$
\times \left[\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]
$$

$$
+ \|f(x)\|_1 |m_{\xi,W} - 1|
$$

holds. Additionally, as $\xi \to 0^+$, we obtain that R.H.S. of (92) goes to zero.
iii) *For $n = 0$, let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L_p(\mathbb{R})$, and the rest as above*

in this section. Then

$$\|E_{0,W}(f,x)\|_p \leq \omega_r(f,\xi)_p \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} \qquad (93)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right.$$

$$\left. + \|f(x)\|_p |m_{\xi,W} - 1| \right]$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (93) goes to zero.
iv) *For $n = 0$ and $p = 1$, the inequality*

$$\|E_{0,W}(f,x)\|_1 \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{r} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right) \omega_r(f,\xi)_1 \qquad (94)$$

$$+ \|f(x)\|_1 |m_{\xi,W} - 1|$$

holds. Hence, as $\xi \to 0^+$, we obtain that R.H.S. of (94) goes to zero.

Furthermore in [3], the authors gave their fractional approximation results as follows

Theorem 23 *Let $f \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$, with $\left\|f^{(m)}\right\|_{\infty} < \infty$, $0 < \xi \leq 1$. Then*
i)

$$\left\| P_{r,\xi}^*(f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \overline{\delta}_k c_{k,\xi}^*}{k!} \right\|_{\infty} \qquad (95)$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\,\Gamma(\gamma+k+1)\xi^k} S_{\gamma,k}^1 \right]$$
$$\cdot \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r\left(D_{x-}^{\gamma} f, \xi\right), \omega_r\left(D_{*x}^{\gamma} f, \xi\right) \right] \right\},$$

where

$$S_{\gamma,k}^1 := \frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{\gamma+k} e^{-\frac{|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}} \qquad (96)$$

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is uniformly bounded for all $\xi \in (0, 1]$.

ii)

$$\left\| W_{r,\xi}^* (f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \overline{\delta}_k p_{k,\xi}^*}{k!} \right\|_\infty \tag{97}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)! \Gamma(\gamma + k + 1) \xi^k} S_{\gamma,k}^2 \right] \cdot \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f, \xi \right), \omega_r \left(D_{*x}^\gamma f, \xi \right) \right] \right\},$$

where

$$S_{\gamma,k}^2 := \frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{\gamma+k} e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \tag{98}$$

is uniformly bounded for all $\xi \in (0, 1]$.

iii) *For* $\beta > \frac{\gamma+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$,

$$\left\| Q_{r,\xi}^* (f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \overline{\delta}_k q_{k,\xi}^*}{k!} \right\|_\infty \tag{99}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)! \Gamma(\gamma + k + 1) \xi^k} S_{\gamma,k}^3 \right] \cdot \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f, \xi \right), \omega_r \left(D_{*x}^\gamma f, \xi \right) \right] \right\},$$

where

$$S_{\gamma,k}^3 := \frac{\sum_{\nu=-\infty}^{\infty} |\nu|^{\gamma+k} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}} \tag{100}$$

is uniformly bounded for all $\xi \in (0, 1]$.

In [3], the authors also demonstrated their fractional results for the error terms as

Theorem 24 *Let* $f \in C^m (\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$, *with* $\left\| f^{(m)} \right\|_\infty < \infty$, $0 < \xi \leq 1$, *and* $\overline{\delta}_k$ *be as in* (26). *Then*

i)

$$\|E_{\gamma,P}(f)\|_\infty \tag{101}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^k} S_{\gamma,k}^4\right]$$
$$\cdot \sup_{x\in\mathbb{R}} \left\{\max\left[\omega_r\left(D_{x-}^\gamma f,\xi\right),\omega_r\left(D_{*x}^\gamma f,\xi\right)\right]\right\} + \|f\|_\infty \left|m_{\xi,P}-1\right|,$$

where

$$S_{\gamma,k}^4 := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} |\nu|^{\gamma+k}\, e^{-\frac{|\nu|}{\xi}}}{1+2\xi e^{-\frac{1}{\xi}}} \tag{102}$$

is uniformly bounded for all $\xi \in (0,1]$.

ii)

$$\|E_{\gamma,W}(f)\|_\infty \tag{103}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\,\xi^k} S_{\gamma,k}^5\right]$$
$$\cdot \sup_{x\in\mathbb{R}} \left\{\max\left[\omega_r\left(D_{x-}^\gamma f,\xi\right),\omega_r\left(D_{*x}^\gamma f,\xi\right)\right]\right\} + \|f\|_\infty \left|m_{\xi,W}-1\right|,$$

where

$$S_{\gamma,k}^5 := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} |\nu|^{\gamma+k}\, e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1-\operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right)+1} \tag{104}$$

is uniformly bounded for all $\xi \in (0,1]$.

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2 Main Results

Here we study important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ.

Let $f : \mathbb{R} \to \mathbb{C}$ be Borel measurable complex valued function such that $f = f_1 + i f_2$ where $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are real valued Borel measurable functions and $i = \sqrt{-1}$. Additionally, assume that $f_1, f_2 \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, and $0 < \xi \le 1$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{105}$$

we define the complex generalized discrete Picard operators as

$$P_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{106}$$

Here we observe that

$$P_{r,\xi}^*(f;x) = P_{r,\xi}^*(f_1;x) + i P_{r,\xi}^*(f_2;x), \tag{107}$$

$$\left| P_{r,\xi}^*(f;x) - f(x) \right| \tag{108}$$
$$\le \left| P_{r,\xi}^*(f_1;x) - f_1(x) \right| + \left| P_{r,\xi}^*(f_2;x) - f_2(x) \right|,$$

$$\left\| P_{r,\xi}^*(f;x) - f(x) \right\|_\infty \tag{109}$$
$$\le \left\| P_{r,\xi}^*(f_1;x) - f_1(x) \right\|_\infty + \left\| P_{r,\xi}^*(f_2;x) - f_2(x) \right\|_\infty,$$

and for $p \ge 1$,

$$\left\| P_{r,\xi}^*(f;x) - f(x) \right\|_p \tag{110}$$
$$\le \left\| P_{r,\xi}^*(f_1;x) - f_1(x) \right\|_p + \left\| P_{r,\xi}^*(f_2;x) - f_2(x) \right\|_p.$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{111}$$

we define the complex generalized discrete Gauss-Weierstrass operators as

$$W_{r,\xi}^*(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{112}$$

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Here we observe that,

$$W^*_{r,\xi}(f;x) = W^*_{r,\xi}(f_1;x) + iW^*_{r,\xi}(f_2;x), \tag{113}$$

$$\begin{aligned}&\left|W^*_{r,\xi}(f;x) - f(x)\right| \\ \leq\ &\left|W^*_{r,\xi}(f_1;x) - f_1(x)\right| + \left|W^*_{r,\xi}(f_2;x) - f_2(x)\right|,\end{aligned} \tag{114}$$

$$\begin{aligned}&\left\|W^*_{r,\xi}(f;x) - f(x)\right\|_\infty \\ \leq\ &\left\|W^*_{r,\xi}(f_1;x) - f_1(x)\right\|_\infty + \left\|W^*_{r,\xi}(f_2;x) - f_2(x)\right\|_\infty,\end{aligned} \tag{115}$$

and for $p \geq 1$,

$$\begin{aligned}&\left\|W^*_{r,\xi}(f;x) - f(x)\right\|_p \\ \leq\ &\left\|W^*_{r,\xi}(f_1;x) - f_1(x)\right\|_p + \left\|W^*_{r,\xi}(f_2;x) - f_2(x)\right\|_p.\end{aligned} \tag{116}$$

