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ON AN ABSTRACT NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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Abstract. In this paper we prove the existence, uniqueness and other properties of mild solutions of a nonlinear Volterra integrodifferential equation with nonlocal condition in Banach space. Our analysis is based on C_0-semigroup theory, Banach fixed point theorem and the integral inequality established by B. G. Pachpatte.

Key words: Volterra integrodifferential,fixed point theorem, continuous dependence, Pachpatte's inequality, Nonlocal condition.
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1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $B = C([t_0, b]; X)$ be Banach space of all continuous functions from $[t_0, b]$ into X, endowed with the norm

$$\|x\|_B = \sup\{\|x(t)\| : x \in B\}, \quad 0 \le t_0 \le t \le b.$$

In the present paper, we study the existence, uniqueness and other properties of mild solutions of a nonlocal problem of the form:

$$x'(t) + Ax(t) = f(t, x(t), \int_{t_0}^{t} a(t,s)k(s, x(s))ds), \quad t \in [t_0, b], \tag{1.1}$$

$$x(t_0) + g(x) = x_0, \tag{1.2}$$

where $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\ t \ge 0$, on a Banach space X and functions $f : [t_0, b] \times X \times X \to X$, $g : B \to X$, $k : [t_0, b] \times X \to X$, $a : [t_0, b] \times [t_0, b] \to R$ are continuous and x_0 is a given element of X.

The nonlocal condition, which is a generalization of the classical initial, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski [4]. Existence of mild, strong and classical solutions for differential and integrodifferential equations in abstract spaces with nonlocal conditions has received much attention in recent years. We refer to the papers of Byszewski [4, 5], Balachandran and Chandrasekaran [1], K. Balachandran[2], S. Karunanithi and S. Chandrasekaran [15], Y. Lin and J. H. Liu [17] and Zuomao Yan [27].

The equation of these type or their special forms commonly come across in almost all phases of physics and other areas od applied mathematics, see, for example [6, 11, 19, 20, 24] and the references listed therein. The problems of existence, uniqueness and other properties of solutions of various special forms of (1.1)–(1.2) have been studied by using different techniques during last few years see, [3, 9, 12, 13, 14, 16, 18, 22, 23, 25, 26] and the references given therein. Our general formulation of (1.1)–(1.2) is an attempt to generalize the results of [2, 7, 8, 10, 12, 22, 24, 26].

The paper is organized as follows. In section 2, we present the preliminaries and hypotheses. Section 3 deals with main results and in Section 4, we discuss the continuous dependence and other properties of the solutions. Finally, in Section 5, we study the boundedness, asymptotic behaviour and growth of solutions.

2. Preliminaries and Hypotheses

Before proceeding to the main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\ t \geq 0$, on a Banach space X. The set of bounded linear operators $T(t), \quad t \in R_+ = [0, \infty)$ is a C_0-semigroup on X if

(i). $T(t+s) = T(t)T(s) = T(s)T(t), \quad t, s \geq 0,$
(ii). $T(0) = I$ the identity operator,
(iii). $T(\cdot)$ is strongly continuous in $t \in R_+$,
(iv). $\|T(t)\| \leq Me^{wt}$ for some $M \geq 1$ and real w and $t \in R_+$

(see, Martin [18], p.276).

Definition 2.1. Let $-A$ is the infinitesimal generator of a C_0-semigroup $T(t),\ t \geq 0$, on a Banach space X. The function $x \in B$ given by

$$x(t) = T(t-t_0)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau)ds, \quad t \in [t_0, b], \tag{2.1}$$

is called the mild solution of the problem (1.1)–(1.2).

We require the following Lemma known as the Pachpatte's inequality in our further discussion.

Lemma 2.2. *(see, [21],p. 758) Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on R^+, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)\Big[u(s) + \int_0^s q(\tau)u(\tau)d\tau\Big]ds,$$

holds for all $t \in R^+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0\Big[1 + \int_0^t p(s)\exp\Big(\int_0^s \big(p(\tau) + q(\tau)\big)d\tau\Big)\Big]ds,$$

holds for all $t \in R^+$.

Let us denote

$$L_1 = \max_{t_0 \le t \le b} \|f(t,0,0)\|,$$
$$K_1 = \max_{t_0 \le s,\ t \le b} \|k(t,0)\|,$$
$$G = \max_{x \in B} \|g(x)\|.$$

We list the following hypotheses for our convenience.

(H_1) $g : B \to X$ and there exists a constant $G_1 > 0$ such that

$$\|g(x_1) - g(x_2)\| \le G_1 \|x_1(t) - x_2(t)\|$$

for $x_1,\ x_2 \in B$.

(H_2) $f : [t_0, b] \times X \times X \to X$ is continuous and there exist constants $L > 0$ such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

for $t \in [t_0, b]$ and $x_i,\ y_i \in X,\ i = 1, 2$.

(H_3) $k : [t_0, b] \times X \to X$ is continuous and there exists constant $K > 0$ such that

$$\|k(t, x_1) - k(t, x_2)\| \le K(\|x_1 - x_2\|),$$

for $t \in [t_0, b]$ and $x_i \in X,\ i = 1, 2$.

(H_4) $a : [t_0, b] \times [t_0, b] \to R$ is continuous and there exists constant $N > 0$ such that

$$|a(t, s)| \le N, \quad \text{for} \quad t, s \ge 0.$$

3. Existence and Uniqueness

Now we first prove the result of existence and uniqueness of mild solutions.

Theorem 3.1. *Assume that the hypotheses $(H_1) - (H_4)$ hold. Then problem (1.1)–(1.2) has a unique mild solution on $[t_0, b]$.*

Proof. We use the Banach contraction principle to prove the existence and uniqueness of the mild solution to (1.1)–(1.2). Let $E_r = \{x \in B : \|x\| \le r\}$, where $r \ge [1 - (MLb + MLKNb^2)]^{-1}[M(\|x_0\| + G) + MLNK_1b^2 + ML_1b]$ with $\Big[MG_1 + MLb + MLNKb^2\Big] < 1$, and define an operator on the Banach space B by

$$(Fx)(t) = T(t - t_0)[x_0 - g(x)] + \int_{t_0}^{t} T(t - s) f\Big(s, x(s), \int_{t_0}^{s} a(s, \tau) k(\tau, x(\tau)) d\tau\Big) ds, \quad t \in [t_0, b]. \tag{3.1}$$

Firstly, we show that the operator F maps E_r into itself. For this by using assumptions, we have

$$\begin{aligned}
\|(Fx)(t)\| &\le M[\|x_0\| + G] + \int_{t_0}^{t} M \|f\Big(s, x(s), \int_{t_0}^{s} a(s, \tau) k(\tau, x(\tau)) d\tau\Big)\| ds \\
&\le M[\|x_0\| + G] + M \int_{t_0}^{t} \Big[\|f\Big(s, x(s), \int_{t_0}^{s} a(s, \tau) k(\tau, x(\tau)) d\tau\Big) - f(s, 0, 0)\| + \|f(s, 0, 0)\|\Big] ds \\
&\le M[\|x_0\| + G] \\
&\quad + M \int_{t_0}^{t} \Big[L\|x(s)\| + L \int_{t_0}^{s} |a(s, \tau)| \|k(\tau, x(\tau)) - k(\tau, 0) + k(\tau, 0)\| d\tau + L_1\Big] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq M[\|x_0\| + G] + M\int_{t_0}^{t}\Big[Lr + LNKr(s-t_0) + LNK_1(s-t_0) + L_1\Big]ds\\
&\leq M[\|x_0\| + G] + M\Big[Lrb + LNKrb^2 + LNK_1b^2 + L_1b\Big]\\
&= [M(\|x_0\| + G) + MLNK_1b^2 + ML_1b] + (MLb + MLNKb^2)r\\
&\leq [1 - (MLb + MLNKb^2)]r + (MLb + MLNKb^2)r = r, \qquad (3.2)
\end{aligned}
$$

for $x \in B$. The equation (3.2) shows that the operator F maps B into itself.

Now for every $x_1,\ x_2 \in E$ and $t \in [t_0, b]$, we obtain

$$
\begin{aligned}
&\|(Fx_1)(t) - (Fx_2)(t)\|\\
&\leq \|T(t-t_0)\|\|g(x_1) - g(x_2)\| + \int_{t_0}^{t}\|T(t-s)\Big[f\big(s, x_1(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x_1(\tau))d\tau\big)\\
&\quad - f\big(s, x_2(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x_2(\tau))d\tau\big)\Big]\|ds\\
&\leq MG_1\|x_1 - x_2\|_B + M\int_{t_0}^{t}\Big[L\|x_1 - x_2\|_B + LNK\|x_1 - x_2\|_B(s-t_0)\Big]ds\\
&\leq MG_1\|z_1 - z_2\|_B + M\Big[L(t-t_0) + LNKb^2\Big]\|x_1 - x_2\|_B\\
&\leq \Big[MG_1 + MLb + MLNKb^2\Big]\|x_1 - x_2\|_B. \qquad (3.3)
\end{aligned}
$$

If we take $q = MG_1 + MLb + MLNKb^2$, then

$$\|Fx_1 - Fx_2\|_B \leq q\|x_1 - x_2\|_B$$

with $0 < q < 1$. This shows that the the operator F is a contraction on the complete metric space B. By the Banach fixed point theorem, the function B has a unique fixed point in the space B and this point is the mild solution of problem (1.1)–(1.2) on $[t_0, b]$.

□

The following theorem shows the uniqueness of solutions to (1.1)–(1.2) without the existence part.

Theorem 3.2. *Suppose that the hypotheses* $(H_1) - (H_4)$ *hold. Then the* (1.1)–(1.2) *has at most one solution on* $[t_0, b]$.

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of (1.1)–(1.2) and $u(t) = \|x_1(t) - x_2(t)\|$, $t \in [t_0, b]$. Then by hypotheses, we have

$$
\begin{aligned}
u(t) &\leq \|T(t-t_0)\|\|g(x_1) - g(x_2)\| + \int_{t_0}^{t}\|T(t-s)\Big[f\big(s, x_1(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x_1(\tau))d\tau\big)\\
&\quad - f\big(s, x_2(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x_2(\tau))d\tau\big)\Big]\|ds\\
&\leq MG_1\|x_1(t) - x_2(t)\| + \int_{t_0}^{t} ML\Big[\|x_1(s) - x_2(s)\| + \int_{t_0}^{s} NK\|x_1(\tau) - x_2(\tau)\|d\tau\Big]ds\\
&\leq MG_1u(t) + \int_{t_0}^{t} ML\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds,
\end{aligned}
$$

which implies

$$u(t) \le \int_{t_0}^{t} \frac{ML}{1-MG_1}\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds. \tag{3.4}$$

Now a suitable application of Lemma 2.2 with $u_0 = 0$ to (3.5) yields

$$\begin{aligned}\|x_1(t)-x_2(t)\| &\le 0\Big[1+\int_{t_0}^{t}\frac{ML}{1-MG_1}\exp\Big\{\int_{t_0}^{s}(\frac{ML}{1-MG_1}+NK)d\tau\Big\}\Big]ds\\ &= 0\Big[1+\frac{ML}{ML+NK(1-MG_1)}\Big(\exp\Big\{\frac{ML+NK(1-MG_1)}{1-MG_1}(t-t_0)\Big\}-1\Big)\Big]\\ &= 0.\end{aligned} \tag{3.5}$$

From (3.5), we have $x_1(t) = x_2(t)$ for $t \in [t_0, b]$. Thus there is at most one solution to (1.1)–(1.2) on $[t_0, b]$. □

4. Continuous Dependence

In this section we study the continuous dependence of solutions to (1.1) on the given initial data, and the function f involved therein. Also we show the continuous dependence of solutions of equations of the form (1.1) on certain parameters.

The following theorem concerning the continuous dependence of solutions to (1.1)–(1.2) on the given initial conditions.

Theorem 4.1. *Suppose that the hypotheses* $(H_1) - (H_4)$ *hold. Suppose that the functions* x_1 *and* x_2 *satisfy the equation* (1.1) *for* $t_0 \le t \le b$ *with* $x_1(t_0) + g(x_1) = x_0^*$ *and* $x_2(t_0) + g(x_2) = x_0^{**}$ *respectively, and* $x_1(t), x_2(t) \in B$. *Then*

$$\|x_1(t)-x_2(t)\| \le \frac{M}{1-MG_1}\|x_0^*-x_0^{**}\|\Big[1+\frac{ML}{ML+NK(1-MG_1)}\Big(\exp\Big\{\frac{ML+NK(1-MG_1)}{1-MG_1}b\Big\}-1\Big)\Big].$$

Proof. The details of the proof of Theorem 4.1 follow by by similar arguments as in the proof of Theorem 3.2 with suitable modification. Hence we omit the details. □

Now we consider the initial value problem (1.1)–(1.2) and the corresponding initial-value problem

$$y'(t) + Ax(t) = \overline{f}(t, y(t), \int_{t_0}^{t} a(t,s)k(s,y(s))ds), \quad t \in [t_0, b], \tag{4.1}$$

$$y(t_0) + \overline{g}(y) = y_0, \tag{4.2}$$

where $\overline{f} : [t_0, b] \times X \times X \to X$, $\overline{g} : B \to X$, $k : [t_0, b] \times X \to X$, $a : [t_0, b] \times [t_0, b] \to R$ are continuous and y_0 is a given element of X.

The following theorem deals with the closeness of solutions of the initial value problem (1.1)–(1.2) and initial value problem (4.1)–(4.2).

Theorem 4.2. *Suppose that the hypotheses* $(H_1) - (H_4)$ *hold and there exist constants* $\epsilon_1 > 0$, $\delta_1 > 0$, $\delta_2 > 0$ *such that*

$$\|f(t,u,v) - \overline{f}(t,u,v)\| \le \epsilon_1, \tag{4.3}$$

$$\|x_0 - y_0\| \leq \delta_1, \; \|g(u) - \overline{g}(u)\| \leq \delta_2, \tag{4.4}$$

where x_0, g, f *and* $y_0, \overline{g}, \overline{f}$ *are as in* (1.1)–(1.2) *and* (4.1)–(4.2). *Let* $x(t)$ *and* $y(t)$ *be respectively, solutions of* (1.1)–(1.2) *and* (4.1)–(4.2) *on* $[t_0, b]$. *Then the solution* $x(t)$ *of the initial value problem* (1.1)–(1.2) *depends continuously on the functions involved therein.*

Proof. Let $u(t) = \|x(t) - y(t)\|$ for $t \in [t_0, b]$. From the hypotheses, we have

$$\begin{aligned} u(t) &\leq M\|x_0 - y_0\| + M\|g(y) - \overline{g}(y)\| + M\|g(x) - g(y)\| \\ &\quad + \int_{t_0}^{t} M\|f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau) - f(s, y(s), \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau))d\tau)\|ds \\ &\quad + \int_{t_0}^{t} M\|f(s, y(s), \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau))d\tau) - \overline{f}(s, y(s), \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau))d\tau)\|ds \\ &\leq M(\delta_1 + \delta_2) + MG_1 u(t) + \int_{t_0}^{t} ML\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds + \int_{t_0}^{t} M\epsilon_1 ds \\ &\leq M(\delta_1 + \delta_2 + \epsilon_1 b) + MG_1 u(t) + \int_{t_0}^{t} ML\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds, \end{aligned}$$

which implies

$$u(t) \leq \frac{M(\delta_1 + \delta_2 + \epsilon_1 b)}{1 - MG_1} + \int_{t_0}^{t} \frac{ML}{1 - MG_1}\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds, \tag{4.5}$$

Now an application of Lemma 2.2 (with $u_0 = \dfrac{M(\delta_1 + \delta_2 + \epsilon_1 b)}{1 - MG_1}$), known as Pachpatte's inequality, to (4.5), yields that for $t_0 \leq t \leq b$,

$$\begin{aligned} \|x(t) - y(t)\| &\leq \frac{M(\delta_1 + \delta_2 + \epsilon_1 b)}{1 - MG_1}\Big[1 + \int_{t_0}^{t} \frac{ML}{1 - MG_1} \exp\Big\{\int_{t_0}^{s} (\frac{ML}{1 - MG_1} + NK)d\tau\Big\}ds\Big] \\ &\leq \frac{M(\delta_1 + \delta_2 + \epsilon_1 b)}{1 - MG_1}\Big[1 + \frac{ML}{ML + NK(1 - MG_1)}\Big(\exp\Big\{\frac{ML + NK(1 - MG_1)}{1 - MG_1}b\Big\} - 1\Big)\Big]. \end{aligned} \tag{4.6}$$

This shows that the solution $x(t)$ of the initial value problem (1.1)–(1.2) depends continuously on the functions involved therein. □

Remark 4.3. The result given in Theorem 4.2 relates the solutions of the initial value problem (1.1)–(1.2) and of initial value problem (4.1)–(4.2) in the sense that if f is close to $\overline{f}$, x_0 is close to y_0, and g is close to $\overline{g}$, then not only the solutions of the initial value problem (1.1)–(1.2) and of initial value problem (4.1)–(4.2) are close to each other, but also depend continuously on the functions involved therein.

Next, consider the initial value problem (1.1)–(1.2) together with

$$y'(t) + Ay(t) = \overline{f}_k(t, y(t), \int_{t_0}^{t} a(t,s)k(s, y(s)))ds, \tag{4.7}$$

$$y(t_0) + \overline{g}_k(y) = \alpha_k, \tag{4.8}$$

for $k = 1, 2, \cdots$, where $\overline{f}_k \in C([t_0, b] \times X \times X, X)$, $\overline{g}_k \in C(B, X)$, $k \in C([t_0, b] \times X, X)$, $a \in C([t_0, b] \times [t_0, b], R)$ and α_k is a sequence in X.

As an immediate consequence of Theorem 4.2, we have the following corollary.

Corollary 4.4. *Suppose that the hypotheses* $(H_1)-(H_4)$ *hold and there exist nonnegative constants* $\epsilon_k,\ \delta_k,\ \overline{\delta}_k \quad (k=1,2,\cdots)$ *such that*

$$\|f(t,u,v)-\overline{f}_k(t,u,v)\| \leq \epsilon_k, \tag{4.9}$$

$$\|x_0-\alpha_k\| \leq \delta_k,\ \ \|g(u)-\overline{g}_k(u)\| \leq \overline{\delta}_k, \tag{4.10}$$

with $\epsilon_k \to 0$ *and* $\delta_k,\ \overline{\delta}_k \to 0$ *as* $k\to\infty$. *If* $x(t)$ *and* $y_k(t)$ $(k=1,2,\cdots)$ *are respectively the solutions of* (1.1)–(1.2) *and* (4.7)–(4.8) *on* $[t_0,b]$, *then* $y_k(t)\to x(t)$ *as* $k\to\infty$ *on* $[t_0,b]$.

Proof. For $k=1,2,\cdots$, the conditions of of Theorem 4.2 hold. As an application of Theorem 4.2 and Lemma 2.2 yields

$$\|y_k(t)-x(t)\| \leq \frac{M(\delta_k+\overline{\delta}_k+\epsilon_k b)}{1-MG_1}\Big[1+\frac{ML}{ML+NK(1-MG_1)}\Big(\exp\Big\{\frac{ML+NK(1-MG_1)}{1-MG_1}b\Big\}-1\Big)\Big], \tag{4.11}$$

for every $t_0\leq t\leq b$. As $k\to\infty$, the required result follows from (4.11). □

Remark 4.5. The result obtained in Corollary 4.4 provide sufficient conditions to ensure that the solutions of initial value problem (4.7)–(4.8) will converge to the solutions of initial value problem (1.1)–(1.2).

Now, we consider the integrodifferential equations:

$$x'(t)+Ax(t)=\overline{f}(t,x(t),\int_{t_0}^{t}a(t,s)k(s,x(s))ds,\mu_1), \tag{4.12}$$

$$x'(t)+Ax(t)=\overline{f}(t,x(t),\int_{t_0}^{t}a(t,s)k(s,x(s))ds,\mu_2), \tag{4.13}$$

for $t\in[t_0,b]$, where $\overline{f}\in C([t_0,b]\times X\times X\times R,X)$, and with the initial condition given by (1.2).

The following theorem states the continuous dependence of solutions to (4.12)-(1.2) and (4.13)-(1.2) on parameters.

Theorem 4.6. *Assume that hypotheses* $(H_1),(H_3)$ *and* (H_4) *hold and there exists positive constant* L_1 *such that*

$$\|\overline{f}(t,x,y,\mu_1)-\overline{f}(t,\bar{x},\bar{y},\mu_2)\| \leq L_1\Big[\|x-\bar{x}\|+\|y-\bar{y}\|+|\mu_1-\mu_2|\Big].$$

Let $x(t)$ *and* $y(t)$ *be the solutions of* (4.12) *with* (1.2) *and* (4.13) *with* (1.2) *respectively. Then*

$$\|x(t)-y(t)\| \leq \frac{ML_1b|\mu_1-\mu_2|}{1-MG_1}\Big[1+\frac{ML}{ML+NK(1-MG_1)}\Big(\exp\Big\{\frac{ML+NK(1-MG_1)}{1-MG_1}b\Big\}-1\Big)\Big],$$

for $t_0\leq t\leq b$.

Proof. Let $u(t) = \|x(t) - y(t)\|$ for $t \in [t_0, b]$. Now, by using the hypotheses, we have

$$\begin{aligned} u(t) &\le MG_1\|x(t) - y(t)\| + \int_{t_0}^{t} M\Big\|\Big[\overline{f}(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau, \mu_1) \\ &\quad - \overline{f}(s, y(s), \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau))d\tau, \mu_2)\Big]\Big\|ds \\ &\le MG_1u(t) + \int_{t_0}^{t} ML_1\Big[\|x(s) - y(s)\| + \int_{t_0}^{s} NK\|x(\tau) - y(\tau)\|d\tau + |\mu_1 - \mu_2|\Big]ds \\ &\le MG_1u(t) + \int_{t_0}^{t} ML_1\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau + |\mu_1 - \mu_2|\Big]ds \\ &\le ML_1b|\mu_1 - \mu_2| + MG_1u(t) + \int_{t_0}^{t} ML_1\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds, \end{aligned}$$

which implies

$$u(t) \le \frac{ML_1b|\mu_1 - \mu_2|}{1 - MG_1} + \int_{t_0}^{t} \frac{ML_1}{1 - MG_1}\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds. \tag{4.14}$$

Now an application of Lemma 2.2 (with $u_0 = \dfrac{ML_1b|\mu_1 - \mu_2|}{1 - MG_1}$), known as Pachpatte's inequality, to (4.14), yields

$$\|x(t) - y(t)\| \le \frac{ML_1b|\mu_1 - \mu_2|}{1 - MG_1}\Big[1 + \frac{ML}{ML + NK(1 - MG_1)}\Big(\exp\Big\{\frac{ML + NK(1 - MG_1)}{1 - MG_1}b\Big\} - 1\Big)\Big]. \tag{4.15}$$

□

A slight variant of Theorem 4.2 is given the following theorem.

Theorem 4.7. *Assume that hypotheses $(H_1), (H_3)$ and (H_4) hold and there exists constant $\overline{L}$ such that*

$$\|f(t, x, y) - \overline{f}(t, \bar{x}, \bar{y})\| \le \overline{L}\Big[\|x - \bar{x}\| + \|y - \bar{y}\|\Big],$$

where $\overline{L} \ge 0$, and that the condition (4.4) hold. Let $x(t)$ and $y(t)$ be the solutions of (1.1)–(1.2) and (4.1)–(4.2) respectively. Then

$$\|x(t) - y(t)\| \le \frac{M(\delta_1 + \delta_2)}{1 - MG_1}\Big[1 + \frac{M\overline{L}}{M\overline{L} + NK(1 - MG_1)}\Big(\exp\Big\{\frac{M\overline{L} + NK(1 - MG_1)}{1 - MG_1}b\Big\} - 1\Big)\Big],$$

for $t_0 \le t \le b$.

Proof. Let $u(t) = \|x(t) - y(t)\|$ for $t \in [t_0, b]$. Now, by using the hypotheses, we have

$$\begin{aligned} u(t) &\le M\|x_0 - y_0\| + M\|g(y) - \overline{g}(y)\| + M\|g(x) - g(y)\| \\ &\quad + \int_{t_0}^{t} M\Big\|\Big[f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau) - \overline{f}(s, y(s), \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau))d\tau)\Big]\Big\|ds \\ &\le M(\delta_1 + \delta_2) + MG_1u(t) + \int_{t_0}^{t} M\overline{L}\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds, \end{aligned}$$

which implies

$$u(t) \le \frac{M(\delta_1 + \delta_2)}{1 - MG_1} + \int_{t_0}^{t} \frac{M\overline{L}}{1 - MG_1}\Big[u(s) + \int_{t_0}^{s} NKu(\tau)d\tau\Big]ds. \tag{4.16}$$

Now an application of Lemma 2.2 (with $u_0 = \dfrac{M(\delta_1+\delta_2)}{1-MG_1}$), known as Pachpatte's inequality, to (4.16), yields

$$\|x(t)-y(t)\| \le \frac{M(\delta_1+\delta_2)}{1-MG_1}\Big[1+\frac{M\overline{L}}{M\overline{L}+NK(1-MG_1)}\Big(\exp\Big\{\frac{M\overline{L}+NK(1-MG_1)}{1-MG_1}b\Big\}-1\Big)\Big]. \quad (4.17)$$

□

5. Boundedness and Growth of Solutions

In this section the boundedness, asymptotic behaviour and growth of the solutions of equations (1.1)–(1.2) are investigated.

We need the following definitions in our subsequent discussion.

Definition 5.1. The solution $x(t)$ of equations (1.1)–(1.2) is said to be exponentially asymptotically stable, if there exist positive constants M and α such that the inequality

$$\|x(t)\| \le M(\|x_0\|+G)e^{-\alpha(t-t_0)}, \quad t \ge t_0,$$

holds for $(\|x_0\|+G)$ sufficiently small.

Definition 5.2. The solution $x(t)$ of equations (1.1)–(1.2) is said to be uniformly slowly growing if, and only if, for every $\alpha > 0$ there exists a constant M, possibly depending on α, such that the inequality

$$\|x(t)\| \le M(\|x_0\|+G)e^{\alpha(t-t_0)}, \quad t \ge t_0,$$

holds for $(\|x_0\|+G) < \infty$.

The following theorem contains the estimate on the solution of the initial value problem (1.1)–(1.2).

Theorem 5.3. *Assume that the hypothesis* (H_4) *holds. Let* k *and* f *satisfy*

$$\|k(t,x(t))\| \le p_1(t)\|x(t)\|, \quad (5.1)$$

and

$$\|f(t,x(t),y(t))\| \le p_2(t)[\|x(t)+\|y(t)\|], \quad (5.2)$$

for all $t \in [t_0,\infty)$, $x, y \in B$, *where* $p_1(t)$ *and* $p_2(t)$ *are real valued nonnegative continuous functions defined on* $[t_0,\infty)$ *such that*

$$\int_{t_0}^{\infty} p_1(t)dt < \infty, \qquad \int_{t_0}^{\infty} p_2(t)dt < \infty, \quad (5.3)$$

then all solutions of the initial value problem (1.1)–(1.2) *are bounded on* R_+.

Proof. Let $x(t) = T(t-t_0)[x_0-g(x)] + \int_{t_0}^{t} T(t-s)f\Big(s,x(s),\int_{t_0}^{s} a(s,\tau)k(\tau,x(\tau))d\tau\Big)ds$ be a solution of (1.1)–(1.2) on R_+. Using conditions (5.1), (5.2), and hypothesis, we have

$$\|x(t)\| \le M[\|x_0\|+G] + \int_{t_0}^{t} M\|f\Big(s,x(s),\int_{t_0}^{s} a(s,\tau)k(\tau,x(\tau))d\tau\Big)\|ds$$

$$\leq M[\|x_0\| + G] + M\int_{t_0}^{t} p_2(s)\Big[\|x(s)\| + \int_{t_0}^{s} |a(s,\tau)|\|k(\tau, x(\tau))\|d\tau\Big]ds$$
$$\leq M[\|x_0\| + G] + M\int_{t_0}^{t} p_2(s)\Big[\|x(s)\| + \int_{t_0}^{s} Np_1(\tau)\|x(\tau)\|d\tau\Big]ds.$$

Applying Lemma 2.2 with $u(t) = \|x(t)\|$, and $u_0 = M[\|x_0\| + G]$, we get

$$\|x(t)\| \leq M[\|x_0\| + G]\Big[1 + \int_{t_0}^{t} Mp_2(s)\exp\Big\{\int_{t_0}^{s}\Big(Mp_2(\tau) + Np_1(\tau)\Big)d\tau\Big\}ds\Big]. \tag{5.4}$$

Thus, in view of condition (5.3), the boundedness of the solution $x(t)$ follows. This completes the proof of the theorem. □

Remark 5.4. It is important to note that the Theorem 5.3 proves not only the boundedness, but also the stability of $x(t)$, if $\|x_0\| + G$ is small enough.

Next theorem deals the asymptotic behaviour of solution of the initial value problem (1.1)–(1.2).

Theorem 5.5. *Assume that the hypothesis (H_4) holds and $\|T(t-s)\| \leq C_1e^{-\alpha(t-s)}$, where C_1 is a nonnegative constant.. Let k and f satisfy*

$$\|k(t, x(t))\| \leq p_1(t)\|x(t)\|, \tag{5.5}$$

and

$$\|f(t, x(t), y(t))\| \leq p_2(t)e^{-\alpha t}[\|x(t) + \|y(t)\|], \tag{5.6}$$

for all $t \in [t_0, \infty)$, $\alpha > 0$ is a constant, where $p_1(t)$ and $p_2(t)$ are as in Theorem 5.3 with condition (5.3). Then all solutions of (1.1)–(1.2) approach to zero as $t \to \infty$.

Proof. Let $x(t) = T(t-t_0)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s)f\Big(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau\Big)ds$ be a solution of (1.1)–(1.2) on R_+. Using conditions (5.5), (5.6), and hypothesis (H_4), we obtain

$$\|x(t)\| \leq C_1e^{-\alpha(t-t_0)}[\|x_0\| + G] + \int_{t_0}^{t} C_1e^{-\alpha(t-s)}\|f\Big(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau\Big)\|ds$$
$$\leq C_1e^{-\alpha(t-t_0)}[\|x_0\| + G] + \int_{t_0}^{t} C_1e^{-\alpha(t-s)}e^{-\alpha s}p_2(s)\Big[\|x(s)\| + \int_{t_0}^{s} Np_1(\tau)\|x(\tau)\|d\tau\Big]ds$$
$$\leq C_1e^{-\alpha t}e^{\alpha t_0}[\|x_0\| + G] + \int_{t_0}^{t} C_1e^{-\alpha t}e^{\alpha s}e^{-\alpha s}p_2(s)\Big[\|x(s)\| + \int_{t_0}^{s} Np_1(\tau)\|x(\tau)\|d\tau\Big]ds. \tag{5.7}$$

Multiplying both sides of (5.7) by $e^{\alpha t}$ and using fact $e^{-\alpha t} < 1$, we obtain

$$\|x(t)\|e^{\alpha t} \leq C_1e^{\alpha t_0}[\|x_0\| + G] + \int_{t_0}^{t} C_1p_2(s)e^{\alpha s}\|x(s)\|ds + \int_{t_0}^{t} C_1p_2(s)\int_{t_0}^{s} Np_1(\tau)e^{\alpha\tau}\|x(\tau)\|d\tau ds. \tag{5.8}$$

Applying Lemma 2.2 with $u(t) = \|x(t)\|e^{\alpha t}$, and $u_0 = C_1e^{\alpha t_0}[\|x_0\| + G]$, we get

$$\|x(t)\|e^{\alpha t} \leq C_1e^{\alpha t_0}[\|x_0\| + G]\Big[1 + \int_{t_0}^{t} C_1p_2(s)\exp\Big\{\int_{t_0}^{s}\Big(C_1p_2(\tau) + Np_1(\tau)\Big)d\tau\Big\}ds\Big]. \tag{5.9}$$

Using the condition (5.3), we get

$$\|x(t)\|e^{\alpha t} \leq L_2,$$

where $L_2 > 0$ is a constant. Thus, as $t \to \infty$, the solution of (1.1)–(1.2) approaches to zero. This completes the proof of the theorem. $\square$

Theorem 5.6. *Assume that the hypothesis* (H_4) *holds and* $\|T(t-s)\| \leq C_1 e^{\alpha(t-s)}$, *where* C_1 *is a nonnegative constant. Let* k *and* f *satisfy*

$$\|k(t, x(t))\| \leq p_1(t) e^{-\alpha t} \|x(t)\|, \tag{5.10}$$

and

$$\|f(t, x(t), y(t))\| \leq p_2(t) e^{-\alpha t} [\|x(t) + \|y(t)\|], \tag{5.11}$$

for all $t \in [t_0, \infty)$, $\alpha > 0$ *is a constant, where* $p_1(t)$ *and* $p_2(t)$ *are as in Theorem 5.3 with condition* (5.3). *Then all solutions of* (1.1)–(1.2) *are slowly growing.*

Proof. Let $x(t) = T(t-t_0)[x_0 - g(x)] + \int_{t_0}^{t} T(t-s) f\Big(s, x(s), \int_{t_0}^{s} a(s,\tau) k(\tau, x(\tau)) d\tau\Big) ds$ be a solution of (1.1)–(1.2) on R_+. Using conditions (5.10), (5.11), and hypothesis (H_4), we obtain

$$\begin{aligned}
\|x(t)\| &\leq C_1 e^{\alpha(t-t_0)}[\|x_0\| + G] + \int_{t_0}^{t} C_1 e^{\alpha(t-s)} \|f\Big(s, x(s), \int_{t_0}^{s} a(s,\tau) k(\tau, x(\tau)) d\tau\Big)\| ds \\
&\leq C_1 e^{\alpha(t-t_0)}[\|x_0\| + G] + \int_{t_0}^{t} C_1 e^{\alpha(t-s)} e^{-\alpha s} p_2(s) \Big[\|x(s)\| + \int_{t_0}^{s} N p_1(\tau) e^{-\alpha\tau} \|x(\tau)\| d\tau\Big] ds \\
&\leq C_1 e^{\alpha t} e^{-\alpha t_0}[\|x_0\| + G] + \int_{t_0}^{t} C_1 e^{\alpha t} e^{-\alpha s} e^{-\alpha s} p_2(s) \Big[\|x(s)\| + \int_{t_0}^{s} N p_1(\tau) e^{-\alpha\tau} \|x(\tau)\| d\tau\Big] ds.
\end{aligned} \tag{5.12}$$

Multiplying both sides of (5.12) by $e^{-\alpha t}$ and using fact $e^{-\alpha t} < 1$, we obtain

$$\|x(t)\| e^{-\alpha t} \leq C_1 e^{-\alpha t_0}[\|x_0\| + G] + \int_{t_0}^{t} C_1 p_2(s) e^{-\alpha s} \|x(s)\| ds + \int_{t_0}^{t} C_1 p_2(s) \int_{t_0}^{s} N p_1(\tau) e^{-\alpha\tau} \|x(\tau)\| d\tau ds. \tag{5.13}$$

Applying Lemma 2.2 with $u(t) = \|x(t)\| e^{-\alpha t}$, and $u_0 = C_1 e^{-\alpha t_0}[\|x_0\| + G]$, we get

$$\|x(t)\| e^{-\alpha t} \leq C_1 e^{-\alpha t_0}[\|x_0\| + G]\Big[1 + \int_{t_0}^{t} C_1 p_2(s) \exp\Big\{\int_{t_0}^{s} \Big(C_1 p_2(\tau) + N p_1(\tau)\Big) d\tau\Big\} ds\Big]. \tag{5.14}$$

Using the condition (5.3) in (5.14), we get

$$\|x(t)\| e^{-\alpha t} \leq L_3,$$

where $L_3 > 0$ is a constant. Thus, as $t \to \infty$, the solutions of (1.1)–(1.2) are slowly growing. This completes the proof of the theorem. $\square$

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FIXED POINTS AND ORTHOGONAL STABILITY OF FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN SPACES

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Abstract. In this paper, by using the fixed point method, we investigate the orthogonal stability of orthogonally Jensen additive functional equation, the orthogonally cubic functional equation and the orthogonally quartic functional equation in non-Archimedean normed spaces.

1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y)$, where $\langle x, y\rangle = 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

Pinsker [52] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [63] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation

$$f(x + y) = f(x) + f(y), \quad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [25]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [59] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [60] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of Rätz ([59]).

Suppose that X is a real vector space with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

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(O1) *totality* of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;

(O2) *independence*: if $x, y \in X - \{0\}$ and $x \perp y$, then x and y are linearly independent;

(O3) *homogeneity*: if $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) *Thalesian property*: if P is a 2-dimensional subspace of X, $x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an *orthogonality space*. By an *orthogonality normed space* we mean an orthogonality space having a normed structure.

Some interesting examples are as follows:

(1) The trivial orthogonality on a vector space X defined by (O_1) and, for any non-zero elements $x, y \in X$, $x \perp y$ if and only if x and y are linearly independent.

(2) The ordinary orthogonality on an inner product space $(X, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(3) The Birkhoff-James orthogonality on a normed space $(X, \| \cdot \|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called *symmetric* if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly, Examples (1) and (2) are symmetric, but Example (3) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[4], [9, 8, 21, 31]).

The stability problem of functional equations was originated from the following question of Ulam [65]:

Under what condition does there is an additive mapping near an approximately additive mapping?

In 1941, Hyers [27] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [54] extended the theorem of Hyers by considering the unbounded Cauchy difference

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (\varepsilon > 0,\, p \in [0,1)).$$

During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. Refer to [18, 28, 33, 45, 58] and references therein for detailed information on stability of functional equations.

Ger and Sikorska [24] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$, namely, they showed that, if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and for some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was Skof [62] by proving that, if f is a mapping from a normed space X into a Banach space Y satisfying

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \varepsilon$$

for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x)-g(x)\| \leq \frac{\varepsilon}{2}$. Cholewa [15] extended the Skof's theorem by replacing X by an abelian group G. Skof's result was later generalized by Czerwik [16] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [17, 51, 50], [55]–[57]).

The orthogonally quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x \perp y$$

was first investigated by Vajzović [66] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Cho et al. [12], Drljević [22], Fochi [23], Moslehian [41, 42] and Szabó [64] generalized this result.

In [32], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

Let X be an orthogonality space and Y a real Banach space. A mapping $f : X \to Y$ is called *orthogonally cubic* if it satisfies the orthogonally cubic functional equation (0.3).

In [37], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*. For more results on the quartic functional equations, see [3, 10, 14, 16, 22, 37, 40, 50, 55, 61, 64].

Let X be an orthogonality space and Y a Banach space. A mapping $f : X \to Y$ is called *orthogonally quartic* if it satisfies the orthogonally quartic functional equation (0.4).

In 1897, Hensel [26] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [19, 35, 36, 46]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that, for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$;

(2) $|rs| = |r||s|$;
(3) $|r + s| \le \max\{|r|, |s|\}$.

Definition 1.2. ([44]) Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(1) $||x|| = 0$ if and only if $x = 0$;
(2) $||rx|| = |r|||x||$ for all $r \in K$ and $x \in X$;
(3) The strong triangle inequality (ultrametric), namely,

$$||x + y|| \le max\{||x||, ||y||\}$$

for all $x, y \in X$.
Then $(X, ||\cdot||)$ is called a *non-Archimedean space.*

Due to the fact that

$$||x_n - x_m|| \le \max\{||x_{j+1} - x_j|| : m \le j \le n-1\} \quad (n > m).$$

Definition 1.3. A sequence $\{x_n\}$ is a *Cauchy sequence* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a *complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

Recently, some authors ([13, 14, 44, 46, 50]) investigated the stability of the functional inequalities, the ACQ functional equations and the generalized quadratic functional equations in non-Archimedean spaces.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following conditions:
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4. [5, 20] *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \ge n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th.M. Rassias [29] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 34, 39, 48, 49, 53]).

In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of the orthogonally Jensen additive functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \tag{1.3}$$

the orthogonally Jensen quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y), \tag{1.4}$$

the orthogonally cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.5}$$

and the orthogonally quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y) \tag{1.6}$$

for all x, y with $x \perp y$, where $\perp$ is the orthogonality in the sense of Rätz, in on-Archimedean normed spaces.

Throughout this paper, assume that $(X, \perp)$ is an orthogonality non-Archimedean space and that $(Y, \|\cdot\|_Y)$ is a non-Archimedean Banach space. Assume that $|2| \neq 1$.

2. **Stability of the orthogonally Jensen additive functional equation**

In this section, applying some ideas from [24, 28], we deal with the stability problem for the orthogonally Jensen additive functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.1}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_Y \leq \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha}{1-\alpha}\varphi(x, 0) \tag{2.3}$$

for all $x \in X$*.*

Proof. Putting $y = 0$ in (2.2), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_Y \leq \varphi(x, 0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{1}{|2|}\varphi(2x, 0) \leq \alpha \cdot \varphi(x, 0) \tag{2.5}$$

for all $x \in X$. Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x, 0),\ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [38]).

Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then we have

$$\|g(x) - h(x)\|_Y \leq \varphi(x, 0)$$

for all $x \in X$ and so

$$\|Jg(x) - Jh(x)\|_Y = \left\|\frac{1}{2}g(2x) - \frac{1}{2}h(2x)\right\|_Y \leq \alpha\varphi(x, 0)$$

for all $x \in X$. Thus $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$, which means that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for all $g, h \in S$. It follows from (2.5) that $d(f, Jf) \leq \alpha$. By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L(2x) = 2L(x) \tag{2.6}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that L is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - L(x)\|_Y \quad \leq \quad \mu\varphi(x, 0)$$

for all $x \in X$;

(2) $d(J^n f, L) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{\alpha}{1-\alpha}.$$

This implies that the inequalities (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\left\|2L\left(\frac{x+y}{2}\right) - L(x) - L(y)\right\|_Y &= \lim_{n\to\infty} \frac{1}{|2|^n}\|2f(2^{n-1}(x+y)) - f(2^n x) - f(2^n y)\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{|2|^n}\varphi(2^n x, 2^n y) \leq \lim_{n\to\infty} \frac{|2|^n \alpha^n}{|2|^n}\varphi(x, y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So

$$2L\left(\frac{x+y}{2}\right) - L(x) - L(y) = 0$$

for all $x, y \in X$ with $x \perp y$. Hence $L : X \to Y$ is an orthogonally Jensen additive mapping. This completes the proof. $\square$

From now on, in Corollaries, assume that $(X, \perp)$ is an orthogonality non-Archimedean normed space.

Corollary 2.2. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_Y \leq \theta(\|x\|^p + \|y\|^p) \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{|2|^p\theta}{|2| - |2|^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-1}$ and we get the desired result. $\square$

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying* (2.2) *and $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\varphi(x, y) \leq \frac{\alpha}{2}\varphi(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{1-\alpha}\varphi(x, 0) \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. It follows from (2.4) that $d(f, Jf) \leq 1$. So

$$d(f, L) \leq \frac{1}{1-\alpha}.$$

Thus we obtain the inequality (2.8). The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.4. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally Jensen additive mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{|2|^p \theta}{|2|^p - |2|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{1-p}$ and we get the desired result. $\square$

3. Stability of the orthogonally Jensen quadratic functional equation

In this section, applying some ideas from [24, 28], we deal with the stability problem for the orthogonally Jensen quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |4|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.1}$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right\|_Y \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{\alpha}{1-\alpha}\varphi(x, 0) \tag{3.3}$$

for all $x \in X$.

Proof. Putting $y=0$ in (3.2), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\|_Y \le \varphi(x,0) \tag{3.4}$$

for all $x\in X$ since $x\perp 0$. So

$$\left\|f(x)-\frac{1}{4}f(2x)\right\|_Y \le \frac{1}{|4|}\varphi(2x,0)\le \alpha\cdot\varphi(x,0) \tag{3.5}$$

for all $x\in X$. By the same reasoning as in the proof of Theorem 2.1, one can obtain an orthogonally Jensen quadratic mapping $Q:X\to Y$ defined by

$$\lim_{n\to\infty}\frac{1}{4^n}f\left(2^n x\right)=Q(x)$$

for all $x\in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=\frac{1}{4}g\left(2x\right)$$

for all $x\in X$. It follows from (3.5) that $d(f,Jf)\le\alpha$. So

$$d(f,Q)\le\frac{\alpha}{1-\alpha}.$$

Thus we obtain the inequality (3.3). This completes the proof. □

Corollary 3.2. *Let θ be a positive real number and p a real number with $p>2$. Let $f:X\to Y$ be a mapping satisfying*

$$\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\|_Y \le \theta(\|x\|^p+\|y\|^p) \tag{3.6}$$

for all $x,y\in X$ with $x\perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q:X\to Y$ such that

$$\|f(x)-Q(x)\|_Y\le\frac{2^p\theta}{4-2^p}\|x\|^p$$

for all $x\in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x,y)=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$ with $x\perp y$. Then we can choose $\alpha=|2|^{p-2}$ and we get the desired result. □

Theorem 3.3. *Let $f:X\to Y$ be a mapping satisfying (3.2) and $f(0)=0$ for which there exists a function $\varphi:X^2\to[0,\infty)$ such that*

$$\varphi(x,y)\le\frac{\alpha}{|4|}\varphi\left(2x,2y\right)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{1-\alpha}\varphi(x, 0) \tag{3.7}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$. It follows from (3.4) that $d(f, Jf) \leq 1$. So we obtain the inequality (3.7). The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. $\square$

Corollary 3.4. *Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be a mapping satisfying (3.6). Then there exists a unique orthogonally Jensen quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|2|^p\theta}{|2|^p - |4|}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{2-p}$ and we get the desired result. $\square$

4. **Stability of the orthogonally cubic functional equation**

In this section, applying some ideas from [24, 28], we deal with the stability problem for the orthogonally cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 4.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |8|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\|_Y \\ &\qquad \leq \varphi(x, y) \end{aligned} \tag{4.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \leq \frac{1}{|16| - |16|\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (4.1), we get

$$\|2f(2x) - 16f(x)\|_Y \le \varphi(x, 0) \tag{4.2}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\|f(x) - \frac{1}{8}f(2x)\right\|_Y \le \frac{1}{|16|}\varphi(x, 0)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$. The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Corollary 4.2. *Let θ be a positive real number and p a real number with $p > 3$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\|_Y \\ &\qquad \le \theta(\|x\|^p + \|y\|^p) \end{aligned} \tag{4.3}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \le \frac{\theta}{|2|(|8| - |2|^p)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-3}$ and we get the desired result. □

Theorem 4.3. *Let $f : X \to Y$ be a mapping satisfying* (4.1) *and $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\varphi(x, y) \le \frac{\alpha}{|8|}\varphi(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\|_Y \le \frac{\alpha}{|16| - |16|\alpha}\varphi(x, 0) \tag{4.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$. It follows from (4.2) that $d(f, Jf) \leq \frac{\alpha}{|16|}$. So we obtain the inequality (4.4). The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. $\square$

Corollary 4.4. *Let θ be a positive real number and p a real number with $0 < p < 3$. Let $f : X \to Y$ be a mapping satisfying (4.3). Then there exists a unique orthogonally cubic mapping $C : X \to Y$ such that*

$$\|f(x) - C(x)\|_Y \leq \frac{\theta}{|2|(|2|^p - |8|)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.3 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{3-p}$ and we get the desired result. $\square$

5. Stability of the orthogonally quartic functional equation

Applying some ideas from [24, 28], we deal with the stability problem for the orthogonally quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$$

for all $x, y \in X$ with $x \perp y$.

Theorem 5.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |16|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y)\|_Y \\ &\leq \varphi(x, y) \end{aligned} \tag{5.1}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\|_Y \leq \frac{1}{|32| - |32|\alpha}\varphi(x, 0)$$

for all $x \in X$.

Proof. Putting $y = 0$ in (5.1), we get

$$\|2f(2x) - 32f(x)\|_Y \le \varphi(x, 0) \tag{5.2}$$

for all $x \in X$, since $x \perp 0$. So

$$\left\| f(x) - \frac{1}{16} f(2x) \right\|_Y \le \frac{1}{|32|} \varphi(x, 0)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16} g\,(2x)$$

for all $x \in X$. The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Corollary 5.2. *Let θ be a positive real number and p a real number with $p > 4$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} &\|f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y)\|_Y \\ &\qquad \le \theta(\|x\|^p + \|y\|^p) \end{aligned} \tag{5.3}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\|_Y \le \frac{\theta}{|2|(|16| - |2|^p)} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.1 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{p-4}$ and we get the desired result. □

Theorem 5.3. *Let $f : X \to Y$ be a mapping satisfying (5.1) and $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\varphi(x, y) \le \frac{\alpha}{|16|} \varphi\,(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that

$$\|f(x) - P(x)\|_Y \le \frac{\alpha}{|32| - |32|\alpha} \varphi\,(x, 0) \tag{5.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$. It follows from (5.2) that $d(f, Jf) \leq \frac{\alpha}{|32|}$. So we obtain the inequality (5.4). The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Corollary 5.4. *Let θ be a positive real number and p a real number with $0 < p < 4$. Let $f : X \to Y$ be a mapping satisfying (5.3). Then there exists a unique orthogonally quartic mapping $P : X \to Y$ such that*

$$\|f(x) - P(x)\|_Y \leq \frac{\theta}{|2|(|2|^p - |16|)}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.3 by taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$. Then we can choose $\alpha = |2|^{4-p}$ and we get the desired result. □

Conclusions

Using the fixed point method, we have proved the Hyers-Ulam stability of the orthogonally Jensen additive functional equation, of the orthogonally Jensen quadratic functional equation, of the orthogonally cubic functional equation and of the orthogonally quartic functional equation in non-Archimedean Banach spaces.

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ON THE GENERALIZED SUMUDU TRANSFORMS

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Abstract

In this paper, we extend the Sumudu transform to a context of distributions and obtain many of its properties in the sense of distributions of compact support. In more general case, we establish a generalization of the cited transform to a space of integrable Boehmians .

Keywords: Distribution Space; Sumudu Transform; Convolution Theorem; Compact Support; Boehmian Spaces.

1 INTRODUCTION

To solve differential equations, various integral transforms were extensively used, in theory and applications, such as Laplace, Fourier, Hankel and convolution transforms, to name but a few. Such transforms have been studied in $[1, 2, 3, 4, 5, 6]$, [9] and [10] .

In the sequence of these transformations, in early 90's, Watugala [11] introduced a new integral transform, namely, the Sumudu transform and further applied it to the solution of ordinary differential equations. For further details see [11] . Its main advantages is that it solves problems without resorting to a new frequency domain, since it preserves scales and units properties.

Having scale and unit-preserving properties, the Sumudu transform may solve intricate problems in engineering mathematics and applied sciences without resorting to a new frequency domain.

However, despite the potential presented by this new operator, only few theoretical investigations have appeared in the literature, over a fifteen-years period. Most of the available transform theory books, do not refer the Sumudu transform, even in the recent well-know comprehensive handbooks. Perhaps, it is because no transform urder this name was declared until the early 90's of the previous century.

Weerakoon in [12] has discussed the Sumudu transform of partial differential equations applying the complex invesion formula to the solution of partial differential equations. In [7] , Belgacem et al., show that the Sumudu transform has deeper connections with the Laplace transform. However, the approach here is somewhat different and interesting. This paper aims at extending the Sumudu transform to a certain space of distributions of compact support and possibley derive certain theorems. This, as we believe, will open a new avenue of the investigations of the transform of generalized functions, such as, distributions, ultradistributions, and possible Boehmian spaces in a way similar to that of the rest of integral transforms.

2 THE SUMUDU TRANSFORMATION

For a function $f(t)$, *the Sumudu transform is defined by* [11]

$$\boldsymbol{M}(u) \cong \boldsymbol{M}(f(t);u) \cong \int_{\mathbb{R}_+} \frac{1}{u} \exp\left(-\frac{t}{u}\right) f(t)\,dt, \tag{2.1}$$

provided the integral exists. The sufficient conditions for the above integral to exist are that $f(t), t \geq 0$, is piecewise continuous and of an exponential order. On the other hand, over a set of functions

$$\boldsymbol{A} = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0,\infty), j = 1, 2, ... \right\},$$

the Sumudu transform is defined by

$$\boldsymbol{M}(u) \cong \int_{\mathbb{R}_+} f(ut)\, e^{-t} dt, u \in (-\tau_1, \tau_2). \tag{2.2}$$

The Sumudu transform, in (2.2) , is, sometimes, reduced to (2.1) after a suitable change of variables. In various papers, it has been shown to be a theoretical dual of the Laplace transform. That is, the Laplace and Sumudu transforms exhibit a duality relation expressed as

$$\boldsymbol{M}\left(\frac{1}{s}\right) = s\boldsymbol{N}(s), \boldsymbol{N}\left(\frac{1}{u}\right) = u\boldsymbol{M}(u), \tag{2.3}$$

where $\boldsymbol{M}, \boldsymbol{N}$ are the Sumudu and Laplace transforms of f, respectively. The duality in (2.3) is known as Laplace-Sumudu duality, which is illustrated by the fact that the Sumudu and Laplace transforms interchange the images of the Dirac and Heaviside functions as

$$\boldsymbol{M}[\boldsymbol{H}(t);u] = \boldsymbol{N}[\delta(t);u] = 1,$$

and

$$\boldsymbol{M}[\delta(t);u] = \boldsymbol{N}[\boldsymbol{H}(t);u] = \frac{1}{u}.$$

Similarly, the duality is also illustrated since both of the transforms interchange the images of $\sin t$ and $\cos t$ in the formulae

$$\boldsymbol{M}[\cos t;u] = \boldsymbol{N}[\sin t;u] = \frac{1}{1+u^2} \text{ and } \boldsymbol{M}[\sin t;u] = \boldsymbol{N}[\cos t;u] = \frac{u}{1+u^2}.$$

The following, are the general properties of the Sumudu transform (2.1) which can easily be drawn from the substitution method and properties of definite integrals.

(*i*) *If* k_1 *and* k_2 *are non-negative real numbers and* $\boldsymbol{M}_1$ *and* $\boldsymbol{M}_2$ *are the corresponding Sumudu transforms of* f_1 *and* f_2, *respectively, then*

$$\boldsymbol{M}((k_1 f_1(t) + k_2 f_2(t));u) = k_1 \boldsymbol{M}_1(u) + k_2 \mathbf{M}_2(u).$$

$(ii)\, \boldsymbol{M}\left(f\left(kt\right);u\right)=\boldsymbol{M}\left(ku\right), k\in\mathbb{R}_{+}.$

$(iii)\, \lim_{t\to 0} f\left(t\right)=\lim_{u\to 0}\boldsymbol{M}\left(u\right)=f\left(0\right)$, where $\boldsymbol{M}\left(u\right)$ *is the Sumudu transform of* f. For more discussion, reader is referred to [7], [11] and [12] and, references cited therein.

3 DISTRIBUTIONAL SUMUDU TRANSFORMATION

Let $\boldsymbol{K}$ be a compact subset of $\mathbb{R}_{+}$. Denote by $\boldsymbol{E}\left(\mathbb{R}_{+}\right)$ the space of all complex-valued infinitely smooth functions on $\mathbb{R}_{+}$ (with arbitrary support) such that

$$\sup_{t\in\mathbf{K}}\left|\mathcal{D}_{t}^{k}\phi\left(t\right)\right|<\infty, \tag{3.1}$$

where k is a non-negative integer and, $\mathcal{D}_{t}^{k}=\dfrac{d^{k}}{dt^{k}}$.

A sequence $\left(\phi_{j}\right)$ of functions $\phi_{j}\in\boldsymbol{E}\left(\mathbb{R}_{+}\right)$ is said to converge in the sense of $\boldsymbol{E}\left(\mathbb{R}_{+}\right)$ if and only if for every fixed k the sequence $\left(\mathcal{D}_{t}^{k}\phi_{j}\right)$ converges uniformly on every compact subset of $\mathbb{R}_{+}$. The space $\boldsymbol{E}\left(\mathbb{R}_{+}\right)$ is complete under convergence in the sense of $\boldsymbol{E}\left(\mathbb{R}_{+}\right)$. The strong dual $\boldsymbol{E}^{'}\left(\mathbb{R}_{+}\right)$ of $\boldsymbol{E}\left(\mathbb{R}_{+}\right)$ consists of distributions of compact support. Indeed, as the definition in (2.1) shows, the kernel of the Sumudu transform $e^{-t/u}/u$ is smooth and satisfies

$$\sup_{t\in\boldsymbol{K}}\left|\mathcal{D}_{t}^{k}\frac{e^{-t/u}}{u}\right|=\sup_{t\in\boldsymbol{K}}\left|\left(-1\right)^{k}\frac{e^{-t/u}}{u^{k+1}}\right|<\infty \tag{3.2}$$

as $\boldsymbol{K}$ ranges over compact subsets of $\mathbb{R}_{+}$ for every positive real u.

Hence, if $f\in\boldsymbol{E}^{'}\left(\mathbb{R}_{+}\right)$ (f is a distribution of compact support), (3.2) suggests we define the distributional Sumudu transform of the distribution f of compact support as

$$\hat{\boldsymbol{M}}\left(u\right)\cong\left\langle f\left(t\right),e^{-t/u}/u\right\rangle, \tag{3.3}$$

for an arbitrary positive real u $\left(u\in\mathbb{R}_{+}\right)$.

Theorem 3.1 *The distributional Sumudu transform is linear .*

Proof of this theorem is straightforward consequence of (3.3). Thus, we avoid the details.

Theorem 3.2 *If* $f\in\boldsymbol{E}^{'}\left(\mathbb{R}_{+}\right)$ *and* $g\left(t\right)=\left\{\begin{array}{ll} f\left(t-\tau\right), t\geq\tau \\ 0, \quad t<\tau\end{array}\right\}$, *then*

$$\hat{\boldsymbol{M}}_{2}\left(u\right)=e^{-\tau/u}\hat{\boldsymbol{M}}_{1}\left(u\right),$$

where $\hat{\boldsymbol{M}}_{1}$ and $\hat{\boldsymbol{M}}_{2}$ are the distributional Sumudu transforms of f and g , respectively.

Proof As it appears from the defined function, $g\in\boldsymbol{E}^{'}\left(\mathbb{R}_{+}\right)$. Therefore, in view of the translation property of distributions through τ $\left[14, \text{p.}26\right]$, we get

$$\begin{aligned}\hat{\boldsymbol{M}}_2(u) &= \left\langle f(t-\tau), e^{-t/u}/u \right\rangle \\ &= \left\langle f(t), e^{-(t+\tau)/u}/u \right\rangle \\ &= e^{-\tau/u}\hat{\boldsymbol{M}}_1(u).\end{aligned}$$

Hence, the theorem.

Theorem 3.3 *Let $f \in \boldsymbol{E}'(\mathbb{R}_+)$ and $\hat{\boldsymbol{M}}(u)$ be the Sumudu transform of f, then*

$$D_u^k \hat{\boldsymbol{M}}(u) = \left\langle f(t), \mathcal{D}_u^k \left(e^{-t/u}/u\right)\right\rangle, \tag{3.4}$$

where $\mathcal{D}_u^k = \dfrac{d^k}{du^k}$ stands for the k-th derivative with respect to u.

Proof We attempt to prove the theorem by induction on k. For $k = 0$, the case is reduced to (3.3).

To proceed to the induction step, we assume the theorem apply for $(k-1)$-derivatives. i.e.

$$\mathcal{D}_u^{k-1}\hat{\boldsymbol{M}}(u) = \left\langle f(t), \mathcal{D}_u^{k-1}\left(e^{-t/u}/u\right)\right\rangle.$$

Let u be fixed and $\Delta u \neq 0$, then

$$(1/\Delta u)\left(\mathcal{D}_u^{k-1}\hat{\boldsymbol{M}}(u+\Delta u) - \mathcal{D}_u^{k-1}\hat{\boldsymbol{M}}(u)\right) - \left\langle f(t), \mathcal{D}_u^k\left(e^{-t/u}/u\right)\right\rangle = \langle f(t), \theta_{\Delta u}(t)\rangle,$$

where

$$\theta_{\Delta u}(t) = (1/\Delta u)\left(\mathcal{D}_u^{k-1}\left(e^{-(t/(u+\Delta u))}/(u+\Delta u)\right) - \mathcal{D}_u^{k-1}\left(e^{-t/u}/u\right)\right) - \mathcal{D}_u^k\left(e^{-t/u}/u\right). \tag{3.5}$$

To complete the proof of the theorem we are merely required to establish that $\theta_{\Delta u}(t) \to 0$ as $\Delta u \to 0$, in the sense of topology of $\boldsymbol{E}(\mathbb{R}_+)$.

