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MATRIX–VALUED WAVELETS AND MULTIRESOLUTION ANALYSIS

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ABSTRACT. We introduce the notions of matrix–valued wavelet set and matrix–valued multiresolution analysis (A-MMRA) associated with a fixed dilation given by an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $n \geq 1$. These are generalizations of the corresponding notions defined by Xia and Suter in 1996 for the case where $d = 1$ and A is the dyadic dilation. We show several properties of orthonormal sequences of translates by integers of matrix–valued functions, focusing on those related to the structure of A-MMRA's and their connection with matrix–valued wavelet sets. Further, we present a strategy for constructing matrix–valued wavelet sets from a given A-MMRA and, in addition, we characterize those matrix–valued wavelet sets which may be built from an A-MMRA.

1. Introduction

Given a fixed expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 1$, such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, we introduce the notion of matrix–valued wavelet and matrix–valued multiresoltuion analysis associated to A in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $n \geq 1$. A linear map A is said to be expansive if all (complex) eigenvalues of A have modulus greater than 1. The subject of this paper is the study of such wavelets and multiresolution analyses. Our starting point is the paper by Xia and Suter [22] where the notion of matrix–valued wavelet and matrix–valued multiresolution analysis have been introduced and studied for the case of $d = 1$ and dyadic dilations. Subsequently, and in this particular context, there appeared several papers related to matrix–valued multiresolution analyses and matrix–valued wavelets and their construction, e.g. [25], [1], [23], [28]. The notion of matrix–valued multiresolution analysis and matrix–valued wavelets when $d = 1$ and A may be any arbitrary integer dilation were introduced in [6], where a necessary and sufficient condition for the existence of matrix–valued wavelets and an algorithm for constructing compactly supported matrix–valued wavelets associated with an integer dilation factor m are presented. For the case $m = 4$ see [4].

Relaxing requirements, the articles [21], [5], [11], [8] study biorthogonal matrix–valued wavelets where $d = 1$ and A is the dyadic dilation.

Since matrix–valued function spaces $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ are related to video imaging, we generalize results in [27] to these spaces with the purpose of showing that the ideas developed there for scalar–valued wavelets and multiresolution analysis fit perfectly in this context. That is our motivation for writing this article.

Key words and phrases. matrix–valued function spaces, Fourier transform, multiresolution analysis, wavelet set.

2010 Mathematics Subject Classification: 42C40.

This work is organized as follows. In Section 2 we present the definitions and notation that will be used. Section 3 contains several properties of orthonormal sequences of integer translates of a function in a matrix–valued space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, focusing on those related to multiresolution analyses and their connection with wavelet sets. Section 4 is devoted to the study of matrix–valued wavelet sets in a signal space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, associated with a dilation given by an expansive linear map A. In addition, as a method for constructing these matrix–valued wavelet sets we introduce the notion of vector–valued multiresolution analysis associated with an expansive linear map A (A-MMRA). Futher, we study the structure of A-MMRA's, present a strategy for constructing matrix–valued wavelet sets and characterize those sets constructed from a given A-MMRA. Our results are given in the context of Fourier space.

2. Notation and basic definitions

The sets of integers, real and complex numbers will be denoted by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. The d–fold product of the interval $[0, 1)$ with itself will be denoted $\mathbb{T}^d$. Thus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 1$.

Unless otherwise indicated, I_n, $n \geq 1$, will denote the $n \times n$ identity matrix and $\mathbf{0}_n$ will denote the $n \times n$ null matrix.

Given an $n \times n$, $n \geq 1$, complex matrix M, $a_{ml} \in \mathbb{C}$ will denote the element on the m-th row and the l-th column of M. The complex vector space of all $n \times n$ complex matrices M will be denoted by $\mathcal{M}_n(\mathbb{C})$. Recall that a matrix $M \in \mathcal{M}_n(\mathbb{C})$ is said to be unitary if $MM^* = I_n$ where M^* is the transpose of the complex conjugate of M.

Let

$$l^2(\mathbb{N}, \mathbb{C}^{n\times n}) := \{\mathbf{M} = \{M_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_n(\mathbb{C}) : \|\mathbf{M}\| = (\sum_{m,l=1}^{n} \sum_{k\in\mathbb{N}} |a_{ml}(k)|^2)^{1/2} < \infty\}.$$

The space $l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$ is similarly defined.

All functions considered in this paper will be assumed to be measurable.

Given $d, n \geq 1$, by $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ we will denote the space

$$\{\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \cdots & f_{1n}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) & \cdots & f_{2n}(\mathbf{x}) \\ & & \cdots & \\ f_{n1}(\mathbf{x}) & f_{n2}(\mathbf{x}) & \cdots & f_{nn}(\mathbf{x}) \end{pmatrix} : f_{ml} \in L^2(\mathbb{R}^d), m, l = 1, ..., n\}.$$

We will also write $\mathbf{f}(\mathbf{x}) = (f_{ml}(\mathbf{x}))_{m,l=1,\ldots,n}$. The spaces $L^p(E, \mathbb{C}^{n\times n})$, $1 \leq p < \infty$, where E is a measurable set in $\mathbb{R}^d$ are defined similarly by replacing $\mathbb{R}^d$ and 2 by the E and p respectively. If we write $\mathbf{f} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$ we will also mean that $\mathbf{f}$ is defined on the whole space $\mathbb{R}^d$ as a $\mathbb{Z}^d$-periodic matrix–valued function.

Given $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $\|\mathbf{f}\|$, will denote the Frobenius norm defined by (see [22])

$$\|\mathbf{f}\| := (\sum_{m,l=1}^{n} \int_{\mathbb{R}^d} |f_{ml}(\mathbf{x})|^2 d\mathbf{x})^{1/2}. \tag{1}$$

The integral of a matrix–valued function $\mathbf{f}$, $\int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})d\mathbf{x}$, is defined by

$$\int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})d\mathbf{x} := (\int_{\mathbb{R}^d} f_{ml}(\mathbf{x})d\mathbf{x})_{m,l=1,\dots,n}.$$

The Fourier transform of a matrix–valued function f will be denoted by $\widehat{f}$. For $f \in L^1(\mathbb{R}^d, \mathbb{C}^{n\times n}) \cap L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$

$$\widehat{\mathbf{f}}(\mathbf{t}) := \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\mathbf{t}}d\mathbf{x}.$$

For two matrix-valued functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$,

$$\langle \mathbf{f}, \mathbf{g}\rangle := \int_{\mathbb{R}^d} \mathbf{f}(\mathbf{x})\mathbf{g}^*(\mathbf{x})d\mathbf{x} \tag{2}$$

and

$$[\mathbf{f}, \mathbf{g}](\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathbf{f}(\mathbf{t}+\mathbf{k})\mathbf{g}^*(\mathbf{t}+\mathbf{k}).$$

Note that $\langle\cdot,\cdot\rangle$ is matrix–valued and therefore it is not an inner product. It has the following properties:

(a) For every $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$,

$$\langle \mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{g}, \mathbf{f}\rangle^*;$$

(b) For every $\mathbf{f}, \mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ and every $M_1, M_2 \in \mathcal{M}_n(\mathbb{C})$,

$$\langle M_1\mathbf{f} + M_2\mathbf{h}, \mathbf{g}\rangle = M_1 \langle \mathbf{f}, \mathbf{g}\rangle + M_2 \langle \mathbf{h}, \mathbf{g}\rangle.$$

Moreover, the scalar Plancherel formula implies that also in the matrix–valued case

$$\langle \mathbf{f}, \mathbf{g}\rangle = \left\langle \widehat{\mathbf{f}}, \widehat{\mathbf{g}}\right\rangle.$$

It is also readily seen that

$$\|\mathbf{f}\| = (\text{trace } \langle \mathbf{f}, \mathbf{f}\rangle)^{1/2}.$$

Given an invertible map $M : \mathbb{R}^n \to \mathbb{R}^n$, for every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $\mathbf{D}_j^M$ and the translation operator $\mathbf{T_k}$ are defined on $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ by

$$\mathbf{D}_j^M\mathbf{f}(\mathbf{t}) := d_M^{j/2}\mathbf{f}(M^j\mathbf{t}) \quad \text{and} \quad \mathbf{T_k}\mathbf{f}(\mathbf{t}) := \mathbf{f}(\mathbf{t}+\mathbf{k}),$$

where $d_M = |\det M|$. A set $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called *shift-invariant* if $\mathbf{f} \in S$ implies that $\mathbf{T_k f} \in S$ for every $\mathbf{k} \in \mathbb{Z}^n$. Let $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, then

$$\mathbf{T}(\mathbf{F}) := \{\mathbf{T_k f} \;:\; \mathbf{f} \in \mathbf{F}, \mathbf{k} \in \mathbb{Z}^n\} \quad \text{and} \quad S(\mathbf{F}) := \overline{\text{span}}\mathbf{T}(\mathbf{F}),$$

where the closure is in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ then $S(\mathbf{F})$ is called a *finitely generated shift-invariant space* or FSI and the functions $\mathbf{f}_l$, $l = 1, ..., m$ are called the generators of $S(\mathbf{F})$. In this case we will also use the symbols $\mathbf{T}(\mathbf{f}_1, ..., \mathbf{f}_m)$ and $S(\mathbf{f}_1, ..., \mathbf{f}_m)$ to denote $\mathbf{T}(\mathbf{F})$ and $S(\mathbf{F})$ respectively. If $\mathbf{F}$ contains a single element, then $S(\mathbf{F})$ is called a *principal shift-invariant space* or PSI.

Two functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ are said to be orthogonal if $\langle \mathbf{f}, \mathbf{g}\rangle = \mathbf{0}_n$. Further, let V, W be two closed subspaces in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $W \subset V$, then the *orthogonal complement* of W in V is the closed subspace defined by

$$W^\perp = \{\mathbf{g} \in V : \langle \mathbf{g}, \mathbf{f}\rangle = \mathbf{0}_n \quad \forall \mathbf{f} \in W\}.$$

A sequence $\{\mathbf{f}_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called an orthonormal set in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ if

$$\langle \mathbf{f}_k, \mathbf{f}_l \rangle = \begin{cases} I_n & \text{if } k = l \\ \mathbf{0}_n & \text{if } k \neq l. \end{cases} \tag{3}$$

Given a closed subspace S in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, a sequence $\{\mathbf{f}_k\}_{k=1}^{\infty} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is called an *orthonormal basis* for S if it satisfies (3), and moreover, for any $\mathbf{g} \in S$ there exists an unique sequence of constant matrices $\{H_k\}_{k=1}^{\infty} \in l^2(\mathbb{N}, \mathbb{C}^{n\times n})$ such that

$$\mathbf{g}(\mathbf{x}) = \sum_{k=1}^{\infty} H_k \mathbf{f}_k(\mathbf{x}), \qquad \text{for } \mathbf{x} \in \mathbb{R}^d$$

where, for each $\mathbf{x}$, $H_k\mathbf{f}_k(\mathbf{x})$ is the product of the $n \times n$ matrices H_k and $\mathbf{f}_k(\mathbf{x})$, and the convergence for the infinite sum is in the sense of the norm $\|\cdot\|$ defined by (1). It readily follows that for every $k = 1, 2, \ldots,$

$$H_k = \langle \mathbf{g}, \mathbf{f}_k \rangle, \qquad \text{and} \qquad \|\{H_k\}_{k=1}^{\infty}\| = \|\mathbf{g}\|. \tag{4}$$

Given a set of matrix–valued functions $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, its Gramian matrix will be denoted by $\mathbf{G}[\mathbf{f}_1, ..., \mathbf{f}_m](\mathbf{t})$ or $\mathbf{G}_{\mathbf{F}}(\mathbf{t})$ and defined as follows:

$$\mathbf{G}_{\mathbf{F}}(\mathbf{t}) := \left([\widehat{\mathbf{f}}_l, \widehat{\mathbf{f}}_j](\mathbf{t})\right)_{l,j=1}^{m}.$$

3. Orthonormal bases of translates

In this section we show several properties on orthonormal sequences of integral translates of functions in a matrix–valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. We focus on those properties closely related to matrix–valued wavelets and matrix–valued multiresoltion analyses, concepts that will be discussed in the next section. Most of the properties presented here are well known in the scalar–valued function space context (cf. e.g. [27]).

The following lemma generalizes a result in [22].

Lemma 1. *Let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. Then $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ if and only if $\mathbf{G}_{\mathbf{F}}(\mathbf{t}) = I_{nm}$ a.e.*

Proof. Let us prove the necessity. By the orthonormality of $\mathbf{T}(\mathbf{F})$, given $j, p \in \{1, ..., m\}$ and $\mathbf{k} \in \mathbb{Z}^d$ we have

$$\int_{\mathbb{R}^d} \mathbf{f}_j(\mathbf{x})\mathbf{f}_p^*(\mathbf{x} - \mathbf{k})d\mathbf{x} = \delta(j, p)\delta(\mathbf{k}, \mathbf{0})I_n, \tag{5}$$

where $\delta(\alpha, \beta) = 1$ if $\alpha = \beta$ and $\delta(\alpha, \beta) = 1$ if $\alpha \neq \beta$. By Plancherel's formula,

$$\begin{aligned} \delta(j,p)\delta(\mathbf{k},\mathbf{0})I_n &= \int_{\mathbb{R}^d} \widehat{\mathbf{f}}_j(\mathbf{t})\widehat{\mathbf{f}}_p^{\,*}(\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t} \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \int_{[-1/2,1/2]^d+\mathbf{k}} \widehat{\mathbf{f}}_j(\mathbf{t})\widehat{\mathbf{f}}_p^{\,*}(\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t} \\ &= \int_{[-1/2,1/2]^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} [\widehat{\mathbf{f}}_j, \widehat{\mathbf{f}}_p](\mathbf{t})e^{2\pi i\mathbf{k}}d\mathbf{t}, \qquad \forall \mathbf{k} \in \mathbb{Z}^d. \end{aligned} \tag{6}$$

This implies that $[\widehat{\mathbf{f}}_j, \widehat{\mathbf{f}}_p](\mathbf{t}) = \delta(j,p)I_n$ a.e. on $\mathbb{R}^d$, whence the assertion follows.

Conversely, note that the orthonormality of $\mathbf{T}(\mathbf{F})$ follows immediately from $\mathbf{G}_{\mathbf{F}}(\mathbf{t}) = I_{nm}$ a.e., (5) and (6). □

Lemma 2. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *and assume that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *Then, a matrix–valued function* $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *belongs to* S *if and only if there are* $\mathbb{Z}^d$*-periodic functions* $\mathbf{H}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$, *such that*

$$\widehat{\mathbf{g}}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_j(\mathbf{t})\widehat{\mathbf{f}}_j(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d, \tag{7}$$

and

$$\| \mathbf{g} \|^2 = \sum_{j=1}^{m} \| \mathbf{H}_j \|^2 \tag{8}$$

Proof. Suppose that $\mathbf{g} \in S$, then we may represent it in terms of the orthonormal basis $\mathbf{T}(\mathbf{F})$ as

$$\mathbf{g}(\mathbf{x}) = \sum_{j=1}^{m} \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} \mathbf{f}_j(\mathbf{x} - \mathbf{k}), \tag{9}$$

where $\{H_{j,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$, $j \in \{1, ..., m\}$, and the convergence of the sum is in the sense of the norm $\|\cdot\|$ defined by (1). Thus, taking the Fourier transform in (9) we obtain (7) with $\mathbf{H}_j(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$.

From (9) and (4) we deduce that $\| g \| = \| \{H_{j,\mathbf{k}}\}_{j=1,...,m,\mathbf{k}\in\mathbb{Z}^d} \|$. Since $\| \mathbf{H}_j \| = \| \{H_{j,\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d} \|$, equation (8) follows.

Conversely, assume that (7) holds. Since $\mathbf{H}_j(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} H_{j,\mathbf{k}} e^{-2\pi i \mathbf{k}\cdot\mathbf{t}}$ with $H_{j,\mathbf{k}} \in l^2(\mathbb{Z}^d, \mathbb{C}^{n\times n})$, we deduce that (9) is satisfied in the sense of convergence in norm, and therefore $\mathbf{g} \in S$. □

Lemma 3. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ *and* $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ *be in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *Assume that* $\mathbf{T}(\mathbf{G})$ *and* $\mathbf{T}(\mathbf{F})$ *are orthonormal sequences in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, *and that there are* $\mathbb{Z}^d$*-periodic functions* $\mathbf{H}_{l,j} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$, $l = 1, ..., p$, *such that*

$$\widehat{\mathbf{g}_l}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\widehat{\mathbf{f}_j}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d \qquad l = 1, ..., p. \tag{10}$$

Then

$$\sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{r,j}(\mathbf{t}) = I_n \delta(l, r) \qquad \text{a.e. on } \mathbb{R}^d \qquad l, r \in \{1, ..., p\}. \tag{11}$$

Proof. Since both sequences are orthonormal, given $l, r \in \{1, ..., p\}$, (3) yields

$$\begin{aligned} I_n\delta(l, r) &= [\widehat{\mathbf{g}_l}, \widehat{\mathbf{g}_r}](\mathbf{t}) = [\sum_{j=1}^{m} \mathbf{H}_{l,j}\widehat{\mathbf{f}_j}, \sum_{j=1}^{m} \mathbf{H}_{r,j}\widehat{\mathbf{f}_j}](\mathbf{t}) \\ &= \sum_{j=1}^{m}\sum_{q=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})[\widehat{\mathbf{f}_j}, \widehat{\mathbf{f}_q}](\mathbf{t})\mathbf{H}^*_{r,q}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{r,j}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d. \end{aligned}$$

□

We are now ready to prove

Proposition 1. *Let $p \leq m$ and let S be a closed subspace of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. Let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ and $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ be such that $\mathbf{T}(\mathbf{F})$ and $\mathbf{T}(\mathbf{G})$ belong to S. If $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S, then $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in S if and only if there exists a matrix $\mathbf{Q}(\mathbf{t}) := \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{np,nm}$ where $h_{q,r} \in L^2(\mathbb{T}^d)$, which satisfies $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. $\mathbf{t}$ on $\mathbb{R}^d$ and also,*

$$\big(\widehat{\mathbf{g}}_1(\mathbf{t}), ..., \widehat{\mathbf{g}}_p(\mathbf{t})\big)^T = \mathbf{Q}(\mathbf{t})\big(\widehat{\mathbf{f}}_1(\mathbf{t}), ..., \widehat{\mathbf{f}}_m(\mathbf{t})\big)^T \quad \textit{a.e. on } \mathbb{R}^d. \tag{12}$$

The $\mathbb{Z}^d$-periodic matrix

$$\mathbf{Q}(\mathbf{t}) = \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{np,nm}$$

will be called a *transition matrix* from the sequence $\mathbf{T}(\mathbf{F})$ to the sequence $\mathbf{T}(\mathbf{G})$.

Proof. To prove the necessity we proceed as follows: Since $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S and $\mathbf{T}(\mathbf{G}) \subset S$, Lemma 2 tells us that there are $\mathbf{H}_{l,j} \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$ and $l = 1, ..., p$, such that

$$\widehat{\mathbf{g}_l}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\widehat{\mathbf{f}_j}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d \qquad l = 1, ..., p. \tag{13}$$

Let $\mathbf{Q}(\mathbf{t})$ be the $np \times nm$ block matrix $\mathbf{Q}(\mathbf{t}) := \big(\mathbf{H}_{l,j}(\mathbf{t})\big)_{l,j=1}^{p,m}$, and for $q = 1, \dots np$, let $\mathbf{v}_q(\mathbf{t}) = (h_{q,1}(\mathbf{t}), ..., h_{q,nm}(\mathbf{t}))$, $q = 1, ..., np$, be the q-th row of $\mathbf{Q}(\mathbf{t})$. (Note that every $h_{q,r}$ belongs to $L^2(\mathbb{T}^d)$). Then, (11) implies that the vectors $\{\mathbf{v}_q(\mathbf{t}) : q \in \{1, ..., np\}\}$ are orthonormal a.e.$(\mathbf{t})$. Thus, setting $\mathbf{Q}(\mathbf{t}) := (h_{q,r}(\mathbf{t}))_{q,r=1}^{np,nm}$ we conclude that $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$. Finally, note that (13) readily implies (12).

To prove the sufficiency, for any $l \in \{1, ..., p\}$ and $j \in \{1, ..., m\}$ let $\mathbf{H}_{l,j}$ in $L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$ be defined by $\mathbf{H}_{l,j} := \big(h_{q,r}\big)_{q=(l-1)n+1, r=(j-1)n+1}^{ln,jn}$. Then (12) yields (13). In addition, the assumption $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. on $\mathbb{R}^d$ implies that

$$\sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{b,j}(\mathbf{t}) = I_n\delta(l, b) \qquad \text{a.e. on } \mathbb{R}^d \qquad l, b \in \{1, ..., p\}.$$

We complete the proof by showing that the Gramian associated to $\mathbf{G}$ is the unitary matrix a.e. on $\mathbb{R}^d$ and applying Lemma 1. For $l \in \{1, ..., p\}$ and $b \in \{1, ..., m\}$ we have:

$$\begin{aligned}
[\widehat{\mathbf{g}_l}, \widehat{\mathbf{g}_b}](\mathbf{t}) &= [\sum_{j=1}^{m} \mathbf{H}_{l,j}\widehat{\mathbf{f}_j}, \sum_{j=1}^{m} \mathbf{H}_{b,j}\widehat{\mathbf{f}_j}](\mathbf{t}) \\
&= \sum_{j=1}^{m}\sum_{q=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})[\widehat{\mathbf{f}_j}, \widehat{\mathbf{f}_q}](\mathbf{t})\mathbf{H}^*_{b,q}(\mathbf{t}) = \sum_{j=1}^{m} \mathbf{H}_{l,j}(\mathbf{t})\mathbf{H}^*_{b,j}(\mathbf{t}) = I_n\delta(l, b)
\end{aligned}$$

a.e. on $\mathbb{R}^d$, and the assertion follows. □

Proposition 2. *Assume that $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}$ and $\mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_p\}$ are functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If $\mathbf{T}(\mathbf{F})$ and $\mathbf{T}(\mathbf{G})$ are orthonormal bases of the same closed subspace $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, then $m = p$.*

Proof. By the symmetry in the notation we may assume, without loss of generality, that $p > m$. Since $\mathbf{T}(\mathbf{F})$ is an orthonormal basis for S and $\mathbf{T}(\mathbf{G}) \subset S$, we infer from Proposition 1 that there exists an $np \times nm$ matrix $\mathbf{Q}(\mathbf{t})$ such that $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. This means that the np vectors defined by the rows of the matrix $\mathbf{Q}(\mathbf{t})$

are orthonormal in the complex vector space $\mathbb{C}^{nm}$. Since $nm < np$, we get a contradiction. $\square$

Proposition 3. *Asssume that the matrix–valued functions*

$$\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}, \mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$$

are such that $\mathbf{T}(\mathbf{F})$ *and* $\mathbf{T}(\mathbf{G})$ *are orthonormal sequences in a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *If* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis for* S *and there exists a matrix* $\mathbf{Q}(\mathbf{t}) := \big(h_{l,j}(\mathbf{t})\big)_{l,j=1}^{nm}$, *where* $h_{l,j} \in L^2(\mathbb{T}^d)$, *such that* $\mathbf{Q}(\mathbf{t})$ *is unitary a.e.* $(\mathbf{t})$ *on* $\mathbb{R}^d$ *and (12) holds, then* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis for* S.

Proof. According to Proposition 1, $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in S. Thus it suffices to show that $S = \overline{\mathrm{span}}\mathbf{T}(\mathbf{G})$. The hypotheses imply that we only need to check that $S \subset \mathbf{T}(\mathbf{G})$. Let $\mathbf{h} \in S$ then, by Lemma 2 there exist $\mathbf{H}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $j = 1, ..., m$, such that

$$\widehat{\mathbf{h}}(\mathbf{t}) = \big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\big(\widehat{\mathbf{f}_1}(\mathbf{t}), ..., \widehat{\mathbf{f}_m}(\mathbf{t})\big)^T \qquad \text{a.e. on } \mathbb{R}^d.$$

Thus, by (12)

$$\begin{aligned} \widehat{\mathbf{h}}(\mathbf{t}) &= \big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\mathbf{Q}^*(\mathbf{t})\big(\widehat{\mathbf{g}_1}(\mathbf{t}), ..., \widehat{\mathbf{g}_m}(\mathbf{t})\big)^T \\ &= \big(\mathbf{L}_1(\mathbf{t}), ..., \mathbf{L}_m(\mathbf{t})\big)\big(\widehat{\mathbf{g}_1}(\mathbf{t}), ..., \widehat{\mathbf{g}_m}(\mathbf{t})\big)^T \qquad \text{a.e. on } \mathbb{R}^d, \end{aligned}$$

where $\mathbf{L}_j(\mathbf{t}) = \big(\mathbf{v}_{(j-1)n+1}(\mathbf{t}), ..., \mathbf{v}_{jn}(\mathbf{t})\big)$ is the $n \times nm$ matrix such that $\mathbf{v}_l$ is the l-th column vector of the matrix $\big(\mathbf{H}_1(\mathbf{t}), ..., \mathbf{H}_m(\mathbf{t})\big)\mathbf{Q}^*(\mathbf{t})$. Observe that for every $j \in \{1, ..., m\}$ the entries of the matrix $\mathbf{L}_j$ are $\mathbb{Z}^d$-periodic functions. Applying the Minkowski and Hölder inequalities, we conclude that $\mathbf{L}_j \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, and the conclusion follows by another application of Lemma 2. $\square$

A straightforward consequence of the preceding propositions is the following.

Corollary 1. *Let* $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_m\}, \mathbf{G} = \{\mathbf{g}_1, ..., \mathbf{g}_m\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *and* $\mathbf{T}(\mathbf{G})$ *are orthonormal sequences in a closed subspace* $S \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. *If* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis for* S, *then* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis for* S.

Proof. By Proposition 1, there exists a matrix $\mathbf{Q}(\mathbf{t}) := \big(h_{q,r}(\mathbf{t})\big)_{q,r=1}^{nm}$ where $h_{q,r} \in L^2(\mathbb{T}^d)$, which satisfies $\mathbf{Q}(\mathbf{t})\mathbf{Q}^*(\mathbf{t}) = I_{np}$ a.e. $(\mathbf{t})$ on $\mathbb{R}^d$ and also (12) holds. Thus, the proof is finished by Proposition 3. $\square$

4. Wavelets and Multiresolution analysis

In what follows we will assume that A is an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. Here and further we use the same notation for a linear map on $\mathbb{R}^d$ and its matrix with respect to the canonical base.

In this section we introduce the notions of matrix–valued wavelet set and matrix–valued multiresolution analysis (A-MMRA) associated with a dilation given by a fixed map A as above in a signal space $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, $d, n \geq 1$. These definitions generalize the matrix–valued wavelet and matrix–valued multiresolution analysis notions defined in [22] when $d = 1$ and A is the dyadic dilation. We study the structure of an A-MMRA, present a strategy to construct matrix–valued wavelet sets associated with a fixed dilation A and characterize the matrix–valued wavelet sets constructed from a given A-MMRA.

Given an expansive linear map A, a matrix-valued wavelet set associated with A is a finite set of functions $\{\Psi_1, ..., \Psi_s\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that the system

$$\{\mathbf{D}_A^j \mathbf{T}_\mathbf{k} \Psi_r : r = 1, 2..., s,\ j \in \mathbb{Z},\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonomal basis for $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$.

A general method for constructing matrix–valued wavelet sets on $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is related to the concept of matrix–valued multiresolution analysis associated with A (A-MMRA): Given an expansive linear map A as above, we define an A-MMRA as a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ that satisfies the following conditions:

(i) For every $j \in \mathbb{Z}$, $V_j \subset V_{j+1}$;
(ii) For every $j \in \mathbb{Z}$, $\mathbf{f}(\mathbf{x}) \in V_j$ if and only if $\mathbf{f}(A\mathbf{x}) \in V_{j+1}$;
(iii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$;
(iv) There exists a function $\Phi \in V_0$, called a *scaling function*, such that

$$\{\ \mathbf{T}_\mathbf{k} \Phi(\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^n\ \}$$

is an orthonormal basis for V_0.

To construct a matrix–valued wavelet set associated with a dilation map A from an A-MMRA with scaling function Φ, we denote by W_j the orthogonal complement of V_j in V_{j+1}. Thus, by condition (i), we have $V_{j+1} = W_j \oplus V_j$. Moreover, condition (iii) implies that $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n}) = \oplus_{j\in\mathbb{Z}} W_j$.

Observe that by condition (ii) we have

$$\forall j \in \mathbb{Z}, \qquad \mathbf{f}(\cdot) \in W_0 \Leftrightarrow \mathbf{f}(A^j \cdot) \in W_j. \tag{14}$$

Thus, to find a matrix–valued wavelet set from an A-MMRA, it will suffice to construct a set of functions $\{\Psi_1, ..., \Psi_s\} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that the system

$$\{\mathbf{T}_\mathbf{k} \Psi_r : r = 1, 2..., s,\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonomal basis for W_0, for then

$$\{\mathbf{D}_A^j \mathbf{T}_\mathbf{k} \Psi_r : r = 1, 2..., s,\ \mathbf{k} \in \mathbb{Z}^d\},$$

is an orthonormal basis of W_j.

We now focus on how to construct orthonormal bases of integer translates for the subspaces V_1 and W_0. For this purpose we study the structure of the subspaces V_j and W_j.

Let us recall that if $A : \mathbb{R}^d \to \mathbb{R}^d$ is an expansive linear map such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, then the quotient group $\mathbb{Z}^d/A(\mathbb{Z}^d)$ is well defined. We will denote by $\Delta_A \subset \mathbb{Z}^d$ a full collection of representatives of the cosets of $\mathbb{Z}^d/A(\mathbb{Z}^d)$. There are exactly d_A cosets (see [10] and [24, p. 109]). Let $\Delta_A = \{\mathbf{q}_i\}_{i=0}^{d_A-1}$ where $\mathbf{q}_0 = \mathbf{0}$.

Note that, if $l \in \{0, 1, 2, ...\}$, then $l = ad_A + i$, where $a \in \{0, 1, 2, ...\}$ and $i \in \{0, 1, ...d_A - 1\}$.

We have:

Theorem 1. *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear map such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$. Let $\mathbf{F} = \{\mathbf{f}_0, ..., \mathbf{f}_{m-1}\}$ be a set of functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of a closed subspace V of $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$, and let $U = \{\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n}) : \mathbf{f}(A^{-1}\cdot) \in V\}$. If*

$$\mathbf{g}_l := d_A^{1/2} \mathbf{f}_a(A\mathbf{x} + \mathbf{q}_i), \qquad l \in \{0, ..., md_A - 1\}$$

then $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ *is an orthonormal basis of* U. *Moreover, any set of functions* $\mathbf{G}$ *in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{G})$ *is an orthonormal basis of* U *has exactly* md_A *functions.*

Proof. Since $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence, a trivial change of variables shows that $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ is an orthogonal sequence. Further, since $\Delta_A = \{\mathbf{q}_i\}_{i=0}^{d_A-1}$ is a full collection of representatives of the cosets of $\mathbb{Z}^d/A(\mathbb{Z}^d)$, given $a \in \{0, ..., m-1\}$ and $\mathbf{k} \in \mathbb{Z}^d$ we have that there exist unique $l \in \{0, ..., m-1\}$ and $\mathbf{r} \in \mathbb{Z}^d$ such that $\mathbf{D}_A\mathbf{T}_\mathbf{k}\mathbf{f}_a = \mathbf{T}_\mathbf{r}\mathbf{g}_l$. Thus $\mathbf{T}(\mathbf{g}_0, ..., \mathbf{g}_{md_A-1})$ is an orthonormal basis of U.

Since the set $\{\mathbf{g}_0, ..., \mathbf{g}_{md_A-1}\}$ has exactly md_A functions, Proposition 2 implies that every other set of functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of U has exactly md_A functions. □

Theorem 1 yields

Theorem 2. *Let* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be a scaling function in an A-MMRA,* $\{V_j : j \in \mathbb{Z}\}$. *If*

$$\Theta_i := d_A^{1/2}\Phi(A\mathbf{x} + \mathbf{q}_i), \qquad i = 0, 1, ..., d_A - 1, \tag{15}$$

then $\mathbf{T}(\Theta_0, ..., \Theta_{d_A-1})$ *is an orthonormal basis of* V_1.

Using Theorem 1 we can deduce some properties of the subspaces V_j. We have the following.

Theorem 3. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA. Then*

(a) *If* $j > 0$, *then there exists a finite set* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j.

(b) *If* $j \geq 0$, *then any set* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j *has exactly* d_A^j *functions.*

(c) *If* $j < 0$, *then there is no set of functions* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* V_j.

(d) *If* $j \neq 0$, *then there is no function* $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{f})$ *is an orthonormal basis of* V_j.

Proof. To prove (a), let Φ be a scaling function in the A-MMRA. According to Theorem 2, there exists a set of exactly d_A functions, $\mathbf{F}_1$, such that $\mathbf{T}(\mathbf{F}_1)$ is an orthonormal basis of V_1. Thus for $j \geq 0$ the existence of a set of exactly d_A^j functions, $\mathbf{F}_j$, such that $\mathbf{T}(\mathbf{F}_j)$ is an orthonormal basis of V_j follows by repeated application of Theorem 1.

From (a), for $j \geq 0$ the set $\mathbf{F}_j$ has exactly d_A^j functions; thus (b) follows from Proposition 2.

We now prove (c). Let $m := d_A^{-j}$. By repeated application of Theorem 1 we conclude that there are functions $f_0, \dots f_{m-1}$ such that $T(f_0, \dots f_{m-1})$ is a basis of V_0. Since A is expansive, we know that $d_A > 1$; thus $m > 1$, which is a contradiction of (b).

Finally, if $j < 0$ (d) follows from (c), whereas if $j > 0$, (d) follows from (b). □

The following two corollaries are immediate consequences of Theorem 3.

Corollary 2. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA, let* $s > 0$ *and* $U_j := V_{j+s}$. *Then* $\{U_j : j \in \mathbb{Z}\}$ *is a sequence of closed subspaces in* $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *satisfying the conditions (i), (ii), (iii) in the definition of A-MMRA, and also, there exists a set* *of functions* $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *such that* $\mathbf{T}(\mathbf{F})$ *is an orthonormal basis of* U_0 *and it has exactly* d_A^s *functions.*

Corollary 3. *Let* $\{V_j : j \in \mathbb{Z}\}$ *be an A-MMRA. Then* V_j *is a proper subset of* V_{j+1} *for every* $j \in \mathbb{Z}$*.*

Proof. Assume that there is $j \in \mathbb{Z}$ such that $V_j = V_{j+1}$, then by the definition of A-MMRA we have $V_j = V_{j+s}$ for every $s \in \mathbb{Z}$. Thus, in particular $V_0 = V_1$ and which is impossible by the condition (b) in Theorem 3. □

We have the following characterization of matrix–valued wavelet sets constructed from an A-MMRA:

Theorem 4. *Let* $\Phi \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be a scaling function in an A-MMRA,* $\{V_j : j \in \mathbb{Z}\}$*, and let* $\Theta_0, ..., \Theta_{d_A-1} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ *be such that* $\mathbf{T}(\Theta_0, ...\Theta_{d_A-1})$ *is an orthonormal basis of* V_1*. The following propositions are equivalent:*

(a) $\{\Psi_1, ..., \Psi_{d_A-1}\}$ *is a matrix–valued wavelet set constructed from the given A-MMRA.*

(b) *There is an* $n\, d_A \times n\, d_A$ *matrix* $\mathbf{Q}(\mathbf{t})$ *of* $\mathbb{Z}^d$*-periodic functions and unitary a.e. on* $\mathbb{R}^d$ *such that*

$$\big(\widehat{\Phi}(\mathbf{t}), \widehat{\Psi}_1(\mathbf{t}), ..., \widehat{\Psi}_{d_A-1}(\mathbf{t})\big)^T := \mathbf{Q}(\mathbf{t})\big(\widehat{\Theta}_0(\mathbf{t}), \widehat{\Theta}_1(\mathbf{t}), ..., \widehat{\Theta}_{d_A-1}(\mathbf{t})\big)^T \quad a.e.\ on\ \mathbb{R}^d.$$

Proof. Let us prove (a) ⇒ (b). The condition (a) means that $\mathbf{T}(\Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of W_0 where W_0 is defined as the orthogonal complement of V_0 in V_1. Further, since $\mathbf{T}(\Phi)$ is an orthonormal basis of V_0 then $\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of V_1. Thus the conditions (b) follows from Proposition 1.

We now prove (b) ⇒ (a). According to Proposition 1, we know that

$$\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$$

is an orthonormal sequence in V_1, and further, by Proposition 3, we know that $\mathbf{T}(\Phi, \Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of V_1. Thus, since $\mathbf{T}(\Phi)$ is an orthonormal basis of V_0 and $V_1 = W_0 \oplus V_0$ then $\mathbf{T}(\Psi_1, ..., \Psi_{d_A-1})$ is an orthonormal basis of W_0. Thus, we conclude that $\{\Psi_1, ..., \Psi_{d_A-1}\}$ is a matrix–valued wavelet set constructed from the A-MMRA. □

We now proceed to describe a strategy for constructing a matrix–valued wavelet set associated to a dilation A from a given A-MMRA with a scaling function Φ. According to Theorem 2 the functions $\Theta_0, \dots, \Theta_{d_A-1}$ defined by (15) are such that $\mathbf{T}(\Theta_0, \dots, \Theta_{d_A-1})$ is an orthonormal basis of V_1. Furthermore, since $\Phi \in V_0 \subset V_1$, Lemma 2 implies that there are $\mathbb{Z}^d$-periodic matrix–valued functions $\mathbf{H}_l \in L^2(\mathbb{T}^d, \mathbb{C}^{n\times n})$, $l = 0, ..., d_A - 1$, such that

$$\widehat{\Phi}(\mathbf{t}) = \sum_{l=0}^{d_A-1} \mathbf{H}_l(\mathbf{t})\widehat{\Theta_l}(\mathbf{t}) \qquad \text{a.e. on } \mathbb{R}^d.$$

Moreover, Lemma 3 implies that

$$\sum_{l=0}^{d_A-1} \mathbf{H}_l(\mathbf{t})\mathbf{H}_l^*(\mathbf{t}) = I_n \qquad \text{a.e. on } \mathbb{R}^d, \tag{16}$$

If we denote by $\mathbf{J}_0$ the $n \times nd_A$ matrix of functions defined by

$$\mathbf{J}_0(\mathbf{t}) = (\mathbf{H}_0(\mathbf{t}), ..., \mathbf{H}_{d_A-1}(\mathbf{t}))$$

and by $\mathbf{v}_q(\mathbf{t})$, $q = 1, ..., n$ the vector in the complex vector space $\mathbb{C}^{nd_A}$ defined by the value at $\mathbf{t}$ of the q-th row in the matrix $\mathbf{J}_0(\mathbf{t})$, the equality (16) implies that the vectors $\{\mathbf{v}_q(\mathbf{t}) : q = 1, ..., n\}$ are a.e. orthonormal. Note that it is possible to construct a $nd_A \times nd_A$ matrix $\mathbf{Q}(\mathbf{t})$ of $\mathbb{Z}^d$-periodic functions, a.e. unitary in $\mathbb{R}^d$, such that for $q = 1, \ldots, n$ the q-th row is given by the function vector $\mathbf{v}_q(\mathbf{t})$. The construction of such a matrix can be done by the Gram–Schmidt orthogonalization process. If $\mathbf{H}_1(\mathbf{t})$ is symmetric, another method for the completion of a unitary matrix is given in [17]. Finally, if $\Psi_1, ..., \Psi_{d_A-1} \in L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is defined by

$$\big(\widehat{\Phi}(\mathbf{t}), \widehat{\Psi}_1(\mathbf{t}), \cdots, \widehat{\Psi}_{d_A-1}(\mathbf{t})\big)^T = \mathbf{Q}(\mathbf{t})\big(\widehat{\Theta}_0(\mathbf{t}), \widehat{\Theta}_1(\mathbf{t}), \ldots, \widehat{\Theta}_{d_A-1}(\mathbf{t})\big)^T \quad \text{a.e. on } \mathbb{R}^d,$$

and applying Theorem 4, we conclude that $\{\Psi_1, ..., \Psi_{d_A-1}\}$ is a matrix–valued wavelet set constructed from the given A-MMRA.

We have therefore proved the following.

Theorem 5. *Given an expansive linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and given an A-MMRA, then there exists a set of matrix–valued functions $\{\Psi_1, ..., \Psi_{d_A-1}\}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ which is a matrix–valued wavelet set constructed from such an A-MMRA.*

Recalling that a set of matrix–valued functions $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ is a matrix–valued wavelet set constructed from an A-MMRA, $\{V_j : j \in \mathbb{Z}\}$, if and only if $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of the subspace W_0 defined as the orthogonal complement of V_0 in V_1, then the following is a corollary of Theorem 5 and Proposition 2.

Corollary 4. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let W_0 denote the orthogonal complement of V_0 in V_1. Then there exists a set of matrix–valued functions $\mathbf{F} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F})$ is an orthonormal basis of W_0, and any set of matrix–valued functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of W_0 has exactly $d_A - 1$ matrix–valued functions.*

From Corollary 4, (14), and Theorem 1, we obtain the following.

Corollary 5. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let W_j denote the orthogonal complement of V_j in V_{j+1}. Then, for every $j \in \{0, 1, 2, ...\}$ there exists a set of matrix–valued functions $\mathbf{F}_j \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{F}_j)$ is an orthonormal basis of W_j, and any set of functions $\mathbf{G}_j$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G}_j)$ is an orthonormal basis of W_j has exactly $(d_A - 1)d_A^j$ matrix–valued functions.*

Let us continue with the study of the structure of subspaces V_j and W_j.

Theorem 6. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_{d_A-1}\}$ be a set of matrix–valued functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If there exists an integer $l < 0$ such that $\mathbf{F} \subset V_l$, then $\mathbf{F}$ cannot be a matrix–valued wavelet set.*

Proof. If $\mathbf{F}$ is a matrix–valued wavelet set then $\mathbf{T}(\mathbf{F})$ is an orthonormal sequence in V_l. Thus, applying Theorem 1 with the expansive linear map A^l, we see that there exist a set of $(d_A - 1)d_A^l$ matrix–valued functions $\mathbf{G}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal sequence in V_0. Moreover, according to the definition of V_0 and Proposition 2, the number $(d_A - 1)d_A^l$ must be less or equal to 1. Since $d_A \geq 2$, we have a contradiction. □

Theorem 7. *Let $\{V_j : j \in \mathbb{Z}\}$ be an A-MMRA and let $\mathbf{F} = \{\mathbf{f}_1, ..., \mathbf{f}_{d_A-1}\}$ be a set of matrix–valued functions in $L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$. If there exists an integer $l \neq 0$ such that $\mathbf{F} \subset W_l$, then $\mathbf{F}$ cannot be a matrix–valued wavelet set.*

Proof. Assume that $\mathbf{F}$ is a matrix–valued wavelet set. If $l < 0$, since W_l is a proper subset of V_0 it follows that $d_A - 1$ must be less of equal to 1 and this is impossible. On the other hand, if $l > 0$, Corollary 5 implies that every set of matrix–valued functions $\mathbf{G} \subset L^2(\mathbb{R}^d, \mathbb{C}^{n\times n})$ such that $\mathbf{T}(\mathbf{G})$ is an orthonormal basis of W_l must have exactly $(d_A - 1)d_A^l$ matrix–valued functions. Since $d_A < (d_A - 1)d_A^l$ we get a contradiction in this case as well. □

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The Random Motion on the Sphere Generated by the Laplace-Beltrami Operator

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Abstract

Using the Laplace-Beltrami operator we construct the Brownian motion process on the n-dimensional sphere, $n = 1, 2, 3$. Then we evaluate explicitly certain quantities for this process. We start with the transition density and continue with the calculation of some probabilistic quantities regarding the exit times of specific domains possessing certain symmetries.

Key words and phrases: n-dimensional sphere, Laplace-Beltrami operator, Brownian motion, transition densities, exit times.

1 Introduction

1.1 The n-Sphere

Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The n-dimensional sphere S^n with center $(c_1, ..., c_{n+1})$ and radius $a > 0$ is the set of all points $x \in \mathbb{R}^{n+1}$ satisfying

$$(x_1 - c_1)^2 + \cdots + (x_{n+1} - c_{n+1})^2 = a^2$$

The most interesting case in applications is, of course, the case $n = 2$. For the sake of comparison we will also discuss the cases $n = 1$ (i.e. the circle) and $n = 3$. In some cases we will even consider the case of general n.

1.2 Stereographic Projection Coordinates

Consider the n-sphere, $n \geq 2$,

$$x_1^2 + \cdots + x_n^2 + (x_{n+1} - a)^2 = a^2$$

To each point $(x_1, ..., x_n, x_{n+1})$ of this sphere, other than its "north pole" $N = (0, ..., 0, 2a)$ we associate the coordinates

$$\xi_1 = \frac{2ax_1}{2a - x_{n+1}}, \ \ldots \ , \xi_n = \frac{2ax_n}{2a - x_{n+1}}$$

Given the coordinates $(\xi_1, ..., \xi_n)$ of a point on the sphere with Cartesian coordinates $(x_1, ..., x_n, x_{n+1})$, we have

$$x_1 = \frac{4a^2\xi_1}{\xi_1^2 + \cdots + \xi_n^2 + 4a^2}, \ \ldots \ , x_n = \frac{4a^2\xi_n}{\xi_1^2 + \cdots + \xi_n^2 + 4a^2}, \ x_{n+1} = \frac{2a\left(\xi_1^2 + \cdots + \xi_n^2\right)}{\xi_1^2 + \cdots + \xi_n^2 + 4a^2}.$$

1.3 Spherical Coordinates

The points of the n-sphere

$$x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = a^2$$

may also be described in spherical coordinates $(\theta_1, ..., \theta_{n-1}, \varphi)$ as follows:

- For $n = 1$, $x_1 = a\cos\varphi$, $x_2 = a\sin\varphi$, where $0 \leq \varphi < 2\pi$.
- For $n = 2$, $(\theta_1 = \theta)$ $x_1 = a\cos\theta\sin\varphi$, $x_2 = a\sin\theta\sin\varphi$, $x_3 = a\cos\varphi$, where $0 \leq \theta < 2\pi$ and $0 \leq \varphi \leq \pi$.
- For $n = 3$, $x_1 = a\cos\theta_1\sin\theta_2\sin\varphi$, $x_2 = a\sin\theta_1\sin\theta_2\sin\varphi$, $x_3 = a\cos\theta_2\sin\varphi$, $x_4 = a\cos\varphi$, where $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_2 \leq \pi$, and $0 \leq \varphi \leq \pi$.
- In general for $n \geq 4$
  $x_1 = a\cos\theta_1\sin\theta_2\sin\theta_3 ... \sin\theta_{n-1}\sin\varphi$, $x_2 = a\sin\theta_1\sin\theta_2\sin\theta_3 ... \sin\theta_{n-1}\sin\varphi$,
  $x_k = a\cos\theta_{k-1}\sin\theta_k ... \sin\theta_{n-1}\sin\varphi$, for $k = 3, 4, ..., n$
  and $x_{n+1} = a\cos\varphi$, where $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi$,
  for $i = 2, 3, ..., n-1$, and $0 \leq \varphi \leq \pi$.

1.4 The Laplace-Beltrami Operator

In spherical coordinates: The Laplace-Beltrami operator of a smooth function f on S^1 is

$$\Delta_1 f = \frac{1}{a^2} \cdot \frac{\partial^2 f}{\partial \varphi^2}$$

The Laplace-Beltrami operator of a smooth function f on S^2 is

$$\Delta_2 f = \frac{1}{a^2 \sin\varphi} \left(\frac{f_{\theta\theta}}{\sin\varphi} + f_\varphi \cos\varphi + f_{\varphi\varphi} \sin\varphi \right)$$

In the case where f is independent of θ we have

$$\Delta_2 f = \frac{1}{a^2} \left(f_{\varphi\varphi} + f_\varphi \cot\varphi \right)$$

The Laplace-Beltrami operator of a smooth function f on S^3 is

$$\Delta_3 f = \frac{1}{a^2 \sin^2\varphi} \left[\frac{1}{\sin^2\theta_2} \cdot \frac{\partial^2 f}{\partial \theta_1^2} + \frac{1}{\sin\theta_2} \cdot \frac{\partial}{\partial \theta_2} \left(\frac{\partial f}{\partial \theta_2} \sin\theta_2 \right) + \frac{\partial}{\partial \varphi} \left(\frac{\partial f}{\partial \varphi} \sin^2\varphi \right) \right]$$

and if f is independent of θ_1 and θ_2,

$$\Delta_3 f = \frac{1}{a^2} \left(f_{\varphi\varphi} + 2 f_\varphi \cot\varphi \right)$$

In stereographic projection coordinates: The Laplace-Beltrami operator of a smooth function f on S^n, $n \geq 2$ is

$$\Delta_n f = \frac{\left(\xi_1^2 + \cdots + \xi_n^2 + 4a^2\right)^2}{16a^4} \left[\sum_{i=1}^{n} \frac{\partial^2 f}{\partial \xi_i^2} - \frac{2(n-2)}{\left(\xi_1^2 + \cdots + \xi_n^2 + 4a^2\right)} \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial \xi_i} \right]$$

In particular, for $n = 2$ we get

$$\Delta_2 f = \frac{\left(\xi_1^2 + \xi_2^2 + 4a^2\right)^2}{16a^4} \left(\frac{\partial^2 f}{\partial \xi_1^2} + \frac{\partial^2 f}{\partial \xi_2^2} \right)$$

1.5 Brownian motion on S^n (starting at $x \in S^n$)

The Brownian motion on S^n is a diffusion (Markov) process $X_t, t \geq 0$, on S^n whose transition density is a function $P(t, x, y)$ on $(0, \infty) \times S^n \times S^n$ satisfying

$$\frac{\partial P}{\partial t} = \frac{1}{2} \Delta_n P,$$

$$P(t, x, y) \to \delta_x(y) \quad \text{as} \quad t \to 0^+$$

where Δ_n is the Laplace-Beltrami operator of S^n acting on the x-variables and $\delta_x(y)$ is the delta mass at x, i.e. $P(t, x, y)$ is the **heat kernel** of S^n. The heat kernel exists, it is unique, positive, and smooth in (t, x, y) [4].

1.5.1 Further Properties of the Heat Kernel $P(t,x,y)$

It is well known that $p(t,x,y)$ satisfies the following properties [4]

1. Symmetry: $P(t,x,y) = P(t,y,x)$
2. The semigroup identity: For any $s \in (0,t)$,
$$P(t,x,y) = \int_{S^n} P(s,x,z)P(t-s,z,y)d\mu(z)$$
where $d\mu$ is the n-th dimensional surface area.
3. For all $t > 0$ and $x \in S^n$
$$\int_{S^n} P(t,x,y)d\mu(y) = 1$$
4. As $t \to \infty$, $P(t,x,y)$ approaches the uniform density on S^n, i.e.
$$\lim_{t\to\infty} P(t,x,y) = \frac{1}{A_n}$$
where A_n is nth dimensional surface area of S^n with radius a. It is well known that [8]
$$A_n = \frac{2\pi^{\frac{n+1}{2}} a^n}{(\frac{n-1}{2})!}, \qquad \text{for } n \text{ odd}$$
$$A_n = \frac{2^n(\frac{n}{2}-1)!\pi^{\frac{n}{2}} a^n}{(n-1)!}, \qquad \text{for } n \text{ even.}$$

Finally, the symmetry of S^n implies that $P(t,x,y)$ depends only on t and $d(x,y)$, the distance between x and y. Thus in spherical coordinates it depends on t and the angle φ between x and y. Hence
$$P(t,x,y) = p(t,\varphi),$$
where $p(t,\varphi)$ satisfies
$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_n p = \frac{1}{2a^2}\left[(n-1)\cot\varphi \cdot \frac{\partial p}{\partial \varphi} + \frac{\partial^2 p}{\partial \varphi^2}\right]$$
and
$$\lim_{t\to 0^+} aA_{n-1}p(t,\varphi)\cdot \sin^{n-1}\varphi = \delta(\varphi).$$
Here $\delta(\cdot)$ is the standard Dirac delta function on $\mathbb{R}$.

2 Explicit Form of the Heat Kernel

Reminder (Poisson Summation Formula)**.** Let $f(x)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ consists of the set of all infinitely differentiable functions f on $\mathbb{R}$ so that f and all its derivatives $f^{(l)}$ are rapidly decreasing, in the sense that
$$\sup_{x\in\mathbb{R}} |x|^k \left|f^{(l)}(x)\right| < \infty \quad \text{for every} \quad k,l \geq 0.$$

Then

$$\sum_{n\in\mathbb{Z}} f(x+2\pi n) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} F(n)\exp(inx),$$

where $F(\xi)$ is the Fourier transform of $f(x)$, i.e.

$$F(\xi) = \int_{-\infty}^{+\infty} f(x)\exp(-i\xi x)dx, \quad \xi\in\mathbb{R}.$$

For example, if

$$f(x) = \exp(-Ax^2+Bx), \quad A>0, \quad B\in\mathbb{C},$$

then

$$F(\xi) = \sqrt{\frac{\pi}{A}}\exp\left(\frac{(i\xi-B)^2}{4A}\right)$$

2.1 The Case of S^1

Proposition 2.1 *The transition density function of the Brownian motion $X_t, t\geq 0$ on S^1 with radius a is the function*

$$p(t,\varphi) = \frac{1}{2\pi a}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{n^2t}{2a^2}+in\varphi\right).$$

Equivalently

$$p(t,\varphi) = \frac{1}{\pi a}\sum_{n\in\mathbb{N}}\left[\exp\left(-\frac{n^2t}{2a^2}\right)\cos(n\varphi)\right]-\frac{1}{2\pi a}$$

and

$$p(t,\varphi) = \frac{1}{\sqrt{2\pi t}}\sum_{n\in\mathbb{Z}}\exp\left(-\frac{a^2}{2t}\left(\varphi-2\pi n\right)^2\right).$$

Proof. If

$$p(t,\varphi) = \frac{1}{\pi a}\sum_{n\in\mathbb{N}}\left[\exp\left(-\frac{n^2t}{2a^2}\right)\cos(n\varphi)\right]-\frac{1}{2\pi a},$$

then

$$\frac{\partial p(t,\varphi)}{\partial t} = -\frac{1}{2\pi a^3}\sum_{n\in\mathbb{N}} n^2\cos(n\varphi)\exp\left(-\frac{n^2t}{2a^2}\right) \tag{2.1}$$

and

$$\frac{\partial^2 p(t,\varphi)}{\partial\varphi^2} = -\frac{1}{\pi a}\sum_{n\in\mathbb{N}} n^2\cos(n\varphi)\exp\left(-\frac{n^2t}{2a^2}\right). \tag{2.2}$$

Therefore

$$\frac{\partial p(t,\varphi)}{\partial t} = \frac{1}{2a^2}\frac{\partial^2 p(t,\varphi)}{\partial\varphi^2}.$$

We will now show that

$$\lim_{t\to 0^+} ap(t,\varphi) = \delta(\varphi).$$

If $\varphi \in (0, 2\pi)$, then

$$\lim_{t \to 0^+} ap(t, \varphi) = 0. \tag{2.3}$$

Next we observe that

$$\int_0^{2\pi} ap(t, \varphi) d\varphi = \frac{1}{\pi} \int_0^{2\pi} \sum_{n \in \mathbb{N}} \left[\exp\left(-\frac{n^2 t}{2a^2} \right) \cos(n\varphi) \right] d\varphi - 1. \tag{2.4}$$

For $t > 0$ let us consider the functions

$$f_n : [0, 2\pi] \to \mathbb{R}, \quad n \in \mathbb{N},$$

with

$$f_n(\varphi) = \cos(n\varphi) \exp\left(-\frac{n^2 t}{2a^2} \right).$$

Notice that $f_n(\varphi)$ are integrable functions on $[0, 2\pi]$. Furthermore

$$\sum_{n=1}^{+\infty} f_n(\varphi)$$

converges uniformly on $[0, 2\pi]$ because

$$|f_n(\varphi)| \le \exp\left(-\frac{n^2 t}{2a^2} \right)$$

and the series

$$\sum_{n=1}^{\infty} \exp\left(-\frac{n^2 t}{2a^2} \right)$$

converges. Therefore (2.4) gives

$$\int_0^{2\pi} ap(t, \varphi) d\varphi = -1 + \frac{1}{\pi} \sum_{n \in \mathbb{N}} \exp\left(-\frac{n^2 t}{2a^2} \right) \int_0^{2\pi} \cos(n\varphi) d\varphi,$$

thus

$$\int_0^{2\pi} ap(t, \varphi) d\varphi = 1, \text{ for every } t > 0. \tag{2.5}$$

Therefore from (2.4) and (2.5)

$$\lim_{t \to 0^+} ap(t, \varphi) = \delta(\varphi)$$

and this complete the proof. ∎

2.2 The Case of S^2

Let $X_t,\ t \ge 0$ be the Brownian motion on a 2-dimensional sphere S^2 of radius a. The transition density function $p(t, \varphi)$ of X_t is the unique solution of

$$\frac{\partial p}{\partial t} = \frac{1}{2a^2 \sin\varphi} \left(\frac{\partial^2 p(t, \varphi)}{\partial \varphi^2} \sin\varphi + \frac{\partial p}{\partial \varphi} \cos\varphi \right) \tag{2.6}$$

and

$$\lim_{t\to 0^+} 2\pi a^2 \sin(\varphi) p(t,\varphi) = \delta(\varphi). \tag{2.7}$$

The solution to the diffusion equation

$$\frac{\partial K(t,\varphi)}{\partial t} = \frac{1}{\sin\varphi}\left(\cos\varphi \frac{\partial K(t,\varphi)}{\partial \varphi} + \sin\varphi \frac{\partial^2 K(t,\varphi)}{\partial \varphi^2}\right) \tag{2.8}$$

with initial condition

$$\lim_{t\to 0^+} 2\pi \sin(\varphi) K(t,\varphi) = \delta(\varphi) \tag{2.9}$$

is given by the function (see [3])

$$K(t,\varphi) = \frac{1}{4\pi}\sum_{n\in\mathbb{N}}(2n+1)\exp\left(-n(n+1)\sqrt{2t}\right) P_n^0(\cos\varphi). \tag{2.10}$$

Here $P_n^0(\cdot)$ is the associated Legendre polynomials of order zero, i.e.

$$P_n^0(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}\left[(x^2-1)^n\right]. \tag{2.11}$$

This fact implies the following

Proposition 2.2 *The transition density function of the Brownian motion X_t, $t \geq 0$, on S^2 with radius a it is given by the function*

$$p(t,\varphi) = \frac{1}{4\pi a^2}\sum_{n\in\mathbb{N}}(2n+1)\exp\left(-\frac{n(n+1)\sqrt{t}}{a}\right) P_n^0(\cos\varphi). \tag{2.12}$$

2.3 The Case of S^3

Proposition 2.3 *Let X_t, $t \geq 0$ be the Brownian motion on a 3-dimensional sphere S^3 of radius a. The transition density function $p(t,\varphi)$ of X_t is given by*

$$p(t,\varphi) = \frac{\exp\left(\frac{t}{2a^2}\right)}{(2\pi t)^{3/2}\sin\varphi}\sum_{n\in\mathbb{Z}}(\varphi+2n\pi)\exp\left(-\frac{(\varphi+2n\pi)^2 a^2}{2t}\right),$$

where $\mathbb{Z}$ is the set of all integers. Equivalently

$$p(t,\varphi) = -\frac{i}{4\pi^2 a^3 \sin\varphi}\sum_{n\in\mathbb{Z}} n\exp\left(-\frac{t(n^2-1)}{2a^2} + i\varphi n\right)$$

and

$$p(t,\varphi) = \frac{1}{2\pi^2 a^3 \sin\varphi}\sum_{n\in\mathbb{N}} n\sin(n\varphi)\exp\left(-\frac{t(n^2-1)}{2a^2}\right).$$

Furthermore $p(t,\varphi)$ is analytic about $\varphi = 0$ and $\varphi = \pi$. In fact

$$p(t,0) = \lim_{\varphi\to 0^+} p(t,\varphi) = \frac{1}{2\pi^2 a^3}\sum_{n\in\mathbb{N}} n^2\exp\left(-\frac{t(n^2-1)}{2a^2}\right)$$

and

$$p(t,\pi) = \lim_{\varphi \to \pi^-} p(t,\varphi) = \frac{1}{2\pi^2 a^3} \sum_{n\in\mathbb{N}} n^2 (-1)^n \exp\left(-\frac{t(n^2-1)}{2a^2}\right).$$

Reminder. The ϑ_3 function of Jacobi is

$$\vartheta_3(z,r) = 1 + 2\sum_{n=0}^{\infty} \exp\left(i\pi r n^2\right)\cos(2nz),$$

where $r \in \mathbb{C}$ with $\operatorname{Im}\{r\} > 0$. It follows that

$$p(t,\varphi) = -\frac{1}{4\pi^2 a^3 \sin\varphi} \exp\left(\frac{t}{2a^2}\right) \frac{\partial}{\partial\varphi} \vartheta_3\left(\frac{\varphi}{2}, \frac{ti}{2a^2\pi}\right).$$

Sketch of Proof. First we will prove that $p(t,\varphi)$ satisfies the differential equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta_3 p.$$

After that, we will show that

$$\lim_{t\to 0^+} 4\pi a^3 \sin^2(\varphi) p(t,\varphi) = \delta(\varphi).$$

For arbitrarily small $\epsilon > 0$, let

$$I_\epsilon = \int_0^\epsilon 4\pi a^3 \sin^2(\varphi) p(t,\varphi) d\varphi.$$

We have

$$\lim_{t\to 0^+} I_\epsilon = \lim_{t\to 0^+} \frac{4\pi a^3 \exp\left(\frac{t}{2a^2}\right)}{(2t\pi)^{3/2}} \left(\int_0^\epsilon \varphi \sin(\varphi) \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi\right.$$
$$\left. + \sum_{n\in\mathbb{Z}^*} \int_0^\epsilon (\varphi + 2n\pi)\sin(\varphi) \exp\left(-\frac{(\varphi+2n\pi)^2 a^2}{2t}\right) d\varphi\right),$$

where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. However

$$\left|\sum_{n\in\mathbb{Z}^*} \int_0^\epsilon (\varphi + 2n\pi)\sin(\varphi) \exp\left(-\frac{(\varphi+2n\pi)^2 a^2}{2t}\right) d\varphi\right| \le \sum_{n\in\mathbb{Z}^*} \int_0^\epsilon (2|n|+1)\pi \exp\left(-\frac{n^2\pi^2 a^2}{2t}\right) d\varphi$$

and

$$\sum_{n\in\mathbb{Z}^*} \int_0^\epsilon (2|n|+1)\pi \exp\left(-\frac{n^2\pi^2 a^2}{2t}\right) d\varphi = \epsilon \sum_{n\in\mathbb{Z}^*} (2|n|+1)\pi \exp\left(-\frac{n^2\pi^2 a^2}{2t}\right),$$

which converges to 0 as $t \to 0^+$, by Lebesgue's Dominated Convergence Theorem. Therefore

$$\lim_{t\to 0^+} I_\epsilon = \lim_{t\to 0^+} \frac{4\pi a^3 \exp\left(\frac{t}{2a^2}\right)}{(2t\pi)^{3/2}} \int_0^\epsilon \varphi \sin\varphi \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi.$$

By using the Laplace method for integrals [1]

$$\int_0^{\epsilon} \varphi \sin(\varphi) \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi \sim \int_0^{\epsilon} \varphi^2 \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi \sim \int_0^{\infty} \varphi^2 \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi,$$

as $t \to 0^+$. Here $A \sim B$ means that $\frac{A}{B} \to 1$. Hence

$$\lim_{t \to 0^+} I_\epsilon = \lim_{t \to 0^+} \frac{4\pi a^3 \exp\left(\frac{t}{2a^2}\right)}{(2t\pi)^{3/2}} \int_0^{\infty} \varphi^2 \exp\left(-\frac{\varphi^2 a^2}{2t}\right) d\varphi,$$

or, for

$$u = \frac{\varphi a}{\sqrt{t}}$$

$$\lim_{t \to 0^+} I_\epsilon = \lim_{t \to 0^+} 2 \exp\left(\frac{t}{2a^2}\right) \int_0^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du.$$

i.e.

$$\lim_{t \to 0^+} I_\epsilon = 1. \tag{2.13}$$

Furthermore, for every $t > 0$, we have

$$I = \int_0^{\pi} 4\pi a^3 \sin^2(\varphi) p(t, \varphi) d\varphi, \tag{2.14}$$

hence,

$$I = \frac{4\pi a^3 \exp\left(\frac{t}{2a^2}\right)}{(2t\pi)^{3/2}} \int_0^{\pi} \sum_{n \in \mathbb{Z}} (\varphi + 2n\pi) \sin(\varphi) \exp\left(-\frac{(\varphi + 2n\pi)^2 a^2}{2t}\right) d\varphi.$$

The series

$$\sum_{n \in \mathbb{Z}} (\varphi + 2n\pi) \sin(\varphi) \exp\left(-\frac{(\varphi + 2n\pi)^2 a^2}{2t}\right)$$

converges uniformly on $[0, \pi]$ for every $t > 0$, because

$$\left| (\varphi + 2n\pi) \sin(\varphi) \exp\left(-\frac{(\varphi + 2n\pi)^2 a^2}{2t}\right) \right| \leq 2|n|\pi \exp\left(-\frac{n^2 \pi^2 a^2}{2t}\right)$$

and the series

$$\sum_{n \in \mathbb{Z}} M_n,$$

where

$$M_n = 2|n|\pi \exp\left(-\frac{n^2 \pi^2 a^2}{2t}\right)$$

converges. Therefore (2.14), implies that

$$I = \frac{4\pi a^3 \exp\left(\frac{t}{2a^2}\right)}{(2t\pi)^{3/2}} \sum_{n \in \mathbb{Z}} \int_0^{\pi} (\varphi + 2n\pi) \sin(\varphi) \exp\left(-\frac{(\varphi + 2n\pi)^2 a^2}{2t}\right) d\varphi.$$

Hence

$$I = \frac{a \exp\left(\frac{t}{2a^2}\right)}{\sqrt{2t\pi}} \sum_{n \in \mathbb{Z}} \int_0^{\pi} [\exp(i\varphi) + \exp(-i\varphi)] \exp\left(-\frac{(\varphi + 2n\pi)^2 a^2}{2t}\right) d\varphi. \tag{2.15}$$

Let $u = \varphi + 2n\pi$, then (2.15) gives

$$I = \frac{a}{\sqrt{2t\pi}} \left(\frac{\sqrt{2t\pi}}{2a} + \frac{\sqrt{2t\pi}}{2a} \right) = 1$$

for every $t > 0$. In particular

$$\lim_{t \to 0^+} \int_0^\pi 4\pi a^3 \sin^2(\varphi) p(t, \varphi) d\varphi = 1. \tag{2.16}$$

From (2.13) and (2.16) we have that that

$$\lim_{t \to 0^+} 4\pi a^3 \sin^2(\varphi) p(t, \varphi) d\varphi = \delta(\varphi)$$

and this complete the proof. ■

3 Stochastic Differential Equation (SDE) of X_t in Local Coordinates

In spherical coordinates:

The Brownian motion on S^1 satisfies the SDE

$$dX_t = \frac{1}{a} dB_t.$$

The Brownian motion on S^2 satisfies the SDE

$$dX_t = \left(0, \frac{\cos\varphi}{2a^2 \sin\varphi}\right) dt + \begin{bmatrix} \frac{1}{a\sin\varphi} & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

The Brownian motion on S^3 satisfies the SDE

$$dX_t = \left(0, \frac{\cos\theta_2}{2a^2 \sin\theta_2 \sin^2\varphi}, \frac{\cos\varphi}{a^2 \sin\varphi}\right) dt + \begin{bmatrix} \frac{1}{a\sin\theta_2 \sin\varphi} & 0 & 0 \\ 0 & \frac{1}{a\sin\varphi} & 0 \\ 0 & 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}.$$

In stereographic projection coordinates: The Brownian motion on S^2 satisfies the SDE

$$dX_t = \frac{\xi_1^2 + \xi_2^2 + 4a^2}{4a^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}.$$

The Brownian motion on S^3 satisfies the SDE

$$dX_t = -\frac{\left(\xi_1^2 + \xi_2^2 + \xi_3^2 + 4a^2\right)}{16a^4} (\xi_1, \xi_2, \xi_3)\, dt + \frac{\left(\xi_1^2 + \xi_2^2 + \xi_3^2 + 4a^2\right)}{4a^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}.$$

The Brownian motion on $S^n, n \geq 2$ satisfies the SDE

$$dX_t = (2-n) \frac{\left(\xi_1^2 + \cdots + \xi_n^2 + 4a^2\right)}{16a^4} (\xi_1, \cdots, \xi_n)\, dt + \frac{\left(\xi_1^2 + \ldots + \xi_n^2 + 4a^2\right)}{4a^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ \vdots \\ dB_n(t) \end{bmatrix}.$$

4 Expectations of exit times

Let X_t be the Brownian motion in S^n and $D \subset S^n$. The random variable

$$T = \inf\{t \geq 0 | \ X_t \notin D\}$$

is called the (first) exit time D.
Reminder. If

$$u(x) = E^x[T],$$

(here the superscript x indicates that $X_0 = x$) then $u(x)$ satisfies

$$\frac{1}{2}\Delta_n u = -1$$

$$u|_{\partial D} = 0$$

Let $\varphi_1, \varphi_2 \in [0, 2\pi)$, $\varphi_1 < \varphi_2$. Consider the set

$$D = (\varphi_1, \varphi_2).$$

If X_t is the Brownian motion on S^1 starting at the point $\varphi \in D$, then

$$E^\varphi[T] = a^2 (\varphi - \varphi_1)(\varphi_2 - \varphi)$$

Let $\varphi_0 \in (0, \pi)$ be fixed. We consider the set D in S^n, $n \geq 2$, such that

$$D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi) | \ \varphi \in [0, \varphi_0)\}.$$

If X_t is the Brownian motion on S^n starting at the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D$$

then

$$E^A[T] = u(\varphi) = 2a^2 \int_\varphi^{\varphi_0} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx$$

Notice that $u(\varphi)$ is an elementary function.
For $\underline{n = 2}$ we obtain

$$E^A[T] = 2a^2 \ln\left(\frac{1 + \cos\varphi}{1 + \cos\varphi_0}\right).$$

For $\underline{n = 3}$ we obtain

$$E^A[T] = a^2 (\varphi \cot\varphi - \varphi_0 \cot\varphi_0).$$

Let $\varphi_1, \varphi_2 \in (0, \pi)$, $\varphi_1 < \varphi_2$. Consider the set D in $S^n, n \geq 2$,

$$D = \{(\theta_1, \ldots, \theta_{n-1}, \varphi) | \ \varphi \in (\varphi_1, \varphi_2)\}.$$

If X_t is the Brownian motion on S^n starting at the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D$$

then

$$E^A[T] = 2a^2 \left[\int_{\varphi}^{\varphi_1} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx + \frac{\int_{\varphi_1}^{\varphi_2} \frac{\int_0^x (\sin\omega)^{n-1} d\omega}{(\sin x)^{n-1}} dx}{\int_{\varphi 1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx} \cdot \int_{\varphi_1}^{\varphi} \frac{1}{(\sin x)^{n-1}} dx \right].$$

For $\underline{n=2}$ we obtain

$$E^A[T] = \frac{4a^2}{\ln\left(\frac{\tan(\varphi_2/2)}{\tan(\varphi_1/2)}\right)} \left[\ln\left(\frac{\cos(\varphi_1/2)}{\cos(\varphi_2/2)}\right) \ln\left(\frac{\sin(\varphi/2)}{\sin(\varphi_1/2)}\right) - \ln\left(\frac{\cos(\varphi_1/2)}{\cos(\varphi/2)}\right) \ln\left(\frac{\sin(\varphi_2/2)}{\sin(\varphi_1/2)}\right) \right].$$

For $\underline{n=3}$ we obtain

$$E^A[T] = \frac{a^2 \left[(\varphi - \varphi_1)\cot\varphi\cot\varphi_1 + (\varphi_1 - \varphi_2)\cot\varphi_1\cot\varphi_2 + (\varphi_2 - \varphi)\cot\varphi_2\cot\varphi \right]}{\cot\varphi_1 - \cot\varphi_2}.$$

Notice that the formulas for $n = 2$ and $n = 3$ are quite different.

5 Hitting Probabilities

Let X_t be the Brownian motion in S^n, $D \subset S^n$, and T its exit time.
Reminder. Let $\Gamma \subset D$ and

$$u(x) = P^x\{X_T \in \Gamma\},$$

then $u(x)$ satisfies

$$\Delta_n u = 0$$

$$u|_{\Gamma} = 1, \qquad u|_{\partial D \setminus \Gamma} = 0$$

Consider the subset $D = (\varphi_1, \varphi_2)$ of S^1, $0 < \varphi_1 < \varphi_2 < 2\pi$. If $\Gamma_1 = \{\varphi_1\}$, then

$$P^{\varphi}\{X_T \in \Gamma_1\} = \frac{\varphi_2 - \varphi}{\varphi_2 - \varphi_1}$$

Let $\varphi_1, \varphi_2 \in (0, \pi)$, $\varphi_1 < \varphi_2$. Consider the set D in $S^n, n \geq 2$,

$$D = \{ (\theta_1, \ldots, \theta_{n-1}, \varphi) |\ \varphi \in (\varphi_1, \varphi_2) \}$$

and the point

$$A = (\theta_1, \ldots, \theta_{n-1}, \varphi) \in D.$$

If $\Gamma_1 = \{(\theta_1, \ldots, \theta_{n-1}, \varphi_1)\}$, then

$$P^A\{X_T \in \Gamma_1\} = \frac{\int_{\varphi}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx}{\int_{\varphi_1}^{\varphi_2} \frac{1}{(\sin x)^{n-1}} dx}.$$

For $\underline{n=2}$ we obtain

$$P^A\{X_T \in \Gamma_1\} = \frac{\ln\left(\frac{\tan\left(\frac{\varphi_2}{2}\right)}{\tan\left(\frac{\varphi}{2}\right)}\right)}{\ln\left(\frac{\tan\left(\frac{\varphi_2}{2}\right)}{\tan\left(\frac{\varphi_1}{2}\right)}\right)}.$$

For $\underline{n = 3}$ we obtain

$$P^A\{X_T \in \Gamma_1\} = \frac{\cot\varphi - \cot\varphi_2}{\cot\varphi_1 - \cot\varphi_2} = \frac{\sin\varphi_1 \sin(\varphi_2 - \varphi)}{\sin\varphi \sin(\varphi_2 - \varphi_1)}.$$

Let D be domain on S^2 whose stereographic coordinate description is

$$D = \{(\xi_1, \xi_2) | \ b < \xi_2 < c\},$$

i.e. D is the domain bounded by two circles passing through the north pole. If $A = (\xi_1, \xi_2) \in D$ and

$$\Gamma_1 = \{(\xi_1, b) | \ \xi_1 \in \mathbb{R}\},$$

then

$$P^A\{X_T \in \Gamma_1\} = \frac{c - \xi_2}{c - b}. \tag{5.1}$$

6 The Moment Generating Function of T

Reminder. Assume that $\lambda > -\lambda_1/2$, where λ_1 is the first Dirichlet eigenvalue of $D \subset S^n$. If

$$u(x) = E^x[e^{-\lambda T}],$$

then $u(x)$ satisfies

$$\frac{1}{2}\Delta_n u = \lambda u$$
$$u|_{\partial D} = 1$$

Suppose $D \subset S^1$ is the domain

$$D = (\varphi_1, \varphi_2), \qquad 0 \leq \varphi_1 < \varphi_2 < 2\pi.$$

Then, for $\varphi \in (\varphi_1, \varphi_2)$

$$E^\varphi[e^{-\lambda T}] = \frac{\sinh\left(a\sqrt{2\lambda}(\varphi_2 - \varphi)\right) + \sinh\left(a\sqrt{2\lambda}(\varphi - \varphi_1)\right)}{\sinh\left(a\sqrt{2\lambda}(\varphi_2 - \varphi_1)\right)}$$

provided

$$\lambda > -\frac{\pi^2}{2a^2(\varphi_2 - \varphi_1)^2}$$

Let X_t be the Brownian motion on S^2 starting at the point

$$A = (\theta, \varphi) \in D,$$

where D is the domain

$$D = \{\, (\theta, \varphi) | \ \theta \in [0, 2\pi), \quad \text{and} \quad \varphi \in [0, \varphi_0)\} \,.$$

Then

$$E^A[\exp(-\lambda T)] = \frac{P_\nu(\cos\varphi)}{P_\nu(\cos\varphi_0)},$$

where ν is such that $\nu(\nu+1) = -2a^2\lambda$ and $P_\nu(\cdot)$ is the Legendre function

$$P_\nu(z) = P_{-\nu-1}(z) = \frac{1}{\pi}\int_0^\pi \left(z + \sqrt{z^2-1}\,\cos\phi\right)^\nu d\phi,$$

where the multiple-valued function $\left(z + \sqrt{z^2-1}\,\cos\phi\right)^\nu$ is to be determined in such a way that for $\phi = \pi/2$ it is equal to (the principal value of) z^ν (which is, in particular, real for positive z and real ν).

Let X_t be the Brownian motion on S^n starting at the point $A \in D$, where

$$D = \{(\theta_1, \dots, \theta_{n-1}, \varphi) |\ \theta_1 \in [0, 2\pi), \theta_i \in [0, \pi] \quad \text{for} \quad i = 2, \dots, n-1 \quad \text{and} \quad \varphi \in [0, \varphi_0)\}.$$

Then

$$E^A[\exp(-\lambda T)] = \frac{(\sin\varphi)^{1-\frac{n}{2}}\, P_\nu^\mu(\cos\varphi)}{(\sin\varphi_0)^{1-\frac{n}{2}}\, P_\nu^\mu(\cos\varphi_0)}, \tag{6.1}$$

where

$$\nu = \frac{1}{2}\left(\sqrt{(n-1)^2 - 8a^2\lambda} - 1\right) \quad \text{and} \quad \mu = \frac{1}{2}(n-2).$$

The function $P_\nu^\mu(\cdot)$ is the associated Legendre function

$$P_\nu^\mu(z) = \frac{1}{\Gamma(-\nu)\Gamma(\nu+1)}\left(\frac{1+z}{1-z}\right)^{\mu/2} \sum_{n=0}^\infty \frac{\Gamma(n-\nu)\Gamma(n+\nu+1)}{\Gamma(n+1-\mu)n!}\left(\frac{1-z}{z}\right)^n.$$

Here $\Gamma(\cdot)$ denotes the Gamma function.

7 The Reflection Principle

We will discuss the reflection principle on S^2. Everything extends easily to S^n.

Notation. For every point $A = (x_1, x_2, x_3) \in S^2$ we denote by $\hat{A}$ the symmetric of A with respect to the x_1x_2-plane. In other words

$$\hat{A} = (x_1, x_2, -x_3) \in S^2$$

Theorem 7.1 *Let X_t, $t \geq 0$, be the Brownian motion on S^2 starting at the point $A = (\theta, \varphi)$ (in spherical coordinates). We assume that $A \in D$, where D is the lower hemisphere, i.e.*

$$D = \{(\theta, \varphi) |\ \theta \in [0, 2\pi) \quad \textit{and} \quad \varphi \in (\pi/2, \pi]\}$$

If

$$T = \inf\{t \geq 0 |\ X_t \notin D\},$$

then

$$P^A\{T < t\} = 2P^A\{X_t \notin D\}.$$

Sketch of Proof.

$$P^A\{T < t\} = P^A\{T < t, X_t \notin D\} + P^A\{T < t, X_t \in D\}.$$

However, if $X_t \notin D$, then, of course, $T < t$. Thus

$$P^A\{T < t, X_t \notin D\} = P^A\{X_t \notin D\}.$$

On the other hand, if we set

$$\tilde{X}_t = \begin{cases} X_t, & \text{if} \quad t \leq T \\ \hat{X}_t, & \text{if} \quad t > T \end{cases}$$

then, by the strong Markov property of X_t

$$P^A\{T < t, X_t \in D\} = P^A\left\{T < t, \tilde{X}_t \in D\right\},$$

but $\tilde{X}_t \in D$ if and only if $X_t \notin D$. Hence,

$$P^A\left\{T < t, \tilde{X}_t \in D\right\} = P^A\{T < t, X_t \notin D\} = P^A\{X_t \notin D\}$$

and

$$P^A\{T < t, X_t \in D\} = P^A\{X_t \notin D\}.$$

Therefore $P^A\{T < t\} = 2P^A\{X_t \notin D\}$. ■

7.1 Applications of the Reflection Principle

The reflection principle can help to calculate the distribution functions of certain exit times.

Let X_t be the Brownian motion on S^2 starting at the south pole S, where $S = (0, \pi)$ in spherical coordinates. If D is the lower hemisphere and T its exit time, then

$$P^S\{T < t\} = 1 - \sum_{n=0}^{\infty}(-1)^n \exp\left(-\frac{(2n+1)^2\sqrt{t}}{a}\right) \cdot \frac{(2n)!(2n+3)}{2^{2n+1}n!}$$

The case of S^1:

Let X_t be the Brownian motion on S^1 starting at $\varphi \in D = (\pi, 2\pi)$. If T is the exit time of D, then

$$P^{\varphi}\{T < t\} = 1 + \frac{4}{\pi}\sum_{n \text{ odd}}\frac{1}{n}\exp\left(-\frac{n^2 t}{2a^2}\right)\sin(n\varphi)$$

The case of S^3:

Let X_t be the Brownian motion on S^3 starting at the south pole S, where $S = (0, 0, \pi)$ in spherical coordinates. If D is the lower hemisphere, namely

$$D = \left\{(\theta_1, \theta_2, \varphi) \in S^3 \;\middle|\; \theta_1 \in [0, 2\pi), \theta_2 \in [0, \pi], \varphi \in (\pi/2, \pi]\right\}$$

and T the exit time of D, then

$$P^S\{T < t\} = 1 + \frac{16}{\pi}\sum_{n=1}^{\infty}(-1)^n n^2 \exp\left(-\frac{(4n^2-1)t}{2a^2}\right).$$

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Quantitative Estimates in the Overconvergence of Some Multivariate Singular Integrals*

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Abstract

In this paper we obtain quantitative estimates in the overconvergence phenomenon in polystrips in $\mathbb{C}^m$, the weighted and non-weighted cases, for some multivariate singular integrals of Picard, Poisson-Cauchy and Gauss-Weierstrass.

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1 Introduction

Let $\xi_1, \xi_2, \ldots, \xi_m > 0$ and $f; \mathbb{R}^m \to \mathbb{R}$ be continuous on $\mathbb{R}^m$, $m \in \mathbb{N}$. If f is 2π- periodic or non periodic and bounded on $\mathbb{R}^m$ or of some exponential or polynomial growth on $\mathbb{R}^m$, the following integrals are well defined :

$$P_{\xi_1,\ldots,\xi_m}(f)(x_1,\ldots,x_m)$$
$$= \frac{1}{\Pi_{j=1}^m (2\xi_j)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1+t_1,\ldots,x_m+t_m)\, \Pi_{j=1}^m e^{-|t_j|/\xi_j} dt_1 \ldots dt_m,$$

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$$Q_{\xi_1,\ldots,\xi_m}(f)(x_1,\ldots x_m)$$
$$= \frac{\prod_{j=1}^{m}\xi_j}{\pi^m}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{f(x_1+t_1,\ldots,x_m+t_m)}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)}dt_1\ldots dt_m,$$
$$R_{\xi_1,\ldots,\xi_m}(f)(x_{1,\ldots,x_m}) = \left(\frac{2}{\pi}\right)^m\prod_{j=1}^{m}\xi_j^3\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}$$
$$\frac{f(x_1+t_1,\ldots,x_m+t_m)}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}dt_1\ldots dt_m,$$

and

$$W_{\xi_1,\ldots,\xi_m}(f)(x_1,\ldots,x_m)$$
$$= \frac{1}{\pi^{m/2}\prod_{j=1}^{m}\xi_j^{1/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(x_1+t_1,\ldots,x_m+t_m)\prod_{j=1}^{m}e^{-t_j^2/\xi_j}dt_1\ldots dt_m.$$

Here $P_{\xi_1,\ldots,\xi_m}(f)(x_1,\ldots x_m)$ is called of Picard type, $Q_{\xi_1,\ldots\xi_m}(f)(x_1,\ldots x_m)$, $R_{\xi_1,\ldots,\xi_m}(f)(x_1,\ldots x_m)$ are called Poisson-Cauchy type and the singular integral $W_{\xi_1,\ldots,\xi_m}(f)$, is called of Gauss-Weierstrass type.

The approximation of $f(x_1,...,x_m)$ by the above singular integrals in the case of real variables as $\xi_j \to 0$, $j = 1,...,m$, was studied in [1].

A quite natural problem would be the study of the overconvergence phenomenon for these singular integrals in polystrips, that is the approximation of the continuous function $f(z_1,\ldots z_m)$ by the complex singular integrals obtained by replacing $x_j \in \mathbb{R}$ by $z_j \in \mathbb{C}$, $j = 1,\ldots,m$ in the above formulae of definition. This case for $m = 1$ was made in [2].

The aim of the present article is to extend the results from [2] for general $m \in \mathbb{N}$.

2 Main Result

The first main result follows.

Theorem 1. *Let $d_1,\ldots,d_m > 0$ and suppose that $f : \times_{j=1}^{m}S_{d_j} \to \mathbb{C}$ is bounded and uniformly continuous in the multivariate strip $\times_{j=1}^{m}S_{d_j}$, where $S_{d_j} = \{z = x+iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| \le d_j\}$.*

(i) Denoting

$$P_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m) = \frac{1}{\prod_{j=1}^{m}(2\xi_j)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f(z_1+t_1,\ldots,z_m+t_m)$$
$$\cdot\prod_{j=1}^{m}e^{-|t_j|/\xi_j}dt_1,\ldots dt_m$$

for all $\xi_j > 0$ and $z_j \in S_{d_j}$, $j = 1,\ldots,m$, we have

$$|P_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m) - f(z_1,\ldots,z_m)| \le (m+1)\omega_1(f;\xi_1,\ldots,\xi_m)_{\times_{j=1}^{m}S_{d_j}},$$

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where for $\delta_j \geq 0,\ j = 1, \ldots, m,$ *we define*

$$\omega_1 \left(f; \delta_1, \ldots, \delta_m\right)_{\times_{j=1}^m S_{d_j}}$$

$$= \sup \{|f(u_1, \ldots, u_m) - f(v_1, \ldots, v_m)| : |u_j - v_j| \leq \delta_j, \text{ with } u_j,\ v_j \in S_{d_j},\ \text{for } j = 1, \ldots, m\}.$$

(ii) *Denoting*

$$R_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) = \left(\frac{2}{\pi}\right)^m \prod_{j=1}^m \xi_j^3 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f\left(z_1 + t_1, \ldots, z_m + t_m\right)}{\prod_{j=1}^m (t_j^2 + \xi_j^2)^2} dt_1 \ldots dt_m,$$

for $\xi_j > 0$ *and* $z_j \in S_{d_j},\ j = 1, \ldots, m,$ *we have*

$$|R_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) - f(z_1, \ldots, z_m)| \leq \left(2^m + \frac{2m}{\pi}\right) \omega_1 \left(f; \xi_1, \ldots, \xi_m\right)_{\times_{j=1}^m S_{d_j}}.$$

(iii) *Denoting*

$$W_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) = \frac{1}{\pi^{m/2} \prod_{j=1}^m \xi_j^{1/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(z_1 + t_1, \ldots, z_m + t_m\right) \prod_{j=1}^m e^{-t_j^2/\xi_j} dt_1 \ldots dt_m,$$

for $\xi_j > 0$ *and* $z_j \in S_{d_j},\ j = 1, \ldots, m,$ *we have*

$$|W_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) - f(z_1, \ldots, z_m)| \leq \left(2^m + \frac{m}{\sqrt{\pi}}\right) \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}}.$$

(iv) *Denoting*

$$Q_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) = \frac{\prod_{j=1}^m \xi_j}{\pi^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{f\left(z_1 + t_1, \ldots, z_m + t_m\right)}{\prod_{j=1}^m (t_j^2 + \xi_j^2)} dt_1 \ldots dt_m,$$

for $\xi_j > 0$ *and* $z_j \in S_{d_j},\ j = 1, \ldots, m,$ *and supposing in addition, that* f *is of Lipschitz class* $(\alpha_1, \ldots, \alpha_m) \in (0, 1)^m$ *in* $\times_{j=1}^m S_{d_j}$*, that is there exists a constant* $\mathcal{M} > 0$ *such that*

$$|f(u_1, \ldots, u_m) - f(v_1, \ldots, v_m)| \leq \mathcal{M} \left(\sum_{j=1}^m |u_j - v_j|^{\alpha_j}\right),$$

for all $u_j,\ v_j \in S_{d_j},\ j=1,\dots,m,$ *we obtain*

$$|Q_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)-f(z_1,\dots,z_m)| \le C\mathcal{M}\left[\sum_{j=1}^{m}\xi_j^{\alpha_j}\right],$$

where

$$C=\frac{2}{\pi}\max_{j=\overline{1,m}}\left\{\int_0^\infty \frac{v^{\alpha_j}}{v^2+1}\,dv\right\}.$$

Proof. (i) If $z_j \in S_{d_j},\ j=1,\dots,m,$ then clearly for all $t\in\mathbb{R}$ we have $z_j+t\in S_{d_j}$ and since f is bounded on $\times_{j=1}^m S_{d_j}$ (denote the bound by $\mathcal{M}(f)$) it clearly follows $|P_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)|\le\mathcal{M}(f)$ for all $(z_1,\dots,z_m)\in\times_{j=1}^m S_{d_j}$. Therefore $P_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)$ exists for all $(z_1,\dots,z_m)\in\times_{j=1}^m S_{d_j}$. Also, the uniform continuity of f on $\times_{j=1}^m S_{d_j}$ implies that

$$\lim_{\xi_1,\dots,\xi_m\to 0}\omega_1(f;\xi_1,\dots,\xi_m)_{\times_{j=1}^m S_{d_j}}=0.$$

For all $(z_1,\dots,z_m)\in\times_{j=1}^m S_{d_j}$ we have

$$|P_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)-f(z_1,\dots,z_m)|$$

$$\le \frac{1}{\prod_{j=1}^m(2\xi_j)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|f(z_1+t_1,\dots,z_m+t_m)-f(z_1,\dots,z_m)|$$

$$\cdot\prod_{j=1}^m e^{-|t_j|/\xi_j}\,dt_1\dots dt_m$$

$$\le \frac{1}{\prod_{j=1}^m(2\xi_j)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\omega_1(f;|t_1|,\dots,|t_m|)_{\times_{j=1}^m S_{d_j}}$$

$$\cdot\prod_{j=1}^m e^{-|t_j|/\xi_j}\,dt_1\dots dt_m$$

$$=\frac{2^m}{\prod_{j=1}^m(2\xi_j)}\int_0^\infty\cdots\int_0^\infty\omega_1\left(f;t_1\frac{\xi_1}{\xi_1},\dots,t_m\frac{\xi_m}{\xi_m}\right)_{\times_{j=1}^m S_{d_j}}$$

$$\cdot\prod_{j=1}^m e^{-t_j/\xi_j}dt_1\dots dt_m\le\frac{1}{\prod_{j=1}^m\xi_j}\omega_1(f;\xi_1,\dots,\xi_m)_{\times_{j=1}^m S_{d_j}}\int_0^\infty\cdots\int_0^\infty$$

$$\left(1+\sum_{j=1}^m\frac{t_j}{\xi_j}\right)\prod_{j=1}^m e^{-t_j/\xi_j}\,dt_1,\dots,dt_m$$

$$=\omega_1(f;\xi_1,\dots,\xi_m)_{\times_{j=1}^m S_{d_j}}\left[\frac{1}{\prod_{j=1}^m\xi_j}\int_0^\infty\cdots\int_0^\infty\prod_{j=1}^m e^{-t_j/\xi_j}\,dt_1\dots dt_m\right.$$

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$$+\frac{1}{\prod_{j=1}^{m}\xi_j}\int_0^{\infty}\cdots\int_0^{\infty}\left(\sum_{j^*=1}^{m}\frac{t_{j^*}}{\xi_{j^*}}\right)\left(\prod_{j=1}^{m}e^{-t_j/\xi_j}\right)\,dt_1\ldots dt_m\Bigg]$$

$$=(m+1)\omega_1\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^{m}S_{d_j}},$$

proving the claim.

(ii) We obtain

$$|R_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m)-f(z_1,\ldots,z_m)|\le\left(\frac{2}{\pi}\right)^m\prod_{j=1}^{m}\xi_j^3$$

$$\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{(|f(z_1+t_1,\ldots,z_m+t_m)-f(z_1,\ldots,z_m)|)}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}\,dt_1\ldots dt_m$$

$$\le\left(\frac{2}{\pi}\right)^m\prod_{j=1}^{m}\xi_j^3\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\omega_1\left(f;|t_1|,\ldots,|t_m|\right)_{\times_{j=1}^{m}S_{d_j}}}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}dt_1\ldots dt_m$$

$$=2^m\left(\left(\frac{2}{\pi}\right)^m\prod_{j=1}^{m}\xi_j^3\right)\int_0^{\infty}\cdots\int_0^{\infty}\frac{\omega_1\left(f;\xi_1\frac{t_1}{\xi_1},\ldots,\xi_m\frac{t_m}{\xi_m}\right)_{\times_{j=1}^{m}S_{d_j}}}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}dt_1\ldots dt_m$$

$$\le\omega_1\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^{m}S_{d_j}}\left(\frac{2^{2m}\prod_{j=1}^{m}\xi_j^3}{\pi^m}\right)$$

$$\cdot\int_0^{\infty}\cdots\int_0^{\infty}\frac{\left(1+\sum_{j=1}^{m}\frac{t_j}{\xi_j}\right)}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}\,dt_1\ldots dt_m$$

$$=\omega_1\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^{m}S_{d_j}}\left(\frac{2^{2m}\prod_{j=1}^{m}\xi_j^3}{\pi^m}\right)\left[\int_0^{\infty}\cdots\int_0^{\infty}\frac{d\,t_1\ldots d\,t_m}{\prod_{j=1}^{m}(t_j^2+\xi_j^2)^2}\right.$$

$$\left.+\sum_{j^*=1}^{m}\int_0^{\infty}\cdots\int_0^{\infty}\frac{t_{j^*}}{\xi_{j^*}}\frac{1}{\prod_{j=1}^{m}(t_{j^2}+\xi_j^2)^2}\,dt_1\ldots dt_m\right]$$

$$=\omega_1\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^{m}S_{d_j}}$$

$$\cdot\left[2^m+\left(\frac{2^{2m}\prod_{j=1}^{m}\xi_j^3}{\pi^m}\right)\sum_{j^*=1}^{m}\left\{\left(\int_0^{\infty}\frac{t_{j^*}}{\xi_{j^*}}\frac{d\,t_{j^*}}{(t_{j^*}+\xi_{j^*}^2)^2}\right)\right.\right.$$

$$\left.\left.\cdot\prod_{\substack{j=1\\j\ne j^*}}^{m}\left(\int_0^{\infty}\frac{t_j}{(t_j^2+\xi_j^2)^2}\right)\right\}\right]$$

$$=\omega_1\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^{m}S_{d_j}}\left[2^m+\frac{2^{2m}\prod_{j=1}^{m}\xi_j^3}{\pi^m}\sum_{j^*=1}^{m}\left\{\left(\frac{1}{2\xi_{j^*}^3}\right)\prod_{\substack{j=1\\j\ne j^*}}^{m}\frac{\pi}{4\xi_j^3}\right\}\right]$$

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$$= \omega_1 (f; \xi_1, \ldots, \xi_m)_{\times_{j=1}^m S_{d_j}} \left[2^m + \frac{2m}{\pi} \right],$$

proving the claim.

(iii) We further have

$$|W_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m) - f(z_1,\ldots,z_m)|$$

$$\leq \frac{1}{\pi^{m/2} \prod_{j=1}^m \xi_j^{1/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |f(z_1+t_1,\ldots,z_m+t_m) - f(z_1,\ldots,z_m)|$$

$$\cdot \prod_{j=1}^m e^{-t_j^2/\xi_j} \, dt_1 \ldots dt_m$$

$$\leq \frac{1}{\pi^{m/2} \prod_{j=1}^m \xi_j^{1/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_1 (f; |t_1|, \ldots, |t_m|)_{\times_{j=1}^m S_{d_j}}$$

$$\cdot \prod_{j=1}^m e^{-t_j^2/\xi_j} \, dt_1 \ldots dt_m$$

$$= \frac{2^m}{\pi^{m/2} \prod_{j=1}^m \xi_j^{1/2}} \int_0^{\infty} \cdots \int_0^{\infty} \omega_1 \left(f; \sqrt{\xi_1} \frac{t_1}{\sqrt{\xi_1}}, \ldots, \sqrt{\xi_m} \frac{t_m}{\sqrt{\xi_m}} \right)_{\times_{j=1}^m S_{d_j}}$$

$$\cdot \prod_{j=1}^m e^{-t_j^2/\xi_j} \, dt_1 \ldots dt_m$$

$$\leq \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m} \right)_{\times_{j=1}^m S_{d_j}} \frac{2^m}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_0^{\infty} \cdots \int_0^{\infty} \left(1 + \sum_{j=1}^m \frac{u_j}{\sqrt{\xi_j}} \right)$$

$$\cdot \prod_{j=1}^m e^{-u_{j^2}/\xi_j} \, du_1 \ldots du_m$$

$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m} \right)_{\times_{j=1}^m S_{d_j}} \left[\frac{1}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_0^{\infty} \cdots \int_0^{\infty} \right.$$

$$\cdot \prod_{j=1}^m e^{-u_j^2/\xi_j} \, du_1 \ldots du_m$$

$$\left. + \frac{1}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_0^{\infty} \cdots \int_0^{\infty} \left(\sum_{j^*=1}^m \frac{u_{j^*}}{\sqrt{\xi_{j^*}}} \right) \prod_{j=1}^m e^{-u_j^2/\xi_j} \, du_1 \ldots du_m \right]$$

$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m} \right)_{\times_{j=1}^m S_{d_j}}$$

$$\left[1 + \frac{1}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \sum_{j^*=1}^m \int_0^{\infty} \cdots \int_0^{\infty} \frac{u_{j^*}}{\sqrt{\xi_{j^*}}} \prod_{j=1}^m e^{-(u_j/\sqrt{\xi_j})^2} \, du_1 \ldots du_m \right]$$

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$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}}$$

$$\left[1 + \frac{1}{\pi^{m/2}} \sum_{j^*=1}^m \left[\left(\int_0^\infty \frac{u_{j^*}}{\sqrt{\xi_{j^*}}} \; e^{-\left(u_{j^*}/\sqrt{\xi_{j^*}}\right)^2} \; \frac{du_{j_*}}{\sqrt{\xi_{j_*}}}\right)\right.\right.$$

$$\left.\left. \cdot \prod_{\substack{j=1\\ j\neq j^*}}^m \left(\int_0^\infty e^{-\left(u_j/\sqrt{\xi_j}\right)^2} \; \frac{du_j}{\sqrt{\xi_j}}\right)\right]\right]$$

$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}}$$

$$\left[1 + \frac{1}{\pi^{m/2}} \sum_{j^*=1}^m \left[\left(\int_0^\infty u_{j^*} \; e^{-u_{j^*}^2} du_{j^*}\right) \prod_{\substack{j=1\\ j\neq j^*}}^m \left(\int_0^\infty e^{-u_j^2} du_j\right)\right]\right]$$

$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}} \left[1 + \frac{1}{\pi^{m/2}} \sum_{j^*=1}^m \left[\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right)^{m-1}\right]\right]$$

$$= 2^m \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}} \left[1 + \frac{1}{\pi^{m/2}} \cdot \frac{m}{2^m} \pi^{\frac{m-1}{2}}\right]$$

$$= \omega_1 \left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}} \left[2^m + \frac{m}{\sqrt{\pi}}\right],$$

proving the claim.

(iv) We observe that

$$|Q_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m) - f(z_1,\ldots,z_m)| \leq \frac{\prod_{j=1}^m \xi_j}{\pi^m}$$

$$\cdot \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \left(\frac{|f(z_1+t_1,\ldots,z_m+t_m) - f(z_1,\ldots,z_m)|}{\prod_{j=1}^m (t_j^2+\xi_j^2)}\right) \; dt_1 \ldots dt_m$$

$$\leq \mathcal{M} \frac{\prod_{j=1}^m \xi_j}{\pi^m} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \left(\frac{\sum_{j=1}^m |t_j|^{\alpha_j}}{\prod_{j=1}^m (t_j^2+\xi_j^2)}\right) dt_1 \ldots dt_m$$

$$= \mathcal{M} \left(\frac{\prod_{j=1}^m (2\xi_j)}{\pi^m}\right) \int_0^\infty \cdots \int_0^\infty \left(\frac{\sum_{j^*=1}^m t_{j^*}^{\alpha_{j^*}}}{\prod_{j=1}^m (t_j^2+\xi_j^2)}\right) dt_1 \ldots dt_m$$

$$= \mathcal{M} \sum_{j^*=1}^m \left[\left[\prod_{\substack{j=1\\ j\neq j^*}}^m \left(\frac{2\xi_j}{\pi}\right) \left(\int_0^\infty \frac{d\,t_j}{(t_j^2+\xi_j^2)}\right)\right] \left(\frac{2\xi_{j^*}}{\pi} \int_0^\infty \frac{t_{j^*}^{\alpha_{j^*}}}{(t_{j^*}^2+\xi_{j^*}^2)} dt_{j^*}\right)\right]$$

$$= \mathcal{M} \sum_{j^*=1}^m \left[\frac{2\xi_{j^*}}{\pi} \int_0^\infty \frac{t_{j^*}^{\alpha_{j^*}}}{(t_{j^*}^2+\xi_{j^*}^2)} dt_{j^*}\right]$$

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$$= \mathcal{M} \sum_{j^*=1}^{m} \left[\frac{2\xi_{j^*}^{\alpha_{j^*}}}{\pi} \left(\int_0^\infty \frac{u^{\alpha_{j^*}}}{u^2+1}\, du \right) \right] \leq \mathcal{C}\mathcal{M} \left(\sum_{j^*=1}^{m} \xi_{j^*}^{\alpha_{j^*}} \right),$$

where

$$\mathcal{C} = \frac{2}{\pi} \cdot \max_{j^*=1,\ldots,m} \left\{ \int_0^\infty \frac{u^{\alpha_{j^*}}}{u^2+1}\, du \right\} < \infty,$$

finishing the proof of the theorem. □

In what follows for $P_{n,\xi_1,\ldots,\xi_m}(f;(z_1,\ldots,z_m)$ and $W_{\xi_1,\ldots,\xi_m}(f)(z_1,\ldots,z_m)$ we will consider the weighted approximation on $\times_{j=1}^m S_{d_j}$, which seems to be more natural because $\times_{j=1}^m S_{d_j}$ is unbounded in $\mathbb{C}^m$, $m \in \mathbb{N}$. For this purpose, first we need some general notations. Let $w : \times_{j=1}^m S_{d_j} \to \mathbb{R}_+$ be a continuous weighted functions in $\times_{j=1}^m S_{d_j}$, with the properties that $w(z_1,\ldots,z_m) > 0$ for any $(z_1,\ldots,z_m) \in \times_{j=1}^m S_{d_j}$ and $\lim_{\substack{|z_j|\to\infty,\\ j=1,\ldots,m}} w(z_1,\ldots,z_m) = 0$.

Define the space

$$C_w\left(\times_{j=1}^m S_{d_j}\right)$$

$$= \left\{ f : \times_{j=1}^m S_{d_j} \to \mathbb{C}; f \text{ is continuous in } \times_{j=1}^m S_{d_j} \text{ and } \|f\|_w < \infty \right\},$$

where

$$\|f\|_w := \sup\left\{ w(z_1,\ldots,z_m)|f(z_1,\ldots,z_m)|; (z_1,\ldots,z_m) \in \times_{j=1}^m S_{d_j} \right\}.$$

Also, for $f \in C_w\left(\times_{j=1}^m S_{d_j}\right)$ define the weighted modulus of continuity

$$\omega_{1,w}\left(f;\delta_1,\ldots,\delta_m\right)_{\times_{j=1}^m S_{d_j}}$$

$$= \sup\{ w(z_1,\ldots,z_m) \left| f(z_1+h_1,\ldots,z_m+h_m) - f(z_1,\ldots,z_m) \right|;$$
$$(z_1,\ldots,z_m) \in \times_{j=1}^m S_{d_j},\ h_j \in \mathbb{R} \text{ with } |h_j| \leq \delta_j,\ j=1,\ldots,m \}$$

Remark. The last modulus of continuity has the properties:

a) it is increasing as a function of each $\delta_j, j = 1,\ldots,m$,

b) $\omega_{1,w}\left(f;0,\ldots,0\right)_{\times_{j=1}^m S_{d_j}} = 0$,

c)

$$\omega_{1,w}\left(f;\lambda_1\delta_1,\ldots,\lambda_m\delta_m\right)_{\times_{j=1}^m S_{d_j}} \leq \left(1 + \sum_{j=1}^m \lambda_j\right) \omega_{1,w}\left(f;\delta_1,\ldots,\delta_m\right)_{\times_{j=1}^m S_{d_j}},$$

for all $\lambda_j \geq 0,\ j = 1,\ldots,m$.

We present

Theorem 2. *Let $d_j > 0,\ j = 1,\ldots,m$, and suppose that $f : \times_{j=1}^m S_{d_j} \to \mathbb{C}$ is continuous in $\times_{j=1}^m S_{d_j}$. Let the Freud-type weight $w(z_1,\ldots,z_m) = \prod_{j=1}^m e^{-q_j|z_j|}$ with $q_j > 0$, fixed, $j = 1,\ldots,m$; and $f \in C_w\left(\times_{j=1}^m S_{d_j}\right)$.*

Then

(i)

$$\|P_{\xi_1,\ldots,\xi_m}(f) - f\|_w \leq (m+1)\omega_{n,w}\left(f;\xi_1,\ldots,\xi_m\right)_{\times_{j=1}^m S_{d_j}},$$

for all $0<\xi_j<\frac{1}{q_j},\ j=1,\dots,m,$ *and*

(ii)

$$\|W_\xi^*(f)-f\|_w \le \left(2^m+\frac{m}{\sqrt{\pi}}\right)\omega_{1,w}\left(f;\sqrt{\xi_1},\dots,\sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}},$$

for all $0<\xi_j<1,\ j=1,\dots,m.$

Proof. The continuity of f in $\times_{j=1}^m S_{d_j}$ immediately implies the continuity of $P_{\xi_1,\dots,\xi_m}(f)$ and $W_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)$.

(i) In addition we have

$$|w(z_1,\dots,z_m)P_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)|$$

$$=\left|\frac{1}{\prod_{j=1}^m(2\xi_j)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}w(z_1+t_1,\dots,z_m+t_m)f(z_1+t_1,\dots,t_m)\right.$$

$$\left.\cdot\frac{w(z_1,\dots,z_m)}{w(z_1+t_1,\dots,z_m+t_m)}\prod_{j=1}^m e^{-|t_j|/\xi_j}\,dt_1\dots dt_m\right|$$

$$\le\|f\|_w\prod_{j=1}^m\frac{1}{2\xi_j}\int_{-\infty}^{\infty}e^{|t_j|\left(q_j-\frac{1}{\xi_j}\right)}\,dt_j\le\left(\prod_{j=1}^m C_{\xi_j,q_j}\right)\|f\|_w,$$

where

$$C_{\xi_j,q_j}=\frac{1}{2\xi_j}\int_{-\infty}^{\infty}e^{|t_j|\left(q_j-\frac{1}{\xi_j}\right)}dt_j<\infty,$$

for all $j=1,\dots,m$.

Passing to supremum over all $(z_1,\dots,z_m)\in\times_{j=1}^m S_{d_j}$, it follows that $\|P_{\xi_1,\dots,\xi_m}(f)\|_w<\infty$, that is $P_{\xi_1,\dots,\xi_m}(f)\in C_w\left(\times_{j=1}^m S_{d_j}\right)$, for $0<\xi_j<\frac{1}{q_j},\ j=1,\dots,m$.

Next for all $z_j\in S_{d_j},\ j=1,\dots,m$, we derive

$$w(z_1,\dots,z_m)|P_{\xi_1,\dots,\xi_m}(f)(z_1,\dots,z_m)-f(z_1,\dots,z_m)|$$

$$=\frac{w(z_1,\dots,z_m)}{\prod_{j=1}^m(2\xi_j)}\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(f(z_1+t_1,\dots,z_m+t_m)-f(z_1,\dots,z_m))\right.$$

$$\left.\cdot\prod_{j=1}^m e^{-|t_j|/\xi_j}dt_1\dots dt_m\right|\le\frac{1}{\prod_{j=1}^m(2\xi_j)}$$

$$\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}w(z_1,\dots,z_m)\,|f(z_1+t_1,\dots,z_m+t_m)-f(z_1,\dots,z_m)|$$

$$\cdot\prod_{j=1}^m e^{-|t_j|/\xi_j}\,dt_1\dots dt_m$$

$$\leq \frac{1}{\prod_{j=1}^{m}(2\xi_j)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_{1,w}\left(f; |t_1|, \ldots, |t_m|\right)_{\times_{j=1}^{m} S_{d_j}}$$

$$\cdot \prod_{j=1}^{m} e^{-|t_j|/\xi_j} dt_1 \ldots dt_m$$

$$= \frac{1}{\prod_{j=1}^{m}\xi_j} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \omega_{1,w}\left(f; \xi_1 \frac{t_1}{\xi_1}, \ldots, \xi_m \frac{t_m}{\xi_m}\right)_{\times_{j=1}^{m} S_{d_j}}$$

$$\cdot \prod_{j=1}^{m} e^{-t_j/\xi_j} dt_1 \ldots dt_m,$$

$$\leq \frac{\omega_{1,w}\left(f; \xi_1, \ldots, \xi_m\right)_{\times_{j=1}^{m} S_{d_j}}}{\prod_{j=1}^{m}\xi_j} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \left(1 + \sum_{j=1}^{m} \frac{t_j}{\xi_j}\right)$$

$$\cdot \prod_{j=1}^{m} e^{-t_j/\xi_j} dt_1 \ldots dt_m = (m+1)\omega_{1,w}\left(f; \xi_1, \ldots, \xi_m\right)_{\times_{j=1}^{m} S_{d_j}},$$

proving the claim.

(ii) Next we observe

$$|w(z_1, \ldots, z_m) W_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m)|$$

$$= \left| \frac{1}{\pi^{m/2}\prod_{j=1}^{m}\xi_j^{1/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w(z_1 + t_1, \ldots, z_m + t_m) f(z_1 + t_1, \ldots, z_m + t_m) \right.$$

$$\left. \cdot \frac{w(z_1, \ldots, z_m)}{w(z_1 + t_1, \ldots, z_m + t_m)} \prod_{j=1}^{m} e^{-t_j^2/\xi_j} dt_1 \ldots dt_m \right|$$

$$\leq \|f\|_w \prod_{j=1}^{m} \frac{1}{\sqrt{\pi \xi_j}} \int_{-\infty}^{\infty} e^{|t_j|(q_j - |t_j|/\xi_j)}\, dt_j$$

$$= 2^m \|f\|_w \prod_{j=1}^{m} \left(\frac{1}{\sqrt{\pi \xi_j}} \int_{0}^{\infty} e^{t_j(q_j - t_j/\xi_j)}\, dt_j \right).$$

But we can write

$$\int_{0}^{\infty} e^{t_j(q_j - t_j/\xi_j)}\, dt_j = \int_{0}^{q_j+1} e^{t_j(q_j - t_j/\xi_j)}\, dt_j + \int_{q_j+1}^{\infty} e^{t_j(q_j - t_j/\xi_j)}\, dt_j.$$

For $0 < \xi_j \leq 1$ and $t_j \geq q_j + 1$ we get $t_j\left(q_j - t_j/\xi_j\right) \leq -t_j$ and $e^{t_j(q_j - t_j/\xi_j)} \leq e^{-t_j}$, which implies

$$\int_{q_j+1}^{\infty} e^{t_j(q_j - t_j/\xi_j)}\, dt_j \leq \int_{q_j+1}^{\infty} e^{-t_j} dt_j = e^{-(q_j+1)},$$

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for $j = 1, \ldots, m$.

In conclusion, from the above considerations we get

$$|w(z_1, \ldots, z_m) W_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m)| \leq C_m \|f\|_w \,,$$

for some $C_m > 0$.

Passing to sup over $(z_1, \ldots, z_m) \in \times_{j=1}^m S_{d_j}$, it follows that $W_{\xi_1,\ldots,\xi_m}(f) \in C_w\left(\times_{j=1}^m S_{d_j}\right)$, for all $0 < \xi_j \leq 1,\ j = 1, \ldots, m$.

For the estimate, for all $(z_1, \ldots, z_m) \in \times_{j=1}^m S_{d_j}$ we find

$$w(z_1, \ldots, z_m) \left| W_{\xi_1,\ldots,\xi_m}(f)(z_1, \ldots, z_m) - f(z_1, \ldots, z_m) \right|$$

$$= \frac{w(z_1, \ldots, z_m)}{\pi^{m/2} \Pi_{j=1}^m \xi_j^{1/2}} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f(z_1 + t_1, \ldots, z_m + t_m) - f(z_1, \ldots, z_m)) \cdot \prod_{j=1}^m e^{-t_j^2/\xi_j}\, dt_1 \ldots dt_m \right|$$

$$\leq \frac{1}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w(z_1, \ldots, z_m) |f(z_1{+}t_1, \ldots, z_m{+}t_m){-}f(z_1, \ldots, z_m)| \cdot \prod_{j=1}^m e^{-t_j^2/\xi_j}\, dt_1 \ldots dt_m$$

$$\leq \frac{1}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_{1,w}\left(f; |t_1|, \ldots, |t_m|\right)_{\times_{j=1}^m S_{d_j}}, \cdot \prod_{j=1}^m e^{-t_j^2/\xi_j}\, dt_1 \ldots dt_m$$

$$= \frac{2^m}{\prod_{j=1}^m \sqrt{\pi \xi_j}} \int_0^{\infty} \cdots \int_0^{\infty} \omega_{1,w}\left(f; \sqrt{\xi_1}\frac{t_1}{\sqrt{\xi_1}}, \ldots, \sqrt{\xi_m}\frac{t_m}{\sqrt{\xi_m}}\right)_{\times_{j=1}^m S_{d_j}} \cdot \prod_{j=1}^m e^{-t_j^2/\xi_j}\, dt_1 \ldots dt_m$$

$$\leq \left(\omega_{1,w}\left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_1}\right)_{\times_{j=1}^m S_{d_j}}\right) \left(\frac{2^m}{\prod_{j=1}^m \sqrt{\pi \xi_j}}\right) \int_0^{\infty} \cdots \int_0^{\infty} \left(1 + \sum_{j=1}^m \frac{t_j}{\sqrt{\xi_j}}\right) \prod_{j=1}^m e^{-t_{j^2}/\xi_j}\, dt_1 \ldots dt_m$$

$$= \left(2^m + \frac{m}{\sqrt{\pi}}\right) \omega_{1,w}\left(f; \sqrt{\xi_1}, \ldots, \sqrt{\xi_m}\right)_{\times_{j=1}^m S_{d_j}}$$

proving the claim and finishing the proof of theorem. $\square$

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A note on strong differential superordinations using Sălăgean and Ruscheweyh operators

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Abstract

In the present paper we establish several strong differential superordinations regardind the new operator SR^m defined by convolution product of the extended Sălăgean operator and Ruscheweyh derivative, $SR^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$, $SR^m f(z,\zeta) = (S^m * R^m) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $R^m f(z,\zeta)$ denote the extended Ruscheweyh derivative, $S^m f(z,\zeta)$ is the extended Sălăgean operator and $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

Keywords: strong differential superordination, convex function, best subordinant, extended differential operator, convolution product.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

We also extend the well known differential operators to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [5].

Definition 1.1 *[1] For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the operator S^m is defined by $S^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} S^0 f(z,\zeta) &= f(z,\zeta), \\ S^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ S^{m+1} f(z,\zeta) &= z\left(S^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2 *[1] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $S^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} j^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.3 *[1] For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ (m+1) R^{m+1} f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)'_z + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.4 *[1] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [4].

Definition 1.5 *[4] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.6 *[4] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.5 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.7 *[3] We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim_{z \to y} f(z,\zeta) = \infty\}$, and are such that $f'_z(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.8 *[3] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$,then $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and is the best subordinant.*

Lemma 1.9 *[3] Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $Re\ \gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and $q(z,\zeta) + \frac{1}{\gamma} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is the best subordinant.*

2 Main results

Definition 2.1 *[2] Let $m \in \mathbb{N} \cup \{0\}$. Denote by SR^m the operator given by the Hadamard product (the convolution product) of the extended Sălăgean operator S^m and the extended Ruscheweyh operator R^m, $SR^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$SR^m f(z,\zeta) = (S^m * R^m) f(z,\zeta).$$

Remark 2.2 *[2] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $SR^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} j^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Theorem 2.3 *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, $\mathrm{Re}\, c > -2$, and suppose that $(SR^m f(z,\zeta))'_z$ is univalent in $U \times \overline{U}$, $(SR^m F(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{1}$$

then

$$q(z,\zeta) \prec\prec (SR^m F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{n z^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant.

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Proof. We have $z^{c+1}F(z,\zeta) = (c+2)\int_0^z t^c f(t,\zeta)\,dt$ and differentiating it, with respect to z, we obtain $(c+1)F(z,\zeta) + zF'_z(z,\zeta) = (c+2)f(z,\zeta)$ and $(c+1)SR^mF(z,\zeta) + z(SR^mF(z,\zeta))'_z = (c+2)SR^mf(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Differentiating the last relation with respect to z we have

$$(SR^mF(z,\zeta))'_z + \frac{1}{c+2}z(SR^mF(z,\zeta))''_{z^2} = (SR^mf(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{2}$$

Using (2), the strong differential superordination (1) becomes

$$h(z,\zeta) \prec\prec (SR^mF(z,\zeta))'_z + \frac{1}{c+2}z(SR^mF(z,\zeta))''_{z^2}. \tag{3}$$

Denote

$$p(z,\zeta) = (SR^mF(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{4}$$

Replacing (4) in (3) we obtain $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2}zp'_z(z,\zeta)$, $\quad z \in U$, $\zeta \in \overline{U}$.

Using Lemma 1.8 for $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec (SR^mF(z,\zeta))'_z,\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.4 *Let* $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$, *where* $\beta \in [0,1)$. *Let* $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z \in U$, $\zeta \in \overline{U}$, $\operatorname{Re} c > -2$, *and suppose that* $(SR^mf(z,\zeta))'_z$ *is univalent in* $U \times \overline{U}$, $(SR^mF(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ *and*

$$h(z,\zeta) \prec\prec (SR^mf(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{5}$$

then

$$q(z,\zeta) \prec\prec (SR^mF(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}}\int_0^z \frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z \in U$, $\zeta \in \overline{U}$. *The function q is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.3 and considering $p(z,\zeta) = (SR^mF(z,\zeta))'_z$, the strong differential superordination (5) becomes $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + \frac{1}{c+2}zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.8 for $\gamma = c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z \frac{\zeta+(2\beta-\zeta)t}{1+t}t^{\frac{c+2}{n}-1}dt = 2\beta - \zeta + \frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}}\int_0^z \frac{t^{\frac{c+2}{n}-1}}{t+1}dt \prec\prec (SR^mF(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$.

The function q is convex and it is the best subordinant. ■

Theorem 2.5 *Let* $q(z,\zeta)$ *be a convex function in* $U \times \overline{U}$ *and let* $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2}zq'_z(z,\zeta)$, *where* $z \in U$, $\zeta \in \overline{U}$, $\operatorname{Re} c > -2$. *Let* $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z \in U$, $\zeta \in \overline{U}$, *and suppose that* $(SR^mf(z,\zeta))'_z$ *is univalent in* $U \times \overline{U}$, $(SR^mF(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ *and*

$$h(z,\zeta) \prec\prec (SR^mf(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{6}$$

then

$$q(z,\zeta) \prec\prec (SR^mF(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. *The function q is the best subordinant.*

Proof. We obtain that
$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{7}$$
Differentiating (7), with respect to z, we have $(c+1)F(z,\zeta)+zF'_z(z,\zeta)=(c+2)f(z,\zeta)$ and
$$(c+1)\,SR^mF(z,\zeta)+z\left(SR^mF(z,\zeta)\right)'_z=(c+2)\,SR^mf(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}. \tag{8}$$
Differentiating (8) with respect to z we have
$$\left(SR^mF(z,\zeta)\right)'_z+\frac{1}{c+2}z\left(SR^mF(z,\zeta)\right)''_{z^2}=\left(SR^mf(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{9}$$
Using (9), the strong differential superordination (6) becomes
$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)\prec\prec\left(SR^mF(z,\zeta)\right)'_z+\frac{1}{c+2}z\left(SR^mF(z,\zeta)\right)''_{z^2}. \tag{10}$$
Denote
$$p(z,\zeta)=\left(SR^mF(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{11}$$
Replacing (11) in (10) we obtain $h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)$, $z\in U,\zeta\in\overline{U}$.

Using Lemma 1.9 for $\gamma=c+2$, we have
$$q(z,\zeta)\prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)\prec\prec\left(SR^mF(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U},$$
where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is the best subordinant. ∎

Theorem 2.6 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(SR^mf(z,\zeta)\right)'_z$ is univalent and $\frac{SR^mf(z,\zeta)}{z}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*
$$h(z,\zeta)\prec\prec\left(SR^mf(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}, \tag{12}$$
then
$$q(z,\zeta)\prec\prec\frac{SR^mf(z,\zeta)}{z}, \quad z\in U,\ \zeta\in\overline{U},$$
where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta)=\frac{SR^mf(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^ma_j^2(\zeta)z^j}{z}=1+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^ma_j^2(\zeta)z^{j-1}$. Evidently $p\in\mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp'_z(z,\zeta)=\left(SR^mf(z,\zeta)\right)'_z$.

Then (12) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.8 for $\gamma=1$, we have
$$q(z,\zeta)\prec\prec p(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \quad \text{i.e. } q(z,\zeta)\prec\prec\frac{SR^mf(z,\zeta)}{z}, \quad z\in U,\ \zeta\in\overline{U},$$
where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant. ∎

Corollary 2.7 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\le\beta<1$. Let $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(SR^mf(z,\zeta)\right)'_z$ is univalent and $\frac{SR^mf(z,\zeta)}{z}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*
$$h(z,\zeta)\prec\prec\left(SR^mf(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}, \tag{13}$$
then
$$q(z,\zeta)\prec\prec\frac{SR^mf(z,\zeta)}{z}, \quad z\in U,\ \zeta\in\overline{U},$$
where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.6 and considering $p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}$, the strong differential superordination (13) becomes $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.8 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e. $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt\prec\prec\frac{SR^m f(z,\zeta)}{z}$, $z\in U$, $\zeta\in\overline{U}$.

The function q is convex and it is the best subordinant. ■

Theorem 2.8 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)$. If $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, suppose that $(SR^m f(z,\zeta))_z'$ is univalent, $\frac{SR^m f(z,\zeta)}{z}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)\prec\prec\left(SR^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{14}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^j}{z}=1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^{j-1}$. Evidently $p\in\mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp_z'(z,\zeta)=\left(SR^m f(z,\zeta)\right)_z'$, $z\in U$, $\zeta\in\overline{U}$, and (14) becomes $q(z,\zeta)+zq_z'(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.9 for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec\frac{SR^m f(z,\zeta)}{z},\ z\in U,\ \zeta\in\overline{U},$$

and q is the best subordinant. ■

Theorem 2.9 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{15}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta)=\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}=\frac{z+\sum_{j=n+1}^{\infty}C_{m+j}^{m+1}j^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^j}=\frac{1+\sum_{j=n+1}^{\infty}C_{m+j}^{m+1}j^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^{j-1}}$. Evidently $p\in\mathcal{H}^*[1,n,\zeta]$.

We have $p_z'(z,\zeta)=\frac{\left(SR^{m+1}f(z,\zeta)\right)_z'}{SR^m f(z,\zeta)}-p(z,\zeta)\cdot\frac{\left(SR^m f(z,\zeta)\right)_z'}{SR^m f(z,\zeta)}$ and $p(z,\zeta)+zp_z'(z,\zeta)=\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)_z'$.

Then (15) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.8 for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.10 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\le\beta<1$. Let $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$ is univalent, $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z,\qquad z\in U,\ \zeta\in\overline{U},\tag{16}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}$, the strong differential superordination (16) becomes $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.8 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e. $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$.

The function q is convex and it is the best subordinant. ■

Theorem 2.11 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. If $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, suppose that $\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$ is univalent, $\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z)}\in\mathcal{H}[1,n]\cap Q$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z,\qquad z\in U,\ \zeta\in\overline{U},\tag{17}$$

then

$$q(z,\zeta)\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta)=\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}=\frac{z+\sum_{j=n+1}^{\infty}C^{m+1}_{m+j}j^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}C^{m}_{m+j-1}j^{m}a_j^2(\zeta)z^j}=\frac{1+\sum_{j=n+1}^{\infty}C^{m+1}_{m+j}j^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}C^{m}_{m+j-1}j^{m}a_j^2(\zeta)z^{j-1}}$. Evidently $p\in\mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp'_z(z,\zeta)=\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'_z$, $z\in U$, $\zeta\in\overline{U}$, and (17) becomes $q(z,\zeta)+zq'_z(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.9 for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}\quad q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U},$$

and q is the best subordinant. ■

Theorem 2.12 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $m\in\mathbb{N}\cup\{0\}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\frac{1}{z}SR^{m+1}f(z,\zeta)$ is univalent and $(SR^m f(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\frac{1}{z}SR^{m+1}f(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},\tag{18}$$

then

$$q(z,\zeta)\prec\prec(SR^m f(z,\zeta))'_z,\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{m+1}{nz^{\frac{m+1}{n}}}\int_0^z h(t,\zeta)t^{\frac{m+1}{n}-1}dt$. The function q is convex and it is the best subordinant.

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Proof. With notation $p(z,\zeta) = (SR^m f(z,\zeta))'_z = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m j^{m+1} a_j^2(\zeta) z^{j-1}$ and $p(0,\zeta) = 1$, we obtain for $f(z) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, $p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta) = \frac{1}{z} SR^{m+1} f(z,\zeta)$. Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

Then (18) becomes $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.8 for $\gamma = m+1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z,\quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.13 *Let $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta) z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \le \beta < 1$. Let $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$ and suppose that $\frac{1}{z} SR^{m+1} f(z,\zeta)$ is univalent and $(SR^m f(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \frac{1}{z} SR^{m+1} f(z,\zeta),\qquad z \in U,\ \zeta \in \overline{U}, \tag{19}$$

then

$$q(z,\zeta) \prec\prec (SR^m f(z,\zeta))'_z,\qquad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)(m+1)}{nz^{\frac{m+1}{n}}} \int_0^z \frac{t^{\frac{m+1}{n}-1}}{1+t} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.12 and considering $p(z,\zeta) = (SR^m f(z,\zeta))'_z$, the strong differential superordination (19) becomes $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.8 for $\gamma = m+1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z t^{\frac{m+1}{n}-1} \frac{\zeta+(2\beta-\zeta)t}{1+t} dt = 2\beta - \zeta + \frac{2(\zeta-\beta)(m+1)}{nz^{\frac{m+1}{n}}} \int_0^z \frac{t^{\frac{m+1}{n}-1}}{1+t} dt \prec\prec (SR^m f(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$.

The function q is convex and it is the best subordinant. ■

Theorem 2.14 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta)$. If $m \in \mathbb{N} \cup \{0\}$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$, suppose that $\frac{1}{z} SR^{m+1} f(z,\zeta)$ is univalent, $(SR^m f(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta) \prec\prec \frac{1}{z} SR^{m+1} f(z,\zeta),\qquad z \in U,\ \zeta \in \overline{U}, \tag{20}$$

then

$$q(z,\zeta) \prec\prec (SR^m f(z,\zeta))',\qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta) = (SR^m f(z,\zeta))'_z = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m j^{m+1} a_j^2(\zeta) z^{j-1}$.