$iii)$ Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}, \tag{117}$$

we define the complex generalized discrete Poisson-Cauchy operators as

$$Q^*_{r,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu)\right)\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{118}$$

Here we observe that

$$Q^*_{r,\xi}(f;x) = Q^*_{r,\xi}(f_1;x) + iQ^*_{r,\xi}(f_2;x), \tag{119}$$

$$\begin{aligned}&\left|Q^*_{r,\xi}(f;x) - f(x)\right| \\ \leq\ &\left|Q^*_{r,\xi}(f_1;x) - f_1(x)\right| + \left|Q^*_{r,\xi}(f_2;x) - f_2(x)\right|,\end{aligned} \tag{120}$$

$$\begin{aligned}&\left\|Q^*_{r,\xi}(f;x) - f(x)\right\|_\infty \\ \leq\ &\left\|Q^*_{r,\xi}(f_1;x) - f_1(x)\right\|_\infty + \left\|Q^*_{r,\xi}(f_2;x) - f_2(x)\right\|_\infty,\end{aligned} \tag{121}$$

and for $p \geq 1$,

$$\begin{aligned}&\left\|Q^*_{r,\xi}(f;x) - f(x)\right\|_p \\ \leq\ &\left\|Q^*_{r,\xi}(f_1;x) - f_1(x)\right\|_p + \left\|Q^*_{r,\xi}(f_2;x) - f_2(x)\right\|_p.\end{aligned} \tag{122}$$

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Observe that for $c \in \mathbb{C}$ constant we have

$$P_{r,\xi}^* (c; x) = W_{r,\xi}^* (c; x) = Q_{r,\xi}^* (c; x) = c. \tag{123}$$

We assume that for $x \in \mathbb{R}$, the operators $P_{r,\xi}^* (f_j; x)$, $W_{r,\xi}^* (f_j; x)$, and $Q_{r,\xi}^* (f_j; x) \in \mathbb{R}$ where $j = 1, 2$. This is the case when $\|f_j\|_{\infty,\mathbb{R}} < \infty$ for $j = 1, 2$.

$iv)$ When

$$\mu_\xi(\nu) := \mu_{\xi,P}(\nu) := \frac{e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}, \tag{124}$$

we define the complex generalized discrete non-unitary Picard operators as

$$P_{r,\xi}(f; x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{125}$$

Here $\mu_{\xi,P}(\nu)$ has mass

$$m_{\xi,P} := \frac{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}. \tag{126}$$

We observe that

$$\frac{\mu_{\xi,P}(\nu)}{m_{\xi,P}} = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{127}$$

which is the probability measure (105) defining the operators $P_{r,\xi}^*$.

$v)$ When

$$\mu_\xi(\nu) := \mu_{\xi,W}(\nu) := \frac{e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}, \tag{128}$$

we define the complex generalized discrete non-unitary Gauss-Weierstrass operators as

$$W_{r,\xi}(f; x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{129}$$

Here $\mu_{\xi,W}(\nu)$ has mass

$$m_{\xi,W} := \frac{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}. \tag{130}$$

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We observe that

$$\frac{\mu_{\xi,W}(\nu)}{m_{\xi,W}} = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{131}$$

which is the probability measure (111) defining the operators $W_{r,\xi}^*$.

We state our first result as follows

Theorem 25 *Let* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$. *Here* $f_j \in C^n(\mathbb{R})$, $f_j^{(n)} \in C_u(\mathbb{R})$, $j = 1, 2$, *and* $n \in \mathbb{N}$. *Then*

i)

$$\left\| P_{r,\xi}^*(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_{\infty,x} \tag{132}$$

$$\leq \left(\frac{\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi)}{n!} \right) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right),$$

ii)

$$\left\| W_{r,\xi}^*(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_{\infty,x} \tag{133}$$

$$\leq \left(\frac{\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi)}{n!} \right) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right),$$

iii) *for* $\alpha \in \mathbb{N}$ *and* $\beta > \frac{n+r+1}{2\alpha}$,

$$\left\| Q_{r,\xi}^*(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_{\infty,x} \tag{134}$$

$$\leq \left(\frac{\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi)}{n!} \right) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \right).$$

Proof. By Theorems 1, 14, 15. ∎

For the case of $n = 0$, we give the following result

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Corollary 26 *Let* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$. *Here* $f_j \in C_u(\mathbb{R})$ *for* $j = 1, 2$. *Then*

i)

$$\left\| P_{r,\xi}^* (f; x) - f(x) \right\|_{\infty, x} \tag{135}$$

$$\leq (\omega_r(f_1, \xi) + \omega_r(f_2, \xi)) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \right),$$

ii)

$$\left\| W_{r,\xi}^* (f; x) - f(x) \right\|_{\infty, x} \tag{136}$$

$$\leq (\omega_r(f_1, \xi) + \omega_r(f_2, \xi)) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \right),$$

iii) *for* $\alpha \in \mathbb{N}$ *and* $\beta > \frac{r+1}{2\alpha}$,

$$\left\| Q_{r,\xi}^* (f; x) - f(x) \right\|_{\infty, x} \tag{137}$$

$$\leq (\omega_r(f_1, \xi) + \omega_r(f_2, \xi)) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \right).$$