Let j be a non-negative integer. In light of (3.5), we have

$$\theta_{\Delta u}^{(j)}(t) = (1/\Delta u)\int_{u-t}^{u-t+\Delta u}\int_{u-t}^{y}\mathcal{D}_\xi^{j+k-1}\left(e^{-t/\xi}/\xi\right)d\xi dy. \tag{3.6}$$

Let $\gamma = \{\xi : u-t-|\Delta u| < \xi < u-t+|\Delta u|\}$. Consequently, from (3.6), togather with simple calculation, we have

$$\left|\theta_{\Delta u}^{(j)}(t)\right| \leq \frac{|\Delta u|}{2}\sup_{\xi\in\gamma}\left|D_\xi^{j+k-1}\left(e^{-t/\xi}/\xi\right)\right| \to 0 \text{ uniformly as } \Delta u \to 0,$$

on compact subsets of $\mathbb{R}_+$. This completes the proof of the theorem.

Theorem 3.4 *Let* $f \in \boldsymbol{E}'(\mathbb{R}_+)$ *and* $\hat{\boldsymbol{M}}(u)$ *be the distributional Sumudu transform of* f*, then*

$$\hat{\boldsymbol{M}}\left(t^n \mathcal{D}_t^n f(t); u\right) = u^n \mathcal{D}_u^n \hat{\boldsymbol{M}}(u)$$

Proof In consideration of the properties of Sumudu transformation and (3.3), we get

$$\begin{aligned} \mathcal{D}_u^n \hat{\boldsymbol{M}}(u) &= \mathcal{D}_u^n \left\langle f(t), e^{-t/u}/u \right\rangle \\ &= \mathcal{D}_u^n \left\langle f(tu), e^{-t} \right\rangle \end{aligned}$$

i.e.

$$\begin{aligned} \mathcal{D}_u^n \hat{\boldsymbol{M}}(u) &= \mathcal{D}_u^n \left\langle f(tu), e^{-t} \right\rangle \\ &= \left\langle t^n \mathcal{D}_t^n f(ut), e^{-t} \right\rangle \end{aligned}$$

i.e.

$$\begin{aligned} \mathcal{D}_u^n \hat{\boldsymbol{M}}(u) &= \frac{1}{u^n} \left\langle (ut)^n \mathcal{D}_t^n f(tu), e^{-t} \right\rangle \\ &= \frac{1}{u^n} \hat{\boldsymbol{M}}\left(t^n \mathcal{D}_t^n f(t); u\right) \end{aligned}$$

This complete the proof of the theorem.

Theorem 3.5 *Let* $f \in \boldsymbol{E}'(\mathbb{R}_+)$ *and* $\hat{\boldsymbol{M}}(u)$ *be its corresponding distributional Sumudu transform, then*

$$\hat{\boldsymbol{M}}\left[e^{at} f(t); u\right] = \left(1/(1-au)\right) \hat{\boldsymbol{M}}\left(u/(1-au)\right).$$

Proof This theorem is, indeed, an obvious result of Equation 2.1 and the basic properties of differentiation.

Theorem 3.6 *Let* $f \in \boldsymbol{E}'(\mathbb{R}_+)$ *and* $\hat{\boldsymbol{M}}(u)$ *is the Sumudu transform of* f*, then*

$$\hat{\boldsymbol{M}}\left(f(at); u\right) = \hat{\boldsymbol{M}}(au).$$

Proof Applying the property of change under scale of Sumudu transforms, our theorem can be easily established.

Following is a theorem, which deals with multiplication of a distribution $f(t)$ by a positive power of t .

Theorem 3.7 *Let* $f \in \boldsymbol{E}'(\mathbb{R}_+)$ *and* $\hat{\boldsymbol{M}}(u)$ be *the distributional Sumudu transform of* f*, then*

(*i*) $\hat{\boldsymbol{M}}\left(tf(t); u\right) = u^2 \mathcal{D}_u^1 \hat{\boldsymbol{M}}(u) + u\hat{\boldsymbol{M}}(u)$.
(*ii*) $\hat{\boldsymbol{M}}\left(t^2 f(t); u\right) = u^4 \mathcal{D}_u^2 \hat{\boldsymbol{M}}(u) + 4u^3 \mathcal{D}_u^1 \hat{\boldsymbol{M}}(u) + 2u^2 \hat{\boldsymbol{M}}(u)$

Proof We prove Part (*i*) of the theorem since the second part is similar .

Let $f \in \boldsymbol{E}'(\mathbb{R}_+)$ then employing (3.3) and Theorem 3.3 we have

$$\mathcal{D}_u^1 \hat{\boldsymbol{M}}(u) = \left\langle f(t), \mathcal{D}_u^1 \left(e^{-t/u}/u\right) \right\rangle.$$

With the aid of the rules of differentiation, simple calculations yield

$$\begin{aligned} \mathcal{D}_u^1 \hat{\boldsymbol{M}}(u) &= \left(1/u^2\right) \left\langle tf(t), e^{-t/u}/u \right\rangle - (1/u) \left\langle f(t), e^{-t/u}/u \right\rangle \\ &= \left(1/u^2\right) \hat{\boldsymbol{M}}(tf(t); u) - (1/u) \hat{\boldsymbol{M}}(u). \end{aligned}$$

Equivalently

$$\hat{\boldsymbol{M}}(tf(t); u) = u^2 \mathcal{D}_u^1 \hat{\boldsymbol{M}}(u) + u \hat{\boldsymbol{M}}(u).$$

Fortunately, we may proceed as in (i) to derive Part (ii) of the theorem. Detailed proof is avoided.

It will be interesting to know that Theorem 3.7 can be easily extended to multiplication by $t^n, n \in \mathbb{N}$. The desired proof of the extended result can, then, be automatically constructed with the help of the principle of mathmatical induction on n.

4 THE GENERALIZED CONVOLUTION OF THE SUMUDU TRANSFORMATION

Let f and g be distributions of compact support in $\boldsymbol{E}'(\mathbb{R}_+)$, then the convolution of f and g is defined by

$$\langle (f * g)(t), \phi(t) \rangle = \langle f(t), \langle g(\tau), \phi(t+\tau) \rangle \rangle, \tag{4.1}$$

for every test function $\phi \in \boldsymbol{E}(\mathbb{R}_+)$. Indeed, the above definition is meaningful provided that

$$\psi(t) = \langle g(\tau), \phi(t+\tau) \rangle$$

belongs to $\boldsymbol{E}(\mathbb{R}_+)$.

Theorem 4.1 *Let $g \in \boldsymbol{E}'(\mathbb{R}_+)$ and $\phi \in \boldsymbol{E}(\mathbb{R}_+)$. If*

$$\psi(t) = \langle g(\tau), \phi(t+\tau) \rangle,$$

then ψ is infinitely differentiable and

$$\mathcal{D}_t^k \psi(t) = \left\langle g(\tau), \mathcal{D}_t^k \phi(t+\tau) \right\rangle, k = 1, 2, \ldots.$$

Proof of the above desired result can be established by an argument similar to that obtained for Theorem 4.5.1 from [10, p.p.130] and, thus, we avoid the same.

Theorem 4.2 (**The Convolution Theorem**) *If $\hat{\boldsymbol{M}}_1(u)$ and $\hat{\boldsymbol{M}}_2(u)$ are the distributional Sumudu transforms of f and g, respectively, then*

$$\hat{\boldsymbol{M}}((f * g)(t); u) = u \hat{\boldsymbol{M}}_1(u) \hat{\boldsymbol{M}}_2(u).$$

Proof Employing the translation property and Equation 4.1, we get

$$\begin{aligned}\hat{\boldsymbol{M}}\left(\left(f*g\right)\left(t\right);u\right) &= \left\langle \left(f*g\right)\left(t\right), e^{-t/u}/u\right\rangle \\ &= \left\langle f\left(t\right), \left\langle g\left(\tau\right), e^{-(t+\tau)/u}/u\right\rangle\right\rangle \\ &= u\left\langle f\left(t\right), e^{-t/u}/u\right\rangle \left\langle g\left(\tau\right), e^{-\tau/u}/u\right\rangle \\ &= u\hat{\boldsymbol{M}}_1\left(u\right)\hat{\boldsymbol{M}}_2\left(u\right).\end{aligned}$$

Thus, the theorem is completely proved.

Theorem 4.3 *Let* $f, g \in \boldsymbol{E}^{'}\left(\mathbb{R}_+\right)$ *and* $\hat{\boldsymbol{M}}\left(g\left(\tau\right);u\right), \hat{\boldsymbol{M}}\left(f\left(t\right);u\right)$ *be their respective distributional Sumudu transforms, then*

$(i)\ \hat{\boldsymbol{M}}\left(\left[\mathcal{D}_t^m\left(f*g\right)\left(t\right);u\right]\right)u\hat{\boldsymbol{M}}\left(\mathcal{D}_t^m f\left(t\right);u\right)\hat{\boldsymbol{M}}\left(g\left(\tau\right);u\right),$
$(ii)\ \hat{\boldsymbol{M}}\left(\mathcal{D}_t^m\left(f*g\right)\left(t\right);u\right) = u\hat{\boldsymbol{M}}\left(f\left(t\right);u\right)\hat{\boldsymbol{M}}\left(\mathcal{D}_t^m g\left(t\right);u\right).$

Proof We intend to prove Part (i) of the theorem since the proof of the second part is similar. With the aid of the fact

$$\mathcal{D}^m\left(f*g\right) = f^{(m)}*g = g*f^{(m)},$$

we obtain

$$\begin{aligned}\hat{\boldsymbol{M}}\left(\mathcal{D}_t^m\left(f*g\right)\left(t\right);u\right) &= \left\langle \mathcal{D}_t^m\left(f*g\right)\left(t\right), \frac{e^{-t/u}}{u}\right\rangle \\ &= \left\langle \mathcal{D}_t^m f\left(t\right), \left\langle g\left(\tau\right), e^{-(t+\tau)/u}/u\right\rangle\right\rangle \\ &= u\left\langle \mathcal{D}_t^m f\left(t\right), e^{-t/u}\right\rangle \left\langle g\left(\tau\right), e^{-\tau/u}/u\right\rangle \\ &= u\hat{\boldsymbol{M}}\left(\mathcal{D}_t^m f\left(t\right);u\right)\hat{\boldsymbol{M}}\left(g\left(\tau\right);u\right)\end{aligned}$$

Proof of Part (ii) is analogous. This completes the proof of the theorem

5 CONSTRUCTION OF BOEHMIANS

One of the most youngest generalizations of functions, and more particularly of distributions, is the theory of Boehmians. The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [8]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose support is bounded from the left. In a concrete case, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions.

The construction of Boehmians is similar to the construction of the field of quotients and in some cases, it gives just the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as space $\boldsymbol{C}$(the space of continous functions) with the operations of pointwise additions and convolution.

Let $\boldsymbol{G}$ be a linear space and $\boldsymbol{S}$ be a subspace of $\boldsymbol{G}$. We assume that to each pair of elements $f \in \boldsymbol{G}$ and $\phi \in \boldsymbol{S}$, is assigned the product $f \circledast g$ such that the following conditions are satisfied:

(1) If $\phi, \psi \in \boldsymbol{S}$, then $\phi \circledast \psi \in \boldsymbol{S}$ and $\phi \circledast \psi = \psi \circledast \phi$.

(2) If $f \in \boldsymbol{G}$ and $\phi, \psi \in \boldsymbol{S}$, then $(f \circledast \phi) \circledast \psi = f \circledast (\phi \circledast \psi)$.

(3) If $f, g \in \boldsymbol{G}, \phi \in \boldsymbol{S}$ and $\lambda \in \mathbb{R}$, then

$$(f + g) \circledast \phi = f \circledast \phi + g \circledast \phi \text{ and } \lambda (f \circledast \phi) = (\lambda f) \circledast \phi.$$

Let Δ be a family of sequences from $\boldsymbol{S}$, such that

Δ_1 If $f, g \in \boldsymbol{G}, (\delta_n) \in \Delta$ and $f \circledast \delta_n = g \circledast \delta_n \, (n = 1, 2, ...)$, then $f = g$.

Δ_2 If$(\phi_n), (\delta_n) \in \Delta$,then $(\phi_n \circledast \psi_n) \in \Delta$.

Elements of Δ will be called *delta sequences.* Consider the class $\boldsymbol{A}$ of pair of sequences defined by

$$\boldsymbol{A} = \left\{ ((f_n), (\phi_n)) : (f_n) \subseteq \boldsymbol{G}^N, (\phi_n) \in \Delta \right\},$$

for each $n \in \mathbb{N}$. An element $((f_n), (\phi_n)) \in \boldsymbol{A}$ is called a quotient of sequences, denoted by f_n/ϕ_n, if $f_i \circledast \phi_j = f_j \circledast \phi_i, \forall i, j \in \mathbb{N}$. Two quotients of sequences f_n/ϕ_n and g_n/ψ_n are said to be *equivalent,* $f_n/\phi_n \sim g_n/\psi_n$, if $f_i \circledast \psi_j = g_j \circledast \phi_i, \forall i, j \in \mathbb{N}$.

The relation $\sim$ is an equivalent relation on $\boldsymbol{A}$ and hence, splits $\boldsymbol{A}$ into equivalence classes. The equivalence class containing f_n/ϕ_n is denoted by $[f_n/\phi_n]$. These equivalence classes are called *Boehmians* and the *space of all Boehmians* is denoted by $\boldsymbol{B}$.The sum of two Boehmians and multiplication by a scalar can be defined in a natural way

$$[f_n/\phi_n] + [g_n/\psi_n] = [((f_n \circledast \psi_n) + (g_n \circledast \phi_n)) / (\phi_n \circledast \psi_n)]$$

and

$$\alpha [f_n/\phi_n] = [\alpha f_n/\phi_n], \alpha \in \mathbb{C}.$$

The operation $\circledast$ and the differentiation are defined by

$$[f_n/\phi_n] \circledast [g_n/\psi_n] = [(f_n \circledast g_n) / (\phi_n \circledast \psi_n)]$$

and

$$\mathcal{D}^\alpha [f_n/\phi_n] = [\mathcal{D}^\alpha f_n/\phi_n].$$

Many a time, $\boldsymbol{G}$ is equipped with a notion of convergence. The relationship between the notion of convergence and $\circledast$ are given by:

(4) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\boldsymbol{G}$ *and,* $\phi \in \boldsymbol{S}$ *is any fixed element, then*

$$f_n \circledast \phi \to f \circledast \phi \text{ in } \boldsymbol{G} \, (\text{as } n \to \infty).$$

(5) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\boldsymbol{G}$ *and* $(\delta_n) \in \Delta$, *then*

$$f_n \circledast \delta_n \to f \text{ in } \boldsymbol{G} \, (\text{as } n \to \infty).$$

The operation $\circledast$ can be extended to $\boldsymbol{B} \times \boldsymbol{S}$ by : *If* $[f_n/\delta_n] \in \boldsymbol{B}$ *and* $\phi \in \boldsymbol{S}$*, then* $[f_n/\delta_n] \circledast \phi = [(f_n \circledast \phi) / \delta_n]$.

In $\boldsymbol{B}$, two types of convergence, $\delta-$**convergence** and $\Delta-$**convergence**, are defined as follows:

($\delta-$ **convergence**) *A sequence of Boehmians* (β_n) *in* $\boldsymbol{B}$ *is said to be* $\delta-$*convergent to a Boehmian* β *in* $\boldsymbol{B}$, *denoted by* $\beta_n \xrightarrow{\delta} \beta$, *if there exists a delta sequence* (δ_n) *such that*

$$(\beta_n \circledast \delta_n), (\beta \circledast \delta_n) \in \boldsymbol{G}, \forall k, n \in \mathbb{N},$$

and

$$(\beta_n \circledast \delta_k) \to (\beta \circledast \delta_k) \text{ as } n \to \infty, \text{ in } \boldsymbol{G}, \text{ for every } k \in \mathbb{N}.$$

The following is equivalent for the statement of $\delta-$*convergence:*

$\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ *in* $\boldsymbol{B}$ *if and only if there is* $f_{n,k}, f_k \in \boldsymbol{G}$ *and* $\delta_k \in \Delta$ *such that* $\beta_n = [f_{n,k}/\delta_k], \beta = [f_k/\delta_k]$ *and for each* $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \mathbf{G}.$$

($\Delta-$ **convergence**) *A sequence of Boehmians* (β_n) *in* $\boldsymbol{B}$ *is said to be* $\Delta-$*convergent to a Boehmian* β *in* $\boldsymbol{B}$, *denoted by* $\beta_n \xrightarrow{\Delta} \beta$, *if there exists a* $(\delta_n) \in \Delta$ *such that* $(\beta_n - \beta) \circledast \delta_n \in \boldsymbol{G}, \forall n \in \mathbb{N}$, *and* $(\beta_n - \beta) \circledast \delta_n \to 0$ *as* $n \to \infty$ *in* $\boldsymbol{G}$.

6 INTEGRABLE BOEHMIANS FOR SUMUDU TRANSFORMS

Let $\boldsymbol{L}^1$ be the space of all Lebesgue integrable functions on the positive real line. With the convolution product, and as in [9] , a sequence $(\delta_n)_{n=1}^{\infty}$ of continuous real functions over $\mathbb{R}_+$ is called a delta sequence if and only if

(i) $\displaystyle\int_{\mathbb{R}_+} \delta_n(x) = 1, n \in \mathbb{N};$

(ii) $\displaystyle\int_{\mathbb{R}_+} |\delta_n| < M$, for all $n \in \mathbb{N}$ and some positive $M \in \mathbb{R}_+$;

(iii) $\displaystyle\int_{|x|>\varepsilon} |\delta_n(x)|\, dx \to 0$ as $n \to \infty$ for every $\varepsilon > 0$.

The space of all integrable Boehmians is denoted by $\boldsymbol{B}_{\boldsymbol{L}^1}$, which is a convolution algebra with the following operations

$$\lambda [f_n/\delta_n] = [\lambda f_n/\delta_n],$$

$$[f_n/\delta_n] + [g_n/\xi_n] = [(f_n * \xi_n + g_n * \delta_n)/\delta_n * \xi_n],$$

and

$$[f_n/\delta_n] * [g_n/\xi_n] = [f_n * g_n/\delta_n * \xi_n].$$

(see [9] for further discussion).

Lemma 6.1 *If* $[f_n/\delta_n] \in \boldsymbol{B}_{\boldsymbol{L}^1}$, *then the sequence*

$$\boldsymbol{M}(f_n(t); u) = \int_{\mathbb{R}_+} \frac{1}{u} \exp\left(-\frac{t}{u}\right) f_n(t)\, dt$$

converges uniformly on each compact set $\boldsymbol{K}$ *in* $\mathbb{R}_+$.

Proof If (δ_n) is a sequence, then $\hat{\delta}_n \left(\hat{\delta}_n = \boldsymbol{M}\left(\delta_n\left(t\right);u\right)\right)$ converges uniformly on each compact subset to the function $\frac{1}{u}$. Hence, for each $\boldsymbol{K}, \hat{\delta}_n > 0$ on $\boldsymbol{K}$ for almost $k \in \mathbb{N}$ and

$$\begin{aligned} \boldsymbol{M}\left(f_n\left(t\right);u\right) &= \boldsymbol{M} f_n\left(u\right) \frac{\hat{\delta}_k}{\hat{\delta}_k} = \frac{\boldsymbol{M}\left(f_n * \delta_k\right)}{u\hat{\delta}_k} = \frac{\boldsymbol{M}\left(f_k * \delta_n\right)}{u\hat{\delta}_k} \\ &= \frac{\boldsymbol{M} f_k}{\hat{\delta}_k} \hat{\delta}_n \text{ on } \boldsymbol{K}. \end{aligned}$$

i.e. $\boldsymbol{M} f_n \to \frac{\boldsymbol{M} f_k}{u\hat{\delta}_k}$ as $n \to \infty$ on $\boldsymbol{K}$ $\left(\boldsymbol{M}\delta_n\left(u\right) \to \frac{1}{u} \text{ as } n \to \infty\right)$.

Based on this result, we define the Sumudu transform of an integrable Boehmian as

$$\boldsymbol{\Psi}\left[f_n/\phi_n\right] = \lim f_n, \tag{6.1}$$

where the limit ranges over compact subsets of $\mathbb{R}_+$. Thus, the Sumudu transform of an integrable Boehmian is a continuous function.

As a next step, we claim the concept in (6.1) is well-defined. For, let $\beta_1 = [f_n/\phi_n]$ and $\beta_2 = [g_n/\psi_n]$ be in $\boldsymbol{B}_{\boldsymbol{L}^1}$ such that $\beta_1 = \beta_2$. Then, $[f_n/\phi_n] = [g_n/\psi_n]$ implies $f_n * \psi_m = g_m * \phi_n, m, n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \boldsymbol{M}\left(f_n * \psi_m\right) &= \boldsymbol{M}\left(g_m * \phi_n\right) \\ &= \boldsymbol{M}\left(g_n * \phi_m\right). \end{aligned}$$

From Theorem 4.2 and (6.1) we have $\lim \boldsymbol{M} f_n = \lim \boldsymbol{M} g_n$, over compact subsets of $(0, \infty)$ .i.e.

$$\boldsymbol{\Psi}\left[f_n/\phi_n\right] = \boldsymbol{\Psi}\left[g_n/\phi_n\right].$$

Theorem 6.2 *Let $F, G \in \boldsymbol{B}_{\boldsymbol{L}^1}$, then*

(i) $\boldsymbol{\Psi}\left(\lambda F\right) = \lambda \boldsymbol{\Psi} F$ *and* $\boldsymbol{\Psi}\left(F + G\right) = \boldsymbol{\Psi} F + \boldsymbol{\Psi} G$
(ii) $\boldsymbol{\Psi}\left(F * G\right) = u \boldsymbol{\Psi} F \boldsymbol{\Psi} G$,
(iii) $\boldsymbol{\Psi}\left(F^{(n)}\right) = \frac{(-1)^k}{u^k} \boldsymbol{\Psi} F$,
(iv)If $\boldsymbol{\Psi} F = 0$ *then* $F = 0$,
(v) If $\Delta - \lim F_n = F$ *then* $\boldsymbol{\Psi} F_n \to \boldsymbol{\Psi} F$ *unifomly on each compact set.*

Proof Properties $(i) - (vi)$ can be directly established from the corresponding properties of the Sumudu transform.

Part (vii) can be proved in a manner similar to that of [9, Theorem 2, Part (f)]. The theorem is, thus, completely proved.

Remark: *The extended Sumudu transform $\boldsymbol{\Psi}$ is a continuous mapping from $\boldsymbol{B}_{\boldsymbol{L}^1}$* into the space of continuous functions *in sense of δ-convergence.*

Proof Let $\beta_n \xrightarrow{\delta} \beta$ $(n \to \infty)$ in $\boldsymbol{B}_{\boldsymbol{L}^1}$, then there is $f_{n,k}, f_k \in \boldsymbol{L}^1$ and $\delta_k \in \Delta$ such that $\beta_n = [f_{n,k}/\delta_k]\,, \beta = [f_k/\delta_k]$ and for each $k \in \mathbb{N}$,

$$f_{n,k} \to f_k \text{ as } n \to \infty \text{ in } \boldsymbol{L}^1.$$

Continuity condition of the Sumudu transform justifies that $\boldsymbol{M}\,(f_{n,k}) \to \boldsymbol{M}\,(f_k)$ as $n \to \infty$ and therefore $\lim_{n\to\infty} \boldsymbol{M}\,(f_{n,k}) \to \lim_{n\to\infty} \boldsymbol{M}\,(f_k)\,.$ Hence $\boldsymbol{\Psi}\,[f_{n,k}/\delta_k] \xrightarrow{\delta}$ $\boldsymbol{\Psi}\,[f_k/\delta_k]$ as $n \to \infty$. This proves the above remark.

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LÉVY-KHINCHIN TYPE FORMULA FOR ELEMENTARY DEFINITIZABLE FUNCTIONS ON HYPERGROUPS

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Abstract. *Our main task in this article is to give an integral representation for the so called elementary definitizable functions defined on a hypergroup K. Firstly, we construct an argument kernel on the cross product $K \times K^*$ (K^* the set of all characters on K), then we study the conditions that guarantee the existence of some integrations having an integrand parts as a function of the constructed kernel. Finally, we give a Lévy- Khinchin type formula for elementary definitizable functions defined on the hypergroup K. Moreover, as an application we give the integral form for elementary definitizable functions defined on the polynomial hypergroup .*

Keywords. Hypergroup; definitizable function; Positive definite function.

1. Introduction.

The notion of an abstract algebraic hypergroup has its origins in the studies of E. Marty and H. S. Wall in the 1930s, and harmonic analysis on hypergroups dates back to J. Delsarte's and B. M. Levitan's work during the 1930s and 1940s, but the substantial development had to wait till the 1970s when Dunkl [6], Jewett [10] and Spector [21] put hypergroups in the right setting for harmonic analysis. A hypergroup is a locally compact space on which the space of finite regular Borel measure has a convolution structure preserving the probability measures. Such a structure can arise in several ways in harmonic analysis. The class of hypergroups includes the class of locally compact topological semigroups. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but commutative hypergroups with an involution and compact hypergroups with an involution have a Haar measure(Spector[21] and Jewett[10]). Maserick and Youssfi [15] gave a Lévy-Khinchin type formula for the so called elementary definitizable functions defined on a semigroup S. Okb El-Bab et al. gave Lévy-Khinchin type formula for the so called strongly negative definite functions defined on the product of two hypergroups[17]. Here in we will give an integral representation for the elementary definitizable function defined on a hypergroup K. The paper is organized as follows. In §2 we introduce the definition of the elementary definitizable function on hypergroup K. In §3 we construct an argument kernel on the cartesian product $K \times K^*$, then we give some properties of this kernel. In §4 we discusses the possibility of finding Lévy measures that guarantees the existence of some integration, having an integrand parts as a function of our constructed kernels, with respect to these measures. After discussing integrability conditions of some functions with respect to Lévy measures in §4 we present a general integral representation theorem (Theorem 13) in §5 for a class of definitizable functions on a hypergroup K. Finally §6 gives an application of our results for polynomial hypergroups. Let $M(K)$ denote the space of all bounded Radon measures on K, $M^1(K)$ be the subset of all probability measures and δ_x be the point measure of $x \in K$. $C(K)$ denotes the space of continous functions on K. Throughout the sequel, K will denote a multiplicative commutative hypergroup with identity e and involution $^-$. In this case there exists an (up to normalization) unique Haar measure $w \in M(K)$ which is characterized by $w(f) = w({}_xf)$ for all $f \in C_c(K)$ and $x \in K$. For each $x, y \in K$ we write

$$ {}_xf(y) = f(x * y) := \int_K f d(\delta_x * \delta_y), \qquad \mu * f(x) := \int_K f(z^- * x) d\mu(z) \tag{1} $$

and

$$ f * g(x) := \int_K f(x * y) g(y^-) dw(y) = \int_K f(y) g(y^- * x) dw(y) \tag{2} $$

Here f, g are measurable functions on K and $\mu \in M(K)$, and the letter equality holds whenever one of f,g is $\sigma-$finite[10, Theorem5.1D].

2. Definitizable functions on K .

A locally bounded measurable function $\chi : K \to \mathbb{C}$ is called a semicharacter if $\chi(e) = 1$ and $\chi(x * y^-) = \chi(x)\overline{\chi(y)}$ for all $x, y \in K$. Every bounded semicharacter is called a character. If the character is not locally null then (see [3, Proposition 1.4.33]) it must be continuous. The dual K^* of K is just the set of continuous characters with the compact-open topology in which case K^* must be locally compact. In this paper we will be concerned with continuous characters on hypergroups. Throughout the sequel χ_0 will denote a character on K which never assumes the value zero. For each $x \in K$, we define the shift operator E_y by $E_y\Phi(x) = \Phi(x * y)$ for all $x, y \in K$ and $\Phi \in \mathbb{C}^K$. The complex span $\mathbb{A}$ of all such operators is a commutative algebra with identity $E_1 = I$ and involution $(\sum \alpha_i E_{x_i})^- = \sum \overline{\alpha_i} E_{x_i^-}$. The hypergroup K is embedded in $\mathbb{A}$ as a Hamel basis. The algebra constructed in this way is isomorphic to the L_1-algebra constructed by Hewitt-Zuckermann [9]. A locally bounded measurable function $\Phi : K \to \mathbb{C}$ is said to be positive definite if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \Phi(x_i * x_j^-) \geq 0 \tag{3}$$

for all choice of $x_1, x_2, ..., x_n \in K, c_1, c_2, ..., c_n \in \mathbb{C}$ and $n \in \mathbb{N}$. Many properties of positive definite and some related functions can be found in [16-18] and [8].

Definition 1. Let k be a non-negative integer. It is not hard to see that the singleton $\{\chi_0\}$ is precisely the set of all characters χ such that $T\chi = O$ for all $T \in \mathbb{A}$ having the form

$$T = (T_1...T_k)(T_1...T_k)^- \qquad where \qquad T_1, ..., T_k \in ker\chi_0 \tag{4}$$

where $ker\chi_0$ denote the space of all operators $R \in \mathbb{A}$ such that $R\chi_0 = 0$. We will denote by $\wp_0(k, \chi_0)$ the class of all hermitian functions $\Phi : K \to \mathbb{C}$ which satisfy

(a) $T\Phi$ is positive definite for all operators T having the form (4).

(b) $T\Phi$ is $|\chi_0|$-bounded for all such T i.e., there exists a constant c such that $T\Phi(x) \leq c|\chi_0(x)|$ for all $x \in K$.

We shall call the elements of the class $\wp_0(k, \chi_0)$ the elementary definitizable functions.

3.Construction of an argument kernel On $K \times K^*$.

Parthasarthy et al. [21] proved the existence of a kernel $\theta(x, \chi)$ (called herein an argument kernel) for an arbitrary abelian group G . Forst [7] used this kernel to establish a Lévy-Khinchin integral representation for conditionally positive definite functions. Sásvari [20] observed that truncations of the power series expansion of $exp(\theta(x, \chi))$ yielded kernels he used to extend Forst's result to a considerably larger class of functions on G. The left x-translate of f is written ${}_xf(y) = f(x * y)$ as seen above. In Bloom and Heyer[2], Definition 2.5 the concept of uniform continuity was introduced, in terms of these translates, and it was shown that continuous functions with compact support are indeed uniformly continuous. For the work that follows we need to extend this idea.

Definition 2. A locally bounded measurable function f is called left locally uniformly continuous at $x_0 \in X$ if there exists a neighbourhood U of x_0 such that for every $\varepsilon > 0$ there exists a neighbourhood V of the identity e satisfying

$$|f(y * x) - f(x)| < \varepsilon$$

for all $x \in U$, $y \in V$.

Theorem 2.6 of Bloom and Heyer[3] shows that a continuous function is left locally uniformly continuous at every point in X. Depending on the preliminary result of Jewett ([10], Theorem 6.2E)) Bloom and Heyer [3] Proved that every function that is left locally uniformly continuous at x_0 is in fact continuous on a neighbourhood of x_0. This result will help us to prove part (b) of the following theorem as well as the following Lemma will do for part (d).

Let H be a closed subhypergroup of K and let w_H be a fixed Haar measure on H. The canonical homomorphism of K onto K/H is denoted π.

Lemma 3. For every compact subset $C \subseteq K/H$ there exists a compact $G \subseteq K$ such that $\pi(G) = C$.

Proof. Let U be a compact neighbourhood of e in K. Since π is an open and continuous mapping, $\pi(U)$ is a compact neighbourhood of e in K/H. There exist finitely many points $x_1, ..., x_n \in K$ such that

$$C \subseteq \bigcup_{i=1}^{n} (\pi(x_i) + \pi(U))$$

The set

$$G = (\bigcup_{i=1}^{n} (x_i + U)) \bigcap \pi^{-1}(C)$$

is compact and $\pi(G) = C$.

Theorem 4. There is a kernel $\rho_{\chi_0} : K \times K^* \to \mathbb{R}$ satisfying each of the following:

(a) If $(x, \chi) \in K \times K^*$ then $\rho_{\chi_0}(x^-, \chi) = -\rho_{\chi_0}(x, \chi)$

(b) For each $x \in K$, the function $\rho_{\chi_0}(x, .)$ of the second variable is measurable and bounded on K^* and continuous on some neighborhood of of χ_0.

(c) For each $\chi \in K^*$, the function $\rho_{\chi_0}(., \chi)$ of the first variable is a homomorphism from $(K, .)$ to $(\mathbb{R}, +)$.

(d) For each $x \in K$, there is a neighborhood V_x of χ_0 such that $\chi(x) \neq 0$ on V_x and $\chi(x)/\chi_0(x) = |\chi(x)/\chi_0(x)|$ exp $i\rho_{\chi_0}(x, \chi)$ for all $\chi \in V_x$.

Proof . Firstly, we will assume that χ_0 is identically 1. As pointed out of of Jewett [10], for each subhypergroup H of K the cosets $x * H(x \in K)$ form a partition of K. Let H denote the subhypergroup of almost hermitian elements of K i.e., $H = \{x \in K : x * u = x * u^- \quad forall \quad u \in K \quad with \quad u = u^-\}$. Then the quotient K/H is a commutative hypergroup and the inverse $[x]^{-1}$ of the canonical image [x] of $x \in K$ in K/H is well defined by $[x^-]$. Dividing this group by the subhypergroup of all elements with finite order, we obtain a maximal independent subhypergroup M of K/H. Let $\hat{x} \in M$ denote the canonical image of $x \in K$. We form a maximal $\mathbb{Z}$-free system $\{\lambda_k; \quad k \in \Lambda\ \}$ such that every $x \in K$ admits a representation of the form

$$\hat{x}^n = \Pi_{k\in\Lambda}\hat{\lambda}_k^{n_k} \quad (n, n_k \in \mathbb{Z}, n > 0), \tag{5}$$

where the rational numbers n_k/n are unique. Let arg(z) denote the measurable extension of that branch of the argument function whose range lies in the half open interval $(-\pi, \pi]$ such that arg(0):=0. Hence ρ as defined by

$$\rho(x, \chi) := \sum_{k\in\Lambda} \frac{n_k}{n} arg(\chi(\lambda_k)) \qquad forfixed \qquad \lambda_k \in \hat{\lambda}_k \tag{6}$$

is well defined and clearly satisfies (a) and (c) as well as the measurability and boundedness conditions of (b). As pointed out in [3] the argument presented by Parthasarathy et al [19] for the group case and Maserick [13] for the semigroup case extends to the hypergroup setting to establish the existence of an open neighborhood $\aleph'_x of \chi_0$ such that

$$arg(x, \chi) := \sum_{k\in\Lambda} \frac{n_k}{n} arg(\chi(\lambda_k)) \qquad forall \qquad \chi \in \aleph'_x \tag{7}$$

Then $\aleph_x := \aleph'_x \cap \{\chi \in K^* : |\chi(x) - e| < 1\}$ satisfies (d) as well as the continuity assertion in (b) when χ_0 is identically 1. The general assertion follows upon setting $\rho_{\chi_0}(x, \chi) = \rho(x, \frac{\chi}{\chi_0})$ since the map $\chi \to \frac{\chi(x)}{\chi_0(x)}$ is a homeomorphism of K^* onto it self. This complete the proof.

We call any kernel satisfying (a)-(d) above an argument kernel.

4. Some integrability conditions .

The function space $\mathbb{C}^K$ can be algebraically identified with the dual $\mathbb{A}^*$ of $\mathbb{A}$ via $T \to (T\phi)(e)(\phi \in \mathbb{C}^K)$ and $(T \in \mathbb{A})$. This map topologically identifies $\mathbb{C}^K$ equipped with the topology of simple convergence and $\mathbb{A}^*$ equipped with the topology of weak*-convergence. For each fourth root of unity $\sigma \in \mathbb{C}$ and $x \in K$, we define an element $\Delta_{x,\sigma} \in \mathbb{A}$ by $\Delta_{x,\sigma} := 1 - \frac{1}{2}[\sigma E_x/\chi_0(x) + \overline{\sigma} E_{x^-}/\chi_0(x^-)]$, where $E_x\Phi(y) = \Phi(x * y), \quad x, y \in K \quad and \quad \Phi \in \mathbb{C}^K$ and for convenience set $\Delta_x = \Delta_{x,1}$. Suppose that G be a generator set for K in the sense that every element of K is a finite product of members of $G \cup G^*$. Let m be a positive integer we will denote by $L^m(\underline{K})$ the collection of all measures w satisfying

$$\int_{\underline{K}\setminus\chi_0} \prod_{j=1}^{m} \Delta_{x_j}\chi(e)dw(\chi) < \infty \qquad forall \qquad x_1, ..., x_m \in K \tag{8}$$

where $\underline{K}$ be the set of all characters χ on K such that $|\chi(x)| \leq |\chi_0(x)|$ for all $x \in K$. By mathematical induction and using Holder inequality we easily get the following generalization of Cauchy Schwartz inequality

$$\int \psi_1...\psi_m d\mu \leq (\int \psi_1^m d\mu ... \int \psi_m^m d\mu)^{\frac{1}{m}} \tag{9}$$

where $\mu \in M(\underline{K})$ and $\{\psi_i\}_1^m$ be a non negative measurable functions with respect to μ. This inequality help us to prove the following Proposition:

Proposition 5. If span(K)=$\mathbb{A}$, then the following are equivalent:

(a) $w \in L^m(\underline{K})$;

(b) $\int_{\underline{K}\setminus\{\chi_0\}}(\Delta_x\chi(e))^m dw(\chi) \qquad forall \qquad x \in K$;

(c) $\int_{\underline{K}\setminus\chi_0} \prod_{j=1}^{m} \Delta_{x_j}\chi(e)dw(\chi) < \infty \qquad forall \qquad x_1, ..., x_m \in G$;

(d) $\int_{\underline{K}\setminus\{\chi_0\}}(\Delta_x\chi(e))^m dw(\chi) \qquad forall \qquad x \in G$.

Proof. (A) The equivalence of the pairs (a),(b) and (c),(d) follows from inequality (9). Clearly (a) implies (c), thus it sufficient to prove that (d) implies (b). Firstly, we assume that K is a Hamel base for $\mathbb{A}$ and prove that (d) implies the formally stronger condition

$$\int_{\underline{K}\setminus\{\chi_0\}} (\chi(t))^m dw(\chi) < \infty \qquad forall \qquad t \in ker\{\chi_0\} \tag{10}$$

Since, every element of G lies in the linear span of the set $\{e, \Delta_{x,1} - \Delta_{x,i}\}$, then every $t \in ker\{\chi_0\}$ is of the form

$$t = \sum_{(m,n)} \alpha(m,n)t(m,n), \tag{11}$$

where

$$t(m,n) = \prod_j (\Delta_{x_j})^{m_j}(e - \Delta_{x_j,i})^{n_j} \tag{12}$$

and $\alpha(m,n) \in \mathbb{C}, x_j \in G$ and $m = (m_j), n = (n_j)$ (j=1,...,p) are sequences of nonnegative integers which are not both null. For each $\chi \in \underline{K}$ define $\chi^*|K \to \mathbb{C}$ by $\chi^*(x) = \chi(x^-)$ for each $x \in K$. But K assumed to be base for $\mathbb{A}$ so χ^* extends to a linear functional on $\mathbb{A}$. Let t_0 be the sub-sum of the terms in the summand in (11) such that $|n| = \sum_j n_j$ is odd and set $t_e = t - t_0$ then

$$0 \leq \chi^*(t) + \chi^*(t_e) = -\chi(t_0) + \chi(t_e) \qquad forall \qquad \chi \in \underline{K}$$

hence, $\chi(t_0) \leq \chi(t_e)$. Therefore for each $t \in ker\{\chi_0\}$ there exists a constant C such that

$$|\chi(t)| \leq C \sum_j (\Delta_{x_j})\chi(e), \qquad \chi \in \underline{K}. \tag{13}$$

Using Minkowski's inequality shows that $\chi \to \chi(t)$ is a member of the Lebesgue space $L^m(\mu)$ whenever t satisfies (d).

(B)Generally, K can isomorphically embedded in $L^1(K)$ via the map $x \to E_x$ and the image of K is a Hamel base for $L^1(K)$. Since K spans $\mathbb{A}$, so $\mathbb{A}$ homomorphic image of $L^1(K)$ under the map π defined by $\pi(\sum_j \alpha_j E_{x_j}) = \sum_j \alpha_j x_j$, where $x_j \in K$. The adjoint map π^* is a homomorphism mapping $\underline{K}$ onto the set $\underline{K}^*$ of hermitian multiplicative linear functionals $\pi(\chi)$ satisfying

$$(\pi^*(\chi))(1 - \frac{1}{2}[\sigma E_x/(\pi^*(\chi_0))(x) + \overline{\sigma} E_{x^-}/(\pi^*(\chi_0))(x^-)]) \geq 0 \qquad forall \qquad x \in G.$$

From (A), the contraction measure $\pi^* o\mu$ satisfies the four conditions of the Theorem.

Assume that W_x is a compact neighborhood of χ_0 which satisfies (d) of the above Theorem as well as

$$-log|\chi(x)/\chi_0(x)| \leq 2(\Delta_x \chi)(e). \tag{14}$$

Since $-log|z| \leq -log|Re(z)| \leq 2(1 - Re(z))$ whenever $|1-z| \leq 0.5$, there does indeed exist such a neighborhood of χ_0.

Lemma 6. There exists a finite subset $\{x_j : j = 1, ..., m\}$ of K and a constant L such that the inequality

$$(-log|\chi(x)/\chi_0(x)|)^p |\rho_{\chi_0}(x,\chi)|^q \leq L^{p+q}\{(\Delta_x\chi)(e)\}^p \{\sum_j \{(\Delta_{x_j}\chi)(e) + |1 - (\Delta_{x_j,i}\chi)(e)|\}\}^q$$

is valid for every $\chi \in W_x$ and every pair of non-negative integers p and q.

Proof. Recall from Theorem4 that there exists a maximal independent subhypergroup M of K/H. Let $\hat{x} \in M$ denote the canonical image of $x \in K$, there in we found a maximal $\mathbb{Z}$-free system $\{\hat{\lambda}_k\}$ such that every $x \in K$ admits a representation of the form

$$\hat{x}^n = \Pi_{k=1}^{m} \hat{\lambda}_k^{n_k}$$

Fix $\chi \in W_x, \quad r_j = |\frac{\chi(\lambda_j)}{\chi_0(\lambda_j)}|$ and $\theta_j = arg(\frac{\chi(\lambda_j)}{\chi_0(\lambda_j)})$ where $-\pi < arg(.) < \pi$ and arg(0):=0. Using

$$|r_j sin(\theta_j)|^2 \leq 2(1 - r_j cos(\theta_j))$$

we easily get

$$|\theta_j - r_j sin(\theta_j)| \leq |\theta_j|(1 - r_j cos(\theta_j))$$

hence

$$|\theta_j| \leq |r_j sin(\theta_j)| + 4(1 - r_j cos(\theta_j))$$

squaring both sides gives

$$\theta_j^2 \leq 26(\Delta_{\lambda_k})(\chi(e))$$

Let $l = max\{\frac{|n_j|}{n}\}_j$. Then

$$|\rho_{\chi_0}(x, \chi)| = |\sum_j (\frac{n_j}{n}\theta_j)| \leq \sqrt{26} l \sum_j \sqrt{\Delta_{\lambda_j}\chi(e)}$$

from which the Cauchy-Schwartz inequality implies

$$|\rho_{\chi_0}(x, \chi)|^2 \leq 26 l^2 m \sum_j \Delta_{\lambda_j}\chi(e)$$

The assertion now follows from the definition of W_x setting $L = max[2, 26l^2m]$ and $x_j = \lambda_j$.

It is clear that $\underline{K}$ is a compact subset of K^* relative to the topology of pointwise convergence.

Denote by $L_k(\underline{K})$ the set of all positive Radon measures w on $\underline{K}\backslash\{\chi_0\}$ such that

$$\int_{\underline{K}\backslash\{\chi_0\}} (T\chi)(e) dw(\chi) < \infty \tag{15}$$

for all operators T having the form (4).

Theorem 7. If p, q is any pair of non-negative integers satisfying $p + q > 2k - 1$ and $\mu \in L_k(\underline{K})$, then

$$\int_{W_x \backslash \chi_0} \{(-log|\chi(x)/\chi_0(x)|)^p |\rho_{\chi_0}(x, \chi)|^q\} dw(\chi) < \infty, \qquad forall \qquad x \in K. \tag{16}$$

Proof. If p and q satisfy the hypothesis , let L be as Lemma 6. Then

$$(-log|\chi(x)/\chi_0(x)|)^p|\rho_{\chi_0}(x,\chi)|^q \leq L^{p+q}\{(\Delta_x\chi)(e)\}^p\{\sum_j\{(\Delta_{x_j}\chi)(e)+|1-(\Delta_{x_j,i}\chi)(e)|\}\}^q,$$

for all $\chi \in \underline{K}$, the multinomial theorem gives a decomposition of the summation on the right hand side as a sum of products of terms of degree $p+q$. Therefore the left hand side of the inequality is dominated by a finite sum of integrable functions, so the integrability assertion of the Theorem follows.

Let $\Omega = \{(x,\chi) \in K \times K^* : \chi(x) \neq 0\}$, we define the kernel $log_{\chi_0}(x,\chi) = log|\chi(x)/\chi_0(x)| + i\rho_{\chi_0(x,\chi)}$. Then for each $\chi \in K^*$, $log(.,\chi)$ is a homomorphism from $(K,*)$ to $(\mathbb{R},+)$. suppose that for each variables y,z and all integers k we denote by $Exp_k(y,z)$ the minimal truncated exponential kernel of order k i.e.,

$$Exp_k(y+z) = \sum_j \frac{1}{j!}\sum_{m_j}\binom{j}{m_j}y^{m_j}z^{j-m_j}$$

where the indices j and m_j are nonnegative integers satisfying $0 \leq m_j \leq j$ and $m_j + \frac{1}{2}(j-m_j) < k$. If we replace y by $log|\chi(x)/\chi_0(x)|$ and z by $i\rho_{\chi_0}(x,\chi)$ where $(x,\chi) \in K * \underline{K}$ then we denote $Exp_k(y+z)$ by $Exp_k(x,\chi)$. By virtue of the above Theorem and the result obtained in [15], we get the following Corollaries

Corollary 8. Let V be a closed neighbourhood of χ_0 and k be a positive integer greater than 0. If $x \in K$ such that $|\chi(x)| \neq 0$ for all $\chi \in V$, then

$$\int_{V\backslash\chi_0} |\chi(x) - \chi_0(x)Exp_k(x,\chi)|dw(\chi) < \infty \tag{17}$$

for every $w \in L_k(\underline{K})$.

Corollary 9. If $Ex_k(x,\chi)$ is any truncation of the power series expansion for $exp(x,\chi)$ which includes all terms of $Exp_k(x,\chi)$, then

$$\int_{V\backslash\chi_0} |\chi(x) - \chi_0(x)Ex_k(x,\chi)|dw(\chi) < \infty \tag{18}$$

for every $w \in L_k(\underline{K})$.

5. Integral Representation Theorems .

Elementary definitizable functions allows in many cases a representation in terms of a local part and an integral term. Here we establish such a representation for a commutative hypergroups. In $\mathbb{C}^K$ we introduce the translation operators A_x for $x \in K$ by the formula

$$(A_x\Phi)(y) = \Phi(x^*y), \quad y \in K, \quad \Phi \in \mathbb{C}^K$$

Since $A_{xy} = A_xA_y$ for $x, y \in K$, the complex linear span of these operators is an algebra $\mathbb{B}$, the algebra of shift operators. For arbitrary function $f \in \mathbb{C}^K$ the subspace

$$T(f) := \{Af | A \in \mathbb{B}\}$$

is invariant under each operator $A \in \mathbb{B}$. The rank of $f \in \mathbb{C}^K$ will define by

$$rk(f) = dimA(f).$$

Since $A(f)$ is independent of the involution, so is the rank of f. To say that $rk(f) = n < \infty$ means that there exist $x_1, ..., x_n \in K$ such that $A_{x_1}f, ..., A_{x_n}f$ is a basis for $A(f)$. In this case there exist functions $a_1, ...a_n$ on K such that

$$f(xt) = \sum_{i=1}^{n} a_i(x)A_{x_i}f(t), x, t \in K$$

Conversely, if $f \in \mathbb{C}^K$ is such that there exist complex-valued functions $a_i, b_i, i = 1, ..., n$ satisfying

$$f(xt) = \sum_{i=1}^{n} a_i(x)b_i(t), x, t \in K \tag{19}$$

then it is clear that f is of finite rank. Furthermore, if n is the smallest number for which a representation of the form (19) exists, then $rk(f) = n$ and $a_i, i = 1, ..., n$ as well as $b_i, i = 1, ..., n$ form a basis for $A(f)$. From the representation (19) it follows that the set of functions $f \in \mathbb{C}^K$ of finite rank is an algebra of functions on K, stable under complex conjugation

Definition 10 . A non-zero function $f \in \mathbb{C}^K$ of finite rank is called elementary if $A(f)$ contains exactly one character function χ. We then say that f is associated with χ.

Theorem 11 . Let $f \in \mathbb{C}^K$ be elementary of rank $\leq n$ and associated with a character χ. If $A_1, ..., A_n \in \mathbb{B}$ are n shift operators satisfying $A_i\chi = 0, \quad i = 1, ..., n$, then $A_1...A_nf = 0$.

Proof . We first note that $A\chi = A\chi(e)\chi$ for any $A \in \mathbb{B}$, so $A\chi = 0$ if and only if $A\chi(e) = 0$. The proof will proceed by induction after n. For $n = 1$ then $A(f) = \mathbb{C}\chi$

and there is nothing to prove. Assume the result established for $n-1$ and let $A_n \in \mathbb{B}$ satisfy $A_n\chi = 0$. Then $\Phi = A_n f$ is of rank$\leq n-1$ because $A(\Phi) = A_n(A(f))$ is a proper subspace of $A(f)$ since $\chi \in A(f)$ and $A_n\chi = 0$. If Φ is non-zero, it is clearly elementary associated with χ, so by induction hypothesis

$$A_1...A_{n-1}\Phi = A_1...A_n f.$$

In the following we shall denote by N_{χ_0} the set of all compact neighbourhood of χ_0 and for $V \in N_{\chi_0}$ we let K_V denote the subhypergroup of all members $x \in K$ such that $\chi(x) \neq 0$ for all $\chi \in V$. Setting $\Lambda_V := span\{E_x : x \in K_V\}$ we obtain the following:

Corollary 12.

$$T\chi_0 Ex_k(.,\chi) = 0 \quad on \quad K_V, \tag{20}$$

for all $T \in \Lambda_V$ of the form (4). If $\chi = \chi_0$ then

$$T\chi_0 \sum_{j=0}^{2k-1} \frac{1}{j!}(i\rho_{\chi_0}(x,\chi))^j = 0 \quad on \quad K,$$

for all $T \in \mathbb{A}$ of the form (4).

Our main result is the following theorem, whose reduction to discrete group generalizes the result of Sásvari [20, Satz 4.6] and whose reduction to arbitrary semigroups extends known generalization of the classical Lévy- Khinchin formula(cf[1,14] and [15]).

Theorem 13 . A hermitian function $\Phi \in \mathbb{C}^K$ belongs to the class $\wp_0(k,\chi_0)$ if and only if Φ admits the Lévy- Khinchin type integral representation

$$\Phi(x) = \int_{V\setminus\chi_0} \{\chi(x) - \chi_0(x)Ex_k(x,\chi)\}dw(\chi) + \int_{\underline{K}\setminus V} \chi(x)dw(\chi) + \alpha_V(x) \tag{21}$$

for all $x \in K$, $V \in V_{\chi_0}$, where $w \in L_k(\underline{K})$ and $\alpha_V : K_V \to \mathbb{C}$ is a correction function satisfying the functional equation

$$T\alpha_V = (T\alpha_V)(1)\chi_0 \quad on \quad K_V,$$

for all $T \in \Lambda_V$ of the form (4).

Proof . The if part is a direct consequence of formula (20). Assume that $\Phi \in \wp_0(k, \chi_0)$ and let T be an operator having the form (4). Since $T\Phi$ is positive definite and $|\chi_0|$ bounded, by Bloom and Ressel[4] we can find a unique Haar measure w_T on $\underline{K}$ such that

$$T\Phi(x) = \int_{\underline{K}} \chi(x) dw_T(\chi) \qquad forall \qquad x \in K.$$

For such an operator T let $O_T := \{\chi \in \underline{K} : T\chi(e) \neq 0\}$. The family $\{O_T\}$ is an open covering of the locally compact space $\underline{K}\backslash\{\chi_0\}$. since $\underline{K}$ is compact, we have the uniqueness of the representing measure. Therefore, $R(\chi)dw_T = T(\chi)dw_R$, for all operator R, T having the form (4), so that the compatibility condition

$$w_T|_{O_R \cap O_T} = w_R|_{O_R \cap O_T}$$

is satisfied. It follows that there exists a unique Haar measure w on $\underline{K}\backslash\{\chi_0\}$ such that for all T as above we have

$$T\Phi(x) = \chi_0(x) w_T(\{\chi_0\}) + \int_{\underline{K}} T\chi(x) dw(\chi) \qquad forall \qquad x \in K$$

In particular for any such T, the function $\chi \to T\chi(e)$ is w-integrable so that $w \in L_k(\underline{K})$. On setting

$$\alpha_V(x) = \Phi(x) - \int_{V\backslash\chi_0} \{\chi(x) - \chi(x)_0 Ex_k(x, \chi)\} dw(\chi) - \int_{\underline{K}\backslash V} \chi(x) dw(\chi)$$

for all $x \in K_V$, $V \in N_{\chi_0}$, and applying formula (20), we see that the correction function α_V satisfies

$$(T\alpha_V)(x) = (T\Phi)(x) - \int_{\underline{K}} (T\chi)(x) dw(\chi) = \chi_0(x) w_T(\{\chi_0\})$$

for all $x \in \Lambda_V$ of the form (4), from which the desired properties of α_V follow. This completes the proof.

6. Polynomial Hypergroups .

In [11,13] Lasser demonstrated a close relationship between certain hypergroups on $\mathbb{N}_0$ and certain orthogonal polynomial sequences and discussed the basic properties concerning these hypergroups, he called them polynomial hypergroups. Let $\{R_n\}_{n\in\mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation of the type

$$R_1(x)R_n(x) = \alpha_n R_{n+1}(x) + \beta_n R_n(x) + \gamma_n R_{n-1}(x)$$

for $n \in \mathbb{N}$ with starting polynomials $R_0(x) = 1$ and $R_1(x) = 1/\alpha_0(x - \beta_0)$ and $\alpha_n > 0$, $\beta_n \geq 0$ for all $n \in \mathbb{N}_0$ and $\gamma_n \geq 0$ for all $n \in \mathbb{N}$. Let the polynomials be normalized at $x = 1$, i.e., $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. By the orthogonality of the polynomial sequence it follows immediately that there exist coefficients $g(n, m; s) \in \mathbb{R}$ with

$$R_n(x)R_m(x) = \sum_{s=|n-m|}^{|n+m|} g(n, m; s)R_s(x)$$

Suppose $g(n, m; s) \geq 0$ for all $n, m, s \in \mathbb{N}_0$. A polynomial sequence with these properties generates a hypergroup structure on $\mathbb{N}_0$. We can obtain a Banach algebra structure by considering the weighted space $l^1(\mathbb{N}_0, w)$ where

$$w(0) = 1, \quad w(1) = \frac{1}{\gamma_1}, \quad w(n) = \frac{\alpha_1\alpha_2 \ldots \alpha_{n-1}}{\gamma_1\gamma_2 \ldots \gamma_n}$$

with translation operators given by

$$T_n\beta(m) = \sum_{s=|n-m|}^{|n+m|} g(n, m; s)\beta(s)$$

The pair $(\mathbb{N}_0; w)$ is called the polynomial hypergroup generated by $(R_n)_{n\in\mathbb{N}}$ and we say that $(R_n)_{n\in\mathbb{N}}$ induces a polynomial hypergroup. Since the linearization coefficient g(n,m;s)are nonnegative then we can define a convolution structure on $\mathbb{N}_0$ via

$$\delta_m * \delta_n = \sum_{s=|n-m|}^{|n+m|} g(n, m; s)\delta_s$$

with this convolution, δ_0 as unit and the identity involution $\mathbb{N}_0$ becomes a discrete hypergroups. Polynomial hypergroups are special cases of discrete hypergroups, cf.[12]. Lasser [11], Proposition 4 showed that the continuous semicharacters of $\mathbb{N}_0$ are given by $\alpha_x : n \to R_n(x)$ where $x \in \mathbb{R}$, and

$$\mathbb{N}_0^* = \{\alpha_x : x \in \mathbb{R} \quad and \quad \alpha_x \quad is \quad bounded\}.$$

Under the homomorphism between $\mathbb{R}$ and the continuous semicharacters on $\mathbb{N}_0$ given by $x \to \alpha_x$, the plancherel measure π on $\mathbb{N}_0^*$ can be identified with the orthogonality measure of (R_n), and supp$\pi \subset [-1, 1]$. The dual space of polynomial hypergroups can be identified with a compact subset of $\mathbb{R}$. Explicitly, the space $\mathbb{N}_0^*$ of all hermitian characters of the hypergroup $\mathbb{N}_0$ is homeomorphic to

$$\mathbb{D}_s = \{x \in \mathbb{R} : |R_n(x)| \leq 1 \quad forall \quad n \in \mathbb{N}_0\}$$

and the space of all characters

$$\mathbb{D} = \{z \in \mathbb{C} : |R_n(z)| \leq 1 \quad forall \quad n \in \mathbb{N}_0\}$$

An example for a polynomial hypergroup is provided by the Jacobi polynomials. Normalizing $p_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomials induce a polynomial hypergroup if $\beta \leq \alpha$ and $\alpha + \beta + 1 \geq 0$.

For $m \in \mathbb{N}_0$, we define a translation operator $T_m : l(\mathbb{N}_0) \to l(\mathbb{N}_0)$ by

$$(\phi_n)_{n\in\mathbb{N}_0} \mapsto (\sum_{s=|n-m|}^{|n+m|} g(n,m;s)\phi(s))_{n\in\mathbb{N}_0}$$

The following characterization of these functions can be found e.g. in [3].

Lemma 14. Let $(R_n)_{n\in\mathbb{N}_0}$ induce a polynomial hypergroup. A function χ is a character of $(\mathbb{N}_0; w)$ if and only if there exists a $z \in \mathbb{C}$ such that

$$\chi(n) = R_n(z), \qquad forall \qquad n \in \mathbb{N}_0$$

In particular, the set of all bounded characters is homeomorphic to $\mathbb{D}$.

Let $\chi_0 = 1$ be the constant character on $\mathbb{N}_0$. Then the corresponding $\underline{K}$ is the interval [-1,1]. The log-kernel in this example is given by

$$Log_{\chi_0}(n,x) = nlog|x|, \quad for \quad n \in \mathbb{N}_0 \quad and \quad x \in \mathbb{R}\backslash\{0\}$$

Suppose Φ belongs to the class of elementary definitizable function on the polynomial hypergroup $(\mathbb{N}_0, w)$. Let $0 < \epsilon < 1$ and $V = [-1, -\epsilon] \cup [\epsilon, 1]$. Then we have $K_V = K$ applying Theorem 13 to the polynomial hypergroup, the neighbourhood V and Φ, guarantees that Φ admits an integral representation of the form

$$\Phi(n) = \int_{[-1,-\epsilon]\cup[\epsilon,1]} \{R_n(x) - 1 - \sum_{j=1}^{2k-1} \frac{(nlog|x|)^j}{j!}\} dw(x) + \int_{|x|<\epsilon} R_n(x) dw(x) + w_\epsilon(n)$$

where $w \in L_k([-1,1])$ and $w_\epsilon := w_V$ is a correction sequence satisfying

$$((x-1)^{2k} * w_\epsilon)(n) = ((x-1)^{2k} * w_\epsilon)(0) \geq 0, \quad n \in \mathbb{N}_0.$$

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*-REGULARITY OF OPERATOR SPACE PROJECTIVE TENSOR PRODUCT OF C*-ALGEBRAS

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ABSTRACT. The Banach *-algebra $A\widehat{\otimes}B$, the operator space projective tensor product of C^*-algebras A and B, is shown to be *-regular if Tomiyama's property (F) holds for $A\otimes_{\min}B$ and $A\otimes_{\min}B = A\otimes_{\max}B$, where $\otimes_{\min}$ and $\otimes_{\max}$ are the injective and the projective C^*-cross norm, respectively. However, $A\widehat{\otimes}B$ has a unique C^*-norm if and only if $A\otimes B$ has. We also discuss the property (F) of $A\widehat{\otimes}B$.