Differentiating with respect to z, we obtain $p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta) = \frac{1}{z} SR^{m+1} f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and (20) becomes $q(z,\zeta) + \frac{1}{m+1} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{m+1} z p'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Using Lemma 1.9 for $\gamma = m+1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt \prec\prec (SR^m f(z,\zeta))',\ z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant. ■

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Certain strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator

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Abstract

In the present paper we establish several strong differential superordinations regardind the new operator DR_λ^m defined by convolution product of the extended generalized Sălăgean operator and Ruscheweyh derivative, $DR_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $R^m f(z,\zeta)$ denote the extended Ruscheweyh derivative, $D_\lambda^m f(z,\zeta)$ is the extended generalized Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$, with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, is the class of normalized analytic functions.

Keywords: strong differential superordination, convex function, best subordinant, extended differential operator, convolution product.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$, with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

We also extend the well known differential operators to the new class of analytic functions $\mathcal{A}_{n\zeta}^*$ introduced in [10].

Definition 1.1 *[5] For $f \in \mathcal{A}_\zeta^*$, $\lambda \ge 0$ and $m \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} D_\lambda^0 f(z,\zeta) &= f(z,\zeta), \\ D_\lambda^1 f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f_z'(z,\zeta) = D_\lambda f(z,\zeta), ..., \\ D_\lambda^{m+1} f(z,\zeta) &= (1-\lambda) D_\lambda^m f(z,\zeta) + \lambda z \left(D_\lambda^m f(z,\zeta)\right)_z' = D_\lambda\left(D_\lambda^m f(z,\zeta)\right), \quad z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2 *[5] If $f \in \mathcal{A}_\zeta^*$ and $f(z) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $D_\lambda^m f(z,\zeta) = z + \sum_{j=2}^\infty [1+(j-1)\lambda]^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.3 *[4] For $f \in \mathcal{A}_\zeta^*$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f_z'(z,\zeta), ..., \\ (m+1) R^{m+1} f(z,\zeta) &= z \left(R^m f(z,\zeta)\right)_z' + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

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Remark 1.4 *[4] If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [9].

Definition 1.5 *[9] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.6 *[9] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.5 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.7 *[6] We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim_{z\to y} f(z,\zeta) = \infty\}$, and are such that $f_z'(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.8 *[6] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta)$ is univalent in $U \times \overline{U}$ and $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and is the best subordinant.*

Lemma 1.9 *[6] Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} z q_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $Re\ \gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta)$ is univalent in $U \times \overline{U}$ and $q(z,\zeta) + \frac{1}{\gamma} z q_z'(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is the best subordinant.*

2 Main results

Definition 2.1 *[2] Let $\lambda \geq 0$ and $m \in \mathbb{N} \cup \{0\}$. Denote by DR_λ^m the operator given by the Hadamard product (the convolution product) of the extended generalized Sălăgean operator D_λ^m and the extended Ruscheweyh operator R^m, $DR_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta).$$

Remark 2.2 *[2] If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then*
$DR_\lambda^m f(z,\zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m [1 + (j-1)\lambda]^m a_j^2(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Remark 2.3 *For $\lambda = 1$ we obtain the Hadamard product SR^m ([1], [3], [7], [8]) of the extended Sălăgean operator S^m and extended Ruscheweyh operator R^m.*

Theorem 2.4 *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $\lambda \geq 0$, $f(z,\zeta) \in \mathcal{A}_\zeta^*$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, $\mathrm{Re}\, c > -2$, and suppose that $\left(DR_\lambda^m f(z,\zeta)\right)_z'$ is univalent in $U \times \overline{U}$, $\left(DR_\lambda^m F(z,\zeta)\right)_z' \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{1}$$

then

$$q(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}} \int_0^z h(t,\zeta) t^{c+1} dt$. The function q is convex and it is the best subordinant.

Proof. We have $z^{c+1}F(z,\zeta) = (c+2)\int_0^z t^c f(t,\zeta)\,dt$ and differentiating it, with respect to z, we obtain $(c+1)F(z,\zeta)+zF'_z(z,\zeta) = (c+2)f(z,\zeta)$ and $(c+1)DR_\lambda^m F(z,\zeta)+z\left(DR_\lambda^m F(z,\zeta)\right)'_z = (c+2)DR_\lambda^m f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Differentiating the last relation with respect to z we have

$$\left(DR_\lambda^m F(z,\zeta)\right)'_z + \frac{1}{c+2}z\left(DR_\lambda^m F(z,\zeta)\right)''_{z^2} = \left(DR_\lambda^m f(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{2}$$

Using (2), the strong differential superordination (1) becomes

$$h(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z + \frac{1}{c+2}z\left(DR_\lambda^m F(z,\zeta)\right)''_{z^2}. \tag{3}$$

Denote

$$p(z,\zeta) = \left(DR_\lambda^m F(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{4}$$

Replacing (4) in (3) we obtain $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.8 for $n=1$ and $\gamma=c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.5 *Let* $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$, *where* $\beta\in[0,1)$. *Let* $m\in\mathbb{N}$, $\lambda\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_\zeta$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re} c > -2$, *and suppose that* $\left(DR_\lambda^m f(z,\zeta)\right)'_z$ *is univalent in* $U\times\overline{U}$, $\left(DR_\lambda^m F(z,\zeta)\right)'_z \in \mathcal{H}^*[1,1,\zeta]\cap Q^*$ *and*

$$h(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}, \tag{5}$$

then

$$q(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta) = 2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{z^{c+2}}\int_0^z \frac{t^{c+1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function q is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta) = \left(DR_\lambda^m F(z,\zeta)\right)'_z$, the strong differential superordination (5) becomes $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt = \frac{c+2}{z^{c+2}}\int_0^z \frac{\zeta+(2\beta-\zeta)t}{1+t}t^{c+1}dt = 2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.6 *Let* $q(z,\zeta)$ *be a convex function in* $U\times\overline{U}$ *and let* $h(z,\zeta) = q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)$, *where* $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re} c > -2$.

Let $m\in\mathbb{N}$, $\lambda\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_\zeta$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, *and suppose that* $\left(DR_\lambda^m f(z,\zeta)\right)'_z$ *is univalent in* $U\times\overline{U}$, $\left(DR_\lambda^m F(z,\zeta)\right)'_z\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$ *and*

$$h(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}, \tag{6}$$

then

$$q(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z, \quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. *The function q is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta) = \left(DR_\lambda^m F(z,\zeta)\right)'_z$, $z\in U$, $\zeta\in\overline{U}$, the strong differential superordination (6) becomes $h(z,\zeta) = q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta) \prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Using Lemma 1.9 for $n=1$ and $\gamma=c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \left(DR_\lambda^m F(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta) = \frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is the best subordinant. ■

Theorem 2.7 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$ and suppose that $\left(DR^m_\lambda f(z,\zeta)\right)'_z$ is univalent and $\frac{DR^m_\lambda f(z,\zeta)}{z} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(DR^m_\lambda f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{7}$$

then

$$q(z,\zeta) \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta) = \frac{DR^m_\lambda f(z,\zeta)}{z} = \frac{z+\sum_{j=2}^\infty C^m_{m+j-1}[1+(j-1)\lambda]^m a_j^2(\zeta) z^j}{z} = 1+\sum_{j=2}^\infty C^m_{m+j-1}\left[1+(j-1)\lambda\right]^m a_j^2(\zeta) z^{j-1}$. Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = \left(DR^m_\lambda f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$.

Then (7) becomes $h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.8 for $n=1$ and $\gamma=1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.8 *Let $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\lambda \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$ and suppose that $\left(DR^m_\lambda f(z,\zeta)\right)'_z$ is univalent and $\frac{DR^m_\lambda f(z,\zeta)}{z} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(DR^m_\lambda f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{8}$$

then

$$q(z,\zeta) \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z,\zeta) = \frac{DR^m_\lambda f(z,\zeta)}{z}$, the strong differential superordination (8) becomes $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt = \frac{1}{z}\int_0^z \frac{\zeta+(2\beta-\zeta)t}{1+t}dt$ $= 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z) \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z}$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.9 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. If $\lambda \geq 0$, $m \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_\zeta$, suppose that $\left(DR^m_\lambda f(z,\zeta)\right)'_z$ is univalent, $\frac{DR^m_\lambda f(z,\zeta)}{z} \in \mathcal{H}^*[1,1,\zeta] \cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(DR^m_\lambda f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{9}$$

then

$$q(z,\zeta) \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta) = \frac{DR^m_\lambda f(z,\zeta)}{z} = \frac{z+\sum_{j=2}^\infty C^m_{m+j-1}[1+(j-1)\lambda]^m a_j^2(\zeta) z^j}{z} = 1+\sum_{j=2}^\infty C^m_{m+j-1}\left[1+(j-1)\lambda\right]^m a_j^2(\zeta) z^{j-1}$. Evidently $p \in \mathcal{H}^*[1,1,\zeta]$.

Differentiating, we obtain $p(z,\zeta) + zp'_z(z,\zeta) = \left(DR^m_\lambda f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, and (9) becomes $q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Using Lemma 1.9 for $n=1$ and $\gamma=1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e.}\ q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{DR^m_\lambda f(z,\zeta)}{z},\ z \in U, \zeta \in \overline{U},$$

and q is the best subordinant. ■

Theorem 2.10 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $\lambda\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_\zeta^*$ and suppose that $\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z',\qquad z\in U,\ \zeta\in\overline{U},\tag{10}$$

then

$$q(z,\zeta)\prec\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta)=\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}=\frac{z+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^j}=\frac{1+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^{j-1}}$. Evidently $p\in\mathcal{H}^*[1,1,\zeta]$.
We have $p_z'(z,\zeta)=\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)_z'}{DR_\lambda^m f(z,\zeta)}-p(z,\zeta)\cdot\frac{(DR_\lambda^m f(z,\zeta))_z'}{DR_\lambda^m f(z,\zeta)}$. Then $p(z,\zeta)+zp_z'(z,\zeta)=\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'$.

Then (10) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\quad\text{i.e.}\quad q(z,\zeta)\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.11 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$. Let $\lambda\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_\zeta^*$ and suppose that $\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'$ is univalent, $\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{11}$$

then

$$q(z,\zeta)\prec\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta)=\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}$, the strong differential superordination (11) becomes $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e. $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt$ $=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z)\prec\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.12 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)$. If $\lambda\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_\zeta^*$, suppose that $\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'$ is univalent, $\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)\prec\prec\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z',\qquad z\in U,\ \zeta\in\overline{U},\tag{12}$$

then

$$q(z,\zeta)\prec\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta)=\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}=\frac{z+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^j}=\frac{1+\sum_{j=2}^\infty C_{m+j}^{m+1}[1+(j-1)\lambda]^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=2}^\infty C_{m+j-1}^{m}[1+(j-1)\lambda]^{m}a_j^2(\zeta)z^{j-1}}$. Evidently $p\in\mathcal{H}^*[1,1,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp'_z(z,\zeta)=\left(\frac{zDR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)'_z$, $z\in U$, $\zeta\in\overline{U}$, and (12) becomes $q(z,\zeta)+zq'_z(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Using Lemma 1.9 for $n=1$ and $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt\prec\prec\frac{DR_\lambda^{m+1}f(z,\zeta)}{DR_\lambda^m f(z,\zeta)},\ z\in U,\zeta\in\overline{U},$$

and q is the best subordinant. ■

Theorem 2.13 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $\lambda\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_\zeta^*$ and suppose that $\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$ is univalent and $(DR_\lambda^m f(z,\zeta))'_z\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{13}$$

then

$$q(z,\zeta)\prec\prec(DR_\lambda^m f(z,\zeta))'_z,\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z h(t,\zeta)t^{\frac{(m-1)\lambda+1}{\lambda}}dt$. The function q is convex and it is the best subordinant.

Proof. With notation $p(z,\zeta)=(DR_\lambda^m f(z,\zeta))'_z=1+\sum_{j=2}^\infty C_{m+j-1}^m[1+(j-1)\lambda]^m ja_j^2(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^\infty a_j(\zeta)z^j$,
$p(z,\zeta)+zp'_z(z,\zeta)=\frac{m+1}{\lambda z}DR_\lambda^{m+1}f(z,\zeta)-\left(m-1+\frac{1}{\lambda}\right)(DR_\lambda^m f(z,\zeta))'_z-\frac{m(1-\lambda)}{\lambda z}DR_\lambda^m f(z,\zeta)$ and
$p(z,\zeta)+\frac{\lambda}{m\lambda+1}zp'_z(z,\zeta)=\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$. Evidently $p\in\mathcal{H}^*[1,1,\zeta]$.

Then (13) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+\frac{\lambda}{m\lambda+1}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=m+\frac{1}{\lambda}$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)\prec\prec(DR_\lambda^m f(z,\zeta))'_z,\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z h(t,\zeta)t^{\frac{(m-1)\lambda+1}{\lambda}}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.14 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$. Let $\lambda\geq 0$, $m\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_\zeta^*$ and suppose that $\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$ is univalent, $(DR_\lambda^m f(z,\zeta))'_z\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{14}$$

then

$$q(z,\zeta)\prec\prec(DR_\lambda^m f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)(m\lambda+1)}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z\frac{t^{\frac{m\lambda+1}{\lambda}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.13 and considering $p(z,\zeta)=(DR_\lambda^m f(z,\zeta))'_z$, the strong differential superordination (14) becomes $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.8 for $n=1$ and $\gamma=\frac{m\lambda+1}{\lambda}$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e. $q(z,\zeta)=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z h(t,\zeta)t^{\frac{(m-1)\lambda+1}{\lambda}}dt$ $=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z t^{\frac{(m-1)\lambda+1}{\lambda}}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)(m\lambda+1)}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z\frac{t^{\frac{m\lambda+1}{\lambda}-1}}{1+t}dt\prec\prec(DR_\lambda^m f(z,\zeta))'_z$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant. ■

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Theorem 2.15 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+\frac{\lambda}{m\lambda+1}zq_z'(z,\zeta)$, $\lambda\geq 0$, $m,n\in\mathbb{N}$. If $f(z,\zeta)\in\mathcal{A}_{n\zeta}^*$, suppose that $\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$ is univalent and $\left(DR_\lambda^m f(z,\zeta)\right)_z'\in\mathcal{H}^*[1,1,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+\frac{\lambda}{m\lambda+1}zq_z'(z,\zeta)\prec\prec \tag{15}$$
$$\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta),\ z\in U,\zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec\left(DR_\lambda^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z h(t,\zeta)\,t^{\frac{(m-1)\lambda+1}{\lambda}}dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta)=\left(DR_\lambda^m f(z,\zeta)\right)'=1+\sum_{j=2}^{\infty}C_{m+j-1}^m\left[1+(j-1)\lambda\right]^m ja_j^2(\zeta)z^{j-1}$.

Differentiating, we obtain $p(z,\zeta)+zp_z'(z,\zeta)=\frac{m+1}{\lambda z}DR_\lambda^{m+1}f(z,\zeta)-\left(m-1+\frac{1}{\lambda}\right)\left(DR_\lambda^m f(z,\zeta)\right)_z'-\frac{m(1-\lambda)}{\lambda z}DR_\lambda^n f(z,\zeta)$ and $p(z,\zeta)+\frac{\lambda}{m\lambda+1}zp_z'(z,\zeta)=\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and (15) becomes $q(z,\zeta)+\frac{\lambda}{m\lambda+1}zq_z'(z,\zeta)\prec\prec p(z,\zeta)+\frac{\lambda}{m\lambda+1}zp_z'(z,\zeta)$, $z\in U,\zeta\in\overline{U}$. Using Lemma 1.9 for $n=1$ and $\gamma=m+\frac{1}{\lambda}$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)=\frac{m\lambda+1}{\lambda z^{\frac{m\lambda+1}{\lambda}}}\int_0^z h(t,\zeta)\,t^{\frac{(m-1)\lambda+1}{\lambda}}dt\prec\prec\left(DR_\lambda^m f(z,\zeta)\right)_z'$, $z\in U$, $\zeta\in\overline{U}$, and q is the best subordinant. ■

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A NEW FIXED POINT THEOREM FOR *m* MAPINGS ON *m* COMPLETE METRIC SPACES

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Abstract: A new fixed point theorem for $m \in N$ mappings, satisfying implicit relations, was proved. This result generalizes and unifies several of well-known fixed point theorems for one, two and three mappings and extends then to an arbitrary number m of mappings.

Keywords: Cauchy sequence, complete metric space, fixed point, implicit relation.

Mathematics Subject Classification: 47H10, 54H25

1. INTRODUCTION

In [1], [2], [8], [11] etc, are proved fixed point theorems on metric spaces for mappings satisfying implicit relations.

In this paper, we will prove a related fixed point theorem for m mappings on m metric spaces, $m-1$ of mappings must be continuous. This result generalizes and unifies the theorems of Rhoades [10], Banach [3], Kannan [7], Bianchini [4], Reich [9], Fisher [5], Jain et al [6], etc. and in the same time, it extends them for an arbitrary number m of mappings.

These generalizations and unifications have been done using a wide class of implicit relations.

In [10], [5] and [6], the following theorems are proved.

Theorem 1.1 (Rhoades [10]) *Let* (X,d) *be a complete metric space and* $T: X \to X$ *a self map of X. If for some* $c \in [0,1)$ *we have*

$$d(Tx,Ty) \le c\max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

for all $x, y \in X$ *, then* T *has a unique fixed point* α *in X.*

Theorem 2.2 Fisher [5]) **Let** (X,d) *and* (Y,ρ) *are complete metric spaces and* $S: X \to Y$, $R: Y \to X$ *be two maps, at least one of them being continuous. If for some* $c \in (0,1)$ *the following inequalities are satisfied:*

$$d(RSx, RSx') \le c\max\{d(x,x'), d(x,RSx), d(x',RSx'), \rho(Sx,Sx')\}$$

$$\rho(SRy, SRy') \le c\max\{\rho(y,y'), \rho(y,SRy), \rho(y',SRy'), d(Ry,Ry')\}$$

for all $x,x'\in X$; $y,y'\in Y$, *then RS has a unique fixed point* $\alpha\in X$ *and SR has a unique fixed point* $\beta\in Y$. *Moreover,* $S\alpha=\beta$ *and.* $R\beta=\alpha$

Theorem 1.3 (Jain et al [6]) *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces. If* T *is a continuous mapping of* X *into* Y, S *is a continuous mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx,RSTx')\le c\max\{d(x,x'),d(x,RSTx),d(x',RSTx')\rho(Tx,Tx'),\sigma(STx,STx')\}$$

$$\rho(TRSy,TRSy')\le c\max\{\rho(y,y'),\rho(y,TRSy),\rho(y',TRSy')\sigma(Sy,Sy'),d(RSy,RSy')\}$$

$$\sigma(STRz,STRz')\le c\max\{\sigma(z,z'),\sigma(z,STRz),\sigma(z',STRz'),d(Rz,Rz'),\rho(TRz,TRz')\}$$

for all $x,x'\in X; y,y'\in Y$ *and* $z,z'\in Z$ *where* $0\le c<1$. *Then* RST *has a unique fixed point* u *in* X, TRS *has a unique fixed point* v *in* Y *and* STR *has a unique fixed point* w *in* Z. *Further,* $Tu=v, Sv=w$ *and* $Rw=u$.

2. MAIN RESULTS

Before stating the main theorem we define new classes implicit functions, whose role will be crucial.

Let $R_+=[0,+\infty)$. We denote by Φ_k, for $k\ge 4$, the set of all functions with k variables $\varphi: R_+^k\to R$ satisfying the properties:

(a). φ is upper semi-continuous in each coordinate variable $t_1,t_2,...,t_k$

(b). If $\varphi(u,v,v,u,v_1,v_2,...,v_{k-4})\le 0$ or $\varphi(u,v,u,v,v_1,v_2,...,v_{k-4})\le 0$ or $\varphi(u,u,v,v,v_1,v_2,...,v_{k-4})\le 0$, for all $u,v,v_1,v_2,...,v_{k-4}\ge 0$, then there exists a real constant $0\le c<1$ such that $u\le c\max\{v,v_1,v_2,...,v_{k-4}\}$.

Every such function will be called a Φ_k*-function* with constant *c*.

Note: If $k_1<k_2$, then $\Phi_{k_1}\subset\Phi_{k_2}$. In the case $k=4$, we have

$$\varphi(u,v,v,u,v_1,v_2,...,v_{k-4})=\varphi(u,v,v,u) \text{ and } \max\{v,v_1,v_2,...,v_{k-4}\}=v$$

Example 2.1 The function $\varphi(t_1,t_2,...,t_k)=t_1^p-c\max\{t_2^p,t_3^p,...,t_k^p\}$, where $0\le c<1$ and $p>0$, is Φ_k-function with constant c.

Proof: (a) is clear since φ is continuous.

Suppose that $u,v,v_1,v_2,...,v_{k-4}\ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,...,v_{k-4})=u^p-c\max\{v^p,v^p,u^p,v_1^p,v_2^p,...,v_{k-4}^p\}=$$
$$=u^p-c\max\{u^p,v^p,v_1^p,v_2^p,...,v_{k-4}^p\}\le 0$$

If $u^p\ge\max\{v^p,v_1^p,v_2^p,...,v_{k-4}^p\}$, then $u^p\le c\max\{u^p,v^p,v_1^p,v_2^p,...,v_{k-4}^p\}=cu^p<u^p$, a contradiction. Therefore,

$$u^p\le c\max\{v^p,v_1^p,v_2^p,...,v_{k-4}^p\} \text{ and so } u\le c_1\max\{v,v_1,v_2,...,v_{k-4}\}$$

where $c_1=\sqrt[p]{c}<1$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,...,v_{k-4})\le 0$ or

$$\varphi(u,u,v,v,v_1,v_2,...,v_{k-4}) \le 0 \text{ , then } u \le c_1 \max\{v,v_1,v_2,...,v_{k-4}\}.$$

The proof of (b) is completed.

Example 2.2 The function $\varphi(t_1,t_2,...,t_k) = t_1 - (a_2t_2^p + a_3t_3^p + ... + a_kt_k^p)^{1/p}$, where $p > 0$ and $0 \le a_i, \sum_{i=2}^{k} a_i < 1, \quad i = 2,3,...,k$, is Φ_k-function with constant $c = \sum_{i=2}^{k} a_i$

Proof: (a) is clear since φ is continuous. Suppose that $u,v,v_1,v_2,...,v_{k-4} \ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,...,v_{k-4}) = u - (a_2v^p + a_3v^p + a_4u^p + a_5v_1^p ... + a_kv_{k-4}^p)^{1/p} \le 0$$

If $u^p \ge \max\{v^p, v_1^p, v_2^p,...,v_{k-4}^p\}$, then

$$u \le (a_2v^p + a_3v^p + a_4u^p + a_5v_1^p + ... + a_kv_{k-4}^p)^{1/p} \le (a_2u^p + a_3u^p + a_4u^p + a_5u^p ... + a_ku^p)^{1/p} =$$

$$= [(a_2 + a_3 + a_4 + ... + a_k)u^p]^{1/p} = (a_2 + a_3 + a_4 + ... + a_k)^{1/p}u = cu < u \text{ ,}$$

a contradiction, where $c = (a_2 + a_3 + a_4 + ... + a_k)^{1/p} < 1$. Therefore,

$u \le [(a_2 + a_3 + a_4 + ... + a_k)\max\{v^p, v_1^p, v_2^p,...,v_{k-4}^p\}]^{1/p} = c\max\{v,v_1,v_2,...,v_{k-4}\}$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,...,v_{k-4}) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,...,v_{k-4}) \le 0$, then $u \le c_1 \max\{v,v_1,v_2,...,v_{k-4}\}$. The proof of (b) is completed.

We denote by $\mathbb{F}_k$ the set of all continuous functions with k variables $f : R_+^k \to R$ satisfying the properties:

(a'). f is non decreasing in respect with each variable.

(b'). $f(t,t,...,t) \le t,\ t \in R_+$

Every such function will be called a $\mathbb{F}_k$ *-function.*

Denote $I_k = \{1,2,...,k\}$. Some examples of $\mathbb{F}_k$-function are as follows:

1. $f(t_1,t_2,...,t_k) = \max\{t_1,t_2,...,t_k\}$
2. $f(t_1,t_2,...,t_k) = [\max\{t_1t_2, t_2t_3,...,t_{k-1}t_k, t_kt_1\}]^{1/2}$
3. $f(t_1,t_2,...,t_k) = [\max\{t_1^p, t_2^p,...,t_k^p\}]^{1/p}$, $p > 0$
4. $f(t_1,t_2,...,t_k) = \dfrac{a_1t_1 + a_2t_2 + ... + a_kt_k}{a_1 + a_2 + ... + a_k}$, where $p > 0$ and $a_i \ge 0$

The following relationship between $\mathbb{F}_{k-1}$-functions and Φ_k-functions holds:

Lemma 2.3 If $f \in \mathbb{F}_{k-1}$ *and* $0 \le c < 1$, then the function $\varphi(t_1,t_2,...,t_k) = t_1 - cf(t_2,t_3,...,t_k)$ *is* Φ_k *-function with constant c*

Proof. **(a)** is clear since φ is continuous. Suppose that $u,v,v_1,v_2,...,v_{k-4} \ge 0$ and then

$$\varphi(u,v,v,u,v_1,v_2,...,v_{k-4}) = u - cf(v,v,u,v_1,...,v_{k-4}) \le o \qquad (*)$$

We have $u \le \max\{v,v_1,v_2,...,v_{k-4}\}$ since in contrary, (if $u > \max\{v,v_1,v_2,...,v_{k-4}\}$), by using the properties of f we get: $f(v,v,u,v_1,...,v_{k-4}) \le f(u,u,...,u) \le u$ and by (*) it follows $u \le cu < u$, a contradiction. Therefore, after replacing the coordinates of the point

$(v,v,u,v_1,v_2,...,v_{k-4})$ by $\max\{v,v_1,v_2,...,v_{k-4}\}$ and using the properties of f we get $u \le c\max\{v,v_1,v_2,...,v_{k-4}\}$. Similarly, if $\varphi(u,v,u,v,v_1,v_2,...,v_{k-4}) \le 0$ or $\varphi(u,u,v,v,v_1,v_2,...,v_{k-4}) \le 0$, then $u \le c\max\{v,v_1,v_2,...,v_{k-4}\}$. The proof of (b) is completed.

The above lemma gives us the possibility to establish other functions of type Φ_k:

Example 2.4 $\varphi(t_1,t_2,...,t_k) = t_1 - c[\max\{t_2t_3,t_3t_4,...,t_{k-1}t_k\}]^{1/2}$, where $0 \le c < 1$.

Example 2.5 $\varphi(t_1,t_2,...,t_k) = t_1 - c\dfrac{t_2+t_3+...+t_k}{k-1}$, where $0 \le c < 1$ etc.

Now, we prove the following theorem for *m* mappings on *m* metric spaces.

Theorem 2.6 Let (X_i,d_i) be m complete metric spaces and T_i m mappings such that $T_i : X_i \to X_{i+1}$ *for* $i = 1,2,...,m-1$, $T_m : X_m \to X_1$ and from which $(m-1)$ are continuous. If satisfying the inequalities:

$$\varphi_1\begin{pmatrix} d_1\left(T_mT_{m-1}...T_2T_1x_1, T_mT_{m-1}...T_2T_1x_1'\right), d_1\left(x_1,x_1'\right), d_1\left(x_1,T_mT_{m-1}...T_1x_1\right), \\ d_1\left(x_1',T_mT_{m-1}...T_1x_1'\right), d_2\left(T_1x_1,T_1x_1'\right),\ \mathrm{d}_3\left(T_2T_1x_1,T_2T_1x_1'\right),..., \\ d_m\left(T_{m-1}T_{m-2}...T_1x_1,T_{m-1}T_{m-2}...T_1x_1'\right) \end{pmatrix} \le 0 \qquad (1)$$

for all $x_1, x_1' \in X_1$

$$\varphi_2\begin{pmatrix} d_2\left(T_1T_m...T_3T_2x_2,T_1T_m...T_3T_2x_2'\right), d_2\left(x_2,x_2'\right), d_2\left(x_2,T_1T_m...T_2x_2\right), \\ d_2\left(x_2',T_1T_m...T_2x_2'\right), d_3\left(T_2x_2,T_2x_2'\right), d_4\left(T_3T_2x_2,T_3T_2x_2'\right),..., \\ d_m\left(T_{m-1}T_{m-2}...T_2x_2,T_{m-1}T_{m-2}...T_2x_2'\right), d_1\left(T_mT_{m-1}...T_2x_2,T_mT_{m-1}...T_2x_2'\right) \end{pmatrix} \le 0 \qquad (2)$$

for all $x_2, x_2' \in X_2$*, in general*

$$\varphi_i\begin{pmatrix} d_i\left(T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_i,T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_i'\right), d_i\left(x_i,x_i'\right), \\ d_i\left(x_i,T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_i\right), d_i\left(x_i',T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_i'\right), \\ d_{i+1}\left(T_ix_i,T_ix_i'\right),...,d_m\left(T_{m-1}T_{m-2}...T_ix_i,T_{m-1}T_{m-2}...T_ix_i'\right), d_1\left(T_mT_{m-1}...T_ix_i,T_mT_{m-1}...T_ix_i'\right), \\ d_2\left(T_1T_mT_{m-1}...T_ix_i,T_1T_mT_{m-1}...T_ix_i'\right),...,d_{i-1}\left(T_{i-2}T_{i-3}...T_1T_mT_{m-1}...T_ix_i,T_{i-2}T_{i-3}...T_1T_mT_{m-1}...T_ix_i'\right) \end{pmatrix} \qquad (i)$$

for all $x_i, x_i' \in X_i$ *for* $i = 3,...,m-1$*, and*

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$$\varphi_m\begin{pmatrix} d_m\left(T_{m-1}T_{m-2}...T_1T_mx_m, T_{m-1}T_{m-2}...T_1T_mx_m^{'}\right), d_m\left(x_m, x_m^{'}\right), d_m\left(x_m, T_{m-1}T_{m-2}...T_1T_mx_m\right), \\ d_m\left(x_m^{'}, T_{m-1}T_{m-2}...T_1T_mx_m^{'}\right), d_1\left(T_mx_m, T_mx_m^{'}\right), d_2\left(T_1T_mx_m, T_1T_mx_m^{'}\right), ..., \\ d_{m-1}\left(T_{m-2}T_{m-3}...T_1T_mx_m, T_{m-2}T_{m-3}...T_1T_mx_m^{'}\right) \end{pmatrix} \tag{m}$$

for all $x_m, x_m^{'} \in X_m$, *where* $\varphi_i \in \Phi_{m+3}$ *for* $i = 1, 2, ..., m$.

Then the maps

$$T_mT_{m-1}...T_1,\ T_1T_mT_{m-1}...T_2,\ \ldots, T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_i\ ,\ldots, T_{m-1}T_{m-2}...T_1T_m$$

have unique fixed point $\alpha_1 \in X_1$, $\alpha_2 \in X_2, \ldots, \alpha_i \in X_i, \ldots,\ \alpha_m \in X_m$, *respectively.*

Further,

$$T_i\alpha_i = \alpha_{i+1}\ \ \textit{for}\ i = 1, ..., m-1\ \textit{and}\ T_m\alpha_m = \alpha_1$$

Proof. Let $x_0^{(1)}$ be an arbitrary point in X_1. We define the sequences $\left\{x_n^{(1)}\right\}, \left\{x_n^{(2)}\right\}, \ldots, \left\{x_n^{(i)}\right\}, \ldots, \left\{x_n^{(m)}\right\}$ *in* $X_1, X_2, \ldots, X_i, \ldots, X_m$ respectively, as follows:

$$x_n^{(1)} = (T_mT_{m-1}...T_1)^n x_0^{(1)}, x_n^{(2)} = T_1x_{n-1}^{(1)}, ..., x_n^{(i)} = T_{i-1}x_n^{(i-1)}, ..., x_n^{(m)} = T_{m-1}x_n^{(m-1)}, n \in N$$

We prove that $\left\{x_n^{(i)}\right\}$ are Cauchy sequences for i=1, 2… m. Denote

$$d_n^{(i)} = d_i(x_n^{(i)}, x_{n+1}^{(i)}), i = 1, 2, \ldots, m.$$

We will assume that $x_n^{(i)} \neq x_{n+1}^{(i)}$ for all *n*. Otherwise, if $x_n^{(1)} = x_{n+1}^{(1)}$ for some *n*, then $x_{n+1}^{(i)} = x_{n+2}^{(i)}$ *for i=2, 3,…, m* and we could put $\alpha_i = x_{n+1}^{(i)}$.

Applying the inequality (2) for $x_2 = x_{n-1}^{(2)}$ *and* $x_2' = x_n^{(2)}$, we have:

$$\varphi_2\begin{pmatrix} d_2\left(T_1T_m...T_3T_2x_{n-1}^{(2)}, T_1T_m...T_2x_n^{(2)}\right), d_2\left(x_{n-1}^{(2)}, x_n^{(2)}\right), d_2\left(x_{n-1}^{(2)}, T_1T_m...T_2x_{n-1}^{(2)}\right), \\ d_2\left(x_n^{(2)}, T_1T_m...T_2x_n^{(2)}\right), d_3\left(T_2x_{n-1}^{(2)}, T_2x_n^{(2)}\right), d_4\left(T_3T_2x_{n-1}^{(2)}, T_3T_2x_n^{(2)}\right), ..., \\ d_m\left(T_{m-1}T_{m-2}...T_2x_{n-1}^{(2)}, T_{m-1}T_{m-2}...T_2x_n^{(2)}\right), d_1\left(T_mT_{m-1}...T_2x_{n-1}^{(2)}, T_mT_{m-1}...T_2x_n^{(2)}\right) \end{pmatrix} =$$

$$= \varphi_2\left(d_n^{(2)}, d_{n-1}^{(2)}, d_{n-1}^{(2)}, d_n^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, ..., d_{n-1}^{(m)}, d_{n-1}^{(1)}\right) \leq 0$$

and from **(b)**, we have $d_n^{(2)} \leq c\max\left\{d_{n-1}^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, ..., d_{n-1}^{(m)}, d_{n-1}^{(1)}\right\}$ or

$$d_n^{(2)} \leq c\max\left\{d_{n-1}^{(1)}, d_{n-1}^{(2)}, d_{n-1}^{(3)}, d_{n-1}^{(4)}, ..., d_{n-1}^{(m)}\right\} \tag{2'}$$

We have denoted by $c = \max\{c_1, c_2, ..., c_m\}$, where c_i is the constant of Φ_k-function φ_i.

Applying inequality (i) for $x_i = x_{n-1}^{(i)}$ *and* $x_i' = x_n^{(i)}$, we obtain:

$$\varphi_i\begin{pmatrix} d_i\left(T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_{n-1}^{(i)},T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_n^{(i)}\right),d_i\left(x_{n-1}^{(i)},x_n^{(i)}\right),\\ d_i\left(x_{n-1}^{(i)},T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_{n-1}^{(i)}\right),d_i\left(x_n^{(i)},T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_ix_n^{(i)}\right),d_{i+1}\left(T_ix_{n-1}^{(i)},T_ix_n^{(i)}\right),\\ d_{i+2}\left(T_{i+1}T_ix_{n-1}^{(i)},T_{i+1}T_ix_n^{(i)}\right),...,d_m\left(T_{m-1}T_{m-2}...T_ix_{n-1}^{(i)},T_{m-1}T_{m-2}...T_ix_n^{(i)}\right),\\ d_1\left(T_mT_{m-1}...T_ix_{n-1}^{(i)},T_mT_{m-1}...T_ix_n^{(i)}\right),d_2\left(T_1T_mT_{m-1}...T_ix_{n-1}^{(i)},T_1T_mT_{m-1}...T_ix_n^{(i)}\right),...,\\ d_{i-1}\left(T_{i-2}T_{i-3}...T_1T_mT_{m-1}...T_ix_{n-1}^{(i)},T_{i-2}T_{i-3}...T_1T_mT_{m-1}...T_ix_n^{(i)}\right) \end{pmatrix}=$$

$$=\varphi_i(d_n^{(i)},d_{n-1}^{(i)},d_{n-1}^{(i)},d_n^{(i)},d_{n-1}^{(i+1)},d_{n-1}^{(i+2)},...,d_{n-1}^{(m)},d_{n-1}^{(1)},d_n^{(2)},...,d_n^{(i-1)})\le 0$$

and from **(b)**, we have

$$d_n^i \le c\max(d_{n-1}^{(i)},d_{n-1}^{(i+1)},d_{n-1}^{(i+2)},...,d_{n-1}^{(m)},d_{n-1}^{(1)},d_n^{(2)},d_n^{(3)},...,d_n^{(i-1)}) \qquad (*)$$

By (*), for i=3 we have:

$$d_n^3 \le c\max(d_{n-1}^{(3)},d_{n-1}^{(4)},...,d_{n-1}^{(m)},d_{n-1}^{(1)},d_n^{(2)})$$

By this inequality and (2’) it follows:

$$d_n^{(3)} \le c\max\left\{d_{n-1}^{(1)},d_{n-1}^{(2)},d_{n-1}^{(3)},d_{n-1}^{(4)},...,d_{n-1}^{(m)}\right\} \qquad (3')$$

In similar way, for i=4,5,…,m-1 and by the inequalities (2’),(3’),...,((i-1)’) we get:

$$d_n^{(i)} \le c\max\left\{d_{n-1}^{(1)},d_{n-1}^{(2)},d_{n-1}^{(3)},d_{n-1}^{(4)},...,d_{n-1}^{(m)}\right\}\ \ i=2,3,...,m-1 \qquad \text{(i)}$$

Applying inequality (m) for $x_m = x_{n-1}^{(m)}$ *and* $x'_m = x_n^{(m)}$, we have:

$$\varphi_m\begin{pmatrix} d_m\left(T_{m-1}T_{m-2}...T_1T_mx_{n-1}^m,T_{m-1}T_{m-2}...T_1T_mx_n^m\right),d_m(x_{n-1}^m,x_n^m),d_m\left(x_{n-1}^m,T_{m-1}T_{m-2}...T_1T_mx_{n-1}^m\right),\\ d_m\left(x_n^m,T_{m-1}T_{m-2}...T_1T_mx_n^m\right),d_1\left(T_mx_{n-1}^m,T_mx_n^m\right),d_2\left(T_1T_mx_{n-1}^m,T_1T_mx_n^m\right),...,\\ d_{m-1}\left(T_{m-2}T_{m-3}...T_1T_mx_{n-1}^m,T_{m-2}T_{m-3}...T_1T_mx_n^m\right) \end{pmatrix}=$$

$$=\varphi_m(d_n^{(m)},d_{n-1}^{(m)},d_{n-1}^{(m)},d_n^{(m)},d_{n-1}^{(1)},d_n^{(2)},d_n^{(3)},...,d_n^{(m-1)})$$

and from **(b)**, we have:

$$d_n^m \le c\max(d_{n-1}^{(m)},d_{n-1}^{(1)},d_n^{(2)},d_n^{(3)},...,d_n^{(m-1)})$$

By this inequality and by (2’),(3’),…, ((m-1)’) it follows:

$$d_n^{(m)} \le c\max(d_{n-1}^{(1)},d_{n-1}^{(2)},d_{n-1}^{(3)},...,d_{n-1}^{(m)}) \qquad (m')$$

Applying inequality (1) for $x_1 = x_{n-1}^{(1)}$ *and* $x'_1 = x_n^{(1)}$, we have:

$$\varphi_1\left(\begin{array}{l} d_1\left(T_mT_{m-1}...T_2T_1x_{n-1}^{(1)},T_mT_{m-1}...T_2T_1x_n^{(1)}\right),d_1\left(x_{n-1}^{(1)},x_n^{(1)}\right),d_1\left(x_{n-1}^{(1)},T_mT_{m-1}...T_1x_{n-1}^{(1)}\right),\\ d_1\left(x_n^{(1)},T_mT_{m-1}...T_1x_n^{(1)}\right),d_2\left(T_1x_{n-1}^{(1)},T_1x_n^{(1)}\right),\ \mathrm{d}_3\left(T_2T_1x_{n-1}^{(1)},T_2T_1x_n^{(1)}\right),...,\\ d_m\left(T_{m-1}T_{m-2}...T_1x_n^{(1)},T_{m-1}T_{m-2}...T_1x_n^{(1)}\right)\end{array}\right)=$$

$$=\varphi_1(d_n^{(1)},d_{n-1}^{(1)},d_{n-1}^{(1)},d_n^{(1)},d_n^{(2)},d_n^{(3)},...,d_n^{(m)})\le 0$$

and from **(b)** we have:

$$d_n^{(1)}\le c\max\{d_{n-1}^{(1)},d_n^{(2)},d_n^{(3)},...,d_n^{(m)}\}$$

By this inequality and by (2’),(3’),…,(m’) it follows:

$$d_n^{(1)}\le c\max\{d_{n-1}^{(1)},d_{n-1}^{(2)},d_{n-1}^{(3)},...,d_{n-1}^{(m)}\} \qquad (1')$$

It now follows from (1’),(2’),…,(m’) that for large enough m

$$\begin{aligned} d_n^{(i)}=d_i(x_n^{(i)},x_{n+1}^{(i)})&\le c\max\{d_{n-1}^{(1)},d_{n-1}^{(2)},d_{n-1}^{(3)},...,d_{n-1}^{(m)}\}\\ &\le c^2\max\{d_{n-2}^{(1)},d_{n-2}^{(2)},d_{n-2}^{(3)},...,d_{n-2}^{(m)}\}\\ &\le c^3\max\{d_{n-3}^{(1)},d_{n-3}^{(2)},d_{n-3}^{(3)},...,d_{n-3}^{(m)}\}\\ &...\\ &\le c^{n-1}\max\{d_1^{(1)},d_1^{(2)},d_1^{(3)},...,d_1^{(m)}\}=c^{n-1}l \end{aligned}$$

where

$$l=\max\{d_1^{(1)},d_1^{(2)},d_1^{(3)},...,d_1^{(m)}\}$$

Since $0\le c<1$, it follows that $\{x_n^{(i)}\}$ are Cauchy sequences with the limits α_i *in* X_i *for* $i=1,2,...,m.$

Now suppose that T_i for $i=1,2,...,m-1$ are continuous, we have

$$\lim_{n\to\infty}x_n^{(2)}=\lim_{n\to\infty}T_1x_{n-1}^{(1)}\Rightarrow T_1\alpha_1=\alpha_2 \text{ and } \lim_{n\to\infty}x_{n+1}^{(i+1)}=\lim_{n\to\infty}T_ix_n^{(i)}\Rightarrow T_i\alpha_i=\alpha_{i+1}$$

for $i=2,3,...,m-1$

Later we will show that $T_m\alpha_m=\alpha_1$.

To prove that α_1 is a fixed point of $T_mT_{m-1}...T_1$

Using the inequality (1) for $x_1=\alpha_1$ *and* $x_1^{'}=x_{n-1}^{(1)}$, we obtain:

$$\varphi_1\left(\begin{array}{l} d_1\left(T_mT_{m-1}...T_2T_1\alpha_1,x_n^{(1)}\right),d_1\left(\alpha_1,x_{n-1}^{(1)}\right),d_1\left(\alpha_1,T_mT_{m-1}...T_1\alpha_1\right),d_1\left(x_{n-1}^{(1)},x_n^{(1)}\right),\\ d_2\left(T_1\alpha_1,T_1x_{n-1}^{(1)}\right),\ \mathrm{d}_3\left(T_2T_1\alpha_1,T_2T_1x_{n-1}^{(1)}\right),...,d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1,T_{m-1}T_{m-2}...T_1x_{n-1}^{(1)}\right)\end{array}\right)\le 0$$

Letting n tend to infinity and using (a) and the continuality of T_i for i=1,2,…,m-1, we have

$$\varphi_1\begin{pmatrix} d_1\left(T_mT_{m-1}...T_2T_1\alpha_1,\alpha_1\right),d_1\left(\alpha_1,\alpha_1\right),d_1\left(\alpha_1,T_mT_{m-1}...T_1\alpha_1\right),d_1\left(\alpha_1,\alpha_1\right), \\ d_2\left(T_1\alpha_1,T_1\alpha_1\right),\ \mathrm{d}_3\left(T_2T_1\alpha_1,T_2T_1\alpha_1\right),...,d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1,T_{m-1}T_{m-2}...T_1\alpha_1\right) \end{pmatrix}=$$

$$=\varphi_1\left(d_1\left(T_mT_{m-1}...T_2T_1\alpha_1,\alpha_1\right),0,d_1\left(\alpha_1,T_mT_{m-1}...T_1\alpha_1\right),0,0,...,0\ \right)\le 0$$

and from **(b)**, we have: $d_1\left(T_mT_{m-1}...T_2T_1\alpha_1,\alpha_1\right)\le c\max\{0,0,...,0\}=0$

Thus $T_mT_{m-1}...T_1\alpha_1=\alpha_1$ and so α_1 is a fixed point of $T_mT_{m-1}...T_1$.

We now have

$$T_1T_mT_{m-1}...T_2\alpha_2=T_1T_mT_{m-1}...T_2T_1\alpha_1=T_1\alpha_1=\alpha_2$$

In general

$$T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_i\alpha_i=T_{i-1}(T_{i-2}...T_1T_mT_{m-1}...T_iT_{i-1}\alpha_{i-1})=T_{i-1}\alpha_{i-1}=\alpha_i\,,\ i=2,3,...,m$$

Hence α_i are fixed points of

$$T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_i\ ,\ i=2,3,...,m$$

We now prove the uniqueness of the fixed point α_i. Let us prove for α_1.

Suppose that $T_mT_{m-1}...T_1$ has a second fixed point $\alpha_1^{'}\ne\alpha_1$. Using the inequality (1) for $x_1=\alpha_1$ *and* $x_1^{'}=\alpha_1^{'}$ we have:

$$\varphi_1\begin{pmatrix} d_1\left(T_mT_{m-1}...T_2T_1\alpha_1,T_mT_{m-1}...T_2T_1\alpha_1^{'}\right),d_1(\alpha_1,\alpha_1^{'}),d_1\left(\alpha_1,T_mT_{m-1}...T_1\alpha_1\right),d_1\left(\alpha_1^{'},T_mT_{m-1}...T_1\alpha_1^{'}\right), \\ d_2\left(T_1\alpha_1,T_1\alpha_1^{'}\right),\ \mathrm{d}_3\left(T_2T_1\alpha_1,T_2T_1\alpha_1^{'}\right),...,d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1,T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right) \end{pmatrix}=$$

$$=\varphi_1\begin{pmatrix} d_1(\alpha_1,\alpha_1^{'}),d_1(\alpha_1,\alpha_1^{'}),0,0,d_2\left(T_1\alpha_1,T_1\alpha_1^{'}\right),\mathrm{d}_3\left(T_2T_1\alpha_1,T_2T_1\alpha_1^{'}\right),..., \\ d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1,T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right) \end{pmatrix}$$

And from (b), we have:

$$d_1(\alpha_1,\alpha_1^{'})\le c\max\begin{Bmatrix} d_2\left(T_1\alpha_1,T_1\alpha_1^{'}\right),\mathrm{d}_3\left(T_2T_1\alpha_1,T_2T_1\alpha_1^{'}\right),..., \\ d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1,T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right) \end{Bmatrix} \qquad (1'')$$

In similar way, applying the inequality (2) for $x_2=T_1\alpha_1$ and $x_2^{'}=T_1\alpha_1^{'}$, by the property (b) of φ_2 and taking in consideration (1’’) we obtain:

$$d_2\left(T_1\alpha_1,T_1\alpha_1^{'}\right)\le c\max\begin{Bmatrix} d_3\left(T_2T_1\alpha_1,T_2T_1\alpha_1^{'}\right),d_4\left(T_3T_2T_1\alpha_1,T_3T_2T_1\alpha_1^{'}\right),..., \\ d_m\left(T_{m-1}T_{m-2}...T_2T_1\alpha_1,T_{m-1}T_{m-2}...T_2T_1\alpha_1^{'}\right), \end{Bmatrix} \qquad (2'')$$

Similarly, applying the inequality (i) for $x_i=T_{i-1}T_{i-2}...T_2T_1\alpha_1$ and $x_i^{'}=T_{i-1}T_{i-2}...T_2T_1\alpha_1^{'}$ and using these inequalities $(1''),(2''),...,((i-1)''),$ we have

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$$d_i\left(T_{i-1}T_{i-2}...T_1\alpha_1, T_{i-1}T_{i-2}...T_1\alpha_1^{'}\right)\le$$
$$\le c\max\begin{pmatrix} d_{i+1}\left(T_iT_{i-1}...T_1\alpha_1, T_iT_{i-1}...T_1\alpha_1^{'}\right), d_{i+2}\left(T_{i+1}T_i...T_1\alpha_1, T_{i+1}T_i...T_1\alpha_1^{'}\right) \\ ,..., d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1, T_{m-1}T_{m-2}...T_i\alpha_1^{'}\right)\end{pmatrix} \qquad \text{(i'')}$$

By (i'') for i=m-1 we get:

$$d_{m-1}\left(T_{m-2}T_{m-3}...T_1\alpha_1, T_{m-2}T_{m-3}...T_1\alpha_1^{'}\right)\le cd_m\left(T_{m-1}T_{m-2}...T_1\alpha_1, T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right) \quad \text{((m-1)'')}$$

Applying the inequality (m), for $x_m = T_{m-1}, T_{m-2},...,T_1\alpha_1$, $x_m^{'} = T_{m-1}, T_{m-2},...,T_1\alpha_1^{'}$ and using the property (b) of φ_m and these inequalities $((m-1)''), ((m-2)''),...,(1''),$ we now have

$$d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1, T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right)\le cd_m\left(T_{m-1}T_{m-2}...T_1\alpha_1, T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right) \qquad \text{(m'')}$$

and so

$$d_m\left(T_{m-1}T_{m-2}...T_1\alpha_1, T_{m-1}T_{m-2}...T_1\alpha_1^{'}\right)=0 \qquad (\circ)$$

Returning back and using $(\circ), ((m-1)''), ((m-2)''),...,(1'')$ we get:

$$d_1\left(\alpha_1, \alpha_1^{'}\right)=0 \Leftrightarrow \alpha_1=\alpha_1^{'}$$

And so, $\alpha_1=\alpha_1^{'}$, then the uniqueness of α_1 is proved.

Similarly, it can be proved that α_i is the unique fixed point of $T_{i-1}T_{i-2}...T_1T_mT_{m-1}...T_i$, for *i*=2, 3… *m*.

We finally prove that also we have $T_m\alpha_m=\alpha_1$. To do this, note that

$$T_m\alpha_m = T_m(T_{m-1}T_{m-2}...T_1T_m\alpha_m) = T_mT_{m-1}T_{m-2}...T_1(T_m\alpha_m)$$

and so, $T_m\alpha_m$ is a fixed point of $T_mT_{m-1}T_{m-2}...T_1$. Since α_1 is the unique fixed point, it follows that $T_m\alpha_m=\alpha_1$. This completes the proof of the theorem.

3. COROLLARIES

The next corollary follows from Theorem 2.6 in the case $m=1$ and $T_1=T$

Corollary 3.1. *Let* (X,d) *be a complete metric space and* $T: X\to X$ *a self map of X. If for some* $c\in[0,1)$ *we have*

$$\varphi(d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty))\le 0$$

for all $x,y\in X$ *and* $\varphi\in\Phi_4$, *then* T *has a unique fixed point* α *in X.*

This corollary is a generalization of Rhoades theorem [10]

The next corollary follows from corollaries 3.1 in the case

$$\varphi(t_1,t_2,t_3,t_4)=t_1-cf(t_2,t_3,t_4), \text{ where } f\in\mathbb{F}_3$$

Corollary 3.2. *Let (X,d) be a complete metric space and $T: X \to X$ a self map of X. If for some $c\in[0,1)$ we have*

$$d(Tx,Ty) \le cf\{d(x,y),d(x,Tx),d(y,Ty)\}$$

for all $x,y\in X$ and $f \in \mathbb{F}_3$, then T has a unique fixed point α in X.

For different expressions of φ, in the Corollary 3.1, and of f, in the Corollary 3.2, we get different theorems. For example:

For $\varphi(t_1,t_2,t_3,t_4)=t_1-c\max\{t_2,t_3,t_4\}$ we have the Theorem 1.1 (Rhoades theorem [10]).

For $\varphi(t_1,t_2,t_3,t_4)=t_1-ct_2$ we have the Banach theorem [3]

For $\varphi(t_1,t_2,t_3,t_4)=t_1-c\dfrac{t_3+t_4}{2}$ we have the Kannan theorem [7]

For $\varphi(t_1,t_2,t_3,t_4)=t_1-c\max\{t_3,t_4\}$ we have the Bianchini theorem [4]

For $\varphi(t_1,t_2,t_3,t_4)=t_1-\dfrac{at_2+bt_3+ct_4}{a+b+c}$ where a, b, c are nonnegative numbers such that, $a+b+c<1$, we have the Reich theorem [9].

The next corollary follows from Theorem 2.6 in the case $m=2$, $T_1=S$ and $T_2=R$

Corollary 3.3 *Let (X,d) and (Y,ρ) are complete metric spaces and $S:X\to Y$, $R:Y\to X$ are two maps, at least one of them being continuous. If the following inequalities are satisfied:*

$$\varphi_1(d(RSx,RSx'),d(x,x'),d(x,RSx),d(x',RSx'),\rho(Sx,Sx'))\le 0$$

$$\varphi_2(\rho(SRy,SRy'),\rho(y,y'),\rho(y,SRy),\rho(y',SRy'),d(Ry,Ry'))\le 0$$

for all $x,x'\in X$; $y,y'\in Y$ and $\varphi_1,\varphi_2\in\Phi_5$, then RS has a unique fixed point $\alpha\in X$ and SR has a unique fixed point $\beta\in Y$. Moreover, $S\alpha=\beta$ and. $R\beta=\alpha$

This corollary is a generalization of Fisher theorem [5].

The next corollary follows from corollaries 3.3 in the case

$$\varphi_i(t_1,t_2,t_3,t_4,t_5)=t_1-c_i f_i(t_2,t_3,t_4,t_5)\text{, where } f_i\in\mathbb{F}_4 \text{ for i=1,2}$$

Corollary 3.4 *Let (X,d) and (Y,ρ) are complete metric spaces and $S:X\to Y$, $R:Y\to X$ be two maps, at least one of them being continuous. If the following inequalities are satisfied:*

$$d(RSx,RSx')\le c_1 f_1(d(x,x'),d(x,RSx),d(x',RSx'),\rho(Sx,Sx'))$$

$$\rho\left(SRy,SRy'\right)\le c_2 f_2(\rho\left(y,y'\right),\rho\left(y,SRy\right),\rho\left(y',SRy'\right),d\left(Ry,Ry'\right))$$

for all $x,x'\in X$; $y,y'\in Y$ *and* $f_1,f_2\in\mathbb{F}_4$, *then RS has a unique fixed point* $\alpha\in X$ *and SR has a unique fixed point* $\beta\in Y$. *Moreover,* $S\alpha=\beta$ *and.* $R\beta=\alpha$

For the different expressions of φ_1 and φ_2 in the Corollary 3.3 and of f_1 *and* f_2 in the Corollary 3.4, we get different theorems. For example:

For $\varphi_1=\varphi_2=\varphi$, where $\varphi\left(t_1,t_2,t_3,t_4,t_5\right)=t_1-c\max\left\{t_2,t_3,t_4,t_5\right\}$, we have the Theorem 1.2 (Fisher theorem [5]).

For $\varphi_1\left(t_1,t_2,t_3,t_4,t_5\right)=t_1-\dfrac{a_1t_2+a_2t_3+a_3t_4+a_4t_5}{a_1+a_2+a_3+a_4}$ and

$\varphi_2\left(t_1,t_2,t_3,t_4,t_5\right)=t_1-\dfrac{b_1t_2+b_2t_3+b_3t_4+b_4t_5}{b_1+b_2+b_3+b_4}$, we have an extension of Reich Theorem [9] from one metric space to two metric spaces.

Corollary 3.5. *Let* (X,d) *and* (Y,ρ) *are complete metric spaces and* $S:X\to Y$, $R:Y\to X$ *be two maps, at least one of them being continuous. If the following inequalities are satisfied*:

$$d\left(RSx,RSx'\right)\le a_1 d(x,x')+a_2 d\left(x,RSx\right)+a_3 d\left(x',RSx'\right)+a_4\rho\left(Sx,Sx'\right)$$

$$\rho\left(SRy,SRy'\right)\le b_1\rho(y,y')+b_2\rho\left(y,SRy\right)+b_3\rho\left(y',SRy'\right)+b_4 d\left(Ry,Ry'\right)$$

for all $x,x'\in X$; $y,y'\in Y$, *where* $a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4$ *are nonnegative numbers such that* $0\le a_1+a_2+a_3+a_4<1,0\le b_1+b_2+b_3+b_4<1$, *then RS has a unique fixed point* $\alpha\in X$ *and SR has a unique fixed point* $\beta\in Y$. *Moreover,* $S\alpha=\beta$ *and* $R\beta=\alpha$.

Remark 3.6 In case $m=2$, in similar way we obtained the above corollary, we can obtained analogues corollaries which extend the Banach, Kannan, Bianchin theorems etc. from one metric space to two metric spaces.

The next corollary follows from Theorem 2.6 in the case $m=3$, $T_1=T$, $T_2=S$ and $T_3=R$:

Corollary 3.7 *Let* $(X,d),(Y,\rho)$ *and* (Z,σ) *be complete metric spaces and* $T:X\to Y$, $\mathrm{S}:\mathrm{Y}\to\mathrm{Z}$ and $R:Z\to X$ *be three maps, at least one of them being continuous. If the following inequalities are satisfied*

$$\varphi_1\left(d(RSTx,RSTx'),d(x,x'),d(x,RSTx),d(x',RSTx'),\rho(Tx,Tx'),\sigma(STx,STx')\}\right)\le 0$$

$$\varphi_2\left(\rho(TRSy,TRSy'),\rho(y,y'),\rho(y,TRSy),\rho(y',TRSy'),\sigma(Sy,Sy'),d(RSy,RSy')\right)\le 0$$

$$\varphi_3\left(\sigma(STRz,STRz'),\sigma(z,z'),\sigma(z,STRz),\sigma(z',STRz'),d(Rz,Rz'),\rho(TRz,TRz')\right)\le 0$$

for all $x, x' \in X; y, y' \in Y; z, z' \in Z$ *and* $\varphi_1, \varphi_2, \varphi_3 \in \Phi_6$, *then* RST *has a unique fixed point* u *in* X , TRS *has a unique fixed point* v *in* Y *and* STR *has a unique fixed point* w *in* Z . *Further,* $Tu = v, Sv = w$ *and* $Rw = u$.

This corollary is a generalization of Jain et all theorem [6].

In the special case *for* $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$ where $\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c\max\{t_2, t_3, t_4, t_5, t_6\}$,

we have the theorem 1.2 (Jain et all theorem [6]).

Remark 3.8 From the corollary 3.6 we can obtain other propositions determined by the form of implicit relations φ_1, φ_2 and φ_3. For example the corollary which extended the theorem of Reich [6], from one metric space to three metric spaces (in similar way as above we extended to two metric spaces), etc.

At last, we would like to emphasize the fact that all the above corollaries does not hold only for 1, 2 and 3 metric spaces, but also for an arbitrary number of metric spaces.

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Direct and converse theorems for generalized Bernstein polynomials based on the $q-$integers

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Abstract

In this paper, we discuss the convergence properties of a new generalization of $q-$ Bernstein polynomials. The uniform convergence is established, and estimates for the rate of convergence are provided. We also prove that if the rate of convergence for these operators is $C/[n]_{q_n}^{\gamma}$, then $f \in \mathrm{Lip}(\gamma, [0,1])$, for some positive constant C and exponent $0 < \gamma < 1$.

Keywords: $q-$ Bernstein polynomials, rate of convergence, inverse theorem.

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1 Introduction and definitions

In some branches of mathematics (for example, approximation theory, probability theory, number theory, the solution of integral and differential equations), Bernstein polynomials have important applications. The reasons for these applications are the simple structure and important properties of the Bernstein polynomials (for example, shape preserving, reproducing linear functions, degree reducing on the set of polynomials, interpolating f at both end points of $[0,1]$). The nth Bernstein polynomial for a given function f on $[0,1]$ is defined as

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{k,n}(x) \tag{1}$$

$$\text{where} \qquad P_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le x \le 1, \quad n = 1,2,3,\ldots$$

If $f \in C[0,1]$, then the sequence $\{B_n(f;x)\}$ converges uniformly to $f(x)$ on $[0,1]$.

In [5], J. D. Cao introduced the following generalized Bernstein polynomials (which we call the Cao polynomials): for $f \in C[0,1]$, the Cao polynomials of f are

$$C_{n,s_n}(f;x) = \frac{1}{s_n} \sum_{k=0}^{n} \sum_{i=0}^{s_n-1} f(\frac{k+i}{n+s_n-1}) P_{k,n}(x), \quad 0 \le x \le 1, \quad n = 1,2,3,\ldots \tag{2}$$

where $\{s_n\}$ is a sequence of *natural numbers* with the properties $\lim_{n\to\infty} \frac{s_n}{n} = 0$ and $s_n \ge 1$. When $s_n = 1$, we recover the Bernstein polynomials.

Before introducing the new operators based on $q-$ integers, we recall the following definitions of the $q-$ calculus (quantum calculus). Given a value of $q > 0$ and any nonnegative integer n, we define the $q-$ integer $[n]_q$ as

$$[n]_q = \begin{cases} (1-q^n)/(1-q) & \text{if} \quad q \ne 1 \\ n & \text{if} \quad q = 1 \end{cases}$$

and the $q-$ factorial $[n]_q!$ as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q & \text{if} \quad n \geq 1 \\ 1 & \text{if} \quad n = 0 \end{cases}.$$

For integers n and k, with $0 \leq k \leq n$, the $q-$ binomial coefficients are then defined as follows (see [9]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Due to the intensive development of the $q-$ calculus, generalizations of Bernstein polynomials based on the $q-$ integers have emerged. In [16], G. M. Phillips proposed the $q-$ Bernstein polynomials (which we also call the Phillips polynomials): for each positive integer n, and $f \in C[0,1]$, the $q-$ Bernstein polynomial of f is

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x), \qquad 0 \leq x \leq 1, \quad n = 1, 2, 3, \ldots \tag{3}$$

where an empty product is taken to equal 1. Note that the Phillips polynomials reduce to the Bernstein polynomials when we choose $q = 1$ in (3). Explicit expressions for $B_{n,q}(t^r;x)$ for $r = 0, 1, 2$ can be obtained by direct calculation, giving

$$\begin{aligned} B_{n,q}(1;x) &= 1 \\ B_{n,q}(t;x) &= x \\ B_{n,q}(t^2;x) &= x^2 + \frac{x(1-x)}{[n]_q}. \end{aligned} \tag{4}$$

In recent years, the $q-$ Bernstein polynomials and some generalizations have been studied by several researchers because of their potential applications in approximation theory and numerical analysis, and some features of the Bernstein polynomials are inherited by the $q-$ Bernstein polynomials, especially for $0 < q < 1$ (see [4], [10]-[20]).

In this paper, as Phillips has done for the Bernstein polynomials, we consider similar modification of the Cao polynomials:

$$B_{n,\alpha_n}^{q}(f;x) = \frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} f\Big(\frac{[k]_q + i}{[n]_q + \alpha_n - 1}\Big) P_{k,n,q}(x), \tag{5}$$

$$\text{where} \qquad P_{k,n,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x), \ 0 \leq x \leq 1, \ n = 1, 2, 3, \ldots$$

where α_n is a positive real number and $[\alpha_n]$, as usual, denotes the greatest integer less than α_n. There should be no confusion between the two notations $[\alpha_n]$ and $[n]_q$. When $\alpha_n = 1$, we recover the Phillips polynomials; when $q = 1$ and $\alpha_n = s_n$ we recover the Cao polynomials.

Note that throughout the paper, we always assume that $\{q_n\}$ is a sequence of real numbers such that $0 < q_n < 1$ for all n and $\lim\limits_{n\to\infty} q_n = 1$, and where $\{\alpha_n\}$ is a sequence of positive *real numbers* with the properties $\lim\limits_{n\to\infty} \frac{\alpha_n}{[n]_{q_n}} = 0$ and $[\alpha_n] \geq 1$.

The outline of this paper is as follows: in Section 2, we give the convergence properties of the polynomials (5). Section 3 deals with the rate of convergence, using different methods. Finally, we prove an inverse theorem for the polynomials (5).

Now, we give the following basic definitions:

Definition 1.1. The $q-$ derivative of a function f with respect to x is

$$D_q\left(f(x)\right)=\frac{f(qx)-f(x)}{qx-x},$$

and this is also known as the Jackson derivative.

The $q-$ derivative 'product rule' is $D_q\left(f(x)g(x)\right)=D_q\left(f(x)\right)g(qx)+f(x)\,D_q\left(g(x)\right)$. Note also that $D_q\left(x^n\right)=[n]_q x^{n-1}$.

Definition 1.2. The $q-$ integral of a function f is

$$\int_a^b f(x)d_qx=(1-q)b\sum_{m=0}^{\infty}q^m f(bq^m)-(1-q)a\sum_{m=0}^{\infty}q^m f(aq^m).$$

Note that as $q\to 1$, the $q-$ derivative and the $q-$ integral approach the usual derivative and the Riemann integral, respectively.

Definition 1.3. For $f\in C[0,1]$ and $\delta>0$, the usual modulus of continuity is defined by

$$\omega(f;\delta)=\sup_{\substack{|x-y|\le\delta\\ x,y\in[0,1]}}|f(x)-f(y)|\,.$$

It is known that $\lim_{\delta\to 0}\omega(f;\delta)=0$ and, for any $\lambda>0$, $\omega(f;\lambda\delta)\le(1+\lambda)\omega(f;\delta)$.

Definition 1.4. For $\delta>0$ and $C^2[0,1]=\{g\in C[0,1]:\ g',g''\in C[0,1]\}$, Peetre's $K-$ functional is defined as

$$K(f;\delta)=\inf_{g\in C^2[0,1]}\left\{\|f-g\|_{C[0,1]}+\delta\,\|g\|_{C^2[0,1]}\right\}$$

where $\|.\|_{C^2[0,1]}$ is the uniform norm on $C^2[0,1]$, defined by

$$\|g\|_{C^2[0,1]}=\left\|g'\right\|_{C[0,1]}+\left\|g''\right\|_{C[0,1]}.$$

2 Approximation properties of $B_{n,\alpha_n}^{q_n}$

From the definition (5) of $B_{n,\alpha_n}^{q_n}(f;x)$ and the first equation in (4), we have

$$B_{n,\alpha_n}^{q_n}(1;x)=1. \tag{6}$$

To construct our approximation theorem for the sequence $\{B_{n,\alpha_n}^{q_n}\}$, we need the following lemma.

Lemma 2.1. If the operator $B_{n,\alpha_n}^{q_n}$ is defined by (5), then

$$\begin{aligned} B_{n,\alpha_n}^{q_n}(t;x) &= \frac{[n]_{q_n}}{[n]_{q_n}+\alpha_n-1}x+\frac{[\alpha_n]-1}{2\left([n]_{q_n}+\alpha_n-1\right)}, && (7)\\ B_{n,\alpha_n}^{q_n}(t^2;x) &= \frac{[n]_{q_n}^2-[n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x^2+\frac{[\alpha_n]\,[n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x+\frac{([\alpha_n]-1)(2\,[\alpha_n]-1)}{6([n]_{q_n}+\alpha_n-1)^2}. && (8)\end{aligned}$$

Proof. Let us consider the case when f is the function $t\mapsto t$; then

$$\begin{aligned} B_{n,\alpha_n}^{q_n}(t;x) &= \frac{1}{[\alpha_n]}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1}\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1}P_{k,n,q_n}(x)\\ &= \frac{1}{[\alpha_n]}\frac{1}{[n]_{q_n}+\alpha_n-1}\left\{[\alpha_n]\,[n]_{q_n}\sum_{k=0}^{n}\frac{[k]_{q_n}}{[n]_{q_n}}P_{k,n,q_n}(x)\right.\\ &\left.\quad+\frac{1}{2}([\alpha_n]-1)\,[\alpha_n]\sum_{k=0}^{n}P_{k,n,q_n}(x)\right\}.\end{aligned}$$

Using (4), we obtain

$$B_{n,\alpha_n}^{q_n}(t;x)=\frac{[n]_{q_n}}{[n]_{q_n}+\alpha_n-1}x+\frac{[\alpha_n]-1}{2\left([n]_{q_n}+\alpha_n-1\right)}.$$

Taking $f:t\mapsto t^2$, we find

$$\begin{aligned}
B_{n,\alpha_n}^{q_n}(t^2;x) &= \frac{1}{[\alpha_n]}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1}\left(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1}\right)^2 P_{k,n,q_n}(x)\\
&= \frac{1}{[\alpha_n]}\frac{1}{([n]_{q_n}+\alpha_n-1)^2}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1}\left([k]_{q_n}^2+2[k]_{q_n}i+i^2\right)P_{k,n,q_n}(x)\\
&= \frac{1}{[\alpha_n]}\frac{1}{([n]_{q_n}+\alpha_n-1)^2}\\
&\quad\times\left\{[\alpha_n][n]_{q_n}^2\sum_{k=0}^{n}\frac{[k]_{q_n}^2}{[n]_{q_n}^2}P_{k,n,q_n}(x)+[\alpha_n]\left([\alpha_n]-1\right)[n]_{q_n}\sum_{k=0}^{n}\frac{[k]_{q_n}}{[n]_{q_n}}P_{k,n,q_n}(x)\right.\\
&\quad\left.+\frac{[\alpha_n]\left([\alpha_n]-1)(2[\alpha_n]-1\right)}{6}\sum_{k=0}^{n}P_{k,n,q_n}(x)\right\}
\end{aligned}$$

From (4) it follows that

$$\begin{aligned}
B_{n,\alpha_n}^{q_n}(t^2;x) &= \frac{1}{[\alpha_n]}\frac{1}{([n]_{q_n}+\alpha_n-1)^2}\\
&\quad\left\{[\alpha_n][n]_{q_n}^2\left(x^2+\frac{x(1-x)}{[n]_{q_n}}\right)+[\alpha_n]\left([\alpha_n]-1\right)[n]_{q_n}x\right.\\
&\quad\left.+\frac{[\alpha_n]\left([\alpha_n]-1)(2[\alpha_n]-1\right)}{6}\right\}\\
&= \frac{[n]_{q_n}^2-[n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x^2+\frac{[\alpha_n][n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x+\frac{([\alpha_n]-1)(2[\alpha_n]-1)}{6([n]_{q_n}+\alpha_n-1)^2}.
\end{aligned}$$

Combining (6) and Lemma 2.1, we have the following main result.

Theorem 2.2. For $f\in C[0,1]$, the polynomials $B_{n,\alpha_n}^{q_n}(f;x)$ converge uniformly to $f(x)$ on $[0,1]$ as $n\to\infty$.

Proof. From (6) - (8), we have

$$\begin{aligned}
B_{n,\alpha_n}^{q_n}(1;x) &= 1,\\
B_{n,\alpha_n}^{q_n}(t;x) &= \frac{[n]_{q_n}}{[n]_{q_n}+\alpha_n-1}x+\frac{[\alpha_n]-1}{2\left([n]_{q_n}+\alpha_n-1\right)},\\
B_{n,\alpha_n}^{q_n}(t^2;x) &= \frac{[n]_{q_n}^2-[n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x^2+\frac{[\alpha_n][n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}x+\frac{([\alpha_n]-1)(2[\alpha_n]-1)}{6([n]_{q_n}+\alpha_n-1)^2}
\end{aligned}$$

and we obtain

$$\begin{aligned}
\left\|B_{n,\alpha_n}^{q_n}(1;x)-1\right\|_{C[0,1]} &= 0,\\
\left\|B_{n,\alpha_n}^{q_n}(t;x)-x\right\|_{C[0,1]} &\leq \left|\frac{[n]_{q_n}}{[n]_{q_n}+\alpha_n-1}-1\right|+\frac{\alpha_n}{[n]_{q_n}},\\
\left\|B_{n,\alpha_n}^{q_n}(t^2;x)-x^2\right\|_{C[0,1]} &\leq \left|\frac{[n]_{q_n}^2-[n]_{q_n}}{([n]_{q_n}+\alpha_n-1)^2}-1\right|+\frac{1}{3}\frac{\alpha_n^2}{[n]_{q_n}^2}+\frac{\alpha_n}{[n]_{q_n}}
\end{aligned}$$

from condition $\lim\limits_{n\to\infty} \frac{\alpha_n}{[n]_{q_n}} = 0$, we get

$$\lim_{n\to\infty} \left\| B^{q_n}_{n,\alpha_n}(t^r;x) - x^r \right\|_{C[0,1]} = 0, \quad r = 0,1,2.$$

The proof of uniform convergence is then completed by applying the well-known Korovkin theorem (see [1]).

3 Rate of convergence

Theorem 3.1. For any $f \in C[0,1]$,

$$\left| B^{q_n}_{n,\alpha_n}(f;x) - f(x) \right| \le 2\omega\left(f; \frac{\sqrt{4\alpha_n^2 + [n]_{q_n}}}{[n]_{q_n}} \right)$$

where $\omega(f;\cdot)$ is the modulus of continuity as given in Definition 1.3.

Proof. Using the relation $\frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} f(x) P_{k,n,q_n}(x) = f(x)$, express the difference between $B^{q_n}_{n,\alpha_n}(f;x)$ and f as

$$B^{q_n}_{n,\alpha_n}(f;x) - f(x) = \frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left[f(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}) - f(x) \right] P_{k,n,q_n}(x)$$

and so

$$\left| B^{q_n}_{n,\alpha_n}(f;x) - f(x) \right| \le \frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left| f(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}) - f(x) \right| P_{k,n,q_n}(x) \tag{9}$$

Letting $y = \dfrac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}$ and $|y - x| = \lambda\delta$, we have $|f(y) - f(x)| \le \omega(f;\lambda\delta) \le (1 + \lambda)\omega(f;\delta)$. Thus $|f(y) - f(x)| \le \left(1 + \frac{|y-x|}{\delta}\right)\omega(f;\delta)$, and hence, by (6)

$$\begin{aligned}
\left| B^{q_n}_{n,\alpha_n}(f;x) - f(x) \right| &\le \frac{1}{[\alpha_n]}\omega(f;\delta) \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left(1 + \frac{\left| \frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1} - x \right|}{\delta} \right) P_{k,n,q_n}(x) \\
&\le \omega(f;\delta) \\
&\quad \times \left\{ 1 + \frac{1}{\delta} \left[\frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right)^2 P_{k,n,q_n}(x) \right]^{1/2} \right\}
\end{aligned}$$

where we have invoked the Cauchy-Schwartz inequality. Expanding the squared term and making use of (6), (7) and (8), we obtain

$$\begin{aligned}
\left| B^{q_n}_{n,\alpha_n}(f;x) - f(x) \right| &\le \omega(f;\delta) \left\{ 1 + \frac{1}{\delta} \right. \\
&\quad \times \left[\frac{[n]^2_{q_n} - [n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} x^2 + \frac{[\alpha_n]\,[n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} x + \frac{([\alpha_n] - 1)(2\,[\alpha_n] - 1)}{6([n]_{q_n} + \alpha_n - 1)^2} \right. \\
&\quad \left. \left. -2x \left(\frac{[n]_{q_n}}{[n]_{q_n} + \alpha_n - 1} x + \frac{[\alpha_n] - 1}{2\,([n]_{q_n} + \alpha_n - 1)} \right) + x^2 \right]^{1/2} \right\}.
\end{aligned} \tag{10}$$

Now, since $0 \leq x \leq 1$ and $1 \leq [\alpha_n] \leq \alpha_n$, by choosing $\delta = \dfrac{\sqrt{4\alpha_n^2 + [n]_{q_n}}}{[n]_{q_n}}$ in (10), we deduce

$$\begin{aligned}
& \frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right)^2 P_{k,n,q_n}(x) \\
= & \left(\frac{[n]_{q_n}^2 - [n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} - \frac{2[n]_{q_n}}{[n]_{q_n} + \alpha_n - 1} + 1 \right) x^2 \\
& + \left(\frac{[\alpha_n]\,[n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} - \frac{[\alpha_n] - 1}{[n]_{q_n} + \alpha_n - 1} \right) x + \frac{([\alpha_n] - 1)(2\,[\alpha_n] - 1)}{6([n]_{q_n} + \alpha_n - 1)^2} \\
= & \frac{(\alpha_n - 1)^2 - [n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} x^2 + \frac{[\alpha_n] + [n]_{q_n} + \alpha_n - [\alpha_n]\,\alpha_n - 1}{([n]_{q_n} + \alpha_n - 1)^2} x + \frac{([\alpha_n] - 1)(2\,[\alpha_n] - 1)}{6([n]_{q_n} + \alpha_n - 1)^2} \\
\leq & \frac{\alpha_n^2}{([n]_{q_n} + \alpha_n - 1)^2} x^2 + \frac{[\alpha_n] + [n]_{q_n} + \alpha_n}{([n]_{q_n} + \alpha_n - 1)^2} x + \frac{[\alpha_n]^2}{3([n]_{q_n} + \alpha_n - 1)^2} \\
\leq & \frac{\alpha_n^2}{([n]_{q_n} + \alpha_n - 1)^2} + \frac{2\alpha_n + [n]_{q_n}}{([n]_{q_n} + \alpha_n - 1)^2} + \frac{\alpha_n^2}{3([n]_{q_n} + \alpha_n - 1)^2} \\
\leq & \frac{4\alpha_n^2 + [n]_{q_n}}{[n]_{q_n}^2} = \delta^2
\end{aligned} \tag{11}$$

so that

$$\left| B_{n,\alpha_n}^{q_n}(f;x) - f(x) \right| \leq 2\,\omega\left(f; \frac{\sqrt{4\alpha_n^2 + [n]_{q_n}}}{[n]_{q_n}} \right)$$

as claimed.

Theorem 3.2. If f is Hölder continuous on $[0,1]$ with exponent $\gamma \in (0,1]$, denoted $f \in \mathrm{Lip}_M(\gamma, [0,1])$ with $0 < \gamma \leq 1$, then

$$\left| B_{n,\alpha_n}^{q_n}(f;x) - f(x) \right| \leq M \left(\frac{\sqrt{4\alpha_n^2 + [n]_{q_n}}}{[n]_{q_n}} \right)^{\gamma}$$

Proof. From (9) we have

$$\begin{aligned}
\left| B_{n,\alpha_n}^{q_n}(f;x) - f(x) \right| & \leq \frac{1}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left| f\left(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}\right) - f(x) \right| P_{k,n,q_n}(x) \\
& \leq \frac{M}{[\alpha_n]} \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left| \frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right|^{\gamma} P_{k,n,q_n}(x)
\end{aligned}$$

by the Hölder condition. Application of Hölder's inequality gives

$$\begin{aligned}
\left| B_{n,\alpha_n}^{q_n}(f;x) - f(x) \right| & \leq \frac{M}{[\alpha_n]} \left\{ \sum_{k=0}^{n} \sum_{i=0}^{[\alpha_n]-1} \left| \frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right|^2 P_{k,n,q_n}(x) \right\}^{\frac{\gamma}{2}} \left\{ \sum_{k=0}^{n} P_{k,n,q_n}(x) \right\}^{\frac{2-\gamma}{2}} \\
& = M \left\{ \sum_{k=0}^{n} \frac{1}{[\alpha_n]^{\frac{2}{\gamma}}} \sum_{i=0}^{[\alpha_n]-1} \left(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right)^2 P_{k,n,q_n}(x) \right\}^{\frac{\gamma}{2}} \\
& \leq M \left\{ \sum_{k=0}^{n} \frac{1}{[\alpha_n]} \sum_{i=0}^{[\alpha_n]-1} \left(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1} - x \right)^2 P_{k,n,q_n}(x) \right\}^{\frac{\gamma}{2}} \\
& \leq M \left(\frac{\sqrt{4\alpha_n^2 + [n]_{q_n}}}{[n]_{q_n}} \right)^{\gamma}
\end{aligned}$$

where we have used (11).

Theorem 3.3. Let $f \in C[0,1]$ and $B_{n,\alpha_n}^{q_n}(f;x)$ be defined by (5). Then

$$\left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| \leq 2K(f;\frac{\delta_n}{2})$$

where $\delta_n = \max\left(\frac{2\alpha_n}{[n]_{q_n}}, \frac{4\alpha_n^2+[n]_{q_n}}{2[n]_{q_n}^2}\right)$.

Proof. Let $g \in C^2[0,1]$. Using Taylor's formula:

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we get

$$B_{n,\alpha_n}^{q_n}(g(t);x) = g(x) + g'(x)B_{n,\alpha_n}^{q_n}(t-x;x) + B_{n,\alpha_n}^{q_n}(\int_x^t (t-u)g''(u)du;x). \tag{12}$$

From (7) and (11), we have $|B_{n,\alpha_n}^{q_n}(t-x;x)| \leq \left|\frac{[n]_{q_n}}{[n]_{q_n}+\alpha_n-1} - 1\right| + \frac{[\alpha_n]}{[n]_{q_n}+\alpha_n-1} \leq \frac{2\alpha_n}{[n]_{q_n}}$, $\left|B_{n,\alpha_n}^{q_n}((t-x)^2;x)\right| \leq \frac{4\alpha_n^2+[n]_{q_n}}{[n]_{q_n}^2}$. Since $\int_x^t (t-u)du = \frac{(t-x)^2}{2}$, we obtain

$$\begin{aligned}\left|B_{n,\alpha_n}^{q_n}(g;x) - g(x)\right| &\leq \left|g'(x)\right|\left|B_{n,\alpha_n}^{q_n}(t-x;x)\right| + \left|B_{n,\alpha_n}^{q_n}(\int_x^t (t-u)g''(u)du;x)\right| \\ &\leq \frac{2\alpha_n}{[n]_{q_n}}\left\|g'\right\|_{C[0,1]} + \frac{4\alpha_n^2+[n]_{q_n}}{2[n]_{q_n}^2}\left\|g''\right\|_{C[0,1]} \\ &\leq \delta_n \left\|g\right\|_{C^2[0,1]}\end{aligned}$$

where $\delta_n = \max\left(\frac{2\alpha_n}{[n]_{q_n}}, \frac{4\alpha_n^2+[n]_{q_n}}{2[n]_{q_n}^2}\right)$.

Now, for $f \in C[0,1]$ and $g \in C^2[0,1]$, we obtain

$$\begin{aligned}\left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| &= \left|B_{n,\alpha_n}^{q_n}(f-g;x) + g(x) - f(x) + B_{n,\alpha_n}^{q_n}(g;x) - g(x)\right| \\ &\leq \left\|f-g\right\|_{C[0,1]}\left|B_{n,\alpha_n}^{q_n}(1;x)\right| + \left\|f-g\right\|_{C[0,1]} + \delta_n\left\|g\right\|_{C^2[0,1]}\end{aligned}$$

From (6), we get

$$\left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| \leq 2\left\|f-g\right\|_{C[0,1]} + \delta_n\left\|g\right\|_{C^2[0,1]}$$

Taking the infimum on the right hand side over all $g \in C^2[0,1]$, we obtain

$$\left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| \leq 2K(f;\frac{\delta_n}{2})$$

and this completes the proof of the theorem.

4 An inverse theorem

Theorem 4.1 (from [9])**.** Suppose $f(x)$ is continuous at $x=0$ and that, for some $\gamma\in[0,1)$ and $A>0$, $x^{\gamma}D_q(f(x))$ is bounded on the interval $(0,A)$. Then, for $a,b\in[0,A]$, we have

$$\int_a^b D_q(f(x))\,d_qx = f(b)-f(a).$$

Note that Theorem 4.1 is an analogue of the fundamental theorem of calculus.

Lemma 4.2 (from [2] p.696)**.** With $h,\delta\in(0,1]$, if $\omega(f;h)\le K_1\left\{\delta^{\gamma}+\frac{h}{\delta}\omega(f;\delta)\right\}$ for some $K_1>0$ and $0<\gamma<1$, then there exists a constant $K_2>0$ such that $\omega(f;h)\le K_2h^{\gamma}$.

Lemma 4.3. For $x\in(0,1)$, the following estimate holds:

$$\left|D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right)\right|\le\frac{1}{1-x}\omega(f;\delta)\left\{[n]_{q_n}+\frac{1}{\delta}\right\}\tag{13}$$

where $\omega(f;.)$ is the modulus of continuity as given in Definition 1.3.

Proof . Taking the q_n- derivative of (5) with respect to x:

$$\begin{aligned}
D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right) &= \frac{1}{[\alpha_n]}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1} f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\begin{bmatrix}n\\k\end{bmatrix}_{q_n} D_{q_n}\left(x^k\prod_{s=0}^{n-k-1}(1-q_n^sx)\right)\\
&= \frac{1}{[\alpha_n]}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1} f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\\
&\quad\times\left\{D_{q_n}\left(x^k\right)\prod_{s=0}^{n-k-1}(1-q_n^{s+1}x)+x^kD_{q_n}\left(\prod_{s=0}^{n-k-1}(1-q_n^sx)\right)\right\}\\
&= \frac{1}{[\alpha_n]}\sum_{k=0}^{n}\sum_{i=0}^{[\alpha_n]-1} f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\begin{bmatrix}n\\k\end{bmatrix}_{q_n}\\
&\quad\times\left\{[k]_{q_n}x^{k-1}\prod_{s=0}^{n-k-1}(1-q_n^{s+1}x)-[n-k]_{q_n}x^k\prod_{s=1}^{n-k-1}(1-q_n^sx)\right\}.
\end{aligned}$$

Using properties of $q-$binomial coefficients, we write

$$\begin{aligned}
D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right) &= \frac{[n]_{q_n}}{[\alpha_n]}\sum_{k=1}^{n}\sum_{i=0}^{[\alpha_n]-1} f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\begin{bmatrix}n-1\\k-1\end{bmatrix}_{q_n} x^{k-1}\prod_{s=0}^{n-k-1}(1-q_n^{s+1}x)\\
&\quad-\frac{[n]_{q_n}}{[\alpha_n]}\sum_{k=0}^{n-1}\sum_{i=0}^{[\alpha_n]-1} f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\begin{bmatrix}n-1\\k\end{bmatrix}_{q_n} x^{k}\prod_{s=1}^{n-k-1}(1-q_n^{s}x)\\
&= \frac{[n]_{q_n}}{[\alpha_n]}\sum_{k=0}^{n-1}\sum_{i=0}^{[\alpha_n]-1}\left\{f(\frac{[k+1]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})-f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\right\}\\
&\quad\times\begin{bmatrix}n-1\\k\end{bmatrix}_{q_n} x^{k}\prod_{s=1}^{n-k-1}(1-q_n^{s}x)
\end{aligned}$$

Upon taking absolute values of both sides and using the modulus of continuity, we obtain

$$\begin{aligned}
\left|D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right)\right| &\le \frac{[n]_{q_n}}{[\alpha_n]}\sum_{k=0}^{n-1}\sum_{i=0}^{[\alpha_n]-1}\left|f(\frac{[k+1]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})-f(\frac{[k]_{q_n}+i}{[n]_{q_n}+\alpha_n-1})\right|\\
&\quad\times\begin{bmatrix}n-1\\k\end{bmatrix}_{q_n} x^{k}\prod_{s=1}^{n-k-1}(1-q_n^{s}x)
\end{aligned}$$

since $0 < x, q_n^s < 1$, we have $(1 - q_n^s x) < 1$. Therefore, we can write $\prod_{s=1}^{n-k-1}(1 - q_n^s x) = \frac{1}{1-x} \prod_{s=0}^{n-k-1}(1 - q_n^s x) \leq \frac{1}{1-x} \prod_{s=0}^{n-k-2}(1 - q_n^s x)$. Hence, we have

$$
\begin{aligned}
\left|D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right)\right| &\leq \frac{[n]_{q_n}}{[\alpha_n]} \frac{1}{1-x} \sum_{k=0}^{n-1} \sum_{i=0}^{[\alpha_n]-1} \left| f(\frac{[k+1]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}) - f(\frac{[k]_{q_n} + i}{[n]_{q_n} + \alpha_n - 1}) \right| P_{k,n-1,q_n}(x) \\
&\leq \frac{[n]_{q_n}}{[\alpha_n]} \frac{1}{1-x} \sum_{k=0}^{n-1} \sum_{i=0}^{[\alpha_n]-1} \omega\left(f; \frac{[k+1]_{q_n} - [k]_{q_n}}{[n]_{q_n} + \alpha_n - 1}\right) P_{k,n-1,q_n}(x) \\
&= \frac{[n]_{q_n}}{1-x} \sum_{k=0}^{n-1} \omega\left(f; \frac{q_n^k}{[n]_{q_n} + \alpha_n - 1}\right) P_{k,n-1,q_n}(x).
\end{aligned}
$$

Since $0 < q_n^k < 1$ and $\alpha_n - 1 > 0$, we get

$$
\begin{aligned}
\left|D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;x)\right)\right| &\leq \frac{[n]_{q_n}}{1-x} \sum_{k=0}^{n-1} \left\{1 + \frac{1}{\delta} \frac{1}{[n]_{q_n} + \alpha_n - 1}\right\} \omega(f;\delta) P_{k,n-1,q_n}(x) \\
&\leq \frac{[n]_{q_n}}{1-x} \omega(f;\delta) \sum_{k=0}^{n-1} \left\{1 + \frac{1}{\delta} \frac{1}{[n]_{q_n}}\right\} P_{k,n-1,q_n}(x) \\
&\leq \frac{1}{1-x} \omega(f;\delta) \left\{[n]_{q_n} + \frac{1}{\delta}\right\}.
\end{aligned}
$$

So the proof is completed.

Theorem 4.4. If $f \in C[0,1]$ is such that

$$
\left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| \leq \frac{C}{[n]_{q_n}^{\gamma}}, \qquad 0 < \gamma < 1,
$$

for some positive constant C and exponent $0 < \gamma < 1$, then $f \in \mathrm{Lip}(\gamma, [0,1])$.

Proof. To prove Theorem 4.4 , we use the same method like in [4]. For any fixed pair of points (x, y) such that $0 < x, y < 1$ and for n large enough, we have

$$
\begin{aligned}
\left|\int_y^x \frac{1}{1-u}\, d_{q_n} u\right| &\leq \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} \left|\int_y^x d_{q_n} u\right| \\
&= \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} \left|(1-q_n)x \sum_{m=0}^{\infty} q_n^m - (1-q_n)y \sum_{m=0}^{\infty} q_n^m\right| \\
&= \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} |x-y| \qquad (14)
\end{aligned}
$$

Combining the estimates (13) and (14), it follows that

$$
\left|\int_y^x D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;u)\right) d_{q_n} u\right| \leq \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} \omega(f;\delta) \left\{[n]_{q_n} + \frac{1}{\delta}\right\} |x-y|
$$

Let $x, y \in (0,1)$ be any points. Using Theorem 4.1, we write

$$
\begin{aligned}
|f(x) - f(y)| &= \left|f(x) - B_{n,\alpha_n}^{q_n}(f;x) + B_{n,\alpha_n}^{q_n}(f;y) - f(y) + \int_y^x D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;u)\right) d_{q_n} u\right| \\
&\leq \left|B_{n,\alpha_n}^{q_n}(f;x) - f(x)\right| + \left|B_{n,\alpha_n}^{q_n}(f;y) - f(y)\right| + \left|\int_y^x D_{q_n}\left(B_{n,\alpha_n}^{q_n}(f;u)\right) d_{q_n} u\right| \\
&\leq \frac{2C}{[n]_{q_n}^{\gamma}} + \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} \omega(f;\delta) \left\{[n]_{q_n} + \frac{1}{\delta}\right\} |x-y|
\end{aligned}
$$

For a fixed n, pick $\delta \in (0,1]$ such that $\frac{1}{[n]_{q_n}} \leq \delta \leq \frac{1+q_n}{[n]_{q_n}}$. Then $\frac{1+q_n}{\delta} \leq \frac{2}{\delta}$, because $0 < q_n < 1$, and consequently

$$|f(x) - f(y)| \leq 2C\delta^\gamma + \frac{3|x-y|}{\delta} \max\left\{\frac{1}{1-x}, \frac{1}{1-y}\right\} \omega(f;\delta).$$

Taking the maximum over all arbitrary pairs $x, y \in (0,1)$ with $|x-y| = h \leq 1$ $(x+h, y+h \leq 1)$, the last inequality gives

$$\omega(f;h) \leq C' \left\{\delta^\gamma + \frac{h}{\delta}\omega(f;\delta)\right\}$$

where $C' = \max\{2C,\ 3/h\}$ and $0 < h, \delta \leq 1$. Lemma 4.2 then tells us that $\omega(f;h) \leq C''h^\gamma$ for some constant C''. The proof of Theorem 4.4 is completed.

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A NEW NON-UNIQUE FIXED POINT THEOREM

ERDAL KARAPINAR

Abstract. In this manuscript, a new non-unique fixed point theorem is proved on usual complete metric spaces. In particular, a self-mapping T on a complete metric space (X,d) has a non-unique fixed point if T satisfies the condition

$ad(Tx,Ty)+b\big[d(x,Tx)+d(y,Ty)\big]+c[d(y,Tx)+d(x,Ty)] \leq sd(x,y)+rd(x,T^2x)$

for all $x,y \in X$ where $0 \leq \frac{s-c}{a+b} < 1$, $a+b \neq 0$, $a+b+c > 0$ and $0 \leq c-r$. It is also shown that the operator T still has a fixed point if one replaces a complete metric space (X,d) with a complete TVS-cone metric space (X,p).

1. Introduction

A certain class of maps in metric spaces with non-unique fixed points is introduced by Ciric ([7]). After this work, many authors (see, e.g. [11, 3, 18, 8, 19]) have studied on this class of maps.

TVS-cone metric spaces are spaces with a metric that takes values in a topological vector space ordered by a cone [10]. If the notion "topological vector space" is replaced by "Banach space" in this definition, then TVS-cone metric spaces are called "Banach valued metric spaces" or "cone metric spaces" (see [12] and also [1, 2, 4, 13, 14, 15, 16, 17]).

In this manuscript, in section 2, a new non-unique fixed point theorem is proved on complete metric space. In section 3, it is proved that this theorem stays valid also on complete TVS-cone metric spaces.

2. A Fixed Point Theorem on Usual Metric Spaces

Theorem 1. *Let (X,d) be a complete metric space. Suppose there exist a,b,c,s,r and $T : X \to X$ satisfies the conditions*

$$0 \leq \frac{s-c}{a+b} < 1, \quad a+b \neq 0, \ a+b+c > 0 \ \textit{and} \ 0 \leq c-r \tag{2.1}$$

$$ad(Tx,Ty)+b\big[d(x,Tx)+d(y,Ty)\big]+c[d(y,Tx)+d(x,Ty)] \leq sd(x,y)+rd(x,T^2x) \tag{2.2}$$

hold for all $x,y \in X$. Then, T has at least one fixed point.

Proof. Let $x_0 \in X$ be arbitrary . Define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} := Tx_n \qquad n = 0,1,2,... \tag{2.3}$$

When we substitute $x = x_n$ and $y = x_{n+1}$ on the inequality (2.2), it implies that

$$\begin{aligned} ad(Tx_n,Tx_{n+1})+b\big[d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})\big]+4c[d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1})] \\ \leq sd(x_n,x_{n+1})+rd(x_n,T^2x_n) \end{aligned} \tag{2.4}$$

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for all a, b, c, s, r that satisfy (2.1). Due to (2.3), the statement (2.4) turns into

$$\begin{aligned} ad(x_{n+1}, x_{n+2}) + b\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big] + c[d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})] \\ \leq sd(x_n, x_{n+1}) + rd(x_n, x_{n+2}). \end{aligned} \tag{2.5}$$

By a simple calculation, one can get

$$(a+b)d(x_{n+1}, x_{n+2}) + (c-r)d(x_n, x_{n+2}) \leq (s-b)d(x_n, x_{n+1}) \tag{2.6}$$

which implies

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}) \tag{2.7}$$

where $k = \frac{s-b}{a+b}$. Due to (2.1), we have $0 \leq k < 1$. Taking account of (2.7), we get inductively

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d(x_0, x_1). \tag{2.8}$$

To show that $\{x_n\}$ is a Cauchy sequence, assume $n > m$. Then by (2.8) and triangular inequality, one can obtain

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n-1} d(x_0, x_1) + \cdots + k^m d(x_0, x_1) \\ &\leq \tfrac{k^m}{1-k} d(x_0, x_1). \end{aligned} \tag{2.9}$$

which concludes that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, the sequence $\{x_n\}$ converges to some element of X, namely z.

To show z is a fixed point of T, it is sufficient to substitute $x = z$ and $y = x_n$ on the inequality (2.2). Indeed,

$$\begin{aligned} ad(Tx_n, Tz) + b\big[d(x_n, Tx_n) + d(z, Tz)\big] + c[d(z, Tx_n) + d(x_n, Tz)] \\ \leq sd(x_n, z) + rd(x_n, T^2 x_n) \end{aligned} \tag{2.10}$$

which is equivalent to

$$\begin{aligned} ad(x_{n+1}, Tz) + b\big[d(x_n, x_{n+1}) + d(z, Tz))\big] + c[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\ \leq sd(x_n, z) + rd(x_n, x_{n+2}). \end{aligned} \tag{2.11}$$

Consequently, $(a+b+c)d(Tz, z) \leq 0$ as $n \to \infty$. Thus, $Tz = z$ as $a+b+c > 0$. □

3. A Fixed Point Theorem on TVS-Cone Metric Spaces

Throughout this section, the pair (E, S) stands for a real Hausdorff locally convex topological vector space E with its generating system of semi-norms S. A non-empty subset P of E is called a cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone P will be assumed to be closed as well. For a given cone P, a partial ordering (denoted by $\leq$: or $\leq_P$) with respect to P can be defined as follow: $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$. Analogously, $x << y$ represent that $y - x \in intP$, where $intP$ denotes the interior of P. Here, the continuity of the algebraic operations in a topological vector space and the properties of the cone imply the following relations:

$$intP + intP \subseteq intP \ and$$

$$\lambda(intP) \subseteq intP \qquad (\lambda > 0).$$

Definition 2. *(See* [6], [9], [10]*) For $c \in intP$, the nonlinear scalarization function $\phi_c : E \to \mathbb{R}$ is defined by*

$$\phi_c(y) = \inf\{t \in \mathbb{R} : y \in tc - P\}, \text{ for all } y \in E.$$

Lemma 3. *(See* [6], [9], [10]*) For each $t \in \mathbb{R}$ and $y \in E$, the following are satisfied:*

(i) $\phi_c(y) < t \Leftrightarrow y \in tc - intP$,
(ii) *if* $y_1 \in y_2 + P$, *then* $\phi_c(y_2) \le \phi_c(y_1)$,
(iii) $\phi_c(y_1 + y_2) \le \phi_c(y_1) + \phi_c(y_2)$, *for all* $y_1, y_2 \in E$.

Definition 4. *Let X be a non-empty set. and E as usual a Hausdorff locally convex topological space. Suppose a vector-valued function $p : X \times X \to E$ satisfies:*

(M1) $0 \le p(x, y)$ *and* $p(x, y) = 0$ *if and only if* $x = y$, *for all* $x, y \in X$,
(M2) $p(x, y) = p(y, x)$ *for all* $x, y \in X$
(M3) $p(x, y) \le p(x, z) + p(z, y)$, *for all* $x, y, z \in X$.

Then, p is called TVS-cone metric on X, and the pair (X, p) is called a TVS-cone metric space (in short, TVS-CMS).

Remark 5. *In* [12]*, the authors considered E as a real Banach space in the definition of TVS-CMS. Thus, a cone metric space (in short, CMS) in the sense of Huang and Zhang* [12] *is a special case of TVS-CMS.*

Lemma 6. *(See* [10]*) Let (X, p) be a TVS-CMS. Then, $d_p : X \times X \to [0, \infty)$ defined by $d_p = \phi_c \circ p$ is a metric.*

Remark 7. *Since a cone metric space (X, p) in the sense of Huang and Zhang* [12]*, is a special case of TVS-CMS, then $d_p : X \times X \to [0, \infty)$ defined by $d_p = \phi_c \circ d$ is also a metric.*

Definition 8. *(See* [10]*) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X.*

(i) $\{x_n\}_{n=1}^{\infty}$ *TVS-cone converges to* $x \in X$ *whenever for every* $0 << c \in E$, *there is a natural number* M *such that* $p(x_n, x) << c$ *for all* $n \ge M$ *and denoted by* $cone - \lim_{n\to\infty} x_n = x$ *(or* $x_n \overset{cone}{\to} x$ *as* $n \to \infty$*),*
(ii) $\{x_n\}_{n=1}^{\infty}$ *TVS-cone Cauchy sequence in* (X, p) *whenever for every* $0 << c \in E$, *there is a natural number* M *such that* $p(x_n, x_m) << c$ *for all* $n, m \ge M$,
(iii) (X, p) *is TVS-cone complete if every sequence TVS-cone Cauchy sequence in* X *is a TVS-cone convergent.*

Lemma 9. *(See* [10]*) Let (X, p) be a TVS-CMS, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ a sequence in X. Set $d_p = \phi_c \circ p$. Then the following statements hold:*

(i) *If* $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in TVS-CMS* (X, p), *then* $d_p(x_n, x) \to 0$ *as* $n \to \infty$,
(ii) *If* $\{x_n\}_{n=1}^{\infty}$ *is Cauchy sequence in TVS-CMS* (X, p), *then* $\{x_n\}_{n=1}^{\infty}$ *is a Cauchy sequence (in usual sense) in* (X, d_p),
(iii) *If* (X, p) *is complete TVS-CMS, then* (X, d_p) *is a complete metric space.*

Remark 10. *Note that the implications in (i) and (ii) of Lemma 9 are also conversely true. (See e.g.* [1]*)*

Proposition 11. *(See [10]) Let (X,p) is complete TVS-CMS and $T : X \to X$ satisfy the contractive condition*

$$p(Tx,Ty) \leq kp(x,y) \tag{3.1}$$

for all $x,y \in X$ and $0 \leq k < 1$. Then, T has a unique fixed point in X. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to fixed point.

Definition 12. *(See [5]) A subset of A of an order vector space (X,τ) via a cone P is said to be $P-$full if for each $x,y \in A$ we have $\{a \in E : x \leq a \leq y\} \subset A$. A cone P of a topological vector space (X,τ) is said to be normal whenever τ has a base of zero consisting of $P-$ full sets.*

Theorem 13. *(See [5]) (a) A cone P of a topological vector space (X,τ) is normal if and only if whenever $\{x_\alpha\}$ and $\{y_\alpha\}$, $\alpha \in \Delta$ are two nets in X with $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha \in \Delta$ and $y_\alpha \to 0$, then $x_\alpha \to 0$.*

(b) Let (X,τ) be an ordered locally convex space. A semi-norm q on X is called monotone if $q(x) \leq q(y)$ for all $x,y \in X$ with $0 \leq x \leq y$ The cone of (X,τ) is normal if and only if τ is generated by a family of monotone $\tau-$ continuous semi-norms.

In particular, if P is a cone of a real Banach space E, then it is called *normal* if there is a number $K \geq 1$ such that for all $x,y \in E$: $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive integer K, satisfying this inequality, is called the normal constant of P.

The following two lemmas generalizes Lemma 1, Lemma 4 and Lemma 5 in [12].

Lemma 14. *(see [2]) Let (X,p) be a cone metric space over a locally convex space (E,S), where S is the family of semi-norms defining the locally convex topology. Let $\{x_n\}$ be a sequence in E. Then*

(i) $x_n \to x$ in (X,p) if and only if $p(x_n,x) \to 0$ in (E,S).

(ii) x_n is Cauchy in (X,p) if and only if $\lim_{m,n\to\infty} p(x_n,x_m) = 0$ in (E,S).

Proof. (i) Suppose $\{x_n\}$ converges to x. Let $\varepsilon > 0$ and $q \in S$ be given, choose $c >> 0$ such that $q(c) < \varepsilon$. This is possible by taking $c = \frac{\varepsilon c_0}{2q(c_0)}$, where c_0 is an interior point of P. Then there is n_0 such that $p(x_n,x) << c$, for all $n > n_0$. Then by normality of the cone P we have $q(p(x_n,x)) \leq q(c) < \varepsilon$ for all $n > n_0$. This means $p(x_n,x) \to 0$ in (E,S). Conversely, suppose that $p(x_n,x) \to 0$ in (E,S). For $c \in E$ with $c >> 0$ find $\delta > 0$ and $\rho \in S$ such that $q(b) < \delta$ implies $b << c$. For this δ and this p find n_0 such that $q(p(x_n,x)) < \delta$ for all $n > n_0$ and so $p(x_n,x) << c$ for all $n > n_0$. Therefore $x_n \to x$ in (X,p).

(ii) The proof is similar to that in (i). □

Lemma 15. *(see [2]) Let (X,p) be a TVS-cone metric space over a normal cone of a locally convex space (E,S), where S is the family of semi-norms defining the locally convex topology. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x$, $y_n \to y$. Then $p(x_n,y_n) \to p(x,y)$ in (E,S).*

Proof. Let $\varepsilon > 0$ and $q \in S$ be given. Choose $c \in E$ with $c >> 0$ such that $q(c) < \dfrac{\varepsilon}{6}$. From $x_n \to x$ and $y_n \to y$, find n_0 such that for all $n > n_0$, $p(x_n,x) << c$ and $p(y_n,y) << c$. Then for all $n > n_0$ we have

$$p(x_n,y_n) \leq p(x_n,x) + p(x,y) + p(y,y_n) \leq p(x,y) + 2c,$$

and

$$p(x,y) \leq p(x,x_n) + p(x_n,y_n) + p(y_n,y) \leq p(x_n,y_n) + 2c.$$

Hence

$$0 \leq p(x,y) + 2c - p(x_n,y_n) \leq 4c$$

and so by the normality of P we obtain

$$q(p(x_n,y_n) - p(x,y)) \leq q(p(x,y) + 2c - p(x_n,y_n)) + q(2c) \leq 6q(c) < \varepsilon.$$

Therefore $p(x_n,y_n) \to p(x,y)$ in (E,S). □

After these preliminary, one can state the analogy of Theorem 1 for the complete TVS-cone metric spaces. The only difference between the proofs of Theorem 1 and the next Theorem is how to show T has a fixed point.

Theorem 16. *Let (X,p) be a complete TVS-cone metric space over normal cone P. Suppose there exist a,b,c,s,r and $T : X \to X$ satisfies the conditions*

$$0 \leq \frac{s-c}{a+b} < 1, \quad a+b \neq 0, \ a+b+c > 0 \text{ and } 0 \leq c-r \tag{3.2}$$

$$ap(Tx,Ty) + b\big[p(x,Tx) + p(y,Ty)\big] + c[p(y,Tx) + p(x,Ty)] \leq sp(x,y) + rp(x,T^2x) \tag{3.3}$$

hold for all $x,y \in X$. Then, T has at least one fixed point.

Proof. Construct a sequence $\{x_n\}$ as in the proof of Theorem 1. Analogously one can get that $\{x_n\}$ is a Cauchy sequence. Due to the completeness of (X,p), $\{x_n\}$ converges to some point in (X,p), namely z.

Regarding Lemma 14 and Lemma 15, one can get the analogy of (2.11) and (2.10) as in the proof of Theorem 1. □

Remark 17. *Theorem 16 stay valid if one change TVS-cone metric space with cone metric space.*

Remark 18. *In* [10]*, it is concluded that some fixed point theorems on usual metric spaces and on TVS-cone metric spaces (in particular, cone metric spaces) are equivalent by using Lemma 3, Lemma 6, Lemma 9 and Remark 10. Note that the analogy of condition (3.3) can not be obtained by applying the nonlinear scalarization mapping (see Definition 2 and Lemma 3). Thus the technique that was used in* [10] *is not applicable for Theorem 16.*

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A simple proof of the Nadler's contraction principle

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Abstract: We give a simple proof of the Nadler's contraction theorem. Moreover, we find the sufficient condition for the uniqueness of fixed points of set–valued contractions.

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1. **Introduction**

In 1922, the Polish mathematician Stefan Banach [1] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. Nadler in 1969 proved a multi–valued extension of the Banach contraction theorem. Many fixed point theorems have been proved by various authors as generalizations to the Nadlers theorem where the contractive nature of the map is weakened along with some additional requirements.

Let (X, d) be a metric space and let $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$ and $x \in X$, define

$$D(x, A) := \inf\{d(x, a); a \in A\}$$

and

$$H(A, B) := \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}.$$

It is easy to see that H is a metric on $CB(X)$. Moreover, $(CB(X), H)$ is complete metric space if (X, d) is a complete metric space. H is called the Hausdorff metric. Note that a point $p \in X$ is said to be a fixed point of multi–valued mapping $T : X \to CB(X)$ if $p \in T(p)$ [2].

In this paper, we will find out the sufficient conditions for the existence and uniqueness of a fixed point for set–valued contraction mapping $T : X \to CB(X)$.

2. Main results

Fundamental contraction Inequality: If $T : X \to CB(X)$ is a contraction with constant $L < 1$ and

$$\inf_{y \in Tx_1} \{d(x_1, y) + d(y, z)\} \leq D(x_1, Tx_1) + D(Tx_1, z), \tag{2.1}$$

for all $x_1, x_2 \in X,\ z \in Tx_2$, then

$$d(x_1, x_2) \leq \frac{1}{1-L}(D(x_1, Tx_1) + D(x_2, Tx_2)) \tag{2.2}$$

for all $x_1, x_2 \in X$.

Proof. It follows from triangle inequality that

$$d(x_1, x_2) \leq d(x_1, y) + d(y, z) + d(z, x_2)$$

for all $x_1, x_2, y, z \in X$. Then by (2.1), we have

$$\begin{aligned} d(x_1, x_2) &\leq \inf_{y \in Tx_1} \{d(x_1, y) + d(y, z)\} + d(z, x_2) \\ &\leq D(x_1, Tx_1) + D(z, Tx_1) + d(z, x_2) \end{aligned}$$

for all $x_1, x_2, z \in X$. On the other hand by (2.1), we obtain that

$$\begin{aligned} \inf_{z \in Tx_2} \{D(z, Tx_1) + d(z, x_2)\} &\leq \inf_{z \in Tx_2} \{d(y, z) + d(z, x_2)\} \\ &\leq D(y, Tx_2) + D(x_2, Tx_2) \end{aligned}$$

for all $x_1, x_2 \in X$, and all $y \in TX_1$. Therefore, we have

$$\begin{aligned} d(x_1, x_2) &\leq D(x_1, Tx_1) + D(y, Tx_2) + D(x_2, Tx_2) \\ &\leq D(x_1, Tx_1) + H(Tx_1, Tx_2) + D(x_2, Tx_2) \\ &\leq D(x_1, Tx_1) + Ld(x_1, x_2) + D(x_2, Tx_2) \end{aligned}$$

and

$$d(x_1, x_2) \leq \frac{1}{1-L}(D(x_1, Tx_1) + D(x_2, Tx_2))$$

for all $x_1, x_2 \in X$. □

Theorem 2.1. *Let (X, d) be a complete metric space. If $T : X \to CB(X)$ is a contraction with constant $L < 1$, then T has a fixed point.*

Moreover, if T satisfies (2.1), then the fixed point of T is unique.

Proof. Let $x_0 \in X$ be arbitrary. Choose an element $x_1 \in X$ such that $x_1 \in Tx_0$. If $x_0 = x_1$, then x_0 is a fixed point of T, and the proof of theorem is complete. Therefore, we suppose that $x_0 \neq x_1$. Let $L_1 \in (L, 1)$ be arbitrary. Since

$$D(x_1, Tx_1) \leq H(Tx_0, Tx_1) < L_1 d(x_0, x_1),$$

then there exists a $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq L_1 d(x_0, x_1).$$

Continuing this process and having chosen x_n in X, we obtain x_{n+1} in X such that $x_{n+1} \in Tx_n$ and $x_n \neq x_{n+1}$ such that

$$d(x_n, x_{n+1}) \leq L_1 d(x_{n-1}, x_n)$$

for all $n \geq 2$. Hence,

$$\begin{aligned} d(x_n, x_m) &\leq \frac{1}{1-L}(d(x_n, x_{n+1}) + d(x_m, x_{m+1})) \\ &\leq \frac{1}{1-L}(L_1^n d(x_0, x_1) + L_1^m d(x_0, x_1)) \end{aligned} \tag{2.3}$$

for all $m > n$. By taking $n, m \to \infty$ in above inequality, it follows that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists some point $z \in X$ such that $x_n \to z$. It follows that

$$\begin{aligned} D(z, Tz) &= D(\lim_{n\to\infty} x_{n+1}, Tz) = \lim_{n\to\infty} D(x_{n+1}, Tz) \\ &\leq \lim_{n\to\infty} H(Tx_n, Tz) \leq \lim_{n\to\infty} L(d(x_n, z)) \\ &\leq \lim_{n\to\infty} d(x_n, z) = 0. \end{aligned}$$

This means that $D(z, Tz) = 0$. In the other words $z \in Tz$ and z is a fixed point of T.

Now suppose that T satisfies the condition (2.1). We show that z is the unique fixed point of T. Let $r \in X$ be another fixed point of T, that is $r \in Tr$. From (2.2), we have

$$d(z, r) \leq \frac{1}{1-L}(D(z, Tz) + D(r, Tr)) = 0$$

therefore $z = r$. □

In inequality (2.3) of Theorem 2.1, if $m \to \infty$, then we have

$$d(x_n, z) \leq \frac{L_1^n}{1-L} d(x_0, x_1).$$

The importance of this latter inequality is as follows. Suppose we are willing to accept an error of ϵ, i.e., instead of the actual fixed point of T we will be satisfied with a point x_n satisfying $d(x_n, z) < \epsilon$, and suppose also that we start our iteration at some point x_0 in X. Since we want $d(x_n, z) < \epsilon$, we just have to pick N so large that $\frac{L_1^N}{1-L} d(x_0, x_1) < \epsilon$. Now the quantity $A = d(x_0, x_1)$ is something that we can compute after the first iteration and we can then compute how large N has to be by taking the log of the above inequality and solving for N (remembering that $\log(L)$ and $\log(L_1)$ are negative). The result is:

Stopping rule : If $A = d(x_0, x_1)$ and

$$N > \frac{\log(\epsilon) + \log(1-L) - \log(A)}{\log(L_1)}$$

then $d(x_N, z) < \epsilon$. From a practical programming point of view, this inequality allows us to express our iterative algorithm with a for loop rather than a while loop, but it has another interesting interpretation. Suppose we take $\epsilon = 10^{-m}$ in our stopping rule inequality. What we see is that the growth of N with m is a constant plus $\frac{m}{|\log(L_1)|}$ more iteration steps. Stated a little differently, if we need N iterative steps to get m

decimal digits of precision, then we need another N to double the precision to $2m$ digits.

Example 2.2. *Let* $X = \{\frac{1}{2}, \frac{1}{4}, ..., \frac{1}{2^n}, ...\} \bigcup \{0, 1\}, d(x, y) = |x - y|$ *for all* $x, y \in X$. *Define mapping* $F : X \to CB(X)$ *as*

$$F(x) = \begin{cases} \{\frac{1}{2^{n+1}}\} & x = \frac{1}{2^n}, \quad n = 0, 1, ..., \\ \{0\} & x = 0. \end{cases}$$

One can show that F *satisfies the conditions of Theorem 2.1, and* 0 *is the unique fixed point of* F.

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Some New Ostrowski Type Inequalities

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Abstract

2000 Mathematics Subject classification : 26D10, 26D15.
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1. Introduction

A. M. Ostrowski proved the following interesting and useful inequality (see[1, p.468])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| < \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\left\| f' \right\|_\infty, \tag{1}$$

for all $x \in [a,b]$, where $f:[a,b] \subseteq \Re \to \Re$ is continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b) \to \Re$ is bounded on (a,b), that is

$$\left\| f' \right\|_\infty = \sup_{x\in(a,b)} \left| f'(x) \right| < \infty.$$

The object of this paper is to present many new inequalities similar to Ostrowskis inequality.

2. New Results

We state and prove the following

Theorem 2.1 *Let* $f, g:[a,b] \subseteq \Re \to \Re$ be *continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f', g':(a,b) \to \Re$ *are bounded on* (a,b), *that is*

$$\left\| f' \right\|_\infty = \sup_{x\in(a,b)} \left| f'(x) \right| < \infty, \quad \left\| g' \right\|_\infty = \sup_{x\in(a,b)} \left| g'(x) \right| < \infty.$$

Suppose further that f *and* g *are non-decreasing. Then*

$$\left| f(x)\,g(x) - \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \right|$$
$$\leq \frac{\left| f(x) \right| \left\| g' \right\|_\infty + \left| g(x) \right| \left\| f' \right\|_\infty}{b-a} \left(\frac{(x-a)^2 + (x-b)^2}{2} \right) \tag{2}$$

Proof.

$$\left| f(x)\,g(x) - \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \right|$$
$$= \left| \frac{1}{b-a}\int_a^b \left(f(x)\,g(x) - f(t)\,g(t) \right) dt \right|$$

$$\le \frac{1}{b-a}\int_a^b \left| \int_t^x \left(f(u)\, g(u) \right)' \, du \right| dt$$

$$= \frac{1}{b-a}\int_a^b \left| \int_t^x \left(f(u)\, g'(u) + f'(u)\, g(u) \right) du \right| dt$$

$$\le \frac{|f(x)| \|g'\|_\infty + |g(x)| \|f'\|_\infty}{b-a} \int_a^b |x-t|\, dt$$

$$= \frac{|f(x)| \|g'\|_\infty + |g(x)| \|f'\|_\infty}{b-a} \left(\int_a^x (x-t)\, dt + \int_x^b (t-x)\, dt \right)$$

$$= \frac{|f(x)| \|g'\|_\infty + |g(x)| \|f'\|_\infty}{b-a} \left(\frac{(x-a)^2 + (x-b)^2}{2} \right).$$

Theorem 2.2. *Let* $f, g : [a,b] \subseteq \Re \to \Re$ be *continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f', g' : (a,b) \to \Re$ *are bounded on* (a,b), *that is*

$$\|f'\|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty, \quad \|g'\|_\infty = \sup_{x\in(a,b)} |g'(x)| < \infty .$$

Suppose further

$$\sup_{x\in(a,b)} \frac{|f(x) - f'(x)|}{g^2(x)} = M, \qquad \sup_{x\in(a,b)} \frac{|g(x) - g'(x)|}{g^2(x)} = N,$$

Then

$$\left| \frac{f(x)}{g(x)} - \frac{1}{b-a}\int_a^b \frac{f(t)}{g(t)}\, dt \right| \le \frac{N\|g'\|_\infty + M\|f'\|_\infty}{b-a} \left(\frac{(x-a)^2 + (x-b)^2}{2} \right). \tag{3}$$

Proof.

$$\left| \frac{f(x)}{g(x)} - \frac{1}{b-a}\int_a^b \frac{f(t)}{g(t)}\, dt \right| = \left| \frac{1}{b-a}\int_a^b \left(\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right) dt \right|$$

$$= \left| \frac{1}{b-a}\int_a^b \int_t^x \left(\frac{f(u)}{g(u)} \right)' du\, dt \right|$$

$$\le \frac{1}{b-a}\int_a^b \left| \int_t^x \left(\frac{f(u)}{g(u)} \right)' du \right| dt$$

$$= \frac{1}{b-a}\int_a^b \left| \int_t^x \left(\frac{g(u)\, f'(u) - f(u)\, g'(u)}{g^2(u)} \right) du \right| dt$$

$$= \frac{1}{b-a}\int_a^b \left| \int_t^x \left(\frac{f'(u)\left(g(u) - g'(u)\right) + g'(u)\left(f'(u) - f(u)\right)}{g^2(u)} \right) du \right| dt$$

$$\le \frac{1}{b-a}\int_a^b \left| \int_t^x \left(\frac{|f'(u)||g(u)-g'(u)| + |g'(u)||f'(u)-f(u)|}{g^2(u)} \right) du \right| dt$$

$$\le \frac{N\|g'\|_\infty + M\|f'\|_\infty}{b-a}\int_a^b |x-t|\,dt$$

$$= \frac{N\|g'\|_\infty + M\|f'\|_\infty}{b-a}\left(\frac{(x-a)^2+(x-b)^2}{2} \right).$$

Theorem 2.3. *Let* $g:[a,b]\subseteq \Re \to \Re$ be *continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $g':(a,b)\to\Re$ *are bounded on* (a,b), *that is*

$$\|g'\|_\infty = \sup_{x\in(a,b)} |g'(x)| < \infty,$$

and let $f:g([a,b])\to\Re$ *be also continuous and has a bounded derivative, that is*

$$\|f'\|_\infty = \sup_{x\in g([a,b])} |f'(x)| < \infty.$$

Then

$$\left| fog(x) - \frac{1}{b-a}\int_a^b fog(t)\,dt \right| \le \frac{\|f'\|_\infty \|g'\|_\infty}{b-a}\left(\frac{(x-a)^2+(x-b)^2}{2} \right). \tag{4}$$

Proof.

$$\left| fog(x) - \frac{1}{b-a}\int_a^b fog(t)\,dt \right| = \left| \frac{1}{b-a}\int_a^b \left(fog(x) - fog(t) \right) dt \right|$$

$$\le \frac{1}{b-a}\int_a^b \left| \int_t^x \left(fog(u) \right)' du \right| dt$$

$$= \frac{1}{b-a}\int_a^b \left| \int_t^x f'(g(u)\, g'(u)\, du \right| dt$$

$$\le \frac{\|f'\|_\infty \|g'\|_\infty}{b-a}\int_a^b \left| \int_t^x du \right| dt$$

$$= \frac{\|f'\|_\infty \|g'\|_\infty}{b-a}\int_a^b |x-t|\,dt$$

$$= \frac{\|f'\|_\infty \|g'\|_\infty}{b-a}\left(\frac{(x-a)^2+(x-b)^2}{2} \right).$$

Theorem 2.4. *Let* $f:[a,b]\subseteq\Re\to\Re$ is continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\Re$ is bounded on (a,b), that is

$$\|f'\|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty.$$

Let $0<a<b.$ *Then*

$$\left| f(xy) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(st)\,dsdt \right| \leq$$

$$\left((b-a)\,x\left((y-a)^2 + (y-b)^2\right) + (b^2 - a^2)\left((x-a)^2 + (x-b)^2\right)\right)\frac{\|f'\|_\infty}{2(b-a)^2} \;. \qquad (5)$$

Proof.

$$\left| f(xy) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(st)\,dsdt \right|$$

$$= \left| \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(f(xy) - f(st)\right)dsdt \right|$$

$$= \left| \frac{1}{(b-a)^2}\int_a^b\int_a^b\int_{st}^{xy} f'(u)\,du\,dsdt \right|$$

$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \left|\int_{st}^{xy} f'(u)\,du\right| dsdt$$

$$\leq \frac{\|f'\|_\infty}{(b-a)^2}\int_a^b\int_a^b |xy - st|\,dsdt$$

$$= \frac{\|f'\|_\infty}{(b-a)^2}\int_a^b\int_a^b |xy - xt + xt - st|\,dsdt$$

$$\leq \frac{\|f'\|_\infty}{(b-a)^2}\int_a^b\int_a^b \left(|x||y-t| + |t||x-s|\right)dsdt$$

$$= \frac{\|f'\|_\infty}{(b-a)^2}\left(\left((b-a)|x|\right)\int_a^b |y-t|\,dt + \left(\int_a^b t\,dt\right)\left(\int_a^b |x-t|\,dt\right)\right)$$

$$\leq \left((b-a)\,x\left((y-a)^2 + (y-b)^2\right) + (b^2 - a^2)\left((x-a)^2 + (x-b)^2\right)\right)\frac{\|f'\|_\infty}{2(b-a)^2}.$$

Theorem 2.5. *Let* $f:[a,b]\subseteq \Re \to \Re$ *is continuous on and bounded on* $[a,b]$ *that is*

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| < \infty.$$

Define F *by*

$$F(x) = \int_a^x f(t)\,dt. \qquad (6)$$

Then

$$\left| F(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{\|f\|_\infty}{2(b-a)}\left((x-a)^2 + (x-b)^2\right). \qquad (7)$$

Proof.

$$\left| F(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| = \left| \frac{1}{b-a}\int_a^b \left(F(x) - F(t)\right)dt \right|$$

$$=\left|\frac{1}{b-a}\int_a^b\int_t^x F'(u)\,dudt\right|$$
$$=\left|\frac{1}{b-a}\int_a^b\int_t^x f(u)\,dudt\right|$$
$$\le\frac{\|f\|_\infty}{b-a}\int_a^b|x-t|\,dt$$
$$=\frac{\|f\|_\infty}{2(b-a)}\left((x-a)^2+(x-b)^2\right).$$

Theorem 2.6. *Let* $f,g:[a,b]\subseteq\Re\to\Re$ be *continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivative* $f',g':(a,b)\to\Re$ *are bounded on* (a,b), *that is*

$$\|f'\|_\infty=\sup_{x\in(a,b)}|f'(x)|<\infty,\ \|g'\|_\infty=\sup_{x\in(a,b)}|g'(x)|<\infty.$$

Then

$$\left|2f(x)g(x)-\frac{g(x)}{b-a}\int_a^b f(t)\,dt-\frac{f(x)}{b-a}\int_a^b g(t)\,dt\right|$$
$$\le\frac{|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty}{2(b-a)}\left((x-a)^2+(x-b)^2\right). \tag{8}$$

Proof.

$$\left|2f(x)g(x)-\frac{g(x)}{b-a}\int_a^b f(t)\,dt-\frac{f(x)}{b-a}\int_a^b g(t)\,dt\right|$$
$$=\left|\frac{g(x)}{b-a}\int_a^b\big(f(x)-f(t)\big)dt+\frac{f(x)}{b-a}\int_a^b\big(g(x)-g(t)\big)dt\right|$$
$$=\left|\frac{g(x)}{b-a}\int_a^b\int_t^x f'(u)\,du\,dt+\frac{f(x)}{b-a}\int_a^b\int_t^x g'(u)\,du\,dt\right|$$
$$\le\frac{|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty}{b-a}\int_a^b|x-t|\,dt$$
$$=\frac{|g(x)|\|f'\|_\infty+|f(x)|\|g'\|_\infty}{2(b-a)}\left((x-a)^2+(x-b)^2\right).$$

Theorem 2.7. *Let* $f,g:[a,b]\subseteq\Re\to\Re$ be *continuous on* $[a,b]$ *and differentiable on* (a,b) *whose derivatives* $f^{(m+1)},g^{(n+1)}:(a,b)\to\Re$ *are bounded on* (a,b), *that is*

$$\left\|f^{(m+1)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(m+1)}(x)\right|<\infty,\ \left\|g^{(n+1)}\right\|_\infty=\sup_{x\in(a,b)}\left|g^{(n+1)}(x)\right|<\infty.$$

Then

$$\left| f^{(m)}(x)\, g^{(n)}(x) + \frac{1}{b-a}\int_a^b \left(f^{(m)}(t)\, g^{(n)}(t) - f^{(m)}(x)\, g^{(n)}(t) - g^{(n)}(x)\, f^{(m)}(t)\right) dt \right|$$
$$\leq \frac{\left\| f^{(m+1)} \right\|_\infty \left\| g^{(n+1)} \right\|_\infty}{3(b-a)} \left((x-a)^3 + (x-b)^3 \right).$$

(9)

Proof.

$$\left| f^{(m)}(x)\, g^{(n)}(x) + \frac{1}{b-a}\int_a^b \left(f^{(m)}(t)\, g^{(n)}(t) - f^{(m)}(x)\, g^{(n)}(t) - g^{(n)}(x)\, f^{(m)}(t)\right) dt \right|$$

$$= \left| \frac{1}{b-a}\int_a^b \left(f^{(m)}(x)\, g^{(n)}(x) + f^{(m)}(t)\, g^{(n)}(t) - f^{(m)}(x)\, g^{(n)}(t) - g^{(n)}(x)\, f^{(m)}(t)\right) dt \right|$$
$$= \left| \frac{1}{b-a}\int_a^b \left(f^{(m)}(x) - f^{(m)}(t)\right)\left(g^{(n)}(x) - g^{(n)}(t)\right) dt \right|$$
$$\leq \left| \frac{1}{b-a}\int_a^b \int_t^x f^{(m+1)}(u)\, du \int_t^x g^{(n+1)}(u)\, du dt \right|$$
$$\leq \frac{\left\| f^{(m+1)} \right\|_\infty \left\| g^{(n+1)} \right\|_\infty}{b-a} \int_a^b (x-t)^2\, dt$$
$$= \frac{\left\| f^{(m+1)} \right\|_\infty \left\| g^{(n+1)} \right\|_\infty}{3(b-a)} \left((x-a)^3 + (x-b)^3 \right).$$

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Logarithmic order and type on some weighted function spaces

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Abstract

We study the space $Q_{K,\omega}(p,q)$, of analytic functions on the unit disk in terms of a nondecreasing function K. The relation between $Q_{K,\omega}(p,q)$ and $\mathcal{B}_{\omega}^{\frac{q+2}{p}}$ spaces, which have attracted considerable attention, is given by studying the growth order of K. The counterpart $Q^{\#}_{K,\omega}(p,q)$ of $Q_{K,\omega}(p,q)$ for the meromorphic case is also considered and investigated. We note that some characterizations of $Q^{\#}_{K,\omega}(p,q)$ and $Q_{K,\omega}(p,q)$ are different.