Proof. By Corollary 2, and the Theorems 14, 15. ∎

In [4], the authors gave

Theorem 27 *Let* $g \in C^{n-1}(\mathbb{R})$, *such that* $g^{(n)}$ *exists,* $n, r \in \mathbb{N}$, $0 < \xi \leq 1$. *Additionally, suppose that for each* $x \in \mathbb{R}$ *the function* $g^{(\tilde{j})}(x + j\nu) \in L_1(\mathbb{R}, \mu_\xi)$ *as a function of* ν, *for all* $\tilde{j} = 0, 1, \ldots, n-1$; $j = 1, \ldots, r$. *Assume that there exist* $\lambda_{\tilde{j}, j} \geq 0$, $\tilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, *with* $\lambda_{\tilde{j}, j} \in L_1(\mathbb{R}, \mu_\xi)$ *such that for each* $x \in \mathbb{R}$ *we have*

$$|g^{(i)}(x + j\nu)| \leq \lambda_{\tilde{j}, j}(\nu), \tag{138}$$

for μ_ξ−*almost all* $\nu \in \mathbb{R}$, *all* $i = 1, \ldots, n$; $j = 1, 2, \ldots, r$. *Then,* $g^{(i)}(x + j\nu)$ *defines a* μ_ξ−*integrable function with respect to* ν *for each* $x \in \mathbb{R}$, *all* $\tilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$.

i) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}$$

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we get

$$\left(P_{r,\xi}^*\left(f;x\right)\right)^{(\tilde{j})} = P_{r,\xi}^*\left(f^{(\tilde{j})};x\right),\tag{139}$$

for all $x \in \mathbb{R}$, and for all $\tilde{j} = 1, \ldots, n$.

ii) *When*

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}$$

we have

$$\left(W_{r,\xi}^*\left(f;x\right)\right)^{(\tilde{j})} = W_{r,\xi}^*\left(f^{(\tilde{j})};x\right),\tag{140}$$

for all $x \in \mathbb{R}$, and for all $\tilde{j} = 1, \ldots, n$.

iii) *Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When*

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}},$$

we obtain

$$\left(Q_{r,\xi}^*\left(f;x\right)\right)^{(\tilde{j})} = Q_{r,\xi}^*\left(f^{(\tilde{j})};x\right),\tag{141}$$

for all $x \in \mathbb{R}$, and for all $\tilde{j} = 1, \ldots, n$.

Now, we present following simultaneous results for the operators $P_{r,\xi}^*$, $W_{r,\xi}^*$, and $Q_{r,\xi}^*$.

Theorem 28 *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$ and $0 < \xi \le 1$. Here $f_j \in C^{n+\rho}$, $n \in \mathbb{N}, \rho \in \mathbb{Z}^+$, $j = 1, 2$, and $f^{(n+\tilde{j})} \in C_u(\mathbb{R})$ where $\tilde{j} = 0, 1, \ldots, \rho$. We consider the assumptions of the Theorem 27 valid for $n = \rho$ there. Then*

i)

$$\left\| \left(P_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!}\delta_k c_{k,\xi}^* \right\|_{\infty,x}\tag{142}$$

$$\le \left(\frac{\omega_r\left(f_1^{(n+\tilde{j})},\xi\right) + \omega_r\left(f_2^{(n+\tilde{j})},\xi\right)}{n!}\right)\left(\frac{\sum\limits_{\nu=-\infty}^{\infty}|\nu|^n\left(1 + \frac{|\nu|}{\xi}\right)^r e^{-\frac{|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}}\right),$$

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ii)

$$\left\| \left(W_{r,\xi}^*(f;x) \right)^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\widetilde{j}+k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_{\infty,x} \tag{143}$$

$$\leq \quad \left(\frac{\omega_r\left(f_1^{(n+\widetilde{j})},\xi\right) + \omega_r\left(f_2^{(n+\widetilde{j})},\xi\right)}{n!} \right) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{-\frac{\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \right),$$

iii)

$$\left\| \left(Q_{r,\xi}^*(f;x) \right)^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\widetilde{j}+k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_{\infty,x} \tag{144}$$

$$\leq \quad \left(\frac{\omega_r\left(f_1^{(n+\widetilde{j})},\xi\right) + \omega_r\left(f_2^{(n+\widetilde{j})},\xi\right)}{n!} \right) \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \right),$$

where $\beta > \frac{n+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$.

Proof. By Theorems 25, 27. ∎

Next, we state our L_p results for the operators $P_{r,\xi}^*$, $W_{r,\xi}^*$, and $Q_{r,\xi}^*$. We start with

Theorem 29 *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$ and $0 < \xi \leq 1$.*
i) *Here $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $j = 1, 2$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left\| P_{r,\xi}^*(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_p \tag{145}$$

$$\leq \quad \xi^{\frac{1}{p}} \left(\frac{\omega_r(f_1^{(n)},\xi)_p + \omega_r(f_2^{(n)},\xi)_p}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \right) \left(M_{p,\xi}^* \right)^{\frac{1}{p}}$$

where

$$M_{p,\xi}^* := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \tag{146}$$

which is uniformly bounded for all $\xi \in (0,1]$.

ii) When $p = 1$, let $f_j \in C^n(\mathbb{R})$, $f_j^{(n)} \in L_1(\mathbb{R})$, $j = 1,2$, and $n \in \mathbb{N} - \{1\}$. Then

$$\left\| P_{r,\xi}^* (f;x) - f(x) - \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_1 \qquad (147)$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n)},\xi)_1 + \omega_r(f_2^{(n)},\xi)_1}{(n-1)!(r+1)} \right) M_{1,\xi}^*$$

holds where $M_{1,\xi}^*$ is defined as in (146).

iii) When $n = 0$, let $f_j \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $j = 1,2$, and $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left\| P_{r,\xi}^* (f;x) - f(x) \right\|_p \leq (\omega_r(f_1,\xi)_p + \omega_r(f_2,\xi)_p) \left(\bar{M}_{p,\xi}^* \right)^{1/p} \qquad (148)$$

where

$$\bar{M}_{p,\xi}^* := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} e^{\frac{-|\nu|}{\xi}}}{\displaystyle\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}} \qquad (149)$$

which is uniformly bounded for all $\xi \in (0,1]$.

iv) When $n = 0$ and $p = 1$, let $f_j \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 1,2$. Then the inequality

$$\left\| P_{r,\xi}^* (f;x) - f(x) \right\|_1 \leq (\omega_r(f_1,\xi)_1 + \omega_r(f_2,\xi)_1) \bar{M}_{1,\xi}^* \qquad (150)$$

holds where $\bar{M}_{1,\xi}^*$ is defined as in (149).