1. INTRODUCTION

The concepts of *-regularity and the uniqueness of C^*-norm have been extensively studied in Harmonic analysis for L^1-group algebras by J. Biodol [6], D. Poguntke [20], Barnes [3], and others. Barnes in [4] studied these concepts in the context of BG^*-algebras. These results on tensor products were further improved by Hauenschild, Kaniuth and Voigt [7].

Recall that for C^*-algebras A and B, and u an element in the algebraic tensor product $A\otimes B$, the operator space projective tensor norm is defined to be

$$\|u\|_\wedge = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\| : u = \alpha(x\otimes y)\beta\},$$

where $\alpha \in M_{1,pq}$, $\beta \in M_{pq,1}$, $x \in M_p(A)$ and $y \in M_q(B)$, $p, q \in \mathbb{N}$, and $x\otimes y = (x_{ij}\otimes y_{kl})_{(i,k),(j,l)} \in M_{pq}(A\otimes B)$. The completion of $A\otimes B$ with respect to this norm is called the operator space projective tensor product of A and B, and is denoted by $A\widehat{\otimes}B$. It is well known that $A\widehat{\otimes}B$ is a Banach *-algebra under the natural involution [8], [14], and is a C^*-algebra if and only if A or B is $\mathbb{C}$. One of the main results about *-regularity obtained in [7], [19] was that the Banach space projective tensor product of C^*-algebras A and B is *-regular if their algebraic tensor product has a unique C^*-norm and $A\otimes_{\min}B$ has Tomiyama's property (F). In Section 2, we prove this result for the operator space projective tensor product. We also show that $A\widehat{\otimes}B$ has a unique C^*-norm if and only if $A\otimes B$ has. Using these results, we obtain several Banach *-algebras which are not *-regular, e.g. $C^*_r(F_2)\widehat{\otimes}C^*_r(F_2)$, $B(H)\widehat{\otimes}B(H)$, and $B(H)/K(H)\widehat{\otimes}B(H)/K(H)$, where $C^*_r(F_2)$ is the C^*-algebra associated to the left regular representation of the free group F_2 on two generators and H an infinite-dimensional separable Hilbert space,

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whereas the Banach *-algebras $C^*(G_1)\widehat{\otimes}C^*(G_2)$, G_1 and G_2 are locally compact Hausdorff topological groups and G_1 is amenable, $K(H)\widehat{\otimes}K(H)$, $B(H)\widehat{\otimes}K(H)$, and $B(H)\widehat{\otimes}K(H)+K(H)\widehat{\otimes}B(H)$ are *-regular. Section 3 deals with the *-regularity of $A\widehat{\otimes}A$ with the reverse involution. Finally, we introduce the notion of property (F) for $A\widehat{\otimes}B$, and prove that if the Banach *-algebra $A\widehat{\otimes}B$ has spectral synthesis in the sense of [12] then it satisfies property (F).

2. *-Regularity And Unique C^*-norm

Throughout this paper, all *-representations of *-algebras are assumed to be normed and for any *-algebra A, $Id(A)$ denotes the space of all two-sided closed ideals of A. Recall that a *-algebra A is called a G^*-algebra if, for every $a \in A$, $\gamma_A(a)$ defined by

$$\gamma_A(a) := \sup\{\|\pi(a)\| : \ \pi \text{ a }^*\text{-representation of } A\},$$

is finite. This γ_A is the largest C^*-seminorm on A; and the reducing ideal A_R of A (or *-radical) is defined as $A_R = \{a \in A : \gamma_A(a) = 0\}$. Denote by $C^*(A)$, the completion of A/A_R in the C^*-norm induced by γ_A. $C^*(A)$ together with the natural mapping $\varphi : A \to C^*(A)$ $(a \to a + A_R)$ is called the enveloping C^*-algebra of A. If $A_R = \{0\}$, i.e. if the points of A are separated by its *-representations, then we say that A is *-reduced(or *-semisimple). Clearly, every Banach *-algebra is a G^*-algebra. Also A_R, in this case, is a norm closed *-ideal of A and the quotient Banach *-algebra A/A_R is automatically *-reduced.

For a *-algebra A, let $Prim^*(A)$ denote the set of all primitive ideals of A, i.e. the set of kernels of topologically irreducible *-representations of A. For a non-empty subset E of $Prim^*(A)$, kernel of E is defined to be

$$k(E) = \bigcap\{P : P \in E\},$$

and for any subset J of A, hull of J relative to $Prim^*(A)$ is defined to be

$$h^*(J) = \{P \in Prim^*(A) : P \supseteq J\}.$$

We endow $Prim^*(A)$ with the hull-kernel topology (hk-topology), that is, for each subset E of $Prim^*(A)$, its closure is $\overline{E} = h^*k(E)$. If A is a C^*-algebra then we usually write $Prim(A)$ instead of $Prim^*(A)$.

In a similar manner, one can define the hk-topology on $Prime(A)$, the space of all prime ideals of A. Also recall that a *-representation π of a *-algebra A is called factorial if $\pi(A)''$ (i.e., von Neumann algebra generated by $\pi(A)$) is a factor. The set of kernels of factorial *-representations of A is called the factorial ideal space of A and is denoted by $Fac(A)$. It is well known that the kernel of a factorial *-representation of a *-algebra A is a (closed) prime ideal, so that one can introduce the hull-kernel topology on $Fac(A)$.

Definition 2.1. ([19]) *A G^*-algebra A is said to be *-regular if the continuous surjection $\breve{\varphi} : Prim(C^*(A)) \to Prim^*(A)$ $(P \to \varphi^{-1}(P))$ is a homeomorphism, where $\varphi : A \to C^*(A)$ is the C^*-enveloping map of A.*

Equivalently, from ([17], Proposition 1.3), a Banach *-algebra A is *-regular if and only if for any two non-degenerate *-representations π and ρ of A, the inclusion $\ker\pi \subseteq \ker\rho$ will imply that $\|\rho(a)\| \leq \|\pi(a)\|$ for all $a \in A$.

Now, we proceed to show that the *-regularity of $A\widehat{\otimes}B$. For this, we first look at the structure of $Fac(A\widehat{\otimes}B)$. The proof of the following result is on lines of ([2], Proposition 4.2) but for sake of completeness, we outline the proof.

PROPOSITION 2.2. *A proper closed ideal K of $A\widehat{\otimes}B$ is factorial if and only if $K = A\widehat{\otimes}J + I\widehat{\otimes}B$ for some $I \in Fac(A)$ and $J \in Fac(B)$.*

Proof. Assume that $K = A\widehat{\otimes}J + I\widehat{\otimes}B$ for some $I \in Fac(A)$ and $J \in Fac(B)$. Since I and J are factorial ideals there exist factorial *-representations $\pi_1 : A \to B(H_1)$ and $\pi_2 : B \to B(H_2)$ such that $I = \ker(\pi_1)$ and $J = \ker(\pi_2)$. Let $\pi = \pi_1 \otimes_{\min} \pi_2 \circ i$, where $\pi_1 \otimes_{\min} \pi_2$ is a *–representation of $A \otimes_{\min} B$ on $H_1 \otimes H_2$ [21] and $i : A\widehat{\otimes}B \to A \otimes_{\min} B$ is an injective algebra *-homomorphism [9]. Clearly, π is a *-representation of $A\widehat{\otimes}B$ and we have

$$\pi(A\widehat{\otimes}B)'' = \pi(A \otimes B)'' = \pi_1 \otimes_{\min} \pi_2(A \otimes B)'' = (\pi_1(A) \otimes \pi_2(B))''$$

and so by Tomita's Commutant Theorem we get $\pi(A\widehat{\otimes}B)'' = \pi_1(A)''\overline{\otimes}\pi_2(B)''$, where $\overline{\otimes}$ denotes the tensor product of von Neumann algebras. Since $\pi_1(A)''$ and $\pi_2(B)''$ are factors, so is $\pi(A\widehat{\otimes}B)''$ by ([5], Proposition III. 1.5.10). Thus $\ker\pi \in Fac(A\widehat{\otimes}B)$. Also, by the definition of π, we have $\ker\pi \supseteq A\widehat{\otimes}J + I\widehat{\otimes}B$. Now, we claim that $K = \ker\pi$. Consider the quotient map $q : A\widehat{\otimes}B \to A/I\widehat{\otimes}B/J$ with $\ker q = K$([10], Lemma 2). Since π is *-representation of $A\widehat{\otimes}B$, $\ker\pi$ is closed ideal of $A\widehat{\otimes}B$ and $\ker\pi \supseteq \ker q$. Suppose that the inclusion is strict. Then $q(\ker\pi)$ is a closed ideal of $A/I\widehat{\otimes}B/J$ by ([10], Lemma 2). Clearly, $q(\ker\pi)$ is a non-zero closed ideal, so it must contain a non-zero elementary tensor, say $(a+I)\otimes(b+J)$, by ([11], Proposition 3.6). Hence $a \otimes b \in \ker\pi$, i.e. $\pi(a \otimes b) = 0$. Thus $\pi_1(a) \otimes \pi_2(b) = 0$, so that either $\pi_1(a) = 0$ or $\pi_2(b) = 0$, a contradiction. Thus $K = \ker\pi$.

For the converse, let $K = \ker(\pi)$ for some factorial *-representation π of $A\widehat{\otimes}B$ on a Hilbert space H. By ([21], Lemma 4.1), there exist representations $\pi_1 : A \to B(H)$ and $\pi_2 : B \to B(H)$ such that $\pi(a\otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a \in A$ and $b \in B$. Since π is a factorial *-representation of $A\widehat{\otimes}B$ and $\pi(A\widehat{\otimes}B)'' = \pi(A \otimes B)''$, so π is factorial *-representation of $A \otimes B$ also. Therefore, π_1 and π_2 are factorial *-representations of A and B by ([5], Theorem II.9.2.1.). Let $I = \ker\pi_1$ and $J = \ker\pi_2$. Clearly, $A\widehat{\otimes}J + I\widehat{\otimes}B \subseteq \ker\pi$. Suppose that the inclusion is strict. Then, as done earlier, we obtain $a \in A\backslash I$ and $b \in B\backslash J$ such that $a \otimes b \in \ker\pi$ and hence $\pi_1(a)\pi_2(b) = 0$. Since $\pi_1(a) \in \pi_1(A)''$, $\pi_2(b) \in \pi_1(A)'$ and $\pi_1(A)''$ is a factor, so by ([21], Proposition 4.20) it follows that either $\pi_1(a) = 0$ or $\pi_2(b) = 0$, a contradiction. Hence $\ker\pi = A\widehat{\otimes}J + I\widehat{\otimes}B$. □

Now recall that $A \otimes_{\min} B$ satisfies Tomiyama's property (F) if the family $\{\phi \otimes_{\min} \varphi : \phi \in P(A),\ \varphi \in P(B)\}$ separates all closed ideals of $A \otimes_{\min}$

B, where $P(A)$ and $P(B)$ denote the set of all pure states of A and B, respectively. For C^*-algebras A and B, it is known that $\|x\|_{\max} \leq \|x\|_{\wedge}$ for all $x \in A \otimes B$. So, there is a contractive homomorphism $i : A\widehat{\otimes}B \to A\otimes_{\max}B$ such that $i(a \otimes b) = a \otimes b$, for all $a \in A$, $b \in B$. Let q be the canonical quotient map from $A \otimes_{\max} B$ onto $A \otimes_{\min} B$. Then, by [9], $q \circ i$ is the canonical injection map from $A\widehat{\otimes}B$ to $A \otimes_{\min} B$. In particular, i is injective.

PROPOSITION 2.3. *For C^*-algebras A and B, if $A \otimes_{\min} B = A \otimes_{\max} B$ and $A \otimes_{\min} B$ has Tomiyama's property (F). Then the mapping $K \to i^{-1}(K)$ is a homeomorphism from $Prim(A \otimes_{\max} B)$ onto $Prim^*(A\widehat{\otimes}B)$, where i is the canonical map from $A\widehat{\otimes}B$ to $A \otimes_{\max} B$.*

Proof. We first claim that that the map $K \to i^{-1}(K)$ is a homeomorphism from $Fac(A\otimes_{\min}B)$ onto $Fac(A\widehat{\otimes}B)$. Consider the map $\phi : Id(A)\times Id(B) \to Id(A\widehat{\otimes}B)$ defined by $\phi(I,J) = A\widehat{\otimes}J + I\widehat{\otimes}B$, $I \in Id(A)$, $J \in Id(B)$. Then Proposition 2.2 shows that the map ϕ maps $Fac(A) \times Fac(B)$ onto $Fac(A\widehat{\otimes}B)$. Also consider the maps $\phi_1 : Id(A) \times Id(B) \to Id(A \otimes_{\min} B)$ $((I,J) \to \ker(q_I \otimes_{\min} q_J))$ and $\phi_2 : Id(A \otimes_{\min} B) \to Id(A\widehat{\otimes}B)$ $(K \to K \cap A\widehat{\otimes}B)$. By ([21], Proposition 4.13) and ([5], Proposition III 1.5.10), ϕ_1 maps $Fac(A) \times Fac(B)$ into $Fac(A \otimes_{\min} B)$. Since $A \otimes_{\min} B$ has property (F) so, by ([7], Lemma 2.5), ϕ_1 is homeomorphism from $Fac(A) \times Fac(B)$ onto $Fac(A\otimes_{\min}B)$. Also, ϕ_2 maps $Fac(A\otimes_{\min}B)$ into $Fac(A\widehat{\otimes}B)$ as every factorial *-representation of $A\otimes_{\min}B$ restricts to a factorial *-representation of $A\widehat{\otimes}B$. Since the factorial ideals are proper so it follows from ([12], Proposition 1.1(v)) that the map ϕ is homeomorphism from $Fac(A)\times Fac(B)$ onto $Fac(A\widehat{\otimes}B)$. Now, note that a following diagram commutes:

$$\begin{tikzcd} Fac(A) \times Fac(B) \arrow[d, "\phi_1"'] \arrow[dr, "\phi"] & \\ Fac(A \otimes_{\min} B) \arrow[r, "\phi_2"'] & Fac(A\widehat{\otimes}B) \end{tikzcd}$$

i.e. $\phi = \phi_2 \circ \phi_1$ by ([12], Lemma 2.8), and so ϕ_2 is homeomorphism from $Fac(A \otimes_{\min} B)$ onto $Fac(A\widehat{\otimes}B)$. Since the primitive ideals are factorial, so $j := \phi_2|_{Prim(A\otimes_{\min}B)}$ is homeomorphism from $Prim(A \otimes_{\min} B)$ into $Prim(A\widehat{\otimes}B)$. In fact, j is onto. To see this, let $P = \ker \pi \in Prim(A\widehat{\otimes}B)$, where π is an irreducible *-representation of $A\widehat{\otimes}B$ on a Hilbert space H. Thus there exist representations $\pi_1 : A \to B(H)$ and $\pi_2 : B \to B(H)$ such that $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a \in A$ and $b \in B$. Therefore, it follows from ([21], Proposition 4.7) that there exists a unique representation $\rho : A \otimes_{\max} B \to B(H)$ such that $\rho(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a \in A$ and $b \in B$, which further gives us $\pi(a \otimes b) = \rho \circ i(a \otimes b)$, where $i : A\widehat{\otimes}B \to A \otimes_{\max} B$ is an injective map. One can easily show that π and

$\rho \circ i$ agree on $A \otimes B$, and so by continuity $\pi = \rho \circ i$. Since $i(A\widehat{\otimes}B)$ is $\|\cdot\|_{\max}$-dense in $A \otimes_{\max} B$ and so ρ is irreducible. Also $P = \ker \rho \circ i = j(\ker \rho)$, and hence the result follows. $\square$

A G^*-algebra A has a unique C^*-norm if the Gelfand-Naimark norm γ_{A/A_R} is the only C^*-norm that can be defined on A/A_R. Note that the G^*-algebra $A \otimes B$ is *-regular if and only if it has a unique C^*-norm and $A \otimes_{\min} B$ has Tomiyama's property (F)([19], Theorem 10.5.36).

Next, we discuss the *-regularity of $A\widehat{\otimes}B$.

THEOREM 2.4. *Let A and B be C^*-algebras, and suppose that $A \otimes_{\min} B$ has Tomiyama's property (F) and $A \otimes_{\min} B = A \otimes_{\max} B$. Then $A\widehat{\otimes}B$ is *-regular. In particular, if A or B is nuclear then $A\widehat{\otimes}B$ is *-regular.*

PROOF: As $i(A\widehat{\otimes}B)$ is $\|\cdot\|_{\max}$- dense in $A \otimes_{\max} B$, so by ([19], Theorem 10.1.11(c)) we get a unique *-homomorphism $C^*(i) : C^*(A\widehat{\otimes}B) \to C^*(A \otimes_{\max} B) = A \otimes_{\max} B$ that makes the following diagram commutative:

$$\begin{tikzcd}
A\widehat{\otimes}B \arrow[d, "\varphi^{A\widehat{\otimes}B}"'] \arrow[dr, "i"] & \\
C^*(A\widehat{\otimes}B) \arrow[r, "C^*(i)"'] & A \otimes_{\max} B
\end{tikzcd}$$

with $C^*(i)$ surjective. Also, by ([16], Theorem 4.8), $C^*(i)$ is an isometric isomorphism from $C^*(A\widehat{\otimes}B)$ onto $A \otimes_{\max} B$.

Since the canonical map $i : A\widehat{\otimes}B \to A \otimes_{\max} B$ is a *-homomorphism, so ([19], Theorem 10.5.6) gives us a continuous map $\breve{i} : Prim(A \otimes_{\max} B) \to Prim^*(A\widehat{\otimes}B)$ defined by $\breve{i}(P) = i^{-1}(P)$ for all $P \in Prim(A \otimes_{\max} B)$, and a commutative diagram:

$$\begin{tikzcd}
Prim(A \otimes_{\max} B) \arrow[d, "\breve{C^*}(i)"'] \arrow[dr, "\breve{i}"] & \\
Prim(C^*(A\widehat{\otimes}B)) \arrow[r, "\breve{\varphi}^{A\widehat{\otimes}B}"'] & Prim^*(A\widehat{\otimes}B)
\end{tikzcd}$$

i.e., $\breve{i} = \breve{\varphi}^{A\widehat{\otimes}B} \circ \breve{C^*}(i)$.

In order to show that $A\widehat{\otimes}B$ is *-regular, suppose that H is a closed subset of $Prim(C^*(A\widehat{\otimes}B))$. Since $\breve{C^*}(i)$ is a continuous map, so $(\breve{C^*}(i))^{-1}(H)$ is closed in $Prim(A \otimes_{\max} B)$. By the given hypothesis and Proposition 2.3, $\breve{i}$ is a homeomorphism from $Prim(A \otimes_{\max} B)$ onto $Prim^*(A\widehat{\otimes}B)$. So $\breve{i}((\breve{C^*}(i))^{-1}(H))$ is a closed subset of $Prim^*(A\widehat{\otimes}B)$. We now claim that

$\breve{C^*}(i)$ is a bijective map. It can be seen easily, using the bijectivity of $C^*(i)$, that $\breve{C^*}(i)$ is an injective map. To see the surjectivity, let $P \in Prim(C^*(A\widehat{\otimes}B))$ then $P = \ker\pi$, π is an irreducible *-representation of $C^*(A\widehat{\otimes}B)$. Let $\tilde{\pi} := \pi \circ C^*(i)^{-1}$, then clearly $\ker\pi = \breve{C^*}(i)(\ker\tilde{\pi}))$ and $\tilde{\pi}$ is an irreducible *-representation of $A\otimes_{\max}B$. So $\breve{i} = \breve{\varphi}^{A\widehat{\otimes}B} \circ \breve{C^*}(i)$ implies that $\breve{\varphi}^{A\widehat{\otimes}B}(H)$ is a closed subset of $Prim^*(A\widehat{\otimes}B)$; note that $\breve{\varphi}^{A\widehat{\otimes}B}$ is injective since $\breve{i}$ and $\breve{C^*}(i)$ both are bijective. Hence the result follows. $\square$

Example 2.5. *For an amenable locally compact Hausdorff topological group G_1 and infinite dimensional separable Hilbert space H, $C^*(G_1)$ and $K(H)$ are nuclear, so $C^*(G_1)\widehat{\otimes}C^*(G_2)$, $B(H)\widehat{\otimes}K(H)$, $K(H)\widehat{\otimes}B(H)$, and $K(H)\widehat{\otimes}K(H)$, are *-regular Banach algebras, where G_2 is a locally compact Hausdorff topological group. Therefore, by (* [19], Lemma 10.5.22)*, $B(H)\widehat{\otimes}K(H) + K(H)\widehat{\otimes}B(H)$ is also *-regular. Thus, every proper closed ideal of $B(H)\widehat{\otimes}B(H)$ is *-regular.*

We now give a partial converse of the above theorem. For this, we first note a result from ([4], Proposition 2.4) namely: a reduced BG^*-algebra A has a unique C^*-norm if and only if for every non-zero closed ideal I of $C^*(A)$, $I \cap A$ is non-zero.

It is known, from [1], that if I is a non-zero closed ideal in $A \otimes_{\min} B$ then I contains a non-zero elementary tensor. However, this may not be true for $A \otimes_{\max} B$ if $\|\cdot\|_{\min} \neq \|\cdot\|_{\max}$ on $A \otimes B$. This fact is essentially used in the following theorem.

Theorem 2.6. *For C^*-algebras A and B, $A\widehat{\otimes}B$ has a unique C^*-norm if and only if $A \otimes B$ has. In particular, if the Banach *-algebra $A\widehat{\otimes}B$ is *-regular then $A \otimes B$ has a unique C^*-norm.*

Proof: Suppose that $A\widehat{\otimes}B$ has a unique C^*-norm. In order to show the uniqueness of C^*-norm on $A\otimes B$, it is enough to show that $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A \otimes B$. Suppose, on the contrary, $\|\cdot\|_{\min} \neq \|\cdot\|_{\max}$ on $A \otimes B$. Then $\ker q$ is a non-zero closed ideal of $A \otimes_{\max} B$, where q is the canonical *-homomorphism from $A\otimes_{\max}B$ onto $A\otimes_{\min}B$. As in Theorem 2.4, we obtain a unique isometric *-isomorphism $C^*(i)$ from $C^*(A\widehat{\otimes}B)$ onto $A \otimes_{\max} B$. Now consider the map $\phi : Id(A \otimes_{\max} B) \to Id(C^*(A\widehat{\otimes}B))$ given by $\phi(I) = C^*(i)^{-1}(I)$ for all $I \in Id(A \otimes_{\max} B)$. Clearly, this map ϕ is injective so $C^*(i)^{-1}(\ker q)$ is a non-zero closed ideal of $C^*(A\widehat{\otimes}B)$. Since $A\widehat{\otimes}B$ is *-reduced, so $C^*(i)^{-1}(\ker q) \cap A\widehat{\otimes}B$ is a non-zero closed ideal of $A\widehat{\otimes}B$. Thus, by ([11], Proposition 3.6), it would contain a non-zero elementary tensor, say $a\otimes b$, which further gives $C^*(i)(a\otimes b) \in \ker q$, i.e. $a\otimes b = 0$, a contradiction. Hence $A\otimes B$ has a unique C^*-norm. Converse follows by the same argument as that in ([19], Corollary 10.5.38). $\square$

From [7], $C_r^*(F_2) \otimes C_r^*(F_2)$ does not have unique C^*-norm, so $C_r^*(F_2)\widehat{\otimes} C_r^*(F_2)$ is not a *-regular Banach algebra, where $C_r^*(F_2)$ is the C^*-algebra associated to the left regular representation of the free group F_2 on two generators. Note that $C_r^*(F_2)$ is non-nuclear simple C^*-algebra [5]. Similarly, for any C^*-algebra A without the weak expectation property of Lance and for a free group F_∞ on an infinite set of generators, $C^*(F_\infty)\widehat{\otimes}A$ is not *-regular by ([18], Proposition 3.3). Also, by ([13], Corollary 3.1), for an infinite dimensional separable Hilbert space H, $B(H)\otimes_{\min}B(H) \neq B(H)\otimes_{\max}B(H)$ so $B(H)\widehat{\otimes}B(H)$ is not *-regular.

Corollary 2.7. *For an infinite dimensional separable Hilbert space H, $B(H)/K(H)\widehat{\otimes}B(H)/K(H)$ is not a *-regular Banach algebra.*

Proof: From [10], we know that there exists an isometric isomorphism ϕ from $A := (B(H)\widehat{\otimes}B(H))/(B(H)\widehat{\otimes}K(H)+K(H)\widehat{\otimes}B(H))$ to $B(H)/K(H)\widehat{\otimes} B(H)/K(H)$, satisfying $\phi(x+(B(H)\widehat{\otimes}K(H)+K(H)\widehat{\otimes}B(H))) = q\widehat{\otimes}q(x)$, for all $x \in B(H)\widehat{\otimes}B(H)$, where q is the quotient map from $B(H)$ to $B(H)/K(H)$. Clearly, this map ϕ is a bijective algebra *-homomorphism. It is known from [10] that the primitive ideals of $B(H)\widehat{\otimes}B(H)$ are $\{0\}$, $B(H)\widehat{\otimes}K(H)$, $K(H)\widehat{\otimes}B(H)$, and $B(H)\widehat{\otimes}K(H) + K(H)\widehat{\otimes}B(H)$. So A has only one primitive ideal $(B(H)\widehat{\otimes}K(H) + K(H)\widehat{\otimes}B(H))/(B(H)\widehat{\otimes}K(H) + K(H)\widehat{\otimes}B(H))$. Thus A is *-reduced. Now suppose that $B(H)/K(H)\widehat{\otimes}B(H)/K(H)$ is *-regular. Then, clearly A is *-regular and so is $B(H)\widehat{\otimes}B(H)$ by ([19], Theorem 10.5.15(d)), a contradiction. □

3. Reverse Involution

Let A be a C^*-algebra. On the Banach algebra $A\widehat{\otimes}A$, with the usual multiplication, define the involution on an elementary tensor as $(a \otimes b)^* = b^* \otimes a^*$ for all $a, b \in A$. This extends to $A\widehat{\otimes}A$, by the definition of operator space projective tensor norm, and $A\widehat{\otimes}A$ becomes a Banach *-algebra with this isometric involution, denoted by $A\widehat{\otimes}_r A$.

Theorem 3.1. *For a C^*-algebra A, $A\widehat{\otimes}_r A$ is *-regular.*

Proof: Let π and ρ be non-degenerate *-representations of $A\widehat{\otimes}_r A$, on the same Hilbert space H, with $\ker\pi \subseteq \ker\rho$. Suppose first that A has an identity 1. Define $\pi_1(a) := \pi(a \otimes 1)$ and $\pi_2(a) := \pi(1 \otimes a)$, $a \in A$; clearly π_1 and π_2 are bounded representations from A into $B(H)$ satisfying $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$ for all $a, b \in A$, and $\pi_1(a^*) = \pi_2(a)^*$ for all $a \in A$. Since every *-representation of a Banach *-algebra into a C^*-algebra is contractive and that $\widehat{\otimes}$ is a cross norm so, for a self-adjoint element $h \in A$, we get $\|\exp(it\pi_1(h))\| = 1$ for all $t \in \mathbb{R}$. Thus $\pi_1(h)$ is a self-adjoint element of $B(H)$. Let $a \in A$, so $a = h + ik$, where h and k are self-adjoint elements of A. One can verify that $\pi_1(a^*) = \pi_1(a)^*$. This shows that π_1 is a *-representation of A and $\pi_1(a)^* = \pi_2(a)^*$ for all $a \in A$, and thus $\pi_1(a) = \pi_2(a)$ for all $a \in A$. But $\pi(a \otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a)$, so $\pi(a \otimes b) = \pi_1(ab) = \pi_2(ba)$, for all $a, b \in A$; similarly, π_2 is also a *-representation of A. In a similar manner, we can define *-representations ρ_1 and ρ_2 of A satisfying $\rho(a\otimes b) = \rho_1(a)\rho_2(b) = \rho_2(b)\rho_1(a)$ for all $a, b \in A$. Arguing as above, we have $\rho(a \otimes b) = \rho_1(ab) = \rho_2(ba)$ for all $a, b \in A$. Clearly,

$\ker \pi_1 \subseteq \ker \rho_1$. It follows easily that π_1, ρ_1, π_2, ρ_2, all are non-degenerate *-representations of A. Therefore, by ([17], Proposition 1.3), we have

$$\|\rho(a \otimes b)\| = \|\rho_1(ab)\| \leq \|\pi_1(ab)\| = \|\pi(a \otimes b)\| \text{ for all } a, b \in A.$$

Now for any $x = \sum_{i=1}^{n} a_i \otimes b_i$ in $A \otimes A$, clearly we have $\left\|\rho\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right\| \leq \left\|\pi\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\right\|$. Since $(A \otimes A, \|\cdot\|_\wedge)$ is dense in $A \widehat{\otimes}_r A$ and *-representation from the Banach *-algebra $A \widehat{\otimes}_r A$ to $B(H)$ is norm reducing, it follows easily that $\|\rho(x)\| \leq \|\pi(x)\|$ for all $x \in A \widehat{\otimes}_r A$. Hence if A is a unital C^*-algebra, $A \widehat{\otimes}_r A$ is *-regular by ([17], Proposition 1.3).

If A does not have identity, consider the unitization A_e of A. Clearly, A is a closed ideal of A_e. Therefore, $A \widehat{\otimes}_r A$ is a closed ideal of $A_e \widehat{\otimes}_r A_e$ by ([14], Theorem 5). In fact, it is a *-ideal of $A_e \widehat{\otimes}_r A_e$. Thus $A_e \widehat{\otimes}_r A_e$ is *-regular, so is $A \widehat{\otimes}_r A$ by ([19], Theorem 10.5.15). □

4. PROPERTY (F) FOR THE OPERATOR SPACE PROJECTIVE TENSOR PRODUCT OF C^*-ALGEBRAS

Tomiyama [22] defined the concept of property (F) for the minimal tensor product of C^*-algebras. Following [22], we define the property (F) for the operator space projective tensor product of C^*-algebras and show that if $A \widehat{\otimes} B$, for any C^*-algebras A and B, has spectral synthesis in the sense of [12] then $A \widehat{\otimes} B$ satisfies property (F). We also show that weak spectral synthesis and spectral synthesis in the sense of [12] coincides on $A \widehat{\otimes} B$.

DEFINITION 4.1. *Let A and B be C^*-algebras. We say that $A \widehat{\otimes} B$ satisfies property (F) if the family $\{\phi \widehat{\otimes} \varphi : \phi \in P(A),\ \varphi \in P(B)\}$, where $P(A)$, $P(B)$ denote the pure states of A and B respectively, separates all the closed ideals of $A \widehat{\otimes} B$.*

From [9], it is known that, for any C^*-algebras A and B, the canonical map $i' : A \widehat{\otimes} B \to A \otimes_{\min} B$ is an injective *-homomorphism, so that we can regard $A \widehat{\otimes} B$ as a *-subalgebra of $A \otimes_{\min} B$. Let I be a closed ideal in $A \widehat{\otimes} B$ and $I_{\min}$ be the closure of $i'(I)$ in $A \otimes_{\min} B$. Now associate two closed ideals, I_l and I^u, with I defined as $I_l =$ closure of the span of all elementary tensors of I in $A \widehat{\otimes} B$, $I^u = I_{\min} \cap A \widehat{\otimes} B$, known as the lower and upper ideal associated with I, respectively. Clearly, $I_l \subseteq I \subseteq I^u$. Following [12], we say that a closed ideal I of $A \widehat{\otimes} B$ is spectral if $I_l = I = I^u$.

Following lemma can be proved using ([14], Theorem 6).

LEMMA 4.2. *Let I and J be closed ideals in $A \widehat{\otimes} B$ such that $I_{\min} = J_{\min}$. Then $J_l \subseteq I \subseteq J^u$ and $I_l \subseteq J \subseteq I^u$.*

We now relate the property (F) of the operator space projective tensor product of C^*-algebras to the spectral synthesis of the closed sets of its primitive ideal space. For this, recall that for any Banach *-algebra A, a closed subset E of $Prim^*(A)$ is called spectral if $k(E)$ is the only closed ideal in A with hull equal to E, and we say that a Banach *-algebra A has spectral synthesis if every closed subset of $Prim^*(A)$ is spectral. From [12], for any C^*-algebras A and B, the Banach *-algebra $A\widehat{\otimes}B$ has spectral synthesis if and only if every closed ideal of $A\widehat{\otimes}B$ is spectral.

Theorem 4.3. *Let A and B be C^*-algebras, and suppose that $A\widehat{\otimes}B$ has spectral synthesis. Then $A\widehat{\otimes}B$ satisfies property (F).*

Proof: Let I and J be non-zero distinct closed ideals in $A\widehat{\otimes}B$. Then, by ([9], Corollary 1), $I_{\min}$ and $J_{\min}$ are non-zero closed ideals in $A \otimes_{\min} B$. Suppose that $I_{\min} = J_{\min}$. Using Lemma 4.2 and the fact that $A\widehat{\otimes}B$ has spectral synthesis, we get $I \subseteq J$, $J \subseteq I$, and thus $I = J$, a contradiction. This shows that $I_{\min}$ and $J_{\min}$ are non-zero distinct closed ideals in $A\otimes_{\min}B$. Now choose an irreducible *-representation π of $A\otimes_{\min}B$ such that $\pi(I_{\min}) = 0$ and $\pi(J_{\min}) \neq 0$. Set $\tilde{\pi} := \pi \circ i'$ then $\tilde{\pi}$ is an irreducible *-representation of $A\widehat{\otimes}B$ [10], and clearly $\tilde{\pi}(I) = 0$, $\tilde{\pi}(J) \neq 0$. Let us denote the restriction of π to A and B by π_1 and π_2, respectively; and $\tilde{\pi}_1$, $\tilde{\pi}_2$ be the restrictions of $\tilde{\pi}$ to A and B.

Define a map $\theta_\pi : \pi(A \otimes B) \to \pi_1(A) \otimes \pi_2(B)$ as $\theta_\pi(\pi(a \otimes b)) = \pi_1(a) \otimes \pi_2(b)$, $a \in A$, $b \in B$. Then θ_π can be extended to a homomorphism $\tilde{\theta_\pi}$ from $\pi(A \otimes_{\min} B)$ onto $\pi_1(A) \otimes_{\min} \pi_2(B)$ and $\tilde{\theta_\pi} \circ \pi = \pi_1 \otimes_{\min} \pi_2$ (see [7] and [16] for details). Note that $\tilde{\pi}(A\widehat{\otimes}B) \subseteq \pi(A \otimes_{\min} B)$. Since $\tilde{\pi}(J) \neq 0$, so choose $x \in J$ such that $\tilde{\pi}(x) \neq 0$. Suppose that $\tilde{\theta_\pi}(\tilde{\pi}(u)) = 0$ for all $u \in J$. In particular, $\tilde{\theta_\pi}(\tilde{\pi}(x)) = 0$, that is, $\pi_1 \otimes_{\min} \pi_2(i'(x)) = 0$. Since both $\pi_1\widehat{\otimes}\pi_2$ and $(\pi_1 \otimes_{\min} \pi_2) \circ i'$ agree on $A \otimes B$, so by continuity $\pi_1\widehat{\otimes}\pi_2(x) = 0$, where $\pi_1\widehat{\otimes}\pi_2$ is the extension of $\pi_1 \otimes \pi_2$. Now we claim that $\tilde{\pi}_1 = \pi_1$ and $\tilde{\pi}_2 = \pi_2$. If A and B are unital then $\tilde{\pi}_1(a) = \tilde{\pi}(a \otimes 1) = \pi(a \otimes 1) = \pi_1(a)$, for all $a \in A$, giving that $\tilde{\pi}_1 = \pi_1$; similarly $\tilde{\pi}_2 = \pi_2$. In the general case, if $\{e_\lambda\}$ and $\{f_\mu\}$ are the bounded approximate identities for A and B, respectively, then for any $a \in A$, $\tilde{\pi}_1(a) =$s–lim $\tilde{\pi}(a \otimes f_\mu) =$s–lim $\pi(a \otimes f_\mu) = \pi_1(a)$ [21], where s–lim denotes the strong limit. Thus $\tilde{\pi}_1 = \pi_1$, similarly $\tilde{\pi}_2 = \pi_2$.

Using ([10], Theorem 7), $\ker\tilde{\pi} = A\widehat{\otimes}I_2 + I_1\widehat{\otimes}B = \ker q_{I_1}\widehat{\otimes}q_{I_2}$, where $I_1 = \ker\tilde{\pi}_1$, $I_2 = \ker\tilde{\pi}_2$. Also, by ([7], Lemma 2.1), $\ker\pi \subseteq \ker\pi_1 \otimes_{\min} \pi_2$, giving that $\ker\pi \cap (A\widehat{\otimes}B) \subseteq \ker\pi_1 \otimes_{\min} \pi_2 \cap (A\widehat{\otimes}B)$, in other words, $\ker\tilde{\pi} \subseteq \ker\tilde{\pi}_1\widehat{\otimes}\tilde{\pi}_2$; that is, $\ker q_{I_1}\widehat{\otimes}q_{I_2} \subseteq \ker\tilde{\pi}_1\widehat{\otimes}\tilde{\pi}_2$. Suppose that the inclusion is strict. Let $K = \ker\tilde{\pi}_1\widehat{\otimes}\tilde{\pi}_2$, then $q_{I_1}\widehat{\otimes}q_{I_2}(K)$ is a non-zero closed ideal of $A/I_1\widehat{\otimes}B/I_2$ by ([10], Lemma 2). So it must contain a non-zero elementary tensor, say $(a + I_1) \otimes (b + I_2)$([11], Proposition 3.6). Hence $a \otimes b \in K$, i.e., $\tilde{\pi}_1\widehat{\otimes}\tilde{\pi}_2(a \otimes b) = 0$. So $\pi_1(a) \otimes \pi_2(b) = 0$, i.e. either $\pi_1(a) = 0$ or $\pi_2(b) = 0$, a contradiction. Thus $\ker\tilde{\pi}_1\widehat{\otimes}\tilde{\pi}_2 = \ker q_{I_1}\widehat{\otimes}q_{I_2} = \ker\tilde{\pi}$. Therefore, $x \in \ker\tilde{\pi}$, which is not true. So $\tilde{\theta_\pi}(\tilde{\pi}(J)) \neq 0$. Also note that $\tilde{\theta_\pi}(\tilde{\pi}(J)) = \pi_1\widehat{\otimes}\pi_2(J)$ and $\overline{\tilde{\theta_\pi}(\tilde{\pi}(J))}$, closure is taken with respect to min-norm, is a closed ideal in $\pi_1(A) \otimes_{\min} \pi_2(B)$. Therefore, there exist $\phi \in P(\pi_1(A))$ and $\varphi \in P(\pi_2(B))$ such that $\phi \otimes_{\min} \varphi(\overline{\tilde{\theta_\pi}(\tilde{\pi}(J))}) \neq 0$ [21], so $\phi \otimes_{\min} \varphi(\tilde{\theta_\pi}(\tilde{\pi}(J))) \neq 0$, which further gives $(\phi\circ\pi_1)\otimes_{\min}(\varphi\circ\pi_2)(i'(J)) \neq 0$. Let $\sigma_1 = \phi\circ\pi_1$ and $\sigma_2 = \varphi\circ\pi_2$, then $\sigma_1 \otimes_{\min} \sigma_2(i'(J)) \neq 0$, $\sigma_1 \in P(A)$, $\sigma_2 \in P(B)$. It is easy to see that both the maps $(\sigma_1 \otimes_{\min} \sigma_2)\circ i'$, $\sigma_1\widehat{\otimes}\sigma_2$ are continuous on $A\widehat{\otimes}B$ and agree on $A \otimes B$, giving that $\sigma_1\widehat{\otimes}\sigma_2(J) \neq 0$. Obviously $\sigma_1\widehat{\otimes}\sigma_2(I) = 0$. Hence $A\widehat{\otimes}B$ has property (F). $\square$

Remark 4.4. (i) *If A or B has finitely many closed ideals then $A\widehat{\otimes}B$ has spectral synthesis* [12]. *Thus, it satisfies property (F). In particular, $B(H)\widehat{\otimes}B(H)$, $K(H)\widehat{\otimes}K(H)$ and $C_0(X)\widehat{\otimes}B(H)$ satisfy property (F).*
(ii) *One can also prove that if $A \otimes_h B$ has spectral synthesis then it satisfies property (F), details can be worked out as in Theorem 4.3.*

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CHEBYSHEV CARDINAL FUNCTIONS FOR SOLUTIONS OF TRANSPORT EQUATION

PARIA SATTARI SHAJARI AND KARIM IVAZ

Abstract. In this paper we use Chebyshev cardinal functions to solve transport equation. The method consists of expanding the required approximate solution as the elements of Chebyshev cardinal functions. Using the operational matrix of derivative, the problem is reduced to a set of algebraic equations. Some numerical examples are included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces very accurate results.

1. Introduction

In this paper we use Chebyshev cardinal functions to solve transport equation [1] for the linear hyperbolic scalar equation:

$$L(u) = u_t + cu_x = a(x,t) \tag{1.1}$$

where c is a positive constant. Also we consider linear advection-diffusion equation of the form:

$$\widehat{L}(u) = u_t + cu_x - \nu u_{xx} = a(x,t) \tag{1.2}$$

where $\nu > 0$ is the diffusion coefficient. Initial conditions and inflow boundary conditions are provided in the usual way,

$$u(x,0) = f(x) \tag{1.3}$$

$$u(0,t) = g(t) \tag{1.4}$$

and the Neumann condition

$$\frac{\partial u}{\partial x}|_{x=L} = p(t). \tag{1.5}$$

However, the numerical method we are going to present here for the simple linear case will be significant enough to enable many interesting conclusions to be drawn.

The general theory on hyperbolic equations and conservation laws has already generated an enormous amount of literature (see for instance [3], [1]). The relevance of advection-dominated problems is also testified to by a number of recent papers dealing with a variety of approximating methods and numerical schemes [4]-[11].

Key words and phrases. Advection-diffusion equation, Chebyshev cardinal functions, transport equation.

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2. CHEBYSHEV CARDINAL FUNCTIONS

Let us consider an independent variable t defined in the [-1, 1] interval. The Chebyshev polynomial $T_N(t)$ of the first kind and of degree N (see for example [2]) is defined by the formula

$$T_N(t) = cos(Narccos(t)).$$

It is evident that $T_N(t)$ has N zeros in the [-1, 1] interval and they are located at the points

$$t_k = cos(\pi(k - \frac{1}{2})/N), \quad k = 1, 2, \ldots, N$$

The Chebyshev polynomials can be also generated from the recurrence relation

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t)$$

with $T_0(t) = 1$ and $T_1(t) = t$. The Chebyshev polynomials of the second kind, denoted as $U_n(t)$, are obtained from the same recurrence relation but with different starting values: $U_1(t) = 0$ and $U_0(t) = 1$.

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [2]:

$$C_j(t) = \frac{T_{N+1}(t)}{T'_{N+1}(t_j)(t - t_j)} j = 1, 2, \ldots, N + 1$$

where $T_{N+1}(t)$ is the first kind Chebyshev function of order $N + 1$ in [-1, 1], i.e., $t_j, j = 1, 2, \ldots, N + 1$ are the zeros of $T_{N+1}(t)$. By direct computation it is easily seen that

$$C_j(t) = \frac{\prod_{i=1, i\neq j}^{N+1}(t - t_i)}{\prod_{i=1, i\neq j}^{N+1}(t_j - t_i)}$$

We change the variable $x = (t + 1)/2$ to use these functions on $[0, 1]$. Now any function $f(x)$ on [0, 1] can be approximated as

$$f(x) = \sum_{j=1}^{N+1} f(x_j)C_j(x) = F^T \phi_N(x) \tag{2.1}$$

where $x_j, j = 1, 2, \ldots, N + 1$ are the shifted points of $t_j, j = 1, 2, \ldots, N + 1$ by transform $x = (t + 1)/2$,

$$F = [f(x_1), f(x_2), \ldots, f(x_{N+1})]^T$$

and

$$\phi_N(x) = [C_1(x), C_2(x), \ldots, C_{N+1}(x)]^T.$$

Also a function $u(x, t)$ of two independent variables defined for $0 \leq x \leq 1$ and $0 \leq t \leq 1$ may be expanded in terms of double Chebyshev cardinal functions as [2]

$$u(x, t) = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} u(x_i, t_j)C_i(t)C_j(x) = \phi_N^T(t)A\phi_N(x) \tag{2.2}$$

where

$$A = (a_{i,j}), \quad a_{i,j} = u(x_j, t_i)$$

It is easy to check that the differential of vector ϕ_N can be expressed as

$$\phi'_N = D\phi_N$$

where

$$D = (d_{i,j}) \quad d_{i,j} = C'_i(x_j)$$

3. Implementation of the method

Using Eq. (2.1) we approximate the functions $f(x)$, $g(t)$ and $p(t)$ as:

$$\begin{aligned} f(x) &= F^T\phi_N(x) \\ g(t) &= G^T\phi_N(t) \\ p(t) &= P^T\phi_N(t) \\ a(x,t) &= \phi_N^T(t)A\phi_N(x) \end{aligned} \tag{3.1}$$

Considering,

$$u(x,t) = \phi_N^T(t)U\phi_N(x)$$

we obtain

$$u_t(x,t) = \phi_N^T(t)D^TU\phi_N(x) \tag{3.2}$$

$$u_x(x,t) = \phi_N^T(t)UD\phi_N(x) \tag{3.3}$$

$$u_{xx}(x,t) = \phi_N^T(t)UD^2\phi_N(x) \tag{3.4}$$

substituting (3.1)-(3.4) into (1.2)-(1.5) we obtain following system

$$\begin{aligned} \phi_N^T(t)\left(D^TU + cUD - \nu UD^2 - A\right)\phi_N(x) &= 0 \\ \phi_N^T(0)U\phi_N(x) &= F^T\phi_N(x) \\ \phi_N^T(t)U\phi_N(0) &= G^T\phi_N(t) = \phi_N^T(t)G \\ \phi_N^T(t)UD\phi_N(L) &= P^T\phi_N(t) = \phi_N^T(t)P \end{aligned} \tag{3.5}$$

setting $x = x_i$, $t = t_j$, and considering the characteristic equations $\phi_i(x_j) = \delta_{ij}$ for $i, j = 1, \ldots, N+1$ we get

$$\begin{aligned} D^TU + cUD - \nu UD^2 - A &= 0 \\ \phi_N^T(0)U &= F^T \\ U\phi_N(0) &= G^T\phi_N(t) = G \\ UD\phi_N(L) &= P^T\phi_N(t) = P. \end{aligned} \tag{3.6}$$

The system (3.6) has $(N+1)^2 + 3(N+1)$ equations and $(N+1)^2$ unknowns. Now, how can we choose $(N+1)^2$ equations from $(N+1)^2 + 3(N+1)$ equations of system (3.6)? Only one of the choices is true selection. Hence, in the first equation of system (3.5) we set $x = x_2, ..., x_N$ and $t = t_2, ..., t_{N+1}$, in the second we set $x = x_1, ..., x_{N+1}$ and finally in the third and forth equation of (3.5) we set $t = x_2, ..., x_{N+1}$ to obtain $(N-1)N + N + 1 + N + N$ Now, we can solve this system to obtain u.

4. Numerical experiment

In this section we give examples to show the efficiency of the method.

Example 4.1. *Consider the equation (1.2) with* $c = 1$, ν, $a(x,t) = e^{x+t}$ *and boundary conditions (1.3)-(1.5) with* $f(x) = e^x$, $g(t) = e^t$ *and* $p(t) = e^t$ *on* $[0,1]$. *The exact solution of this equation is* $u(x,t) = e^{x+t}$. *Table 1 shows the exact solution and approximate solution and the absolute value of errors for* $N = 5$ *at node points. This table shows the efficiency of the method.*

TABLE 1. Numerical and exact solutions for $N = 5$.

(x_i, t_j)	exact solution	approximate solution	error
(0.0245, 0.0245)	1.0501609978	1.0501295658	0.000031432
(0.2061, 0.0245)	1.2593290960	1.2592610944	0.000068001
(0.5000, 0.0245)	1.6895660855	1.6894596188	0.000106467
(0.7939, 0.0245)	2.2667891708	2.2666778658	0.000111305
(0.9755, 0.0245)	2.7182818284	2.7181995759	0.000082252
(0.0245, 0.2061)	1.2593290960	1.25928346207	0.000045634
(0.2061, 0.2061)	1.5101587046	1.51003698304	0.000121722
(0.5000, 0.2061)	2.02608908118	2.02595187254	0.000137209
(0.7939, 0.2061)	2.71828182846	2.71829112477	0.00000929631
(0.9755, 0.2061)	3.2597015170	3.25994112416	0.000239607
(0.0245, 0.5000)	1.6895660855	1.68953019568	0.0000358899
(0.2061,0.5000)	2.0260890811	2.02614148852	0.0000524073
(0.5000, 0.5000)	2.7182818284	2.7186178102	0.000335982
(0.7939, 0.5000)	3.6469551944	3.64772746858	0.000772274
(0.9755, 0.5000)	4.37334541818	4.37449069415	0.00114528
(0.0245, 0.7939)	2.26678917081	2.26673975775	0.00114528
(0.2061, 0.7939)	2.71828182846	2.71835404076	0.0000722123
(0.5000, 0.7939)	3.6469551944	3.64743408074	0.000478886
(0.7939, 0.7939)	4.89290037946	4.89398484749	0.00108447
(0.9755, 0.7939)	5.86745444227	5.86904285408	0.00158841
(0.0245, 0.9755)	2.71828182846	2.71823024588	0.0000515826
(0.2061, 0.9755)	3.25970151706	3.25983798953	0.000136472
(0.5000, 0.9755)	4.37334541818	4.374010683	0.000665265
(0.7939, 0.9755)	5.86745444227	5.86887808231	0.00142364
(0.9755, 0.9755)	7.03611742774	7.0381561356	0.00203871

TABLE 2. The absolute value of errors for $N = 5$.

0.0005	0.0192	0.0408	0.0049	0.0029	0.0030
0.0004	0.0045	0.0102	0.0116	0.0121	0.0127
0.0001	0.0010	0.0028	0.0055	0.0068	0.0070
0.0001	0.0006	0.0015	0.0020	0.0024	0.0025
0.0000	0.0001	0.0004	0.0010	0.0013	0.0013
0.0000	0.0002	0.0006	0.0005	0.0006	0.0007

Example 4.2. *Consider the equation (1.2) with* $c = 1$, $\nu = 1$, $a(x,t) = x^3 - 4xt + 3x^2t + t^2$ *and boundary conditions (1.3)-(1.5) with* $f(x) = 0$, $g(t) = 0$ *and* $p(t) = t^2 + 3t$ *on* $[0, 1]$. *The exact solution of this equation is* $u(x,t) = xt^2 + x^3t$. *Table 2 shows the the absolute value of errors for* $N = 5$ *at node points. This table shows the efficiency of the method.*

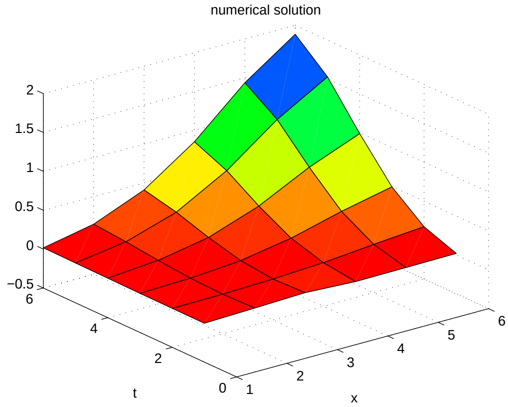

Figure 1. Numerical solution of example 2.

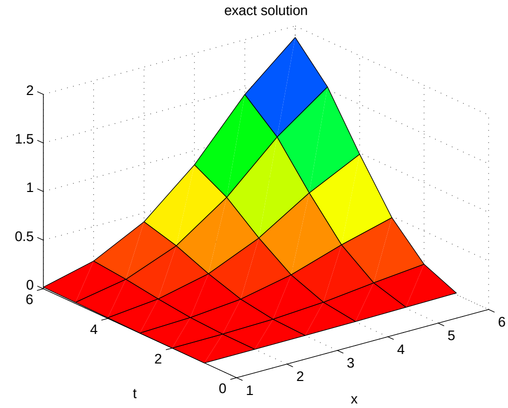

Figure 2. Exact solution of example 2.

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Multiple Positive solutions for boundary value problem of nonlinear fractional differential equation

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In this paper, we study a boundary value problem of nonlinear fractional differential equation. Existence and positivity results of solutions are obtained. Two examples are given to show the effectiveness of our works.

Keywords: *Positive solution, Fractional Caputo derivative, Banach Contraction principle, Guo-Krasnoselskii Theorem, Avery and Peterson fixed point Theorem.*

2000 Mathematics Subject Classification: *Primary 05C38, 15A15; Secondary 05A15, 15A18.*

1 Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see [2,6,14,15,18,19] and references therein. Positive solutions for ordinary differential equations and fractional differential equations also have been considered by many authors, e.g. [4,7,20,21,22], where the major tool in finding positive solutions for both fractional and ordinary differential equations have been fixed point theorems.

In [22] , using fixed point theorems on cones, Zhang investigated the existence and multiplicity of positive solutions of the following problem:

$$^{c}D_{0^+}^{\alpha}u(t) = f(t, u(t)), 0 < t < 1, 1 < \alpha \leq 2$$

$$u(0) + u'(0) = 0, u(1) + u'(1) = 0.$$

By means of the Schauder fixed point theorem and fixed point index theory, Bai [4] discussed the existence of positive solutions for the fractional boundary value problem

$$^cD_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, 0 < t < 1, 1 < \alpha \leq 2$$

$$u(0) = 0, \beta u(\eta) = u(1).$$

In [20]the autors study the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$^cD_{0+}^{\alpha} u(t) = h(t)f(t, u(t)), 0 < t < 1, 3 < \alpha \leq 4$$

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

The main objective of this paper is to investigate the existence and multiplicity of positive solutions of the boundary value problem (P1):

$$^cD_{0+}^{q} u(t) + f(t, u(t)) = 0, 0 < t < 1,$$

$$u(0) = u''(0) = 0, u(1) = u(\xi).$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $2 < q < 3$, $0 < \xi < 1$.

We obtain our main result by using the fixed point theorem on a cone preserving operator on an ordered Banach space that will be defined in Section 2. First we obtain an integral representation of the solution by the corresponding Green's function.

2 Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 1 *[1] The Riemann -Liouville fractional integral of order $\alpha > 0$ of a function $g \in C([a, b]$ is defined by*

$$I_{a+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2 *[1] The caputo fractional derivative of order $\alpha > 0$ of*

$g \in AC^n[a,b]$ is defined by

$$^cD_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$ ($[\alpha]$ is the entire part of α).

Lemma 3 *[1] Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$, then the following relations hold:*

$$^cD_{a^+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \beta > n,$$

and

$$^cD_{a^+}^{\alpha} t^k = 0, k = 0, 1, 2, ..., n-1.$$

Lemma 4 *[1] Assume that $u \in C^n[a,b]$, then*

$$I_{a^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},$$

where, $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., n$, and $n = [\alpha] + 1$.

Lemma 5 *[1] Let p, $q \geq 0$, $f \in L_1([a,b]$. Then*

$$I_{0^+}^{P} I_{0^+}^{q} f(t) = I_{0^+}^{P+q} f(t) = I_{0^+}^{q} I_{0^+}^{P} f(t),$$

and

$$^cD_{a^+}^{q} I_{0^+}^{q} f(t) = f(t), \forall t \in [a,b].$$

Lemma 6 *[1] Let $\beta > \alpha > 0$, $f \in L_1([a,b]$. Then for all $t \in [a,b]$ we have*

$$^cD_{a^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\beta-\alpha} f(t).$$

Denote by $L^1([0,1], \mathbb{R})$ the Bananch space of lesbegue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norme $\|Y\|_{L^1} = \int_0^1 |Y(t)| dt$.

Lemma 7 *Given $y \in C([0,1])$, and $2 < q < 3$, the unique solution of fractional problem (P_0)*

$$\begin{cases} ^cD_{0^+}^{q} u(t) + y(t) = 0, \ 0 < t < 1 \\ u(0) = u''(0) = 0, u(1) = u(\xi), 0 < \xi < 1, \end{cases}$$

is given by

$$u(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t,s) y(s) ds,$$

where

$$G(t,s) = \begin{cases} \frac{t(1-s)^{q-1}}{1-\xi} - \frac{t(\xi-s)^{q-1}}{1-\xi} - (t-s)^{q-1}, 0 \leq s \leq \min(t,\xi) \leq 1 \\ \frac{t(1-s)^{q-1}}{1-\xi} - (t-s)^{q-1}, \xi \leq s \leq t \\ \frac{t(1-s)^{q-1}}{1-\xi} - \frac{t(\xi-s)^{q-1}}{1-\xi}, t \leq s \leq \xi \\ \frac{t(1-s)^{q-1}}{1-\xi}, \max(t,\xi) \leq s \leq 1. \end{cases} \tag{2.1}$$

Proof. Using Lemmas 1 and 2 we have

$$u(t) = -I_{0+}^{q} y(t) + C + Bt + At^2,$$

from the conditions $u(0) = u''(0) = 0$, we obtain $C = A = 0$, and the condition $u(1) = u(\xi)$ implies

$$B = \frac{1}{1-\xi} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} y(s)ds,$$

so $u(t)$ can be written as

$$u(t) = -I_{0+}^{q} y(t) + \frac{1}{1-\xi} \int_0^1 \frac{t(1-s)^{q-1}}{\Gamma(q)} y(s)ds - \int_0^\xi \frac{t(\xi-s)^{q-1}}{\Gamma(q)} y(s)ds,$$

where G is defined by (2.1) . The proof is complete. ∎

Lemma 8 *For all $s, t \in [0,1]$, the Green fonction $G(t,s)$ is non negative, continuous and satisfies*
$i) G(t,s) \leq \frac{(1-s)^{q-1}}{(1-\xi)}$,
$ii) G(t,s) \geq t(\zeta - \zeta^{q-1}) \frac{(1-s)^{q-1}}{(1-\xi)}$.

Proof. It is easy to check that $G(t,s)$ is non negative, continuous and satisfies (i). So we prove that (ii) is true.
For $0 \leq s \leq \min(t,\xi) \leq 1$ we have

$$\begin{aligned} G(t,s) &= \frac{t(1-s)^{q-1}}{1-\xi} - \frac{t(\xi-s)^{q-1}}{1-\xi} - (t-s)^{q-1} \\ &\geq t(\zeta - \zeta^{q-1}) \frac{(1-s)^{q-1}}{(1-\xi)} \end{aligned}$$

By using an analogous argument, we can conclude that for all $s, t \in [0,1]$, $G(t,s) \geq t(\zeta - \zeta^{q-1}) \frac{(1-s)^{q-1}}{(1-\xi)}$.
The proof is complete. ∎

Lemma 9 *If $f \in C([0,1], \mathbb{R}_+)$, then, the solution of problem (P_1) satisfies*

$$\min_{t\in[\tau,1]} u(t) \geq \tau(\xi - \xi^{q-1}) \|u\|.$$

Proof. By Lemma 7, u can be expressed by

$$u(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t,s) f(s,u(s))ds \leq \frac{1}{\Gamma(q)} \int_0^1 \frac{(1-s)^{q-1}}{(1-\xi)} f(s,u(s))ds$$

then

$$\begin{aligned} \|u\| &= \max_{0\leq t\leq 1} |u(t)| = \max_{0\leq t\leq 1} \frac{1}{\Gamma(q)} \int_0^1 G(t,s) f(s,u(s))ds \\ &\leq \frac{1}{\Gamma(q)} \int_0^1 \frac{(1-s)^{q-1}}{(1-\xi)} f(s,u(s))ds. \end{aligned}$$

Also, we have

$$\begin{aligned} u(t) &\geq \frac{1}{\Gamma(q)} \frac{\xi - \xi^{q-1}}{(1-\xi)} t \int_0^1 (1-s)^{q-1} f(s,u(s)) ds \\ &\geq t(\xi - \xi^{q-1}) \|u\| \end{aligned}$$

therefore

$$\min_{\tau \leq t \leq 1} u(t) \geq \tau(\xi - \xi^{q-1}) \|u\| .$$

■

Theorem 10 *(Guo-krasnosel'skii fixed point Theorem on cone)[9]. Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let*

$A : K \cap (\overline{\Omega_2}/\Omega_1) \to K$
be a completely continuous operator such that
i)$\|Au\| \leq \|u\|$, $u \in K\cap\ \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in K\cap\ \partial\Omega_2$, or
ii)$\|Au\| \geq \|u\|$, $u \in K\cap\ \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in K\cap\ \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega_2}/\Omega_1)$.