1 Introduction

Let $\mathbb{D}=\{z:|z|<1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [4]) is defined as follows:

$$\mathcal{B}=\{f: f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty\},$$

the little Bloch space $\mathcal{B}_0$ (cf. [4]) is a subspace of $\mathcal{B}$ consisting of all $f\in\mathcal{B}$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0.$$

The Dirichlet space is defined by

$$\mathcal{D}=\{f: f \text{ analytic in } \mathbb{D} \text{ and } \int_{\mathbb{D}}|f'(z)|^2 dA(z)<\infty\},$$

where $dA(z)$ is the Euclidean area element $dxdy$. Let $0<q<\infty$. Then the Besov-type spaces

$$\mathbf{B^q}=\left\{f: f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^q \left(1-|z|^2\right)^{q-2}(1-|\varphi_a(z)|^2)^2 dA(z)<\infty\right\}$$

are introduced and studied intensively by Stroethoff (cf. [13]). Here, $\varphi_a(z)$ stands for the Möbius transformation of $\mathbb{D}$ given by

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z}, \text{ where } a\in\mathbb{D}.$$

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In 1994, Aulaskari and Lappan [4] introduced a class of holomorphic functions, the so called Q_p-spaces as follows:

$$Q_p = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)dA(z) < \infty \right\},$$

where $0 < p < \infty$ and the weight function

$$g(z,a) = \log \frac{1-\bar{a}z}{a-z}$$

is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. The weight function $g(z,a)$ is actually Green's function in $\mathbb{D}$ with pole at $a \in \mathbb{D}$.

For a point $a \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a,r) = \varphi_a(rD)$.
The pseudo-hyperbolic disk $D(a,r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [13]). Let A denote the normalized Lebesgue area measure on $\mathbb{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbb{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2} r^2.$$

Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function f analytic in $\mathbb{D}$ belongs to the space $Q_K(p,q)$ (cf. [19]), if

$$\|f\|^p_{Q_K(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q K(g(z,a))dA(z) < \infty.$$

Using the above mentioned function K, several authors have been studied some classes of holomorphic and meromorphic function spaces (see [2, 7, 8, 10, 11, 15, 16, 17, 18, 19, 20] and others).
Now, given a reasonable function $\omega : (0,1] \to (0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ (see [6]) is defined as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$(1-|z|)|f'(z)| \le C\omega(1-|z|), \quad z \in \mathbb{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.
Now, we introduce the following definitions:

Definition 1.1 *For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}^\alpha_\omega$ if*

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| < \infty.$$

Definition 1.2 *For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the little weighted Bloch space $\mathcal{B}^\alpha_{\omega,0}$ if*

$$\|f\|_{\mathcal{B}^\alpha_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha}{\omega(1-|z|)} |f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$.

The logarithmic order (log-order) of the function $K(r)$ is defined as

$$\rho = \limsup_{r\to\infty} \frac{\ln^+ \ln^+ K(r)}{\ln r},$$

where $\ln^+ x = \max\{\ln x, 0\}$. If $0 < \rho < \infty$, the logarithmic type (log-type) of the function $K(r)$ is defined as

$$\sigma = \limsup_{r\to\infty} \frac{\ln^+ K(r)}{r^\rho}.$$

Note that if f is an entire function, then the growth order of f is just the log-order of $M(r)$, the maximum modulus function of f.

Definition 1.3 *[10, 11] For a nondecreasing function $K : [0, \infty) \to [0, \infty)$, $0 < p < \infty$, $-2 < q < \infty$ and for a given reasonable function $\omega : (0, 1] \to (0, \infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p, q)$ if*

$$\|f\|^p_{Q_{K,\omega}(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|)^q K\ \ g(z,a)}{\omega^p(1-|z|)}\, dA(z) < \infty.$$

Definition 1.4 *For a given reasonable function $\omega : (0, 1] \to (0, \infty)$, and let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and an analytic function f in $\mathbb{D}$ is said to belong to the spaces $F_\omega(p, q, s)$ if*

$$\|f\|^p_{F_\omega(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega^p(1-|z|)}\, dA(z) < \infty.$$

Moreover, if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega^p(1-|z|)}\, dA(z) = 0,$$

then $f \in F_{\omega,0}(p, q, s)$.

Since every Möbius map φ can be written as $\varphi(z) = e^{i\theta}\varphi_a(z)$, where θ is real.
We assume throughout the paper that

$$\int_0^1 (1-r^2)^{-2} K(\log \frac{1}{r}) r dr < \infty.$$

We can define an auxiliary function as follows:

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty.$$

Remark 1.1 *It should be remarked that our $Q_{K,\omega}(p, q)$ classes are more general than many classes of analytic functions. If $\omega \equiv 1$, we obtain $Q_K(p, q)$ type spaces (cf. [18] and [19]). If $q = p = 2$, and $\omega(t) = t$, we obtain Q_K spaces as studied recently in [7, 8, 12, 16, 17, 20] and others. If $q = p = 2$, $\omega(t) = t$ and $K(t) = t^p$, we obtain Q_p spaces as studied in [4, 5, 21] and others. If $\omega \equiv 1$ and $K(t) = t^s$, then $Q_{K,\omega} = F(p, q, s)$ classes (cf. [1, 22]).*

2 Analytic classes $Q_{K,\omega}(p, q)$

We first give some basic properties of analytic $Q_{K,\omega}(p, q)$ spaces.

Proposition 2.1 *Let $K : [0, \infty) \to [0, \infty)$ be a nondecreasing function and $\omega : (0, 1] \to (0, \infty)$, where $\omega(\lambda t) = \lambda\omega(t)$. For $0 < p < \infty$ and $-2 < q < \infty$, we have that the spaces $Q_{K,\omega}(p, q)$ are subset of the weighted Bloch space $\mathcal{B}_\omega^{\frac{q+2}{p}}$.*

Proof: For a fixed $r \in (0, 1)$ and $a \in \mathbb{D}$, let

$$E(a, r) = \left\{ z \in \mathbb{D},\ |z - a| < r(1 - |a|) \right\}.$$

Also, suppose that $f \in Q_{K,\omega}(p, q)$, we obtain

$$\begin{aligned}
\|f\|^p_{Q_{K,\omega}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\, dA(z) \\
&\ge \int_{\mathbb{D}} |f'(z)|^p \ \ 1-|z|^2 \ \ ^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\ge \int_{D(a,r)} |f'(z)|^p \ \ 1-|z|^2 \ \ ^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\ge K(\log \frac{1}{r}) \int_{D(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} dA(z) \\
&\ge K(\log \frac{1}{r}) \int_{E(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} dA(z).
\end{aligned}$$

We know that $E(a,r) \subset D(a,r)$ and for any $z \in E(a,r)$, we have

$$(1-r)(1-|a|) \leq 1-|z| \leq (1+r)(1-|a|),$$

then,

$$(1-|z|)^p \geq (1-r)^p(1-|a|)^p \;, \;\; \forall \;\; p > 0.$$

Now, since we assume that ω is non-decreasing, then we obtain that

$$\begin{aligned}\|f\|^p_{Q_{K,\omega}(p,q)} &\geq K(\log\frac{1}{r})\int_{E(a,r)} |f'(z)|^p \frac{(1-|z|^2)^q}{\omega^p(1-|z|)} dA(z)\\ &\geq \frac{K(\log\frac{1}{r})(1-r)^p(1-|a|)^q}{\omega^p((1-r)(1-|a|))}\int_{E(a,r)} |f'(z)|^p dA(z)\end{aligned}$$

Since $|f'(z)|^p$ is a subharmonic function, then

$$\int_{E(a,r)} |f'(z)|^p \, dA(z) \geq |E(a,r)| \; |f'(a)|^p = r^2(1-|a|)^2|f'(a)^{\;p}.$$

Then we obtain

$$\begin{aligned}\|f\|^p_{Q_{K,\omega}(p,q)} &\geq \frac{K(\log\frac{1}{r})(1-r)^{p-1}(1-|a|)^{q+2}}{\omega^p((1-r)(1-|a|))}|f'(a)|^p\\ &\geq C_1\frac{K(\log\frac{1}{r})(1-r)^p(1-|a|)^{q+2}}{\omega^p((1-|a|))}|f'(a)|^p\end{aligned}$$

which implies that

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \leq \frac{\|f\|^p_{Q_{K,\omega}(p,q)}}{C_1(1-r)^{p-1}K(\log\frac{1}{r})}. \tag{1}$$

Our proposition is therefore established.

Next, we give the following proposition.

Proposition 2.2 *If the log-order ρ and the log-type σ of a nondecreasing function $K(r)$ satisfy one of the following conditions:*
(i) $\rho > 1$
(ii) $\rho = 1$ and $0 < \sigma < \infty$,
then $\|f\|^p_{Q_{K,\omega}(p,q)} \subset \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}$, where $0 < p < \infty$, $-2 < q < \infty$ and $\omega : (0,1] \to (0,\infty)$.

Proof: By Proposition 2.1, it suffices to show that each non-constant weighted α-Bloch function f can not belong to the spaces $Q_{K,\omega}(p,q)$.
In fact, if either the log-order ρ of $K(r)$ is greater than 0, or the log-order ρ of $K(r)$ equals 1 and the log-type σ of $K(r)$ is greater than 2, then there exists a sequence $\{r_n\}$ with $r_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\ln^+\ln^+ K(r_n)}{\ln r_n} = \rho > 1 \tag{2}$$

or

$$\sigma = \lim_{n\to\infty} \frac{\ln^+ K(r_n)}{r_n} = \lambda > 0 \tag{3}$$

In the case (2) or (3), we obtain

$$\lim_{n\to\infty} \frac{K(r_n)}{e^{\lambda r_n}} = const. \tag{4}$$

Let f be a non-constant weighted α-Bloch function. Then

$$\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} = \sup_{z\in\mathbb{D}}\{\frac{(1-|z|^2)^q}{\omega^p(1-|z|)}|f'(z)|^p : z \in \mathbb{D}\} \neq 0.$$

However, by (1) and (4) we have

$$\begin{aligned}\|f\|^p_{Q_{K,\omega}(p,q)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^p \;\; 1-|z|^2 \;\;^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)\\ &\geq \pi\|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}}(1-t_n)^p K(\log\frac{1}{t_n}) \not\to \infty.\end{aligned}$$

Hence, $f \in Q_{K,\omega}(p,q)$.

Theorem 2.1 *For each nondecreasing function $K : [0, \infty) \to [0, \infty)$ and $\omega : (0, 1] \to (0, \infty)$, satisfying both of the following:*
Condition A. There exist a constant $p > 1$ such that

$$\lim_{n\to\infty} \frac{K(r)}{r^p} = c \neq 0.$$

Condition B. The log-order ρ and the log-type σ satisfy one of the following cases:
(i) $0 \leq \rho < 1$
(ii) $\rho = 1$ and $0 < \sigma < \infty$.
Then $Q_{K,\omega}(p, q) = \mathcal{B}_\omega^{\frac{q+2}{p}}$.

Proof: Let

$$\lim_{r\to\infty} \frac{K(r)}{r^p} = C \neq 0.$$

for some $p \in (1, \infty)$. Then there exists a fixed $r_1 \in (0, 1)$ such that

$$\frac{c}{2} \leq \frac{K(r)}{r^p} \leq c + 1, \quad 0 < r < r_1. \tag{5}$$

We may choose $r_0 \in (0, 1)$ such that

$$z \in \mathbb{D} \backslash D(a, r_0) \Rightarrow g(z, a) = \log \frac{1}{|\varphi_a(z)|} < r_1. \tag{6}$$

Now we first suppose that $f \in Q_{K,\omega}(p, q)$ with

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{K(g(z, a))}{\omega^p(1 - |z|)} dA(z) = C,$$

and write

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z) \\
&= \int_{D(a,r_0)} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z) \\
&+ \int_{\mathbb{D}\backslash D(a,r_0)} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z) \\
&= I_1 + I_2.
\end{aligned} \tag{7}$$

Since $Q_{K,\omega}(p, q) \subset \mathcal{B}_\omega^{\frac{p+2}{q}}$ from Proposition 2.1, we have

$$\begin{aligned}
I_1 = &\int_{D(a,r_0)} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z) \\
\leq \ & \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \int_{D(a,r_0)} \ \ 1 - |z|^2 \ \ {}^{-2} \ \ \log \frac{1}{\varphi_a(z)} \ \ {}^p dA(z) \\
= \ & 2\pi \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \int_0^{r_0} r(1 - r^2)^{-2} \ \ \log \frac{1}{r} \ \ {}^p dr \\
= \ & 2\pi \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} I(r_0, p),
\end{aligned} \tag{8}$$

where the integral $I(r_0, p) < \infty$ for $0 < r_0 < 1$ and $1 < p < \infty$. On the other hand, by (5) and (6)

$$\begin{aligned}
I_2 = &\int_{\mathbb{D}\backslash D(a,r_0)} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z). \\
\leq \ & \frac{2}{c} \int_{\mathbb{D}\backslash D(a,r_0)} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{K(g(z, a))}{\omega^p(1 - |z|)} dA(z) \\
\leq \ & \frac{2C}{c} < \infty.
\end{aligned} \tag{9}$$

Consequently, by (7), (8), and (9),

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \ \ 1 - |z|^2 \ \ {}^q \frac{(g(z, a))^p}{\omega^p(1 - |z|)} dA(z) \leq \sup\{I_1 + I_2\} < \infty.$$

Thus $f \in Q_{K,\omega}(p,q)$. Hence $Q_{K,\omega}(p,q) \subset \mathcal{B}_\omega^{\frac{p+2}{q}}$. Since $f \in Q_{K,\omega}(p,q)$, f must be a weighted Bloch function in $\mathbb{D}$, and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q \frac{(1-|\varphi_a(z)|^2)^p}{\omega^p(1-|z|)} dA(z) = C < \infty.$$

It follows from (5) and (6) that

$$\begin{aligned} J_2 = & \int_{\mathbb{D}\setminus D(a,r_0)} |f'(z)|^p \left(1-|z|^2\right)^q \frac{K\left(\log \frac{1}{\varphi_a(z)}\right)}{\omega^p(1-|z|)} dA(z) \\ \leq & (c+1)\int_{\mathbb{D}\setminus D(a,r_0)} |f'(z)|^p \left(1-|z|^2\right)^q \frac{\left(\log \frac{1}{\varphi_a(z)}\right)^p}{\omega^p(1-|z|)} dA(z) \\ = & (c+1)C. \end{aligned} \tag{10}$$

Since $f \in Q_{K,\omega}(p,q)$, f must be weighted Bloch function in $\mathbb{D}$. Similar to (8) we have

$$\begin{aligned} J_1 = & \int_{D(a,r_0)} |f'(z)|^p \left(1-|z|^2\right)^q \frac{K\left(\log \frac{1}{\varphi_a(z)}\right)}{\omega^p(1-|z|)} dA(z) \\ \leq & 2\pi \|f\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \int_0^{r_0} (1-r^2)^{-2} K(\log \frac{1}{r}) r dr \\ \leq & \frac{2\pi \mathcal{B}_\omega^{\frac{q+2}{p}}}{(1-r_0^2)^2} \int_0^{r_0} K(\log \frac{1}{r}) r dr \end{aligned} \tag{11}$$

Now we show that the integral $\int_0^{r_0} K(\log \frac{1}{r}) r dr$ in (11) is convergent. Setting $t = \log \frac{1}{r}$, we have

$$J(K) = \int_0^{r_0} K(\log \frac{1}{r}) r dr = \int_{t_0}^{+\infty} \frac{K(t)}{e^{2t}} dt.$$

If $K(t)$ satisfies condition (i), then for given $\epsilon > 0$ with $\rho + \epsilon < 1$, there exists $t_1 > t_0$ such that

$$K(t) < e^{t\rho+\epsilon} < e^t, \quad t \geq t_1.$$

Therefore,

$$\begin{aligned} J(K) &= \int_{t_0}^{t_1} \frac{K(t)}{e^{2t}} dt + \int_{t_1}^{+\infty} \frac{K(t)}{e^{2t}} dt \\ &\leq \int_{t_0}^{t_1} \frac{K(t)}{e^{2t}} dt + \int_{t_1}^{+\infty} \frac{1}{e^{2t}} dt < \infty. \end{aligned} \tag{12}$$

If $K(t)$ satisfies condition (ii), then for given $\epsilon > 0$ with $0 < \sigma + 2\varepsilon < 2$, there exists $t_2 > t_0$ such that

$$K(t) < e^{(\sigma+\epsilon)t} < e^{(2-\varepsilon)t}, \quad t \geq t_2.$$

Thus

$$\begin{aligned} J(K) &= \int_{t_0}^{t_2} \frac{K(t)}{e^{2t}} dt + \int_{t_2}^{+\infty} \frac{K(t)}{e^{2t}} dt \\ &\leq \int_{t_0}^{t_2} \frac{K(t)}{e^{2t}} dt + \int_{t_2}^{+\infty} \frac{e^{(2-\varepsilon)t}}{e^{2t}} dt \\ &= \int_{t_0}^{t_2} \frac{K(t)}{e^{2t}} dt + \int_{t_2}^{+\infty} e^{-\varepsilon t} dt < \infty. \end{aligned}$$

Therefore, by (10) and (11), we get

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) = \sup_{a\in\mathbb{D}} \{J_1 + J_2\} < \infty, \tag{13}$$

which implies that $f \in Q_{K,\omega}(p,q)$. The proof is therefore completed.
By the proof of Theorem 2.1, we have

Corollary 2.1 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. If $K(r)$ satisfies Condition A in Theorem 2.1 for some $p, 1 < p < \infty$, then $Q_{K,\omega}(p,q) \subset \mathcal{B}_{\omega}^{\frac{q+2}{p}}$.*

Remark 2.1 *In the case $p > 1$, we know that $Q_{K,\omega}(p,q) = \mathcal{B}_{\omega}^{\frac{q+2}{p}}$ (see [11]), but now we have always that $Q_{K,\omega}(p,q) \subset \mathcal{B}_{\omega}^{\frac{q+2}{p}}$.*

Corollary 2.2 *Let $K : [0,\infty) \to [0,\infty)$ be a bounded and nondecreasing function. If*

$$\lim_{r\to 0} \frac{K(r)}{r^p} = C \neq \infty.$$

holds for some $p > 1$, then $Q_{K,\omega}(p,q) = \mathcal{B}_{\omega}^{\frac{q+2}{p}}$.

3 Meromorphic classes $Q^{\#}_{K,\omega}(p,q)$

For a meromorphic function f a natural analogue of $|f'(z)|$ is the spherical derivative

$$f^{\#}(z) = \frac{|f'(z)|}{(1+|f(z)|^2)}.$$

Corresponding to the definitions of the spaces $F_\omega(p,q,s)$ and $F_{\omega,0}(p,q,s)$, we define the classes $F^{\#}_{\omega}(p,q,s)$ and $F^{\#}_{\omega,0}(p,q,s)$ as follows:

Definition 3.1 *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. A function f meromorphic in $\mathbb{D}$ is said to belong to the class $F^{\#}_{\omega}(p,q,s)$ if*

$$\|f\|^p_{F^{\#}_{\omega}p,q,s} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \ f^{\#}(z)\ ^{p}\ 1-|z|^2\ ^{q} \frac{g^s(z,a)}{\omega^p(1-|z|)} dA(z) < \infty.$$

Moreover, if

$$\|f\|^p_{F^{\#}_{\omega,0}p,q,s} = \lim_{|a|\to 1^-} \int_{\mathbb{D}} \ f^{\#}(z)\ ^{p}\ 1-|z|^2\ ^{q} \frac{g^s(z,a)}{\omega^p(1-|z|)} dA(z) = 0,$$

then $f \in F^{\#}_{\omega}(p,q,s)$.

Therefore we define the classes $M^{\#}_{\omega}(p,q,s)$ and $M^{\#}_{\omega,0}(p,q,s)$ as follows.

Definition 3.2 *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. A function f meromorphic in $\mathbb{D}$ is said to belong to the class $M^{\#}_{\omega}(p,q,s)$ if*

$$\|f\|^p_{M^{\#}_{\omega}p,q,s} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \ f^{\#}(z)\ ^{p}\ 1-|z|^2\ ^{q} \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} dA(z) < \infty.$$

Moreover, if

$$\|f\|^p_{M^{\#}_{\omega,0}p,q,s} = \lim_{|a|\to 1^-} \int_{\mathbb{D}} \ f^{\#}(z)\ ^{p}\ 1-|z|^2\ ^{q} \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} dA(z) = 0,$$

then $f \in M^{\#}_{\omega}(p,q,s)$.

Let $\mathcal{N}^{\alpha}_{\omega}$ be the class of all normal functions in $\mathbb{D}$. We recall that a function f meromorphic in $\mathbb{D}$ is called to be normal if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|)} f^{\#}(z) < \infty.$$

We will need the following theorem in future:

Theorem 3.1 *[14, 15] Let $2 \le q < \infty$, and let $0 < p < \infty$ and let $0 < r < 1$. Then a function f meromorphic in $\mathbb{D}$ is normal if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \ f^{\#}(z)\ ^{q}\ 1-|z|^2\ ^{q-2}\ g(z,a)\ ^{p} dA(z) < \infty.$$

Now, we give the following theorem.

Theorem 3.2 *Let $0 < p < \infty$, $0 < q < \infty$ and let $0 < s < \infty$ and let $0 < r < 1$. Then a function f meromorphic in $\mathbb{D}$ is normal if and only if*

$$F^{\#}_{\omega}(p,q,s)(f) = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \quad f^{\#}(z) \quad ^{p} \quad 1-|z|^2 \quad ^{q} \frac{(g(z,a))^s}{\omega^p(1-|z|)}\, dA(z) < \infty.$$

Proof: The proof of this theorem is very similar to Theorem 3.1, so it will be omitted.

In the corresponding way to the analytic case, we define the meromorphic classes $Q^{\#}_{K,\omega}(p,q)$ as follows.

Definition 3.3 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$ a function f meromorphic in $\mathbb{D}$ is said to belong to the classes $Q^{\#}_{K,\omega}(p,q)$ if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \quad f^{\#}(z) \quad ^{p} \quad 1-|z|^2 \quad ^{q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty. \tag{14}$$

Remark 3.1 *Similar to the analytic case, if we take $\omega \equiv 1$ and $K(t) = t^s$ for $0 \le s < \infty$, then $Q^{\#}_{K,\omega}(p,q) = F^{\#}(p,q,s)$ (see [15]), the corresponding meromorphic of $F(p,q,s)$ spaces. If we take $K(t) = t^p, q = 0$ and $\omega \equiv 1$, then $Q^{\#}_{K,\omega}(p,q) = Q^{\#}_p$. When $\omega \equiv 1$ and $p = 2$ and $q = 0$, we obtain $Q^{\#}_K$ space (see [3, 7, 14]).*

Definition 3.4 *[14] A function f meromorphic in $\mathbb{D}$ is said to be a spherical Bloch function, denoted by $f \in \mathcal{B}^{\#}$, if there exists an r, $0 < r < 1$, such that*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^{\#}(z))^2\, dA(z) < \infty.$$

It is easy to see that a normal function is a spherical Bloch function, that is, $\mathcal{N} \subset \mathcal{B}^{\#}$, but the converse is not true. A counterexample can be found in [9].

Proposition 3.1 *For each nondecreasing function $K : [0,\infty) \to [0,\infty)$, the classes $Q^{\#}_{K,\omega}(p,q)$ are subsets of the spherical Bloch classes $\mathcal{B}^{\#}{}_{\omega^p}^{\frac{q+2}{p}}$, where $0 < p < \infty$.*

Proof: We can prove the proposition by making the obvious modifications to the proof of Proposition 2.1.

Theorem 3.3 *Let $K : [0,\infty) \to [0,\infty)$ be a bounded and nondecreasing function and let f is a normal function. If*

$$\lim_{r\to 0} \frac{K(r)}{r^s} = c < \infty$$

holds for some $0 < s < \infty$, then $f \in Q^{\#}_{K,\omega}(p,q)$, where $2 \le p < \infty$.

Proof: Suppose that

$$\lim_{r\to 0} \frac{K(r)}{r^s} = c < \infty$$

holds for some $0 < s < \infty$. Then there exists a fixed $r_1 \in (0,1)$ such that $\frac{K(r)}{r^s} \le c+1$ for $0 < r < r_0$, we may take $r_0 \in (0,1)$ such that both

$$g(z,a) = \log \frac{1}{|\varphi_a(z)|} < r_1$$

and

$$\log \frac{1}{|\varphi_a(z)|} \le c_1(1-|\varphi_a(z)|^2)$$

hold for constant $c_1 > 0$ whenever $z \in \mathbb{D}\backslash D(a,r_0)$. Now, since

$$\begin{aligned}
\|f\|^p_{Q^{\#}_{K,\omega}(p,q)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \quad f_0{}^{\#}(z) \quad ^{p} \quad 1-|z|^2 \quad ^{q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&= \sup_{D(a,r_0)} \int_{D(a,r_0)} \quad f_0^{\#}(z) \quad ^{p} \quad 1-|z|^2 \quad ^{q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&+ \sup_{\mathbb{D}\backslash D(a,r_0)} \int_{\mathbb{D}} \quad f_0^{\#}(z) \quad ^{p} \quad 1-|z|^2 \quad ^{q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z)
\end{aligned}$$

For $a \in \mathbb{D}$ and r_0 as above, then using Theorem 3.2, we have that

$$\begin{aligned}
L_2 &= \int_{\mathbb{D}\setminus D(a,r_0)} f_0^{\#}(z)^{\;\;p} \;\; 1-|z|^2 {}^{\;\;q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\le (c+1)(c_1)^s \int_{\mathbb{D}\setminus D(a,r_0)} f_0^{\#}(z)^{\;\;p} \;\; 1-|z|^2 {}^{\;\;q} \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} dA(z) \\
&\le (c+1)(c_1)^s F_{\omega}^{\#}(p,q,s) < \infty. \qquad (15)
\end{aligned}$$

On the other hand, since K is bounded, there exists a constant C_2 such that $K(r) \le C_2$ for all $r : 0 < r < \infty$. Thus

$$\begin{aligned}
L_1 &= \int_{D(a,r_0)} f_0^{\#}(z)^{\;\;p} \;\; 1-|z|^2 {}^{\;\;q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) \\
&\le \frac{C_2}{(1-r_0^2)^s} \int_{D(a,r_0)} f_0^{\#}(z)^{\;\;p} \;\; 1-|z|^2 {}^{\;\;q} \frac{(1-|\varphi_a(z)|^2)^s}{\omega^p(1-|z|)} dA(z) \\
&= \frac{C_2}{(1-r_0^2)^s} F_{\omega}^{\#}(p,q,s). \qquad (16)
\end{aligned}$$

Therefore, by (15) and (16), we have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} f_0^{\#}(z)^{\;\;p} \;\; 1-|z|^2 {}^{\;\;q} \frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)} dA(z) = \sup_{a\in\mathbb{D}}\{L_1+L_2\} < \infty.$$

Thus $f_0 \in \|f\|^p_{Q^{\#}{}_{k,\omega}(p,q)}$, the proof of our theorem is completed.

Finally, we consider the harmonic counterpart of $Q_{K,\omega}(p,q)$ as follows:

Definition 3.5 *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. A function u real-valued harmonic in $\mathbb{D}$ is said to belong to the space $Q_{Kh,\omega}(p,q)$ if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\nabla u(z)|^p \;\; 1-|z|^2 {}^{\;\;q} \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty,$$

where $\nabla u(z) = (u_x, u_y)$ is the gradient of u and $|\nabla u(z)| = \sqrt{u_x^2+u_y^2}$, $\quad 0 < p < \infty$ and $-2 < q < \infty$.
Note that if we take $K(r) = r^p$ for $0 < p < \infty$, then $Q_{Kh,\omega}(p,q)$ is a harmonic weighted Bloch space $\mathcal{B}^{\alpha}_{h,\omega}$, where

$$\mathcal{B}^{\alpha}_{h,\omega} = \{u : u \text{ harmonic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}|\nabla u(z)|}{\omega(1-|z|)} < \infty\}.$$

It is easy to see that some corresponding results to Propositions 2.1, 2.2, and Theorem 2.1 are also true for $Q_{Kh,\omega}(p,q)$ and the proof are similar to those of them.

Remark 3.2 *It should be remarked that our results in this paper are completely different from that introduced by Wulan et. al in [19], since the authors in [19] introduced their characterizations without use of the logarithmic order and logarithmic type of a nondecreasing function. On the other hand , we introduced some results in the sense of meromorphic and harmonic functions.*

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SOME FIXED POINT THEOREM FOR MAPPING ON COMPLETE G-CONE METRIC SPACES

DURAN TURKOGLU AND NURCAN BILGILI

ABSTRACT. In this paper we introduce G-cone metric spaces and we give some properties about this spaces. Further we prove some fixed point results for mapping satisfying sufficient conditions on complete G-cone metric space,then we showed that if the G-cone metric space (X, G) is symmetric ,the existence and uniqueness of these fixed point results follow from well-known theorems in cone metric space (X, d_G),where (X, d_G) is the cone metric space which defined from the G-cone metric space (X, G).

1. INTRODUCTION

Cone metric spaces were introduced by Huang and Zhang in [4]. The authors there described convergence in cone metric spaces and introduced completeness. Then they proved some fixed point theorems of contractive mappings on cone metric spaces. Some definitions and topological concepts were generalized in [2] and they proved there that every cone metric space is a topological space, they also generalized the concept of diametrically contractive mappings and proved some fixed point theorems in cone metric spaces. Furthermore, cone metric spaces were studied by many authors (see [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

On the other hand, during the sixties, the notion of 2-metric space introduced by Gähler (see [14, 15])as a generalization of usual notion of metric space (X, d).But different authors proved that there is no relation between these two functions, for instance, Ha et al.in [16] show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D.thesis introduce a new class of generalized metric space called D-metric spaces .([17, 18])

In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [18, 19]). He claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

But in 2003 Zead Mustafa and Brailey Sims demonstrated in [20] that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, so, were introduced more appropriate notion of generalized metric space in [21] and they proved some fixed point results for mapping satisfying sufficient conditions on complete G-metric space.

In this paper we introduce G-cone metric spaces, we give some properties about this spaces and we prove some fixed point results for mapping satisfying sufficient conditions on complete G-cone metric spaces.

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2. G-cone metric spaces

In this section we shall define G-cone metric spaces and prove some properties.

Let E always be a real Banach space and P a subset of E. P is called a cone if and only if

(i) P is closed, nonempty, and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbf{R}$, $a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P. The cone P is called normal if there is a number $K \geq 1$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space, P is cone in E with $intP \neq \varnothing$ and $\leq$ is partial ordering with respect to P.

Definition 1. *Let X be a nonempty set , and let $G : X \times X \times X \longrightarrow E$ be a function satisfying the following properties:*

$(G1)$ $G(x, y, z) = \theta$ if and only if $x = y = z$;

$(G2)$ $\theta < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, (symmetry in all three variables);

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized cone metric , or, more specifically, a G -cone metric on X, and the pair (X, G) is called a G-cone metric space.

Definition 2. *Let (X, G) be a G-cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) << c$, then (x_n) is said to be G-cone convergent to x in X and x is limit of (x_n). We denote this by $\lim\limits_{n,m \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ $(n, m \longrightarrow \infty)$*

Lemma 1. *Let (X, G) be a G-cone metric space. Then for $c \in E$ with $\theta << c$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in intP$.*

Proof. Since $c >> \theta$, then $c \in intP$. Hence find $\delta > 0$ such that $N_\delta\ (c) = \{x \in E : \|x - c\| < \delta\} \subset intP$. (Since$\|c - c\| = \theta < \delta$, then $c \in N_\delta\ (c)$ so $N_\delta\,(c) \neq \varnothing$). Now if $\|x\| < \delta$ then $\|x - c + c\| < \delta$ hence, $|-1|\,\|x - c + c\| = \|-x + c - c\| = \|(c - x) - c\| < \delta$ then $c - x \in N_\delta\,(c) \subset intP$ and so, $c - x \in intP$. □

Lemma 2. *Let (X, G) be a G-cone metric space, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. Then (x_n) is G-cone convergent to x if and only if $G(x, x_n, x_m) \longrightarrow \theta$ $(n, m \longrightarrow \infty)$.*

Proof. Suppose that (x_n) is G-cone convergent to x. For every real $\varepsilon > 0$, choose $c \in E$ with $\theta << c$ and $K\|c\| < \varepsilon$. Then there is $N \in \mathbf{N}$, for all $n, m \geq N$, $G(x, x_n, x_m) << c$. Since P is a normal cone with normal constant K, when $n, m \geq N$, $\|G(x, x_n, x_m)\| \leq K\|c\| < \varepsilon$. This means $G(x, x_n, x_m) \longrightarrow \theta$ $(n, m \longrightarrow \infty)$ or $\lim\limits_{n,m \longrightarrow \infty} G(x, x_n, x_m) = \theta$.

Conversely, suppose that $G(x, x_n, x_m) \longrightarrow \theta$ $(n, m \longrightarrow \infty)$. From Lemma 1 for every $c \in E$ with $\theta << c$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in intP$. For this δ there is $N \in \mathbf{N}$, such that for all $n, m \geq N$, $\|G(x, x_n, x_m)\| < \delta$. So $c - G(x, x_n, x_m) \in intP$. This means $G(x, x_n, x_m) << c$. Therefore (x_n) is G-cone convergent to x. □

Lemma 3. *Let (X, G) be a G-cone metric space, P be a normal cone with normal constant K. Then the following are equivalent.*

(1) *(x_n) is G-cone convergent to x.*
(2) *$G(x_n, x_n, x) \longrightarrow \theta$, as $n \longrightarrow \infty$.*
(3) *$G(x_n, x, x) \longrightarrow \theta$, as $n \longrightarrow \infty$.*
(4) *$(x_m, x_n, x) \longrightarrow \theta$, as $m, n \longrightarrow \infty$.*

Proof. (1) $\Longrightarrow$ (2) :

Suppose that (x_n) is G-cone convergent to x. From Lemma 2, $G(x, x_n, x_m) \longrightarrow \theta$ $(n, m \longrightarrow \infty)$. If we choose $m = n$, then $G(x, x_n, x_n) \longrightarrow \theta$ $(n \longrightarrow \infty)$. From $(G4)$, $G(x, x_n, x_n) = G(x_n, x_n, x)$ so $G(x_n, x_n, x) \longrightarrow \theta$, as $n \longrightarrow \infty$.

(2) $\Longrightarrow$ (3) :

Suppose that $G(x_n, x_n, x) \longrightarrow \theta$, as $n \longrightarrow \infty$. If we choose $a = x_n$ in $(G5)$, then $G(x, x_n, x) \leq G(x, x_n, x_n) + G(x_n, x_n, x)$. From $(G4)$, $G(x_n, x_n, x) = G(x, x_n, x_n)$ and since $G(x_n, x_n, x) \longrightarrow \theta$, as $n \longrightarrow \infty$, for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that for all $n \geq N$,$\|G(x_n, x_n, x)\| < \frac{\varepsilon}{2K}$. Since P is a normal cone with normal constant K,

$$\begin{aligned} \|G(x, x_n, x)\| &\leq K\|G(x, x_n, x_n) + G(x_n, x_n, x)\| \\ &= K(\|G(x, x_n, x_n)\| + \|G(x_n, x_n, x)\|) \\ &< K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon. \end{aligned}$$

From $(G4)$, $G(x_n, x, x) = G(x, x_n, x)$ so for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that for all $n \geq N$, $\|G(x_n, x, x)\| < \varepsilon$.

This means $G(x_n, x, x) \longrightarrow \theta$, as $n \longrightarrow \infty$.

(3) $\Longrightarrow$ (4) :

Suppose that $G(x_n, x, x) \longrightarrow \theta$,as $n \longrightarrow \infty$. If we choose $a = x$ in $(G5)$, then $G(x_m, x_n, x) \leq G(x_m, x, x) + G(x, x_n, x)$. Since $G(x_n, x, x) \longrightarrow \theta$, as $n \longrightarrow \infty$ and from $(G4)$,$G(x_m, x, x) \longrightarrow \theta$ and $G(x_n, x, x) = G(x, x_n, x) \longrightarrow \theta$. So for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $\|G(x_m, x, x)\| < \frac{\varepsilon}{2K}$, $\|G(x, x_n, x)\| < \frac{\varepsilon}{2K}$. Since P is a normal cone with normal constant K,

$$\begin{aligned} \|G(x_m, x_n, x)\| &\leq K\|G(x_m, x, x) + G(x, x_n, x)\| \\ &= K(\|G(x_m, x, x)\| + \|G(x, x_n, x)\|) \\ &< K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon. \end{aligned}$$

So for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $\|G(x_m, x_n, x)\| < \varepsilon$. This means $G(x_m, x_n, x) \longrightarrow \theta$, as $m, n \longrightarrow \infty$.

(4) $\Longrightarrow$ (1) :

Suppose that $G(x_m, x_n, x) \longrightarrow \theta$, as $m, n \longrightarrow \infty$. From $(G4)$,$G(x_m, x_n, x) = G(x, x_n, x_m)$. So from Lemma 2, (x_n) is G-cone convergent to x. □

Lemma 4. *Let (X, G) be a G-cone metric space, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. If (x_n) is G-cone convergent to x and (x_n) is G-cone convergent to y, then $x = y$. That is the limit of (x_n) is unique.*

Proof. From $x_n \longrightarrow x$ and $x_n \longrightarrow y$ $(n, m \longrightarrow \infty)$, for any $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) << c$ and $G(y, x_n, x_m) << c$. If we choose $a = x_m$ in $(G5)$, then

$$\begin{aligned} G(x, x_n, y) &\leq G(x, x_m, x_m) + G(x_m, x_n, y) \\ &\leq c + c = 2c \end{aligned}$$

and we choose $a = x_n$ in $(G5)$, then

$$\begin{aligned} G(x, x, y) &\leq G(x, x_n, x_n) + G(x_n, x, y) \\ &\leq c + 2c = 3c. \end{aligned}$$

Since P is a normal cone with normal constant K, $\|G(x, x, y)\| \leq 3K\|c\|$. Since c is arbitrary $G(x, x, y) = \theta$, therefore $x = y$. □

Definition 3. *Let (X, G) be a G-cone metric space. Let (x_n) be a sequence in X. If for every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) << c$, then (x_n) is called a G-cone Cauchy sequence in X.*

Lemma 5. *Let (X, G) be a G-cone metric space, P be a normal cone with normal constant K. Let (x_n) be a sequence in X. Then (x_n) is a G-cone Cauchy sequence in X if and only if $G(x_n, x_m, x_l) \longrightarrow \theta$ $(n, m, l \longrightarrow \infty)$.*

Proof. Suppose that (x_n) is a G-cone Cauchy sequence in X. For every real $\varepsilon > 0$, choose $c \in E$ with $\theta << c$ and $K\|c\| < \varepsilon$. Then there is $N \in \mathbf{N}$, for all $n, m, l \geq N$,$G(x_n, x_m, x_l) << c$. Since P is a normal cone with normal constant K, when $n, m, l \geq N$, $\|G(x_n, x_m, x_l)\| \leq K\|c\| < \varepsilon$. This means $G(x_n, x_m, x_l) \longrightarrow \theta$ $(n, m, l \longrightarrow \infty)$.

Conversely, suppose that $G(x_n, x_m, x_l) \longrightarrow \theta$ $(n, m, l \longrightarrow \infty)$. From Lemma 1 for every $c \in E$ with $\theta << c$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in intP$. For this δ there is $N \in \mathbf{N}$, such that for all $n, m, l \geq N$, $\|G(x_n, x_m, x_l)\| < \delta$. So $c - G(x_n, x_m, x_l) \in intP$.

This means $G(x_n, x_m, x_l) << c$. Therefore (x_n) is a G-cone Cauchy sequence in X. □

Definition 4. *Let (X, G) be a G-cone metric space. If every G-cone Cauchy sequence is G-cone convergent in X, then X is called a complete G-cone metric space.*

Lemma 6. *Let (X, G) be a G-cone metric space, (x_n) be a sequence in X. If (x_n) is G-cone convergent to x in X, then (x_n) is a G-cone Cauchy sequence in X.*

Proof. Suppose that (x_n) is G-cone convergent to x in X. For any $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x, x_n, x_m) << \frac{c}{2}$ and $G(x, x_m, x_l) << \frac{c}{2}$. If we choose $a = x$ in $(G5)$, then $G(x_n, x_m, x_l) \leq G(x_n, x, x) + G(x, x_m, x_l)$. Also from $(G4)$ and $(G3)$, $G(x_n, x, x) = G(x, x, x_n) \leq G(x, x_m, x_n) = G(x, x_n, x_m)$, then $G(x_n, x, x) \leq G(x, x_n, x_m) << \frac{c}{2}$. This means $G(x_n, x_m, x_l) \leq G(x_n, x, x) + G(x, x_m, x_l) << \frac{c}{2} + \frac{c}{2} = c$. Therefore (x_n) is a G-cone Cauchy sequence in X. □

Lemma 7. *Let (X, G) be a G-cone metric space. Then the following are equivalent.*

(1) *(x_n) is a G-cone Cauchy sequence in X.*

(2) *For every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x_n, x_m, x_m) << c$.*

Proof. $(1) \Longrightarrow (2)$:

Suppose that (x_n) is a G-cone Cauchy sequence in X. So for every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) << c$. If we choose $l = m$, then for every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m \geq N$,

$G(x_n, x_m, x_m) << c$.

$(2) \Longrightarrow (1)$:

Suppose that for every $c \in E$ with $\theta << c$ there is $N \in \mathbf{N}$ such that for all $n, m \geq N$,$G(x_n, x_m, x_m) << c$. If we choose arbitrary $c \in E$ with $\theta << c$ and $a = x_m$ in $(G5)$, then $G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l)$.

For this arbitrary $c \in E$ with $\theta << c$, there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_m) << \frac{c}{2}$ and $G(x_l, x_m, x_m) << \frac{c}{2}$.

From $(G4)$, $G(x_m, x_m, x_l) << \frac{c}{2}$. Then $G(x_n, x_m, x_l) << \frac{c}{2} + \frac{c}{2} = c$.

Hence for every $c \in E$ with $\theta << c$, there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$,$G(x_m, x_m, x_l) << c$. This means (x_n) is a G-cone Cauchy sequence in X. □

Definition 5. *Let (X, G) and $\left(X^1, G^1\right)$ be two G-cone metric spaces and let $f : (X, G) \longrightarrow \left(X^1, G^1\right)$ be a function, then f is said to be G-cone continuous at a point $a \in X$ if and only if , given $\theta << c$, there exists $\theta << c_0$ such that $x, y \in X$; and $G(a, x, y) << c_0$ implies $G^1(f(a), f(x), f(y)) << c$. A function f is G-cone continuous at X if and only if it is G-cone continuous at all $a \in X$.*

Lemma 8. *Let (X, G) and $\left(X^1, G^1\right)$ be two G-cone metric spaces. Then a function $f : (X, G) \longrightarrow \left(X^1, G^1\right)$ is G-cone continuous at a point $x \in X$ if and only if it is G-cone sequently continuous at x; that is , whenever (x_n) is G-cone convergent to x, $(f(x_n))$ is G-cone convergent to $f(x)$.*

Proof. Suppose that a function $f : X \longrightarrow X^1$ is G-cone continuous at a point $x \in X$. And suppose that (x_n) is G-cone convergent to x. We choose arbitrary $c \in E$ with $\theta << c$. From (x_n) is G-cone convergent to x, for any $c_1 \in E$ with $\theta << c_1$, there exists $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) << c_1$. And from a function $f : X \longrightarrow X^1$ is G-cone continuous at a point $x \in X$, for $c \in E$ with $\theta << c$, there exists $\theta << c_0$; and $G(x, x_n, x_m) << c_0$ implies $G^1(f(x), f(x_n), f(x_m)) << c$. This means $(f(x_n))$ is G-cone convergent to $f(x)$. Therefore f is G-cone sequently continuous at x.

Conversely, suppose that f is G-cone sequently continuous at x but f is not G-cone continuous at x. Then there is $c \in E$ with $\theta << c$ and for every $c_0 \in E$ with $\theta << c_0$, all $x, y, z \in X$; $G(x, y, z) << c_0$ implies $c << G^1(f(x), f(y), f(z))$. Then if we choose (x_n) is a sequence in X, for any $c_0 \in E$ with $\theta << c_0$, $x_n, x_m \in (x_n) \subset X$; $G(x, x_n, x_m) << c_0$. Especially if we choose for all $n, m \geq N$, $G(x, x_n, x_m) << c_0$ too. This means (x_n) is G-cone convergent to x. Since f is not G-cone continuous at x, $c << G^1(f(x), f(x_n), f(x_m))$. This means $(f(x_n))$ is not G-cone convergent to $f(x)$. This is a contradiction with f is G-cone sequently continuous at x. Therefore f is G-cone continuous at x. □

Definition 6. *A G-cone metric space (X, G) is called symmetric G-cone metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

Lemma 9. *Let (X, G) be a G-cone metric space, P be a normal cone with normal constant K. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proof. From Lemma 8, be G-cone continuous and be G-cone sequently continuous are equivalent in G-cone metric spaces. Hence, if G is jointly sequently continuous in all three of its variables, it is jointly continuous in all three of its variables. Suppose that (x_n),(y_n),(z_n) are sequences in X and (x_n) is G-cone convergent to x in X, (y_n) is G-cone convergent to y in X, (z_n) is G-cone convergent to z in X. For every $\varepsilon > 0$, choose $c \in E$ with $\theta << c$ and $\|c\| < \frac{\varepsilon}{6K+3}$. From (x_n) is G-cone convergent to x, (y_n) is G-cone convergent to y, (z_n) is G-cone convergent to z, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) << c$, $G(y, y_n, y_m) << c$, $G(z, z_n, z_m) << c$. We have

$$\begin{aligned} G(x,y,z) &\leq G(x,x_n,x_n) + G(x_n,y,z) \\ &= G(x,x_n,x_n) + G(y,x_n,z) \\ &\leq G(x,x_n,x_n) + G(y,y_n,y_n) + G(y_n,x_n,z) \\ &= G(x,x_n,x_n) + G(y,y_n,y_n) + G(z,y_n,x_n) \\ &\leq G(x,x_n,x_n) + G(y,y_n,y_n) + G(z,z_n,z_n) + G(z_n,y_n,x_n) \\ &= G(x,x_n,x_n) + G(y,y_n,y_n) + G(z,z_n,z_n) + G(x_n,y_n,z_n) \end{aligned}$$

Hence $G(x, y, z) \leq G(x, x_n, x_n) + G(y, y_n, y_n) + G(z, z_n, z_n) + G(x_n, y_n, z_n)$. And similarly we have $G(x_n, y_n, z_n) \leq G(x_n, x, x) + G(y_n, y, y) + G(z_n, z, z) + G(x, y, z)$. From $(G3)$,$G(x_n, x, x) = G(x, x, x_n) \leq G(x, x_n, x_m) << c$ and $G(x, x_n, x_n) = G(x_n, x_n, x) \leq G(x_n, x, x_m) = G(x, x_n, x_m) << c$.Then

$$\begin{aligned} G(x,x_n,x_n) &< < c \text{ and } G(x_n,x,x) << c, \\ G(y,y_n,y_n) &< < c \text{ and } G(y_n,y,y) << c, \\ G(z,z_n,z_n) &< < c \text{ and } G(z_n,z,z) << c. \end{aligned}$$

Hence $G(x, y, z) \leq 3c + G(x_n, y_n, z_n)$ and $G(x_n, y_n, z_n) \leq 3c + G(x, y, z)$. Then

$$\begin{aligned} \theta &\leq G(x_n,y_n,z_n) + 3c - G(x,y,z) \\ &\leq 3c + G(x,y,z) + 3c - G(x,y,z) = 6c. \end{aligned}$$

Since P is a normal cone with normal constant K,

$$\begin{aligned} \|G(x_n,y_n,z_n) - G(x,y,z)\| &\leq \|G(x_n,y_n,z_n) - G(x,y,z) + 3c - 3c\| \\ &\leq \|G(x_n,y_n,z_n) - G(x,y,z) + 3c\| + \|3c\| \\ &\leq 6K\|c\| + 3\|c\| = (6K+3)\|c\| < \varepsilon. \end{aligned}$$

Therefore $G(x_n, y_n, z_n) \longrightarrow G(x, y, z)$ $(n \longrightarrow \infty)$. □

Lemma 10. *Every G-cone metric space (X, G) will define a cone metric space (X, d_G) by*

$$d_G(x,y) = G(x,y,y) + G(y,x,x), \forall x,y \in X \tag{2.1}$$

Note that if (X, G) is symmetric G-cone metric space, then

$$d_G(x,y) = 2G(x,y,y), \forall x,y \in X \tag{2.2}$$

However, if (X,G) is not symmetric, then it holds by the G-cone metric properties that

$$\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 3G(x,y,y), \forall x,y \in X \tag{2.3}$$

and that in general these inequalities cannot be improved.

Proof. Suppose that (X,G) is G-cone metric space. And we choose $d_G(x,y) = G(x,y,y) + G(y,y,x)$, $\forall x,y \in X$. If $d_G(x,y)$ satisfies

$(d1)$, $(d2)$, $(d3)$, then it is a cone metric.

$(d1)$ For all $x,y \in X$ with $x \neq y$, $d_G(x,y) = G(x,y,y) + G(y,x,x)$. From $(G2)$, $\theta < G(x,y,y)$ and $\theta < G(y,x,x)$; for all $x,y \in X$ with $x \neq y$. Therefore $\theta < d_G(x,y)$; for all $x,y \in X$ with $x \neq y$.

From $(G1)$, $G(x,y,y) = \theta$ and $G(y,x,x) = \theta$ if and only if $x = y$. Hence $d_G(x,y) = \theta$ if and only if $x = y$.

$(d2)$ For all $x,y \in X$, $d_G(x,y) = G(x,y,y)+G(y,x,x) = G(y,x,x)+G(x,y,y) = d_G(y,x)$. Hence for all $x,y \in X$, $d_G(x,y) = d_G(y,x)$.

$(d3)$ For all $x,y,z \in X$,

$$\begin{aligned} d_G(x,y) &= G(x,y,y) + G(y,x,x) \\ &\le G(x,z,z) + G(z,y,y) + G(y,z,z) + G(z,x,x) \\ &= G(x,z,z) + G(z,x,x) + G(z,y,y) + G(y,z,z) \\ &= d_G(x,z) + d_G(z,y). \end{aligned}$$

From $(d1)$, $(d2)$ and $(d3)$, $d_G(x,y)$ is a cone metric and (X,d_G) is a cone metric space.

Now suppose that (X,G) is a symmetric G-cone metric space. Then for all $x,y \in X$,

$d_G(x,y) = G(x,y,y) + G(y,x,x) = 2G(x,y,y)$.

And suppose that if (X,G) is not symmetric, for all $x,y \in X$, $3G(x,y,y) < d_G(x,y) < \frac{3}{2}G(x,y,y)$.

Then $6G(x,y,y) < 2d_G(x,y) < 3G(x,y,y)$ hence from

$$\begin{aligned} 2d_G(x,y) &< 3G(x,y,y), \\ 2[G(x,y,y) + G(y,x,x)] &< 3G(x,y,y), \\ 2G(y,x,x) &< G(x,y,y). \end{aligned}$$

This is a contradiction, since, from $(G5)$, for all $x,y \in X$, $G(x,y,y) = G(y,x,y) \le G(y,x,x) + G(x,x,y) = 2G(y,x,x)$.

Similarly from

$$\begin{aligned} 6G(x,y,y) &< 2d_G(x,y), \\ 6G(x,y,y) &< 2[G(x,y,y) + G(y,x,x)], \\ 4G(x,y,y) &< 2G(y,x,x), \\ 2G(x,y,y) &< G(y,x,x). \end{aligned}$$

This is a contradiction too, since, from $(G5)$, for all $x,y \in X$, $G(y,x,x) = G(x,y,x) \le G(x,y,y) + G(y,y,x) = 2G(x,y,y)$.

These contradictions imply that for all $x,y \in X$, $\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 3G(x,y,y)$. □

Definition 7. *A G-cone metric space (X, G) is G-cone complete (or complete G-cone) metric space if every G-cone Cauchy sequence in (X, G) is G-cone convergent in (X, G).*

Lemma 11. *A G-cone metric space (X, G) is complete G-cone metric space if and only if (X, d_G) is complete cone metric space.*

Proof. Suppose that (X, G) is complete G-cone metric space and (x_n) is a Cauchy sequence in (X, d_G). We choose arbitrary $c \in E$ with $\theta << c$. From (x_n) is a Cauchy sequence in (X, d_G), there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $d_G(x_n, x_m) << \frac{c}{2}$. Then $d_G(x_n, x_m) = G(x_n, x_m, x_m) + G(x_m, x_n, x_n) << \frac{c}{2}$.

Hence for all $n, m, l \geq N$, $G(x_n, x_m, x_l) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x_l) << \frac{c}{2} + \frac{c}{2} = c$.

This means (x_n) is a G-cone Cauchy sequence in (X, G).Since (X, G) is complete G-cone metric space, (x_n) is G-cone convergent to x in (X, G). Then for any $c \in E$ with $\theta << c$, there is $N_1 \in \mathbf{N}$ such that for all $n, m \geq N_1, G(x_n, x_m, x) << \frac{c}{2}$. Hence from Lemma 3 for any $c \in E$ with $\theta << c$, there is $N_1 \in \mathbf{N}$ such that for all $n \geq N_1, d_G(x_n, x) = G(x_n, x, x) + G(x, x_n, x_n) << \frac{c}{2} + \frac{c}{2} = c$. This means (x_n) is convergent to x in (X, d_G). Therefore (X, d_G) is complete cone metric space.

Conversely suppose that (X, d_G) is complete cone metric space and (x_n) is a G-cone Cauchy sequence in (X, G). We choose arbitrary $c \in E$ with $\theta << c$. From (x_n) is a G-cone Cauchy sequence in (X, G), there is $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) << \frac{c}{2}$. Hence for all $n, m = l \geq N$, $d_G(x_n, x_m) = G(x_n, x_m, x_m) + G(x_m, x_n, x_n) << \frac{c}{2} + \frac{c}{2} = c$.

This means (x_n) is a Cauchy sequence in (X, d_G). Since (X, d_G) is complete cone metric space, (x_n) is convergent to x in (X, d_G).

Then for any $c \in E$ with $\theta << c$, there is $N_1 \in \mathbf{N}$ such that for all $n \geq N_1$, $d_G(x_n, x) << \frac{c}{2}$. Then $d_G(x_n, x) = G(x_n, x, x) + G(x, x_n, x_n) << \frac{c}{2}$.

Hence from $(G5)$, for any $c \in E$ with $\theta << c$, there is $N_1 \in \mathbf{N}$ such that for all $n, m \geq N_1$,

$G(x_n, x_m, x) \leq G(x_n, x_m, x_m) + G(x_m, x_m, x) \leq G(x_n, x, x) + G(x, x_m, x_m) + G(x_m, x_m, x) << \frac{c}{2} + \frac{c}{2} = c$.

This means (x_n) is G-cone convergent to x in (X, G). Therefore (X, G) is complete G-cone metric space. □

3. Main results

Here we start our work with the following theorem.

Theorem 1. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping satisfying one of the following conditions*

$$\begin{aligned} G(T(x), T(y), T(z)) \leq \{&aG(x, y, z) + bG(x, T(x), T(x)) \\ &+cG(y, T(y), T(y)) + dG(z, T(z), T(z))\} \end{aligned} \tag{3.1}$$

or

$$\begin{aligned} G(T(x), T(y), T(z)) \leq \{&aG(x, y, z) + bG(x, x, T(x)) \\ &+cG(y, y, T(y)) + dG(z, z, T(z))\} \end{aligned} \tag{3.2}$$

for all $x, y, z \in X$ where $0 \leq a + b + c + d < 1$, then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G-cone continuous at u.

Proof. Suppose that T satisfies condition (3.1), then for all $x, y \in X$, we have

$$G(Tx, Ty, Ty) \leq aG(x, y, y) + bG(x, Tx, Tx) + (c+d)\, G(y, Ty, Ty),$$

(3.3)

$$G(Ty, Tx, Tx) \leq aG(y, x, x) + bG(y, Ty, Ty) + (c+d)\, G(x, Tx, Tx)$$

Suppose that (X, G) is symmetric, then by definition of cone metric (X, d_G) and (2.2), we get for all $x, y \in X$,

$$\begin{aligned} d_G(Tx, Ty) &\leq a d_G(x, y) + \frac{c+d+b}{2} d_G(x, Tx) \\ &\quad + \frac{c+d+b}{2} d_G(y, Ty) \end{aligned} \tag{3.4}$$

In this line, since $0 \leq a+b+c+d < 1$, then the existence and uniqueness of the fixed point follows from well-known theorem in cone metric space (X, d_G).(see [9])

However, if (X, G) is not symmetric then by definition of cone metric (X, d_G) and (2.3), we get for all $x, y \in X$.

$$\begin{aligned} d_G(Tx, Ty) &\leq a d_G(x, y) + \frac{2(c+d+b)}{3} d_G(x, Tx) \\ &\quad + \frac{2(c+d+b)}{3} d_G(y, Ty) \end{aligned} \tag{3.5}$$

Then the cone metric condition gives no information about this map since $0 < a + \frac{2(c+d+b)}{3} + \frac{2(c+d+b)}{3}$ need not be less than 1. But this can be proved by G-cone metric.

Let $x_0 \in X$ be an arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$. By (3.1), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) \\ &\quad + (c+d)\, G(x_n, x_{n+1}, x_{n+1}), \end{aligned} \tag{3.6}$$

then

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1}, x_n, x_n). \tag{3.7}$$

Let $q = \frac{a+b}{1-(c+d)}$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$. So,

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n). \tag{3.8}$$

Continuing in the same argument, we will get

$$G(x_n, x_{n+1}, x_{n+1}) \leq q^n G(x_0, x_1, x_1). \tag{3.9}$$

Moreover, for all $n, m \in \mathbf{N}$; $n < m$, we have by rectangle inequality

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(q^n + q^{n+1} + \ldots + q^{m-1}\right) G(x_0, x_1, x_1) \\ &\leq \frac{q^n}{1-q} G(x_0, x_1, x_1), \end{aligned} \tag{3.10}$$

Since P is a normal cone with normal constant K,

$$\|G(x_n, x_m, x_m)\| \leq K \frac{q^n}{1-q} \|G(x_0, x_1, x_1)\|.$$

This means $\lim G(x_n, x_m, x_m) = \theta$, as $n, m \longrightarrow \infty$. Thus, from Lemma 7, (x_n) is G-cone Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that (x_n) is G-cone converge to u.

Suppose that $Tu \neq u$, then

$$\begin{aligned} G(x_n, T(u), T(u)) &\leq aG(x_{n-1}, u, u) + bG(x_{n-1}, x_n, x_n) \\ &\quad + (c+d) G(u, T(u), T(u)), \end{aligned} \tag{3.11}$$

Since P is a normal cone with normal constant K,

$$\begin{aligned} \|G(x_n, T(u), T(u))\| &\leq K(a \|G(x_{n-1}, u, u)\| + b \|G(x_{n-1}, x_n, x_n)\| \\ &\quad + (c+d) \|G(u, T(u), T(u))\| .) \end{aligned}$$

And using the fact that (x_n) is G-cone Cauchy sequence, (x_n) is G-cone converge to u, then

$$\|G(x_n, T(u), T(u))\| \leq (c+d) \|G(u, T(u), T(u))\| .$$

This contradiction implies that $u = Tu$. To prove uniqueness, suppose that $u \neq v$ such that $Tv = v$, then

$$\begin{aligned} G(u, v, v) &\leq aG(u, v, v) + bG(u, T(u), T(u)) \\ &\quad + (c+d) G(v, T(v), T(v)) \\ &= aG(u, v, v) \end{aligned} \tag{3.12}$$

Since P is a normal cone with normal constant K,$\|G(u, v, v)\| \leq Ka \|G(u, v, v)\|$. This contradiction implies that $u = v$. To show that T is G-cone continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim\limits_{n \longrightarrow \infty} (y_n) = u$. We can deduce that

$$\begin{aligned} G(u, T(y_n), T(y_n)) &\leq aG(u, y_n, y_n) + bG(u, T(u), T(u)) \\ &\quad + (c+d) G(y_n, T(y_n), T(y_n)) \\ &= aG(u, y_n, y_n) + (c+d) G(y_n, T(y_n), T(y_n)), \end{aligned} \tag{3.13}$$

and since $G(y_n, T(y_n), T(y_n)) \leq G(y_n, u, u) + G(u, T(y_n), T(y_n))$, we have that

$$\begin{aligned} G(u, T(y_n), T(y_n)) &\leq \frac{a}{1-(c+d)} G(u, y_n, y_n) \\ &\quad + \frac{c+d}{1-(c+d)} G(y_n, u, u). \end{aligned}$$

Since P is a normal cone with normal constant K,

$$\begin{aligned} \|G(u, T(y_n), T(y_n))\| &\leq K\left(\frac{a}{1-(c+d)} \|G(u, y_n, y_n)\| \right. \\ &\quad \left. + \frac{c+d}{1-(c+d)} \|G(y_n, u, u)\|\right). \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, from which we see that $G(u, T(y_n), T(y_n)) \longrightarrow \theta$ and so, by Lemma 8, $T(y_n) \longrightarrow u = Tu$. It is proved that T is G-cone continuous at u.

If T satisfies condition (3.2), then the argument is similar to that above. However, to show that the sequence (x_n) is G-cone Cauchy, we start with

$$\begin{aligned} G(x_n, x_n, x_{n+1}) &\leq aG(x_{n-1}, x_{n-1}, x_n) + (b+c) G(x_{n-1}, x_{n-1}, x_n) \\ &\quad + dG(x_n, x_n, x_{n+1}), \end{aligned} \tag{3.14}$$

then

$$G(x_n, x_n, x_{n+1}) \leq \frac{a+b+c}{1-d} G(x_{n-1}, x_{n-1}, x_n). \tag{3.15}$$

Let $q = \frac{a+b+c}{1-d}$, then $0 \leq q < 1$ since $0 \leq a+b+c+d < 1$.

Continuing in the same way, we find that

$$G(x_n, x_n, x_{n+1}) \leq q^n G(x_0, x_0, x_1). \tag{3.16}$$

Then for all $n, m \in \mathbf{N}$; $n < m$, we have by repeated use of the rectangle inequality $G(x_n, x_n, x_m) \leq \frac{q^n}{1-q} G(x_0, x_0, x_1)$. □

Corollary 1. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping satisfying one of the following conditions:*

$$\begin{aligned} G(T^m(x), T^m(y), T^m(z)) \quad \leq \quad & \{aG(x,y,z) + bG(x, T^m(x), T^m(x)) \\ & + cG(y, T^m(y), T^m(y)) + dG(z, T^m(z), T^m(z))\} \end{aligned} \tag{3.17}$$

or

$$\begin{aligned} G(T^m(x), T^m(y), T^m(z)) \quad \leq \quad & \{aG(x,y,z) + bG(x, x, T^m(x)) \\ & + cG(y, y, T^m(y)) + dG(z, z, T^m(z))\}, \end{aligned} \tag{3.18}$$

for all $x, y, z \in X$, where $0 \leq a+b+c+d < 1$. Then T has a unique fixed point (say u), and T^m is G-cone continuous at u.

Proof. From the previous theorem, we see that T^m has a unique fixed point (say u), that is, $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 2. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some constant $0 \leq k < 1$ and for all $x, y, z \in X$, there exists*

$$u \in \{G(x, T(x), T(x)), G(y, T(y), T(y)), G(z, T(z), T(z))\}$$

such that

$$G(T(x), T(y), T(z)) \leq ku \tag{3.19}$$

or there exists

$$w \in \{G(x, x, T(x)), G(y, y, T(y)), G(z, z, T(z))\}$$

such that

$$G(T(x), T(y), T(z)) \leq kw, \tag{3.20}$$

then T has a unique fixed point (say u), and T is G-cone continuous at u.

Proof. Suppose that T satisfies condition (3.19), then for some constant $0 \leq k < 1$ and for all $x, y \in X$, there exists

$$u \in \{G(x, Tx, Tx), G(y, Ty, Ty)\}$$

such that

$$\begin{aligned} G(Tx, Ty, Ty) &\leq ku, \\ G(Ty, Tx, Tx) &\leq ku. \end{aligned} \tag{3.21}$$

Suppose that (X,G) is symmetric, then by definition of the cone metric (X,d_G) and (2.2), there exists $v=2u\in\{d_G(x,Tx),d_G(y,Ty)\}$ such that $d_G(Tx,Ty)\leq kv,\forall x,y\in X$. Since $0\leq k<1$, then the existence and uniqueness of the fixed point follows from a theorem in cone metric space (X,d_G). (see [9])

However, if (X,G) is not symmetric, then by definition of the metric (X,d_G) and (2.3), there exists $v\in\{d_G(x,Tx),d_G(y,Ty)\}$ such that

$$d_G(Tx,Ty)\leq\frac{4k}{3}v,\forall x,y\in X. \tag{3.22}$$

The cone metric condition gives no information about this map since $\frac{4k}{3}$ need not be less than 1, but we will proof it by G-cone metric. Let $x_0\in X$ be an arbitrary point, and define the sequence (x_n) by $x_n=T^n(x_0)$. By (3.19), there exists $u\in\{G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1})\}$ such that

$$\begin{aligned} G(x_n,x_{n+1},x_{n+1}) &\leq ku \\ &= k(Gx_{n-1},x_n,x_n)\quad(\text{since } 0\leq k<1). \end{aligned} \tag{3.23}$$

Continuing in the same argument, we will find

$$G(x_n,x_{n+1},x_{n+1})\leq k^nG(x_0,x_1,x_1). \tag{3.24}$$

For all $n,m\in\mathbf{N}$; $n<m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n,x_m,x_m) &\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2}) \\ &\quad +G(x_{n+2},x_{n+3},x_{n+3})+...+G(x_{m-1},x_m,x_m) \\ &\leq \left(k^n+k^{n+1}+...+k^{m-1}\right)G(x_0,x_1,x_1) \\ &\leq \frac{k^n}{1-k}G(x_0,x_1,x_1). \end{aligned} \tag{3.25}$$

Since P is a normal cone with normal constant K,

$$\|G(x_n,x_m,x_m)\|\leq K\frac{k^n}{1-k}\|G(x_0,x_1,x_1)\|.$$

Then, $\lim G(x_n,x_m,x_m)=\theta$, as $n,m\longrightarrow\infty$, and thus, from Lemma 7, (x_n) is G-cone Cauchy sequence. Due to the completeness of (X,G), there exists $u\in X$ such that $(x_n)\longrightarrow u$.

Suppose that $Tu\neq u$, then there exists

$$t\in\{G(x_{n+1},x_{n+2},x_{n+2}),G(u,T(u),T(u))\}$$

such that $G(x_{n+1},T(u),T(u))\leq kt$. Since P is a normal cone with normal constant K,

$$\|G(x_{n+1},T(u),T(u))\|\leq Kk\|t\|$$

and by taking the limit as $n\longrightarrow\infty$, and using the fact that (x_n) is G-cone Cauchy sequence, (x_n) is G-cone converge to u, we get that $\|G(u,T(u),T(u))\|\leq Kk\|\theta\|=0$ or $\|G(u,T(u),T(u))\|\leq Kk\|G(u,T(u),T(u))\|$. Then these implie that $u=Tu$.

To prove uniqueness, suppose that $u\neq v$ such that $Tv=v$, then there exists $p\in\{G(v,v,v),G(u,u,u)\}$ such that $G(u,v,v)\leq kp=\theta$ which implies that $u=v$.

To show that T is G-cone continuous at u, let $(y_n)\subseteq X$ be a sequence such that $\lim\limits_{n\longrightarrow\infty}(y_n)=u$, then there exists $q\in\{G(u,T(u),T(u)),G(y_n,T(y_n),T(y_n))\}$ such that

$$G(u,T(y_n),T(y_n))\leq kq. \tag{3.26}$$

Then $G(u,T(y_n),T(y_n)) \leq kG(u,T(u),T(u)) = \theta$ implies that $T(y_n) = u$, $\forall n \in \mathbf{N}$. Or $G(u,T(y_n),T(y_n)) \leq kG(y_n,T(y_n),T(y_n))$ .But,

$$G(y_n,T(y_n),T(y_n)) \leq G(y_n,u,u) + G(u,T(y_n),T(y_n))$$

then $G(u,T(y_n),T(y_n)) \leq \left(\frac{k}{1-k}\right) G(y_n,u,u)$. Since P is a normal cone with normal constant K, $\|G(u,T(y_n),T(y_n))\| \leq K\left(\frac{k}{1-k}\right)\|G(y_n,u,u)\|$. Taking the limit as $n \longrightarrow \infty$, from which we see that $G(u,T(y_n),T(y_n)) \longrightarrow \theta$, and so by Lemma 8, $T(y_n) \longrightarrow u = Tu$. So these implie that T is G-cone continuous at u. ☐

Corollary 2. *Let (X,G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some $m \in \mathbf{N}$, some constant $0 \leq k < 1$ and for all $x,y,z \in X$, there exists*

$$u \in \{G(x,T^m(x),T^m(x)), G(y,T^m(y),T^m(y)), G(z,T^m(z),T^m(z))\}$$

such that

$$G(T^m(x),T^m(y),T^m(z)) \leq ku \tag{3.27}$$

or there exists

$$w \in \{G(x,x,T^m(x)), G(y,y,T^m(y)), G(z,z,T^m(z))\}$$

such that

$$G(T^m(x),T^m(y),T^m(z)) \leq kw, \tag{3.28}$$

then T has a unique fixed point (say u), and T^m is G-cone continuous at u.

Proof. We use the same argument as in Corollary 1. ☐

Theorem 3. *Let (X,G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some constant $0 \leq k < 1$ and for all $x,y \in X$, there exists*

$$u \in \{G(x,T(y),T(y)), G(y,T(x),T(x)), G(y,T(y),T(y))\}$$

such that

$$G(T(x),T(y),T(y)) \leq ku \tag{3.29}$$

or there exists

$$w \in \{G(x,x,T(y)), G(y,y,T(x)), G(y,y,T(y))\}$$

such that

$$G(T(x),T(y),T(y)) \leq kw, \tag{3.30}$$

then T has a unique fixed point (say u) and T is G-cone continuous at u.

Proof. Suppose that T satisfies condition (3.29) then for some constant $0 \leq k < 1$ and for all $x,y \in X$, there exists

$$\begin{aligned} u_1 &\in \{G(x,Ty,Ty), G(y,Tx,Tx), G(y,Ty,Ty)\}, \\ u_2 &\in \{G(x,Ty,Ty), G(y,Tx,Tx), G(x,Tx,Tx)\} \end{aligned}$$

such that

$$G(Tx,Ty,Ty) \leq ku_1,$$

(3.31)

$$G\left(Ty,Tx,Tx\right)\leq ku_{2}.$$

Suppose that (X,G) is symmetric, then by definition of the cone metric (X,d_G) and (2.2), there exists

$$v\in\left\{d_{G}\left(x,Ty\right),d_{G}\left(y,Tx\right),\frac{1}{2}d_{G}\left(x,Tx\right),\frac{1}{2}d_{G}\left(y,Ty\right)\right\}$$

such that

$$d_{G}\left(Tx,Ty\right)\leq kv,\forall x,y\in X \tag{3.32}$$

Since $0\leq k<1$, then the existence and uniqueness of the fixed point follows from a theorem in cone metric space (X,d_G). However, if (X,G) is not symmetric, then by definition of the metric (X,d_G) and (2.3), there exists

$$v\in\left\{d_{G}\left(x,Ty\right),d_{G}\left(y,Tx\right),\frac{1}{2}d_{G}\left(x,Tx\right),\frac{1}{2}d_{G}\left(y,Ty\right)\right\}$$

such that

$$d_{G}\left(Tx,Ty\right)\leq\frac{4k}{3}v,\forall x,y\in X. \tag{3.33}$$

The cone metric condition gives no information about this map since $\frac{4k}{3}$ need not be less than 1, but we will proof it by G-cone metric. Let $x_0\in X$ be an arbitrary point, and define the sequence (x_n) by $x_n=T^n\left(x_0\right)$. By 3.29, there exists

$$u_{3}\in\left\{G\left(x_{n-1},x_{n+1},x_{n+1}\right),G\left(x_{n},x_{n+1},x_{n+1}\right)\right\}$$

such that

$$\begin{aligned} G\left(x_{n},x_{n+1},x_{n+1}\right) &\leq ku_{3}\\ &= kG\left(x_{n-1},x_{n+1},x_{n+1}\right)\ \text{(since } 0\leq k<1). \end{aligned} \tag{3.34}$$

So,

$$G\left(x_{n},x_{n+1},x_{n+1}\right)\leq kG\left(x_{n-1},x_{n+1},x_{n+1}\right), \tag{3.35}$$

and using there exists

$$u_{4}\in\left\{G\left(x_{n-2},x_{n+1},x_{n+1}\right),G\left(x_{n},x_{n-1},x_{n-1}\right),G\left(x_{n},x_{n+1},x_{n+1}\right)\right\}$$

such that

$$G\left(x_{n-1},x_{n+1},x_{n+1}\right)\leq ku_{4}, \tag{3.36}$$

then,

$$G\left(x_{n},x_{n+1},x_{n+1}\right)\leq k^{2}u_{4}. \tag{3.37}$$

Continuing in this procedure, we will have

$$G\left(x_{n},x_{n+1},x_{n+1}\right)\leq k^{n}T_{n}, \tag{3.38}$$

where

$$T_{n}\in\left\{G\left(x_{i},x_{j},x_{j}\right);for\ all\ i,j\in\{0,1,...,n+1\}\right\}.$$

Then for all $n,m\in\mathbf{N}$; $n<m$, we have by rectangle inequality

$$\begin{aligned} G\left(x_{n},x_{m},x_{m}\right) &\leq G\left(x_{n},x_{n+1},x_{n+1}\right)+G\left(x_{n+1},x_{n+2},x_{n+2}\right)\\ &\quad +G\left(x_{n+2},x_{n+3},x_{n+3}\right)+...+G\left(x_{m-1},x_{m},x_{m}\right)\\ &\leq k^{n}T_{n}+k^{n+1}T_{n+1}+...+k^{m-1}T_{m-1} \end{aligned} \tag{3.39}$$

Since P is a normal cone with normal constant K,

$$\begin{aligned}\|G(x_n,x_m,x_m)\| &\leq K\left\|k^nT_n+k^{n+1}T_{n+1}+...+k^{m-1}T_{m-1}\right\| \\ &\leq K\left\{k^n\|T_n\|+k^{n+1}\|T_{n+1}\|+...+k^{m-1}\|T_{m-1}\|\right\},\end{aligned}$$

If we choose $T_0=\max\{\|T_i\|;for\ all\ i=n,...,m-1\}$ then

$$\begin{aligned}\|G(x_n,x_m,x_m)\| &\leq K\left(k^n+k^{n+1}+...+k^{m-1}\right)T_0 \\ &\leq K\frac{k^n}{1-k}T_0.\end{aligned}$$

Then, $\lim G(x_n,x_m,x_m)=\theta$, as $n,m\longrightarrow\infty$, and thus, from Lemma (1.11), (x_n) is G-cone Cauchy sequence. Since (X,G) is G-cone complete then there exists $u\in X$ such that (x_n) is G-cone converge to u. Suppose that $Tu\neq u$, then by (3.29), there exists

$$t\in\{G(x_{n-1},T(u),T(u)),G(u,x_{n+1},x_{n+1}),G(u,T(u),T(u))\}$$

such that

$$G(x_n,T(u),T(u))\leq kt. \tag{3.40}$$

Taking the limit as $n\longrightarrow\infty$, we get $G(u,T(u),T(u))=\theta$ implies $Tu=u$ or

$$G(u,T(u),T(u))\leq kG(u,T(u),T(u))$$

, this contradiction implies that $u=Tu$. To prove the uniqueness, suppose that $u\neq v$ such that $Tv=v$. So, by (3.29), there exists $t\in\{G(u,v,v),G(v,u,u)\}$ such that

$$G(u,v,v)\leq kt\Rightarrow G(u,v,v)\leq kG(v,u,u). \tag{3.41}$$

Again we will find $G(v,u,u)\leq kG(u,v,v)$, so

$$G(u,v,v)\leq k^2G(u,v,v), \tag{3.42}$$

since P is a normal cone with normal constant K,

$$\|G(u,v,v)\|\leq Kk^2\|G(u,v,v)\|$$

, since $0\leq k<1$, this implies that $u=v$. To show that T is G-cone continuous at u, let $(y_n)\subseteq X$ be a sequence such that $\lim\limits_{n\longrightarrow\infty}(y_n)=u$, then by (3.29), there exists

$$t\in\{G(u,T(y_n),T(y_n)),G(y_n,T(u),T(u)),G(y_n,T(y_n)T(y_n))\}$$

such that

$$G(u,T(y_n),T(y_n))\leq kt. \tag{3.43}$$

But,

$$G(y_n,T(y_n),T(y_n))\leq G(y_n,u,u)+G(u,T(y_n)T(y_n))$$

so, $G(u,T(y_n),T(y_n))\leq\frac{k}{1-k}G(y_n,u,u)$.

Since P is a normal cone with normal constant K,

$$\|G(u,T(y_n),T(y_n))\|\leq K\frac{k}{1-k}\|G(y_n,u,u)\|.$$

Taking the limit as $n\longrightarrow\infty$, from which we see that $G(u,T(y_n),T(y_n))\longrightarrow\theta$ and so, by Lemma (1.13), we have $T(y_n)\longrightarrow u=Tu$ which implies that T is G-cone continuous at u. □

Corollary 3. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some constant $0 \leq k < 1$ and for all $x, y, z \in X$, there exists*

$$u \in \left\{ \begin{array}{c} G(x, T(y), T(y)), G(x, T(z), T(z)), \\ G(y, T(x), T(x)), G(y, T(z), T(z)), \\ G(z, T(x), T(x)), G(z, T(y), T(y)) \end{array} \right\}$$

such that

$$G(T(x), T(y), T(z)) \leq ku, \tag{3.44}$$

there exists

$$w \in \left\{ \begin{array}{c} G(x, x, T(y)), G(x, x, T(z)), \\ G(y, y, T(x)), G(y, y, T(z)), \\ G(z, z, T(x)), G(z, z, T(y)) \end{array} \right\}$$

such that

$$G(T(x), T(y), T(z)) \leq kw. \tag{3.45}$$

Then T has a unique fixed point (say u) and T is G-cone continuous at u.

Proof. If we let $z = y$ in conditions (3.44) and (3.45), then they become conditions (3.29) and (3.30), respectively, in Theorem 3; so the proof follows from Theorem 3. □

Corollary 4. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some $m \in \mathbf{N}$, some constant $0 \leq k < 1$ and for all $x, y, z \in X$, there exists*

$$\begin{aligned} u &\in \left\{ \begin{array}{c} G(x, T^m(y), T^m(y)), G(x, T^m(z), T^m(z)), \\ G(y, T^m(x), T^m(x)), G(y, T^m(z), T^m(z)), \\ G(z, T^m(x), T^m(x)), G(z, T^m(y), T^m(y)) \end{array} \right\}, \\ w &\in \left\{ \begin{array}{c} G(x, x, T^m(y)), G(x, x, T^m(z)), \\ G(y, y, T^m(x)), G(y, y, T^m(z)), \\ G(z, z, T^m(x)), G(z, z, T^m(y)) \end{array} \right\} \end{aligned}$$

, $t_1 \in \{G(x, T^m(y), T^m(y)), G(y, T^m(x), T^m(x)), G(y, T^m(y), T^m(y))\}$ such that

$$\begin{aligned} G(T^m(x), T^m(y), T^m(z)) &\leq ku, \\ G(T^m(x), T^m(y), T^m(z)) &\leq kw, \\ G(T^m(x), T^m(y), T^m(y)) &\leq kt_1. \end{aligned} \tag{3.46}$$

or $t_2 \in \{G(x, x, T^m(y)), G(y, y, T^m(x)), G(y, y, T^m(y))\}$ such that

$$G(T^m(x), T^m(y), T^m(y)) \leq kt_2. \tag{3.47}$$

Then T has a unique fixed point (say u) and T^m is G-cone continuous at u.

Proof. The proof follows from Theorem 3, Corollary 3, and from an argument similar to that used in Corollary 1. □

Theorem 4. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping. If for some constant $0 \leq k < 1$ and for all $x, y \in X$, there exists*

$$u \in \{G(x, T(y), T(y)), G(y, T(x), T(x))\}$$

such that

$$G(T(x), T(y), T(y)) \leq ku \tag{3.48}$$

or there exists

$$w \in \{G(x, x, T(y)), G(y, y, T(x))\}$$

such that

$$G(T(x), T(y), T(y)) \leq kw, \tag{3.49}$$

then T has a unique fixed point (say u) and T is G-cone continuous at u.

Proof. Since whenever the mapping satisfies condition (3.48), or (3.49), then it satisfies condition (3.44), or (3.45), respectively, in Theorem 3. Then the proof follows from Theorem 3. □

Corollary 5. *Let (X, G) be a complete G-cone metric space, P be a normal cone with normal constant K and let $T : X \longrightarrow X$ be a mapping satisfying one of the following conditions :*

$$G(T(x), T(y), T(y)) \leq a\{G(x, T(y), T(y)) + G(y, T(x), T(x))\} \tag{3.50}$$

or

$$G(T(x), T(y), T(y)) \leq a\{G(x, x, T(y)) + G(y, y, T(x))\} \tag{3.51}$$

for all $x, y \in X$, where $a \in \left[0, \frac{1}{2}\right)$, then T has a unique fixed point (say u,i.e.,$Tu = u$), and T is G-cone continuous at u.

Proof. Suppose that T satisfies condition (3.50), then we have

$$\begin{gathered} G(Tx, Ty, Ty) \leq a\{G(y, Tx, Tx) + G(x, Ty, Ty)\}, \\ G(Ty, Tx, Tx) \leq a\{G(x, Ty, Ty) + G(y, Tx, Tx)\}, \end{gathered} \tag{3.52}$$

for all $x, y \in X$.

Suppose that (X, G) is symmetric, then by definition of the cone metric (X, d_G) and (2.2), we get

$$d_G(Tx, Ty) \leq a\{d_G(x, Ty) + d_G(y, Tx)\}, \forall x, y \in X. \tag{3.53}$$

Since $0 \leq 2a < 1$, then the existence and uniqueness of the fixed point follow from a theorem in cone metric space (X, d_G).

However, if (X, G) is not symmetric, then by definition of the metric (X, d_G) and (2.3), we have

$$d_G(Tx, Ty) \leq \frac{4a}{3} d_G(x, Ty) + \frac{4a}{3} d_G(y, Tx), \forall x, y \in X. \tag{3.54}$$

So, the cone metric space (X, d_G) gives no information about this map since $\frac{8a}{3}$ need not be less than 1. But this can be proved by G-cone metric.

Let $x_0 \in X$ be arbitrary point, and define the sequence (x_n) by $x_n = T^n(x_0)$, then by (3.50), we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq a\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} \\ &= aG(x_{n-1}, x_{n+1}, x_{n+1}). \end{aligned} \tag{3.55}$$

But

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + aG(x_n, x_{n+1}, x_{n+1}), \tag{3.56}$$

thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a}{1-a} G(x_{n-1}, x_n, x_n). \tag{3.57}$$

Let $k = \frac{a}{1-a}$, hence $0 \leq k < 1$ then continue in this procedure, we will get that

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1).$$

For all $n, m \in \mathbf{N}$; $n < m$, we have by rectangle inequality that

$$\begin{aligned} G(x_n, x_m, x_m) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \\ &\leq \left(k^n + k^{n+1} + \dots + k^{m-1}\right) G(x_0, x_1, x_1). \\ &\leq \frac{k^n}{1-k} G(x_0, x_1, x_1). \end{aligned} \tag{3.58}$$

Since P is a normal cone with normal constant K,

$$\|G(x_n, x_m, x_m)\| \leq K \frac{k^n}{1-k} \|G(x_0, x_1, x_1)\|.$$

Then,$\lim G(x_n, x_m, x_m) = \theta$, as $n, m \longrightarrow \infty$, and thus, from Lemma (1.11), (x_n) is G-cone Cauchy sequence. Due to the completeness of (X, G), there exists $u \in X$ such that $(x_n) \longrightarrow u$.

Suppose that $Tu \neq u$, then

$$G(x_n, T(u), T(u)) \leq a \{G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)\}.$$

Since P is a normal cone with normal constant K,

$$\|G(x_n, T(u), T(u))\| \leq Ka \|G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)\|$$

and by taking the limit as $n \longrightarrow \infty$, and using the fact that (x_n) is G-cone Cauchy sequence, (x_n) is G-cone converge to u, we get that

$$\|G(u, T(u), T(u))\| \leq Ka \|G(u, T(u), T(u))\|.$$

This contradiction implies that $u = Tu$.

To prove uniqueness, suppose that $u \neq v$ such that $Tv = v$, then

$$G(u, v, v) \leq a \{G(u, v, v) + G(v, u, u)\}$$

so

$$G(u, v, v) \leq \left(k = \frac{a}{1-a}\right) G(v, u, u), \tag{3.59}$$

again by the same argument, we can verify that $G(u, v, v) \leq k^2 G(u, v, v)$, since P is a normal cone with normal constant K,

$$\|G(u, v, v)\| \leq Kk^2 \|G(u, v, v)\|.$$

This contradiction implies that $u = v$.

To show that T is G-cone continuous at u, let $(y_n) \subseteq X$ be a sequence such that $\lim (y_n) = u$, then

$$G(u, T(y_n), T(y_n)) \leq a \{G(u, T(y_n), T(y_n)) + G(y_n, T(u), T(u))\}, \tag{3.60}$$

and so

$$G(u, T(y_n), T(y_n)) \leq \left(\frac{a}{1-a}\right) G(y_n, T(u), T(u)).$$

Since P is a normal cone with normal constant K,

$$\|G(u,T(y_n),T(y_n))\| \leq \left(\frac{a}{1-a}\right)\|G(y_n,T(u),T(u))\|.$$

Taking the limit as $n \longrightarrow \infty$, from which we see that $G(u,T(y_n),T(y_n)) \longrightarrow \theta$. By Lemma (8), we have $T(y_n) \longrightarrow u = Tu$ which implies that T is G-cone continuous at u. □

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Stability of an additive–quadratic functional equation in menger probabilistic normed spaces

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Abstract. In this paper, we establish the generalized Hyres–Ulam–Rassias stability of the mixed type additive and quadratic functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+2f(3x)-2f(x)$$

in menger probabilistic normed spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [30] in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*,d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon>0$, does there exist a $\delta>0$, such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\epsilon$ for all $x\in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [16] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f:E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\le\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T:E\to E'$ such that

$$\|f(x)-T(x)\|\le\delta$$

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for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. In 1978, Th. M. Rassias [26] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Z. Gajda [9] answered the question for the case $p > 1$, which was rased by Rassias. This new concept is known as Hyers–Ulam–Rassias stability of functional equations (see [1], [10, 15, 17] and [24, 25]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

is related to symmetric bi–additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi–additive function B such that $f(x) = B(x, x)$ for all x (see [1, 18]). The bi–additive function B is given by

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)) \tag{1.2}$$

Hyers–Ulam–Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \to B$, where A is normed space and B Banach space [29] (see also [2, 4] and [13]).
We mention here the papers [7, 8, 11] and [12] concerning the stability of the mixed type functional equations.

The generalized Hyers–Ulam–Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [6], [19], [20], [21], [22] and [23]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M. Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$, such that F is left–continuous and non–decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point–wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.1. ([27]). *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous t-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz $t-$norm). Recall (see [14]) that if T is a $t-$norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, $T^n_{i=1} x_i$ is defined recurrently by $T^1_{i=1} x_i = x_1$ and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n)$ for $n \ge 2$. $T^\infty_{i=n} x_i$ is defined as $T^\infty_{i=1} x_{n+i}$. It is known ([3], [14]) that for the Lukasiewicz $t-$norm the following implication holds:

$$\lim_{n\to\infty} (T_L)^\infty_{i=1} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty. \tag{1.3}$$

Definition 1.2. *A* Menger Probabilistic normed space *(briefly, PN–space) is a triple* (X, μ, T), *where* X *is a nonempty set,* T *is a continuous* $t-norm$, *and* μ *is a mapping from* X *into* D^+ *such that, the following conditions hold:*
$(PN1)$ $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$;
$(PN2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ *for all* $x \in X$, $\alpha \neq 0$;
$(PN3)$ $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ *for all* $x, y \in X$ *and* $t, s \geq 0$.

Clearly every Menger PN–space is probabilistic metric space having a metrizable uniformity on X if $\sup_{a<1} T(a,a) = 1$.

Definition 1.3. *Let* (X, μ, T) *be a Menger PN–space.*
(1) *A sequence* $\{x_n\}$ *in* X *is said to be* convergent *to* x *in* X *if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists positive integer* N *such that* $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ *whenever* $n \geq N$.
(2) *A sequence* $\{x_n\}$ *in* X *is called* Cauchy sequence *if, for every* $\epsilon > 0$ *and* $\lambda > 0$, *there exists positive integer* N *such that* $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ *whenever* $n \geq m \geq N$.
(3) *A Menger PN–space* (X, μ, T) *is said to be* complete *if and only if every Cauchy sequence in* X *is convergent to a point in* X.

Theorem 1.4. *If* (X, μ, T) *is a Menger PN–space and* $\{x_n\}$ *is a sequence such that* $x_n \to x$, *then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ *almost everywhere.*

Recently Shakeri and others [28] proved the stability of cubic functional equation on Menger PN-spaces. In this study, the stability of a mixed type additive–quadratic functional equation

$$f(3x+y) + f(3x-y) = f(x+y) + f(x-y) + 2f(3x) - 2f(x) \tag{1.4}$$

in the setting of Menger probabilistic normed spaces is established.

2. Main Results

Throughout this section, $(X, \nu, \mathbb{R})$ will be Menger PN–space and (Y, μ, T) will be a complete Menger PN–space. First we establish the stability of additive–quadratic functional equation (1.4) for even functions.

Theorem 2.1. *Let* $f : X \to Y$ *be an even function with* $f(0) = 0$ *for which there is* $\rho : X \times X \to D^+$ *(* $\rho(x,y)$ *is denoted by* $\rho_{x,y}$*) with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{2.1}$$

for all $x, y \in X$ *and all* $t > 0$ *and*

$$\begin{aligned}\lim_{n\to\infty} T_{i=1}^{\infty}(T[T(\rho_{\frac{2^{i+n-1}x}{4},\frac{2^{i+n-1}x}{4}}(2^{2(i+n-1)}t), \rho_{\frac{2^{i+n-1}x}{4},\frac{2^{i+n-1}x}{4}}(2^{2(i+n-1)}t))\\ ,T(\rho_{\frac{2^{i+n-1}x}{4},\frac{5.2^{i+n-1}x}{4}}(2^{2(i+n-1)}t), \rho_{\frac{2^{i+n-1}x}{4},\frac{-3.2^{i+n-1}x}{4}}(2^{2(i+n-1)}t))]) = 1\end{aligned} \tag{2.2}$$

and

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^{2n}t) = 1 \tag{2.3}$$

for all $x, y \in X$ *and all* $t > 0$, *then there exists a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}\mu_{Q(x)-f(x)}(t) \geq T_{i=1}^{\infty}(T[T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(2^{2(i-1)}t), \rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(2^{2(i-1)}t))\\ ,T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(2^{2(i-1)}t), \rho_{\frac{2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(2^{2(i-1)}t))]),\end{aligned} \tag{2.4}$$

for all $x \in X$ *and all* $t > 0$.

Proof. By replacing y by $x+y$ in (2.1), we get

$$\mu_{f(4x+y)+f(2x-y)-f(2x+y)-f(y)-2f(3x)+2f(x)}(t) \geq \rho_{x,x+y}(t) \tag{2.5}$$

for all $x, y \in X$. If we Replace y by $-y$ in (2.5), we get

$$\mu_{f(4x-y)+f(2x+y)-f(2x-y)-f(y)-2f(3x)+2f(x)}(t) \geq \rho_{x,x-y}(t) \tag{2.6}$$

for all $x, y \in X$. If we add (2.5) to (2.6), we have

$$\mu_{f(4x+y)+f(4x-y)-2f(y)-4f(3x)+4f(x)}(t) \geq T(\rho_{x,x+y}(\frac{t}{2}), \rho_{x,x-y}(\frac{t}{2})). \tag{2.7}$$

Letting $y=0$ in (2.7), we get the inequality

$$\mu_{2f(4x)-4f(3x)+4f(x)}(t) \geq T(\rho_{x,x}(\frac{t}{2}), \rho_{x,x}(\frac{t}{2})) \tag{2.8}$$

for all $x \in X$. Once again By letting $y = 4x$ in (2.7), we obtain the inequality

$$\mu_{f(8x)-2f(4x)-4f(3x)+4f(x)}(t) \geq T(\rho_{x,5x}(\frac{t}{2}), \rho_{x,-3x}(\frac{t}{2})) \tag{2.9}$$

for all $x \in X$. It follows from (2.8) and (2.9) that

$$\mu_{f(8x)-4f(4x)}(t) \geq T(T(\rho_{x,x}(\frac{t}{4}), \rho_{x,x}(\frac{t}{4})), T(\rho_{x,5x}(\frac{t}{4}), \rho_{x,-3x}(\frac{t}{4}))) \tag{2.10}$$

for all $x \in X$. If we replace x by $\frac{x}{4}$ in (2.10), we get

$$\mu_{f(2x)-4f(x)}(t) \geq T(T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4})), T(\rho_{\frac{x}{4},\frac{5x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{-3x}{4}}(\frac{t}{4}))) \tag{2.11}$$

for all $x \in X$. Let

$$\psi_{x,x}(t) = T(T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4})), T(\rho_{\frac{x}{4},\frac{5x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{-3x}{4}}(\frac{t}{4}))) \tag{2.12}$$

for all $x \in X$, then we obtain

$$\mu_{f(2x)-4f(x)}(t) \geq \psi_{x,x}(t) \tag{2.13}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{f(2x)}{2^2}-f(x)}(t) \geq \psi_{x,x}(2^2 t) \tag{2.14}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}}-\frac{f(2^k x)}{2^{2k}}}(t) \geq \psi_{2^k x,2^k x}(2^{2(k+1)}t) \tag{2.15}$$

for all $x \in X$ and all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}\mu_{\frac{f(2^n x)}{2^{2n}}-f(x)}(t) &\geq T_{k=0}^{n-1}(\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}}-\frac{f(2^k x)}{2^{2k}}}(t)) \geq T_{k=0}^{n-1}(\psi_{2^k x,2^k x}(2^{2(k+1)}t)) \\ &= T_{i=1}^{n}(\psi_{2^{i-1}x,2^{i-1}x}(2^{2i}t))\end{aligned} \tag{2.16}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$, we replace x with $2^m x$ in (2.16) to find that

$$\mu_{\frac{f(2^{n+m}x)}{2^{2(n+m)}}-\frac{f(2^m x)}{2^{2m}}}(t) \geq T_{i=1}^{n}(\psi_{2^{i+m-1}x,2^{i+m-1}x}(2^{2(i+m)}t)). \tag{2.17}$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$

for all $x \in X$. Now, we show that Q is a quadratic map. Replacing x, y with $2^n x$ and $2^n y$, respectively, in (2.1), it follows that

$$\mu_{\frac{f(3.2^n x+2^n y)}{2^{2n}}+\frac{f(3.2^n x-2^n y)}{2^{2n}}-\frac{f(2^n x+2^n y)}{2^{2n}}-\frac{f(2^n x-2^n y)}{2^{2n}}-2\frac{f(3.2^n x)}{2^{2n}}+2\frac{f(2^n x)}{2^{2n}}}(t) \geq \rho_{2^n x, 2^n y}(2^{2n} t). \tag{2.18}$$

Taking the limit as $n \to \infty$, we find that Q satisfies (1.4) for all $x, y \in X$. By evenness of Q it follows that $Q : X \to Y$ is quadratic (see [5]).

To prove (2.4), take the limit as $n \to \infty$ in (2.16) and by (2.12). Finally, to prove the uniqueness of the quadratic function Q subject to (2.4), let us assume that there exists a quadratic function Q' which satisfies (2.4). Since $Q(2^n x) = 2^{2n} Q(x)$ and $Q'(2^n x) = 2^{2n} Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.4) it follows that

$$\begin{aligned}\mu_{Q(x)-Q'(x)}(2t) &= \mu_{Q(2^n x)-Q'(2^n x)}(2^{2n+1} t)\\ &\geq T(\mu_{Q(2^n x)-f(2^n x)}(2^{2n} t), \mu_{f(2^n x)-Q'(2^n x)}(2^{2n} t))\\ &\geq T(T_{i=1}^{n}(\psi_{2^{i+n-1}x, 2^{i+n-1}x}(2^{2(i+n)} t)), T_{i=1}^{n}(\psi_{2^{i+n-1}x, 2^{i+n-1}x}(2^{2(i+n)} t)))\end{aligned} \tag{2.19}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (2.19), we find that $Q = Q'$. □

Theorem 2.2. *Let $f : X \to Y$ be an odd function with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{2.20}$$

for all $x, y \in X$ and all $t > 0$ and

$$\begin{aligned}\lim_{n\to\infty} T_{i=1}^{\infty}(T[T(\rho_{\frac{2^{i+n-1}x}{4}, \frac{2^{i+n-1}x}{4}}(2^{i+n-2}t), \rho_{\frac{2^{i+n-1}x}{4}, \frac{3.2^{i+n-1}x}{4}}(2^{i+n-2}t))\\ , T(\rho_{\frac{2^{i+n-1}x}{4}, \frac{2^{i+n-1}x}{4}}(2^{i+n-2}t), \rho_{\frac{2^{i+n-1}x}{4}, \frac{5.2^{i+n-1}x}{4}}(2^{i+n-2}t))]) = 1\end{aligned} \tag{2.21}$$

and

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^n t) = 1 \tag{2.22}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}\mu_{A(x)-f(x)}(t) \geq T_{i=1}^{\infty}(T[T(\rho_{\frac{2^{i-1}x}{4}, \frac{2^{i-1}x}{4}}(2^{i-2}t), \rho_{\frac{2^{i-1}x}{4}, \frac{3.2^{i-1}x}{4}}(2^{i-2}t))\\ , T(\rho_{\frac{2^{i-1}x}{4}, \frac{2^{i-1}x}{4}}(2^{i-2}t), \rho_{\frac{2^{i-1}x}{4}, \frac{5.2^{i-1}x}{4}}(2^{i-2}t))]),\end{aligned} \tag{2.23}$$

for all $x \in X$ and all $t > 0$.

Proof. By letting $y = x$ in (2.20), we get

$$\mu_{f(4x)-2f(3x)+2f(x)}(t) \geq \rho_{x,x}(t) \tag{2.24}$$

for all $x \in X$. If we let $y = 3x$ in (2.20), we get by the oddness of f,

$$\mu_{f(6x)-2f(3x)-f(4x)+2f(x)+f(2x)}(t) \geq \rho_{x,3x}(t) \tag{2.25}$$

for all $x \in X$. It follows from (2.24) and (2.25) that

$$\mu_{f(6x)-2f(4x)+f(2x)}(t) \geq T(\rho_{x,x}(\frac{t}{2}), \rho_{x,3x}(\frac{t}{2})) \tag{2.26}$$

for all $x \in X$. Once again, by letting $y = 5x$ in (2.20), we get by the oddness of f,

$$\mu_{f(8x)-f(2x)-f(6x)+f(4x)-2f(3x)+2f(x)}(t) \geq \rho_{x,5x}(t) \tag{2.27}$$

for all $x \in X$. By (2.24) and using (2.26) and (2.27), we obtain

$$\mu_{f(8x)-2f(4x)}(t) \geq T[T(\rho_{x,x}(\frac{t}{4}), \rho_{x,3x}(\frac{t}{4})), T(\rho_{x,x}(\frac{t}{4}), \rho_{x,5x}(\frac{t}{4}))] \tag{2.28}$$

for all $x \in X$. If we replace x by $\frac{x}{4}$ in (2.28), we get that

$$\mu_{f(2x)-2f(x)}(t) \geq T[T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{3x}{4}}(\frac{t}{4})), T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{5x}{4}}(\frac{t}{4}))] \tag{2.29}$$

for all $x \in X$. Let

$$\phi_{x,x}(t) = T[T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{3x}{4}}(\frac{t}{4})), T(\rho_{\frac{x}{4},\frac{x}{4}}(\frac{t}{4}), \rho_{\frac{x}{4},\frac{5x}{4}}(\frac{t}{4}))] \tag{2.30}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{f(2x)-2f(x)}(t) \geq \phi_{x,x}(t) \tag{2.31}$$

for all $x \in X$ and $t > 0$. Thus, we have

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq \phi_{x,x}(2t) \tag{2.32}$$

for all $x \in X$. This implies that

$$\mu_{\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}}(t) \geq \phi_{2^k x,2^k x}(2^{k+1}t) \tag{2.33}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Thus we have

$$\begin{aligned} \mu_{\frac{f(2^n x)}{2^n}-f(x)}(t) \geq T_{k=0}^{n-1}(\mu_{\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}}(t)) &\geq T_{k=0}^{n-1}(\phi_{2^k x,2^k x}(2^{k+1}t)) \\ &= T_{i=1}^{n}(\phi_{2^{i-1}x,2^{i-1}x}(2^i t)) \end{aligned} \tag{2.34}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^n}\}$, we replace x with $2^m x$ in (2.34) to find that

$$\mu_{\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^m x)}{2^m}}(t) \geq T_{i=1}^{n}(\phi_{2^{i+m-1}x,2^{i+m-1}x}(2^{i+m}t)). \tag{2.35}$$

Since the right hand side of the inequality tends to 1 as m and n tend to infinity, the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence. Therefore, we may define $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in X$. Now, we show that A is a additive map. Replacing x, y with $2^n x$ and $2^n y$ respectively in (2.20), it follows that

$$\mu_{\frac{f(3.2^n x+2^n y)}{2^n}+\frac{f(3.2^n x-2^n y)}{2^n}-\frac{f(2^n x+2^n y)}{2^n}-\frac{f(2^n x-2^n y)}{2^n}-2\frac{f(3.2^n x)}{2^n}+2\frac{f(2^n x)}{2^n}}(t) \geq \rho_{2^n x,2^n y}(2^n t). \tag{2.36}$$

Taking the limit as $n \to \infty$, we find that A satisfies (1.4) for all $x, y \in X$. Therefore the mapping $A : X \to Y$ is additive (see [5]). To prove (2.23), take the limit as $n \to \infty$ in (2.34) and by (2.30). Finally, to prove the uniqueness of the additive function A subject to (2.23), let us assume that there exists a additive function A' which satisfies (2.23). Since $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (2.23) it follows that

$$\begin{aligned} \mu_{A(x)-A'(x)}(2t) &= \mu_{A(2^n x)-A'(2^n x)}(2^{n+1}t) \\ &\geq T(\mu_{A(2^n x)-f(2^n x)}(2^n t), \mu_{f(2^n x)-A'(2^n x)}(2^n t)) \\ &\geq T(T_{i=1}^{n}(\phi_{2^{i+n-1}x,2^{i+n-1}x}(2^{i+n}t)), T_{i=1}^{n}(\phi_{2^{i+n-1}x,2^{i+n-1}x}(2^{i+n}t))) \end{aligned} \tag{2.37}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (2.37), we find that $A = A'$. □

Theorem 2.3. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{2.38}$$

for all $x, y \in X$ and all $t > 0$ and

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}(T[T(T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4})))]) = 1\\
&= \lim_{n\to\infty} T_{i=1}^{\infty}T_{i=1}^{\infty}(T[T(T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^{i}t}{4})))])
\end{aligned} \tag{2.39}$$

and

$$\lim_{n\to\infty} T(\rho_{2^n x,2^n y}(2^n t), \rho_{2^n x,2^n y}(2^n t)) = 1 = \lim_{n\to\infty} T(\rho_{2^n x,2^n y}(2^{2n} t), \rho_{2^n x,2^n y}(2^{2n} t)) \tag{2.40}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
\mu_{f(x)-A(x)-Q(x)}(t) \geq T\{&T_{i=1}^{\infty}(T[T(T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4})))])\\
&, T_{i=1}^{\infty}(T[T(T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{i}t}{4}))\\
&, T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^{i}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^{i}t}{4})))])\}
\end{aligned} \tag{2.41}$$

for all $x \in X$ and all $t > 0$.

Proof. Let

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\mu_{f_e(3x+y)+f_e(3x-y)-f_e(x+y)-f_e(x-y)-2f_e(3x)+2f_e(x)}(t) \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t)) \tag{2.42}$$

for all $x, y \in X$. Hence, in view of Theorem 2.1, there exists a unique quadratic function $Q : X \to Y$ such that

$$\begin{aligned}\mu_{f_e(x)-Q(x)}(t) \geq T_{i=1}^{\infty}(T[T(T(&\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^{2(i-1)}t}{4}))))]).\end{aligned} \tag{2.43}$$

Let

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\mu_{f_o(3x+y)+f_o(3x-y)-f_o(x+y)-f_o(x-y)-2f_o(3x)+2f_o(x)}(t) \geq T(\rho_{x,y}(t), \rho_{-x,-y}(t)) \tag{2.44}$$

for all $x, y \in X$. From Theorem 2.2, it follows that there exist a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}\mu_{f_o(x)-A(x)}(t) \geq T_{i=1}^{\infty}(T[T(T(&\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^i t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^i t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{3.2^{i-1}x}{4}}(\frac{2^i t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-3.2^{i-1}x}{4}}(\frac{2^i t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{2^{i-1}x}{4}}(\frac{2^i t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-2^{i-1}x}{4}}(\frac{2^i t}{4}))\\ &,T(\rho_{\frac{2^{i-1}x}{4},\frac{5.2^{i-1}x}{4}}(\frac{2^i t}{4}), \rho_{\frac{-2^{i-1}x}{4},\frac{-5.2^{i-1}x}{4}}(\frac{2^i t}{4}))))]).\end{aligned} \tag{2.45}$$

Now it is obvious that (2.41) holds true for all $x \in A$, and the proof of theorem is complete. □

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ON NEW CLASSES OF DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

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Abstract. In this paper we introduce new double sequence spaces via Orlicz function and examine some properties of the resulting these spaces.

1. INTRODUCTION

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

Recall in [11] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle [16] .An Orlicz function M is said to satisfy $\Delta_2 - condition$ for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_i) : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) < \infty, \text{ for some } r > 0 \right\},$$

which is a Banach space normed by

$$\|(x_i)\| = \inf \left\{ r > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{r}\right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta [5] , Esi [1 − 2] , Esi and Et [3] , Parashar and Choudhary [12] and many others.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{ij})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > n$, [4] . We shall write more briefly as "$P-convergent$".

The initial works on double sequences is found in Bromwich [14] . Later on it was studied by Hardy [8] , Moricz [6] , Moricz and Rhoades [7] and many others.

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Hardy [8] introduced the notion of regular convergence for double sequences. The concept of paranormed sequences was studied by Nakano [9] and Simons [13] at the initial stage. Later on it was studied by many others.

The double sequence $x = (x_{ij})$ is bounded if there exists a positive number M such that $|x_{ij}| < M$ for all $i, j \in \mathbb{N}$. Let l_∞^2 the space of all bounded double such that

$$\|x_{ij}\|_{(\infty,2)} = \sup_{i,j} |x_{ij}| < \infty.$$

Throughout the paper, $w^2(X)$ denotes the spaces of all double sequences in X, where (X, q) denotes a seminormed space, seminormed by q. The zero double sequence is denoted by θ in X.

2.*DEFINITIONS AND BACKGROUND*

Let P_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements and $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \le (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The sequence space $m(\phi)$ introduced by Sargent [15] is defined as follows:

$$m(\phi) = \left\{ x = (x_k) : \ \|x_k\|_{m(\phi)} = \sup_{s \ge 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{i \in \sigma} |x_i| < \infty \right\}$$

and studied some of its properties and obtained its relationship with the space l_p.

A double sequence space E is said to be solid or normal if $(\alpha_{ij} x_{ij}) \in E$, whenever $(x_{ij}) \in E$ for all double sequences (α_{ij}) of scalars such that $|\alpha_{ij}| \le 1$ for all $i, j \in \mathbb{N}$.

A double sequence space E is said to be symmetric if $(x_{ij}) \in E$ implies $\left(x_{\pi(i)\pi(j)}\right) \in E$, where π is a permutation of the elements of $\mathbb{N}$.

Let $K = \{(i_n, j_k) : \ n, k \in \mathbb{N}; \ i_1 < i_2 < i_3 < .... \text{ and } j_1 < j_2 < j_3 < ...\} \subset \mathbb{N} \times \mathbb{N}$ and E be a double sequence space. A $K-$*step space* of E is a sequence space

$$\lambda_K^E = \{(x_{i_n j_k}) : \ (x_{ij}) \in E\}.$$

A canonical pre-image of a sequence $(x_{i_n j_k}) \in E$ is a sequence $(y_{ij}) \in E$ defined as follows:

$$y_{ij} = \{ \begin{array}{c} x_{ij}, \text{ if } (i,j) \in K \\ 0, \quad \text{otherwise} \end{array} .$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A double sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.

Lemma. A double sequence space E is solid implies E is monotone.

Let P_{st} denotes the class of all subsets of $\mathbb{N} \times \mathbb{N}$, those do not contain more that $s \times t$ elements. Throughout the paper $\{\phi_{n,m}\}$ represent a non-decreasing double sequence of real numbers such that $n\phi_{n+1,m} \le (n+1)\phi_{n,m}$ and $m\phi_{n,m+1} \le (m+1)\phi_{n,m}$. In this paper we introduce the following double sequence spaces: Let M be an Orlicz function and a $p = (p_{ij})$ be a bounded double sequence of positive real numbers such that $0 < H_o = \inf_{i,j} p_{ij} \leq p_{ij} \leq \sup_{i,j} p_{ij} = H < \infty$, then

$$l_{\infty}^{2}(M,q,p) = \left\{ x = (x_{ij}) \in w^{2}(X) : \sup_{i,j} \left[M\left(q\left(\frac{x_{ij}}{r} \right) \right) \right]^{p_{ij}} < \infty, \text{ for some } r > 0 \right\},$$

$$l_{p}^{2}(M,q) = \left\{ x = (x_{ij}) \in w^{2}(X) : \sum_{i,j=1,1}^{\infty} \left[M\left(q\left(\frac{x_{ij}}{r} \right) \right) \right]^{p_{ij}} < \infty, \text{ for some } r > 0 \right\},$$

$$m^{2}(M,\phi,q,p) = \left\{ x = (x_{ij}) \in w^{2}(X) : \sup_{s,t \geq 1, \sigma \in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{x_{ij}}{r} \right) \right) \right]^{p_{ij}} < \infty, \text{ for some } r > 0 \right\}.$$

The following inequality will be used throughout the paper

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq \max\left(1, 2^{H-1}\right) \left(|a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}} \right)$$

where a_{ij} and b_{ij} are complex numbers and $H = \sup_{i,j} p_{ij} < \infty$.

It is easy to see that the double sequence space $l_{p}^{2}(M,q)$ is a seminormed space, seminormed by

$$g\left((x_{ij})\right) = \inf \left\{ r^{\frac{p_{ij}}{J}} > 0 : \sum_{i,j=1,1}^{\infty} M\left(q\left(\frac{x_{ij}}{r} \right) \right) \leq 1 \right\}, \text{ where } J = max\left(1, 2^{H-1}\right)$$

3.*MAIN RESULTS*

In this section we prove some results involving the double sequence spaces $m^{2}(M,\phi,q,p)$, $l_{p}^{2}(M,q)$ and $l_{\infty}^{2}(M,q,p)$.

Theorem 3.1. $m^{2}(M,\phi,q,p)$ and $l_{\infty}^{2}(M,q,p)$ are linear spaces.

Proof. Let $(x_{ij}), (y_{ij}) \in m^{2}(M,\phi,q,p)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists positive numbers r_1 and r_2 such that

$$\sup_{s,t \geq 1, \sigma \in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{x_{ij}}{r_1} \right) \right) \right]^{p_{ij}} < \infty$$

and

$$\sup_{s,t \geq 1, \sigma \in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{y_{ij}}{r_2} \right) \right) \right]^{p_{ij}} < \infty.$$

Let $r_3 = \max\left(2|\alpha| r_1, 2|\beta| r_2\right)$. Since M is non-decreasing convex function and q is a seminorm, we have

$$\sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{\alpha x_{ij} + \beta y_{ij}}{r_3} \right) \right) \right]^{p_{ij}}$$

$$\leq \sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{\alpha x_{ij}}{r_3} \right) + q\left(\frac{\beta y_{ij}}{r_3} \right) \right) \right]^{p_{ij}}$$

$$\leq max\left(1, 2^{H-1}\right) \sum_{(i,j) \in \sigma} \left[M\left(q\left(\frac{x_{ij}}{r_1} \right) \right) \right]^{p_{ij}} + max\left(1, 2^{H-1}\right) \left[M\left(q\left(\frac{y_{ij}}{r_2} \right) \right) \right]^{p_{ij}}.$$

So,

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M\left(q\left(\frac{\alpha x_{ij}+\beta y_{ij}}{r_3}\right)\right)\right]^{p_{ij}}$$

$$\leq \sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M\left(q\left(\frac{x_{ij}}{r_1}\right)\right)\right]^{p_{ij}}+\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M\left(q\left(\frac{y_{ij}}{r_2}\right)\right)\right]^{p_{ij}}$$
$$<\infty,$$

therefore $(\alpha x_{ij}+\beta y_{ij})\in m^2(M,\phi,q,p)$. Hence $m^2(M,\phi,q,p)$ is a linear space.

The proof for the case $l_\infty^2(M,q,p)$ is a routine work in view of the above proof.

Theorem 3.2. The space $m^2(M,\phi,q,p)$ is a seminormed space, seminormed by

$$h((x_{ij}))=\inf\left\{r^{\frac{p_{ij}}{J}}>0:\ \sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}}{r}\right)\right)\leq 1\right\},\ \text{where } J=max\left(1,2^{H-1}\right).$$

Proof. Clearly $h((x_{ij}))\geq 0$ for all $(x_{ij})\in m^2(M,\phi,q,p)$ and $h(\theta)=0$. Let $r_1>0$ and $r_2>0$ be such that

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}}{r_1}\right)\right)\leq 1$$

and

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{y_{ij}}{r_2}\right)\right)\leq 1.$$

Let $r=r_1+r_2$. Then we have

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}+y_{ij}}{r}\right)\right)$$

$$=\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}+y_{ij}}{r_1+r_2}\right)\right)$$

$$\leq \sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left\{\frac{r_1}{r_1+r_2}M\left(q\left(\frac{x_{ij}}{r_1}\right)\right)+\frac{r_2}{r_1+r_2}M\left(q\left(\frac{y_{ij}}{r_2}\right)\right)\right\}$$

$$\leq\left(\frac{r_1}{r_1+r_2}\right)\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}}{r_1}\right)\right)$$
$$+\left(\frac{r_2}{r_1+r_2}\right)\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{y_{ij}}{r_2}\right)\right)$$
$$\leq 1.$$

Since the $r's$ are nonnegative, so we have

$$h((x_{ij})+(y_{ij}))=\inf\left\{r^{\frac{p_{ij}}{J}}>0:\ \sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}M\left(q\left(\frac{x_{ij}+y_{ij}}{r}\right)\right)\leq 1\right\}$$

$$\leq \inf\left\{ r_1^{\frac{p_{ij}}{J}} > 0 : \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} M\left(q\left(\frac{x_{ij}}{r_1}\right)\right) \leq 1 \right\}$$

$$+ \inf\left\{ r_2^{\frac{p_{ij}}{J}} > 0 : \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} M\left(q\left(\frac{x_{ij}}{r_2}\right)\right) \leq 1 \right\}$$

$$= h\left((x_{ij})\right) + h\left((y_{ij})\right).$$

Next for $\lambda \in \mathbb{C}$, without loss of generality, let $\lambda \neq 0$, then

$$h\left((\lambda x_{ij})\right) = \inf\left\{ r^{\frac{p_{ij}}{J}} > 0 : \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} M\left(q\left(\frac{\lambda x_{ij}}{r}\right)\right) \leq 1 \right\}$$

$$= \inf\left\{ \left(|\lambda|\,\rho\right)^{\frac{p_{ij}}{J}} > 0 : \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} M\left(q\left(\frac{x_{ij}}{\rho}\right)\right) \leq 1 \right\}, \text{where } \rho = \frac{r}{|\lambda|}$$

Hence we get

$$h\left((\lambda x_{ij})\right) \leq \max\left(1, |\lambda|\right) h\left((x_{ij})\right).$$

This completes the proof of the theorem.

The proof of the following result is a consequence of the above theorem.

Proposition 3.3. The double sequence space $l_\infty^2\left(M, q, p\right)$ is a seminormed space, seminormed by

$$f\left((x_{ij})\right) = \inf\left\{ r^{\frac{p_{ij}}{J}} > 0 : \sup_{i,j} M\left(q\left(\frac{x_{ij}}{r}\right)\right) \leq 1 \right\}, \text{ where } J = max\left(1, 2^{H-1}\right)$$

Theorem 3.4. $m^2\left(M, \phi, q, p\right) \subset m^2\left(M, \psi, q, p\right)$ if and only if $\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}} < \infty$.

Proof. Let $\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}} < \infty$ and $(x_{ij}) \in m^2\left(M, \phi, q, p\right)$. Then

$$\sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty, \text{ for some } r > 0.$$

So,

$$\sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\psi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} \leq \left(\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}}\right) \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

Therefore $(x_{ij}) \in m^2\left(M, \psi, q, p\right)$. Hence $m^2\left(M, \phi, q, p\right) \subset m^2\left(M, \psi, q, p\right)$.

Conversely, let $m^2\left(M, \phi, q, p\right) \subset m^2\left(M, \psi, q, p\right)$. Suppose that $\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}} = \infty$. Then there exists a sequence of natural numbers (s_{ij}) such that $P\!-\!\lim_{i,j} \frac{\phi_{s_i,t_j}}{\psi_{s_i,t_j}} = \infty$. Let $(x_{ij}) \in m^2\left(M, \phi, q, p\right)$. Then there exists $r > 0$ such that

$$\sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

Now, we have

$$\sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\psi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} \geq \left(\sup_{i,j} \frac{\phi_{s_i,t_j}}{\psi_{s_i,t_j}}\right) \sup_{i,j\geq 1,\sigma\in P_{s_i t_j}} \frac{1}{\phi_{s_i,t_j}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} = \infty.$$

Therefore $(x_{ij}) \notin m^2\left(M, \psi, q, p\right)$. This is a contradiction. Hence $\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}} < \infty$.

The following result is a consequence of Theorem 3.4.

Corollary 3.5. Let M be an Orlicz function. Then $m^2(M,\phi,q,p) = m^2(M,\psi,q,p)$ if and only if $\sup_{s,t}\frac{\phi_{s,t}}{\psi_{s,t}} < \infty$ and $\sup_{s,t}\frac{\psi_{s,t}}{\phi_{s,t}} < \infty$ for all $s,t = 1,2,....$

Theorem 3.6. Let M, M_1, M_2 be Orlicz functions satisfying Δ_2-*condition*. Then

(i) $m^2(M_1,\phi,q,p) \subset m^2(MoM_1,\phi,q,p)$,

(ii) $m^2(M_1,\phi,q,p) \cap m^2(M_2,\phi,q,p) \subset m^2(M_1+M_2,\phi,q,p)$.

Proof. **(i)** Let $(x_{ij}) \in m^2(M_1,\phi,q,p)$. Then there exists $r > 0$ such that

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M_1\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

Let $0 < \varepsilon < 1$ and δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t < \delta$. Let $y_{ij} = M_1\left(q\left(\frac{x_{ij}}{r}\right)\right)$ and for any $\sigma \in P_{st}$, let

$$\sum_{(i,j)\in\sigma}[M(y_{ij})]^{p_{ij}} = \sum_{1}[M(y_{ij})]^{p_{ij}} + \sum_{2}[M(y_{ij})]^{p_{ij}}$$

where the first summation is over $y_{ij} \leq \delta$ and the second is over $y_{ij} > \delta$. By the remark we have

(3.1)
$$\sum_{1}M(y_{ij}) \leq \max\left(1,[M(1)]^H\right)\sum_{1}(y_{ij})^{p_{ij}} \leq \max\left(1,[M(2)]^H\right)\sum_{1}(y_{ij})^{p_{ij}}$$

For $y_{ij} > \delta$

$$y_{ij} < y_{ij}\delta^{-1} \leq 1 + y_{ij}\delta^{-1},$$

since M is non-decreasing and convex, so

$$M(y_{ij}) < M\left(1+y_{ij}\delta^{-1}\right) < \frac{1}{2}M(2) + \frac{1}{2}M\left(2y_{ij}\delta^{-1}\right).$$

Since M satisfies $\Delta_2 - $*condition*, so

$$M(y_{ij}) < \frac{K}{2}y_{ij}\delta^{-1}M(2) + \frac{K}{2}y_{ij}\delta^{-1}M(2) = Ky_{ij}\delta^{-1}M(2).$$

Hence,

$$\sum_{2}[M(y_{ij})]^{p_{ij}} \leq \max\left(1,\left[K\delta^{-1}M(2)\right]^H\right)\sum_{2}(y_{ij})^{p_{ij}}. \tag{3.2}$$

By (3.1) and (3.2) we have $(x_{ij}) \in m^2(MoM_1,\phi,q,p)$. Thus $m^2(M_1,\phi,q,p) \subset m^2(MoM_1,\phi,q,p)$.

(ii) Let $(x_{ij}) \in m^2(M_1,\phi,q,p) \cap m^2(M_2,\phi,q,p)$. Then there exists $r > 0$ such that

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M_1\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty$$

and

$$\sup_{s,t\geq 1,\sigma\in P_{st}}\frac{1}{\phi_{s,t}}\sum_{(i,j)\in\sigma}\left[M_2\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

The rest of the proof follows from the equality

$$\sum_{(i,j)\in\sigma}\left[(M_1+M_2)\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}}$$

$$\leq \max\left(1, 2^{H-1}\right) \sum_{(i,j)\in\sigma} \left[M_1\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} + \max\left(1, 2^{H-1}\right) \sum_{(i,j)\in\sigma} \left[M_2\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}}.$$

This completes the proof.

Taking $M_1(x) = x$ in above theorem, we have the following result.

Corollary 3.7. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^2(\phi, q, p) \subset m^2(M, \phi, q, p)$.

From Theorem 3.4. and Corollary 3.7., we have:

Corollary 3.8. Let M be an Orlicz function satisfying $\Delta_2 - condition$, then $m^2(\phi, q, p) \subset m^2(M, \psi, q, p)$ if and only if $\sup_{s,t} \frac{\phi_{s,t}}{\psi_{s,t}} < \infty$.

Theorem 3.9. The double space $m^2(M, \phi, q, p)$ is solid and symmetric.

Proof. Let $(x_{ij}) \in m^2(M, \phi, q, p)$. Then

$$\sup_{s,t\geq 1, \sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty. \tag{3.3}$$

Let (λ_{ij}) be a double sequence of scalars with $|\lambda_{ij}| \leq 1$ for all $i, j \in \mathbb{N}$. Then the result follows from (3.3) and the following inequality

$$\sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{\lambda_{ij}x_{ij}}{r}\right)\right)\right]^{p_{ij}} \leq \sum_{(i,j)\in\sigma} \left[|\lambda_{ij}| M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} \qquad \text{(by the Remark)}$$

$$\leq \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}}.$$

The symmetricity of the space follows from the definition of the double space $m^2(M, \phi, q, p)$ and symmetric double sequence space.

The following result follows from Theorem 3.9 and the Lemma.

Corollary 3.10. The double space $m^2(M, \phi, q, p)$ is monotone.

The proof of the following result is a routine work.

Proposition 3.11. The double spaces $l_p^2(M, q)$ and $l_\infty^2(M, q, p)$ are solid and as such are monotone.

Theorem 3.12. $l_p^2(M, q) \subset m^2(M, \phi, q, p) \subset l_\infty^2(M, q, p)$.

Proof. Let $(x_{ij}) \in l_p^2(M, q)$. Then we have

$$\sum_{i,j=1,1}^{\infty} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty, \text{ for some } r > 0. \tag{3.4}$$

Since $\{\phi_{n,m}\}$ is monotonic increasing, so we have

$$\frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} \leq \frac{1}{\phi_{1,1}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

Hence

$$\sup_{s,t\geq 1, \sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty.$$

Thus $(x_{ij}) \in m^2(M, \phi, q, p)$. Therefore $l_p^2(M, q) \subset m^2(M, \phi, q, p)$. Next let $(x_{ij}) \in m^2(M, \phi, q, p)$. Then we have

$$\sup_{s,t\geq 1, \sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty, \text{ for some } r > 0.$$

So,

$$\sup_{i,j\in\mathbb{N}} \frac{1}{\phi_{1,1}} \left[M\left(q\left(\frac{x_{ij}}{r}\right)\right)\right]^{p_{ij}} < \infty, \text{ for some } r>0 \text{ (on taking cardinality of } \sigma \text{ to be 1)}.$$

Therefore $(x_{ij}) \in l_\infty^2(M,q,p)$. Hence $m^2(M,\phi,q,p) \subset l_\infty^2(M,q,p)$. This completes the proof.

Theorem 3.13.(i) $m^2(M,\phi,q,p) = l_p^2(M,q)$ if and only if $\sup_{s,t\geq 1}\phi_{s,t} < \infty$.

(ii) $m^2(M,\phi,q,p) = l_\infty^2(M,q,p)$ if and only if $\sup_{s,t\geq 1}\frac{st}{\phi_{s,t}} < \infty$.

Proof.(i) It is clear that $m^2(M,\psi,q,p) = l_p^2(M,q)$ when $\psi_{s,t} = 1$ for all $s,t \in \mathbb{N}$. By Theorem 3.4., $m^2(M,\phi,q,p) \subset m^2(M,\psi,q,p)$ if and only if $\sup_{s,t}\frac{\phi_{s,t}}{\psi_{s,t}} < \infty$ i.e. $\sup_{s,t}\phi_{s,t} < \infty$. By Theorem 3.11., $m^2(M,\phi,q,p) = l_p^2(M,q)$ if and only if $\sup_{s,t}\phi_{s,t} < \infty$.

(ii) We have $m^2(M,\psi,q,p) = l_\infty^2(M,q)$ if $\psi_{s,t} = st$ for all $s,t\in\mathbb{N}$. By Theorem 3.4. and Theorem 3.11., it follows that $m^2(M,\phi,q,p) = l_\infty^2(M,q,p)$ if and only if $\sup_{s,t\geq 1}\frac{st}{\phi_{s,t}} < \infty$.

This completes the proof.

The proof of the following result is routine work.

Proposition 3.14. Let M be an Orlicz function, q_1 and q_2 be seminorms. Then

(i) $m^2(M,\phi,q_1,p) \cap m^2(M,\phi,q_2,p) \subset m^2(M,\phi,q_1+q_2,p)$,

(ii) If q_1 is stronger than q_2, then $m^2(M,\phi,q_1,p) \subset m^2(M,\phi,q_2,p)$,

(iii) $l_\infty^2(M,q_1,p) \cap l_\infty^2(M,q_2,p) \subset l_\infty^2(M,q_1+q_2,p)$,

(iv) If q_1 is stronger than q_2, then $l_\infty^2(M,q_1,p) \subset l_\infty^2(M,q_2,p)$,

(v) $l_p^2(M,q_1) \cap l_p^2(M,q_2) \subset l_p^2(M,q_1+q_2)$,

(vi) If q_1 is stronger than q_2, then $l_p^2(M,q_1) \subset l_p^2(M,q_2)$.

4. *PARTICULAR CASES*

If one considers a normed linear space $(X,\|.\|)$ instead of a seminormed space (X,q), then one will get $m^2(M,\phi,\|.\|,p)$, which will be a normed linear space, normed by

$$\|(x_{ij})\| = \inf\left\{ r^{\frac{p_{ij}}{J}} > 0 : \sup_{s,t\geq 1,\sigma\in P_{st}} \frac{1}{\phi_{s,t}} \sum_{(i,j)\in\sigma} M\left(\left\|\frac{x_{ij}}{r}\right\|\right) \leq 1\right\}, \text{where } J = max\left(1,2^{H-1}\right).$$

The double space $m^2(M,\phi,\|.\|,p)$ will be a solid, monotone and symmetric space. Further most of the results proved in the previous section will be true for this space too.