Proof. By Theorems 5, 6, 18, and Propositions 7, 8. ∎

For the operators $W_{r,\xi}^*$, we obtain

Theorem 30 Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$ and $0 < \xi \leq 1$.

i) Here $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $j = 1,2$, $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left\| W_{r,\xi}^* (f;x) - f(x) - \sum_{k=1}^n \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_p \qquad (151)$$

$$\leq \quad \xi^{\frac{1}{p}} \left(\frac{\omega_r(f_1^{(n)},\xi)_p + \omega_r(f_2^{(n)},\xi)_p}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}} \right) \left(N_{p,\xi}^* \right)^{\frac{1}{p}}$$

where

$$N_{p,\xi}^* := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi} \right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}}}{\displaystyle\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \qquad (152)$$

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which is uniformly bounded for all $\xi \in (0, 1]$.

ii) *When $p = 1$, let $f_j \in C^n(\mathbb{R})$, $f_j^{(n)} \in L_1(\mathbb{R})$, $j = 1, 2$, and $n \in \mathbb{N} - \{1\}$. Then*

$$\left\| W_{r,\xi}^* (f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_1 \qquad (153)$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1}{(n-1)!(r+1)} \right) N_{1,\xi}^*$$

holds where $N_{1,\xi}^*$ is defined as in (152).

iii) *When $n = 0$, let $f_j \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $j = 1, 2$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left\| W_{r,\xi}^* (f; x) - f(x) \right\|_p \leq (\omega_r(f_1, \xi)_p + \omega_r(f_2, \xi)_p) \left(\bar{N}_{p,\xi}^* \right)^{1/p} \qquad (154)$$

where

$$\bar{N}_{p,\xi}^* := \frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} e^{\frac{-\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}} \qquad (155)$$

which is uniformly bounded for all $\xi \in (0, 1]$.

iv) *When $n = 0$ and $p = 1$, let $f_j \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 1, 2$. Then the inequality*

$$\left\| W_{r,\xi}^* (f; x) - f(x) \right\|_1 \leq (\omega_r(f_1, \xi)_1 + \omega_r(f_2, \xi)_1) \bar{N}_{1,\xi}^* \qquad (156)$$

holds where $\bar{N}_{1,\xi}^*$ is defined as in (155).

Proof. By Theorems 5, 6, 19, and Propositions 7, 8. ∎

For the operators $Q_{r,\xi}^*$, we get

Theorem 31 *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$ and $0 < \xi \leq 1$.*

i) *Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $j = 1, 2$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > \frac{p(r+n)+1}{2\alpha}$, $\alpha \in \mathbb{N}$. Then*

$$\left\| Q_{r,\xi}^* (f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_p \qquad (157)$$

$$\leq \quad \xi^{\frac{1}{p}} \left(\frac{\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \right) \left(S_{p,\xi}^* \right)^{\frac{1}{p}}$$

where

$$S_{p,\xi}^* := \frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi} \right)^{rp+1} - 1 \right) |\nu|^{np-1} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}} \qquad (158)$$

is uniformly bounded for all $\xi \in (0, 1]$.

ii) *When $p = 1$, let $f_j \in C^n(\mathbb{R})$, $f_j^{(n)} \in L_1(\mathbb{R})$, $j = 1, 2$, $n \in \mathbb{N} - \{1\}$, and $\beta > \frac{r+n+1}{2\alpha}$. Then*

$$\left\| Q_{r,\xi}^* (f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_1 \tag{159}$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1}{(n-1)! \, (r+1)} \right) S_{1,\xi}^*$$

holds where $S_{1,\xi}^*$ is defined as in (158).

iii) *When $n = 0$, let $f_j \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $j = 1, 2$, $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\beta > \frac{p(r+2)+1}{2\alpha}$. Then*

$$\left\| Q_{r,\xi}^* (f; x) - f(x) \right\|_p \leq (\omega_r(f_1, \xi)_p + \omega_r(f_2, \xi)_p) \left(\bar{S}_{p,\xi}^* \right)^{1/p} \tag{160}$$

where

$$\bar{S}_{p,\xi}^* := \frac{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}}{\displaystyle\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha} \right)^{-\beta}} \tag{161}$$

which is uniformly bounded for all $\xi \in (0, 1]$.

iv) *When $n = 0$ and $p = 1$, let $f_j \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 1, 2$, $\beta > \frac{r+3}{2\alpha}$. The inequality*

$$\left\| Q_{r,\xi}^* (f; x) - f(x) \right\|_1 \leq (\omega_r(f_1, \xi)_1 + \omega_r(f_2, \xi)_1) \, \bar{S}_{1,\xi}^* \tag{162}$$

holds where $\bar{S}_{1,\xi}^*$ is defined as in (161).

Proof. By Theorems 5, 6, 20, and Propositions 7, 8. ∎

Now, we give our simultaneous results for L_p norm. For the case of $p > 1$, we have

Theorem 32 *Here $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$ and $0 < \xi \leq 1$. Let $f_j \in C^{n+\rho}(\mathbb{R})$, with $f_j^{(n+\tilde{j})} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then*

i)

$$\left\| \left(P_{r,\xi}^*(f; x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_{p,x} \tag{163}$$

$$\leq \quad \xi^{\frac{1}{p}} \left(\frac{\omega_r(f_1^{(n+\tilde{j})}, \xi)_p + \omega_r(f_2^{(n+\tilde{j})}, \xi)_p}{((n-1)!) \, (q(n-1)+1)^{\frac{1}{q}} \, (rp+1)^{\frac{1}{p}}} \right) \left(M_{p,\xi}^* \right)^{\frac{1}{p}},$$

ii)

$$\left\|\left(W_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!}\delta_k p_{k,\xi}^*\right\|_{p,x} \quad (164)$$

$$\leq \quad \xi^{\frac{1}{p}}\left(\frac{\omega_r(f_1^{(n+\tilde{j})},\xi)_p + \omega_r(f_2^{(n+\tilde{j})},\xi)_p}{((n-1)!)\left(q(n-1)+1\right)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}\right)\left(N_{p,\xi}^*\right)^{\frac{1}{p}},$$

iii) *for* $\beta > \frac{p(n+r)+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\|\left(Q_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!}\delta_k q_{k,\xi}^*\right\|_{p,x} \quad (165)$$

$$\leq \quad \xi^{\frac{1}{p}}\left(\frac{\omega_r(f_1^{(n+\tilde{j})},\xi)_p + \omega_r(f_2^{(n+\tilde{j})},\xi)_p}{((n-1)!)\left(q(n-1)+1\right)^{\frac{1}{q}}(rp+1)^{\frac{1}{p}}}\right)\left(S_{p,\xi}^*\right)^{\frac{1}{p}}.$$

Proof. By Theorems 9, 27, 29-31. ∎

Now, we give our results for the special case of $n = 0$.