Theorem 11 *(Avery and Peterson fixed point Theorem)[3]. Let P be a cone in a real Banach space E. Let φ and Φ be continuous, nonnegative and convex functionals on P Λ be a continuous nonnegative and concave functional on P and Ψ be continuous and nonnegative functional on P satisfying $\Psi(ku) \leq k\|u\|$ for $0 \leq k \leq 1$. Define the sets*

$P(\varphi, d) = \{u \in P, \varphi(u) < d\}$,
$P(\varphi, \Lambda, b, d) = \{u \in P, b \leq \Lambda(u), \varphi(u) \leq d\}$,
$P(\varphi, \Phi, \Lambda, b, c, d) = \{u \in P, b \leq \Lambda(u), \Phi(u) \leq c, \varphi(u) \leq d\}$,
$R(\varphi, \Psi, a, d) = \{u \in P, a \leq \Psi(u), \varphi(u) \leq d\}$. For M and d positive numbers we have$\Lambda(u) \leq \Psi(u)$and $\|u\| \leq M\varphi(u)$ for any $u \in \overline{P(\varphi, d)}$.

Assume $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$ is a completely continuous and there exist positive numbers a, b and c with $a < b$ such that
$(S_1) : \{u \in P(\varphi, \Phi, \Lambda, b, c, d), \Lambda(u) > b\} \neq \emptyset$ and $\Lambda(Tu) > b$
for $u \in P(\varphi, \Phi, \Lambda, b, c, d)$.
$(S_2) : \Lambda(Tu) > b$ for $u \in P(\varphi, \Lambda, b, d)$ with $\Phi(Tu) > c$.
$(S_3) : 0 \notin R(\varphi, \Psi, a, d)$ and $\Psi(Tu) < a$ for $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = a$.

Then T has at least three positive fixed points $u_1, u_2, u_3 \in \overline{P(\varphi, d)}$ such that $\varphi(u_i) \leq d$, for $i = 1, 2, 3$. $b < \Lambda(u_1), a < \Psi(u_2)$, with $\Lambda(u_2) < b$ and $\Psi(u_3) < a$.

3 Main results

Denote by $E = C([0,1], \mathbb{R})$ the Banach space of all continuous real functions on $[0,1]$ endowed with the norm

$\|u\| = \max_{t \in [0,1]} |u(t)|$ and P be the cone defined by

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$P = \left\{ u \in E, u(t) \geq 0, 0 \leq t \leq 1, \min_{\tau \leq t \leq 1} u(t) \geq \tau(\xi - \xi^{q-1}) \|u\| \right\}.$

Define the integral operator $T : E \to E$ by

$$Tu(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s)f(u(s))ds, \forall t \in [0,1].$$

We define some important constants:

$A_0 = \lim_{u \to 0^+} \frac{f(u)}{u}$, $A_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$. The case $A_0 = 0$ and $A_\infty = \infty$ is called superlinear case and the case $A_0 = \infty$ and $A_\infty = 0$ is called sublinear case.

Theorem 12 *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $a \in C([0,1], \mathbb{R}_+)$ and $\int_0^1 (1-s)^{q-1} a(s)ds \neq 0$,then the problem (P1) has at least one positif solution in the both cases superlinear as well as sublinear.*

Proof. We apply Guo-krasnosel'skii fixed point Theorem on cone.

Let u in P, in view of nonnegativeness and continuity of functions G (t ,s) and f, we conclude that $Tu \geq 0, t \in [0,1]$ and continuous and $T(P) \subset P$.

i)Let $Br = \{u \in P, \|u\| \leq r\}$ be bounded. Since f is continuous, then there exists a constant k such

$\max_{t \in [0,1]} |a(t)f(u(t))| = k$. For any $u \in Br$ and by applying Lemma 8 we obtain

$$|Tu(t)| \leq \frac{k}{\Gamma(q+1)(1-\xi)}$$

hence T is uniformly bounded.

(ii)T is equicontinuous.

Since G (t,s) is continuous on $[0,1] \times [0,1]$, it is uniformly continuous on $[0,1] \times [0,1]$. Thus , for fixed $s \in [0,1]$ and for any $\varepsilon > 0$, there exist a constant $\delta > 0$, such that for any t_1, $t_2 \in [0,1]$, $|t_1 - t_2| < \delta$,we have

$$|G(t_1,s) - G(t_2,s)| \leq \frac{\varepsilon \Gamma(q)}{k},$$

since

$$|Tu(t_1) - Tu(t_2)| \leq \frac{1}{\Gamma(q)} \int_0^1 |G(t_1,s) - G(t_2,s)| \, a(s)f(u(s))ds,$$

we obtain

$$|Tu(t_1) - Tu(t_2)| \leq \varepsilon.$$

Consequently $T(B_r)$ is equicontinuous, by means of the Arzela-Ascoli Theorem we conclude that T is completely continuous.

Now we prove the superlinear case. Since $A_0 = 0$, then for any $A > 0$ there exists $\delta > 0$, such that for any u, $0 < u \leq \delta$, then $f(u) \leq Au$.

Set $\Omega_1 = \{u \in E : \|u\| < \delta\}$. let $u \in P \cap \partial\Omega_1$, then we have

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\leq \frac{A\|u\|}{\Gamma(q)(1-\xi)} \int_0^1 (1-s)^{q-1}a(s)ds \end{aligned}$$

if we choose $A = \frac{\Gamma(q)(1-\xi)}{\int_0^1 (1-s)^{q-1}a(s)ds}$, then

$$\|Tu\| \leq \|u\|.$$

On the other hand since $A_\infty = \infty$, we deduce that for any $\varepsilon > 0$ there exists $\gamma > 0$, such that for any u, $u \geq \gamma$ then $f(u) \geq \varepsilon u$.

Setting $R = \max\left\{2\delta, \frac{1}{\tau(\xi-\xi^{q-1})}\gamma\right\}$ and $\Omega_2 = \{u \in E : \|u\| < R\}$,

then $\overline{\Omega_1} \subset \Omega_2$ and for $u \in P \cap \partial\Omega_2$ we have

$$\begin{aligned} Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s)f(u(s))ds \\ &\geq \frac{\tau^2}{\Gamma(q)}(\xi - \xi^{q-1})^2 \varepsilon \|u\| \int_0^1 (1-s)^{q-1}a(s)ds, \end{aligned}$$

Choosing $\varepsilon = \frac{\Gamma(q)}{\tau^2(\xi-\xi^{q-1})^2 \int_0^1 (1-s)^{q-1}a(s)ds}$ we get $\|Tu\| \geq \|u\|$ for all $u \in P \cap \partial\Omega_2$. The second part of Theorem implies that T has a fixed point in $P \cap (\overline{\Omega_2}/\Omega_1)$, that means that T has at least a positive solution in $P \cap (\overline{\Omega_2}/\Omega_1)$. Arguing as above, we prove the sublinear case. The proof is complete. ■

Let us introduce the following functionals. Define on P, the nonnegative, continuous and concave functional Λ by $\Lambda(u) = \min_{t\in[\tau,1]} u(t)$, then $\Lambda(u) \leq \|u\|$. Define the nonnegative, continuous and convex functionals φ and Φ on P by $\varphi(u) = \Phi(u) = \|u\|$ and the nonnegative continuous functional Ψ on P by $\Psi(u) = \|u\|$, then $\Psi(ku) \leq k\|u\|$ for $0 \leq k \leq 1$.

Theorem 13 *Let $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $a \in (C[0,1], \mathbb{R}_+)$ Suppose there exist positive constants a, b, c, d, μ et L such that $a < b, \mu > \frac{1}{\Gamma(q)(1-\xi)} \int_0^1 (1-s)^{q-1}a(s)ds$,*

$L < \frac{\tau(\xi-\xi^{q-1})}{\Gamma(q)(1-\xi)} \int_0^1 (1-s)^{q-1}a(s)ds$ and
i)$f(u) \leq \frac{d}{\mu}$ for $u \in [0,d]$
ii)$f(u) \geq \frac{b}{L}$ for $u \in [b, \frac{b}{\tau(\xi-\xi^{q-1})}]$
iii)$f(u) \leq \frac{a}{\mu}$ for $u \in [0,a]$.

The Problem (P1) has at least three positive solutions $u_1, u_2, u_3 \in \overline{P(\varphi,d)}$ such that $\varphi(u_i) \leq d$, for $i = 1,2,3$. $b < \Lambda(u_1), a < \Psi(u_2)$, with $\Lambda(u_2) < b$ and $\Psi(u_3) < a$.

Proof. To prove the existence of three positive solutions, we apply Theorem 11. Proceeding analogoustly as in the proof of theorem 12, we prove that the maping T is completely continuous on $\overline{P(\varphi,d)}$.

1)Let $u \in \overline{P(\varphi, d)}$, then $\|u\| \leq d$, we have

$$\begin{aligned}
\varphi(Tu) &= \|Tu\| = \max_{t\in[0,1]} \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s)f(u(s))ds \\
&\leq \frac{1}{\Gamma(q)} \int_0^1 \frac{(1-s)^{q-1}}{(1-\xi)} a(s)f(u(s))ds \\
&\leq \frac{d}{\mu} \frac{1}{\Gamma(q)(1-\xi)} \int_0^1 (1-s)^{q-1} a(s)ds \leq d,
\end{aligned}$$

hence $Tu \in \overline{P(\varphi, d)}$.

2)(S_1) holds i.e. $\left\{u \in P(\varphi, \Phi, \Lambda, b, \frac{b}{\tau(\xi-\xi^{q-1})}, d), \Lambda(u) > b\right\} \neq \emptyset$
and $\Lambda(Tu) > b$ for $u \in P(\varphi, \Phi, \Lambda, b, \frac{b}{\tau(\xi-\xi^{q-1})}, d)$.
We choose $u(t) = \frac{b}{\tau(\xi-\xi^{q-1})}$. It is easy to see that
$u(t) = \frac{b}{\tau(\xi-\xi^{q-1})} \in P(\varphi, \Phi, \Lambda, b, \frac{b}{\tau(\xi-\xi^{q-1})}, d)$,
$\Lambda(u) = \min_{t\in[\tau,1]} u(t) = \frac{b}{\tau(\xi-\xi^{q-1})} > b$,
and so $\left\{u \in P(\varphi, \Phi, \Lambda, b, \frac{b}{\tau(\xi-\xi^{q-1})}, d), \Lambda(u) > b\right\} \neq \emptyset$.
Hence if $u \in P(\varphi, \Phi, \Lambda, b, \frac{b}{\tau(\xi-\xi^{q-1})}, d)$, then $b \leq u(t) \leq \frac{b}{\tau(\xi-\xi^{q-1})}$, and

$$\begin{aligned}
\Lambda(Tu) &= \min_{t\in[\tau,1]} Tu(t) \geq \frac{\tau}{\Gamma(q)} \frac{(\xi-\xi^{q-1})}{(1-\xi)} \int_0^1 (1-s)^{q-1} a(s)f(u(s))ds \\
&\geq \frac{\tau}{\Gamma(q)} \frac{(\xi-\xi^{q-1})}{(1-\xi)} \frac{b}{L} \int_0^1 (1-s)^{q-1} a(s)ds > b,
\end{aligned}$$

This shows that condition (S_1) is satisfied.

3)For $u \in P(\varphi, \Lambda, b, d)$ with $\Phi(Tu) = \|Tu\| > \frac{b}{\tau(\xi-\xi^{q-1})}$, we get

$$\Lambda(Tu) = \min_{t\in[\tau,1]} Tu(t) \geq \tau(\xi - \xi^{q-1}) \|Tu\| > b,$$

so (S_2) holds true.

We finally show that (S_3) is satisfied. Suppose that $u \in R(\varphi, \Psi, a, d)$ then $0 < a < \|u\| \leq d$, then $0 \notin R(\varphi, \Psi, a, d)$.

Let $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = \|u\| = a$, then by using assumption (iii) it yields

$$\begin{aligned}
\Psi(Tu) &= \max_{t\in[0,1]} \frac{1}{\Gamma(q)} \int_0^1 G(t,s)a(s)f(u(s))ds \\
&\leq \frac{1}{\Gamma(q)} \int_0^1 \frac{(1-s)^{q-1}}{(1-\xi)} a(s)f(u(s))ds \\
&\leq \frac{a}{\mu} \frac{1}{\Gamma(q)(1-\xi)} \int_0^1 (1-s)^{q-1} a(s)ds < a,
\end{aligned}$$

then (S_3) holds.

Therefore, an application of Theorem 11 implies that the boundary value problem (P1)has at least three positive solutions u_1, u_2 and $u_3 \in \overline{P(\varphi, d)}$ such that $\varphi(u_i) \leq d$, for $i = 1, 2, 3$. $b < \Lambda(u_1), a < \Psi(u_2)$, with $\Lambda(u_2) < b$ and $\Psi(u_3) < a$.

The proof is complete. ■

Example 14 *Consider the boundary value problem:*

$$\begin{aligned} {}^cD_{0^+}^{\frac{8}{3}}u(t) + (1-t)\frac{u^2}{4} &= 0, 0 < t < 1, \\ u(0) &= u''(0) = 0, u(1) = u(\xi). \end{aligned}$$

$$u(0) = u''(0) = 0, u(1) = u(\xi).$$

We have $\int_0^1 (1-s)^{\frac{8}{3}} ds = \frac{3}{11} \neq 0$, $\lim\limits_{u\to 0} \frac{f(u)}{u} = \lim\limits_{u\to 0} \frac{u^2}{4u} = 0$ and $\lim\limits_{u\to\infty} \frac{f(u)}{u} = \lim\limits_{u\to\infty} \frac{u^2}{4u} = \infty$, then by applying Theorem 12 we deduce that there exist at least one positive solution.

Example 15 *Let us consider the fractional boundary value problem:*

$$\begin{aligned} {}^cD_{0^+}^{\frac{9}{4}}u(t) + a(t)f(u(t)) &= 0, 0 < t < 1, \\ u(0) &= u''(0) = 0, u(1) = u(\xi). \end{aligned}$$

where $a(t) = 1 - t$ and

$$f(u) = \begin{cases} u^2, 0 \leq u \leq 1, \\ 125u - 124, 1 \leq u \leq 2, \\ 126, u \geq 2. \end{cases}$$

It is easy to see that $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $a \in (C[0,1], \mathbb{R}_+)$, Let us check the assumptions of Theorem 13 for $\tau = \frac{11}{12}$ and $\xi = \frac{1}{2}$

$\mu > \frac{1}{\Gamma(\frac{9}{4})(1-\frac{1}{2})} \int_0^1 (1-s)^{\frac{9}{4}} ds = 0.1604$

$L < \frac{\frac{11}{12}(\frac{1}{2} - (\frac{1}{2})^{\frac{5}{4}})}{\Gamma(\frac{9}{4})(1-\frac{1}{2})} \int_0^1 (1-s)\frac{9}{4} ds = 0.0234.$

If we choose $\mu = 1, L = 0.02, a = 1, b = 2, d \geq 126$ and $c = 27.426$, then the assumptions of Theorem 13 are satisfied, hence there exist at least three positive solutions u_1, u_2 and $u_3 \in \overline{P(\varphi, d)}$ such that $\|u_i\| \leq 126$, for $i = 1, 2, 3$. $2 < \min\limits_{t\in[\frac{11}{12},1]} u_1(t), 1 < \|u_2\|$, with $\min\limits_{t\in[\frac{11}{12},1]} u_2(t) < 2$ and $\|u_3\| < 1$.

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Coupled fixed point theorems in cone metric spaces for a general class of $G-$contractions

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Abstract

In this paper, we obtain some coupled fixed point theorems in cone metric spaces by assuming that the cone has nonempty interior as well as employing an hybrid contractive-type condition. Our theorems generalize, extend and improve some recently announced results in the literature. In particular, our results extend the recent results of Akram et al from fixed point setting ($A-$contractions) to the corresponding notion of coupled fixed points ($G-$contractions).

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1. Introduction

Chang and Ma [6] introduced the concept of coupled fixed points. Since then, the notion has been of great interest to many researchers in fixed point theory.

Bhaskar and Lakshmikantham [5] established a coupled fixed point theorem in a metric space endowed with partial order by employing a weak contractive type condition. The result of [5] has also been generalized and extended by Lakshmikantham and Ciric [11]. Huang and Zhang [10] extended the notion of metric spaces by considering vector-valued metrics (that is, cone metrics) with vaues in an ordered real Banach space and then established some fixed point theorems in cone metric spaces. In the recent times, several papers including Sabetghadam et al [19] have been devoted to the study of the concepts of coupled fixed points in cone metric spaces. We refer to [1, 3, 7, 10, 12, 13, 14, 15, 16, 17] and others in the reference section for detail.

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We consider the following definitions which shall be required in the sequel:

Definition 1.1: Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a *cone* in E if it satisfies the following:
(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a,\ b \in \mathbb{R},\ a,\ b \geq 0$ and $x,\ y \in P \implies ax + by \in P$;
(iii) $x \in P$ and $-x \in P \implies x = 0$.
For a given cone $P \subset E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. If $y - x \in$ int P, we write $x \prec\prec y$ (where int P denotes the interior of P). Also, we use $x < y$ if $x \leq y$ and $x \neq y$.

We shall subsequently take E as a real Banach space, P a cone and $\leq$ is the partial ordering with respect to P.

Definition 1.2: Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d\colon X \times X \to E$ satisfies the following conditions:
(i) $0 \preceq d(x, y),\ \forall\ x,\ y \in X$ and $d(x, y) = 0 \iff x = y$;
(ii) $d(x, y) = d(y, x),\ \forall\ x,\ y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(z, y),\ \forall\ x,\ y,\ z \in X$.
Then, d is called a *cone metric* on X, and the pair (X, d) is called a *cone metric space.*

Definition 1.3: Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $T\colon X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.
For the definitions above, we refer to [5, 8, 10, 11, 19].

Definition 1.4 [10, 19]: Let (X, d) be a cone metric space. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x \in X$. Then,
(i) $\{x_n\}_{n=1}^{\infty}$ converges to x, that is, $\lim\limits_{n\to\infty} x_n = x$, if for every $c \in E$ with $0 \prec\prec c$ there exists a natural number N such that $d(x_n, x) \prec\prec c$ for all $n \succeq N$;
(ii) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for every $c \in E$ with $0 \prec\prec c$, there exists a natural number N such that $d(x_n, x_m) \prec\prec c$ for all $n,\ m \succeq N$.
A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point $x \in X$.

Huang and Zhang [10] proved some existence and uniqueness theorems for selfmappings $T\colon X \to X$ using various contractive conditions as well as imposing normality condition on the cone P. However, some of the results of [10] were later improved by Rezapour and Hamlbarani [18] by removing the normality condition on the cone.

In this paper, we assume that E is a real Banach space, $P \subset E$ is a cone with int. $P \neq \phi$ and $\preceq$ is a partial order with respect to P. The relations $P +$ int P $\subseteq$ P and λint P $\subseteq$ int P $(\lambda > 0)$ always hold.

Our purpose in this paper is to obtain some coupled fixed point theorems in cone metric spaces by assuming that the cone has nonempty interior as well as employing an hybrid contractive-type condition. Our theorems generalize, extend and improve some recently announced results in the literature. In particular, our results extend the recent results of Akram et al [2] from fixed point setting ($A-$contractions) to the corresponding notion of coupled fixed points ($G-$contractions).

Definition 1.5 (Akram et al [2]): A selfmap $T\colon X \to X$ of a metric space (X, d) is said to be *A-contraction* if it satisfies the condition:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)),\ \forall\ x,\ y \in X, \tag{A}$$

and some $\alpha \in A$, where A is the set of all functions $\alpha\colon \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying
(i) α is continuous on the set $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3$);
(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$, or $a \leq \alpha(b, a, b)$, or $a \leq \alpha(b, b, a),\ \forall\ a,\ b \in \mathbb{R}_+$.

Following the concept of Akram et al [2] for fixed point theorems, we extend the notion to coupled fixed point consideration by obtaining new class of operators more general than those studied by [5, 11, 19].

Definition 1.6: A mapping $T\colon X \times X \to X$ of a cone metric space (X, d) is said to be a $G-$*contraction* if it satisfies the condition:

$$d(T(x, y), T(u, v)) \preceq \alpha(d(x, u), d(y, v), d(x, T(x, y)), d(u, T(u, v)), \tag{G}$$

$\forall\ x,\ y,\ u,\ v \in X$ and some $\alpha \in G$, where G is the set of all functions $\alpha\colon E^4 \to E$ satisfying:
(i) α is continuous on the set E^4 (with respect to the $E-$metric);
(ii) there exists some $k \in [0, 1)$ such that $a \preceq kb$ whenever $a \preceq \alpha(b, c, b, a)$, or, $a \preceq \alpha(b, b, c, a)$, or, $a \preceq \alpha(a, c, b, b),\ \forall\ a,\ b,\ c \in E$.

2. Main results

Theorem 2.1: Let (X, d) be a complete cone metric space with nonempty interior and let $T\colon X \times X \to X$ be a mapping satisfying condition (G), for each $x,\ y,\ u,\ v \in X$. Then, T has a unique coupled fixed point.

Proof: Choose $(x_0, y_0) \in X \times X$ and set $x_1 = T(x_0, y_0),\ y_1 = T(y_0, x_0)$, and in general, $x_{n+1} = T(x_n, y_n),\quad y_{n+1} = T(y_n, x_n)$.
Therefore, we have by condition (G) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\preceq \alpha(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(x_{n-1}, T(x_{n-1}, y_{n-1})), d(x_n, T(x_n, y_n))) \\ &= \alpha(d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})) \\ &\preceq kd(x_{n-1}, x_n). \end{aligned} \tag{1}$$

Similarly, using condition (G) again yields

$$\begin{aligned} d(y_n, y_{n+1}) &= d(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\preceq \alpha(d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(y_{n-1}, T(y_{n-1}, x_{n-1})), d(y_n, T(y_n, x_n))) \\ &= \alpha(d(y_{n-1}, y_n), d(x_{n-1}, x_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})) \\ &\preceq k d(y_{n-1}, y_n). \end{aligned} \tag{2}$$

Let $q_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$.
Then, we have from (1) and (2) that

$$q_n \preceq k q_{n-1}. \tag{3}$$

Thus, we have from (3) that

$$0 \preceq q_n \preceq k q_{n-1} \preceq k^2 q_{n-2} \preceq \cdots \preceq k^n q_0. \tag{4}$$

If $q_0 = 0$, then (x_0, y_0) is a coupled fixed point of T.
Suppose that $q_0 \succ 0$. Then, for each $r \in \mathbb{N}$, we obtain by (4) and the repeated application of triangle inequality that

$$\begin{aligned} d(x_n, x_{n+r}) + d(y_n, y_{n+r}) &\preceq [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] \\ &\quad + \cdots + [d(x_{n+r-1}, x_{n+r}) + d(y_{n+r-1}, y_{n+r})] \\ &\preceq q_n + q_{n+1} + \cdots + q_{n+r-1} \\ &\preceq \tfrac{k^n(1-k^r)q_0}{1-k} \preceq \tfrac{k^n q_0}{1-k} \to \ 0 \text{ as n} \to \infty. \end{aligned} \tag{5}$$

It follows from (5) that for $c \in E,\ 0 \prec\prec c$ and large n, we have $\frac{k^n q_0}{1-k} \prec\prec c$, thus $d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \preceq c$.
Therefore, $\{x_n\},\ \{y_n\}$ are Cauchy sequences in (X, d).
Now, since $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} y_n = y^*$, for $c \in E,\ 0 \prec\prec c$, there exists $N \in \mathbb{N}$ such that

$$d(x_N, x^*) \prec\prec \frac{c}{2} \text{ and } d(y_N, y^*) \prec\prec \frac{c}{2}, \tag{6}$$

for all $n \geq N$.
By condition (G) again, we get

$$\begin{aligned} d(x^*, T(x^*, y^*)) &\preceq d(x^*, x_{N+1}) + d(x_{N+1}, T(x^*, y^*)) \\ &= d(x^*, x_{N+1}) + d(T(x_N, y_N), T(x^*, y^*)) \\ &\preceq d(x^*, x_{N+1}) + \alpha\left(d(x_N, x^*), d(y_N, y^*), d(x_N, T(x_N, y_N)), d(x^*, T(x^*, y^*))\right) \\ &= d(x^*, x_{N+1}) + \alpha\left(d(x_N, x^*), d(y_N, y^*), d(x_N, x_{N+1}), d(x^*, T(x^*, y^*))\right). \end{aligned} \tag{7}$$

Also, by the first part of (6),

$$d(x_N, x_{N+1}) \preceq d(x_N, x^*) + d(x^*, x_{N+1}) \prec\prec \frac{c}{2} + \frac{c}{2} = c. \tag{8}$$

Using (6) and (8) in (7), we obtain

$$d(x^*, T(x^*, y^*)) \prec\prec \frac{c}{2} + \alpha\left(\frac{c}{2}, \frac{c}{2}, c, d(x^*, T(x^*, y^*))\right) \preceq \frac{c}{2} + \frac{kc}{2} \preceq \frac{c}{2} + \frac{c}{2} = c,$$

from which it follows that $d(x^*, T(x^*, y^*)) = 0$, that is, $T(x^*, y^*) = x^*$.
Similarly, using condition (G) again gives

$$\begin{aligned} d(y^*, T(y^*, x^*)) &\preceq d(y^*, y_{N+1}) + d(y_{N+1}, T(y^*, x^*)) \\ &= d(y^*, y_{N+1}) + d(T(y_N, x_N), T(y^*, x^*)) \\ &\preceq d(y^*, y_{N+1}) + \alpha(d(y_N, y^*), d(x_N, x^*), d(y_N, T(y_N, x_N)), d(y^*, T(y^*, x^*))) \\ &= d(y^*, y_{N+1}) + \alpha(d(y_N, y^*), d(x_N, x^*), d(y_N, y_{N+1}), d(y^*, T(y^*, x^*))). \quad (9) \end{aligned}$$

Also, by the second part of (6),

$$d(y_N, y_{N+1}) \preceq d(y_N, y^*) + d(y^*, y_{N+1}) \prec\prec \frac{c}{2} + \frac{c}{2} = c. \qquad (10)$$

Using (6) and (10) in (9), we have

$$d(y^*, T(y^*, x^*)) \prec\prec \frac{c}{2} + \alpha\left(\frac{c}{2}, \frac{c}{2}, c, d(y^*, T(y^*, x^*))\right) \preceq \frac{c}{2} + \frac{kc}{2} \preceq \frac{c}{2} + \frac{c}{2} = c,$$

from which it follows that $d(y^*, T(y^*, x^*)) = 0$, that is, $T(y^*, x^*) = y^*$.
Therefore, (x^*, y^*) is a coupled fixed point of T.
We now prove the uniqueness of coupled fixed point of T : Suppose that (x^*, y^*) and (x', y') are two coupled fixed points of T. Then, by condition (G), we obtain

$$\begin{aligned} d(x^*, x') &= d(T(x^*, y^*), T(x', y')) \\ &\preceq \alpha(d(x^*, x'), d(y^*, y'), d(x^*, T(x^*, y^*), d(x', T(x', y')) \\ &= \alpha(d(x^*, x'), d(y^*, y'), 0, 0) \preceq k\ .\ 0 = 0, \end{aligned}$$

which gives $d(x^*, x') = 0 \iff x^* = x'$.
Again, by condition (G), we have

$$\begin{aligned} d(y^*, y') &= d(T(y^*, x^*), T(y', x')) \\ &\preceq \alpha(d(y^*, y'), d(x^*, x'), d(y^*, T(y^*, x^*), d(y', T(y', x')) \\ &= \alpha(d(y^*, y'), d(x^*, x'), 0, 0) \preceq k\ .\ 0 = 0, \end{aligned}$$

which gives $d(y^*, y') = 0 \iff y^* = y'$.
Therefore, $d(x', x^*) = d(y', y^*)$, proving the uniqueness of the coupled fixed point of T.

Theorem 2.2: Let (X, d) be a complete cone metric space with nonempty interior and $\{T_n\}_{n=1}^{\infty}$ defined by $T_n\colon X \times X \to X$ $(n = 1, 2, \cdots)$, be a sequence of mappings satisfying the contractivity condition

$$d(T_i(x, y), T_j(u, v)) \preceq \alpha(d(x, u), d(y, v), d(x, T_i(x, y)), d(u, T_j(u, v)),\ i,\ j \in \mathbb{N}, \quad (G')$$

$\forall\ x,\ y,\ u,\ v \in X$ and some $\alpha \in G$, for each $x,\ y,\ u,\ v \in X$. Then, the members of $\{T_n\}_{n=1}^{\infty}$ have a unique common coupled fixed point.

Proof: Choose $(x_0, y_0) \in X \times X$ and define $\{x_n\},\ \{y_n\}$ by

$$x_{n+1} = T_{n+1}(x_n, y_n), \quad y_{n+1} = T_{n+1}(y_n, x_n).$$

Therefore, we have by condition (G') that

$$\begin{aligned} d(x_1, x_2) &= d(T_1(x_0, y_0), T_2(x_1, y_1)) \\ &\preceq \alpha(d(x_0, x_1), d(y_0, y_1), d(x_0, T_1(x_0, y_0)), d(x_1, T_2(x_1, y_1))) \\ &= \alpha(d(x_0, x_1), d(y_0, y_1), d(x_0, x_1), d(x_1, x_2)) \\ &\preceq kd(x_0, x_1). \end{aligned} \tag{11}$$

Also,

$$\begin{aligned} d(x_2, x_3) &= d(T_2(x_1, y_1), T_3(x_2, y_2)) \\ &\preceq \alpha(d(x_1, x_2), d(y_1, y_2), d(x_1, T_2(x_1, y_1)), d(x_2, T_3(x_2, y_2))) \\ &= \alpha(d(x_1, x_2), d(y_1, y_2), d(x_1, x_2), d(x_2, x_3)) \\ &\preceq kd(x_1, x_2). \end{aligned} \tag{12}$$

In general, we have from above that

$$d(x_n, x_{n+1}) \preceq kd(x_{n-1}, x_n). \tag{13}$$

Again, by condition (G'), we have

$$\begin{aligned} d(y_1, y_2) &= d(T_1(y_0, x_0), T_2(y_1, x_1)) \\ &\preceq \alpha(d(y_0, y_1), d(x_0, x_1), d(y_0, T_1(y_0, x_0)), d(y_1, T_2(y_1, x_1))) \\ &= \alpha(d(y_0, y_1), d(x_0, x_1), d(y_0, y_1), d(y_1, y_2)) \\ &\preceq kd(y_0, y_1). \end{aligned} \tag{14}$$

Also,

$$d(y_2, y_3) \preceq kd(y_1, y_2). \tag{15}$$

Similarly, we obtain, in general that

$$d(y_n, y_{n+1}) \preceq kd(y_{n-1}, y_n). \tag{16}$$

Again, By letting $q_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ as in Theorem 2.1, we obtain from (13) and (16) that

$$0 \preceq q_n \preceq kq_{n-1} \preceq k^2 q_{n-2} \preceq \cdots \preceq k^n q_0,$$

from which it follows that for $c \in E,\ 0 \prec\prec c$ and large n, we have as in Theorem 2.1 that $d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \preceq c$. Thus, showing that $\{x_n\},\ \{y_n\}$ are Cauchy

sequences in (X, d).
Now, since $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$, for $c \in E,\ 0 \prec\prec c$, there exists $N \in \mathbb{N}$ such that

$$d(x_N, x^*) \prec\prec \frac{c}{2} \text{ and } d(y_N, y^*) \prec\prec \frac{c}{2}, \tag{$\star$}$$

for all $n \geq N$.
By condition (G') again, we get

$$\begin{aligned} d(x^*, T_n(x^*, y^*)) &\preceq d(x^*, x_{s+1}) + d(x_{s+1}, T_n(x^*, y^*)) \\ &= d(x^*, x_{s+1}) + d(T_{s+1}(x_s, y_s), T_n(x^*, y^*)) \\ &\preceq d(x^*, x_{s+1}) + \alpha\left(d(x_s, x^*), d(y_s, y^*), d(x_s, T_{s+1}(x_s, y_s)), d(x^*, T_n(x^*, y^*))\right) \\ &= d(x^*, x_{N+1}) + \alpha\left(d(x_s, x^*), d(y_s, y^*), d(x_s, x_{s+1}), d(x^*, T_n(x^*, y^*))\right). \quad (17) \end{aligned}$$

Also, by the first part of $(\star)$,

$$d(x_s, x_{s+1}) \preceq d(x_s, x^*) + d(x^*, x_{s+1}) \prec\prec \frac{c}{2} + \frac{c}{2} = c. \tag{18}$$

Using $(\star)$ and (18) in (17), we obtain

$$d(x^*, T_n(x^*, y^*)) \prec\prec \frac{c}{2} + \alpha\left(\frac{c}{2}, \frac{c}{2}, c, d(x^*, T_n(x^*, y^*))\right) \preceq \frac{c}{2} + \frac{kc}{2} \preceq \frac{c}{2} + \frac{c}{2} = c,$$

from which it follows that $d(x^*, T_n(x^*, y^*)) = 0$, that is, $T_n(x^*, y^*) = x^*$.
Similarly, using condition (G') again and obtaining similar inequalities as above as well as using $(\star)$ yield

$$d(y^*, T_n(y^*, x^*)) \prec\prec \frac{c}{2} + \alpha\left(\frac{c}{2}, \frac{c}{2}, c, d(y^*, T_n(y^*, x^*))\right) \preceq \frac{c}{2} + \frac{kc}{2} \preceq \frac{c}{2} + \frac{c}{2} = c,$$

from which it follows that $d(y^*, T_n(y^*, x^*)) = 0$, that is, $T_n(y^*, x^*) = y^*$.
Therefore, (x^*, y^*) is a common coupled fixed point of $\{T_n\}_{n\in\mathbb{N}}$.
We now prove the uniqueness of common coupled fixed point of T_n : Suppose that (x^*, y^*) and (x', y') are two common coupled fixed points of $\{T_n\}$, that is, $T_i(x^*, y^*) = x^*,\ T_i(y^*, x^*) = y^*,$ and $T_j(x', y') = x',\ T_j(y', x') = y'$. Then, by condition (G'), we obtain

$$\begin{aligned} d(x^*, x') &= d(T_i(x^*, y^*), T_j(x', y')) \\ &\preceq \alpha(d(x^*, x'), d(y^*, y'), d(x^*, T_i(x^*, y^*), d(x', T_j(x', y')) \\ &= \alpha(d(x^*, x'), d(y^*, y'), 0, 0) \preceq k\ .\ 0 = 0, \end{aligned}$$

which gives $d(x^*, x') = 0 \iff x^* = x'$.
Again, by condition (G'), we obtain $d(y^*, y') = 0 \iff y^* = y'$.
Therefore, $d(x', x^*) = d(y', y^*)$, proving the uniqueness of the common coupled fixed point of $\{T_n\}_{n\in\mathbb{N}}$.

Remark 2.1: Theorem 2.1 and Theorem 2.2 generalize, extend and improve all the results of Sabetghadam et al [19].

Example 2.1: Consider the mapping $T: X \times X \to X$ satisfying condition (G) with $\alpha(d(x,u), d(y,v), d(x,T(x,y)), d(u,T(u,v)) = kd(x,u) + \mu d(y,v),\ x,\ y,\ u,\ v \in X,$ $k \geq 0,\ \mu \geq 0,\ k+\mu < 1.$
That is, we have

$$d(T(x,y), T(u,v)) \preceq kd(x,u) + \mu d(y,v),\ x,\ y,\ u,\ v \in X, \tag{19}$$

$k \geq 0,\ \mu \geq 0,\ k+\mu < 1.$
Condition (19) above is condition (2.1) of Sabetghadam et al [19] (see Theorem 2.2 of that paper). Condition (19) has just been recently nomenclated as $(k,\mu)-$contraction condition in Olatinwo [16].
Let $E = \mathbb{R}^+,\ X = C[a,b]$:=Space of real-valued continuous functions on $[a,b]$. Define $T: X \times X \to X$ by

$$T(f,h)(t) = \phi(t) + \lambda \int_a^t f(s)ds + \psi(t) - \gamma \int_a^t h(s)ds,$$

where $f,\ h \in C[a,b],\ \lambda,\ \gamma \in \mathbb{R},\ \phi$ and ψ are continuous functions on a subset of $\mathbb{R}$.
We claim that T satisfies (19) with $k = |\lambda|(b-a)$ and $\mu = |\gamma|(b-a),\ k+\mu < 1.$
Let $d(x,y) = \max\{|x(s) - y(s)|\ :\ x,\ y \in C[a,b],\ s \in [a,b]\}.$

$$\begin{aligned} |T(f,h)(t) - T(u,v)(t)| &= |\lambda \textstyle\int_a^t (f(s) - u(s))ds + \gamma \int_a^t (v(s) - h(s))ds| \\ &\leq |\lambda| \textstyle\int_a^t |f(s) - u(s)|ds + |\gamma| \int_a^t |h(s) - v(s)|ds. \end{aligned}$$

Therefore,

$$\max_{t\in[a,b]}|T(f,h)(t) - T(u,v)(t)| \leq |\lambda| \int_a^t \max_{s\in[a,b]}|f(s) - u(s)|ds + |\gamma| \int_a^t \max_{s\in[a,b]}|h(s) - v(s)|ds,$$

from which it follows that

$$d(T(f,h)(t), T(u,v)(t)) \leq |\lambda|(b-a)d(f,u) + |\gamma|(b-a)d(h,v). \tag{20}$$

In (20), by letting $k = |\lambda|(b-a)$ and $\mu = |\gamma|(b-a),\ k+\mu < 1$, then we have that the mapping T satisfies the $(k,\mu)-$contraction condition (19).

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NON-LINEAR SYMMETRIC POSITIVE SYSTEMS

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Abstract. Given a symmetric positive operator L and a non-linear maximal operator G, for h in $L^2(\mathbb{R}^n)$, it is shown that there is a unique strong solution u in $L^2(\mathbb{R}^n)$ such that $Lu + Gu = h$ under the semi-admissible boundary condition $\beta_- u = 0$.

Key words and phrases: Symmetric positive operator, Symmetric positive systems, Maximal monotone operators, Weak solutions, Strong solutions.

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1. Introduction

The existence and uniqueness Theorems of weak and strong solutions for symmetric positive systems has been studied in [4].

In [5], the theory of symmetric positive systems has been used to prove the uniqueness and existence for a special ODE, and in [3] the author gives conditions to show that a strong solution of a symmetric positive system as defined in [4] belongs to H_s.

Recently in [6], it is shown that the self-adjoint Neumann problem, the non self-adjoint Neumann problem and the non self-adjoint Dirichlet problem have a unique strong solution by using the theory of symmetric positive systems.

In this paper we study the non-linear perturbations to symmetric positive systems via the theory of monotone maximal operators. That is, we show that there is a strong solution u in $L^2(\mathbb{R}^n)$ under the sum of two operators L and G, where L is a symmetric positive operator and G is not necessarily linear. That is, for a given h in $L^2(\mathbb{R}^n)$, we give conditions to show that there is a unique strong solution u in $L^2(\mathbb{R}^n)$ so that $Lu + Gu = h$, under the semi-admissible condition $\beta_- u = 0$.

Most of the theory given in this paper is based on [4] and [6]. So, for the reader's convenience, we summarize this theory in the following section.

2. Notations and Definitions

Consider a bounded region Ω in $\mathbb{R}^m$. Let k be a positive integer, and for each $\rho = 1, 2, \ldots, m$, each $\lambda = 1, 2, \ldots, k$, and each $\nu = 1, 2, \ldots, k$, let $\lambda^\rho_{\lambda\nu} : \overline{\Omega} \to \mathbb{R}$ be a function of class C_1 in $\overline{\Omega}$. Also, for each $\lambda = 1, 2, \ldots, k$, and each $\nu = 1, 2, \ldots, k$, let $\gamma^\rho_{\lambda\nu} : \Omega \to \mathbb{R}$ be a continuous function on Ω.

Note 1. *We will consider the following convention: Every time and index is repeated, it will be understood that we will add over this index.*

Definition 1. *For any function $u : \Omega \to \mathbb{R}^k$ of class C_1, define the differential operator L as*

$$Lu = 2\alpha^\rho \frac{\partial u}{\partial x_\rho} + \gamma u, \tag{1}$$

where α^ρ and γ are the $k \times k$ matrices defined by: $\quad \alpha^\rho = (\alpha^\rho_{\lambda\nu}) \quad$ *and* $\quad \gamma = (\gamma_{\lambda\nu})$.

Definition 2. *The differential operator L defined in (1) is said to be symmetric positive if*

1) The matrix α^ρ is symmetric for any $\rho = 1, 2, \ldots, m$.

2) The symmetric part of the matrix $\xi = \gamma - \dfrac{\partial \alpha^\rho}{\partial x_\rho}$ is positive definite.

Definition 3. *Consider a function $f : \Omega \to \mathbb{R}^k$ such that $f \in L^2(\mathbb{R})$. The system*

$$\begin{cases} Lu(x) = f(x) \quad \text{if} \quad x \in \Omega \\ \beta_-(x)u(x) = 0 \quad \text{if} \quad x \in \partial\Omega, \end{cases} \tag{2}$$

is said to be symmetric positive if

1) The differential operator L is symmetric positive.

2) For each $x \in \partial\Omega$ there are two $k \times k$ matrices $\beta_+(x)$ and $\beta_-(x)$ such that the matrix $\beta(x) \equiv \eta_\rho(x)\alpha^\rho(x)$ can be written as $\beta(x) = \beta_+(x) + \beta_-(x)$, and where the symmetric part of *the matrix* $\beta_+(x) - \beta_-(x)$ *is non-negative, where* $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ *is the unit normal vector in each point of* $\partial\Omega$.

In this case the boundary condition $\beta_-(x)u(x) = 0$ *is called semi-admissible for the differential operator* L.

Moreover, if the matrices $\beta_+(x)$ *and* $\beta_-(x)$ *satisfy that for each function* $u : \Omega \to \mathbb{R}^k$ *there are two functions* $u_+ : \Omega \to \mathbb{R}^k$ *and* $u_- : \Omega \to \mathbb{R}^k$ *so that* $u(x) = u_+(x) + u_-(x)$ *and* $\beta_+(x)u_-(x) = 0 = \beta_-(x)u_+(x)$*, then the condition* $\beta_-(x)u(x) = 0$ *is called an admissible boundary condition for the differential operator* L.

Definition 4. *Consider a function* $f : \Omega \to \mathbb{R}^k$ *such that* $f \in L^2(\mathbb{R})$. *We say that the function* $u : \Omega \to \mathbb{R}^k$ *where* $u \in L^2(\mathbb{R})$ *is a weak solution of the differential equation* $Lu = f$ *under the semi-admissible boundary condition* $\beta_- u = 0$*, if for each function* $v \in C^1(\overline{\Omega})$*, where* $\beta_+^t v = 0$ *we have*

$$\int_\Omega u \cdot (L^* v) = \int_\Omega f \cdot v$$

In this case, β_+^t *is the transpose of* β_+*, and where* L^* *is the adjoint of* L *given by:*

$$L^* = -2\alpha^\rho \frac{\partial}{\partial x_\rho} - \frac{\partial \alpha^\rho}{\partial x_\rho} + \xi^t.$$

The following two Theorems related with existence and uniqueness for weak solutions are given in [4].

Theorem 1. *(Existence) If the system (2) is symmetric positive, then for any* $f \in L^2(\Omega)$*, this system has a weak solution*

Theorem 2. *(Uniqueness) If the system (2) is symmetric positive, then for any* $f \in L^2(\Omega)$*, this system has at most one weak solution in* $C_1(\Omega)$.

Note 2. *The proofs of the last two Theorems are based on the following classical inequality: There is* $C > 0$ *such that if* u *is in the domain of* L *and* $\beta_- u = 0$*, then*

$$(Lu, u) \geq C\|u\|^2 \tag{3}$$

Definition 5. *We say that $u \in L^2(\Omega)$ is a strong solution for $Lu = f$, where f is in $L^2(\Omega)$ if there is a sequence of functions $u^\nu \in C^1$ where $\|u^\nu - u\|_{L^2} \to 0$ as $\nu \to \infty$ such that $\|Lu^\nu - f\|_{L^2} \to 0$ as $\nu \to \infty$ and $\beta_- u^\nu = 0$.*

In this case, we will assume that Ω is a region whose boundary is a manifold with edges. That is, we will suppose that there are set $U_1, U_2, \ldots, U_n$ in $\mathbb{R}^m$ such that

$$\Omega = \bigcup_{i=1}^{n} U_i, \quad \overline{\Omega} = \bigcup_{i=1}^{n} \overline{U_i},$$

and each U_i satisfies one of the following properties:

1) $\overline{U_i} \subset \Omega$

2) $\overline{U_i} \cap \partial\Omega \neq \emptyset$, and there is a diffeomorphism

$$\phi_i : \overline{U_i} \to \left\{ (x_1, x_2, \ldots, x_m) \mid x_1^2 + x_2^2 + \cdots + x_m^2 \leq 1 \quad \text{and} \quad x_m \leq 0 \right\}$$

such that $\phi_i(z) = (x_1, x_2, \ldots, x_{m-1}, 0)$ if $z \in \partial\Omega$

3) $\overline{U_i} \cap \partial\Omega \neq \emptyset$, and there is a diffeomorphism

$$\phi_i : \overline{U_i} \to \left\{ (x_1, \ldots, x_m) \mid x_1^2 + \cdots + x_m^2 \leq 1 \quad \text{and} \quad x_1 \leq 0, \ldots, x_m \leq 0 \right\},$$

and either

$\phi_i(z) = (x_1, x_2, \ldots, x_{m-1}, 0)$, or $\phi_i(z) = (x_1, x_2, \ldots, 0, x_m)$, $\cdots$, or $\phi_i(z) = (0, x_2, \ldots, x_m)$ if $z \in \partial\Omega$

The sets U_i will be called patches, and if U_i satisfies 1), the set U_i will be called an interior patch.

Then we have the following Theorem.

Theorem 3. *For each $f \in L^2(\Omega)$ the system (2) has a strong solution if the boundary of Ω is a manifold with edges and for each non interior patch there is a set of operators of first order D_σ, where $\sigma = 0, 1, \ldots, m$, with*

$$D_\sigma = d_\sigma^\tau \frac{\partial}{\partial x_\tau} + d_\sigma, \quad \tau = 1, 2, \ldots, m,$$

where the numbers d_σ^τ and the matrices d_σ are functions in C^1 that satisfy the following conditions:

1) If $x \in \partial\Omega$, then $d_\sigma^\tau(x)\eta_\tau(x) = 0$.

2) Each operator $d^\tau \frac{\partial}{\partial x_\tau} + d$ is a linear combination of the operators D_σ with C^0 coefficients and where d^τ and d are in C^0, so that $d^m = 0$ in $\partial\Omega$

3) $D_\sigma L - LD_\sigma$ is a linear combination of the operators D_τ and L. That is, $D_\sigma L - LD_\sigma = p_\sigma^\tau D_\tau + t_\sigma L$, where p_σ^τ are matrices in C^0 and t_τ are matrices in C^1.

4) $D_\sigma\beta_- - \beta_- D_\sigma$ is a linear combination of the operators D_τ and β_-. That is,

$$D_\sigma\beta_- - \beta_- D_\sigma = q_\sigma^\tau D_\tau + t_\sigma^{\partial\Omega}\beta_- \ ,$$

where

$$t_\sigma^{\partial\Omega} = t_\sigma + \frac{\partial}{\partial x^m} d_\sigma^m$$

5) $\underset{1}{\nu} + \underset{1}{\nu}' \geq 0$, where $\underset{1}{\nu}\underset{1}{u} = \{\nu u_\sigma + q_\sigma^\tau u_\tau\}$ for a given compose system $\underset{1}{\nu} = \{u_\sigma\}$.

Proof. See [4]. □

Remark 1. *The existence of the tangential operators D_σ, where $\sigma = 0, 1, \ldots, m$ exist when there exist matrices σ_λ and τ_λ in C^0 such that*

$$\frac{\partial\alpha^m}{\partial x_\lambda} = \tau_\lambda\alpha^m + \alpha^m\sigma_\lambda, \quad \lambda = 1, 2, \ldots, m-1$$

Remark 2. *We have the following choices for σ_λ and τ_λ:*

1) If the matrix α^m is non-singular, we can take

$$\sigma_\lambda = (\alpha^m)^{-1}\frac{\partial\alpha^m}{\partial x_\lambda}, \quad and \quad \tau_\lambda = 0$$

2) If the matrix α^m is singular, then we ask for the matrices α^m in different points of Ω to be similar. That is, we need the existence of a non-singular matrix $W(x)$ in C^1 such that $\alpha^m(x) = W(x)\alpha^m(x_0)W^t(x)$ so that we can take

$$\sigma_\lambda = \tau_\lambda^t, \quad \textit{and} \quad \tau_\lambda = \frac{\partial W}{\partial x_\lambda} W^{-1}$$

3) In the case that α^m is a constant matrix, we can take $\sigma_\lambda = 0 = \tau_\lambda$

Theorem 4. *If $\beta_- u = 0$ is a semi-admissible boundary condition to the system (2), then each strong solution to (2) is unique.*

Proof. See [6]. □

3. Maximal monotone operators

In this section, we will give the definition of maximal monotone operators and we will see some of its properties. This theory is based in [2].

Let H be a Hilbert space over $\mathbb{R}$, and let $P(H)$ be the set of parts of H. Then we have the following definitions.

Definition 6. *Let $A : H \to P(H)$ be a multi-valued operator.*

1) The domain of A is defined as the set $D(A) = \{x \in H \mid Ax \neq \emptyset\}$

2) The image of A is the set

$$R(A) = \bigcup_{x \in H} Ax$$

3) Let $A : H \to P(H)$ and $B : H \to P(H)$ be two multi-valued operators, and let α, β in $\mathbb{R}$. Then $\alpha A + \beta B : H \to P(H)$ is the multi-valued operator defined as the set

$$(\alpha A + \beta B)(x) = \{\alpha u + \beta v \mid u \in Ax \quad \textit{and} \quad v \in Bx\}, \textit{ and } D(\alpha A + \beta B) = D(A) \cap D(B)$$

4) The graph of A is defined as the set $G(A) = \{(x, y) \in H \times H \mid y \in Ax\}$

5) We say that $x \in A^{-1}y$ if and only if $y \in Ax$, and $D(A^{-1}) = R(A)$

Definition 7. *A multi-valued operator $A : H \to P(H)$ is said to be monotone if for any x, y in $D(A)$, we have $\langle Ax - Ay, x - y \rangle \geq 0$.*

Definition 8. *We say that $A : H \to P(H)$ is a maximal monotone operator if $G(A)$ is not a proper subset of the graphs of some monotone operator.*

Definition 9. *We say that the operator $A : H \to P(H)$ is hemi-continuous if for any x in H and any ξ in H, we have $A((1-t)x + t\xi) \to A(x) \quad as \quad t \to 0$*

Proposition 1. *Let $A : H \to P(H)$ be a maximal monotone operator. If there is x_0 in H such that*

$$\lim_{|x|\to\infty} \frac{\langle A^o x, x - x_0\rangle}{|x|} = +\infty$$

for x in $D(A)$, then A is onto. In this case, $A^o x = Proy_{Ax} 0$.

Proof. See ([2], Corollary 2.4). □

Proposition 2. *Let A be a monotone univalent operator where $D(A) = H$. If A is hemi-continuous, then A is a maximal monotone operator.*

Proof. See ([2], Proposition 2.4). □

Proposition 3. *Let A and B be two maximal monotone operators. If $IntD(A) \cap D(B) \neq \emptyset$, then $A + B$ is a maximal monotone operator and $\overline{D(A) \cap D(B)} = \overline{D(A)} \cap \overline{D(B)}$.*

Proof. See ([2], Corollary 2.7). □

4. Main results

In this section we will show the existence and uniqueness of strong solutions under the operator $L+G$ satisfying some semi-admissible conditions. In this case, L is a symmetric positive operator and G is a maximal monotone operator. First we have the following result:

Lemma 1. *Let $g : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous function such that $|g(x)| \leq M|x| + N$, where M and N are positive constants. For each $u : \Omega \to \mathbb{R}^k$ with u in $L^2(\Omega)$, we define $G(u)(x) = g(u(x))$. If $m(\Omega) < \infty$ where Ω is a bounded region in $\mathbb{R}^k$, then the operator $G : D(G) \to L^2(\Omega)$ is hemi-continuous and $D(G) = L^2(\Omega)$.*

Proof. Note that $G(u)$ is measurable since u is measurable and g is continuous.

Let us proof first that $D(G) = L^2(\Omega)$. Consider then u in $L^2(\Omega)$. Thus,

$$\int_\Omega |g(u(x))|^2 dx \leq \int_\Omega (M|u(x)| + N)^2 dx = \int_\Omega (M^2|u(x)|^2 + 2MN|u(x)| + N^2) dx$$

$$\leq M^2 \int_\Omega |u(x)|^2 dx + 2MN \left(\int_\Omega |u(x)|^2 dx \right)^{\frac{1}{2}} (m(\Omega))^{\frac{1}{2}} + N^2 m(\Omega) < \infty.$$

This means that $g(u(x))$ is in $L^2(\Omega)$. Hence $D(G) = L^2(\Omega)$.

Now, let us prove that G is hemi-continuous. So, consider u_1, u_2 in $L^2(\Omega)$, and let $\{t_n\}$ be a sequence in $[0,1]$ such that $t_n \to \overline{t}$ as $n \to \infty$. Then

$$\begin{aligned}
&\Big| g(t_n u_1(\xi) + (1-t_n)u_2(\xi)) - g(\overline{t}u_1(\xi) + (1-\overline{t})u_2(\xi)) \Big|^2 \\
&\leq \big[(M|t_n u_1(\xi) + (1-t_n)u_2(\xi)| + N) + \big(M|\overline{t}u_1(\xi) + (1-\overline{t})u_2(\xi)| + N \big) \big]^2 \\
&\leq \big[M|t_n u_1(\xi)| + M|(1-t_n)u_2(\xi)| + M|\overline{t}u_1(\xi)| + M|(1-\overline{t})u_2(\xi)| + 2N \big]^2 \\
&\leq [M|u_1(\xi)| + M|u_2(\xi)| + M|u_1(\xi)| + M|u_2(\xi)| + 2N]^2 \\
&= [2M|u_1(\xi)| + 2M|u_2(\xi)| + 2N]^2 .
\end{aligned} \tag{4}$$

If we set $|S_n|^2 = \Big[g(t_n u_1(\xi) + (1-t_n)u_2(\xi)) - g(\overline{t}u_1(\xi) + (1-\overline{t})u_2(\xi)) \Big]^2$, then since g is a continuous function, it follows that $S_n \to 0$ as $n \to \infty$ pointwise. Then by the last row in (4) and the dominated convergence Theorem, $\|S_n\|^2 \to 0$ as $n \to \infty$. Then G is hemi-continuous.

This proves Lemma 1. □

Corollary 1. *Let g and G be as in Lemma 1. If moreover g is a non-decreasing function, then $G : L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator.*

Proof. From Lemma 1, the operator G is hemicontinous and $D(G) = L^2(\Omega)$. Now, since g is a non-decreasing function, it follows that G is monotone. Hence, G is univalent. Thus, from Proposition 2, G is maximal monotone.

This proves Corollary 1. □

Theorem 5. *Let $f : \Omega \to \mathbb{R}^k$ be a function such that $f \in L^2(\mathbb{R})$, and consider the symmetric positive system given in (2). Suppose that there are tangential operators $\{D_\sigma\}$ that satisfy the hypotheses of Theorem 3, and $G : L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator so that $IntD(G) \cap D(L) \neq \emptyset$, then the system*

$$\begin{cases} Lu(x) + Gu(x) = f(x) \quad \text{if} \quad x \in \Omega \\ \beta_-(x)u(x) = 0 \quad \text{if} \quad x \in \partial\Omega \end{cases} \tag{5}$$

has a unique strong solution.

Proof. Let us prove first that L with the semi-admissible boundary condition $\beta_-(x)u(x) = 0$ is a maximal monotone operator.

Note that $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator if and only if $L^{-1} : D(L^{-1}) \subset L^2(\Omega) \to L^2(\Omega)$ is a maximal monotone operator. Now from Proposition 2, the operator L^{-1} is a maximal monotone operator if L^{-1} is univalent, monotone, hemi-continuous, and $D(L^{-1}) = L^2(\Omega)$.

1) From the uniqueness Theorem (Theorem 4), the operator L^{-1} is univalent.

2) Let f, g be two functions in the range of L, and let u, v be in $L^2(\Omega)$ strong solutions of $Lu = f$ and $Lv = g$ respectively with $\beta_- u = 0$ and $\beta_- v = 0$. Then from (3),

$$\left\langle f - g, L^{-1}f - L^{-1}g \right\rangle = \langle Lu - Lv, u - v\rangle = \langle L(u - v), u - v\rangle \geq C\|u - v\|^2 \geq 0,$$

where $C > 0$. This means that L^{-1} is monotone.

3) Let f, u be in $L^2(\Omega)$ such that u is a strong solution of $Lu = f$ with $\beta_- u = 0$. Then from (3), we have $C\|u\|^2 \leq \langle Lu, u\rangle = \langle f, u\rangle \leq \|f\|\,\|u\|$. Hence, $C\|u\| \leq \|f\|$. Therefore, $\|L^{-1}f\| = \|u\| \leq \dfrac{1}{C}\|f\|$. This means that L^{-1} is bounded. Therefore, since L^{-1} is a bounded linear operator, it follows that L^{-1} is continuous.

4) From Theorem 3, it follows that $Lu = f$ with $\beta_- u = 0$ is onto. This means that $D(L^{-1}) = L^2(\Omega)$. This proves that L^{-1} is a maximal monotone operator. Hence, L is a maximal monotone operator.

On the other hand, we will show that $L + G$ is a maximal monotone operator.

We know that $D(G) = L^2(\Omega)$ and $D(L) \subset L^2(\Omega)$, then

$$Int(D(G)) \cap D(L) = Int(L^2(\Omega)) \cap D(L) = L^2(\Omega) \cap D(L) \neq \emptyset.$$

Then by Proposition 3, we have that $L + G$ is a maximal monotone operator.

Finally, we will show that the operator $L + G$ is onto. For this, take u in $L^2(\Omega)$ and let us consider

$$\lim_{\|u\|\to\infty} \frac{\langle (L + G)^o u, u - 0\rangle}{\|u\|}.$$

Note that since $L + G$ is univalent, $(L + G)^o(u) = Proy_{(L+G)(u)}0 = (L + G)(u)$.

Thus, for $L+G$ we have

$$\frac{\langle Lu+Gu,u\rangle}{\|u\|}=\frac{\langle Lu,u\rangle}{\|u\|}+\frac{\langle Gu,u\rangle}{\|u\|}=\frac{\langle Lu,u\rangle}{\|u\|}+\frac{\langle G(u)-G(0)+G(0),u\rangle}{\|u\|}$$
$$=\frac{\langle Lu,u\rangle}{\|u\|}+\frac{\langle G(u)-G(0),u\rangle}{\|u\|}+\frac{\langle G(0),u\rangle}{\|u\|}.$$

Now since G is a monotone operator, it follows that $\langle G(u)-G(0),u\rangle\geq 0$, and since

$$\frac{\langle G(0),u\rangle}{\|u\|}\leq\frac{\|G(0)\|\,\|u\|}{\|u\|}=\|G(0)\|,$$

then from (3), we have

$$\frac{\langle Lu+Gu,u\rangle}{\|u\|}\geq\frac{C\|u\|^2}{\|u\|}+0-\|G(0)\|=C\|u\|-\|G(0)\|\to+\infty$$

as $\|u\|\to+\infty$. Then by Proposition 1, the operator $L+G$ is onto.