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On the convergence of variational iteration method for the solution of partial differential equations of fractional order

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Abstract

The variational iteration method has been widely used to handle the linear and nonlinear problems. The main property of this method is its flexibility and ability to solve nonlinear equations accurately and conveniently. In this paper, we present an alternative approach of the method, then we study the convergence analysis for nonlinear partial differential equations with fractional derivative in the Caputo's sense. Our emphasis is to address the sufficient condition for convergence and the error estimate. Illustrative experiments are investigated to verify the convergency of the results and to show the efficiency of the method.

Keywords: Variational iteration method; Fractional partial differential equations; Caputo's derivative; Nonlinear model; Convergence analysis; Error estimate.

2010 Mathematics Subject Classification: 35A15; 35R11; 70K75; 65M12; 65M15.

1 Introduction

The variational iteration method(VIM)was developed in 1999 by He[1-8]. This method is now widely used by many researchers to study linear and nonlinear problems. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications. It is based on Lagrange multiplier and it has the merits of simplicity and easy execution. It was shown by many researchers[9-17] that this method is more powerful than existing techniques such as the Adomian decomposition method, perturbation method, and etc. The VIM gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The Adomian decomposition method suffers from the complicated computational work that is needed for the

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derivation of Adomian polynomials for nonlinear terms. The VIM has no specific requirements, such as linearization, small parameters, and etc. for nonlinear operators. Therefore, the VIM can overcome the foregoing restrictions and limitations of perturbation techniques, such that it provides us with a possibility to analyze strongly nonlinear problems.

On the other hand, the VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution[12-16]. Moreover, the power of the method gives it a wider applicability in handling a huge number of analytical and numerical applications. The VIM was successfully applied to, for example, wave equations[17-23], fractional differential equations[2, 24-34], nonlinear problems arising in engineering[1, 35-40] and other autonomous ordinary and partial differential equations.

The VIM, which is thoroughly used by many researchers, gives rapidly convergent successive approximations and handles the linear and nonlinear problems in a similar manner. In the present work, we aim to study the convergence of the method for nonlinear partial differential equations of fractional order. The convergence of the method for nonlinear problems with derivative of integer order and to address the sufficient condition is presented in [41].

The paper is organized as follows: Section 2 gives the notation and basic definitions of the fractional calculus. In section 3, an alternative approach of the VIM for solving nonlinear partial differential equations of fractional order is discussed. In section 4, convergence analysis of the VIM is presented. In section 5, convergence results for solving nonlinear partial differential equations of fractional order is introduced. Illustrative experiments are given in section 6 with a conclusion in section 7.

2 Basic definitions of the fractional calculus

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 1. *A real function f(x), $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty]$, and it is said to be in the space C_μ^m iff $f^m \in C_\mu$, $m \in N$.*

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as:*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad x > 0,$$
$$I^0 f(x) = f(x).$$

Properties of the operator I^α can be found in [42, 43]. We mention only the following: For $f \in C_\mu$, $\mu \geq 0$ and $\gamma > -1$:

i. $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$,
ii. $I^\alpha I^\beta f(x) = I^\beta I^\alpha f(x)$,
iii. $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity[44].

Definition 3. *The fractional derivative of f(x) in the Caputo's sense is defined as*

$$D^\alpha f(x) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) dt,$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, *and* $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1. *If* $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ *and* $f \in C_\mu^m$, $\mu \geq -1$, *then*

i. $D^\alpha I^\alpha f(x) = f(x)$,
ii. $I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$

3 An alternative approach of the VIM

The VIM, which provides an approximate solution, is applied to various nonlinear problems[17-40, 45]. In this section, we present an alternative approach of VIM. This approach can be implemented, in a reliable and efficient way, to handle the nonlinear partial differential equations of fractional order in the following form:

$$Lu(x,t) + Nu(x,t) = g(x,t), \qquad t > 0, \tag{1}$$

where the linear operator L is defined as $L = \frac{d^\alpha}{dt^\alpha}$, is the Caputo fractional derivative operator of order α, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$. N is a nonlinear operator and g(x,t) is a known analytic function, subject to the initial conditions:

$$u^{(k)}(x,0) = f_k(x), \qquad k = 0, 1, \cdots, m-1.$$

According to variational iteration method, we construct the correction functional approximately for equation (1) as,

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \left[\lambda(\tau) \left(\frac{\partial^m}{\partial \tau^m} u_k(x,\tau) + N\widetilde{u}_k(x,\tau) - g(x,\tau) \right) \right] d\tau, \tag{2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory. Here, we apply restricted variations to nonlinear term Nu. In this case, we can easily determine the multiplier. Making the above functional stationary, noticing that $\delta \widetilde{u}_k = 0$,

$$\delta u_{k+1}(x,t) = \delta u_k(x,t) + \delta \int_0^t \left[\lambda(\tau) \left(\frac{\partial^m}{\partial \tau^m} u_k(x,\tau) - g(x,t) \right) \right] d\tau,$$

yields the following Lagrange multipliers,

$$\lambda = -1 \ , \quad \text{for} \quad \text{m=1},$$

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$\lambda = \tau - t$, for m=2,

and in general,

$$\lambda = \frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1}, \quad for \quad m \geq 1. \tag{3}$$

Therefore, substituting equation (3) into functional (2), we obtain the following iteration formula:

$$u_{k+1}(x,t) = u_k(x,t) + \int_0^t \frac{(-1)^m(\tau - t)^{m-1}}{(m-1)!} \left(Lu_k(x,t) + Nu_k(x,t) - g(x,t)\right) d\tau. \tag{4}$$

Now, define the operator A[u] as,

$$A[u] = \int_0^t \left[\frac{(-1)^m}{(m-1)!}(\tau - t)^{m-1} \left(Lu(x,\tau) + Nu(x,\tau) - g(x,\tau)\right)\right] d\tau, \tag{5}$$

and define the components ν_k, $k = 0, 1, \cdots$, as

$$\begin{cases} \nu_0 = u_0, \\ \nu_1 = A[\nu_0], \\ \nu_2 = A[\nu_0 + \nu_1], \\ \quad \vdots \\ \nu_{k+1} = A[\nu_0 + \nu_1 + \cdots + \nu_k], \end{cases} \tag{6}$$

then, consequently, we have $u(x,t) = \lim_{k\to\infty} u_k(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$. Therefore, as a result, the solution of problem (1) can be derived, using equations (5) and (6), in the series form,

$$u(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t). \tag{7}$$

The zeroth(initial) approximation $\nu_0 = u_0$ can be freely chosen if it satisfies the initial conditions of the problem. The success of the method depends on the proper selection of the initial approximation ν_0. However, using the initial values $u^{(k)}(x,0) = f_k(x),\ k = 0, 1, \cdots, m-1$; are preferably used for the selective zeroth approximation ν_0 as will be seen later. In our alternative approach we select the initial approximation ν_0 as:

$$\nu_0 = \sum_{k=0}^{\infty} \frac{f_k(x)}{k!} t^k. \tag{8}$$

For the approximation purposes, we approximate the solution $u(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$, by the nth-order truncated series $\sum_{k=0}^{n} \nu_k(x,t)$.

4

4 Convergence analysis

In this section, we study the convergence of the variational iteration method, according to the alternative approach of the VIM presented in the previous section, when applied to problem (1). The sufficient conditions for convergence and the error estimate are presented. The main results are proposed in the following theorems.

Theorem 1. *Let A[u], defined in equation (5), be an operator from a Hilbert space H to H. The series solution* $u(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$*, defined in (7), converges if there exist* $0 < \gamma < 1$ *such that*

$$\| A[\nu_0 + \nu_1 + \cdots + \nu_{k+1}] \| \leq \gamma \| A[\nu_0 + \nu_1 + \cdots + \nu_k] \|,$$

(that is $\| \nu_{k+1} \| \leq \gamma \| \nu_k \|$*),* $\forall k \in \mathbb{N} \cup \{0\}$.

Theorem 1 is a special case of Banach fixed point theorem which is used in [46], as a sufficient condition, to study the convergence of VIM for some partial differential equations.

Proof. See [41]. □

Theorem 2. *If the series solution* $u(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$*, defined in (7) converges, then it is an exact solution of the nonlinear problem (1).*

Proof. Suppose that the series solution (7) converges, say $\Phi(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$, then we have:

$$\begin{aligned} \lim_{j\to\infty} \nu_j &= 0, \\ \sum_{j=0}^{n} [\nu_{j+1} - \nu_j] &= \nu_{n+1} - \nu_0, \end{aligned}$$

and so,

$$\sum_{j=0}^{\infty} [\nu_{j+1} - \nu_j] = \lim_{j\to\infty} \nu_j - \nu_0 = -\nu_0. \tag{9}$$

Applying the operator $L = \frac{\partial^\alpha}{\partial t^\alpha}$, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, to both sides of equation (9), then from equation (8), we obtain:

$$\sum_{j=0}^{\infty} L[\nu_{j+1} - \nu_j] = -L[\nu_0] = 0. \tag{10}$$

On the other hand, from definition (6), we have:

$$L[\nu_{j+1} - \nu_j] = L\left(A[\nu_0 + \cdots + \nu_j] - A[\nu_0 + \cdots + \nu_{j-1}]\right),$$

when $j \geq 1$, and so, using definition (5), we get:

$$\begin{aligned} L[\nu_{j+1}-\nu_j] &= L\{\int_0^t [\frac{(-1)^m}{(m-1)!}(\tau-t)^{m-1}(L[\nu_0+\cdots+\nu_j]-L[\nu_0+\cdots+\nu_{j-1}] \\ &+ N[\nu_0+\nu_1+\cdots+\nu_j]-N[\nu_0+\nu_1+\cdots+\nu_{j-1}])]d\tau\}, \quad j \geq 1. \end{aligned}$$

According to definition of operator $L = \frac{\partial^\alpha}{\partial t^\alpha}$, then the above equation becomes as,

$$L[\nu_{j+1}-\nu_j] = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1} \frac{d^m}{dy^m}\{\int_0^y [\frac{(-1)^m}{(m-1)!}(\tau-y)^{m-1}(L[\nu_j]$$

$$+N[\nu_0+\nu_1+\cdots+\nu_j]-N[\nu_0+\nu_1+\cdots+\nu_{j-1}])]d\tau\}dy. \tag{11}$$

Now, the operator A[u], defined in (6), gives the mth-fold integral of Lu(x,t)+Nu(x,t)-g(x,t). Since the differential operator $L' = \frac{d^m}{dy^m}$ of order m is left inverse to mth-fold integral operator, then equation (11) becomes as:

$$\begin{aligned} L[\nu_{j+1}-\nu_j] &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1}(L[\nu_j]+N[\nu_0+\cdots+\nu_j]-N[\nu_0+\cdots+\nu_{j-1}])dy \\ &= I^{m-\alpha}(L[\nu_j]+N[\nu_0+\cdots+\nu_j]-N[\nu_0+\cdots+\nu_{j-1}]). \end{aligned}$$

Consequently, we have:

$$\sum_{j=0}^{n} L[\nu_{j+1}-\nu_j] = I^{m-\alpha}\left[\sum_{j=0}^{n}(L[\nu_j]+N[\nu_0+\cdots+\nu_j]-N[\nu_0+\cdots+\nu_{j-1}])\right]. \tag{12}$$

Now, applying the operator $L'' = \frac{\partial^{m-\alpha}}{\partial t^{m-\alpha}}$, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, which is fractional derivative operator in the Caputo's sense of order $(m-\alpha)$, and is left inverse to Riemann-Liouville fractional integral operator of order $(m-\alpha)$, to both sides of equation (12), then equation (12) becomes as:

$$\begin{aligned} L''\left(\sum_{j=0}^{n} L[\nu_{j+1}-\nu_j]\right) &= (L[\nu_0]+N[\nu_0]-g(x,t) \\ &+ L[\nu_1]+N[\nu_0+\nu_1]-N[\nu_0] \\ &+ L[\nu_2]+N[\nu_0+\nu_1+\nu_2]-N[\nu_0+\nu_1] \\ &\vdots \\ &+ L[\nu_n]+N[\nu_0+\cdots+\nu_n]-N[\nu_0+\cdots+\nu_{n-1}]). \end{aligned}$$

Consequently, we have:

$$L''\left(\sum_{j=0}^{\infty} L[\nu_{j+1}-\nu_j]\right) = L[\sum_{j=0}^{\infty}\nu_j]+N[\sum_{j=0}^{\infty}\nu_j]-g(x,t). \tag{13}$$

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From equations (10) and (13), we can observe that $\Phi(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$ is an exact solution of problem (1). This completes the proof of theorem 2. □

Theorem 3. *Assume that the series solution $\sum_{k=0}^{\infty} \nu_k(x,t)$, defined in (7), is convergent to the solution u(x,t). If the truncated series $\sum_{k=0}^{j} \nu_k(x,t)$ is used as an approximation to the solution u(x,t) of problem (1), then the maximum error, $E_j(x,t)$ is estimated as,*

$$E_j(x,t) \leq \frac{1}{1-\gamma}\gamma^{j+1} \parallel \nu_0 \parallel.$$

Proof. See [41]. □

In summary, theorems 1 and 2 state that the variational iteration solution of nonlinear problem (1), obtained using the iteration formula's (4) or (6), converge to an exact solution under the condition that there exist $0 < \gamma < 1$ such that

$$\parallel A[\nu_0 + \nu_1 + \cdots + \nu_{k+1}] \parallel \leq \gamma \parallel A[\nu_0 + \nu_1 + \cdots + \nu_k] \parallel,$$

(that is $\parallel \nu_{k+1} \parallel \leq \gamma \parallel \nu_k \parallel$), $\forall k \in \mathbb{N} \cup \{0\}$.

In other words, If we define for every $i \in \mathbb{N} \cup \{0\}$, the parameters,

$$\beta_i = \begin{cases} \dfrac{\parallel \nu_{j+1} \parallel}{\parallel \nu_j \parallel}, & \parallel \nu_j \parallel \neq 0, \\ \\ 0, & \parallel \nu_j \parallel = 0, \end{cases}$$

then the series solution $\sum_{k=0}^{\infty} \nu_k(x,t)$ of problem (1) converges to the exact solution u(x,t), when $0 \leq \beta_i < 1$, $\forall k \in \mathbb{N} \cup \{0\}$. Moreover, as stated in theorem 3, the maximum absolute truncation error is estimated to be

$$\parallel u(x,t) - \sum_{k=0}^{j} \nu_k(x,t) \parallel \leq \frac{1}{1-\beta}\beta^{j+1} \parallel \nu_0 \parallel,$$

where $\beta = \max\{\beta_i;\ i = 0, 1, \cdots, j\}$.

Remark:

If the first finite β_i; $i = 0, 1, \cdots, l$, are not less than one and $\beta_i < 1$ for $i > l$, then, of course, the series solution $\sum_{k=0}^{\infty} \nu_k(x,t)$ of problem (1) converges to the exact solution. In other words, the first finite terms do not affect the convergence of series solution.This is because that, following the theorem 1, we have:

$$\parallel S_n - S_j \parallel = \frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j-l+1} \parallel \nu_{l+1} \parallel,$$

and since $0 < \gamma < 1$, for $n \geq j$ and fixed l, we get:

$$\lim_{n,j\to\infty} \parallel S_n - S_j \parallel = 0.$$

7

In this case, the convergence of VIM approach depends on β_i, for $i > l$.

5 Convergence results

Consider the nonlinear partial differential equation of fractional order,

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + Nu(x,t) = g(x,t), \quad t > 0, \tag{14}$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, N is a nonlinear operator, g(x,t) is a known analytic function and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo's fractional derivative of order α. The initial conditions of problem (14) are given in terms of the field variable with their integer order as,

$$u^{(k)}(x,0) = f_k(x), \qquad k = 0, 1, \cdots, m-1.$$

Following the same analysis presented in the previous section, we can show that the variational iteration solution $u(x,t) = \sum_{k=0}^{\infty} \nu_k(x,t)$ obtained using the iteration formula,

$$\begin{cases} \nu_0 = \displaystyle\sum_{k=0}^{\infty} \frac{f_k(x)}{k!} t^k, \\ \nu_{k+1} = \int_0^t \frac{(-1)^m(\tau-t)^{m-1}}{(m-1)!}\left(\frac{\partial^\alpha}{\partial \tau^\alpha}(\nu_0 + \cdots + \nu_k)(x,\tau) + N(\nu_0 + \cdots + \nu_k)(x,\tau) - g(x,\tau)\right)d\tau, \end{cases} \tag{15}$$

converges to the solution of problem (14) if there exist $0 < \gamma < 1$ such that $\| \nu_{k+1} \| \leq \gamma \| \nu_k \|$, $\forall k \in \mathbb{N} \cup \{0\}$.

6 Illustrative experiments

In this section, we apply the proposed alternative approach of VIM to solve two nonlinear problems. Then we examine the convergence condition for each of them.

Experiment 1. Consider the nonlinear partial differential equation of fractional order in the Caputo's sense as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right), \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \alpha \leq 2, \tag{16}$$

subject to the initial conditions:

$$u(x,0) = x^2, \quad u_t(x,0) = -2x^2.$$

In view of equations (15), the iteration formula of problem (16) can be constructed as:

$$\begin{cases}\nu_0 = x^2(1-2t),\\ \nu_{k+1} = \int_0^t (\tau - t)(\frac{\partial^\alpha}{\partial \tau^\alpha}(\nu_0+\cdots+\nu_k)(x,\tau) - \frac{\partial}{\partial x}[(\nu_0+\cdots+\nu_k)(x,\tau)\frac{\partial}{\partial x}(\nu_0+\cdots+\nu_k)(x,\tau)])d\tau.\end{cases}$$

Using the above iteration formula, we obtain the following successive approximations:
$\nu_1(x,t) = x^2(2t^4 - 4t^3 + 3t^2)$,

$\nu_2(x,t) = x^2 \frac{4}{15}[(-\frac{45}{2}\alpha^2 - 675 + \frac{495}{2}\alpha)\, t^{4-\alpha} + (540 - 90\alpha)t^{5-\alpha} - 180t^{6-\alpha}$

$+ \left(-\frac{120}{7}t^7 - 5t^9 + t^{10} - 15t^3 + \frac{87}{4}t^6 + \frac{75}{4}t^4 - \frac{45}{2}t^5 + \frac{45}{4}t^2 + \frac{45}{4}t^8\right)\Gamma(7-\alpha)]/\Gamma(7-\alpha)$,

$\nu_3(x,t) = 6x^2(-32\Gamma(12-\alpha)t^{(7-2\alpha)} + 56\Gamma(12-\alpha)t^{(6-2\alpha)} + 161280\Gamma(9-2\alpha)t^{(11-\alpha)}$
$+5512320\Gamma(9-2\alpha)t^{(7-\alpha)} - 8316000\Gamma(9-2\alpha)t^{(4-\alpha)} + 8\Gamma(12-\alpha)t^{(8-2\alpha)}$
$+8\Gamma(12-\alpha)t^{(7-2\alpha)}\alpha + 4\Gamma(12-\alpha)t^{(6-2\alpha)}\alpha^2 - 26448\Gamma(9-2\alpha)t^{(7-\alpha)}\alpha^3$
$+24\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha^6 + 4044032\Gamma(9-2\alpha)t^{(6-\alpha)}\alpha + 6650\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^5$
$+20160\Gamma(9-2\alpha)t^{(9-\alpha)}\alpha^2 + 696\Gamma(9-2\alpha)t^{(7-\alpha)}\alpha^4 + 375144\Gamma(9-2\alpha)t^{(7-\alpha)}\alpha^2$
$-5880096\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha - 5760\Gamma(9-2\alpha)t^{(6-\alpha)}\alpha^4 - 1224\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha^5$
$+1788576\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha^2 - 287640\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha^3 + 672245\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^3$
$-423360\Gamma(9-2\alpha)t^{(9-\alpha)}\alpha + 128\Gamma(9-2\alpha)t^{(6-\alpha)}\alpha^5 - 280\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^6$
$+3840\Gamma(9-2\alpha)t^{(8-\alpha)}\alpha^3 - 3088120\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^2 + 1148160\Gamma(9-2\alpha)t^{(8-\alpha)}$
$-115200\Gamma(9-2\alpha)t^{(8-\alpha)}\alpha^2 + 103040\Gamma(9-2\alpha)t^{(6-\alpha)}\alpha^3 - 915840\Gamma(9-2\alpha)t^{(6-\alpha)}\alpha^2$
$+5\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^7 + 7788300\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha - 2353872\Gamma(9-2\alpha)t^{(7-\alpha)}\alpha$
$+25800\Gamma(9-2\alpha)t^{(5-\alpha)}\alpha^4 - 86800\Gamma(9-2\alpha)t^{(4-\alpha)}\alpha^4 + 80640\Gamma(9-2\alpha)t^{(10-\alpha)}\alpha$
$-30\Gamma(12-\alpha)t^{(6-2\alpha)}\alpha - 7096320\Gamma(9-2\alpha)t^{(6-\alpha)} + 2217600\Gamma(9-2\alpha)t^{(9-\alpha)}$
$-3801600\Gamma(9-2\alpha)t^{(8-\alpha)} - 887040\Gamma(9-2\alpha)t^{(10-\alpha)} + 7983360\Gamma(9-2\alpha)t^{(5-\alpha)})$
$/(\Gamma(12-\alpha)\Gamma(9-2\alpha))$,
$\vdots$

By computing β_i's for this experiment, when $0 < t < 1,\ 0 < x < 1$ and $0.5 < \alpha \leq 1$, we have:

$$\beta_i = \frac{\|\nu_{i+1}\|}{\|\nu_i\|} < 1, \quad i \geq 0.$$

This confirms that the variational approach of problem (16), converges to the exact solution.
For the other values of x and t, convergence approach is the same.

Experiment 2. Consider the nonlinear advection partial differential equation of fractional order in the Caputo's sense as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) + \left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) = x + xt^2, \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \tag{17}$$

subject to the initial condition:
$u(x,0) = 0$.
In view of equations (15), the iteration formula of problem (17) can be constructed as:

$$\begin{cases} \nu_0 = 0, \\ \nu_{k+1} = -\int_0^t (\frac{\partial^\alpha}{\partial \tau^\alpha}(\nu_0 + \cdots + \nu_k)(x,\tau) + [(\nu_0 + \cdots + \nu_k)(x,\tau)\frac{\partial}{\partial x}(\nu_0 + \cdots + \nu_k)(x,\tau)] - x - x\tau^2)d\tau. \end{cases}$$

Using the above iteration formula, we obtain the following successive approximations:

$\nu_1(x,t) = x(t + \frac{t^3}{3})$,

$\nu_2(x,t) = -x\left(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{t^7}{63} + \frac{2t^5}{15} - t\right)$,

$\nu_3(x,t) = -\frac{8x}{59535}[-\frac{59535}{2}((\alpha-3)(\alpha-5)^2(-6+\alpha)^2(\alpha-7)^2(\alpha-8)^2(\alpha-4)^2\Gamma(7-2\alpha)$
$-8(\alpha-\frac{5}{2})\Gamma(9-\alpha)^2)(\alpha-10)(\alpha-3)(\alpha-12)(\alpha-\frac{9}{2})(\alpha-\frac{7}{2})t^{(5-2\alpha)} + (\alpha-\frac{5}{2})$
$(-119070\Gamma(7-2\alpha)(\alpha-10)(\alpha-8)^2(\alpha-3)(\alpha-7)^2(\alpha-12)(\alpha-\frac{9}{2})(\alpha-4)(\alpha-5)^2$
$(\alpha-6)^2t^{(7-2\alpha)} + (-119070\Gamma(7-2\alpha)(\alpha-10)(\alpha-12)(\alpha-5)^2(\alpha-6)^2(\alpha-7)^2(\alpha-8)^2$
$t^{(9-2\alpha)} + (119070\Gamma(7-2\alpha)(\alpha-12)(\alpha-3)(\alpha-4)(\alpha-5)(\alpha-6)(\alpha-7)(\alpha-8)(\alpha-10)t^{(2-\alpha)}$
$-1890\Gamma(7-2\alpha)(\alpha-8)(\alpha-7)(\alpha-12)(\alpha^2-7\alpha+\frac{144}{5})(\alpha-5)(\alpha-6)t^{(10-\alpha)}$
$238140(\alpha-\frac{5}{2})\Gamma(7-2\alpha)(\alpha-8)(\alpha-7)(\alpha-12)(\alpha-5)(\alpha-6)t^{(4-\alpha)}$
$+39690\Gamma(7-2\alpha)(\alpha-12)(\alpha-7)(\alpha-8)(\alpha^3-12\alpha+59\alpha-144)t^{(6-\alpha)}$
$-15876\Gamma(7-2\alpha)(\alpha-12)(\alpha^5-25\alpha^4+240\alpha^3-1085\alpha^2+2219\alpha-1170)t^{(8-\alpha)}$
$-3780\Gamma(7-2\alpha)(\alpha-5)(\alpha-6)(\alpha-7)(\alpha-8)t^{(12-\alpha)} + (476280(\alpha-\frac{5}{2})(\alpha-3)(\alpha-2)t^{(3-2\alpha)}$
$+\Gamma(7-2\alpha)t(t^{14} + \frac{252}{13}t^{12} + 15876t^4 + \frac{2142}{55}t^{10} - 1008t^8 - 3591t^6 + 59535t^2 - 59535))$
$\Gamma(9-\alpha)(\alpha-12))(\alpha-10))\Gamma(9-\alpha)(\alpha-\frac{9}{2}))(\alpha-\frac{7}{2}))]/$
$((\alpha-10)(\alpha-12)\Gamma(9-\alpha)^2(2\alpha+5)(2\alpha-9)(2\alpha-7)\Gamma(7-2\alpha))$,

$\vdots$

By computing β_i's for this experiment, when $0 < t < 1,\ 0 < x < 1$ and $0.5 < \alpha \leq 1$, we have:

$$\beta_i = \frac{\|\nu_{i+1}\|}{\|\nu_i\|} < 1, \quad i \geq 0.$$

This confirms that the variational approach of problem (17), converges to the exact solution.
For the other values of x and t, convergence approach is the same.

7 Conclusion

There are two main goals that we aimed in this paper. The first is presenting an alternative approach of the variational iteration method to handle nonlinear partial differential equations with fractional derivative in the Caputo's sense. The second is studying the convergence analysis and addressing the sufficient condition for convergence. These two goals are achieved. We have studied the convergence of the presented approach and the main results are given in theorems 1-3. Furthermore, we have examined the convergence condition of VIM solution for the nonlinear problems. The VIM gives several successive approximations through using the iteration of the correction functional without any restrictive assumptions or transformation and hence the procedure is direct and straightforward. The VIM proved to be easy to use and provides an efficient method for handling nonlinear problems, where a few approximation can be used to achieve the higher order accuracy. Moreover, VIM reduces the size of calculations and

facilitates the computational work when compared with Adomian decomposition method or perturbation techniques.

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LATTICTIC RANDOM STABILITY OF AN ADDITIVE-QUADRATIC FUNCTIONAL EQUATION BY FIXED POINT METHOD

CHOONKIL PARK AND REZA SAADATI*

ABSTRACT. In [39], Th.M. Rassias proved that the norm defined over a real vector space V is induced by an inner product if and only if for a fixed positive integer l

$$2l\left\|\frac{1}{2l}\sum_{i=1}^{2l}x_i\right\|^2+\sum_{i=1}^{2l}\left\|x_i-\frac{1}{2l}\sum_{j=1}^{2l}x_j\right\|^2=\sum_{i=1}^{2l}\|x_i\|^2$$

holds for all $x_1,\cdots,x_{2l}\in V$. For the above equality, we can define the following functional equation

$$2lf\left(\frac{1}{2l}\sum_{i=1}^{2l}x_i\right)+\sum_{i=1}^{2l}f\left(x_i-\frac{1}{2l}\sum_{j=1}^{2l}x_j\right)=\sum_{i=1}^{2l}f(x_i). \tag{1}$$

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation (1) in complete lattictic random normed spaces.

1. Introduction

Probability theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g. population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence, etc. The random topology proves to be a very useful tool to deal with such situations where the use of classical theories breaks down. The usual uncertainty principle of Werner Heisenberg leads to a generalized uncertainty principle, which has been motivated by string theory and non-commutative geometry. In strong quantum gravity regime space-time points are determined in a random manner. Thus impossibility of determining the position of particles gives the space-time a random structure. Because of this random structure, position space representation of quantum mechanics breaks down and therefore a generalized normed space of quasi-position eigenfunction is required. Hence, one needs to discuss on a new family of random norms. There are many situations where the norm of a vector is not possible to be found and the concept of random norm seems to be more suitable in

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such cases, that is, we can deal with such situations by modeling the inexactness through the random norm.

The stability problem of functional equations was originated from a question of Ulam [52] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [38] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [38] has provided a lot of influence in the development of what we call the *generalized Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [51] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [10] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [16], [18], [19], [35]–[37], [40]–[47]).

In [**?**], Park, Lee and Shin proved that an even mapping $f : V \to W$ satisfies the functional equation (1) if and only if the even mapping $f : V \to W$ is quadratic. Moreover, they proved the generalized Hyers-Ulam stability of the quadratic functional equation (1) in real Banach spaces.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 12] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$; (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5], [6], [30], [32], [33], [34]).

2. Preliminaries

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied by Alsina [1], Mirmostafaee, Mirzavaziri and Moslehian [29, 30], Miheţ and Radu [23], Miheţ, Saadati and Vaezpour [24, 25], Baktash et. al [3] and Saadati et. al. [48].

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, i.e., a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of latticetic random distribution functions, denoted by Δ_L^+, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that F is left continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

$D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ_L^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([8]) A *triangular norm* (t–norm) on L is a mapping $\mathcal{T} : (L)^2 \longrightarrow L$ satisfying the following conditions:

(a) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (boundary condition);

(b) $(\forall (x, y) \in (L)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);

(c) $(\forall (x, y, z) \in (L)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);

(d) $(\forall (x, x', y, y') \in (L)^4)(x \leq_L x' \text{ and } y \leq_L y' \Longrightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (monotonicity).

Let $\{x_n\}$ be a sequence in L converges to $x \in L$ (equipped order topology). The t-norm $\mathcal{T}$ is said to be a *continuous t–norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y),$$

for each $y \in L$.

Definition 2.2. ([8]) A continuous t–norm $\mathcal{T}$ on $L = [0,1]^2$ is said to be *continuous t–representable* if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1]^2$ are continuous t–representable.

Define the mapping $\mathcal{T}_\wedge$ from L^2 to L by:

$$\mathcal{T}_\wedge(x, y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases}$$

Recall (see [20], [21]) that if $\{x_n\}$ is a given sequence in L, $(\mathcal{T}_\wedge)_{i=1}^n x_i$ is defined recurrently by $(\mathcal{T}_\wedge)_{i=1}^1 x_i = x_1$ and $(\mathcal{T}_\wedge)_{i=1}^n x_i = \mathcal{T}_\wedge((\mathcal{T}_\wedge)_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L}$ and $\mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L$, then $\mathcal{N}$ is called an involutive negation. In the following $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

Definition 2.3. A *latticetic random normed space* is a triple $(X, \mu, \mathcal{T}_\wedge)$, where X is a vector space and μ is a mapping from X into D_L^+ such that the following conditions hold:

(LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(LRN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$;
(LRN3) $\mu_{x+y}(t+s) \geq_L \mathcal{T}_\wedge(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note that from (LRN2) it follows $\mu_{-x}(t) = \mu_x(t) \quad (x \in X, t \geq 0)$.

Example 2.4. Let $L = [0,1] \times [0,1]$ and operation $\leq_L$ be defined by:

$$L = \{(a_1, a_2) : (a_1, a_2) \in [0,1] \times [0,1] \text{ and } a_1 + a_2 \leq 1\},$$

$$(a_1, a_2) \leq_L (b_1, b_2) \iff a_1 \leq b_1,\ a_2 \geq b_2, \quad \forall a = (a_1, a_2), b = (b_1, b_2) \in L.$$

Then $(L, \leq_L)$ is a complete lattice (see [8]). In this complete lattice, we denote its units by $0_L = (0, 1)$ and $1_L = (1, 0)$. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) =$

$(\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0,1] \times [0,1]$ and μ be a mapping defined by

$$\mu_x(t) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big), \quad \forall t \in \mathbb{R}^+, .$$

Then $(X, \mu, \mathcal{T})$ is a latticetic random normed spaces.

If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space, then

$$\mathcal{V} = \{V(\varepsilon, \lambda) : \varepsilon >_L 0_{\mathcal{L}}, \lambda \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}, V(\varepsilon, \lambda) = \{x \in X : F_x(\varepsilon) >_L \mathcal{N}(\lambda)\}$$

is a complete system of neighborhoods of null vector for a linear topology on X generated by the norm F.

Definition 2.5. Let $(X, \mu, \mathcal{T}_\wedge)$ be a latticetic random normed spaces.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mu_{x_n - x}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if, for every $t > 0$ and $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) >_L \mathcal{N}(\varepsilon)$ whenever $n \geq m \geq N$.

(3) A latticetic random normed spaces $(X, \mu, \mathcal{T}_\wedge)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.6. *If $(X, \mu, \mathcal{T}_\wedge)$ is a latticetic random normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Proof. The proof is the same as classical random normed spaces, see [49]. □

3. Generalized Hyers-Ulam stability of the functional equation (1): an odd case

For a given mapping $f : X \to Y$, we define

$$Cf(x_1, \cdots, x_{2l}) := 2lf\left(\frac{1}{2l}\sum_{i=1}^{2l} x_i\right) + \sum_{i=1}^{2l} f\left(x_i - \frac{1}{2l}\sum_{j=1}^{2l} x_j\right) - \sum_{i=1}^{2l} f(x_i)$$

for all $x_1, \cdots, x_{2l} \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x_1, \cdots, x_{2l}) = 0$ in complete LRN-space: an odd case.

Theorem 3.1. *Let X be a linear space, $(Y, \mu, \mathcal{T}_\wedge)$ a complete* LRN*-space and Φ a mapping from X^{2l} to D_L^+ ($\Phi(x_1, ..., x_{2l})$ is denoted by $\Phi_{x_1,...,x_{2l}}$) and*

$$\Psi_x := \Phi_{\underbrace{x, \cdots, x}_{l\ times}, \underbrace{0, \cdots, 0}_{l\ times}}$$

be functions such that there exists an $R < 1$ with

$$\Phi_{x_1,\cdots,x_{2l}}\left(\frac{R}{2}t\right) \geq_L \Phi_{2x_1,\cdots,2x_{2l}}(t)$$

for all $x_1, \cdots, x_{2l} \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Cf(x_1,\cdots,x_{2l})}(t) \geq_L \Phi_{x_1,\cdots,x_{2l}}(t) \tag{2}$$

for all $x_1, \cdots, x_{2l} \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq_L \Psi_x((l - lR)t) \tag{3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x_1 = \cdots = x_l = x$ and $x_{l+1} = \cdots = x_{2l} = 0$ in (2), we get

$$\mu_{2lf(\frac{x}{2})-lf(x)}(t) \geq_L \Phi_{\underbrace{x,\cdots,x}_{l\text{ times}},\underbrace{0,\cdots,0}_{l\text{ times}}}(t) = \Psi_x(t) \tag{4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{r \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(rt) \geq_L \Psi_x(t),\ \forall x \in X, \forall t > 0\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 of [23].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \Psi_x(t)$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(R\varepsilon t) &= \mu_{2g(\frac{x}{2})-2h(\frac{x}{2})}(R\varepsilon t) \\
&= \mu_{g(\frac{x}{2})-h(\frac{x}{2})}\left(\frac{R}{2}\varepsilon t\right) \\
&\geq_L \Psi_{\frac{x}{2}}\left(\frac{Rt}{2}\right) \geq_L \Psi_x\left(\frac{Rt}{2}\right) \\
&= \Psi_x(t)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq R\varepsilon$. This means that

$$d(Jg, Jh) \leq Rd(g, h)$$

for all $g, h \in S$.

It follows from (4) that $d(f, Jf) \leq \frac{1}{l}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(i) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{5}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (5) such that there exists a $r \in (0, \infty)$ satisfying

$$\mu_{f(x)-A(x)}(rt) \geq_L \Psi_x(t)$$

for all $x \in X$;

(ii) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(iii) $d(f, A) \leq \frac{1}{1-R} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{l - lR}.$$

This implies that the inequality (2) holds.

By (1),

$$\mu_{2^n Cf\left(\frac{x_1}{2^n}, \cdots, \frac{x_{2l}}{2^n}\right)}(2^n t) \geq_L \Phi_{\frac{x_1}{2^n}, \cdots, \frac{x_{2l}}{2^n}}(t)$$

for all $x_1, \cdots, x_{2l} \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{2^n Cf\left(\frac{x_1}{2^n}, \cdots, \frac{x_{2l}}{2^n}\right)}(t) \geq_L \Phi_{x_1, \cdots, x_{2l}}\left(\frac{t}{R^n}\right)$$

for all $x_1, \cdots, x_{2l} \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \Phi_{x_1, \cdots, x_{2l}}\left(\frac{t}{R^n}\right) = 1_{\mathcal{L}}$$

for all $x_1, \cdots, x_{2l} \in X$ and all $t > 0$,

$$\mu_{CA(x_1, \cdots, x_{2l})}(t) = 1_{\mathcal{L}}$$

for all $x_1, \cdots, x_{2l} \in X$ and all $t > 0$. Thus $CA(x_1, \cdots, x_{2l}) = 0$. Since A is odd, it follows from Lemma 2.1 of [**?**] that the mapping $A : X \to Y$ is additive, as desired. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$ and let $(X, \mu, \min)$ be a a complete* LRN*-space in which μ is a distribution function on $D^+_{[0,1]} = D^+$ i.e., $L = [0,1]$ (see [49]). Let $f : X \to X$ be an odd mapping satisfying*

$$\mu_{Cf(x_1,\cdots,x_{2l})}(t) \geq \frac{t}{t + \theta \sum_{j=1}^{2l} \|x_j\|^p} \tag{6}$$

for all $x_1, \cdots, x_{2l} \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\Phi_{x_1,\cdots,x_{2l}}(t) := \frac{t}{t + \theta \sum_{j=1}^{2l} \|x_j\|^p}$$

for all $x_1, \cdots, x_{2l} \in X$. Then we can choose $R = 2^{1-p}$ and we get the desired result. □

Theorem 3.3. *Let X be a linear space, $(Y, \mu, \mathcal{T}_\wedge)$ a complete* LRN*-space and Φ a mapping from X^{2l} to D^+_L ($\Phi(x_1, ..., x_{2l})$ is denoted by $\Phi_{x_1,...,x_{2l}}$) and*

$$\Psi_x := \Phi_{\underbrace{x,\cdots,x}_{l\ times},\underbrace{0,\cdots,0}_{l\ times}}$$

be functions such that there exists an $R < 1$ with

$$\Phi_{x_1,\cdots,x_{2l}}(2Rt) \geq_L \Phi_{\frac{x_1}{2},\cdots,\frac{x_{2l}}{2}}(t)$$

for all $x_1, \cdots, x_{2l} \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Cf(x_1,\cdots,x_{2l})}(t) \geq_L \Phi_{x_1,\cdots,x_{2l}}(t) \tag{7}$$

for all $x_1, \cdots, x_{2l} \in X$ and all $t > 0$. Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq_L \Psi_x\left(\frac{(l - lR)}{R} t\right) \tag{8}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2} g(2x)$$

for all $x \in X$.

It follows from (7) that

$$\mu_{2lf(x)-lf(2x)}(t) \geq_L \Psi_{2x}(t)$$

for all $x \in X$ and all $t > 0$. Thus

$$\mu_{f(x)-\frac{1}{2}f(2x)}\left(\frac{2R}{2l}t\right) \geq_L \Psi_x(t)$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{R}{l}$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$ and let $(X, \mu, \min)$ be a a complete* LRN*-space in which μ is a distribution function on $D^+_{[0,1]} = D^+$ i.e., $L = [0, 1]$ (see [49]). Let $f : X \to X$ be an odd mapping satisfying (6). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to X$ such that*

$$\mu_{f(x)-A(x)}(t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\Phi_{x_1,\cdots,x_{2l}}(t) := \frac{t}{t + \theta\sum_{j=1}^{2l}\|x_j\|^p}$$

for all $x_1, \cdots, x_{2l} \in X$. Then we can choose $R = 2^{p-1}$ and we get the desired result. □

4. Generalized Hyers-Ulam stability of the functional equation (1): an even case

In this section, using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x_1, \cdots, x_{2l}) = 0$ in complete LRN-spaces: an even case.

Theorem 4.1. *Let X be a linear space, $(Y, \mu, \mathcal{T}_\wedge)$ a complete* LRN*-space and Φ a mapping from X^{2l} to D^+_L ($\Phi(x_1, ..., x_{2l})$ is denoted by $\Phi_{x_1,...,x_{2l}}$) and*

$$\Psi_x := \Phi_{\underbrace{x,\cdots,x}_{l\ times},\underbrace{0,\cdots,0}_{l\ times}}$$

be functions such that there exists an $R < 1$ with

$$\Phi_{x_1,\cdots,x_{2l}}\left(\frac{R}{4}t\right) \geq_L \Phi_{2x_1,\cdots,2x_{2l}}(t)$$

for all $x_1, \cdots, x_{2l} \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2). Then $Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq_L \Psi_x((l - lR)t) \tag{9}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x_1 = \cdots = x_l = x$ and $x_{l+1} = \cdots = x_{2l} = 0$ in (2), we get

$$\mu_{4lf(\frac{x}{2})-lf(x)}(t) \geq_L \Phi_{\underbrace{x,\cdots,x}_{l\text{ times}},\underbrace{0,\cdots,0}_{l\text{ times}}}(t) = \Psi_x(t) \tag{10}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq_L \Psi_x(t)$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(R\varepsilon t) &= \mu_{4g(\frac{x}{2})-4h(\frac{x}{2})}(R\varepsilon t) \\ &= \mu_{g(\frac{x}{2})-h(\frac{x}{2})}\left(\frac{R}{4}\varepsilon t\right) \geq_L \Psi_{\frac{x}{2}}\left(\frac{Rt}{4}\right) \\ &\geq_L \Psi_x(t)\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq R\varepsilon$. This means that

$$d(Jg,Jh) \leq Rd(g,h)$$

for all $g, h \in S$.

It follows from (10) that $d(f,Jf) \leq \frac{1}{l}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(i) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{11}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (11) such that there exists a $r \in (0,\infty)$ satisfying

$$\mu_{f(x)-Q(x)}(rt) \geq_L \Psi_x(t)$$

for all $x \in X$;

(ii) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(iii) $d(f,Q) \leq \frac{1}{1-R} d(f,Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{1}{l-lR}.$$

This implies that the inequality (9) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 4.2. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let Y be a normed vector space with norm $\|\cdot\|$ and let $(X, \mu, \min)$ be a a complete* LRN*-space in which μ is a distribution function on $D^+_{[0,1]} = D^+$ i.e., $L = [0,1]$ (see [49]). Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (6). Then $Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(2^p-4)t}{(2^p-4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.1 by taking

$$\Phi_{x_1,\cdots,x_{2l}}(t) := \frac{t}{t + \theta \sum_{j=1}^{2l} \|x_j\|^p}$$

for all $x_1, \cdots, x_{2l} \in X$. Then we can choose $R = 2^{2-p}$ and we get the desired result. □

Similarly, we can obtain the following. We will omit the proof.

Theorem 4.3. *Let X be a linear space, $(Y, \mu, \mathcal{T}_\wedge)$ a complete* LRN*-space and Φ a mapping from X^{2l} to D^+_L ($\Phi(x_1, ..., x_{2l})$ is denoted by $\Phi_{x_1,...,x_{2l}}$) and*

$$\Psi_x := \Phi_{\underbrace{x,\cdots,x}_{l\ times},\underbrace{0,\cdots,0}_{l\ times}}$$

be functions such that there exists an $R < 1$ with

$$\Phi_{x_1,\cdots,x_{2l}}(4Rt) \geq_L \Phi_{\frac{x_1}{2},\cdots,\frac{x_{2l}}{2}}(t)$$

for all $x_1, \cdots, x_{2l} \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2). Then $Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq_L \Psi_x((l - lR)t)$$

for all $x \in X$ and all $t > 0$.

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let Y be a normed vector space with norm $\|\cdot\|$ and let $(X, \mu, \min)$ be a a complete* LRN*-space in which μ is a distribution function on $D^+_{[0,1]} = D^+$ i.e., $L = [0,1]$ (see [49]). Let $f : X \to Y$ be* *an even mapping satisfying $f(0) = 0$ and (6). Then $Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$\mu_{f(x)-Q(x)}(t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 4.3 by taking

$$\Phi_{x_1,\cdots,x_{2l}}(t) := \frac{t}{t + \theta\sum_{j=1}^{2l}\|x_j\|^p}$$

for all $x_1, \cdots, x_{2l} \in X$. Then we can choose $R = 2^{p-2}$ and we get the desired result. $\square$

5. Conclusion

Remark 5.1. Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (2) in which

$$\Phi_{x_1,\cdots,x_{2n}}(t) := \frac{t}{t + \varphi(x_1, \cdots, x_{2n})},$$

and

$$\mu_x(t) = \frac{t}{t + \|x\|}.$$

By a similar method to the proof of Theorem 3.3, one can show that if there exists an $R < 1$ such that

$$\varphi(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}) \leq 2R\varphi\left(\underbrace{\frac{x}{2}, \cdots, \frac{x}{2}}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}\right)$$

for all $x \in X$, then there exists a unique Jensen additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{R}{n - nR}\varphi(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}})$$

for all $x \in X$.

Remark 5.2. Let $f : X \to Y$ be an even mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (2) in which

$$\Phi_{x_1,\cdots,x_{2n}}(t) := \frac{t}{t + \varphi(x_1, \cdots, x_{2n})},$$

$$\mu_x(t) = \frac{t}{t + \|x\|}.$$

and $f(0) = 0$ such that

$$\sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x_1, \cdots, 2^j x_{2n}) < \infty$$

for all $x_1, \cdots, x_{2n} \in X$. By a similar method to the proof of Theorem 4.3, one can show that if there exists an $R < 1$ such that

$$\varphi(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}) \leq 4R\varphi\left(\underbrace{\frac{x}{2}, \cdots, \frac{x}{2}}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}}\right)$$

for all $x \in X$, then there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) - Q(x)\| \leq \frac{R}{n - nR}\varphi(\underbrace{x, \cdots, x}_{n \text{ times}}, \underbrace{0, \cdots, 0}_{n \text{ times}})$$

for all $x \in X$.

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ON $m\mathcal{I}_g$-NORMAL SPACES

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ABSTRACT. In this paper, characterizations and properties of $m\mathcal{I}_g$-normal space are given. Moreover, Urysohn's Lemma on $m\mathcal{I}_g$-normal spaces is proved.

1. INTRODUCTION AND PRELIMINARIES

Ideals were studied by Kuratowski [6] and Vaidyanathaswamy [15]. Their applications have been researched intensively (see [2], [4]).

The concept of minimal spaces was introduced by Popa and Noiri [13], which is a generalization of topological spaces. They given some characterizations and properties of m-regular spaces and m-normal spaces. The concept of ideal minimal spaces was studied by Ozbakir and Yildirim [9]. They introduced $m\mathcal{I}_g$-closed sets in ideal minimal spaces and investigated some basic properties.

The purpose of this paper is to study separation properties in ideal minimal spaces. We introduce the concept of $m\mathcal{I}_g$-normal spaces and give their characterizations and properties. In addition, we prove Urysohn's lemma on $m\mathcal{I}_g$-normal spaces.

Throughout this paper, spaces always mean minimal spaces or ideal minimal spaces on which no separation axiom is assumed, and maps are onto. N denotes the set of all natural integers. 2^X denotes the power set of X. If $\mathcal{U} \subset 2^X$, $Y \subset X$ and $x \in X$, $\mathcal{U}_Y$ denotes $\{U \bigcap Y : U \in \mathcal{U}\}$, $\mathcal{U}(x)$ denotes $\{U \in \mathcal{U} : x \in U\}$, $\mathcal{U}^c$ denotes $\{X - U : U \in \mathcal{U}\}$.

2. PRELIMINARIES

Let X be a set and $\mathcal{I}, \mathcal{M} \subset 2^X$. $\mathcal{I}$ is called an ideal on X if $\mathcal{I}$ satisfies the following conditions:

(1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

$\mathcal{M}$ is called a minimal structure on X if $\emptyset, X \in \mathcal{M}$. By $(X, \mathcal{M})$, we denote a set X with a minimal structure $\mathcal{M}$ on X. Simply we call $(X, \mathcal{M})$ a minimal space (briefly m-space) [13]. Elements in $\mathcal{M}$ are called m-open subsets and the complements are called m-closed subsets.

Let $(X, \mathcal{M})$ be a m-space. For $A \subset X$, the m-closure of A and the m-interior of A are defined as the following [6]:

$$int_m(A) = \bigcup\{U : U \subset A, U \in \mathcal{M}\},$$

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$$cl_m(A) = \bigcap\{F : F \supset A, X - F \in \mathcal{M}\}.$$

Definition 2.1 ([3])**.** *A m-space $(X, \mathcal{M})$ is called having property $[\mathcal{U}]$ if the union of every subfamily of $\mathcal{M}$ belongs to $\mathcal{M}$.*

If $\mathcal{I}$ is an ideal on X and $\mathcal{M}$ is a minimal structure on X, then $(X, \mathcal{M}, \mathcal{I})$ is called an ideal m-space.

Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. A set operator $(.)^* : 2^X \longrightarrow 2^X$, called a local map [1] of A with respect to $\mathcal{M}$ and $\mathcal{I}$, is defined as follows: for any $A \subset X$,

$$A^*(\mathcal{I}, \mathcal{M}) = \{x \in X : V \cap A \notin \mathcal{I} \ for\ every\ V \in \mathcal{M}(x)\}.$$

For $A \subset X$, the m-closure of A and the m-interior of A in X are defined as the following [6]:

$cl^*_m(A) = A \cup A^*(\mathcal{I}, \mathcal{M}),$
$int^*_m(A) = X - cl^*_m(X - A).$

When there is no chance for confusion, we will simply write $i_m A$ for $int_m(A)$, $c_m A$ for $cl_m(A)$, A^* for $A^*(\mathcal{I}, \mathcal{M})$, $i^*_m A$ for $int^*_m(A)$ and $c^*_m A$ for $cl^*_m(A)$, respectively.

Lemma 2.2 ([3], [12])**.** *Let $(X, \mathcal{M})$ be a m-space and $A, B \subset X$. Then the following properties hold.*

(1) *If $A \subset B$, then $c_m A \subset c_m B$ and $i_m A \subset i_m B$.*
(2) *$c_m\emptyset = \emptyset, c_m X = X, i_m\emptyset = \emptyset$ and $i_m X = X$.*
(3) *If $A \in \mathcal{M}$, then $i_m A \subset A$; if $A \in \mathcal{M}^c$, then $A \subset c_m A$.*
(4) *$i_m i_m A = i_m A$ and $c_m c_m A = c_m A$.*
(5) *$X - c_m A = i_m(X - A)$ and $X - i_m A = c_m(X - A)$.*
(6) *$i_m(A \cap B) = i_m A \cap i_m B$ and $i_m(A \cup B) \supset i_m A \cup i_m B$.*
(7) *$c_m(A \cup B) = c_m A \cup c_m B$ and $c_m(A \cap B) \subset c_m A \cap c_m B$.*
(8) *$x \in cl_m(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in \mathcal{M}(x)$.*

Lemma 2.3 ([14])**.** *Let $(X, \mathcal{M})$ be a m-space and $A \subset X$, where $(X, \mathcal{M})$ has the property $[\mathcal{U}]$. Then the following properties hold.*

(1) *$i_m A = A$ if and only if $A \in \mathcal{M}$.*
(2) *$c_m A = A$ if and only if $A \in \mathcal{M}^c$.*
(3) *$i_m A \in \mathcal{M}$ and $c_m A \in \mathcal{M}^c$.*

Definition 2.4 ([9])**.** *Let $(X, \mathcal{M})$ be a m-space and $A \subset X$. Then A is called*

(1) *mg-closed in X if $c_m A \subset U$ whenever U is m-open and $A \subset U$.*
(2) *mg-open in X if $X - A$ is mg-closed in X.*

Definition 2.5 ([9])**.** *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $A \subset X$. Then A is called*

(1) *$m*$-closed in X if $c^*_m A = A$, and $m*$-open in X if $X - A$ is $m*$-closed in X.*
(2) *$m\mathcal{I}_g$-closed in X if $A^*_m \subset U$ whenever U is m-open and $A \subset U$, and $m\mathcal{I}_g$-open in X if $X - A$ is $m\mathcal{I}_g$-closed in X.*

Obviously, m-closed sets $\Rightarrow$ $m*$-closed sets and
m-closed sets $\Rightarrow$ mg-closed sets $\Rightarrow$ $m\mathcal{I}_g$-closed sets $\Leftarrow$ $m*$-closed sets.

Definition 2.6. *A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called*

(1) *m-continuous [7] if $f^{-1}(V)$ is m-open in X for each $V \in \mathcal{N}$.*
(2) *m-closed [11] if $f(F)$ is m-closed in Y for any $X - F \in \mathcal{M}$.*

Definition 2.7 ([11]). *A m-space $(X, \mathcal{M})$ is called m-normal (resp. mg-normal) if for each pair consisting of disjoint m-closed subsets (resp. mg-closed subsets) A and B, there exist disjoint m-open subsets U and V such that $A \subset U$ and $B \subset V$.*

3. On $m\mathcal{I}_g$-normal spaces

Definition 3.1. *A ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is called $m\mathcal{I}_g$-normal, if for each pair consisting of disjoint $m\mathcal{I}_g$-closed subsets A and B, there exist disjoint m-open subsets U and V such that $A \subset U$ and $B \subset V$.*

Clearly, if $\mathcal{I} = \{\emptyset\}$, then $m\mathcal{I}_g$-normal spaces and mg-normal spaces coincide.

Every $m\mathcal{I}_g$-normal space is m-normal. But the following Example 3.2 illustrates that m-normal spaces need not be $m\mathcal{I}_g$-normal.

Example 3.2. *Let $X = \{a, b, c, d\}$, $\mathcal{M} = \{\emptyset, \{b\}, \{a, b\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $\mathcal{M}^c = \{\emptyset, \{b\}, \{c, d\}, \{a, c, d\}, X\}$.*

Pick $A = \{b\}$ and $B = \{c, d\}$ (resp. $A = \{b\}$ and $B = \{a, c, d\}$), then there exist disjoint m-open subsets $U = \{a, b\}$ and $V = \{a, c, d\}$ such that $A \subset U$ and $B \subset V$. Hence X be m-normal space.

Put $C = \{b, c\}$ and $D = \{a, d\}$, then we have $C^*_m = \{b\}$ and $D^*_m = \{c, d\}$. This implies that C and D are disjoint $m\mathcal{I}_g$-closed subsets of X. But there not exists m-open subset containing C. Hence X be not $m\mathcal{I}_g$-normal space.

The following Theorem 3.3 gives some characterizations of $m\mathcal{I}_g$-normal spaces.

Theorem 3.3. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space. Then the following are equivalent.*

(1) X is $m\mathcal{I}_g$-normal;

(2) For each $m\mathcal{I}_g$-closed subset A and $m\mathcal{I}_g$-open subset U containing A, there exists m-open subset V containing A such that $A \subset V \subset c_m V \subset U$;

(3) For each pair consisting of disjoint $m\mathcal{I}_g$-closed subsets A and B, there exists m-open subsets U containing A such that $c_m U \cap B = \emptyset$;

(4) For each pair consisting of disjoint $m\mathcal{I}_g$-closed subsets A and B, there exist a m-open subset U containing A and a m-open subset V containing B such that $c_m U \cap c_m V = \emptyset$.

Proof. (1)$\Rightarrow$ (2). Let A be a $m\mathcal{I}_g$-closed set and U be a $m\mathcal{I}_g$-open set containing A. Then we have $X - U$ is $m\mathcal{I}_g$-closed set and $A \cap (X - U) = \emptyset$. Since X is a $m\mathcal{I}_g$-normal space, then there exist disjoint m-open subsets U_1 and V_1 such that $A \subset U_1$ and $X - U \subset V_1$. This implies that $X - V_1 \subset U$. Since $U_1 \cap V_1 = \emptyset$, then we obtain $c_m U_1 \subset X - V_1$. Set $V = U_1$, then $c_m V \subset X - V_1 \subset U$. Therefore, $A \subset V \subset c_m V \subset X - V_1 \subset U$.

(2)$\Rightarrow$ (3). Let A and B are disjoint $m\mathcal{I}_g$-closed subsets of X. Then we have $A \subset X - B$ and $X - B$ is $m\mathcal{I}_g$-open. By (2), there exists m-open subset U containing A such that $A \subset U \subset c_m U \subset X - B$. Therefore, $c_m U \cap B = \emptyset$.

(3)$\Rightarrow$ (4). Let A and B are disjoint $m\mathcal{I}_g$-closed subsets of X. By (3), there exists m-open subset U containing A such that $c_m U \cap B = \emptyset$. Since $c_m U$ is m-closed in X, then $c_m U$ is $m\mathcal{I}_g$-closed. By (3), there exists m-open subsets V containing B such that $c_m U \cap c_m V = \emptyset$.

(4)$\Rightarrow$ (1). This is obvious. □

If $\mathcal{I} = \{\emptyset\}$ in Theorem 3.3, then we have the following Corollary 3.4.

Corollary 3.4. *Let $(X, \mathcal{M})$ be a m-space. Then the following are equivalent.*

(1) *X is mg-normal;*

(2) *For each mg-closed subset A and mg-open subset U containing A, there exists m-open subset V containing A such that $A \subset V \subset c_m V \subset U$;*

(3) *For each pair consisting of disjoint mg-closed subsets A and B, there exists m-open subsets U containing A such that $c_m U \cap B = \emptyset$;*

(4) *For each pair consisting of disjoint mg-closed subsets A and B, there exist a m-open subset U containing A and a m-open subset V containing B such that $c_m U \cap c_m V = \emptyset$.*

Lemma 3.5. *Let $(X, \mathcal{M}, \mathcal{I})$ be a $m\mathcal{I}_g$-normal space. For each pair consisting of disjoint $m\mathcal{I}_g$-closed subsets A and B, there exist disjoint m-open subsets U and V such that $c_m^* A \subset U$ and $c_m^* B \subset V$.*

Proof. Let A and B are disjoint $m\mathcal{I}_g$-closed subsets of X. By Theorem 3.3, there exist a m-open subsets U containing A and a m-open subsets V containing B such that $c_m U \cap c_m V = \emptyset$. Since A is $m\mathcal{I}_g$-closed, then we have $A_m^* \subset U$. Hence $A_m^* \cup A = c_m^* A \subset U$. Similarly $c_m^* B \subset U$. □

Theorem 3.6. *Let $(X, \mathcal{M}, \mathcal{I})$ be a $m\mathcal{I}_g$-normal space. For each $m\mathcal{I}_g$-closed subset A and $m\mathcal{I}_g$-open subset U containing A, there exists m-open subset V such that $A \subset c_m^* A \subset V \subset i_m^* U \subset U$.*

Proof. Let A be a $m\mathcal{I}_g$-closed and U be a $m\mathcal{I}_g$-open containing A in X. Then A and $X - U$ are disjoint $m\mathcal{I}_g$-closed in X. By Lemma 3.5, there exist disjoint m-open subsets V_1 and V_2 such that $c_m^* A \subset V_1$ and $c_m^*(X - U) \subset V_2$. Thus we have $X - i_m^* U = c_m^*(X - U) \subset V_2$. Hence $X - V_2 \subset i_m^* U$. Since $V_1 \cap V_2 = \emptyset$, then we obtain $V_1 \subset X - V_2$. Therefore, $A \subset c_m^* A \subset V_1 \subset X - V_2 \subset i_m^* U \subset U$. □

If $\mathcal{I} = \{\emptyset\}$ in Theorem 3.6, then we have the following Corollary 3.7.

Corollary 3.7. *Let $(X, \mathcal{M})$ be a mg-normal space. For each mg-closed subset A and mg-open subset U containing A, there exists m-open subset V such that $A \subset c_m A \subset V \subset i_m U \subset U$.*

4. Properties of $m\mathcal{I}_g$-normal spaces

Recall that if $\mathcal{I}$ is an ideal of $(X, \mathcal{M})$ and $Y \subset X$, then $\mathcal{I}_Y$ is an ideal of $(Y, \mathcal{M}_Y)$ (see [8]). So $(Y, \mathcal{M}_Y, \mathcal{I}_Y)$ is also an ideal m-space. Thus, $(Y, \mathcal{M}_Y, \mathcal{I}_Y)$ is called a subspace of $(X, \mathcal{M}, \mathcal{I})$.

Lemma 4.1. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $A \subset Y \subset X$. Then*

$$A_m^*(\mathcal{I}_Y, \mathcal{M}_Y) = A_m^*(\mathcal{I}, \mathcal{M}) \cap Y.$$

Proof. Let $y \in A_m^*(\mathcal{I}, \mathcal{M}) \cap Y$, for every $V \in \mathcal{M}_Y(y)$. Then there exists $U \in \mathcal{M}$ such that $V = U \cap Y$. Thus we have $U \in \mathcal{M}(y)$. This implies that $U \cap A \notin \mathcal{I}$. Since $\mathcal{I}_Y \subset \mathcal{I}$ and $U \cap Y \cap A = U \cap A$, then we obtain $V \cap A = U \cap Y \cap A \notin \mathcal{I}_Y$. Hence $y \in A_m^*(\mathcal{I}_Y, \mathcal{M}_Y)$ and $A_m^*(\mathcal{I}, \mathcal{M}) \cap Y \subset A_m^*(\mathcal{I}_Y, \mathcal{M}_Y)$.

Conversely. Since $A_m^*(\mathcal{I}_Y, \mathcal{M}_Y) = \{y \in Y : V \cap A \notin \mathcal{I}_Y \; for \; every \; V \in \mathcal{M}_Y(y)\}$, then $A_m^*(\mathcal{I}_Y, \tau_Y) \subset Y$. Let $y \in A_m^*(\mathcal{I}_Y, \mathcal{M}_Y)$, for every $U \in \mathcal{M}(y)$. Then $U \cap Y \in \mathcal{M}_Y(y)$. This implies that $(U \cap Y) \cap A \notin \mathcal{I}_Y$. Since $(U \cap Y) \cap A = U \cap A$, then we have $U \cap A \notin \mathcal{I}$. Hence $y \in A_m^*(\mathcal{I}, \mathcal{M})$ and $A_m^*(\mathcal{I}_Y, \mathcal{M}_Y) \subset A_m^*(\mathcal{I}, \mathcal{M}) \cap Y$. □

Lemma 4.2. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space and $A \subset Y \subset X$. If A is a $m(\mathcal{I}_Y)_g$-closed subset of $(Y, \mathcal{M}_Y, \mathcal{I}_Y)$ and Y is $m*$-closed in X, then A is a $m\mathcal{I}_g$-closed subset of $(X, \mathcal{M}, \mathcal{I})$.*

Proof. Let $A \subset U \in \mathcal{M}$, then $A \subset U \cap Y \in \mathcal{M}_Y$. Since A is $m(\mathcal{I}_Y)_g$-closed in Y, then $A^*_m(\mathcal{I}_Y, \mathcal{M}_Y) \subset U \cap Y$. By Lemma 4.1, we have $A^*_m(\mathcal{I}, \mathcal{M}) \cap Y \subset U \cap Y$. Since $A \subset Y$ and Y is $m*$-closed in X, then $A^*_m(\mathcal{I}, \mathcal{M}) \subset c^*_m A \subset c^*_m Y = Y$. Thus we obtain $A^*_m(\mathcal{I}, \mathcal{M}) \subset U \cap Y$. Therefore $A^*_m(\mathcal{I}, \mathcal{M}) \subset U$ and A is $m\mathcal{I}_g$-closed in X. □

Theorem 4.3. *Let $(X, \mathcal{M}, \mathcal{I})$ be a $m\mathcal{I}_g$-normal space and $Y \subset X$. If Y be a $m*$-closed subset of X, then $(Y, \mathcal{M}_Y, \mathcal{I}_Y)$ is a $m(\mathcal{I}_Y)_g$-normal space.*

Proof. Let A and B are disjoint $m(\mathcal{I}_Y)_g$-closed subsets of Y. Since Y be a $m*$-closed in X, then A and B are disjoint $m\mathcal{I}_g$-closed subsets of X by Lemma 4.2. Since $(X, \mathcal{M}, \mathcal{I})$ is $m\mathcal{I}_g$-normal space, then there exist disjoint m-open subsets U and V such that $A \subset U$ and $B \subset V$. Hence, $A \subset U \cap Y \in \mathcal{M}_Y$ and $B \subset V \cap Y \in \mathcal{M}_Y$. Note that $(U \cap Y) \cap (V \cap Y) = \emptyset$. Therefore $(Y, \mathcal{M}_Y, \mathcal{J})$ is $m(\mathcal{I}_Y)_g$-normal space. □

Lemma 4.4. *Let $(X, \mathcal{M}, \mathcal{I})$ be an ideal m-space satisfying the property $[\mathcal{U}]$ and B, $Y \subset X$. If B is a $m\mathcal{I}_g$-closed in X and Y is m-open and m-closed in X, then $B \cap Y$ is a $m(\mathcal{I}_Y)_g$-closed subset of $(Y, \mathcal{M}_Y, \mathcal{I}_Y)$.*

Proof. Let $B \cap Y \subset U \in \mathcal{M}_Y$, then $B \subset U \cup (X - Y)$. $U \in \mathcal{M}_Y$ and $Y \in \mathcal{M}$ imply $U \in \mathcal{M}$. Since Y is also a m-closed in X, then $U \cup (X - Y) \in \mathcal{M}$. Since X has property $[\mathcal{U}]$ and B is a $m\mathcal{I}_g$-closed in X, then $B^*_m(\mathcal{I}, \mathcal{M}) \subset U \cup (X - Y)$. By Lemma 4.1, $(B \cap Y)^*_m(\mathcal{I}_Y, \mathcal{M}_Y) = (B \cap Y)^*_m(\mathcal{I}, \mathcal{M}) \cap Y \subset B^*_m(\mathcal{I}, \mathcal{M}) \cap Y \subset (U \cup (X - Y)) \cap Y = U$. Therefore $B \cap Y$ is $m(\mathcal{I}_Y)_g$-closed in Y. □

Lemma 4.5 ([5]). *If every $\mathcal{I}_\alpha$ is an ideal of X_α, then $\{\bigcup_{\alpha \in \bigwedge} I_\alpha : I_\alpha \in \mathcal{I}_\alpha\}$ is an ideal of $\bigcup_{\alpha \in \bigwedge} X_\alpha$.*

Let $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in \Lambda\}$ be a family of pairwise disjoint m-space, i.e., $X_\alpha \cap X_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. Put $X = \bigcup_{\alpha \in \Lambda} X_\alpha$, $\mathcal{M} = \{\bigcup_{\alpha \in \Lambda} A_\alpha : A_\alpha \in \mathcal{M}_\alpha\}$. Obviously, $\emptyset, X \in \mathcal{M}$. So $\mathcal{M}$ is a minimal structure on X.

For $A \subset X$, we claim $A \in \mathcal{M}$ if and only if $A \cap X_\alpha \in M_\alpha$ for each $\alpha \in \bigwedge$. In fact. Suppose $A \in \mathcal{M}$. Then $A = \bigcup_{\beta \in \bigwedge} A_\beta$, where $A_\beta \in M_\beta$ $(\beta \in \Lambda)$. So for each $\alpha \in \bigwedge$, $A \cap X_\alpha = (\bigcup_{\beta \in \bigwedge} A_\beta) \cap X_\alpha = \bigcup_{\beta \in \bigwedge} (A_\beta \cap X_\alpha)$. If $\beta \neq \alpha$, then $A_\beta \cap X_\alpha \subset X_\beta \cap X_\alpha = \emptyset$. Thus $A \cap X_\alpha = A_\alpha \cap X_\alpha = A_\alpha \in M_\alpha$. On the other hand. $A = A \cap X = A \cap (\bigcup_{\alpha \in \bigwedge} X_\alpha) = \bigcup_{\alpha \in \bigwedge} (A \cap X_\alpha)$. By $A \cap X_\alpha \in M_\alpha$, $A \in \mathcal{M}$. Thus, the m-space $(X, \mathcal{M})$ is called the mini sum of $\{(X_\alpha, \mathcal{M}_\alpha) : \alpha \in \Lambda\}$. We denote $(X, \mathcal{M})$ by $\bigoplus_{\alpha \in \bigwedge} X_\alpha$.

It is easy to prove that every X_α are both m-open and m-closed in $\bigoplus_{\alpha \in \bigwedge} X_\alpha$.

Obviously, if every m-space $(X_\alpha, \mathcal{M}_\alpha)$ satisfies the property $[\mathcal{U}]$, then $\bigoplus_{\alpha \in \bigwedge} X_\alpha$ also satisfies the property $[\mathcal{U}]$.

Let $\{(X_\alpha, \mathcal{M}_\alpha, \mathcal{I}_\alpha) : \alpha \in \bigwedge\}$ be a family of pairwise disjoint ideal m-spaces, i.e., $X_\alpha \cap X_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. Put $X = \bigcup_{\alpha\in\Lambda} X_\alpha$, $\mathcal{M} = \{\bigcup_{\alpha\in\Lambda} A_\alpha : A_\alpha \in \mathcal{M}_\alpha\}$, $\mathcal{I} = \{\bigcup_{\alpha\in\bigwedge} I_\alpha : I_\alpha \in \mathcal{I}_\alpha\}$. By Lemma 4.5, $(X, \mathcal{M}, \mathcal{I})$ is a ideal m-space. For $A \subset X$, $A \in \mathcal{M}$ if and only if $A \cap X_\alpha \in M_\alpha$ for each $\alpha \in \bigwedge$. Thus, $\langle X, \mathcal{M}, \mathcal{I}\rangle$ is called the ideal mini sum of $\{(X_\alpha, \mathcal{M}_\alpha, \mathcal{I}_\alpha) : \alpha \in \Lambda\}$. We also denote $(X, \mathcal{M}, \mathcal{I})$ by $\bigoplus_{\alpha\in\bigwedge} X_\alpha$.

Theorem 4.6. *Let $\{(X_\alpha, \mathcal{M}_\alpha, \mathcal{I}_\alpha) : \alpha \in \Lambda\}$ be a family of pairwise disjoint ideal m-spaces satisfying the property $[\mathcal{U}]$ and let $\mathcal{I} = \{\bigcup_{\alpha\in\bigwedge} I_\alpha : I_\alpha \in \mathcal{I}_\alpha\}$. Then $\bigoplus_{\alpha\in\bigwedge} X_\alpha$ is a $m\mathcal{I}_g$-normal space if and only if every X_α is a $m(\mathcal{I}_\alpha)_g$-normal space.*

Proof. The proof of Necessity follows from Theorem 4.3.

Sufficiency. Denote $X = \bigoplus_{\alpha\in\bigwedge} X_\alpha$.

Let A and B be disjoint $m\mathcal{I}_g$-closed subsets of X. Then for any $\alpha \in \Lambda$, $A \cap X_\alpha$ and $B \cap X_\alpha$ are disjoint $m\mathcal{I}_g$-closed subsets of X_α by Lemma 4.4. Since X_α is $m(\mathcal{I}_\alpha)_g$-normal, then there exist disjoint m-open subsets U_α and V_α of X_α such that $A \cap X_\alpha \subset U_\alpha$ and $B \cap X_\alpha \subset V_\alpha$.

Clearly,

$$A = A \bigcap (\bigcup_{\alpha\in\bigwedge} X_\alpha) \subset U = \bigcup_{\alpha\in\bigwedge} U_\alpha,$$
$$B = B \bigcap (\bigcup_{\alpha\in\bigwedge} X_\alpha) \subset V = \bigcup_{\alpha\in\bigwedge} V_\alpha.$$

If $\alpha \neq \beta$, then $U_\alpha \cap V_\beta \subset X_\alpha \cap X_\beta = \emptyset$. Thus for any $\alpha, \beta \in \Lambda$, $U_\alpha \cap V_\beta = \emptyset$. Hence $U \bigcap V = \bigcap_{\alpha,\ \beta\in\Lambda} (U_\alpha \cap V_\beta) = \emptyset$.

Since every X_α is m-open in X and X has property $[\mathcal{U}]$, then U and V are m-open in X. Therefore X is $m\mathcal{I}_g$-normal. □

Lemma 4.7 ([10])**.** *If $f : X \to Y$ is a map, $A \subset X$ and $B \subset Y$, then $f^{-1}(B) \subset A$ if and only if $B \subset Y - f(X - A)$.*

Lemma 4.8. *Let $f : (X, \mathcal{M}, \mathcal{I}) \to (Y, \mathcal{N}, \mathcal{J})$ be a map with $f^{-1}(\mathcal{J}) \subset \mathcal{I}$. Then*

(1) If f is a m-continuous map, then $f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M}) \subset f^{-1}(B^*_m(\mathcal{J}, \mathcal{N}))$ for any $B \subset Y$.

(2) If f is a m-continuous and m-closed map and B is a $m\mathcal{J}_g$-closed subset of Y, then $f^{-1}(B)$ is a $m\mathcal{I}_g$-closed subset of X.

Proof. (1) Suppose that $f(f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M})) - B^*_m(\mathcal{J}, \mathcal{N}) \neq \emptyset$, pick

$$y \in f(f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M})) - B^*_m(\mathcal{J}, \mathcal{N}).$$

Thus we have $y \in f(f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M}))$. This implies that $y = f(x)$ for some $x \in f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M})$. Since $y \notin B^*_m(\mathcal{J}, \mathcal{N})$, then $V \cap B \in \mathcal{J}$ for some $V \in \mathcal{N}(y)$. Since f is m-continuous, then $f^{-1}(V) \in \mathcal{N}(x)$. Thus $f^{-1}(V) \cap f^{-1}(B) \notin \mathcal{I}$. But, $V \cap B \in \mathcal{J}$ implies that $f^{-1}(V) \cap f^{-1}(B) = f^{-1}(V \cap B) \in f^{-1}(\mathcal{J}) \subset \mathcal{I}$. This is a contradiction. Hence $f(f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M})) - B^*_m(\mathcal{J}, \mathcal{N}) = \emptyset$. Therefore, $f(f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M})) \subset B^*_m(\mathcal{J}, \mathcal{N})$ and $f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M}) \subset f^{-1}(B^*_m(\mathcal{J}, \mathcal{N}))$.

(2) Let $f^{-1}(B) \subset U \in \mathcal{M}$, then $B \subset Y - f(X - U)$ by Lemma 4.7. Since f is m-closed, then $Y - f(X - U) \in \mathcal{N}$. Since B is $m\mathcal{J}_g$-closed in Y, then $B^*_m(\mathcal{J}, \mathcal{N}) \subset Y - f(X - U)$. By Lemma 4.7, $f^{-1}(B^*_m(\mathcal{J}, \mathcal{N})) \subset U$. By (1), $f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M}) \subset f^{-1}(B^*_m(\mathcal{J}, \mathcal{N}))$. Hence $f^{-1}(B)^*_m(\mathcal{I}, \mathcal{M}) \subset U$. Therefore, $f^{-1}(B)$ is $m\mathcal{I}_g$-closed in X. □

Theorem 4.9. *Let $f : (X, \mathcal{M}, \mathcal{I}) \to (Y, \mathcal{N}, \mathcal{J})$ be a m-continuous and m-closed map with $f^{-1}(\mathcal{J}) \subset \mathcal{I}$. If X is $m\mathcal{I}_g$-normal, then Y is $m\mathcal{J}_g$-normal.*

Proof. Let A and B are disjoint $m\mathcal{J}_g$-closed subsets of Y, then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $m\mathcal{I}_g$-closed subsets of X by Lemma 4.8. Since X is $m\mathcal{I}_g$-normal, then exist disjoint m-open subsets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Lemma 4.7, $A \subset Y - f(X - U)$ and $B \subset Y - f(X - V)$. Note that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint m-open subsets of Y. Hence Y is $m\mathcal{J}_g$-normal. □

If $\mathcal{I} = \{\emptyset\}$ in Theorem 4.9, then we have the following Corollary 4.10.

Corollary 4.10. *If X is mg-normal and $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a m-continuous and m-closed map, then Y is mg-normal.*

5. Urysohn's lemma on $m\mathcal{I}_g$-normal spaces.

Below we give Urysohn's lemma on $m\mathcal{I}_g$-normal spaces.

Theorem 5.1. *An ideal m-space $(X, \mathcal{M}, \mathcal{I})$ is $m\mathcal{I}_g$-normal if and only if for each pair of disjoint $m\mathcal{I}_g$-closed subsets A and B of X, there exists a m-continuous mapping $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Sufficiency. Suppose for each pair of disjoint $m\mathcal{I}_g$-closed subsets A and B of X, there exists a m-continuous mapping $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Put $U = f^{-1}([0, 1/2))$, $V = f^{-1}((1/2, 1])$, then U and V are disjoint m-open subsets of X such that $A \subset U$ and $B \subset V$. Hence X is $m\mathcal{I}_g$-normal.

Necessity. Suppose X is $m\mathcal{I}_g$-normal. For each pair of disjoint $m\mathcal{I}_g$-closed subsets A and B of X, $A \subset X - B$, where A is $m\mathcal{I}_g$-closed and $X - B$ in X is $m\mathcal{I}_g$-open in X. By Theorem 3.3, there exists an m-open subset $U_{1/2}$ of X such that

$$A \subset U_{1/2} \subset c_m U_{1/2} \subset X - B.$$

Since $A \subset U_{1/2}$, A is $m\mathcal{I}_g$-closed in X and $U_{1/2}$ is $m\mathcal{I}_g$-open in X, then there exists an m-open subset $U_{1/4}$ of X such that $A \subset U_{1/4} \subset c_m U_{1/4} \subset U_{1/2}$ by Theorem 3.3. Since $c_m U_{1/2} \subset X - B$, $c_m U_{1/2}$ is $m\mathcal{I}_g$-closed in X and $X - B$ is $m\mathcal{I}_g$-open in X, then there exists an m-open subset $U_{3/4}$ of X such that $c_m U_{1/2} \subset U_{3/4} \subset c_m U_{3/4} \subset X - B$ by Theorem 3.3. Thus, there exist two m-open subsets $U_{1/2}$ and $U_{3/4}$ of X such that

$$A \subset U_{1/4} \subset c_m U_{1/4} \subset U_{1/2} \subset c_m U_{1/2} \subset U_{3/4} \subset c_m U_{3/4} \subset X - B.$$

We get a family $\{U_{m/2^n} : 1 \leq m < 2^n, n \in N\}$ of m-open subsets of X, denotes $\{U_{m/2^n} : 1 \leq m < 2^n, n \in N\}$ by $\{U_\alpha : \alpha \in \Gamma\}$. $\{U_\alpha : \alpha \in \Gamma\}$ satisfies the following condition:

(1) $A \subset U_\alpha \subset c_m U_\alpha \subset X - B$,

(2) if $\alpha < \alpha'$, then $c_m U_\alpha \subset U_{\alpha'}$.

We define $f : X \to [0, 1]$ as follows:

$$f(x) = \begin{cases} inf\{\alpha \in \Gamma : x \in U_\alpha\}, & \text{if } x \in U_\alpha \text{ for some } \alpha \in \Gamma, \\ 1, & \text{if } x \notin U_\alpha \text{ for any } \alpha \in \Gamma. \end{cases}$$

For each $x \in A$, $x \in U_\alpha$ for any $\alpha \in \Gamma$ by (1), so $f(x) = inf\{\alpha \in \Gamma : x \in U_\alpha\} = inf\Gamma = 0$. Thus, $f(A) = \{0\}$.

For each $x \in B$, $x \notin X - B$ implies $x \notin U_\alpha$ for any $\alpha \in \Gamma$ by (1), so $f(x) = 1$. Thus, $f(B) = \{1\}$.

We have to show f is m-continuous.

For $x \in X$ and $\alpha \in \Gamma$, we have the following Claim:

Claim 1: if $f(x) < \alpha$, then $x \in U_\alpha$.

Suppose $f(x) < \alpha$, then $inf\{\alpha \in \Gamma : x \in U_\alpha\} < \alpha$, so there exists $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha_1 < \alpha$. By (2), $c_m U_{\alpha_1} \subset U_\alpha$.Notice that $x \in U_{\alpha_1}$. Hence $x \in U_\alpha$.

Claim 2: if $f(x) > \alpha$, then $x \notin c_m U_\alpha$.

Suppose $f(x) > \alpha$, then there exists $\alpha_1 \in \Gamma$ such that $\alpha < \alpha_1 < f(x)$. Notice that $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ implies $\alpha_1 \geq inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$. Thus, $\alpha_1 \notin \{\alpha \in \Gamma : x \in U_\alpha\}$. So $x \notin U_{\alpha_1}$. By (2), $c_m U_\alpha \subset U_{\alpha_1}$. Hence $x \notin c_m U_\alpha$.

Claim 3: if $x \notin c_m U_\alpha$, then $f(x) \geq \alpha$.

Suppose $x \notin c_m U_\alpha$, we claim that $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Otherwise, there exists $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha \geq \beta$. $x \notin c_m U_\alpha$ implies $\alpha \notin \{\alpha \in \Gamma : x \in U_\alpha\}$. So $\alpha \neq \beta$. Thus $\alpha > \beta$. By (2), $cU_\beta \subset U_\alpha$. So $x \notin \beta$, contridiction. Therefore $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Hence $\alpha \leq inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$.

For $x_0 \in X$, if $f(x_0) \in (0,1)$, suppose V is an m-open neighborhood of $f(x_0)$ in $[0,1]$, then there exists $\varepsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V \bigcap (0,1)$. Pick $\alpha', \alpha" \in \Gamma$ such that

$$0 < f(x_0) - \epsilon < \alpha' < f(x_0) < \alpha" < f(x_0) + \epsilon < 1.$$

By **Claim 1** and **Claim 2**, $x_0 \in U_\alpha"$, $x_0 \notin c_m U'_\alpha$. Put $U = U_\alpha" - c_m U'_\alpha$, then U is an m-open neighborhood of x_0 in X.

We will prove that $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$. if $y \in f(U)$, then $y = f(x)$ for some $x \in U$. $x \in U$ implies that $x \in U_\alpha"$ and $x \notin c_m U'_\alpha$. Since $x \in U_\alpha"$, then $\alpha" \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Thus, $\alpha" \geq inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$. Notice that $\alpha" < f(x_0) + \epsilon$. Therefore $f(x) < f(x_0) + \epsilon$. Since $x \notin c_m U'_\alpha$, then $f(x) \geq \alpha'$ by **Claim 3**. Notice that $f(x_0) - \epsilon < \alpha'$. Therefore $f(x) > f(x_0) - \epsilon$. Hence, $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Therefore, $f(U) \subset V$. This implies f is m-continuous at x_0.

if $f(x_0) = 0,$ or 1, the proof that f is m-continuous at x_0 is similar. □

If $\mathcal{I} = \{\emptyset\}$ in Theorem 5.1, then we have the following Corollary 5.2.

Corollary 5.2. *A m-space $(X, \mathcal{M})$ is mg-normal if and only if for each pair of disjoint mg-closed subsets A and B of X, there exists a m-continuous mapping $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

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Impact of (E. A.) Property on Common Fixed Point Theorems in Fuzzy Metric Spaces

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Abstract: In this paper, it is observed that the property (E.A.) relatively relaxes the required containment of range of one mapping into the range of other which is utilized to construct the sequence of joint iterates to prove common fixed point theorems. As a consequence, several previously known fixed point theorems in fuzzy metric spaces are generalized and improved.

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1. Introduction and Preliminaries

It was proved to be a turning point not only for mathematics but also for applications of mathematics in diverse real world problems when notion of fuzzy set was introduced by Zadeh [23] with a view to represent the vagueness in everyday life. By now the fuzzy sets are generalized in several ways which includes intutionistic fuzzy set, soft fuzzy sets and most probably some more in years to come. In mathematical programming problems are expressed as optimizing some goal function given certain constraints suggested by some concrete practical situation. There do exist real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using fuzzy sets [20]. In fact the richness of applications of fuzzy mathematics has engineered the all round development of fuzzy mathematics. The study of fuzzy metric space has also been carried out in several ways (see [5] and [10]). George and Veeramani [7] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [11] with a view to obtain a Hausdorff topology on fuzzy metric space which has found very fruitful applications in quantum particle physics, particularly in connection with both string and ε^{∞} theory (see [6] and references cited therein). Many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [2], [4], [14], [17], [18], [19], [21] and [22].

As patterned by Jungck [8], a metrical common fixed point theorem is comprised of conditions on completeness, continuity, commutability and suitable containment of ranges of the involved mappings besides an appropriate contraction condition. In the process of improving an existing common fixed point theorem, the researchers of this domain are required to improve to one or more of these constituent conditions.

In this paper, we observe that the property (E. A.) relatively relaxes the containment of range of one mapping into the range of other required to construct the sequence of joint iterates to prove common fixed point theorems. As a consequence, several previously known fixed point theorems in fuzzy metric spaces are generalized and improved (e.g. [3, 14, 19, 21]).

Definition1.1. ([23]) Let X be any set. A fuzzy set G in X is a function with domain X and values in [0, 1].

Definition1.2. ([11]) A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t- norm if it satisfies the following conditions:

(I) $*$ is associative and commutative,

(II) $*$ is continuous,

(III) $a * 1 = a$ for every $a \in [0, 1]$,

(IV) $a * b \le c * d$ if $a \le c$ and $b \le d$ for all $a,b,c,d \in [0, 1]$.

Definition1.3. ([7]) A triplet $(X, M, *)$ is a fuzzy metric space whenever X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying (for every $x, y, z \in X$ and $s, t > 0$) the following conditions:

(I) $M(x, y, t) > 0$

(II) $M(x, y, t) = 1$ iff $x = y$,

(III) $M(x, y, t) = M(y, x, t)$,

(IV) $M(x, y, t) * M(y, z, t) \le M(x, z, t+s)$,

(V) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Note that, M (x, y, t) can be realized as the measure of nearness between *x* and *y* with respect to *t*. It is known that M(x, y, .) is nondecreasing for all $x, y \in X$.

Let $(X, M,*)$ is a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}$. The collection $\{B(x, r, t): x \in X, 0 < r < 1, t > 0\}$ is a neighbourhood system for a topology τ on X induced by fuzzy metric M. This topology is Hausdorff and first countable.

Definition1.4. ([7]) A sequence $\{x_n\}$ in X converges to x if and only if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0 \in N$, $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.

Remark 1.1. ([7]). Let (X, d) be a metric space. If we define $a*b = ab$ for all $a, b \in [0, 1]$ and $M_d(x, y, t) = t/(t + d(x,y))$ for every $(x, y, t) \in X \times X \times (0, +\infty)$, then $(X, M_d, *)$ is a fuzzy metric. The fuzzy metric space $(X, M_d, *)$ is complete iff the metric space (X, d) is complete.

Definition1.5. ([18]) A pair of self-mappings (f, g) defined on a fuzzy metric space $(X, M, *)$ is said to be compatible (or asymptotically commuting) if for all $t > 0$,

$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

Definition1.6. The pair (f, g) is called non-compatible, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, but either $\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1$ or the limit does not exists.

Definition1.7. ([2]) A pair of self-mappings (f, g) defined on a fuzzy metric pace $(X, M, *)$ is said to satisfy the property (E. A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

It is clear from the above definitions that a compatible as well as non-compatible pair satisfies the property (E. A.).

For further details on the property (E. A.), one can consult [1].

Definition1.8. ([9]) A pair of self mappings (f, g) defined on a fuzzy metric space $(X, M, *)$ is called weakly compatible if they commute at their coincidence point i.e. fx = gx implies that fgx = gfx.

The following definitions will be utilized to state various results in Section 2.

Definition1.9. ([4]) Let $(X, M, *)$ be a fuzzy metric space and $f, g: X \to X$ be a pair of maps. The map f is called a fuzzy contraction with respect to map g if there exists an upper semi continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(\tau) < \tau$ for every $\tau > 0$ such that

$$\frac{1}{M(fx, fy, t)} - 1 \leq \phi\left(\frac{1}{m(f,g,x,y,t)} - 1\right)$$

for every $x, y \in X$ and each $t > 0$, where $m(f, g, x, y, t) = \min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}$.

Definition1.10. ([4]) Let $(X, M, *)$ be a fuzzy metric space and $f, g: X \to X$ be a pair of maps. The map f is called a fuzzy k-contraction with respect to g if there exists $k \in (0, 1)$, such that

$$\frac{1}{M(fx, fy, t)} - 1 \leq k\left(\frac{1}{m(f,g,x,y,t)} - 1\right)$$

for every $x, y \in X$ and each $t > 0$, where $m(f, g, x, y, t) = \min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}$.

Definition 1.11. Let f, g, S and T be four self mappings of a fuzzy metric space (X, M,*). Then the mappings f and g are called a generalized fuzzy contraction with respect to S and T if there exists an upper semi continuous function r: $(0,\infty) \to (0,\infty)$, with $r(\tau) < \tau$ for every $\tau > 0$, such that

$$\frac{1}{M(fx,gy,t)} - 1 \le r\left(\frac{1}{\min\{M(Sx,Ty,t),M(fx,Sx,t),M(gy,Ty,t)\}} - 1\right) \text{-------------(1.11.1.)}$$

for each x, y $\in$ X and t > 0.

2. Results

Using the property (E. A.), we prove the following theorem for two pairs of mappings under relatively lighter conditions.

Theorem 2.1. Let f, g, S and T be four self mappings of a fuzzy metric space(X, M, *). Suppose that

(I) the pair (f, S) (or (g, T)) satisfies the property (E. A.),

(II) *f*(X)$\subset$T(X) (or *g*(X)$\subset$S(X)),

(III) S(X) (or T(X)) is a closed subset of X and

(IV) *f* and *g* are the generalized fuzzy contraction with respect to S and T.

Then

(a) the pairs (*f*, S) and (*g*, T) have a point of coincidence, whereas

(b) *f*, *g*, S, and T have a unique common fixed point provided both the pairs (f, S) and (g, T) are weakly compatible.

Proof: If the pair (*f*, S) enjoys the property (E.A.), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in X.$$

Since f(X) $\subset$ T(X), hence for every $\{x_n\}$ there exists $\{y_n\}$ in X such that $fx_n = Ty_n$. Therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = z$. Thus, in all we have $\lim_{x\to\infty} fx_n \to z$, $\lim_{x\to\infty} Sx_n \to z$ and $\lim_{x\to\infty} Ty_n \to z$. Now, we assert that $\lim_{n\to\infty} gy_n \to z$. If not, then using (1.11.1), we have

$$\frac{1}{M(fx_n,gy_n,t)} - 1 \le r\left(\frac{1}{\min\{M(Sx_n,Ty_n,t),M(fx_n,Sx_n,t),M(gy_n,Ty_n,t)\}} - 1\right)$$

which on making n→∞, reduces to

$$\frac{1}{M(z,gy_n,t)}-1\le r\left(\frac{1}{M(z,gy_n,t)}-1\right)$$

$$<\left(\frac{1}{M(z,gy_n,t)}-1\right)$$

a contradiction. Hence $\lim\limits_{n\to\infty} gy_n \to z$ which implies that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z \in X$$

Since S(X) is a closed subspace of X, therefore $\lim\limits_{n\to\infty} Sx_n = z \in S(X)$ and henceforth there exists a point $u \in X$ such that Su = z. Now we assert that fu = Su. If not, then using (1.11.1), we have

$$\frac{1}{M(fu,gy_n,t)}-1\le r\left(\frac{1}{\min\{M(Su,Ty_n,t),M(fu,Su,t),M(gy_n,Ty_n,t)\}}-1\right)$$

which on making $n\to\infty$, reduces to

$$\frac{1}{M(fu,z,t)}-1\le r\left(\frac{1}{M(fu,z,t)}-1\right)$$

for every *t*>0 which is a contradiction yielding thereby *fu*=S*u.* Therefore, *u* is a coincidence point of the pair (*f*, S).

Since f(X) $\subset$ T(X) and $fu \in f(X)$, there exists w $\in$X such that fu = Tw. Now, we assert that *gw* = T*w*. If not, then again using (1.11.1) we have

$$\left(\frac{1}{M(Tw,gw,t)}-1\right)=\left(\frac{1}{M(fu,gw,t)}-1\right)\le r\left(\frac{1}{\min\{M(Su,Tw,t),M(fu,Su,t),M(gw,Tw,t)\}}-1\right)$$

$$<\left(\frac{1}{M(Tw,gw,t)}-1\right)$$

for every *t*>0 which is again a contradiction as earlier. Thus, it follows that g*w* = T*w* which shows that *w* is a coincidence point of the pair (*g*, T). Since the pair (*f*, S) is weakly compatible and *f*u=Su, hence *f*z=*f*Su=S*f*u=Sz.

Now, we assert that *z* is a common fixed point of the pair (f, S). Suppose that $fz \neq z$, then using (1.11.1) we have

$$\begin{aligned}\left(\frac{1}{M(fz,z,t)}-1\right)&=\left(\frac{1}{M(fz,gw,t)}-1\right)\\&\leq r\left(\frac{1}{\min\{M(Sz,Tw,t),M(fz,Sz,t),M(gw,Tw,t)\}}-1\right)\\&<\left(\frac{1}{M(fz,z,t)}-1\right)\end{aligned}$$

for all t > 0 implying thereby fz=g*w*=z. Now, using the weak compatibility of the pair (g, T) together with contractive condition (1.11.1), we get gz=z=Tz. Hence, *z* is a common fixed point of both the pairs (*f*, S) and (*g*, T). Uniqueness of the common fixed point *z* is a consequence of contractive condition (1.11.1.).

By setting *f*= *g* and S = T, we deduces the following corollary for a pair of mappings.

Corollary 2.1. Let (*f*, S) be a pair of self mapping of a fuzzy metric space (X, M, *), such that

(I) the pair (*f*, S) satisfies the property (E.A.),

(II) *f*(X)$\subset$S(X),

(III) S(X) is a closed subset of X and

(IV) *f* is the fuzzy contraction with respect to S.

Then

(a) the pair (*f*, S) has a point of coincidence, whereas

(b) *f* and S has a unique common fixed point provided that the pair (*f*, S) is weakly compatible

However, on the lines of Imdad and Ali [13], we can prove a more natural theorem for a pair of mappings which runs as follows. Notice that this theorem cannot be derived from Theorem 2.1.

Theorem 2.2. Let (*f*, S) be a pair of self-mappings of a fuzzy metric space (X, M, *) (where * is any continuous t-norm) such that

(I) (*f*, S) satisfies the property (E. A.),

(II) *f* is a fuzzy contraction with respect to S,

(III) *f*(X) is a closed subspace of X.

Then

(a) the pair (f, S) has a point of coincidence, whereas

(b) the pair (f, S) has a unique common fixed point provided it is weakly compatible.

Proof: In view of the condition (I), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z, \text{ for same } z \in X.$$

Suppose f(X) is a closed subspace of X, then every convergent sequence of points in f(X) finds a limit in f(X). Therefore

$$\lim_{n\to\infty} fx_n = z = fa = \lim_{n\to\infty} Sx_n; \quad \text{for some } a \in X,$$

which in turn yields that $z = fa \in S(X)$. Now we assert that $fa = Sa$. Otherwise $fa \neq Sa$ implies $M(fa, Sa, t) < 1$. Using, (II) for large value of n, $\forall\ t > 0$, we have

$$\frac{1}{M(fa, fx_n, t)} - 1 \le \phi\left(\frac{1}{m(f, S, a, x_n, t)} - 1\right)$$

Now, (by corollary 7 of Grabiec, M. [3]) $\lim_{h\to\infty} M(fa, fx_n, t) = M(fa, Sa, t)$

and consequently, we get

$$\begin{aligned}
\frac{1}{M(fa, Sa, t)} - 1 &= \lim_{n\to\infty}\left(\frac{1}{M(fa, fx_n, t)} - 1\right) \\
&\le \lim_{n\to\infty} \varphi\left(\frac{1}{\min\left\{M(Sa, Sx_n, t), M(fa, Sa, t), M(fx_n, Sx_n, t)\right\}} - 1\right) \\
&\le \varphi\left(\frac{1}{M(fa, Sa, t)} - 1\right) \\
&< \left(\frac{1}{M(fa, Sa, t)} - 1\right)
\end{aligned}$$

a contradiction. Hence $fa = Sa = z$ which shows that a is a coincidence point of the pair (f, S). Since the pair (f, S) is weakly compatible, therefore fz = fSa = Sfa = Sz.

Now, we assert that $fz = z$. If not, then $M(fz, z, t) < 1$. On using (II), we have

$$
\begin{aligned}
\frac{1}{M(fz,z,t)}-1=\frac{1}{M(fz,\,fa,t)}-1 \\
&\le \varphi\left(\frac{1}{\min\left\{M(Sz,Sa,t),M(fz,Sz,t),M(fa,Sa,t)\right\}}-1\right) \\
&\le \varphi\left(\frac{1}{M(fz,z,t)}-1\right) \\
&< \left(\frac{1}{M(fz,z,t)}-1\right)
\end{aligned}
$$

for every $t > 0$, it follows that $fz = z$ which shows that z is common fixed point of f and S.

The uniqueness of the common fixed point is an easy consequence of fuzzy contraction condition (II).

Example2.1: Let X = [-1, 1] and $a * b = ab$ $\forall$ a, b $\in$ [0, 1]. For x, y $\in$ X and t > 0, let

M(x, y, t) = $\dfrac{t}{t+d(x,y)}$, then (X, M,$*$) is a fuzzy metric space. Define mappings f, S: X $\rightarrow$ X by

$$
f(x)=\begin{cases}\frac{1}{10} & if\ \ x=-1 \\ 0 & if\ \ -1<x<1 \\ \frac{1}{11} & if\ \ x=1\end{cases} \qquad and \qquad S(x)=\begin{cases}\frac{1}{2} & if\ \ x=-1 \\ 0 & if\ \ -1<x<1 \\ \frac{-1}{2} & if\ \ x=1\end{cases}
$$

Then the pair (f, S) satisfies all the conditions of Theorem 2.2 with $\phi(\tau) = k\tau$, where k =1/2 and has a unique common fixed point x = 0 which also remains a point of discontinuity for component maps. It may be noted that in this example neither f(X) is contained in S(X) nor S(X) is contained in f(X). Thus, this example highlights the utility of the property (E. A.) in common fixed point considerations in completely relaxing the condition on containment of range of one mapping into the range of other up to a pair of mappings.

Setting $\phi(\tau) = k\tau$, in Theorem 2.2, with $k \in (0,1)$, we get the following:

Corollary 2.2 Let (f, S) be a pair of self mappings of a fuzzy metric space (X, M,*) (where * is any continuous t-norm) such that,

(I) the pair (f, S) satisfies the property (E. A.),

(II) f is a fuzzy k- contraction with respect to S and

(III) f(X) is a closed subspace of X. Then

(a) the pair (f, S) has a point of coincidence, whereas

(b) the pair (f, S) has a unique common fixed point provided it is weakly compatible.

Further, if we set

$$M(x,y,t)=\frac{t}{t+d(x,y)} \text{ and } \min\{M(Sx,Sy,t),M(fx,Sx,t),M(fy,Sy,t)\}=M(Sx,Sy,t)$$

in Corollary 2.2, then we deduce the following result which is an improvement over a classical result contained in Jungck [8]. A multitude of such metrical fixed point theorems is available in Imdad and Ali [14].

Corollary 2.3. Let f and S be self mappings of a metric space(X, d) such that,

(I) the pair (f, S) satisfies the property (E. A.),

(II) d (fx, fy) $\leq k$ d (Sx, Sy), $k \in (0,1)$,

(III) f(X) is a closed subspace of X.

Then

(a) the pair (f, S) has a point of coincidence, whereas

(b) the pair (f, S) has a unique common fixed point provided it is weakly compatible.

3. Common Fixed Point Theorems via Implicit Functions

We recall the following implicit function was defined and studied by Imdad and Ali [14] and the same was further utilized to prove some common fixed point theorems. In order to describe this implicit function, let Ψ denote the family of all continuous functions F: $[0, 1]^4 \to$ R satisfying the following conditions:

F_1: For every u > 0, v ≥ 0 with F(u, v, u, v) ≥ 0 or F(u, v, v, u) ≥ 0, we have u > v.

F_2: F(u, u, 1, 1)< 0, $\forall$ u > 0.

Example 3.1: Define F: $[0, 1]^4 \to$ R as F(t_1, t_2, t_3, t_4) = t_1 - ϕ(min{ t_2, t_3, t_4}). where ϕ:[0, 1] $\to$[0, 1] is a continuous function such that ϕ(s) > s for 0 < s < 1. Then F$\in\Psi$.

Example 3.2: Define F(t_1, t_2, t_3, t_4): $[0, 1]^4 \to$ R as F(t_1, t_2, t_3, t_4)= t_1 –k min{t_2, t_3, t_4}, where k > 1.

Example 3.3: Define F(t_1, t_2, t_3, t_4): $[0, 1]^4 \to$ R as F(t_1, t_2, t_3, t_4)= t_1 –k t_2- min{t_3, t_4}, where k > 0.

Example 3.4: Define F(t_1, t_2, t_3, t_4): $[0, 1]^4 \to$ R as F(t_1, t_2, t_3, t_4)= t_1 –a t_2-bt_3-c t_4 where a>1 and b, c ≥ 0 (≠1).

Example 3.5: Define F(t_1, t_2, t_3, t_4): $[0, 1]^4 \to$R as F(t_1, t_2, t_3, t_4)= t_1 –a t_2-b(t_3+ t_4) where a>1 and b ≥ 0 (≠1).

Example 3.6: Define F(t_1, t_2, t_3, t_4): $[0, 1]^4 \to$ R as F(t_1, t_2, t_3, t_4)= t_1^3 –k$t_2t_3t_4$ where k > 1.

With a view to generalize some common fixed point theorems contained in Imdad and Ali [12] , we prove the following fixed point theorem which in turn generalizes several previously known results due to Chugh and Kumar [3], Vasuki [22] , Turkoglu et al. [20], U. Mishra et al [21] and some others.

Theorem 3.1. Let *f, g*, S and T be self mapping of a fuzzy metric space(X, M, *). Suppose that

(I) the pair (*f*, S) (or (*g*, T)) satisfies the properly (E. A.),

(II) f(X)$\subset$T(X) (or g(X)$\subset$S(X)),

(III) S(X) (or T(X)) is a closed subset of X and

(IV) $F(M(fx, gy, t), M(Sx, Ty, t), M(Sx, fx, t), M(gy, Ty, t)) \geq 0$ --------------------- (3.1.1)

for all distinct x, y$\in$ X and t > 0, where $F \in \Psi$.

Then

(a) the pair (*f*, S) as well as (*g*, T) have a point of coincidence each, whereas

(b) *f, g*, S, and T have a unique common fixed point provided both the pairs (*f*, S) and (*g*, T) are weakly compatible.

Proof. If the pair (*f*, S) enjoys the property (E.A.), then there exists a sequence { x_n } in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z\in X.$$

Since f(X) $\subset$ T(X), hence for each {x_n} there exists a {y_n} in X such that $fx_n = Ty_n$. Therefore, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Ty_n = z$. Thus, in all we have $fx_n \to$z, $Sx_n \to$z and $Ty_n \to z$. Now, we assert that $gy_n \to z$.

If not, then using (3.1.1), we have

$$F(M(fx_n, gy_n, t), M(Sx_n Ty_n, t), M(fx_n, Sx_n, t), M(gy_n, Ty_n, t)) \geq 0,$$

which on making n→∞, reduces to

$$F(\lim_{n\to\infty} M(z, gy_n, t), 1, 1, \lim_{n\to\infty} M(gy_n, z, t)) \geq 0.$$

Implying thereby $\lim_{n\to\infty} M(z, gy_n, t)$ >1, a contradiction. Hence $\lim_{n\to\infty} gy_n = z$ and in all

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z \in X.$$

If S(X) is a closed subspace of X, then $\lim_{n\to\infty} Sx_n = z \in S(X)$. Therefore, there exists a point $u \in X$ such that S*u* = z. Now we assert that *fu* = S*u*. If not, then using (3.1.1), we have

$$F(M(fu, gy_n, t), M(Su, Ty_n, t), M(fu, Su, t), M(gy_n, Ty_n, t)) \geq 0$$

which on making $n \to \infty$ reduces to

$$F(M(fu, z, t), M(Su, z, t), M(fu, Su, t), M(z, z, t)) \geq 0$$

$$F(M(fu, z, t), 1, M(fu, Su, t), 1) \geq 0$$

a contradiction to F_1. Hence fu=Su. Therefore, u is a coincidence point of the pair (*f*, S). Since f(X) $\subset$ T(X) and $fu \in f(X)$, there exists w $\in$X such that fu = Tw.

Now, we assert that *gw* = T*w*. If not, then again using (3.1.1), we have

$$F(M(fu, gw, t), M(Su, Tw, t), M(fu, Su, t), M(gw, Tw, t)) \geq 0$$

i.e.

$$F(M(Tw, gw, t), 1, 1, M(gw, Tw, t)) \geq 0.$$

Implying thereby, M(Tw, gw, t)>1, a contradiction. Hence T*w*=g*w*, which shows that *w* is a coincidence point of the pair (*g*, T). Since the pair (*f*, S) is weakly compatible and *f*u=Su, hence *fz*=*f*Su=S*f*u=Sz.

Now, we assert that *z* is a common fixed point of the pair (*f,* S). Suppose *fz≠z*. Then using (3.1.1.),

we have

$$F(M(fz, gw, t), M(Sz, Tw, t), M(fz, Sz, t), M(gw, Tw, t)) \geq 0$$

or

$$F(M(fz, z, t), M(fz, z, t), 1, 1) \geq 0$$

which contradict (F_2). Hence *fz*=*z*. Now, using the notion of the weak compatibility of the pair (*g*, T) and inequality (3.1.1), we get *gz*=z=T*z.* Hence z is a common fixed point of both the pairs (*f*, S) and (*g*, T). Uniqueness of *z* is an easy consequence of contractive condition (3.1.1).

Example 3.7: Let (X, M,$*$) be a fuzzy metric space as defined in the example 2.1 (above) where X = [2, 20]. Now, define mappings f, g, S and T by putting f2 = 2, fx = 3, if x > 2, S2 = 2, Sx = 6, if x > 2

gx = 2, if x = 2 or x > 5, gx = 5, if 2 < x ≤ 5 and

T2 = 2, Tx = 12, if 2 < x ≤ 5, Tx = (x+1)/3, if x > 5.

We also define $F(t_1, t_2, t_3, t_4) = t_1 - \phi(\min\{t_2, t_3, t_4\})$ with $\phi(a) = (7a/3a+4) > a$ *where* $a = \frac{1}{1+\frac{d(Sx,Ty)}{t}}$. Then *f*, *g*, S and T satisfy all the conditions of the theorem 3.1 and have a unique common fixed point x = 2.

Here it is worth noting that none of the theorems contained in (Chugh and Kumar [3], Vasuki [22] , Turkoglu et. al [20], U. Mishra et. al [21], Imdad and Ali [12]) can be used in the context of this example as Theorem 3.1 never require containment condition in respect of ranges ($g(X) \subset S(X)$) of the involved mappings.

Remark 3.1. The conclusion of theorem 3.1 remains true if for all distinct $x, y \in X$, condition (3.1.1) is replaced by one of the followings conditions:

(I) $M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})$,

where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(s) > s$ for all $0 < s < 1$.

(II) $M(Ax, By, t) \geq k(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})$, where $k > 1$.

(III) $M(Ax, By, t) \geq kM(Sx, Ty, t) + \min\{M(Sx, Ax, t), M(By, Ty, t)\}$, where $k > 0$.

(IV) $M(Ax, By, t) \geq aM(Sx, Ty, t) + bM(Sx, Ax, t) + cM(By, Ty, t)$, where $a > 1$ and $b, c \geq 0 (\neq 1)$.

(V) $M(Ax, By, t) \geq aM(Sx, Ty, t) + b[M(Sx, Ax, t) + M(By, Ty, t)]$, where $a > 1$ and $b \geq 0 (\neq 1)$.

(VI) $M(Ax, By, t) \geq kM(Sx, Ty, t)M(Sx, Ax, t)M(By, Ty, t)$, where $k > 1$.

Proof: The proof of the corollaries corresponding to contraction conditions (I)-(VI) follows from Theorem 3.1 and Examples 3.1-3.6.

Remark 3.2: The version of Theorem 3.1 corresponding to condition (I) is a result due to Imdad and Ali [12] whereas results corresponding to various conditions (II – VI) presents a sharpened form of Corollary2 of Imdad and Ali [13]. Similarly, from these conditions, we can also deduce generalized versions of certain results contained in [3, 18, 21, 22].

On the other hand, Singh and Jain [19] also utilized similar implicit function to prove some fixed point theorems in fuzzy metric spaces. In order to describe such implicit function, let Φ be the set of all real continuous function ϕ: $[0, 1]^4 \to R$, non decreasing in first argument and satisfying the following conditions:

ϕ_1***:*** For u, v ≥ 0, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies that u ≥ v.

ϕ_2***:*** $\phi(u, u, 1, 1) \geq 0$ implies that u ≥ 1.

Example 3.8: Define $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. Then, $\phi \in \Phi$.

Before stating our next result, it may be pointed out that above mentioned two classes of functions Φ and Ψ are independent classes. To substantiate this claim, note that implicit function $F(t_1, t_2, t_3, t_4) = t_1 - k \min\{t_2, t_3, t_4\}$, where k > 1 (belonging to Ψ) does not belongs to Φ as F (u,u,1,1) < 0 implies u > 0 whereas implicit function $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$ (belonging to Φ) does not belongs to Ψ as F(u, v, u, v)=0 implies u=v instead of u > v.

The following theorem is a substantial improvement over Theorem 3.1 contained in Singh and Jain [19].

Theorem 3.2. Let f, g, S and T be self mappings of a fuzzy metric space (X, M, *) which satisfies the inequalities

$$\phi\big(M(fx, gy, kt), M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, kt)\big) \geq 0$$

and

$$\phi(M(fx, gy, kt), M(Sx, Ty, t), M(fx, Sx, kt), M(gy, Ty, t)) \geq 0$$

for all distinct *x*, *y* $\in$ X and t > 0, $k \in (0,1)$ where $\phi \in \Phi$.

Suppose that

(I) the pair (*f*, S) (or (*g*, T)) satisfies the properly (E.A.),

(II) *f*(X) $\subset$ T(X) (or *g*(X) $\subset$ S(X)),

(III) S(X) (or T(X)) is a closed subset of X and

(a) Then the pairs (*f*, S) and (*g*, T) have a point coincidence each, whereas

(b) *f*, *g*, S and T have a unique common fixed point provided both the pairs (*f*, S) and (*g*, T) are weakly compatible.

Proof: The proof can be completed on the lines of Theorem 3.1 (above), hence details are omitted.

Corollary 3.1: Let *f*, *g*, S and T be four self mappings of a fuzzy metric space (X, M, *) such that f^m, g^n, S^p and T^q satisfy the condition (3.1.1). If the pairs (f^m, S^p) and (g^n, T^q) share the common properly (E.A.) and $S^p(X)$ as well as $T^q(X)$ are a closed subspaces of X, then f, g, S and T have a unique common fixed point provided the pairs (f^m, S^p) and (g^n, T^q) commute.

Remark.3.3. Results similar to Corollary 3.1 can be outlined in the context of Theorem 3.2.

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Best $p-$Simultaneous Approximaton in $L^p(\mu, X)$.

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Abstract

Let X be a Banach space and G be a closed subspace of X. Let us denote $L^p(\mu, X)$, $1 \leq p < \infty$, the Banach space of all $X-$valued strongly measurable functions f on a measure space (Ω, Σ, μ) such that $\int_\Omega \|f(s)\|^p \, d\mu < \infty$. In this paper we show that if G is separable and (Ω, Σ, μ) is $\sigma-$finte complete measure space, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$ if and only if G is simultaneously proximinal in X.

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1 Introduction

Let X be a Banach space, G a closed subspace of X, and (Ω, Σ, μ) be a measure space.

Definition 1.1 Let (M, d) be a metric space. A Borel measurable function from Ω to M is called strongly measurable if it is the pointwise limit of a sequence of simple Borel measurable functions from Ω to M.

For $1 \leq p < \infty$, $L^p(\mu, X)$ is the Banach space consisting of (equivalent classes of) strongly measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(s)\|^p \, d\mu$ is finite, with the usual norm

$$\|f\|_p = \left(\int_\Omega \|f(s)\|^p \, d\mu \right)^{\frac{1}{p}}.$$

If X is the Banach space of real numbers, we simply write $L^p(\mu)$. For $A \in \Sigma$ and a strongly measurable function $f : \Omega \to X$, we write χ_A for the charateristic function of A and $\chi_A f$ denotes the function $\chi_A f\ (s) = \chi_A(s) f(s)$. In particular, for $x \in X$, $\chi_A x(s) = \chi_A(s) x$.

For a finite number of elements $x_1, x_2, ..., x_m$ in X, we set

$$d_p(\{x_i : 1 \leq i \leq m\}, G) = \inf_{g \in G} \left(\sum_{i=1}^{m} \|x_i - g\|^p \right)^{\frac{1}{p}}.$$

G is said to be simultaneously proximinal under L^p-norm if for any finite number of elements $x_1, x_2, ..., x_m$ in X there exists at least $y \in G$ such that

$$\left(\sum_{i=1}^{m} \|x_i - y\|^p \right)^p = d_p(\{x_i : 1 \leq i \leq m\}, G).$$

The element y is called a best simultaneous approximation of $x_1, x_2, ..., x_m$ in G. Of course, for $m = 1$ the preceding concepts are just best approximation and proximinality.

The theory of best simultaneous approximation has been invistigated by many authors. Most of the work done has dealt with the space of continuous functions with values in a Bnach space e.g. [4, 8, 18]. Some recent results for best simultaneous approximation in $L^p(\mu, X)$ have been obtained in [5 - 7,14, 16, 19]. In [7], it is shown that if G is a reflexive subspace of a Banach space X, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$. In [5], it is shown that if G is L^p-summand of a Banach space X, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$. It is the aim of this paper to show that if G is a closed separable subspace, then $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$ if and only if G is simultaneously proximinal in X.

2 Preliminary Results

Throughout this section X is a Banach space and G is a closed subspace of X. Let $f_1, f_2, ..., f_m$ be any finite number of elements in $L^p(\mu, X)$, $1 \leq p < \infty$, and set

$$\phi(s) = d_p(\{f_i(s) : 1 \leq i \leq m\}, G).$$

In [16], It is shown that if (Ω, Σ, μ) is a finite measure space, then $\phi \in L^p(\mu)$ and

$$\left(\int_\Omega |\phi(s)|^p \, d\mu \right)^{\frac{1}{p}} \leq d_p\left(\{f_i : 1 \leq i \leq m\},\ L^p(\mu, X) \right) \leq 2 \left(\int_\Omega |\phi(s)|^p \, d\mu \right)^{\frac{1}{p}},$$

with equality of the first and second parts holds when $f_1, f_2, ..., f_m$ are simple fuctions. This inequality has been put in a better form in [5] :

$$\left(\int_\Omega |\phi(s)|^p \, d\mu \right)^{\frac{1}{p}} \leq d_p\left(\{f_i : 1 \leq i \leq m\},\ L^p(\mu, X) \right) \leq 2^{\frac{p-1}{p}} \left(\int_\Omega |\phi(s)|^p \, d\mu \right)^{\frac{1}{p}},$$

with equality of the first and second parts holds when $f_1, f_2, ..., f_m$ are simple fuctions or $p = 1$.

The following Theorem gives much better formula that holds for any measure space (Ω, Σ, μ).

Theorem 2.1 Let (Ω, Σ, μ) be a measure space, $f_1, f_2, ..., f_m$ be any finite number of elements in $L^p(\mu, X)$, and $\phi(s)$ as defined above. Then, $\phi \in L^p(\mu)$ and

$$d_p(\{f_i : 1 \leq i \leq m\}, \ L^p(\mu, G)) = \left(\int_\Omega |\phi(s)|^p \, d\mu \right)^{\frac{1}{p}}.$$

Proof. Since $f_1, f_2, ..., f_m \in L^p(\mu, X)$, there exist sequences of simple functions

$$(f_{(i,n)})_{n=1}^{\infty}, i = 1, 2, ..., m,$$

such that

$$\lim \left\| f_{(i,n)}(s) - f_i(s) \right\| = 0,$$

for $i = 1, 2, ..., m$, and for almost all s. We may write

$$f_{(i,n)} = \sum_{j=1}^{k(n)} \chi_{A(n,j)}(\cdot) \, x_{(i,n,j)}, \quad i = 1, 2, ..., m,$$

$\sum_{j=1}^{k(n)} \chi_{A(n,j)}(\cdot) = 1$, and that $\mu(A(n,j)) > 0$. Then

$$d_p\left(\left\{f_{(i,n)}(s) : 1 \leq i \leq m\right\}, \ G\right) = \sum_{j=1}^{k(n)} \chi_{A(n,j)} d_p\left(\left\{x_{(i,n,j)} : 1 \leq i \leq m\right\}, \ G\right)$$

and by the continuity of d_p

$$\lim d_p\left(\left\{f_{(i,n)}(s) : 1 \leq i \leq m\right\}, \ G\right) = d_p(\{f_i(s) : 1 \leq i \leq m\}, \ G),$$

for almost all s. Thus ϕ is measurable and $\phi \in L^p(\mu)$.

Now, for any $h \in L^p(\mu, G)$,

$$\begin{aligned} \int_\Omega d_p(\{f_i(s) : 1 \leq i \leq m\}, \ G)^p \, d\mu &\leq \int_\Omega \sum_{i=1}^{m} \|f_i(s) - h(s)\|^p \, d\mu \\ &= \sum_{i=1}^{m} \|f_i - h\|_p^p \end{aligned}$$

Hence

$$\left(\int_\Omega d_p(\{f_i(s) : 1 \leq i \leq m\}, \ G)^p \, d\mu \right)^{\frac{1}{p}} \leq d_p(\{f_i : 1 \leq i \leq m\}, \ L^p(\mu, G)).$$

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To prove the reverse inequality, let $\epsilon > 0$ be given and $w_i,\ i = 1, 2, ..., m$, be simple functions in $L^p(\mu, X)$ such that

$$\|f_i - w_i\|_p < \frac{\epsilon}{3m^{\frac{1}{p}}}.$$

We may write $w_i = \sum_{k=1}^{\ell} \chi_{A_k}(\cdot)\ x_{(i,k)},\ \sum_{k=1}^{\ell} \chi_{A_k}(\cdot) = 1$, and that $\mu(A_k) > 0$. Since $w_i \in L^p(\mu, X)$ for all i, we have $\left\|x_{(i,k)}\right\| \mu(A_k) < \infty$ for all k and i. If $\mu(A_k) < \infty$, select $h_k \in G$ so that

$$\sum_{i=1}^{m} \left\|x_{(i,k)} - h_k\right\|^p < d_p\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right)^p + \frac{\epsilon^p}{3^p \mu(A_k)},$$

for all k. If $\mu(A_k) = \infty$, put $h_k = 0$.

Let $g = \sum_{k=1}^{\ell} \chi_{A_k}(\cdot)\, h_k$. It is clear that $g \in L^p(\mu, G)$. Then

$$\begin{aligned}
d_p\left(\{f_i : 1 \le i \le m\},\ L^p(\mu, G)\right) &\le \left(\sum_{i=1}^{m} \|f_i - g\|_p^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^{m} \|f_i - w_i + w_i - g\|_p^p\right)^{\frac{1}{p}} \\
&\le \left(\sum_{i=1}^{m} \|f_i - w_i\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m} \|w_i - g\|_p^p\right)^{\frac{1}{p}} \\
&< \left(\sum_{i=1}^{m} \left(\frac{\epsilon^p}{3^p m}\right)\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m} \|w_i - g\|_p^p\right)^{\frac{1}{p}} \\
&= \frac{\epsilon}{3} + \left(\sum_{i=1}^{m} \sum_{k=1}^{\ell} \mu(A_k) \left\|x_{(i,k)} - h_k\right\|^p\right)^{\frac{1}{p}} \\
&= \frac{\epsilon}{3} + \left(\sum_{k=1}^{\ell} \sum_{i=1}^{m} \mu(A_k) \left\|x_{(i,k)} - h_k\right\|^p\right)^{\frac{1}{p}} \\
&= \frac{\epsilon}{3} + \left(\sum_{k=1}^{\ell} \mu(A_k) \sum_{i=1}^{m} \left\|x_{(i,k)} - h_k\right\|^p\right)^{\frac{1}{p}} \\
&\le \frac{\epsilon}{3} + \left(\sum_{k=1}^{\ell} \mu(A_k)\, d_p\left(\left\{x_{(i,k)} : 1 \le i \le m\right\},\ G\right)^p + \frac{\epsilon^p}{3^p}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\epsilon}{3}+\left(\sum_{k=1}^{\ell}\mu\left(A_k\right)d_p\left(\left\{x_{(i,k)}:1\leq i\leq m\right\},\ G\right)^p\right)^{\frac{1}{p}}+\left(\frac{\epsilon^p}{3^p}\right)^{\frac{1}{p}} \\
&= \frac{2\epsilon}{3}+\left(\int_{\Omega}\sum_{k=1}^{\ell}\chi_{A_k}\left(s\right)d_p\left(\left\{x_{(i,k)}:1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}} \\
&= \frac{2\epsilon}{3}+\left(\int_{\Omega}d_p\left(\left\{w_i\left(s\right):1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}} \\
&\leq \frac{2\epsilon}{3}+\left(\int_{\Omega}\left[d_p\left(\left\{f_i\left(s\right):1\leq i\leq m\right\},\ G\right)+\left(\sum_{i=1}^{m}\left\|f_i\left(s\right)-w_i\left(s\right)\right\|^p\right)^{\frac{1}{p}}\right]^p d\mu\right)^{\frac{1}{p}}
\end{aligned}$$

Using triangle inequality in $L^p(\mu)$ so that the right hand side becomes

$$\begin{aligned}
&\leq \frac{2\epsilon}{3}+\left[\begin{array}{c}\left(\int_{\Omega}d_p\left(\left\{f_i\left(s\right):1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}} \\ +\left(\int_{\Omega}\sum_{i=1}^{m}\left\|f_i\left(s\right)-w_i\left(s\right)\right\|^p d\mu\right)^{\frac{1}{p}}\end{array}\right] \\
&\leq \frac{2\epsilon}{3}+\left(\int_{\Omega}d_p\left(\left\{f_i\left(s\right):1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{m}\left\|f_i-w_i\right\|_p^p\right)^{\frac{1}{p}} \\
&\leq \frac{2\epsilon}{3}+\left(\int_{\Omega}d_p\left(\left\{f_i\left(s\right):1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}}+\frac{\epsilon}{3} \\
&\leq \epsilon+\left(\int_{\Omega}d_p\left(\left\{f_i\left(s\right):1\leq i\leq m\right\},\ G\right)^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}$$

This ends the proof.

Corollary 2.2 Let (Ω,Σ,μ) be a measure space, $f_1,f_2,...,f_m$ be any finite number of elements in $L^p(\mu,X)$. Let $g:\Omega\to G$ be a measurable function such that $g(s)$ is a best simultaneous approximation of $f_1(s),f_2(s),...,f_n(s)$ for almost all s. Then g is a best simultaneous approximation of $f_1,f_2,...,f_n$ in $L^p(\mu,G)$ (and therefore $g\in L^p(\mu,G)$).

Proof. Assume that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s.Then

$$\left(\sum_{i=1}^{m} \|f_i(t) - g(t)\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} \|f_i(t) - z\|^p\right)^{\frac{1}{p}},$$

for almost all s, and for all $z \in G$. Then, set $z = 0$ and use triangle inequality of L^p-norm,

$$\left(\sum_{i=1}^{m} \|g(t)\|^p\right)^{\frac{1}{p}} \leq 2\left(\sum_{i=1}^{m} \|f_i(t)\|^p\right)^{\frac{1}{p}},$$

for almost all s, therefore $g \in L^p(\mu, G)$. By Theorem 2.1,

$$\begin{aligned} d_p(\{f_i : 1 \leq i \leq m\}, L^p(\mu, G))^p &= \int_\Omega d_p(\{f_i(s) : 1 \leq i \leq m\}, G)^p \, d\mu \\ &= \int_\Omega \sum_{i=1}^{m} \|f_i(s) - g(s)\|^p \, d\mu \\ &= \sum_{i=1}^{m} \|f_i - g\|_p^p. \end{aligned}$$

Therefore g is a best simultaneous approximation for $f_1, f_2, .., f_m$ in $L^p(\mu, G)$.

The condition in Corollary 2.2 is sufficient ; $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_m(s)$ for almost all s in G, implies g is a best simultaneous approximation of $f_1, f_2, ..., f_m$ in $L^p(\mu, G)$. In fact we have the following theorem :

Theorem 2.3. Let (Ω, Σ, μ) be a measure space. Then, $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$ if and only if for any finite number of elements $f_1, f_2, ..., f_m$ in $L^p(\mu, X)$, there exists $g \in L^p(\mu, G)$ such that $g(s)$ is a best simultaneous approximation of $f_1(s), f_2(s), ..., f_n(s)$ for almost all s.

Proof. Sufficiency of the condition is an immediate consequence of Corollary 2.2. We will show the necessity. Assume that $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$ and let $f_1, f_2, ..., f_m$ be any finte number of elements in $L^p(\mu, X)$. Then, there exists $g \in L^p(\mu, G)$ such that

$$\begin{aligned} \sum_{i=1}^{m} \|f_i - g\|_p^p &= d_p(\{f_i : 1 \leq i \leq m\}, L^p(\mu, G))^p \\ &= \int_\Omega d_p(\{f_i(s) : 1 \leq i \leq m\}, G)^p \, d\mu, \end{aligned}$$

hence

$$\int_{\Omega} \left(\sum_{i=1}^{m} \|f_i(s) - g(s)\|^p - d_p\left(\{f_i(s) : 1 \le i \le m\},\ G\right)^p \right) d\mu = 0.$$

Thus

$$\sum_{i=1}^{m} \|f_i(s) - g(s)\|^p = d_p\left(\{f_i(s) : 1 \le i \le m\},\ G\right)^p,$$

for allmost all s.

3 Main Result.

Let (Ω, Σ, μ) be a measure space and X be a Banach space. We say that $f : \Omega \to X$ is measurable in the classical sense if $f^{-1}(O)$ is measurable for every open set $O \subset X$.

The following lemmas will be used to prove our main result.

Lemma 3.1 [21] . Let (Ω, Σ, μ) be a complete measure space and X be a Banach space. If f is a strongly measurable functon from Ω to X, then f is measurable in the classical sense.

Lemma 3.2 [21] . Let (Ω, Σ, μ) be a complete measure space and X be a Banach space. If $f : \Omega \to X$ is measurable in the classical sense and has essetially seprable range, then f is srongly measurable.

Let Φ be a set-valued mapping, taking each piont of a measurable space Ω into a subset of a metric space X. We say that Φ is weakly measurable if $\Phi^{-1}(O)$ is measurable in Ω whenever O is open in X. Here we have put, for any $A \subset X$,

$$\Phi^{-1}(A) = \{s \in \Omega : \Phi(s) \cap A \neq \phi\}.$$

The following theorem is due to Kuratowski [15], it is known as Measurable Selection Theorem.

Theorem 3.3 [15] . Let Φ be a weakly measurable set-valued map which carries each piont of a measurable space Ω to a closed nonvoid subset of a complete seprable metric space X. Then Φ has a measurable selection; i.e., there exists a function $f : \Omega \to X$ such that $f(s) \in \Phi(s)$ for each $s\epsilon\Omega$ and $f^{-1}(O)$ is measurable in Ω whenever O is open in X.