Proposition 33 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$ *and* $0 < \xi \leq 1$. *Let* $f_j^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $j = 1, 2$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *We consider the assumptions of Theorem 27 as valid for* $n = \rho$ *there. Then for all* $\tilde{j} = 0, 1, \ldots, \rho$, *we have*
i)

$$\left\|\left(P_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x)\right\|_{p,x} \leq \left(\omega_r(f_1^{(\tilde{j})},\xi)_p + \omega_r(f_2^{(\tilde{j})},\xi)_p\right)\left(\bar{M}_{p,\xi}^*\right)^{\frac{1}{p}}, \quad (166)$$

ii)

$$\left\|\left(W_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x)\right\|_{p,x} \leq \left(\omega_r(f_1^{(\tilde{j})},\xi)_p + \omega_r(f_2^{(\tilde{j})},\xi)_p\right)\left(\bar{N}_{p,\xi}^*\right)^{\frac{1}{p}}, \quad (167)$$

iii) *for* $\beta > \frac{p(r+2)+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\|\left(Q_{r,\xi}^*(f;x)\right)^{(\tilde{j})} - f^{(\tilde{j})}(x)\right\|_{p,x} \leq \left(\omega_r(f_1^{(\tilde{j})},\xi)_p + \omega_r(f_2^{(\tilde{j})},\xi)_p\right)\left(\bar{S}_{p,\xi}^*\right)^{\frac{1}{p}}. \quad (168)$$

Proof. By Theorems 27, 29-31 and Proposition 10. ∎

For the special case of $p = 1$, we obtain

Theorem 34 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$ *and* $0 < \xi \leq 1$. *Let* $f_j \in C^{n+\rho}(\mathbb{R})$, *with* $f_j^{(n+\tilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N}-\{1\}$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem 27 as valid for* $n = \rho$ *there. Then for all*

$\tilde{j} = 0, 1, \ldots, \rho,$ we have

i)

$$\left\| \left(P_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!} \delta_k c_{k,\xi}^* \right\|_{1,x} \qquad (169)$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n+\tilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\tilde{j})}, \xi)_1}{(n-1)!\,(r+1)} \right) M_{1,\xi}^*,$$

ii)

$$\left\| \left(W_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!} \delta_k p_{k,\xi}^* \right\|_{1,x} \qquad (170)$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n+\tilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\tilde{j})}, \xi)_1}{(n-1)!\,(r+1)} \right) N_{1,\xi}^*,$$

iii) for $\beta > \frac{n+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(\tilde{j}+k)}(x)}{k!} \delta_k q_{k,\xi}^* \right\|_{1,x} \qquad (171)$$

$$\leq \quad \xi \left(\frac{\omega_r(f_1^{(n+\tilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\tilde{j})}, \xi)_1}{(n-1)!\,(r+1)} \right) S_{1,\xi}^*.$$

Proof. By Theorems 11, 27, 29-31. ∎

For $p = 1$ and $n = 0$, we give

Proposition 35 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$ *and* $0 < \xi \leq 1$. *Let* $f^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem 27 as valid for* $n = \rho$ *there. Then for all* $\tilde{j} = 0, 1, \ldots, \rho$, *we have*

i)

$$\left\| \left(P_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) \right\|_{1,x} \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_1 + \omega_r(f_2^{(\tilde{j})}, \xi)_1 \right) \bar{M}_{1,\xi}^*, \qquad (172)$$

ii)

$$\left\| \left(W_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) \right\|_{1,x} \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_1 + \omega_r(f_2^{(\tilde{j})}, \xi)_1 \right) \bar{N}_{1,\xi}^*, \qquad (173)$$

iii) for $\beta > \frac{r+3}{2\alpha}$, $\alpha \in \mathbb{N}$

$$\left\| \left(Q_{r,\xi}^*(f;x) \right)^{(\tilde{j})} - f^{(\tilde{j})}(x) \right\|_{1,x} \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_1 + \omega_r(f_2^{(\tilde{j})}, \xi)_1 \right) \bar{S}_{1,\xi}^*. \qquad (174)$$

Proof. By Theorems 27, 29-31 and Proposition 12. ∎

Next, we give the fractional approximation results for the operators $P_{r,\xi}^*$, $W_{r,\xi}^*$, and $Q_{r,\xi}^*$. We start with

Theorem 36 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$ *and* $0 < \xi \le 1$. *Let* $f_j \in C^m(\mathbb{R})$, $m = \lceil \gamma \rceil$, $\gamma > 0$, *with* $\left\| f_j^{(m)} \right\|_\infty < \infty$, $j = 1, 2$. *Then*

i)

$$\left\| P_{r,\xi}^*(f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \delta_k c_{k,\xi}^*}{k!} \right\|_\infty \qquad (175)$$

$$\le \left[\sum_{k=0}^{r} \frac{r!}{(r-k)! \Gamma(\gamma+k+1) \xi^k} S_{\gamma,k}^1 \right]$$

$$\cdot \left(\sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_1, \xi \right), \omega_r \left(D_{*x}^\gamma f_1, \xi \right) \right] \right\} \right.$$

$$\left. + \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_2, \xi \right), \omega_r \left(D_{*x}^\gamma f_2, \xi \right) \right] \right\} \right),$$

where $S_{\gamma,k}^1$ *is defined as in* (96).

ii)

$$\left\| W_{r,\xi}^*(f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \delta_k p_{k,\xi}^*}{k!} \right\|_\infty \qquad (176)$$

$$\le \left[\sum_{k=0}^{r} \frac{r!}{(r-k)! \Gamma(\gamma+k+1) \xi^k} S_{\gamma,k}^2 \right]$$

$$\cdot \left(\sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_1, \xi \right), \omega_r \left(D_{*x}^\gamma f_1, \xi \right) \right] \right\} \right.$$

$$\left. + \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_2, \xi \right), \omega_r \left(D_{*x}^\gamma f_2, \xi \right) \right] \right\} \right),$$

where $S_{\gamma,k}^2$ *is defined as in* (98).

iii) *For $\beta > \frac{\gamma+r+1}{2\alpha}$, $\alpha \in \mathbb{N}$,*

$$\left\| Q_{r,\xi}^* (f) - f - \sum_{k=1}^{m-1} \frac{f^{(k)} \delta_k q_{k,\xi}^*}{k!} \right\|_\infty \tag{177}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)! \Gamma(\gamma + k + 1) \xi^k} S_{\gamma,k}^3 \right]$$

$$\cdot \left(\sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_1, \xi \right), \omega_r \left(D_{*x}^\gamma f_1, \xi \right) \right] \right\} \right.$$

$$\left. + \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^\gamma f_2, \xi \right), \omega_r \left(D_{*x}^\gamma f_2, \xi \right) \right] \right\} \right),$$

where $S_{\gamma,k}^3$ is defined as in (100).