This proves Theorem 5. □

5. Applications

Semi-linear heat equation

We will show that the semi-linear heat equation has a unique strong solution. That is, we have the following result:

Theorem 6. *Let $\Gamma=\{x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n \mid 0\leq x_j\leq 1; j=1,2,\ldots,m\}$, and for $T>0$, consider t in $[0,T]$. If $g:\mathbb{R}\to\mathbb{R}$ is a continuous function such that $(g(u)-g(v))(u-v)\geq C(u-v)^2$ with u,v in $\mathbb{R}$ and C a constant, and there are real numbers M and N so that $|g(x)|\leq M|x|+N$, then given h in $L^2(\Omega)$, where $\Omega=\Gamma\times[0,T]$, the semi-linear heat problem has a unique strong solution $\phi:\mathbb{R}^{m+1}\to\mathbb{R}$ satisfying:*

$$\begin{cases}\dfrac{\partial\phi}{\partial t}-\dfrac{\partial^2\phi}{\partial x^2}+g(\phi)=h(x,t) \quad \textit{if} \quad (x,t)\in\Gamma\times[0,T]\\ \phi(x_1,x_2,\ldots,x_{n-1},0,t)=0,\ \phi(x_1,x_2,\ldots,1,x_n,t)=0,\\ \vdots\\ \phi(0,x_2,\ldots,x_{n-1},x_n,t)=0,\ \phi(1,x_2,\ldots,x_{n-1},x_n,t)=0,\\ \phi(x,0)=0, \quad \textit{where} \quad x\in\Gamma \quad \textit{and} \quad t\in[0,T].\end{cases}\tag{6}$$

Proof. Let λ be in $\mathbb{R}$, and consider the following functions:

$$u_0 = e^{-\lambda t}\phi, u_1 = e^{-\lambda t}\frac{\partial \phi}{\partial x_1}, u_2 = e^{-\lambda t}\frac{\partial \phi}{\partial x_2}, \ldots, u_m = e^{-\lambda t}\frac{\partial \phi}{\partial x_m}.$$

Then the first equation in (6) can be written as:

$$\left(A_0 \frac{\partial}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial}{\partial x_j} + B \right) u + D = H,$$

where

1) A_0 is the $(m+1)\times(m+1)$ matrix whose entries are $0's$ except in the 1×1 entry, where the element is 1.

2) A_j is the $(m+1)\times(m+1)$ matrix whose elements are $0's$ except in the entries $1\times(j+1)$ and $(j+1)\times 1$, where the elements are $-1's$, and where $j = 0, 1, 2, \ldots, m$.

3) B is the $(m+1)\times(m+1)$ diagonal matrix, where the element in the 1×1 entry is λ and the remaining elements are $1's$.

4) u is the $(m+1)\times 1$ matrix, where its elements are $u_0, u_1, u_2, \ldots, u_m$.

5) D is the $(m+1)\times 1$ matrix, where the 1×1 element is $e^{-\lambda t}g(e^{\lambda t}u_0)$, and the remaining elements are $0's$.

6) H is the $(m+1)\times 1$ matrix, where the 1×1 element is $e^{-\lambda t}h$, and the remaining elements are $0's$.

Note that if we define B' as the matrix B with $\frac{\lambda}{2}$ instead of λ, then we set D' as the matrix D with $\frac{\lambda}{2}u_0 + e^{-\lambda t}g(e^{\lambda t}u_0)$ instead of $e^{-\lambda t}g(e^{\lambda t}u_0)$, so that the first equation in (6) can now be written as:

$$\left(A_0 \frac{\partial}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial}{\partial x_j} + B' \right) u + D' = H. \tag{7}$$

Now, if we define

$$L = A_0 \frac{\partial}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial}{\partial x_j} + B', \tag{8}$$

and

$$G = \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_m \end{pmatrix} \quad \text{so that} \quad Gu = \begin{pmatrix} G_0(u_0) \\ G_1(u_1) \\ \vdots \\ G_m(u_m) \end{pmatrix} = D', \tag{9}$$

then the equation (7) is equivalent to $Lu + Gu = H$.

Thus, we have the following three Claims:

Claim 1. *The operator G is monotone.*

Proof. Note that $G_0(u_0) = \frac{\lambda}{2}u_0 + e^{-\lambda t}g(e^{\lambda t}u_0)$ and $G_i(u_i) = 0$ for $i = 1, 2, \ldots, m$. Thus, for G_0 we have

$$\left\langle G_0(u_0) - G_0(v_0), u_0 - v_0 \right\rangle = \int_\Omega \left(\frac{\lambda}{2}u_0 + e^{-\lambda t}g(e^{\lambda t}u_0) - \frac{\lambda}{2}v_0 - e^{-\lambda t}g(e^{\lambda t}v_0) \right)(u_0 - v_0)$$
$$= \frac{\lambda}{2}\int_\Omega (u_0 - v_0)^2 + \int_\Omega e^{-2\lambda t}\left(g(e^{\lambda t}u_0) - (e^{\lambda t}v_0)\right)\left(e^{\lambda t}u_0 - e^{\lambda t}v_0\right)$$
$$\geq \frac{\lambda}{2}\int_\Omega (u_0 - v_0)^2 + \int_\Omega e^{-2\lambda t}C\left(e^{\lambda t}u_0 - e^{\lambda t}v_0\right)^2 = \left(\frac{\lambda}{2} + C\right)\int_\Omega (u_0 - v_0)^2 \geq 0 \quad \text{if} \quad \frac{\lambda}{2} \geq -C.$$

Hence, The operator G is monotone if $\lambda \geq -2C$.

This completes the proof of Claim 1. □

Claim 2. *The domain of G_0 is $L^2(\Omega)$.*

Proof. Let u_0 be in $L^2(\Omega)$. Then

$$\int_\Omega \left|e^{-\lambda t}g(e^{\lambda t}u_0)\right|^2 \leq \int_\Omega e^{-2\lambda t}\left(M|e^{\lambda t}u_0| + N\right)^2$$
$$= M^2\int_\Omega |u_0|^2 + 2MN\int_\Omega e^{-\lambda t}|u_0| + N^2\int_\Omega e^{-2\lambda t}$$
$$\leq M^2\int_\Omega |u_0|^2 + 2MN\int_\Omega |u_0| + N^2\int_\Omega 1 < \infty.$$

On the other hand, since $\frac{\lambda}{2}u_0$ is in $L^2(\Omega)$, it follows that $D(G_0) = L^2(\Omega)$.

This proves Claim 2. □

Claim 3. *The operator G_0 is hemicontinuous.*

Proof. Let $\{t_n\}$ be a sequence in $[0,1]$ such that $t_n \to \bar{t}$ as $n \to \infty$, and $u_0, v_0 \in L^2(\Omega)$. Then

$$\begin{aligned}
&\left| \frac{\lambda}{2}(\bar{t}u_0 + (1-\bar{t})v_0) + e^{-\lambda t} g\left[e^{\lambda t}(\bar{t}u_0 + (1-\bar{t})v_0)\right] \right. \\
&\left. - \frac{\lambda}{2}(t_n u_0 + (1-t_n)v_0) - e^{-\lambda t} g\left[e^{\lambda t}(t_n u_0 + (1-t_n)v_0)\right] \right|^2 \\
&\leq \left\{ \left|\frac{\lambda}{2}(\bar{t}u_0 + (1-\bar{t})v_0)\right| + e^{-\lambda t}\left(M\left|e^{\lambda t}(\bar{t}u_0 + (1-\bar{t})v_0)\right| + N\right) \right. \\
&\left. + \left|\frac{\lambda}{2}(t_n u_0 + (1-t_n)v_0)\right| + e^{-\lambda t}\left(M\left|e^{\lambda t}(t_n u_0 + (1-t_n)v_0)\right| + N\right) \right\}^2 \\
&\leq \left\{ (|\lambda| + 2M)|u_0| + (|\lambda| + 2M)|v_0| + 2N \right\}^2.
\end{aligned}$$

Following the same argument used after (4), it can be proved that G_0 is hemi-continuous.

This proves Claim 3. □

Hence, by Claims 1, 2, 3, the operators $G_0, G_1, \ldots, G_m$ are monotone, hemi-continuous, univalent, and $D(G_i) = L^2(\Omega)$ for $i = 0, 1, \ldots, m$. Then by Proposition 2, the operators $G_0, G_1, \ldots, G_m$ are maximal monotone. Therefore, the operator G is maximal monotone.

On the other hand, since L is a positive symmetric operator for $\lambda > 0$, with the semi-admissible boundary condition $\beta_- u = 0$, then by Theorem 5, the problem (6) has a unique strong solution.

This completes the proof of Theorem 6. □

Hyperbolic non-linear problem

Now, we will prove that by using the theory of symmetric positive systems, the problem given in [1] has a unique strong solution. That is, we have the following result:

Theorem 7. *Suppose that $\delta_1(x)$ and $\delta_2(x)$ are C^1 functions in $\overline{\Omega}$, where $\Omega = \{(x,t) | 0 \leq x \leq 1,\ t \in [0, +\infty]\}$, such that $\delta_i(x) \geq \epsilon_i$, where ϵ_i are positive constants for $i = i, 2$, and $|\delta_i(x)| \leq M_i$ a. e., where M_i are positive constants for $i = 1, 2$. Also, suppose that $A_i(x, \phi_i(x,t))$ are continuous, monotone and measurable functions satisfying $|A_i(x, \zeta_i)| \leq a_i|\zeta_i| + b_i$, where a_i, b_i are positive constants for $i = 1, 2$. Then given $f_i(x,t)$ in $L^2(\Omega)$ for $i = 1, 2$, the following system has a unique strong solution.*

$$\begin{cases} \delta_1(x)\dfrac{\partial \phi_1}{\partial t} - \dfrac{\partial \phi_2}{\partial x} + A_1(x, \phi_1(x,t)) = f_1(x,t) \\ \delta_2(x)\dfrac{\partial \phi_2}{\partial t} - \dfrac{\partial \phi_1}{\partial x} + A_2(x, \phi_2(x,t)) = f_2(x,t) \end{cases} \tag{10}$$

Proof. For λ in $\mathbb{R}$, let $u_i = e^{-\lambda t}\phi_i$ for $i = 1, 2$. Then (10) is equivalent to:

$$\left[\delta(x)\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + \lambda\delta(x)\right] u + D = h, \tag{11}$$

where

$$\delta(x) = \begin{bmatrix} \delta_1(x) & 0 \\ 0 & \delta_2(x) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} e^{-\lambda t}A_1(x,\phi_1) \\ e^{-\lambda t}A_2(x,\phi_2) \end{bmatrix},$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{and} \quad h = \begin{bmatrix} e^{-\lambda t}f_1(x,t) \\ e^{-\lambda t}f_2(x,t) \end{bmatrix}.$$

Now, if we let

$$\alpha^1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} \delta_1(x) & 0 \\ 0 & \delta_2(x) \end{bmatrix}, \quad \text{and} \quad \gamma = \begin{bmatrix} \lambda\delta_1(x) & 0 \\ 0 & \lambda\delta_2(x) \end{bmatrix},$$

then

1) The matrices α^1, α^2 are symmetric.

2) The symmetric part of

$$\xi = \gamma - \frac{\partial\alpha^1}{\partial x} - \frac{\partial\alpha^2}{\partial t} = \begin{bmatrix} \lambda\delta_1(x) & 0 \\ 0 & \lambda\delta_2(x) \end{bmatrix}$$

is positive definite if $\lambda\delta_1(x) > 0$ and $\lambda^2\delta_1(x)\delta_2(x) > 0$. But this is true if $\lambda > 0$ since $\delta_1(x) > 0$ and $\delta_2(x) > 0$.

3) For each (x,t) in Ω and for $\eta = (\eta_1, \eta_2)$ as the exterior normal unit vector to each point in $\partial\Omega$, the matrix

$$\beta(x) = \eta_1(x)\alpha^1(x) + \eta_2(x)\alpha^2(x) = \begin{bmatrix} \eta_2\delta_1(x) & -\eta_1 \\ -\eta_1 & \eta_2\delta_2(x) \end{bmatrix}$$

can be written as $\beta = \beta_+ + \beta_-$, where

$$\beta_+ = \begin{bmatrix} 0 & 0 \\ -\eta_1 & 0 \end{bmatrix} \quad \text{and} \quad \beta_- = \begin{bmatrix} \eta_2\delta_1(x) & -\eta_1 \\ 0 & \eta_2\delta_2(x) \end{bmatrix}.$$

Thus, the symmetric part of

$$\beta_+ - \beta_- = \begin{bmatrix} -\eta_2\delta_1(x) & \eta_1 \\ -\eta_1 & -\eta_2\delta_2(x) \end{bmatrix}$$

is non-negative if $\eta_2 \leq 0$ and $\eta_2^2\delta_1(x)\delta_2(x) \geq 0$. But this is true if $\eta_2 \leq 0$.

So, for $\eta = (\pm 1, 0)$ we write

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{as} \quad u = u_+ + u_-, \quad \text{where} \quad u_+ = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_- = \begin{bmatrix} 0 \\ u_2 \end{bmatrix}$$

so that

$$\beta_+ u_- = \begin{bmatrix} 0 & 0 \\ \pm 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta_- u_+ = \begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\beta_- u = \begin{bmatrix} 0 & \pm 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \pm u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{if} \quad u_2 = 0.$$

That is, the boudary condition $\beta_- u = 0$ is admissible if $\phi(x,t) = 0$.

Now, in the direction $\eta = (0, -1)$ we write

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{as} \quad u = u_+ + u_-, \quad \text{where} \quad u_+ = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_- = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

so that

$$\beta_+ u_- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \beta_- u_+ = \begin{bmatrix} -\delta_1(x) & 0 \\ 0 & -\delta_2(x) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$\beta_- u = \begin{bmatrix} -\delta_1(x) & 0 \\ 0 & -\delta_2(x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\delta_1(x)u_1 \\ -\delta_2(x)u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $u_1 = 0$ and $u_2 = 0$. That is the boundary condition $\beta_- u = 0$ is admissible if $\phi_1(x,t) = 0 = \phi_2(x,t)$.

Hence, if we take

$$L = \delta(x)\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + \lambda\delta(x) \quad \text{and} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

then the problem

$$\begin{cases} L\,u(x,t) = h(x,t) \\ \beta_-(x,t)\,u(x,t) = 0 \end{cases} \tag{12}$$

is a symmetric positive system for (x,t) in Ω, and where

$$h(x,t) = \begin{bmatrix} e^{-\lambda t} f_1(x,t) \\ e^{-\lambda t} f_2(x,t) \end{bmatrix},$$

and since $\beta_- u = 0$ is and admissible boundary condition, it follows from Theorems 1 and 2 that (12) has a unique weak solution for each h in $L^2(\Omega)$.

In order to show the existence of a strong solution, let us now write (11) as

$$L\,u + G\,u = h,$$

where

$$Lu = \left[\delta(x)\frac{\partial}{\partial t} + B\frac{\partial}{\partial x} + \frac{\lambda}{2}\delta(x)\right]u, \tag{13}$$

$$Gu = \begin{bmatrix} \frac{\lambda}{2}\delta_1(x)u_1 + e^{-\lambda t}A_1(x, e^{\lambda t}u_1) \\ \frac{\lambda}{2}\delta_2(x)u_2 + e^{-\lambda t}A_2(x, e^{\lambda t}u_2) \end{bmatrix} = \begin{bmatrix} G_1(u_1) \\ G_2(u_2) \end{bmatrix}, \tag{14}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \text{and where} \quad h = \begin{bmatrix} e^{-\lambda t}f_1(x,t) \\ e^{-\lambda t}f_2(x,t) \end{bmatrix}.$$

Note that the new operator L defined in (13) is still a symmetric positive operator since $\lambda > 0, \delta_1(x) > 0$ and $\delta_2(x) > 0$. Besides, the boundary condition $\beta_- u = 0$ is semi-admissible.

Now, for the operator G defined in (14) we have the following Claims.

Claim 4. *The operator $G : D(G) \subset L^2(\Omega) \to L^2(\Omega)$ is monotone.*

Proof. Let us consider both, G_1 and G_2. Then from (14) and for u_i and v_i in $L^2(\Omega)$ with $i = 1, 2$,

$$\begin{aligned}
&\Big\langle G_i(u_i) - G_i(v_i), u_i - v_i \Big\rangle = \int_\Omega [G_i(u_i) - G_i(v_i)][u_i - v_i] \\
&\int_\Omega \left[\frac{\lambda}{2}\delta_i(x)u_i + e^{-\lambda t}A_i(x, e^{\lambda t}u_i) - \frac{\lambda}{2}\delta_i(x)v_i - e^{-\lambda t}A_i(x, e^{\lambda t}v_i)\right][u_i - v_i] \\
&= \int_\Omega \frac{\lambda}{2}\delta_i(x)(u_i - v_i)^2 + \int_\Omega e^{-2\lambda t}\left[A_i(x, e^{\lambda t}u_i) - A_i(x, e^{\lambda t}v_i)\right]\left[e^{\lambda t}u_i - e^{\lambda t}v_i\right].
\end{aligned}$$

Now, if the functions $A_i : \mathbb{R}^2 \to \mathbb{R}$ satisfy that for $i = 1, 2$,

$$\left[A_i(x, e^{\lambda t}u_i) - A_i(x, e^{\lambda t}v_i)\right]\left[e^{\lambda t}u_i - e^{\lambda t}v_i\right] \geq C_i\left[e^{\lambda t}u_i - e^{\lambda t}v_i\right]^2,$$

where C_i are positive constants, then

$$\begin{aligned}
&\Big\langle G_i(u_i) - G_i(v_i), u_i - v_i \Big\rangle \geq \int_\Omega \frac{\lambda}{2}\delta_i(x)(u_i - v_i)^2 + \int_\Omega C_i e^{-2\lambda t}\left(e^{\lambda t}u_i - e^{\lambda t}v_i\right)^2. \\
&= \left(\frac{\lambda}{2}\epsilon_i + C_i\right)\int_\Omega (u_i - v_i)^2.
\end{aligned}$$

Thus G_i is monotone if $\frac{\lambda}{2}\epsilon_i + C_i \geq 0$. That is, the operator G is monotone if $\frac{\lambda}{2}\epsilon_i \geq -C_i$ for $i = 1, 2$. This proves Claim 4. $\square$

Claim 5. *The domain of the operator G is $L^2(\Omega)$.*

Proof. Let u_i be in $L^2(\Omega)$ with $i = 1, 2$. Then from (14)

$$\int_\Omega \left|\frac{\lambda}{2}\delta_i(x)u_i\right|^2 \leq M_i^2\left|\frac{\lambda}{2}\right|^2\int_\Omega |u_i|^2 < \infty.$$

On the other hand since $|A_i(x,\zeta_i)| \leq a_i|\zeta_i| + |b_i|$, it follows that $e^{-\lambda t}A_i(x, e^{\lambda t}u_i)$ is in $L^2(\Omega)$. Thus, $D(G) = L^2(\Omega)$.

This proves Claim 5. □

Claim 6. *The operator $G : L^2(\Omega) \to L^2(\Omega)$ is hemi-continuous.*

Proof. Let t_n be a sequence in $[0,1]$ such that $t_n \to \bar{t}$, and take u_i, v_i in $L^2(\Omega)$ for $i = 1,2$. Then

$$\begin{aligned}
&\left\|G_i[(1-t_n)v_i + t_n u_i] - G_i[(1-\bar{t})v_i + \bar{t}u_i]\right\|^2 \\
&= \int_\Omega \Big|\frac{\lambda}{2}\delta_i(x)[(1-t_n)v_i + t_n u_i] + e^{-\lambda t}A_i\left(x, e^{\lambda t}[(1-t_n)v_i + t_n u_i]\right) \\
&- \frac{\lambda}{2}\delta_i(x)[(1-\bar{t})v_i + \bar{t}u_i] - e^{-\lambda t}A_i\left(x, e^{\lambda t}[(1-\bar{t})v_i + \bar{t}u_i]\right)\Big|^2 \\
&\leq \int_\Omega \Big[\frac{\lambda}{2}M_i|(1-t_n)v_i + t_n u_i| + e^{-\lambda t}\left(a_i|e^{\lambda t}[(1-t_n)v_i + t_n u_i]| + b_i\right) \\
&+ \frac{\lambda}{2}M_i|(1-\bar{t})v_i + \bar{t}u_i| + e^{-\lambda t}\left(a_i|e^{\lambda t}[(1-\bar{t})v_i + \bar{t}u_i]| + b_i\right)\Big]^2 \\
&\leq \int_\Omega \Big[(\lambda M_i + 2a_i)|v_i| + (\lambda M_i + 2a_i)|u_i| + 2b_i\Big]^2.
\end{aligned}$$

Following the same argument used after (4), it can be proved that G_i is hemi-continuous. Hence, G is hemi-continuous.

This proves Claim 6. □

Finally, since $IntD(G) \cap D(L) \neq \emptyset$, and since G is hemi-continuous, it follows from Proposition 2 that G is a maximal monotone operator. Thus, from Theorem 5, the problem (10) has a unique strong solution.

Remark 3. *The existence of the strong solution for problem (10) is based on the existence of the matrices σ_1 and τ_1 in C^0 such that*

$$\frac{\partial \alpha^2}{\partial x} = \tau_1\alpha^2 + \alpha^2\sigma_1,$$

where

$$\alpha^2 = \begin{bmatrix} \delta_1(x) & 0 \\ 0 & \delta_2(x) \end{bmatrix}$$

is a symmetric non-singular matrix. Then from part 1) of Remark 2, take

$$\alpha^2 = \begin{bmatrix} \frac{1}{\delta_1(x)}\frac{\partial}{\partial x}\delta_1(x) & 0 \\ 0 & \frac{1}{\delta_2(x)}\frac{\partial}{\partial x}\delta_2(x) \end{bmatrix} \quad \textit{and} \quad \tau_1 = 0.$$

This completes the proof of Theorem 7. □

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ON EXPECTATION OF SOME PRODUCTS OF WICK POWERS

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ABSTRACT. In this paper we give a new expression for the expectation of Wick products used in the literature. We compare this expression with that obtained by I. E. Segal in [12] p. 452, which has been used to treat differential equations involving Wick powers.

1. Introduction

The Wick products (cf. Dyson [6] and Wick [13]) have been useful in practical quantum mechanics, specially in applications to Feynman graph theory, where Bose and Fermi fields play an important role in defining Feymann propagators as the Fock (vacuum) state of ordinary products of them (cf. [3, Section 17.4], [4, Section 17] and [9, Section 4-2]).

In a more explicit mathematical form, and according to the ideas of Bratteli and Robinson (cf. [5, Section 5.2]), one considers Wick products over the CCR algebra.

Here, we obtain a new expression for the expectation of Wick products. We compare this expression with that obtained by I. E. Segal in [12, p 452]. This formula was used by Segal concerning his studies on differential equations including Wick powers (cf. [12, Corollary 2.3 and 2.4]).

Our treatment benefits immeasurably from the interpretation of Wick products given by Segal (cf. [11] and [12, Section 1.4]), Segal et al. [1, Section 7.2] and Jorgensen et al. [10].

For recent studies treating Wick powers on CCR algebra see [2], [7] and [8].

2. The expectation of products of Wick powers

Throughout this paper we will use the terminology and the notations of monograph [1] and paper [12]. Recall some notions and basic results.

Let $\mathbf{L}$ be a real linear space endowed with a nondegenerate anti-symmetric bilinear form $A(\cdot,\cdot)$, having pure imaginary values. Denote by $\mathbf{E}$ the (infinitesimal) Weyl algebra over $(\mathbf{L}, A)$, which is the associative complex algebra generated by $\mathbf{L}$ and an unit e, with the relation $zz' - z'z = -iA(z,z')e$, for arbitrary z, z' in $\mathbf{L}$ [1, Definition, p. 175]. Consider on $\mathbf{L}$ an admissible complex structure and let E be the associated vacuum normal. Then there exists a unique mapping $:\cdot:$, called renormalization map, from monomials in $\mathbf{E}$ to $\mathbf{E}$ verifying certain conditions. This mapping extends to a linear mapping of $\mathbf{E}$ into itself which is uniquely determined by some properties.

In Theorem 3 we give an expression for the expectations $E(:z_1^n:\cdots:z_t^n:)$.

Notice that by definition of E one has that $E(e) = 1$, where e is the unit and, by [12, Corollary 5], if $z_1, \ldots, z_t \in \mathbf{L} + i\mathbf{L}$, then $E(:z_1 \cdots z_t:) = 0$.

Observe that given $z \in \mathbf{L} + i\mathbf{L}$, by [12, Theorem 1.3 (a)], it is clear that

$$:z^n:z =: z^{n+1}: + n:z^{n-1}:E(z^2)\,. \tag{1}$$

We have found part (1) of the following proposition in *http://eom.springer.de/w/w097870.htm*, however we have included the proof in order to be self contained.

Proposition 1. *Let $z \in \mathbf{L} + i\mathbf{L}$ and $n \in \mathbb{N}$. Then the following properties hold:*

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1

$$\textit{(1)}\ : z^n := \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!\,2^m} z^{n-2m} E(z^2)^m,$$

$$\textit{(2)}\ z^n = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{m!\,(n-2m)!\,2^m} : z^{n-2m} : E(z^2)^m \ ,$$

where $\left[\frac{n}{2}\right]$ *denotes the integer part of* $\frac{n}{2}$.

Proof. (1) Let us prove it by induction. For $n = 1$ is clear. Suppose that the equality is true for $n-1$ and assume that $n-1$ is even. Let us prove it for n. Then $\frac{n-1}{2} = k$ with $k \in \mathbb{N}$, hence $[\frac{n-1}{2}] = [\frac{n}{2}] = k$, $[\frac{n-2}{2}] = k-1$ and by [12, Theorem 1.3 (a)] and the induction hypothesis we have

$$\begin{aligned}
: z^n : \ &=: z^{n-1} : z - (n-1) : z^{n-2} : E(z^2)\\
&= \sum_{m=0}^{[\frac{n-1}{2}]} (-1)^m \frac{(n-1)!}{m!(n-1-2m)!2^m} z^{n-1-2m} E(z^2)^m z\\
&\quad -(n-1) \sum_{m=0}^{[\frac{n-2}{2}]} (-1)^m \frac{(n-2)!}{m!(n-2-2m)!2^m} z^{n-2-2m} E(z^2)^{m+1}\\
&= z^n + \sum_{m=1}^{k} (-1)^m \frac{(n-1)!}{m!(n-1-2m)!2^m} z^{n-2m} E(z^2)^m\\
&\quad - \sum_{m=1}^{k} (-1)^{m-1} \frac{(n-1)!}{(m-1)!(n-2m)!2^{m-1}} z^{n-2m} E(z^2)^m\\
&= z^n + \sum_{m=1}^{k} (-1)^m (n-1)! \left(\frac{1}{m!(n-1-2m)!2^m} + \frac{1}{(m-1)!(n-2m)!2^{m-1}} \right) z^{n-2m} E(z^2)^m\\
&= \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!\,2^m} z^{n-2m} E(z^2)^m \ .
\end{aligned}$$

If $n-1$ is odd, then the proof is similar to the even case.

(2) Let us prove it by induction. For $n = 1$ is clear. Suppose that the equality is true for $n-1$. Assume that $n-1$ is odd. Then $n = 2k$ with $k \in \mathbb{N}$, $[\frac{n}{2}] = k$ and $[\frac{n-1}{2}] = k-1$. By [12, Theorem 1.3 (a)] and the induction hypothesis we have that

$$\begin{aligned}
z^n \ &= z^{n-1} z = \sum_{m=0}^{k-1} \frac{(n-1)!}{m!\,(n-1-2m)!\,2^m} : z^{n-1-2m} : z E(z^2)^m\\
&= \sum_{m=0}^{k-1} \frac{(n-1)!}{m!\,(n-1-2m)!\,2^m} \left(: z^{n-2m} : + (n-1-2m) : z^{n-2-2m} : E(z^2)\right) E(z^2)^m\\
&= \sum_{m=0}^{k-1} \frac{(n-1)!}{m!\,(n-1-2m)!\,2^m} : z^{n-2m} : E(z^2)^m\\
&\quad + \sum_{m=0}^{k-1} \frac{(n-1)!\,(n-1-2m)}{m!\,(n-1-2m)!\,2^m} : z^{n-2-2m} : E(z^2)^{m+1}\\
&=: z^n : + \sum_{m=1}^{k-1} \frac{(n-1)!}{m!\,(n-1-2m)!\,2^m} : z^{n-2m} : E(z^2)^m\\
&\quad + \sum_{m=1}^{k} \frac{(n-1)!(n-2m+1)}{(m-1)!(n-2m+1)!2^{m-1}} : z^{n-2m} : E(z^2)^m
\end{aligned}$$

$$=: z^n : + \sum_{m=1}^{k-1} (n-1)! \left(\frac{1}{m!(n-1-2m)!2^m} + \frac{1}{(m-1)!(n-2m)!2^{m-1}} \right) : z^{n-2m} : E(z^2)^m$$
$$+ \tfrac{(n-1)!}{(k-1)!\, 2^{k-1}} : z^0 : E(z^2)^k$$

$$= \sum_{m=0}^{k} \frac{n!}{m!\,(n-2m)!\,2^m} : z^{n-2m} : E(z^2)^m \ .$$

The case $n-1$ even is similar to the above. □

Denote

$$n!! = \begin{cases} n(n-2)\ldots 5.3.1 & \text{if } n > 0 \text{ and odd} \\ n(n-2)\ldots 4.2 & \text{if } n > 0 \text{ and even} \\ 1 & \text{if } n = -1, 0 \ . \end{cases}$$

Corollary 2. *Let $z \in \mathbf{L} + i\mathbf{L}$ and $n \in \mathbb{N}$. Then the following property holds:*

$$E(z^n) = \begin{cases} (n-1)!!\ E(z^2)^{\frac{n}{2}} & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \ . \end{cases} \tag{2}$$

Proof. Using part (2) of Proposition 1 and the linearity of the functional E we have

$$E(z^n) = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{m!\,(n-2m)!\,2^m} E(: z^{n-2m} :) E(z^2)^m \ .$$

Notice that $E(: z^{n-2m} :) = 0$ for all $n - 2m \neq 0$. Assume that n is even, that is $n = 2k$ for some $k \in \mathbb{N}$. So,

$$E(z^n) = \frac{(2k)!}{k!\ 2^k} E(z^2)^k = (2k-1)!!\ E(z^2)^k = (n-1)!!\ E(z^2)^{\frac{n}{2}} \ .$$

The case n odd is clear. □

The following theorem gives a formula of the expectation of products of the form $: z_1^n : \cdots : z_t^n :$.

Theorem 3. *Let $z_1, \ldots, z_t \in \mathbf{L} + i\mathbf{L}$ and $n \in \mathbb{N}$ with nt even. Then*

$$E(: z_1^n : \cdots : z_t^n :) = \sum_{\substack{0 \le l_k \le [\frac{n}{2}] \\ 0 \le k \le t}} \sum_{\substack{1 \le i_h < j_h < 2s \\ 1 \le h \le s}} \frac{(-1)^{l_1 + \cdots + l_t}\ n!^t\ E(\tilde{z}_{i_1} \tilde{z}_{j_1}) \cdots E(\tilde{z}_{i_s} \tilde{z}_{j_s})}{l_1! \cdots l_t!\ (n-2l_1)! \cdots (n-2l_t)!\ 2^{l_1 + \cdots + l_t}} E(z_1^2)^{l_1} \cdots E(z_t^2)^{l_t} \tag{3}$$

where $2s = nt - 2(l_1 + \cdots + l_t)$, the second sum ranges over $i_1 < i_2 < \cdots < i_s$ and $\tilde{z}_h = z_j$ for $(j-1)n - 2(l_0 + \cdots + l_{j-1}) + 1 \le h \le jn - 2(l_1 + \cdots + l_j)$ and $1 \le j \le t$ with $l_0 = 0$.

Proof. By part (1) of Proposition 1 we have that

$$E(: z_1^n : \cdots : z_t^n :) = \sum_{\substack{0 \le l_k \le [\frac{n}{2}] \\ 0 \le k \le t}} \frac{(-1)^{l_1 + \cdots + l_t}\ n!^t\ E(z_1^{n-2l_1} \cdots z_t^{n-2l_t})}{l_1! \cdots l_t!\ (n-2l_1)! \cdots (n-2l_t)!\ 2^{l_1 + \cdots + l_t}} E(z_1^2)^{l_1} \cdots E(z_t^2)^{l_t} \ . \tag{4}$$

By Wick's Theorem [12, Theorem 1.3 (b)], we obtain

$$E(z_1^{n-2l_1} \cdots z_t^{n-2l_t}) = \sum_{\substack{1 \le i_h < j_h < 2s \\ 1 \le h \le s}} E(\tilde{z}_{i_1} \tilde{z}_{j_1}) \cdots E(\tilde{z}_{i_s} \tilde{z}_{j_s}) \ ,$$

where the sum ranges over $i_1 < i_2 < \cdots < i_s$ and the $\tilde{z}_h$ are defined as the statement. So, we get the desire result. □

It is clear that if nt is odd in Theorem 3, then $E(: z_1^n : \cdots : z_t^n :) = 0$.

Lemma 4. *Let $z \in \mathbf{L} + i\mathbf{L}$ and $n, m \in \mathbb{N}$ with $m \leq n$. Then*

$$E(: z^n : z^m) = \begin{cases} 0 & \text{if } m < n \\ \\ n!\, E(z^2)^n & \text{if } m = n\,. \end{cases} \tag{5}$$

Proof. By (1) we have

$$\begin{aligned} : z^n : z^m &= \big(: z^{n+1} : + n : z^{n-1} E(z^2)\big)\, z^{m-1} \\ &= \big(: z^{n+2} : + (n+1) : z^n : E(z^2) + n(: z^n : + (n-1) : z^{n-2} : E(z^2))E(z^2)\big)\, z^{m-2}\,. \end{aligned}$$

Repeating the property (1) and using the linearity of the renormalization we obtain the result. □

In the next corollary we present a particular case of Theorem 3.

Corollary 5. *Let $z \in \mathbf{L} + i\mathbf{L}$ and $n, t \in \mathbb{N}$ with nt even. Then the following property holds:*

$$E(: z^n :^t) = \sum_{\substack{0 \leq l_k \leq [\frac{n}{2}] \\ 0 \leq k \leq t}} \frac{(-1)^{l_1 + \cdots + l_t}\; n!^t (nt - 2(l_1 + \cdots l_t) - 1)!!}{l_1! \cdots l_t!\; (n - 2l_1)! \cdots (n - 2l_t)!\; 2^{l_1 + \cdots l_t}} E(z^2)^{\frac{nt}{2}}\,.$$

In particular, if $t = 2$ then $E(: z^n :^2) = n!\, E(z^2)^n$.

Proof. It is a consequence of (4) of Theorem 3 and Corollary 2.

Suppose that $t = 2$. By Proposition 1 we have

$$: z^n :^2 =: z^n : \left(\sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!\,2^m} z^{n-2m} E(z^2)^m \right)\,. \tag{6}$$

Taking the expectation in (6) and using Lemma 4 we obtain

$$\begin{aligned} E(: z^n :^2) &= E\left(: z^n : \left(\sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!\,2^m} z^{n-2m} E(z^2)^m \right)\right) \\ &= \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{n!}{m!\,(n-2m)!\,2^m} E(: z^n : z^{n-2m}) E(z^2)^m \\ &= n!\, E(z^2)^n\,. \end{aligned}$$

□

Remark 6. In [12, p. 452] the author assures that if $z_1, \ldots, z_t \in \mathbf{L} + i\mathbf{L}$ with t even and $n \in \mathbb{N}$, then

$$E(: z_1^n : \cdots : z_t^n :) = \sum_{q \in Q} \prod_{i<j} E(z_i z_j)^{q(i,j)}\,, \tag{7}$$

where Q is the set of all the functions $q(i,j)$ having nonnegative integer values, defined for $i, j = 1, 2..., t$ and $i \neq j$, and such that $\sum_i q(i,j) = n$ for all j and $q(i,j) = q(j,i)$. If we compare the expectation values obtained by Segal's formula (7) and by Corollary 5, we find a gap (see Table 1).

Table 1. Comparation between the expectation values by Segal's formula (7) and by Corollary 5.

Expectation	**By Segal's formula (7)**	**By Corollary 5**
$E(: z^n :^2)$	$E(z^2)^n$	$n!E(z^2)^n$
$E(: z^2 :^4)$	$6E(z^2)^4$	$60E(z^2)^4$
$E(: z^3 :^4)$	$10E(z^2)^6$	$3348E(z^2)^6$

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Solving nonlinear Klein-Gordon equation with high accuracy multiquadric quasi-interpolation scheme

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Abstract

In this paper, we present a numerical method for solving the nonlinear Klein-Gordon equation. This method is based on the multiquadric quasi-interpolation operator $\mathcal{L}_{\mathcal{W}_2}$. In the present scheme, the third order convergence finite difference method is used to disrcretize the temporal derivative. Then, the unknown function and its spatial derivatives are approximated by the multiquadric quasi-interpolation operator $\mathcal{L}_{\mathcal{W}2}$. Further, by using collocation method, the approximated solution of the equation is obtained. This method is applied on some test experiments and the numerical results have been compared with the exact solutions and the solutions in [1, 2]. The L_∞, L_2 and root-mean-square (RMS) errors of the solutions show the efficiency and the accuracy of the method.

Keywords: Nonlinear Klein-Gordon equation; Multiquadric quasi-interpolation scheme; Collocation scheme; Radial basis function.

2010 Mathematics Subject Classification: 35K61; 97N50; 65N35; 33E99.

1 Introduction

In this paper, we concentrate on the numerical solution of one of the well known equation named as Klein-Gordon equation:

$$u_{tt} + \mu u_{xx} + F(u) = f(x,t), \qquad x \in \Omega = [a,b] \subset \mathbb{R},\ 0 < t \leqslant T, \tag{1}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= g_1(x), \quad x \in \Omega, \\ u_t(x,0) &= g_2(x), \quad x \in \Omega, \end{aligned} \tag{2}$$

and the boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \tag{3}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t, μ is a known constant, $F(u)$ is the nonlinear force such that $\frac{\partial F}{\partial u} \geqslant 0$ and $g_1(x)$, $g_2(x)$ and $g(x,t)$ are known functions.

In the present work, the numerical approximation of the following nonlinear Klein-Gordon equation:

$$u_{tt} + \mu u_{xx} + \alpha u + \beta u^k = f(x,t), \tag{4}$$

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is considered wherein $k = 2$ or $k = 3$ and α and β are known constants. In this case, the nonlinear force $F(u)$ is equal to $\alpha u + \beta u^k$.

The nonlinear Klein-Gordon equation appears in different application areas, including differential geometry and relativistic field theory, and it also appears in other physical applications, such as the propagation of fluxons in the Josephson junctions, the motion of rigid pendula attached a stretched wire, and dislocations in crystals [3, 4].

Various numerical techniques have been presented to solve Klein-Gordon equation such as finite difference [5], finite element, spectral and pseudo-spectral methods [6, 7]. Although these methods have been widely used but they usually require the construction and update of a mesh and hence bring inconvenience during computation. To avoid the mesh generation, meshless methods have attracted the attention of researchers. These methods are based on radial basis functions (RBFs). The idea of using RBFs for solving partial differential equations (PDEs) was first proposed by Kansa (1990) [8, 9]. The kansa's method is applied for solving of different kinds of nonlinear PDEs such as Burgers' equation [10], Sine-Gordon equation [11], Klein-Gordon equation [2], Korteweg-de Vries (KdV) equation [12] etc..

In most of the known methods for solving PDEs using RBFs, one must resolve a linear system of equations at each time step. Hon and Wu [13], Wu [14, 15] and others have provided some successful examples using MQ quasi-interpolation scheme for solving differential equations. Beatson and Powell [16] and Wu and Schaback [17] proposed other univariate MQ quasi-interpolations. In [18, 19], Chen and Wu used MQ quasi-interpolation to solve Burgers' equation and hyperbolic conservation laws. Also, Xiao et al. [20] presented the numerical method based on Chen and Wu's method for solving the third-order KdV equation. Recently, Jiang et al. [21] have introduced a new multi-level univariate MQ quasi-interpolation approach with high approximation order compared with initial MQ quasi-interpolation scheme. This method is based on inverse multiquadric (IMQ) RBF interpolation, and Wu and Schaback's MQ quasi-interpolation operator $\mathcal{L}_D$ that have the advantages of high approximation order.

The aim of this paper is to present a numerical scheme to solve the nonlinear Klein-Gordon equation that is based on Jiang et al. MQ quasi-interpolation operator $\mathcal{L}_{\mathcal{W}_2}$. This paper is arranged as follows. In Section 2, we describe a brief information about the MQ quasi-interpolation scheme. In Section 3, the method is applied on the nonlinear Klein-Gordon equation. In Section 4, the results of four numerical experiments are presented and compared with the analytical solutions and the results in [1,2]. Finally, a brief discussion and conclusion is presented in Section 5.

2 The MQ quasi-interpolation scheme

In this section, some elementary knowledge about three univariate MQ quasi-interpolation schemes, namely, $\mathcal{L}_\mathcal{D}, \mathcal{L}_\mathcal{W}$ and $\mathcal{L}_{\mathcal{W}_2}$ are presented. For more details about MQ quasi-interpolation operators, see [16–19].

For a given interval $\Omega = [a, b]$ and a finite set of distinct points

$$a = x_0 < x_1 < \ldots < x_N = b, \quad h = \max_{1 \leqslant i \leqslant N} (x_i - x_{i-1}).$$

quasi-interpolation of a univariate function $f : [a, b] \to \mathbb{R}$ has the form

$$\mathcal{L}(f) = \sum_{i=0}^{N} f(x_i)\phi_i(x),$$

where each function $\phi_i(x)$ is a linear combination of the MQs

$$\psi_i(x) = \sqrt{c^2 + (x - x_i)^2},$$

and $c \in \mathbb{R}^+$ is a shape parameter. In [17], Wu and Scheback presented the univariate MQ quasi-interpolation operator $\mathcal{L}_\mathcal{D}$ that is defined as

$$\mathcal{L}_\mathcal{D} f(x) = \sum_{i=0}^{N} f(x_i)\widetilde{\psi}_i(x), \tag{5}$$

where

$$\widetilde{\psi}_0(x) = \frac{1}{2} + \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\widetilde{\psi}_1(x) = \frac{\psi_2(x) - \psi_1(x)}{2(x_2 - x_1)} - \frac{\psi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\widetilde{\psi}_i(x) = \frac{\psi_{i+1}(x) - \psi_i(x)}{2(x_{i+1} - x_i)} - \frac{\psi_i(x) - \psi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 2 \leqslant i \leqslant N-2, \tag{6}$$

$$\widetilde{\psi}_{N-1}(x) = \frac{(x_N - x) - \psi_{N-1}(x)}{2(x_N - x_{N-1})} - \frac{\psi_{N-1}(x) - \psi_{N-2}(x)}{2(x_{N-1} - x_{N-2})},$$

and

$$\widetilde{\psi}_N(x) = \frac{1}{2} + \frac{\psi_{N-1}(x) - (x_N - x)}{2(x_N - x_{N-1})}.$$

In RBFs interpolation, high approximation order can be gotten by increasing the number of interpolation centers but one has to solve unstable linear system of equations. By using MQ quasi-interpolation scheme, one can avoid this problem, whereas, the approximation order is not good. Therefore, Jiang et al. [21] defined two MQ quasi-interpolation operators denoted as $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{W}_2}$, which pose the advantages of RBFs interpolation and MQ quasi-interpolation scheme. The process of MQ quasi-interpolation of $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{W}_2}$ are as follows which is described in [21].

Suppose that $\{x_{k_j}\}_{j=1}^{\bar{N}}$ is a smaller set from the given points $\{x_i\}_{i=0}^{N}$ where $\bar{N}$ is a positive integer satisfying $\bar{N} < N$ and $0 = k_0 < k_1 < \ldots < k_{\bar{N}+1} = N$. Using the IMQ-RBF, the second derivative of $f(x)$ can be approximated by RBF interpolant $S_{f''}$ as

$$S_{f''}(x) = \sum_{j=1}^{\bar{N}} \lambda_j \bar{\varphi}(|x - x_{k_j}|), \tag{7}$$

where

$$\bar{\varphi}(r) = \frac{s^2}{(s^2 + r^2)^{3/2}},$$

and $s \in \mathbb{R}^+$ is a shape parameter. The coefficients $\{\lambda_j\}_{j=1}^{\bar{N}}$ are uniquely determined [22] by the interpolation condition

$$S_{f''}(x_{k_i}) = \sum_{j=1}^{\bar{N}} \lambda_j \bar{\varphi}(|x_{k_i} - x_{k_j}|) = f''(x_{k_i}), \quad 1 \leqslant i \leqslant \bar{N}. \tag{8}$$

Since, the Eq. (8) is solvable [22], so

$$\lambda = A_X^{-1}.f''_X, \tag{9}$$

where

$$X = \{x_{k_1}, \ldots, x_{k_{\bar{N}}}\}, \qquad \lambda = [\lambda_1, \ldots, \lambda_{\bar{N}}]^T, \qquad A_X = [\bar{\varphi}(|x_{k_i} - x_{k_j}|)], \qquad f''_X = [f''(x_{k_1}), \ldots, f''(x_{k_{\bar{N}}})]^T.$$

By using f and the coefficient λ defined in Eq. (9), a function $e(x)$ is constructed in the form

$$e(x) = f(x) - \sum_{i=1}^{\bar{N}} \lambda_i \sqrt{s^2 + (x - x_{k_i})^2}. \tag{10}$$

Then the MQ quasi-interpolation operator $\mathcal{L}_{\mathcal{W}}$ by using $\mathcal{L}_{\mathcal{D}}$ defined by Eqs. (5) and (6) on the data $(x_i, e(x_i))_{0\leqslant i\leqslant N}$ with the shape parameter c is defined as follows:

$$\mathcal{L}_{\mathcal{W}} f(x) = \sum_{i=1}^{\bar{N}} \lambda_i \sqrt{s^2 + (x - x_{k_i})^2} + \mathcal{L}_{\mathcal{D}} e(x). \tag{11}$$

The shape parameters c and s should not be the same constant in Eq. (11).

In Eq. (8), $f''_{x_{k_j}}$ can be replaced as

$$f''_{x_{k_j}} = \frac{f(x_{k_j+1}) - 2f(x_{k_j}) + f(x_{k_j-1})}{h_2^2}, \text{ with } h_2 = \frac{b-a}{\bar{N}},$$

when the data $(x_{k_i}, f(x_{k_i}))_{0\leqslant i\leqslant \bar{N}}$ are given, and $(x_i)_{0\leqslant i\leqslant N}$ are equally spaced points. So, if f''_X in Eq. (9) replaced by

$$f''_X = [f''_{x_{k_1}}, \ldots, f''_{x_{k_{\bar{N}}}}]^T, \tag{12}$$

the quasi-interpolation operator defined by Eqs. (10) and (11) is denoted by $\mathcal{L}_{\mathcal{W}_2}$. In this case, $\mathcal{L}_{\mathcal{W}_2}$ defined as follows:

$$\mathcal{L}_{\mathcal{W}_2} f(x) = \sum_{i=1}^{\bar{N}} \lambda_i \sqrt{s^2 + (x - x_{k_i})^2} + \sum_{i=0}^{N} (f(x_i) - \sum_{j=1}^{\bar{N}} \lambda_j \sqrt{s^2 + (x_i - x_{k_j})^2}) \widetilde{\psi}_i(x). \tag{13}$$

For more information about the properties and accuracy of $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{W}2}$, see [21]. In this paper, we use the MQ quasi-interpolation operator $\mathcal{L}_{\mathcal{W}_2}$ with $h_2 = 2h$.

3 The numerical method

In this section, the numerical scheme is presented for solving the Klein-Gordon equation (1) by using the MQ quasi-interpolation $\mathcal{L}_{\mathcal{W}_2}$. In our approach, the MQ quasi-interpolation approximates the solution function and the spatial derivatives of the differential equation and the fourth order finite difference approximation employs for discretizing of the temporal derivative similar to work that Rashidinia did in [1].

According of the fourth order finite difference, the term $u_{tt}^n = u_{tt}(x, t_n)$, $t_n = n\Delta t$, can be arranged as

$$u_{tt}^n \cong \frac{\delta_t^2}{(\Delta t)^2(1+\delta_t^2)} u^n + O((\Delta t)^2), \tag{14}$$

where $\delta_t^2 = u^{n+1} - 2u^n + u^{n-1}$.
Substituting Eq. (14) into Eq. (1) yields the following time discretized form of Klein-Gordon equation:

$$\delta_t^2 u^n + \mu(\Delta t)^2(1+\gamma\delta_t^2)u_{xx}^n + (\Delta t)^2(1+\gamma\delta_t^2)F^n = (\Delta t)^2(1+\gamma\delta_t^2)f^n, \tag{15}$$

where $f^n = f(x, t_n)$ and $F^n = F(u^n) = \alpha u^n + \beta(u^n)^k$.
After some arrangements, Eq. (15) can be written in the following form:

$$(\frac{1}{\gamma(\Delta t)^2} + \alpha)\ u^{n+1} + \mu\ u_{xx}^{n+1} + \beta\ (u^{n+1})^k = \chi(x), \tag{16}$$

where

$$\chi(x) = (\frac{2}{\gamma(\Delta t)^2} - \alpha\kappa)u^n - (\frac{1}{\gamma(\Delta t)^2} + \alpha)u^{n-1} - \mu[u_{xx}^n + u_{xx}^{n-1}] - \beta[\kappa(u^n)^k + (u^{n-1})^k],$$

and $\kappa = \frac{1-2\gamma}{\gamma}$.

Assuming that there are a total of $N+1$ interpolation points, u^n can be approximated by

$$u^n(x)=\sum_{i=1}^{\bar N}\alpha_i\sqrt{s^2+(x-x_{k_i})^2}+\sum_{i=0}^{N}(u_i^n-\sum_{j=1}^{\bar N}\alpha_j\sqrt{s^2+(x_i-x_{k_j})^2})\widetilde{\psi}_i(x), \tag{17}$$

where $u_i^n=u(x_i,t_n)$. Now, substituting Eq. (17) into Eqs. (16) and (3) and applying collocation method yield the following equations:

$$\begin{aligned}&(\frac{1}{\gamma(\Delta t)^2}+\alpha)\sum_{i=1}^{\bar N}\lambda_i\sqrt{s^2+(x_l-x_{k_i})^2}+\sum_{i=0}^{N}(u_i^n-\sum_{j=1}^{\bar N}\lambda_j\sqrt{s^2+(x_i-x_{k_j})^2})\widetilde{\psi}_i(x_l)+\mu\ (\sum_{i=1}^{\bar N}\lambda_i\sqrt{s^2+(x_l-x_{k_i})^2}\\&+\sum_{i=0}^{N}(u_i^{n+1}-\sum_{j=1}^{\bar N}\lambda_j\sqrt{s^2+(x_i-x_{k_j})^2})\widetilde{\psi}_i^{''}(x_l))+\beta(\sum_{i=1}^{\bar N}\lambda_i\sqrt{s^2+(x_l-x_{k_i})^2}+\sum_{i=0}^{N}(\tilde u_i^{n+1}-\sum_{j=1}^{\bar N}\lambda_j\sqrt{s^2+(x_i-x_{k_j})^2})\\&\widetilde{\psi}_i(x_l))^k=\chi(x_l),\qquad l=1,\ldots,N-1,\end{aligned} \tag{18}$$

and

$$\sum_{i=1}^{\bar N}\lambda_i\sqrt{s^2+(x_l-x_{k_i})^2}+\sum_{i=0}^{N}(u_i^n-\sum_{j=1}^{\bar N}\lambda_j\sqrt{s^2+(x_i-x_{k_j})^2})\widetilde{\psi}_i(x_l)=g^{n+1}(x_l),\qquad l=0,N, \tag{19}$$

where $g^{n+1}(x_l)=g(x_l,t_{n+1})$ and

$$\chi(x_l)=(\frac{2}{\gamma(\Delta t)^2}-\alpha\kappa)u^n(x_l)-(\frac{1}{\gamma(\Delta t)^2}+\alpha)u^{n-1}(x_l)-\mu[u_{xx}^n(x_l)+u_{xx}^{n-1}(x_l)]-\beta[\kappa(u^n(x_l))^k+(u^{n-1}(x_l))^k].$$

The coefficients $\{\lambda_i\}_{i=1}^{\bar N}$ are determined by the solvable linear system

$$\sum_{i=1}^{\bar N}\lambda_i\bar\varphi(|x_{k_j}-x_{k_i}|)=\frac{u_{k_j+1}^n-2u_{k_j}^n+u_{k_j-1}^n}{h_2^2},\qquad j=1,\ldots,\bar N. \tag{20}$$

At $n=1$, according to the initial conditions that was introduced in (2) and approach that Rashidinia did in [1] based on Taylor series, we apply the following assumptions

$$u^0(x)=u(x,0)=g_1(x),$$

and

$$\begin{aligned}u^1(x)=&g_1(x)+\Delta t g_2(x)-\frac{(\Delta t)^2}{2!}[\mu u_{xx}+F-f]^0-\frac{(\Delta t)^3}{3!}[\mu g_2^{''}(x)+F_t-f_t+F_uu_t]^0+\mu\frac{(\Delta t)^4}{4!}[\mu u_{xxxx}+\\&F_xu_x+F_{xx}-f_{xx}+F_{xu}u_x+u_x(F_{xu}+F_{uu}u_x)]^0+O((\Delta t)^5).\end{aligned}$$

In each time step (i.e. time step $n+1$), at first we set $\tilde u_j^{n+1}=u_j^n$. Having this, Eqs. (18) and (19) are solved as a system of linear algebraic equations for unknowns u_j^{n+1}; $j=0,1,\ldots,N$. Then, we recompute $\tilde u_j^{n+1}=u_j^{n+1}$ where u_j^{n+1} as we illustrate, can be obtained by solving Eqs. (18) and (19). Now, at each time level, we iterate calculating $\tilde u_j^{n+1}$ and solving the approximation values of the unknown, until the tolerance of any two latest iterations is not bigger than 10^{-8}, i.e. a predictor-corrector scheme is adopted in each time level, then one can move on to the next time level.

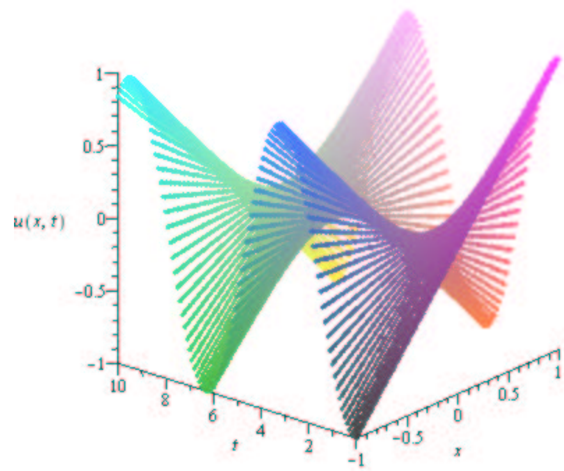

Figure 1:
The graph of the estimated solution up to $t = 10$ with $\Delta t = 0.0001$ and $N = 10$ of experiment 1.

4 The numerical experiments

Four experiments are studied to investigate the robustness and the accuracy of the proposed method. We compare the numerical results of the Klein-Gordon equation by using presented scheme with the analytical solutions and solutions in [1,2]. These methods include Thin Plate Splines (TPS) RBF collocation method [2] and cubic B-spline collocation method (CBS) [1]. We denote our scheme by MQQI. The L_2, L_∞ and RMS errors which are defined by

$$\mathrm{L}_2 = \sqrt{h_m \sum_{j=0}^{M} (\breve{u}^n(x_j) - u^n(x_j))^2},$$

$$\mathrm{L}_\infty = \max_{0 \leqslant j \leqslant M} |\breve{u}^n(x_j) - u^n(x_j)|,$$

$$\mathrm{RMS} = \sqrt{(\sum_{j=0}^{M} (\breve{u}^n(x_j) - u^n(x_j))^2)/M},$$

are used to measure the accuracy of our scheme where $\breve{u}$ is the approximation solution and M and h_m are the number and the space of points that used to compute the errors, respectively.

In this paper, we propose the following values for the shape parameters c and s :

$$c = |\sigma \log(h)h|, \qquad s = |\sigma \log(h2)h2|, \tag{21}$$

where σ is an input parameter. Our numerical observations show that the accuracy of the solution depends on the magnitude of σ in such a way that the error drops to a minimum by adjusting the value of σ so that the numerical solution provides a reasonable approximation to the exact solution.

The computations associated with our experiments are performed in Maple 14 on a PC with a CPU of 2.4 GHZ.

Table 1
The comparison of the L_∞, L_2 and RMS errors of our method with the results of [1,2] at different times of experiment 1.

Time	1	3	5	7	10
MQQI; $N = 10$, $M = 100$ and $\Delta t = 0.0001$					
L_∞	6.0719E-11	1.1651E-10	1.4196E-10	1.1799E-10	3.3761E-11
L_2	4.1778E-10	7.6003E-10	1.0234E-09	9.2925E-10	1.9398E-10
RMS	4.1571E-11	7.5626E-11	1.0182E-10	9.2464E-11	1.9302E-11
Time	1	3	5	10	20
CBSM [1]; $N = 100$ and $\Delta t = 0.0001$					
L_∞	4.7698E-13	2.8899E-13	4.3667E-13	8.4593E-13	3.1344E-12
L_2	1.8986E-12	1.0680E-12	1.5686E-12	5.3244E-12	4.4324E-11
RMS	2.6850E-14	1.5105E-13	2.2184E-12	5.9738E-12	2.2332E-12
Time	1	3	5	7	10
TPSM [2]; $N = 100$ and $\Delta t = 0.0001$					
L_∞	1.2540E-05	1.5554E-05	3.3792E-05	3.7753E-05	1.3086E-05
L_2	6.5422E-05	1.1717E-04	2.2011E-04	2.5892E-04	7.9854E-05
RMS	6.5097E-06	1.1659E-05	2.1902E-05	2.5763E-05	7.9458E-06

Experiment 1. In this experiment, the Klein-Gordon equation (1) is considered with $\mu = 1$, $f(x,t) = -x\cos(t) + x^2\cos(t)$ in interval $-1 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u^2$; so that the values of constants in (16) are $\alpha = 0$, $\beta = 1$ and $k = 2$. The initial conditions are given by

$$u(x,0) = x, \qquad -1 \leqslant x \leqslant 1,$$

$$u_t(x,0) = 0, \qquad -1 \leqslant x \leqslant 1.$$

The exact solution is given in [23] as

$$u(x,t) = x\cos(t).$$

The boundary function $g(x,t)$ can be extracted from the exact solution. The L_2, L_∞ and RMS errors in the solutions with $\Delta t = 0.001$ and $\Delta t = 0.0001$, $\sigma = 0.815$, $N = 10$, $M = 100$ and $\gamma = \frac{1}{12}$ are listed in Table 1 and compared with the results in [1,2]. The space-time graphs of the estimated solution is drawn in Fig. 1. Table 1 indicates that the proposed method requires less nodes to attain the accuracy of the CBSM [1] and TPSM [2]. Also, it show that this scheme performs better than TPSM.

Experiment 2. Consider the Klein-Gordon equation (1) with $\mu = 1$, $f(x,t) = 6xt(x^2 - t^2) + x^6t^6$ in interval $0 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u^2$ wherein α, β and k are considered as 0, 1 and 2, respectively. The initial conditions are given by

$$u(x,0) = 0, \qquad 0 \leqslant x \leqslant 1,$$

$$u_t(x,0) = 0, \qquad 0 \leqslant x \leqslant 1.$$

The exact solution is given in [23] as

$$u(x,t) = x^3t^3.$$

The boundary function $g(x,t)$ can be extracted from the exact solution. Table 2 shows the L_2, L_∞ and RMS errors in the solutions with $\Delta t = 0.001$, $\sigma = 0.815$, $N = M = 50$ and $\gamma = \frac{1}{12}$. Our numerical results are compared with the results in [1,2]. Moreover, the space-time graph of the estimated solution is drawn in Fig. 2. Table 2 shows that our scheme has better accuracy than TPSM whereas we use time step $\Delta t = 0.001$ and the time step $\Delta t = 0.0001$ is used in TPSM [2].