Theorem 3.4. Let X be a Banach space and G be a closed seprable subspace of X, and (Ω, Σ, μ) is $\sigma-$finite complete measure space. Then the following are equivalent :

1. G is simultaneously proximinal in X.
2. $L^p(\mu, G)$ is simultaneously proximinal in $L^p(\mu, X)$.

Proof. $(2) \Rightarrow (1)$: Let $x_1, x_2, ..., x_m$ be any finite number of elements in X. Since (Ω, Σ, μ) is $\sigma-$finite, we can assume that $\Omega = \underset{n \in N}{\cup} A_n$ such that $\mu(A_n) < \infty$ for all $n \in N$. Then there must be $k_0 \in N$ such that $0 < \mu(A_{k_0}) < \infty$. Define $f_{x_i} : \Omega \to X$, $i = 1, 2, ..., m$, by

$$f_{x_i}(s) = \mu(A_{k_0})^{\frac{1}{p}-1} \chi_{A_{k_0}}(s)\, x_i,$$

for all $s \in \Omega$. Then $f_{x_i} \in L^p(\mu, X)$ for all i. By the assumption, there exists $f_0 \in L^p(\mu, G)$ such that

$$\left(\sum_{i=1}^{m} \|f_{x_i} - f_0\|_p^p\right)^{1/p} = d_p\left(\{f_{x_i} : 1 \leq i \leq m\}, L^p(\mu, G)\right).$$

Then

$$\begin{aligned}
\left(\sum_{i=1}^{m} \|f_{x_i} - f_0\|_p^p\right)^{1/p} &\leq \left(\sum_{i=1}^{m} \left\|f_{x_i} - \mu(A_{k_0})^{\frac{1}{p}-1} \chi_{A_{k_0}} g\right\|_p^p\right)^{1/p} \\
&= \mu(A_{k_0})^{\frac{1}{p}-1} \left(\sum_{i=1}^{m} \left\|\chi_{A_{k_0}} x_i - \chi_{A_{k_0}} g\right\|_p^p\right)^{1/p} \\
&= \mu(A_{k_0})^{\frac{1}{p}-1} \left(\int_{A_{k_0}} \left(\sum_{i=1}^{m} \|x_i - g\|^p\right) d\mu\right)^{1/p} \\
&= \mu(A_{k_0})^{\frac{2}{p}-1} \left(\sum_{i=1}^{m} \|x_i - g\|^p\right)^{1/p},
\end{aligned}$$

for all $g \in G$. By Theorem 2.3 , $f_0(s)$ is a best simultaneous approximation of $f_{x_1}(s), f_{x_2}(s), ..., f_{x_m}(s)$ for almost all s. Then

$$\left(\sum_{i=1}^{m} \|f_{x_i}(s) - f_0(s)\|^p\right)^{1/p} \leq \left(\sum_{i=1}^{m} \|f_{x_i}(s) - h(s)\|^p\right)^{1/p},$$

for almost all s and for any strongly measurable function $h : \Omega \to G$, hence

$f_0 = \chi_{A_{k_0}} f_0$. Put $x_0 = \int\limits_{A_{k_0}} f_0 (s)\, d\mu$. Then

$$\begin{aligned}
\left(\sum_{i=1}^{m} \left\| x_i - \frac{x_0}{\mu(A_{k_0})^{\frac{1}{p}}} \right\|^p \right)^{1/p} &= \mu(A_{k_0})^{-\frac{1}{p}} \left(\sum_{i=1}^{m} \left\| \int\limits_{A_{k_0}} f_{x_i}(s)\, d\mu - \int\limits_{A_{k_0}} f_0(s)\, d\mu \right\|^p \right)^{1/p} \\
&\leq \mu(A_{k_0})^{-\frac{1}{p}} \left(\sum_{i=1}^{m} \left(\int\limits_{A_{k_0}} \| f_{x_i}(s) - f_0(s) \|\, d\mu \right)^p \right)^{1/p} \\
&\leq \mu(A_{k_0})^{-\frac{1}{p}} \left[\sum_{i=1}^{m} \left(\left(\int\limits_{A_{k_0}} \| f_{x_i}(s) - f_0(s) \|^p\, d\mu \right)^{1/p} (\mu(A_{k_0}))^{1-1/p} \right)^p \right]^{1/p} \\
&= \mu(A_{k_0})^{1-\frac{2}{p}} \left(\sum_{i=1}^{m} \| f_{x_i} - f_0 \|_p^p\, d\mu \right)^{1/p} \\
&\leq \left(\sum_{i=1}^{m} \| x_i - g \|^p \right)^{1/p},
\end{aligned}$$

for all $g \in G$. Hence $\frac{1}{\mu(A_{k_0})^{\frac{1}{p}}} x_0$ is a best simultaneous approximation of $x_1, x_2, ..., x_m$ in G.

(1) $\Rightarrow$ (2) : Let $f_1, f_2, ..., f_m$ be any finite number of elements in $L^p(\mu, X)$. For each $s \in \Omega$ define

$$\Phi(s) = \left\{ g \in G : \left(\sum_{i=1}^{m} \| f_i(s) - g \|^p \right)^{1/p} = d_p\left(\{ f_i(s) : 1 \leq i \leq m \},\ G \right) \right\}.$$

For each $s \in \Omega$, $\Phi(s)$ is closed, bounded, and nonvoid subset of G. We shall show that Φ is weakly measurable. Let O be an open set in X, the set

$$\Phi^{-1}(O) = \{ s \in \Omega : \Phi(s) \cap O \neq \phi \}$$

can be also be described as

$$\Phi^{-1}(O) = \{ s \in \Omega : \inf_{g \in G} \left(\sum_{i=1}^{m} \| f_i(s) - g \|^p \right)^{1/p} = \inf_{g \in O} \left(\sum_{i=1}^{m} \| f_i(s) - g \|^p \right)^{1/p} \}.$$

Since (Ω, Σ, μ) is complete, f_i is measurable in the classical sence for $i = 1, 2, ..., m$ by Lemma 3.1. Since subtraction in G, sum, and the norm in X are continuous, then the map

$$s \rightarrow \inf_{g \in A} \left(\sum_{i=1}^{m} \| f_i(s) - g \|^p \right)^{1/p}$$

is measurable for any set A. It follows that $\Phi^{-1}(O)$ is measurable. By Theorem 3.3, Φ has a measurable selection; i.e., there exists a function $f: \Omega \to G$ such that $f(s) \in \Phi(s)$ for each $s\epsilon\Omega$ and f is measurable in the classical sense. By Lemma 3.2, f is strongly measurable. Hence f is a best simultaneous approximation for $f_1, f_2, ..., f_m$ in $L^p(\mu, G)$ by Corollary 2.2.

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WHEN AN EXPOSED POINT OF A CONVEX SET IS A REMOTAL POINT

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ABSTRACT. Let X be a Banach space and Y be a closed bounded subset of X. We call Y remotal in X if for every $x \in X$ there exists some $y \in Y$ such that $\|x-y\| \geq \|x-z\| \quad \forall z \in Y$. It is called uniquely remotal in X if for every $x \in X$ there is a unique $y \in Y$ such that $\|x-y\| \geq \|x-z\| \quad \forall z \in Y$. In this paper we study the remotality of exposed points of convex sets in Banach spaces.

1. INTRODUCTION

Let X be a Banach space and E be a closed bounded set of X. For $x \in X$, let

$$D(x,E) = \sup_{e \in E} \|x-e\| .$$

This supremum need not be attained. If $\forall x \in X$, there exists some $e \in E$ such that $D(x,E) = \|x-e\|$, then E is called remotal. For
$x \in X$, we set

$$F(x,E) = \{e \in E : \|x-e\| = D(x,E)\}$$

The point e is called a farthest point of E from x. The study of remotal sets goes back to the sixties. Edelstein (1966) showed that if X is a uniformly convex Banach space, then the set of points in X that has farthest points in E are dense in X. The importance of the study of remotal sets comes from its ties to geometry of Banach spaces. Such study is little difficult and less developed. On the contrary, closest points are well studied and well established. The problem of remotality in $L^p(I,X)$ $1 \leq p < \infty$, was initiated by Khalil and Al-Sharif [2]. Then it was developed by Sababhah and Khalil [5]. One of the standing conjectures in the theory of remotal sets is "Every uniquely remotal set in a Banach space is a singleton". So much work was done to solve such a conjecture, We refer to Astaneh [1], Narang [3], and Sababhah and Khalil [6,7] and , for more and references on such a problem.

Another problem in the theory of remotal sets is" When is a boundary point of a set a farthest point from some point in X?" It is the object of this paper to study such a problem.

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2. Remotality Of Exposed Points

In this section we introduce new results that partially answer the following primary question " when an exposed point must be a farthest point from some point in X ?".

Lemma 2.1 : There exist convex sets with extreme points that are not remotal points.

Proof :

Let $X = R^2$. Let S_1 be the unit circle in X when endowed with the Euclidian norm and let S_2 be the unit circle in X when endowed with the sup norm. Let E be the convex hull of the union of $\{(x, y) \in S_1 : x \geq 0\}$ and $\{(x, y) \in S_2 : x \leq 0\}$. It is clear that the point $A(0, 1)$ is an extreme point of E. We assert that A is not a remotal point. That is, we show that $A \notin F(x, E)$ for any $x \in X$, X being endowed with the Euclidian norm. To do so, we use Lemma 2.1. Thus, suppose on the way of contrary that a circle $x^2 + y^2 + ax + by + c = 0$ exists such that A belongs to the circle and such that E is inside the circle. For A to be on the circle, we need to have $c = -b - 1$. Hence, the equation of the circle becomes $x^2 + y^2 + ax + by = b + 1$. Since E lies inside the circle, $B(-1, 1), C(-1, -1)$ and $D(1, 0)$ must be inside the circle. Now, for $B$to be inside the circle we must have $a \geq 1$ and for D to be inside the circle we need to have $a \leq b$. Now let (x, y) be any point on the right half of the unit circle. That is $x^2 + y^2 = 1$ and $x \geq 0$. The question is “Does (x, y) lie inside the circle $x^2 + y^2 + ax + by = b + 1$?That is Does (x, y) satisfy $x^2 + y^2 + ax + by \leq b + 1$?. Since $x^2 + y^2 = 1$, we need to check whether $ax + by \leq b$ or not. Thus the question reduces to: What is the maximum value of $ax + by$ subject to the conditions $x^2 + y^2 = 1$and $x \geq 0$? If we use the method of Lagrange multipliers we find that the point $(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})$ is on the right half of the unit circle which lies outside the circle $x^2 + y^2 + ax + by = b + 1$.This shows that no circle passing through A can satisfy the conditions of Lemma 2.1. Hence, A is not a remotal point of E.

Definition 2.2 : an exposed point e of a convex set E is a boundary point of E such that a functional

$f \in X^*$ exists with the properties $f(e) = \alpha$ and $f(x) < \alpha \forall x \in E \backslash \{e\}$.

In other words, it means that a hyperplane H exists such that e is the only element of E that belongs to H , and E lies entirely on one side of H.

Lemma 2.3 : There are closed bounded convex sets in Hilbert spaces whose exposed points are not necessarily remotal points.

Proof : Let X be R^2 endowed with Euclidean norm and let

$$E_0 = \{(\pm\frac{1}{n}, \frac{1}{n^3}) : n \in \mathbb{N}\}$$

If E is the closed convex hull of E_0, then clearly E lies above the x-axis which touches E uniquely at $(0, 0)$. That is, $(0, 0)$ is an exposed point of E. We assert that $(0, 0)$ is not a remotal point for E. Observe that if $(x, y) \in X$ then

$$\| (x,y) - (\pm\tfrac{1}{n}, \tfrac{1}{n^3}) \|^2 = (x^2+y^2) + (\tfrac{1}{n^2} + \tfrac{1}{n^6}) + (\pm\tfrac{2}{n}x - \tfrac{2}{n^3}y)$$

$$= (x^2+y^2) + (\tfrac{1}{n^6} \pm \tfrac{2}{n}x) + (\tfrac{1}{n^2} - \tfrac{2}{n^3}y).$$

Observe that $n \in \mathbb{N}$ can be chosen so that $(\frac{1}{n^2} - \frac{2}{n^3}y) > 0$ for a given y. As for $(\frac{1}{n^6} \pm \frac{2}{n}x)$,one can manage to have the plus or the minus n, according to whether x is negative or positive, in order to make $(\frac{1}{n^6} \pm \frac{2}{n}x) > 0$. These observations together tell us that

$$\| (x,y) - (\pm\tfrac{1}{n}, \tfrac{1}{n^3}) \|^2 > x^2 + y^2 = \| (x,y) - (0,0) \| .$$

That is, $(0,0)$ cannot be a farthest point in E from (x,y). In other words, $(0,0)$ is not a remotal point of E.

Definition 2.4 Let X be a normed space and E be a bounded closed subset of X. For $e \in \partial E$(in $spanE$), we define $C(E,e) = \{\lambda(x-e) : \lambda > 0, x \in X\}$. This is called the cone generated by with E vertex e.

Example 2.5: Let $E = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Then $C(E,(0,0))$ is the first quadrant.

Example 2.6: Let $E = \{(x,y) : x^2+y^2 = 1\}$. Then $C(E,(0,0)) = \mathbb{R}^2$.

Remark 2.7. We should remark that any two dimensional normed space X have an equivalent norm that makes it isometrically isomorphic to $\mathbb{R}^2$ with the Euclidean norm. The angle between two rays in X, is the angle between them when X is mapped onto $\mathbb{R}^2$ isometrically.

Definition 2.8. Let X be a normed space and E be a bounded closed subset of X. If e is an exposed point of E define

$$\theta(C(E,e),H) = \inf\ \{\theta(x,y) : x \in H\ ,\ y \in C(E,e)\}$$

where H is the hyper plane exists by e, and $\theta(x,y)$ is the angle between the two segment $[e,x,-]$ and $[e,y,-]$ as in Remark 4.3..

Definition 2.9: Let E be a closed convex bounded set of a Banach space X, and e be an exposed point of E. The point e is called cone remotal if and only if there is a subset B of E and x in E such that $D(x,E) = D(x,B\ = \|x-e\|$ and $\theta(C(B,e),H) > 0$.

Remark 2.10: It is obvious from definition that if e is a cone remotal point then e is a remotal point.

Theorem 2.11: Let E be a closed bounded convex set of a Banach space X. Assume that e is a remotal exposed point of E. Then e is cone remotal.

Proof:

Assume that e is a remotal exposed point of E. Thus there exits x in X such that $D(x,E) = \|x-e\|$. Consider $[x,e] \cap E$. Then $B = [z,e] \subset E$. Now, any hyper plane supporting E at e has a positive angle with $[z,e]$. Hence $\theta(C(B,e),H) > 0$.Since $D(x,E) = D(x,B) = \|x-e\|$, then e is cone remotal. This completes the proof.

Lemma 2.12: Let X be a reflexive Banach space, and $H = \{z : f(z) = 0\}$ for some $f \in X^*$. Then there exits an $x \in X$ such that $\|x\| = d(x,H)$.

Proof. Let z be any element in X. Since X is reflexive, then H is proximinal. Hence there exists $y \in H$ such that $\|z-y\| \leq \|z-h\|$ for all $h \in H$. Now, let $x = z - y$. Since H is a subspace, then $y + w \in H$ for all $w \in H$. hence $\|x\| = \|z-y\| \leq \|z-(y+w)\| = \|x-w\|$ for all $w \in H$. Hence $\|x\| = d(x,H)$.

Lemma 2.13 : Let X be any normed space, and $T : X \to X$ be defined by $T(x) = x + a$ for some fixed a in X. Then T preserves extreme and exposed points of convex sets of X.

Proof. We prove the case of extreme points. So, let E be a closed convex set in a normed space X, and e be an extreme point of E.

We claim that $e + a$ is an extreme point of $E + a = \{x + a : x \in E\}$. Indeed, if possible, assume $e + a$ is not an extreme point of $E + a$. Then

$e + a = \frac{1}{2}[(x+a) + (y+a)]$ for some $x, y \in E$. But that implies $e = \frac{1}{2}(x+y)$, and e is not an extreme point.

Theorem 2.14. Let X be a uniformly convex Banach space, and E be a closed bounded convex set of X. If e is an exposed point of E such that $\theta(C(E,e),H) > 0$, for some hyperplane supporting E at e, then e is a remotal point of E.

Proof. Since e is an exposed point of E, there is at least one hyperplane H that supports E at e only. That is $H \cap E = \{e\}$. However, since X is reflexive, one can use Lemma 4.1, and Lemma 4.2 to assume without any loss of generality that $e = 0$.

Again, since X is reflexive, then H is proximinal. Further, being a subspace, then for every point $z \in H$, there exists a point $x \in X$

such that $\|x - z\| = d(x,H)$. Thus, there is some $x \in H$ such that $\|x\| = d(x,H)$. Further, we can choose the x on the same side of the set E

Now, $\|x\| \leq \|x-z\|$ for all $z \in H$. But this implies that $\|tx\| \leq \|tx - z\|$ for all $z \in H$ and all $t > 0$. Indeed $\|tx\| = t\|x\| \leq t\|x-z\| = \|tx - tz\|$ for all $z \in H$

Since H is a subspace, then $tz \in H$. Further, and $w \in H$ is of the form $w = t\frac{w}{t} = tz$ for some $z \in H$. Hence $\|tx\| \leq \|tx - z\|$ for all $z \in H$. Now consider, $B[tx, \|tx\|]$. These balls touches the hyperplane H at 0 for all $t > 0$. Now, if for some $t > 0$, $E \subset B[tx, \|tx\|]$, then E is remotal.

If possible assume that there is no $t > 0$, such that $E \subset B[tx, \|tx\|]$. This implies there is a sequence t_n such that for each n, there is

some $e_n \in E$ and $\|t_n x - e_n\| > \|t_n x\|$. Further, since X is uniformly convex, $d(e_n, H) \to 0$, for otherwise, some ball $B[tx, \|tx\|]$ (for some t, will contain all of the $e_n^{,s}$). But in this case, $\theta(C(E,e),H) = 0$.

This contradicts the assumption. Hence E is remotal.

We end this paper with the following questions

Question1. Can one give a necessary and sufficient conditions for an exposed point in reflexive spaces to be remotal?

Question 2. Can one prove Theorem 2.14 in general Banach spaces?

Question 3. When a boundary point in a closed convex bounded set in a Banach space is a remotal point?

Question 4. Can one improve Theorem 2.14?

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DOUBLE MULTIPLIER SEQUENCE SPACES OF FUZZY NUMBERS DEFINED BY A SEQUENCE OF ORLICZ FUNCTION

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Abstract. In this paper we introduce some double sequence spaces of Fuzzy numbers defined by a sequence of Orlicz function $\mathcal{M} = (M_{kl})$ and a multiplier function $u = (u_{kl})$. We also make an efforts to prove some topological properties and inclusion relations between these spaces.

1. Introduction

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a Fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use Fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make Fuzzy sets a valuable mathematical tool. The concepts of Fuzzy sets and Fuzzy set operations were first introduced by Zadeh [7] and subsequently several authors have discussed various aspects of the theory and applications of Fuzzy sets such as Fuzzy topological spaces, similarity relations and Fuzzy orderings, Fuzzy measures of Fuzzy events, Fuzzy mathematical programming. Matloka [9] introduced bounded and convergent sequences of Fuzzy numbers and studied some of their properties.
In particular the concept of Fuzzy topology has very important applications in quantum particle physics, especially in the connections with both string and ϵ^{∞} theory which were given and studied by El Naschie [8]. In [12], Nanda studied on sequences of Fuzzy numbers and showed that the set of all convergent sequences of Fuzzy numbers forms a complete metric space.
The idea of statistical convergence of a sequence was introduced by Fast [5]. Statistical convergence was generalized by Buck [11] and studied by other authors, using a regular nonnegative summability matrix A in place of Cesaro matrix. The existing literature on statistical convergence appears to have been restricted to real or complex analysis, but at the first time Nurray and Savas [4] extended the idea to apply the sequences of Fuzzy numbers. For more details on Fuzzy sequence spaces see ([2], [3]) and references therein.

2. Definitions and preliminaries

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to define the following

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sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. Also l_M is a Banach space with the norm

$$\|x\| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Also, it was shown in [6] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A Fuzzy number is a Fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions :

(1) X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;
(2) X is Fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1 - \lambda)y) \geq \min[X(x), X(y)]$;
(3) X is upper semi-continuous;
(4) the closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

The set of all upper-semi continuous, normal, convex Fuzzy real numbers is denoted by $R(I)$.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ compact and convex } \}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a, \lambda \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (not separable) metric space.
For $0 < \alpha \leq 1$, the α-level set

$$X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$$

is a nonempty compact convex, subset of $\mathbb{R}^n$, as is the support X^0. Let $L(\mathbb{R}^n)$ denote the set of all Fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in \mathbb{R}$, in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$. Define for each $1 \leq q < \infty$

$$d_q(X,Y) = \Big\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha \Big\}^{\frac{1}{q}}$$

and $d_\infty(X,Y) = \sup_{0 \leq \alpha \leq 1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X,Y) = \lim_{q \to \infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover d_q is a complete, separable and locally compact metric space [10].
In this paper, d will denote d_q with $1 \leq q \leq \infty$.

A Fuzzy double sequence is a double infinite array of Fuzzy numbers. We denote a Fuzzy double sequence by (X_{mn}), where X_{mn}'s are Fuzzy numbers for each $m, n \in \mathbb{N}$. By $s''(F)$ we denote the set of all double sequences of Fuzzy numbers.

Definition ([1]) A double sequence $X = (X_{kl})$ of Fuzzy numbers is said to be convergent in the Pringsheim's sense or P-convergent to a Fuzzy number X_0, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(X_{kl}, X_0) < \epsilon \ \text{ for } \ k, l > N,$$

where $\mathbb{N}$ is the set of natural numbers, and we denote by $P - \lim X = X_0$. The number X_0 is called the Pringsheim limit of X_{kl}.
More exactly we say that a double sequence (X_{kl}) converges to a finite number X_0 if X_{kl} tend to X_0 as both k and l tends to ∞ independently of one another.

Let $\mathcal{M} = (M_{kl})$ be a sequence of Orlicz function, $p = (p_{kl})$ be a factorable double sequence of strictly positive real numbers and $u = (u_{kl})$ be any double sequence of positive real numbers. We now define the following classes of sequences in this paper:

$$W''(\mathcal{M}, u, p, F) = \Big\{ X = (X_{kl}) \in s''(F) : P\text{--}\lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl} \Big(M_{kl} \Big(\frac{d(X_{kl}, X_0)}{\rho} \Big) \Big)^{p_{kl}} = 0, \text{ for some } \ \rho > 0 \ \text{ and } \ X_0 \in R(I) \Big\},$$

$$W''_0(\mathcal{M}, u, p, F) = \Big\{ X = (X_{kl}) \in s''(F) : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl} \Big(M_{kl} \Big(\frac{d(X_{kl}, \bar{0})}{\rho} \Big) \Big)^{p_{kl}} = 0, \text{ for some } \ \rho > 0 \ \Big\}$$

and

$$W''_\infty(\mathcal{M}, u, p, F) = \Big\{ X = (X_{kl}) \in s''(F) : \sup_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl} \Big(M_{kl} \Big(\frac{d(X_{kl}, \bar{0})}{\rho} \Big) \Big)^{p_{kl}} < \infty, \text{ for some } \ \rho > 0 \ \Big\}.$$

where

$$\bar{0}(t) = \begin{cases} 1, & if \ \ t = 0, \\ 0, & \text{otherwise} \end{cases}$$

If $X \in W''(\mathcal{M}, u, p, F)$ we say that X is double strongly convergent with respect to the sequences of Orlicz function $\mathcal{M}$. In this case we write $X_{kl} \to X_0(W''(\mathcal{M}, u, p, F))$. When

$\mathcal{M}(X) = X$, then the family of sequences defined above become $W''[u,p,F]$, $W''_0[u,p,F]$ and $W''_\infty[u,p,F]$ respectively, which are presented as follows:

$$W''[u,p,F] = \Big\{X = (X_{kl}) \in s''(F) : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho}\Big)^{p_{kl}} = 0, \text{ for some } \rho > 0 \text{ and } X_0 \in R(I)\Big\},$$

$$W''_0[u,p,F] = \Big\{X = (X_{kl}) \in s''(F) : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(\frac{d(X_{kl}, \bar{0})}{\rho}\Big)^{p_{kl}} = 0, \text{ for some } \rho > 0 \Big\}$$

and

$$W''_\infty[u,p,F] = \Big\{X = (X_{kl}) \in s''(F) : \sup_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(\frac{d(X_{kl}, \bar{0})}{\rho}\Big)^{p_{kl}} < \infty, \text{ for some } \rho > 0 \Big\}.$$

(i) If $p_{kl} = 1$ for all $k,l \in \mathbb{N}$, then
$W''(\mathcal{M},u,p,F) = W''(\mathcal{M},u,F), W''_0(\mathcal{M},u,p,F) = W''_0(\mathcal{M},u,F)$, and $W''_\infty(\mathcal{M},u,p,F) = W''_\infty(\mathcal{M},u,F)$,

(ii) If $\mathcal{M} = M_{kl}(x) = x$ for all k,l and $p_{kl} = 1$ for all $k,l \in \mathbb{N}$, then
$W''(\mathcal{M},u,p,F) = W''(u,F), W''_0(\mathcal{M},u,p,F) = W''_0(u,F)$, and $W''_\infty(\mathcal{M},u,p,F) = W''_\infty(u,F)$,

(iii) If $p_{kl} = 1$ for all $k,l \in \mathbb{N}$, and $u_{kl} = 1$ for all k,l then
$W''(\mathcal{M},u,p,F) = W''(\mathcal{M},F), W''_0(\mathcal{M},u,p,F) = W''_0(\mathcal{M},F)$, and $W''_\infty(\mathcal{M},u,p,F) = W''_\infty(\mathcal{M},F)$.

In this paper we introduce some double multiplier sequence spaces of Fuzzy numbers defined by a sequence of Orlicz function.

3. Main Results

Theorem 3.1. Let the sequence (p_{kl}) be bounded, then

$$W''_0(\mathcal{M},u,p,F) \subset W''(\mathcal{M},u,p,F) \subset W''_\infty(\mathcal{M},u,p,F).$$

Proof. The inclusion $W''_0(\mathcal{M},u,p,F) \subset W''(\mathcal{M},u,p,F)$ is easy and is therefore omitted. Let $X \in W''(\mathcal{M},u,p,F)$. Then we have

$$\begin{aligned}
\frac{1}{mn}\sum_{k,l=1,1}^{mn} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},\bar{0})}{2\rho}\Big)\Big)^{p_{kl}} &\le \frac{D}{mn}\sum_{k,l=1,1}^{mn}\frac{1}{2^{p_{kl}}}u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big)^{p_{kl}} \\
&+ \frac{D}{mn}\sum_{k,l=1,1}^{mn}\frac{1}{2^{p_{kl}}}u_{kl}\Big(M_{kl}\Big(\frac{d(X_0,\bar{0})}{\rho}\Big)\Big)^{p_{kl}} \\
&\le \frac{D}{mn}\sum_{k,l=1,1}^{mn} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big)^{p_{kl}} \\
&+ D\max\Big(1, \sup u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},\bar{0})}{\rho}\Big)\Big)^{H}\Big),
\end{aligned}$$

where $\sup p_{kl} = H$ and $D = \max(1, 2^{H-1})$. Thus we have $X \in W''_{\infty}(\mathcal{M}, u, p, F)$.

Theorem 3.2. If $p_{kl} > 0$ and X is double strongly convergent to X_0, with respect to the sequence of Orlicz function $\mathcal{M}$, that is $X_{kl} \to X_0(W''(\mathcal{M}, u, p, F))$, then X_0 is unique.

Proof. Let $\lim p_{kl} = l > 0$ and suppose that $X_{kl} \to X_0(W''(\mathcal{M}, u, p, F))$, and $X_{kl} \to X_1(W''(\mathcal{M}, u, p, F))$. Then there exists ρ_1 and ρ_2 such that

$$P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho_1}\Big)\Big)^{p_{kl}} = 0$$

and

$$P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho_2}\Big)\Big)^{p_{kl}} = 0.$$

Let $\rho = \max(2\rho_1, 2\rho_2)$. Then we have

$$\begin{aligned} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_1, X_0)}{\rho}\Big)\Big)^{p_{kl}} &\leq \frac{D}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho_1}\Big)\Big)^{p_{kl}} \\ &+ \frac{D}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_1)}{\rho_2}\Big)\Big)^{p_{kl}} \\ &\to 0, (m, n \to \infty) \end{aligned}$$

where $\sup p_{kl} = H$ and $D = \max(1, 2^{H-1})$. Thus

$$P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_1, X_0)}{\rho}\Big)\Big)^{p_{kl}} = 0.$$

Also, since clearly

$$P - \lim_{k,l} u_{kl}\Big(M_{kl}\Big(\frac{d(X_1, X_0)}{\rho}\Big)\Big)^{p_{kl}} = u_{kl}\Big(M_{kl}\Big(\frac{d(X_1, X_0)}{\rho}\Big)\Big)^{l} = 0,$$

we have

$$u_{kl}\Big(M_{kl}\Big(\frac{d(X_1, X_0)}{\rho}\Big)\Big)^{l} = 0.$$

Finally, we get $X_0 = X_1$. This completes the proof.

Theorem 3.3. (i) If $0 < \inf p_{kl} < p_{kl} < 1$, then $W''(\mathcal{M}, u, p, F) \subset W''(\mathcal{M}, u, F)$.
(ii) If $1 \leq p_{kl} \leq \sup_{k,l} p_{kl} < \infty$, then $W''(\mathcal{M}, u, F) \subset W''(\mathcal{M}, u, p, F)$.

Proof. (i) Let $X \in W''(\mathcal{M}, u, p, F)$. Since $0 < \inf p_{kl} \leq 1$, we obtain the following:

$$\frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho}\Big)\Big) \leq \frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho}\Big)\Big)^{p_{kl}}.$$

Thus $X \in W''(\mathcal{M}, u, F)$.
(ii) Let $p_{kl} \geq 1$ for each (k, l) and $\sup_{k,l} p_{kl} < \infty$. Also let $X \in W''(\mathcal{M}, u, F)$, then for each $0 < \epsilon < 1$ there exists a positive integer N such that

$$\frac{1}{mn} \sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl}, X_0)}{\rho}\Big)\Big) \leq \epsilon < 1$$

for all $m, n \geq N$. This implies that

$$\frac{1}{mn}\sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big)^{p_{kl}} \leq \frac{1}{mn}\sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big).$$

Thus we have $X \in W''(\mathcal{M}, u, p, F)$.

Theorem 3.4. If $0 < \inf p_{kl} < p_{kl} \leq \sup_{k,l} p_{kl} < \infty$, then $W''(\mathcal{M}, u, p, F) = W''(\mathcal{M}, u, F)$.

Proof. The proof is trivial.

Theorem 3.5. Let $0 \leq p_{kl} \leq q_{kl}$ and let $\frac{q_{kl}}{p_{kl}}$ be bounded. Then

$$W''(\mathcal{M}, u, q, F) \subset W''(\mathcal{M}, u, p, F).$$

Proof. It is easy to prove so we omit the details.

4. Statistical convergence.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \lim_{m,n} \inf \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case the double sequence $(\frac{K_{m,n}}{mn})_{m,n=1,1}^{\infty,\infty}$ has a limit in the Pringsheim's sense then we say that K has a double natural density as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero).
In [3] Savas and Mursaleen defined the statistical analogue for double sequences $X = (X_{kl})$ of Fuzzy numbers as follows:

Definition A double sequence $X = (X_{kl})$ of Fuzzy numbers is said to be statistically convergent to X_0 provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{nm} |\{(j, k); j \leq m \ \text{ and } \ k \leq n : d(X_{kl}, X_0) \geq \epsilon\}| = 0.$$

In this case, we write $st_2 - \lim_{k,l} X_{kl} = X_0$ and we denote the set of all double statistically convergent sequences of Fuzzy numbers by $st_2(F)$ and denote the set of P-statistically null sequence by $(st_2)_0(F)$.

Theorem 4.1. If $\mathcal{M}$ is a sequence of Orlicz function then $W''(\mathcal{M}, u, F) \subset (st_2)(F)$.

Proof. Suppose $X \in W''(\mathcal{M}, u, F)$ and $\epsilon > 0$, then for every n and m, we obtain the following inequality:

$$\begin{aligned}\sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big) &\geq \sum_{d(X_{kl},X_0)\geq\epsilon\&k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big)\\ &\geq M(\epsilon)|\{k\leq m, l\leq n : d(X_{kl},X_0)\geq\epsilon\}|\end{aligned}$$

from which it follows that $X \in (st_2)(F)$.

Theorem 4.2. If $\mathcal{M}$ is a sequence of Orlicz function then $W''(\mathcal{M},u,F) = (st_2)(F)$.

Proof. By theorem (4.1), it is sufficient to show that $W''(\mathcal{M},u,F) \supset (st_2)(F)$.
Let $X \in (st_2)(F)$ and $\epsilon > 0$, then for every n and m, we have

$$\begin{aligned}\sum_{k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big) &= \sum_{d(X_{kl},X_0)<\epsilon\&k,l=1,1}^{m,n} u_{kl}\Big(M_{kl}\Big(\frac{d(X_{kl},X_0)}{\rho}\Big)\Big)\\ &\leq M(\epsilon)|\{k\leq m, l\leq n : d(X_{kl},X_0)<\epsilon\}|\end{aligned}$$

from which it follows that $X \in W''(\mathcal{M},u,F)$.

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Nonresonance on the boundary and strong solutions of elliptic equations with nonlinear boundary conditions

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Abstract

We deal with the solvability of linear second order elliptic partial differential equations with nonlinear boundary conditions by imposing asymptotic nonresonance conditions of nonuniform type with respect to the Steklov spectrum on the boundary nonlinearity. Unlike some recent approaches in the literature for problems with nonlinear boundary conditions, we cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *trace spaces* in order to prove the existence of strong solutions. The proofs are based on *a priori* estimates for possible solutions to a homotopy on suitable trace spaces and topological degree arguments.

1 Introduction

This paper is concerned with existence results for strong solutions of second order elliptic partial differential equations with nonlinear boundary conditions of the form

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad &\text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} = g(x,u) \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class C^2, $\partial/\partial\nu := \nu \cdot \nabla$ is the outward (unit) normal derivative on $\partial\Omega$, $c \in L^p(\Omega)$, $p > N$, where $c(x) \geq 0$ a.e. in Ω with strict inequality on a subset of Ω of positive measure, and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function with at most linear growth (see below). The case where $c \equiv 0$ (the original Steklov problem concerning harmonic functions) will also be considered; the reader is referred to Remarks 3 and 4 at the end of the paper.
As aforementioned, throughout this paper the boundary nonlinearity $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following two conditions.

For every constant $r > 0$, there is a constant $K = K(r) > 0$ such that

$$|g(x,u) - g(y,v)| \leq K\left(|x-y| + |u-v|\right) \tag{1.2}$$

for all $x, y \in \partial\Omega$ and all $u, v \in \mathbb{R}$ with $u, v \in [-r, r]$.
There are constants $a, b > 0$ such that

$$|g(x,u)| \leq a + b|u| \tag{1.3}$$

for all $(x,u) \in \partial\Omega \times \mathbb{R}$.
By a (strong) solution to Eq.(1.1) we mean a function $u \in W_p^2(\Omega)$ which satisfies (1.1) (the second equality in (1.1) being satisfied in the sense of trace). The reader is referred for instance to [2, 3, 4, 5, 6, 14] for the definitions and properties of Sobolev trace-spaces used in this paper.
We are mainly interested in the case when the boundary nonlinearity g interacts in some sense with two consecutive eigenvalues of the linear problem

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad &\text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} = \mu u \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\mu \in \mathbb{R}$ is a spectral parameter on the boundary which was first introduced on a disk in [15] and, more recently, significantly extended in [1, 9]. More specifically, we consider the case when the nonlinear ratio $g(x,u)/u$ asymptotically stays between two consecutive eigenvalues, but need not be uniformly bounded away from these eigenvalues as was previously required in the literature (see e.g. [1] for the linear case and [10, 11] for both the linear and nonlinear problems, and references therein). To the best of our knowledge this appears to be the first time this sort of conditions are considered between any consecutive Steklov eigenvalues when one has a trace-nonlinearity, unlike the case when one has a reaction nonlinearity in the differential equation (see e.g. [13] and references therein). It should be pointed out that, in this case, we work in a completely different setting since trace-spaces are considered in order to obtain the required *a priori* estimates for the nonlinear problem.
Unlike some recent approaches in the literature for problems with nonlinear boundary conditions, we cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *trace spaces* in order to prove the existence of strong solutions. In Section 2 below, we state the main result and introduce the nonlinear functional analytic setting. The proofs are based on *a priori* estimates (derived herein) for possible solutions to a homotopy on suitable trace spaces and topological degree arguments. Remarks are given at the end of the paper to shed more light on the main result and discuss its variants.

2 Nonuniform Nonresonance

In this section we impose conditions on the asymptotic behavior of the 'slopes' of the boundary nonlinearity $g(x,u)$, i.e., on $g(x,u)/u$ as $|u| \to \infty$. These conditions are of nonuniform type since the asymptotic ratio $g(x,u)/u$ need not be (uniformly) bounded away from consecutive Steklov eigenvalues. We mention that in all the results below, the boundary nonlinearity $g(x,u)$ may be replaced by $g(x,u) + h(x)$ where $h \in W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$.

In order to prove our results, we take a different approach which is based on topological degree theory on suitable boundary-trace spaces. This is in contrast with some recent approaches where variational methods were used for problems with nonlinear boundary conditions. The main result of this paper is given in the following existence theorem. (The case $c \equiv 0$ will be discussed at the end of the paper; see Remarks 3 and 4.)

Theorem 1 (Nonuniform nonresonance between consecutive Steklov eigenvalues) *Assume there are functions $\alpha, \beta \in L^\infty(\partial\Omega)$ such that*

$$\mu_j \leq \alpha(x) \leq \liminf_{|u|\to\infty} \frac{g(x,u)}{u} \leq \limsup_{|u|\to\infty} \frac{g(x,u)}{u} \leq \beta(x) \leq \mu_{j+1}$$

uniformly for a.e. $x \in \partial\Omega$ with

$$\oint (\alpha(x) - \mu_j)\varphi^2 > 0 \ \forall \varphi \in E_j \setminus \{0\} \quad \text{and} \quad \oint (\mu_{j+1} - \beta(x))\psi^2 > 0 \ \forall \psi \in E_{j+1} \setminus \{0\},$$

where, for $i \in \mathbb{N}$, E_i denotes the (finite-dimensional) Steklov nullspace associated with the Steklov eigenvalue μ_i. Then, the nonlinear equation (1.1) *has at least one (strong) solution $u \in W^2_p(\Omega)$.*

In contrast to some recent approaches in the literature for problems with nonlinear boundary conditions, we first cast the problem in terms of nonlinear compact perturbations of the identity on appropriate *boundary-trace spaces* as follows.
Set $\sigma := (\mu_j + \mu_{j+1})/2$, we consider the homotopy

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad & \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} - \sigma u = \lambda[-\sigma u + g(x,u)] \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\lambda \in [0,1]$; or equivalently,

$$\begin{aligned} -\Delta u + c(x)u = 0 \quad & \text{a.e. in } \Omega, \\ \frac{\partial u}{\partial \nu} = (1-\lambda)\sigma u + \lambda g(x,u) \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Note that for $\lambda = 0$ we have a linear problem which admits only the trivial solution since σ is in the resolvent of the linear Steklov problem (see e.g. [1, 9]). Whereas, for $\lambda = 1$, we have Eq.(1.1).
We define the linear (Steklov) boundary operator

$$\mathcal{B}: \ \mathrm{Dom}(\mathcal{B}) \subset W^2_p(\Omega) \Subset W^{1-1/p}_p(\partial\Omega) \to W^{1-1/p}_p(\partial\Omega) \quad \text{by}$$

$$\mathcal{B}u := \frac{\partial u}{\partial \nu} - \sigma u,$$

where

$$\mathrm{Dom}(\mathcal{B}) := \{u \in W^2_p(\Omega)) : -\Delta u + c(x)u = 0 \text{ a.e. in } \Omega\}.$$

Here, the compact 'containment' $W_p^2(\Omega) \Subset W_p^{1-1/p}(\partial\Omega)$ must be understood in the sense of trace; i.e., the trace operator $W_p^2(\Omega) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$ is a compact linear operator (see e.g. [5]).
We now define the nonlinear (Nemytskiĭ) operator

$$\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$$

by

$$\mathcal{N}u = -\sigma u + g(\cdot, u).$$

Eq.(1.1) is then equivalent to finding $u \in \mathrm{Dom}(\mathcal{B})$ such that

$$\mathcal{B}u = \mathcal{N}u. \tag{2.3}$$

Whereas the homotopy Eq.(2.1) is equivalent to

$$\mathcal{B}u = \lambda\mathcal{N}u, \quad \lambda \in [0,1], \quad u \in \mathrm{Dom}(\mathcal{B}). \tag{2.4}$$

From the above definitions, we deduce the following properties for the linear operator $\mathcal{B}$ and the nonlinear operator $\mathcal{N}$. Observe first that $\mathrm{Dom}(\mathcal{B}) := \{u \in W_p^2(\Omega)) : -\Delta u + c(x)u = 0 \text{ a.e. in } \Omega\}$ is a closed linear subspace of $W_p^2(\Omega)$, and that the linear operator $\mathcal{B} :\ \mathrm{Dom}(\mathcal{B}) \to W_p^{1-1/p}(\partial\Omega)$ is continuous, one-to-one and onto. Thus, it is a Fredholm operator of index zero since the nullspace $\mathrm{Ker}(\mathcal{B}) = \{0\}$ and the range $R(\mathcal{B}) = W_p^{1-1/p}(\partial\Omega)$. Owing to the compactness of the trace operator $\mathrm{Dom}(\mathcal{B}) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$, we deduce that

$$\mathcal{K} := \mathcal{B}^{-1} : W_p^{1-1/p}(\partial\Omega) \to \mathrm{Dom}(\mathcal{B}) \hookrightarrow W_p^{1-1/p}(\partial\Omega)$$

is a compact linear operator from $W_p^{1-1/p}(\partial\Omega)$ into $W_p^{1-1/p}(\partial\Omega)$.
Since the function g is locally Lipschitz and $W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$ (through the surjectivity of the trace operator $W_p^1(\Omega) \to W_p^{1-1/p}(\partial\Omega)$ and the imbedding $W_p^1(\Omega) \subset\subset C(\overline{\Omega})$ for $p > n$), it follows that the nonlinear operator $\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$ is continuous, and therefore $\mathcal{K}\mathcal{N} : W_p^{1-1/p}(\partial\Omega) \to W_p^{1-1/p}(\partial\Omega)$ is a nonlinear compact (i.e., completely continuous) operator. Thus, Eq.(2.4) is equivalent to

$$u = \lambda\mathcal{K}\mathcal{N}u, \quad \text{with } \lambda \in [0,1] \text{ and } u \in W_p^{1-1/p}(\partial\Omega); \tag{2.5}$$

which shows that, for each $\lambda \in [0,1]$, the operator $\lambda\mathcal{K}\mathcal{N}$ is a nonlinear compact perturbation of the identity on $W_p^{1-1/p}(\partial\Omega)$. It suffices to show that $\mathcal{K}\mathcal{N}$ has a fixed point u in $W_p^{1-1/p}(\partial\Omega)$. (Notice that, by the properties of $\mathcal{K}$, it follows that such a fixed point u belongs necessarily to $\mathrm{Dom}(\mathcal{B})$. Hence, $u \in W_p^2(\Omega)$ and is a (strong) solution of the nonlinear equation (1.1).)
For this purpose, we show that all possible solutions to the homotopy (2.1) (equivalently, (2.2) and (2.4)) are uniformly bounded in $W_p^{1-1/p}(\partial\Omega)$ independently of $\lambda \in [0,1]$ (actually we show that they are bounded in $W_p^2(\Omega)$ also), and then use topological degree theory to show existence of a strong solution. We first prove the following lemma which provides intermediate *a priori* estimates.

Lemma 1 *Assume that the conditions in Theorem 1 are met. Then all possible solutions to the homotopy (2.2) are (uniformly) bounded in $H^1(\Omega)$ independently of $\lambda \in [0,1]$.*

Proof. Suppose the conclusion of the lemma does not hold. Then, there are sequences $\{u_n\} \subset H^1(\Omega)$ and $\{\lambda_n\} \subset [0,1]$ such that $||u_n||_c \to \infty$ and

$$\int \nabla u_n \nabla v + \int c(x) u_n v = \oint (1-\lambda_n)\sigma u_n v + \oint \lambda_n g(x,u_n) v \quad \text{for all } v \in H^1(\Omega). \tag{2.6}$$

Set $v_n = \dfrac{u_n}{||u_n||_c}$. One sees that v_n is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence (relabeled) v_n which converges weakly to v_0 in $H^1(\Omega)$, and v_n converges strongly to v_0 in $L^2(\partial\Omega)$. Without loss of generality $\lambda_n \to \lambda_0 \in [0,1]$. Due to the at most linear growth condition on the boundary nonlinearity g, it follows that $\dfrac{g(x,u_n)}{||u_n||_c}$ is bounded in $L^2(\partial\Omega)$. Using the fact that $L^2(\partial\Omega)$ is a reflexive Banach space, we get that $\dfrac{g(x,u_n)}{||u_n||_c}$ converges weakly to g_0 in $L^2(\partial\Omega)$. Dividing (2.6) by $||u_n||_c$ we get that

$$\int \nabla v_n \nabla v + \int c(x) v_n v = (1-\lambda_n)\sigma \oint v_n v + \lambda_n \oint \frac{g(x,u_n)}{||u_n||_c} v \quad \text{for all } v \in H^1(\Omega). \tag{2.7}$$

Going to the limit as $n \to \infty$, we have that

$$\int \nabla v_0 \nabla v + \int c(x) v_0 v = (1-\lambda_0)\sigma \oint v_0 v + \lambda_0 \oint g_0 v \quad \text{for all } v \in H^1(\Omega), \tag{2.8}$$

Taking $v = v_0$ in (2.8) we get

$$||v_0||_c^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0. \tag{2.9}$$

Now, taking $v = \dfrac{u_n}{||u_n||_c}$ in (2.7), we get that $1 = ||v_n||_c^2 = (1-\lambda_n)\sigma \oint v_n^2 + \lambda_n \oint \dfrac{g(x,u_n)}{||u_n||_c} v_n$. Taking the limit as $n \to \infty$ and using (2.9) and the fact that $\dfrac{g(x,u_n)}{||u_n||_c}$ converges weakly to g_0 in $L^2(\partial\Omega)$ and v_n converges strongly to v_0 in $L^2(\partial\Omega)$, we have that

$$||v_0||_c^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0 = 1. \tag{2.10}$$

Now, we want to show that $v_0 = 0$; which will lead to a contradiction. From (2.7), notice that v_0 is a weak solution of the following linear equation

$$\begin{cases} -\Delta u + c(x)u = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = (1-\lambda_0)\sigma u + \lambda_0 g_0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Let us mention here that Eq.(2.11) implies that $\lambda_0 \neq 0$. Otherwise, since σ is in the Steklov resolvent, we deduce that $v_0 = 0$; which contradicts the fact that $||v_0||_c^2 = 1$.

In order to bring out all the properties of the function v_0, we need to analyze a little bit more carefully the function $(1-\lambda_0)\sigma v_0(x)+\lambda_0 g_0(x)$. Let us denote by $k(x)$ the function defined by

$$k(x)=\begin{cases}(1-\lambda_0)\sigma+\lambda_0\dfrac{g_0(x)}{v_0(x)} & \text{if } v_0(x)\neq 0,\\ 0 & \text{if } \ v_0(x)=0.\end{cases}$$

From the definition of σ and the conditions in Theorem 1, it turns out that

$$\mu_j\leq\alpha(x)\leq k(x)\leq\beta(x)\leq\mu_{j+1}\ \text{ for } v_0(x)\neq 0. \tag{2.12}$$

Therefore, v_0 is a weak solution to the linear equation

$$\begin{cases}-\Delta u+c(x)u=0 & \text{a.e. in }\Omega,\\ \dfrac{\partial u}{\partial\nu}=k(x)u & \text{on }\partial\Omega,\end{cases} \tag{2.13}$$

that is;

$$\int\nabla v_0\nabla v+\int c(x)v_0v=\oint k(x)v_0v\quad\text{for all } v\in H^1(\Omega). \tag{2.14}$$

We claim that this implies that either $v_0\in E_j$ or $v_0\in E_{j+1}$ only (see Lemma 2 below). Let us assume for the time being that this holds and finish the proof.
If $v_0\in E_j$, then taking $v=v_0$ in (2.14) we have that $\mu_j\oint v_0^2=||v_0||_c^2=\oint k(x)v_0^2$. Using (2.12), we get that $\oint(\alpha(x)-\mu_j)v_0^2\leq 0$. Since $\oint(\alpha(x)-\mu_j)\varphi^2>0$ for all $\varphi\in E_j\setminus\{0\}$, we conclude that $v_0=0$; which contradicts the fact that $||v_0||_c^2=1$.
Similarly, if $v_0\in E_{j+1}$, then taking again $v=v_0$ in (2.14), we get that $\oint(\mu_{j+1}-\beta(x))v_0^2\leq 0$. Since $\oint(\mu_{j+1}-\beta(x))\psi^2>0$ for all $\varphi\in E_{j+1}\setminus\{0\}$, we conclude that $v_0=0$; which contradicts the fact that $||v_0||_c^2=1$ again.
Thus, all possible solutions of the homotopy (2.2) are (uniformly) bounded in $H^1(\Omega)$ independently of $\lambda\in[0,1]$. The proof is complete.
The following lemma provide some useful information about the function v_0 that was used in the proof of the preceding lemma.

Lemma 2 *If u is a (nontrivial) weak solution of Eq.(2.13) with $\mu_j\leq\alpha(x)\leq k(x)\leq\beta(x)\leq\mu_{j+1}$, then either $u\in E_j$ or $u\in E_{j+1}$.*

Proof. Since u is (also) a weak solution, it satisfies

$$\int\nabla u\nabla v+\int c(x)uv=\oint k(x)uv\quad\text{for all } v\in H^1(\Omega). \tag{2.15}$$

Observe that $u\in[H_0^1(\Omega)]^\perp$. Hence, $u=\theta+\omega$, where $\theta\in\oplus_{l\leq j}E_l$ and $\theta\in\oplus_{l\geq j+1}E_l$. We know from the properties of the Steklov eigenfunctions (see e.g. [1, 9]) that

$$||\theta||_c^2\leq\mu_j\oint\theta^2\ \text{for all }\theta\in\oplus_{l\leq j}E_l\quad\text{and}\quad||\omega||_c^2\geq\mu_{j+1}\oint\omega^2\ \text{for all }\omega\in\oplus_{l\geq j+1}E_l. \tag{2.16}$$

Taking $v = \theta - \omega$ in (2.15), we get that

$$\int |\nabla\theta|^2 + c(x)\theta^2 - \int |\nabla\omega|^2 + c(x)\omega^2 = \oint k(x)\theta^2 - \oint k(x)\omega^2. \tag{2.17}$$

Using (2.16), we obtain that $\oint (k(x) - \mu_j)\theta^2 + \oint (\mu_{j+1} - k(x))\omega^2 \leq 0$. Therefore,

$$\oint (k(x) - \mu_j)\theta^2 = 0 \quad \text{and} \quad \oint (\mu_{j+1} - k(x))\omega^2 = 0.$$

Let $S_1 := \{x \in \partial\Omega : \theta(x) \neq 0\}$ and $S_2 := \{x \in \partial\Omega : \omega(x) \neq 0\}$. It follows that

$$k(x) = \mu_j \text{ a.e. on } S_1 \quad \text{and} \quad k(x) = \mu_{j+1} \text{ a.e on } S_2. \tag{2.18}$$

If $meas(S_1 \cap S_2) > 0$, we have that $\mu_j = k(x) = \mu_{j+1}$ for a.e. $x \in S_1 \cap S_2$, which cannot happen since $\mu_j \neq \mu_{j+1}$.
Now assume that $meas(S_1 \cap S_2) = 0$; that is; $\omega(x) = 0$ a.e. on S_1 and $\theta(x) = 0$ a.e. on S_2. If $\theta \not\equiv 0$, then taking $v = \theta$ in (2.15) and using the c-orthogonality, we get that

$$\int |\nabla\theta|^2 + c(x)\theta^2 = \oint k(x)\theta^2 + \oint k(x)\omega\theta = \oint k(x)\theta^2.$$

Since $k(x) \geq \mu_j$, we have that $||\theta||_c^2 \geq \mu_j \oint \theta^2$. It follows from (2.16) that $||\theta||_c^2 = \mu_j \oint \theta^2$; which implies that $\theta \in E_j$.
Similarly, if $\omega \not\equiv 0$, then taking $v = \omega$ in (2.15) and using the c-orthogonality, we get that

$$\int |\nabla\omega|^2 + c(x)\omega^2 = \oint k(x)\omega^2 + \oint k(x)\omega\theta = \oint k(x)\omega^2.$$

Since $k(x) \leq \mu_{j+1}$, we have that $||\omega||_c^2 \leq \mu_{j+1} \oint \omega^2$. It follows from (2.16) that $||\omega||_c^2 = \mu_{j+1} \oint \omega^2$; which implies that $\omega \in E_{j+1}$. Thus, $u = \theta + \omega$ with $\theta \in E_j$ and $\omega \in E_{j+1}$.
Finally, we claim that the function u cannot be written in the form $u = \theta + \omega$ where $\theta \in E_j \backslash \{0\}$ and $\omega \in E_{j+1} \backslash \{0\}$. Indeed, suppose that this does not hold; that is, $u = \theta + \omega$ with $\theta \in E_j \backslash \{0\}$ and $\omega \in E_{j+1} \setminus \{0\}$. Then, by taking $v = \theta - \omega$ in (2.15), we again get (2.17). Since $\theta \in E_j$ and $\omega \in E_{j+1}$ and $\alpha(x) \leq k(x) \leq \beta(x)$ a.e. on $\partial\Omega$, we deduce that

$$\oint (\alpha(x) - \mu_j)\theta^2 \leq 0 \quad \text{and} \quad \oint (\mu_{j+1} - \beta(x))\omega^2 \leq 0;$$

which contradicts the fact that $\oint (\alpha(x) - \mu_j)\varphi^2 > 0$ for all $\varphi \in E_j \setminus \{0\}$ and $\oint (\mu_{j+1} - \beta(x))\psi^2 > 0$ for all $\psi \in E_{j+1} \setminus \{0\}$. Thus, either $u \in E_j$ or $u \in E_{j+1}$. The proof is complete.
We are now in a position to prove the main result stated above.
Proof of Theorem 1 Let $(\lambda, u) \in [0,1] \times W_p^{1-1/p}(\partial\Omega)$ be a solution to the homotopy (2.5) (equivalently (2.2)). Since $\int \nabla u \nabla v + \int c(x)uv = 0$ for all $v \in C_0^1(\Omega)$, and the trace of

$u \in W_p^{1-1/p}(\partial\Omega) \subset C(\partial\Omega)$, it follows from Theorem 13.1 in [6, pp. 199-200] (also see [3, 4]) that there is a constant $c_0 > 0$ (independent of u) such that $\sup_{\Omega} |u(x)| \leq c_0 |u|_{H^1(\Omega)}$, and so $\max_{\overline{\Omega}} |u(x)| \leq c_0 |u|_{H^1(\Omega)}$ by continuity of u on $\overline{\Omega}$. From Lemma 1 above and the (local Lipschitz) continuity of g we deduce that that $\max_{\partial\Omega} |\partial u/\partial\nu| = \max_{\partial\Omega} |(1-\lambda)\sigma u + \lambda g(\cdot, u)|$ is bounded independently of u and λ. Actually, we deduce from Theorem 2 in [7, p. 1204] that $|u|_{C^1(\overline{\Omega})}$ is bounded (independently of u and λ). Therefore, the continuity of the trace operator $C^1(\overline{\Omega}) \subset W_p^1(\Omega) \to W_p^{1-1/p}(\partial\Omega)$ and Lemma 1 herein imply that there is a constant $c_1 > 0$ (independent of u and λ) such that

$$|u|_{W_p^{1-1/p}(\partial\Omega)} < c_1, \tag{2.19}$$

for all possible solutions to the homotopy (2.5) (or equivalently (2.2)).
Now, by the homotopy invariance property of the topological degree (see e.g. [8, 12]), it follows that

$$1 = deg(I, B_{c_1}(0), 0) = deg(I - \mathcal{KN}, B_{c_1}(0), 0) \neq 0,$$

where $B_{c_1}(0) \subset W_p^{1-1/p}(\partial\Omega)$ is the ball of radius $c_1 > 0$ centered at the origin. Thus, by the existence property of the topological degree (see e.g. [8, 12]), the (nonlinear) operator $\mathcal{KN}$ has a fixed point in $W_p^{1-1/p}(\partial\Omega)$ (which is also in $W_p^2(\Omega)$ as aforementioned). The proof is complete.

Remark 1 Notice that, since g is locally Lipschitz, it follows from (2.19) and the boundary condition in the homotopy (2.2) that $|u|_{W_p^{2-1/p}(\partial\Omega)} \leq c_2$ for some constant $c_2 > 0$ independent of u and λ. Therefore, $|u|_{W_p^2(\Omega)} \leq c_3$ for some constant $c_3 > 0$.

Remark 2 The case $\mu_j = \mu_1$ more clearly illustrates the fact that the nonresonance conditions in Theorem 1 are genuinely of nonuniform type. Indeed, in this case $E_1 \setminus \{0\}$ contains only (continuous) functions which are either positive or negative on $\overline{\Omega}$. The condition that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with $\oint (\alpha(x) - \mu_1)\varphi^2 > 0$ for all $\varphi \in E_1 \setminus \{0\}$ is equivalent to saying that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. Thus $\alpha(x)$ need not be (uniformly) bounded away from μ_1.

Remark 3 Our main result, Theorem 1 herein, still holds true when $c \equiv 0$. (This Laplace's equation is the original linear equation which was considered by Steklov on a disk in [15].) Indeed, a modification is needed in the proof of Lemma 1 as follows. We proceed as in that proof with $||\cdot||_c$ replaced by $||\cdot||_1$ (here $||\cdot||_1$ denotes the standard $H^1(\Omega)$-norm), and $v_n = u_n/||u_n||_1$ up to the equation (2.8). Taking $v = v_0$ in (2.8) we now get

$$\int |\nabla v_0|^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0. \tag{2.20}$$

Now, taking $v = u_n/||u_n||_1$ in (2.7) where $||u_n||_c$ is replaced by $||u_n||_1$, we get that

$$\int |\nabla v_n|^2 = (1-\lambda_n)\sigma \oint v_n^2 + \lambda_n \oint \frac{g(x, u_n)}{||u_n||_1} v_n. \tag{2.21}$$

Taking the limit as $n \to \infty$ and using (2.20) and the fact that $\dfrac{g(x,u_n)}{||u_n||_1}$ converges weakly to g_0 in $L^2(\partial\Omega)$ and v_n converges strongly to v_0 in $L^2(\partial\Omega)$, we have that

$$\lim_{n\to\infty} \int |\nabla v_n|^2 = (1-\lambda_0)\sigma \oint v_0^2 + \lambda_0 \oint g_0 v_0 = \int |\nabla v_0|^2.$$

This implies that

$$||v_0||_1^2 = \int |\nabla v_0|^2 + \int v_0^2 = \lim_{n\to\infty} \left(\int |\nabla v_n|^2 + \int v_n^2 \right) = \lim_{n\to\infty} ||v_n||_1^2 = 1. \tag{2.22}$$

We now proceed as in the proof of Lemma 1 after Eq.(2.10) to show that $v_0 = 0$; which is a contradiction with (2.22).
The proof of Lemma 2 also needs to be modified as follows. The norm $||\cdot||_c$ is now replaced by the $H^1(\Omega)$-equivalent norm $||\cdot||$ defined by

$$||u||^2 := \int |\nabla u|^2 + \oint u^2 \quad \text{for } u \in H^1(\Omega).$$

(See e.g. [1, pp. 333–334].) By using the decomposition of $H^1(\Omega)$ given in [1, Theorem 7.3, p. 337], we now proceed with the arguments used in the proof of Lemma 2 herein to reach its conclusion.

Remark 4 Note that the case $c \equiv 0$ even more clearly illustrates the fact that the nonresonance conditions in Theorem 1 are genuinely of nonuniform type. Indeed, in this case $\mu_1 = 0$ and E_1 contains only constant functions. The condition that $\alpha(x) \geq \mu_1$ a.e. on $\partial\Omega$ with $\oint (\alpha(x) - \mu_1)\varphi^2 > 0$ for all $\varphi \in E_1 \setminus \{0\}$ is equivalent to $\alpha(x) \geq 0$ a.e. on $\partial\Omega$ with strict inequality on a subset of $\partial\Omega$ of positive measure. Thus $\alpha(x)$ need not be (uniformly) bounded away from $\mu_1 = 0$. Actually, a careful analysis of the proofs of lemmas 1 and 2 shows that, in this case, one can drop the requirement that $\alpha(x) \geq 0$ a.e. on $\partial\Omega$ and require only that $\oint \alpha(x) > 0$. Thus, a 'crossing' of the zero eigenvalue on a subset of $\partial\Omega$ of positive measure is allowed; that is, $\alpha(x)$ could be negative on a subset of $\partial\Omega$ of positive measure.

Remark 5 Our main result remains valid if one considers an equation with a more general linear part with variable coefficients; that is,

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x)u = 0 & \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where now $\partial/\partial\nu := \nu \cdot A\nabla$ is the (unit) outward conormal derivative. The matrix $A(x) := \big(a_{ij}(x)\big)$ is symmetric with $a_{ij} \in C^{0,1}(\overline{\Omega})$ such that there is a constant $\gamma > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$\langle A(x)\xi, \xi \rangle \geq \gamma |\xi|^2 \quad \text{on } \overline{\Omega}.$$

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On Hybrid (A, η, m)-proximal Point Algorithm Frameworks for Solving General Operator Inclusion Problems[1]

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Abstract. The purpose of this paper is to introduce and study a new class of hybrid (A, η, m)-proximal point algorithms with errors for solving general nonlinear operator inclusion problems in Hilbert spaces based on (A, η, m)-monotonicity framework. Furthermore, by using the generalized resolvent operator technique associated with the (A, η, m)-monotone operators, we discuss the approximation solvability of the operator inclusion problems and the convergence rate of iterative sequences generated by the algorithm.
Key Words: (A, η, m)-monotonicity, hybrid (A, η, m)-proximal point algorithm, nonlinear operator inclusion problem, generalized resolvent operator technique, convergence rate.
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1 Introduction

In 2006, Lan [1] first introduced a new concept of (A, η)-monotone (so called (A, η, m)-maximal monotone [2]) operators, which generalizes the (H, η)-monotonicity, A-monotonicity and other existing monotone operators as special cases, and studied some properties of (A, η)-monotone operators and defined resolvent operators associated with (A, η)-monotone operators. Further, some (systems of) variational inequalities, nonlinear (random or parametric) operator inclusions, nonlinear (set-valued) inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relations to Nash equilibrium problems. See, for example, [1-16] and the references therein.

On the other hand, Pennanen [17] has shown using the over-relaxed proximal point algorithm and applying a similar approach to Rockafellar [18] by restricting M^{-1} to be locally Lipschitz continuous and by strengthening error tolerance that the sequence converges linearly to a solution of the following general nonlinear operator inclusion problem in Hilbert space $\mathcal{X}$: Find $x \in \mathcal{X}$ such that

$$0 \in M(x), \tag{1.1}$$

where $M : \mathcal{X} \to 2^{\mathcal{X}}$ is a set-valued operator and $2^{\mathcal{X}}$ denotes the family of all the nonempty subsets of $\mathcal{X}$.

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We remark that for appropriate and suitable choices of M and $\mathcal{X}$, one can know that a number of known variational inequalities, variational inclusions and corresponding optimization problems can be unified into the special cases of the problem (1.1), which provide us a general and unified framework for studying a wide range of interesting and important problems arising in mathematics, physics, engineering sciences, optimal control and economics finance, etc. For more details, see [1-7, 9, 10, 13-16] and the references therein, and the following examples.

Example 1.1. Suppose that $A : \mathcal{X} \to \mathcal{X}$ is r-strongly η-monotone, and $f : \mathcal{X} \to \mathbb{R}$ is locally Lipschitz such that ∂f, the subdifferential, is m-relaxed η-monotone with $r - m > 0$. Clearly, we have

$$\langle x - y, \eta(x, y)\rangle \geq (r - m)\|x - y\|^2,$$

where $x \in A(x) + \partial f(x)$ and $y \in A(y) + \partial f(y)$ for all $x, y \in \mathcal{X}$. Thus, $A + \partial f$ is η-pseudomonotone, which is indeed, η-maximal monotone. This is equivalent to stating that $A + \partial f$ is (A, η, m)-monotone and the problem (1.1) becomes to finding $x \in \mathcal{X}$ such that

$$0 \in A(x) + \partial f(x).$$

Example 1.2. Consider the following convex optimization problem with bound constraints:

$$\begin{array}{l} \min f(x), \\ s.t. \quad x \in \Omega, \end{array} \tag{1.2}$$

where $\Omega = \{x \in \mathbb{R}^l \,|\, d \leq x \leq h\}$, $\mathbb{R} = (-\infty, +\infty)$ and $f : \Omega \to \mathbb{R}$ is convex and continuously differentiable. From the Karush-Kuhn-Tucher conditions, we see that x^* is an optimal solution to the problem (1.2) if and only if x^* satisfies

$$\begin{array}{ll} \partial f/\partial x_i^* \geq 0, & x_i^* = d_i, \\ \partial f/\partial x_i^* = 0, & x_i^* \in (d_i, h_i), \\ \partial f/\partial x_i^* \leq 0, & x_i^* = h_i. \end{array} \tag{1.3}$$

The problem (1.3) is equivalent to the following variational inequality:

$$(x - x^*)^T \nabla f(x^*) \geq 0, \quad \forall x \in \Omega,$$

where $\nabla f(x)$ is the gradient of f.

Recently, Verma [14] developed a general framework for a hybrid proximal point algorithm using the notion of (A, η)-monotonicity and explored convergence analysis for this algorithm in the context of solving the nonlinear inclusion problem (1.1) along with some results on the resolvent operator corresponding to (A, η)-monotonicity. Verma [13, 15] introduced a general framework for the over-relaxed A-proximal point algorithm based on the A-maximal monotonicity and pointed out "the over-relaxed A-proximal point algorithm is of interest in the sense that it is quite application-oriented, but nontrivial in nature". Very recently, Xia and Huang [16] investigated a general iterative algorithm, which consists of an inexact proximal point step followed by a suitable orthogonal projection onto a hyperplane and proved the convergence of the algorithm for a pseudomonotone mapping with weakly upper semicontinuity and weakly compact and convex values. Further, the authors also analyzed the convergence rate of the iterative sequence under some suitable conditions.

Motivated and inspired by the above works, in this paper, we introduce and study a new class of hybrid (A, η, m)-proximal point algorithms with errors for solving the general nonlinear operator inclusion problem (1.1) in Hilbert spaces based on (A, η, m)-monotonicity framework. Furthermore, by using the generalized resolvent operator technique associated with the (A, η, m)-monotone operators, we shall discuss the approximation solvability of the operator inclusion problems and the convergence rate of iterative sequences generated by the algorithm.

2 Preliminaries

Definition 2.1. Let $A : \mathcal{X} \to \mathcal{X}$ and $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be two single valued operators. Then

(i) A is δ-strongly monotone, if there exists a positive constant δ such that

$$\langle A(x) - A(y), x - y\rangle \geq \delta\|x - y\|^2, \quad \forall x,\, y \in \mathcal{X};$$

(ii) A is r-strongly η-monotone, if there exists a positive constant r such that

$$\langle A(x) - A(y), \eta(x, y)\rangle \geq r\|x - y\|^2, \quad \forall x,\, y \in \mathcal{X};$$

(iii) A is κ-Lipschitz continuous, if there exists a constant $\kappa > 0$ such that

$$\|A(x) - A(y)\| \leq \kappa\|x - y\|^2, \quad \forall x,\, y \in \mathcal{X};$$

(iv) η is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \qquad \forall x, y \in \mathcal{X}.$$

Definition 2.2. Let $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $A : \mathcal{X} \to \mathcal{X}$ be two single-valued operators. Then set-valued operator $M : \mathcal{X} \to 2^{\mathcal{X}}$ is said to be

(i) m-relaxed η-monotone if there exists a constant $m > 0$ such that

$$\langle x - y, \eta(x, y)\rangle \geq -m\|x - y\|^2, \quad \forall x, y \in \mathcal{X}, x \in M(x), y \in M(y);$$

(ii) (A, η, m)-maximal monotone if M is m-relaxed η-monotone and $R(A+\rho M) = \mathcal{X}$ for every $\rho > 0$.

Remark 2.1. (A, η, m)-monotonicity (so-called (A, η)-monotonicity [1], (A, η)-maximal relaxed monotonicity [5, 14]) includes (H, η)-monotonicity, H-monotonicity, A-monotonicity, maximal η-monotonicity, classical maximal monotonicity (see [1-5, 7, 9, 12-16]). Further, we note that the idea of this extension is so close to the idea of extending convexity to invexity introduced by Hanson in [19], and the problem studied in this paper can be used in invex optimization and also for solving the variational-like inequalities as a direction for further applied research, see, related works in [11, 12] and the references therein.

Lemma 2.1. ([1]) Let $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be τ-Lipschitz continuous, $A : \mathcal{X} \to \mathcal{X}$ be r-strongly η-monotone and $M : \mathcal{X} \to 2^{\mathcal{X}}$ be (A, η, m)-maximal monotone. Then the resolvent operator defined by

$$R_{\rho,M}^{A,\eta}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in \mathcal{X}$$

is $\frac{\tau}{r-\rho m}$-Lipschitz continuous.

3 Proximal Point Algorithms and Convergence

In this section, we shall introduce a new class of hybrid (A, η, m)-proximal point algorithms for solving the problem (1.1).

Definition 3.1. An operator M^{-1}, the inverse of $M : \mathcal{X} \to \mathcal{X}$, is (ι, t)-Lipschitz continuous at 0 if for any $t \geq 0$, there exist a constant $\iota \geq 0$ and a solution x^* of $0 \in M(x)$ (equivalently $x^* \in M^{-1}(0)$) such that

$$\|x - x^*\| \leq \iota \|w - 0\|, \forall x \in M^{-1}(w) \text{ and } w \in B_t = \{w | \, \|w\| \leq t, w \in \mathcal{X}, t > 0\}.$$

Lemma 3.1. Let $\mathcal{X}$ be a real Hilbert space, $A : \mathcal{X} \to \mathcal{X}$ be r-strongly η-monotone and $M : \mathcal{X} \to 2^{\mathcal{X}}$ be (A, η, m)-maximal monotone. Then the following statements are mutually equivalent:

(i) An element $x \in \mathcal{X}$ is a solution to (1.1).

(ii) For $x \in \mathcal{X}$, we have

$$x = R_{\rho,M}^{A,\eta} A(x),$$

where $R_{\rho_n,M}^{A,\eta}(x) = (A + \rho M)^{-1}(x)$.

Algorithm 3.1. *Step* 1. Choose an arbitrary initial point $x_0 \in \mathcal{X}$.

Step 2. Choose sequences $\{\alpha_n\}$, $\{\epsilon_n\}$ and $\{\rho_n\}$ such that for $n \geq 0$, $\{\alpha_n\}$, $\{\epsilon_n\}$ and $\{\rho_n\}$ are three sequences in $[0, \infty)$ satisfying

$$\sum_{i=0}^{\infty} \sigma_i < \infty, \quad \alpha = \limsup_{n \to \infty} \alpha_n < 1, \quad \rho_n \to \rho \in (0, \frac{r}{m}),$$

and $\{e_n\}$ is error sequence in $\mathcal{X}$ to take into account a possible inexact computation of the operator point, which satisfies $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Step 3. Let $\{x_n\} \subset \mathcal{X}$ be generated by the following iterative procedure

$$A(x_{n+1}) = (1 - \alpha_n)A(x_n) + \alpha_n y_n + e_n, \tag{3.1}$$

and y_n satisfies

$$\|y_n - A(R_{\rho_n,M}^{A,\eta}(A(x_n)))\| \leq \epsilon_n \|y_n - A(x_n)\|,$$

where $n \geq 0$, $R_{\rho_n,M}^{A,\eta} = (A + \rho_n M)^{-1}$ and $\rho_n > 0$ is a constant.

Step 5. If x_n, y_n and e_n $(n = 0, 1, 2, \cdots)$ satisfy (3.2) to sufficient accuracy, stop; otherwise, set $k := k + 1$ and return to *Step* 2.

Remark 3.1. We note that Algorithm 3.1 is differ from the algorithm of Theorem 3.4 associated with (A, η)-maximal monotonicity in [14].

Theorem 3.1. Let M and $\mathcal{X}$ be the same as in problem (1.1). If, in addition,

(i) $A : \mathcal{X} \to \mathcal{X}$ is r-strongly η-monotone and κ-Lipschitz continuous, and $\eta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is τ-Lipschitz continuous;

(ii) the iterative sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded;

(iii) there exists a constant ρ such that

$$0 < \rho < \frac{r - \kappa\tau}{m},$$

then the sequence $\{x_n\}$ converges linearly to a solution x^* of the problem (1.1) with convergence rate

$$\theta = 1 - \alpha(1 - \frac{\kappa\tau}{r - \rho m}) < 1.$$

Proof. Let x^* be a solution of problem (1.1). Then for all $\rho_n > 0$ and $n \geq 0$, by Lemma 3.1, now we know

$$A(x^*) = (1 - \alpha_n)A(x^*) + \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x^*))), \tag{3.2}$$

Let

$$A(z_{n+1}) = (1 - \alpha_n)A(x_n) + \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x_n))) + e_n, \forall n \geq 0.$$

It follows from the assumptions of the theorem, Lemma 2.1 and (3.2) that

$$\begin{aligned}
&\|A(z_{n+1}) - A(x^*)\| \\
&= \|(1 - \alpha_n)A(x_n) + \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x_n))) + e_n \\
&\quad -(1 - \alpha_n)A(x^*) - \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x^*)))\| \\
&\leq (1 - \alpha_n)\|A(x_n) - A(x^*)\| \\
&\qquad +\alpha_n\kappa\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - R^{A,\eta}_{\rho_n,M}(A(x^*))\| + \|e_n\| \\
&\leq (1 - \alpha_n)\|A(x_n) - A(x^*)\| + \frac{\alpha_n\kappa\tau}{r - \rho_n m}\|A(x_n) - A(x^*)\| + \|e_n\| \\
&= \theta_n\|A(x_n) - A(x^*)\| + \|e_n\|,
\end{aligned} \tag{3.3}$$

where

$$\theta_n = 1 - \alpha_n + \frac{\alpha_n\kappa\tau}{r - \rho_n m}.$$

Since $A(x_{n+1}) = (1-\alpha_n)A(x_n)+\alpha_n y_n+e_n$, $A(x_{n+1})-A(x_n) = \alpha_n[y_n-A(x_n)]+e_n$. Thus, we have

$$\begin{aligned}
\|A(x_{n+1}) - A(z_{n+1})\| &= \alpha_n\|y_n - A(R^{A,\eta}_{\rho_n,M}(A(x_n)))\| \\
&\leq \alpha_n\epsilon_n\|y_n - A(x_n)\| \\
&= \epsilon_n\|A(x_{n+1}) - A(x_n) - e_n\| \\
&\leq \epsilon_n\|A(x_{n+1}) - A(x_n)\| + \epsilon_n\|e_n\|.
\end{aligned} \tag{3.4}$$

Next, we estimate using (3.3) and (3.4) that

$$\begin{aligned}
&\|A(x_{n+1}) - A(x^*)\| \\
&\le \|A(x_{n+1}) - A(z_{n+1})\| + \|A(z_{n+1}) - A(x^*)\| \\
&\le \epsilon_n\|A(x_{n+1}) - A(x_n)\| + \theta_n\|A(x_n) - A(x^*)\| + (1+\epsilon_n)\|e_n\| \\
&\le \epsilon_n\|A(x_{n+1}) - A(x^*)\| + (\epsilon_n + \theta_n)\|A(x_n) - A(x^*)\| + (1+\epsilon_n)\|e_n\|.
\end{aligned}$$

This implies that

$$\|A(x_{n+1}) - A(x^*)\| \le \frac{\theta_n + \epsilon_n}{1-\epsilon_n}\|A(x_n) - A(x^*)\| + \frac{1+\epsilon_n}{1-\epsilon_n}\|e_n\|. \tag{3.5}$$

Since A is κ-Lipschitz continuous and r-strongly η-monotone (and hence, $\|A(x) - A(y)\| \ge \frac{r}{\tau}\|x-y\|, \forall x, y \in \mathcal{X}$), it follows from (3.5) and $\sum_{n=0}^{\infty}\|e_n\| < \infty$ that the $\{x_n\}$ converges linearly to a solution x^* for θ_n.

Hence, we have

$$\limsup_{n\to\infty}\frac{\theta_n + \epsilon_n}{1-\epsilon_n} = \limsup_{n\to\infty}\theta_n = 1 - \alpha(1 - \frac{\kappa\tau}{r - \rho m}),$$

where $\alpha = \limsup_{n\to\infty}\alpha_n$, $\rho_n \to \rho$. This completes the proof. □

Theorem 3.2. Assume that A, M, η and $\mathcal{X}$ are the same as in Theorem 3.1 and condition (ii) in Theorem 3.1 holds. If, in addition,

(iv) M^{-1} is $(\iota, t)-$Lipschitz continuous at 0;

(v) for $\gamma > \frac{1}{2}$, $n \ge 0$ and $x^* \in \mathcal{X}$,

$$\begin{aligned}
&\langle A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*))), A(x_n) - A(x^*)\rangle \\
&\ge \gamma\|A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*)))\|^2,
\end{aligned}$$

(vi) there exists a constant ρ such that

$$\iota\sqrt{\kappa^2 - r^2\tau^{-2}(2\gamma-1)} < \rho < \frac{r}{m}, \quad \kappa\tau > r\sqrt{2\gamma-1},$$

then the sequence $\{x_n\}$ converges linearly to a solution x^* of (1.1) with convergence rate

$$\theta = 1 - \alpha + \alpha\kappa d < 1,$$

where $d = \frac{\tau\iota}{\sqrt{r^2\iota^2(2\gamma-1)+\tau^2\rho^2}}$.

Proof. Let x^* be a zero of M. we infer from Lemma 3.1 that any solution of (1.1) is a fixed point of $R^{A,\eta}_{\rho_n,M} \circ A$, i.e., $R^{A,\eta}_{\rho_n,M}(A(x^*)) = x^*$. For $J_{\rho_n} = A - A \circ R^{A,\eta}_{\rho_n,M} \circ A$, and under the assumptions (including (vi)), it follows that $A(x_n) - A(R^{A,\eta}_{\rho_n,M}(A(x_n))) \to 0$. Since $\rho_n^{-1}J_{\rho_n}(x_n) \in M(R^{A,\eta}_{\rho_n,M}(A(x_n)))$, this implies $R^{A,\eta}_{\rho_n,M}(A(x_n)) \in M^{-1}(\rho_n^{-1}J_{\rho_n}(x_n))$. Thus, applying the Lipscitz continuity of M^{-1} by setting $w = \rho_n^{-1}J_{\rho_n}(x_n)$ and $z = \rho_n^{-1}J_{\rho_n}(x_n) \in M(R^{A,\eta}_{\rho_n,M}(A(x_n)))$, we have

$$\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - x^*\| \le \iota\|\rho_n^{-1}J_{\rho_n}(x_n)\|. \tag{3.6}$$

On the other hand, since $J_{\rho_n}(x^*) = A(x^*) - A(R^{A,\eta}_{\rho_n,M}(A(x^*))) = A(x^*) - A(x^*) = 0$, $\rho_n^{-1} J_{\rho_n}(x^*) = 0$ and it follows from condition (v) the r-strong η-monotonicity of A (and hence, A being $\frac{r}{\tau}$-expanding) that

$$
\begin{aligned}
&\|J_{\rho_n}(x_n) - J_{\rho_n}(x*)\|^2 = \|A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*))) - (A(x_n) - A(x^*))\|^2 \\
&\leq \|A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*))\|^2 + \|A(x_n) - A(x^*)\|^2 \\
&\quad -2\langle A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*)), A(x_n) - A(x^*)\rangle \\
&\leq -(2\gamma - 1)\|A(R^{A,\eta}_{\rho,M}(A(x_n))) - A(R^{A,\eta}_{\rho,M}(A(x^*))\|^2 + \|A(x_n) - A(x^*)\|^2 \\
&\leq -\frac{r^2(2\gamma - 1)}{\tau^2}\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - R^{A,\eta}_{\rho_n,M}(A(x^*))\|^2 + \|A(x_n) - A(x^*)\|^2 \\
&= \|A(x_n) - A(x^*)\|^2 - \frac{r^2(2\gamma - 1)}{\tau^2}\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - x^*\|^2. \qquad (3.7)
\end{aligned}
$$

Combining (3.6) with (3.7), we have

$$
\begin{aligned}
&\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - x^*\|^2 \\
&\leq \frac{\iota^2}{\rho_n^2}\|J_{\rho_n}(x_n) - J_{\rho_n}(x^*)\|^2 \\
&\leq \frac{\iota^2}{\rho_n^2}\|A(x_n) - A(x^*)\|^2 - \frac{r^2\iota^2(2\gamma - 1)}{\tau^2\rho_n^2}\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - x^*\|^2,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - R^{A,\eta}_{\rho_n,M}(A(x^*))\| &= \|R^{A,\eta}_{\rho_n,M}(A(x_n)) - x^*\| \\
&\leq d_n(\|A(x_n) - A(x^*)\|, \qquad (3.8)
\end{aligned}
$$

where

$$
d_n = \frac{\tau\iota}{\sqrt{r^2\iota^2(2\gamma - 1) + \tau^2\rho_n^2}}.
$$

On the other hand, let

$$
A(z_{n+1}) = (1 - \alpha_n)A(x_n) + \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x_n))) + e_n, \forall n \geq 0.
$$

Then, from (3.3) and (3.8), it follows that

$$
\begin{aligned}
&\|A(z_{n+1}) - A(x^*)\| \\
&= \|(1 - \alpha_n)A(x_n) + \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x_n))) + e_n \\
&\quad -(1 - \alpha_n)A(x^*) - \alpha_n A(R^{A,\eta}_{\rho_n,M}(A(x^*)))\| \\
&\leq (1 - \alpha_n)\|A(x_n) - A(x^*)\| + \alpha_n\kappa\|R^{A,\eta}_{\rho_n,M}(A(x_n)) - R^{A,\eta}_{\rho_n,M}(A(x^*))\| + \|e_n\| \\
&\leq \theta_n\|A(x_n) - A(x^*)\| + \|e_n\|,
\end{aligned}
$$

where

$$
\theta_n = 1 - \alpha_n + \alpha_n\kappa d_n.
$$

The rest of proof can be obtained from (3.5) and the proof of Theorem 3.1 and so it is omitted. □

Remark 3.2. (1) We note that the conditions in Theorems 3.1 and 3.2 are simple and satisfied lightly. Further, if $\kappa = 1$ in Theorems 3.1 and 3.2 or $\gamma = 1$ in Theorem 3.2, that is, A is nonexpansive, or $e^n = 0$ $(n \geq 0)$ in Algorithm 3.1, then the conclusions of Theorems 3.1 and 3.2 also hold.

(2) By the same methods as in Theorem 3.2 of [15], we can obtain the corresponding conclusions of Theorem 3.2 when $e^n = 0$ $(n \geq 0)$ in Algorithm 3.1 with $\alpha_n \geq 0 (n \geq 1)$, which can be found in other research.

Remark 3.3. The corresponding results can be shown when M is (H, η)-monotonicity, H-monotonicity, A-monotonicity, maximal η-monotonicity and classical maximal monotonicity, respectively. That is, the results presented in this paper improve and generalize the corresponding results of recent works.

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ON WEIGHTED VARIABLE EXPONENT LORENTZ-KARAMATA SPACES

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ABSTRACT. In this paper, a characterization of the non-singular measurable transformations T from X into itself and complex-valued measurable functions u on X inducing weighted composition operators is obtained and subsequently their compactness and closedness of the range on the weighted variable exponent Lorentz-Karamata spaces $L^w_{p(\cdot),q(\cdot);b}(X,\Sigma,\mu)$ are completely identified where (X,Σ,μ) is a $\sigma-$finite measure space and $p(t)$, $q(t)$ are variable exponents.

1. INTRODUCTION AND PRELIMINARIES

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by D.E.Edmunds, R.Kerman and L.Pick in [EKP]. By using Karamata theory, they introduced Lorentz-Karamata spaces and compared quasinorms on these spaces. Also J.S.Neves studied Lorentz-Karamata $(L-K)$ spaces $L_{p,q;b}(R,\mu)$ in [N] where $p,q\in(0,\infty]$, b is a slowly varying function on $[1,\infty)$ and (R,μ) is a measure space. These spaces cover the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$ (introduced in [EGO]), Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$ (introduced in [BR]), Zygmund spaces $L^{p}(\log L)^{\alpha}(R)$ (introduced in [BS,Z]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^{p}(R)$ under convenient choices of slowly varying functions and parameters p, q. In [EE,N], it is proved that $L_{p,q;b}(R,\mu)$ space endowed with a convenient norm, is a rearrangment-invariant Banach function space and has an associate space $L_{p',q';b^{-1}}(R,\mu)$ if (R,μ) is a resonant measure space, $p\in(1,\infty)$ and $q\in[1,\infty]$. Also it is showed that when $p\in(1,\infty)$ and $q\in[1,\infty)$, $L-K$ spaces have absolutely continuous norm.

For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A\precsim B$ means that $A\le cB$, for some positive constant c independent of the variables in the expressions A and B. If $A\precsim B$ and $B\precsim A$, we will write $A\approx B$ and say that A and B are equivalent.

Definition 1. *A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on $(0,\infty)$ in the sense of Karamata if, for each $\varepsilon>0$, $t^{\varepsilon}b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon}b(t)$ is equivalent to a non-increasing function on $(0,\infty)$.*

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1

Given a s.v. function b on $(0,\infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right) \quad \text{for all } t > 0. \tag{1.1}$$

It is known that any slowly varying function b on $(0,\infty)$ is equivalent to a slowly varying continuous function $\widetilde{b}$ on $(0,\infty)$. Consequently, without loss of generality, we assume that all slowly varying functions in question are continuous functions in $(0,\infty)$ [GO]. We shall need the following property of s.v. functions, for which we refer to [N, Lemma 3.1].

Lemma 1. *Let b be a slowly varying function on $(0,\infty)$.*
(i) Let $r \in \mathbb{R}$. Then b^r is a slowly varying function on $(0,\infty)$ and $\gamma_b^r(t) = \gamma_{b^r}(t)$ for all $t > 0$.
(ii) Given positive numbers ε and κ, $\gamma_b(\kappa t) \approx \gamma_b(t)$, i.e., there are positive constants c_ε and C_ε such that

$$c_\varepsilon min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \le \gamma_b(\kappa t) \le C_\varepsilon max\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \tag{1.2}$$

for all $t > 0$.

(iii) Let $\alpha > 0$. Then

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)d\tau \approx t^{\alpha}\gamma_b(t) \quad \text{and} \quad \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)d\tau \approx t^{-\alpha}\gamma_b(t) \tag{1.3}$$

for all $t > 0$.

The detailed study of Karamata theory, properties and examples of slowly varying functions can be found in [EKP, EE, N, S].

Let (X, Σ, μ) stand for a σ-finite measure space, w be a weight function , i.e. a measurable, locally bounded function on X, satisfying $w(x) \ge 1$ for all $x \in X$ and χ_A be characteristic function of a set A.

In [EKS], the authors introduced Lorentz spaces $^{p(\cdot),q(\cdot)}$ with variable exponents $p(t), q(t)$ and proved the boundedness of singular integral and fractional type operators in these spaces. They also showed the fundamental properties of (weighted) variable exponent Lorentz spaces. According to the notation of this paper, let $p(t)$ be a measurable function with $0 \le a < p_- = \inf\limits_{t\in[0,\infty]} p(t) \le p_+ = \sup\limits_{t\in[0,\infty]} p(t) < \infty$ and P_a be the class of these functions. By $\wp([0,\infty])$, they denote the class of all functions $p \in L^\infty([0,\infty])$ such that there exist the limits

$$p(0) = \lim_{t\to 0} p(t) \quad \text{and} \quad p(\infty) = \lim_{t\to\infty} p(t)$$

and the (decay) conditions of log-type

$$|p(t) - p(0)| \le \frac{C}{\ln|t|} \ \text{ for } |t| \le \frac{1}{2}\ , \ |p(t) - p(\infty)| \le \frac{C}{\ln(e + |t|)}$$

are satisfied with a constant $C > 0$. They also denote $\wp_a([0,\infty]) = \wp([0,\infty]) \cap P_a$.

Now, let us take the measure $wd\mu$. Let f be a complex-valued measurable function defined on σ-finite measure space $(X, \Sigma, wd\mu)$. Then the distribution function of f is defined as

$$\mu_{f,w}(s) = w\{x \in X : |f(x)| > s\} =_{\{x\in X:\ |f(x)|>s\}} w(x)\,d\mu(x)\,,\ s \ge 0. \tag{1.4}$$

The nonnegative rearrangement of f is given by

$$f_w^*(t) = \sup\{s > 0 : \mu_{f,w}(s) > t\}, \ t \geq 0, \tag{1.5}$$

where we assume that $\sup \phi = 0$. Also the average (maximal) function of f on $(0,\infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t}\int_0^t f_w^*(s)\,ds. \tag{1.6}$$

Note that $\lambda_{f,w}(\cdot), f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are nonincreasing and right continuous functions.

Definition 2. *Let $p,q \in P_0([0,\infty])$ and let b be a slowly varying function on $(0,\infty)$. The weighted variable exponent Lorentz-Karamata space $L^w_{p(\cdot),q(\cdot);b}(X,\Sigma,\mu)$ is defined to be the set of all functions such that*

$$\|f\|^w_{p(\cdot),q(\cdot);b} := \left\| t^{\frac{1}{p(t)}-\frac{1}{q(t)}}\gamma_b(t)\, f_w^{**}(t) \right\|_{q(\cdot);(0,\infty)} \tag{1.7}$$

is finite. Here $\|\cdot\|_{q(\cdot);(0,\infty)}$ stands for variable exponent $L_{q(\cdot)}$ norm over the interval $(0,\infty)$.

After this point, $L^w_{p(\cdot),q(\cdot);b}(X)$ will be used for $L^w_{p(\cdot),q(\cdot);b}(X,\Sigma,\mu)$. It is easy to show that (by the same argument in [EE,Theorem 3.4.41], [EKS], [N]) $L^w_{p(\cdot),q(\cdot);b}(X)$ space endowed with a convenient norm (1.7), is a rearrangment-invariant Banach function space and have absolutely continuous norm when $p,q \in \wp_1([0,\infty])$, $p(0) > 1$ and $p(\infty) > 1$. It is clear that, $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces contain the characteristic functions of every measurable subset of X with finite measure and hence, by linearity, every $wd\mu-$simple function. In this case, with a little thought, it is easy to obtain that the set of simple functions is dense in $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces since these spaces have absolutely continuous norm.

Let $T : X \to X$ be a measurable ($T^{-1}(E) \in \Sigma$, for any $E \in \Sigma$) and non-singular transformation ($w\left(T^{-1}(E)\right) = 0$ whenever $w(E) = 0$) and u a complex-valued function defined on X. We define a linear transformation $W = W_{u,T}$ on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces into the linear space of all complex-valued measurable functions by

$$W_{u,T}(f)(x) = u(T(x))\, f(T(x)) \tag{1.8}$$

for all $x \in X$ and $f \in L^w_{p(\cdot),q(\cdot);b}(X)$. If W is bounded with range in $L^w_{p(\cdot),q(\cdot);b}(X)$, then it is called a *weighted composition operator* on $L^w_{p(\cdot),q(\cdot);b}(X)$. If $u \equiv 1$, then $W \equiv C_T : f \to f \circ T$ is called a *composition operator* induced by T. If T is the identity mapping, then $W \equiv M_u : f \to u \cdot f$ is a *multiplication operator* induced by u. The study of these operators acting on Lebesgue and Lorentz spaces has been made in [C, JP, SM] and [ADV1, ADV2, KK1, KK2], respectively.

In the next part of this paper, we will characterize the boundedness, compactness and closedness of the range of the weighted composition operators on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces for $p,q \in \wp_1([0,\infty])$, $p(0) > 1$ and $p(\infty) > 1$.

2. Results

Theorem 1. *Let $(X,\Sigma,wd\mu)$ be a σ-finite measure space and $u : X \to \mathbb{C}$ a measurable function. Let $T : X \to X$ be a non-singular measurable transformation such* *that the Radon-Nikodym derivative* $f_T = wd\mu\left(T^{-1}\right)/wd\mu$ *is in* $L^\infty(\mu)$. *Then*

$$W_{u,T} : f \to u \circ T \cdot f \circ T \tag{2.1}$$

is bounded on $L^w_{p(\cdot),q(\cdot);b}(X)$ *if* $u \in L^\infty(\mu)$.

Proof. Suppose that $\|f_T\|_\infty = k$. The distribution function of $Wf = W_{u,T}(f) = u \circ T \cdot f \circ T$ is found that

$$\begin{aligned} \mu_{Wf,w}(s) &= w\{x \in X : |u(T(x)) f(T(x))| > s\} \\ &\leq wT^{-1}\{x \in X : \|u\|_\infty \ |f(x)| > s\} \\ &\leq kw\{x \in X : \|u\|_\infty \ |f(x)| > s\} = k\mu_{\|u\|_\infty \ f,w}(s). \end{aligned} \tag{2.2}$$

Hence for each $t \geq 0$, by (2.2) we get

$$\left\{s > 0 : \mu_{\|u\|_\infty \ f,w}(s) \leq \frac{t}{k}\right\} \subseteq \{s > 0 : \mu_{Wf,w}(s) \leq t\} \tag{2.3}$$

and

$$\begin{aligned} (Wf)^*_w(t) &= \inf\{s > 0 : \mu_{Wf,w}(s) \leq t\} \\ &\leq \inf\left\{s > 0 : \mu_{\|u\|_\infty \ f,w}(s) \leq \frac{t}{k}\right\} \\ &= \inf\left\{s > 0 : w\{x \in X : \|u\|_\infty \ |f(x)| > s\} \leq \frac{t}{k}\right\} \\ &= \|u\|_\infty f^*_w\left(\frac{t}{k}\right). \end{aligned} \tag{2.4}$$

Also, we write that $(Wf)^{**}_w(t) \leq \|u\|_\infty f^{**}_w\left(\frac{t}{k}\right)$ by (2.4). Therefore,

$$\begin{aligned} \|Wf\|^w_{p(\cdot),q(\cdot);b} &= \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \gamma_b(t) (Wf)^{**}_w(t) \right\|_{q(\cdot);(0,\infty)} \\ &\leq \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \gamma_b(t) \|u\|_\infty f^{**}_w\left(\frac{t}{k}\right) \right\|_{q(\cdot);(0,\infty)} \\ &\lesssim \|u\|_\infty k^{\frac{1}{p_-}} \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} \gamma_b(t) f^{**}_w(t) \right\|_{q(\cdot);(0,\infty)} \\ &= k^{\frac{1}{p_-}} \|u\|_\infty \|f\|^w_{p(\cdot),q(\cdot);b} \end{aligned} \tag{2.5}$$

can be written by (1.2). Consequently, W is a bounded operator on $L^w_{p(\cdot),q(\cdot);b}(X)$ and $\|W\| \lesssim k^{\frac{1}{p_-}} \|u\|_\infty$ by (2.5). □

Remark 1. *The above theorem is also valid for* $u \in L^\infty\left(w\left(T^{-1}\right)\right)$, *i.e.* $u \circ T \in L^\infty(\mu)$.

Theorem 2. *Let* u *be a complex-valued measurable function and* $T : X \to X$ *be a non-singular measurable transformation such that* $T(E_\varepsilon) \subseteq E_\varepsilon$ *for all* $\varepsilon > 0$, *where* $E_\varepsilon = \{x \in X : |u(x)| > \varepsilon\}$. *If* $W_{u,T}$ *is bounded on* $L^w_{p(\cdot),q(\cdot);b}(X)$, *then* $u \in L^\infty(\mu)$.

Proof. Let us assume that $u \notin L^\infty(\mu)$. Then the set $E_n = \{x \in X : |u(x)| > n\}$ has a positive measure for all $n \in \mathbb{N}$. Since $T(E_n) \subseteq E_n$ or equivalently $\chi_{E_n} \leq \chi_{T^{-1}(E_n)}$, we write that

$$\begin{aligned} \{x \in X : |\chi_{E_n}(x)| > s\} &\subseteq \left\{x \in X : \left|\chi_{T^{-1}(E_n)}(x)\right| > s\right\} \\ &\subseteq \left\{x \in X : \left|u(T(x)) \chi_{T^{-1}(E_n)}(x)\right| > ns\right\} \end{aligned} \tag{2.6}$$

and so

$$
\begin{aligned}
(2.7) \quad (W\chi_{E_n})_w^*(t) &= \inf\left\{s>0:\mu_{W\chi_{E_n},w}(s)\le t\right\}\\
&= \inf\left\{s>0: w\left\{x\in X:\ |W\chi_{E_n}(x)|>s\right\}\le t\right\}\\
&= \inf\left\{s>0: w\left\{x\in X:\ |u(T(x))\chi_{E_n}(T(x))|>s\right\}\le t\right\}\\
&= n\inf\left\{s>0: w\left\{x\in X:\ \left|u(T(x))\chi_{T^{-1}(E_n)}(x)\right|>ns\right\}\le t\right\}\\
&\ge n\inf\left\{s>0: w\left\{x\in X:\ |\chi_{E_n}(x)|>s\right\}\le t\right\}=n(\chi_{E_n})_w^*(t).
\end{aligned}
$$

Thus we have $(W\chi_{E_n})_w^{**}(t)\ge n(\chi_{E_n})_w^{**}(t)$ for all $t>0$ by (2.7). This gives us the contradiction that $\|W\chi_{E_n}\|_{p(\cdot),q(\cdot);b}^w\ge n\|\chi_{E_n}\|_{p(\cdot),q(\cdot);b}^w$. □

If we combine Theorem 1 and Theorem 2, then we have the following theorem.

Theorem 3. *Let u be a complex-valued measurable function and $T:X\to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_T=wd\mu\left(T^{-1}\right)/wd\mu$ is in $L^\infty(\mu)$ and $T(E_\varepsilon)\subseteq E_\varepsilon$ for all $\varepsilon>0$, where $E_\varepsilon=\{x\in X:|u(x)|>\varepsilon\}$. Then $W_{u,T}$ is bounded on $L_{p(\cdot),q(\cdot);b}^w(X)$ if and only if $u\in L^\infty(\mu)$.*

Now, we are ready to discuss the compactness and the closed range of the weighted composition operator $W=W_{u,T}:f\to u\circ T\cdot f\circ T$ on $L_{p(\cdot),q(\cdot);b}^w(X)$ spaces. Let $T:X\to X$ be a non-singular measurable transformation with the Radon-Nikodym derivative $f_T=wd\mu\left(T^{-1}\right)/wd\mu$. If $f_T\in L^\infty(\mu)$ with $\|f_T\|_\infty=k$, then we get

$$
\begin{aligned}
(2.8) \quad (Wf)_w^*(kt) &= \inf\left\{s>0:\mu_{Wf,w}(s)\le kt\right\}\\
&= \inf\left\{s>0: w\left\{x\in X:|u(T(x))f(T(x))|>s\right\}\le kt\right\}\\
&= \inf\left\{s>0: wT^{-1}\left\{x\in X:|(u\cdot f)(x)|>s\right\}\le kt\right\}\\
&\le \inf\left\{s>0: w\left\{x\in X:|(u\cdot f)(x)|>s\right\}\le t\right\}=(M_uf)_w^*(t)
\end{aligned}
$$

and similarly $(Wf)_w^{**}(kt)\le(M_uf)_w^{**}(t)$ for all $f\in L_{p(\cdot),q(\cdot);b}^w(X)$ and $t>0$. Therefore, by (1.2), we obtain

$$
\begin{aligned}
(2.9) \qquad \|Wf\|_{p(\cdot),q(\cdot);b}^w &= \left\|z^{\frac{1}{p(z)}-\frac{1}{q(z)}}\gamma_b(z)(Wf)_w^{**}(z)\right\|_{q(\cdot);(0,\infty)}\\
&\lesssim k^{\frac{1}{p_-}}\left\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}\gamma_b(t)(M_uf)_w^{**}(t)\right\|_{q(\cdot);(0,\infty)}\\
&= k^{\frac{1}{p_-}}\|M_uf\|_{p(\cdot),q(\cdot);b}^w.
\end{aligned}
$$

Now, if f_T is bounded away from zero on S, i.e. $f_T>\delta$ almost everywhere for some $\delta>0$, then

$$
(2.10) \qquad w\left(T^{-1}(E)\right)=\int_E f_Twd\mu\ge\delta w(E)
$$

for all $E\in\Sigma$, $E\subseteq S$, where $S=\{x:u(x)\neq0\}$. Therefore, we have

$$
(2.11) \qquad \|Wf\|_{p(\cdot),q(\cdot);b}^w\ge\delta^{\frac{1}{p}}\|M_uf\|_{p(\cdot),q(\cdot);b}^w.
$$

Hence for each $f\in L_{p(\cdot),q(\cdot);b}^w(X)$, we have

$$
(2.12) \qquad \|Wf\|_{p,q;b}^w\approx\|M_uf\|_{p,q;b}^w
$$

whenever $f_T \in L^{\infty}(\mu)$ and bounded away from zero. By [HKK, Theorem 2.4] and (2.12), we can write the following theorem:

Theorem 4. *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^{\infty}(\mu)$ and is bounded away from zero. Let u be a complex-valued measurable function and $W_{u,T}$ is bounded on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces. Then the following are equivalent:*
(i) $W_{u,T}$ is compact,
(ii) M_u is compact,
(iii) $L^w_{p(\cdot),q(\cdot);b}(u,\varepsilon)$ are finite dimensional for each $\varepsilon > 0$, where

$$L^w_{p(\cdot),q(\cdot);b}(u,\varepsilon) = \left\{ f\chi_{(u,\varepsilon)} : f \in L^w_{p(\cdot),q(\cdot);b}(X) \right\} \text{ and } (u,\varepsilon) = \{x \in X : |u(x)| \geq \varepsilon\}.$$

We know that $W_{u,T} = C_T M_u$ and $wd\mu$ is atomic. Therefore, if we use [KK1, Theorem 3.1] for $W_{u,T}$ on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces, then we get the following theorem:

Theorem 5. *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^{\infty}(\mu)$ and u be a complex-valued measurable function with $u \in L^{\infty}(\mu)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be all the atoms of X with $w(A_n) > 0$ for all $n \in \mathbb{N}$. Then $W_{u,T}$ is compact on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces if $wd\mu$ is purely atomic and*

$$c_n = \frac{w\left(T^{-1}(A_n)\right)}{w(A_n)} \to 0.$$

Theorem 6. *If $wd\mu$ is non-atomic and $W_{u,T}$ is bounded on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces, then $W_{u,T}$ is compact if and only if $u \cdot f_T = 0$ almost everywhere.*

Proof. Let us assume that $W = W_{u,T}$ is compact. If $u \cdot f_T \neq 0$ a.e., then there exists $c \geq 1$, such that the set

$$E = \left\{ x \in X : |u(x)| \text{ and } f_T(x) > \frac{1}{c} \right\} \tag{2.13}$$

has positive measure. Since $wd\mu$ is non- atomic, we can find a decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable subsets of E such that $w(E_n) = \frac{a}{2^n}$, $0 < a < w(E)$. Now, if we construct a sequence such that $e_n = \frac{\chi_{E_n}}{\|\chi_{E_n}\|^w_{p(\cdot),q(\cdot);b}}$, then it is easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is bounded in $L^w_{p(\cdot),q(\cdot);b}(X)$. For $m,n \in \mathbb{N}$, let $m = 2n$. Then we have

$$\begin{aligned}
(We_n - We_m)^*_w\left(\frac{t}{c}\right) &= \inf\left\{ s > 0 : \mu_{We_n - We_m, w}(s) \leq \frac{t}{c} \right\} \\
&= \inf\left\{ s > 0 : w\{x \in X : |u(T(x))\, e_n(T(x)) - u(T(x))\, e_m(T(x))| > s\} \leq \frac{t}{c} \right\} \\
&= \inf\left\{ s > 0 : wT^{-1}\{z \in E_n : |u(z)|\, |e_n(z) - e_m(z)| > s\} \leq \frac{t}{c} \right\} \\
&\geq \inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > sc\} \leq t\} \\
&= \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > s\} \leq t\} \\
&\geq \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n \backslash E_m : |e_n(z) - e_m(z)| > s\} \leq t\}
\end{aligned}$$

for all $t \geq 0$. This gives us that

$$(We_n - We_m)^*_w\left(\frac{t}{c}\right) \geq \frac{\left(\chi_{E_n \backslash E_m}\right)^*_w(t)}{c\,\|\chi_{E_n}\|^w_{p(\cdot),q(\cdot);b}} \tag{2.14}$$

and so

$$\|We_n - We_m\|^w_{p(\cdot),q(\cdot);b} \gtrsim \frac{1}{c^2}\left(\frac{w(E_n \backslash E_m)}{w(E_n)}\right)^{\frac{1}{p_+}} \geq \varepsilon \tag{2.15}$$

for some $\varepsilon > 0$ and large values of n by (ii) and (iii) of Lemma 1. Thus the sequence $\{We_n\}_{n\in\mathbb{N}}$ doesn't admit a convergent subsequence which conradicts the compactness of W. Hence $u \cdot f_T = 0$ a.e.
The converse implication is obvious. □

Theorem 7. *Let $T : X \to X$ be a non-singular measurable transformation with f_T in $L^\infty(\mu)$ and bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces. Then $W_{u,T}$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on the support of u.*

Proof. Suppose that $W = W_{u,T}$ has closed range. Therefore there exists an $\varepsilon > 0$ such that $\|Wf\|^w_{p(\cdot),q(\cdot);b} \geq \varepsilon\|f\|^w_{p(\cdot),q(\cdot);b}$ for all $f \in L^w_{p(\cdot),q(\cdot);b}(S)$ where S is the support of u and $L^w_{p(\cdot),q(\cdot);b}(S) = \left\{f\chi_S : f \in L^w_{p(\cdot),q(\cdot);b}(X)\right\}$. Now, let us choose $\delta > 0$ such that $k^{\frac{1}{p_-}}\delta < \varepsilon$ where $k = \|f_T\|_\infty$. Assume that the set $E = \{x \in X : |u(x)| < \delta\}$ has positive measure, i.e. $0 < w(E) < \infty$. Then $\chi_E \in L^w_{p(\cdot),q(\cdot);b}(S)$ and

$$\begin{aligned}\|W\chi_E\|^w_{p(\cdot),q(\cdot);b} &\lesssim k^{\frac{1}{p_-}}\|u\cdot\chi_E\|^w_{p(\cdot),q(\cdot);b} \leq k^{\frac{1}{p_-}}\delta\|\chi_E\|^w_{p(\cdot),q(\cdot);b} \\ &< \varepsilon\|\chi_E\|^w_{p(\cdot),q(\cdot);b}\end{aligned}$$

by (2.9). This conradiction says that $|u(x)| \geq \delta$ a.e. on the support of u.
Conversely, assume that there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on S. Since f_T is bounded away from zero, we can write that $f_T > m$ for some $m > 0$. By using this fact and (2.11), we get

$$\|Wf\|^w_{p(\cdot),q(\cdot);b} \geq m^{\frac{1}{p_+}}\|u\cdot f\|^w_{p(\cdot),q(\cdot);b} \geq m^{\frac{1}{p_+}}\delta\|f\|^w_{p(\cdot),q(\cdot);b} \tag{2.16}$$

for all $f \in L^w_{p(\cdot),q(\cdot);b}(S)$. Therefore W has closed range because $ker(W) = L^w_{p(\cdot),q(\cdot);b}(X)$. □

Corollary 1. *If $T^{-1}(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$ and $W_{u,T}$ has closed range, then $|u(x)| \geq \delta$ a.e. on S, the support of u for some $\delta > 0$.*

Using the equivalence (2.12) and [ADV1, Theorem 4.1], we can state the following theorem:

Theorem 8. *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^\infty(\mu)$ and is bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded on $L^w_{p(\cdot),q(\cdot);b}(X)$ spaces. Then the following are equivalent:*
(i) $W_{u,T}$ has closed range,
(ii) M_u has closed range,
(iii) $|u(x)| \geq \delta$ a.e. for some $\delta > 0$ on S, the support of u.

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AN ANALYTICAL APPROACH TO THE STUDY OF THE FRACTAL BEHAVIOR OF TCP TRAFFIC

YINGDONG LU

ABSTRACT. We propose an analytical approach for the study of fractal behavior of telecommunication traffic following Transmission Control Protocol (TCP) through a mathematical model proposed by Gilbert and Karloff [7]. Through this approach, we are able to calculate the Hausdorff dimensions of some traffic data sets under mild conditions, thus partially answers the conjectures in Gilbert and Karloff [7].

1. INTRODUCTION

Fractal behavior of TCP (Transmission Control Protocol) traffic has drawn considerable attentions recently in efforts to better understand the dynamics of communication networks. Many phenomena related to fractal sets, such as self-similarity and chaotic behavior, are observed from TCP traffic data, see, e.g. [4] and [13] and references therein for related theoretical developments and experimental illustrations. In Gilbert & Karloff [7], a simple mathematical model is proposed for the purpose of capturing two important features of TCP, the additive-increase, multiplicative decrease (AIMD) mechanism and feedback control. It is a two-parameter (the depletion rate and the buffer size) model for the dynamics of the transit rates of two resources that are competing for a common buffer under TCP. The main result in [7] is that, for many selected parameters, the state space of the system, which is formed by the transition rates of different sources, has fractal box counting dimension, hence is a fractal. It is conjectured that this should hold for almost all parameters. Although it is simple, the model does capture the key dynamic features of the TCP traffic. It can shed light into more sophisticated study of the complex nature of TCP, and also serve as a building block for further hierarchical study of the TCP behaviors. In this note, we will try to partially answer this conjecture. To be specific, we consider a conceptually same mathematical model, but will take a rather different approach. Instead of estimate the lower and upper box dimension directly as in the case of Gilbert & Karloff [7], we calculate the Hausdorff dimension of the attractor through an analytical approach. More specifically, we identify the contraction maps that cause the self-similar behavior, then use covering arguments, a basic technique in geometric measure theory to compute the Hausdorff dimension of the self-similar set.

It is well-known that the box counting dimension and the Hausdorff dimension are not equivalent, although they coincide in many cases. In general, we know that,

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for any set E,

$$\dim_{\text{Hausdorff}}(E) \leq \dim_{\text{boxlower}}(E) \leq \dim_{\text{boxupper}}(E)$$

where $\dim_{\text{Hausdorff}}(E)$ denotes the Hausdorff dimension of E, $\dim_{\text{boxlower}}(E)$ and $\dim_{\text{boxupper}}(E)$ denote the lower and upper box counting dimension of E, respectively. However, for the purpose of categorizing the fractal behavior of the TCP traffics, we believe that the it is adequate to show that the Hausdorff dimension of the state space is fractional.

Furthermore, this establishes connection between the study of fractal behavior of the TCP with the rich and fast-growing field of geometric analysis, it should induce not only many more study on the behavior of TCP, but the analysis of actions upon TCP.

In Sec. 2, we will describe in detail the mathematical model for TCP, and present the statement of our main result; in Sec. 3, we will present the detailed analysis relate to the calculation of the Hausdorff dimension of the attractor.

2. The TCP Model

Let us introduce in detail the mathematical model of TCP. It is a two-parameter model. A (B, d)-model refers to a system of the following: a recipient of traffic with buffer size B and depleting rate d; the contents are created by two sources, labeled 1 and 2, at rates of r_1 and r_2 respectively. At each time unit, the following events will take place chronically,

- Depletion: $(d \wedge B)$ units of buffer will be drained at the recipient;
- Additive-increase: $r_i = r_i + 1$.
- Randomly select $i \in \{1, 2\}$.
- Multiplicative-decrease(I): if $b+r_i > B$, then $r_i = r_i/2$, otherwise, $b = b+r_i$.
- Multiplicative-decrease(II): let $j = 3 - i$, if $b + r_j > B$, then $r_j = r_j/2$, otherwise, $b = b + r_j$.

There are some slight differences between our model and the one studied in Gilbert & Karloff [7]. In our model, when a source sends the contents to the recipient, which is called "fire", if the buffer is full, the content will be lost, instead of stored partially as in [7]. There are many different implementations for TCP, see, e.g. [6], both models can be found in practical treatment of unsuccessfully transmitted content, however, in most of the implements, the partially transmitted content will discard it like we modeled here. Mathematically, to model the original Gilbert & Karloff model precisely will add a small nonlinear perturbation to our model, and the main results can still be expected to hold.

Let Ω denote the set of all possible values of (r_1, r_2). It is conjectured in [7] that when $B > d$ and $d > 2$, Ω has fractal box counting dimension, hence, is fractal. To facilitate the analysis, we want to equip Ω with a self-similar structure.

Now consider the quadruplet (r_1, r_2, b, I), where I is an indicator function for the event that source 1 fires first. Each iteration is equivalent to apply the following piece-wise linear map upon $[1, B) \times [1, B) \times (0, B] \times \{0, 1\}$,

$$T(r_1, r_2, b, I) = \begin{cases} (\frac{r_1}{2}, r_2, b + r_2, I) & \{r_1 + b > B, r_2 + b \leq B, I = 1\} \\ & \cup\{r_1 + r_2 + b > B, r_2 + b \leq B, I = 0\} \\ (r_1, \frac{r_2}{2}, b + r_1, I) & \{r_1 + b \leq B, r_1 + r_2 + b > B, I = 1\} \\ & \cup\{r_1 + b \leq B, r_2 + b > B, I = 0\} \\ (\frac{x}{2}, \frac{y}{2}, b, I) & \{r_1 + b > B, r_2 + b > B\} \\ (r_1, r_2, b + r_1 + r_2, I) & \{r_1 + r_2 + b \leq B\} \end{cases}$$

Note that, for the ease of exposition, our coordinate is different from that in [7] by a translation, this, evidently, will not affect the validity of the result.

There are usually noises and measurement errors involved in studying network behaviors, therefore, we add a small displacement at each coordinate to extend our basic model. So our self-similar map can be as general as the following

$$T^\epsilon(r_1, r_2, b, I) = \begin{cases} (\frac{r_1}{2} + \epsilon_1, r_2, b + r_2, I) & \{r_1 + b > B, r_2 + b \leq B, I = 1\} \\ & \cup\{r_1 + r_2 + b > B, r_2 + b \leq B, I = 0\} \\ (r_1, \frac{r_2}{2} + \epsilon_2, b + r_1, I) & \{r_1 + b \leq B, r_1 + r_2 + b > B, I = 1\} \\ & \cup\{r_1 + b \leq B, r_2 + b > B, I = 0\} \\ (\frac{r_1}{2} + \epsilon_3, \frac{r_2}{2} + \epsilon_4, b, I) & \{r_1 + b > B, r_2 + b > B\} \\ (r_1 + \epsilon_5, r_2 + \epsilon_6, b + r_1 + r_2, I) & \{r_1 + r_2 + b < B\}. \end{cases}$$

In [7], statistics of (r_1, r_2) are taken after a large number of buffer filling, 100 is the exact number presented in the paper. Similarly, we will only observe a subset of Ω. Therefore, our goal is to calculate the Hausdorff dimension for $\tilde{\Omega}$, the self-similar set under map $(T^\epsilon)^K$, for a large constant K whose value is to be determined in the following section. Our main result is,

Theorem 1. *The Hausdorff dimension of* $\tilde{\Omega}$ *is* s, $2 < s < 3$.

Notice that our state space includes the buffer size, instead of just the transition rates as in Gilbert & Karloff. For the purpose of demonstrating the fractal behavior of TCP, this result is adequate.

3. Calculation of the Hausdorff Dimension

3.1. **Results from Geometric Measure Theory.** The following definition of the Hausdorff dimension is taking from Kigami [8]. For further details on related geometric measure theory, please see the rest of the book and standard references in the area, such as, [3], [12] and [10].

Definition 2. Let (X, d) be a metric space. For any compact set K, the diameter of K, denoted by $\mathrm{diam}(K) := \sup_{x,y \in K} d(x, y)$. For any bounded set $A \subset X$, define,

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{i\geq 1} \mathrm{diam}(E_i)^s : A \subset \cup_{i\geq 1} E_i, \mathrm{diam}(E_i) \leq \delta\},$$

and

$$\mathcal{H}^s(A) = \limsup_{\delta \downarrow 0} \mathcal{H}^s_\delta(A).$$

It can be easily shown that,

$$\sup\{s : \mathcal{H}^s(E) = \infty\} = \inf\{s : \mathcal{H}^s(E) = 0\},$$

and it is called the Hausdorff dimension of E, and denoted as $\dim_H E$.

The key to the calculation of the Hausdorff dimension is the following classic result first proved by Moran[11] and then rediscovered by Hutchinson[5]. It is presented in the same form as that on pp. 30 of [8]. First, we need to define some notations.

Definition 3. For a natural number N, define the set of words of length m as

$$W_m = \{w_1, w_2 \dots w_m : w_i \in \{1, 2, \dots, N\}\}.$$

For any $w = w_1 w_2 \dots w_m \in W_m$, $K_w = f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_m}(K)$. $W_* = \cup_{m \geq 0} W_m$ with $W_0 = \{\emptyset\}$.

Definition 4. For a vector $\mathbf{r} = (r_1, r_2, \dots, r_N)$, satisfies $r_i \in (0,1)$ for all $i = 1, 2, \dots, N$ and $a \in (0,1)$, let,

$$\Lambda(\mathbf{r}, a) = \{w : w = w_1 w_2 \dots, w_m \in W_*, r_{w_1 w_2 \dots w_{m-1}} > a \geq r_m\},$$

where $r_v = r_{v_1} r_{v_2} \dots r_{v_k}$ for $v = v_1 v_2 \dots v_k \in W_k$.

We also use $\mathcal{B}(K, d)$ to denote the Borel algebra of K with metric topology defined by the metric d.

Theorem 5. Suppose that there exist $\mathbf{r} = (r_1, r_2, \cdots, r_N)$ with $0 < r_i < 1$, and positive constants c_1, c_2, C_* and M such that,

$$\operatorname{diam}(K_w) \leq c_1 r_w, \tag{1}$$

for all $w \in W_*$ and

$$\#\{w : w \in \Lambda(\mathbf{r}a), d(x, K_w) \leq c_2 a\} \leq M, \tag{2}$$

for any $x \in K$ and any $a \in (0, c_*)$. Then there exist constants $c_3, c_4 > 0$ such that for any $A \in \mathcal{B}(K, d)$,

$$c_3 \nu(A) \leq \mathcal{H}^\alpha(A) \leq c_4 \nu(A), \tag{3}$$

where ν is a self-similar measure on K with weight r_i^α and α is the unique positive number that satisfies $\sum_{i=1}^N r_i^\alpha = 1$. In particular, $0 < \mathcal{H}^\alpha(K) < \infty$ and $\dim_H(K, d)$.

Remark: These two conditions are quite intuitive. Condition (1) reflects the nature of the contraction, that is the diameter of the image of a map of word w can always be bounded by r_w; condition (2) roughly says that for all $w \in \Lambda(\mathbf{r}a)$, except a finite few($\leq M$), there are always points that can be away from K_w, hence guarantees that the images of the maps do not cluster together.

3.2. Hausdorff Dimension Calculation through Projection.

We will use the techniques similar to that in Sec. 3 of [2] to compute the Hausdorff dimension. Since the map involved are all linear, we can apply a linear change of coordinates to guarantee that all the maps are from $[0,1] \times [0,1]$ to $[0,1] \times [0,1]$.

First, consider the following maps,

$$T_1(x, y) = (\frac{x}{2}, y), T_2(x, y) = (x, \frac{y}{2}), T_3(x, y) = (\frac{x}{2}, \frac{y}{2}).$$

Theorem 5 enables us to compute the Hausdorff dimension of the invariant set under these three maps.

Proposition 6. The Hausdorff dimension of the invariant set under $T_i, i = 1, 2, 3$, is s, which satisfies,

$$\left(\frac{1}{2}\right)^s + 2\left(\sqrt{\frac{5}{8}}\right)^s = 1. \tag{4}$$

Proof Since the maps are linear, the two conditions in Theorem 5 are very easy to verify. From the set-up of the mapping, we can see that $\sqrt{5/8}$, $\sqrt{5/8}$ and $1/2$ are the proper choices of c_i, $i = 1, 2, 3$, respectively. □

Remark Notice that equation (4) implies $s \approx 3.37944$. In the arguments below, we will need to study the projection of this set, for that purpose, we need to study a set that has Hausdorff dimension that is lower than that of the space it is embedded in, which is 2. This can be achieved by extracting a subset in the following way.

For an integer $K > 0$, we consider the K-th convolution of maps $T_i, i = 1, 2, 3$, thus, we have $\frac{(K+3)(K+2)}{2}$ different maps $T_{ij}^{(K)}, i = 0, 1, \cdots K, j = 0, 1 \cdots, K - i$, with

$$r_{ij} = (1/2)^i(\sqrt{5/8}^j)(\sqrt{5/8}^{K-i-j}) = (1/2)^i(\sqrt{5/8}^{K-i}).$$

After some simple algebra, we can see that there exists an integer K_0, when $K \geq K_0$, $\sum r_i^s = 1$ will have roots in the interval of $(1, 2)$. Because,

$$\sum r_{ij}^s = \sum_{i=0}^{K}(K-i)(1/2)^{is}(\sqrt{5/8})^{s(K-i)},$$

is decreasing with respect to K, and we can easily identify K_0 such that

$$\sum r_{ij}^2 = \sum_{i=0}^{K_0}(K_0-i)(1/4)^i(5/8)^{(K_0-i)} < 1.$$

Following the same argument in [2], it can be seen that the result also holds for maps with some small displacement, that is, if we are considering the following maps,

$$\hat{T}_{ij}^{(K)} = T_{ij}^{(K)} + c_{ij},$$

for $c_{ij} > 0$ such that $T_{ij}^{(K)}([0, 1] \times [0, 1]) \subset [0, 1] \times [0, 1]$. Therefore,

Proposition 7. There exists an integer K_0 such that for any $K \geq K_0$, the Hausdorff dimension of the invariant set under $\hat{T}_{ij}^{(K)}, i = 1, 2, 3$ is $s(K)$ and $1 < s(K) < 2$.

Now recall the definition of $T^\epsilon(r_1, r_2, b, I)$, when (r_1, r_2, b) sits in the following three domains,

- $\{r_1 + b > B, r_2 + b \leq B, I = 1\} \cup \{r_1 + r_2 + b > B, r_2 + b \leq B, I = 0\}$,
- $\{r_1 + b \leq B, r_1 + r_2 + b > B, I = 1\} \cup \{r_1 + b \leq B, r_2 + b > B, I = 0\}$,
- $\{r_1 + b > B, r_2 + b > B\}$,

the maps T^ϵ is in the same form of those of $T_i(x, y) + a_i$ for some displacements a_i, respectively, again after a linear transformation of the underlying domain. In the fourth domain, $\tilde{\Omega} = \{r_1 + r_2 + b < B\}$, T^ϵ is essentially just a displacement. Furthermore, it is easy to see that there exists $M_0 > 0$, such that for each point $x \in \tilde{\Omega}$, there exists $M \leq M_0$, and $(T^\epsilon)^M(x) \notin \tilde{\Omega}$. Therefore, there exist $K_1 > 0$, in fact, $K_1 > M_0 * K_0$, such that for any $K > K_1$, $(T^\epsilon)^K$ are the piecewise linear maps of the form $\hat{T}_{ij}^{(K)}$.

In addition, we observe that the structure for the case of $I = 1$ and $I = 0$ is essentially the same.

Combined the above two facts, we can see that, for $I = 0$ and $I = 1$, the self-similar set of the map $(T^\epsilon)^K$, $K > K_1$ is the same as that of the following map.

$$T(x, y, z) = \{T_{ij}^{(K)}(x, y), z + a^i x + b^i y, \quad (x, y, z) \in D_{ij}\}, \tag{5}$$

where z represents the content level in the buffer, and D_{ij} is the subset to which T_{ij}^K applies. Hence, our problem is reduced to that of calculating the Hausdorff dimension of the self-similar set (attractor) of these maps.

Now define the following maps on $[0, 1]^5$,

$$S(x, y, z, w_1, w_2) = \{T_{ij}^{(K)}(x, y), z + a^i x + b^i y, T_{ij}^{(K)}(w_1, w_2), \quad (x, y, z) \in D_{ij}\}.$$

For each value z, the image of the above mapping is the the effect of the three maps, Theorem 5 can be applied, hence, and the image has Hausdorff dimension of s.

Define $U_{t_1,t_2}(x, y, z, w_1, w_2) = (x + t_1 w_1, y + t_2 w_2, z)$. Let $D = [0, 1]^2 \times [1, B] \times [0, 1]^2$ and $D' = [0, 1]^2 \times [1, B]$; then the following diagram is commutative,

$$\begin{tikzcd} D \arrow[r, "S(x,y,z,w_1,w_2)"] \arrow[d, "U_{t_1,t_2}"'] & D \arrow[d, "U_{t_1,t_2}"'] \\ D' \arrow[r, "T(x,y,z)"'] & D' \end{tikzcd}$$

Thus, letting F be the attractor of S, $U_t(F)$ is the attractor of T, and $c_{ij} = a_i t_1 + b_j t_2$. Since a_i, b_j, t_1 and t_2 can be arbitrary selected, we can always make this hold. By properties of Hausdorff dimension under projection, we can conclude that, for any fixed z_0, $\dim_H(U_{t_1,t_2}(F) \cap \{z = z_0\}) = s$.

Theorem 8. (Falconer [1], Theorem 5.12) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, then,

$$\mathcal{H}^{s+t}(A \times B) \geq b\mathcal{H}^s(A)\mathcal{H}^t(B),$$

for a constant $b > 0$.

An immediate consequence of the theorem is that for any subsets A and B of $\mathbb{R}$,

$$\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B).$$

From the above theorem, we can conclude that $\dim_H(U_{t_1,t_2}(F)) \geq 1 + s$ for almost all t. Meanwhile, a direct covering argument, along with the definition of the Hausdorff dimension, can imply that $\dim_H(U_{t_1,t_2}(F)) \leq 1 + s$. To be more specific, for any integer j, the attractor can be covered by 3^j sets, which are in the form of $E_k \times [0, 1], k = 1, 2, \cdots, 3^j$, and E_k has area $\lambda_1, \lambda_2, \cdots, \lambda_{3^j}$. From the definition of the Hausdorff dimension and the fact that $\dim_H(U_{t_1,t_2}(F) \cap \{z = z_0\}) = s$, we know that $\sum_{k=1}^{3^j} \lambda_k^s \leq 1$ for j large enough. Therefore, we can conclude that, $\dim_H(U_{t_1,t_2}(F)) \leq 1 + s$. This concludes that, overall, the Hausdorff dimension of $U_{t_1,t_2}(F)$ is $1 + s$.

Theorem 9. *The Hausdorff dimension of the attractor for map $T(x, y, z)$ is $1 + s$.*

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UNIFORM CONTINUITY OF THE ABSTRACT WAVELET TRANSFORM

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Abstract. Given two abstract locally compact topological groups A and B, where A is Abelian, the abstract wavelet transform for a function $f \in L^2(A)$ with respect to an admissible function $h \in L^2(A)$ is defined so that the left and right uniform continuity of the wavelet transform holds because of the left and right uniform continuity of f.

Key words and phrases: left and right uniform continuity, admissible function, abstract wavelet transform, inversion formula, convolution.

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1. Introduction

The continuous wavelet transform with respect to an "admissible" function has been used to detect singularities of functions in the Hilbert space $L^2(\mathbb{R})$, see [3]. In higher dimensions, see [4].

In this paper, we follow the locally compact topological groups point of view to define the wavelet transform, where our group G is given as the product of two locally compact topological groups A and B, by means of a square integrable, irreducible, and unitary representation acting on the Hilbert space $L^2(A)$, where this representation depends of two parameters $a \in A$, and $b \in B$ so that the inversion formula is obtained for a given function $f \in L^2(A)$. In this case, the left and right uniform continuity of $f \in L^2(A)$ implies the left and right uniform continuity of the abstract wavelet transform. Also, under the existence of $\lim_{(a,b)\to(a_1,e_B)}(L_h f)(a,b)$ for any a_1 in a neighborhood of e_A, we have the left and right uniform continuity of $f \in L^2(A)$, where e_A is the identity in A and e_B is the identity in B. In order to prove these results we apply must of the theory given in [5]. So, for the reader's convenience, we summarize this theory in the following section.

2. Notations and definitions

Let us begin by defining a homomorphism for an Abelian locally compact topological group. So, consider two locally compact topological groups A and B where A is Abelian. Now, for each $b \in B$ consider the map $\Gamma_b : A \to A$, such that for $a \in A$, $a \to \Gamma_b(a)$ is a homeomorphism. Note that the homomorphism Γ from B into the group of all automorphisms of A given by $(a,b) \to \Gamma_b(a)$ is continuous on $A \times B$ to A.