Proof. By Theorems 13, 23. ∎

Let $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + i f_2$. We define the following error quantities:

$$\begin{aligned} E_{0,P}(f, x) \quad &: \quad = P_{r,\xi}(f; x) - f(x) \tag{178} \\ &= \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} - f(x), \end{aligned}$$

$$\begin{aligned} E_{0,W}(f, x) \quad &: \quad = W_{r,\xi}(f; x) - f(x) \tag{179} \\ &= \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x + j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf} \left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} - f(x). \end{aligned}$$

Additionally, we introduce the errors $(n \in \mathbb{N})$:

$$E_{n,P}(f, x) := P_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \tag{180}$$

and

$$E_{n,W}(f, x) := W_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^k e^{-\frac{\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf} \left(\frac{1}{\sqrt{\xi}} \right) \right) + 1}. \tag{181}$$

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We observe that

$$|E_{0,P}(f,x)| \tag{182}$$
$$\leq |E_{0,P}(f_1,x)| + |E_{0,P}(f_2,x)|,$$

$$|E_{0,W}(f,x)| \tag{183}$$
$$\leq |E_{0,W}(f_1,x)| + |E_{0,W}(f_2,x)|.$$

Next, we give following results for errors $E_{0,P}(f,x)$ and $E_{0,W}(f,x)$.

Corollary 37 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$. *Let* $f_j \in C_u(\mathbb{R})$ *for* $j = 1,2$. *Then*

i)

$$|E_{0,P}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} (\omega_r(f_1,|\nu|) + \omega_r(f_2,|\nu|))\, e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) \tag{184}$$
$$+ (|f_1(x)| + |f_2(x)|)\, |m_{\xi,P} - 1|,$$

ii)

$$|E_{0,W}(f,x)| \leq \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} (\omega_r(f_1,|\nu|) + \omega_r(f_2,|\nu|))\, e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi}\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right) \tag{185}$$
$$+ (|f_1(x)| + |f_2(x)|)\, |m_{\xi,W} - 1|.$$

Proof. By Corollary 16, (182), and (183). ∎

We also observe that

$$\|E_{n,P}(f,x)\|_{\infty} \tag{186}$$
$$\leq \|E_{n,P}(f_1,x)\|_{\infty} + \|E_{n,P}(f_2,x)\|_{\infty}$$

and

$$\|E_{n,W}(f,x)\|_{\infty} \tag{187}$$
$$\leq \|E_{n,W}(f_1,x)\|_{\infty} + \|E_{n,W}(f_2,x)\|_{\infty}.$$

For $E_{n,P}$ and $E_{n,W}$, we present

Theorem 38 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$. *Let* $f_j \in C^n(\mathbb{R})$ *with* $f_j^{(n)} \in C_u(\mathbb{R})$, $j = 1,2$, $n \in \mathbb{N}$, *and* $\|f_j\|_{\infty,\mathbb{R}} < \infty$. *Then*

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i)

$$\|E_{n,P}(f,x)\|_{\infty,x} \tag{188}$$

$$\leq \frac{\left(\omega_r(f_1^{(n)},\xi) + \omega_r(f_2^{(n)},\xi)\right)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}}\right)$$

$$+ \left(\|f_1\|_{\infty,\mathbb{R}} + \|f_2\|_{\infty,\mathbb{R}}\right) |m_{\xi,P} - 1|,$$

ii)

$$\|E_{n,W}(f,x)\|_{\infty,x} \tag{189}$$

$$\leq \frac{\left(\omega_r(f_1^{(n)},\xi) + \omega_r(f_2^{(n)},\xi)\right)}{n!} \left(\frac{\sum\limits_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi}\right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1}\right)$$

$$+ \left(\|f_1\|_{\infty,\mathbb{R}} + \|f_2\|_{\infty,\mathbb{R}}\right) |m_{\xi,W} - 1|.$$

In the above inequalities (188) *-* (189)*, the ratios of sums in their R.H.S. are uniformly bounded with respect to* $\xi \in (0,1]$*.*

Proof. By Theorem 17, (186), and (187). ∎

Furthermore, for $p \geq 1$, we obtain that

$$\|E_{n,P}(f,x)\|_p \tag{190}$$
$$\leq \|E_{n,P}(f_1,x)\|_p + \|E_{n,P}(f_2,x)\|_p$$

and

$$\|E_{n,W}(f,x)\|_p \tag{191}$$
$$\leq \|E_{n,W}(f_1,x)\|_p + \|E_{n,W}(f_2,x)\|_p.$$

Next, we state our L_p approximation results for the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$. We start with

Theorem 39 i) *Let* $p,q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*,* $n \in \mathbb{N}$ *such that* $np \neq 1$*,*

$f_j \in L_p(\mathbb{R})$, $j = 1, 2$, and the rest as above in this section. Then

$$\|E_{n,P}(f,x)\|_p \qquad (192)$$

$$\leq \left(\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p \right)$$

$$\times \left(\frac{\xi^{\frac{1}{p}} \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \right)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{p}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \left(\|f_1(x)\|_p + \|f_2(x)\|_p \right) |m_{\xi,P} - 1|$$

holds.

ii) When $p = 1$, let $f_j \in C^n(\mathbb{R})$, $f_j \in L_1(\mathbb{R})$, $f_j^{(n)} \in L_1(\mathbb{R})$, and $n \in \mathbb{N} - \{1\}$. Then

$$\|E_{n,P}(f,x)\|_1 \qquad (193)$$

$$\leq \xi \frac{\left(\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1 \right)}{(n-1)! (r+1)}$$

$$\times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \left(\|f_1(x)\|_1 + \|f_2(x)\|_1 \right) |m_{\xi,P} - 1|$$

holds.

iii) When $n = 0$, let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $f_j \in L_p(\mathbb{R})$, and the rest as

above in this section. Then

$$\|E_{0,P}(f,x)\|_p \leq (\omega_r(f_1,\xi)_p + \omega_r(f_2,\xi)_p) \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}} \quad (194)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \right)^{1/p}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \left(\|f_1(x)\|_p + \|f_2(x)\|_p \right) |m_{\xi,P} - 1|$$

holds.

iv) *When $n = 0$ and $p = 1$, the inequality*

$$\|E_{0,P}(f,x)\|_1 \leq (\omega_r(f_1,\xi)_1 + \omega_r(f_2,\xi)_1) \quad (195)$$

$$\times \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{r} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right)$$

$$+ (\|f_1(x)\|_1 + \|f_2(x)\|_1) |m_{\xi,P} - 1|$$

holds.