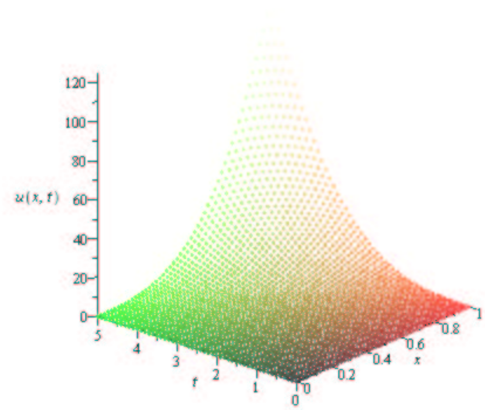

Figure 2:
The graph of the estimated solution up to $t = 5$ with $\Delta t = 0.001$ and $N = 50$ of experiment 2.

Table 2
The comparison of the L_∞, L_2 and RMS errors of our method with the results of [1, 2] at different times of experiment 2.

Time	1	2	3	4	5
MQQI; $N = 50$, $M = 50$ and $\Delta t = 0.001$					
L_∞	2.5756E-05	2.0580E-04	6.5461E-04	1.4417E-03	2.5914E-03
L_2	6.5854E-05	6.0958E-04	1.4182E-03	2.4832E-03	3.8577E-03
RMS	9.2213E-06	8.5358E-05	1.9858E-04	3.4772E-04	5.4019E-04
CBSM [1]; $N = 50$ and $\Delta t = 0.0001$					
L_∞	5.5733E-14	3.0198E-13	3.5829E-12	5.1088E-12	7.2456E-11
L_2	1.4257E-13	8.7463E-13	1.0177E-11	1.7568E-11	3.0183E-10
RMS	2.0162E-14	1.2369E-13	1.4392E-12	2.4846E-12	4.2685E-12
TPSM [2]; $N = 50$ and $\Delta t = 0.0001$					
L_∞	1.1012E-05	1.6496E-04	5.9728E-04	1.8264E-03	3.6915E-03
L_2	5.4998E-05	1.1522E-03	3.2588E-03	9.8191E-03	1.9139E-02
RMS	5.4725E-06	1.1465E-04	3.2426E-04	9.7704E-04	1.9044E-03

Experiment 3. In this experiment, we consider the Klein-Gordon equation (1) with $\mu = 2.5$, $f(x,t) = 0$ in interval $0 \leqslant x \leqslant 1$ and the nonlinear force $F(u) = u + 1.5u^3$ wherein α, β and k are considered as 1, 1.5 and 3. The initial conditions are given by

$$u(x,0) = B\tan(Kx), \qquad 0 \leqslant x \leqslant 1,$$

$$u_t(x,0) = BCK\sec^2(Kx), \qquad 0 \leqslant x \leqslant 1.$$

The exact solution is given in [24] as

$$u(x,t) = B\tan(K(x+Ct)),$$

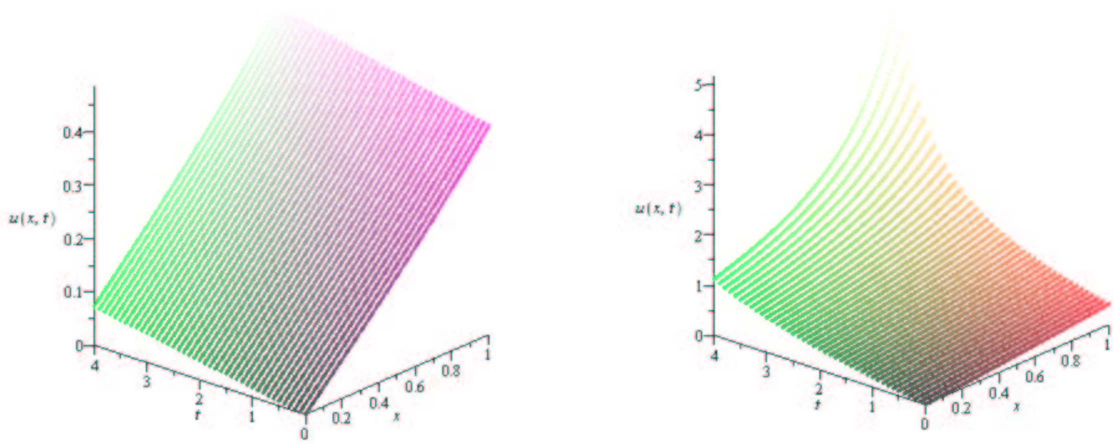

Figure 3:
The graph of the estimated solution up to $t = 4$ with $\Delta t = 0.001$ and $N = 30$ of experiment 3, $C = 0.05$ (Left), $C = 0.5$ (Right).

Table 3
The comparison of the L_∞, L_2 and RMS errors of our method with the results of [1, 2] with $C = 0.05$ at different times of experiment 3.

Time	1	2	3	4
MQQI; $N = 30$, $M = 100$ and $\Delta t = 0.001$				
L_∞	2.9293E-06	3.1901E-06	3.5819E-06	4.1980E-06
L_2	1.5566E-05	2.0814E-05	2.3670E-05	2.0758E-05
RMS	1.5489E-06	2.0711E-06	2.3553E-06	2.0655E-06
CBSM [1]; $N = 100$ and $\Delta t = 0.001$				
L_∞	1.1986E-08	2.4733E-08	2.8958E-08	1.9916E-08
L_2	7.5619E-08	1.7997E-07	2.0797E-07	1.4058E-07
RMS	7.5619E-08	1.7997E-08	2.0797E-08	1.4058E-08
TPSM [2]; $N = 100$ and $\Delta t = 0.001$				
L_∞	3.6497E-07	3.8952E-07	4.2123E-07	4.5928E-07
L_2	1.7861E-06	1.5383E-06	1.7275E-06	2.0097E-06
RMS	1.7772E-07	1.5306E-07	1.7190E-07	1.9997E-07

where $B = \sqrt{\frac{\alpha}{\beta}}$ and $K = \sqrt{\frac{\alpha}{-2\mu + C^2}}$. The boundary function $g(x,t)$ can be extracted from the exact solution. In Tables 3 and 4, the L_2, L_∞ and RMS errors in the solutions are listed with $\Delta t = 0.001$, $\sigma = 1$, $N = 30$, $M = 100$, $\gamma = \frac{1}{12}$ and $C = 0.05$ and $C = 0.5$. Tables 3 and 4 indicate that the proposed method requires less nodes to attain the accuracy of the CBSM [1] and TPSM [2].

Experiment 4. Consider the nonlinear Klein-Gordon equation (1) with $\mu = 1$ and the nonlinear force $F(u) = u + u^3$ in interval $-1 \leqslant x \leqslant 1$. In this case, the constants α, β and k are considered as 1, 1 and 3, respectively. The initial conditions are given by

$$u(x,0) = x^2 \cosh(x), \qquad -1 \leqslant x \leqslant 1,$$

Table 4
The comparison of the L_∞, L_2 and RMS errors of our method with the results of [1, 2] with $C = 0.5$ at different times of experiment 3.

Time	1	2	3	4
MQQI; $N = 30$, $M = 100$ and $\Delta t = 0.001$				
L_∞	7.6149E-06	2.1557E-05	1.0041E-04	2.2122E-03
L_2	3.9375E-05	1.0820E-04	4.2058E-04	6.4696E-03
RMS	3.9179E-06	1.0766E-05	4.1849E-05	6.4375E-04
CBSM [1]; $N = 100$ and $\Delta t = 0.001$				
L_∞	2.6949E-08	8.7462E-08	3.0903E-07	1.9394E-06
L_2	1.9015E-07	6.3813E-07	2.2341E-06	1.3439E-05
RMS	1.9015E-08	6.3813E-08	2.2344E-07	1.3439E-06
TPSM [2]; $N = 100$ and $\Delta t = 0.001$				
L_∞	5.9964E-06	2.1973E-05	9.0893E-05	8.2945E-04
L_2	4.0761E-05	1.5769E-04	6.4792E-04	5.3572E-03
RMS	4.0559E-06	1.5691E-05	6.4470E-05	5.3306E-04

Table 5
The comparison of the L_∞, L_2 and RMS errors of our method with the results of [1, 2] at different times of experiment 4.

Time	1	2	3	4	5
MQQI; $N = 50$, $M = 50$ and $\Delta t = 0.001$					
L_∞	2.5756E-05	2.0580E-04	6.5461E-04	1.4417E-03	2.5914E-03
L_2	6.5854E-05	6.0958E-04	1.4182E-03	2.4832E-03	3.8577E-03
RMS	9.2213E-06	8.5358E-05	1.9858E-04	3.4772E-04	5.4019E-04
CBSM [1]; $N = 50$ and $\Delta t = 0.0001$					
L_∞	3.5666E-06	3.1949E-06	3.9619E-06	5.6889E-06	6.3356E-06
L_2	2.5993E-05	2.2013E-05	2.3990E-05	2.9542E-05	3.2638E-05
RMS	2.5930E-06	2.2013E-06	2.3909E-06	2.9542E-06	2.9092E-06
TPSM [2]; $N = 50$ and $\Delta t = 0.0001$					
L_∞	5.0705E-05	5.0260E-04	2.0612E-03	6.5720E-03	1.9067E-02
L_2	2.9474E-04	2.7082E-03	9.7246E-03	2.7881E-02	7.7337E-02
RMS	2.0789E-05	1.9102E-04	6.8592E-04	1.9666E-03	5.4549E-03

$$u_t(x, 0) = x^2 \sinh(x), \qquad -1 \leqslant x \leqslant 1.$$

The exact solution is given as

$$u(x, t) = x^2 \cosh(x + t).$$

The boundary function $g(x, t)$ can be extracted from the exact solution. Table 5 shows the L_2, L_∞ and RMS errors in the solutions with $\Delta t = 0.001$, $\sigma = 0.815$, $N = M = 50$ and $\gamma = \frac{1}{12}$. We compare our results with the results in [1, 2]. Also, the space-time graph of the estimated solution is drawn in Fig. 4. Table 5 shows that our scheme has better accuracy than TPSM whereas we use the time step $\Delta t = 0.001$ and the time step $\Delta t = 0.0001$ is used in TPSM [2].

5 Conclusion

In this paper, a numerical scheme based on high accuracy MQ quasi-interpolation scheme has been applied to solve the nonlinear Klein-Gordon equation with quadratic and cubic nonlinearity. The numerical results which are

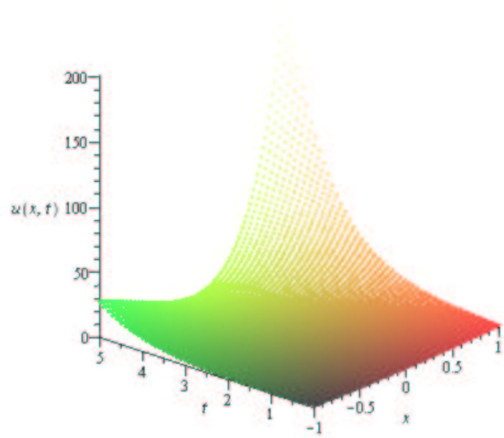

Figure 4:
The graph of the estimated solution up to $t = 5$ with $\Delta t = 0.001$ and $N = 50$ of experiment 4.

given in the previous section demonstrate the good accuracy of the present scheme. Also, the Tables show that this scheme performs better than TPS method and requires less nodes to attain accuracy. Moreover, we have used bigger time step Δt, in comparison with [2]. Therewith, we would like to emphasize that, the scheme introduced in this paper can be well studied for any other nonlinear PDEs.

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A note on strong differential subordinations using Sălăgean operator and Ruscheweyh derivative

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Abstract

In the present paper we establish several strong differential subordinations regardind the extended new operator L_α^m, given by $L_\alpha^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$, $L_\alpha^m f(z,\zeta) = (1-\alpha)R^m f(z,\zeta) + \alpha S^m f(z,\zeta)$, where $R^m f(z,\zeta)$ denote the extended Ruscheweyh derivative, $S^m f(z,\zeta)$ is the extended Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions. A number of interesting consequences of some of these strong subordination results are discussed. For functions belonging to the class $SL_m(\delta,\alpha,\zeta)$, $\delta \in [0,1)$, $\alpha \geq 0$ and $m \in \mathbb{N}$, of analytic functions in $U \times \overline{U}$, which are investigated in this paper, the author derives several interesting strong differential subordination results. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

Keywords: strong differential subordination, univalent function, convex function, best dominant, extended differential operator.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let
$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$
where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and
$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\},$$
for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [5] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [7], [6].

Definition 1.1 *[7] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.2 *[7] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta)=H(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f\left(U\times\overline{U}\right)\subset H\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma 1.3 *[4] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $Re\gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta)+\frac{1}{\gamma}zp'(z,\zeta)\prec\prec h(z,\zeta),$$

then

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

where $g(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$ is convex and it is the best dominant.

Lemma 1.4 *[4] Let $g(z,\zeta)$ be a convex function in U, for all $\zeta\in\overline{U}$, and let*

$$h(z,\zeta)=g(z,\zeta)+n\alpha zg'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $\alpha>0$ and n is a positive integer. If

$$p(z,\zeta)=g(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots,\quad z\in U,\ \zeta\in\overline{U},$$

is holomorphic in U, for all $\zeta\in\overline{U}$, and

$$p(z,\zeta)+\alpha zp'(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

then

$$p(z,\zeta)\prec\prec g(z,\zeta)$$

and this result is sharp.

We extend the Sălăgean operator [9] and the Ruscheweyh derivative [8] to the new class of analytic functions $\mathcal{A}^*_\zeta$ introduced in [6].

Definition 1.5 *[4] For $f\in\mathcal{A}^*_{n\zeta}$, $n,m\in\mathbb{N}$, the extended operator S^m is defined by $S^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned}S^0f(z,\zeta)&=f(z,\zeta)\\ S^1f(z,\zeta)&=zf'(z,\zeta)\\ &\ldots\\ S^{m+1}f(z,\zeta)&=z\left(S^mf(z,\zeta)\right)',\quad z\in U,\ \zeta\in\overline{U}.\end{aligned}$$

Remark 1.6 *[4] If $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^\infty a_j(\zeta)z^j$, then $S^mf(z,\zeta)=z+\sum_{j=n+1}^\infty j^ma_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.*

Definition 1.7 *[4] For $f\in\mathcal{A}^*_{n\zeta}$, $n,m\in\mathbb{N}$, the extended operator R^m is defined by $R^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned}R^0f(z,\zeta)&=f(z,\zeta)\\ R^1f(z,\zeta)&=zf'(z,\zeta)\\ &\ldots\\ (m+1)R^{m+1}f(z,\zeta)&=z\left(R^mf(z,\zeta)\right)'+mR^mf(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.\end{aligned}$$

Remark 1.8 *[4] If $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^\infty a_j(\zeta)z^j$, then $R^mf(z,\zeta)=z+\sum_{j=n+1}^\infty C^m_{m+j-1}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.*

2 Main results

We also extend the differential operator L_α^m studied in [1], [2], [3] to the new class of analytic functions $\mathcal{A}_{n\zeta}^*$.

Definition 2.1 *[4] Let $\alpha \geq 0$, $m \in \mathbb{N}$. Denote by L_α^m the extended operator given by $L_\alpha^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$,*

$$L_\alpha^m f(z,\zeta) = (1-\alpha) R^m f(z,\zeta) + \alpha S^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 2.2 *[4] If $f \in \mathcal{A}_{n\zeta}^*$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then*
$L_\alpha^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Definition 2.3 *Let $\delta \in [0,1)$, $\alpha \geq 0$ and $n, m \in \mathbb{N}$. A function $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$ is said to be in the class $SL_m(\delta,\alpha,\zeta)$ if it satisfies the inequality*

$$Re\left(L_\alpha^m f(z,\zeta)\right)_z' > \delta, \qquad z \in U,\ \zeta \in \overline{U}. \tag{2.1}$$

Theorem 2.4 *The set $SL_m(\delta,\alpha,\zeta)$ is convex.*

Proof. Let the functions

$$f_k(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_{jk}(\zeta) z^j, \qquad \text{for } k = 1,2, \quad z \in U,\ \zeta \in \overline{U},$$

be in the class $SL_m(\delta,\alpha,\zeta)$. It is sufficient to show that the function

$$h(z,\zeta) = \eta_1 f_1(z,\zeta) + \eta_2 f_2(z,\zeta)$$

is in the class $SL_m(\delta,\alpha,\zeta)$, with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

Since $h(z,\zeta) = z + \sum_{j=n+1}^{\infty} (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) z^j$, $\quad z \in U$, $\zeta \in \overline{U}$, then

$$L_\alpha^m h(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) z^j,\ z \in U, \zeta \in \overline{U}. \tag{2.2}$$

Differentiating with respect to z (2.2) we obtain
$\left(L_\alpha^m h(z,\zeta)\right)_z' = 1 + \sum_{j=n+1}^{\infty} \left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] (\eta_1 a_{j1}(\zeta) + \eta_2 a_{j2}(\zeta)) j z^{j-1}$, $z \in U$, $\zeta \in \overline{U}$.

Hence

$$\begin{aligned} Re\left(L_\alpha^m h(z,\zeta)\right)_z' &= 1 + Re\left(\eta_1 \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{j1}(\zeta) z^{j-1}\right) \\ &+ Re\left(\eta_2 \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{j2}(\zeta) z^{j-1}\right). \end{aligned} \tag{2.3}$$

Taking into account that $f_1, f_2 \in SL_m(\delta,\alpha,\zeta)$ we deduce

$$Re\left(\eta_k \sum_{j=n+1}^{\infty} j\left[\alpha j^m + (1-\alpha) C_{m+j-1}^m\right] a_{jk}(\zeta) z^{j-1}\right) > \eta_k(\delta - 1), \quad k = 1,2. \tag{2.4}$$

Using (2.4) we get from (2.3)

$$Re\left(L_\alpha^m h(z,\zeta)\right)_z' > 1 + \eta_1(\delta-1) + \eta_2(\delta-1) = \delta, \quad z \in U,\ \zeta \in \overline{U},$$

which is equivalent that $SL_m(\delta,\alpha,\zeta)$ is convex. ∎

Theorem 2.5 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, $c>0$. If $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in SL_m(\delta,\alpha,\zeta)$ and $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, then*

$$(L_\alpha^m f(z,\zeta))'_z \prec\prec h(z,\zeta)\,,\quad z\in U,\ \zeta\in\overline{U}, \tag{2.5}$$

implies

$$(L_\alpha^m F(z,\zeta))'_z \prec\prec g(z,\zeta)\,,\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{2.6}$$

Differentiating (2.6), with respect to z, we have $(c+1)F(z,\zeta)+zF'_z(z,\zeta)=(c+2)f(z,\zeta)$ and

$$(c+1)L_\alpha^m F(z,\zeta)+z(L_\alpha^m F(z,\zeta))'_z=(c+2)L_\alpha^m f(z,\zeta)\,,\qquad z\in U,\ \zeta\in\overline{U}. \tag{2.7}$$

Differentiating (2.7) with respect to z we have

$$(L_\alpha^m F(z,\zeta))'_z+\frac{1}{c+2}z(L_\alpha^m F(z,\zeta))''_{z^2}=(L_\alpha^m f(z,\zeta))'_z\,,\qquad z\in U,\ \zeta\in\overline{U}. \tag{2.8}$$

Using (2.8), the strong differential subordination (2.5) becomes

$$(L_\alpha^m F(z,\zeta))'_z+\frac{1}{c+2}z(L_\alpha^m F(z,\zeta))''_{z^2}\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)\,. \tag{2.9}$$

Denote

$$p(z,\zeta)=(L_\alpha^m F(z,\zeta))'_z\,,\qquad z\in U,\ \zeta\in\overline{U}. \tag{2.10}$$

Replacing (2.10) in (2.9) we obtain

$$p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)\prec\prec g(z,\zeta)+\frac{1}{c+2}zg'_z(z,\zeta)\,,\qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.4 we have

$$p(z,\zeta)\prec\prec g(z,\zeta)\,,\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}\quad (L_\alpha^m F(z,\zeta))'_z\prec\prec g(z,\zeta)\,,\ z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. ■

Theorem 2.6 *Let $h(z,\zeta)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$, $\delta\in[0,1)$ and $c>0$. If $\alpha\geq 0$, $m\in\mathbb{N}$ and I_c is given by Theorem 2.5, then*

$$I_c\left[SL_m(\delta,\alpha,\zeta)\right]\subset SL_m(\delta^*,\alpha,\zeta)\,, \tag{2.11}$$

where $\delta^*=2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right)$ and $\beta(x)=\int_0^1\frac{t^{x+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.5 we get from the hypothesis of Theorem 2.6 that

$$p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)\prec\prec h(z,\zeta)\,,$$

where $p(z,\zeta)$ is defined in (2.10).

Using Lemma 1.3 we deduce that

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

that is

$$\left(L_\alpha^m F(z,\zeta)\right)'_z \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where

$$\begin{aligned} g(z,\zeta) &= \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z t^{\frac{c+2}{n}-1}\frac{\zeta+(2\delta-\zeta)t}{1+t}dt = \\ & (2\delta-\zeta)+\frac{2(c+2)(\zeta-\delta)}{nz^{\frac{c+2}{n}}}\int_0^z \frac{t^{\frac{c+2}{n}-1}}{1+t}dt. \end{aligned}$$

Since g is convex and $g\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \text{Re }\left(L_\alpha^m F(z,\zeta)\right)'_z \geq \min_{|z|=1}\text{Re } g(z,\zeta) = \text{Re } g(1,\zeta) = \delta^* = \\ 2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right). \end{aligned} \tag{2.12}$$

From (2.12) we deduce inclusion (2.11). ∎

Theorem 2.7 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\alpha\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(L_\alpha^m f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.13}$$

holds, then

$$\frac{L_\alpha^m f(z,\zeta)}{z} \prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. By using the properties of the extended operator L_α^m, we have

$$L_\alpha^m f(z,\zeta) = z+\sum_{j=n+1}^{\infty}\left(\alpha j^m+(1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j, \qquad z\in U,\ \zeta\in\overline{U}.$$

Consider $p(z,\zeta)=\frac{L_\alpha^m f(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}\left(\alpha j^m+(1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j}{z}=1+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots$, $z\in U$, $\zeta\in\overline{U}$.

Let $L_\alpha^m f(z,\zeta)=zp(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Differentiating with respect to z, we obtain $\left(L_\alpha^m f(z,\zeta)\right)'_z = p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then (2.13) becomes

$$p(z,\zeta)+zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta)+zg'_z(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.4, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}, \quad \text{i.e.} \quad \frac{L_\alpha^m f(z,\zeta)}{z} \prec\prec g(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}.$$

∎

Theorem 2.8 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $\alpha \geq 0$, $n,m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(L^m_\alpha f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.14}$$

holds, then

$$\frac{L^m_\alpha f(z,\zeta)}{z} \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt$ is convex and it is the best dominant.

Proof. With notation $p(z,\zeta)=\frac{L^m_\alpha f(z,\zeta)}{z}=1+\sum_{j=n+1}^{\infty}\left(\alpha j^m+(1-\alpha)C^m_{m+j-1}\right)a_j(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, $p(z,\zeta)+zp'_z(z,\zeta)=\left(L^m_\alpha f(z,\zeta)\right)'_z$.

We have $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Since $p(z,\zeta)\in \mathcal{H}^*[1,n,\zeta]$, using Lemma 1.3, for $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{L^m_\alpha f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

Corollary 2.9 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U\times\overline{U}$, $0\leq\beta<1$. If $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$\left(L^m_\alpha f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.15}$$

then

$$\frac{L^m_\alpha f(z,\zeta)}{z} \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where g is given by $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function g is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.8 and considering $p(z,\zeta)=\frac{L^m_\alpha f(z,\zeta)}{z}$, the strong differential subordination (2.15) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.3 for $\gamma=1$, we have $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.,

$$\frac{L^m_\alpha f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt$$

$$=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt, \quad z \in U,\ \zeta \in \overline{U}.$$

■

Theorem 2.10 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\alpha\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.16}$$

holds, then

$$\frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $L^m_\alpha f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C^m_{m+j-1}\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} = \frac{z+\sum_{j=n+1}^{\infty}\left(\alpha j^{m+1}+(1-\alpha)C^{m+1}_{m+j}\right)a_j(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left(\alpha j^m+(1-\alpha)C^m_{m+j-1}\right)a_j(\zeta)z^j} = \frac{1+\sum_{j=n+1}^{\infty}\left(\alpha j^{m+1}+(1-\alpha)C^{m+1}_{m+j}\right)a_j(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}\left(\alpha j^m+(1-\alpha)C^m_{m+j-1}\right)a_j(\zeta)z^{j-1}}$.

We have $p'_z(z,\zeta) = \frac{\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z}{L^m_\alpha f(z,\zeta)} - p(z,\zeta) \cdot \frac{(L^m_\alpha f(z,\zeta))'_z}{L^m_\alpha f(z,\zeta)}$. Then $p(z,\zeta) + zp'_z(z,\zeta) = \left(\frac{zL^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)}\right)'_z$.

Relation (2.16) becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and by using Lemma 1.4 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{L^{m+1}_\alpha f(z,\zeta)}{L^m_\alpha f(z,\zeta)} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.11 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z + \frac{(1-\alpha)\, mz(R^m f(z,\zeta))''_{z^2}}{m+1} \prec\prec h(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{2.17}$$

holds, then

$$[L^m_\alpha f(z,\zeta)]'_z \prec\prec g(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

This result is sharp.

Proof. By using the properties of the extended operator L^m_α, we obtain

$$L^{m+1}_\alpha f(z,\zeta) = (1-\alpha) R^{m+1} f(z,\zeta) + \alpha S^{m+1} f(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}. \tag{2.18}$$

Then (2.17) becomes

$$\left((1-\alpha)R^{m+1}f(z,\zeta) + \alpha S^{m+1} f(z,\zeta)\right)'_z + \frac{(1-\alpha)\, mz\,(R^m f(z,\zeta))''_{z^2}}{m+1} \prec\prec h(z,\zeta),$$

with $z \in U$, $\zeta \in \overline{U}$.

After a short calculation, we obtain

$(1-\alpha)(R^m f(z,\zeta))'_z + \alpha (S^m f(z,\zeta))'_z + z\left((1-\alpha)(R^m f(z,\zeta))''_{z^2} + \alpha (S^m f(z,\zeta))''_{z^2}\right) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Let

$$p(z,\zeta) = (1-\alpha)(R^m f(z,\zeta))'_z + \alpha (S^m f(z,\zeta))'_z = (L^m_\alpha f(z,\zeta))'_z \tag{2.19}$$

$$= 1 + \sum_{j=n+1}^{\infty} \left(\alpha j^{m+1} + (1-\alpha)\, jC^m_{m+j-1}\right) a_j(\zeta) z^{j-1} = 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots.$$

Using the notation in (2.19), the strong differential subordination becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta).$$

By using Lemma 1.4, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.} \quad (L^m_\alpha f(z,\zeta))'_z \prec\prec g(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

and this result is sharp. ■

Theorem 2.12 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $\alpha \geq 0$, $n,m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$[L^{m+1}_{\alpha} f(z,\zeta)]'_z + \frac{(1-\alpha)\, mz\, (R^m f(z,\zeta))''_{z^2}}{m+1} \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \tag{2.20}$$

holds, then

$$(L^m_{\alpha} f(z,\zeta))'_z \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

where $g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt$ is convex and it is the best dominant.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have $L^m_{\alpha} f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C^m_{m+j-1}\right) a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.
Consider $p(z,\zeta) = (L^m_{\alpha} f(z,\zeta))'_z = 1 + \sum_{j=n+1}^{\infty} \left(\alpha j^{m+1} + (1-\alpha) j C^m_{m+j-1}\right) a_j(\zeta) z^{j-1} \in \mathcal{H}^*[1,n,\zeta]$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = [L^{m+1}_{\alpha} f(z,\zeta)]'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$, $z \in U$, $\zeta \in \overline{U}$.

Then $[L^{m+1}_{\alpha} f(z,\zeta)]'_z + \frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1} \prec\prec h(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.3, for $\gamma = 1$, we obtain $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $(L^m_{\alpha} f(z,\zeta))'_z \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ∎

Corollary 2.13 *Let $h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}$ a convex function in $U \times \overline{U}$, $0 \leq \beta < 1$. If $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$[L^{m+1}_{\alpha} f(z,\zeta)]'_z + \frac{(1-\alpha)\, mz\, (R^m f(z,\zeta))''_{z^2}}{m+1} \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \tag{2.21}$$

then

$$(L^m_{\alpha} f(z,\zeta))'_z \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U},$$

where g is given by $g(z,\zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$, $\zeta \in \overline{U}$. The function g is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.12 and considering $p(z,\zeta) = (L^m_{\alpha} f(z,\zeta))'_z$, the strong differential subordination (2.21) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}, \quad z \in U, \ \zeta \in \overline{U}.$$

By using Lemma 1.3 for $\gamma = 1$, we have $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.,

$$(L^m_{\alpha} f(z,\zeta))'_z \prec\prec g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{\zeta + (2\beta-\zeta)\, t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \quad z \in U, \ \zeta \in \overline{U}.$$

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New Iterative Algorithms with Errors for Approximating Zeroes of m-accretive Operators[1]

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Abstract. The purpose of this paper is to construct a new class of iterative sequence with errors and to prove convergence of the iterative sequence with errors to a zero of m-accretive operators in Banach spaces. Our results improve and generalize the corresponding results of recent works.

Key Words: Uniformly smooth Banach space, iterative algorithm with errors, m-accretive operator, resolvent operator and convergence.

2000 MR Subject Classification 47H06, 47J25

1 Introduction

Let X be a real Banach space, C be a nonempty closed convex subset of X, $A : X \to X$ be an m-accretive operator (possibly multivalued) and $J_r = (I + rA)^{-1}$ be the resolvent of A for all $r > 0$. The following iterative schemes is well-known:

$$\begin{aligned} &x_0 = u \in X, \\ &x_{n+1} = J_{r_n} x_n, \quad n \geq 0, \end{aligned} \tag{1.1}$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of (1.1) has been studied by many authors. See, for example, [2–4, 10, 12, 15, 16, 19, 20] and the references therein.

On the other hand, Halpern [9] and Mann [14] introduced the following iterative schemes for approximating fixed points of nonexpansive mappings T of X into itself, respectively:

$$\begin{aligned} &x_0 \in X, \\ &x_{n+1} = \alpha_n u + (1 - \alpha_n) T(x_n), \quad n \geq 0, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &x_0 \in X, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n), \quad n \geq 0, \end{aligned} \tag{1.3}$$

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where $u \in C$ and $\{\alpha_n\}$ is a sequence in [0,1]. The iterative schemes (1.2) and (1.3) have been studied extensively by Takahashi [21] and others (see the references therein). Further, Kim and Xu [11] studied the sequence generated by the algorithm (1.2) when $T = J_{r_n}$ for $r_n > 0$.

Let C be a nonempty closed convex subset of X such that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I+rA)$. We can consider the following corresponding iterative schemes to (1.2) and (1.3), respectively:

$$x_{n+1} = P(\alpha_n x + (1-\alpha_n) J_{r_n} x_n + f_n), \quad n \geq 0,$$

and

$$x_{n+1} = P(\alpha_n x_n + (1-\alpha_n) J_{r_n} x_n + f_n), \quad n \geq 0,$$

where P is a nonexpansive retraction of X onto C and f_n is the term showing a computational error.

Recently, Cho et al. [6] studied a new iterative scheme

$$x_{n+1} = \alpha_n u + \beta_n J_{r_n} x_n + \gamma_n P e_n, \quad n \geq 0, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$, $\{e_n\}$ is a sequence in X and $\{r_n\}$ is a sequence in $(0,\infty)$ satisfying some conditions.

Motivated and inspired by (1.1)-(1.4) and recent works of [5, 8, 13, 17], in this paper, we introduce a new class of terative schemes with errors

$$\begin{aligned} z_n &= \lambda_n x_n + \mu_n J_{r_n} x_n + \nu_n h_n, \\ y_n &= a_n x_n + b_n J_{r_n} z_n + c_n g_n, \\ x_{n+1} &= \alpha_n u + \beta_n J_{r_n} y_n + \gamma_n f_n, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ are sequences in $(0,1)$, $\{f_n\}$ $\{g_n\}$, $\{h_n\}$ are sequences in X and $\{r_n\}$ is a sequence in $(0,\infty)$ satisfying some conditions and show that, if $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to a zero of A.

2 Preliminaries

Throughout this paper, let X be a real Banach space with norm $\|\cdot\|$ and let X^* denote the dual space of X. We denote the value of $y^* \in X^*$ at $x \in X$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in X, we denote strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$.

The (*normalized*) *duality mapping* J from X into 2^{X^*} is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for any $x \in X$. The norm of X is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in X : \|x\| = 1\}$. The space X is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. It is known that X is smooth if and only if each duality mapping J is single-valued.

Let C be a closed convex subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of X is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D. Let D be a subset of C. Then a mapping P of D into itself is said to be a *retraction* if $P^2 = P$. A subset D of C is said to be a *nonexpansive retract* of C if there exists a nonexpansive retraction of C onto D. Further, a map $Q : C \to D$ is sunny ([18]) provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows ([7, 18]): If X is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Q(x), J(y - Q(x)) \rangle \leq 0 \tag{2.2}$$

for all $x \in C$ and $y \in D$.

Reich [19] showed that, if X is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows:

Lemma 2.1. ([19]) *Let X be a uniformly smooth Banach space and $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C : x \mapsto tu + (1 - t)tx$ converges strongly as $t \to 0$ to a fixed point of T. Define a mapping $Q : C \to F(T)$ by $Q(u) = s - \lim_{t \to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$ and*

$$\langle u - Q(u), J(z - Q(u)) \rangle \leq 0$$

for all $u \in C$ and $z \in F(T)$.

Let I denote the identity operator on X. An operator $A : X \to X$ with domain $D(A) = \{z \in X : A(z) \neq \emptyset\}$ and range $R(A) = \cup\{A(z) : z \in D(A)\}$ is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in A(x_i)$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2) \rangle \geq 0.$$

If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for any $x_i \in D(A)$, $y_i \in A(x_i)$, $i = 1, 2$, and $r > 0$.

If A is accretive, then we can define a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by

$$J_r = (I + rA)^{-1}$$

for each $r > 0$, which is called the *resolvent* of A. We also define the *Yosida approximation* A_r by

$$A_r = \frac{1}{r}(I - J_r).$$

An accretive operator A is said to be *m-accretive* if $R(I + rA) = X$ for any $r > 0$ ([1]).

The set of zeros of A is denoted by F , that is,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that, if A is m-accretive, then $J_r : X \to X$ is nonexpansive and $F(J_r) = F = A^{-1}(0)$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$. We assume that C is convex.

On the other hand, we know that $A_r(x) \in A(J_r x)$ for all $x \in R(I + rA)$ and $\|A_r(x)\| \le \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$.

Lemma 2.2. ([23]) *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers satisfying*

$$a_{n+1} \le (1 - t_n)a_n + t_n b_n + c_n, \quad n \ge 0,$$

where $\{t_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are two nonnegative real numbers sequences such that

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\sum_{n=0}^{\infty} t_n = \infty$, $\sum\limits_{n=0}^{\infty} c_n < \infty$,

(ii) *either* $\limsup_{n\to\infty} b_n \le 0$ *or* $\sum_{n=0}^{\infty} t_n b_n < \infty$.

Then $a_n \to 0$ as $n \to \infty$.

Lemma 2.3. *In a Banach space X, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $x, y \in X$, where $j(x + y) \in J(x + y)$.

Lemma 2.4. ([1]) *For all $\lambda > 0$, $\mu > 0$ and $x \in X$, $J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)J_\lambda x\Big)$.*

Lemma 2.5. *A Banach space X is uniformly smooth if and only if the duality map J is the single-valued and norm-to-norm uniformly continuous on bounded sets of X.*

3 Main Results

In this section, we shall give some strong convergence theorems of the iterative sequence $\{x_n\}$ with errors defined by (1.5) to a zero of the accretive operator A in Banach spaces.

Theorem 3.1. *Let X be a uniformly smooth Banach space and A be an m-accretive operator in X such that $A^{-1}(0) \ne \emptyset$. Assume that the sequences $\{\alpha_n\}$,*

$\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$ in $(0,1)$, $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ in X and $\{r_n\}$ in $(0,\infty)$ satisfy the following conditions:

(i) $\alpha_n+\beta_n+\gamma_n=1$, $a_n+b_n+c_n=1$, $\lambda_n+\mu_n+\nu_n=1$;

(ii) $\sum\limits_{n=0}^{\infty}\alpha_n=+\infty$, $\alpha_n\to 0$, $\gamma_n\to 0$ *and there exist* α,γ *such that* $\alpha_n>\alpha>0$ *and* $\gamma_n>\gamma>0$, $\sum\limits_{n=0}^{\infty}\gamma_n<+\infty, c_n\to 0, \nu_n\to 0$;

(iii) $r_n\to\infty$ *as* $n\to\infty$;

(iv) $\sum_{n=1}^{\infty}|r_n-r_{n-1}|<\infty$, $\sum_{n=1}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty$, $\sum_{n=1}^{\infty}|\nu_n-\nu_{n-1}|<\infty$, $\sum_{n=1}^{\infty}|a_n-a_{n-1}|<\infty$, $\sum_{n=1}^{\infty}|c_n-c_{n-1}|<\infty$, $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$ *and* $|\gamma_n-\gamma_{n-1}|<\infty$;

(v) $\{f_n\}$, $\{g_n\}$ *and* $\{h_n\}$ *are bounded.*

Then, for any given $u\in C$, *the sequence* $\{x_n\}$ *defined by* (1.5) *converges strongly to a zero of* A.

Proof. We proceed with the following four steps.

Step 1. We observe that $\{x_n\}$ is bounded. Indeed, if we take a fixed element $p\in A^{-1}(0)$, and set

$$M=\max\{\|x_0-p\|,\|u-p\|,\sup_{n\ge 0}\|f_n-p\|,\sup_{n\ge 0}\|g_n-p\|,\sup_{n\ge 0}\|h_n-p\|\},$$

then we have $\|x_0-p\|\le M$ and $\|u-p\|\le M$. Assume that $\|x_n-p\|\le M$ for some positive integer n. Then, by using (1.5), we have

$$\begin{aligned}\|z_n-p\|&\le\lambda_n\|x_n-p\|+\mu_n\|J_{r_n}x_n-p\|+\nu_n\|h_n-p\|\\&\le(1-\nu_n)\|x_n-p\|+\nu_n\|h_n-p\|\\&\le M,\end{aligned}$$

$$\begin{aligned}\|y_n-p\|&\le a_n\|x_n-p\|+b_n\|J_{r_n}z_n-p\|+c_n\|g_n-p\|\\&\le a_n\|x_n-p\|+b_n\|z_n-p\|+c_n\|g_n-p\|\\&\le(a_n+b_n+c_n)M\\&=M,\end{aligned}$$

$$\begin{aligned}\|x_{n+1}-p\|&\le\alpha_n\|u-p\|+\beta_n\|J_{r_n}y_n-p\|+\gamma_n\|f_n-p\|\\&\le\alpha_n\|u-p\|+\beta_n\|y_n-p\|+\gamma_n\|f_n-p\|\\&\le(\alpha_n+\beta_n+\gamma_n)M\\&=M.\end{aligned}$$

Thus, by induction, we assert that $\|x_n-p\|\le M$ for all $n\ge 0$ and hence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$. Further, $\{J_{r_n}x_n\}$, $\{J_{r_n}y_n\}$ and $\{J_{r_n}z_n\}$ are also bounded since J_{r_n} is nonexpansive. As a result, we obtain, by the condition (ii),

$$\|x_{n+1}-J_{r_n}y_n\|=\alpha_n\|u-J_{r_n}y_n\|+\gamma_n\|f_n-J_{r_n}y_n\|\to 0\quad\text{as}\quad n\to\infty.\tag{3.1}$$

Step 2. We prove $\|x_{n+1}-x_n\|\to 0$. It follows from (1.5) that

$$\begin{aligned}\|z_n-z_{n-1}\|\le{}&\lambda_n\|x_n-x_{n-1}\|+\mu_n\|J_{r_n}x_n-J_{r_{n-1}}x_{n-1}\|\\&+\nu_n\|h_n-h_{n-1}\|+|\lambda_n-\lambda_{n-1}|\cdot\|x_{n-1}-J_{r_{n-1}}x_{n-1}\|\\&+|\nu_n-\nu_{n-1}|\cdot\|J_{r_{n-1}}x_{n-1}-h_{n-1}\|.\end{aligned}\tag{3.2}$$

Lemma 2.4 implies that

$$J_{r_n}x_n = J_{r_{n-1}}\Big(\frac{r_{n-1}}{r_n}x_n + \Big(1-\frac{r_{n-1}}{r_n}\Big)J_{r_n}x_n\Big).$$

By the assumption (iii) on $\{r_n\}$, without loss of generality, we assume that $\epsilon < r_{n-1} < r_n$ for some $\epsilon > 0$ for all $n \geq 1$. Then we have

$$\begin{aligned}
&\|J_{r_n}x_n - J_{r_{n-1}}x_{n-1}\| \\
&\leq \Big\|\frac{r_{n-1}}{r_n}x_n + \Big(1-\frac{r_{n-1}}{r_n}\Big)J_{r_n}x_n - x_{n-1}\Big\| \\
&= \Big\|\frac{r_{n-1}}{r_n}(x_n - x_{n-1}) + \Big(1-\frac{r_{n-1}}{r_n}\Big)(J_{r_n}x_n - x_{n-1})\Big\| \\
&\leq \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}x_n - x_{n-1}\|.
\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\leq (\lambda_n + \mu_n)\|x_n - x_{n-1}\| + \mu_n\frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}x_n - x_{n-1}\| \\
&\quad + \nu_n\|h_n - h_{n-1}\| + |\lambda_n - \lambda_{n-1}| \cdot \|x_{n-1} - J_{r_{n-1}}x_{n-1}\| \\
&\quad + |\nu_n - \nu_{n-1}| \cdot \|J_{r_{n-1}}x_{n-1} - h_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + \nu_n\|h_n - h_{n-1}\| \\
&\quad + M_1(|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\nu_n - \nu_{n-1}|),
\end{aligned} \tag{3.4}$$

where M_1 is a constant such that

$$M_1 > \max\Big\{\frac{\|J_{r_n}x_n - x_{n-1}\|}{\epsilon}, \|x_{n-1} - J_{r_{n-1}}x_{n-1}\|, \|J_{r_{n-1}}x_{n-1} - h_{n-1}\|\Big\}.$$

Similarly, we have

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\leq a_n\|x_n - x_{n-1}\| + b_n\|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| \\
&\quad + c_n\|g_n - g_{n-1}\| + |a_n - a_{n-1}| \cdot \|x_{n-1} - J_{r_{n-1}}z_{n-1}\| \\
&\quad + |c_n - c_{n-1}| \cdot \|J_{r_{n-1}}z_{n-1} - g_{n-1}\| \\
&\leq a_n\|x_n - x_{n-1}\| + b_n(\|z_n - z_{n-1}\| \\
&\quad + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}z_n - z_{n-1}\|) \\
&\quad + c_n\|g_n - g_{n-1}\| + |a_n - a_{n-1}| \cdot \|x_{n-1} - J_{r_{n-1}}z_{n-1}\| \\
&\quad + |c_n - c_{n-1}| \cdot \|J_{r_{n-1}}z_{n-1} - g_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + M_1(|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\nu_n - \nu_{n-1}|) \\
&\quad + \frac{r_n - r_{n-1}}{\epsilon}\|J_{r_n}z_n - z_{n-1}\| \\
&\quad + |a_n - a_{n-1}| \cdot \|x_{n-1} - J_{r_{n-1}}z_{n-1}\| \\
&\quad + |c_n - c_{n-1}| \cdot \|J_{r_{n-1}}z_{n-1} - g_{n-1}\| \\
&\quad + c_n\|g_n - g_{n-1}\| + \nu_n\|h_n - h_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + c_n\|g_n - g_{n-1}\| + \nu_n\|h_n - h_{n-1}\| \\
&\quad + M_2(2|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| \\
&\quad + |\nu_n - \nu_{n-1}| + |a_n - a_{n-1}| + |c_n - c_{n-1}|),
\end{aligned} \tag{3.5}$$

and it follows from the assumption (ii) that

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|\alpha_n u + \beta_n J_{r_n} y_n + \gamma_n f_n \\
&\quad - (\alpha_{n-1} u + \beta_{n-1} J_{r_{n-1}} y_{n-1} + \gamma_{n-1} f_{n-1})\| \\
&\le \beta_n \|J_{r_n} y_n - J_{r_{n-1}} y_{n-1}\| \\
&\quad + \gamma_n \|f_n - f_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{r_{n-1}} y_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \cdot \|J_{r_{n-1}} y_{n-1} - f_{n-1}\| \\
&\le \beta_n (\|y_n - y_{n-1}\| + \frac{r_n - r_{n-1}}{\epsilon} \|J_{r_n} y_n - y_{n-1}\|) \\
&\quad + \gamma_n \|f_n - f_{n-1}\| + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{r_{n-1}} y_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \cdot \|J_{r_{n-1}} y_{n-1} - f_{n-1}\| \\
&\le \beta_n \|x_n - x_{n-1}\| + \beta_n (c_n \|g_n - g_{n-1}\| + \nu_n \|h_n - h_{n-1}\|) \\
&\quad + M_2 (2|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| \\
&\quad + |\nu_n - \nu_{n-1}| + |a_n - a_{n-1}| + |c_n - c_{n-1}|) \\
&\quad + \frac{r_n - r_{n-1}}{\epsilon} \|J_{r_n} y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \cdot \|u - J_{r_{n-1}} y_{n-1}\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \cdot \|J_{r_{n-1}} y_{n-1} - f_{n-1}\| + \gamma_n \|f_n - f_{n-1}\| \\
&\le [1 - (\alpha_n + \gamma_n)] \|x_n - x_{n-1}\| + M_3 (3|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| \\
&\quad + |\nu_n - \nu_{n-1}| + |a_n - a_{n-1}| + |c_n - c_{n-1}| \\
&\quad + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) \\
&\quad + (\gamma_n \|f_n - f_{n-1}\| + c_n \|g_n - g_{n-1}\| + \nu_n \|h_n - h_{n-1}\|) \\
&\le [1 - (\alpha_n + \gamma_n)] \|x_n - x_{n-1}\| \\
&\quad + (\alpha_n + \gamma_n) \cdot \frac{\gamma_n \|f_n - f_{n-1}\| + c_n \|g_n - g_{n-1}\| + \nu_n \|h_n - h_{n-1}\|}{\alpha + \gamma} \\
&\quad + M_3 (3|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\nu_n - \nu_{n-1}| \\
&\quad + |a_n - a_{n-1}| + |c_n - c_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|),
\end{aligned}
\tag{3.6}
$$

where M_2 and M_3 are constants such that

$$
\begin{aligned}
M_2 &> \max\Big\{\frac{\|J_{r_n} z_n - z_{n-1}\|}{\epsilon}, \|x_{n-1} - J_{r_{n-1}} z_{n-1}\|, \|J_{r_{n-1}} z_{n-1} - g_{n-1}\|, M_1\Big\}. \\
M_3 &> \max\Big\{\frac{\|J_{r_n} y_n - y_{n-1}\|}{\epsilon}, \|u - J_{r_{n-1}} y_{n-1}\|, \|J_{r_{n-1}} y_{n-1} - f_{n-1}\|, M_2\Big\}.
\end{aligned}
$$

By the assumptions (i)-(iii), we have that

$$
\lim_{n\to\infty} (\alpha_n + \gamma_n) = 0, \quad \sum_{n=1}^{\infty} (\alpha_n + \gamma_n) = \infty,
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{\infty} (3|r_n - r_{n-1}| + |\lambda_n - \lambda_{n-1}| + |\nu_n - \nu_{n-1}| \\
&\qquad + |a_n - a_{n-1}| + |c_n - c_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|) < \infty.
\end{aligned}
$$

Hence it follows from Lemma 2.2 and (3.6) that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.7}$$

Step 3. $\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \le 0$, where $z = \lim_{t\to\infty} J_t u$, which is guaranteed by Lemma 2.2. It follows from (1.5) that

$$\begin{aligned}
&\|J_{r_n}x_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + \|J_{r_n}y_n - J_{r_n}x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + \|y_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| \\
&\quad + b_n\|x_n - J_{r_n}z_n\| + c_n\|x_n - g_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + b_n\|x_n - J_{r_n}x_n\| \\
&\quad + b_n\|J_{r_n}x_n - J_{r_n}z_n\| + c_n\|x_n - g_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + b_n\|x_n - J_{r_n}x_n\| \\
&\quad + b_n\|x_n - z_n\| + c_n\|x_n - g_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + b_n\|x_n - J_{r_n}x_n\| \\
&\quad + b_n\mu_n\|x_n - J_{r_n}x_n\| + b_n\nu_n\|x_n - h_n\| + c_n\|x_n - g_n\|,
\end{aligned}$$

that is,

$$\begin{aligned}
&(1 - b_n - b_n\mu_n)\|J_{r_n}x_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}y_n\| + b_n\nu_n\|x_n - h_n\| + c_n\|x_n - g_n\|.
\end{aligned}$$

From (3.1), (3.7) and the condition (ii), we get

$$\|J_{r_n}x_n - x_n\| \to 0.$$

Taking a fixed number r such that $\epsilon > r > 0$, it follows from Lemma 2.4 that

$$\begin{aligned}
&\|J_{r_n}x_n - J_r x_n\| \\
&\le \left\|J_r\Big(\frac{r}{r_n}x_n + \Big(1 - \frac{r}{r_n}\Big)J_{r_n}x_n\Big) - J_r x_n\right\| \\
&\le (1 - \frac{r}{r_n})\|x_n - J_{r_n}x_n\| \\
&\le \|x_n - J_{r_n}x_n\|
\end{aligned}$$

and so

$$\begin{aligned}
\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\
&\le \|J_{r_n}x_n - x_n\| + \|J_{r_n}x_n - x_n\| \\
&\le 2\|J_{r_n}x_n - x_n\| \to 0.
\end{aligned}$$

Since in a uniformly smooth Banach space, the sunny nonexpansive retract Q from X onto the fixed point set $F(J_r)(= F = A^{-1}(0))$ of J_r is unique, it must be obtained from Reich's theorem (Lemma 2.1). Namely,

$$Q(u) = s - \lim_{t\to 0} z_t$$

for all $u \in X$, where $t \in (0,1)$ and z_t solves the fixed point equation

$$z_t = tu + (1-t)J_r z_t.$$

Thus we have

$$\|z_t - x_n\| = \|(1-t)(J_r z_t - x_n) + t(u - x_n)\|.$$

It follows from Lemma 2.3 that

$$\begin{aligned}\|z_t - x_n\|^2 &\leq (1-t)^2\|J_r z_t - x_n\|^2 + 2t\langle u - x_n, J(z_t - x_n)\rangle \\ &\leq (1 - 2t + t^2)\|z_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,\end{aligned}$$

where

$$f_n(t) = (2\|z_t - x_n\| + \|x_n - J_r x_n\|)\|x_n - J_r x_n\| \to 0 \tag{3.8}$$

as $n \to \infty$. It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{3.9}$$

Letting $n \to \infty$ in (3.9) and noting (3.8) yield

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{t}{2}\Gamma, \tag{3.10}$$

where $\Gamma > 0$ is a constant such that $\Gamma \geq \|z_t - x_n\|^2$ for all $t \in (0,1)$ and $n \geq 1$. Letting $t \to 0$, it follows from (3.10) that

$$\lim_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq 0$$

and so, for any $\epsilon > 0$, there exists a positive number δ_1 such that, for any $t \in (0, \delta_1)$,

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leq \frac{\epsilon}{2}. \tag{3.11}$$

On the other hand, $z_t \to q$ as $t \to 0$ and it follows from Lemma 2.5 that there exists $\delta_2 > 0$ such that, for any $t \in (0, \delta_2)$,

$$\begin{aligned}&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\leq |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\ &\quad + |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\leq |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \\ &\leq \frac{\epsilon}{2}.\end{aligned}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. For all $t \in (0, \delta)$, we have

$$\langle u - Q(u), J(x_n - Q(u))\rangle \leq \langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2},$$

that is,

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \leq \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (3.11) that

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u - Q(u), J(x_n - Q(u))\rangle \le 0. \tag{3.12}$$

Step 4. We shall show that $x_n \to Q(u)$ as $n \to \infty$.

In fact, by using Lemma 2.3 again we obtain

$$\begin{aligned}
&\|x_{n+1} - Q(u)\|^2 \\
&= \|\beta_n(J_{r_n}y_n - Q(u)) + \alpha_n(u - Q(u)) + \gamma_n(f_n - Q(u))\|^2 \\
&\le \beta_n^2\|J_{r_n}y_n - Q(u)\|^2 \\
&\quad + 2\langle \alpha_n(u - Q(u)) + \gamma_n(f_n - Q(u)), J(x_{n+1} - Q(u))\rangle \\
&\le (1 - (\alpha_n + \gamma_n))\|x_n - Q(u)\|^2 \\
&\quad + 2(\alpha_n + \gamma_n)\langle u - Q(u), J(x_{n+1} - Q(u))\rangle \\
&\quad + 2\gamma_n\langle f_n - Q(u), J(x_{n+1} - Q(u))\rangle.
\end{aligned}$$

By Lemma 2.2 and (3.12), now we know that $\|x_n - Q(u)\| \to 0$. This completes the proof. □

Remark 3.1. when $\gamma_n = 0$ or $c_n = 0$ or $\nu_n = 0$ in (1.5), we can also obtain the corresponding results. Our results improve and generalize the corresponding results of recent works. For more detail, see [6, 11, 22, 23] and the references therein.

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NUMERICAL METHODS TO SOLVING OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. One of priorities direction in numerical mathematics is the investigation of the numerical solution of integro-differential equations. As is known, many vital tasks such as research in the field of atomic physics, ecology, geophysics, to extended infectious diseases and, etc. reduced to solving of integro-differential equations. Here, applied forward-jumping methods to solving initial- value problem for Volterra integro- differential equations. Constructed concrete methods, which are used to solving any model problems.

1. INTRODUCTION

In construction of the mathematical models for various processes of natural science, we are faced with the finding the solution of integro-differential equations (see [1]). It is known that V.Volterra in the make up of model for some problems in the theory elasticity received a new type of equations, which he called integro-differential (see [2, pp. 22-33]). V.Volterra shown that many problems of ecology, geophysics and etc reduced to the finding of the solution of integro-differential equations with variable boundary. Consider the following initial value problem for nonlinear integro-differential equation of Volterra type:

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds, \ y(x_0) = y_0, \quad x_0 \le s \le x \le X. \tag{1}$$

Suppose that the problem (1) has a unique continuous solution $y(x)$defined on the interval $[x_0, X]$. To find the numerical solutions of the problem (1), divide the segment $[x_0, X]$ into N equal parts with the mesh points $x_i = x_0 + ih(i = 0, 1, 2, ...)$. Here the parameter $h > 0$ is the step size. Denoted by y_i- approximate, and after $y(x_i)$- the exact values of the solution of the problem (1) at the mesh point x_i $(i = 0, 1, 2, ...)$.

Note that finding approximate solutions of the problem (1) the scientists involved, beginning with the works of V.Volterra. V.Volterra to solving of the problem (1) used the method of quadratures. Using the quadrature method to solving of the problem (1) engaged in works written by many scholars (see e.g. [1] - [9]). The main idea of the quadrature method is to replace the integral with the integral

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sum:

$$\vartheta(x_n) = \int\limits_{x_0}^{x_n} K(x_n,\, s,\, y(s))ds = h\sum_{i=0}^{n} a_i K(x_n,\, x_i,\, y_i) + R_n. \tag{2}$$

Here the variables a_i $(i = 0,\, 1,\, ...,\, n)$ are the coefficients but R_n is the remainder term of the quadrature formula. As is follows from (2), on the calculation of the values of $\vartheta(x_n + h)$, the quantities of the conversion to calculation of the function $K(x,\, s,\, y)$are increases. Hence we find that in using the quadrature method volume of computing work is increases with the values of variation n. In [10], proposed the multistep method permutation to use the constant value of computational work on each step, and also constructed specific stable methods, which has the order of accuracy$p = 2\,[k/2] + 2$. The limitation obtained by Dahlquist for stable multistep methods can be written as $p \le 2[k/2]+2$ (see [11]). Because consider to constructed stable forward-jumping methods with the order of accuracy$p > 2\,[k/2] + 2$. To solving of Volterra integral equations by using this scheme, considered in [12]. Here applied the forward-jumping methods to solving of the problem (1). To this end, the problem (1) let us write in the following form:

$$y' = f(x,\, y) + \vartheta(x),\;\; y(x_0) = y_0, \tag{3}$$

$$\vartheta(x) = \int\limits_{0}^{x} K(x,\, s,\, y(s))ds. \tag{4}$$

By the replacement solving of the problem (1) is reduced to solving of the system consisting from equations (4) and of the problem (3), involved many researchers.

Consider the following forward-jumping method:

$$\sum_{i=0}^{k-m} \alpha_i \vartheta_{n+i} = h\sum_{i=0}^{k} \beta_i \vartheta'_{n+i} \quad (m > 0). \tag{5}$$

After applying the method (5) to the solving of the problem (3) we have:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} \; + h\sum_{i=0}^{k} \beta_i \vartheta_{n+i} \quad (m > 0), \tag{6}$$

If applied the method (5) to solving equation (4)then we have:

$$\sum_{i=0}^{k-l} \alpha_i \vartheta_{n+i} = h\sum_{j=0}^{k}\sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j},\, x_{n+i},\, y_{n+i}) \quad (l > 0). \tag{7}$$

Consider the investigation of the methods (6) and (7).

2. The construction and application of the forward-jumping methods to solving integro-differential equations of Volterra type

Note that the methods (6) and (7) by using any schemes can be written in the several form as the single method, one of which as the following:

$$\begin{aligned} y_{n+k-m} = \varphi(&x_n, \ldots, x_{n+k},\, y_{n+k}, \ldots, y_{n+k-m-1},\, \bar{y}_{n+k-m}, \ldots, \bar{y}_{n+k}, \\ &\vartheta_{n+k}, \ldots, \vartheta_{n+k-m-1},\, \bar{\vartheta}_{n+k-m}, \ldots, \bar{\vartheta}_{n+k}), \end{aligned} \tag{8}$$

here we assume that the values of variables $\bar{y}_{n+k-m+\nu}$, $\bar{\vartheta}_{n+k-m+\nu}$ $\quad (\nu = 0,\, 1,\, ...,\, m)$ are found by any ways. Method (8) is used for increasing the order of accuracy of the approximate values y_{n+k-m} of the solution the problem's (1) at the point x_{n+k-m}. As is known, the using of forward –jumping methods has some difficulties. To address these shortcomings, one can be use the predictor-corrector methods with special structures (see e.g. [13], [14]). Easy to prove that if a method (5) is stable, then (see e.g. [14]):

$$p \le k + m + 1 \quad (0 < m \le 3k). \tag{9}$$

In particular, consider the following method:

$$y_{n+2} = (11y_n + 8y_{n+1})/19 + h(10f_n + 57f_{n+1} + 24f_{n+2} - f_{n+3})/57 \tag{10}$$

$$(y' = f(x,\, y)).$$

For calculation the values y_{n+2} by the method (10) must be known the approximately values of quantities, y_{n+1}, y_{n+2} and y_{n+3}. Here the difficulty to contained in the computation of item y_{n+3}, which will be determine by the methods (10) in the next step. Depends from the choosing of the scheme to calculate the values y_{n+3} in the method (10), one can be receive A-stable methods. Indeed, if we use the following method

$$y_{n+3} = y_{n+2} + h(23f_{n+2} - 16f_{n+1} + 5f_n)/12, \tag{11}$$

in the method (10), then in result receive A-stable method, which will have the degree $p = 5$. Note that the corresponding forward-jumping methods is not A-stable. However, by using the forward-jumping methods constructed A-stable methods with the degree $p = k + m + 1$ (see[15]).