Definition 1. *Define G as the product of A and B. That is, consider $G = A \times B = \{(a,b)\,|\, a \in A,\ \text{and}\ b \in B\}$, and in G define*

$$(a,b)(a',b') = (a\Gamma_b(a'), bb'). \tag{1}$$

Then with this product G becomes a group, where $e_G = (e_A, e_B)$ is the identity (e_A is the identity in A and e_B is the identity in B), and where $(a,b)^{-1} = (\Gamma_{b^{-1}}(a^{-1}), b^{-1})$ is the inverse of (a,b) in G.

Note also that $G = A \times B$ is a locally compact topological group. Then we will denote by $d\mu_G(a,b)$ the left Haar measure on G, the left Haar measure on A by $d\mu_A(a)$ and the left Haar measure in B by $d\mu_B(b)$.

Then we have the following Lemma.

Lemma 1. *The function $\cdot : G \times A \to A$ given by $(a,b) \cdot x = a\Gamma_b(x)$ is an action of G on A where $(a,b) \in G$ and $x \in A$.*

Proof. See [5]. □

In our case (see [5]), for $G = A \times B$, there is a positive continuous homomorphism $\eta : G \to (0, \infty)$ so that for any $h \in C_0(A)$ we have

$$\int_A h\left((a,b)^{-1} \cdot x\right) d\mu_A(x) = \eta(a,b) \int_A h(x) d\mu_A(x). \tag{2}$$

Formula (2) can be extended for all $h \in L^1(A)$, see [6]. So, from now on consider $\eta : G \to (0, \infty)$ satisfying (2) for $h \in L^1(A)$.

Definition 2. *For $h \in L^2(A)$ define the following operators*

$$1) \quad (J_a h)(x) = \frac{1}{\sqrt{\eta(a, e_B)}} h[(a, e_B)^{-1} \cdot x],$$

where $(a, e_B) \in G, x \in A$, and $a \in A$.

$$2) \quad (T_b h)(x) = \frac{1}{\sqrt{\eta(e_A, b)}} h[(e_A, b)^{-1} \cdot x],$$

where $(e_A, b) \in G, x \in A$, and $b \in B$.

Then we have the following Lemma.

Lemma 2. *For the operators J_a and T_b,*

1) $J_a^* = J_a^{-1} = J_{a^{-1}}$, *where* $a \in A$.

2) $T_b^* = T_b^{-1} = T_{b^{-1}}$, *where* $b \in B$.

Proof. See [5]. □

Definition 3. *Let $\mathcal{G}$ be a locally compact topological Abelian group, and let $\mathbf{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. We say that the function $\rho : \mathcal{G} \to \mathbf{T}$ is a character on $\mathcal{G}$ if ρ is a continuous homomorphism.*

Definition 4. *Given a locally compact topological Abelian group $\mathcal{G}$, we define the dual group of $\mathcal{G}$ as*

$$\widehat{\mathcal{G}} = \{\rho : \mathcal{G} \to \mathbf{T} \mid \rho \textit{ is a character}\}$$

In this case we denote $\rho(g) = \langle g, \rho \rangle$ where $g \in \mathcal{G}$ and $\rho \in \widehat{\mathcal{G}}$.

Note that $\widehat{\mathcal{G}}$ is clearly an Abelian group under pointwise multiplication $(\rho_1\rho_2)(g) = \rho_1(g)\rho_2(g)$. Its identity element is the constant function **1** and the inverse element is $\rho^{-1}(g) = \overline{\rho(g)} = \rho(g^{-1})$.

The dual group of a locally compact topological Abelian group is used to define an abstract version of the Fourier transform.

Definition 5. *Given $h \in L^1(\mathcal{G})$, the Fourier transform of h is the function $\widehat{h} : \widehat{\mathcal{G}} \to \mathbb{C}$ defined by*

$$\widehat{h}(\rho) = \int_{\mathcal{G}} h(g)\overline{\rho(g)}\,d\mu_{\mathcal{G}}(g), \tag{3}$$

where the integral is relative to the left Haar measure on $\mathcal{G}$.

Definition 6. *For (a,b) in $G = A \times B$, define the two parameter family of operators $U(a,b) = J_a T_b$. Note that $U(a,b)$ acts on the Hilbert space $L^2(A)$ by:*

$$\begin{aligned}(U(a,b)h)(x) &= (J_a T_b h)(x) = (J_a(T_b h))(x) = \frac{1}{\sqrt{\eta(a,e_B)}}(T_b h)\left[(a,e_B)^{-1}\cdot x\right] \\ &= \frac{1}{\sqrt{\eta(a,e_B)}}\frac{1}{\sqrt{\eta(e_A,b)}}h\left[(e_a,b)^{-1}(a,e_B)^{-1}\cdot x\right] \\ &= \frac{1}{\sqrt{\eta((a,e_B)(e_A,b))}}h\left[((a,e_B)(e_A,b))^{-1}\cdot x\right] = \frac{1}{\sqrt{\eta(a,b)}}h\left[(a,b)^{-1}\cdot x\right].\end{aligned}$$

Lemma 3. *The left Haar measure on $G = A \times B$ is given by :*

$$d(a,b) = \frac{1}{\eta(a,b)}d\mu_A(a)d\mu_B(b).$$

Proof. See [5]. □

Definition 7. *A function h in $L^2(A)$ is said to be admissible if*

$$\int_G |\langle h, U(a,b)h\rangle|^2 d(a,b) < \infty.$$

Lemma 4. *Let h be in $L^1(A) \cap L^2(A)$. If $\mu(B) < \infty$, then*

$$C_h \equiv \int_B |\widehat{h}(\rho \circ \Gamma_b)|^2 d\mu_B(b)$$

is uniformly bounded for $\rho \in \widehat{A}$.

Proof. See [5]. □

Definition 8. *Given* (a, b) *in* $G = A \times B$ *and* h *admissible in* $L^2(A)$*, the abstract wavelet transform with respect to* h *is defined as the linear operator*

$$L_h(a, b) : L^2(A, d\mu_A) \to L^2(G, d(a, b))$$

such that for any f *in* $L^2(A)$ *we have*

$$(L_h f)(a, b) = \langle f, U(a, b)h \rangle_{L^2(A)} .$$

That is,

$$(L_h f)(a, b) = \int_A f(x)\overline{(U(a, b)h)\,(x)}d\mu_A(x) = \int_A f(x)\frac{1}{\sqrt{\eta(a, b)}}\overline{h((a, b)^{-1} \cdot x)}d\mu_A(x). \quad (4)$$

Now, in order to get back the function f from the abstract wavelet transform $(L_h f)(a, b)$, we will apply the Grossmann-Morlet-Paul theorem [2], where the hypotheses for the representation $U(a, b)$ are: unitary, irreducible and strongly continuous. In this case, our representation satisfies these conditions (see [5]).

Lemma 5. *For any* f *in* $L^2(A)$ *and an admissible non-zero function* h *in* $L^2(A)$*, we have the following identity in the weak sense* (see [5]).

$$f = \frac{1}{C_h}\int_G (L_h f)(a, b)\, U(a, b)h\, d(a, b). \quad (5)$$

Definition 9. *If* $f, g \in L^1(A)$*, then the convolution of* f *and* g *is defined as the function*

$$(f * g)(x) = \int_A f(y)g(y^{-1}x)d\mu_A(y), \quad (6)$$

where $x, y \in A$

Lemma 6. *If* $h \in C_0(A)$ *is admissible and* $f \in L^2(A)$*, then*

$$(L_h f)(a, b) = \frac{1}{\sqrt{\eta(a, e_B)}}\left[f * (T_b\overline{h})^{\sim}\right](a), \quad (7)$$

where $\sim$ *means* $\psi^{\sim}(x) = \psi(x^{-1})$.

Proof. See [5]. □

3. Left and right uniformly continuous

Definition 10. *For a function* $h : A \to \mathbb{C}$*, define the left and right translations of* h *by*

$$(I_y h)(x) = h(y^{-1}x) \quad \textit{and} \quad (R_y h)(x) = h(xy), \quad \textit{where} \quad y, x \in A.$$

Definition 11. *For a function* $h : A \to \mathbb{C}$*, we say that:*

a) h *is left uniformly continuous if for any* $\epsilon > 0$ *there is an open neighborhood* U *of* e_A *such that for any* $y \in U$*, we have* $|(I_y h)(x) - h(x)| < \epsilon$ *for any* $x \in A$.

b) h *is right uniformly continuous if for any* $\epsilon > 0$ *there is an open neighborhood* U *of* e_A *such that for any* $y \in U$*, we have* $|(R_y h)(x) - h(x)| < \epsilon$ *for any* $x \in A$.

Definition 12. *We say that the wavelet transform* $(L_h f)(a, b)$ *is*

a) left uniformly continuous on A *if for any* $\epsilon > 0$ *there is an open neighborhood* U *of* e_A *such that if* $y \in U$*, then* $|(L_h I_y f)(a, b) - (L_h f)(a, b)| < \epsilon$ *for any* $a \in A$.

b) right uniformly continuous A *if for any* $\epsilon > 0$ *there is an open neighborhood* U *of* e_A *such that if* $y \in U$*, then* $|(L_h R_y f)(a, b) - (L_h f)(a, b)| < \epsilon$ *for any* $a \in A$.

Since the left and right uniformly continuous of $f : A \to \mathbb{C}$ under the wavelet transform $(L_h f)(a, b)$ act on $a \in A$, we have the following result.

Lemma 7. *Given* $f \in L^2(A)$ *and* h *admissible in* $C_0(A)$*, for any* $y \in A$*,*

a) $(L_h I_y f)(a, b) = \sqrt{\eta(y, e_B)}\,(L_h f)(y^{-1}a, b)$

b) $(L_h R_y f)(a, b) = \sqrt{\eta(y^{-1}, e_B)}\,(L_h f)(ay, b)$

Proof.

a) On one hand, by Lemma 2 and since $J_{a_1 a_2} = J_{a_1} J_{a_2}$ (see [5]), then

$$\sqrt{\eta(y, e_B)}\,(L_h f)(y^{-1}a, b) = \sqrt{\eta(y, e_B)}\,\left\langle f, J_{y^{-1}a} T_b h \right\rangle$$

$$= \sqrt{\eta(y, e_B)}\,\left\langle f, J_{y^{-1}} J_a T_b h \right\rangle = \sqrt{\eta(y, e_B)}\,\left\langle J_y f, J_a T_b h \right\rangle$$

$$= \sqrt{\eta(y, e_B)}\,(L_h J_y f)(a, b).$$

But on the other hand, by Definition 2

$$
\begin{aligned}
(J_y f)(x) &= \frac{1}{\sqrt{\eta(y, e_B)}}\, f((y, e_B)^{-1} \cdot x) \\
&= \frac{1}{\sqrt{\eta(y, e_B)}}\, f((\Gamma_{e_B^{-1}}(y^{-1}), e_B^{-1}) \cdot x) \\
&= \frac{1}{\sqrt{\eta(y, e_B)}}\, f((y^{-1}, e_B) \cdot x) \\
&= \frac{1}{\sqrt{\eta(y, e_B)}}\, f(y^{-1} x) = \frac{1}{\sqrt{\eta(y, e_B)}}\, (I_y f)(x).
\end{aligned}
$$

This proves part a). Part b) can be proved in a similar way.

This completes the proof of Lemma 7. □

4. Main Results

We will see first that by using (7) we will have the continuity of the abstract wavelet transform $L_h f$. So, we have the following result.

Theorem 1. *Suppose $f \in L^1(A) \cap L^2(A)$ and $h \in C_0(A)$ is admissible. Then $L_h f$ is continuous at $(a, b) \in G$.*

Proof. Note that from Lemma 6,

$$
(L_h f)(a, b) = \frac{1}{\sqrt{\eta(a, e_B)}} \left[f * (T_b \overline{h})^{\sim} \right](a), \tag{8}
$$

and since $\eta : G \to (0, \infty)$ is continuous, $f \in L^1(A)$ and $h \in L^\infty(A)$, it follows from Proposition 2.39 [1] that $f * (T_b \overline{h})^{\sim}$ is continuous.

Hence $L_h f$ is continuous at $(a, b) \in G$. □

Before we give our next main result we need the following two Lemmas.

Lemma 8. *If $h \in L^2(A)$ is admissible, and $f \in L^2(A)$, then the abstract wavelet transform can be written as*

$$
(L_h f)(a, b) = \int_A f((a, b) \cdot x) \frac{1}{\sqrt{\eta[(a, b)^{-1}]}}\, \overline{h}(x)\, d\mu_A(x). \tag{9}
$$

Proof. Note that since $T_b^* = T_b^{-1} = T_{b^{-1}}$ and $J_a^* = J_a^{-1} = J_{a^{-1}}$ (Lemma 2), it follows that

$$
\begin{aligned}
(L_h f)(a,b) &= \langle f, U(a,b)h\rangle = \langle f, J_a T_b h\rangle = \langle J_a^* f, T_b h\rangle \\
&= \left\langle T_b^* J_a^* f, h\right\rangle = \left\langle T_b^{-1} J_a^{-1} f, h\right\rangle = \langle T_{b^{-1}} J_{a^{-1}} f, h\rangle .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(L_h f)(a,b) &= \langle T_{b^{-1}} J_{a^{-1}} f, h\rangle \\
&= \int_A \left(T_{b^{-1}} J_{a^{-1}} f\right)(x)\overline{h}(x)d\mu_A(x) = \int_A T_{b^{-1}}\left(J_{a^{-1}} f\right)(x)\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta(e_A, b^{-1})}}(J_{a^{-1}} f)\left[(e_A, b^{-1})^{-1}\cdot x\right]\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta(e_A, b^{-1})}}\frac{1}{\sqrt{\eta(a^{-1}, e_B)}} f\left[(a^{-1}, e_B)^{-1}(e_A, b^{-1})^{-1}\cdot x\right]\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta[(a,b)^{-1}]}} f\left[\left[(e_A, b^{-1})(a^{-1}, e_B)\right]^{-1}\cdot x\right]\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta[(a,b)^{-1}]}} f\left[\left(e_A\Gamma_{b^{-1}}(a^{-1}), b^{-1}e_B\right)^{-1}\cdot x\right]\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta[(a,b)^{-1}]}} f\left[\left(\Gamma_{b^{-1}}(a^{-1}), b^{-1}\right)^{-1}\cdot x\right]\overline{h}(x)d\mu_A(x) \\
&= \int_A \frac{1}{\sqrt{\eta[(a,b)^{-1}]}} f\left[(a,b)\cdot x\right]\overline{h}(x)d\mu_A(x).
\end{aligned}
$$

□

Lemma 9. *If $h \in L^2(A)$ is admissible, and $f \in L^2(A)$, then for any $y \in A$,*

$$
\begin{aligned}
&1) \quad I_y(L_h f)(a,b) = L_h(I_y f)(a,b) \\
&2) \quad R_y(L_h f)(a,b) = L_h(R_y f)(a,b)
\end{aligned}
$$

Proof.

1) Note that from (7),

$$
I_y(L_h f)(a,b) = I_y\frac{1}{\sqrt{\eta(a, e_B)}}\left[f * (T_b\overline{h})^{\sim}\right](a).
$$

Since $I_y(f_1 * f_2) = (I_y f_1 * f_2)$ see [1], it follows that

$$I_y(L_h f)(a,b) = \frac{1}{\sqrt{\eta(a,e_B)}}\left[I_y f * (T_b\overline{h})^{\sim}\right](a) = L_h(I_y f)(a,b).$$

2) The result holds since A is Abelian.

□

The following result shows that the left and right uniform continuity of a function f implies the left and right uniform continuity of the abstract wavelet transform $L_h f$.

Theorem 2. *Suppose that $h \in C_0(A)$ is admissible, and for $f \in L^2(A)$ consider $(\mathcal{W}_h f)(a,b) = [\eta(a,b)]^{-\frac{1}{2}}(L_h f)(a,b)$. If f is left uniformly continuous on A, then $(\mathcal{W}_h f)(a,b)$ is left uniformly continuous on A, and if f is right uniformly continuous, then $(\mathcal{W}_h f)(a,b)$ is right uniformly continuous on A.*

Proof. We give the proof for left uniformly continuous, the argument for right uniformly continuous is similar.

Let $\epsilon > 0$ be given. Since $f \in L^2(A)$ is left uniformly continuous on A, there is an open neighborhood U of e_A such that for any $y \in U$, it follows that $|(I_y f)(x) - f(x)| < \epsilon$ for any $x \in A$. Then for y in U we have from (4),

$$\begin{aligned}
&\left|(\mathcal{W}_h I_y f)(a,b) - (\mathcal{W}_h f)(a,b)\right| \\
&= [\eta(a,b)]^{-\frac{1}{2}}\left|(L_h I_y f)(a,b) - (L_h f)(a,b)\right| \\
&= [\eta(a,b)]^{-\frac{1}{2}}\left|\frac{1}{\sqrt{\eta(a,b)}}\int_A [(I_y f)(x) - f(x)]\,\overline{h}\left((a,b)^{-1}\cdot x\right) d\mu_A(x)\right| \\
&< \frac{\epsilon}{\eta(a,b)}\int_A \left|h((a,b)^{-1}\cdot x)\right|\, d\mu_A(x).
\end{aligned}$$

It follows from [5] that since $|h| \in C_0(A)$,

$$|(\mathcal{W}_h I_y f)(a,b) - (\mathcal{W}_h f)(a,b)| < \frac{\epsilon}{\eta(a,b)}\eta(a,b)\int_A |h(x)|\, d\mu_A(x) = \epsilon\|h\|_1.$$

□

In the previous result if we drop the condition that $h \in C_0(A)$, so that if $h \in L^1(A) \cap L^2(A)$, then we have the existence of the limit of $|(\mathcal{W}_h f)(a,b)|$ as $(a,b) \to (a_1, e_B)$ for any a_1 in an open neighborhood of e_A. That is, we have the following result.

Theorem 3. *Suppose that $h \in L^1(A) \cap L^2(A)$ is admissible, and for $f \in L^2(A)$, consider $(\mathcal{W}_h f)(a,b) = [\eta(a,b)]^{-\frac{1}{2}} (L_h f)(a,b)$. If f is left or right uniformly continuous on A, then*

$$\lim_{(a,b)\to(a_1,e_B)} |(\mathcal{W}_h f)(a,b)| \quad \textit{exists for any } a_1 \textit{ in an open neighborhood of } e_A.$$

Proof. Let $\epsilon > 0$ be given. Since $f \in L^2(A)$ is left uniformly continuous on A, there is an open neighborhood U of e_A such that for any $y \in U$, it follows that $|(I_y f)(x) - f(x)| < \epsilon$ for any $x \in A$.

On the other hand from (9),

$$(L_h f)(a,b) = \int_A f((a,b)\cdot x) \frac{1}{[\eta(a,b)]^{-\frac{1}{2}}} \overline{h}(x)\, d\mu_A(x).$$

Hence, for a, a_1 and x in U,

$$|(\mathcal{W}_h f)(a,b)| \le \int_A \left| f\left(a^{-1} a \Gamma_b(x)\right) - f\left(a \Gamma_b(x)\right)\right| \, |h(x)|\, d\mu_A(x)$$

$$+ \int_A |f\left(\Gamma_b(x)\right) - f\left(\Gamma_b(a)\right)| \, |h(x)|\, d\mu_A(x)$$

$$+ \int_A |f\left(\Gamma_b(a)\right)| \, |h(x)|\, d\mu_A(x).$$

Note that since $f \in L^2(A)$ is left uniformly continuous on A, it follows that the first integral is less than $\epsilon \|h\|_1$. For the second integral note that since $f \circ \Gamma_b$ is continuous on A, it follows that for a, x in U, this integral is also less than $\epsilon \|h\|_1$. Finally note that the third integral converges to $|f(a_1)|\, \|h\|_1$ as $(a,b) \to (a_1, e_B)$. Thus,

$$\lim_{(a,b)\to(a_1,e_B)} |(\mathcal{W}_h f)(a,b)| \quad \text{exists for any } a_1 \text{ in a neighborhood of } e_A.$$

The proof for right uniformly continuous is similar. □

Lemma 10. *Let f be in $L^2(A)$. Suppose h in $C_0(A)$ is admissible. For (a,b) in $A \times B$, let $(\mathcal{L}_h f)(a,b) = \eta(a,b)^{-\frac{3}{2}}(L_h f)(a,b)$, and suppose that*

$$L(a_1) \equiv \lim_{(a,b)\to(a_1,e_B)} (\mathcal{L}_h f)(a,b)$$

exists for each a_1 in an open neighborhood containing the closed neighborhood $\overline{U}$ of e_A. Then for each x in an open neighborhood U of e_A, the function

$$(\mathcal{I}_h f)(x,b) = \begin{cases} (\mathcal{L}_h f)(x,b) & \text{if} \quad b \neq e_B \\ L(x) & \text{if} \quad b = e_B \end{cases}$$

is continuous on $\overline{U} \times B$.

Proof. Let (x_1, b_1) be in $U \times B$.

1) If $b_1 \neq e_B$, then

$$(\mathcal{I}_h f)(x,b) = (\mathcal{L}_h f)(x,b) = \eta(x,b)^{-\frac{3}{2}}(L_h f)(x,b).$$

Note that since the function $\eta : A \times B \to (0,\infty)$ is continuous, and by Theorem 1, $(L_h f)(a,b)$ is continuous on $A \times B$, it follows that $\mathcal{I}_h f$ is continuous at $(x_1, b_1) \in \overline{U} \times B$.

2) If $b_1 = e_B$, then

$$\lim_{(x,b)\to(x_1,e_B)} (\mathcal{I}_h f)(x,b) = \lim_{(a,b)\to(x_1,e_B)} (\mathcal{L}_h f)(a,b) = L(x_1) = (\mathcal{I}_h f)(x_1, e_B).$$

This shows that $\mathcal{I}_h f$ is continuous on $U \times B$. $\square$

The following result shows that if limit of $(\mathcal{L}_h f)(a,b)$ exists when $(a,b) \to (a_1, e_B)$ for any a_1 in an open neighborhood of e_A, then the right and left uniform continuity of f holds.

Theorem 4. *Let f be in $L^2(A)$. Suppose $h \neq 0$ in $C_0(A)$ is admissible. For (a,b) in $A \times B$, let $(\mathcal{L}_h f)(a,b) = \eta(a,b)^{-\frac{3}{2}}(L_h f)(a,b)$, and suppose that*

$$L(a_1) \equiv \lim_{(a,b)\to(a_1,e_B)} (\mathcal{L}_h f)(a,b)$$

exists for each a_1 *in an open neighborhood containing the closed neighborhood* $\overline{U}$ *of* e_A. *Then if* $\displaystyle\int_U \int_{B\setminus V} \frac{1}{[\eta(a,b)]^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a)$ *exists in an open neighborhood containing the closed neighborhood* $\overline{V}$ *of* e_B, *and where* $supp\, h \subset U$, *then* f *is right and left uniformly continuous.*

Proof. Since h in $L^2(A)$ has compact support, there is an open neighborhood U of e_A such that $supp\, h((a,b)^{-1} \cdot x) \subset U$, and because the inversion formula (5),

$$f(x) = \frac{1}{C_h} \int_U \int_B (L_h f)(a,b) \frac{1}{\eta(a,b)^{\frac{3}{2}}} h\left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a),$$

for the proper convergence of f, where $x \in A$.

Now, we will split the integral over B into two parts so that for a closed neighborhood $\overline{V}$ of e_B,

$$\begin{aligned} f(x) &= \frac{1}{C_h} \int_U \int_V (L_h f)(a,b) \frac{1}{\eta(a,b)^{\frac{3}{2}}} h\left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a) \\ &+ \frac{1}{C_h} \int_U \int_{B\setminus V} (L_h f)(a,b) \frac{1}{\eta(a,b)^{\frac{3}{2}}} h\left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a) \end{aligned}$$

Note that since h in $L^2(A)$ has compact support, it follows that h is right and left uniformly continuous (see [1]). Then for any $\epsilon > 0$ there is a neighborhood W of e_A such that for any $y \in W$ and since $h \circ \Gamma_{b^{-1}}$ is left uniformly continuous, it follows that

$$\begin{aligned} &|f(xy) - f(x)| \\ &\leq \frac{1}{C_h} \int_U \int_V |(L_h f)(a,b)| \frac{1}{\eta(a,b)^{\frac{3}{2}}} \left|h\left[(a,b)^{-1} \cdot xy\right] - h\left[(a,b)^{-1} \cdot x\right]\right| d\mu_B(b) d\mu_A(a) \\ &+ \frac{1}{C_h} \int_U \int_{B\setminus V} |(L_h f)(a,b)| \frac{1}{\eta(a,b)^{\frac{3}{2}}} \left|h\left[(a,b)^{-1} \cdot xy\right] - h\left[(a,b)^{-1} \cdot x\right]\right| d\mu_B(b) d\mu_A(a) \\ &< \frac{\epsilon}{C_h} \int_U \int_V |(L_h f)(a,b)| \frac{1}{\eta(a,b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a) \\ &+ \frac{\epsilon}{C_h} \int_U \int_{B\setminus V} \|f\|_2 \, \|h\|_2 \, \frac{1}{\eta(a,b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a) \\ &= \frac{\epsilon}{C_h} \int_U \int_V |(\mathcal{I}_h f)(a,b)| d\mu_B(b) d\mu_A(a) \\ &+ \frac{\epsilon}{C_h} \|f\|_2 \, \|h\|_2 \left(\int_U \int_{B\setminus V} \frac{1}{\eta(a,b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a) \right). \end{aligned}$$

Since $\mathcal{I}_h f$ is continuous on $\overline{U} \times \overline{V}$ (see Lemma 10), and by hypothesis $\int_U \int_{B\setminus V} \frac{1}{\eta(a,b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a) < \infty$, it follows that f is right uniformly continuous.

The proof for left uniformly continuous is similar. □

Finally, if we drop the condition of the existence of limit of $(L_h f)(a,b)$ when $(a,b) \to (a_1, e_B)$ and instead we have that $(L_h f)(a,b)$ is right and left uniformly continuous, then f is right and left uniformly continuous. That is, we have the following result.

Theorem 5. *Let f be in $L^2(A)$. Suppose $h \neq 0$ in $C_0(A)$ is admissible, and for (a,b) in $A \times B$, let $(\mathcal{L}_h f)(a,b) = [\eta(a,b)]^{-\frac{3}{2}} (L_h f)(a,b)$. If $(\mathcal{L}_h f)(a,b)$ is right and left uniformly continuous on A and $\int_U \int_{B\setminus V} \frac{1}{[\eta(a,b)]^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a)$ exists in an open neighborhood containing the closed neighborhood $\overline{V}$ of e_B, and where $supp\, h[(a,b)^{-1} \cdot x] \subset U$ for some closed neighborhood $\overline{U}$ of e_A, then f is right and left uniformly continuous.*

Proof. Because the inversion formula (5),

$$f(x) = \frac{1}{C_h} \int_U \int_B (L_h f)(a,b) \frac{1}{[\eta(a,b)]^{\frac{3}{2}}} \overline{h} \left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a),$$

for the proper convergence of f, where $x \in A$, and where $supp\, h[(a,b)^{-1} \cdot x] \subset U$ for some open neighborhood containing the closed neighborhood $\overline{U}$ of e_A.

Now, for a closed neighborhood $\overline{V}$ of e_B, we split the integral over $b \in B$ into two parts to get

$$\begin{aligned} f(x) =& \frac{1}{C_h} \int_U \int_V (L_h f)(a,b) \frac{1}{[\eta(a,b)]^{\frac{3}{2}}} \overline{h} \left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a) \\ &+ \frac{1}{C_h} \int_U \int_{B\setminus V} (L_h f)(a,b) \frac{1}{[\eta(a,b)]^{\frac{3}{2}}} \overline{h} \left[(a,b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a). \end{aligned}$$

For the first double integral, consider the change of variable $w = (a,b)^{-1} \cdot x$. Then $(a,b) \cdot w = x$, which means $a\Gamma_b(w) = x$. Hence, $a = x\Gamma_{b^{-1}}(w)$. Thus, $d\mu_A(a) = d\mu_A(\Gamma_{b^{-1}}(w))$ since $d\mu_A$ is left invariant. Hence,

$$f(x) = \frac{1}{C_h} \int_{U'} \int_V (L_h f)(x\Gamma_{b^{-1}}(w), b) \frac{1}{\eta(x\Gamma_{b^{-1}}(w), b)^{\frac{3}{2}}} \bar{h}(w) d\mu_B(b) d\mu_A(\Gamma_{b^{-1}}(w))$$
$$+ \frac{1}{C_h} \int_U \int_{B \setminus V} (L_h f)(a, b) \frac{1}{\eta(a, b)^{\frac{3}{2}}} \overline{h} \left[(a, b)^{-1} \cdot x\right] d\mu_B(b) d\mu_A(a),$$

where $supp\, h \subset U'$, for a closed neighborhood $\overline{U'}$ of e_A.

Using now the fact that $(\mathcal{L}_h f)(a, b)$ is left uniformly continuous on A, then from Lemma 9, and since

$$(I_y f)(x) = \frac{1}{C_h} \int_{U'} \int_V (L_h I_y f)(x\Gamma_{b^{-1}}(w), b) \frac{1}{\eta(x\Gamma_{b^{-1}}(w), b)^{\frac{3}{2}}} \bar{h}(w) d\mu_B(b) d\mu_A(\Gamma_{b^{-1}}(w))$$
$$+ \frac{1}{C_h} \int_U \int_{B \setminus V} (L_h f)(a, b) \frac{1}{\eta(a, b)^{\frac{3}{2}}} \overline{h} \left[(a, b)^{-1} \cdot y^{-1} x\right] d\mu_B(b) d\mu_A(a),$$

then for any $\epsilon > 0$ there is a neighborhood W of e_A such that for any $y \in W$,

$$\left|(\mathcal{L}_h I_y f)(x\Gamma_{b^{-1}}(w), b) - (\mathcal{L}_h f)(x\Gamma_{b^{-1}}(w), b)\right| < \epsilon,$$

for any $x \in A$.

Also, since h is in $C_0(A)$, it follows that h is right and left uniformly continuos, see [1].

Hence,

$$|I_y f(x) - f(x)| \leq \frac{\epsilon}{C_h} \int_{U'} \int_V |h(w)| d\mu_B(b) d\mu_A(\Gamma_{b^{-1}}(w))$$
$$+ \frac{\epsilon}{C_h} \|f\|_2 \|h\|_2 \left(\int_U \int_{B \setminus V} \frac{1}{\eta(a, b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a) \right).$$

Since h is in $C_0(A)$, and $\displaystyle\int_U \int_{B \setminus V} \frac{1}{\eta(a, b)^{\frac{3}{2}}} d\mu_B(b) d\mu_A(a)$ exists, it follows that f is left uniformly continuous.

The proof for right uniformly continuous is similar. □

Example. Let us consider the additive group $A = \mathbb{R}$ with identity $e_{\mathbb{R}} = 0$, and the multiplicative group $B = \mathbb{R} \setminus \{0\}$ with identity $e_{\mathbb{R} \setminus \{0\}} = 1$.

Now, let us take the homomorphism Γ from $\mathbb{R} \setminus \{0\}$ to the group of all automorphisms of $\mathbb{R}$. That is, for each $s \in \mathbb{R} \setminus \{0\}$, the map $\Gamma_s : \mathbb{R} \to \mathbb{R}$ is defined as $\Gamma_s(k) = sk$, where $k \in \mathbb{R}$.

In this case, the product in $G = \mathbb{R} \times \mathbb{R} \setminus \{0\} = \{(k,s) | k \in \mathbb{R}, s \in \mathbb{R} \setminus \{0\}\}$ is defined as

$$(k,s)(k',s') = (k + \Gamma_s(k'), ss') = (k + sk', ss').$$

Note that with this product, G becomes a locally compact topological group where the identity is $e_G = (e_{\mathbb{R}}, e_{\mathbb{R}\setminus\{0\}}) = (0,1)$, and where the inverse of (k,s) is given by

$$(k,s)^{-1} = (\Gamma_{s^{-1}}(-k), s^{-1}) = (-s^{-1}k, s^{-1}).$$

Moreover, G acts on $\mathbb{R}$ with the following action $\cdot : G \times \mathbb{R} \to \mathbb{R}$ given by

$$(k,s) \cdot x = k + \Gamma_s(x) = k + sk, \quad \text{where} \quad x \in \mathbb{R}.$$

On the other hand, the function $\eta : G \to (0, \infty)$ satisfying

$$\int_{\mathbb{R}} h((k,s)^{-1} \cdot x) d\mu_{\mathbb{R}}(x) = \eta(k,s) \int_{\mathbb{R}} h(x) d\mu_{\mathbb{R}}(x)$$

is given by $\eta(k,s) = |s|$, where $h \in L^1(\mathbb{R})$.

Also note that the left Haar measure is given by $d\mu_{\mathbb{R}}(x) = dx$, and $(k,s)^{-1} \cdot x$ means

$$(k,s)^{-1} \cdot x = (-s^{-1}k, s^{-1}) \cdot x = -s^{-1}k + s^{-1}x = \frac{x-k}{s}.$$

Now, for $(k,s) \in G$ define the family of two operator $U(k,s) = J_k T_s$, where for $h \in L^2(\mathbb{R})$

1) $(J_k h)(x) = \frac{1}{\sqrt{\eta(k,1)}} h((k,1)^{-1} \cdot x) = h(x-k)$, where $x \in \mathbb{R}$ and $k \in \mathbb{R}$, and

2) $(T_s h)(x) = \frac{1}{\sqrt{\eta(0,s)}} h((0,s)^{-1} \cdot x) = \frac{1}{\sqrt{|s|}} h\left(\frac{x}{s}\right)$, where $x \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$.

Then $U(k,s)$ is a unitary representation of G acting on $L^2(\mathbb{R})$ by

$$[U(k,s)h](x) = (J_k T_s h)(x) = (T_s h)(x-k) = \frac{1}{\sqrt{|s|}} h\left(\frac{x-k}{s}\right).$$

Moreover since the left Haar measure on $\mathbb{R} \setminus \{0\}$ is $d\mu_{\mathbb{R}\setminus\{0\}} = \frac{1}{|s|} ds$, it follows that the left Haar measure on G is

$$d(k,s) = \frac{1}{\eta(k,s)} d\mu_{\mathbb{R}}(k) d\mu_{\mathbb{R}\setminus\{0\}}(s) = \frac{1}{|s|} dk \frac{1}{|s|} ds = \frac{1}{s^2} dk ds.$$

Then, the admissibility condition for $h \in L^2(\mathbb{R})$ becomes

$$C_h = \int_{\mathbb{R}\setminus\{0\}} |\widehat{h}(\rho \circ \Gamma_s)|^2 d\mu_{\mathbb{R}^+}(s) = \int_{\mathbb{R}\setminus\{0\}} |\widehat{h}(\rho s)|^2 \frac{1}{s} ds,$$

where $\rho \in \mathbb{R}$. Thus, if we take $y = \rho s$, we get

$$C_h = \int_{\mathbb{R}\setminus\{0\}} |\widehat{h}(y)|^2 \frac{1}{|y|} dy.$$

Hence, for $(k, s) \in G$ and h admissible in $L^2(\mathbb{R})$, the continuous wavelet transform for $f \in L^2(\mathbb{R})$ with respect to h is given by

$$(L_h f)(k, s) = \langle f, J_k T_s h\rangle = \int_{\mathbb{R}} f(x) \frac{1}{|s|^{\frac{1}{2}}} \bar{h}\left(\frac{x-k}{s}\right) dx,$$

and the inverse formula is given by

$$f = \frac{1}{C_h} \int_G (L_h f)(k, s) U(k, s) h d(k, s),$$

where the equality holds in the weak sense.

1) Then because of Lemma 10 we have the following result.

Suppose $f \in L^2(\mathbb{R})$, and that h in $C_0(\mathbb{R})$ is admissible. For (k, s) in $G = \mathbb{R} \times \mathbb{R}^+$, let $(\mathcal{L}_h f)(k, s) = \eta(k, s)^{-\frac{3}{2}} (L_h f)(k, s) = s^{-\frac{3}{2}} (L_h f)(k, s)$, and suppose that

$$L(k_1) \equiv \lim_{(k,s)\to(k_1,1)} (\mathcal{L}_h f)(k, s)$$

exists for each k_1 in an open neighborhood containing the closed neighborhood $\overline{U}$ of 0. Then for each $x \in U$, the function

$$(\mathcal{I}_h f)(x, s) = \begin{cases} (\mathcal{L}_h f)(x, s) & \text{if} \quad s \neq 1 \\ L(x) & \text{if} \quad s = 1 \end{cases}$$

is continuous on $\overline{U} \times \mathbb{R} \setminus \{0\}$.

This shows an example of Lemma 10.

2) Now, according to Theorem 2

If $f \in L^2(\mathbb{R})$ is uniformly continuous on $\mathbb{R}$, then for any $\epsilon > 0$, there is $\delta > 0$ such that if $|y| < \delta$, then $|f(x-y) - f(x)| < \epsilon$ for any $x \in \mathbb{R}$.

Hence, for $(\mathcal{W}_h f)(k,s) = |s|^{-\frac{1}{2}}(L_h f)(k,s)$ and for $y \in \mathbb{R}$ where $|y| < \delta$,

$$|(\mathcal{W}_h I_y f)(k,s) - (\mathcal{W}_h f)(k,s)| \le |s|^{-\frac{1}{2}} \int_{\mathbb{R}} |(I_y f)(x) - f(x)|\, |s|^{-\frac{1}{2}} \left| h\left(\frac{x-k}{s}\right) \right| dx$$
$$= |s|^{-\frac{1}{2}} \int_{\mathbb{R}} |f(x-y) - f(x)|\, |s|^{-\frac{1}{2}} \left| h\left(\frac{x-k}{s}\right) \right| dx < \frac{\epsilon}{|s|} \int_{\mathbb{R}} \left| h\left(\frac{x-k}{s}\right) \right| dx$$
$$= \frac{\epsilon}{|s|} \int_{\mathbb{R}} |h(z)|\, |s| dz = \epsilon \|h\|_1.$$

This shows an example for Theorem 2.

3) Now, with respect to Theorem 3 we have the following result.

Suppose $f \in L^2(\mathbb{R})$ is uniformly continuous in a neighborhood of $x = 0$ in $\mathbb{R}$. If $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is admissible, then $\lim_{(k,s)\to(k_1,1)} |(\mathcal{W}_h f)(k,s)|$ exists for any k_1 in a neighborhood of $x = 0$.

So, if $\epsilon > 0$ is given and if f is uniformly continuous in an open neighborhood of $x = 0$ containing the closed interval $[-\Delta, \Delta]$, where $\Delta > 0$, then for k and k_1 in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$, there is $\delta(\epsilon) > 0$ such that if $|k - k_1| < \delta(\epsilon)$, then $|f(k) - f(k_1)| < \epsilon$.

Note that if $y = s^{-1}(x-k)$, then

$$(\mathcal{W}_h f)(k,s) = |s|^{-\frac{1}{2}} \int_{\mathbb{R}} |s|^{\frac{1}{2}} f(k+sy)\overline{h(y)} dy.$$

Hence, for s sufficiently small so that $k + sy$ and $k_1 + sy$ are in $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$,

$$|(\mathcal{W}_h f)(k,s)| \le \int_{\mathbb{R}} |f(k+sy) - f(k+sy-k)|\, |h(y)dy$$
$$+ \int_{\mathbb{R}} |f(sy) - f(sk)|\, |h(y)|dy + \int_{\mathbb{R}} |f(sk)|\, |h(y)|dy$$
$$< \epsilon\, \|h\|_1 + \epsilon\, \|h\|_1 + \int_{\mathbb{R}} |f(sk)|\, |h(y)\|dy.$$

Note that the third term tends to $|f(k_1)|\, \|h\|_1$ as $(k,s) \to (k_1, 1)$. Thus,

$$\lim_{(k,s)\to(k_1,1)} |(\mathcal{W}_h f)(k,s)| \quad \text{exists} \quad \text{and} \lim_{(k,s)\to(k_1,1)} |(\mathcal{W}_h f)(k,s)| = |f(k_1)| \, \|h\|_1.$$

This shows an example of Theorem 3.

4) On the other hand, because of Theorem 4, we have the following result.

Let $f \in L^2(\mathbb{R})$. Suppose $h \neq 0$ in $C_0(\mathbb{R})$ is admissible. For (k,s) in $\mathbb{R} \times \mathbb{R} \setminus \{0\}$, let $(\mathcal{L}_h f)(k,s) = s^{-\frac{3}{2}}(L_h f)(k,s)$ for $s \neq 0$, and suppose that $L(k_1) \equiv \lim_{(k,s)\to(k_1,1)}(\mathcal{L}_h f)(k,s)$ exists for each k_1 in an open neighborhood containing the closed interval $[k_0 - \Delta, k_0 + \Delta]$, where $\Delta > 0$. Then if $\int_{|s|>1} s^{-\frac{3}{2}} ds$ exists, then f is left and right uniformly continuous in a neighborhood of k_0.

Then, let $\epsilon > 0$ be given. Note that if $supp\ h \subset [-L, L]$ for some $L > 0$, then $\int_{-L}^{L}\int_{|s|>1} s^{-\frac{3}{2}} ds dy$ exists. So if $k, k_1 \in [k_0 - \Delta, k_0 + \Delta]$, there is $\delta(\epsilon) > 0$ so that if $|k - k_1| < \delta(\epsilon)$, then for s sufficiently small, $|s^{-1}(x-k) - s^{-1}(x-k_1)| < \delta(\epsilon)$, and since $h \in C_0^\infty$,

$$\begin{aligned}
|f(k) - f(k_1)| \leq & \frac{1}{C_h}\int_{-L}^{L}\int_{-1}^{1} |(L_h f)(k,s)| s^{-\frac{3}{2}} \left| h\left(\frac{x-k}{s}\right) - h\left(\frac{x-k_1}{s}\right)\right| dsdk \\
& + \frac{1}{C_h}\int_{-L}^{L}\int_{|s|>1} |(L_h f)(k,s)| s^{-\frac{3}{2}} \left| h\left(\frac{x-k}{s}\right) - h\left(\frac{x-k_1}{s}\right)\right| dsdk \\
& < \frac{\epsilon}{C_h}\int_{-L}^{L}\int_{-1}^{1} |(L_h f)(k,s)| s^{-\frac{3}{2}} \, dsdk \\
& + \frac{\epsilon}{C_h}\int_{-L}^{L}\int_{|s|>1} \|f\|_2 \, \|h\|_2 s^{-\frac{3}{2}} \, dsdk.
\end{aligned}$$

Since $(\mathcal{L}_h f)(k,s) = s^{-\frac{3}{2}}(L_h f)(k,s)$ is continuous on $[-L, L] \times [-1, 1]$, and $\int_{|s|>1} \|f\|_2 \, \|h\|_2 \, s^{-\frac{3}{2}} \, ds$ exists, it follows that f is uniformly continuous in a neighborhood of k_0.

This shows an example of Theorem 4.

5) Finally, according to Theorem 5, we have the following result.

Suppose that $f \in L^2(\mathbb{R})$ and consider h admissible in $C_0(\mathbb{R})$. For (k,s) in $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$, let $(\mathcal{L}_h f)(k,s) = s^{-\frac{3}{2}}(L_h f)(k,s)$. If $(\mathcal{L}_h f)(k,s)$ is uniformly continuous on $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$. To see that this result holds, choose $L > 0$ such that $supp\, h(s^{-1}(x-k)) \subset [-L, L]$. Then because of the inversion formula (5), the function f can be written as

$$
\begin{aligned}
f(x) &= \frac{1}{C_h} \int_{-L}^{L} \int_{-1}^{1} (L_h f)(k,s)\, |s|^{-\frac{3}{2}}\, \overline{h}\left(\frac{x-k}{s}\right) ds\, dk \\
&+ \frac{1}{C_h} \int_{-L}^{L} \int_{|s|>1} (L_h f)(k,s)\, |s|^{-\frac{3}{2}}\, \overline{h}\left(\frac{x-k}{s}\right) ds\, dk
\end{aligned}
$$

For the first double integral consider the change of variable $k = x - sw$. Then

$$
\begin{aligned}
f(x) &= \frac{1}{C_h} \int_{\frac{x+L}{|s|}}^{\frac{x-L}{|s|}} \int_{-1}^{1} (L_h f)(x-sw,s)\, |s|^{-\frac{3}{2}}\, \overline{h}(w)\, ds\, (-|s|) dw \\
&+ \frac{1}{C_h} \int_{-L}^{L} \int_{|s|>1} (L_h f)(k,s)\, |s|^{-\frac{3}{2}}\, \overline{h}\left(\frac{x-k}{s}\right) ds\, dk
\end{aligned}
$$

On the other hand, if $(\mathcal{L}_h f)(k,s)$ is uniformly continuous on $\mathbb{R}$, then for any $\epsilon > 0$ there is $\delta > 0$ such that if $|y| < \delta$, then $|(\mathcal{L}_h f)(x-y-sw,s) - (\mathcal{L}_h f)(x-sw,s) < \epsilon$ for any $x \in \mathbb{R}$. Also, since $h \in C_0(\mathbb{R})$, it follows that h is uniformly continuous on $\mathbb{R}$. Hence, for $y \in \mathbb{R}$ such that $|y| < \delta$,

$$
\begin{aligned}
&|f(x-y) - f(x)| \\
&\leq \frac{1}{C_h} \int_{\frac{x-L}{|s|}}^{\frac{x+L}{|s|}} \int_{-1}^{1} |(L_h I_y f)(x-sw,s) - (L_h f)(x-sw,s)|\, |s|^{-\frac{3}{2}}\, |h(w)|\, ds\, (|s|) dw \\
&+ \frac{1}{C_h} \int_{-L}^{L} \int_{|s|>1} |(L_h f)(k,s)|\, |s|^{-\frac{3}{2}} \left| h\left(\frac{x-y-k}{s}\right) - h\left(\frac{x-k}{s}\right) \right| ds\, dk.
\end{aligned}
$$

Because of the definition of $(\mathcal{L}_h f)(k,s)$, we have

$$
\begin{aligned}
|f(x-y) - f(x)| &\leq \frac{\epsilon}{C_h} \int_{\frac{x-L}{|s|}}^{\frac{x+L}{|s|}} \int_{-1}^{1} |h(w)|\, ds\, (|s|) dw \\
&\qquad + \frac{\epsilon}{C_h} \|f\|_2\, \|h\|_2 \int_{-L}^{L} \int_{|s|>1} |s|^{-\frac{3}{2}}\, ds\, dk.
\end{aligned}
$$

Since both double integrals exist, it follows that f is uniformly continuous on $\mathbb{R}$.

This shows an example of Theorem 5.

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Invariant Sets for The Systems of Difference Equations

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Abstract

In this work we establish the connection between the invariant sets of the systems of differential equations and the corresponding systems of difference equations in terms of the zeros of positively definite Lyapunov's functions.
Keywords:Difference equations, Lyapunov's function, Invariant set, Stability.

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1 Introduction

The results of the existence of invariant sets for systems of differential equations in terms of the zeros of non-negatively definite Lyapunov's functions have been established in the monograph of A.M.Samoilenko [1]. Later these results were generalized for the systems of non-autonomous differential equations and the equations with random perturbations [2],[3]. As shown in these works, the zeros of some non-negatively definite Lyapunov's function $V(t,x)$, $t \geq 0$, $x \in D \subset R^m$ comprise an invariant set under the condition that their projection on R^n is compact in D and the derivative of $V(t,x)$ with respect to the system is non-positive definite. The work [4] addressed the issue whether the existence of a non-negative definite Lyapunov's function is a sufficient condition of the existence and stability of invariant sets. The reverse Samoilenko's theorems have been obtained, which are the analogies of the reverse Lyapunov's theorems in the stability theory [5].

In this paper we will establish similar results for the difference equations. The connection between the invariant sets of the systems of differetial and corresponding difference equations in terms of the zeros of positive definite Lyapunov's functions has been studied as well.

2 Main result

Consider the system of differential equations :

$$\frac{dx}{dt} = X(t,x) \tag{1}$$

and the corresponding system of difference equations

$$x_{n+1} = x_n + X(n,x), \tag{2}$$

where $n \in Z^+, x \in D \subset R^m$. The function $X(n,x)$ is defined for$x \in D$, $n \in Z^+$, continuous and Lipschitz with respect to x. Let $M \subset Z^+ \times R^m$, and M_{n_0} be the cut of M with the hyperplane $n = n_{0,}$ $n_0 \in Z^+$.

Definition .1 A set M is called the positively invariant set of the system (2) if the solution $x_n(x)$ of (2) such that $x_n(x_0) \in M_{n_0}$ possesses the property: $x_n(x) \in M_n$ for $n \in Z^+$.

Definition .2 A positive invariant set M of the system (2) is called stable for $n \geq n_0$ if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, n) > 0$ such that

$$\rho(x_{n_0}, M_{n_0}) < \delta,$$

implies

$$\rho(x_n(x), M_n) < \epsilon$$

for $n \in Z^+$,where $\rho(x_n(x), M_n) = \min_{y \in M_n} \|x_n(x) - y\|$ is the distance of $x_n(x)$ to M_n.

Let D_1 be a bounded domain, contained in D together with its neighborhood, and $\bar{D}_1$ be the closure of D_1.

Definition .3 The function $V(x)$, defined on $\bar{D}_1$ is said to be of a sign-constant in D_1 if for any $x \in \bar{D}_1$, the nonzero values of the function $V(x)$ are of the same sign. A sign-constant function in D_1 is called sign -definite in D_1 if the set of its zeros is nonempty and compact in D_1.

Denote $x_n(n_0, x)$ to be the solution of the system (2) such that $x_{n_0}(n_0, x) = x$, $x_n(n_0, x) \in D$ for any $n \in Z^+$.
The following theorem holds.

Theorem .1 Suppose the system (2) has a positive definite stable manifold M for $n \geq 0$, $x \in D$ such that $\mathrm{Pr}_{R^m} M$ is compact in D.

Then in some neighborhood D_1 of the set M , there exists a sequence of functions $V_n(x)$ with the properties:

1) $V_n(x)$ is positive definite uniformly in $n \geq 0$:

$$\inf_{k \geq 0; x: \rho(x, M_k) > \epsilon} V_k(x) = V_\epsilon > 0, \tag{3}$$

for any $\epsilon > 0$.

2) $\Delta V_n(x)$ is negative semidefinite:

$$\Delta V_n(x) = V_{n+1}(x_{n+1}) - V_n(x_n) \leq 0,$$

for $n \geq 0$, $x \in D_1$.

3) The set of zeros of $V_n(x)$ coincides with M, that is ,

$$M = \{(n, x) : V_n(x) = 0, n \geq 0, x \in D_1\}.$$

Proof. Consider the following sequence of functions

$$V_n(x) = \sup_{k \geq n} \rho^2(x_k(n, x), M_k),$$

■

Then

$$\inf_{k \geq 0; x: \rho(x, M_k) > \epsilon} V_k(x) = \inf_{k \geq 0; x: \rho(x, M_k) > \epsilon} \sup_{n \geq k} \rho^2(x_n(k, x), M_n) \geq \inf_{k \geq 0; x: \rho(x, M_k) > \epsilon} \rho^2(x_n(k, x), M_n) \geq \epsilon^2 > 0.$$

Thus, we have

$$\inf_{k\geq 0; x:\rho(x,M_k)>\epsilon} V_k(x) = V_\epsilon > 0 \quad for\ any \quad \epsilon > 0.$$

Then

$$\triangle V_n(x) = V_{n+1}(x_{n+1}) - V_n(x_n) = \sup_{k\geq n+1} \rho^2(x_k(n,x), M_k) - \sup_{k\geq n} \rho^2(x_k(n,x), M_k) \leq 0$$

since the second set contains more elements.
To show (3) we use the fact that M is a positively stable manifold. Then for any $k \geq 0$, we have

$$\rho(x_k(n,x), M_k) = 0.$$

Therefore $V_n(x) = 0$. Furthermore, if $V_n(x) = 0$, then

$$\sup_{k\geq n} \rho^2(x_k(n,x), M_k) = 0,$$

so,

$$\sup_{k\geq n} \rho(x_k(n,x), M_k) = 0.$$

Then $x_k(n,k) \in M_k$ for any $k \geq 0$, and hence M is a positively stable set.This concludes the proof of the theorem.

The further analysis is devoted to the study of the connection between the invariant sets of the differential and the corresponding difference equations. The work [4] gave the conditions on the existence and the stability of the invariant set for the system of differential equations in the terms of the sign constant Lyapunov's functions for the system of difference equations. In the present work we consider the reverse problem.That is, we obtain the conditions of the existence and the stability of an invariant set for the system of difference equations in terms of the sign constant Lyapunov's functions for the system of differential equations.

Consider an autonomous system of differential equations

$$\frac{dx}{dt} = X(x) \tag{4}$$

and the corresponding system of difference equations

$$x_{n+1} = x_n + hX(x_n), \tag{5}$$

where $n \geq 0, x \in D \subset R^m, h > 0$ is the step of the difference equation, and the function $X(x)$ is bounded and Lipschitz in D.
Then the following theorem holds.

Theorem .2 Suppose that there exists a positive definite smooth function $V = V(x)$ such that $\frac{\partial V}{\partial x}$ is Lipschitz and

$$\dot{V} = \dot{V}(x) = \frac{\partial V(x)}{\partial x} X(x)$$

is non-positive definite function in D. Then the system (5) has a positive definite stable set if the functions $V(x)$ and $\dot{V}(x)$ are of different signs.

Proof. Suppose we have $V(x) \geq 0$ and $\overset{\bullet}{V}(x) \geq 0$ for $x \in \bar{D}_1$. It follows from ([1],p.68) that the set

$$V(x) = 0, \ x \in D_1,$$

is a positively invariant set for the system (4). ■

Since $V(x)$ is positive definite, the set of its zeros is nonempty and compact. The set of the zeros of $\overset{\bullet}{V}(x)$ is nonempty and compact as well, since $\overset{\bullet}{V}(x)$ is non-positive definite in D.
Consider the following function

$$V_h(x) = \left\{ \begin{array}{cc} V(x), & \overset{\bullet}{V}(x) < 0 \\ 0, & \overset{\bullet}{V}(x) = 0 \end{array} \right\}. \tag{6}$$

Denote by

$$N_0^h = \{x : V_h(x) = 0\}$$

the set of zeros of $V_h(x)$.
Due to the choice of $V_h(x)$,

$$N_0^h = \{x : V(x) = 0 \ \} \cup \left\{ x : \overset{\bullet}{V}(x) = 0 \right\}.$$

Then it follows from [1, p.66] that

$$N_0^h = \{x : V_h(x) = 0 \ \} = \left\{ x : \overset{\bullet}{V}(x) = 0 \right\}.$$

Since $V(x) \in C^1(D_1)$, it has a derivative in any direction and at any point:

$$\lim_{h \to 0} \frac{V(x + hX(x)) - V(x)}{h} \leq 0$$

for any $x \in D_1$.
This implies that there exists a step $h_0 > 0$, such that for any $h < h_0$ the following inequality holds

$$\triangle V(x) = V(x + hX(x)) - V(x) \leq 0.$$

Note that we have a pointwise convergence, that is

$$\frac{V(x + hX(x)) - V(x)}{h} \to \overset{\bullet}{V}(x), \text{ as } h \to 0.$$

Now, we show that the step h_0 is independent of x . Let

$$f_h(x) = \frac{V(x + hX(x)) - V(x)}{h}.$$

Since the function $V(x)$ is Lipschitz and $X(x)$ is bounded, we get an estimate for $|f_h(x)|$, that is

$$|f_h(x)| = \left| \frac{V(x + hX(x)) - V(x)}{h} \right| = \frac{1}{h} \left| V(x + hX(x)) - V(x) \right| \leq |X(x)| \leq C.$$

Moreover, since $\dfrac{\partial V}{\partial x}$ is Lipschitz and $X(x)$ is bounded, we get the estimate

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{h}\left[\frac{\partial V(x+hX(x))}{\partial x}(I+h\frac{\partial X}{\partial x})-\frac{\partial V(x)}{\partial x}\right]\\ &= \frac{1}{h}\left[(\frac{\partial V(x+hX(x))}{\partial x}-\frac{\partial V(x)}{\partial x})+h\frac{\partial V(x+hX(x))}{\partial x}\frac{\partial X}{\partial x})\right]\le C_1.\end{aligned}$$

Since h can be chosen arbitrary small,we conclude that $x+hX(x)\in D$ for $x\in D_1$.

Suppose $f_h(x)$ which satisfies the Lipschitz condition, is pointwise convergent and equicontinuous. Then the function $f_h(x)$ is uniformly convergent,that is:

$$f_h(x)\to f(x), \quad \text{as} \quad h\to 0,$$

that is, $\forall\epsilon>0$, $\exists N=N(\epsilon)\in N$, $\forall n\ge N$ and $\forall x\in D$, we have

$$|f_{h_n}(x)-f(x)|<\epsilon.$$

Assume on the contrary that $f_h(x)$ is not uniformly convergent. This means that there exists an $\epsilon>0$ and the sequence $\{x_n\}$, $x_n\to x_0$, $n\to\infty$, such that

$$|f_{h_n}(x_n)-f(x_n)|\ge\epsilon.$$

On the other hand ,we have:

$$\begin{aligned}|f_{h_n}(x_n)-f(x_n)| &= |f_{h_n}(x_n)-f_{h_n}(x_0)+f_{h_n}(x_0)-f(x_0)+f(x_0)-f(x_n)|\\ &\le |f_{h_n}(x_n)-f_{h_n}(x_0)|+|f_{h_n}(x_0)-f(x_0)|+|f(x_0)-f(x_n)|.\end{aligned}$$

Now, we give estimate of the terms in the last inequality. Since $f_h(x)$ is Lipschitz continuous, and $x_n\to x_0$, $n\to\infty$, we have

$$|f_{h_n}(x_n)-f_{h_n}(x_0)|\le L|x_n-x_0|\to 0.$$

Similary, we get

$$|f(x_n)-f(x_0)|\le L|x_n-x_0|.$$

But the pointwise convergence implies that the second term converges to zero, which is a contradiction.

Hence [6] implies that the system (5) has a positively invariant stable set, and the statment of the theorem follows.

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VORONOVSKAYA TYPE RESULT FOR q-DERIVATIVE OF q-BASKAKOV OPERATORS

ALI ARAL AND TUNCER ACAR

Abstract. The present article deals with the obtaining a Voronovskaya type result for q-derivative of q-Baskakov operators attached to functions in polynomial weighted space. For this aim, we give the central moments for q-Baskakov operators. By using these results and some properties of q-calculus we give limits of some central moments.

1. Introduction

Let f be a real valued function defined on $[0, \infty)$. In [2], a new q-analogue of Baskakov operators belonging to f introduced as follows,

$$\begin{aligned} \mathcal{B}_{n,q}(f,x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}(q,x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right). \end{aligned} \tag{1.1}$$

where

$$\mathcal{P}_{n,k}(q,x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}}.$$

for $q > 0$ and each positive integer n.

While for $q = 1$ these operators coincide with the classical ones, for $q \neq 1$ the new operators possess interesting properties. As in the classical Baskakov operators, the q-Baskakov operators have some shape-preserving properties and monotonicity property for convex function. In spite of the fact that the q-Baskakov operators are not polynomials, they preserve the linear space of polynomials of degree m. Also, the operator $\mathcal{B}_{n,q}(f)$ becomes an approximation process, that is $\mathcal{B}_{n,q}(f)$ converges to f uniformly on any compact subinterval of the semi-axis $[0, \infty)$ and have rate of convergence in weighted space, (see [2]). Direct global estimates for q-Baskakov operators, using Ditzian-Totik modulus of smoothness, proved in [12]. All the above mentioned properties of these new operators motivate us for the current work.

In this paper we firstly obtain a recurrence formula, including q-derivative for the q-Baskakov operators which used to calculate q-central moment and obtain an estimate for the q-central moments of the q-Baskakov operators in the case $q > 0$. Then using these q-central moments, a Voronovskaya-type theorem for q-derivative of q-Baskakov operators is given. Central moment theorem for q-Bernstein operators was established by Videnskii [11]. These moments were generalized by Mahmudov

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[10]. The central moments and their estimates for classical linear positive operators can be found [5]. Also, Voronovskaya-type theorems for derivatives of the Bernstein-Chlodowsky polynomials and the Szasz-Mirakyan operators given in [7]. In 2009 Aral and Gupta [3] introduced another generalization of Baskakov operators based on the q-integers called q-Baskakov operators.

2. Definitions and preliminary results

Let us now recall the definition of a q-integer. Given a value of $q>0$ and any non-negative integer n, we define $[n]_q$ as

$$[n]_q=\begin{cases}(1-q^n)/(1-q) & ,\ q\neq 1\\ n & ,\ q=1\end{cases}.$$

and call $[n]_q$ a q-integer. Note that $[n]_q$ is a continuous function of q. We next define $[n]_q!$, where n is a non-negative integer, as

$$[n]_q!=\begin{cases}[n]_q[n-1]_q...[1]_q & ,\ n=1,2,...\\ 1 & ,\ n=0\end{cases}$$

and call $[n]_q!$ a q-factorial. The q-binomial coefficient $\begin{bmatrix} n\\ k\end{bmatrix}$ is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q:=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

for $n\geq k\geq 1$, and has the value 1 when $k=0$ and value zero otherwise (see [1]). Note that $[n]_q$ is a polynomial in q and, less obviously,the q-binomial coefficients are also polynomials, known as Gaussian polynomials.

We recall that the $q-$derivative operator $\mathcal{D}_q$ is given by

$$\mathcal{D}_q f(x):=\frac{f(qx)-f(x)}{(q-1)x},\ x\neq 0 \tag{2.1}$$

and $\mathcal{D}_q f(x)|_{x=0}:=f'(0)$.

Higher $q-$derivatives are

$$\mathcal{D}_q^0 f:=f,\ \mathcal{D}_q^n f:=\mathcal{D}\left(\mathcal{D}_q^{n-1}f\right),\quad n=1,\ 2,\ 3,\ldots.$$

Note that

$$\lim_{q\to 1}\mathcal{D}_q^n f(x))=f^{(n)}(x),\ n=1,\ 2,\ 3,\ldots. \tag{2.2}$$

if f is n times differentiable.

The $q-$analogue of Leibnitz rule is

$$\mathcal{D}_q(f(x)g(x))=g(x)\mathcal{D}_q f(x)+f(qx)\mathcal{D}_q g(x) \tag{2.3}$$

and as a symmetrically, we can write the this rule in the form

$$\mathcal{D}_q(f(x)g(x))=g(qx)\mathcal{D}_q f(x)+f(x)\mathcal{D}_q g(x).$$

Also $q-$derivative of a quotient is

$$\mathcal{D}_q\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\mathcal{D}_q f(x)-f(x)\mathcal{D}_q g(x)}{g(x)g(qx)}. \tag{2.4}$$

For $a \in \mathbb{R}$, $q-$analogue of $(a+x)^n$ is given by

$$(a+x)_q^n = \prod_{s=0}^{n-1}(a+q^s x).$$

Also $q-$derivative of $(a+x)_q^n$ is

$$\mathcal{D}_q\,(a+x)_q^n = [n]_q\,(a+qx)^{n-1}.$$

From the above definition it is obvious that a continuous function on an interval, which does not include 0 is continuous q-differentiable. We say that a function f is q-differentiable on a real I if for any $q \in (0,1)$ the q-derivative of f exists and is finite in every $x \in I$.

For details see [6].

Lemma 1. ([2])*For $n,\ k \geq 0$, we have*

$$\mathcal{D}_q\left[\frac{x^k}{(1+x)_q^{n+k}}\right] = [k]_q\,\frac{x^{k-1}}{(1+x)_q^{n+k}} - q^k\,[n+k]_q\,\frac{x^k}{(1+x)_q^{n+k+1}}. \tag{2.5}$$

Lemma 2. *Let $m \in \mathbb{N}$. If we define the central moment as*

$$L_{n,m}\,(x,q) = \mathcal{B}_{n,q}((t-x)_q^m\,,x)$$

then we have

$$L_{n,0}\,(x,q) = 1,\ \ L_{n,1}\,(x,q) = 0,\ \ L_{n,2}\,(x,q) = \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right)$$

and the recurrence relations:

$$\begin{aligned} L_{n,m+1}\,(qx,q) \quad &= \quad \frac{qX}{[n]_q}\mathcal{D}_q L_{n,m}\,(x,q) \\ &\quad +q\frac{[m]_q}{[n]_q}x\left\{(1+x) - x[n]_q[m-1]_q\,(1-q)^2\right\}L_{n,m-1}\,(qx,q) \\ &\quad +x\left(1-q^2\right)[m]_q L_{n,m}\,(qx,q)\,, \end{aligned}$$

where $X = x\,(1+x)\,.$

Proof. It is easy to show that $L_{n,0}\,(x,q) = 1$, $L_{n,1}\,(x,q) = 0$, $L_{n,2}\,(x,q) = \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right)$ by using the definition of $L_{n,m}\,(x,q)\,.$ Based on (2.3) we have

$$\begin{aligned} \mathcal{D}_q L_{n,m}\,(x,q) \quad &= \quad \sum_{k=0}^{\infty} -\,[m]_q\,\mathcal{P}_{n,k}\,(q,qx)\left(\frac{[k]_q}{q^{k-1}[n]_q} - qx\right)_q^{m-1} \\ &\quad +\sum_{k=0}^{\infty}\mathcal{D}_q\mathcal{P}_{n,k}\,(q,x)\left(\frac{[k]_q}{q^{k-1}[n]_q} - x\right)_q^{m}. \end{aligned} \tag{2.6}$$

From (2.5) we can write

$$\mathcal{D}_q\mathcal{P}_{n,k}\,(q,x) = \frac{[k]_q}{x}\mathcal{P}_{n,k}\,(q,x) - \frac{[n+k]_q}{(1+x)}\mathcal{P}_{n,k}\,(q,qx)$$

and

$$\begin{aligned} x\left(1+q^{n+k}x\right)\mathcal{D}_q P_{n,k}\left(q;x\right) &= [k]_q\left(1+q^{n+k}x\right)\mathcal{P}_{n,k}\left(q,x\right)-q^k x\left[n+k\right]_q \mathcal{P}_{n,k}\left(q,x\right) \\ &= \left([k]_q - xq^k\left[n\right]_q\right)\mathcal{P}_{n,k}\left(q,x\right). \end{aligned}$$

Thus

$$X\mathcal{D}_q\mathcal{P}_{n,k}\left(q,x\right)=\left(\frac{[k]_q}{q^k\left[n\right]_q}-x\right)[n]_q\,\mathcal{P}_{n,k}\left(q,qx\right). \tag{2.7}$$

Substituting (2.7) into the (2.6), we can write:

$$\begin{aligned} X\mathcal{D}_q L_{n,m}\left(x,q\right) &= -X\sum_{k=0}^{\infty}[m]_q\,\mathcal{P}_{n,k}\left(q,qx\right)\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m-1} \\ &\quad+\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-x\right)_q^{m}\left(\frac{[k]_q}{q^k\left[n\right]_q}-x\right). \end{aligned}$$

Since

$$\left(\frac{[k]_q}{q^k\left[n\right]_q}-x\right)=\frac{1}{q}\left(\frac{[k]_q}{q^{k-1}\left[n\right]_q}-qx-q^m x+q^m x\right)$$

we have

$$\begin{aligned} X\mathcal{D}_q L_{n,m}\left(x,q\right) &= -X\sum_{k=0}^{\infty}[m]_q\,\mathcal{P}_{n,k}\left(q,qx\right)\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m-1} \\ &\quad+\frac{1}{q}\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-x\right)_q^{m+1} \\ &\quad+x\left(q^{m-1}-1\right)\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-x\right)_q^{m}. \end{aligned}$$

On the other hand, using the equality

$$\left(\frac{[k]_q}{q^{k-1}[n]_q}-x\right)_q^{m}=\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m}+\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m-1}x\left(q^m-1\right)$$

we get

$$\begin{aligned} X\mathcal{D}_q L_{n,m}\left(x,q\right) &= -X\sum_{k=0}^{\infty}[m]_q\,\mathcal{P}_{n,k}\left(q,qx\right)\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m-1} \\ &\quad+\frac{1}{q}\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m+1} \\ &\quad+\frac{1}{q}x\left(q^{m+1}-1\right)\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m} \\ &\quad+x\left(q^{m-1}-1\right)\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m} \\ &\quad+x^2\left(q^m-1\right)\left(q^{m-1}-1\right)\sum_{k=0}^{\infty}\mathcal{P}_{n,k}\left(q,qx\right)[n]_q\left(\frac{[k]_q}{q^{k-1}[n]_q}-qx\right)_q^{m-1}. \end{aligned}$$

If we rewrite the equality we have

$$\begin{aligned} X\mathcal{D}_q L_{n,m}(x,q) &= -X[m]_q L_{n,m-1}(qx,q) + \frac{[n]_q}{q} L_{n,m+1}(qx,q) \\ &\quad + \frac{[n]_q}{q} x\left(q^{m+1}-1\right) L_{n,m}(qx,q) + x\left(q^{m-1}-1\right)[n]_q L_{n,m}(qx,q) \\ &\quad + x^2\left(q^m-1\right)\left(q^{m-1}-1\right)[n]_q L_{n,m-1}(qx,q) \end{aligned}$$

and

$$\begin{aligned} L_{n,m+1}(qx,q) &= \frac{q}{[n]_q} X\mathcal{D}_q L_{n,m}(x,q) \\ &\quad + q\frac{[m]_q}{[n]_q} x\left\{(1+x) - x[n]_q[m-1]_q(1-q)^2\right\} L_{n,m-1}(qx,q) \\ &\quad + x\left(1-q^2\right)[m]_q L_{n,m}(qx,q). \end{aligned}$$

□

For $m=2$ and $m=3$, we have following results:

Now we may present explicit formula for q-central moments of orders 3 and 4.

Corollary 1. *The following equalities hold :*

i) $L_{n,3}(x,q) = X_q\left[1 + \frac{x}{q}\left(\frac{1+q}{q} + q(1-q^n)(1+2q+q^2)\right)\right]$

ii) $L_{n,4}(qx,q) = X_q\left\{\frac{1}{[n]_q} + [2]_q x\left(\frac{1}{q^2[n]_q} - (1+q) + \frac{[3]_q x}{q^5[n]_q}\right)\right.$
$+(1+\frac{1}{q})(\frac{[3]_q x^2}{q^3} - \frac{[3]_q x^2}{q} + \frac{[2]_q x}{q}) + [3]_q x\left[(1+\frac{x}{q}) - \frac{x}{q}(1-q^n)\left(1-q^2\right)\right]$
$\left.+\frac{x}{q}(q\left(1-q^2\right) + \left(1-q^4\right))\left[1 + \frac{x}{q}(\frac{1+q}{q} + (1-q^n)(1+2q+q^2)\right]\right\},$
where $X_q = \frac{x}{[n]_q^2}(1+\frac{x}{q})$.

Lemma 3. *Assume that $q_n \in (0;1)$; $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For every $x \in [0,\ \infty)$, there hold*

i) $\lim_{n\to\infty} \frac{1}{q_n X_{q_n}} \frac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} L_{n,3}(x,q_n) = \frac{1+2x(3-2a)}{2} \lim_{n\to\infty} \mathcal{D}_{q_n}^2 f(x),$

ii) $\lim_{n\to\infty} \frac{1}{q_n X_{q_n}} \frac{\mathcal{D}_{q_n}^3 f(x)}{[3]_{q_n}!} L_{n,4}(x,q_n) = x(1+x) \lim_{n\to\infty} \mathcal{D}_{q_n}^3 f(x),$

iii) $\lim_{n\to\infty} \frac{1}{q_n X_{q_n}} x\left(q_n^2-1\right) L_{n,2}(x,q_n) \frac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} = x(1-a) \lim_{n\to\infty} \mathcal{D}_{q_n}^2 f(x),$

iv) $\lim_{n\to\infty} \frac{1}{q_n X_{q_n}} x\left(q_n^3-1\right) L_{n,3}(x,q_n) \frac{\mathcal{D}_{q_n}^3 f(x)}{[3]_{q_n}!} = 0.$

Proof. We only calculate *i)*. Others are similar.

i) Using the Lemma 2, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{q_n X_{q_n}} \frac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} L_{n,3}(x,q_n) \\ &= \lim_{n\to\infty} \frac{1}{q_n} \frac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} \left(1 + \frac{x}{q_n}(\frac{1+q_n}{q_n} + q_n(1-q_n^n)(1+2q_n+q_n^2)\right) \\ &= \frac{1+2x(3-2a)}{2} \lim_{n\to\infty} \mathcal{D}_{q_n}^2 f(x). \end{aligned}$$

□

Using (2.2), we have following results:

Corollary 2. *Assume that $f^{''}(x)$ and $f^{(3)}(x)$ exist for $x \in [0, \infty)$ and $q_n \in (0;1)$; $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For every $x \in [0, \infty)$, there hold*

$$
\begin{aligned}
i)\ & \lim_{n\to\infty} \tfrac{1}{q_n X_{q_n}} \tfrac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} L_{n,3}(x, q_n) = \tfrac{1+2x(3-2a)}{2} f^{''}(x), \\
ii)\ & \lim_{n\to\infty} \tfrac{1}{q_n X_{q_n}} \tfrac{\mathcal{D}_{q_n}^3 f(x)}{[3]_{q_n}!} L_{n,4}(x, q_n) = x(1+x) f^{(3)}(x), \\
iii)\ & \lim_{n\to\infty} \tfrac{1}{q_n X_{q_n}} x\left(q_n^2 - 1\right) L_{n,2}(x, q_n) \tfrac{\mathcal{D}_{q_n}^2 f(x)}{[2]_{q_n}} = x(1-a) f^{''}(x), \\
iv)\ & \lim_{n\to\infty} \tfrac{1}{q_n X_{q_n}} x\left(q_n^3 - 1\right) L_{n,3}(x, q_n) \tfrac{\mathcal{D}_{q_n}^3 f(x)}{[3]_{q_n}!} = 0.
\end{aligned}
$$

Lemma 4. *For a fixed $m \in \mathbb{N}$, $\mathcal{B}_{n,q}((t-x)_q^m, x)$ is a polynomial in x of degree less than or equal to m in $\frac{1}{[n]_q}$. The minimum degree of $\frac{1}{[n]_q}$ is $\left\lfloor \frac{m+1}{2} \right\rfloor$. Moreover it can be written in the following form*

$$
L_{n,m}(x, q) = \frac{1}{[n]_q^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \sum_{k=0}^{m} a_{k,m,n}(q)\, x^k,
$$

Proof. By Lemma 2 it is true for $m = 1$. Assuming that it is true. By the recurrence formula in Lemma 2 we have

$$
\begin{aligned}
L_{n,m+1}(qx, q) \quad = \quad & \frac{qX}{[n]_q} \mathcal{D}_q \left(\frac{1}{[n]_q^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \sum_{k=0}^{m} a_{k,m,n}(q)\, x^k \right) \\
& + q \frac{[m]_q}{[n]_q} x \left\{ (1+x) - x(1-q^n)\left(1-q^{m-1}\right) \right\} L_{n,m-1}(qx, q) \\
& + x \left\{ q\left(1-q^{m-1}\right) + \left(1-q^{m+1}\right) \right\} L_{n,m}(qx, q)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{qX}{[n]_q} \mathcal{D}_q \left(\frac{1}{[n]_q^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \sum_{k=0}^{m} a_{k,m,n}(q)\, x^k \right) \\
& + q \frac{[m]_q}{[n]_q} x \left\{ (1+x) - x [n]_q [m-1]_q (1-q)^2 \right\} \frac{1}{[n]_q^{\left\lfloor \frac{m}{2} \right\rfloor}} \sum_{k=0}^{m-1} a_{k,m-1,n}(q)\, q^k x^k \\
& + x [2]_q [m]_q (1-q) \frac{1}{[n]_q^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \sum_{k=0}^{m} a_{k,m,n}(q)\, q^k x^k
\end{aligned}
$$

and

$$
\begin{aligned}
L_{n,m+1}(qx,q) &= \frac{qX}{[n]_q^{\left\lfloor\frac{m+1}{2}\right\rfloor+1}}\sum_{k=0}^{m} a_{k,m,n}(q)\,[k]_q\,x^{k-1} \\
&\quad + \frac{[m]_q}{[n]_q^{1+\left\lfloor\frac{m}{2}\right\rfloor}}\sum_{k=0}^{m-1} a_{k,m-1,n}(q)\,q^{k+1}x^{k+1} \\
&\quad + \left\{1-[n]_q\,[m-1]_q\,(1-q)^2\right\}\frac{[m]_q}{[n]_q^{1+\left\lfloor\frac{m}{2}\right\rfloor}}\sum_{k=0}^{m-1} a_{k,m-1,n}(q)\,q^{k+1}x^{k+2} \\
&\quad + [2]_q\,[m]_q\,(1-q)\,\frac{1}{[n]_q^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\sum_{k=0}^{m} a_{k,m,n}(q)\,q^{k}x^{k+1}.
\end{aligned}
$$

Using the $\frac{x}{q}$ place of x we obtain

$$
\begin{aligned}
L_{n,m+1}(x,q) &= \frac{1}{[n]_q^{\left\lfloor\frac{m+1}{2}\right\rfloor+1}}\sum_{k=0}^{m} a_{k,m,n}(q)\,[k]_q\,\frac{x^k}{q^{k-1}} \\
&\quad + \frac{1}{[n]_q^{\left\lfloor\frac{m+1}{2}\right\rfloor+1}}\sum_{k=1}^{m+1} a_{k-1,m,n}(q)\,[k-1]_q\,\frac{x^k}{q^{k-1}} \\
&\quad + \frac{[m]_q}{[n]_q^{1+\left\lfloor\frac{m}{2}\right\rfloor}}\sum_{k=1}^{m} a_{k-1,m-1,n}(q)\,x^k \\
&\quad + \left\{1-[n]_q\,[m-1]_q\,(1-q)^2\right\}\frac{[m]_q}{[n]_q^{1+\left\lfloor\frac{m}{2}\right\rfloor}}\sum_{k=2}^{m+1} a_{k-2,m-1,n}(q)\,\frac{x^k}{q} \\
&\quad + [2]_q\,[m]_q\,(1-q)\,\frac{1}{[n]_q^{\left\lfloor\frac{m+1}{2}\right\rfloor}}\sum_{k=1}^{m+1} a_{k-1,m,n}(q)\,\frac{x^k}{q} \\
&:= \frac{1}{[n]_q^{\left\lfloor\frac{m+2}{2}\right\rfloor}}\sum_{k=0}^{m+1} a_{k,m+1,n}(q)\,x^k
\end{aligned}
$$

$$
\begin{aligned}
a_{k,m+1,n}(q) &= \frac{1}{[n]^{\alpha}}a_{k,m,n}(q)\,\frac{[k]_q}{q^{k-1}} + \frac{1}{[n]^{\alpha}}a_{k-1,m,n}(q)\,\frac{[k-1]_q}{q^{k-1}} \\
&\quad + \frac{[m]_q}{[n]_q^{\alpha}}a_{k-1,m-1,n}(q) + \frac{[m]_q}{q\,[n]_q^{\alpha}}\left\{1-[n]_q\,[m-1]_q\,(1-q)^2\right\}a_{k-2,m-1,n}(q) \\
&\quad + \frac{[m]_q}{q\,[n]_q^{\alpha}}\,[2]_q\,(1-q)\,a_{k-1,m,n}(q),
\end{aligned}
$$

where

$$
\begin{aligned}
a_{-1,m,n}(q) &= a_{-1,m-1,n}(q) = a_{-2,m-1,n}(q) = 0, \quad 0 \le k \le m+1, \\
a_{m+1,m,n}(q) &= a_{m,m-1,n}(q) = 0, \quad \alpha = 1 + \left\lfloor\frac{m+1}{2}\right\rfloor - \left\lfloor\frac{m+2}{2}\right\rfloor \\
\alpha &= 0 \text{ or } \alpha = 1.
\end{aligned}
$$

which proves the lemma. □

3. Voronovskaya type theorem for q-derivatives

The following is the main convergence result of this paper. Theorem 1 is devoted to the Voronovskaya-type theorem for the q-derivative of the q-Baskakov operators.

We consider the weighted space

$$C_2\left[0,\infty\right) := \left\{ f \in C\left[0,\infty\right) : \lim_{x\to\infty} \frac{f\left(x\right)}{1+x^2} \text{ uniformly continuous and bounded on } \left[0,\infty\right) \right\}$$

and

$$C_2^r\left[0,\infty\right) := \left\{ f \in C_2\left[0,\infty\right) : D_q^r f \in C_2\left[0,\infty\right) \right\}$$

where $C\left[0,\infty\right)$ is the space of continuous function on $\left[0,\infty\right)$. The norm in this space is

$$\left\|f\right\|_2 := \sup_{x\geq 0} \frac{\left|f\left(x\right)\right|}{1+x^2}$$

Theorem 1. *Let $q = q_n$ satisfies $0 < q_n < 1$ also let $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For each $f \in C_2^3\left[0,\infty\right)$ and for every $x \geq 0$ we have*

$$\lim_{n\to\infty} \left[n\right]_{q_n} \left\{ \mathcal{D}_{q_n} B_{n,q_n}(f, \frac{x}{q_n}) - \frac{\mathcal{D}_{q_n} f\left(x\right)}{q_n} \right\} = \frac{1+2x\left(3-2a\right)}{2} \lim_{n\to\infty} \mathcal{D}_{q_n}^2 f\left(x\right) + x\left(1+x\right) \lim_{n\to\infty} \mathcal{D}_{q_n}^3 f\left(x\right).$$

Proof. Using q-derivative, we can write

$$\mathcal{D}_q B_{n,q}(f,x) = \left[n\right]_q \sum_{k=0}^{\infty} q^k \mathcal{P}_{n+1,k}^q\left(x\right) \left(f\left(\frac{\left[k+1\right]_q}{q^k \left[n\right]_q} \right) - f\left(\frac{\left[k\right]_q}{q^{k-1}\left[n\right]_q} \right) \right).$$

We will consider two cases. For $x = 0$, we have

$$\mathcal{D}_q B_{n,q}(f,0) = \left[n\right]_q \left(f\left(\frac{1}{\left[n\right]_q} \right) - f\left(0\right) \right).$$

Since

$$\lim_{n\to\infty} \left[n\right]_{q_n} \left\{ \left[n\right]_{q_n} \left(f\left(\frac{1}{\left[n\right]_{q_n}} \right) - f\left(0\right) \right) - \frac{f'\left(0\right)}{q_n} \right\} = \frac{f''\left(0\right)}{2}$$

we have desired result for $x = 0$.

So let $x > 0$. By simple calculations we have

$$\begin{aligned} \mathcal{D}_q \frac{1}{\left(1+\frac{x}{q}\right)_q^{n+k}} &= \frac{-\frac{\left[n+k\right]_q}{q}\left(1+x\right)_q^{n+k-1}}{\left(1+\frac{x}{q}\right)_q^{n+k}\left(1+x\right)_q^{n+k}} \\ &= -\frac{\left[n+k\right]_q}{q} \frac{1}{\left(1+\frac{x}{q}\right)_q^{n+k+1}}. \end{aligned}$$

Then by (2.3) it follows that

$$
\begin{aligned}
\mathcal{D}_q B_{n,q}(f,\frac{x}{q}) &= \sum_{k=1}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}-k} [k]_q \frac{x^{k-1}}{\left(1+\frac{x}{q}\right)_q^{n+k}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&\quad -\sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right]_q q^{\frac{k(k-1)}{2}} \frac{[n+k]_q}{q} \frac{x^k}{\left(1+\frac{x}{q}\right)_q^{n+k+1}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}\left(q,\frac{x}{q}\right)\left(\frac{[k]_q}{x} - \frac{q^{k-1}[n+k]_q}{1+q^{n+k-1}x}\right) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right).
\end{aligned}
$$

Using the equalities

$$[n+k]_q = [n]_q + q^n [k]_q$$

and

$$q^k \left(1+\frac{x}{q}\right) \mathcal{P}_{n,k}\left(q,\frac{x}{q}\right) = \left(1+q^{n+k-1}x\right) \mathcal{P}_{n,k}(q,x)$$

we can write

$$
\begin{aligned}
\mathcal{D}_q B_{n,q}(f,\frac{x}{q}) &= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}\left(q,\frac{x}{q}\right)\left(\frac{[k]_q - xq^{k-1}[n]_q}{x\left(1+q^{n+k-1}x\right)}\right) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&= \sum_{k=0}^{\infty} \mathcal{P}_{n,k}(q,x)\left(\frac{[k]_q - xq^{k-1}[n]_q}{q^k x\left(1+\frac{x}{q}\right)}\right) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&= \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} \sum_{k=0}^{\infty} \mathcal{P}_{n,k}(q,x)\left(\frac{[k]_q}{q^{k-1}[n]_q} - x\right) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\
&= \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} B_{n,q}((t-x)f(t),x). \qquad (3.1)
\end{aligned}
$$

Using the q-Taylor's formula

$$f(t) = \sum_{j=0}^{3} \frac{\mathcal{D}_q^j f(x)}{\Gamma_q(j+1)} (t-x)_q^j + (t-x)_q^3 h(t,x,q)$$

$$
\begin{aligned}
= \; & f(x) + (t-x)\mathcal{D}_q f(x) + (t-x)(t-qx)\frac{\mathcal{D}_q^2 f(x)}{[2]_q} + (t-x)(t-qx)\left(t-q^2x\right)\frac{\mathcal{D}_q^3 f(x)}{[3]_q!} \\
& + (t-x)_q^3 h(t,x,q)
\end{aligned}
$$

we can write

$$
\begin{aligned}
(t-x)f(t) &= (t-x)f(x) + (t-x)_q^2 \mathcal{D}_q f(x) + x(q-1)(t-x)\mathcal{D}_q f(x) \\
&\quad + (t-x)_q^3 \frac{\mathcal{D}_q^2 f(x)}{[2]_q!} + x\left(q^2-1\right)(t-x)_q^2 \frac{\mathcal{D}_q^2 f(x)}{[2]_q}
\end{aligned}
$$

$$+(t-x)_q^4 \frac{\mathcal{D}_q^3 f(x)}{[3]_q!} + x\left(q^3-1\right)(t-x)_q^3 \frac{\mathcal{D}_q^3 f(x)}{[3]_q!}$$
$$+(t-x)_q^4 h(t,x,q) + x\left(q^3-1\right)(t-x)_q^3 h(t,x,q).$$

where $h(\cdot,x,q) \in C_2^3[0,\infty)$ and $\lim_{t\to x} h(t,x,q)=0$.

Using this equality and (3.1) we have

$$\begin{aligned}
\mathcal{D}_q B_{n,q}(f,\frac{x}{q}) &= \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} B_{n,q}((t-x)f(t),x) \\
&= \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} \mathcal{D}_q f(x) L_{n,2}(x,q)
\end{aligned}$$

$$\begin{aligned}
&+\frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} \frac{\mathcal{D}_q^2 f(x)}{[2]_q} L_{n,3}(x,q) + \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} x\left(q^2-1\right) L_{n,2}(x,q) \frac{\mathcal{D}_q^2 f(x)}{[2]_q} \\
&+\frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} \frac{\mathcal{D}_q^3 f(x)}{[3]_q!} L_{n,4}(x,q) + \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} x\left(q^3-1\right) L_{n,3}(x,q) \frac{\mathcal{D}_q^3 f(x)}{[3]_q!} \\
&+\frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} B_{n,q}((t-x)_q^4 h(t,x,q),x) + x\left(q^3-1\right) \frac{[n]_q}{qx\left(1+\frac{x}{q}\right)} B_{n,q}((t-x)_q^3 h(t,x,q),x),x)
\end{aligned}$$

Since $L_{n,2}(x,q) = \frac{x}{[n]_q}\left(1+\frac{1}{q}x\right)$ we can write

$$[n]_q \left\{ \mathcal{D}_q B_{n,q}(f,\frac{x}{q}) - \frac{\mathcal{D}_q f(x)}{q} \right\} = \frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} \mathcal{D}_q f(x) \left\{ L_{n,2}(x,q) - \frac{x}{[n]_q}\left(1+\frac{x}{q}\right) \right\}$$

$$\begin{aligned}
&+\frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} \frac{\mathcal{D}_q^2 f(x)}{[2]_q!} L_{n,3}(x,q) + \frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} x\left(q^2-1\right) L_{n,2}(x,q) \frac{\mathcal{D}_q^2 f(x)}{[2]_q!} \\
&+\frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} \frac{\mathcal{D}_q^3 f(x)}{[3]_q!} L_{n,4}(x,q) + \frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} x\left(q^3-1\right) L_{n,3}(x,q) \frac{\mathcal{D}_q^3 f(x)}{[3]_q!} \\
&+K_n(x,q),
\end{aligned}$$

where

$$\begin{aligned}
&K_n(x,q) \\
= &\frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} B_{n,q}\left((t-x)_q^4 h(t,x,q),x\right) + \frac{[n]_q^2}{q\left(1+\frac{x}{q}\right)} \left(q^3-1\right) B_{n,q}\left((t-x)_q^3 h(t,x,q),x\right).
\end{aligned}$$

Based on the Corollary 1, in order to complete the proof we need to show that we have for $x>0$

$$\lim_{n\to\infty} K_n(x,q_n) = 0.$$

From Cauchy -Schwarz inequality we can write

$$\begin{aligned} & K_n(x,q) \\ = \; & \frac{[n]_q^2}{qx\left(1+\frac{x}{q}\right)} B_{n,q}\left((t-x)_q^4 h(t,x,q), x\right) + \frac{[n]_q^2}{q\left(1+\frac{x}{q}\right)}\left(q^3-1\right) B_{n,q}\left((t-x)_q^3 h(t,x,q), x\right) \\ \leq \; & \sqrt{B_{n,q}\left(h^2(t,x,q), x\right)} \sqrt{\frac{[n]_q^4}{q^2x^2\left(1+\frac{x}{q}\right)^2} B_{n,q}\left((t-x)_q^4 (t-x)_q^4, x\right)} \\ & + \sqrt{B_{n,q}\left(h^2(t,x,q), x\right)} \sqrt{\frac{[n]_q^4}{q^2\left(1+\frac{x}{q}\right)^2} \left(q^3-1\right)^2 B_{n,q}\left((t-x)_q^3 (t-x)_q^3, x\right)}. \end{aligned}$$

We know from [2] that

$$\lim_{n\to\infty} B_{n,q_n}\left(h^2(t,x,q_n), x\right) = 0.$$

We note that $(t-x)_q^4 (t-x)_q^4$ can be rewrite as a summation of $(t-x)_q^4, .., (t-x)_q^8$. Using Lemma 2 we can show that the limit of $\frac{[n]_{q_n}^4}{q_n^2 x^2 \left(1+\frac{x}{q_n}\right)^2} B_{n,q_n}\left((t-x)_{q_n}^4 (t-x)_{q_n}^4, x\right)$ is finite for each $x>0$. Similar situation is true for the last term of above inequality. This proves the theorem. □

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SEMIGROUP ALGEBRAS AND THEIR WEAK MODULE AMENABILITY

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ABSTRACT. In the present paper, the weak module amenability for a Banach algebra is investigated and it is proven that for an inverse semigroup S with the set of idempotents E, $\ell^1(S)$ is $\ell^1(E)$-weakly module amenable. Also, it is shown that $C^*(S)$, the enveloping C^*-algebra of $\ell^1(S)$, is weakly module amenable as a $C^*(E)$-module with trivial left action.

1. INTRODUCTION

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [5], who termed a commutative Banach algebra $\mathcal{A}$"weakly amenable" if every continuous derivation from $\mathcal{A}$ into a symmetric Banach $\mathcal{A}$-bimodule is zero. Johnson in [13] generalized this notion for arbitrary Banach algebras and showed that for any locally compact group G, the group algebra $L^1(G)$ is weakly amenable (refer to [8] for a short proof). This fact is not true for semigroups though. For instance, if $\mathcal{C}$ is the bicyclic inverse semigroup, then $\ell^1(\mathcal{C})$ is not weakly amenable [6].

The notion of weak module amenability of Banach algebras is defined in [4] and studied for their second dual in [2]. The main result of [4] is that, $\ell^1(S)$ is weakly module amenable as an $\ell^1(E)$-module when S is commutative. It is also proved in [2] that if E is an upward directed set and $\ell^1(E)$ acts on $\ell^1(S)$ trivially from left, then $\ell^1(S)$ is weakly module amenable.

In this paper, we modified the definition of the weak module amenability for a Banach algebra $\mathcal{A}$ which is a Banach module over Banach algebra $\mathcal{O}$, and we show that upward directed condition (the condition D_1 of Duncan and Namioka [10]) for E is strong and in fact, it is redundant for the inverse semigroup algebra $\ell^1(S)$ to be weakly module amenable as an $\ell^1(E)$-module. Also, we find the relation between weak module amenability of $\mathcal{A}$ and weak amenability of $\mathcal{A}/J$, where J is the closed ideal of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{O}$. If $C^*(S)$ is the enveloping C^*-algebra of $\ell^1(S)$, then $C^*(S)$ is weakly module amenable as a $C^*(E)$-module. As an example, we show that Clifford semigroup algebra $\ell^1(C)$ is $\ell^1(E_C)$-weakly module amenable.

2. NOTATIONS

Throughout this paper, $\mathcal{A}$ and $\mathcal{O}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathcal{O}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathcal{O}).$$

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Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathcal{O}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x) = (a\cdot\alpha)\cdot x,\ (\alpha\cdot x)\cdot a = \alpha\cdot(x\cdot a) \quad (a \in \mathcal{A}, \alpha \in O, x \in X)$$

with similar actions for the right or two-sided. Then we say that X is a Banach $\mathcal{A}$-$\mathcal{O}$-module. If moreover

$$\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathcal{O}, x \in X)$$

then X is called a *commutative* $\mathcal{A}$-$\mathcal{O}$-module. Note that when $\mathcal{A}$ acts on itself by algebra multiplication, it is not in general a Banach $\mathcal{A}$-$\mathcal{O}$-module, as we have not assumed the compatibility condition

$$a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b \quad (\alpha \in \mathcal{O}, a, b \in \mathcal{A}).$$

If $\mathcal{A}$ is a commutative $\mathcal{O}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathcal{O}$-module.

A bounded map $D : \mathcal{A} \longrightarrow X$ is called a $\mathcal{O}$*-module derivation* if

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha$$

and

$$D(a \pm b) = D(a) \pm D(b), \ \ D(ab) = D(a) \cdot b + a \cdot D(b),$$

for all $\alpha \in \mathcal{O}$ and $a, b \in \mathcal{A}$. If X is a commutative $\mathcal{A}$-$\mathcal{O}$-module, then each $x \in X$ defines a module derivation $D_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. These are called the *inner* module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathcal{O}$-module) if for any commutative Banach $\mathcal{A}$-$\mathcal{O}$-module X, each $\mathcal{O}$-module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [1].

Consider the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathcal{O}}\mathcal{A}$ which is isomorphic to the quotient space $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$, where I is the closed ideal of the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form $a\cdot\alpha\otimes b - a\otimes\alpha\cdot b$ for $\alpha \in \mathcal{O}, a, b \in \mathcal{A}$ [16]. Also consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a\cdot\alpha)b - a(\alpha\cdot b)$ for $\alpha \in \mathcal{O}, a, b \in \mathcal{A}$. Then I and J are $\mathcal{A}$-submodules and $\mathcal{O}$-submodules of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ and $\mathcal{A}$ respectively, and the quotients $\mathcal{A}\widehat{\otimes}_{\mathcal{O}}\mathcal{A}$ and $\mathcal{A}/J$ are $\mathcal{A}$-modules and $\mathcal{O}$-modules. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathcal{O}$-modules with the compatible actions when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.

3. Main results

Definition 3.1. *The Banach algebra $\mathcal{A}$ is called weakly module amenable (as an $\mathcal{O}$-module) if for any subset Y of $\mathcal{A}^*$ which is $\mathcal{A}$-submodule and commutative Banach $\mathcal{O}$-submodule, each module derivation from $\mathcal{A}$ to Y is inner.*

In the following result, we find an equivalent definition for weak module amenability when $\mathcal{A}/J$ is a commutative $\mathcal{A}$-$\mathcal{O}$-module.

Proposition 3.2. *If $\mathcal{A}/J$ is a commutative Banach $\mathcal{O}$-module, then the following are equivalent:*

(i) *Every module derivation from $\mathcal{A}$ to $J^\perp$ is inner.*

(ii) *$\mathcal{A}$ is weakly module amenable.*

Proof. $(i) \Rightarrow (ii)$: For each $\alpha \in \mathcal{O}, a, b \in \mathcal{A}, y \in Y$, we have

$$\langle y, (a \cdot \alpha)b - a(\alpha \cdot b)\rangle = \langle y, (a \cdot \alpha)b\rangle - \langle y, a(\alpha \cdot b)\rangle = \langle b \cdot y, a \cdot \alpha\rangle - \langle y \cdot a, \alpha \cdot b\rangle$$

$$
\begin{aligned}
&= \langle \alpha \cdot (b \cdot y), a\rangle - \langle (y \cdot a) \cdot \alpha, b\rangle = \langle (b \cdot y) \cdot \alpha, a\rangle - \langle \alpha \cdot (y \cdot a), b\rangle \\
&= \langle b \cdot (y \cdot \alpha), a\rangle - \langle (\alpha \cdot y) \cdot a, b\rangle = \langle y \cdot \alpha, ab\rangle - \langle \alpha \cdot y, ab\rangle \\
&= \langle y \cdot \alpha - \alpha \cdot y, ab\rangle = 0.
\end{aligned}
$$

We have used the commutativity of $\mathfrak{A}$-submodule Y in the above equalities. It follows from the linearity and continuity of y as a functional on $\mathcal{A}$ that $Y \subseteq J^{\perp}$.
$(ii) \Rightarrow (i)$: This follows from the fact that $Y = J^{\perp}$ is a commutative Banach submodule of $\mathcal{A}^*$. Δ

Note that if $\mathcal{O} = \mathbb{C}$ (the set of complex numbers), then $J = \{0\}$ and so, the above definition and Johnson's definition [13] in the classical case for the weak amenability of Banach algebras coincide.

Let $\mathcal{O}$ be a commutative Banach algebra with character space $\Phi_{\mathcal{O}}$ and let $\mathcal{A}$ be a commutative Banach $\mathcal{O}$-bimodule with compatible actions. Let $\varphi \in \Phi_{\mathcal{O}} \cup \{0\}$. Then $\mathcal{O}$ is an $\mathcal{O}$-bimodule with actions:

$$\alpha \cdot \beta = \beta \cdot \alpha = \varphi(\alpha)\beta \quad (\alpha, \beta \in \mathcal{O}).$$

Now consider the linear map $\psi : \mathcal{A} \longrightarrow \mathcal{O}$ such that

$$\psi(ab) = \psi(a)\psi(b), \quad \psi(a \cdot \alpha) = \varphi(\alpha)\psi(a) \quad (a \in \mathcal{A},\, \alpha \in \mathcal{O}).$$

The map ψ induces an $\mathcal{A}$-module structure on $\mathcal{O}$ by the following action:

$$\alpha \cdot a = a \cdot \alpha = \varphi \circ \psi(a)\alpha \quad (a \in \mathcal{A},\, \alpha \in \mathcal{O}).$$

With the above structures $\mathcal{O}$ becomes a commutative Banach $\mathcal{A}$-$\mathcal{O}$-module and we denote it by $\mathcal{O}_{(\varphi,\psi)}$. A bounded module derivation from $\mathcal{A}$ into $\mathcal{O}_{(\varphi,\psi)}$ is called a *point module derivation* on $\mathcal{A}$ at (φ, ψ).

It is proved in [7, Proposition 1.3] that if there is a non-zero, continuous point derivation on the Banach algebra $\mathcal{A}$, then it is to be failed weak amenability. The following result is its module version.

Proposition 3.3. *Let $\mathcal{A}$ be a commutative $\mathcal{O}$-module and let $\mathcal{O}$ be a commutative Banach algebra. If $\mathcal{A}$ is weakly module amenable, then there is no non-zero bounded module point derivation on $\mathcal{A}$.*

Proof. Let φ and ψ be as above. By [4, Proposition 2.4] there is no non-zero bounded point derivation at (φ, ψ) on $\mathcal{A}$ when $\varphi = 0$. Now assume that $\varphi \neq 0$ and $d : \mathcal{A} \longrightarrow \mathcal{O}_{(\varphi,\psi)}$ is a non-zero bounded module point derivation at (φ, ψ). Then the map $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ defined by $D(a) = d(a) \cdot (\varphi \circ \psi)$ is a bounded module derivation because for each $a, b, c \in \mathcal{A}$ we get

$$
\begin{aligned}
\langle D(ab), c\rangle &= \langle d(ab) \cdot (\varphi \circ \psi), c\rangle \\
&= \varphi \circ \psi(a)\varphi \circ \psi(c)d(b) + \varphi \circ \psi(b)\varphi \circ \psi(c)d(a) \\
&= \langle D(a) \cdot b, c\rangle + \langle a \cdot D(b), c\rangle.
\end{aligned}
$$

Since D is a module derivation, there exists $f \in \mathcal{A}^*$ such that for every $a \in \mathcal{A}$, we have

$$d(a)(\varphi \circ \psi)(a) = \langle D(a), a\rangle = \langle a \cdot f - f \cdot a, a\rangle = \langle f, a^2 - a^2\rangle = 0.$$

Hence $d = 0$ on the set $K = \{a \in \mathcal{A} : \psi(a) \notin Ker(\varphi)\}$. If $a \in \mathcal{A}\backslash K$ and $b \in K$, then $2d(\psi(a)) = d(\psi(a+b)) + d(\psi(a-b)) = 0$. Therefore $d = 0$ which is a contradiction.Δ

We say that $\mathcal{O}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathcal{O}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), where f is a continuous linear functional on $\mathcal{O}$. The following lemma is proved in [3, Lemma 3.1].

Lemma 3.4. *Let $\mathcal{A}$ be a Banach algebra and Banach $\mathcal{O}$-module with compatible actions, and J_0 be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathcal{O}$ and $a \in \mathcal{A}$ we have $a \cdot \alpha - \alpha \cdot a \in J_0$, i.e., $\mathcal{A}/J_0$ is a commutative Banach $\mathcal{O}$-module.*

The above Lemma shows that when $\mathcal{O}$ acts on $\mathcal{A}$ trivially from left or right, then the actions of $\mathcal{O}$ on $\mathcal{A}/J$ from both sides are trivial, that is

$$\alpha \cdot (a + J) = (a + J) \cdot \alpha = f(\alpha)a + J, \quad (a \in \mathcal{A}, \alpha \in \mathcal{O}).$$

Thus the actions of $\mathcal{O}$ on $\mathcal{A}/J$ are trivial.

Recall that a left Banach $\mathcal{A}$-module X is called a *left essential* $\mathcal{A}$-module if the linear span of $\mathcal{A} \cdot X = \{a \cdot x : a \in \mathcal{A}, x \in X\}$ is dense in X. Right essential $\mathcal{A}$-modules and (two-sided) essential $\mathcal{A}$-bimodules are defined similarly.

Theorem 3.5. *Let $\mathcal{A}$ be a Banach $\mathcal{O}$-module and Y be a commutative $\mathcal{A}$-$\mathcal{O}$-submodule of $\mathcal{A}^*$, and let $\mathcal{A}/J$ has a left or right identity and $\mathcal{O}$ acts trivially on $\mathcal{A}$ from left. Consider the following statements:*

(i) *Every bounded derivation $D : \mathcal{A} \longrightarrow Y$ is inner;*
(ii) *Every module derivation $D : \mathcal{A} \longrightarrow Y$ is inner;*
(iii) *Every module derivation $D : \mathcal{A}/J \longrightarrow Y$ is inner;*
(iv) *$\mathcal{A}/J$ is weakly amenable;*
(v) *$\mathcal{A}$ is weakly module amenable.*

Then (ii) is equivalent to (iii) and (iv) is equivalent to (v). In addition, if $\mathcal{A}$ is a right essential $\mathcal{O}$-module, then (i) implies (ii).

Proof. Since $\mathcal{A}$ is an essential right $\mathcal{O}$-module then for each $a \in \mathcal{A}$, there is a sequence $(E_n) \subseteq \mathcal{A}\cdot\mathcal{O}$ such that $\lim_n E_n = a$. Suppose that $E_n = \sum_{m=1}^{K_n} a_{n,m} \cdot \alpha_{n,m}$ for some finite sequences $(a_{n,m})_{m=1}^{m=K_n} \subseteq \mathcal{A}$ and $(\alpha_{n,m})_{m=1}^{m=K_n} \subseteq \mathcal{O}$. Let $\lambda \in \mathbb{C}$. Then for each module derivation $D : \mathcal{A} \longrightarrow Y$, we have

$$\begin{aligned} D(\lambda E_n) &= D(\lambda \textstyle\sum_{m=1}^{K_n} a_{n,m} \cdot \alpha_{n,m}) = \sum_{m=1}^{K_n} D(a_{n,m} \cdot (\lambda\alpha_{n,m})) \\ &= \textstyle\sum_{m=1}^{K_n} D(a_{n,m}) \cdot (\lambda\alpha_{n,m}) = \sum_{m=1}^{K_n} \lambda D(a_{n,m} \cdot \alpha_{n,m}) = \lambda D(E_n) \end{aligned}$$

and so by the continuity of D, $D(\lambda a) = \lambda D(a)$. Thus ($i$) implies ($ii$).
(ii) $\Longleftrightarrow$ (iii): The $\mathcal{O}$-commutativity of Y and the compatibility of actions of $\mathcal{A}$ and $\mathcal{O}$ on Y show that $J \cdot Y = Y \cdot J = 0$. Hence, the following module actions

$$(a + J) \cdot y := a \cdot y, \quad y \cdot (a + J) := y \cdot a \quad (y \in Y, a \in \mathcal{A}),$$

are well-defined and so, Y is a Banach $\mathcal{A}/J$-module. Assume that $D : \mathcal{A} \longrightarrow Y$ is a module derivation. Define $\widetilde{D} : \mathcal{A}/J \longrightarrow Y$ via $\widetilde{D}(a + J) = D(a)$. For each $\alpha \in \mathcal{O}$

and $a, b \in \mathcal{A}$ we have

$$\begin{aligned} D((a\cdot\alpha)b - a(\alpha\cdot b)) &= D((a\cdot\alpha)b) - D(a(\alpha\cdot b)) \\ &= D(a\cdot\alpha)\cdot b + (a\cdot\alpha)\cdot D(b) - D(a)\cdot(\alpha\cdot b) - a\cdot D(\alpha\cdot b) \\ &= (D(a)\cdot\alpha)\cdot b - D(a)\cdot(\alpha\cdot b) + (a\cdot\alpha)\cdot D(b) - a\cdot(\alpha\cdot D(b)) = 0. \end{aligned}$$

On the other hand, since J is a closed ideal, the restriction of D to J is zero. Therefore $\widetilde{D}$ is well-defined. By hypothesis, that is inner, and so D is inner. Now, if $D : \mathcal{A}/J \longrightarrow Y$ is a module derivation, then the module derivation $\widetilde{D} : \mathcal{A} \longrightarrow Y$ defined by $\widetilde{D}(a) = D(a + J)$ is inner.
$(v) \Longrightarrow (iv)$: Let $D : \mathcal{A}/J \longrightarrow (\mathcal{A}/J)^* = J^\perp$ and let $\tilde{D} : \mathcal{A} \longrightarrow J^\perp$ be defined by $\tilde{D}(a) = D(a + J)$ for $a \in \mathcal{A}$. By Lemma 3.4, for each $\alpha \in \mathcal{O}$ and $a, b \in \mathcal{A}$, we get

$$\tilde{D}(\alpha\cdot a) = D(\alpha\cdot a + J) = D(f(\alpha)a + J) = f(\alpha)D(a + J) = \alpha\cdot\tilde{D}(a).$$

Similarly, we can obtain $\tilde{D}(a\cdot\alpha) = \tilde{D}(a)\cdot\alpha$. Also for $a, b \in \mathcal{A}$ we have $\tilde{D}(a \pm b) = \tilde{D}(a) \pm \tilde{D}(b)$ and $\tilde{D}(ab) = \tilde{D}(a)\cdot b + a\cdot\tilde{D}(b)$. Thus $\tilde{D}$ is a module derivation. Hence, there exists $y \in Y$ such that $\tilde{D}(a) = a\cdot y - y\cdot a$. Therefore $D(a + J) = (a + J)\cdot y - y\cdot(a + J)$ and so, D is inner.
$(iv) \Longrightarrow (v)$: Since $\mathcal{A}/J$ is a $\mathcal{O}$-module commutative, it is enough to prove that each module derivation $D : \mathcal{A} \longrightarrow J^\perp$ is inner. Now, we define $\widetilde{D} : \mathcal{A}/J \longrightarrow (A/J)^*$ via $\widetilde{D}(a + J) = D(a)$. By the proof of the part $(i) \Longrightarrow (ii)$, $\widetilde{D}$ is $\mathbb{C}$-linear and thus, D is inner. Δ

One should remember that an *inverse semigroup* is a discrete semigroup S such that for each $s \in S$, there is a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. Elements of the form ss^* are called *idempotents* of S and denoted by E_S or E, if there is no risk of confusion. In this section we show that if S is an inverse semigroup with the set of idempotents E, then $\ell^1(S)$ is weakly module amenable (as an $\ell^1(E)$-module). Suppose that S is an inverse semigroup with the set idempotents E, endowed with the partial order

$$e \le d \Longleftrightarrow ed = e \quad (e, d \in E).$$

Theorem V.1.2 of [12] shows that E is a commutative subsemigroup of S . Thus $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions [1]. Here we let $\ell^1(E)$ acts on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e\cdot\delta_s = \delta_s,\ \delta_s\cdot\delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E). \tag{3.1}$$

In this case, the ideal J (see section 2) is the closed linear span of $\{\delta_{set} - \delta_{st} : s, t \in S, e \in E\}$. Now, we consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \ \ (s, t \in S).$$

For an inverse semigroup S, the quotient $S/{\approx}$ is a discrete group (see [3] and [14]). As in [15, Theorem 3.3], we may observe that $\ell^1(S)/J$ is isomorphic to $\ell^1(S/\approx)$. One can see that $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \ (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

It is proved in [4, Theorem 3.1] that if S is a commutative inverse semigroup, then $\ell^1(S)$ is always weakly module amenable as an $\ell^1(E)$-module when $\ell^1(E)$ acts on $\ell^1(S)$ with the usual actions.

Recall that $\mathcal{C}$ is a Clifford semigroup if it is an inverse semigroup in which every idempotent is central, that is $es = se$ for all $s \in \mathcal{C}$ and $e \in E_{\mathcal{C}}$.

Example 3.6. Let $\mathcal{C}$ be a Clifford semigroup. Then $\ell^1(\mathcal{C})$ is $\ell^1(E_{\mathcal{C}})$-module with the following actions:

$$\delta_e \cdot \delta_s = \delta_s \cdot \delta_e = \delta_{es} \quad (s \in \mathcal{C}, e \in E_{\mathcal{C}}).$$

By [4, Theorem 3.1], $\ell^1(\mathcal{C})$ is weakly module amenable. Note that in the proof of [4, Theorem 3.1], it is just used the fact that each idempotent is central.

In the upcoming theorem, we prove Amini and Bagha's result [4, Theorem 3.1] but without the condition of commutativity for S, where we consider $\ell^1(S)$ as an $\ell^1(E)$-module with trivial left action. In fact, we show that the hypothesis on E being upward directed in [2, Corollary 3.5] can be eliminated and $\ell^1(S)$ is always weakly module amenable.

Theorem 3.7. *Let S be an inverse semigroup with the set of idempotents E. Then $\ell^1(S)$ is weakly module amenable.*

Proof. With the actions $\ell^1(E)$ on $\ell^1(S)$ considered in (3.1), $\mathcal{A} = \ell^1(S)$ is always a right essential $\ell^1(E)$-module. Indeed if $f \in \ell^1(S)$, we have

$$f = \sum_{s \in S} f(s)\delta_s = \sum_{s \in S} f(s)\delta_s * \delta_{s^*s} = \sum_{s \in S} f(s)\delta_s \cdot \delta_{s^*s},$$

which belongs to the closed linear span of $\ell^1(S) \cdot \ell^1(E) = \{\delta_s \cdot \delta_e : e \in E, s \in S\}$. Since $S/\approx$ is a discrete group, the group algebra $\ell^1(S/\approx)$ has an identity. By using [8, Theorem 1] and Theorem 3.5 we can obtain the desired result. Δ

Let S be an inverse semigroup with the set of idempotents E. If $C^*(S)$ is the enveloping C^*-algebra of $\ell^1(S)$ (see [9] and [11]), then by continuity of the action of $\ell^1(E)$ on $\ell^1(S)$ extends to an action of $C^*(E)$ on $C^*(S)$, we have the following result.

Theorem 3.8. *Let S be an inverse semigroup with the set of idempotents E, then $C^*(S)$ is weakly module amenable as a $C^*(E)$-module with trivial left action.*

Proof. The C^*-algebra $C^*(S)$ is weakly amenable [7, Theorem 2.1]. Also, it has a bounded approximate identity and so, it is a right essential $C^*(E)$-module. Now, the result follows from the proof of the part $(i) \Longrightarrow (ii)$ of Theorem 3.5. Δ

Example 3.9. Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by a and b, that is

$$\mathcal{C} = \{a^m b^n : m, n \geq 0\}, \ \ (a^m b^n)^* = a^n b^m.$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{a^n b^n : n = 0, 1, ...\}$ with the following order:

$$a^n b^n \leq a^m b^m \Longleftrightarrow m \leq n.$$

It is shown in [3] that $\ell^1(\mathcal{C})$ is $\ell^1(E_{\mathcal{C}})$-module amenable, and so it is weakly module amenable, but it is not even weakly amenable [6, Theorem 2.8].

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The stability of solution for a class of third order nonlinear vector differential equations

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Abstract: In this paper , the stability of trivial solution for a class of third order nonlinear vector differential equations is studied by using the Lyapunov's second method. Some sufficient conditions are given ensuring the stability of the zero solution of this equation.

1. Introduction

In 2007, Yan[1] investigated the stability of solutions to nonlinear third order differential equations:

$$\dddot{x} + a\ddot{x} + f(\dot{x}) + g(x) = 0$$

and

$$\dddot{x} + f(\ddot{x}) + b\ddot{x} + g(x) = 0,$$

and in the same year, Zhang and Si [2] studied asymptotic stability of the zero solution of scalar differential equation

$$\dddot{x} + g(\dot{x})\ddot{x} + f(x,\dot{x}) + h(x) = 0.$$

More recently, Tunç and Ateş[3,4] investigate the stability and boundedness of solutions of non-linear vector differential equations:

$$\dddot{X} + F(X,\dot{X},\ddot{X})\ddot{X} + B(t)\dot{X} + H(X) = P(t,X,\dot{X},\ddot{X})$$

and

$$\dddot{X} + F(t,X,\dot{X},\ddot{X})\ddot{X} + G(\dot{X}) + cX = P(t,X,\dot{X},\ddot{X}),$$

respectively.

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We discusse here stability of the trivial solution $X = 0$ of vector differential equation of the form:

$$\dddot{X} + G(\dot{X})\ddot{X} + F(X, \dot{X}) + H(X) = 0 . \tag{1}$$

The associated system of (1) is

$$\begin{cases} \dot{X} = Y, \dot{Y} = Z \\ \dot{Z} = -G(Y)Z - F(X, Y) - H(X) \end{cases}, \tag{2}$$

where $X \in R^n$, G is a continuous symmetric matrix function; F, H are $n-$vector functions, $F(X,0) = 0,$ and $H(0) = 0$. Let $J_H(X)$ denote the Jacobian matrix corresponding to the function $H(X)$, that is

$$J_H(X) = (\partial h_i / \partial x_j) \;,\; (i, j = 1,2,...,n),$$

where $(x_1, x_2,...,x_n)$ and $(h_1, h_2,...,h_n)$ are the components of X and H, respectively. The symbol $\langle X, Y \rangle$ corresponding to any pair X, Y in R^n stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$.

It should be noted that our equation, (1), is an $n-$ dimensional generalization of equation studied by Zhang and Si [2].

2. Preliminaries

Before introducing our main result, we state some basic information which will be required in future. Consider the non-autonomous differential system

$$\frac{dx}{dt} = F(t, x) , \tag{3}$$

where x is an n-vector, $t \in [0, \infty)$. Suppose that $F(t,x)$ is continuous in (t,x) on $I \times D$, where I denotes the interval $0 \le t < \infty$ and D is a connected open set in $\Re^n$, $\Re^n$ denotes Euclidean n-space.

Theorem 1. Suppose that $F(t,0)=0$ in (3) and there exists a Lyapunov function $V=V(t,x)$ defined on $0\le t<\infty$, $\|x\|<H$, $H>0$, which satisfies the following conditions;

(i) $V(t,0)=0$,

(ii) $c(\|x\|)\le V(t,x)$, where $c(r)\in CIP$ (CIP denotesthe families of continuous increasing and positive definite functions).

(iii) $\dot{V}(t,x)\le 0$.

Then, the solution $x(t)\equiv 0$ of system (3) is stable.

Proof: See Yoshizawa [5].

Lemma. Let A be a real symmetric $n\times n$-matrix. Then for any $X\in\Re^n$

$$\delta_a\|X\|^2\le\langle AX,X\rangle\le\Delta_a\|X\|^2 ,$$

where δ_a and Δ_a are respectively, the least and greatest eigenvalues of the matrix A.

Proof. See Bellman [6].

3. Main result

Theorem 2. Let all the basic assumptions imposed on G, F and H hold. In addition, we assume that there exist positive constants a and b such that the following conditions hold:

$$\frac{a}{2}\le\lambda_i(G(Y)) , \qquad \sum_{i=1}^{n}\frac{b}{2}y_i^{\,2}\le\sum_{i=1}^{n}F_i(X,Y)y_i ,$$

$$0<\lambda_i(J_H(X))\le ab \quad\text{and}\quad \lambda_i(J_F(X,Y)|X)\le 0$$

$(i=1,2,...,n)$.

Then, the solution $X=0$ of Eq. (1) is stable.

Proof. To prove Theorem 2, we introduce a Lyapunov function $V=V(X,Y,Z)$, which is defined by:

$$V(X,Y,Z)=a\int_0^1\langle H(\sigma X),X\rangle d\sigma+\langle H(X),Y\rangle+\frac{1}{2}\langle aY+Z,aY+Z\rangle$$

$$+a\int_0^1\langle\sigma G(\sigma Y)Y,Y\rangle d\sigma-\frac{1}{2}\langle aY,aY\rangle+\int_0^1\langle\sigma F(X,\sigma Y),Y\rangle d\sigma .$$

This function can be rearranged as the following;

$$V(X,Y,Z) = a\int_0^1 \langle H(\sigma X), X\rangle d\sigma - \frac{1}{2b}\langle H(X), H(X)\rangle + \frac{b}{2}\langle Y + \frac{1}{b}H(X), Y + \frac{1}{b}H(X)\rangle$$

$$+\int_0^1 \langle(\sigma F(X,\sigma Y) - \frac{b}{2}IY), Y\rangle d\sigma + \frac{1}{2}\langle aY + Z, aY + Z\rangle$$

$$+\int_0^1 \langle a(G(\sigma Y) - \frac{a}{2}I)Y, Y\rangle d\sigma. \tag{4}$$

It is clear from (4) that

$$V(0,0,0) = 0.$$

On the other hand, since

$$H(0) = 0,\ \frac{\partial}{\partial\sigma}H(\sigma X) = J_H(\sigma X)X,$$

then

$$H(X) = \int_0^1 J_H(\sigma X)X d\sigma.$$

By using the assumption $0 < \lambda_i(J_H(X)) \le ab,$ it follows that

$$a\int_0^1 \langle H(\sigma X), X\rangle d\sigma = a\int_0^1\int_0^1 \langle \sigma_1 J_H(\sigma_1\sigma_2 X)X, X\rangle d\sigma_2 d\sigma_1$$

$$\le a\int_0^1\int_0^1 \langle \sigma_1 abX, X\rangle d\sigma_2 d\sigma_1 = a^2 b\langle X, X\rangle = a^2 b\|X\|^2$$

and

$$\frac{1}{2b}\langle H(X), H(X)\rangle = \frac{1}{2b}\int_0^1 \langle J_H(\sigma X)X, J_H(\sigma X)X\rangle d\sigma$$

$$\le \frac{1}{2}a^2 b\int_0^1 \langle X, X\rangle d\sigma = \frac{1}{2}a^2 b\|X\|^2.$$

In view of Theorem 2 and the above discussion, we obtain

$$V \ge \frac{1}{2}a^2 b\|X\|^2 + \frac{1}{2}\|aY + Z\|^2 + \frac{b}{2}\|Y + b^{-1}H(X)\|^2 \ge 0.$$

Now, let $(X(t), Y(t), Z(t))$ be any solution of system (2).
Differentiating the function $V(X,Y,Z)$ with respect to t along system (2), we get

$$\dot{V} = -\langle G(Y)Z, Z\rangle + \langle aZ, Z\rangle - \langle F(X,Y), aY\rangle + \langle J_H(X)Y, Y\rangle$$

$$+\int_0^1 \langle \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma .$$

Recall that

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle d\sigma = \int_0^1 \sigma\langle J_H(\sigma X)Y, X\rangle d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma$$

$$= \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle H(\sigma X), Y\rangle d\sigma + \int_0^1 \langle H(\sigma X), Y\rangle d\sigma$$

$$= \sigma\langle H(\sigma X), Y\rangle\Big|_0^1 = \langle H(X), Y\rangle ,$$

$$\frac{d}{dt}\int_0^1 \langle \sigma G(\sigma Y)Y, Y\rangle d\sigma = \int_0^1 \langle \sigma G(\sigma Y)Z, Y\rangle d\sigma$$

$$+\int_0^1 \sigma^2\langle J_G(\sigma Y)YZ, Y\rangle d\sigma + \int_0^1 \sigma\langle G(\sigma Y)Y, Z\rangle d\sigma$$

$$= \int_0^1 \langle \sigma G(\sigma Y)Z, Y\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma G(\sigma Y)Z, Y\rangle d\sigma$$

$$= \sigma^2\langle G(\sigma Y)Z, Y\rangle\Big|_0^1 = \langle G(Y)Z, Y\rangle$$

and

$$\frac{d}{dt}\int_0^1 \langle \sigma F(X,\sigma Y), Y\rangle d\sigma = \int_0^1 \langle \sigma (F(X,\sigma Y), Z\rangle d\sigma$$

$$+\int_0^1 \langle \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma + \int_0^1 \langle \sigma^2 J_F((X,\sigma Y) \mid Y)Y, Z\rangle d\sigma$$

$$= \int_0^1 \langle \sigma (F(X,\sigma Y), Z\rangle d\sigma + \int_0^1 \langle \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma$$

$$+\int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle F(X,\sigma Y), Z\rangle d\sigma$$

$$= \sigma\langle F(X,\sigma Y), Z\rangle\Big|_0^1 + \int_0^1 \langle \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma$$

$$= \langle F(X,Y), Z\rangle + \int_0^1 \langle \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma .$$

In view of the assumptions of Theorem 2 and the above discussion, we get,

$$\dot{V} \le -\left[\lambda_i(G(Y)) - a\right] \|Z\|^2 - \left[ab - \lambda_i(J_H(X))\right] \|Y\|^2$$

$$+ \int_0^1 \sigma J_F((X,\sigma Y) \mid X)Y, Y\rangle d\sigma \le 0 .$$

Thus, all the conditions of Theorem 1 holds.
Then, the solution $X = 0$ of Eq. (1) is stable.

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SQUARE-MEAN ALMOST PERIODIC SOLUTIONS TO SOME CLASSES OF NONAUTONOMOUS STOCHASTIC EVOLUTION EQUATIONS WITH FINITE DELAY

PAUL BEZANDRY AND TOKA DIAGANA

ABSTRACT. In this paper, using the well-known Krasnoselskii fixed point theorem, we obtain the existence of a square-mean almost periodic solution to some classes of nonautonomous stochastic evolution equations with finite delay.

1. INTRODUCTION

Let $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t : t \in \mathbb{R}\}$, that is, a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$.

In recent years, the existence of almost periodic (respectively, periodic) solutions to semilinear evolution equations has received much attention by many authors (see, e.g. [3], [8], [16], and [33] and the references therein) due to their significance and applications in physics, mathematical biology, control theory, and others. This interest also arises from a need to extend well-known results on stochastic ordinary differential equations to a class of stochastic partial differential equations.

In their recent paper [9], the authors studied the existence of square-mean almost periodic solutions to the class of nonautonomous stochastic differential equations

$$dX(t) = \mathcal{A}(t)X(t)\,dt + M(t, X(t))\,dt + N(t, X(t))\,d\mathbb{W}(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $(\mathcal{A}(t))_{t\in\mathbb{R}}$ is a family of densely defined closed linear operators, $M : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{H})$ and $N : \mathbb{R} \times L^2(\Omega, \mathbb{H}) \to L^2(\Omega, \mathbb{L}_2^0)$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}$ is a Wiener process. For that, Bezandry and Diagana assumed that the family of linear operators $\mathcal{A}(t)$ satisfy the well-known Acquistapace-Terreni conditions [3], which in fact do guarantee the existence of an evolution family $\mathcal{T} = \{V(t,s)\}_{t\geq s}$ associated with the family of linear operators $\mathcal{A}(t)$. The main result in [9] was then subsequently utilized to study the existence of square-mean almost periodic solutions to some parabolic stochastic partial differential equations.

In this paper, our approach to this problem is somewhat different from that used in [9]. We consider a more general setting, that is, we make extensive use of intermediate space techniques and the Krasnoselskii fixed point theorem to study and obtain the existence of square-mean almost periodic solutions to the class of

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nonautonomous stochastic differential equations with finite delay given by

$$d\Big[X(t) + F_1(t, X_t)\Big] = \mathcal{A}(t)X(t)dt + F_2(t, X_t)dt + F_3(t, X_t)d\mathbb{W}(t) \tag{1.2}$$

where $(\mathcal{A}(t))_{t\in\mathbb{R}}$ is a family of closed linear operators on $\mathbb{H}$ satisfying Acquistapace-Terreni conditions, the history X_t defined by $X_t(\theta) := X(t+\theta)$ for each $\theta \in [-\tau, 0]$, and the functions $F_1 : \mathbb{R} \times L^2(\Omega, C_\tau) \to L^2(\Omega, \mathbb{H}^t_\beta)$ $(0 \le \alpha < 0.5 < \beta < 1)$, $F_i(i = 2,\, 3) : \mathbb{R} \times L^2(\Omega, C_\tau) \mapsto L^2(\Omega, \mathbb{H})$ are jointly continuous satisfying some additional conditions, and $\mathbb{W}$ is a $\mathbb{R}$-valued Brownian motion with the real number line as time parameter. (Here $\mathbb{H}^t_\beta$ is an interpolation space and for $0 \le \alpha < 1$, $C_{\tau,\,\alpha} := C([-\tau, 0], \mathbb{H}_\alpha)$ is the Banach space of all continuous functions from $[-\tau, 0]$ in $\mathbb{H}_\alpha$.)

In view of the above, there exists an evolution family $\mathcal{U} = \{U(t, s)\}_{t\ge s}$ associated with the family of closed linear operators $\mathcal{A}(t)$. Assuming that the evolution family $\mathcal{U} = \{U(t, s)\}_{t\ge s}$ is exponentially dichotomic (hyperbolic) and under some additional assumptions, it will be shown that Eq. (1.2) has a square-mean almost periodic solution. It is worth mentioning that the main result of this paper generalizes, to some extent, most of known results on square-mean almost periodic solutions to autonomous and nonautonomous differential equations, especially those in [8], [9], and [10].

The paper is organized as follows: Section 2 is devoted to preliminaries facts related to the existence of an evolution family. Some preliminary results on intermediate spaces are also stated there. In Section 3, basic definitions and results on the concept of square-mean almost periodic functions are given. In Section 4, we first state a key technical lemma due to Diagana [14] and next make use of it to prove the main result. In Section 5, we illustrate our main result by studying the existence of square-mean almost periodic solutions to the one-dimensional stochastic heat equation with finite delay.