Proof. By Theorem 21 and (190). ∎

We have also

Theorem 40 i) *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$ such that $np \neq 1$,*

$f_j \in L_p(\mathbb{R})$, $j = 1, 2$, and the rest as above in this section. Then

$$\|E_{n,W}(f,x)\|_p \tag{196}$$

$$\leq \left(\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p \right)$$

$$\times \left(\frac{\xi^{\frac{1}{p}} \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \right)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]$$

$$+ \left(\|f_1(x)\|_p + \|f_2(x)\|_p \right) |m_{\xi,W} - 1|$$

holds.

ii) For $p = 1$, let $f_j \in C^n(\mathbb{R}), f_j \in L_1(\mathbb{R})$, $f_j^{(n)} \in L_1(\mathbb{R})$, $j = 1, 2$, and $n \in \mathbb{N} - \{1\}$. Then

$$\|E_{n,W}(f,x)\|_1 \tag{197}$$

$$\leq \xi \frac{\left(\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1 \right)}{(n-1)! \, (r+1)}$$

$$\times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]$$

$$+ \left(\|f_1(x)\|_1 + \|f_2(x)\|_1 \right) |m_{\xi,W} - 1|$$

holds.

iii) For $n = 0$, let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $f_j \in L_p(\mathbb{R})$, and the rest as

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above in this section. Then

$$\|E_{0,W}(f,x)\|_p \leq (\omega_r(f_1,\xi)_p + \omega_r(f_2,\xi)_p) \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} \quad (198)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi} \left(1 - \text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]$$

$$+ \left(\|f_1(x)\|_p + \|f_2(x)\|_p \right) |m_{\xi,W} - 1|$$

holds.

iv) *For $n = 0$ and $p = 1$, the inequality*

$$\|E_{0,W}(f,x)\|_1 \leq \left(\omega_r(f_1^{(n)},\xi)_1 + \omega_r(f_2^{(n)},\xi)_1 \right) \quad (199)$$

$$\times \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{r} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \text{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right)$$

$$+ \left(\|f_1(x)\|_p + \|f_2(x)\|_p \right) |m_{\xi,W} - 1|$$

holds.

Proof. By Theorem 22, and (191). ∎

Next, we demonstrate our simultaneous approximation results for the derivatives of the errors $E_{0,P}$, $E_{0,W}$, $E_{n,P}$, and $E_{n,W}$. We start with

Corollary 41 *Let $f_j^{(\tilde{j})} \in C_u(\mathbb{R})$, $\tilde{j} = 0, 1, \ldots, \rho$, $\rho \in \mathbb{Z}^+$, and $0 < \xi \leq 1$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then for all $\tilde{j} = 0, 1, \ldots, \rho$, we have*

i)

$$\left| (E_{0,P}(f,x))^{(\tilde{j})} \right| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(\omega_r(f_1^{(\tilde{j})}, |\nu|) + \omega_r(f_2^{(\tilde{j})}, |\nu|) \right) e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right) \quad (200)$$

$$+ \left(\left| f^{(\tilde{j})}(x) \right| + \left| f^{(\tilde{j})}(x) \right| \right) |m_{\xi,P} - 1|,$$

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ii)

$$
\left| (E_{0,W}(f,x))^{(\tilde{j})} \right| \leq \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(\omega_r(f_1^{(\tilde{j})}, |\nu|) + \omega_r(f_2^{(\tilde{j})}, |\nu|) \right) e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} \right) \tag{201}
$$
$$
+ \left(\left| f^{(\tilde{j})}(x) \right| + \left| f^{(\tilde{j})}(x) \right| \right) |m_{\xi,W} - 1|.
$$

Proof. By Corollary 37, and Theorem 27. ∎

We have also

Theorem 42 *Let $f_j \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}$, $\rho \in \mathbb{Z}^+$ and $f_j^{(n+\tilde{j})} \in C_u(\mathbb{R})$, $\tilde{j} = 0, 1, \ldots, \rho$, $0 < \xi \leq 1$, and $\left\| f^{(\tilde{j})} \right\|_{\infty,\mathbb{R}} < \infty$. We consider the assumptions of Theorem 27 valid for $n = \rho$. Then for all $\tilde{j} = 0, 1, \ldots, \rho$, we have*
i)

$$
\left\| (E_{n,P}(f,x))^{(\tilde{j})} \right\|_{\infty,x} \tag{202}
$$
$$
\leq \frac{\left(\omega_r(f_1^{(n+\tilde{j})}, \xi) + \omega_r(f_2^{(n+\tilde{j})}, \xi) \right)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi} \right)^r e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right)
$$
$$
+ \left(\left\| f_1^{(\tilde{j})} \right\|_{\infty,\mathbb{R}} + \left\| f_2^{(\tilde{j})} \right\|_{\infty,\mathbb{R}} \right) |m_{\xi,P} - 1|,
$$

and
ii)

$$
\left\| (E_{n,W}(f,x))^{(\tilde{j})} \right\|_{\infty,x} \tag{203}
$$
$$
\leq \frac{\left(\omega_r(f_1^{(n+\tilde{j})}, \xi) + \omega_r(f_2^{(n+\tilde{j})}, \xi) \right)}{n!} \left(\frac{\sum_{\nu=-\infty}^{\infty} |\nu|^n \left(1 + \frac{|\nu|}{\xi} \right)^r e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} \right)
$$
$$
+ \left(\left\| f_1^{(\tilde{j})} \right\|_{\infty,\mathbb{R}} + \left\| f_2^{(\tilde{j})} \right\|_{\infty,\mathbb{R}} \right) |m_{\xi,W} - 1|.
$$

Proof. By Theorems 27, 38. ∎

Next, we give following L_p approximation results. We obtain

Theorem 43 i) *Let $f_j \in C^{n+\rho}(\mathbb{R})$, with $f_j^{(n+\tilde{j})} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $\tilde{j} = 0, 1, \dots, \rho \in \mathbb{Z}^+$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $np \neq 1$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then for all $j = 0, 1, \dots, \rho$,*

$$\left\| (E_{n,P}(f,x))^{(\tilde{j})} \right\|_p \tag{204}$$

$$\leq \left(\omega_r(f_1^{(n+\tilde{j})}, \xi) + \omega_r(f_2^{(n+\tilde{j})}, \xi) \right)$$

$$x \left(\frac{\xi^{\frac{1}{p}} \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \right)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{p}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_p + \left\| f_2^{(\tilde{j})}(x) \right\|_p \right) |m_{\xi,P} - 1|$$

holds.

ii) *Let $f_j \in C^{n+\rho}(\mathbb{R})$, with $f_j^{(n+\tilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N} - \{1\}$, $\tilde{j} = 0, 1, \dots, \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then for all $\tilde{j} = 0, 1, \dots, \rho$,*

$$\left\| (E_{n,P}(f,x))^{(\tilde{j})} \right\|_1 \tag{205}$$

$$\leq \xi \frac{\left(\omega_r(f_1^{(n+\tilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\tilde{j})}, \xi)_1 \right)}{(n-1)! (r+1)}$$

$$\times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_1 + \left\| f_2^{(\tilde{j})}(x) \right\|_1 \right) |m_{\xi,P} - 1|$$

holds.