If we use the method (11) in (10), then we have the following:

$$\begin{aligned} y_{n+2} = {} & (11y_n + 8y_{n+1})/19 + h(10f_n + 57f_{n+1} + 24f_{n+2})/57 - \\ & -\frac{h}{57} f(x_{n+3},\, y_{n+2} + h(23f_{n+2} - 16f_{n+1} + 5f_n)/12). \end{aligned} \tag{12}$$

The receive method is implicit and has the degree $p = 5$. For applying the method (12) to solving of some problems one can use the predictor-corrector methods. Now the method (11) replace with the following:

$$y_{n+3} = y_{n+1} + h(7f_{n+2} - 2f_{n+1} + f_n)/3$$

and to use it in the method (10). Then we get an implicit method that which has the degree $p = 5$ and stable. However, it is not A-stable. Consequently, the properties of the methods based on the scheme described above is highly dependent on the choosing of the predictor formula. As the forward-jumping method let us take the following:

$$y_{n+1} = y_n + h(8f_{n+1} + 5f_n)/12 - hf(x_{n+2},\, y_{n+2})/12. \tag{13}$$

If we replace the value $y_{n+2}$by using the following formula:

$$y_{n+2} = y_{n+1} + h(3f_{n+1} - f_n)/2, \tag{14}$$

and use it in the method (13), then in result receive the method that is A-stable, and has the degree$p = 3$. But, if we replace the method (14) to the following:

$$y_{n+2} = 3y_{n+1} - 2y_n + hf_n, \tag{15}$$

then by the application of these method to solving the model problem $y' = \lambda y$, $y(0) = 1$, we have:

$$(12 - 5h\lambda)y_{n+1} = (12 + 7h\lambda + h^2\lambda^2)y_n.$$

Hence, the method obtained after accounting (15) in the method (13) is stable, but not A-stable.

Now consider the application of the forward-jumping method to solving of the problem (1). For this purpose, one can use the problem (3) and the integral equation (4). In these case we have the methods (6) and (7).

As noted above, for using of the method (6), must be known $y_{n+k-m+1}, ..., y_{n+k}$ and the values $\vartheta_{n+k-m+1}, ..., \vartheta_{n+k}$. However, if are known the values $y_{n+k-m+1}, ...,$ y_{n+k}, then one by using the methods such as (7) can be calculated $\vartheta_{n+k-m+1}, ...,$ ϑ_{n+k}. Note that if the method (6) has the degree$p$ then the method by which calculates the values ϑ_{n+k} should be has the order of accuracy not less than $p-1$. Therefore for using the forward-jumping method, with the order of accuracy greater than $k+3$ arising is necessity to change the method (7) to the corresponding forward-jumping methods. Note that the same forward-jumping methods can be applied to solving the problem (3), and equation (4). In this case, if assuming the known's of the values $\bar{y}_m, \bar{\vartheta}_m$ $(m = k-m+1, ..., k)$ then the forward-jumping method can be written as follows:

$$y_{n+k-m} = -\sum_{i=0}^{k-m+1} \bar{\alpha}_i y_{n+i} + h\sum_{i=0}^{k-m+1} \bar{\beta}_i (f_{n+i} + \vartheta_{n+i}) +$$

$$+h\sum_{i=k-m-1}^{k} \bar{\gamma}_i (\bar{f}_{n+i} + \bar{\vartheta}_{n+i}), \tag{16}$$

here $\bar{f}_m = f(x_m, \bar{y}_m)$ $(m = 0, 1, 2, ...)$, and the coefficients $\bar{\alpha}_i, \bar{\beta}_i, \bar{\gamma}_i$ $(i = 0, 1, ..., k-m-1;\ j = k-m+1, ..., k)$ - are real numbers, which are determined by the items α_i, β_i $(i = 0, 1, ..., k)$.

For example, consider a combination of the methods (14) and (13). Then we have:

$$y_{n+1} = y_n + h(8f_{n+1} + 5f_n)/12 - hf(x_{n+2}, y_{n+1} + h(3f_{n+1} - f_n)/2)/12 +$$

$$+h(8\vartheta_{n+1} + 4\vartheta_n - h(K(x_{n+2}, x_{n+1}, y_{n+1}) + K(x_{n+1}, x_{n+1}, y_{n+1})))/12. \tag{17}$$

These method is implicit and its application to solving of some initial value problem, one can use the predictor-corrector scheme. Here as a predictor formula, to proposing the midpoint method:

$$y_{n+1} = y_n + h(f_n + f(x_{n+1}, y_n + hf_n))/2 + h(\vartheta_n + \vartheta_{n+1})/2, \tag{18}$$

$$\vartheta_{n+1} = \vartheta_n + h(K(x_{n+1/2}, x_{n+1/2}, y_n + (h/2)f_n) +$$

$$+K(x_{n+1}, x_{n+1/2}, y_n + (h/2)f_n))/2. \tag{19}$$

Thus constructed forward-jumping method for solving of the problem (1). Obviously, if the functions $f(x, y)$ and $K(x, s, y)$ linear in y, then the method (17) can be applied to solving of the problem (1), otherwise by the method (19) can be determine the value of ϑ_{n+1}, and by the methods (18) the value of item y_{n+1}, then by the received values can be corrected the value of item y_{n+1} by using the method (17).

Note that the method (18) by its structure coincides with the method (16). This method is obtained by using the midpoint rule which to remind a hybrid methods. Hybrid methods have some advantages (see e.g. [15]-[19]). However, their application to solving some practical problems more difficult than forward-jumping methods.

Indeed, consider the following hybrid method

$$y_{n+1} = y_n + h(y'_{n+1/2-\alpha} + y'_{n+1/2+\alpha})\Big/2 \quad (\alpha = \sqrt{6}\Big/3). \tag{20}$$

which is stable and has the order of accuracy $p = 4$. To apply the method (20) to solve some specific problems is needed to be known values of items $y_{n+1/2-\alpha}$ and $y_{n+1/2+\alpha}$. If the values of the parameter α, a rational number, then among the known methods can select suitable formula. For example, when $\alpha = 0$ from the formula (20) receive the midpoint rule, and when $\alpha = 1/2$ we get the trapezoidal method. Note that for the application the following hybrid method to solving of the problem (1)

$$y_{n+1} = y_n + h(3y'_{n+1/3} + y'_{n+1})/4\,, \tag{21}$$

appears necessity to determine the values of the solution of the problem (1) at an intermediate point $x_n + h/3$. Take into account that the method (21) has the order of accuracy $p = 3$, to calculate the values $y_{n+1/3}$ one can use the following scheme

$$\hat{y}_{n+\alpha} = y_n + \alpha h y'_n, \tag{22}$$

$$y_{n+\alpha} = y_n + \alpha h(3y'_n + \hat{y}'_{n+\alpha})\Big/2 \quad (\alpha = 1/3). \tag{23}$$

Obviously by, the scheme (22)-(23) can calculate the values $y_{n+\alpha}$ for any values of the parameter α. However, the order of accuracy of the calculated values must be small, than $O(h^3)$. But if to determine the values of $y_{n+\alpha}$ use the more accurate methods, then appears of necessity to define the values of the variable $y_{n+\beta(\alpha)}$. To this end, consider the following method:

$$y_{n+\alpha} = y_n + \alpha h(y'_n + 4y'_{n+\alpha/2} + y'_{n+\alpha})\Big/6\,. \tag{24}$$

For using the method (24) it is necessity to determine the approximate values of variables $y_{n+\alpha/2}$ and $y_{n+\alpha}$. Consequently, for the using of hybrid methods encounters with some difficulties, which can be solved by the block methods (see e.g. [18]).

If generalize the above mentioned hybrid methods, we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} \;+ h\sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i} \quad (|\nu_i| < 1;\; i = 0, 1, 2, ..., k)\,. \tag{25}$$

Method (25) in the work [18] was used to solving differential equations of first order and in [20] to solving Volterra integral equations.

3. Algorithm for using method (13) and its application to solving some concrete problems

Now by using the method (13) consider to construction a specific algorithm for solving problem (1). For these aims use the scheme of the work [21, page 304].

To approximate the solution of the initial-value problem

$$y' = f(x,\, y) + \int_{x_0}^{x} K(x,\, s,\, y(s))ds, \quad x_0 \le s \le x \le X, \quad y(x_0) = y_0,$$

at N equal spaced numbers in the interval $[x_0,\, X]$:

INPUT end points $x_0,\ X$; integer N; initial condition y_0.

OUTPUT i, x_i, y_i where at the step y_i approximates $y(x_i)$ at the N values of x.

Step 1 set $h = (X - x_0)/N$;

$$x = x_0; \quad z = y_0;$$

OUTPUT $(x,\, z)$.
Step 2 For $i = 1,\, 2,\, ...,\, N$ do Step 3-6
Step 3 Set

$$v_{i+1} = v_i + h(K(x_{i+1/2},\, x_{i+1/2},\, y_i + h(f_i + v_i)/2) +$$
$$+ K(x_{i+1},\, x_{i+1/2},\, y_i + h(f_i + v_i)/2))/2;$$
$$x_{i+1/2} = x_i + h/2; \quad x_{i+1} = x_i + h;$$

Step 4 Set

$$y_{i+1} = y_i + h(f_i + f(x_{i+1},\, y_i + h(f_i + v_i)))/2 + h(v_i + v_{i+1})/2;$$
$$(\text{predict } y_{i+1})$$

Step 5 set

$$y_{i+1} = y_i + h(8f_{i+1} + 5f_i - f(x_{i+2},\, y_i + 2hf_{i+1} + 2hv_{i+1}))/12 +$$
$$+ h(8v_{i+1} + 4v_i - h(K(x_{i+1},\, x_{i+1},\, y_{i+1}) + K(x_{i+2},\, x_{i+1},\, y_{i+1})))/12;$$
$$(\text{correct } y_{i+1})$$

$$x_{i+2} = x_i + 2h;$$

Step 6 OUTPUT $(i,\, x_{i+1},\, y_{i+1})$.
Step 7 STOP.

By the above mentioned concrete method was shown that by the forward-jumping methods one can be to solve of the integral and integro-differential equation. For the applying of the forward-jumping methods to solving problem (1), it is necessary propose the scheme to determine the values of the coefficients of the methods (6) and (7).

We can show that if the order of accuracy for the method is determined by the formulas (6) and (7) equals to p, then its coefficients must satisfy the following conditions:

$$\begin{gathered} \sum_{i=0}^{k-m} \alpha_i = 0; \quad \sum_{i=0}^{k} \beta_i = \sum_{j=0}^{k-m} j\alpha_j \\ \sum_{i=0}^{k} i^{l-1}\beta_i = \sum_{j=0}^{k-m} j^l \alpha_j / l; \;\; l = 1,\, 2,\, ...,\, p. \end{gathered} \tag{26}$$

But the coefficients $\beta_i^{(j)}$ $(i, j = 0, 1, 2, ..., k)$ must satisfy the following conditions:

$$\sum_{j=0}^{k} \beta_i^{(j)} = \sum_{j=0}^{k} \beta_j; \quad i = 0,\, 1,\, 2,\, ...,\, k. \tag{27}$$

For the construction of the method with the higher order of accuracy let us consider application of the next modification of the method (25)

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h \sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i}. \tag{28}$$

For the determined variable $\alpha_i, \beta_i, \gamma_i, \nu_i$ $(i = 0, 1, 2, ..., k)$ suppose here to solving the following system:

$$\begin{gathered}\sum_{i=0}^{k-m} \alpha_i = 0; \quad \sum_{i=0}^{k} (\beta_i + \gamma_i) = \sum_{j=0}^{k-m} i\alpha_i; \\ \sum_{i=0}^{k} (\frac{i^{l-1}}{(l-1)!}\beta_i + \frac{(i+\nu_i)^{l-1}}{(l-1)!}\gamma_i) = \sum_{j=0}^{k-m} \frac{i^l}{l!}\alpha_i, \quad (l = 2, 3, .., p)\,.\end{gathered} \tag{29}$$

It is easy to determine that system (29) for the values $\nu_i = 0$ $(i = 0, 1, ..., k)$ is linear and coincides with known systems that are used to determine the coefficients of the multistep method with constant coefficients. Furthermore, for the conditions $|\nu_0| + |\nu_1| + ... + |\nu_k| \neq 0$, system (29) is nonlinear; by solving it, we determine the coefficients of method (28). In this system, the number of unknowns is equal to $4k + 4$ and the number of equations is equal to $p + 1$. Because system (29) is homogeneous, it always has a trivial solution, but to ensure that system (29) will have a solution that is different from zero, the condition $4k + 4 > p + 1$ must hold. Thus, one can be write the following:

$$p \leq 4k + 2 - m.$$

It is clear that $k - m > 0$. Consequently if $m = 1$ then $k \geq 2$. Method constructed for the value (29) $k = 2$ is stable. Thus we receive that by solving the system consist from the algebraic equation we can be constructed stable method of type (28) with the degree $p \geq 8$.

The methods (28) constructed in the joint of the forward-jumping and hybrid methods. So they have some properties as the forward-jumping methods and hybrid methods. These methods are more accurate than the known. Note that the study of the usual methods for finding the values of solutions of (1) at the point x_{n+1} by using of its values in the previous points. But the forward-jumping methods are used information of the solution the considered problems. It follows that such methods is desirable to apply is to the study of those problems for which the solution is an oscillating function. With the need to solve such problems are faced when dealing with some scientific and engineering problems. For example in the study of the trajectory of guided ballistic missiles.

Among the most popular hybrid methods are symmetric that can be construct built for even values k. For the $k = 2$ symmetrical type of method (28) has the following form:

$$\begin{gathered}y_{n+1} = y_n + h(\beta_2 y'_{n+2} + \beta_1 y'_{n+1} + \beta_0 y'_n) + \\ + h(\gamma_2 y'_{n+1+\alpha} + \gamma_1 y'_{n+1} + \gamma_0 y'_{n+1-\alpha}), \quad 0 < \alpha < 1.\end{gathered} \tag{30}$$

Solving the system of equations with contained from the coefficients of the method (28), we obtain different solutions. For example, considers defining coefficients of the method (30). Then receive the following:

$$\begin{aligned}\beta_0 &= 133.82166032672427, \\ \beta_1 &= 124.2912214645596, \\ \beta_2 &= 0.00001615285793018, \\ \gamma_0 &= 0.5237738917262333, \\ \gamma_1 &= -124.0822376790813, \\ \gamma_1 &= -133.55443415633394,\end{aligned}$$

$$m_0 = 0.5363788961926791,$$
$$m_1 = 1.0001066023537044,$$
$$m_2 = -0.00017423672396165.$$

As the second variant consider case $\beta_2 = 0$ in the method (30). Then receive:

$$\beta_0 = 218.62792919687118,$$
$$\beta_1 = 0.06425150613651433,$$
$$\beta_2 = -0.00000270478210308,$$
$$\gamma_0 = -218.44818963227522,$$
$$\gamma_1 = 0.4184313953445676,$$
$$\gamma_1 = 0.33758023886880617,$$
$$m_0 = -0.00004558491684368,$$
$$m_1 = 0.38645278117498433,$$
$$m_2 = 0.7823087296154607.$$

Note that of negative values the parameter α can get, are not interesting. To find the coefficients of the method (30) we set $k = 2$ in the system (29). Then we have:

$$\begin{aligned}
&\beta_0 + \beta_1 + \beta_2 + \gamma_0 + \gamma_1 + \gamma_2 = 1,\\
&\beta_1 + 2\beta_2 + m_0\gamma_0 + m_1\gamma_1 + m_2\gamma_2 = 1/2,\\
&\beta_1 + 2^2\beta_2 + m_0^2\gamma_0 + m_1^2\gamma_1 + m_2^2\gamma_2 = 1/3,\\
&\beta_1 + 2^3\beta_2 + m_0^3\gamma_0 + m_1^3\gamma_1 + m_2^3\gamma_2 = 1/4,\\
&\beta_1 + 2^4\beta_2 + m_0^4\gamma_0 + m_1^4\gamma_1 + m_2^4\gamma_2 = 1/5,\\
&\beta_1 + 2^5\beta_2 + m_0^5\gamma_0 + m_1^5\gamma_1 + m_2^5\gamma_2 = 1/6,\\
&\beta_1 + 2^6\beta_2 + m_0^6\gamma_0 + m_1^6\gamma_1 + m_2^6\gamma_2 = 1/7,\\
&\beta_1 + 2^7\beta_2 + m_0^7\gamma_0 + m_1^7\gamma_1 + m_2^7\gamma_2 = 1/8,\\
&\beta_1 + 2^8\beta_2 + m_0^8\gamma_0 + m_1^8\gamma_1 + m_2^8\gamma_2 = 1/9,\\
&\beta_1 + 2^9\beta_2 + m_0^9\gamma_0 + m_1^9\gamma_1 + m_2^9\gamma_2 = 1/10.
\end{aligned} \tag{31}$$

Note that the solution of the system (31) is not unique. One of them is the following:

$$\begin{aligned}
&\beta_0 = 0.05082388467541876,\\
&\beta_1 = 0.05137427406610529,\\
&\beta_2 = -0.00000067307130487,\\
&\gamma_0 = 0.35187646010359275,\\
&\gamma_1 = 0.27254332133827536,\\
&\gamma_1 = 0.2733827328879044,\\
&m_0 = 0.49929961804083894,\\
&m_1 = 0.17410576766585883,\\
&m_2 = 0.8247921251524742.
\end{aligned} \tag{32}$$

Now consider the construction algorithm for using of the method (28). For this aim we are used a block by block method, which in single form can be written as follows

Block I

$$\hat{y}_{n+1/2} = y_n + hy'_n/2, \;\; y_{n+1/2} = y_n + h(y'_n + \hat{y}'_{n+1/2})/4, \;\; \bar{y}_{n+1} = y_n + hy'_{n+1/2},$$

$$y_{n+1} = y_n + h(y'_n + 4y'_{n+1/2} + \bar{y}'_{n+1})/6,$$
$$\hat{y}_{n+1/2} = y_n - h(y'_{n+1} - 8y'_{n+1/2} - 5y'_n)/12.$$
$$y_{n+1} = y_n + h(y'_n + 4y'_{n+1/2} + y'_{n+1})/6.$$

Block II

$$\hat{y}_{n+3/2} = y_{n+1} + h(23y'_{n+1} - 16\hat{y}'_{n+1/2} + 5y'_n)/24,$$
$$y_{n+3/2} = y_{n+1} + h(9\hat{y}'_{n+3/2} + 19y'_{n+1} - 5\hat{y}'_{n+1/2} + y'_n)/48, \tag{33}$$
$$y_{n+1/2} = y_n + h(\hat{y}'_{n+3/2} - 5y'_{n+1} + 19\hat{y}'_{n+1/2} + 9y'_n)/48, \tag{34}$$
$$\hat{y}_{n+1} = y_{n+1} + h(8y'_{n+3/2} - 5y'_{n+1/2} + 4y'_{n+1/2} - y'_n)/6,$$
$$\begin{aligned} y_{n+1+\alpha} = y_{n+1} + \frac{h}{720}((96\alpha^5 + 120\alpha^4 + 120\alpha^3 - 60\alpha^2)y'_{n+2} - \\ -(384\alpha^5 + 240\alpha^4 - 480\alpha^2)y'_{n+3/2} + \\ +(576\alpha^5 - 240\alpha^3 + 720\alpha)y'_{n+1} - (384\alpha^5 - 240\alpha^4 + 480\alpha^2)y'_{n+1/2} + \\ +(96\alpha^5 - 120\alpha^4 + 120\alpha^3 + 60\alpha^2)y'_n). \end{aligned} \tag{35}$$
$$y_{n+1/2} = y_n + h(y'_{n+1/2} + y'_n)/24 + 5h(y'_{n+1/4-\alpha/2} + y'_{n+1/4+\alpha/2})/24, \tag{36}$$
$$y_{n+1} = y_n + h(5y_{n+1/2+\alpha/2} + 8y'_{n+1/2} + 5y'_{n+(1-\alpha)/2})/18, \tag{37}$$
$$\begin{aligned} y_{n+2} = y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 + \\ +h(18y'_{n+1+\beta} + 98y'_{n+1} + 64y'_{n+1-\beta})/180. \end{aligned}$$

Any variant of method (30)

The block I proposed for calculating quantity $y_{n+1/2}$, because that is used only one time. But block II is used for all the values of variable n.

For the receive results of more accuracy one can be used twice calculating of methods (36) and (37).

For using formula (35) we must define approximately values of the solution of initial problem in five mesh points, but can be construct formula which has the same order of accuracy with the formula (35) and used approximately values of the solution of initial problem only in three mesh points. For example, the next formula

$$\begin{aligned} y_{n+1/2+\beta} = 4\beta^2(1-2\beta^2)(y_{n+1}+y_n) + (1-4\beta^2)^2 y_{n+1/2} + \\ +2\beta^3(5-12\beta^2)(y_{n+1}-y_n) + h\beta^3(4\beta^2-1)(y'_{n+1}-y'_n) + \\ +h\beta(1-4\beta^2)^2 y'_{n+1/2} - h\beta^2\left(\frac{1}{2}-2\beta^2\right)(y'_{n+1}-y'_n). \end{aligned}$$

Note that the system (26) is a linear system of algebraic equations, which may have a unique solution, but the system (27) has more than one solution, as in which the number of unknowns is greater than the number of equations. From here follows that some of these unknown can be choices in free form. For example, if use the coefficients of the method (13) in the system (27), then obtain any methods for solving equation (4). Here are several of them:

$$\begin{aligned} \vartheta_{n+1} = \vartheta_n + h(4K(x_{n+1}, x_{n+1}, y_{n+1}) + 3K(x_{n+1}, x_n, y_n) + \\ +4K(x_{n+2}, x_{n+1}, y_{n+1}) + 2K(x_{n+2}, x_n, y_n) - K(x_{n+2}, x_{n+2}, y_{n+2}))/12, \end{aligned}$$
$$\begin{aligned} \vartheta_{n+1} = \vartheta_n + h(10K(x_{n+1},\ x_{n+1},\ y_{n+1}) + 6K(x_{n+1}, x_n, y_n) + \\ +6K(x_{n+2}, x_{n+1}, y_{n+1}) + 4K(x_n, x_n, y_n) - K(x_{n+2}, x_{n+2}, y_{n+2}))/24. \end{aligned}$$

For illustrations of the results of this work, applied of the method (13) to solving the following problems by the above described algorithm. Let us consider determine the approximate values of the solution of the following problems:

1. $y' = \int\limits_0^x xs\cos(s^2)ds, \quad 0 \le x \le 2, \quad y(0) = -1/4,$

the step size $h = 0,125$, exact solution for which is: $y(x) = -\cos(x^2)/4$.

2. $y' = 1 + y - x\exp(-x^2) - 2\int\limits_0^x xs\exp(-y^2(s))ds, \quad 0 \le x \le 2, \quad y(0) = 0,$

the step size, $h = 0,05$ and the exact solution : $y(x) = x$.

3. $y' = -x^3/3 + 4\exp(-y)/3 + \frac{4}{3}\int\limits_0^x \frac{s^2}{x}\exp(y(s))ds, \quad 1 \le x \le 3, \quad y(1) = 0,$

the step size $h = 1/125$ and the exact solutions can be written as following form: $y(x) = \ln x$.

4. $y' = \int\limits_0^x \cos s ds$, $0 \le x \le 2$, $y(0) = -1$, the step size $h = 0,125$ and exact solution: $y(x) = -\cos x$.

The obtained results, place in the following table.

Number of example	**Step size**	**Variable x**	**Error of the method (13)**
I	$h = 0,125$	0.25	0.5E-04
		0.50	0.97E-04
		1.00	0.26E-03
		1.50	0.11E-02
		2.00	0.24E-02
II	$h = 0,05$	0.10	0.11E-06
		0.50	0.42E-06
		1.00	0.88E-05
		1.50	0.28E-04
		2.00	0.77E-04
III	$h = 0,05$	1.1	0.72E-05
		1.5	0.17E-04
		2.00	0.40E-05
		2.50	0.12E-03
		3.00	0.68E-03
IV	$h = 0,125$	0.25	0.24E-04
		0.50	0.42E-04
		1.00	0.62E-04
		1.50	0.44E-04
		2.00	0.16E-04

Remark that for the constructed the methods to solving equation (4) with the best properties one can be used results from the work [24]. And for the constructed available methods to solving model problems, some authors after application finite-difference methods suggested using iterative methods (see for example [25]) in which used the midpoint method.

4. Conclusion

Here considered to application one of the little investigation method to solving integro-differential equation, which is called forward-jumping method. These method in the first time was applied to investigation the motion of Halley's comet by Cowell (see [22]), because some authors calles forward-jumping methods as Cowell's methods (see[23, p.293]). Bu using any information about forward-jumping methods shown that the methods of Cowell's type has some advetages. For these aim has redused two forward-jumping methods and constructed algorithm to using one of them. For the comparse forward-jumping method with the known, here have used model problems, which sometimes ago have solving in the known works. Results of these illustration are agrees with the theoretical, it is shown that the forward-jumping methods have the some preference.

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Closed-Form Solutions to Discrete-Time Portfolio Optimization Problems

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Abstract

In this paper, we study some discrete-time portfolio optimization problems. We introduce a discrete-time financial market model. The change in asset prices is modelled in contrast to the continuous-time market model by stochastic difference equations. We provide solutions of these stochastic difference equations. Then we introduce the discrete-time risk measures and the portfolio optimization problems. The main contributions of this paper are the closed-form solutions to the discrete-time portfolio models. For simulation purposes, the discrete-time financial market is often better suited. Several examples illustrating our theoretical results are provided.

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Keywords. Portfolio optimization, discrete-time financial market, mean-variance.

1 Introduction

In [5, 6], the authors solved the continuous-time multi-period Earnings-at-Risk optimization problem

$$\begin{cases} \min\limits_{\varphi\in\mathbb{R}^n} \ \mathrm{EaR}(\varphi) \\ \text{s.t. } \mathbb{E}(X^\varphi(T)) \geq C \end{cases} \tag{1.1}$$

and the continuous-time multi-period mean-variance optimization problem

$$\begin{cases} \min\limits_{\varphi\in\mathbb{R}^n} \ \mathbb{V}\mathrm{ar}(\varphi) \\ \text{s.t. } \mathbb{E}(X^\varphi(T)) \geq C \end{cases} \tag{1.2}$$

with a constant rebalanced portfolio. A standard Black–Scholes financial market was assumed, which was modelled by stochastic differential equations (see [1,4]). In this paper, we consider discrete-time versions of the problems (1.1) and (1.2). In Section 2, we briefly introduce the discrete-time financial market and the portfolio process. In Section 3, we prove some auxiliary results that are needed throughout the paper. Next, in Sections 4–7, we introduce several risk measures and solve the discrete-time one-period mean-Earnings-at-Risk problem, one-period Capital-at-Risk problem, one-period Value-at-Risk problem, and multi-period mean-variance problem.

2 Discrete-Time Financial Market

We construct our portfolio with $n+1$ assets. In our model we are considering discrete trading times on $[0,T]\cap\mathbb{N}_0$, where $T\in\mathbb{N}$. Let us denote the price of asset i at time t with $P_i(t)$ for $i=0,\dots,n$. We have one risk-free asset in our model. Without loss of generality it is asset $i=0$. The risk-free asset is the bank account which pays constant interest with rate r every year. Denote by $P_0(t)$ the price of the risk-free asset at time t. Then P_0 follows the difference equation

$$P_0(t+1)-P_0(t)=P_0(t)r. \tag{2.1}$$

Lemma 2.1 (Solution of (2.1))**.** *The solution of* (2.1) *is given by*

$$P_0(t)=P_0(0)(1+r)^t,\quad t\in\mathbb{N}_0. \tag{2.2}$$

Proof. The relation (2.2) follows easily from (2.1) by induction. □

We now introduce the price processes of the risky assets. These are described by stochastic difference equations. First we need some notation to define the price processes of the risky assets. Let $b=(b_1,\dots,b_n)'$ be the vector with the expected returns of the individual assets, and denote by $\sigma=(\sigma_{ij})_{1\le i,j\le n}$ the $n\times n$-matrix with the stock volatilities. To simplify the calculations, b and σ are assumed to be constant over the time. Now P_i follow the stochastic difference equations

$$P_i(t+1)-P_i(t)=P_i(t)\left(b_i+\sum_{j=1}^{n}\sigma_{ij}\left(B_j(t+1)-B_j(t)\right)\right),\quad i=1,\dots,n, \tag{2.3}$$

where $B(t)$ is a standard n-dimensional Brownian motion.

Lemma 2.2 (Solution of (2.3))**.** *The solution of* (2.3) *is given by*

$$P_i(t)=P_i(0)\prod_{a=0}^{t-1}\left(1+b_i+\sum_{j=1}^{n}\sigma_{ij}\left(B_j(a+1)-B_j(a)\right)\right),\quad t\in\mathbb{N}_0. \tag{2.4}$$

Proof. The relation (2.4) follows easily from (2.3) by induction. □

Now we define the portfolio for our model. With $X^{\varphi}(t)$ we denote the total wealth at time t, and $\varphi_i(t)$ is the fraction of $X^{\varphi}(t)$ invested in asset i at time t. The vector $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))' \in \mathbb{R}^n$ is called the portfolio construction process, and $X^{\varphi}(t)$ is called the wealth process of the portfolio. In this paper we only consider so-called constant rebalanced investment portfolio strategies, i.e., $\varphi(t) \equiv \varphi$ is the same at each time $t \in [0, T] \cap \mathbb{N}_0$. We can calculate the weight of the risk-free asset in the portfolio by

$$\varphi_0 = 1 - \varphi' \mathbf{1}, \quad \text{where} \quad \mathbf{1} = (1, \dots, 1)'.$$

If $\varphi_0 = 1$, then the entire wealth is invested in the risk-free asset ("pure-bond strategy"). The numbers of shares of the assets in our portfolio are

$$N_i(t) = X^{\varphi}(t) \frac{\varphi_i}{P_i(t)}, \quad i = 0, 1, \dots, n. \tag{2.5}$$

Lemma 2.3 (Total wealth). *The wealth of the portfolio at time t is given by*

$$X^{\varphi}(t) = \sum_{i=0}^{n} N_i(t) P_i(t), \quad t \in \mathbb{N}_0. \tag{2.6}$$

Proof. The calculation

$$\sum_{i=0}^{n} N_i(t) P_i(t) \overset{(2.5)}{=} \sum_{i=0}^{n} X^{\varphi}(t) \frac{\varphi_i}{P_i(t)} P_i(t) = X^{\varphi}(t) \sum_{i=0}^{n} \varphi_i = X^{\varphi}(t)$$

shows (2.6). $\square$

The assumptions in this paper are: We have no transaction costs, no consumption over time, and a self-financing portfolio strategy. Now we find the change of portfolio wealth over one period. We obtain a stochastic difference equation.

Lemma 2.4 (Change in portfolio wealth over one period). *We have*

$$X^{\varphi}(t+1) - X^{\varphi}(t) = X^{\varphi}(t) \left(r + \varphi'(b - r\mathbf{1}) + \varphi' \sigma (B(t+1) - B(t)) \right). \tag{2.7}$$

Proof. Using our assumptions, we find

$$\begin{aligned} X^{\varphi}(t+1) - X^{\varphi}(t) &\overset{(2.6)}{=} \sum_{i=0}^{n} N_i(t)(P_i(t+1) - P_i(t)) \\ &\underset{(2.3)}{\overset{(2.1)}{=}} N_0(t) r P_0(t) + \sum_{i=1}^{n} N_i(t) b_i P_i(t) \\ &\quad + \sum_{i=1}^{n} N_i(t) P_i(t) \sum_{j=1}^{n} \sigma_{ij} (B_j(t+1) - B_j(t)) \\ &\overset{(2.5)}{=} r X^{\varphi}(t)(1 - \varphi' \mathbf{1}) + \sum_{i=1}^{n} X^{\varphi}(t) \varphi_i b_i \end{aligned}$$

$$
\begin{aligned}
& +\sum_{i=1}^{n} X^{\varphi}(t)\varphi_i \sum_{j=1}^{n} \sigma_{ij}(B_j(t+1)-B_j(t)) \\
= \quad & X^{\varphi}(t)\left((1-\varphi'\mathbf{1})r+\varphi' b+\varphi'\sigma(B(t+1)-B(t))\right).
\end{aligned}
$$

This shows (2.7). □

Lemma 2.5 (Solution of (2.7)). *The solution of* (2.7) *is given by*

$$X^{\varphi}(t)=X^{\varphi}(0)\prod_{a=0}^{t-1}\left[1+r+\varphi'(b-r\mathbf{1})+\varphi'\sigma\Delta B(a)\right], \quad t\in\mathbb{N}_0. \tag{2.8}$$

Proof. The relation (2.8) follows easily from (2.7) by induction. □

We use the explicit formula (2.8) for $X^{\varphi}(t)$ to calculate expectation and variance of the portfolio. Some simple calculations using the properties of Brownian motion show the following results.

Theorem 2.6 (Expectation and variance of the wealth process). *With*

$$\alpha:=r+\varphi'(b-r\mathbf{1}), \quad c:=\sigma'\varphi, \quad \text{and} \quad x:=X^{\varphi}(0), \tag{2.9}$$

we have

$$X^{\varphi}(t)=x\prod_{a=0}^{t-1}\left[1+\alpha+c'\Delta B(a)\right], \tag{2.10}$$

and therefore

$$\mathbb{E}(X^{\varphi}(t))=x(1+\alpha)^t, \quad t\in\mathbb{N}_0 \tag{2.11}$$

and

$$\mathbb{V}\mathrm{ar}(X^{\varphi}(t))=x^2\left[((1+\alpha)^2+c'c)^t-(1+\alpha)^{2t}\right], \quad t\in\mathbb{N}_0. \tag{2.12}$$

Proof. By (2.9), (2.10) is the same as (2.8). We use (2.10) and the fact that increments of Brownian motion are independent with expectation zero to find

$$\mathbb{E}\left(X^{\varphi}(t)\right)=x\prod_{a=0}^{t-1}\mathbb{E}\left(1+\alpha+\sum_{j=1}^{n}c_j\Delta B_j(a)\right)=x\prod_{a=0}^{t-1}(1+\alpha)=x(1+\alpha)^t.$$

This shows (2.11). Next, using (2.10) and the fact that increments of Brownian motion are independent with expectation zero and variance one, we find

$$
\begin{aligned}
\mathbb{E}((X^{\varphi}(t))^2) &= \mathbb{E}\left(x^2\prod_{a=0}^{t-1}\left[1+\alpha+c'\Delta B(a)\right]^2\right) \\
&= x^2\prod_{a=0}^{t-1}\mathbb{E}\left((1+\alpha)^2+2(1+\alpha)\sum_{j=1}^{n}c_j\Delta B_j(a)+\left(\sum_{j=1}^{n}c_j\Delta B_j(a)\right)^2\right) \\
&= x^2\prod_{a=0}^{t-1}\left((1+\alpha)^2+c'c\right)=x^2\left((1+\alpha)^2+c'c\right)^t.
\end{aligned}
$$

By (2.11) and $\mathbb{V}\mathrm{ar}(X^\varphi(t)) = \mathbb{E}\left((X^\varphi(t))^2\right) - (\mathbb{E}(X^\varphi(t)))^2$, we get (2.12). $\square$

We now introduce the main component of the risk measures used in this paper.

Definition 2.7. For a portfolio φ with wealth $X^\varphi(1)$, we define the risk measure $\mu(\varphi)$ corresponding to the β-quantile of $X^\varphi(1)$ by

$$\mathbb{P}(X^\varphi(1) \le \mu(\varphi)) = \beta, \quad \text{where} \quad \beta \in (0,1). \tag{2.13}$$

In the next lemma we give an explicit expression for $\mu(\varphi)$ for a given β.

Lemma 2.8. *Let $\beta \in (0,1)$. If z_β denotes the β-quantile of the standard normal distribution, then $\mu(\varphi)$ in (2.13) is given by*

$$\mu(\varphi) = x(z_\beta \|\sigma'\varphi\| + 1 + r + \varphi'(b - r\mathbf{1})). \tag{2.14}$$

Proof. Since $X^\varphi(1)$ is standard normally distributed with expectation $x(1+\alpha)$ (see (2.11)) and variance $x^2c'c$ (see (2.12)), it follows that $z_\beta = (\mu(\varphi) - x(1+\alpha))/(x\sqrt{c'c})$, i.e., using (2.9), (2.14) holds. $\square$

3 Auxiliary Results

In this section we provide some simple auxiliary results. For the rest of this paper we assume

$$\sigma \quad \text{is invertible and} \quad b \ne r\mathbf{1}, \quad \text{and let} \quad \Theta := \left\|\sigma^{-1}(b - r\mathbf{1})\right\|. \tag{3.1}$$

We first give the following three properties which are used often in Sections 4–7.

Lemma 3.1. *Assume (3.1). We have*

$$|\varphi'(b - r\mathbf{1})| \le \|\sigma'\varphi\|\,\Theta \quad \textit{for all} \quad \varphi \in \mathbb{R}^n. \tag{3.2}$$

Moreover, if we define

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda \in \mathbb{R},$$

then we have

$$(\varphi^*)'(b - r\mathbf{1}) = \lambda\Theta \tag{3.3}$$

and

$$\|\sigma'\varphi^*\| = |\lambda|. \tag{3.4}$$

Proof. First, we let $\varphi \in \mathbb{R}^n$ and use the Cauchy–Schwarz inequality to obtain

$$|\varphi'(b - r\mathbf{1})| = |(\sigma'\varphi)'(\sigma^{-1}(b - r\mathbf{1}))| \le \|\sigma'\varphi\| \left\|\sigma^{-1}(b - r\mathbf{1})\right\| = \|\sigma'\varphi\|\,\Theta,$$

which shows (3.2). Next, we get

$$\begin{aligned}(\varphi^*)'(b-r\mathbf{1}) &= \lambda\frac{(b-r\mathbf{1})'(\sigma\sigma')^{-1}}{\Theta}(b-r\mathbf{1}) = \lambda\frac{(b-r\mathbf{1})'(\sigma')^{-1}\sigma^{-1}(b-r\mathbf{1})}{\Theta}\\ &= \lambda\frac{(\sigma^{-1}(b-r\mathbf{1}))'\sigma^{-1}(b-r\mathbf{1})}{\Theta} = \lambda\frac{\left\|\sigma^{-1}(b-r\mathbf{1})\right\|^2}{\Theta} = \lambda\Theta,\end{aligned}$$

which shows (3.3). Finally, we obtain

$$\|\sigma'\varphi^*\| = \left\|\sigma'\frac{\lambda(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\Theta}\right\| = \left\|\lambda\frac{\sigma'(\sigma')^{-1}\sigma^{-1}(b-r\mathbf{1})}{\Theta}\right\| = \|\lambda\|\,\frac{\Theta}{\Theta} = |\lambda|,$$

which shows (3.4). □

Next we give a lemma that will be used frequently for the mean-CaR optimization problem (Section 5) and the mean-VaR optimization problem (Section 6). There and for the rest of this paper we assume

$$0 < \beta < \frac{1}{2}, \text{and } z_\beta \text{ is the } \beta\text{-quantile of the standard normal distribution.} \tag{3.5}$$

Lemma 3.2. *Assume* (3.1) *and* (3.5). *Let* $\Psi \in \mathbb{R}$ *be independent of* φ *and let*

$$A := \{\varphi \in \mathbb{R}^n : \ \varphi'(b-r\mathbf{1}) + z_\beta \|\sigma'\varphi\| = \Psi\}.$$

If $\varphi \in A$, *then*

$$\varphi'(b-r\mathbf{1}) \geq \frac{\Psi\Theta}{\Theta - z_\beta} \tag{3.6}$$

and

$$(\Theta + z_\beta)\varphi'(b-r\mathbf{1}) \geq \Psi\Theta. \tag{3.7}$$

Proof. Note first that (3.5) implies $z_\beta < 0$. Let $\varphi \in A$. Then

$$\varphi'(b-r\mathbf{1}) \geq -|\varphi'(b-r\mathbf{1})| \overset{(3.2)}{\geq} -\|\sigma'\varphi\|\,\Theta \overset{(\varphi\in A)}{=} \frac{\Psi - \varphi'(b-r\mathbf{1})}{-z_\beta}\Theta,$$

i.e.,

$$-z_\beta\varphi'(b-r\mathbf{1}) \geq \Psi\Theta - \Theta\varphi'(b-r\mathbf{1}),$$

i.e.,

$$(\Theta - z_\beta)\varphi'(b-r\mathbf{1}) \geq \Psi\Theta,$$

which proves (3.6) since $\Theta - z_\beta > 0$. Next,

$$\varphi'(b-r\mathbf{1}) \leq |\varphi'(b-r\mathbf{1})| \overset{(3.2)}{\leq} \|\sigma'\varphi\|\,\Theta \overset{(\varphi\in A)}{=} \frac{\Psi - \varphi'(b-r\mathbf{1})}{z_\beta}\Theta,$$

i.e.,

$$z_\beta\varphi'(b-r\mathbf{1}) \geq \Psi\Theta - \Theta\varphi'(b-r\mathbf{1}),$$

which proves (3.7). □

Finally, we give a lemma which we use for the multi-period mean-variance problem (Section 7).

Lemma 3.3. *Let $c_1, c_2 \geq 0$ and $T \in \mathbb{N}$ and define $f : [0, \infty) \to \mathbb{R}$ by*

$$f(x) = \left((c_1 + x)^2 + c_2\right)^T - (c_1 + x)^{2T}.$$

Then f is increasing.

Proof. We let $x \geq 0$ and calculate

$$\begin{aligned} f'(x) &= T\left((c_1 + x)^2 + c_2\right)^{T-1} 2(c_1 + x) - 2T(c_1 + x)^{2T-1} \\ &= 2T(c_1 + x)\left[\left((c_1 + x)^2 + c_2\right)^{T-1} - (c_1 + x)^{2T-2}\right] \\ &\geq 2T(c_1 + x)\left[((c_1 + x)^2)^{T-1} - (c_1 + x)^{2T-2}\right] = 0, \end{aligned}$$

which completes the proof. □

4 One-Period Mean-Earnings-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Earnings-at-Risk problem and provide a closed-form solution. The difference between the expected wealth after one period and the risk measure $\mu(\varphi)$ with the same portfolio φ is called *Earnings-at-Risk*.

Definition 4.1 (Earnings-at-Risk)**.** $\mathrm{EaR}(\varphi) := \mathbb{E}(X^{\varphi}(1)) - \mu(\varphi)$.

We solve the optimization problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \ \mathrm{EaR}(\varphi) \\ \text{s.t. } \mathbb{E}(X^{\varphi}(1)) \geq C, \end{cases} \tag{4.1}$$

where $C \in \mathbb{R}$ is the expected terminal wealth at time $T = 1$.

Theorem 4.2 (Closed-form solution of the discrete-time one-period mean-EaR optimization problem)**.** *Assume* (3.1) *and* (3.5)*. The closed-form solution of the one-period mean-Earnings-at-Risk problem* (4.1) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\Theta},$$

where

$$z^+ = \begin{cases} z & \textit{if} \quad z \geq 0 \\ 0 & \textit{if} \quad z < 0 \end{cases} \qquad \textit{for any} \quad z \in \mathbb{R}.$$

The expected wealth after one period is C with Earnings-at-Risk $-xz_\beta\lambda$.

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = -xz_\beta \lambda \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := -xz_\beta \|\sigma' \varphi\| \quad \text{and} \quad A := \left\{ \varphi \in \mathbb{R}^n : \; \varphi'(b - r\mathbf{1}) \geq \frac{C}{x} - 1 - r \right\}.$$

To show this, first note that

$$(\varphi_*)'(b - r\mathbf{1}) \stackrel{(3.3)}{=} \lambda \Theta = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\Theta} \Theta = \left(\frac{C}{x} - 1 - r \right)^+ \geq \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= \frac{-xz_\beta}{\Theta} \|\sigma' \varphi\| \, \Theta \stackrel{(3.2)}{\geq} \frac{-xz_\beta}{\Theta} |\varphi'(b - r\mathbf{1})| \stackrel{(\varphi \in A)}{\geq} \frac{-xz_\beta}{\Theta} \left(\frac{C}{x} - 1 - r \right)^+ \\ &= -xz_\beta \lambda \stackrel{(3.4)}{=} -xz_\beta \|\sigma' \varphi_*\| = g(\varphi_*). \end{aligned}$$

This completes the proof. □

As an immediate consequence of Theorem 4.2 we get that the optimal Earnings-at-Risk is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths. Since the supremum of EaR is infinity and the constraint of (4.1) is unbounded from above, the solution of the corresponding maximum problem is infinity. We denote with $\omega := \mathbb{E}(X^\varphi(1))$ the expected wealth after 1 year. We plug ω into λ given by Theorem 4.2 and get

$$\mathrm{EaR}(\omega) = -xz_\beta \lambda = -xz_\beta \frac{\left(\frac{\omega}{x} - 1 - r\right)^+}{\|\sigma^{-1}(b - r\mathbf{1})\|}.$$

Example 4.3. Let

$$r = 0.05, \quad b = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}$$

and

$$x = 1000, \quad C = 1056, \quad z_\beta = -1.64.$$

Now we calculate

$$\lambda = \frac{\left(\frac{C}{x} - 1 - r\right)^+}{\|\sigma^{-1}(b - r\mathbf{1})\|} = \frac{\frac{1056}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx 0.002384.$$

With that λ we calculate the Earnings-at-Risk for our portfolio with an expected terminal wealth of C as

$$\mathrm{EaR}(\varphi_*) = -xz_\beta\lambda = -1000 \cdot (-1.64) \cdot \lambda \approx 3.908947.$$

This is the minimal Earnings-at-Risk for the portfolio with an expected terminal wealth of 1056 at time 1. By Theorem 4.2, the optimal policy is given by

$$\varphi_* = \frac{\lambda \cdot \begin{pmatrix} 0.041 & 0.0242 & 0.0133 \\ 0.0242 & 0.1016 & 0.018 \\ 0.0133 & 0.018 & 0.0134 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} -0.006349 \\ -0.001753 \\ 0.026322 \end{pmatrix}.$$

This means 2.6322% are invested in asset 3 and the rest is invested risk free. Now we check if the expected wealth at time 1 really is 1056:

$$\mathbb{E}(X^\varphi(1)) = 1000 \left(1 + 0.05 + (\varphi_*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) = 1056.$$

5 One-Period Capital-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Capital-at-Risk problem and provide a closed-form solution. The solution of the continuous-time optimization problem can be found in [3]. The difference between the possible risk-free profit after one period and the risk measure $\mu(\varphi)$ is called *Capital-at-Risk*.

Definition 5.1 (Capital-at-Risk)**.** $\mathrm{CaR}(\varphi) := x(1+r) - \mu(\varphi)$.

We accept a certain amount as Capital-at-Risk and we want to maximize the expected return. We solve the optimization wealth

$$\begin{cases} \max\limits_{\varphi\in\mathbb{R}^n} \mathbb{E}(X^\varphi(1)) \\ \text{s.t. } \mathrm{CaR}(\varphi) = C, \end{cases} \tag{5.1}$$

and we also solve the problem

$$\begin{cases} \min\limits_{\varphi\in\mathbb{R}^n} \mathbb{E}(X^\varphi(1)) \\ \text{s.t. } \mathrm{CaR}(\varphi) = C, \end{cases} \tag{5.2}$$

where C is the CaR at time $T = 1$. An overview of the results given in this section can be found in Table 1.

Table 1: Overview mean-Capital-at-Risk problem

$\Theta + z_\beta$	C	Result	See
< 0	> 0	Found max and min	Theorem 5.2
> 0	> 0	Found min	Theorem 5.3
> 0	< 0	Found min	Theorem 5.4
< 0	< 0	$A = \emptyset$	Theorem 5.5

Theorem 5.2 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 1)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C > 0.$$

The closed-form solution of the one-period mean-Capital-at-Risk problem (5.1) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The corresponding expected wealth after one period is

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + r + \lambda\Theta) \quad \textit{and} \quad \mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \mu\Theta)\,,$$

respectively, with $\mathrm{CaR}(\varphi^*) = \mathrm{CaR}(\varphi_*) = C$*.*

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi^*, \varphi_* \in A$ and

$$x(1 + r + \mu\Theta) = g(\varphi_*) \le g(\varphi) \le g(\varphi^*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := x(1 + r + \varphi'(b - r\mathbf{1}))$$

and

$$A := \left\{\varphi \in \mathbb{R}^n : \ \varphi'(b - r\mathbf{1}) + z_\beta\,\|\sigma'\varphi\| = -\frac{C}{x}\right\}.$$

To show this, first note

$$(\varphi^*)'(b - r\mathbf{1}) + z_\beta\,\|\sigma'\varphi^*\| \underset{(3.4)}{\overset{(3.3)}{=}} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = -\frac{C}{x}$$

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implies $\varphi^* \in A$ and

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \overset{(3.3)}{\underset{(3.4)}{=}} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi_*) &= x(1 + r + (\varphi_*)'(b - r\mathbf{1})) \\ &\overset{(3.3)}{=} x(1 + r + \mu\Theta) = x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta - z_\beta}\right) \\ &\overset{(3.6)}{\leq} x(1 + r + \varphi'(b - r\mathbf{1})) = g(\varphi) \overset{(3.7)}{\leq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta + z_\beta}\right) \\ &= x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi^*)'(b - r\mathbf{1})) = g(\varphi^*). \end{aligned}$$

This completes the proof. $\square$

Theorem 5.3 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 2)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C > 0.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\left(1 + r + \mu\Theta\right)$$

with $\mathrm{CaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 5.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \mu\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \overset{(3.3)}{\underset{(3.4)}{=}} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.6)}{\geq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta - z_\beta}\right) \\ &= x(1 + r + \mu\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. $\square$

Theorem 5.4 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 3)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C < 0.$$

The closed-form solution of problem (5.2) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \lambda\Theta)$$

with $\mathrm{CaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 5.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \underset{(3.4)}{\overset{(3.3)}{=}} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = -\frac{C}{x}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.7)}{\geq} x\left(1 + r - \frac{\frac{C}{x}\Theta}{\Theta + z_\beta}\right) \\ &= x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 5.5 (Closed-form solution to the discrete-time one-period mean-CaR optimization problem, part 4)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C < 0.$$

Then both (5.2) *and the mean-Capital-at-Risk problem* (5.1) *are unsolvable.*

Proof. Let A be the feasible set as in the proof of Theorem 5.2. If $\varphi \in A$, then

$$0 < -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta} \overset{(3.6)}{\leq} \varphi'(b - r\mathbf{1}) \overset{(3.7)}{\leq} -\frac{C}{x}\frac{\Theta}{\Theta + z_\beta} < 0.$$

This contradiction shows $A = \emptyset$, and hence both (5.1) and (5.2) are unsolvable. □

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Example 5.6. We calculate the maximal expected wealth with $\mathrm{CaR} = C$. Let

$$r = 0.05, \quad b = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}$$

and

$$x = 1000, \quad C = 20, \quad z_\beta = -1.64.$$

Then

$$\Theta + z_\beta = \left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64 \approx -0.203859$$

so that all assumptions of Theorem 5.2 are satisfied. Next,

$$\lambda = -\frac{\frac{20}{1000}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64} \approx 0.098107.$$

By Theorem 5.2, the optimal investment strategy is given by

$$\varphi^* = \frac{\lambda \cdot \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} 0.341564 \\ 0.113855 \\ 0.426955 \end{pmatrix}.$$

This means 34.1564% are invested in asset 1, 11.3855% are invested in asset 2, and 42.6955% are invested in asset 3. Now we calculate the expected wealth of this strategy:

$$\mathbb{E}(X^\varphi(1)) = 1000 \cdot \left(1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) \approx 1190.895254.$$

We finally check if the CaR of this strategy really is 20:

$$\mathrm{CaR}(\varphi^*) = -1000 \cdot \left((-1.64) \cdot \lambda + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right) = 20.$$

6 One-Period Value-at-Risk Problem

In this section we introduce the discrete-time one-period mean-Value-at-Risk problem and provide a closed-form solution.

Definition 6.1 (Value-at-Risk)**.** $\mathrm{VaR}(\varphi) := \mu(\varphi)$.

We accept a certain amount as Value-at-Risk and we want to find the portfolio strategy which maximizes our expected wealth. We solve the optimization problem

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \mathrm{VaR}(\varphi) = C, \end{cases} \tag{6.1}$$

and we also solve the problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \mathrm{VaR}(\varphi) = C, \end{cases} \tag{6.2}$$

where C is the VaR at time $T = 1$. An overview of the results given in this section is displayed in Table 2.

Table 2: Overview mean-Value-at-Risk problem

$\Theta + z_\beta$	$\frac{C}{x} - 1 - r$	Result	See
< 0	< 0	Found max and min	Theorem 6.2
> 0	< 0	Found min	Theorem 6.4
> 0	> 0	Found min	Theorem 6.3
< 0	> 0	$A = \emptyset$	Theorem 6.5

Theorem 6.2 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 1)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \textit{and} \quad C < x(1+r).$$

The closed-form solution of the one-period mean-Value-at-Risk problem (6.1) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}.$$

The corresponding expected wealth after one period is

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1+r+\lambda\Theta) \quad \textit{and} \quad \mathbb{E}(X^{\varphi_*}(1)) = x\,(1+r+\mu\Theta)\,,$$

respectively, with $\mathrm{VaR}(\varphi^*) = \mathrm{VaR}(\varphi_*) = C$.

Proof. Using (2.11) for $t = 1$ and (2.14), it suffices to show that $\varphi^*, \varphi_* \in A$ and

$$x(1+r+\mu\Theta) = g(\varphi_*) \leq g(\varphi) \leq g(\varphi^*) = x(1+r+\lambda\Theta) \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := x(1+r+\varphi'(b-r\mathbf{1}))$$

and

$$A := \left\{ \varphi \in \mathbb{R}^n : \ \varphi'(b-r\mathbf{1}) + z_\beta \left\|\sigma'\varphi\right\| = \frac{C}{x} - 1 - r \right\}.$$

To show this, first note

$$(\varphi^*)'(b-r\mathbf{1}) + z_\beta \left\|\sigma'\varphi^*\right\| \underset{(3.4)}{\overset{(3.3)}{=}} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi^* \in A$ and

$$(\varphi_*)'(b-r\mathbf{1}) + z_\beta \left\|\sigma'\varphi_*\right\| \underset{(3.4)}{\overset{(3.3)}{=}} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned}
g(\varphi_*) &= x(1+r+(\varphi_*)'(b-r\mathbf{1})) \overset{(3.3)}{=} x(1+r+\mu\Theta) \\
&= x\left(1+r+\frac{\left(\frac{C}{x}-1-r\right)\Theta}{\Theta - z_\beta}\right) \overset{(3.6)}{\leq} x(1+r+\varphi'(b-r\mathbf{1})) \\
&= g(\varphi) \overset{(3.7)}{\leq} x\left(1+r+\frac{\left(\frac{C}{x}-1-r\right)\Theta}{\Theta + z_\beta}\right) \\
&= x(1+r+\lambda\Theta) \overset{(3.3)}{=} x(1+r+(\varphi^*)'(b-r\mathbf{1})) = g(\varphi^*).
\end{aligned}$$

This completes the proof. $\square$

Theorem 6.3 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 2)**.** *Assume* (3.1)*,* (3.5)*, and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C < x(1+r).$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\mu(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\Theta} \quad \textit{with} \quad \mu = \frac{\frac{C}{x}-1-r}{\Theta - z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1+r+\mu\Theta)$$

with $\mathrm{VaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 6.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \mu\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \|\sigma' \varphi_*\| \underset{(3.4)}{\overset{(3.3)}{=}} \mu\Theta + |\mu| z_\beta = \mu(\Theta - z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.6)}{\geq} x\left(1 + r + \frac{\left(\frac{C}{x} - 1 - r\right)\Theta}{\Theta - z_\beta}\right) \\ &= x(1 + r + \mu\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 6.4 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 3)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta > 0 \quad \textit{and} \quad C > x(1 + r).$$

The closed-form solution of problem (6.2) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\Theta} \quad \textit{with} \quad \lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The expected wealth after one period is

$$\mathbb{E}(X^{\varphi_*}(1)) = x\,(1 + r + \lambda\Theta)$$

with $\mathrm{VaR}(\varphi_*) = C$.

Proof. As in the proof of Theorem 6.2 and with the same g and A, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x(1 + r + \lambda\Theta) \quad \text{for all} \quad \varphi \in A.$$

To show this, first note

$$(\varphi_*)'(b - r\mathbf{1}) + z_\beta \|\sigma' \varphi_*\| \underset{(3.4)}{\overset{(3.3)}{=}} \lambda\Theta + |\lambda| z_\beta = \lambda(\Theta + z_\beta) = \frac{C}{x} - 1 - r$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\begin{aligned} g(\varphi) &= x(1 + r + \varphi'(b - r\mathbf{1})) \overset{(3.7)}{\geq} x\left(1 + r + \frac{\left(\frac{C}{x} - 1 - r\right)\Theta}{\Theta + z_\beta}\right) \\ &= x(1 + r + \lambda\Theta) \overset{(3.3)}{=} x(1 + r + (\varphi_*)'(b - r\mathbf{1})) = g(\varphi_*). \end{aligned}$$

This completes the proof. □

Theorem 6.5 (Closed-form solution to the discrete-time one-period mean-VaR optimization problem, part 4)**.** *Assume* (3.1), (3.5), *and*

$$\Theta + z_\beta < 0 \quad \text{and} \quad C > x(1+r).$$

Then both (6.2) *and the mean-Value-at-Risk problem* (6.1) *are unsolvable.*

Proof. Let A be the feasible set as in the proof of Theorem 6.2. If $\varphi \in A$, then

$$0 < \left(\frac{C}{x} - 1 - r\right)\frac{\Theta}{\Theta - z_\beta} \overset{(3.6)}{\leq} \varphi'(b - r\mathbf{1}) \overset{(3.7)}{\leq} \left(\frac{C}{x} - 1 - r\right)\frac{\Theta}{\Theta + z_\beta} < 0.$$

This contradiction shows $A = \emptyset$, and hence both (6.1) and (6.2) are unsolvable. □

Example 6.6. We calculate the maximal expected wealth with $\mathrm{VaR} = C$. Let r, b, σ, x, and z_β be as in Example 5.6 and let $C = 1030$. Thus $C < x(1+r)$ so that all assumptions of Theorem 6.2 are satisfied. Next,

$$\lambda = \frac{\frac{1030}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\| - 1.64} \approx 0.098107.$$

By Theorem 6.2, the optimal investment strategy is given by

$$\varphi^* = \frac{\lambda \cdot \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx \begin{pmatrix} 0.341564 \\ 0.113855 \\ 0.426955 \end{pmatrix}.$$

This means 34.1564% are invested in asset 1, 11.3855% are invested in asset 2, and 42.6955% are invested in asset 3. Now we calculate the expected wealth of this strategy:

$$\mathbb{E}(X^\varphi(1)) = 1000 \cdot \left(1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}\right) \approx 1190.895254.$$

We finally check if the VaR of this strategy really is 1030:

$$\mathrm{VaR}(\varphi^*) = 1000 \cdot \left((-1.64) \cdot \lambda + 1 + 0.05 + (\varphi^*)' \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix}\right) = 1030.$$

7 Multi-Period Mean-Variance Problem

In this section we introduce the multi-period mean-variance problem (see also [7]) and provide a closed-form solution. We solve the optimization problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \mathbb{V}\mathrm{ar}(X^\varphi(T)) \\ \text{s.t. } \mathbb{E}(X^\varphi(T)) \geq C, \end{cases} \tag{7.1}$$

where C is the expected terminal wealth at time T. We assume that the expected wealth of the investor is greater than the wealth of the risk-free asset.