2. Preliminaries

In this section, we introduce some notations and collect some preliminary results from Diagana [14] that will be used later. Throughout the rest of this paper, $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ stands for a real separable Hilbert space, $\mathcal{A}(t)$ for $t \in \mathbb{R}$ is a family of closed operators on $D(\mathcal{A}(t))$ satisfying the Acquistapace-Terreni conditions (see (H.1)). Moreover, the operators $\mathcal{A}(t)$ are not necessarily densely defined. The functions $F_1 : \mathbb{R} \times C_\tau \to \mathbb{H}^t_\beta$ $(0 \le \alpha < 0.5 < \beta < 1)$, and $F_i(i = 2,\, 3) : \mathbb{R} \times C_\tau \to \mathbb{H}$ are respectively jointly continuous satisfying some additional assumptions.

If L is a linear operator on $\mathbb{H}$, then $\rho(L)$, $\sigma(L)$, $D(L)$, $\ker(L)$, $R(L)$ stand respectively for the resolvent set, spectrum, domain, kernel, and range of L. Moreover, one sets $R(\lambda, L) = (\lambda I - L)^{-1}$ for all $\lambda \in \rho(L)$.

Throughout the rest of the paper, we set $Q(t) = I - P(t)$, if $P(t)$ is a family of projections. If $\mathbb{B}_1, \mathbb{B}_2$ are Banach spaces, then the notation $B(\mathbb{B}_1, \mathbb{B}_2)$ stands for the Banach space of bounded linear operators from $\mathbb{B}_1$ into $\mathbb{B}_2$. When $\mathbb{B}_1 = \mathbb{B}_2$, this is simply denoted $B(\mathbb{B}_1)$.

In the present work we study operators $\mathcal{A}(t)$ on $\mathbb{H}$ subject to the following hypotheses.

Hypothesis (H.1). The family of closed linear operators $\mathcal{A}(t)$ for $t \in \mathbb{R}$ on $\mathbb{H}$ with domain $D = D(\mathcal{A}(t))$ (possibly not densely defined) satisfy the so-called Acquistapace and Terreni conditions, that is:

there exist constants $\zeta \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $L > 0$ and $\mu,\ \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subset \rho(\mathcal{A}(t) - \zeta) \ni \lambda, \quad \left\| R\big(\lambda, \mathcal{A}(t) - \zeta\big)\right\| \leq \frac{K}{1 + |\lambda|} \tag{2.1}$$

and

$$\left\| \big(\mathcal{A}(t) - \zeta\big) R\big(\lambda, \mathcal{A}(t) - \zeta\big) \big[R\big(\zeta, \mathcal{A}(t)\big) - R\big(\zeta, \mathcal{A}(s)\big)\big] \right\| \leq L \big|t - s\big|^{\mu} \big|\lambda\big|^{-\nu} \tag{2.2}$$

for $t,\ s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \Big\{\lambda \in \mathbb{C}\backslash\{0\} : \big|\arg\lambda\big| \leq \theta\Big\}$.

In the particular case when $\mathcal{A}(t)$ has a constant domain $\mathcal{D} = D(\mathcal{A}(t))$, Eq. (2.2) can be replaced with the following: there exist constants L and $0 < \mu \leq 1$ such that

$$\left\| \big(\mathcal{A}(t) - \mathcal{A}(s)\big) R\big(\zeta, \mathcal{A}(r)\big) \right\| \leq L \big|t - s\big|^{\mu}, \quad s, t, r \in \mathbb{R}.$$

It should be mentioned that (H.1) was introduced in the literature by Acquistapace and Terreni [3] for $\zeta = 0$. Among other things, it ensures that there exists a unique evolution family $\mathcal{U} = U(t,s)$ on $\mathbb{H}$ satisfying

(a) $U(t,s)U(s,r) = U(t,r)$ for $t \geq s \geq r \in \mathbb{R}$;
(b) $U(t,t) = I$
(c) $(t,s) \to U(t,s) \in B(\mathbb{H})$ is continuous for $t > s$;
(d) $U(\cdot, s) \in C^1((s,\infty), B(\mathbb{H}))$, $\dfrac{\partial U}{\partial t}(t,s) = \mathcal{A}(t)U(t,s)$ and $\left\|\mathcal{A}(t)^k U(t,s)\right\| \leq C(t-s)^{-k}$ for $0 < t - s \leq 1$, $k = 0,\ 1$; and
(e) $\partial_s^+ U(t,s)x = -U(t,s)\mathcal{A}(s)x$ for $t > s$ and $x \in D(\mathcal{A}(s))$ with $\mathcal{A}(s)x \in \overline{D(\mathcal{A}(s)}$.

It should also be mentioned that the above-mentioned properties were mainly established in [1, Theorem 2.3] and [35, Theorem 2.1], see also [3, 34]. In this case we say that $\mathcal{A}(\cdot)$ generates the evolution family $U(\cdot,\cdot)$.

Definition 2.1. We say that an evolution family $\mathcal{U}(t,s)$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t), t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(1) $U(t,s)P(s) = P(t)U(t,s)$,
(2) the restriction $U_Q(t,s) : Q(s)\mathbb{H} \to Q(t)\mathbb{H}$ of $U(t,s)$ is invertible (and we set $\tilde{U}_Q(s,t) := U_Q(t,s)^{-1}$),
(3) $\left\|U(t,s)P(s)\right\| \leq Ne^{-\delta(t-s)}$ and $\left\|\tilde{U}_Q(s,t)Q(t)\right\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

Here and below $Q = I - P$ for a projection P. Under Acquistpace-Terreni conditions, the family of operators defined by

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & \text{if } t \geq s,\ t,\ s \in \mathbb{R} \\ \\ -\tilde{U}_Q(t,s)Q(s), & \text{if } t < s,\ t,\ s \in \mathbb{R} \end{cases}$$

are called Green function corresponding to U and $P(\cdot)$. If $P(t) = I$ for $t \in \mathbb{R}$, then U is exponentially stable.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $\mathcal{A}(t)$. For details, we refer the reader to Lunardi [24].

Definition 2.2. A linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\zeta \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $M > 0$ such that $S_{\theta,\zeta} \subset \rho(\mathcal{A})$,

$$S_{\theta,\zeta} := \left\{\lambda \in \mathbb{C} : \lambda \neq \zeta, \ \left|\arg(\lambda - \zeta)\right| < \theta\right\}$$
$$\text{and } \left\|R(\lambda, \mathcal{A})\right\| \leq \frac{M}{\left|\lambda - \zeta\right|}, \quad \lambda \in S_{\theta,\zeta}$$

where $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}$ for each $\lambda \in \rho(\mathcal{A})$.

Let $\mathcal{A}$ be a sectorial operator on $\mathbb{H}$ and let $\alpha \in (0, 1)$. Define the real interpolation space

$$\mathbb{H}^{\mathcal{A}}_{\alpha} := \left\{x \in \mathbb{H} : \left\|x\right\|^{\mathcal{A}}_{\alpha} := \sup_{r>0} \left\|r^{\alpha}(\mathcal{A} - \zeta)R(r, \mathcal{A} - \zeta)x\right\| < \infty\right\},$$

which is a Banach space when endowed with the norm $\|\cdot\|^{\mathcal{A}}_{\alpha}$. For convenience we further write

$$\mathbb{H}^{\mathcal{A}}_0 := \mathbb{H}, \quad \left\|x\right\|^{\mathcal{A}}_0 := \left\|x\right\|, \quad \mathbb{H}^{\mathcal{A}}_1 := D(\mathcal{A}) \ \text{ and } \ \left\|x\right\|^{\mathcal{A}}_1 := \left\|(\zeta - \mathcal{A})x\right\|.$$

Moreover, let $\hat{\mathbb{H}}^{\mathcal{A}} := \overline{D(\mathcal{A})}$ of $\mathbb{H}$. In particular, we will frequently be using the following continuous embedding

$$D(\mathcal{A}) \hookrightarrow \mathbb{H}^{\mathcal{A}}_{\beta} \hookrightarrow D\big((\zeta - \mathcal{A})^{\alpha}\big) \hookrightarrow \mathbb{H}^{\mathcal{A}}_{\alpha} \hookrightarrow \hat{\mathbb{H}}^{\mathcal{A}} \subset \mathbb{H}, \tag{2.3}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(\mathcal{A})$ is not dense in the spaces $\mathbb{H}^{\mathcal{A}}_{\alpha}$ and $\mathbb{H}$. However, we have the following continuous injection

$$\mathbb{H}^{\mathcal{A}}_{\beta} \hookrightarrow \overline{D(\mathcal{A})}^{\|\cdot\|^{\mathcal{A}}_{\alpha}} \tag{2.4}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $\mathcal{A}(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{H}^t_{\alpha} := \mathbb{H}^{\mathcal{A}(t)}_{\alpha}, \quad \hat{\mathbb{H}}^t := \hat{\mathbb{H}}^{\mathcal{A}(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms.

The embedding in (2.3) hold with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_{\alpha}$ (see [24]) and hence there is a constant $c(\alpha)$ such that

$$\left\|y\right\|^t_{\alpha} \leq c(\alpha)\left\|y\right\|^{1-\alpha}\left\|\mathcal{A}(t)y\right\|^{\alpha}, \quad y \in D(\mathcal{A}(t)). \tag{2.5}$$

we have the following estimates for the evolution family $\mathcal{U}$.

Lemma 2.3. [7, 14] *For $x \in \mathbb{H}$, $0 \leq \alpha \leq 1$, and $t > s$, the following hold*

(i) *There is a constant $c(\alpha)$ such that*

$$\left\|U(t, s)P(s)x\right\|^t_{\alpha} \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t - s)^{-\alpha}\left\|x\right\|. \tag{2.6}$$

(ii) *There is a constant $m(\alpha)$ such that*

$$\left\|\tilde{U}_Q(s, t)Q(s)x\right\|^s_{\alpha} \leq m(\alpha)e^{-\delta(t-s)}\left\|x\right\|. \tag{2.7}$$

In addition to above, we also need the following assumptions:
Hypothesis (H.2). The domain $D(\mathcal{A}(t) = D$ is constant in $t \in \mathbb{R}$. Moreover, the evolution family $\mathcal{U}$ generated by $\mathcal{A}(\cdot)$ has an exponential dichotomy with constants N, $\delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$. Furthermore, $0 \in \rho(\mathcal{A}(t))$ for each $t \in \mathbb{R}$ and the following holds

$$\sup_{t,\, s \in \mathbb{R}} \Big\| \mathcal{A}(s)\mathcal{A}^{-1}(t) \Big\|_{B(\mathbb{H}, \mathbb{H}_\alpha)} < c_0 \,. \tag{2.8}$$

If $0 \le \mu < \alpha < \beta < 1$, then the nonnegative constants $k(\alpha), k, k',$, and k_2 denote respectively the bounds of the embedding $\mathbb{H}_\beta \hookrightarrow \mathbb{H}_\alpha$, $\mathbb{H}_\alpha \hookrightarrow \mathbb{H}$, $\mathbb{H}_\beta \hookrightarrow \mathbb{H}$, and $\mathbb{H}_\mu \hookrightarrow \mathbb{H}$.

3. Square-Mean Almost periodic Stochastic Processes

For details on this section, we refer the reader to [8, 11] and the references therein. In this paper, we assume that $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Let $\mathbb{W}$ be a Brownian motion on $\mathbb{R}$. It is worth mentioning that $\mathbb{W}$ can be obtained as follows: let $\{\mathbb{W}_i(t),\ t \in \mathbb{R}_+\},\ \ i = 1, 2$, be independent $\mathbb{R}$-valued Brownian motions, then

$$\mathbb{W}(t) = \begin{cases} \mathbb{W}_1(t) & \text{if } t \ge 0, \\ \\ \mathbb{W}_2(-t) & \text{if } t \le 0, \end{cases}$$

is a Brownian motion with the real number line as time parameter. We then let $\mathcal{F}_t = \sigma\{\mathbb{W}(s),\ s \le t\}$.

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space. This setting requires the following preliminary definitions.

Definition 3.1. A stochastic process $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ is said to be continuous whenever

$$\lim_{t \to s} \mathbf{E} \Big\| X(t) - X(s) \Big\|^2 = 0.$$

Definition 3.2. A continuous stochastic process $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ is said to be square-mean almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} \mathbf{E} \Big\| X(t + \tau) - X(t) \Big\|^2 < \varepsilon.$$

The collection of all stochastic processes $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$ which are quadratic mean almost periodic is then denoted by $AP(\mathbb{R}; L^2(\Omega; \mathbb{B}))$.

The next lemma provides some properties of square-mean almost periodic processes.

Lemma 3.3. *If X belongs to $AP(\mathbb{R}; L^2(\Omega; \mathbb{B}))$, then*

(i) *the mapping $t \to \mathbf{E}\big\|X(t)\big\|^2$ is uniformly continuous;*
(ii) *there exists a constant $M > 0$ such that $\mathbf{E}\big\|X(t)\big\|^2 \le M$, for all $t \in \mathbb{R}$.*

Let $\mathbf{CUB}(\mathbb{R};\, L^2(\Omega; \mathbb{B}))$ denote the collection of all stochastic processes $X : \mathbb{R} \to L^2(\Omega; \mathbb{B})$, which are continuous and uniformly bounded. It is then easy to check that $\mathbf{CUB}(\mathbb{R};\, L^2(\Omega; \mathbb{B}))$ is a Banach space when it is equipped with the norm:

$$\big\|X\big\|_\infty = \sup_{t \in \mathbb{R}} \Big(\mathbf{E}\|X(t)\|^2 \Big)^{\frac{1}{2}}.$$

Lemma 3.4. *$AP(\mathbb{R}; L^2(\Omega;\mathbb{B})) \subset \mathbf{CUB}(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ is a closed subspace.*

In view of the above, the space $AP(\mathbb{R}; L^2(\Omega;\mathbb{B}))$ of square-mean almost periodic processes equipped with the norm $\|\cdot\|_\infty$ is a Banach space.

Let $(\mathbb{B}_1, \|\cdot\|_{\mathbb{B}_1})$ and $(\mathbb{B}_2, \|\cdot\|_{\mathbb{B}_2})$ be Banach spaces and let $L^2(\Omega;\mathbb{B}_1)$ and $L^2(\Omega;\mathbb{B}_2)$ be their corresponding L^2-spaces, respectively.

Definition 3.5. A function $F : \mathbb{R} \times L^2(\Omega;\mathbb{B}_1) \to L^2(\Omega;\mathbb{B}_2))$, $(t, Y) \mapsto F(t, Y)$, which is jointly continuous, is said to be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$ where $\mathbb{K} \subset L^2(\Omega;\mathbb{B}_1)$ is any compact subset if for any $\varepsilon > 0$, there exists $l(\varepsilon,\mathbb{K}) > 0$ such that any interval of length $l(\varepsilon,\mathbb{K})$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} \mathbf{E}\Big\|F(t+\tau, Y) - F(t, Y)\Big\|^2_{\mathbb{B}_2} < \varepsilon$$

for each stochastic process $Y : \mathbb{R} \to \mathbb{K}$.

Theorem 3.6. *Let $F : \mathbb{R} \times L^2(\Omega;\mathbb{B}_1) \to L^2(\Omega;\mathbb{B}_2)$, $(t, Y) \mapsto F(t, Y)$ be a square-mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in \mathbb{K}$, where $\mathbb{K} \subset L^2(\Omega;\mathbb{B}_1)$ is compact. Suppose that F is Lipschitz in the following sense:*

$$\mathbf{E}\Big\|F(t, Y) - F(t, Z)\Big\|^2_{\mathbb{B}_2} \le M\ \mathbf{E}\Big\|Y - Z\Big\|^2_{\mathbb{B}_1}$$

for all $Y, Z \in L^2(\Omega;\mathbb{B}_1)$ and for each $t \in \mathbb{R}$, where $M > 0$. Then for any square-mean almost periodic process $\Phi : \mathbb{R} \to L^2(\Omega;\mathbb{B}_1)$, the function $t \mapsto F(t, \Phi(t))$ is square-mean almost periodic.

The present setting requires the following composition of square-mean almost periodic processes.

Theorem 3.7. *Let $F : \mathbb{R} \times L^2(\Omega;\mathbb{B}_1) \to L^2(\Omega;\mathbb{B}_2)$, $(t, Y) \mapsto F(t, Y)$ be a square-mean almost periodic process in $t \in \mathbb{R}$ uniformly in $Y \in K$, where $K \subset L^2(\Omega;\mathbb{B}_1)$ is any compact subset. Suppose that $F(t, \cdot)$ is uniformly continuous on bounded subsets $K' \subset L^2(\Omega;\mathbb{B}_1)$ in the following sense: for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $X, Y \in K'$ and $\mathbf{E}\Big\|X - Y\Big\|^2_{\mathbb{B}_1} < \delta_\varepsilon$, then*

$$\mathbf{E}\Big\|F(t, X) - F(t, Y)\Big\|^2_{\mathbb{B}_2} < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Then for any square-mean almost periodic process $\Phi : \mathbb{R} \to L^2(\Omega;\mathbb{B}_1)$, the function $t \mapsto F(t, \Phi(t))$ is square-mean almost periodic.

Proof. Since $\Phi : \mathbb{R} \to L^2(\Omega;\mathbb{B}_1)$ is a square-mean almost periodic process, for all $\varepsilon > 0$ there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains a τ with the property that

$$\mathbf{E}\Big\|\Phi(t+\tau) - \Phi(t)\Big\|^2_{\mathbb{B}_1} < \varepsilon, \quad \forall t \in \mathbb{R}. \tag{3.1}$$

In addition, $\Phi : \mathbb{R} \to L^2(\Omega;\mathbb{B}_1)$ is bounded, that is, $\sup_{t\in R} \mathbf{E}\Big\|\Phi(t)\Big\|^2_{\mathbb{B}_1} < \infty$. Let $K'' \subset L^2(\Omega;\mathbb{B}_1)$ be a bounded subset such that $\Phi(t) \in K''$ for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}\mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 &\leq 2\mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t+\tau,\Phi(t))\Big\|_{\mathbb{B}_2}^2\\ &+ 2\mathbf{E}\Big\|F(t+\tau,\Phi(t))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2.\end{aligned}$$

Taking into account Eq. (3.1) (take $\delta_\varepsilon = \varepsilon$) and using the uniform continuity of F on bounded subsets of $L^2(\Omega;\mathbb{B}_1)$ it follows that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t+\tau,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \frac{\varepsilon}{4}. \tag{3.2}$$

Similarly, using the square-mean almost periodicity of F it follows that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\Big\|F(t+\tau,\Phi(t))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \frac{\varepsilon}{4}. \tag{3.3}$$

Combining Eq. (3.2) and Eq. (3.3) one obtains that

$$\sup_{t\in\mathbb{R}} \mathbf{E}\Big\|F(t+\tau,\Phi(t+\tau))-F(t,\Phi(t))\Big\|_{\mathbb{B}_2}^2 < \varepsilon,$$

and hence the function $t \mapsto F(t,\Phi(t))$ is square-mean almost periodic. □

4. Existence of Square-Mean Almost Periodic Solutions

This section is devoted to the existence of a square-mean almost periodic solution to the nonautonomous stochastic differential equation Eq. (1.2).

In the sequel, we denote for brevity that for $0 \leq \alpha < 1$, $C_{\tau,\,\alpha} = C([-\tau,0],\mathbb{H}_\alpha)$, the space of all continuous functions from $[-\tau,0]$ into $\mathbb{H}_\alpha$, equipped with the sup norm given by

$$\|z\|_{\tau,\,\alpha} := \Big[\sup_{-\tau\leq\theta\leq 0}\|z(\theta)\|_\alpha^2\Big]^{1/2}.$$

Definition 4.1. A continuous random function, $X : \mathbb{R} \mapsto L^2(\Omega; C_{\tau,\,\alpha})$ is said to be a bounded solution of Eq.(1.2) on $\mathbb{R}$ provided that $s \to \mathcal{A}(s)\Gamma(t,s)F_1(s,X_s)$ is integrable on $(-\infty,t)$ and, $s \to \mathcal{A}(s)\Gamma(t,s)F_1(s,X_s)$ is integrable on (t,∞), and

$$\begin{aligned}X(t) &= -F_1(t,X_t) - \int_{-\infty}^t \mathcal{A}(s)\Gamma(t,s)\,F_1(s,X_s)\,ds + \int_t^\infty \mathcal{A}(s)\Gamma(t,s)\,F_1(s,X_s)\,ds\\ &\quad + \int_{-\infty}^t \Gamma(t,s)\,F_2(s,X_s)\,ds - \int_t^\infty \Gamma(t,s)\,F_2(s,X_s)\,ds\\ &\quad + \int_{-\infty}^t \Gamma(t,s)\,F_3(s,X_s)\,d\mathbb{W}(s) - \int_t^\infty \Gamma(t,s)\,F_3(s,X_s)\,d\mathbb{W}(s)\end{aligned}$$

for each $t \geq s$ and for all $t,s \in \mathbb{R}$.

In this paper, Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 stand respectively for the nonlinear integral operators defined by

$$(\Gamma_1 X)(t) := \int_{-\infty}^t \mathcal{A}(s)\Gamma(t,s)\,\Psi_1(s)\,ds,\quad (\Gamma_2 X)(t) := \int_t^\infty \mathcal{A}(s)\Gamma(t,s)\,\Psi_1(s)\,ds,$$

$$(\Gamma_3 X)(t) := \int_{-\infty}^t \Gamma(t,s)\,\Psi_2(s)\,ds,\quad (\Gamma_4 X)(t) := \int_t^\infty \Gamma(t,s)\,\Psi_2(s)\,ds,$$

$$(\Gamma_5 X)(t) := \int_{-\infty}^{t} \Gamma(t,s)\, \Psi_3(s)\, d\mathbb{W}(s), \quad (\Gamma_6 X)(t) := \int_{t}^{\infty} \Gamma(t,s)\, \Psi_3(s)\, d\mathbb{W}(s),$$

where $\Psi_1(s) = F_1(s, X_s)$, $\Psi_2(s) = F_2(s, X_s)$, and $\Psi_3(s) = F_3(s, X_s)$.

To discuss the existence of square-mean almost periodic solution to Eq. (1.2) we need some additional assumptions. First of all, note that for $0 < \alpha < \beta < 1$, then

$$L^2(\Omega, \mathbb{H}_\beta) \hookrightarrow L^2(\Omega, \mathbb{H}_\alpha) \hookrightarrow L^2(\Omega; \mathbb{H})$$

are continuously embedded and hence therefore exist constants $k_1 > 0$, $k(\alpha) > 0$ such that

$$\mathbf{E}\big\|X\big\|^2 \leq k_1 \mathbf{E}\big\|X\big\|_\alpha^2 \text{ for each } X \in L^2(\Omega, \mathbb{H}_\alpha) \text{ and}$$

$$\mathbf{E}\big\|X\big\|_\alpha^2 \leq k(\alpha)\mathbf{E}\big\|X\big\|_\beta^2 \text{ for each } X \in L^2(\Omega, \mathbb{H}_\beta).$$

(H.3) $R(\zeta, \mathcal{A}(\cdot)) \in B\Big(AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))\Big)$. Moreover, there exists a function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma \in L^1[0,\infty)$ such that for every $\varepsilon > 0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a τ with the property

$$\Big\|\mathcal{A}(t+\tau)\Gamma(t+\tau, s+\tau) - \mathcal{A}(t)\Gamma(t,s)\Big\|_{B(\mathbb{H}_\alpha, \mathbb{H})} \leq \varepsilon\gamma(\big|t-s\big|)$$

for all $s, t \in \mathbb{R}$.

(H.4) Let $0 \leq \alpha < 0.5 < \beta < 1$. Let $F_1 : \mathbb{R} \times L^2(\Omega; C_{\tau,\alpha}) \to L^2(\Omega, \mathbb{H}_\beta)$ be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}$ ($\mathcal{O} \subset L^2(\Omega; C_{\tau,\alpha})$ being any compact subset). Moreover, F_1 is Lipschitz in the following sense: there exists $K_1 > 0$ for which

$$\mathbf{E}\Big\|F_1(t,X) - F_1(t,Y)\Big\|_\beta^2 \leq\ K_1\, \mathbf{E}\Big\|X - Y\Big\|_{\tau,\alpha}^2$$

for all random variables $X, Y \in L^2(\Omega; C_{\tau,\alpha})$ and $t \in \mathbb{R}$;

(H.5) Let $F_i (i = 2,3) : \mathbb{R} \times L^2(\Omega; C_{\tau,\alpha}) \to L^2(\Omega, \mathbb{H})$ be square-mean almost periodic in $t \in \mathbb{R}$ uniformly in $X \in \mathcal{O}_i$ ($\mathcal{O}_i \subset L^2(\Omega; C_{\tau,\alpha})$) being any compact subset). Moreover, $F_i(\mathbb{R} \times B)$ is precompact for each bounded subset B of $L^2(\Omega; C_{\tau,\alpha})$, and locally uniformly continuous, that is, for each r, $\varepsilon > 0$, there is $\eta(r,\varepsilon)$ such that $\mathbf{E}\|F_i(t,X) - F_i(t,Y)\|^2 \leq \varepsilon$ whenever $t \in \mathbb{R}$ and X, $Y \in L^2(\Omega; C_{\tau,\alpha})$ with $\mathbf{E}\|X\|_{\tau,\alpha}^2 < r$, $\mathbf{E}\|Y\|_{\tau,\alpha}^2 < r$ and $\mathbf{E}\|X - Y\|_{\tau,\alpha}^2 < \eta$;

(H.6) For $i = 2$, 3 and for any $\varepsilon > 0$, there is $a > 0$ such that $\mathbf{E}\|F_i(t,X)\|^2 \leq \varepsilon\mathbf{E}\|X\|_{\tau,\alpha}^2$ for all $t \in \mathbb{R}$ and $X \in L^2(\Omega; C_{\tau,\alpha})$ with $\mathbf{E}\|X\|_{\tau,\alpha}^2 \geq a$.

The main result of the present paper will be based upon the use of the well-known fixed point theorem of Krasnoselskii given as follows:

Theorem 4.2. *Let C be a closed, convex, and nonempty subset of a Banach space $\mathcal{B}$. Suppose the (possibly nonlinear) operators L and M map C into $\mathcal{B}$ satisfying*

(a) *for all $u, v \in C$, then $Lu + Mv \in C$;*
(b) *the operator L is a contraction;*
(c) *the operator M is continuous and $M(C)$ is contained in a compact set.*

Then there exists $u \in C$ such that $u = Lu + Mu$.

To prove the main result (Theorem 4.9) we need the technical Lemma 4.3 due to Diagana. For the sake of clarity and completeness, we reproduce its proof here.

Lemma 4.3. [14, Diagana] *Under assumptions* (H.1)—(H.2)*, if* $0 \leq \mu < \alpha < \beta < 1$ *with* $2\alpha > \mu + 1$*, then there exist two constants* $m(\alpha, \beta), n(\alpha, \mu) > 0$ *such that*

$$\left\|\mathcal{A}(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\alpha \leq m(\alpha,\beta)e^{-\delta(s-t)}\left\|x\right\|_\beta \qquad \text{for } t \leq s, \tag{4.1}$$

$$\left\|\mathcal{A}(s)U(t,s)P(s)x\right\|_\alpha \leq n(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\left\|x\right\|_\beta, \qquad \text{for } t > s. \tag{4.2}$$

Proof. Let $x \in \mathbb{H}_\beta$. Since the restriction of $\mathcal{A}(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\left\|\mathcal{A}(s)\widetilde{U}_Q(t,s)Q(s)x\right\|_\alpha &\leq ck(\alpha)\left\|\widetilde{U}_Q(t,s)Q(s)x\right\|_\beta \\
&\leq ck(\alpha)m(\beta)e^{\delta(s-t)}\left\|x\right\| \\
&\leq m(\alpha,\beta)e^{\delta(s-t)}\left\|x\right\|_\beta
\end{aligned}$$

for $t \leq s$ by using Eq. (2.7).

Similarly, for each $x \in \mathbb{H}_\beta$, using (H.2), we obtain

$$\begin{aligned}
\left\|\mathcal{A}(s)U(t,s)P(s)x\right\|_\alpha &= \left\|\mathcal{A}(s)\mathcal{A}(t)^{-1}\mathcal{A}(t)U(t,s)P(s)x\right\|_\alpha \\
&\leq \left\|\mathcal{A}(s)\mathcal{A}(t)^{-1}\right\|_{B(\mathbb{H},\mathbb{H}_\alpha)}\left\|\mathcal{A}(t)U(t,s)P(s)x\right\| \\
&\leq c_0\left\|\mathcal{A}(t)U(t,s)P(s)x\right\| \\
&\leq c_0k\left\|\mathcal{A}(t)U(t,s)P(s)x\right\|_\alpha
\end{aligned}$$

for $t \geq s$.

First of all, note that $\left\|\mathcal{A}(t)U(t,s)\right\|_{B(\mathbb{H},\mathbb{H}_\alpha)} \leq N'(t-s)^{-(1-\alpha)}$ for all t, s such that $0 < t - s \leq 1$ and $\alpha \in [0,1]$.

Letting $t - s \geq 1$, we obtain

$$\begin{aligned}
\left\|\mathcal{A}(t)U(t,s)P(s)x\right\|_\alpha &= \left\|\mathcal{A}(t)U(t,t-1)U(t-1,s)P(s)x\right\|_\alpha \\
&\leq \left\|\mathcal{A}(t)U(t,t-1)\right\|_{B(\mathbb{H},\mathbb{H}_\alpha)}\left\|U(t-1,s)P(s)x\right\| \\
&\leq NN'e^{\delta}e^{-\delta(t-s)}\left\|x\right\| \\
&\leq K_1e^{-\delta(t-s)}\left\|x\right\|_\beta \\
&= K_1e^{-\frac{3\delta}{4}(t-s)}(t-s)^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\left\|x\right\|_\beta.
\end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^\alpha \to 0$ as $t \to \infty$ it follows that there exists $c_4(\alpha) > 0$ such that

$$\left\|\mathcal{A}(t)U(t,s)P(s)x\right\|_\alpha \leq c_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\left\|x\right\|_\beta$$

and hence

$$\left\|\mathcal{A}(s)U(t,s)P(s)x\right\|_\alpha \leq c_0kc_4(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\left\|x\right\|_\beta$$

for all $t, s \in \mathbb{R}$ such that $t - s > 1$.

Now, let $0 < t - s \leq 1$. Using Eq. (2.6) and the fact $2\alpha > \mu + 1$, we obtain

$$\begin{aligned}
\Big\|\mathcal{A}(t)U(t,s)P(s)x\Big\|_\alpha &= \Big\|\mathcal{A}(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\Big\|_\alpha \\
&\leq \Big\|\mathcal{A}(t)U(t,\frac{t+s}{2})\Big\|_{B(\mathbb{H},\mathbb{H}_\alpha)}\Big\|U(\frac{t+s}{2},s)P(s)x\Big\| \\
&\leq k_2\Big\|\mathcal{A}(t)U(t,\frac{t+s}{2})\Big\|_{B(\mathbb{H},\mathbb{H}_\alpha)}\Big\|U(\frac{t+s}{2},s)P(s)x\Big\|_\mu \\
&\leq k_2N'\Big(\frac{t-s}{2}\Big)^{\alpha-1}c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu}e^{-\frac{\delta}{4}(t-s)}\Big\|x\Big\| \\
&\leq k_2k'N'\Big(\frac{t-s}{2}\Big)^{\alpha-1}c(\mu)\Big(\frac{t-s}{2}\Big)^{-\mu}e^{-\frac{\delta}{4}(t-s)}\Big\|x\Big\|_\beta \\
&\leq c_5(\alpha,\mu)(t-s)^{\alpha-1-\mu}e^{-\frac{\delta}{4}(t-s)}\Big\|x\Big\|_\beta \\
&\leq c_5(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\Big\|x\Big\|_\beta.
\end{aligned}$$

Therefore there exists $n(\alpha,\mu) > 0$ such that

$$\Big\|\mathcal{A}(t)U(t,s)P(s)x\Big\|_\alpha \leq n(\alpha,\mu)(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\Big\|x\Big\|_\beta$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. □

Lemma 4.4. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5)*, the integral operators* Γ_1 *and* Γ_2 *defined above map* $AP(\mathbb{R}; L^2(\Omega, \mathbb{H}_\alpha))$ *into itself.*

Proof. Let $X \in AP(\mathbb{R}; L^2(\Omega; C_{\tau,\,\alpha}))$. Since $t \mapsto X_t$ is square-mean almost periodic, using Theorem 3.6 it follows that $\Psi_1(\cdot) = F_1(\cdot, X_t(\cdot))$ is in $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$ whenever $X \in AP(\mathbb{R}; L^2(\Omega; C_{\tau,\alpha}))$.

Let us show that $\Gamma_1 X \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Indeed, since $\Psi_1 \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\beta))$, for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for all ξ there is $\tau \in [\xi, \xi + l(\varepsilon)]$ such that

$$\mathbf{E}\|\Psi_1(t+\tau) - \Psi_1(t)\|_\beta^2 < \nu^2\varepsilon \ \text{ for each } \ t \in \mathbb{R},$$

where $\nu = \dfrac{\delta^{1-\alpha}}{n(\alpha,\mu)\Gamma(1-\alpha)4^{1-\alpha}}$ ($\Gamma(\cdot)$ being the classical gamma function).

Now

$$\begin{aligned}
&\mathbf{E}\Big\|\Gamma_1 X(t+\tau) - \Gamma_1 X(t)\Big\|_\alpha^2 \\
\leq\ & 2\mathbf{E}\Big\|\int_{-\infty}^t \mathcal{A}(s+\tau)\Gamma(t+\tau,s+\tau)[\Psi_1(s+\tau) - \Psi_1(s)]\,ds\Big\|_\alpha^2 \\
&+2\mathbf{E}\Big\|\int_{-\infty}^t \big[\mathcal{A}(s+\tau)\Gamma(t+\tau,s+\tau) - \mathcal{A}(s)\Gamma(t,s)\big]\Psi_1(s)\,ds\Big\|_\alpha^2 \\
\leq\ & 2L_1 + 2L_2.
\end{aligned}$$

Using the estimate in Eq. (4.2) yields

$$\begin{aligned}
L_1 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t}\Big\|\mathcal{A}(s+\tau)\Gamma(t+\tau,s+\tau)[\Psi_1(s+\tau)-\Psi_1(s)]\Big\|\,ds\Big\}^2\\
&\leq n(\alpha,\mu)^2\,\mathbf{E}\Big\{\int_{-\infty}^{t}(t-s)^{-\alpha}\,e^{-\frac{\delta}{4}(t-s)}\Big\|\Psi_1(s+\tau)-\Psi_1(s)\Big\|\,ds\Big\}^2\\
&\leq n(\alpha,\mu)^2\Big(\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\,ds\Big)^2\sup_t\mathbf{E}\Big\|\Psi_1(t+\tau)-\Psi_1(t)\Big\|_\beta^2\\
&\leq n(\alpha,\mu)^2\Big(\frac{\Gamma(1-\alpha)4^{1-\alpha}}{\delta^{1-\alpha}}\Big)^2\sup_t\mathbf{E}\Big\|\Psi_1(t+\tau)-\Psi_1(t)\Big\|_\beta^2<\varepsilon.
\end{aligned}$$

Similarly, using assumption (H.3), it follows that

$$\begin{aligned}
L_2 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t}\Big\|\mathcal{A}(s+\tau)\Gamma(t+\tau,s+\tau)-\mathcal{A}(s)\Gamma(t,s)\Big\|\,\Big\|\Psi_1(s)\Big\|_\alpha\,ds\Big\}^2\\
&\leq \varepsilon^2\Big\{\int_{-\infty}^{t}\gamma(t-s)\,ds\Big\}\Big\{\int_{-\infty}^{t}\gamma(t-s)\mathbf{E}\Big\|\Psi_1(s)\Big\|_\alpha^2\,ds\Big\}\\
&\leq \varepsilon^2\Big(\int_{-\infty}^{t}\gamma(t-s)\,ds\Big)^2\sup_t\mathbf{E}\Big\|\Psi_1(s)\Big\|_\alpha^2=\varepsilon^2k(\alpha)\big\|\gamma\big\|_{L^1}^2\,K_{\Psi_1}\,.
\end{aligned}$$

Therefore,

$$\mathbf{E}\Big\|\Gamma_1X(t+\tau)-\Gamma_1X(t)\Big\|_\alpha^2\leq\Big(2+2k(\alpha)\big\|\gamma\big\|_{L^1}^2\,K_{\Psi_1}\varepsilon\Big)\varepsilon,$$

for each $t\in\mathbb{R}$, and hence $\Gamma_1X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$.

The proof for $\Gamma_2X(\cdot)$ is omitted as it follows along the same line as that of Γ_1X. □

Lemma 4.5. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5)*, the integral operators* Γ_3 *and* Γ_4 *defined above map* $AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. The proof for the square-mean almost periodicity of Γ_4X is similar to that of Γ_3X and hence will be omitted. Note, however, that for Γ_4X, we make use of Eq. (2.7) rather than Eq. (2.6).

Let $X\in AP(\mathbb{R};L^2(\Omega,C_{\tau,\alpha}))$. Clearly, $X_t\in AP(\mathbb{R};L^2(\Omega,C_{\tau,\alpha}))$. Setting $\Psi_2(t)=F_2(t,X_t)$ and using Theorem 3.7 it follows that $\Psi_2\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}))$.

We now show that $\Gamma_3X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$. Indeed, since $\Psi_2\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathbf{E}\Big\|\Psi_2(t+\tau)-\Psi_2(t)\Big\|^2<\kappa^2\cdot\varepsilon\ \text{ for each }\ t\in\mathbb{R},$$

where $\kappa=\dfrac{\delta^{1-\alpha}}{c(\alpha)2^{1-\alpha}\Gamma(1-\alpha)}$ ($\Gamma(\cdot)$ being the classical gamma function).

Now

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_3X)(t+\tau)-(\Gamma_3X)(t)\Big\|_\alpha^2\\
&= \mathbf{E}\Big\|\int_{-\infty}^{t}\Gamma(t+\tau,s+\tau)\Psi_2(s+\tau)\,ds-\int_{-\infty}^{t}\Gamma(t,s)\Psi_2(s)\,ds\Big\|_\alpha^2\\
&\leq 3\mathbf{E}\Big\|\int_0^\infty \Gamma(t+\tau,t-s+\tau)[\Psi_2(t-s+\tau)-\Psi_2(t-s)]\,ds\Big\|_\alpha^2\\
&\quad +3\mathbf{E}\Big\|\int_\varepsilon^\infty \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_2(t-s)\,ds\Big\|_\alpha^2\\
&\quad +3\mathbf{E}\Big\|\int_0^\varepsilon \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_2(t-s)\,ds\Big\|_\alpha^2\\
&\leq 3L_1'+3L_2'+3L_3'.
\end{aligned}$$

Using Eq. (2.6), it follows that

$$\begin{aligned}
L_1' &\leq \mathbf{E}\Big\{\int_0^\infty \Big\|\Gamma(t+\tau,t-s+\tau)[\Psi_2(t-s+\tau)-\Psi_2(t-s)]\Big\|_\alpha ds\Big\}^2\\
&\leq c(\alpha)^2\mathbf{E}\Big\{\int_0^\infty s^{-\alpha}e^{-\frac{\delta}{2}s}\Big\|\Psi_2(t-s+\tau)-\Psi_2(t-s)\Big\|\,ds\Big\}^2\\
&\leq c(\alpha)^2\Big(\int_0^\infty s^{-\alpha}e^{-\frac{\delta}{2}s}\,ds\Big)^2\sup_t \mathbf{E}\Big\|\Psi_2(t+\tau)-\Psi_2(t)\Big\|\\
&\leq c(\alpha)^2\Big(\frac{\Gamma(1-\alpha)2^{1-\alpha}}{\delta^{1-\alpha}}\Big)^2\sup_t \mathbf{E}\Big\|\Psi_2(t+\tau)-\Psi_2(t)\Big\| \leq \varepsilon\,.
\end{aligned}$$

For L_2', we use [25, Proposition 4.4] to obtain

$$\begin{aligned}
L_2' &\leq \mathbf{E}\Big\{\int_\varepsilon^\infty \Big\|\big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_2(t-s)\Big\|_\alpha ds\Big\}^2\\
&\leq \frac{4}{\delta^2}\,\varepsilon^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2.
\end{aligned}$$

The evaluation of the last term is straightforward. We obtain:

$$\begin{aligned}
L_3' &\leq \mathbf{E}\Big\{\int_0^\varepsilon \Big\|\big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_2(t-s)\Big\|_\alpha ds\Big\}^2\\
&\leq 4\,M^2\,\varepsilon^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2.
\end{aligned}$$

Combining these evaluations, we conclude that $\Gamma_3X \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. □

Lemma 4.6. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5)*, the integral operators* Γ_5 *and* Γ_6 *defined above map* $AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$ *into itself.*

Proof. Let $X \in AP(\mathbb{R};L^2(\Omega;C_{\tau,\alpha}))$. Setting $\Psi_3(t)=F_3(t,X_t)$ and using Theorem 3.7 it follows that $\Psi_3 \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$. We claim that $\Gamma_5X \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$. Indeed, since $\Psi_3 \in AP(\mathbb{R};L^2(\Omega;\mathbb{H}))$, for every $\varepsilon>0$ there exists $l(\varepsilon)>0$ such that for all ξ there is $\tau\in[\xi,\xi+l(\varepsilon)]$ with

$$\mathbf{E}\|\Psi_3(t+\tau)-\Psi_3(t)\|^2 < \zeta\cdot\varepsilon \ \text{ for each } \ t\in\mathbb{R}, \tag{4.3}$$

where $\zeta = \dfrac{\delta^{1-2\alpha}}{c(\alpha)^2\,\Gamma(1-2\alpha)}$.

Now using the expression

$$(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t) = \int_{-\infty}^{t} \Gamma(t+\tau, s+\tau)\Psi_3(s+\tau)d\mathbb{W}(s)-\int_{-\infty}^{t} \Gamma(t,s)\Psi_3(s)d\mathbb{W}(s)$$

it follows that

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_5 X)(t+\tau)-(\Gamma_5 X)(t)\Big\|_\alpha^2\\
&\leq\ 3\mathbf{E}\Big\|\int_0^\infty \Gamma(t+\tau, t-s+\tau)[\Psi_3(t-s+\tau)-\Psi_3(t-s)]\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\quad +3\mathbf{E}\Big\|\int_\varepsilon^\infty \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\quad +3\mathbf{E}\Big\|\int_0^\varepsilon \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\leq\ 3L_1''+3L_2''+3L_3''.
\end{aligned}$$

Now,

$$\begin{aligned}
L_1'' &= \mathbf{E}\Big\|\int_0^\infty \Gamma(t+\tau,t-s+\tau)\big[\Psi_3(t-s+\tau)-\Psi_3(t-s)\big]\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\leq c(\alpha)^2\int_0^\infty s^{-2\alpha}e^{-\delta s}\mathbf{E}\Big\|\Psi_3(t-s+\tau)-\Psi_3(t-s)\Big\|^2\,ds\\
&\leq c(\alpha)^2\Big(\int_0^\infty s^{-2\alpha}e^{-\delta s}\,ds\Big)\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t+\tau)-\Psi_3(t)\Big\|^2\\
&\leq c(\alpha)^2\frac{\Gamma(1-2\alpha)}{\delta^{1-2\alpha}}\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t+\tau)-\Psi_3(t)\Big\|^2<\varepsilon\,.
\end{aligned}$$

For L_2'', using [25, Proposition 4.4], it follows that

$$\begin{aligned}
L_2'' &= \mathbf{E}\Big\|\int_\varepsilon^\infty \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\leq \frac{C(\alpha)}{\delta}\,\varepsilon^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t)\Big\|^2\,.
\end{aligned}$$

As to L_3'', it is straightforward. We obtain

$$\begin{aligned}
L_3'' &= \mathbf{E}\Big\|\int_0^\varepsilon \big[\Gamma(t+\tau,t-s+\tau)-\Gamma(t,t-s)\big]\Psi_3(t-s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\leq 4\,C(\alpha)\,M^2\,\varepsilon\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(t)\Big\|^2\,.
\end{aligned}$$

Combining these evaluations, we conclude that $\Gamma_5 X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha))$.

The proof for $\Gamma_6 X(\cdot)$ is similar to that of $\Gamma_5 X(\cdot)$ except that Eq. (2.7) and Eq. (4.1) are used instead of Eq. (2.6) and Eq. (4.2), respectively. □

Consider the nonlinear operator Ξ on the space $\Big(AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha)),\|\cdot\|_{\infty,\alpha}\Big)$ defined by

$$\Xi X=\Xi_1 X+\Xi_2 X\ \text{ for all }\ X\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha)),$$

where

$$
\begin{aligned}
(\Xi_1 X)(t) &= -F_1(t, X_t) - \int_{-\infty}^{t} \mathcal{A}(s)\Gamma(t,s)\, F_1(s, X_s)\, ds + \int_{t}^{\infty} \mathcal{A}(s)\Gamma(t,s)\, F_1(s, X_s))\, ds \\
(\Xi_2 X)(t) &= \int_{-\infty}^{t} \Gamma(t,s)\, F_2(s, X_s))\, ds - \int_{t}^{\infty} \Gamma(t,s)\, F_2(s, X_s)\, ds \\
&\quad + \int_{-\infty}^{t} \Gamma(t,s)\, F_3(s, X_s)\, d\mathbb{W}(s) - \int_{t}^{\infty} \Gamma(t,s)\, F_3(s, X_s)\, d\mathbb{W}(s).
\end{aligned}
$$

In view of Lemma 4.4, Lemma 4.5, and Lemma 4.6, it follows that Ξ maps $AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$ into itself. In order to apply Krasnoselskii's fixed point theorem, we need to construct two mappings: a contraction map and a compact map.

Lemma 4.7. *The operator Ξ_1 is a contraction provided $K(\alpha, \beta, \delta, \mu, K_1) < 1$.*

Proof. Let $X,\ Y \in AP(\mathbb{R}; L^2(\Omega; \mathbb{H}_\alpha))$. Using (H.1)-(H4), we obtain

$$
\begin{aligned}
\mathbf{E}\Big\| F_1(t, X_t) - F_1(t, Y_t) \Big\|_\alpha^2 &\le k(\alpha) K_1 \mathbf{E}\Big\| X_t - Y_t \Big\|_{\tau,\alpha}^2 \\
&\le k(\alpha) \cdot K_1 \Big\| X - Y \Big\|_{\infty,\alpha}^2,
\end{aligned}
$$

which yields

$$
\Big\| F_1(\cdot, X_\cdot) - F_1(\cdot, Y_\cdot) \Big\|_{\infty,\alpha} \le k'(\alpha) \cdot K_1' \Big\| X - Y \Big\|_{\infty,\alpha}.
$$

Now for Γ_1 and Γ_2, we have the following evaluations

$$
\begin{aligned}
&\mathbf{E}\Big\| (\Gamma_1 X)(t) - (\Gamma_1 Y)(t) \Big\|_\alpha^2 \\
&\quad\le \mathbf{E}\Big(\int_{-\infty}^{t} \Big\| \mathcal{A}(s)\Gamma(t,s)[F_1(s, X_s) - F_1(s, Y_s)] \Big\|_\alpha ds \Big)^2 \\
&\quad\le n(\alpha, \mu)^2 \Big(\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{4}(t-s)}\, ds \Big) \times \\
&\qquad \times \Big(\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{4}(t-s)} \mathbf{E}\Big\| F_1(s, X_s) - F_1(s, Y_s) \Big\|_\beta^2 ds \Big) \\
&\quad\le n(\alpha, \mu)^2\ K_1 \Big(\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\frac{\delta}{4}(t-s)}\, ds \Big)^2 \Big\| X - Y \Big\|_{\infty,\alpha}^2,
\end{aligned}
$$

and hence

$$
\Big\| \Gamma_1 X - \Gamma_1 Y \Big\|_{\infty,\alpha} \le n(\alpha, \mu) \cdot K_1' \, \frac{4^{1-\alpha}\Gamma(1-\alpha))}{\delta^{1-\alpha}} \Big\| X - Y \Big\|_{\infty,\alpha}.
$$

Similarly,

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_2 X)(t)-(\Gamma_2 Y)(t)\Big\|_\alpha^2 \\
&\quad\le \mathbf{E}\Big(\int_t^\infty \Big\|\mathcal{A}(s)\Gamma(t,s)[F_1(s,X_s)-F_1(s,Y_s)]\Big\|_\alpha\,ds\Big)^2 \\
&\quad\le m(\alpha,\beta)^2\Big(\int_t^\infty e^{-\delta(s-t)}\,ds\Big)\times \\
&\qquad\times\Big(\int_t^\infty e^{-\delta(s-t)}\mathbf{E}\|F_1(s,X_s)-F_1(s,Y_s)\|_\beta^2\,ds\Big) \\
&\quad\le m(\alpha,\beta)^2\,k(\alpha)\,K_1\,\Big(\int_t^\infty e^{-\delta(s-t)}\,ds\Big)^2\,\Big\|X-Y\Big\|_{\infty,\alpha}^2,
\end{aligned}
$$

and hence,

$$
\Big\|\Gamma_2 X-\Gamma_2 Y\Big\|_{\infty,\alpha}\le\frac{m(\alpha,\beta)\cdot k'(\alpha)\cdot K_1'}{\delta}\,\Big\|X-Y\Big\|_{\infty,\alpha}.
$$

□

Lemma 4.8. *The nonlinear operator Ξ_2 is continuous. Moreover, its image is contained in a compact set.*

Proof. Let us consider the set $V=\big\{X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha)):\|X\|_{\infty,\alpha}^2\le R'\big\}$ for some fixed $R'>0$. For the continuity, let $X^n\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$ be a sequence which converges to some $X\in AP(\mathbb{R};L^2(\Omega,\mathbb{H}_\alpha))$, that is, $\|X^n-X\|_{\infty,\alpha}\to 0$ as $n\to\infty$. It follows from the estimates in Lemma 2.3 that

$$
\begin{aligned}
&\mathbf{E}\Big\|\int_{-\infty}^t \Gamma(t,s)[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2 \\
&\qquad\le\mathbf{E}\Bigg[\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}\,(t-s)}\Big\|F_2(s,X_s^n)-F_2(s,X_s)\Big\|\,ds\Bigg]^2.
\end{aligned}
$$

Now, using the continuity of F_2 and the Lebesgue Dominated Convergence Theorem we obtain that

$$
\mathbf{E}\Big\|\int_{-\infty}^t \Gamma(t,s)[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2\to 0\quad\text{as}\quad n\to\infty\,.
$$

By similar arguments, we can also show that

$$
\mathbf{E}\Big\|\int_t^\infty \Gamma(t,s)[F_2(s,X_s^n)-F_2(s,X_s)]\,ds\Big\|_\alpha^2\to 0\quad\text{as}\quad n\to\infty\,.
$$

For the term containing the Brownian motion $\mathbb{W}$, we use the estimates (4.1) and (4.2) to obtain

$$
\begin{aligned}
&\mathbf{E}\Big\|\int_{-\infty}^t \Gamma(t,s)[F_3(s,X_s^n)-F_3(s,X_s)]\,d\mathbb{W}(s)\Big\|_\alpha^2 \\
&\qquad\le C(\alpha)^2\int_{-\infty}^t (t-s)^{-2\alpha}e^{-\delta\,(t-s)}\mathbf{E}\Big\|F_3(s,X_s^n)-F_3(s,X_s)\Big\|^2\,ds\,.
\end{aligned}
$$

Now, using the continuity of F_3 and the Lebesgue Dominated Convergence Theorem we obtain that

$$\mathbf{E}\Big\| \int_{-\infty}^{t} \Gamma(t,s)[F_3(s,X_s^n) - F_3(s,X_s)]\, d\mathbb{W}(s)\Big\|_\alpha^2 \to 0 \;\text{ as }\; n\to\infty\,.$$

By similar arguments, we can also show that

$$\mathbf{E}\Big\| \int_{t}^{\infty} \Gamma(t,s)[F_3(s,X_s^n) - F_3(s,X_s)]\, d\mathbb{W}(s)\Big\|_\alpha^2 \to 0 \;\text{ as }\; n\to\infty\,.$$

Therefore,

$$\Big\|\Xi_2 X^n - \Xi_2 X\Big\|_{\infty,\alpha} \to 0 \;\text{ as }\; n\to\infty.$$

We now show that Ξ_2 maps V into a compact set; in particular, we show that $\Xi_2(V)$ is an equicontinuous set. Indeed, let $\varepsilon > 0$, $t_1 < t_2$, and $X \in V$ be arbitrary. Now

$$\begin{aligned}
\mathbf{E}\Big\|(\Xi_2 X)(t_2) &- (\Xi_2 X)(t_1)\Big\|_\alpha^2 \\
&\leq 4\mathbf{E}\Big\|(\Gamma_3 X)(t_2) - (\Gamma_3 X)(t_1)\Big\|_\alpha^2 + 4\mathbf{E}\Big\|(\Gamma_4 X)(t_2) - (\Gamma_4 X)(t_1)\Big\|_\alpha^2 \\
&\quad +4\mathbf{E}\Big\|(\Gamma_5 X)(t_2) - (\Gamma_5 X)(t_1)\Big\|_\alpha^2 + 4\mathbf{E}\Big\|(\Gamma_6 X)(t_2) - (\Gamma_6 X)(t_1)\Big\|_\alpha^2\,.
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_3 X)(t_2) - (\Gamma_3 X)(t_1)\Big\|_\alpha^2 \\
&\leq 2\mathbf{E}\Big\|\int_{t_1}^{t_2} U(t_2,s)P(s)\,\Psi_2(s)\,ds\Big\|_\alpha^2 + 2\mathbf{E}\Big\|\int_{-\infty}^{t_1} [U(t_2,s) - U(t_1,s)]P(s)\,\Psi_2(s)\,ds\Big\|_\alpha^2 \\
&= 2\mathbf{E}\Big\|\int_{t_1}^{t_2} U(t_2,s)P(s)\,\Psi_2(s)\,ds\Big\|_\alpha^2 + 2\mathbf{E}\Big\|\int_{-\infty}^{t_1} \Big(\int_{t_1}^{t_2} \frac{\partial U(\tau,s)}{\partial\tau} d\tau\Big) P(s)\,\Psi_2(s)\,ds\Big\|_\alpha^2 \\
&= 2\mathbf{E}\Big\|\int_{t_1}^{t_2} U(t_2,s)P(s)\,\Psi_2(s)\,ds\Big\|_\alpha^2 + 2\mathbf{E}\Big\|\int_{-\infty}^{t_1} \Big(\int_{t_1}^{t_2} \mathcal{A}(\tau)U(\tau,s)P(s)\,\Psi_2(s)\,d\tau\Big)\,ds\Big\|_\alpha^2 \\
&= 2N_1 + 2N_2.
\end{aligned}$$

Clearly,

$$\begin{aligned}
N_1 &\leq \mathbf{E}\Big\{\int_{t_1}^{t_2} \Big\|U(t_2,s)P(s)\,\Psi_2(s)\Big\|_\alpha\, ds\Big\}^2 \\
&\leq c(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2} (t_2-s)^{-\alpha} e^{-\frac{\delta}{2}(t_2-s)}\Big\|\Psi_2(s)\Big\|\, ds\Big\}^2\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_2 &\leq \mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2} \Big\|\mathcal{A}(\tau)U(\tau,s)P(s)\,\Psi_2(s)\Big\|_\alpha\, d\tau\Big)\,ds\Big\}^2 \\
&\leq c_0^2\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2} \Big\|\mathcal{A}(s)U(\tau,s)P(s)\,\Psi_2(s)\Big\|_\alpha\, d\tau\Big)\,ds\Big\}^2 \\
&\leq c_0^2\, n(\alpha,\mu)^2\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2} (\tau-s)^{-\alpha} e^{-\frac{\delta}{2}(\tau-s)}\Big\|\Psi_2(s)\Big\|_\beta\, d\tau\Big)\,ds\Big\}^2\,.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_3 X)(t_2)-(\Gamma_3 X)(t_1)\Big\|_\alpha^2\\
&\le\ 2\,c(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\frac{\delta}{2}(t_2-s)}\Big\|\Psi_2(s)\Big\|\,ds\Big\}^2\\
&\quad +2\,c_0^2\,n(\alpha,\mu)^2\,\mathbf{E}\Big\{\int_{-\infty}^{t_1}\Big(\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-\,\frac{\delta}{2}(\tau-s)}\Big\|\Psi_2(s)\Big\|_\beta\,d\tau\Big)\,ds\Big\}^2\\
&\le\ K(\alpha,\delta,\mu)\,(t_2-t_1)^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2,
\end{aligned}
$$

where $K(\alpha,\delta,\mu)$ is a positive constant.

Similar computations show that

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_4 X)(t_2)-(\Gamma_4 X)(t_1)\Big\|_\alpha^2\\
&\qquad\le\ 2\,c(\alpha)^2\mathbf{E}\Big\{\int_{t_1}^{t_2}e^{-\,\delta(s-t_1)}\Big\|\Psi_2(s)\Big\|\,ds\Big\}^2\\
&\qquad\quad +2\,c_0^2\,m(\alpha,\beta)^2\,\mathbf{E}\Big\{\int_{t_2}^{\infty}\Big(\int_{t_1}^{t_2}e^{-\,\delta(s-\tau)}\Big\|\Psi_2(s)\Big\|_\beta\,d\tau\Big)\,ds\Big\}^2\\
&\qquad\le\ K(\alpha,\delta,\beta)\,(t_2-t_1)^2\sup_{t\in\mathbb{R}}\mathbf{E}\Big\|\Psi_2(t)\Big\|^2,
\end{aligned}
$$

where $K(\alpha,\delta,\beta)$ is a positive constant.

Let us now evaluate $\Gamma_5 X$. We have

$$
\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_5 X)(t_2)-(\Gamma_5 X)(t_1)\Big\|_\alpha^2\\
&\qquad\le\ 2\mathbf{E}\Big\|\int_{t_1}^{t_2}U(t_2,s)P(s)\,\Psi_3(s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\qquad\quad +2\mathbf{E}\Big\|\int_{-\infty}^{t_1}[U(t_2,s)-U(t_1,s)]P(s)\,\Psi_3(s)\,d\mathbb{W}(s)\Big\|_\alpha^2\\
&\qquad=\ 2N_1'+2N_2'.
\end{aligned}
$$

Let us start with the first term. By Ito isometry identity, we have

$$
\begin{aligned}
N_1' &\le\ c(\alpha)^2\Big\{\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}e^{-\,\delta(t_2-s)}\mathbf{E}\Big\|\Psi_3(s)\Big\|^2\,ds\\
&\le\ c(\alpha)^2\Big(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}e^{-\,\delta\,(t_2-s)}\,ds\Big)\sup_{s\in\mathbb{R}}\mathbf{E}\Big\|\Psi_3(s)\Big\|^2.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
N_2' &= \mathbf{E}\Big\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} \frac{\partial}{\partial \tau} U(\tau, s)\, d\tau\Big] P(s)\, \Psi_3(s)\, d\mathbb{W}(s)\Big\|_\alpha^2 \\
&= \mathbf{E}\Big\| \int_{-\infty}^{t_1} \Big[\int_{t_1}^{t_2} \mathcal{A}(\tau) U(\tau, s)\, d\tau\Big] P(s)\, \Psi_3(s)\, d\mathbb{W}(s)\Big\|_\alpha^2 \\
&= \mathbf{E}\Big\| \int_{t_1}^{t_2} \mathcal{A}(\tau) U(\tau, t_1)\Big\{ \int_{-\infty}^{t_1} U(t_1, s) P(s)\, \Psi_3(s)\, d\mathbb{W}(s)\Big\}\, d\tau\Big\|_\alpha^2 \\
&\leq \mathbf{E}\Big[\int_{t_1}^{t_2} \Big\| \int_{-\infty}^{t_1} \mathcal{A}(\tau) U(\tau, s)\, P(s)\, \Psi_3(s)\, d\mathbb{W}(s)\Big\|_\alpha\, d\tau\Big]^2 \\
&\leq c_0^2\, (t_2 - t_1) \int_{t_1}^{t_2} \mathbf{E}\Big\| \int_{-\infty}^{t_1} \mathcal{A}(s) U(\tau, s)\, P(s)\, \Psi_3(s)\, d\mathbb{W}(s)\Big\|_\alpha^2\, d\tau \\
&\leq c_0^2\, n(\alpha, \mu)^2 (t_2 - t_1) \int_{t_1}^{t_2} \Big\{ \int_{-\infty}^{t_1} (\tau - s)^{-2\alpha} e^{-\frac{\delta}{2}(\tau - s)} \mathbf{E}\Big\|\Psi_3(s)\Big\|_\beta^2\, ds\Big\}\, d\tau \\
&\leq c_0^2\, n(\alpha, \mu)^2 (t_2 - t_1)^2 \int_{-\infty}^{t_1} (t_1 - s)^{-2\alpha} e^{-\frac{\delta}{2}(t_1 - s)} \mathbf{E}\Big\|\Psi_3(s)\Big\|_\beta^2\, ds \\
&\leq c_0^2\, n(\alpha, \mu)^2 (t_2 - t_1)^2\, \Gamma\big(1 - 2\alpha\big) \big(\frac{\delta}{2}\big)^{1-2\alpha} \sup_{s \in \mathbb{R}} \mathbf{E}\Big\|\Psi_3(s)\Big\|^2\, ds\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_5 X)(t_2) - (\Gamma_5 X)(t_1)\Big\|_\alpha^2 \\
&\leq\ K\Big[K(\Gamma, \alpha, \delta)\, (t_2 - t_1) + K_1(\Gamma, \alpha, \delta, \mu)\, (t_2 - t_1)^2\Big]\,,
\end{aligned}$$

where $K(\Gamma, \alpha, \delta, \mu)$ is a positive constant depending on Γ, α, δ, and μ.

Similar computations show

$$\begin{aligned}
&\mathbf{E}\Big\|(\Gamma_6 X)(t_2) - (\Gamma_6 X)(t_1)\Big\|_\alpha^2 \\
&\leq\ K\, \Big[K(\alpha, \delta)\, (t_2 - t_1) + K(\alpha, \beta, \delta)\, (t_2 - t_1)^2\Big]\,.
\end{aligned}$$

where $K(\alpha, \delta)$ and $K(\alpha, \beta, \delta)$ are positive constants.

From the theorem of Ascoli-Arzela, it follows that $\Xi_2(V)$ is contained in a compact set. The proof is complete. □

Theorem 4.9. *Suppose assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5)-(H.6) *hold and that* $K(\alpha, \beta, \delta, \mu, K_1) < 1$*, the evolution equation Eq.* (1.2) *has a square-mean almost periodic solution* X *satisfying*

$$X = \Xi_1 X + \Xi_2 X.$$

Proof. Fix $\varepsilon > 0$ and let i= 2, 3. It follows from assumption (H.6) that there exists $r > 0$ such that

$$\mathbf{E}\Big\|F_i(t, Y)\Big\|^2 \leq \varepsilon \mathbf{E}\big\|Y\big\|_{\tau,\alpha}^2 \text{ for all } t \in \mathbb{R} \text{ and } Y \in L^2(\Omega, \mathbb{H}_\alpha) \text{ with } \mathbf{E}\big\|Y\big\|_{\tau,\alpha}^2 > r\,.$$

Setting

$$M_i = \sup\left\{\mathbf{E}\Big\|F_i(t,Y)\Big\|^2 : \ t\in\mathbb{R},\ \mathbf{E}\big\|Y\big\|^2_{\tau,\alpha}\le r\right\}.$$

Therefore,

$$\mathbf{E}\Big\|F_i(t,Y)\Big\|^2 \le M_i + \varepsilon\mathbf{E}\big\|Y\big\|^2_{\tau,\alpha} \ \text{ for all } (t,Y)\in\mathbb{R}\times L^2(\Omega, C_{\tau,\alpha})\,. \tag{4.4}$$

Also, using Lemma 2.3 , Eq. (4.1), Eq. (4.2), Eq. (4.4), and assumption (H.4) we can show that

$$\begin{aligned}
&\mathbf{E}\Big\|(\Xi_1 X)(t)+(\Xi_2 Y)(t)\Big\|^2_\alpha\\
&\le\ 12\left[1+\ n(\alpha,\mu)^2\left(\frac{\Gamma(1-\alpha)\,2^{1-\alpha}}{\delta^{1-\alpha}}\right)^2+\left(\frac{m(\alpha,\beta)}{\delta}\right)^2\right]\left(K_1\,\big\|X\big\|_{\infty,\alpha}+\sup_{t\in\mathbb{R}}\Big\|F_1(t,0)\Big\|^2\right)\\
&+\ 8c(\alpha)^2\Bigg[\left[\left(\frac{\Gamma(1-\alpha)\,2^{1-\alpha}}{\delta^{1-\alpha}}\right)^2+\left(\frac{m(\alpha)}{\delta}\right)^2\right]\Big(M_2+\varepsilon\big\|Y\big\|_{\infty,\alpha}\Big)\\
&+\ \left[K(\alpha)\frac{\Gamma(1-2\alpha)}{\delta^{1-2\alpha}}+\frac{K(\alpha,\beta)}{2\delta}\right]\Big(M_3+\varepsilon\big\|Y\big\|_{\infty,\alpha}\Big)\Bigg]\\
&=\ c_1(\alpha,\beta,\delta,\mu,\Gamma)\left(K_1\,\big\|X\big\|_{\infty,\alpha}+\sup_{t\in\mathbb{R}}\Big\|F_1(t,0)\Big\|^2\right)+c_2(\alpha,\beta,\delta,\Gamma)\Big(M_2+\varepsilon\big\|Y\big\|_{\infty,\alpha}\Big)\\
&+\ c_3(\alpha,\beta,\delta,\Gamma)\Big(M_3+\varepsilon\big\|Y\big\|_{\infty,\alpha}\Big)\,.
\end{aligned}$$

Now, for ε, K_1 small enough, choose R such that

$$c_1(\alpha,\beta,\delta,\mu,\Gamma)\Big(K_1\,R+a\Big)+c_2(\alpha,\beta,\delta,\Gamma)\Big(M_2+\varepsilon\,R\Big)+c_3(\alpha,\beta,\delta,\Gamma)\Big(M_3+\varepsilon\,R\Big)\le R$$

where $a=\sup_{t\in\mathbb{R}}\Big\|F_1(t,0)\Big\|^2$, $c_1(\alpha,\beta,\delta,\mu,\Gamma)$ and $c_i(\alpha,\beta,\delta,\Gamma)$ $(i=2,3)$ are constants depending on α, β , δ, and the classical gamma function $\Gamma(\cdot)$.

Let $W=\left\{Z\in AP(\mathbb{R};L^2(\Omega;\mathbb{H}_\alpha):\big\|Z\big\|_{\infty,\alpha}\le R\right\}$. For X, $Y\in W$, we have

$$\mathbf{E}\Big\|(\Xi_1X)(t)+(\Xi_2Y)(t)\Big\|^2_\alpha\le R\,.$$

Thus $(\Xi_1X)(t)+(\Xi_2Y)(t)\in W$. In view of Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.7, and Lemma 4.8, the proof can be completed by using the Krasnoselskii's fixed point theorem (Theorem 4.2).

□

5. Example

Throughout the rest of this paper, we suppose $0<\alpha<0.5<\beta<1$.

Example 5.1. Fix $T>0$. Let $\mathbb{H}=L^2[0,\pi]$ equipped with its natural topology. Here, for $\mu\in(0,1)$, we take $\mathbb{H}_\mu=\Big(L^2[0,\pi],\mathbb{H}^1_0[0,\pi]\cap\mathbb{H}^2[0,\pi]\Big)_{\mu,\infty}$ equipped with its μ-norm $\|\cdot\|_\mu$.

To illustrate our main result, we study the existence of square-mean almost periodic solutions to the one-dimensional stochastic heat equation with finite delay given by

(5.1)
$$\begin{cases} \partial\Big[\Phi + F_1(t,\Phi_t)\Big] = \Big[\dfrac{\partial^2\Phi}{\partial x^2} + a(t,x)\Phi + F_2(t,\Phi_t)\Big]\partial t + F_3(t,\Phi_t)d\mathbb{W}(t), & \mathbb{R}\times[0,\pi] \\ \Phi(t,0) = \Phi(t,\pi) = 0, & t\in\mathbb{R} \end{cases}$$

where $a : \mathbb{R}\times[0,\pi] \mapsto \mathbb{R}$ is T-periodic in $t \in \mathbb{R}$ uniformly in $x \in [0,\pi]$, and $F_1 : \mathbb{R}\times L^2(\Omega;\mathfrak{B}_\alpha) \mapsto L^2(\Omega,\mathbb{H}_\beta)$ and $F_i(i = 2,3) : \mathbb{R}\times L^2(\Omega,\mathfrak{B}_\alpha) \to L^2(\Omega, L^2[0,\pi])$ are square-mean almost periodic processes.

Define the corresponding linear operator $A(t)$ on $L^2(\Omega, L^2[0,\pi])$ as follows:

$$A(t)\Phi = \frac{\partial^2\Phi}{\partial x^2} + a(t,x)\Phi \quad \text{for all} \quad \Phi\in D(A(t)) = L^2(\Omega,\mathbb{H}^1_0[0,\pi]\cap\mathbb{H}^2[0,\pi]),$$

where $a : \mathbb{R}\times[0,\pi] \mapsto \mathbb{R}$, in addition of being T-periodic satisfies: there exist $\delta_0 \in (0,1)$ such that

$$a(t,x) \le \delta_0$$

for all $t\in\mathbb{R}$ and $x\in[0,\pi]$.

Clearly, $A(t+T) = A(t)$ for all $t\in\mathbb{R}$. Moreover, it is then easy to check that the evolution family $U(t,s)$ associated with $A(t)$ is exponentially stable, which yields dichotomy with dichotomy projections $P(t) = I$ and $Q(t) = 0$. Indeed,

$$U(t,s) = T(t-s)e^{\displaystyle\int_s^t a(\tau,x)d\tau} \quad \text{for all} \quad t\ge s,$$

where $T(t)$ is the analytic semigroup associated with the second-order differential operator

$$A\Phi = \frac{\partial^2\Phi}{\partial x^2} \quad \text{with} \quad D(A) = L^2(\Omega,\mathbb{H}^1_0[0,\pi]\cap\mathbb{H}^2[0,\pi]).$$

Now

$$\|U(t,s)\Phi\| \le e^{-(1-\delta_0)(t-s)}\|\Phi\|, \quad t\ge s.$$

Moreover, $U(t+T,s+T) = U(t,s)$ for all $t\ge s$. Now since $Q(t) = 0$, it follows that $\Gamma(t,s) = U(t,s)$ and hence

$$A(t+T)\Gamma(t+T,s+T) = A(t)\Gamma(t,s).$$

We have

Theorem 5.2. *Under previous assumptions, then the heat equation Eq. (5.1) has a solution* $\Phi\in AP(\mathbb{R}, L^2([0,\pi],\mathbb{H}_\alpha))$.

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Application of homotopy analysis method to fractional order generalized Huxley equation

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Abstract: Based on the homotopy analysis method, a scheme is developed to obtain approximation solution of a fractional order generalized Huxley equation. The fractional derivatives are described by Caputo's sense. Exact and /or approximate analytical solutions of this equations are obtained. The solutions of our model equation are calculated in the form of convergent series with easily computable components.

Key words: Fractional order-nonlinear PDE; Huxley equation; Homotopy analysis method; Homotopy perturbation method; Adomian decomposition method.

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1. Introduction

Nonlinear partial differential equations (NPDEs) are encountered in such various fields as physics, chemistry, biology, mathematics and engineering,. Most nonlinear models of real life problems are still very difficult to solve, either numerically or theoretically. The fractional order generalized Huxley equation

$$D_t^{\alpha}u - u_{xx} = \beta u(1-u^{\delta})(u^{\delta}-\gamma), \qquad 0 \le x \le 1, \quad t \ge 0, \quad 0 < \alpha \le 1 \tag{1}$$

with the initial condition of

$$u\,(x,0) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\,(\sigma\gamma x)]^{\frac{1}{\delta}} \tag{2}$$

describes nerve pulse propagation in nerve fibres and wall motion in liquid crystals. The exact solution of this equation was derived by Wang et al. [1],using nonlinear transformations and is given by

$$u\,(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\sigma\gamma(x + \left\{\frac{(1+\delta-\gamma)\,\rho}{2\,(1+\delta)}\right\}t\right))]^{\frac{1}{\delta}}, \tag{3}$$

where $\sigma = \frac{\delta\rho}{4(1+\delta)}$ and $\rho = \sqrt{4\beta\,(1+\delta)}$.

Many researchers have used various numerical methods to solve equation (1) numerically. Hashim et al. investigated the generalized Huxley equation, using Adomian decomposition method (ADM) [2] and Wazwaz studied the generalized forms of Burgers, Burgers-Kdv and Burgers-Huxley equations [3]. Hashem et al. studied the generalized Burger's-Huxley equation [4], and Estevez investigated non-classical symmetries and the singular modified Burger's and Burger's-Huxley equation [5]. Hashemi et al. solved the generalized Huxley equation by He's homotopy perturbation method (HPM) [6]. Batiha et al. solved the generalized Huxley equation by variational iteration method (VIM) [7].

In this paper, i solve the fractional order generalized Huxley equation by homotopy analysis method (HAM) [8-14]. The results are compared with the available exact solutions and those were obtained by the (ADM) [2] and homotopy perturbation method (HPM) [6].

2. Basic definitions

In this sections, we give some defintions and properties of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. There are many books [15-18] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville definition, Caputo's definition and generalized function approach. For

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the purpose of this paper, the Caputo's definition of the fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equation with Caputo's derivatives take on the traditional form as for integer-order differential equation.

Definition 1. A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$,such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0,\infty)$,and it is said to be in the space C_μ^n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $h \in C_\mu, \mu \geq -1$, is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0) J^0 h(t) = h(t) \tag{4}$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

(1) $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,

(2) $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} h^{(n)}(\tau) d\tau \quad ,\text{for } n-1 < \alpha \leq n, \quad n \in N \ , t > 0, \quad h \in C_{-1}^n. \tag{5}$$

The following are two basic properties of Caputo's fractional Derivative [18]:

(1) Let $h \in C_{-1}^n, n \in \mathbb{N}$. Then $D^\alpha h, 0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}$.

(2) Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and $h \in C_\mu^n, \mu \geq -1$. Then

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^+) \frac{t^k}{k!}. \tag{6}$$

3. The homotopy analysis method (HAM)

We apply the HAM [8-14] to Huxley equation with initial conditions. We consider the following differential equation

$$N[u(x,t)] = 0, \tag{7}$$

where N is a nonlinear operator for this problem, x and t denote independent variables, $u(x,t)$ is an unknown function. By means of the HAM, one first construct zero-order deformation equation

$$(1-q)(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)], \tag{8}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, is an auxiliary linear operator, $u_0(x,t)$ is an initial guess. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{9}$$

Liao [8-14] expanded $\phi(x,t;q)$ in Taylor series with respect to the embedding parameter q, as follows:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{10}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} |_{q=0} \tag{11}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are selected such that the series (10) is convergent at $q = 1$, then we have from (10)

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{12}$$

Let us define the vector

$$\vec{u_n}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}. \tag{13}$$

Differentiating (8) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the mth-order deformation equation

$$\pounds(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}), \tag{14}$$

where

$$\mathcal{R}_m(\vec{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(x,t;q)]}{\partial q^{m-1}}\ |_{q=0}, \tag{15}$$

and

$$\varkappa_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{16}$$

The mth-order deformation Eq. (14) is linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MathLab.

4. Analysis of the method by the HAM

To solve Eq. (1) with an initial condition (2) by means of HAM, we choose the linear operator

$$[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t}, \tag{17}$$

with property $[c] = 0$, where c is a constant. We define a nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - \beta(1+\gamma)\phi^{\delta+1}(x,t;q) + \beta\phi^{2\delta+1}(x,t;q) + \beta\gamma\phi(x,t;q). \tag{18}$$

We construct the zeroth-order deformation equation

$$(1-q)(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)].$$

For $q = 0$ and $q = 1$, we can write

$$\phi(x,t;0) = u_0(x,t) = u(x,0), \qquad \phi(x,t;1) = u(x,t). \tag{19}$$

Thus, we obtain the mth-order deformation equations

$$\pounds(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(\vec{u_{m-1}}),$$

where

$$\mathcal{R}_m(\vec{u_{m-1}}) = \frac{\partial \phi_{m-1}(x,t;q)}{\partial t} - \frac{\partial^2 \phi_{m-1}(x,t;q)}{\partial x^2} + \beta\phi_{m-1}^{2\delta+1}(x,t;q) - \beta(1+\gamma)\phi_{m-1}^{\delta+1}(x,t;q) + \beta\gamma\phi_{m-1}(x,t;q). \tag{20}$$

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [12], the auxiliary function can be determined uniquely $H(t) = 1$.

Now the solution of the mth -order deformation equations (20) for $m \geq 1$ become

$$u_m(x,t) = \varkappa_m u_{m-1}(x,t) + h\ j^{\alpha}\mathcal{R}_m(\vec{u_{m-1}}). \tag{21}$$

So, a few terms of series solution are as follows:

$$u_0\left(x,t\right)=[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}},\tag{22}$$

$$\begin{aligned}u_1\left(x,t\right)=&\frac{-h}{\delta^2\Gamma(\alpha+1)}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta-2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right)\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t^{\alpha}).\end{aligned}\tag{23}$$

According to the HAM, we can conclude that

$$u\left(x,t\right)=u_0\left(x,t\right)+u_1\left(x,t\right)+......\tag{24}$$

Therefore, substituting the values of $u_0\left(x,t\right)$ and $u_1\left(x,t\right)$ from Eqs. (22), (23) into. Eq. (24) yields:

$$\begin{aligned}u\left(x,t\right)=&\frac{-h}{\delta^2\Gamma(\alpha+1)}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta-2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right)\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t^{\alpha})+[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}}.\end{aligned}\tag{25}$$

When $h=-1$ and $\alpha=1$, we obtain

$$\begin{aligned}u\left(x,t\right)=&\frac{1}{\delta^2}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}}(2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh^2\left(\sigma\gamma x\right)\delta-2^{(\frac{-1+\delta}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\tanh\left(\sigma\gamma x\right)\\&+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta(2^{(\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+\delta}{\delta})}\beta\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1+2\delta}{\delta})}\sigma^2\delta+2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{\delta}\delta^2\\&-2^{(-\frac{1}{\delta})}\gamma^{(\frac{1}{\delta})}\beta(2^{(\frac{-1}{\delta})}\gamma^{(\frac{1}{\delta})}((1+\tanh\left(\sigma\gamma x\right))^{\frac{1}{\delta}})^{2\delta}\delta^2)t)+[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left(\sigma\gamma x\right)]^{\frac{1}{\delta}},\end{aligned}\tag{26}$$

which the same as the solution obtained by [6] and [19]. Then find at $h=-1$,and $\alpha=1$

$$u\left(x,t\right)_{\text{HAM}}=u\left(x,t\right)_{\text{ADM}}=u\left(x,t\right)_{\text{HPM}}.$$

Now we shall illustrate the accuracy and efficiency of HAM applied to Eq. (1) compared to the ADM [2] and HPM[6]. For this purpose, i consider the same parameter values for the generalized Huxley equation (1) as considered specifically in [2].

Table 1: Approximate solution of (1) for some α using the 2−term HAM approximation with $h = -0.3, h = -0.7$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 1$.

			$h = -0.3$		$h = -0.7$	
x	t	Exact	$\alpha = 0.2$	$\alpha = 0.75$	$\alpha = 0.2$	$\alpha = 0.75$
0.1	0.05	5.00030E-4	4.99973E-4	5.00009E-4	4.99913E-4	4.99998E-4
	0.1	5.00043E-4	4.99966E-4	5.00003E-4	4.99897E-4	4.99984E-4
	1	5.00268E-4	4.99936E-4	4.99936E-4	4.99827E-4	4.99827E-4
0.5	0.05	5.00101E-4	5.00044E-4	5.00080E-4	4.99984E-4	5.00068E-4
	0.1	5.00113E-4	5.00037E-4	5.00074E-4	4.99968E-4	5.00055E-4
	1	5.00338E-4	5.00007E-4	5.00007E-4	4.99898E-4	4.99898E-4
0.9	0.05	5.00172E-4	5.00114E-4	5.00150E-4	5.00054E-4	5.00139E-4
	0.1	5.00184E-4	5.00108E-4	5.00145E-4	5.00039E-4	5.00125E-4
	1	5.00409E-4	5.00077E-4	5.00078E-4	4.99969E-4	4.99969E-4

Table 2: Approximate solution of (1) for some α using the 2−term HAM approximation with $h = -0.3, h = -0.7$, $\beta = 1$, $\gamma = 0.001$ and $\delta = 3$.

			$h = -0.3$		$h = -0.7$	
x	t	Exact	$\alpha = 0.2$	$\alpha = 0.75$	$\alpha = 0.2$	$\alpha = 0.75$
0.1	0.05	7.93740E-2	7.93649E-2	7.93707E-2	7.93554E-2	7.93688E-2
	0.1	7.93760E-2	7.93639E-2	7.93697E-2	7.93530E-2	7.93667E-2
	1	7.94117E-2	7.93591E-2	7.93591E-2	7.93418E-2	7.93418E-2
0.5	0.05	7.93820E-2	7.93729E-2	7.93780E-2	7.93634E-2	7.93768E-2
	0.1	7.93839E-2	7.93718E-2	7.93777E-2	7.93609E-2	7.93746E-2
	1	7.94196E-2	7.93670E-2	7.93670E-2	7.93497E-2	7.93498E-2
0.9	0.05	7.93899E-2	7.93808E-2	7.93865E-2	7.93713E-2	7.93847E-2
	0.1	7.93919E-2	7.93797E-2	7.93856E-2	7.93688E-2	7.93825E-2
	1	7.94275E-2	7.93749E-2	7.93750E-2	7.93577E-2	7.93577E-2

5. Conclusion

In this paper, the homotopy analysis method HAM was implemented to derive exact and approximate analytical solutions for both linear and nonlinear partial differential equations of fractional order. HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. It was also demonstrated that the Adomian decomposition method ADM and Homotopy perturbation method HPM are a special cases of HAM. Mathematica has been used for computations in this paper.

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ϵ-Pair proximity in normed spaces

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Abstract

Let A and B be nonempty subsets of a metric space X and also $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$, $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. We are going to consider element $x \in A$ and $y \in B$ that is $d(Tx, Sy) \leq \epsilon$. We call (x, y) an ϵ-pair proximity.

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Keywords: ϵ-Pair proximity, ϵ-Pair contraction maps, ϵ-Approximately compact.

1 Introduction

Let T be a self map of a metric space (X, d). Let us look for an approximate solution of $Tx = x$. If there exists a point $z \in X$ such that $d(Tz, z) \leq \epsilon$, where ϵ is a positive number, then z is called an approximate solution of the equation $Tx = x$, or equivalently,$z \in X$ is an approximate fixed point (or ϵ-fixed point) of T. In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate fixed point is more than enough, an approximate solution plays an important role in such situations. The theory of fixed points and consequently of approximate fixed points finds application in mathematical economics, noncooperative game theory, dynamic programming, nonlinear analysis, variational calculus, theory of integro-differential equation and serval other areas of applicable analysis (see, for instance, [5], [9], [10], [14], [15]). Cromme and Diener [7] have found approximate fixed points by generalizing Brouwers fixed point theorem to a discontinuous map, Hou and Chen [11] have extended their results to set valued maps.Espinola and KirK [10] obtained interesting results in product spaces. Tijs er al [15] have discussed approximate fixed point theorems for contractive and non-expensive maps by weakening the conditions on the spaces. R. Branzei et al [5] further extended these results to multifunctions in Banach space. Recently M. Berinde [4] obtained approximate fixed point theorems for operators satisfying Kannan, Chatterjea and Zamfirescu type of conditions on metric spaces. In this paper we study some basic ϵ-pair proximity results in generalized metric spaces. Also we want to obtain relations ϵ-pair proximity and ϵ -fixed point. Throught the paper, suppose X is a metric space and A, B are two nonempty subsets of X.

Definition 1.1. *Let $T: X \longrightarrow X$ and $\epsilon > 0$ and $x \in X$. Then an element $x_0 \in X$ is an approximate fixed point (or ϵ -fixed point) of T if $d(Tx_0, x_0) < \epsilon$.*

2 Main Results

In this section we consider ϵ-pair proximity for two maps $T: A \cup B \to A \cup B$ and $S: A \cup B \to A \cup B$.

Definition 2.1. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. A point (x, y) in $A \times B$ is said to be an ϵ-pair proximity point for (T, S), if*

$$d(Tx, Sy) \leq \epsilon.$$

We say (T, S) has the ϵ-pair proximity property if for every $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$ where

$$P^{\epsilon}_{(T,S)}(A, B) = \{(x, y) \in A \times B : \ d(Tx, Sy) \leq \epsilon\}.$$

Proposition 2.2. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. If for every $(x, y) \in A \times B$*

$$d(T^n(x), S^n(y)) \to 0$$

then (T, S) has the ϵ-pair proximity property .

Proof: Suppose $(x, y) \in A \times B$. Since $d(T^n(x), S^n(y)) \to 0$, for every $\epsilon > 0$

$$\exists n_0 > 0 \ s.t. \ \forall n \geq n_0 : \ d(T^n(x), S^n(y)) < \epsilon$$

Then $d(T(T^{n-1}(x), S(S^{n-1}(y)) < \epsilon$ for every $n \geq n_0$. Put $x_0 = T^{n_0-1}(x)$ and $y_0 = S^{n_0-1}(y))$. Hence $d(T(x_0), S(y_0)) \leq \epsilon$ and $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$. ■

Proposition 2.3. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. Also for every $(x, y) \in A \times B$*

$$d(Tx, Sy) \leq \alpha d(x, y), \qquad 0 < \alpha < 1 \qquad (*)$$

If $x \in A$ is an ϵ-fixed point for T or $y \in B$ is an ϵ-fixed point for S, then (T, S) has the ϵ-pair proximity property.

Proof: Suppose $x \in A$ and $y \in B$

$$d(T(x), S(Tx)) \leq \alpha d(x, Tx) \text{ and } d(S(y), T(Sy)) \leq \alpha d(y, Sy)$$

Therefore, if x is an ϵ-fixed point for T , or y is an ϵ-fixed point for S, then for all $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$.

Proposition 2.4. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. For every $(x, y) \in A \times B$*

$$d(Tx, Sy) \leq \alpha d(x, y), \qquad 0 < \alpha < 1 \quad (*)$$

then for every $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$. Moreover, if $x \in A$ is an ϵ-fixed point T, $y \in B$ is an ϵ-fixed point S and (T, S) satisfy in (*) then $d(x, y) \leq \frac{2\epsilon}{1-\alpha}$.

Proof: Suppose $x \in A$ and $y \in B$

$$\begin{aligned} d(T^n(x), S^n(y)) &= d(T(T^{n-1}(x), S(S^{n-1}(y))) \\ &\leq \alpha d(T^{n-1}(x), S^{n-1}(y)) \\ &\leq \ldots.. \\ &\leq \alpha^{n-1} d(Tx, Sy) \\ &\leq \alpha^n d(x, y). \end{aligned}$$

Therefore $d(T^n(x), S^n(y)) \to 0$ as $n \to \infty$. From Proposition 2.3, (T, S) has the ϵ-pair proximity property. Since

$$\begin{aligned} d(x, y) &\leq d(x, Tx) + d(Tx, Sy) + d(y, Sy) \\ &\leq 2\epsilon + \alpha d(x, y). \end{aligned}$$

Then $d(x, y) \leq \frac{2\epsilon}{1-\alpha}$. ■

Proposition 2.5. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. For every $(x, y) \in A \times B$*

$$d(Tx, Sy) \leq \beta[d(x, Tx) + d(y, Sy)],$$

where $\beta \geq 0$ and $2\beta < 1$. Then

a) *if x is an ϵ-fixed point for T , or y is an ϵ-fixed point for S, then for all $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$.*

b) *if $\{x_n\}$ has a convergent subsequence. Then for every $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$.*

Proof. Part a): Suppose $(x, y) \in A \times B$

$$\begin{aligned} d(Tx, S(Tx)) &\leq \beta[d(x, Tx) + d(Tx, S(Tx))] \\ d(T(Sy), Sy) &\leq \beta[d(Sy, T(Sy)) + d(y, Sy)]. \end{aligned}$$

Therefore

$$\begin{aligned} d(Tx, S(Tx)) &\leq \frac{\beta}{1-\beta} d(x, Tx) \leq d(x, Tx) \qquad (*) \\ d(Sy, T(Sy)) &\leq \frac{\beta}{1-\beta} d(y, Sy) \leq d(y, Sy). \end{aligned}$$

Since x is an ϵ-fixed point for T , or y is an ϵ-fixed point for S, for every $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$.

Part b): Suppose $(x_0, y_0) \in A \times B$ and define

$$x_{n+1} = Sy_n, \; y_{n+1} = Tx_n, n \in \mathbb{N}.$$

Now in the part (*) we set $x = x_n$ and $y = y_{n+1}$. Then

$$d(y_{n+1}, x_{n+2}) \leq kd(x_n, y_{n+1}).$$

Also there exists $s \in R$ such that $k = s^2$. Therefore

$$\begin{aligned} d(y_{n+1}, x_{n+2}) &\leq s^2 d(x_n, y_{n+1}) \\ &\leq s^4 d(x_{n-2}, y_{n+1}) \\ &\leq s^6 d(x_{n-4}, y_{n+1}) \\ &\leq ..... \\ &\leq s^{n+2} d(x_0, y_{n+1}) \; if \; n \; even \; and \; s^{n+1} d(x_1, y_{n+1}) \; if \; n \; odd. \end{aligned}$$

It follows that $d(y_{n+1}, x_{n+2}) = d(T(x_n), S(y_{n+1})) \longrightarrow 0$. If subsequence $\{x_{n_k}\}$ converges to x. Then $d(T(x_{n_k}), S(y_{n_{k+1}})) \longrightarrow 0$. Then

$$\forall \epsilon > 0 \; \exists n_0 \in N \; such \; that \; d(T(x), S(y_{n_{k_0+1}}) < \epsilon.$$

Therefore $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$. ∎

Definition 2.6. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. A pair maps (T,S) is said to be a power contraction if for some $k \in R$,*

$$d(Tx, Sy) \leq k^2 d(A, B) + (1-k)kd(x, y) \quad x \in A, \; y \in B. \quad (*)$$

For example, let $A = \{(x, 0) : x \in [0, 1]\}$ and $B = \{(x, 1) : x \in [0, 1]\}$. Define maps T, S by $T(x, 0) = (\frac{1}{2}, 1)$ and $S(x, 1) = (\frac{1}{2}, 0)$. We show that (T, S) is a power contraction map.

If $x \in A$ and $y \in B$, then $d(x, y) \geq 1$, by archimedean property there exists a $k \in N$ such that $k(d(x, y) - 1) \geq 1$. Since $d(A, B) = 1$ and $d(Tx, Sy) = 1$

$$d(Tx, Sy) \leq k^2 d(A, B) + (1-k)kd(x, y). \blacksquare$$

Proposition 2.7. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$, also pair (T, S) be a power contraction map with $0 < k < 1$. Then for all $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$.*

Proof. Suppose $\epsilon > 0$, $x \in A$ and $y \in B$. Then

$$\begin{aligned} d(Tx, Sy) &\leq k^2 d(A, B) + kd(x, y) - k^2 d(x, y) \\ &= k^2(d(A, B) - d(x, y)) + kd(x, y) \\ &\leq kd(x, y). \end{aligned}$$

From Proposition 2.4, for all $\epsilon > 0$, $P^{\epsilon}_{(T,S)}(A, B) \neq \emptyset$. ■

Proposition 2.8. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps, such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$, also (T, S) be a power contraction with $0 < k < 1$, $(x_0, y_0) \in A \times B$ and T is a continues map and define*

$$x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n, \quad n \in \mathbb{N}.$$

Suppose $\{x_n\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = \frac{k^2}{1-k(1-k)} dist(A, B)$.

Proof.

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(Tx_n, Sy_n) \\ &\leq k^2 d(A, B) + k(1 - k)d(x_n, y_n) \\ &\leq ... \\ &\leq (k^2 + ... + k^n(1 - k)^n)d(A, B) + k^n(1 - k)^n d(x_0, y_0). \end{aligned}$$

Therfore if $n \longrightarrow \infty$, we have $d(x_{n+1}, y_{n+1}) \to \frac{k^2}{1-k(1-k)} dist(A, B)$. Then

$$d(x, Tx) = \frac{k^2}{1 - k(1 - k)} dist(A, B). \ \blacksquare$$

Definition 2.9. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$, and $\epsilon > 0$, we say that B is an ϵ-approximately compact for (T, S), if for all $x \in A$ from the fact that $y_n \in B$ $n = 1, 2, ...$ and $lim_{n\to\infty} d(Tx, Sy_n) \leq \epsilon$ there follows the possibility of selecting from $\{y_n\}$ a subsequence converging to some $y \in B$.*

Proposition 2.10. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$, $0 < \lambda < 1$, $A \bigcap B = \emptyset$ and*

$$d(Tx, Sz) \leq \lambda d(x, Sz) + \lambda dist(B, Sz)$$

for every $(x, z) \in A \times B$. If $dist(Sz, B) \leq 1 - \lambda$. Then there exists a sequence $\{x_n\} \subseteq A$ such that

$$\lim_{n\to\infty} d(Tx_n, Sy) \leq \lambda.$$

Proof. Starting with any $x_0 \in A$ and consider $x_{n+1} = Tx_n,\ x = 0, 1, 2, \cdots$. Then

$$\begin{aligned} d(x_{n+1}, Sy) &= d(Tx_n, Sy) \\ &\leq \lambda(d(x_n, Sy) + \lambda dist(Sy, B)) \\ &\leq \lambda^2(d(x_{n-1}, Sy) + (\lambda + \lambda^2)dist(Sy, B)) \\ &\leq \cdots \\ &\leq \lambda^n d(x_0, Sy) + (\lambda + \lambda^2 + \cdots + \lambda^n)dist(Sy, B) \\ &\leq \lambda^n d(x_0, Sy) + \frac{\lambda}{1-\lambda} dist(Sy, B). \end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(Tx_n, Sy) \leq \lambda. \ \blacksquare$$

Corollary 2.11. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$, $0 < \lambda < 1$, $A \bigcap B = \emptyset$ and*

$$d(Tx, Sz) \leq \lambda d(x, Sz) + \lambda dist(B, Sz)$$

for every $(x, z) \in A \times B$. If $dist(Sz, B) \leq 1-\lambda$. If $\lambda < \epsilon$ and A is an ϵ-approximately compact for (T, S). Then there exists an $a \in A$, such that $(a, y) \in P^{\epsilon}_{(T,S)}(A, B)$.

Proposition 2.12. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be continues maps such that $T(A) \subseteq B$, $S(B) \subseteq A$, B is an ϵ-approximately compact for (T, S) and $\epsilon > 0$. Define set-value function $U : A \to 2^B$*

$$Ux = \{y \in B \ : d(Tx, Sy) \leq \epsilon\}.$$

Then U is upper semicontinuous.

Proof. Suppose N is a closed set in B. We show that $D = \{x \in A : Ux \cap N \neq \emptyset\}$ is closed. If $\{x_n\} \subset D$ such that $x_n \to x \in X$. Therefore there exists sequence $\{g_n\} \subset B$ such that for $n = 1, 2, ..., g_n \in Ux_n \cap N$. Since $g_n \in Ux_n$ then

$$\begin{aligned} d(Tx, Sg_n) &\leq d(Tx, Tx_n) + d(Tx_n, Sg_n) \\ &= d(Tx, Tx_n) + \epsilon. \end{aligned}$$

Since $\lim_{n\to\infty} x_n = x$ and T is continous, then $\lim_{n\to\infty} d(Tx, Sg_n) \leq \epsilon$. Since B is ϵ-approximately compact for (T, S). There exists $g_{n_k} \to g_0 \in B$ and so

$$d(Tx, Sg_0) = d(Tx, \lim Sg_{n_k}) = \lim d(Tx, Sg_{n_k}) \leq \epsilon.$$

Thus $g_0 \in Ux$. Hence $g_0 \in Ux \cap N$ i.e. $x \in D$. Therefore D is closed. $\blacksquare$

Definition 2.13. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be continues maps such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. We define diameter $P^{\epsilon}_{(T,S)}(A, B)$ by*

$$diam(P^{\epsilon}_{(T,S)}(A, B)) = \sup\{d(x, y) : \quad d(Tx, Ty) \leq \epsilon\}.$$

Example 2.14. Suppose $A = \{(X, 0) : \ 0 \leq x \leq 1\}$, $B = \{(x, 1) : \ 0 \leq x \leq 1\}$, $T(x, 0) = (\frac{1}{2}, 1)$ and $S(x, 1) = (\frac{1}{2}, 0)$. Then $P^r_{(T,S)}(A, B) = \emptyset$ if $r \leq 1$ and $P^r_{(T,S)}(A, B) = A \times B$ if $r > 1$. Also $d(T(x, 0), S(y, 1)) = 1$ and $diam(P^r_{(T,S)}(A, B)) = diam(A \times B) = \sqrt{2}$.

Theorem 2.15. *Let $T : A \cup B \to A \cup B$, $S : A \cup B \to A \cup B$ be continues maps such that $T(A) \subseteq B$, $S(B) \subseteq A$ and $\epsilon > 0$. If there exists $k \in [0, 1]$*

$$d(x, Tx) + d(Sy, y) \leq kd(x, y).$$

Then

$$diam(P^\epsilon_{(T,S)}(A, B)) \leq \frac{\epsilon}{1 - k}.$$

Proof. If $(x, y) \in P^\epsilon_{(T,S)}(A, B)$, then

$$\begin{aligned} d(x, y) &\leq d(x, Tx) + d(Tx, Sy) + d(Sy, y) \\ &\leq \epsilon + kd(x, y). \end{aligned}$$

Therefore $d(x, y) \leq \frac{\epsilon}{1-k}$. Then $diam(P^\epsilon_{(T,S)}(A, B)) \leq \frac{\epsilon}{1-k}$. ■

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A Recent Note On The Absolute Riesz Summability Factor Of Infinite Series

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Abstract. In this note we prove a new result concerning absolute summability factor of an infinite series via quasi - f -power increasing sequence, improving some conditions used by Bor [2] and Leindler [4] in recent results. In fact we are giving two improvements to the result of Leindler.

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Key words : Infinite series, absolute summability, summability factor.
2000 (MSC) : 40A05, 40D15, 40F05.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$.

A positive sequence $a = (a_n)$ is said to be quasi $\beta-$power increasing if there exists a constant $K = K(\beta, a) \ge 1$ such that

$$K\, n^{\beta}\, a_n \ge m^{\beta}\, a_m \tag{1.0}$$

holds for all $n \ge m$. If (1.0) stays with $\beta = 0$ then a is called a quasi increasing seq- uence . It should be noted that every almost increasing sequence is a quasi $\beta-$power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking $a_n = n^{-\beta}$.

A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi$-\,f\,-$power increasing sequence, $f = (f_n)$, if there exits a constant $K = K(\alpha, f)$ such that

$$K\ f_n\alpha_n \ge f_m\alpha_m$$

holds for $n \ge m \ge 1$ (see [5]). Clearly if α is quasi-f-power increasing sequence, then $\alpha\, f$ is quasi increasing sequence.

By t_n we denote the nth (C,1) mean of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \ge 1$, if (see[3])

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty .$$

A series $\sum a_n$ with partial sums s_n is said to be summable $\left|\bar{N}, p_n\right|_k, k \geq 1,$ if (see[1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_n - T_{n-1}|^k < \infty ,$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \ as \ n \to \infty$$

and

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v .$$

The following results are proved

Theorem 1.1 [2]. *Let* (X_n) *be a quasi* $\beta-$*power increasing sequence for some* $0 < \beta < 1$, *and* (λ_n) *be a real sequence. If the conditions*

$$\sum_{n=1}^{m} \frac{1}{n} P_n = O(P_m) \tag{1.1}$$

$$\lambda_n X_n = O(1), \tag{1.2}$$

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m), \tag{1.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m), \tag{1.4}$$

and

$$\sum_{n=1}^{\infty} n X_n \left|\Delta^2 \lambda_n\right| < \infty, \quad \left(\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}\right) \tag{1.5}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem 1.2[4]. *If the sequence* (X_n) *is quasi* $\beta-$*power increasing for some* $0 \leq \beta < 1$, (λ_n) *satisfies the conditions*

$$\sum_{n=1}^{m} \lambda_n = o(m) , \tag{1.6}$$

and

$$\sum_{n=1}^{m} |\Delta\lambda_m| = o(m), \tag{1.7}$$

further the conditions

$$\sum_{n=1}^{\infty} n X_n(\beta) \left|\Delta|\Delta\lambda_n|\right| < \infty , \tag{1.8}$$

(1.3) and (1.4) *holds, where* $X_n(\beta)=\max\left(n^\beta X_n, \log n\right)$, $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, *then the series* $\sum a_n\lambda_n$ *is summable* $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

2. Lemmas

Lemma 2.1. *Let* (X_n) *be a quasi* $-$ f $-$*power increasing sequence,* $f = (f_n)$ $= \left(n^\beta \log^\gamma n\right)$ *such that the conditions*

$$\lambda_n \to 0 \ \ as \ \ n \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n X_n \left|\Delta\left|\Delta\lambda_n\right|\right| < \infty, \tag{2.2}$$

are satisfied. Then

$$n X_n \left|\Delta\lambda_n\right| = O(1), \tag{2.3}$$

$$\sum_{n=1}^{m} X_n \left|\Delta\lambda_n\right| < \infty, \tag{2.4}$$

Proof. As $\Delta\lambda_n \to 0$, and $n^\beta \log^\gamma n X_n$ is non-decreasing, we have

$$\begin{aligned}
nX_n\left|\Delta\lambda_n\right| &= n^{1-\beta}\log^{-\gamma} n\, n^\beta \log^\gamma n\, X_n \sum_{v=n}^{\infty}\Delta\left|\Delta\lambda_v\right| \\
&= O(1) n^{1-\beta}\log^{-\gamma} n \sum_{v=n}^{\infty} v^\beta \log^\gamma v X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\sum_{v=n}^{\infty} v^{1-\beta}\log^{-\gamma} v\, v^\beta \log^\gamma v X_v\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&= O(1)\sum_{v=n}^{\infty} vX_v\left|\Delta\left|\Delta\lambda_v\right|\right| = O(1).
\end{aligned}$$

This proves (2.3). To prove (2.4), we observe that

$$\begin{aligned}
\sum_{v=1}^{m} X_v\left|\Delta\lambda_v\right| &= \sum_{v=1}^{m-1}\left(\sum_{r=1}^{v} X_r\right)\Delta\left|\Delta\lambda_v\right| + \left(\sum_{v=1}^{m} X_v\right)\left|\Delta\lambda_m\right| \\
&= O(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v} r^{-\beta}\log^{-\gamma} r\, r^\beta \log^\gamma r X_r\right)\left|\Delta\left|\Delta\lambda_v\right|\right| \\
&\qquad + O(1)\left(\sum_{v=1}^{m} v^{-\beta}\log^{-\gamma} v\, v^\beta \log^\gamma v X_v\right)\left|\Delta\lambda_m\right| \\
&= O(1)\sum_{v=1}^{m-1} v^\beta \log^\gamma v\, X_v \left|\Delta\left|\Delta\lambda_v\right|\right| \sum_{r=1}^{v} r^{-\beta-\epsilon}\log^{-\gamma} r\, r^{\epsilon} \\
&\qquad + O(1) m^\beta X_m \left|\Delta\lambda_m\right| \log^\gamma m \sum_{v=1}^{m} v^{-\beta-\epsilon}\log^{-\gamma} v\, v^{\epsilon}, \quad \epsilon < 1-\beta
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1} v^{\beta}\log^{\gamma} v\, X_v \left|\Delta\right|\Delta\lambda_v\left|\right| v^{\in}\log^{-\gamma} v \sum_{r=1}^{v} r^{-\beta-\in}$$

$$+ O(1) m^{\beta} X_m \left|\Delta\lambda_m\right| \log^{\gamma} m\, m^{\in}\log^{-\gamma} m \sum_{v=1}^{m} v^{-\beta-\in}$$

$$= O(1)\sum_{v=1}^{m} v^{\beta+\in} X_v \left|\Delta\right|\Delta\lambda_v\left|\right| \left(\int_1^v u^{-\beta-\in} du\right)$$

$$+ O(1) m^{\beta+\in} X_m \left|\Delta\lambda_m\right| \left(\int_1^m u^{-\beta-\in} du\right)$$

$$= O(1)\sum_{v=1}^{m} v\, X_v \left|\Delta\right|\Delta\lambda_v\left|\right| + O(1) m\, X_m \left|\Delta\lambda_m\right|$$

$$= O(1).$$

3. Main Result

We state and prove the following new result

Theorem 3.1. *If the sequences* : (X_n) *is quasi* $-\ f\ -$*power increasing,* $f = (f_n) =$ $(n^{\beta}\log^{\gamma} n)$. $0 \le \beta < 1, \gamma \ge 0$, (λ_n) *is a sequence of constants both satisfying conditions (2.1), (2.2) and*

$$\left|\lambda_n\right| X_n = O(1), \tag{3.0}$$

$$\sum_{n=1}^{\infty} \frac{1}{n X_n^{k-1}} \left|t_n\right|^k = O(X_m), \tag{3.1}$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n} \frac{1}{X_n^{k-1}} \left|t_n\right|^k = O(X_m). \tag{3.2}$$

Then the series $\sum a_n\lambda_n$ *is summable* $\left|\overline{N}, p_n\right|_k, k \ge 1.$.

Lemma 3.2. Conditions (3.1) and (3.2) where X_n is non-decreasing are weaker than conditions (1.3) and (1.4) respectively.

Proof. If (1.3) holds, then we have

$$\sum_{n=1}^{m} \frac{\left|s_n\right|^k}{n X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m} \frac{1}{n}\left|s_n\right|^k = O(X_m),$$

while if (23) is satisfied then,

$$\sum_{n=1}^{m} \frac{1}{n}\left|s_n\right|^k = \sum_{n=1}^{m} \frac{1}{n X_n^{k-1}}\left|s_n\right|^k X_n^{k-1}$$

$$= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{|s_v|^k}{vX_v^{k-1}}\right)\Delta X_n^{k-1} + \left(\sum_{n=1}^{m}\frac{|s_n|^k}{nX_n^{k-1}}\right)X_m^{k-1}$$
$$= \mathrm{O}(1)\sum_{n=1}^{m-1}X_n\left|\Delta X_n^{k-1}\right| + \mathrm{O}(X_m)X_m^{k-1}$$
$$= \mathrm{O}(X_{m-1})\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1}-X_n^{k-1}\right)+\mathrm{O}\left(X_m^k\right)$$
$$= \mathrm{O}(X_{m-1})\left(X_m^{k-1}-X_1^{k-1}\right)+\mathrm{O}\left(X_m^k\right)$$
$$= \mathrm{O}\left(X_m^k\right).$$

Therefore (1.3) implies (1.4) but not conversely .

The proof of the other part is similar.

We define the following groups of conditions

Group $A=\{(1.6),(1.7),(1.8)\}$, Group $B=\{(2.1),((2.2),(3.0)\}$

Remark . It may be mentioned that Theorem 3.1 gives two improvements in comparing with Theorem 1.2 in the following sense :

1. Conditions (1.3) and (1.4) are better than conditions (3.1) and (3.2) respectively in the following sense
   (i) Conditions (3.1) and (3.2) where X_n is non-decreasing are weaker than conditions (1.3) and (1.4) respectively (see lemma 3.2) .

   (ii). The more advantage of our conditions is to obtain the desired result without any loss of powers through estimations. As an example the proof via conditions (1.3) and (1.4) impose to deal with $|\lambda_n|^k$ *as* $|\lambda_n|^k=|\lambda_n|^k|\lambda_n|=\mathrm{O}(|\lambda_n|)$, loosing $|\lambda_n|^{k-1}$ as considered to be $\mathrm{O}(1)$. We have no such case via our conditions.
2. The group *B*, which is a subset of the set of conditions of theorem 3.1 is weaker than group *A* which is a subset of the set of conditions of theorem 1.2 in the sense that *B* implies *A* but not conversely .

Proof of Theorem 2.1. Let (T_n) denote the $(\overline{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{r=0}^{v}a_r\lambda_r=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v .$$

Therefore, for $n\geq 1$, we have

$$T_n-T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}va_v\left(\frac{1}{v}P_{v-1}\lambda_v\right),$$

and via Abel's transformation,

$$T_n-T_{n-1} = \frac{n+1}{n}\frac{p_n}{P_n}t_n\lambda_n-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vt_v\Delta\lambda_v$$

$$+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v t_v \lambda_v \frac{1}{v}$$
$$=T_{n1}+T_{n2}+T_{n3}+T_{n4}.$$

To prove the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{nj}\right|^k<\infty,\quad j=1,2,3,4.$$

Applying Holder's inequality,

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n1}\right|^k &= \mathrm{O}(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\left|t_n\right|^k\left|\lambda_n\right|^k\\
&= \mathrm{O}(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{\left|t_n\right|^k}{X_n^{k-1}}\left(X_n^{k-1}\left|\lambda_n\right|\right)^{k-1}\left|\lambda_n\right|\\
&= \mathrm{O}(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{\left|t_n\right|^k}{X_n^{k-1}}\left|\lambda_n\right|\\
&= \mathrm{O}(1)\sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{p_v}{P_v}\frac{\left|t_v\right|^k}{X_v^{k-1}}\right)\left|\Delta\left|\lambda_n\right|\right| + \mathrm{O}(1)\left(\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{\left|t_n\right|^k}{X_n^{k-1}}\right)\left|\lambda_m\right|\\
&= \mathrm{O}(1)\sum_{n=1}^{m-1}X_n\left|\Delta\lambda_n\right| + \mathrm{O}(1)X_m\left|\lambda_m\right|\\
&= \mathrm{O}(1),
\end{aligned}$$

in view of lemma 2.

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n2}\right|^k &= \mathrm{O}(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v\left|t_v\right|^k\left|\lambda_v\right|^k\left(\sum_{v=1}^{n-1}\frac{p_v}{P_{n-1}}\right)^{k-1}\\
&= \mathrm{O}(1)\sum_{v=1}^{m}p_v\left|t_v\right|^k\left|\lambda_v\right|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= \mathrm{O}(1)\sum_{v=1}^{m}\frac{p_v}{P_v}\left|t_v\right|^k\left|\lambda_v\right|^k\\
&= \mathrm{O}(1),\ \text{as in the case of } T_{n1}.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n3}\right|^k &= \mathrm{O}(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\sum_{v=1}^{n-1}P_v^k\frac{\left|t_v\right|}{X_v^{k-1}}^k\left|\Delta\lambda_v\right|\left(\sum_{v=1}^{n-1}X_v\left|\Delta\lambda_v\right|\right)^{k-1}\\
&= \mathrm{O}(1)\sum_{v=1}^{m}P_v^k\frac{\left|t_v\right|^k}{X_v^{k-1}}\left|\Delta\lambda_v\right|\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\\
&= \mathrm{O}(1)\sum_{v=1}^{m}\frac{1}{v}\frac{\left|t_v\right|^k}{X_v^{k-1}}\left(v\left|\Delta\lambda_v\right|\right)\\
&= \mathrm{O}(1)\sum_{v=1}^{m-1}\left(\sum_{r=1}^{v}\frac{1}{r}\frac{\left|t_r\right|^k}{X_r^{k-1}}\right)\Delta\left(v\left|\Delta\lambda_v\right|\right)+\mathrm{O}(1)\left(\sum_{v=1}^{m}\frac{1}{v}\frac{\left|t_v\right|^k}{X_v^{k-1}}\right)m\left|\Delta\lambda_m\right|
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1} X_v \left(-|\Delta\lambda_v| + (v+1)|\Delta|\Delta\lambda_v||\right) + O(1) m X_m |\Delta\lambda_m|$$
$$= O(1)\sum_{v=1}^{m} X_v |\Delta\lambda_v| + O(1)\sum_{v=1}^{m} v X_v |\Delta|\Delta\lambda_v|| + O(1) m X_m |\Delta\lambda_m|$$
$$= O(1).$$

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n4}|^k = O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \sum_{v=1}^{n-1} \frac{P_v}{v} |t_v|^k |\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{P_v}{v}\right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m} \frac{P_v}{v} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} \frac{P_v}{v} |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} \frac{1}{v} \frac{|t_v|^k}{X_v^{k-1}} \left(X_v |\lambda_v|\right)^{k-1} |\lambda_v|$$
$$= O(1) \sum_{v=1}^{m} \frac{1}{v} \frac{|t_v|^k}{X_v^{k-1}} |\lambda_v|$$
$$= O(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \frac{1}{r} \frac{|t_r|^k}{X_r^{k-1}}\right) |\Delta|\lambda_v|| + O(1) \left(\sum_{v=1}^{m} \frac{1}{v} \frac{|t_v|^k}{X_v^{k-1}}\right) |\lambda_m|$$
$$= O(1) \sum_{v=1}^{m} X_v |\Delta\lambda_v| + O(1)\, X_m |\lambda_m|$$
$$= O(1).$$

This completes the proof of the theorem .

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ON HILBERT'S INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Abstract

New general kinds of integral inequalities similar to Hilbert's inequality are presented

Mathematics subject classification : 26D15.
Key words : Hilbert's integral inequality, Holder's inequality, weight function.

1. Introduction

If $f, g \geq 0, \frac{1}{p}+\frac{1}{q}=1, 0<\int_0^\infty f^p(x)\,dx<\infty,\ 0<\int_0^\infty g^q(x)\,dx<\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leq \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int_0^\infty g^q(x)\,dx\right)^{1/q} \qquad (1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Many mathematicians presented generalizations or new kinds of (1). Very recently P. X. Ying and G. Mingzhe proved the following new kind

Theorem 1.1. *Let* $f(x)$ *be a real function. If* $0<\int_0^\infty f^2(x)\,dx<\infty$, *then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}dxdy\right)^2 < \pi^2\left(\left(\int_0^\infty f^p(x)\,dx\right)^2-\left(\int_0^\infty \varpi(x) f^p(x)\,dx\right)^2\right), \qquad (2)$$

where $\varpi(x)=\dfrac{1}{1+\sqrt{x}}-\dfrac{1}{1+x}$.

2. Lemma

The following lemma is needed for our aim

Lemma 2.1. *Let* $h(x,y)$ *be symmetric. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{h(x,y)}F(x,y)\,dxdy=\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{h(x,y)}\,dxdy \qquad (3)$$

where $F(x,y)=1-k(x)+k(y)$.

Proof.

$$\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}F(x,y)\,dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}\big(1-k(x)+k(y)\big)dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}dxdy-\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\,k(x)}{h(x,y)}dxdy+\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\,k(y)}{h(x,y)}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}dxdy-\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\,k(x)}{h(x,y)}dxdy+\int_0^\infty\int_0^\infty \frac{f(y)\,f(x)\,k(x)}{h(y,x)}dydx$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}dxdy-\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\,k(x)}{h(x,y)}dxdy+\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\,k(x)}{h(x,y)}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}dxdy.$$

The object of this paper is to present the following general result

3. Main Results

Theorem 3.1. *Let* $f,g,h,k\ge 0$, *h is homogeneous and symmetric of degree* λ *and* $F(x,y)=1-k(x)+k(y)\ge 0.$ *Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{h(x,y)}dxdy\right)^4$$

$$\le\left(\left(C\int_0^\infty x^{1-\lambda}\ f^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx\right)^2\right)$$

$$\times\left(\left(C\int_0^\infty x^{1-\lambda}\ g^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}g^2(x)\,dx\right)^2\right),\quad (4)$$

where

$$C=\int_0^\infty \frac{t^a}{h(1,t)}dt,\qquad C(x)=\int_0^\infty \frac{k(xt)\,t^a}{h(1,t)}dt,$$

provided the integrals on the RHS do exists.

Proof.

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)\,F(x,y)}{h(x,y)}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}}\times\frac{g(y)\sqrt{F(x,y)}}{\sqrt{h(x,y)}}\,dxdy$$
$$\leq\left(\int_0^\infty\int_0^\infty \frac{f^2(x)F(x,y)}{h(x,y)}\,dxdy\right)^{1/2}\left(\int_0^\infty\int_0^\infty \frac{g^2(y)F(x,y)}{h(x,y)}\,dxdy\right)^{1/2}$$
$$=\sqrt{M}\sqrt{N}.$$

$$M=\int_0^\infty\int_0^\infty \frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}}\left(\frac{y}{x}\right)^{a/2}\times\frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}}\left(\frac{x}{y}\right)^{a/2}dxdy$$
$$\leq\left(\int_0^\infty\int_0^\infty \frac{f^2(x)\,F(x,y)}{h(x,y)}\left(\frac{y}{x}\right)^{a}dxdy\right)^{1/2}\left(\int_0^\infty\int_0^\infty \frac{f^2(x)F(x,y)}{h(x,y)}\left(\frac{x}{y}\right)^{a}dxdy\right)^{1/2}$$
$$=\sqrt{M_1}\sqrt{M_2}.$$

$$M_1=\int_0^\infty\int_0^\infty \frac{f^2(x)\big(1-k(x)+k(y)\big)}{h(x,y)}\left(\frac{y}{x}\right)^{a}dxdy$$
$$=\int_0^\infty f^2(x)\big(1-k(x)\big)\int_0^\infty\frac{(y/x)^a}{h(x,y)}\,dydx+\int_0^\infty f^2(x)\int_0^\infty\frac{k(y)(y/x)^a}{h(x,y)}\,dydx$$
$$=\int_0^\infty x^{1-\lambda}\big(1-k(x)\big)f^2(x)\int_0^\infty\frac{u^a}{h(1,u)}\,dudx$$
$$+\int_0^\infty x^{1-\lambda}f^2(x)\int_0^\infty\frac{u^a k(xu)}{h(1,u)}\,dudx$$
$$=C\int_0^\infty x^{1-\lambda}\,f^2(x)\,dx-\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx.$$

Similarly,

$$M_2=C\int_0^\infty x^{1-\lambda}\,f^2(x)\,dx+\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx.$$

Therefore

$$M^2=\left(C\int_0^\infty x^{1-\lambda}\,f^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx\right)^2,$$

and

$$N^2=\left(C\int_0^\infty x^{1-\lambda}\,g^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}g^2(x)\,dx\right)^2.$$

Applying lemma 2.1 to have

$$\left(\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{h(x,y)}\,dxdy\right)^4=\left(\int_0^\infty\int_0^\infty\frac{f(x)g(y)F(x,y)}{h(x,y)}\,dxdy\right)^4$$

$$\le \left(\left(C\int_0^\infty x^{1-\lambda}\ f^2(x)\,dx \right)^2 - \left(\int_0^\infty \big(Ck(x)-C(x)\big)x^{1-\lambda} f^2(x)\,dx \right)^2 \right)$$

$$\times \left(\left(C\int_0^\infty x^{1-\lambda}\ g^2(x)\,dx \right)^2 - \left(\int_0^\infty \big(Ck(x)-C(x)\big)x^{1-\lambda} g^2(x)\,dx \right)^2 \right).$$

Remark. It may be mentioned that theorem 1.1 follows from corollary 4.1 by putting

$$h(x,y)=x+y,\quad a=-1/2,\quad \lambda=1,\quad k(x)=1/(1+x),$$

as follows

$$C=\int_0^\infty \frac{u^{-1/2}}{1+u}\,du = 2\int_0^\infty \frac{du}{1+u^2}=\pi\ .$$

$$C(x)=\int_0^\infty \frac{k(xu)\,u^{-1/2}}{1+u}\,du=\int_0^\infty \frac{k(xu^2)}{1+u^2}\,du=\int_0^\infty \frac{du}{(1+xu^2)(1+u^2)}=\frac{\pi}{1+\sqrt{x}},$$

and

$$Ck(x)-C(x)=\pi\left(\frac{1}{1+x}-\frac{1}{1+\sqrt{x}}\right).$$

Theorem 3.2. *Let* $f\ge 0,\ 0\le g\le 1,\ \lambda>0$. *Then*

$$\left(\int_0^\infty\!\!\int_0^\infty \frac{f(x)\,f(y)\sqrt{1-g(x)}\,\sqrt{1+g(y)}}{(x+y)^\lambda}\,dxdy\right)^2$$

$$\le B^2\!\left(\tfrac{\lambda}{2},\tfrac{\lambda}{2}\right)\left(\left(\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right)^2-\left(\int_0^\infty x^{1-\lambda} f^2(x)\,g(x)\,dx\right)^2\right).$$

Proof.

$$\int_0^\infty\!\!\int_0^\infty \frac{f(x)\,f(y)\sqrt{1-g(x)}\,\sqrt{1+g(y)}}{(x+y)^\lambda}\,dxdy$$

$$=\int_0^\infty\!\!\int_0^\infty \frac{f(x)\sqrt{1-g(x)}}{(x+y)^{\lambda/2}}\left(\frac{y}{x}\right)^{\frac{\lambda}{4}-\frac{1}{2}}\times\frac{f(y)\sqrt{1+g(y)}}{(x+y)^{\lambda/2}}\left(\frac{x}{y}\right)^{\frac{\lambda}{4}-\frac{1}{2}}dxdy$$

$$\le\left(\int_0^\infty\!\!\int_0^\infty \frac{f^2(x)\big(1-g(x)\big)}{(x+y)^\lambda}\left(\frac{y}{x}\right)^{\frac{\lambda}{2}-1}dxdy\right)^{1/2}\left(\int_0^\infty\!\!\int_0^\infty \frac{f^2(y)\big(1+g(y)\big)}{(x+y)^\lambda}\left(\frac{x}{y}\right)^{\frac{\lambda}{2}-1}dxdy\right)^{1/2}$$

$$=\sqrt{G}\sqrt{H}\ .$$

$$G = \int_0^\infty f^2(x)\left(1-g(x)\right)\left(\int_0^\infty \frac{(y/x)^{\frac{\lambda}{2}-1}}{(x+y)^\lambda}dy\right)dx$$

$$= \int_0^\infty f^2(x)\left(1-g(x)\right)\left(\int_0^\infty \frac{x^{1-\lambda}t^{\frac{\lambda}{2}-1}}{(1+t)^\lambda}dt\right)dx$$

$$= B\left(\tfrac{\lambda}{2},\tfrac{\lambda}{2}\right)\int_0^\infty x^{1-\lambda} f^2(x)\left(1-g(x)\right)dx\,.$$

Similarly

$$H = B\left(\tfrac{\lambda}{2},\tfrac{\lambda}{2}\right)\int_0^\infty y^{1-\lambda} f^2(y)\left(1+g(y)\right)dy\,,$$

and hence

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)\sqrt{1-g(x)}\,\sqrt{1+g(y)}}{\left(x+y\right)^\lambda}dxdy\right)^2 \le GH$$

$$= B^2\left(\tfrac{\lambda}{2},\tfrac{\lambda}{2}\right)\left(\left(\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right)^2 - \left(\int_0^\infty x^{1-\lambda} f^2(x)\,g(x)\,dx\right)^2\right).$$

4. Applications

Corollary 4.1. *Let* $f, h, k \ge 0,$ *h is homogeneous and symmetric of degree* λ *and* $F(x,y) = 1-k(x)+k(y) \ge 0.$ *Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{h(x,y)}dxdy\right)^2$$

$$\le \left(\left(C\int_0^\infty x^{1-\lambda}\; f^2(x)\,dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right)x^{1-\lambda} f^2(x)\,dx\right)^2\right)$$

where

$$C = \int_0^\infty \frac{t^a}{h(1,t)}dt, \qquad C(x) = \int_0^\infty \frac{k(xt)\,t^a}{h(1,t)}dt\,,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from theorem 3.1 by putting $g = f$.

Corollary 4.2. *Let* $f, h, k \ge 0,$ *h is homogeneous and symmetric of degree* λ *and* $F(x,y) = 1-k(x)+k(y) \ge 0.$ *Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,f(y)}{(x+y)^\lambda}dxdy\right)^2$$

$$\leq\left(\left(C\int_0^\infty x^{1-\lambda}\ f^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx\right)^2\right)$$

where

$$C=\int_0^\infty\frac{t^a}{(1+t)^\lambda}dt,\quad C(x)=\int_0^\infty\frac{k(xt)\,t^a}{(1+t)^\lambda}dt,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h(x,y)=(x+y)^\lambda$.

Corollary 4.3. *Let* $f,h,k\geq 0$, *h is homogeneous and symmetric of degree* λ *and* $F(x,y)=1-k(x)+k(y)\geq 0.$ *Then*

$$\left(\int_0^\infty\int_0^\infty\frac{f(x)\,f(y)}{x^\lambda+y^\lambda}dxdy\right)^2$$
$$\leq\left(\left(C\int_0^\infty x^{1-\lambda}\ f^2(x)\,dx\right)^2-\left(\int_0^\infty\big(Ck(x)-C(x)\big)x^{1-\lambda}f^2(x)\,dx\right)^2\right)$$

where

$$C=\int_0^\infty\frac{t^a}{1+t^\lambda}dt,\quad C(x)=\int_0^\infty\frac{k(xt)\,t^a}{1+t^\lambda}dt,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h(x,y)=x^\lambda+y^\lambda$.

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Optimal Source for Maximum Distinguishability in Optical Imaging

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Abstract

In this paper, we formulate an optimal source problem for optical imaging by maximizing eight different distinguishability criteria. Each distinguishability criterion represents one of the eight pairs of function spaces considered. We extend the general concept of finding an optimal current for a particular distinguishability criterion, as in electrical impedance imaging, to an optimal source in optical imaging.

The optimal source depends on the distinguishability criterion which in turn depends on the choice of the function space pairs. We shall provide an analytic framework for computing all these eight different choices of distinguishability criteria. In particular, the adjoints of the relevant linear operators involved in the criteria are derived.

Numerically, we employ the power method to compute the optimal sources, i.e. the dominant eigenfunctions of the associated operators. Numerical experiments are presented to demonstrate the efficacy of different criteria, and a localization measure is used to determine the optimal source profile best discriminating inhomogeneities from a known background.

Keywords: distinguishability criteria, abstract function spaces, optimal source, optical imaging, biomedical imaging

1 Introduction

In optical imaging of highly scattering media such as biological tissue, the media is illuminated by low-energy visible (wavelength from 380 to 750 nm) or near-infrared light (wavelength from 700 to 1200 nm). Light penetrates the medium and interacts with it. The predominant effects are absorption and scattering [12, 3, 10, 16]. The widely accepted photon transport model is the radiative transfer equation [5, 19], an integro-differential equation for the radiance, involving spatially varying scattering and absorption parameters. In practice, a low order diffusion approximation to the radiative transfer equation is often adopted. The approximation is a parabolic differential equation in the time-dependent case, while in the steady-state case or frequency domain, it is an elliptic differential equation [1, 2]. Most existing computational methods for photon transport in highly scattering media such as photon migration in biological tissues are based on the diffusion approximation because of its simplicity compared to the full blown radiative transfer equation [4, 17].

In the experimental setup of non-invasive diffuse optical imaging such as optical tomography and other similar non-invasive imaging modalities, e.g. electrical impedance imaging, one has very little control over the parameters in the domain other than the boundary source. Therefore, the idea

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of using an optimal source to maximize the distinguishability is an important step. For example suppose that $\Lambda(q^+)[f] : X \to Y$ is the source to data map where q^+ represents the true parameters and $\Lambda(q)[f] : X \to Y$ is the source to simulated data map for a guess q of the parameters. Then one may use the following natural distinguishability criterion [18, 7], $\delta_{XY}(f) = \|\Lambda(q)[f] - \Lambda(q^+)[f]\|_Y$ to find the best f that will discriminate between the simulated model and the measured data by solving the following problem,

$$\max_{f \in X} \delta_{XY}(f) = \|\Lambda(q)[f] - \Lambda(q^+)[f]\|_Y \tag{1}$$

where $\Lambda : X \to Y$. The selection of an optimal source by maximizing a particular distinguishability criterion has potential to lead to the following benefits:

(i) to improve image reconstruction in diffuse optical imaging applications. For example, there exist numerous studies in electrical impedance tomography (EIT) that indicate that optimal currents have the potential to improve the image reconstruction [18, 6, 22, 7]. It is expected that the above mentioned developments in EIT will have a similar impact for optical tomography (OT) [9, 14];

(ii) in formulating a statistical hypothesis testing problem to address the question of whether there is a tumor present or not. For example, one of the criteria studied in this paper corresponds to the maximum likelihood ratio test. Therefore our framework will also have applications in signal and image reconstruction [26];

(iii) for extracting features of the source-to-data operator using the outer product of the first few largest eigenvectors as an approximation of the operator. In addition, the characteristic profile of the optimal source may also correlate to the type of inhomogeneity present in the interior;

(iv) in developing a general framework to find optimal source for hybrid imaging modalities. It is expected that hybrid imaging techniques involving optical sources will develop rapidly in the coming decade [15, 29, 28].