iii) *Let $f_j^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\tilde{j} = 0, 1, \dots, \rho \in \mathbb{Z}^+$; $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the assumptions of Theorem 27 as valid for $n = \rho$*

there. Then for all $\tilde{j} = 0, 1, \ldots, \rho$,

$$
\left\| (E_{0,P}(f,x))^{(\tilde{j})} \right\|_p \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_p + \omega_r(f_2^{(\tilde{j})}, \xi)_p \right) \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}} \right)^{\frac{1}{q}} (206)
$$

$$
\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{rp} e^{\frac{-|\nu|}{\xi}} \right)^{1/p}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right]
$$

$$
+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_p + \left\| f_2^{(\tilde{j})}(x) \right\|_p \right) |m_{\xi,P} - 1|
$$

holds.

iv) *Let $f_j^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_1(\mathbb{R})), \tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then for all $\tilde{j} = 0, 1, \ldots, \rho$,*

$$
\left\| (E_{0,P}(f,x))^{(\tilde{j})} \right\|_1 \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_1 + \omega_r(f_2^{(\tilde{j})}, \xi)_1 \right) \tag{207}
$$

$$
\times \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi}\right)^{r} e^{\frac{-|\nu|}{\xi}}}{1 + 2\xi e^{-\frac{1}{\xi}}} \right)
$$

$$
+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_1 + \left\| f_2^{(\tilde{j})}(x) \right\|_1 \right) |m_{\xi,P} - 1|
$$

holds.

Proof. By Theorems 27, 39. ∎

For $E_{n,W}(f,x)$, we have

Theorem 44 i) *Let $f_j \in C^{n+\rho}(\mathbb{R})$, with $f_j^{(n+\tilde{j})} \in L_p(\mathbb{R})$, $n \in \mathbb{N}, \tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $np \neq 1$. We consider the assumptions of Theorem*

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27 as valid for $n = \rho$ there. Then for all $\tilde{j} = 0, 1, \ldots, \rho$,

$$\left\| (E_{n,W}(f,x))^{(\tilde{j})} \right\|_p \tag{208}$$

$$\leq \left(\omega_r(f_1^{(n+\tilde{j})}, \xi) + \omega_r(f_2^{(n+\tilde{j})}, \xi) \right)$$

$$\times \left(\frac{\xi^{\frac{1}{p}} \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}}}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \right)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{rp+1} - 1 \right) |\nu|^{np-1} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_p + \left\| f_2^{(\tilde{j})}(x) \right\|_p \right) |m_{\xi,W} - 1|$$

holds.

ii) *Let $f \in C^{n+\rho}(\mathbb{R})$, with $f^{(n+\tilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N} - \{1\}$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorem 27 as valid for $n = \rho$ there. Then for all $\tilde{j} = 0, 1, \ldots, \rho$, we have*

$$\left\| (E_{n,W}(f,x))^{(\tilde{j})} \right\|_1 \tag{209}$$

$$\leq \xi \frac{\left(\omega_r(f_1^{(n+\tilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\tilde{j})}, \xi)_1 \right)}{(n-1)! \, (r+1)}$$

$$\times \left[\frac{\sum_{\nu=-\infty}^{\infty} \left(\left(1 + \frac{|\nu|}{\xi}\right)^{r+1} - 1 \right) |\nu|^{n-1} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi\xi} \left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{\xi}}\right)\right) + 1} \right]$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_1 + \left\| f_2^{(\tilde{j})}(x) \right\|_1 \right) |m_{\xi,W} - 1|$$

holds.

iii) *Let $f^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the assumptions of Theorem 27 as valid for $n = \rho$*

there. Then for all $\tilde{j} = 0, 1, \ldots, \rho,$

$$\left\| (E_{0,W}(f,x))^{(\tilde{j})} \right\|_p \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_p + \omega_r(f_2^{(\tilde{j})}, \xi)_p \right) \left(\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{q}} (210)$$

$$\times \left[\frac{\left(\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{rp} e^{\frac{-\nu^2}{\xi}} \right)^{\frac{1}{p}}}{\sqrt{\pi \xi} \left(1 - \operatorname{erf} \left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} \right]$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_p + \left\| f_2^{(\tilde{j})}(x) \right\|_p \right) |m_{\xi,W} - 1|$$

holds.

iv) *Let* $f^{(\tilde{j})} \in (C(\mathbb{R}) \cap L_1(\mathbb{R})), \tilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+.$ *We consider the assumptions of Theorem 27 as valid for* $n = \rho$ *there. Then for all* $\tilde{j} = 0, 1, \ldots, \rho,$

$$\left\| (E_{0,W}(f,x))^{(\tilde{j})} \right\|_1 \leq \left(\omega_r(f_1^{(\tilde{j})}, \xi)_1 + \omega_r(f_2^{(\tilde{j})}, \xi)_1 \right) \qquad (211)$$

$$\times \left(\frac{\sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right)^{r} e^{\frac{-\nu^2}{\xi}}}{\sqrt{\pi \xi} \left(1 - \operatorname{erf} \left(\frac{1}{\sqrt{\xi}} \right) \right) + 1} \right)$$

$$+ \left(\left\| f_1^{(\tilde{j})}(x) \right\|_1 + \left\| f_2^{(\tilde{j})}(x) \right\|_1 \right) |m_{\xi,W} - 1|$$

holds.

Proof. By Theorems 27, 40. ∎

Finally, we give the fractional results for the error terms as

Theorem 45 *Here* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2.$ *Let* $f_j \in C^m(\mathbb{R}),$ $m = \lceil \gamma \rceil, \gamma > 0,$ *with* $\left\| f_j^{(m)} \right\|_\infty < \infty, 0 < \xi \leq 1,$ *and* $\bar{\delta}_k$ *be as in* (26). *Then*

i)

$$\|E_{\gamma,P}(f)\|_{\infty} \tag{212}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k} S_{\gamma,k}^4\right]$$

$$\cdot \left(\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_1,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_1,\xi\right)\right]\right\}\right.$$

$$+\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_2,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_2,\xi\right)\right]\right\}\right)$$

$$+\left(\|f_1\|_{\infty}+\|f_2\|_{\infty}\right)|m_{\xi,P}-1|,$$

where $S_{\gamma,k}^4$ is defined as in (102).

ii)

$$\|E_{\gamma,W}(f)\|_{\infty} \tag{213}$$

$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k} S_{\gamma,k}^5\right]$$

$$\cdot \left(\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_1,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_1,\xi\right)\right]\right\}\right.$$

$$+\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_2,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_2,\xi\right)\right]\right\}\right)$$

$$+\left(\|f_1\|_{\infty}+\|f_2\|_{\infty}\right)|m_{\xi,W}-1|,$$

where $S_{\gamma,k}^5$ is defined as in (104).

Proof. By Theorems 13, 24. ∎

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