Therefore, the problem of finding an optimal source with maximum distinguishability is an important and timely subject in optical imaging. In addition, the laser sources in optical imaging are more flexible in terms of tunability and adaptability [24, 25] than, for example, current sources in electrical impedance imaging. Hence, we expect an even stronger impact of using optimal sources in optical imaging. Furthermore, the tremendous growth and intense research in laser sources make a wide range of sources realistically achievable and consequently might improve image resolution [24, 25].

The idea of distinguishability criteria for optimal currents in EIT has been around for more than a decade [18, 13, 6, 22, 7]. However, a thorough investigation in a functional analytic framework involving all of the relevant function space pairs and numerical comparison of respective optimal currents are only getting attention very recently [20]. A careful reading of some of the literature referenced above both in electrical impedance and optical imaging indicate that most of these papers mention the general setup for various appropriate function spaces but then typically choose one of them without fully analyzing the general case. It seems that this is done mainly to avoid cumbersome analytic calculations required to compute the optimal source.

In this paper we will formulate the distinguishability criteria for all eight relevant pairs of the function spaces for the operator of choice and demonstrate the general framework on how to handle them both theoretically and computationally. In the numerical section we will then determine the function space setting most suitable in designing optimal sources for optical tomography experiments.

Let us summarize the current state of the art of optimal sources in diffuse optical imaging. There exist past and recent studies on how to optimize wavelength and frequency in optical image reconstruction [9, 14], as well as on how to optimize location and power for sources and detectors [26]. However, none of these studies simultaneously consider all of the relevant function spaces pairs, and heuristic analysis is carried out on the discretized problem without analyzing the involved function spaces. In this paper, we shall treat the infinite dimensional model and propose an algorithm to compute the optimal source with a finite dimensional approximation for every plausible function space setting in a mathematically rigorous manner. In particular, we shall extend the distinguishability criteria for optimal currents in impedance imaging to optimal sources in diffuse optical imaging, systematically investigate the influence of function spaces on optimal sources both analytically and numerically, and determine the most effective pair of function spaces using a localization measure.

The analytical work and the computational simulations presented in this paper involve eight function space pairs. Therefore in this paper, we have concentrated on the details of the appropriate formulation relevant to our computational setup. We have presented relevant simulations to demonstrate that the solutions for the different distinguishability criteria corresponding to the various function spaces differ. For convenience, in this paper, we only considered the Neumann and Dirichlet boundary conditions; however the analysis can be easily extended to any boundary conditions including Robin boundary conditions. We note that the there are several questions that require further analysis and are beyond the scope of this manuscript: (i) since distinguishability is a local measure, a theoretical and numerical sensitivity analysis of how the solution varies from region to region in the parameter space will be taken up in the future, (ii) the simulations presented in the paper involve mainly model error due to discretization; however we do not take up the issue of effects due to finite dimensional projection or approximation, or the issue of uncertainty analysis.

The rest of the paper is organized as follows. We introduce relevant mathematical models and describe important function spaces in Section 2. In Section 2.2, we discuss the optimal source problem. To numerically calculate the optimal sources of these distinguishability criteria, we determine the adjoint of appropriate operators in Section 3, and in Section 4 we demonstrate the efficacy of our approach using two-dimensional examples of optical imaging source optimization.

2 Mathematical Model and Distinguishability Criteria

In this section, we describe the mathematical model and introduce distinguishability criteria as well as the functional analytical framework for properly formulating these models.

2.1 Mathematical model

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be an open bounded domain with Lipschitz boundary $\partial\Omega$. The diffusion approximation model in the frequency domain for optical tomography can be written as

$$\begin{aligned} -\nabla \cdot (D\nabla u) + (\mu + ik)\, u &= h \quad \text{in } \Omega, \\ \gamma_1 u &= f \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $k = \omega/c$ is the wave number with ω and c representing the frequency of laser modulation and light speed, respectively, and γ_1 is the Neumann trace map, i.e. $\gamma_1 u = D\frac{\partial u}{\partial \nu}$. We further assume that $D \in L^\infty(\Omega)$ is the diffusion coefficient with $0 < D_0 \leq D(x) \leq D_1 < \infty$ for positive real constants D_0 and D_1, $\mu \in L^\infty(\Omega)$ is the absorption coefficient with $0 < \mu_0 \leq \mu(x) \leq \mu_1 < \infty$ for positive real constants μ_0 and μ_1, and f is a source function. We denote by $q = (D, \mu)$ the parameters, and by $F_N^{(k,q)}(h, f)$ the solution, which physically represents the photon density.

2.2 Distinguishability Criterion

Here we introduce the distinguishability criterion. Let us denote the true parameter by q^+, which is unknown, and denote the initial guess by q. Typically, q is a constant function obtained from mean values of typical tissue parameters or an a priori guess of the background. Information about q^+ can be obtained by performing OT experiments, e.g. applying a source f and measuring $\gamma_0 F_N^{(k,q^+)}(0, f)$ on the boundary. Hence $A : X \to Y$ defined by

$$Af = \gamma_0 F_N^{(k,q)}(0, f) - \gamma_0 F_N^{(k,q^+)}(0, f) \tag{3}$$

is an operator measuring the difference between q^+ and q in a certain sense. Note that the term $\gamma_0 F_N^{(k,q^+)}$ can be experimentally obtained without any knowledge of the unknown parameter q^+. Hence in a real life situation, evaluating A amounts to performing an OT experiment with q^+ and numerically evaluating Neumann-Dirichlet trace operators for q.

The inverse optimal source problem can be formulated as maximizing such a discrepancy functional involving the operator A, i.e.

$$\max_{f \in X} \|Af\|_Y,$$

where the space Y is defined on boundary $\partial\Omega$. In either case the inverse source problem depends on the norms of X and Y.

We also consider a second class of discrepancy functionals, which are used to measure distinguishability. Following the approach of [21], we map the measured Dirichlet boundary data back to the domain by solving a Dirichlet forward problem: we introduce

$$Bf = F_N^{(k,q)}(0, f) - F_D^{(k,q)}(0, \gamma_0 F_N^{(k,q^+)}(0, f)) \tag{4}$$

and determine the optimal source by finding $\max_{f \in X} \|Bf\|_Z$, where X is a function space on the boundary $\partial\Omega$ as before, but Z is now a function space on Ω. Note that in practical applications, smoothing of the experimental data $\gamma_0 F_N^{(k,q^+)}(0, f)$ may be required if the abstract space X under consideration requires more smoothness.

Mathematically, the optimal sources amount to dominant eigenfunctions of the operator A or B. Since in general the operator A or B is not self-adjoint for any combination of X and Y or X and Z computing the optimal source requires finding dominant eigenfunctions of A^*A and B^*B. We shall compute the optimal source as the limit of the following iteration

$$f_{n+1} = \frac{A^*Af_n}{\|A^*Af_n\|},$$

or

$$f_{n+1} = \frac{B^*Bf_n}{\|B^*Bf_n\|}$$

in the spirit of the classical power method of numerical linear algebra. The determination of A^* and B^* requires calculating adjoints of the Neumann to Dirichlet map operator and Neumann and Dirichlet solution operator in certain function spaces. However, for the operator considered in this paper – the optical tomography model – is not self-adjoint, and hence the choices of different combinations of function spaces result in rather different optimal sources. We consider four choices of function space pairs for A and four choices of function space pairs for B yielding a total of eight different distinguishability criteria.

2.3 Sesquilinear Weak Formulation

We shall use the following sesquilinear form

$$a^{(k,q)}(u,v) = \int_\Omega D\nabla u \cdot \nabla \bar{v} + (\mu + ik) u\bar{v} dx.$$

The weak formulation of the Neumann forward problem (2) now reads: find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$

$$a^{(k,q)}(u,v) = \int_\Omega h\bar{v}dx + \int_{\partial\Omega} f\gamma_0\bar{v}ds. \tag{5}$$

The measurement g is modeled as the photon density u at the boundary, i.e.,

$$g = u|_{\partial\Omega} = \gamma_0 u \tag{6}$$

where $\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ denotes the Dirichlet trace operator. With the measurement g, we can define a nonhomogeneous Dirichlet boundary value problem as follows,

$$\begin{aligned} -\nabla \cdot (D\nabla u) + (\mu + ik)\, u = h \quad & \text{in } \Omega, \\ \gamma_0 u = g \quad & \text{on } \partial\Omega. \end{aligned} \tag{7}$$

We denote the Dirichlet solution u by $F_D^{(k,q)}(h,g)$. Its weak formulation is given by: find $u \in H^1(\Omega)$ with $\gamma_0 u = g$ such that

$$a^{(k,q)}(u,v) = \int_\Omega h\bar{v}dx \quad \forall v \in H_0^1(\Omega). \tag{8}$$

Observe that $F_D^{(k,q)}(h,g) = F_N^{(k,q)}(h, \gamma_1 F_D^{(k,q)}(h,g))$, and thus by the weak formulation of $F_N^{(k,q)}$, there holds: for any $v \in H^1(\Omega)$

$$a^{(k,q)}(F_D^{(k,q)}(h,g), v) = \int_\Omega h\bar{v}dx + \int_{\partial\Omega} \gamma_1 F_D^{(k,q)}(h,g)\bar{v}ds.$$

This formula will be used several times in Section 3.

We mainly consider boundary value problems (2) and (7) with vanishing h, and the corresponding weak formulations (5) and (8). We will at times consider these boundary value problems but with $k = 0$ or opposite signs for k, which shall be denoted by e.g. $F_N^{(0,q)}(h,f)$ and $F_N^{(-k,q)}(h,f)$. Finally, we let $\mathcal{L}^{(k,q)}u = -\nabla \cdot (D\nabla u) + (\mu + ik)u$.

2.4 Function spaces

To investigate the mathematical models, we shall utilize a Hilbert space framework [11, 8]. First we equip the space $H_q^1(\Omega)$ with an inner product

$$\langle u, v\rangle_{H_q^1(\Omega)} = \int_\Omega D\nabla u \cdot \nabla\bar{v} + \mu u\bar{v}dx \tag{9}$$

where the bar denotes taking complex conjugate. The dual of $H_q^1(\Omega)$ will be denoted by $H_q^{-1}(\Omega)$. Then we define an inner product for $H_q^{1/2}(\partial\Omega)$ by

$$\langle f, g\rangle_{H_q^{1/2}(\partial\Omega)} = \langle F_D^{(0,q)}(0,f), F_D^{(0,q)}(0,g)\rangle_{H_q^1(\Omega)}.$$

By appealing to the weak formulation of $F_D^{(0,q)}(0,f) = F_N^{(0,q)}(0, \gamma_1 F_D^{(0,q)}(0,f))$, we can equivalently write

$$\langle f, g\rangle_{H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} \gamma_1 F_D^{(0,q)}(0,f)\bar{g}ds = \int_{\partial\Omega} f\gamma_1 \overline{F_D^{(0,q)}(0,g)}ds.$$

Similarly, we define the $H_q^{-1/2}(\partial\Omega)$ inner product by

$$\langle f, g\rangle_{H_q^{-1/2}(\partial\Omega)} = \langle F_N^{(0,q)}(0,f), F_N^{(0,q)}(0,g)\rangle_{H_q^1(\Omega)}$$

and equivalently

$$\langle f, g\rangle_{H_q^{-1/2}(\partial\Omega)} = \int_{\partial\Omega} f\gamma_0 \overline{F_N^{(0,q)}(0,g)}ds = \int_{\partial\Omega} \gamma_0 F_N^{(0,q)}(0,f)\bar{g}ds.$$

We would like to point out that the spaces $H_q^{1/2}(\partial\Omega)$ and $H_q^{-1/2}(\partial\Omega)$ defined above are dual to each other, and thus the integrals on the boundary can be understood as duality pairings as usual.

2.5 Riesz representation of the sesquilinear form

The following Riesz representation of the sesquilinear form $a^{(k,q)}(u,v)$ in (5) will be needed in the subsequent sections. By virtue of the Riesz representation theorem for sesquilinear functionals [8], we deduce that there exists a unique continuous bijection $S : H_q^1(\Omega) \to H_q^1(\Omega)$ such that for any $u, v \in H_q^1(\Omega)$ there holds

$$a^{(k,q)}(u,v) = \langle Su, v\rangle_{H_q^1(\Omega)} = \langle u, S^* v\rangle_{H_q^1(\Omega)}. \tag{10}$$

The operator S depends on q and k, and the dependence is often suppressed. The representation operator S and its adjoint S^* can be explicitly characterized. The operator S will play a crucial role in deriving the adjoints of various operators below.

Lemma 1. *The operator S and its adjoint S^* are given by $Su = F_N^{(0,q)}(\mathcal{L}^{(k,q)}u, \gamma_1 u)$ and $S^*v = F_N^{(0,q)}(\mathcal{L}^{(-k,q)}v, \gamma_1 v)$, respectively. In particular, $w = Su$ satisfies*

$$\begin{aligned} -\nabla\cdot(D\nabla w) + \mu w &= -\nabla\cdot(D\nabla u) + (\mu + ik)u \\ \gamma_1 w &= \gamma_1 u. \end{aligned}$$

Proof. Let $w = F_N^{(0,q)}(\mathcal{L}^{(k,q)}u, \gamma_1 u)$. By the definition of the $H^1(\Omega)$ norm and its weak formulation we have that for any $v \in H_q^1(\Omega)$

$$\langle w, v\rangle_{H_q^1(\Omega)} = \int_\Omega \mathcal{L}^{(k,q)}u\bar{v}dx + \int_{\partial\Omega} \gamma_1 u\bar{v}ds = a(u,v).$$

Thus $Su = w = F_N^{(0,q)}(\mathcal{L}^{(k,q)}u, \gamma_1 u)$. The assertion for S^* can be shown similarly. □

The operators S and S^* have some interesting properties. For a detailed discussion see [23]. In subsequent sections we will need the following properties.

Lemma 2. *Let $u, v \in H_q^1(\Omega)$, $f, \tilde{f} \in H_q^{1/2}(\partial\Omega)$, $g \in H_q^{-1/2}(\partial\Omega)$ and $h \in H_q^{-1}(\Omega)$. There hold*

$$\begin{aligned} &a) \quad S^* F_N^{(-k,q)}(h,g) &&= F_N^{(0,q)}(h,g) = S F_N^{(k,q)}(h,g), \\ &b) \quad \gamma_1 S u &&= \gamma_1 u = \gamma_1 S^* u, \\ &c) \quad \langle Su, S^{*-1}v\rangle_{H_q^1(\Omega)} &&= \langle u, v\rangle_{H_q^1(\Omega)}, \end{aligned}$$

$$d)\ f = \gamma_0 S^{-1}(F_D^{(0,q)}(0,\tilde{f})) \iff \tilde{f} = \gamma_0 S F_D^{(k,q)}(0,f).$$

Proof. The first two lines follow directly from Lemma 1, and c) follows from the definition of S and S^*. To see d), we observe from the characterization of S that $\tilde{f} = \gamma_0 S F_D^{(k,q)}(0, f)$ is equivalent to

$$\tilde{f} = \gamma_0 F_N^{(0,q)}(\mathcal{L}^k F_D^{(k,q)}(0, f), \gamma_1 F_D^{(k,q)}(0, f)) = \gamma_0 F_N^{(0,q)}(0, \gamma_1 F_D^{(k,q)}(0, f)),$$

which consequently gives

$$F_D^{(0,q)}(0, \tilde{f}) = F_N^{(0,q)}(0, \gamma_1 F_D^{(k,q)}(0, f)) = S F_D^{(k,q)}(0, f),$$

i.e.

$$F_D^{(k,q)}(0, f) = S^{-1} F_D^{(0,q)}(0, \tilde{f}).$$

Upon taking Dirichlet trace, it gives the desired assertion. The remaining parts follow similarly. □

3 Adjoint Operators

The aim of this paper is to determine optimal source profiles for applications in optical imaging. To this end we compare measurements for the true parameter q^+ with simulated measurements obtained with an a priori guess q. As explained in Section 2, we use a functional analytic framework for characterizing the optimal sources. This leads to the task of solving the problem,

$$\max_{\|f\|_X=1} \|Af\|_Y$$

or

$$\max_{\|f\|_X=1} \|Bf\|_Z$$

or equivalently of finding the largest eigenvalue of A^*A and B^*B. Here A measures the difference between the Neumann-to-Dirichlet maps for q^+ and q. Similarly, B measures the difference between the Dirichlet solution, in the full domain, corresponding to the experimental Neumann-to-Dirichlet data for q^+ and the simulated solution, in the full domain, corresponding to a given guess q.

As we have seen already, we can use different function spaces for the domain of $A : X \to Y$, i.e. $X = L_2(\partial\Omega)$ or $X = H^{-1/2}(\partial\Omega)$, and two choices for the range of A, i.e. $X = L_2(\partial\Omega)$ or $X = H^{1/2}(\partial\Omega)$. Similarly, we can use different function spaces for the domain and range of $B : X \to Z$, i.e. $X = L_2(\partial\Omega)$ or $X = H^{-1/2}(\partial\Omega)$ and $Z = L_2(\Omega)$ or $Z = H^1(\Omega)$. The adjoint

$$A^* : Y \to X \ ,$$

which is determined via its defining relation $\langle Af, g\rangle_Y = \langle f, A^*g\rangle$, depends on the choice of function spaces and accordingly the resulting optimal sources will differ considerably, when computed for those different function space settings. The same holds for the operator B.

First, in Section 3.1. we determine the adjoint operators for the two most natural function space settings for A. The general case for both A and B is discussed in Section 3.2. The computation of adjoint operators is a standard exercise in functional analysis; however, our choice of norms, e.g. $H_q^1(\Omega)$, depends on the parameters q or q^+, which leads to some subtle computations even for the trivial embedding operators.

3.1 Adjoint operators for the classical functionals

In this section we address the computation of the adjoint operators for evaluating the distinguishability criterion based on comparing the Neumann-to-Dirichlet maps for q and q^+, as described in (3):

$$\|Af\|_Y = \|\gamma_0 F_N^{(k,q)}(0,f) - \gamma_0 F_N^{(k,q^+)}(0,f)\|_Y \ .$$

We use the two natural function space settings

$$\gamma_0 F_N^{(k,q)}(0,\cdot) : L_2(\partial\Omega) \to L_2(\partial\Omega)$$

and

$$\gamma_0 F_N^{(k,q)}(0,\cdot) : H_q^{-1/2}(\partial\Omega) \to H_q^{1/2}(\partial\Omega) \ .$$

In the second case, the $H_q^{\pm 1/2}$ norms depend in q, hence we have to consider the embedding operators

$$e_- : H_q^{-1/2}(\partial\Omega) \to H_{q^+}^{-1/2}(\partial\Omega) \text{ and } e_+ : H_{q^+}^{1/2}(\partial\Omega) \to H_q^{1/2}(\partial\Omega)$$

between spaces with parameters $q = (D,\mu)$ and $q^+ = (D^+,\mu^+)$.

Accordingly we must compute the adjoints of $\gamma_0 F_N^{(k,q)}(0,\cdot)$ and $e_+ \circ \gamma_0 F_N^{(k,q^+)}(0,\cdot) \circ e_-$ before obtaining a closed expression for A^*. This adjoint is then used in Section 4 for computing the optimal sources.

We start by computing the adjoints of the Neumann-to-Dirichlet maps.

Lemma 3. *Let $\Lambda(k,q) = \gamma_0 F_N^{(k,q)}(0,\cdot)$ denote the Neumann-to-Dirichlet map. The adjoint of $\Lambda(k,q) : X \to Y$ is given by*

(a) $\Lambda(k,q)^* g = \gamma_0 F_N^{(-k,q)}(0,g)$ *for $X = L^2(\partial\Omega)$ and $Y = L^2(\partial\Omega)$;*

(b) $\Lambda(k,q)^* g = \gamma_1 F_D^{(0,q)}(0, \gamma_0 S^{*-1} F_D^{(0,q)}(0,g))$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^{1/2}(\partial\Omega)$.*

Proof. For the first case, we consider $\langle \Lambda(k,q)f, g\rangle_{L_2(\partial\Omega)}$. By the weak formulations of $F_N^{(-k,q)}(0,g)$ and $F_N^{(0,q)}(0,f)$, we derive

$$\begin{aligned}
\langle \Lambda(k,q)f, g\rangle_{L^2(\partial\Omega)} &= \overline{\langle g, \Lambda(k,q)f\rangle_{L^2(\partial\Omega)}} \\
&= \overline{a^{(-k,q)}(F_N^{(-k,q)}(0,g), F_N^{(k,q)}(0,f))} \\
&= a^{(k,q)}(F_N^{(k,q)}(0,f), F_N^{(-k,q)}(0,g)) \\
&= \langle f, \gamma_0 F_N^{(-k,q)}(0,g)\rangle_{L^2(\partial\Omega)}.
\end{aligned}$$

Hence we obtain $\Lambda(k,q)^* = \Lambda(-k,q)$.

For proving the second case, we start with $\langle \Lambda(k,q)f, g\rangle_{H_q^{1/2}(\partial\Omega)}$ and use the definition of the norm in $H_q^{1/2}(\partial\Omega)$. Moreover, we apply the identity $F_D^{(0,q)}(0, \gamma_0 F_N^{(k,q)}(0,f)) = F_N^{(k,q)}(0,f)$, which follows from observing that the Dirichlet traces as well as the vanishing source terms coincide:

$$\begin{aligned}
\langle \Lambda(k,q)f, g\rangle_{H_q^{1/2}(\partial\Omega)} &= \langle F_D^{(0,q)}(0, \gamma_0 F_N^{(k,q)}(0,f)), F_D^{(0,q)}(0,g)\rangle_{H_q^1(\Omega)} \\
&= \langle F_N^{(k,q)}(0,f), F_D^{(0,q)}(0,g)\rangle_{H_q^1(\Omega)} \\
&= a^{(0,q)}(F_N^{(k,q)}(0,f), F_D^{(0,q)}(0,g)).
\end{aligned}$$

We now apply the definition of S, see (10), and the weak formulation for $F_N^{(k,q)}(0,f)$ to obtain

$$\begin{aligned}\langle \Lambda(k,q)f,g\rangle_{H_q^{1/2}(\partial\Omega)} &= a^{(k,q)}(F_N^{(k,q)}(0,f), S^{*-1}F_D^{(0,q)}(0,g))\\ &= \int_{\partial\Omega} f\gamma_0\overline{S^{*-1}F_D^{(0,q)}(0,g)}ds.\end{aligned}$$

Finally we prepare to use the definition of the norm in $H^{-1/2}$ as follows

$$\begin{aligned}\langle \Lambda(k,q)f,g\rangle_{H_q^{1/2}(\partial\Omega)} &= \int_{\partial\Omega} f\gamma_0\overline{S^{*-1}F_D^{(0,q)}(0,g)}ds\\ &= \langle F_N^{(0,q)}(0,f), F_D^{(0,q)}(0,\gamma_0 S^{*-1}F_D^{(0,q)}(0,g))\rangle_{H_q^1(\Omega)}\\ &= \langle F_N^{(0,q)}(0,f), F_N^{(0,q)}(0,\gamma_1 F_D^{(0,q)}(0,\gamma_0 S^{*-1}F_D^{(0,q)}(0,g)))\rangle_{H_q^1(\Omega)}\\ &= \langle f, \gamma_1 F_D^{(0,q)}(0,\gamma_0 S^{*-1}F_D^{(0,q)}(0,g))\rangle_{H_q^{-1/2}(\partial\Omega)}.\end{aligned}$$

This proves the second assertion.

□

In the next lemma, we explicitly compute the adjoints of the embedding operators.

Lemma 4. *The adjoints of the embedding operators e_- and e_+ are given by*

$$\begin{aligned}e_-^* g &= \gamma_1 F_D^{(0,q)}(0,\gamma_0 F_N^{(0,q^+)}(0,g)),\\ e_+^* g &= \gamma_0 F_N^{(0,q^+)}(0,\gamma_1 F_D^{(0,q)}(0,g)).\end{aligned}$$

Proof. The definition of the $H_{q^+}^{-1/2}(\partial\Omega)$ inner product and the weak formulations of $F_N^{(0,q^+)}(0,f)$ and $F_N^{(0,q)}(0,f)$ give

$$\begin{aligned}\langle e_- f,g\rangle_{H_{q^+}^{-1/2}(\partial\Omega)} &= \langle F_N^{(0,q^+)}(0,f), F_N^{(0,q^+)}(0,g)\rangle_{H_{q^+}^1(\Omega)}\\ &= \int_{\partial\Omega} f\gamma_0\overline{F_N^{(0,q^+)}(0,g)}ds\\ &= \langle F_N^{(0,q)}(0,f), F_D^{(0,q)}(0,\gamma_0 F_N^{(0,q^+)}(0,g))\rangle_{H_q^1(\Omega)}\\ &= \langle F_N^{(0,q)}(0,f), F_N^{(0,q)}(0,\gamma_1 F_D^{(0,q)}(0,\gamma_0 F_N^{(0,q^+)}(0,g)))\rangle_{H_q^1(\Omega)}\\ &= \langle f, \gamma_1 F_D^{(0,q)}(0,\gamma_0 F_N^{(0,q^+)}(0,g))\rangle_{H_q^{-1/2}(\partial\Omega)}.\end{aligned}$$

Now the definition of the $H_q^{1/2}(\partial\Omega)$ inner product and the weak formulations of $F_D^{(0,q)}(0,g) = F_N^{(0,q)}(0,\gamma_1 F_D^{(0,q)}(0,g))$ and $F_N^{(0,q^+)}(0,\gamma_1 F_D^{(0,q)}(0,g))$ imply

$$\begin{aligned}\langle e_- f,g\rangle_{H_q^{1/2}(\partial\Omega)} &= \langle F_D^{(0,q)}(0,f), F_D^{(0,q)}(0,g)\rangle_{H_q^1(\Omega)}\\ &= \int_{\partial\Omega} f\gamma_1\overline{F_D^{(0,q)}(0,g)}ds\\ &= \langle F_D^{(0,q^+)}(0,f), F_N^{(0,q^+)}(0,\gamma_1 F_D^{(0,q)}(0,g))\rangle_{H_{q^+}^1(\Omega)}\\ &= \langle F_D^{(0,q^+)}(0,f), F_D^{(0,q^+)}(0,\gamma_0 F_N^{(0,q^+)}(0,\gamma_1 F_D^{(0,q)}(0,g)))\rangle_{H_{q^+}^1(\Omega)}\end{aligned}$$

$$= \langle f, \gamma_0 F_N^{(0,q^+)}(0, \gamma_1 F_D^{(0,q)}(0, g)) \rangle_{H^{1/2}_{q^+}(\partial\Omega)}.$$

This completes the proof of the lemma. □

We now are in a position to state the main result of this section, an explicit expression of the adjoint A^* of the operator A with the function spaces under consideration.

Theorem 1. *Let $\tilde{E} = \gamma_0 F_N^{(-k,q)}(0,\cdot) - \gamma_0 F_N^{(-k,q^+)}(0,\cdot)$. For the operator as defined in (3) the adjoint of the operator $A : X \to Y$ is given by $A^* : Y \to X$ with A^*g defined as follows*

(a) $A^*g = \tilde{E}g$ *for $X = L^2(\partial\Omega)$ and $Y = L^2(\partial\Omega)$;*

(b) $A^*g = \gamma_1 F_D^{(0,q)}(0,\cdot) \circ \tilde{E} \circ \gamma_1 F_D^{(0,q)}(0,g))$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^{1/2}(\partial\Omega)$.*

Proof. The proof of case (a) follows directly from Lemma 3. For case (b) we must incorporate the embedding operators and consider

$$A = \gamma_0 F_N^{(k,q)}(0,\cdot) - e_+ \circ \gamma_0 F_N^{(k,q^+)}(0,\cdot) \circ e_-.$$

Applying the results of the preceding lemma gives the desired result. □

3.2 Adjoints in the general case

The previous section contained the derivation of the adjoints for the most classical cases. However, we can also consider the second distinguishibility criteria involving B as well as different combinations of function spaces. All together, we have two choices for the operator: A and B as described (3, 4). We also have four combinations of function spaces X and Y. In order to treat all these cases we also need the adjoints of the Neumann and Dirichlet forward operators. We first derive the adjoint for the Neumann forward operator $F_N^{(k,q)}(0,\cdot)$.

Lemma 5. *The adjoint $(F_N^{(k,q)}(0,\cdot))^* : Z \to X$ of the operator $F_N^{(k,q)}(0,\cdot) : X \to Z$ is given by*

(a) $(F_N^{(k,q)}(0,\cdot))^* v = \gamma_0 F_N^{(-k,q)}(v,0)$ *for $X = L^2(\partial\Omega)$ and $Z = L^2(\Omega)$;*

(b) $(F_N^{(k,q)}(0,\cdot))^* v = \gamma_0 S^{*-1} v$ *for $X = L^2(\partial\Omega)$ and $Z = H_q^1(\Omega)$;*

(c) $(F_N^{(k,q)}(0,\cdot))^* v = \gamma_1 F_D^{(0,q)}(0, \gamma_0 S^{*-1} v)$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Z = H_q^1(\Omega)$;*

(d) $(F_N^{(k,q)}(0,\cdot))^* v = \gamma_1 F_D^{(0,q)}(0, \gamma_0 F_N^{(-k,q)}(v,0))$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Z = L^2(\Omega)$.*

Proof. We only prove case (b), the other cases follow by similar arguments. Hence we consider $X = L^2(\partial\Omega)$ and $Z = H_q^1(\Omega)$. By Lemma 1 and case a) of Lemma 2, and the weak formulation of $F_N^{(0,q)}(0,f)$, we arrive at

$$\begin{aligned}\langle F_N^{(k,q)}(0,f), v \rangle_{H_q^1(\Omega)} &= \langle S F_N^{(k,q)}(0,f), S^{*-1} v \rangle_{H_q^1(\Omega)} \\ &= \langle F_N^{(0,q)}(0,f), S^{*-1} v \rangle_{H_q^1(\Omega)} = \int_{\partial\Omega} f \gamma_0 \overline{S^{*-1} v} ds.\end{aligned}$$

□

Next we calculate the adjoint of the Dirichlet operator $F_D^{(k,q)}(0,\cdot)$.

Lemma 6. *The adjoint of the operator $F_D^{(k,q)}(0,\cdot): H_q^{1/2}(\partial\Omega) \to Z$ is given by*

(a) $(F_D^{(k,q)}(0,\cdot))^* v = \gamma_0 S^* F_D^{(-k,q)}(0, \gamma_0 S^{*-1} v)$ *for* $Z = H_q^1(\Omega)$*;*

(b) $(F_D^{(k,q)}(0,\cdot))^* v = \gamma_0 S^* F_D^{(-k,q)}(0, \gamma_0 F_N^{(-k,q)}(v,0))$ *for* $Z = L^2(\Omega)$*.*

Proof. The defining equation of the adjoint operator is

$$\langle F_D^{(k,q)}(0,g), v\rangle_{H_q^1(\Omega)} = \langle g, (F_D^{(k,q)}(0,\cdot))^* v\rangle_{H_q^{1/2}(\partial\Omega)}.$$

By our definition of the $H_q^{1/2}(\partial\Omega)$ norm, we need to transform the scalar product on the left hand side such that both arguments of this scalar product are images of a Dirichlet operator with $k=0$. We start by applying case c) of Lemma 2, the characterization of S in Lemma 1, and weak formulations of $SF_D^{(k,q)}(0,g) = F_N^{(0,q)}(0, \gamma_1 F_D^{(k,q)}(0,g))$ and $F_D^{(k,q)}(0,g) = F_N^{(k,q)}(0, \gamma_1 F_D^{(k,q)}(0,g))$:

$$\begin{aligned}
\langle F_D^{(k,q)}(0,g), v\rangle_{H_q^1(\Omega)} &= \langle SF_D^{(k,q)}(0,g), S^{*-1}v\rangle_{H_q^1(\Omega)} \\
&= \int_{\partial\Omega} \gamma_1 F_D^{(k,q)}(0,g)\gamma_0 \overline{S^{*-1}v}\, ds \\
&= a^{(k,q)}(F_D^{(k,q)}(0,g), F_D^{(-k,q)}(0, \gamma_0 S^{*-1}v)).
\end{aligned}$$

Now the definition of the operator S^* and the weak formulation of $S^* F_D^{(-k,q)}(0, \gamma_0 S^{*-1} v)$ gives

$$\begin{aligned}
\langle F_D^{(k,q)}(0,g), v\rangle_{H_q^1(\Omega)} &= \langle F_D^{(k,q)}(0,g), S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v)\rangle_{H_q^1(\Omega)} \\
&= \int_{\partial\Omega} g\gamma_1 \overline{S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v)}\, ds \\
&= \langle F_D^{(0,q)}(0,g), F_D^{(0,q)}(0, \gamma_0 S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v))\rangle_{H_q^1(\Omega)} \\
&= \langle g, \gamma_0 S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v)\rangle_{H_q^{1/2}(\partial\Omega)}
\end{aligned}$$

by observing the identity $S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v) = F_D^{(0,q)}(0, \gamma_0 S^* F_D^{(-k,q)}(0,\gamma_0 S^{*-1}v))$. This shows the first assertion.

The second assertion follows from similar arguments. $\square$

Next we derive the adjoint for the Neumann-to-Dirichlet operator $\Lambda = \gamma_0 F_N^{(k,q)}(0,\cdot)$ for the missing cases.

Lemma 7. *The adjoint of the Neumann-to-Dirichlet map $\Lambda : X \to Y$ is given by*

(a) $\Lambda^* g = \gamma_0 S^{*-1} F_D^{(0,q)}(0,g)$ *for* $X = L^2(\partial\Omega)$ *and* $Y = H_q^{1/2}(\partial\Omega)$*;*

(b) $\Lambda^* g = \gamma_1 F_D^{(0,q)}(0, \gamma_0 F_N^{(-k,q)}(0,g))$ *for* $X = H_q^{-1/2}(\partial\Omega)$ *and* $Y = L^2(\partial\Omega)$*.*

We omit the proofs, since they use similar arguments to those in Lemma 3.

We now are in a position to state the main result of this section, the adjoint A^* of the operator A and the adjoint B^* of the operator B in the general case. We have two choices for the operators A and B as described in (3, 4), and four combinations of function space pairs X and Y and X and Z. We will frequently need to evaluate expressions involving several forward operators, e.g. $F_D^{(k,q)}(0, \gamma_0 F_N^{(k,q^+)}(0,f))$ which will be expressed as $F_D^{(k,q)}(0,\cdot) \circ \gamma_0 F_N^{(k,q^+)}(0,f))$. For completeness, we repeat the cases of the previous theorem.

Theorem 2. *Let $\tilde{E} = \gamma_0 F_N^{(-k,q)}(0,\cdot) - \gamma_0 F_N^{(-k,q^+)}(0,\cdot)$. For the operator A as defined in (3), the adjoint of the operator $A : X \to Y$ is given by $A^* : Y \to X$ with A^*g defined as follows*

(A1) $A^*g = \tilde{E}g$ *for $X = L^2(\partial\Omega)$ and $Y = L^2(\partial\Omega)$;*

(A2) $A^*g = \tilde{E} \circ \gamma_1 F_D^{(0,q)}(0,g)$ *for $X = L^2(\partial\Omega)$ and $Y = H_q^{1/2}(\partial\Omega)$;*

(A3) $A^*g = \gamma_1 F_D^{(0,q)}(0,\cdot) \circ \tilde{E}g$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Y = L^2(\partial\Omega)$;*

(A4) $A^*g = \gamma_1 F_D^{(0,q)}(0,\cdot) \circ \tilde{E} \circ \gamma_1 F_D^{(0,q)}(0,g))$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^{1/2}(\partial\Omega)$.*

Let $E = I - \gamma_0 F_N^{(-k,q^+)}(0,\cdot) \circ \gamma_1 F_D^{(-k,q)}(0,\cdot)$. For the operator B as defined in (4), the adjoint of the operator $B : X \to Z$ is given by $B^* : Z \to X$ with B^*v defined as follows

(B1) $B^*v = E \circ \gamma_0 F_N^{(-k,q)}(v,0)$ *for $X = L^2(\partial\Omega)$ and $Z = L^2(\Omega)$;*

(B2) $B^*v = E \circ \gamma_0 S^{*-1}v$ *for $X = L^2(\partial\Omega)$ and $Z = H_q^1(\Omega)$;*

(B3) $B^*v = \gamma_1 F_D^{(0,q)}(0,\cdot) \circ E \circ \gamma_0 F_N^{(-k,q)}(v,0)$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Z = L^2(\Omega)$;*

(B4) $B^*v = \gamma_1 F_D^{(0,q)}(0,\cdot) \circ E \circ \gamma_0 S^{*-1}v$ *for $X = H_q^{-1/2}(\partial\Omega)$ and $Z = H_q^1(\Omega)$.*

4 Numerical Simulations

In this section, we present numerical results on the optimal sources obtained by maximizing the discrepancy functionals introduced in (3) and (4) in Section 2. Our computational program is an implementation of the power method on the operator A^*A and B^*B for the cases presented in Theorem 2.

Let us recall the underlying reasoning for computation of the optimal source for maximizing distinguishability criteria. In a given experiment, the true physical parameters q^+ are unknown and we are given an a priori guess q. The distinguishability criteria test whether q can be distinguished from q^+, by searching for a source distribution, which maximizes the "difference" between q^+ and q. We use (3) or (4) as a measure for this difference. Further potential applications of such optimal source designs were discussed in the introduction.

Given the above setup and using different function space configurations we have eight different measures of type (3) or (4). In the following we will present numerical experiments for these cases and develop an ad hoc criterion to select the best optimal source strategy.

4.1 Numerical computation of optimal sources

We use test problems, which consist of a homogeneous background with small inclusions. The domain Ω is a disk of radius 4.3 cm with background diffusion coefficient $D_0 = 5.5$ cm^{-1} and background absorption coefficient $\mu_0 = 0.06$ cm^{-1}, which are representative values for soft tissues [27, 1]. The value of k is taken to be $10^8/299792458$ cm^{-1} [1]. Given the background values we have simulated both single and multiple circular inclusions corresponding to q^+ of radius r_t centered at (r_c, θ_c). We have used constant background values plus some multiple of the background values as a constant over the inclusion. Given the inclusion q^+, we simulated the "synthetic" data. Then we chose the background values as a particular guess for q and computed the distinguishability operator both A and B and their adjoints A^* and B^* to form A^*A and B^*B for the power iteration to find the optimal source f_n at the nth iteration. We have tested the computation of optimal source with a single inclusion at different locations as well as with multiple inclusions. We have also varied the values of the diffusion and absorption coefficients for the inclusions, using values both smaller and larger than the background listed above.

We discretize both the Dirichlet and Neumann problems with the piecewise linear finite element method. The domain is triangulated into a mesh with 4064 finite elements. We have tried finer meshes and our simulations of the various cases reported in this article seem stable with respect to mesh parameter. All computations were performed in MATLAB. The initial guess f_0 for the optimal source is generated randomly, and we compute 500 iterations of the power method in each case, which is reasonable considering that the convergence (in magnitude) in each case is rather fast, usually within 50 iterations.

Before we present the numerical results, we would like to comment on the computations involved in the algorithm. At each iteration, several forward problems are needed, most of which are Dirichlet-to-Neumann or Neumann-to-Dirichlet maps. For all these cases, each incurs two Neumann-to-Dirichlet operators involving q^+, which can be accessed experimentally. The remaining involves only q, and can be evaluated numerically.

Results from a single inclusion of radius $r_t = 0.86$ cm centered at $r_c = 3.2$ cm away from the origin at an angle of $\theta_c = 7\pi/12$, that is $[x_c, y_c] = [r_c \cos(\theta_c), r_c \sin(\theta_c)]$ are shown in Figure 1. The diffusion and absorption coefficients within the inclusion are each $\alpha = 11$ times of those in the background of the medium. Predictably, the optimal source is localized on the part of the boundary closest to the inclusion in all cases. However, it is evident that Case (B4), where $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^1(\Omega)$, gives the sharpest localization if we consider the scale of the vertical axis and quick decay to zero away from the inclusion.

Shown in Figure 2 are results when two inclusions are placed in the medium. The first inclusion is centered at $[x_{c1}, y_{c1}] = [0, 3.44\text{cm}]$ with a radius $r_{t1} = 0.43$ cm, and the second is centered at $[x_{c2}, y_{c2}] = [-2.58\text{cm}, 0]$ with a radius $r_{t2} = 1.075$ cm. The diffusion and absorption coefficients within the inclusions are set to $\alpha_1 = \alpha_2 = 6$ times of those in the background medium. As in the single tumor case, the sources are localized near the inclusions, and Case (B4) shows shows a particularly sharp localization near the tumor that is closest to the boundary, near $\pi/2$. We further note that in the case of a single or very few isolated inclusions, the shape of the optimal source distributions itself can be used as an indicator for the presence of such inclusions. One can determine at least some information about the location and the size of the inclusions.

4.2 Comparison of different cases

While it is often evident by examining the graph that Case (B4) gives the sharpest localization, we present quantitative measures as well to support the observation. For each θ on the boundary, $0 \le \theta < 2\pi$, for which a value $f(\theta)$ is defined in the finite element routine, we compute

$$p(\theta) = \frac{|f(\theta)|}{\int_0^{2\pi} |f(\theta)| d\theta}.$$

We then expect that the mean $\bar{\theta}$ of θ with $p(\theta)$ as the weight should be close to the angle of the inclusion $\hat{\theta}$. For each case we compute

$$Err(\bar{\theta}) = |\bar{\theta} - \hat{\theta}| = \left| \int_0^{2\pi} \theta p(\theta) d\theta - \hat{\theta} \right|.$$

In addition, we compute the variance in each case

$$var(\theta) = \int_0^{2\pi} (\theta - \bar{\theta})^2 p(\theta) d\theta,$$

which gives us a sense of how sharp the localization is.

We have computed $Err(\bar{\theta})$ and $var(\theta)$ for the single inclusion from Figure 1 as well as for an inclusion of radius $r_t = 0.215$ cm centered at an angle of $\theta_c = \pi/3$, $r_c = 3.655$ cm from the origin. The optical parameters within the inclusions for this case were $\alpha = 6$ times of those of the background. The results are given in Table 1. As is evident from the figures, Case (B4) results in the lowest value for $Err(\bar{\theta})$ and for $var(\theta)$. For the majority of the single inclusions near the boundary we have tried, the results were similar. However, for inclusions located near the center of the medium, and thus far from the boundary, Case (B4) does not always give meaningful results. Case (A4), where $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^{1/2}(\partial\Omega)$, or Case (A3), where $X = H_q^{-1/2}(\partial\Omega)$ and $Y = L_q^2(\partial\Omega)$, may be preferable: while the optimal sources that result from inclusions on the boundary are not as localized as in Case (B4), they are often second and third best, and also give meaningful results for inclusions closer to the center. In general, Cases (B3), (B4), (A3) and (A4), all of which have $X = H_q^{-1/2}(\partial\Omega)$, show much sharper peaks closest to tumors than do Cases (B1), (B2), (A1) and (A2), all of which have $X = L_q^2(\partial\Omega)$, suggesting that $X = H_q^{-1/2}(\partial\Omega)$ is a better choice than the alternative.

Case	$7\pi/12$		$\pi/3$	
	$var(\theta)$	$Err(\bar{\theta})$	$var(\theta)$	$Err(\bar{\theta})$
A1	1.8046	0.5128	1.7866	0.5075
A2	1.8046	0.5128	1.7866	0.5075
A3	1.3733	0.3314	1.3107	0.3570
A4	1.1882	0.2908	0.9004	0.2153
B1	2.234	0.6827	2.8914	1.1095
B2	1.6130	0.4571	1.6952	0.4256
B3	1.2035	0.2685	1.0784	0.2934
B4	*0.9582*	*0.2268*	*0.4576*	*0.1218*

Table 1: The localization of the source for inclusions centered at $7\pi/12$ and $\pi/3$.

5 Conclusions

In this paper we have investigated an analytical framework for the inverse optimal source problem. It is based on maximizing certain distinguishability criteria discriminating the unknown physical parameters of the tissue and an initial guess for the parameters. The necessary analytical tools are provided. Numerically, we have proposed an algorithm for computing the optimal source by the power method. The method has been evaluated on a circular media, and a comparison of eight different combination of function spaces is performed. The results indicate that that the choice with $X = H_q^{-1/2}(\partial\Omega)$ and $Y = H_q^1(\Omega)$ gives the best localized optimal source in our computational experiments. We note that the simulations presented in the paper involve mainly model error due to discretization and involve ideal circumstances. Due to its analytical tractability and clear computational framework, the numerical results and the present work is expected to serve as a benchmark case and should be a very good starting point for future research in the area of distinguishability in diffuse optical imaging. The question of which of these spaces is the most localized as well as computationally most stable given other uncertainties present such as measurement errors will be explored in a sequel.

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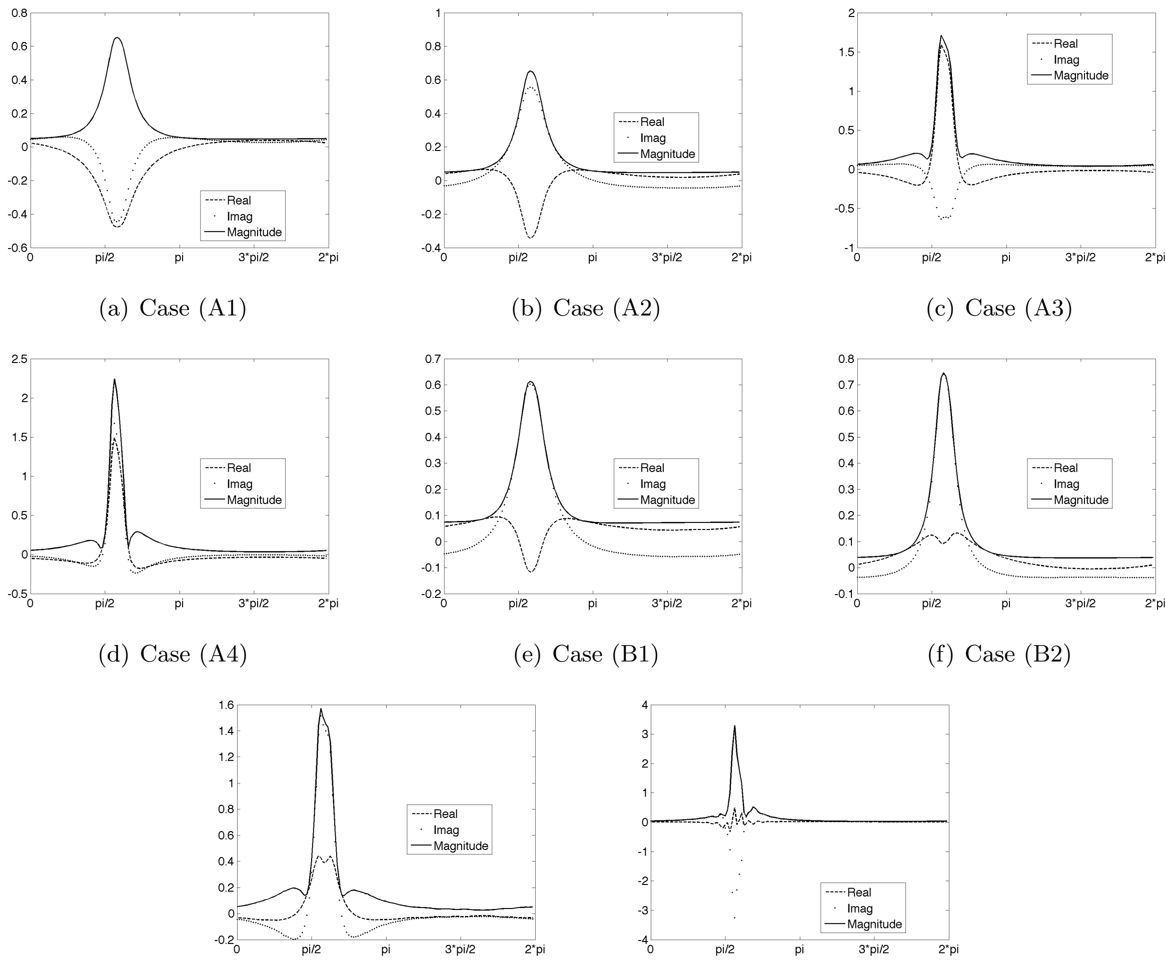
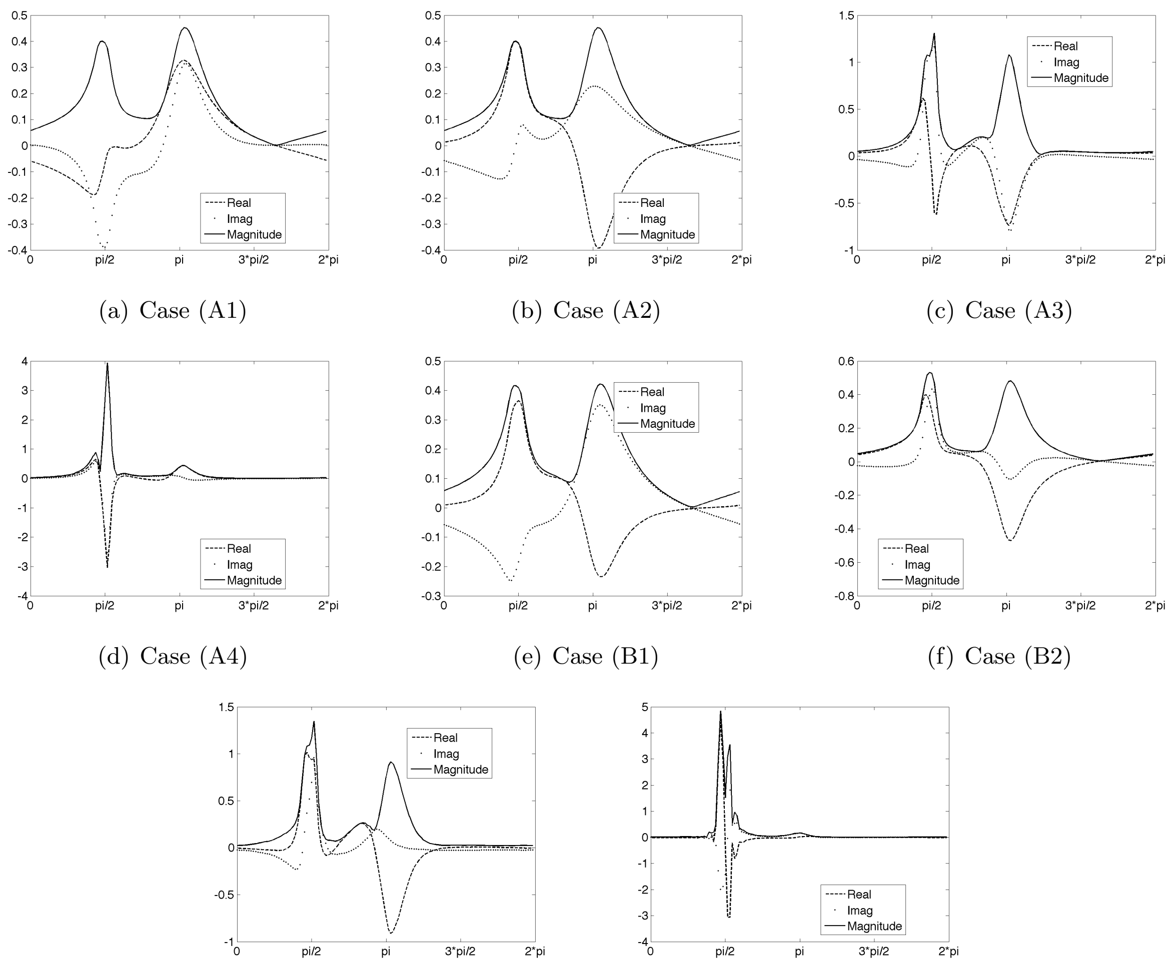


(a) Case (A1) (b) Case (A2) (c) Case (A3)

(d) Case (A4) (e) Case (B1) (f) Case (B2)

(g) Case (B3) (h) Case (B4)

Figure 1: Optimal source results from a single inclusion of radius $r_t = 0.86$ cm centered at $r_c = 3.2$ cm away from the origin at an angle of $\theta_c = 7\pi/12$ are shown. The diffusion and absorption coefficients within the inclusion are each $\alpha = 11$ times of those in the background of the medium. Predictably, the optimal source is localized on the part of the boundary closest to the inclusion in all cases. However, it is evident that Case (B4), where $X = H_q^{-1/2}(\partial\Omega)$ and $Z = H_q^1(\Omega)$, gives the sharpest localization if we consider the scale of the vertical axis and quick decay to zero away from the inclusion.



(a) Case (A1) (b) Case (A2) (c) Case (A3)

(d) Case (A4) (e) Case (B1) (f) Case (B2)

(g) Case (B3) (h) Case (B4)

Figure 2: Optimal source results for inclusions near $\pi/2$ and π when two inclusions are placed in the medium. The first inclusion is centered at $[x_{c1}, y_{c1}] = [0, 3.44\text{cm}]$ with a radius $r_{t1} = 0.43$ cm, and the second is centered at $[x_{c2}, y_{c2}] = [-2.58\text{cm}, 0]$ with a radius $r_{t2} = 1.075$ cm. The diffusion and absorption coefficients within the inclusions are set to $\alpha_1 = \alpha_2 = 6$ times of those in the background medium. As in the single tumor case, the sources are localized near the inclusions, and Case (B4) shows shows a particularly sharp localization near the tumor that is closest to the boundary, near $\pi/2$.

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MULTIRESOLUTION ANALYSIS AND HARR-LIKE WAVELET BASES ON LOCALLY COMPACT GROUPS

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Abstract. The multiresolution analysis (MRA) on certain non-abelian locally compact groups G is considered. Characterizations for a refinable function to generate an MRA in $L^2(G)$ are given. Here, no regularity properties or decay conditions are placed on the scaling functions. MRAs for $L^2(G)$ generated by a self-similar tile as a scaling function are shown and Haar-like wavelet bases are constructed. Concrete examples related to Heisenberg group are provided to illustrate the theorems.

Keywords: multiresolution analysis, non-abelian locally compact group, scaling function, Haar-like wavelet base, Heisenberg group.

1. INTRODUCTION

The theory of wavelets in the Hilbert space $L^2(\mathbb{R}^d)$ has been studied extensively in recent decades. The principal framework for constructing and understanding wavelet bases for the Hilbert space $L^2(\mathbb{R}^d)$ is the concept of multiresolution analysis (MRA) [**4, 8, 10**]. For a general Hilbert space $\mathcal{H}$, the notion of MRA can be formulated with respect to a distinguished affine structure (Π, σ) in the following way [**1**]. Let Π be a countable, discrete subgroup of the group of unitary operators on $\mathcal{H}$ and σ be another unitary operator on $\mathcal{H}$ satisfying $\sigma^{-1}\Pi\sigma\subseteq\Pi$ and $1 < [\Pi : \sigma^{-1}\Pi\sigma] < \infty$. An MRA with the scaling vector (or function) $\phi\in\mathcal{H}$ for the affine structure (Π, σ) is a doubly infinite sequence $\{V_j : j\in\mathbb{Z}\}$ of closed subspaces of $\mathcal{H}$ with the following properties:

(i) $\{u\phi : u\in\Pi\}$ is an orthonormal basis for V_0;

(ii) $V_j = \sigma^j V_0$, for all $j\in\mathbb{Z}$;

(iii) $V_j\subseteq V_{j+1}$, for all $j\in\mathbb{Z}$;

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (the triviality of the intersection);

(v) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = \mathcal{H}$ (the density of the union).

A scaling vector $\phi\in\mathcal{H}$ is called *refinable* if $\phi\in\overline{< \{\sigma(u\phi) : u\in\Pi\} >}$, the closure of the finite linear combinations of the functions from $< \{\sigma(u\phi) : u\in\Pi\} >$. If ϕ is refinable, condition (iii) in the above definition is satisfied. Using the unitary operator σ and the space V_0, we get a sequence $\{V_j : j\in\mathbb{Z}\}$ of nested closed subspaces of $\mathcal{H}$. In order to construct an MRA, the triviality of the intersection and the density of the union become two crucial conditions. We will see in section 3 that the triviality of the intersection is a direct consequence of conditions (i), (ii), and (iii). The question now is: when does the density of the union hold? To answer this question, let us first take a look at the Hilbert space $L^2(\mathbb{R}^d)$.

In $L^2(\mathbb{R}^d)$, the above mentioned affine structure is $\Pi = \{T_k : k\in\mathbb{Z}^d\}$ and $\sigma = \sigma_D$, where T_x is the translation operator defined by $T_x f(\cdot) := f(\cdot - x)$ for any $f\in L^2(\mathbb{R}^d)$ and σ_D is the dilation operator defined by $\sigma_D f(\cdot) = \delta_D^{1/2} f(D\cdot)$ for any $f\in L^2(\mathbb{R}^d)$ with D being the dilation matrix and $\delta_D = |\det(D)|$. A special dilation matrix is $D = 2I$, where I is the identity matrix. Suppose $\phi\in L^2(\mathbb{R}^d)$. Define $V(\phi)$ as $\overline{\{T_k\phi : k\in\mathbb{Z}^d\}}$. Then $V(\phi)$ is the smallest closed shift invariant subspace generated by ϕ, that is, $T_k f\in V(\phi)$ for any $f\in V(\phi)$ and any $k\in\mathbb{Z}^d$. Now define $V_j = {\sigma_{2I}}^j V(\phi)$ for any $j\in\mathbb{Z}$. The sequence of the closed subspaces $\{V_j : j\in\mathbb{Z}\}$ is nested if ϕ is refinable. A scaling function must be a refinable function. But the other way around is not true. Boor, DeVore and Ron [**2**] showed that the refinability of ϕ alone is not enough for ϕ to generate an MRA. For a refinable function to generate an MRA, additional conditions are required. They proved that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$ if and only if ϕ is refinable and $\bigcup_{j\in\mathbb{Z}}\mathrm{supp}(\widehat{\phi_j}) = \mathbb{R}^d$ modulo a null-set, where $\phi_j(\cdot) := {\sigma_{2I}}^j\phi(\cdot) = 2^{dj/2}\phi(2^j\cdot)$, $\widehat{\phi_j}$ is the Fourier transform of ϕ_j, and $\mathrm{supp}(\widehat{\phi_j}) := \{\xi\in\mathbb{R}^d \;:\; \widehat{\phi_j}(\xi)\neq 0\,\}$. They also gave a sufficient condition for the density of the union: $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$ if ϕ is refinable and $\widehat{\phi}$ is nonzero a.e. in some neighborhood of the origin. This sufficient condition can be easily proven. If $\widehat{\phi}$ is nonzero a.e. in some neighborhood of the origin, then we see that $\bigcup_{j\in\mathbb{Z}}\mathrm{supp}(\widehat{\phi_j}) = \mathbb{R}^d$ because $\widehat{\phi_j}(\cdot) = \widehat{\phi}(\cdot/2^j)$.

In this paper, we are interested in MRA defined over a more abstract Hilbert space. More specifically, we are interested in MRA defined over Hilbert spaces of the form $L^2(G)$, where G is some locally compact group. In the case that G is an abelian locally compact group, Dahlke has successfully extended the concept of MRA to $L^2(G)$ [**3**]. The main purpose of this paper is to develop some characterizations of functions which can serve as scaling functions in the more general case where the Hilbert space is $L^2(G)$, where G is a locally compact group, but may or may not be abelian. From the abstract harmonic analysis point of view, [**2**] uses the information on the Plancherel side to describe the qualities of a refinable function that can generate an MRA for the space $L^2(\mathbb{R}^d)$. For a general locally compact group G, it may be impossible to determine the Plancherel measure on the dual space $\widehat{G}$. Thus, the information on the Plancherel side is not available in general in this case. In contrast to [**2**], we only use the concepts coming within the space $L^2(G)$ to develop characterizations of functions that can serve as scaling functions without looking at the Plancherel side. Here, we note that we do not need to assume that the scaling functions have regularity properties, nor impose any decay conditions. To make the argument more general, we only assume that the scaling functions are elements in the space $L^2(G)$.

Analogous to the construction of MRA in the space $L^2(\mathbb{R}^d)$, to build an MRA on a general group G, the group G must have a uniform lattice subgroup Γ and a dilative automorphism α such that $\alpha(\Gamma)\subseteq\Gamma$ and $1 < [\Gamma : \alpha(\Gamma)] < \infty$ (see section 2 for the precise definition). We call (Γ, α) a *scaling system*. The corresponding affine structure on $L^2(G)$ is then provided by (Π, σ_α), where $\Pi = \lambda_G(\Gamma)$, λ_G is the left regular representation of G, σ_α is the unitary operator defined by $\sigma_\alpha f(x) = \delta_\alpha^{1/2} f(\alpha(x))$ and δ_α is the factor by which α scales the Haar measure on G.

An important example is $G = \mathbb{H}$, the three dimensional nilpotent Lie group. $\mathbb{H}$ is realized as follows: $\mathbb{H} = \{(p,q,t) : p,q,t\in\mathbb{R}\}$ with group product given by $(p,q,t)(p',q',t') = (p+p', q+q', t+t'+\frac{1}{2}(pq'-qp'))$. A choice for a uniform lattice in $\mathbb{H}$ is $\Gamma = \{(m,n,l/2) : m,n,l\in\mathbb{Z}\}$ and a compatible dilative automorphism is $\alpha(p,q,t) = (2p, 2q, 2^2 t)$, for all $(p,q,t)\in\mathbb{H}$. This example and variations on it will be used to illustrate our main results later on.

For any locally compact group G and a subset $F\subseteq L^2(G)$, we say that the family F is a *left zero divisor* in $L^2(G)$ if there exists a $g\in L^2(G)$, $g\neq 0$, such that $f*g = 0$, for all $f\in F$. If α is an automorphism of G, a single function $f\in L^2(G)$ is called *α-substantial* if $\{\sigma_\alpha^j f(\cdot) : j\in\mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$.

If $G = \mathbb{R}$ and $F\subseteq L^2(G)$, then F is a left zero divisor in $L^2(G)$ if and only if there exists a measurable subset E of $\mathbb{R}$, of positive Lebesque measure, such that $\widehat{f}(\omega) = 0$, for all $f\in F$ and almost all $\omega\in E$. If α is the automorphism given by $\alpha(t) = 2t$, for all $t\in\mathbb{R}$, then a function $f\in L^2(\mathbb{R})$ is α-substantial if and only if there exists a measurable subset $A\subseteq\mathbb{R}$ such that $\widehat{f}(\omega)\neq 0$, for almost all $\omega\in A$ and $\bigcup_{j\in\mathbb{Z}} 2^j A = \mathbb{R}$. In the particular case that $\widehat{f}(\omega)\neq 0$ for almost all ω in a neighborhood of the origin, then $\bigcup_{j\in\mathbb{Z}} 2^j \operatorname{supp}(\widehat{f}) = \mathbb{R}$. So any such function f is α-substantial.

Despite the lack of tools from Fourier analysis to help us to recognize α-substantial functions in the case of a general locally compact group G, we are able to show that if α is dilative, $f\in L^2(G)$, $f\geq 0$, $f\neq 0$ and is of compact support, then f is α-substantial. This is done in section 3.

Consider a locally compact group G. A subspace X of $L^2(G)$ is called *left shift invariant* if $\lambda_G(\gamma)X\subseteq X$, for all $\gamma\in\Gamma$. For $\phi\in L^2(G)$, let $V(\phi)$ denote the smallest left shift invariant closed subspace of $L^2(G)$ containing ϕ. Define $V_j = \sigma_\alpha^j\, V(\phi)$, for all $j\in\mathbb{Z}$. Then the fact that ϕ is refinable implies that $V_0\subseteq V_1$ (then $V_j\subseteq V_{j+1}$, for all $j\in\mathbb{Z}$). Define $\phi_j(x) = \sigma_\alpha^j \phi(x) = \delta_\alpha^{j/2}\phi(\alpha^j(x))$, for any $x\in G$, $j\in\mathbb{Z}$.

The main results of this paper are summarized in the following theorems.

Theorem 3.1(Triviality of the intersection): *Let G be a locally compact group with a scaling system (Γ, α). Let ϕ be a refinable function of $L^2(G)$ and V_j, $j \in \mathbb{Z}$ defined as above. Suppose that the shifts of ϕ, that is, $\{L_\gamma \phi \,:\, \gamma \in \Gamma\}$, constitute an orthonormal basis for V_0, then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

Theorem 3.5 (Density of the union): *Let ϕ be a refinable function in $L^2(G)$ and V_j, $j \in \mathbb{Z}$ defined as above. Then the following are equivalent:*

(a) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$

(b) $\{\phi_j\}_{j\in\mathbb{Z}}$ *is a left nonzero divisor in* $L^2(G)$

(c) ϕ *is* $\alpha-$*substantial.*

Theorem 4.3: *Let G be a locally compact group and (Γ, α) a scaling system on G. Suppose that there exists a self-similar tile T for (Γ, α) on G. Then $\phi = \chi_T$ will generate an MRA for the space $L^2(G)$.*

Theorem 4.6: *The MRA generated by a self-similar tile will always guarantee a Haar-like orthonormal wavelet basis for the space $L^2(G)$.*

The rest of the paper is arranged as follows. Section 2 provides the basic concepts, definitions of the terms and some basic results on a scaling system and its corresponding affine structure. In section 3, we first prove the intersection triviality theorem. Then we prove several propositions that lead to the proof of theorem 3.5 as stated above. Section 4 concerns with refinable functions of self-similar tile in the space $L^2(G)$, MRAs generated by using these refinable functions as scaling functions, and Haar-like wavelet bases associated with these MRAs. In section 5, we turn to the examples of the Heisenberg group. We draw upon the idea of [**12**] to give an explicit construction of refinable function on the Heisenberg group. This construction will provide examples to illustrate our theorems established in sections 3 and 4.

2. BASIC CONCEPTS

Let G be a locally compact group with left Haar measure m_G. Integration with respect to m_G will be denoted simply by $\int_G f(x)\ dx$, for any appropriate complex-value function f on G. Let $L^2(G)$ denote the Hilbert space of (equivalence classes of) square integrable complex-valued functions on G with inner product: $< f, g >= \int_G f(x)\overline{g(x)}\ dx$, for $f, g \in L^2(G)$. The left regular representation of G is the faithful homomorphism λ_G of G into the unitary group on $L^2(G)$ given by $\lambda_G(x)f(y) = f(x^{-1}y)$, $\forall x, y \in G$, $f \in L^2(G)$. If the group of unitary operators on $L^2(G)$ is endowed with the strong operator topology, then λ_G is continuous. If α is an automorphism (topological homomorphism and algebraic automorphism) of G, then $f \to \int_G f(\alpha(x))\ dx$ is a left invariant integral so there exists a positive constant δ_α so that $\int_G f(x)\ dx = \delta_\alpha \int_G f(\alpha(x))\ dx$, for any appropriate function f on G. This means that α induces a unitary operator σ_α on $L^2(G)$ by $\sigma_\alpha f(x) = \delta_\alpha^{1/2} f(\alpha(x))$, for all $x \in G$, $f \in L^2(G)$. Note that, for any $j \in \mathbb{Z}$, α^j is an automorphism of G and

$$\sigma_\alpha{}^j f(x) = \delta_\alpha^{j/2} f(\alpha^j(x)),\ \text{for all } x \in G,\ f \in L^2(G).$$

An automorphism α of G is called *dilative* if, for any compact $K \subseteq G$ and any neighborhood U of the identity e of G, there exists an $n_0 \in \mathbb{N}$ such that $K \subseteq \alpha^n(U)$, for all $n \geq n_0$. Note that α is dilative implies that $\delta_\alpha > 1$ but the converse is not true.

A subgroup Γ of G is called a uniform lattice in G if Γ is discrete, countable and G/Γ is compact. Suppose Γ is a uniform lattice in G and α is a dilative automorphism of G such that $\alpha(\Gamma) \subseteq \Gamma$ and $1 < [\Gamma : \alpha(\Gamma)] < \infty$. Then we will call (Γ, α) a scaling system (based on G).

For many of the results of this paper, we assume that (Γ, α) is a scaling system. This condition imposes strong restrictions on the group G and the discrete subgroup Γ. We will not fully investigate these restrictions here, but we need to make a few observations.

Proposition 2.1. *Let (Γ, α) be a scaling system. Then the following conditions hold.*

(a) *G is a unimodular group;*

(b) *Γ is not an open subgroup of G;*

(c) *$m_G(\Gamma) = 0$;*

(d) *For any* $j_0 \in \mathbb{Z}$, $\bigcup_{j \geq j_0} \alpha^{-j}(\Gamma)$ *is dense in* G.

Proof. (a) follows from numbers 1.8 to 1.11 on page 21 in [**11**] for example. To see (b), suppose Γ were an open subgroup of G. Then it is a neighborhood containing the e. Since $\alpha(\Gamma) \subseteq \Gamma$, if $n \geq 1$, then $\alpha^n(\Gamma) \subseteq \Gamma$. Since $1 < [\Gamma : \alpha(\Gamma)]$ and α is an automorphism of G, Γ is not all of G. Taking $U = \Gamma$ and $K = \{x\}$ for some $x \in G \backslash \Gamma$, then we would have that $K = \{x\} \subseteq \alpha^n(\Gamma) \subseteq \Gamma$ by the dilative property of α, which is a contradiction to the fact that $x \in G \backslash \Gamma$. Then (c) follows from (b).

For (d), choose a compact subset K of G such that $G = \bigcup_{\gamma \in \Gamma} K\gamma$; this is possible because Γ is a uniform lattice in G. For any $x \in G$ and any neighborhood W of x, let U be a symmetric ($U^{-1} = U$) neighborhood of e such that $Ux \subseteq W$. Since α is dilative, there exists $n_0 \in \mathbb{N}$ such that $j \geq n_0$ implies $K \subseteq \alpha^j(U)$; that is, $\alpha^{-j}(K) \subseteq U$. Then

$$G = \alpha^{-j}(G) = \bigcup_{\gamma \in \Gamma} \alpha^{-j}(K\gamma) = \bigcup_{\gamma \in \Gamma} \alpha^{-j}(K)\alpha^{-j}(\gamma) \subseteq \bigcup_{\gamma \in \Gamma} U\alpha^{-j}(\gamma).$$

Thus, there exists a $\gamma \in \Gamma$ such that $x \in U\alpha^{-j}(\gamma)$. Hence, $x = u\alpha^{-j}(\gamma)$, for some $u \in U$ and $\alpha^{-j}(\gamma) = u^{-1}x \in Ux \subseteq W$. Therefore, $W \bigcap \alpha^{-j}(\Gamma) \neq \emptyset$, for any $j \geq n_0$. Because $\alpha^{-j}(\Gamma) \subseteq \alpha^{-(j+1)}(\Gamma)$ for all j, $\bigcup_{j \geq j_0} \alpha^{-j}(\Gamma)$ is dense in G for any fixed $j_0 \in \mathbb{Z}$. $\square$

If (Γ, α) is a scaling system and $\phi \in L^2(G)$, we use ϕ to generate a family of closed subspaces of $L^2(G)$ in analogy with the role played by a scaling vector in an MRA. Let $V(\phi)$ denote the closed linear span of $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$. For each $j \in \mathbb{Z}$, let $V_j = {\sigma_\alpha}^j V(\phi)$. A function of the form $\lambda_G(\gamma)\phi$ is called a shift of ϕ, so the shifts of ϕ forms an orthonormal basis exactly when $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$ forms an orthonormal basis of $V(\phi)$. Moreover, the fact that ϕ is refinable exactly means $V_0 \subseteq V_1$. So $V_j \subseteq V_{j+1}$, for all $j \in \mathbb{Z}$.

Proposition 2.2. *Let* (Γ, α) *be a scaling system and* $\phi \in L^2(G)$. *Then the following conditions hold.*

(a) ${\sigma_\alpha}^j \lambda_G(\gamma) {\sigma_\alpha}^{-j} = \lambda_G(\alpha^{-j}(\gamma))$, *for all* $j \in \mathbb{Z}$, $\gamma \in \Gamma$,

(b) *If* $\Pi = \lambda_G(\Gamma)$, *then* (Π, σ_α) *is an affine structure on* $L^2(G)$,

(c) $V_j = \overline{< \{\lambda_G(\nu){\sigma_\alpha}^j\phi : \nu \in \alpha^{-j}(\Gamma)\} >}$, *for* $j \in \mathbb{Z}$.

Proof. For $f\in L^2(G)$ and $x\in G$, compute

$$\sigma_\alpha{}^j\lambda_G(\gamma)\sigma_\alpha{}^{-j}f(x) = \delta_\alpha^{j/2}\lambda_G(\gamma)\sigma_\alpha{}^{-j}f(\alpha^j(x)) = \delta_\alpha^{j/2}\sigma_\alpha{}^{-j}f(\gamma^{-1}\alpha^j(x))$$

$$= f(\alpha^{-j}(\gamma^{-1})x) = \lambda_G(\alpha^{-j}(\gamma))f(x).$$

This establishes (a). In particular, $\sigma_\alpha{}^{-1}\lambda_G(\gamma)\sigma_\alpha = \lambda_G(\alpha(\gamma))$. So, if $\Pi = \lambda_G(\Gamma)$, then $\sigma_\alpha{}^{-1}\Pi\sigma_\alpha = \lambda_G(\alpha(\Gamma))$. Since λ_G is a faithful homomorphism, $[\Pi : \sigma_\alpha{}^{-1}\Pi\sigma_\alpha] = [\Gamma : \alpha(\Gamma)]$ and (Π, σ_α) is an affine structure in the sense of [**1**]. So (b) holds and (c) also follows from (a) and the definition of V_j. □

3. THE CHARACTERIZATIONS OF A SCALING FUNCTION

In this section, the first two theorems stated in the introduction are proven. We first prove the triviality of the intersection because it is the direct consequence of refinability and orthogonal shifts. Then we establish the density of the union by considering several propositions. Finally, we show that the space $L^2(G)$ provides an abundant supply of α-substantial functions.

Theorem 3.1. *(Triviality of the intersection)* Let G be a locally compact group with scaling system (Γ, α). Let ϕ be a refinable function of $L^2(G)$ and define $V_j = \sigma_\alpha^j V(\phi)$ for $j\in\mathbb{Z}$. Suppose that the shifts of ϕ, that is, $\{\Gamma_\gamma\phi : \gamma\in\Gamma\}$, constitutes an orthogonal basis for V_0, then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof. Since ϕ is refinable, $V_j\subseteq V_{j+1}$, for all $j\in\mathbb{Z}$, so

$$\bigcap_{j\in\mathbb{Z}} V_j = \bigcap_{n\in\mathbb{N}} V_{-n}.$$

Since ϕ is a unit vector with orthogonal shifts and σ is unitary, for each $j\in\mathbb{Z}$, $\{\sigma^j(\lambda_G(\gamma)\phi) : \gamma\in\Gamma\}$ is a orthonormal basis for V_j. The orthogonal projection P_j of $L^2(G)$ onto V_j is given by

$$P_j f = \sum_{\gamma\in\Gamma} < f, \sigma^j(\lambda_G(\gamma)\phi) > \sigma^j(\lambda_G(\gamma)\phi),\ \forall f\in L^2(G).$$

Let $f\in\bigcap_{j\in\mathbb{Z}} V_j$; so $P_j f = f$, for all $j\in\mathbb{Z}$. Let $\epsilon > 0$ be arbitrary. Select a continuous function of compact support, f_1, so that $\|f - f_1\|_2 < \epsilon$. Then $\|f - P_j f_1\|_2 = \|P_j(f - f_1)\|_2 < \epsilon$ and so $\|f\|_2\leq\|P_j f_1\|_2 + \epsilon$, for any $j\in\mathbb{Z}$.

Let K be a compact subset of G so that $\mathrm{supp}(f_1)\subseteq K$ and let $M = \sup\{|f_1(x)| : x\in K\}$. Let W be a neighborhood of e in G such that $W\bigcap\Gamma = \{e\}$. Let U be another neighborhood of e such that $UU^{-1}\subseteq W$. Then, for any γ_1, $\gamma_2\in\Gamma$, $\gamma_1\neq\gamma_2$ implies $\gamma_1^{-1}U\bigcap\gamma_2^{-1}U = \emptyset$.

Since α is dilative, there exists $n_0\in\mathbb{N}$ such that $K\subseteq\alpha^n(U)$ for any $n\geq n_0$; that is $\alpha^{-n}(K)\subseteq U$ if $n\geq n_0$. Therefore, for γ_1, $\gamma_2\in\Gamma$, with $\gamma_1\neq\gamma_2$ and $n\geq n_0$, we have $\gamma_1^{-1}\alpha^{-n}(K)\bigcap\gamma_2^{-1}\alpha^{-n}(K) = \emptyset$. Let $E_{-n} = \bigcup_{\gamma\in\Gamma}\gamma^{-1}\alpha^{-n}(K)$. If $n\geq n_0$, we have additivity of the characteristic functions,

$$\chi_{E_{-n}} = \sum_{\gamma\in\Gamma}\chi_{\gamma^{-1}\alpha^{-n}(K)}.$$

Now, fix a point $x\in G\backslash\Gamma$. Since Γ is discrete in the relative topology of G, it is a closed subgroup. Thus, there exists a symmetric neighborhood V of e such that $xV\bigcap\Gamma = \emptyset$. Then x is not in $\gamma^{-1}V$, for any $\gamma\in\Gamma$. Using the dilative nature of α again, there exists $n_1\geq n_0$ such that $n\geq n_1$ implies $\alpha^{-n}(K)\subseteq V$. So, for any $n\geq n_1$, $\chi_{E_{-n}}(x) = 0$. Therefore, the sequence $(\chi_{E_{-n}})_{n=1}^{\infty}$ converges to 0 pointwise on $G\backslash\Gamma$, so it converges to 0 pointwise almost everywhere on G, since $m_G(\Gamma) = 0$. For $n\geq n_1$,

$$\begin{aligned}
\|P_{-n}f_1\|_2^2 &= \sum_{\gamma\in\Gamma}| < f_1, \sigma^{-n}(\lambda_G(\gamma)\phi) > |^2 \\
&= \sum_{\gamma\in\Gamma}\delta_\alpha^{-n}|\int_G f_1(x)\overline{\phi(\gamma^{-1}\alpha^{-n}(x))}\,dx|^2 \\
&\leq \sum_{\gamma\in\Gamma}\delta_\alpha^{-n}(\int_G |f_1(x)||\phi(\gamma^{-1}\alpha^{-n}(x))|\,dx)^2 \\
&\leq M^2\sum_{\gamma\in\Gamma}\delta_\alpha^{-n}(\int_G |\chi_K(x)||\phi(\gamma^{-1}\alpha^{-n}(x))|\,dx)^2 \\
&\leq M^2 m_G(K)^2\sum_{\gamma\in\Gamma}\delta_\alpha^{-n}\int_K |\phi(\gamma^{-1}\alpha^{-n}(x))|^2\,dx \\
&= M^2 m_G(K)^2\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}\alpha^{-n}(K)} |\phi(y)|^2\,dy \\
&= M^2 m_G(K)^2\int_G \chi_{E_{-n}}(x)|\phi(y)|^2\,dy \longrightarrow 0, \text{ as } n\rightarrow\infty,
\end{aligned}$$

by the Lebesque Dominated Convergence Theorem. The last inequality in the above calculation is an application of the Cauch-Schartz inequality. Since $\|f\|_2\leq\|P_{-n}f_1\|_2+$

ϵ, for all n, $\|f\|_2\leq\epsilon$ and $\epsilon > 0$ being arbitrary, we conclude that $f = 0$. Therefore, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. □

REMARK 3.2. In Theorem 3.1, one can replace the assumption of orthogonal shifts with the assumption that the shifts of ϕ constitute a frame in V_0, as for the $L^2(\mathbb{R}^d)$ situation, but we preferred to write out the clearer argument with the stronger assumption since we need orthogonal shifts elsewhere.

A subspace X of $L^2(G)$ is called *left translation invariant* if $\lambda_G(x)f\in X$, for all $f\in X$ and $x\in G$. For a family of functions $F\subseteq L^2(G)$. Let $X(F)$ denote the smallest closed left translation invariant subspace of $L^2(G)$ which contains F. Obviously,

$$X(F) = \overline{< \{\lambda_G(x)f : x\in G, f\in F\} >} = \{\lambda_G(x)f : x\in G, f\in F\}^{\perp\perp}.$$

Recall that we call F a left zero divisor in $L^2(G)$ if there exists a nonzero g in $L^2(G)$ such that $f{*}g = 0$, for all $f\in F$.

PROPOSITION 3.3. *Let G be a unimodular locally compact group and let $F\subseteq L^2(G)$. Then $X(F) = L^2(G)$ if and only if F is not a left zero divisor in $L^2(G)$.*

PROOF. For $g\in L^2(G)$, let $g^*(x) = \overline{g(x^{-1})}$, for all $x\in G$. Then $g\longrightarrow g^*$ is a norm preserving conjugate linear bijection of $L^2(G)$ (this is where unimodularity of G is used). Now, for $f, g\in L^2(G)$ and $x\in G$, the following is a standard calculation,

$$\begin{aligned} f{*}g(x) &= \int_G f(y)g(y^{-1}x)\,dy \\ &= \int_G f(y)\overline{g^*(x^{-1}y)}\,dy \\ &= \int_G f(xy)\overline{g^*(y)}\,dy \\ &= < \lambda_G(x^{-1})f, g^* > . \end{aligned}$$

Thus, $f{*}g = 0$, for all $f\in F$ if and only if $g^*\in\{\lambda_G(z)f : z\in G, f\in F\}^\perp = X(F)^\perp$. Therefore, $X(F) = L^2(G)$ if and only if F is not a left zero divisor in $L^2(G)$. □

We are most concerned about the nature of $X(F)$ where $F = \{{\sigma_\alpha}^j\phi : j\in\mathbb{Z}\}$ and ϕ is a refinable function associated with a scaling system.

PROPOSITION 3.4. *Let* (Γ, α) *be a scaling system and let* ϕ *be a refinable function in* $L^2(G)$*. Then* $X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$.

PROOF. According to Proposition 2.2 (c), for any $k\in\mathbb{Z}$, V_k is generated by $\{\lambda_G(\nu)\sigma_\alpha{}^k\phi : \nu\in\alpha^{-k}(\Gamma)\}$. Thus, $V_k\subseteq X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\})$, for all $k\in\mathbb{Z}$. Therefore, $\overline{\bigcup_{j\in\mathbb{Z}} V_j}\subseteq X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\})$. Let $f\in\bigcup_{j\in\mathbb{Z}} V_j$. Then $f\in V_k$, for some k, so $\lambda_G(\nu)f\in V_k$, for all $\nu\in\alpha^{-k}(\Gamma)$. But then $f\in V_j$, for any $j\geq k$, so $\lambda_G(\nu)f\in\bigcup_{j\geq k} V_j$, for any $\nu\in\bigcup_{j\geq k}\alpha^{-j}(\Gamma)$. Because of the nesting properties, $\bigcup_{j\in\mathbb{Z}} V_j$ is invariant under left translations from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$. Therefore, $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is also invariant under left translations from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$.

Now, for any $f\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ and any $x\in G$, use Proposition 2.1 (d) to select a net (ν_β) of elements from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$ such that $\nu_\beta\to x$ in G. Since λ_G is continuous with respect to the strong operator topology, $\lambda_G(\nu_\beta)f\longrightarrow\lambda_G(x)f$. Therefore, $\lambda_G(x)f\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$, for all $x\in G$. Hence, $X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. □

We are now ready for the main theorem characterizing scaling functions for a scaling system (Γ, α). Recall that, for $\phi\in L^2(G)$ and $j\in\mathbb{Z}$, $\sigma_\alpha{}^j\phi(x) = \delta_\alpha^{j/2}\phi(\alpha^j(x))$ and ϕ is α-substantial if and only if $\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$. Combining Proposition 3.3 and 3.4 gives us that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$ if and only if ϕ is refinable and α-substantial. Thus we have the following theorem.

THEOREM 3.5. *(Density of the union)* Let ϕ be a refinable function in $L^2(G)$ and V_j, $j\in\mathbb{Z}$ defined as above. Then the following are equivalent:

(a) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$

(b) $\{\phi_j\}_{j\in\mathbb{Z}}$ *is a left nonzero divisor in* $L^2(G)$

(c) ϕ *is* α*-substantial.*

Next we show that the condition of being α-substantial is not really that imposing.

PROPOSITION 3.6. *Let* G *be a locally compact group with a dilative automorphism* α*. Let* $f\in L^2(G)$ *satisfy* $f\geq 0$*,* $f\neq 0$ *and there exists a compact subset* K *of* G *such that* $f(x) = 0$*, for almost all* $x\in G\setminus K$*. Then* f *is* α*-substantial.*

PROOF. Since f is compactly supported, it is actually in $L^1(G)$. Without loss of generality, assume $\|f\|_1 = 1$. For each $n\in\mathbb{N}$, define $f_n(x) = \delta_\alpha^n f(\alpha^n(x)) = \delta_\alpha^{n/2}{\sigma_\alpha}^n f(x)$, for all $x\in G$. Then $\int_G f_n(x)\,dx = 1$, for all $n\in\mathbb{N}$. Therefore, for any $g\in L^2(G)$,

$$f_n*g(y) - g(y) = \int_G f_n(x)[\lambda_G(x)g(y) - g(y)]dx,$$

for all $y\in G$. Using a version of Minkowski's inequality for integrals instead of sum, see [**5**], VI.11.13 on page 530, we get

$$\begin{aligned}
\|f_n*g - g\|_2 &= \{\int_G |f_n*g(y) - g(y)|^2 dy\}^{1/2} \\
&= \{\int_G | \int_G [\lambda_G(x)g(y) - g(y)]f_n(x)\,dx|^2 dy\}^{1/2} \\
&\le \int_G \{\int_G |\lambda_G(x)g(y) - g(y)|^2 dy\}^{1/2} f_n(x)dx \\
&= \int_G \|\lambda_G(x)g - g\|_2 f_n(x)dx.
\end{aligned}$$

For any $\epsilon > 0$, there exists a neighborhood U of e such that $\|\lambda_G(x)g - g\|_2 < \epsilon$, for all $x\in U$ (this is just the strong operator continuity of λ_G again). Since α is dilative, there exists $n_0\in\mathbb{N}$ such that $\alpha^{-n}(K)\subseteq U$, for all $n\ge n_0$. The support of f_n is contained in $\alpha^{-n}(K)$, so $n\ge n_0$ and $f_n(x)\neq 0$ implies $\|\lambda_G(x)g - g\|_2 < \epsilon$, for almost every $x\in G$. Therefore, $n\ge n_0$ implies

$$\|f_n*g - g\|_2 \le \int_G \|\lambda_G(x)g - g\|_2 f_n(x)\,dx \le \epsilon \int_G f_n(x)\,dx = \epsilon.$$

Thus, $\{f_n : n = 1, 2, 3, \cdots\}$ forms a left approximate identity for the module action of $L^2(G)$ on $L^2(G)$ by convolution.

Clearly ${\sigma_\alpha}^n f*g = 0$ implies $f_n*g = 0$. So ${\sigma_\alpha}^n f*g = 0$, for all $n\in\mathbb{N}$ implies $g = 0$, for all $g\in L^2(G)$. Thus, $\{{\sigma_\alpha}^j f : j\in\mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$. That is, f is α-substantial. $\square$

4. MRAS GENERATED BY SELF-SIMILAR TILES AND HAAR-LIKE WAVELET BASES

This section concerns refinable functions that arise from self-similar tiles in the space $L^2(G)$, the MRAs generated by a self-similar tile as its scaling function and Haar-like wavelet bases associated with the MRAs. Gröchenig and Madych [**7**] considered the scaling system $(\mathbb{Z}^d, D)$ on the group $\mathbb{R}^d$, where D is a matrix with integer entries with all eigenvalues of which have absolute values bigger than 1. They established a connection between self-similar tilings and MRAs that are generated by a characteristic function for its scaling function. Besides developing the basic properties of self-similar tiles for $(\mathbb{Z}^d, D)$, they looked at a variety of interesting examples by choosing different integer matrices in $\mathbb{R}^2$. For example, the matrix $D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has the fractal set known as the twin dragon as a self-similar tile. Self-similar tiles are very often fractal in nature. Following Meyer's recipe, [**7**] also constructed Haar-like wavelet bases using the MRAs generated by self-similar tiles. We have been very much inspired by the results in [**7**].

Let Γ be a uniform lattice in a locally compact group G. A measurable subset T of G is called a tile for G if $m_G(T) < \infty$, $G = \bigcup_{\gamma\in\Gamma} \gamma T$ and $m_G(\gamma T \bigcap T) = 0$, for $\gamma\in\Gamma\backslash\{e\}$. Since Γ is countable, the last condition is equivalent to $\sum_{\gamma\in\Gamma} \chi_T(\gamma^{-1}x) = 1$, for almost all $x\in G$, where χ_A denotes the characteristic function of a subset A of G. The next proposition contains useful observations about tiles.

PROPOSITION 4.1. *Let Γ be a uniform lattice in a locally compact group G. Let T be a tile in G and S a measurable subset of G such that $G = \bigcup_{\gamma\in\Gamma} \gamma S$. Then* (a) *$m_G(T) > 0$,* (b) *$S$ is a tile if and only if $m_G(S) = m_G(T)$.*

PROOF. Since Γ is countable, $m_G(G) = \sum_{\gamma\in\Gamma} m_G(T)$. So $m_G(T) > 0$. For (b), let

$$f(x) = \sum_{\gamma\in\Gamma} \chi_S(\gamma^{-1}x) = \sum_{\gamma\in\Gamma} \chi_{\gamma S}(x) \geq 1,$$

for almost all $x \in G$. Then, for any $\gamma' \in \Gamma$,

$$\begin{aligned} m_G(T) &= \int_{\gamma' T} 1\, dx \leq \int_{\gamma' T} f(x) dx \\ &= \int_G \chi_{\gamma' T}(x) \sum_{\gamma \in \Gamma} \chi_S(\gamma^{-1} x) dx \\ &= \int_G \sum_{\gamma \in \Gamma} \chi_T({\gamma'}^{-1} \gamma x) \chi_S(x) dx \\ &= \int_G \chi_S(x) dx = m_G(S). \end{aligned}$$

Thus, $m_G(T) \leq m_G(S)$ and $m_G(T) = m_G(S)$ if and only if $f(x) = 1$, for almost all $x \in \gamma' T$, for each $\gamma' \in \Gamma$, so, if and only if S is a tile. $\square$

Suppose α is an automorphism of G so that (Γ, α) is a scaling system and T is a tile for G. If $\alpha(T) = \bigcup_{\gamma \in \Gamma_0} \gamma T$, for some subset $\Gamma_0 \subseteq \Gamma$, then we call T a self-similar tile for (Γ, α).

PROPOSITION 4.2. *Let (Γ, α) be a scaling system of a locally compact group G. Suppose that there exists a self-similar tile T for (Γ, α), then the following properties hold:*

(a) *If $\Gamma_0 \subseteq \Gamma$ is such that $\alpha(T) = \bigcup_{\gamma \in \Gamma_0} \gamma T$, then Γ_0 is a complete set of right coset representatives for $\alpha(\Gamma)$ in Γ,*

(b) $[\Gamma : \alpha(\Gamma)] = \delta_\alpha$,

(c) *$\phi = m_G(T)^{-1/2} \chi_T$ is a refinable function in $L^2(G)$.*

PROOF. We begin by proving (b). Let $\gamma_1, \cdots, \gamma_k$ be a complete set of right coset representations for $\alpha(\Gamma)$ in Γ. So Γ is the disjoint union $\bigcup_{i=1}^k \alpha(\Gamma) \gamma_i$.

Since T is a tile for G,

$$G = \bigcup_{\gamma \in \Gamma} \alpha(\gamma) [\bigcup_{i=1}^k \gamma_i T].$$

Applying α^{-1}, we get

$$G = \bigcup_{\gamma \in \Gamma} \gamma \alpha^{-1} [\bigcup_{i=1}^k \gamma_i T] = \bigcup_{\gamma \in \Gamma} \gamma [\bigcup_{i=1}^k \alpha^{-1}(\gamma_i T)].$$

If $S = \bigcup_{i=1}^{k} \alpha^{-1}(\gamma_i T)$, then one easily checks that $m_G(\gamma S \bigcap S) = 0$ if $\gamma \neq e$. Thus S is a tile. So $m_G(S) = m_G(T)$. On the other hand, $m_G(S) = k\delta_\alpha^{-1} m_G(T)$. Therefore, $[\Gamma : \alpha(\Gamma)] = k = \delta_\alpha$ and (b) holds.

For (a), suppose Γ_0 is the subset of Γ such that $\alpha(T) = \bigcup_{\gamma\in\Gamma_0} \gamma T$. Then $\chi_{\alpha(T)}(x) = \sum_{\gamma'\in\Gamma_0} \chi_{\gamma' T}(x)$, for almost all x. Since G is "tiled" by the sets $\{\gamma T : \gamma\in\Gamma\}$, the $\{\chi_{\gamma T} : \gamma\in\Gamma\}$ are mutually orthogonal projections in the commutative von Newmann algebra $L^\infty(G)$ (at least, that is a fancy way of thinking for the following calculation). Each of the following equalities is true for almost every $x\in G$.

$$\begin{aligned}
1 &= \sum_{\gamma\in\Gamma} \chi_{\gamma T}(x) = \sum_{\gamma\in\Gamma} \chi_{\gamma T}(\alpha^{-1}(x)) \\
&= \sum_{\gamma\in\Gamma} \chi_{\alpha(\gamma)\alpha(T)}(x) = \sum_{\nu\in\alpha(\Gamma)} \chi_{\nu\alpha(T)}(x) \\
&= \sum_{\nu\in\alpha(\Gamma)} \chi_{\alpha(T)}(\nu^{-1}x) = \sum_{\nu\in\alpha(\Gamma)} \sum_{\gamma'\in\Gamma_0} \chi_{\gamma' T}(\nu^{-1}x) \\
&= \sum_{\nu\in\alpha(\Gamma)} \sum_{\gamma'\in\Gamma_0} \chi_{(\nu\gamma')T}(x).
\end{aligned}$$

Thus, $\sum_{\gamma\in\Gamma} \chi_{\gamma T}(x) = \sum_{\nu\in\alpha(\Gamma)} (\sum_{\gamma'\in\Gamma_0} \chi_{(\nu\gamma')T}(x))$, for almost every $x\in G$. This implies that each $\gamma\in\Gamma$ has a unique expression of the form $\nu\gamma'$, with $\nu\in\alpha(\Gamma)$ and $\gamma'\in\Gamma_0$. In other words, Γ_0 is a complete set of right coset representations for $\alpha(\Gamma)$ in Γ.

Finally, we prove part (c). If $\phi = m_G(T)^{-1/2}\chi_T$, then $\|\phi\|_2 = 1$, then ϕ has orthogonal shifts and we can see that ϕ is refinable as follows. For any $x\in G$,

$$\begin{aligned}
\sigma_\alpha^{-1}\phi(x) &= \delta_\alpha^{-1/2}\phi(\alpha^{-1}(x)) \\
&= \delta_\alpha^{-1/2} m_G(T)^{-1/2}\chi_T(\alpha^{-1}(x)) \\
&= \delta_\alpha^{-1/2} m_G(T)^{-1/2}\chi_{\alpha T}(x) \\
&= \sum_{\gamma\in\Gamma_0} \delta_\alpha^{-1/2} m_G(T)^{-1/2}\chi_{\gamma T}(x) \\
&= \sum_{\gamma\in\Gamma_0} \delta_\alpha^{-1/2} m_G(T)^{-1/2}\chi_T(\gamma^{-1}x) \\
&= \sum_{\gamma\in\Gamma_0} \delta_\alpha^{-1/2}\lambda_G(\gamma)\phi(x).
\end{aligned}$$

So $\sigma_\alpha{}^{-1}\phi = \sum_{\gamma\in\Gamma_0} \delta_\alpha^{-1/2}\lambda_G(\gamma)\phi$ which implies $\phi = \sum_{\gamma\in\Gamma_0} \delta_\alpha^{-1/2}\sigma_\alpha[\lambda_G(\gamma)\phi]$. Therefore ϕ is refinable. $\square$

From the proof above, we see that the number of right coset representatives for $\alpha(\Gamma)$ in Γ is equal to δ_α. This number will appear later on.

Boor, DeVore and Ron in [**2**] showed that refinability is not enough to generate an MRA in the space $L^2(\mathbb{R}^d)$. Using the results from section 3, we see that whenever we have a refinable function of self-similar tile, an MRA can always be produced by this function as a scaling function in the space $L^2(G)$.

THEOREM 4.3. Let G be a locally compact group and (Γ, α) a scaling system on G. Suppose that there exists a self-similar tile T for (Γ, α) on G. Then $\phi = \chi_T$ is a scaling function, that is, it will generate an MRA for the space $L^2(G)$.

PROOF. The refinability of ϕ guarantees that condition (iii) in the definition holds. Define $V_0 = V(\phi)$, the closure of $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$. It is clear that $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$ is an orthonormal basis for V_0. Thus condition (i) holds. Using the unitary operator σ_α, a sequence of closed subspaces $V_j = \sigma_\alpha^j V_0$ are constructed. Condition (iv) is trivial by Theorem 3.1. By Proposition 3.6, ϕ is α-substantial. Thus the density of the union (v) is also satisfied. Therefore, an MRA for $L^2(G)$ is generated by the self-similar tile χ_T as a scaling function. $\square$

Once an MRA has been built up in the space $L^2(G)$, next we want to construct wavelet basis using the structure provided by the MRA.

Let $\{V_j : j\in\mathbb{Z}\}$ be an MRA in the space $L^2(G)$ with a self-similar tile χ_T as its scaling function for the scaling system (Γ, α). Let W_j be the orthogonal complement V_j in V_{j+1}, that is, $V_{j+1} = V_j \bigoplus W_j$, $j\in\mathbb{Z}$. Then we can decompose $L^2(G)$ as $\bigoplus_{j\in\mathbb{Z}} W_j$. To construct an orthogonal wavelet basis for $L^2(G)$, all we need is to construct an orthogonal basis for W_0. If an orthogonal basis for W_0 can be constructed, then σ_α^j will send this orthogonal basis for W_0 to an orthogonal basis for $W_j, j\in\mathbb{Z}$. Therefore the union of all these bases would give an orthogonal basis for $L^2(G)$ because $L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j$. In the space $L^2(\mathbb{R}^d)$ with a scaling system

$(\mathbb{Z}^d, D)$, if ϕ is a scaling function of an MRA and $q = |det(D)|$, [**9**] showed that there exist $q-1$ functions $\psi_1, \cdots, \psi_{q-1}$ such that $\{T_k\psi_i : k\in\mathbb{Z}^d, i = 1, \cdots, q-1\}$ is an orthogonal basis of W_0, where $\psi_i, i = 1, \cdots, q-1$ satisfy

$$\psi_i(x) = \sum_{k\in\mathbb{Z}^d} a_{ik}|det(D)|^{1/2}\phi(Dx - k), \tag{4.1}$$

with some sequences $\{a_{ik}\}, i = 1, \cdots, q-1$ in $l^2(\mathbb{Z}^d)$. Therefore, $\{\sigma_D^j T_k\psi_i : k\in\mathbb{Z}^d, j\in\mathbb{Z}, i = 1, \cdots, q-1\}$ forms an orthogonal wavelet basis for $L^2(\mathbb{R}^d)$. Since $V_0\subset V_1$, there must exist a sequence $\{a_k\}$ in $l^2(\mathbb{Z}^d)$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k\sigma_D T_k\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k|det(D)|^{1/2}\phi(Dx - k). \tag{4.2}$$

To construct a wavelet basis following Meyer's recipe, one first begins with an MRA with a scaling function ϕ satisfying equation (4.2) and then look for wavelet basis satisfying equation (4.1). For the MRA in $L^2(\mathbb{R}^d)$ generated by a self-similar tile as a scaling function, [**7**] constructed a piecewise constant wavelet basis associated with the scaling system $(\mathbb{Z}^d, D)$ following Meyer's recipe (See [**9**]). It turns out that Meyer's recipe still works in the space $L^2(G)$ if an MRA generated by a scaling function of self-similar tile is available.

Proposition 4.4. *Let χ_T be a self-similar tile associated with a scaling system (Γ, α) and $\{V_j : j\in\mathbb{Z}\}$ be an MRA generated by χ_T. Let $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_{\delta_\alpha}\}$ be a complete set of right coset representatives for $\alpha(\Gamma)$ in Γ. Then the subspace W_0 is a set of functions satisfying $f(x) = \sum_{\gamma\in\Gamma} a_\gamma\sigma_\alpha\lambda_G(\gamma)\chi_T(x)$ with $\{a_\gamma\}$ in $l^2(\Gamma)$ satisfying $\sum_{\gamma'\in\Gamma_0} a_{\alpha(\gamma)\gamma'} = 0$ for all $\gamma\in\Gamma$.*

Proof. A function $f\in W_0\subset V_1$ can be written as

$$f(x) = \sum_{\gamma\in\Gamma} a_\gamma\sigma_\alpha\lambda_G(\gamma)\chi_T(x) = \sum_{\gamma\in\Gamma} a_\gamma\delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x)),$$

for some sequence $\{a_\gamma\}$ in $l^2(\Gamma)$. A function $g\in V_0$ can be written as

$$g(x) = \sum_{\gamma'\in\Gamma} b_{\gamma'}\lambda_G(\gamma')\chi_T(x) = \sum_{\gamma'\in\Gamma} b_{\gamma'}\chi_T({\gamma'}^{-1}x),$$

for some sequence $\{b_{\gamma'}\}$ in $l^2(\Gamma)$. Then

$$\begin{aligned}
\langle f,g\rangle &= \int_G \sum_{\gamma\in\Gamma} a_\gamma \delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x))\overline{\sum_{\gamma'\in\Gamma} b_{\gamma'}\chi_T({\gamma'}^{-1}x)}dx \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}\int_G \chi_{\alpha^{-1}(\gamma T)}(x)\chi_{\gamma' T}(x)dx \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma T)\bigcap\gamma' T) \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma)\alpha^{-1}(T)\bigcap\gamma' T).
\end{aligned}$$

Since $\Gamma = \bigcup_{i=1}^{\delta_\alpha}\alpha(\Gamma)\gamma_i$, which is a union of disjoint right cosets, a sum over the set Γ is equal to the sum over the set $\bigcup_{i=1}^{\delta_\alpha}\alpha(\Gamma)\gamma_i$. Thus the above equals

$$\begin{aligned}
&\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma)\alpha^{-1}(T)\bigcap\gamma' T) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i)\alpha^{-1}(T)\bigcap\gamma' T) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\alpha(T))) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\bigcup_{j=1}^{\delta_\alpha}\gamma_j T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\bigcup_{j=1}^{\delta_\alpha}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\gamma_j T))) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i}\overline{b_\gamma}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i}\overline{b_\gamma}m_G(\alpha^{-1}(T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma} c_\gamma\overline{b_\gamma}m_G(\alpha^{-1}(T)),
\end{aligned}$$

where $c_\gamma = \sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i}$. The third last equality is due to the following basic fact: $m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma)\gamma_j T))$ is either equal to 0 or $m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T))$ because T is a tile. The second last equality holds because G is unimodular. Thus, $\langle f,g\rangle = \delta_\alpha^{1/2}\langle c,b\rangle\delta_\alpha^{-1} = \delta_\alpha^{-1/2}\langle c,b\rangle$. Therefore, $f\in W_0$ if and only if f can be written as $f(x) = \sum_{\gamma\in\Gamma} a_\gamma\delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x))$ and $c_\gamma = \sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i} = 0$. □

PROPOSITION 4.5. *Suppose that* χ_T, (Γ, α), $\{V_j : j\in\mathbb{Z}\}$, *and* Γ_0 *are the same as in Proposition 4.4. Suppose that* $U = (u_{ij})$ *is a* $\delta_\alpha \times \delta_\alpha$ *unitary matrix with all entries on the first row being the same constant* $\delta_\alpha^{-1/2}$. *Then the set of functions* $\{\psi_1, \psi_2, \cdots, \psi_{\delta-1}\}$ *defined by*

$\psi_{i-1}(x) = m_G(T)^{-1/2} \sum_{j=1}^{\delta_\alpha} u_{ij}\sigma_\alpha\lambda_G(\gamma_j)\chi_T(x), i = 2, \cdots, \delta_\alpha$

is a set of mother wavelets for the MRA. That is, the following set

$F = \{\lambda_G(\gamma)\psi_i : \gamma\in\Gamma, i = 1, \cdots, \delta_\alpha - 1\}$

is a complete orthonormal basis for the space W_0.

PROOF. We first show that the set F is an orthogonal system and then prove it is complete.

In the following, we will use Proposition 2.2 (a): $\lambda_G(\gamma)\sigma_\alpha = \sigma_\alpha\lambda_G(\alpha(\gamma))$. For $\lambda_G(\gamma')\psi_{i-1}, \lambda_G(\gamma'')\psi_{j-1}\in F$, then

$$
\begin{aligned}
& \langle\lambda_G(\gamma')\psi_{i-1}, \lambda_G(\gamma'')\psi_{j-1}\rangle \\
= & \langle\lambda_G(\gamma')m_G(T)^{-1/2}\sum_{m=1}^{\delta_\alpha} u_{im}\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')m_G(T)^{-1/2}\sum_{n=1}^{\delta_\alpha} u_{jn}\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\gamma')\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\gamma')\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\sigma_\alpha\lambda_G(\alpha(\gamma'))\lambda_G(\gamma_m)\chi_T, \sigma_\alpha\lambda_G(\alpha(\gamma''))\lambda_G(\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\alpha(\gamma'))\lambda_G(\gamma_m)\chi_T, \lambda_G(\alpha(\gamma''))\lambda_G(\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\alpha(\gamma')\gamma_m)\chi_T, \lambda_G(\alpha(\gamma'')\gamma_n)\chi_T\rangle \\
= & m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}m_G(\alpha(\gamma')\gamma_m T\cap\alpha(\gamma'')\gamma_n T) \\
= & m_G(T)^{-1}\delta(\gamma' - \gamma'')\sum_{m=1}^{\delta_\alpha} u_{im}\overline{u_{jm}}m_G(T) \\
= & \delta(\gamma' - \gamma'')\delta(i - j).
\end{aligned}
$$

Next we show that the set F is complete in W_0. For any $f\in W_0$, we want to prove that if $\langle f, \lambda_G(\gamma)\psi_{i-1}\rangle = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$, then $f = 0$. We see that

$$\begin{aligned}\langle f, \lambda_G(\gamma)\psi_i\rangle &= \langle \sum_{\gamma'\in\Gamma} a_{\gamma'}\sigma_\alpha\lambda_G(\gamma')\chi_T, \lambda_G(\gamma)m_G(T)^{-1/2}\sum_{j=1}^{\delta_\alpha} u_{ij}\sigma_\alpha\lambda_G(\gamma_j)\chi_T\rangle \\ &= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\sigma_\alpha\lambda_G(\gamma')\chi_T, \lambda_G(\gamma)\sigma_\alpha\lambda_G(\gamma_j)\chi_T\rangle \\ &= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\sigma_\alpha\lambda_G(\gamma')\chi_T, \sigma_\alpha\lambda_G(\alpha(\gamma))\lambda_G(\gamma_j)\chi_T\rangle\end{aligned}$$

$$\begin{aligned} &= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\lambda_G(\gamma')\chi_T, \lambda_G(\alpha(\gamma)\gamma_j)\chi_T\rangle \\ &= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}m_G(\gamma'T\bigcap\alpha(\gamma)\gamma_jT) \\ &= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{m=1}^{\delta_\alpha}\sum_{j=1}^{\delta_\alpha} a_{\alpha(\gamma')\gamma_m}\overline{u_{ij}}m_G(\alpha(\gamma')\gamma_mT\bigcap\alpha(\gamma)\gamma_jT) \\ &= m_G(T)^{-1/2}\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}}m_G(\alpha(\gamma)\gamma_mT) \\ &= m_G(T)^{1/2}\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}}.\end{aligned}$$

Thus, $\langle f, \lambda_G(\gamma)\psi_{i-1}\rangle = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$ means $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}} = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$. Since $f\in W_0$ and all entries on the first row of the unitary matrix $U = (u_{ij})$ are constant, proposition 4.4 implies that $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{1m}} = 0$ for any $\gamma\in\Gamma$. Thus, $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}} = 0$ for any $\gamma\in\Gamma$, $i = 1, 2, 3, \cdots, \delta_\alpha$. This shows that, for any $\gamma\in\Gamma$, the vector $(a_{\alpha(\gamma)\gamma_1}, a_{\alpha(\gamma)\gamma_2}, \cdots, a_{\alpha(\gamma)\gamma_{\delta_\alpha}})$ is perpendicular to all rows in the unitary matrix U. So, $a_{\alpha(\gamma)\gamma_m}$ must be 0 for any $\gamma\in\Gamma$ and $m = 1, 2, \cdots, \delta_\alpha$. Therefore, $a_\gamma = 0$ for any $\gamma\in\Gamma$. Hence $f = 0$. That is, The set F is complete in W_0. □

THEOREM 4.6. Given $\psi_{i-1}, i = 2, \cdots, \delta_\alpha$ that are defined in Proposition 4.5, the set $\{\sigma_\alpha^j\psi_i : j\in\mathbb{Z}, i = 1, 2, \cdots, \delta_\alpha - 1\}$ forms a complete orthonormal basis for $L^2(G)$.

5. EXAMPLES ON THE HEISENBERG GROUP

The theorems in sections 3 and 4 hold for general space $L^2(G)$, where G is a locally compact group which includes the Heisenberg group as an important example. In this section, we show examples to illustrate those theorems. All we need to do is to construct the refinable functions of self-similar tile on the Heisenberg group. According to the theorems in sections 3 and 4, the existence of refinable functions of self-similar tile will automatically lead us to build MRAs, hence to create Haar-like wavelet bases on the Heisenberg group.

Let G be the $2d+1$ dimensional Heisenberg group $\mathbb{H}^d$, which is a nilpotent Lie group with underlying manifold $\mathbb{R}^{2d+1}$. We denote points in $\mathbb{H}^d$ by $(\underline{q}, \underline{p}, t)$ with $\underline{q}, \underline{p} \in \mathbb{R}^d$, $t \in \mathbb{R}$, and define the group operation by $(\underline{q}, \underline{p}, t)(\underline{q}', \underline{p}', t') = (\underline{q} + \underline{q}', \underline{p} + \underline{p}', t + t' + \frac{1}{2}(\underline{p} \cdot \underline{q}' - \underline{p}' \cdot \underline{q}))$. Let Γ be the following uniform lattice subgroup in $\mathbb{H}^d$: $\Gamma = \{ (\underline{m}, \underline{n}, l/2) \; : \; \underline{m}, \; \underline{n} \in \mathbb{Z}^d \;, l \in \mathbb{Z}\}$. And let α be a dilative automorphism given by $\alpha(\underline{q}, \underline{p}, t) := (2\underline{q}, 2\underline{p}, 2^2 t)$. Then (Γ, α) forms a scaling system on $\mathbb{H}^d$ with $\delta = [\Gamma : \alpha(\Gamma)] = 2^{2(d+1)}$.

It is known (Folland [**6**]) that every automorphism α of $\mathbb{H}^d$ can be uniquely decomposed as a product of four factors $\alpha_1\alpha_2\alpha_3\alpha_4$, with $\alpha_j \in G_j$ $(j = 1, 2, 3, 4)$, where G_j is defined as follows: G_1 denotes the symplectic group $\mathrm{ps}(d, \mathbb{R})$; G_2 consists of inner automorphisms: $(\underline{a}, \underline{b}, c)(\underline{q}, \underline{p}, t)(\underline{a}, \underline{b}, c)^{-1} = (\underline{q}, \underline{p}, c + \underline{a} \cdot \underline{p} - \underline{b} \cdot \underline{q})$; G_3 consists of dilations $\delta[r]$ defined by $\delta[r](\underline{q}, \underline{p}, t) = (r\underline{q}, r\underline{p}, r^2 t)$; and G_4 consists of two elements, the identity and the automorphism i defined by $i(\underline{q}, \underline{p}, t) = (\underline{p}, \underline{q}, -t)$.

We restrict ourself to constructing special self-similar tiles for (Γ, α) on $\mathbb{H}^d$, where α can be written as $\alpha_1\alpha_3\alpha_4$. That is, $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), r_\alpha t)$, where r_α is some integer and D_α is a dilative automorphism from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$. The fundamental idea to construct such self-similar tiles for (Γ, α) is the following. We decompose the process of construction into two steps: first constructing in the direction $\mathbb{R}^{2d}$, that is, constructing self-similar tiles for the scaling system $(\mathbb{Z}^{2d}, D_\alpha)$. The work in [**7**] provides us details for this. Then, based on the self-similar tile obtained for $(\mathbb{Z}^{2d}, D_\alpha)$, we construct a self-similar tile in the direction $\mathbb{R}$ for (Γ, α). Such a self-similar tile is called a self-similar stacked tile due to the obvious geometric reason.

For simple notation reason, let's use $(\underline{x}, t)$ to denote the element $(\underline{q}, \underline{p}, t)$ in the Heisenberg group, that is, $\underline{x} = (\underline{q}, \underline{p}) \in \mathbb{R}^{2d}$. Then the group law becomes $(\underline{x}, t)(\underline{x'}, t') = (\underline{x} + \underline{x'}, t + t' + S(\underline{x}, \underline{x'}))$ where $S(\underline{x}, \underline{x'}) = ((\underline{q}, \underline{p}), (\underline{q'}, \underline{p'})) = 1/2(\underline{p} \cdot \underline{q'} - \underline{q} \cdot \underline{p'})$ is a skew-symmetric bilinear form from $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ to $\mathbb{R}$.

Let A be a self-similar tile for the scaling system $(\mathbb{Z}^{2d}, D_\alpha)$ on $\mathbb{R}^{2d}$. The existence of A is confirmed by [**7**]. Such an A is measurable. Without loss of generality, we can assume that

$$A \bigcap (\underline{k} + A) = \emptyset \text{ for } \underline{k} \neq 0,\ \underline{k} \in \mathbb{Z}^{2d} \text{ and } \bigcup_{\underline{k} \in \mathbb{Z}^{2d}} (\underline{k} + A) = \mathbb{R}^{2d}, \text{ and } D_\alpha(A) = \bigcup_{i=1}^{s} (\underline{k}_i + A)$$

where $k_1,\ k_2,\ \cdots,\ k_s$ are lattice points that are representatives of distinct cosets in $\mathbb{Z}^{2d}/D_\alpha(\mathbb{Z}^{2d})$. Thus, the Lebesgue measure of A must be 1, see lemma 1 in [**7**]. Since the measure of A is 1 and the disjoint union $\bigcup_{k \in \mathbb{Z}^{2d}} (k + A)$ fill out the whole space $\mathbb{R}^{2d}$, we could arrange a one to one correspondence between the lattice points in $\mathbb{Z}^{2d}$ and the tiles. Or simply speaking, we can assume that each tile only contains one lattice point. For $\underline{x} \in \mathbb{R}^{2d}$, we use $[\underline{x}]_A$ to denote the lattice point that corresponds to the tile which contains $\underline{x}$. Let $< \underline{x} >_A = \underline{x} - [\underline{x}]_A \in A$.

Let F be a bounded measurable real-valued function defined first on A and then extended periodically to the whole space $\mathbb{R}^{2d}$. Thus, we have $F(\underline{x}) = F(< \underline{x} >_A)$. We are going to produce a self-similar tile, denoted by T, for the scaling system (Γ, α) as follows: $T = \{\ (\underline{x}, t) \in \mathbb{H}^d\ :\ \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2\ \}$, where F is to be determined later. We can view $F(\underline{x})$ as a piece of surface over A and think of T as a solid over A bounded between two surfaces $F(\underline{x})$ and $F(\underline{x}) + 1/2$. Thus the volume of T is equal to $1/2$. So we can think of the "thickness" (in the direction of t-axis) of tile T as $1/2$.

For an element $\gamma = (\underline{a}, l/2) \in \Gamma$, the image of T under the left translation by γ is given by $\gamma T = \{\ (\underline{x}, t) \in \mathbb{H}^d\ :\ \underline{x} - \underline{a} \in A, 0 \leq t - l/2 - S(\underline{a}, \underline{x} - \underline{a}) - F(\underline{x}) < 1/2\ \}$. To show that $\bigcup_{\gamma \in \Gamma} \gamma T$ is a tiling of $\mathbb{H}^d$, we need to check two things. (a) $\bigcup_{\gamma \in \Gamma} \gamma T$ is a disjoint union. (b) $\bigcup_{\gamma \in \Gamma} \gamma T$ fills out the whole space $\mathbb{H}^d$. For (a), if $\underline{a} \neq \underline{a'}$, then

$(\underline{a}, l/2)T\bigcap(\underline{a'}, l/2)T = \emptyset$ since the image $(\underline{a}, l/2)T$ of T is in a stack of tiles lying over the tile $(\underline{a}, 0)T$. If l and l' are different integers, then $(\underline{a}, l/2)T$ and $(\underline{a}, l'/2)T$ are two different tiles in one stack located at tile $(\underline{a}, 0)T$, but $(\underline{a}, l/2)T\bigcap(\underline{a}, l'/2)T = \emptyset$ since the thickness for each tile is $1/2$. As for (b), for any $(\underline{x}, t)\in\mathbb{H}^d$, there exists a unique element $\underline{a}\in\mathbb{Z}^{2d}$ such that $\underline{x} - \underline{a}\in A$. And also there exists a unique element $l\in\mathbb{Z}$ with the property $0\leq t - l/2 - S(\underline{a}, \underline{x} - \underline{a}) - F(\underline{x} - \underline{a}) < 1/2$.

Now, we can start constructing a self-similar stacked tiling related to the tile A in $\mathbb{R}^{2d}$. From the explanation above, we know that the key point is to determine the surface described by the equation $t = F(\underline{x})$ on A. We start by choosing

$$\Gamma_0 = \{\ (\underline{k_i}, c)\ :\ i = 1, 2, \cdots, s,\ \text{and } c = 0, 1/2, 1, 3/2, \cdots, (|r_\alpha| - 1)/2\ \}.$$

Then we have

Proposition 5.1. *T is a self-similar stacked tile for (Γ, α) with the above choice of the finite set Γ_0 if and only if the function $F(\underline{x})$ on A satisfies*

$$F(\underline{x}) = \frac{1}{|r_\alpha|}F(< D_\alpha(\underline{x}) >_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A, < D_\alpha(\underline{x}) >_A).$$

Proof. By the choice of Γ_0, we have

$$\begin{aligned}
&\bigcup\nolimits_{\gamma\in\Gamma_0}\gamma T \text{ (disjoint finite union)}\\
= &\ \{\ (\underline{x}, t) : \underline{x}\in\bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A), 0\leq t - S([\underline{x}]_A, < \underline{x} >_A) - F(< \underline{x} >_A) < \frac{|r_\alpha|}{2}\ \}\\
= &\ \{(\underline{x}, t) : \underline{x}\in\bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A), 0\leq\frac{1}{|r_\alpha|}t - \frac{1}{|r_\alpha|}S([\underline{x}]_A, < \underline{x} >_A) - \frac{1}{|r_\alpha|}F(< \underline{x} >_A) < \frac{1}{2}\}.
\end{aligned}$$

Geometrically speaking, there are s stacks of tiles in $\bigcup_{\gamma\in\Gamma_0}\gamma T$. For each stack there are $|r_\alpha|$ tiles with the "thickness" for each tile $1/2$, so the "thickness" for each stack is $|r_\alpha|\times 1/2$. On the other hand,

$$\begin{aligned}
\alpha T &= \alpha\{\ (\underline{x}, t)\in\mathbb{H}^d\ :\ \underline{x}\in A, 0\leq t - F(\underline{x}) < 1/2\ \}\\
&= \{\ (D_\alpha(\underline{x}), r_\alpha t)\in\mathbb{H}^d\ :\ \underline{x}\in A, 0\leq t - F(\underline{x}) < 1/2\ \}\\
&= \{\ (\underline{x}, t)\in\mathbb{H}^d\ :\ {D_\alpha}^{-1}(\underline{x})\in A, 0\leq\frac{t}{|r_\alpha|} - F(D_\alpha^{-1}(\underline{x})) < 1/2\ \}\\
&= \{\ (\underline{x}, t)\in\mathbb{H}^d\ :\ \underline{x}\in D_\alpha(A) = \bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A) \text{ and } 0\leq\frac{t}{|r_\alpha|} - F(D_\alpha^{-1}(\underline{x})) < 1/2\ \}.
\end{aligned}$$

These two sets are equal if and only if

$$F(D_\alpha^{-1}(\underline{x})) = \frac{1}{|r_\alpha|}F(< \underline{x} >_A) + \frac{1}{|r_\alpha|}S([\underline{x}]_A, < \underline{x} >_A).$$

Or equivalently

$$F(\underline{x}) = \frac{1}{|r_\alpha|}F(< D_\alpha(\underline{x}) >_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A, < D_\alpha(\underline{x}) >_A).$$

□

This proposition yields the following theorem.

THEOREM 5.2. For the choice of Γ_0 given above, there exists a unique self-similar stacked tile T for (Γ, α). The function $F(\underline{x})$ is given explicitly by

$$F(\underline{x}) = \sum_{m=1}^{\infty} \frac{1}{|r_\alpha|^m} S([D_\alpha^m(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), < D_\alpha^m(\underline{x}) >_A),$$

where a lattice point $\underline{k}$ mod $(D_\alpha(\mathbb{Z}^{2d}))$ equals the representative of the coset which contains element $\underline{k}$.

PROOF. Define a mapping M from $L^\infty(A)$ to $L^\infty(A)$ by

$$Mf(\underline{x}) = \frac{1}{|r_\alpha|}F(< D_\alpha(\underline{x}) >_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), < D_\alpha(\underline{x}) >_A),$$

where $L^\infty(A)$ is a Banach space with the supermum norm. Given $f,\ g \in L^\infty(A)$, we have

$$\begin{aligned} \|Mf - Mg\|_{L^\infty(A)} &= \|\frac{1}{|r_\alpha|}f(< D_\alpha(\underline{x}) >_A) - \frac{1}{|r_\alpha|}g(< D_\alpha(\underline{x}) >_A)\|_{L^\infty(A)} \\ &\leq \frac{1}{|r_\alpha|}\|f - g\|_{L^\infty(A)}. \end{aligned}$$

So M is a contractive mapping. There exists a unique fixed point, denoted by $F(\underline{x})$. Especially, we have $F = \lim_{m\to\infty} M^m 0$. Thus,

$$F(\underline{x}) = \sum_{m=1}^{\infty} \frac{1}{|r_\alpha^m|} S([D_\alpha^m(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), < D_\alpha^m(\underline{x}) >_A).$$

□

Now we can provide the first example based on Theorem 5.2.

EXAMPLE 5.3. Consider α from $\mathbb{H}^d$ to $\mathbb{H}^d$ defined by $\alpha(\underline{q},\underline{p},t) := (2\underline{q},2\underline{p},2^2t)$. It is clear that α is in G_3. We can write $\alpha(\underline{q},\underline{p},t) = (D_\alpha(\underline{q},\underline{p}), r_\alpha t) = (2(\underline{q},\underline{p}),4t)$. Thus, $r_\alpha = 4$ and D_α is the dilative automorphism on $\mathbb{R}^{2d}$. Let $A = \{\ \underline{x}\in\mathbb{R}^{2d}\ :\ 0\leq x_j < 1, j = 1,2,\cdots,2n\ \}$ denote the "half open and half closed" standard tile in the Euclidean space $\mathbb{R}^{2d}$, where x_j denotes the jth component of $\underline{x}$. Then it is obvious that $\bigcup_{\underline{a}\in\mathbb{Z}^{2d}}(A+\underline{a})$ (disjoint union) fills out the whole space $\mathbb{R}^{2d}$. Clearly, A is a self-similar tile. If we choose $\Gamma_0 = \{\ (\underline{a},b)\ :\ a_j = 0 \text{ or } 1, 1\leq j\leq 2n, b = 0,1/2,1 \text{ or } 3/2\ \}$, then by Theorem 5.2, $T = \{\ (\underline{x},t)\in\mathbb{H}^d\ :\ \underline{x}\in A, 0\leq t-F(\underline{x}) < 1/2\ \}$ is a self similar-tile for (Γ,α) with F defined by

$$\begin{aligned} F(\underline{x}) &= \sum_{m=1}^{\infty}\frac{1}{4^m}S([D_\alpha^m(\underline{x})]_A\text{mod }(D_\alpha^m(\mathbb{Z}^{2d})), < D_\alpha^m(\underline{x}) >_A) \\ &= \sum_{m=1}^{\infty}\frac{1}{4^m}S([2^m\underline{x}]\text{mod}2, < 2^m\underline{x} >), \end{aligned}$$

where $[2^m\underline{x}]$mod 2 means $([2^m x_1]\text{mod } 2, [2^m x_2]\text{mod } 2, \cdots, [2^m x_{2n}]\text{mod } 2)$.

EXAMPLE 5.4. In this example, we choose a different dilative automorphism on $\mathbb{H}^d$ which is defined as follows. $\alpha(\underline{q},\underline{p},t) := (2\underline{q},3\underline{p},6t)$. This α can be decomposed as $\alpha = \alpha_1\alpha_3$, where $\alpha_1(\underline{q},\underline{p},t) := (\sqrt{\frac{2}{3}}\underline{q},\sqrt{\frac{3}{2}}\underline{p},t)$ and $\alpha_3(\underline{q},\underline{p},t) := (\sqrt{6}\underline{q},\sqrt{6}\underline{p},(\sqrt{6})^2t) = (\sqrt{6}\underline{q},\sqrt{6}\underline{p},6t)$. Further, α can be written as $\alpha(\underline{q},\underline{p},t) = (D_\alpha(\underline{q},\underline{p}),6t)$, where D_α is a dilative automorphism from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$ defined by $D_\alpha(\underline{q},\underline{p}) := (2\underline{q},3\underline{p})$. Thus, we have $r_\alpha = 6$. Still using the same Γ as the one used in Example 5.3, we choose Γ_0 as the set $\Gamma_0 = \{(\underline{a},c)\}$, where $a_j = 0$ or 1 for $1\leq j\leq d$, $a_j = 0,\ 1$ or 2 for $d<j\leq 2d$ and $c = 0,\ 1/2,\ 1,\cdots,5/2$. So the set A in Example 5.3 is a self-similar tile for the scaling system $(\mathbb{Z}^{2d},D_\alpha)$ with dilated tile by D_α consisting of 6 original tiles. With this self similar tile in $\mathbb{R}^{2d}$, by Theorem 5.2 we obtain a self-similar stacked tile in $\mathbb{H}^d$:

$$T = \{(\underline{x},t)\in\mathbb{H}^d : \underline{x}\in A, 0\leq t - F(\underline{x}) < 1/2\}$$

with $F(\underline{x})$ constructed by

$$F(\underline{x}) = \sum_{m=1}^{\infty}\frac{1}{6^m}S([D_\alpha{}^m\underline{x}]_A\text{mod }(D_\alpha(\mathbb{Z}^{2d})), < D_\alpha{}^m\underline{x} >_A),$$

where $[D_\alpha^m \underline{x}]_A \mathrm{mod}\ (D_\alpha(\mathbb{Z}^{2d}))$ means $[2^m x_1]$ mod 2, $[2^m x_2]$ mod 2, ..., $[2^m x_n]$ mod 2 and $[3^m x_{n+1}]$ mod 3, $[3^m x_{n+2}]$ mod 3, ..., $[3^m x_{2n}]$ mod 3.

Generally speaking, whenever an automorphism α from $\mathbb{H}^{2d}$ to $\mathbb{H}^{2d}$ can be decomposed as $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), r_\alpha t)$ and there exists a self similar tile A in $\mathbb{R}^{2d}$ associated with D_α, then with this A, we can always construct a self similar tile in $\mathbb{H}^d$ associated with α.

The above functions F serve as scaling functions to generate MRAs for the space $L^2(\mathbb{H}^d)$. Since $F > 0$ has compact support, Proposition 3.6 shows that F is α-substantial. Therefore, F will generate MRAs for $L^2(\mathbb{H}^d)$ by Theorem 3.5. Theorem 4.6 guarantees the existence of Haar-like wavelet bases for the space $L^2(\mathbb{H}^d)$.

6. CONCLUSION

In this paper we are able to give the characterizations for a refinable function that is capable of generating an MRA in the space $L^2(G)$, where G is a locally compact group that does not have to be abelian. In deriving these characterizations, we did not use any information from the Plancherel side. In fact, for a general locally compact group, we may not be able to build the Fourier transform on it. However, in the case that the Fourier transform can be built up in the space $L^2(G)$ for some non-abelian locally compact groups, how can we characterize a refinable function to have a scaling function using the information from the Plancherel side? In particular, if G is a second countable, type I, unimodular locally compact group, do those results obtained by [**2**] in the space $L^2(\mathbb{R}^d)$ mentioned in the introduction still hold in the space $L^2(G)$? The authors intend to explore these questions in their future study.

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