Theorem 7.1 (Closed-form solution of the discrete-time multi-period mean-variance optimization problem)**.** *Assume* (3.1)*. The closed-form solution of the multi-period mean-variance problem* (7.1) *is given by*

$$\varphi_* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbb{1})}{\Theta} \quad \text{with} \quad \lambda = \frac{\sqrt[T]{\frac{C}{x}} - 1 - r}{\Theta}.$$

The expected wealth after T periods is C with variance

$$\mathbb{V}\mathrm{ar}(X^\varphi(T)) = x^2 \left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}} + \lambda^2 \right)^T - \left(\frac{C}{x}\right)^2 \right].$$

Proof. Using (2.11) and (2.12) for $t = T$, it suffices to show that $\varphi_* \in A$ and

$$g(\varphi) \geq g(\varphi_*) = x^2 \left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}} + \lambda^2 \right)^T - \left(\frac{C}{x}\right)^2 \right] \quad \text{for all} \quad \varphi \in A,$$

where

$$g(\varphi) := x^2 \left[\left((1 + r + \varphi'(b - r\mathbb{1}))^2 + \varphi'\sigma\sigma'\varphi \right)^T - (1 + r + \varphi'(b - r\mathbb{1}))^{2T} \right]$$

and

$$A := \left\{ \varphi \in \mathbb{R}^n : \ 1 + r + \varphi'(b - r\mathbb{1}) \geq \sqrt[T]{\frac{C}{x}} \right\}.$$

To show this, first note that

$$\begin{aligned} x\left(1 + r + (\varphi_*)'(b - r\mathbb{1})\right)^T &\overset{(3.3)}{=} x(1 + r + \lambda\Theta)^T \\ &= x\left(1 + r + \frac{\sqrt[T]{\frac{C}{x}} - 1 - r}{\Theta}\Theta\right)^T = C \end{aligned}$$

implies $\varphi_* \in A$. Next, if $\varphi \in A$, then

$$\|\sigma'\varphi\|^2 \overset{(3.2)}{\geq} \frac{|\varphi'(b - r\mathbb{1})|^2}{\Theta^2} \geq \left| \frac{\varphi'(b - r\mathbb{1})}{\Theta} \right|^2 \geq \lambda^2 \tag{7.2}$$

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and thus

$$\begin{aligned}
g(\varphi) &= x^2 \left[\left((1+r+\varphi'(b-r\mathbf{1}))^2 + \varphi'\sigma\sigma'\varphi \right)^T - (1+r+\varphi'(b-r\mathbf{1}))^{2T} \right] \\
&\overset{(7.2)}{\geq} x^2 \left[\left((1+r+\varphi'(b-r\mathbf{1}))^2 + \lambda^2 \right)^T - (1+r+\varphi'(b-r\mathbf{1}))^{2T} \right] \\
&\overset{(\varphi\in A)}{\geq} x^2 \left(\left(1+r+\sqrt[T]{\frac{C}{x}} - 1 - r \right)^2 + \lambda^2 \right)^T \\
&\quad -x^2 \left(1+r+\sqrt[T]{\frac{C}{x}} - 1 - r \right)^{2T} \\
&= x^2 \left[\left(\left(\frac{C}{x} \right)^{\frac{2}{T}} + \lambda^2 \right)^T - \left(\frac{C}{x} \right)^2 \right] \\
&= x^2 \left[\left((1+r+\lambda\Theta)^2 + \lambda^2 \right)^T - (1+r+\lambda\Theta)^{2T} \right] \\
&\underset{(3.4)}{\overset{(3.3)}{=}} x^2 \left[\left((1+r+(\varphi_*)'(b-r\mathbf{1}))^2 + (\varphi_*)'\sigma\sigma'\varphi_* \right)^T \right] \\
&\quad -x^2 (1+r+(\varphi_*)'(b-r\mathbf{1}))^{2T} \\
&= g(\varphi_*),
\end{aligned}$$

where in the second inequality sign we have used Lemma 3.3. □

As an immediate consequence of Theorem 7.1 we get that the mean-variance is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths. Let us denote with $\omega := \mathbb{E}(X^{\varphi}(T))$ the expected wealth after T periods. Now we can plug it into the result of Theorem 7.1 to get

$$\mathbb{V}\mathrm{ar}(\omega) = x^2 \left[\left(\left(\frac{\omega}{x} \right)^{\frac{2}{T}} + \left(\frac{\sqrt[T]{\frac{\omega}{x}} - 1 - r}{\Theta} \right)^2 \right)^T - \left(\frac{\omega}{x} \right)^2 \right].$$

If we know our desirable expected terminal wealth, then we can calculate λ and the portfolio construction strategy. Another way is that we accept a certain amount as variance, and then we calculate ω and set this equal to C. Then we are able to calculate the optimal portfolio.

Example 7.2. Let r, b, σ, and x be as in Example 4.3 and let $C = 1110$ and $T = 2$. Now we calculate

$$\lambda = \frac{\sqrt{\frac{1110}{1000}} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.05 \\ 0.15 \\ 0.25 \end{pmatrix} \right\|} \approx 0.001416.$$

Then we find the variance of our portfolio with expected terminal wealth C as

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\varphi_*) &= x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T-\left(\frac{C}{x}\right)^2\right]\\ &= 1000^2\left[\left(\frac{1110}{1000}+\lambda^2\right)^2-\left(\frac{1110}{1000}\right)^2\right]\approx 4.453.\end{aligned}$$

This is the minimal variance for the portfolio with an expected terminal wealth of 1110 at time 2. By Theorem 7.1, the optimal investment strategy is given by

$$\varphi_* = \frac{\lambda\cdot\begin{pmatrix}0.041 & 0.0242 & 0.0133\\ 0.0242 & 0.1016 & 0.018\\ 0.0133 & 0.018 & 0.0134\end{pmatrix}^{-1}\begin{pmatrix}0.05\\0.15\\0.25\end{pmatrix}}{\left\|\begin{pmatrix}0.2 & 0.01 & 0.03\\ 0.1 & 0.3 & 0.04\\ 0.05 & 0.03 & 0.1\end{pmatrix}^{-1}\begin{pmatrix}0.05\\0.15\\0.25\end{pmatrix}\right\|}\approx\begin{pmatrix}-0.003773\\-0.001042\\0.015641\end{pmatrix}.$$

This means we invest 1.5641% of our initial wealth in asset 3. The rest is invested risk free. Now we check if the expected wealth at time 2 really is 1110:

$$\mathbb{E}(X^{\varphi}(2)) = 1000\cdot\left(1+0.05+(\varphi_*)'\begin{pmatrix}0.05\\0.15\\0.25\end{pmatrix}\right)^2 = 1110.$$

Remark 7.3. The presented results can also be generalized from difference equations to dynamic equations on isolated time scales (see [2]). This will be done in a forthcoming paper of the authors.

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DEDICATED TO 65TH BIRTHDAY OF

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HEINER GONSKA

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On the 65th Birthday of Prof. Dr. dr.h.c. Heiner Gonska

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This issue of "Journal of Applied Functional Analysis" is dedicated to Professor Heiner Gonska who will celebrate his 65th birthday on January 6, 2014.

Heiner Gonska was born and grew up in Gelsenkirchen, Germany, which is located in the region called Ruhrgebiet (meaning the area of the river Ruhr). After finishing high school (Gymnasium) in 1967 in Gelsenkirchen, he studied mathematics, economy, philosophy and education science at the newly founded Ruhr University in Bochum, where he was a student of Hartmut Ehlich and Werner Haußmann. Both scholars had recently come from the south of Germany: Professor Ehlich was the head of the Institute of Numerical Mathematics and Applied Approximation Theory, and, at the same time, director of the data center of the Ruhr University, with Werner Haußmann being his assistant. So Heiner Gonska was present when applied mathematics started to grow in Germany, and universities in the Ruhr area were founded (next to Bochum, the universities of Dortmund, Hagen, Essen and Duisburg) and offered the chance of higher education in this region.

His academic studies were interrupted by military service from 1968 to 1969, and in 1975 Heiner Gonska finished his studies at the Ruhr University of Bochum with a diploma thesis on convergence theorems of Bohman-Korovkin-type for positive linear operators. He followed then Werner Haussmann who, in that year, became a full professor at the University of Duisburg (now University of Duisburg-Essen); the University of Duisburg seems to be a point in Heiner Gonska's life to which he always returned, after quite a few longer absences. While being assistant (Wissenschaftlicher Assistent) to Prof. Simm and Prof. Haußmann from 1975 to 1982, he was at the same time, visiting assistant professor at the Rensselaer Polytechnic Institute, Troy, NY, from 1981 to 1982. Then he was assistant professor (Hochschulassistent) at Duisburg University from 1982 till 1987, and also during this time, from 1983 till 1987, he worked as assistant professor at Drexel University, Philadelphia, PA. From 1987 to 1989, Heiner Gonska was appointed temporary professor at University of Duisburg.

During this time, he finished his doctoral degree in 1979 with a dissertation on quantitative results on the approximation by positive linear operators, supervised by Werner Haußmann and, in 1986, his "Habilitationsschrift" Quantitative Approximation in $C(X)$; two works that greatly influenced many researchers in this topic.

Also while working as an assistant and later professor for mathematics at Duisburg, Troy and Drexel, Gonska became interested in the new topic of computer science and studied this field at the FernUniversität Hagen, Germany, an open university, from 1980 to 1984. During his stays at US universities, he also learned to appreciate a certain brand of computers (at that time hardly known, but very popular nowadays). At his recommendation, when the mathematics department at the University of Duisburg decided to equip the stuff with personal computers, all of them were (and still are) marked with a certain fruit.

After this period of very intensive activity - graduation in mathematics, holding two positions on two continents at the same time over seven years, and studying computer science, Heiner Gonska became a full professor for Theoretical Computer Science at the European Business School, a private university near Frankfurt, Germany, in 1989. He held this position till 1993, when he was appointed full professor at the University of Duisburg. This is the position Heiner Gonska has held since then.

The list of publications of Heiner Gonska contains so far more than 150 papers and contributions. Most of them deal with different topics in approximation theory, some also with applications in computer aided geometric design. His work on positive linear operators, in particular, pushed the research to the very limits of what can be achieved in this area, solved problems that remained open for a long time and formulated further questions and conjectures that inspired many researches around the world.

Next to his research activities, there are other things that are very special to Professor Gonska: one of them is his early interest in Eastern European and Asian works, another one is his awareness of the social aspects of being a mathematician.

Let us start with the second aspect and mention one example of this awareness which is typical for him. In 1989, Heiner Gonska and Ewald Quak, at that time University of Dortmund, realized that they frequently met at different places all over the world, but never in the Ruhr area, where their universities were located at a distance of 50 kilometers from each other. So they came with the idea to bring mathematicians with common interests in approximation theory (and neighboring disciplines such as numerical mathematics) together; this was the starting point of the Oberseminar Rhein-Ruhr. Predominantly, mathematicians were addressed that were in a close regional distance. Since then, every year six to eight such local meetings have regularly been taking place, and now five universities collaborate to make this happen. At the end of each winter semester, a two-day workshop is being organized and attracts people also from the rest of Germany. Heiner Gonska made sure that especially young mathematicians got the chance to present their research results, without the pressure of a large international conference. Also, a lot of German senior experts participated regularly in these workshops, giving friendly advises to the youngsters. It has been almost 25 years now that Heiner contributes to bring people together in a casual setting and help young people start their conference experience. Not only the authors of this contribution, but also many others are very grateful to him for this fruitful experience he enabled.

Gonska's early interest in Eastern European and Asian research activities was - to a certain extend - shared by other members of the Duisburg approximation group as well (Proff. Haußmann, Jetter and Knoop). Being aware of the strong contribution of eastern European mathematicians (Heiner Gonska and his later wife Jutta Meier compiled a first bibliography on Approximation of Functions by Bernstein-type operators in 1983, analyzing piles of Eastern European journals) to approximation theory, he frequently contacted and visited Eastern European scientists, in spite of the Iron Curtain. As early as 1986, his first publications with coauthors from China and Romania appeared. After 1990, he was strongly engaged to help many mathematicians from Eastern Europe to visit him at Duisburg and to collaborate with him and several other people from his group. Aside from these collaborations with individuals, he was among the initiators, organizers and members of the scientific committee of RoGer (Romanian-German Seminar on Approximation Theory and its Applications), a bi-national conference held every second year since 1994. He also was in the scientific committee of two international conferences on Numerical Analysis and Approximation Theory in 2006 and 2010 in Cluj-Napoca, Romania.

His activies yielded, among others, many joint papers with co-authors from China and Eastern Europe. Among the ten habilitations and dissertations he supervised so far, there are two PhD theses he supervised at Babes -Bolyai University from Cluj-Napoca, Romania. Professor Gonska received a price from the Ministery of Education of the People's Republic of China (1993), and an honorary doctorate from the Babes-Bolyai University (1999). His collaboration work was supported by many diverse grants, among which the largest one is the Center of Excellence for Applications of Mathematics, financed by the German Federal Foreign Office and the German Academic Exchange Service (DAAD) with an amount of about one million Euros so far. This center, with Prof. Gonska as coordinator, involves 15 partner universities from ten south-eastern countries.

Heiner's daughters, Jenny and Nadine, are both accomplishing now their academic studies, Nadine being also very successful in sports.

The authors of this article are positive to speak on behalf of all of Professor Gonska's students when expressing our best wishes for the future, in particular for continuing success in his various professional activities and a life full of health and happiness.

Improvement and generalization of some Ostrowski-type inequalities

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Several inequalities of Ostrowski-type available in the literature are generalized and improved. New bounds for the error in some numerical integration rules are derived.

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Key Words and Phrases: Chebyshev functional, Chebyshev-type inequality, Grüss-type inequality, Ostrowski-type inequality, moduli of smoothness.

1 Introduction

In the last years, Grüss- and Ostrowski-type inequalities have attracted much attention, because of their applications in mathematical statistics, econometrics and actuarial mathematics. In this paper, we improve and generalize some Ostrowski-type inequalities involving differentiable mappings. The connection between the Ostrowski and the Grüss inequality is emphasized, explaining in this way the term "Ostrowski-Grüss-type inequalities" used in the literature. Some generalizations of these inequalities, using the least concave majorant of the modulus of continuity and the second order modulus of smoothness, are considered. B. Gavrea and I. Gavrea [6] were the first to observe the possibility of using moduli in this context. The functional given by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \qquad (1.1)$$

where $f, g : [a,b] \to \mathbb{R}$ are integrable functions, is well known in the literature as the Chebyshev functional (see [2]). In 1935, G. Grüss [9] obtained the following result.

Theorem 1 *Let f and g be two functions defined and integrable on $[a,b]$. If $m \leq f(x) \leq M$ and $p \leq g(x) \leq P$ for all $x \in [a,b]$, then we have*

$$|T(f,g)| \leq \frac{1}{4}(M-m)(P-p). \qquad (1.2)$$

The constant 1/4 is the best possible.

In 1882, P.L. Chebyshev [2] obtained the following inequality.

Theorem 2 *If $f, g \in C^1[a,b]$, then*

$$|T(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2 \qquad (1.3)$$

holds, where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

In 1970, A. Ostrowski [12] proved the following result, which is a combination of the Chebyshev and the Grüss results (1.3) and (1.2).

Theorem 3 *If f is Lebesgue integrable on $[a,b]$ satisfying $m \leq f(x) \leq M$, $x \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous with $g' \in L_\infty[a,b]$, then the inequality*

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty \qquad (1.4)$$

holds. The constant $\frac{1}{8}$ is sharp.

Remark 4 *The inequalities (1.2) and(1.4) are known in the literature as Grüss-type inequalities, but we consider them to be of Chebyshev-Grüss-type.*

Another celebrated classical inequality was proved by A. Ostrowski [11] in 1938, which we cite below in the form given by G.A. Anastassiou in 1995 (see [3]).

Theorem 5 *Let f be in $C^1[a,b]$, $x \in [a,b]$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \|f'\|_\infty. \tag{1.5}$$

This inequality gives an upper bound for the approximation of the average value of the function f by the value $f(x)$ at the point $x \in [a,b]$. In 1997, S.S. Dragomir and S. Wang [5] applied Theorem 1 to the mappings $f'(t)$ and $p(x,t) = \begin{cases} t-a, \, t \in [a,x] \\ t-b, \, t \in (x,b] \end{cases}$, obtaining a new result for bounded differentiable mappings, which is known as the Ostrowski-Grüss-type inequality.

Theorem 6 *(see Theorem 2.1 in [5]) Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $Int(I)$ and let $a, b \in Int(I)$ with $a < b$. If $f' \in L_1[a,b]$ and $\gamma \leq f'(x) \leq \Gamma$, $\forall\, x \in [a,b]$, then we have the following inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma).$$

This inequality has been improved by M. Matić et al. ([10]), as shown in the follwing theorem.

Theorem 7 *(see Theorem 6 in [10]) Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping which is differentiable in the interior $Int(I)$ of I, and let $a, b \in Int(I)$ with $a < b$. If $\gamma \leq f'(x) \leq \Gamma$, for $x \in [a,b]$ and some constants $\gamma, \Gamma \in \mathbb{R}$, then, for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq \frac{1}{4\sqrt{3}}(b-a)(\Gamma-\gamma).$$

This result is improved by X.L. Cheng in [4], as follows in the next theorem. He also proved that the constant 1/8 is sharp.

Theorem 8 *(see Theorem 1.5. in [4]) Let the assumptions of Theorem 7 hold. Then, for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq \frac{1}{8}(b-a)(\Gamma-\gamma). \tag{1.6}$$

In [14], N. Ujević proved the following, involving the second derivative of the mapping f.

Theorem 9 *(see Theorem 4 in [14]) Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ an interval, be a twice continuously differentiable mapping in the interior $Int(I)$ of I with*

$f'' \in L_2(a,b)$, and let $a,b \in Int(I)$, $a<b$. Then we have, for all $x \in [a,b]$,

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{(b-a)^{3/2}}{2\pi\sqrt{3}}\|f''\|_2, \tag{1.7}$$

Remark 10 *The above results are known in the literature as Ostrowski-Grüss-type inequalities, because on the left hand-side there are Ostrowski-type expressions present, while the right hand-side looks of Grüss type. Nevertheless, because Grüss-type inequalities involve the Chebyshev functional on the left hand-side, we consider these inequalities to be of modified Ostrowski-type.*

For brevity, in the sequel we will denote

$$\mathcal{M}_x[f] := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right),$$

where $f : [a,b] \to \mathbb{R}$ is an integrable function and $x \in [a,b]$.

The structure of this paper is as follows: in Section 2 we give new bounds and improve some inequalities available in the literature for the functional $\mathcal{M}_x[f]$ (see [4], [14]), involving the least concave majorant of the modulus of continuity and the second order modulus of smoothness. In Section 3, using Peano's theorem, we propose a generalization of some Ostrowski-type inequalities. Finally, in Section 4, we provide new estimates for the error in some numerical integration rules.

2 Ostrowski-type inequalities in terms of the least concave majorant and moduli of smoothness

The aim of this section is to give new inequalities for the functional $\mathcal{M}_x[f]$, involving the second derivative of the mapping f. Also, we will extend the inequalities mentioned in the previous section, by using the least concave majorant of the modulus of continuity, K-functionals and second order moduli of smoothness.

Theorem 11 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with the second derivative bounded on (a,b), i.e., $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then, for all $x \in [a,b]$, we get $|\mathcal{M}_x[f]| \le \dfrac{u(x;a,b)}{b-a}\cdot \|f''\|_\infty$,*
where $u(x;a,b) = \begin{cases} u_1(x;a,b), \ x \in \left[a, \frac{a+b}{2}\right], \\ u_2(x;a,b), \ x \in \left(\frac{a+b}{2}, b\right], \end{cases}$

$$u_1(x;a,b) = \frac{1}{2}\left[-4abx + \frac{a^3}{2} - \frac{8}{3}x^3 + \frac{b^3}{6} + 3bx^2 + \frac{3}{2}ba^2 - 3xa^2 + 5ax^2 + \frac{1}{2}ab^2 - xb^2\right],$$
$$u_2(x;a,b) = \frac{1}{2}\left[4abx - \frac{b^3}{2} + \frac{8}{3}x^3 - \frac{a^3}{6} - 3ax^2 - \frac{3}{2}ab^2 + 3xb^2 - 5bx^2 - \frac{1}{2}ba^2 + xa^2\right].$$

Proof. We define

$$\mathcal{P}(x,t) = \begin{cases} \frac{1}{2}(t+b-2x)(t-a), t \in [a,x], \\ \frac{1}{2}(t+a-2x)(t-b), t \in (x,b]. \end{cases} \tag{2.1}$$

Integrating by parts, we have

$$\int_a^b \mathcal{P}(x,t)f''(t)dt = \int_a^b f(t)dt - (b-a)f(x) + \left(x - \frac{a+b}{2}\right)(f(b)-f(a)).$$

From the above relation, we get

$$|\mathcal{M}_x[f]| = \frac{1}{b-a}\left|\int_a^b \mathcal{P}(x,t)f''(t)dt\right| \le \frac{1}{b-a}\|f''\|_\infty \cdot \int_a^b |\mathcal{P}(x,t)|dt.$$

Since $\int_a^b |\mathcal{P}(x,t)|dt = u(x;a,b)$, the inequality (11) is proved. ■

Remark 12 *Since $u(x;a,b) \le \frac{(b-a)^3}{12}$, from inequality (11) it follows*

$$|\mathcal{M}_x[f]| \le \frac{(b-a)^2}{12} \cdot \|f''\|_\infty.$$

Theorem 13 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with $f'' \in L_2[a,b]$. Then, for all $x \in [a,b]$, we have*

$$|\mathcal{M}_x[f]| \le \frac{\sqrt{\mu(x;a,b)}}{b-a}\|f''\|_2, \text{ where} \tag{2.2}$$

$$\mu(x;a,b) = \frac{1}{4}(b-a)x^4 - \frac{1}{2}(b^2-a^2)x^3 + x^2\left[\frac{1}{2}(ab^2-a^2b) + \frac{1}{3}(b^3-a^3)\right]$$
$$+x\left[\frac{1}{3}\left(ba^3-ab^3\right) + \frac{1}{12}\left(a^4-b^4\right)\right] + \frac{1}{120}(b^5-a^5) + \frac{1}{24}\left(ab^4-ba^4\right) + \frac{1}{12}(a^2b^3-b^2a^3).$$

Proof. If we consider the function $\mathcal{P}(x,t)$ defined in (2.1), it follows

$$|\mathcal{M}_x[f]| = \frac{1}{b-a}\left|\int_a^b \mathcal{P}(x,t)f''(t)dt\right| \le \frac{\|f''\|_2}{b-a}\left[\int_a^b (\mathcal{P}(x,t))^2\,dt\right]^{1/2} = \frac{\sqrt{\mu(x;a,b)}}{b-a}\|f''\|_2.$$

■

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Remark 14 *For $a = 0$ and $b = 1$, the inequalities (1.7) and (2.2), respectively, become*

$$|\mathcal{M}_x[f]| \le \frac{1}{2\pi\sqrt{3}}\|f''\|_2 \approx 0.0919\|f''\|_2, \text{ and}$$

$$|\mathcal{M}_x[f]| \le \sqrt{\frac{1}{120}+\frac{x^4}{4}-\frac{x^3}{2}-\frac{x}{12}+\frac{x^2}{3}}\cdot\|f''\|_2 \le \frac{\sqrt{30}}{60}\|f''\|_2 \approx 0.0913\|f''\|_2,$$

respectively. In this particular case, our estimates are better than N. Ujević's result (1.7).

Theorem 15 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with $f'' \in L_1[a,b]$. Then, for all $x \in [a,b]$, we have*

$$|\mathcal{M}_x[f]| \le \frac{\nu(x;a,b)}{b-a}\cdot\|f''\|_1, \text{ where } \nu(x;a,b) = \begin{cases} \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2, & x \in [x_1,x_2], \\ \frac{1}{2}(x-a)(b-x), & x \in [a,b]\setminus[x_1,x_2], \end{cases} \tag{2.3}$$

for

$$x_1 = \frac{(2+\sqrt{2})a+(2-\sqrt{2})b}{4}, \; x_2 = \frac{(2-\sqrt{2})a+(2+\sqrt{2})b}{4}.$$

Proof. If we consider the function $\mathcal{P}(x,t)$ defined by (2.1), it follows

$$|\mathcal{M}_x[f]| = \frac{1}{b-a}\left|\int_a^b \mathcal{P}(x,t)f''(t)dt\right| \le \frac{\|f''\|_1}{b-a}\sup_{t\in[a,b]}|\mathcal{P}(x,t)| = \nu(x;a,b)\cdot\frac{\|f''\|_1}{b-a}.$$

■

In order to formulate the next result we need the following

Definition 16 *Let $f \in C[a,b]$. If, for $t \in [0,\infty)$, the quantity*

$$\omega(f;t) = \sup\{|f(x)-f(y)|,\ |x-y| \le t\}$$

is the usual modulus of continuity, its least concave majorant is given by

$$\tilde{\omega}(f;t) = \sup\left\{\frac{(t-x)\omega(f;y)+(y-t)\omega(f;x)}{y-x};\ 0 \le x \le t \le y \le b-a, x \ne y\right\}.$$

Let $I = [a,b]$ be a compact interval of the real axis and $f \in C(I)$. In [13], the following result for the least concave majorant is proved:

$$K\left(\frac{t}{2}, f; C[a,b], C^1[a,b]\right) := \inf_{g\in C^1(I)}\left(\|f-g\|_\infty + \frac{t}{2}\|g'\|_\infty\right) = \frac{1}{2}\tilde{\omega}(f;t),\ t \ge 0.$$

Theorem 17 *If $f \in C^1[a,b]$, then*

$$|\mathcal{M}_x[f]| \leq \frac{b-a}{8}\tilde{\omega}\left(f'; \frac{8u(x;a,b)}{(b-a)^2}\right),$$

where the constant $u(x;a,b)$ is defined in Theorem 11.

Proof. Let $A_x : C[a,b] \to \mathbb{R}$ be defined by

$$A_x[f] = \int_a^x \left(t - x + \frac{b-a}{2}\right) f(t)dt + \int_x^b \left(t - x - \frac{b-a}{2}\right) f(t)dt.$$

We have

$$|A_x[f]| \leq \|f\|_\infty \cdot \left\{\int_a^x \left|t - x + \frac{b-a}{2}\right| dt + \int_x^b \left|t - x - \frac{b-a}{2}\right| dt\right\} = \frac{(b-a)^2}{4} \cdot \|f\|_\infty. \tag{2.4}$$

Let $g \in C^1[a,b]$ and $\mathcal{P}(x,t)$ be the mapping defined by (2.1). We obtain

$$\int_a^b \mathcal{P}(x,t)g'(t)dt = -\int_a^x \left(t - x + \frac{b-a}{2}\right) g(t)dt - \int_x^b \left(t - x - \frac{b-a}{2}\right) g(t)dt,$$

namely

$$|A_x[g]| = \left|\int_a^b \mathcal{P}(x,t)g'(t)dt\right| \leq \|g'\|_\infty \cdot \int_a^b |\mathcal{P}(x,t)|dt = \|g'\|_\infty \cdot u(x;a,b). \tag{2.5}$$

From relations (2.4) and (2.5), we have

$$\begin{aligned} |A_x[f]| &= |A_x(f - g + g)| \leq |A_x(f-g)| + |A_x(g)| \\ &\leq \frac{(b-a)^2}{4}\|f - g\|_\infty + \|g'\|_\infty \cdot u(x;a,b) \\ &\leq \frac{(b-a)^2}{4} \inf_{g \in C^1[a,b]} \left\{\|f-g\|_\infty + \frac{4u(x;a,b)}{(b-a)^2}\|g'\|_\infty\right\}. \end{aligned}$$

Therefore,

$$|A_x[f]| \leq \frac{(b-a)^2}{8}\tilde{\omega}\left(f; \frac{8u(x;a,b)}{(b-a)^2}\right). \tag{2.6}$$

If we write (2.6) for the function f', we obtain inequality (17). ∎

Theorem 18 *If $f \in C[a,b]$, then for all $x \in [a,b]$, we have*

$$|\mathcal{M}_x[f]| \leq 3K\left(\frac{u(x;a,b)}{3(b-a)}; f; C[a,b], C^2[a,b]\right), \tag{2.7}$$

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where

$$K\left(t; f; C[a,b], C^2[a,b]\right) := \inf_{g\in C^2[a,b]}\{\|f-g\|_\infty + t\|g''\|_\infty\}.$$

Proof. For any $f \in C[a,b]$, $|\mathcal{M}_x[f]| \le 3\|f\|_\infty$. For $g \in C^2[a,b]$, from Theorem 11 we get

$$|\mathcal{M}_x[g]| \le \frac{u(x;a,b)}{b-a}\|g''\|_\infty.$$

So, for $f \in C[a,b]$ fixed and $g \in C^2[a,b]$ arbitrary, we have

$$\begin{aligned}|\mathcal{M}_x[f]| &\le |\mathcal{M}_x[f-g]| + |\mathcal{M}_x[g]| \le 3\|f-g\|_\infty + \frac{u(x;a,b)}{b-a}\|g''\|_\infty \\ &= 3\{\|f-g\|_\infty + \frac{u(x;a,b)}{3(b-a)}\|g''\|_\infty\}.\end{aligned}$$

Passing to the infimum over $g \in C^2[a,b]$ gives relation (2.7). ∎

An upper bound in terms of the second modulus of smoothness will be considered in the next part of this paper. For brevity, in the sequel we will take $[a,b] = [0,1]$. In order to formulate our result, we need the following lemma (see [12], [8]).

Lemma 19 *For $f \in C[0,1]$, $0 < h \le \frac{1}{2}$ fixed and any $\varepsilon > 0$, there are polynomials $p = p(f,h)$, such that*

$$\|f-p\|_\infty \le \frac{3}{4}\omega_2(f;h) + \varepsilon,\ \|p''\|_\infty \le \frac{3}{2h^2}\omega_2(f;h)$$

hold.

Theorem 20 *If $f \in C[0,1]$, then for all $x \in [0,1]$, the following inequality holds*

$$|\mathcal{M}_x[f]| \le \frac{15}{4}\omega_2\left(f; \sqrt{u(x;0,1)}\right). \tag{2.8}$$

Proof. Let $\varepsilon > 0$ be arbitrarily given. For $f \in C[0,1]$ and $0 < h \le \frac{1}{2}$, we consider the polynomial $p = p(f,h)$ in Lemma 19. Since $p \in C^2[0,1]$, from the proof of Theorem 18 we have

$$\begin{aligned}|\mathcal{M}_x[f]| &\le 3\left\{\|f-p\|_\infty + \frac{u(x;0,1)}{3}\|p''\|_\infty\right\} \\ &\le 3\left\{\frac{3}{4}\omega_2(f;h) + \epsilon + \frac{u(x;0,1)}{2h^2}\omega_2(f;h)\right\}.\end{aligned}$$

Letting ε tend to zero shows that

$$|\mathcal{M}_x[f]| \le 3\left\{\frac{3}{4} + \frac{u(x;0,1)}{2h^2}\right\}\omega_2(f;h),\ \text{where } 0 < h \le \frac{1}{2} \text{ is arbitrary.}$$

Now choose $h = \sqrt{u(x;0,1)} \leq \dfrac{1}{\sqrt{12}}$. This implies relation (2.8). ■

3 A generalized Ostrowski-type inequality

Inequalities for a linear functional in terms of a variety of norms are given in this section. For the particular cases, some inequalities of Ostrowski- and Ostrowski-Grüss-type from the literature are retrieved. Let $L : C^{n+1}[a,b] \to \mathbb{R}$ be a linear functional, with degree of exactness n ($L(e_i) = 0$, $e_i(x) = x^i, i = \overline{0,n}$). Using Peano's theorem, the functional has the following integral representation $L[f] = \int_a^b K(t) f^{(n+1)}(t)dt$, where $K(t) = L\left(\frac{(x-t)_+^n}{n!}\right)$. Denote by $\tilde{L}[f] = L[f] - A\left[f^{(n)}(b) - f^{(n)}(a)\right]$, where $A = \dfrac{1}{b-a}\displaystyle\int_a^b K(t)dt$. Since the functional $\tilde{L}$ has degree of exactness n, it follows $\tilde{L}[f] = \displaystyle\int_a^b \tilde{K}(t) f^{(n+1)}(t)dt$, where $\tilde{K}(t) = \tilde{L}\left(\dfrac{(x-t)_+^n}{n!}\right) = L\left(\dfrac{(x-t)_+^n}{n!}\right) - A = K(t) - A$. From the above relation, we remark that $\displaystyle\int_a^b \tilde{K}(t)dt = 0$.

Remark 21 *a) The functional*

$$L[f] = f(x) - \frac{1}{b-a}\int_a^b f(t)dt \tag{3.1}$$

has degree of exactness $n = 0$ and it holds that

$$\tilde{L}[f] = \mathcal{M}_x[f] := f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right),$$

with $K(t) = \begin{cases} \dfrac{t-a}{b-a}, \ t \in [a,x], \\ \dfrac{t-b}{b-a}, \ t \in (x,b], \end{cases}$ *and* $\tilde{K}(t) = \begin{cases} \dfrac{1}{b-a}\left[t - x + \dfrac{b-a}{2}\right], \ t \in [a,x], \\ \dfrac{1}{b-a}\left[t - x - \dfrac{b-a}{2}\right], \ t \in (x,b]. \end{cases}$

b) The functional

$$L[f] = \frac{1}{2}\left[f(x) + f(a)\right] - \frac{1}{b-a}\int_a^b f(t)dt \tag{3.2}$$

has degree of exactness $n = 0$ and

$$\tilde{L}[f] = \mathcal{N}_x[f] := \frac{1}{2}f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)},$$

with

$$K(t)=\begin{cases}\dfrac{1}{2}-\dfrac{b-t}{b-a}, \ t\in[a,x],\\ -\dfrac{b-t}{b-a}, \ t\in(x,b],\end{cases}\quad \tilde{K}(t)=\begin{cases}\dfrac{1}{b-a}\left[t-\dfrac{x+a}{2}\right], \ t\in[a,x],\\ \dfrac{1}{b-a}\left[t-\dfrac{x+b}{2}\right], \ t\in(x,b].\end{cases}$$

Denote by $W_p^n[a,b]=\left\{f\in C^n[a,b],\ f^{(n)} \text{ absolutely continuous}, \|f^{(n+1)}\|_p<\infty\right\}$, where

$$\|f\|_p:=\left\{\int_a^b |f(x)|^p\,dx\right\}^{\frac{1}{p}}, \quad \text{for} \quad 1\le p<\infty,$$
$$\|f\|_\infty:=\sup_{x\in[a,b]}|f(x)|.$$

Using the following relation

$$\tilde{L}[f]=\int_a^b \tilde{K}(t)f^{(n+1)}(t)dt=\int_a^b\left[K(t)-\frac{1}{b-a}\int_a^b K(t)dt\right]f^{(n+1)}(t)dt \quad (3.3)$$
$$=(b-a)T\left(K,f^{(n+1)}\right),$$

we obtain the next results.

Theorem 22 *Let $f\in W_\infty^n[a,b]$ and $\gamma\le f^{(n+1)}(t)\le\Gamma$, for all $t\in[a,b]$ and some constants $\gamma,\Gamma\in\mathbb{R}$. Then the inequality $\left|\tilde{L}[f]\right|\le\frac{\Gamma-\gamma}{2}\cdot\|\tilde{K}\|_1$ holds.*

Proof. Since $\displaystyle\int_a^b\tilde{K}(t)dt=0$, we have

$$\left|\tilde{L}[f]\right|=\left|\int_a^b\tilde{K}(t)f^{(n+1)}(t)dt\right|=\left|\int_a^b\tilde{K}(t)\left(f^{(n+1)}(t)-\frac{\Gamma+\gamma}{2}\right)dt\right|$$
$$\le\sup_{t\in[a,b]}\left|f^{(n+1)}(t)-\frac{\Gamma+\gamma}{2}\right|\cdot\int_a^b|\tilde{K}(t)|dt\le\frac{\Gamma-\gamma}{2}\cdot\|\tilde{K}\|_1.$$

■

Remark 23 *Using Remark 21 and applying Theorem 22 for the functionals $\mathcal{N}_x[f]$ and $\mathcal{M}_x[f]$, we obtain the results given by X.L. Cheng in [4], namely*

$$|\mathcal{M}_x[f]|\le\frac{1}{8}(b-a)(\Gamma-\gamma),\ |\mathcal{N}_x[f]|\le\frac{(x-a)^2+(b-x)^2}{8(b-a)}(\Gamma-\gamma).$$

Theorem 24 *Let $f \in W_1^n[a,b]$ and $\gamma \leq f^{(n+1)}(t)$, for all $t \in [a,b]$. Then*

$$|\tilde{L}[f]| \leq \|\tilde{K}\|_\infty \cdot \left[\frac{f^{(n)}(b) - f^{(n)}(a)}{b-a} - \gamma \right] (b-a).$$

Proof. Since $\int_a^b \tilde{K}(t)dt = 0$, we have

$$\left|\tilde{L}[f]\right| = \left| \int_a^b \tilde{K}(t) f^{(n+1)}(t) dt \right| = \left| \int_a^b \tilde{K}(t) \left(f^{(n+1)}(t) - \gamma \right) dt \right|$$

$$\leq \sup_{t\in[a,b]} \left|\tilde{K}(t)\right| \cdot \int_a^b \left(f^{n+1}(t) - \gamma \right) dt = \|\tilde{K}\|_\infty \cdot \left[\frac{f^{(n)}(b) - f^{(n)}(a)}{b-a} - \gamma \right] (b-a).$$

■

Theorem 25 *Let $f \in W_1^n[a,b]$ and $f^{(n+1)}(t) \leq \Gamma$, for all $t \in [a,b]$. Then*

$$|\tilde{L}[f]| \leq \|\tilde{K}\|_\infty \cdot \left[\Gamma - \frac{f^{(n)}(b) - f^{(n)}(a)}{b-a} \right] (b-a).$$

Theorem 26 *If $f \in W_2^n[a,b]$, then*

$$|\tilde{L}[f]| \leq (b-a)\sqrt{T(K,K)} \cdot \sqrt{T(f^{(n+1)}, f^{(n+1)})}. \tag{3.4}$$

The inequality (3.4) is sharp, in the sense that the constant $(b-a)\sqrt{T(K,K)}$ cannot be replaced by a smaller ones.

Proof. The inequality (3.4) follows easily from relation (3.3).

To prove that the constant $(b-a)\sqrt{T(K,K)}$ cannot be replaced by a smaller ones, we define the function $F \in C^{n+1}[a,b]$, such that $F^{(n+1)}(x) = K(x)$, $x \in [a,b]$. For the function F, the right hand-side of (3.4) is equal with $(b-a)T(K,K)$ and the left hand-side becomes

$$\left|\tilde{L}[F]\right| = \int_a^b \tilde{K}(t)K(t)dt = \int_a^b \left(K(t) - \frac{1}{b-a} \int_a^b K(t)dt \right) K(t)dt$$

$$= \int_a^b K(t)^2 dt - \frac{1}{b-a} \int_a^b K(t)dt \int_a^b K(t)dt = (b-a)T(K,K).$$

■

Corollary 27 *If $f, g \in C^1[a,b]$, then $|\mathcal{M}_x[f]| \leq \frac{b-a}{\sqrt{12}} \cdot \sqrt{T(f',f')}$, and $|\mathcal{N}_x[f]| \leq \frac{\sqrt{a^2 - 3xa + ab + b^2 - 3bx + 3x^2}}{\sqrt{12}} \cdot \sqrt{T(f',f')}$.*

In [3] P. Cerone and S.S. Dragomir proved the following inequality for the functional $T(f,g)$.

Theorem 28 *Assume that $f:[a,b]\to\mathbb{R}$ is a measurable function on $[a,b]$ and such that $\tilde{f}:=f-\frac{1}{b-a}\int_a^b f(t)dt$, $e\tilde{f}\in L_2[a,b]$, where $e(t)=t, t\in[a,b]$. If $g:[a,b]\to\mathbb{R}$ is absolutely continuous and $g'\in L_2[a,b]$, then we have the inequality*

$$\begin{aligned}|T(f,g)| &\le \frac{2}{b-a}\|g'\|_2\cdot\left[\frac{\int_a^b \tilde{f}(t)^2dt\cdot\int_a^b t^2\tilde{f}(t)^2dt-\left(\int_a^b t\tilde{f}(t)^2dt\right)^2}{\int_a^b \tilde{f}(t)^2dt}\right]^{1/2}\\ &\le \frac{2}{b-a}\|g'\|_2\cdot\|e\tilde{f}\|_2.\end{aligned}$$

Using P. Cerone and S.S. Dragomir's result, a new inequality for the functional $\tilde{L}[f]$ in L_2-norm can be given.

Theorem 29 *Let $f:[a,b]\to\mathbb{R}$ be $(n+2)$-differentiable on the interval (a,b), with $f^{(n+2)}\in L_2[a,b]$. Then*

$$\|\tilde{L}[f]\|\le 2\|f^{(n+2)}\|_2\cdot\left[\frac{\int_a^b \tilde{K}(t)^2dt\cdot\int_a^b t^2\tilde{K}(t)^2dt-\left(\int_a^b t\tilde{K}(t)^2dt\right)^2}{\int_a^b \tilde{K}(t)^2dt}\right]^{1/2}. \tag{3.5}$$

Proof. Since $\tilde{L}[f]=(b-a)T\left(K,f^{(n+1)}\right)$, using Theorem 28, the inequality (3.5) holds. ■

Corollary 30 *Let $f:[0,1]\to\mathbb{R}$ be twice differentiable on the interval $(0,1)$, with $f''\in L_2[0,1]$. Then, for all $x\in[0,1]$, we have*

$$|\mathcal{M}_x[f]|\le\frac{1}{30}\sqrt{-4800x^6+14400x^5-16200x^4+8400x^3-1800x^2+45}\cdot\|f''\|_2, \tag{3.6}$$

$$|\mathcal{N}_x[f]|\le\frac{\sqrt{15}}{30}\sqrt{\frac{40x^6-120x^5+180x^4-160x^3+84x^2-24x+3}{3x^2-3x+1}}\cdot\|f''\|_2. \tag{3.7}$$

Proof. Let us consider the functions K and $\widetilde{K}$, defined in Remark 21 a). Since $\mathcal{M}_x[f]=\tilde{L}[f]$, applying Theorem 29, the inequality (3.6) holds. In a similar

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way, for the functions K and $\widetilde{K}$ defined in Remark 21 b), we prove the inequality (3.7). ■

4 Application to numerical quadrature rules

In this section we give some applications of the Ostrowski-type inequalities. Using the previous results, we derive new error-bounds in some numerical integration rules.

Let $\Delta_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of $[a,b]$, and $\xi = (\xi_0, \xi_1, \cdots, \xi_{n-1})$, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$ be an intermediate point vector. Denote by $Q(f, \Delta_n, \xi) := \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) (f(x_{i+1}) - f(x_i))$ a quadrature rule, where $h_i = x_{i+1} - x_i$, $i = \overline{0, n-1}$.

Theorem 31 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with second derivative bounded on (a,b), i.e., $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then $\left| \int_a^b f(t)dt - Q(f, \Delta_n, \xi) \right| \le \sum_{i=0}^{n-1} u(\xi_i; x_i, x_{i+1}) \cdot \|f''\|_\infty^{(i)}$ holds, where the function u is defined in (11) and $\|f''\|_\infty^{(i)} = \sup_{t\in[x_i,x_{i+1}]} |f(t)|$.*

Theorem 32 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with $f'' \in L_2[a,b]$. Then $\left| \int_a^b f(t)dt - Q(f, \Delta_n, \xi) \right| \le \sum_{i=0}^{n-1} \sqrt{\mu(\xi_i; x_i, x_{i+1})} \|f''\|_2^{(i)}$, where the function μ is defined by (2.2) and $\|f''\|_2^{(i)} = \left(\int_{x_i}^{x_{i+1}} f''(t)^2 dt \right)^{1/2}$.*

Theorem 33 *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on the interval (a,b), with $f'' \in L_1[a,b]$. Then $\left| \int_a^b f(t) - Q(f, \Delta_n, \xi) \right| \le \sum_{i=0}^{n-1} \nu(\xi_i; x_i, x_{i+1}) \cdot \|f''\|_1^{(i)}$, where the function ν is defined in (2.3) and $\|f''\|_1^{(i)} = \int_{x_i}^{x_{i+1}} |f''(t)|dt$.*

Remark 34 *Taking $\xi_i = \frac{x_i + x_{i+1}}{2}$, $i = \overline{0, n-1}$ in the above theorems, the error bounds of the mid-point quadrature rule, defined by $Q_M(f, \Delta_n) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2} \right)$, are obtained. For $\xi_i = x_i$ and $\xi_i = x_{i+1}$, respectively, in the above theorems, bounds for the error in the trapezoid quadrature rule, defined by $Q_T(f, \Delta_n) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2}$, are obtained.*

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Balanced Canavati type Fractional Opial Inequalities

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Here we present L_p, $p > 1$, fractional Opial type inequalities subject to high order boundary conditions. They involve the right and left Canavati type generalised fractional derivatives. These derivatives are mixed together into the balanced Canavati type generalised fractional derivative. This balanced fractional derivative is introduced and activated here for the first time.

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Key Words and Phrases: Opial inequality, fractional inequality, Canavati fractional derivative, boundary conditions.

1 Introduction

This article is inspired by the famous theorem of Z. Opial [10], 1960, which follows.

Theorem 1 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{1}$$

In (1), the constant $\frac{h}{4}$ is the best possible. Inequality (1) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases}$$

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where $c > 0$ is an arbitrary constant.

To prove easier Theorem 1, Beesack [4] proved the following well-known Opial type inequality which is used very commonly.

This is another inspiration to our work.

Theorem 2 *Let $x(t)$ be absolutely continuous in $[0, a]$, and $x(0) = 0$. Then*

$$\int_0^a |x(t)\, x'(t)|\, dt \leq \frac{a}{2} \int_0^a (x'(t))^2\, dt. \tag{2}$$

Inequality (2) is sharp, it is attained by $x(t) = ct$, $c > 0$ is an arbitrary constant.

Opial type inequalities are used a lot in proving uniqueness of solutions to differential equations, also to give upper bounds to their solutions.

By themselves have made a great subject of intensive research and there exists a great literature about them.

Typical and great sources on them are the monographs [1], [2].

We define here the balanced Canavati type fractional derivative and we prove related Opial type inequalities subject to boundary conditions.

These have smaller constants than in other Opial inequalities when using traditional fractional derivatives.

2 Background

Let $\nu > 0$, $n := [\nu]$ (integral part of ν), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. The gamma function Γ is given by $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. Here $[a, b] \subseteq \mathbb{R}$, $x, x_0 \in [a, b]$ such that $x \geq x_0$, where x_0 is fixed. Let $f \in C([a, b])$ and define the left Riemann-Liouville integral

$$\left(J_\nu^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x - t)^{\nu-1} f(t)\, dt, \tag{3}$$

$x_0 \leq x \leq b$. We define the subspace $C_{x_0}^\nu([a.b])$ of $C^n([a, b])$:

$$C_{x_0}^\nu([a, b]) := \left\{ f \in C^n([a, b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0, b]) \right\}. \tag{4}$$

For $f \in C_{x_0}^\nu([a, b])$, we define the left generalized ν-fractional derivative of f over $[x_0, b]$ as

$$D_{x_0}^\nu f := \left(J_{1-\alpha}^{x_0} f^{(n)} \right)', \tag{5}$$

see [2], p. 24, and Canavati derivative in [5].

Notice that $D_{x_0}^\nu f \in C([x_0, b])$.

We need the following generalization of Taylor's formula at the fractional level, see [2], pp. 8-10, and [5].

Theorem 3 *Let $f \in C_{x_0}^{\nu}([a,b])$, $x_0 \in [a,b]$ fixed.*

(i) If $\nu \geq 1$ then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + ... + f^{(n-1)}(x_0)\frac{(x - x_0)^{n-1}}{(n-1)!} \tag{6}$$
$$+ \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0.$$

(ii) If $0 < \nu < 1$ we get

$$f(x) = \left(J_{\nu}^{x_0} D_{x_0}^{\nu} f\right)(x), \quad \text{all } x \in [a,b] : x \geq x_0 \tag{7}$$

We will use (6) and (7).

Furthermore we need:

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (J - x)^{\alpha - 1} f(J)\, dJ, \tag{8}$$

$x \in [a,b]$, see also [3], [6], [7], [8], [11]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f := (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{10}$$

see [3]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [3], we need the following Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

1) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b_-)}{k!} (x - b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{11}$$

2) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x), \quad \forall\, x \in [a,b]. \tag{12}$$

We will use (11) and (12).

We introduce a new concept:

Definition 5 *Let $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{\alpha+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{13}$$

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3 Main Result

We give our main result.

Theorem 6 *Let $f \in C([a,b])$, $\alpha > 0$, $m := [\alpha]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. Assume further that*

$$f^{(k)}(a) = f^{(k)}(b) = 0, \quad k = 0, 1, ..., m-1; \tag{14}$$

$$p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \quad \text{and} \quad \alpha > \frac{1}{q}.$$

(i) Case of $1 < q \leq 2$. Then

$$\int_a^b |f(\omega)| |D^{\alpha} f(\omega)| d\omega \leq \tag{15}$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q d\omega\right)^{\frac{2}{q}}.$$

(ii) Case of $q > 2$. Then

$$\int_a^b |f(\omega)| |D^{\alpha} f(\omega)| d\omega \leq \tag{16}$$

$$\frac{2^{-\left(\alpha+\frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha) \left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b |D^{\alpha} f(\omega)|^q d\omega\right)^{\frac{2}{q}}.$$

(iii) When $p = q = 2$, $\alpha > \frac{1}{2}$, then

$$\int_a^b |f(\omega)| |D^{\alpha} f(\omega)| d\omega \leq \tag{17}$$

$$\frac{2^{-\left(\alpha+\frac{1}{2}\right)} (b-a)^{\alpha}}{\Gamma(\alpha) \left[\sqrt{2\alpha(2\alpha-1)}\right]} \left(\int_a^b |D^{\alpha} f(\omega)|^2 d\omega\right).$$

Remark 7 *Let us say that $\alpha = 1$, then by (17) we obtain*

$$\int_a^b |f(\omega)| |f'(\omega)| d\omega \leq \frac{(b-a)}{4} \left(\int_a^b (f'(\omega))^2 d\omega\right), \tag{18}$$

that is reproving and recovering Opial's inequality (1), see [10], see also Olech's result [9].

Proof. of Theorem 6. Let $x \in \left[a, \frac{a+b}{2}\right]$, we have by assumption $f^{(k)}(a) = 0$, $k = 0, 1, ..., m-1$ and Theorem 3 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} D_a^\alpha f(\tau)\, d\tau. \tag{19}$$

Let $x \in \left[\frac{a+b}{2}, b\right]$, we have by assumption $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$ and Theorem 4 that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} D_{b-}^\alpha f(\tau)\, d\tau. \tag{20}$$

Using Hölder's inequality on (19) we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} |D_a^\alpha f(\tau)|\, d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_a^x \left((x-\tau)^{\alpha-1}\right)^p d\tau\right)^{\frac{1}{p}} \left(\int_a^x |D_a^\alpha f(\tau)|^q\, d\tau\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^x |D_a^\alpha f(\tau)|^q\, d\tau\right)^{\frac{1}{q}}. \tag{21}$$

Set

$$z(x) := \int_a^x |D_a^\alpha f(\tau)|^q\, d\tau, \quad (z(a) = 0).$$

Then

$$z'(x) = |D_a^\alpha f(x)|^q,$$

and

$$|D_a^\alpha f(x)| = (z'(x))^{\frac{1}{q}}, \quad \text{all } a \leq x \leq \frac{a+b}{2}.$$

Therefore by (21) we have

$$|f(\omega)|\, |D_a^\alpha f(\omega)| \leq \frac{1}{\Gamma(\alpha)} \frac{(\omega-a)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(z(\omega)\, z'(\omega)\right)^{\frac{1}{q}}, \tag{22}$$

all $a \leq \omega \leq x \leq \frac{a+b}{2}$.

Next working similarly with (20) we obtain

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (\tau - x)^{\alpha-1} \left|D_{b-}^\alpha f(\tau)\right| d\tau \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_x^b \left((\tau-x)^{\alpha-1}\right)^p d\tau\right)^{\frac{1}{p}} \left(\int_x^b \left|D_{b-}^\alpha f(\tau)\right|^q d\tau\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)}\frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_x^b \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau\right)^{\frac{1}{q}}. \tag{23}$$

Set

$$\lambda(x) := \int_x^b \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau = -\int_b^x \left|D_{b-}^{\alpha}f(\tau)\right|^q d\tau, \quad (\lambda(b)=0).$$

Then

$$\lambda'(x) = -\left|D_{b-}^{\alpha}f(x)\right|^q$$

and

$$\left|D_{b-}^{\alpha}f(x)\right| = (-\lambda'(x))^{\frac{1}{q}}, \quad \text{all } \frac{a+b}{2} \le x \le b.$$

Therefore by (23) we have

$$|f(\omega)|\left|D_{b-}^{\alpha}f(\omega)\right| \le \frac{1}{\Gamma(\alpha)}\frac{(b-\omega)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}(-\lambda(\omega)\lambda'(\omega))^{\frac{1}{q}}, \tag{24}$$

all $\frac{a+b}{2} \le x \le \omega \le b$.

Next we integrate (22) over $[a,x]$ to obtain

$$\int_a^x |f(\omega)|\left|D_a^{\alpha}f(\omega)\right| d\omega \le$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_a^x (\omega-a)^{\frac{p(\alpha-1)+1}{p}}(z(\omega)z'(\omega))^{\frac{1}{q}} d\omega \le$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left(\int_a^x (\omega-a)^{p(\alpha-1)+1} d\omega\right)^{\frac{1}{p}}\left(\int_a^x z(\omega)z'(\omega)d\omega\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\frac{(x-a)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}}\frac{z(x)^{\frac{2}{q}}}{2^{\frac{1}{q}}} =$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x \left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{25}$$

So we have proved

$$\int_a^x |f(\omega)|\left|D_a^{\alpha}f(\omega)\right| d\omega \le$$

$$\frac{2^{-\frac{1}{q}}(x-a)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\left(\int_a^x \left|D_a^{\alpha}f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{26}$$

for all $a \le x \le \frac{a+b}{2}$.

By (26) we get

$$\int_a^{\frac{a+b}{2}} |f(\omega)| |D_a^\alpha f(\omega)| d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}} 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^{\frac{a+b}{2}} |D_a^\alpha f(\omega)|^q d\omega\right)^{\frac{2}{q}}. \tag{27}$$

Similarly we integrate (24) over $[x, b]$ to obtain

$$\int_x^b |f(\omega)| \left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_x^b (b-\omega)^{\frac{p(\alpha-1)+1}{p}} (-\lambda(\omega)\lambda'(\omega))^{\frac{1}{q}} d\omega \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_x^b (b-\omega)^{p(\alpha-1)+1} d\omega\right)^{\frac{1}{p}} \left(\int_x^b -\lambda(\omega)\lambda'(\omega) d\omega\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \frac{(b-x)^{\frac{p(\alpha-1)+2}{p}}}{(p(\alpha-1)+2)^{\frac{1}{p}}} \frac{(\lambda(x))^{\frac{2}{q}}}{2^{\frac{1}{q}}}. \tag{28}$$

We have proved that

$$\int_x^b |f(\omega)| \left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{2^{-\frac{1}{q}} (b-x)^{\frac{p(\alpha-1)+2}{p}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_x^b \left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}, \tag{29}$$

for all $\frac{a+b}{2} \leq x \leq b$.

By (29) we get

$$\int_{\frac{a+b}{2}}^b |f(\omega)| \left|D_{b-}^\alpha f(\omega)\right| d\omega \leq$$

$$\frac{(b-a)^{\frac{(p(\alpha-1)+2)}{p}} 2^{-\left[\frac{p(\alpha-1)+2}{p}+\frac{1}{q}\right]}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_{\frac{a+b}{2}}^b \left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{30}$$

Adding (27) and (30) we get

$$\int_a^b |f(\omega)| |D^\alpha f(\omega)| d\omega \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}}\cdot$$

$$\left[\left(\int_a^{\frac{a+b}{2}} |D_a^\alpha f(\omega)|^q d\omega\right)^{\frac{2}{q}} + \left(\int_{\frac{a+b}{2}}^b \left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}\right] =: (*.) \tag{31}$$

Assume $1 < q \leq 2$, then $\frac{2}{q} \geq 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \cdot$$

$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^\alpha f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} = \tag{32}$$

$$\frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{33}$$

So for $1 < q \leq 2$ we have proved (15).

Assume now $q > 2$, then $0 < \frac{2}{q} < 1$.

Therefore we get

$$(*) \leq \frac{2^{-\left(\alpha+\frac{1}{p}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)} 2^{1-\frac{2}{q}}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \cdot$$

$$\left[\int_a^{\frac{a+b}{2}} \left|D_a^\alpha f(\omega)\right|^q d\omega + \int_{\frac{a+b}{2}}^b \left|D_{b-}^\alpha f(\omega)\right|^q d\omega\right]^{\frac{2}{q}} =$$

$$\frac{2^{-\left(\alpha+\frac{1}{q}\right)} (b-a)^{\left(\frac{p(\alpha-1)+2}{p}\right)}}{\Gamma(\alpha)\left[(p(\alpha-1)+1)(p(\alpha-1)+2)\right]^{\frac{1}{p}}} \left(\int_a^b \left|D^\alpha f(\omega)\right|^q d\omega\right)^{\frac{2}{q}}. \tag{34}$$

So when $q > 2$ we have established (16).

(iii) The case of $p = q = 2$, see (17), is obvious, it derives from (15) immediately. ■

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K-spectral sets: an asymptotic viewpoint

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Dedicated to Heiner Gonska for his 65th anniversary.

Abstract

We discuss several results about K-spectral sets of bounded linear operators on Hilbert space from an asymptotic viewpoint.

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Key Words and Phrases: Spectral sets; K-spectral sets; numerical range; dilations; functional calculi.

1 Introduction

Preamble.

The aim of this note is to highlight, put into context, and review several results about K-spectral sets of continuous linear operators acting on complex Hilbert spaces. We focus on some recent "asymptotic" results which were proved in [8, 6, 7] jointly with Bernhard Beckermann from Lille, Michel Crouzeix from Rennes and Bernard Delyon from Rennes.

Notation.

Throughout this paper H will denote a complex Hilbert space. We denote by $\mathcal{B}(H)$ the C*-algebra of all continuous linear operators on H endowed with the operator norm $\|\cdot\|$ and involution $A \mapsto A^*$ (the Hilbertian adjoint of A). The *spectrum* of A, defined as the set of complex numbers z for which $zI - A$ is not invertible, is denoted by $\sigma(A)$. Here I is the identity operator. The inverse of an invertible operator A is denoted by A^{-1} and $A^{\pm}$ will stand for the operator A or its inverse. The *numerical range* $W(A)$ of A is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}.$$

We refer to the books [27, 20] as basic references for the spectral theory of linear operators.

The open disk of center a and radius r is denoted by $D(a,r)$ and its closure by $\overline{D}(a,r)$. We set $\mathbb{D} = D(0,1)$.

For a (possibly unbounded) set X in the complex plane we denote by $\mathcal{R}(X)$ and $\mathcal{C}(X)$ the algebras of complex-valued bounded rational functions on X, and complex-valued bounded continuous functions on X, respectively, equipped with the supremum norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\}.$$

Dedication and acknowledgements.

Heiner Gonska was one of the coauthors of my first published mathematical paper (about a completely different topic) in 1986, and this note is dedicated to him. I would like to thank Bernd Beckermann, Michel Crouzeix and Bernard Delyon who are the coauthors of the main mathematical results of this manuscript (Theorems 4.2, 5.1, 5.2 and 5.3). The present note is a modified version of talks delivered by the author at several institutions during the last years, including the Erwin Schrödinger Institute in Vienna, the Mathematical Institute of the Romanian Academy in Bucharest and the Université de Lorraine in Metz. I would like to thank these institutions for their support during my visits.

Organization of the paper.

The rest of the paper is organized as follows. In the next section we recall the definition of spectral and K-spectral sets and we present several examples and results. Section 3 deals with the theorem and the problem of Shields [31] about an annulus as a K-spectral set. The next section presents several results about the numerical range of an operator as a K-spectral set. Different results of asymptotic nature are discussed in Section 5. The present note concludes with a discussion about the proofs of some of the results presented here.

2 Spectral and K-spectral sets

The notion of spectral set of a Hilbert space linear operator has been introduced in 1951 by John von Neumann [34]. We refer to two books [27, 23] and one recent survey [5] for more detailed presentations and more information.

Let X be a closed set in the complex plane. Suppose that A is a bounded linear operator acting on the Hilbert space H. Suppose now that the spectrum $\sigma(A)$ of A is included in the closed set X and that $f = p/q \in \mathcal{R}(X)$ denotes a rational function with poles off X. As the poles of the rational function f are outside of X, the operator $f(A)$ is naturally defined as $f(A) = p(A)q(A)^{-1}$ or, equivalently, by the Riesz holomorphic functional calculus [27].

Definition 2.1. For a fixed constant $K > 0$, the set X is said to be a K-**spectral** set for A if the spectrum $\sigma(A)$ of A is included in X and the inequality $\|f(A)\| \leq K\|f\|_X$ holds for every $f \in \mathcal{R}(X)$. The set X is a **spectral** set for A if it is a K-spectral set with $K = 1$.

Example 2.2. (Normal operators.) Suppose that A is a normal operator, that is A commutes with its adjoint A^*. The spectral theorem for normal operators implies that $\sigma(A)$ is spectral for A. More generally, the same result holds true if A is subnormal [11]. It was proved by Stampfli [33] that if the spectrum of an operator is K-spectral for some $K \geq 1$, then the operator possess a nontrivial closed invariant subspace. The case $K = 1$ of spectral sets has been previously proved by Jim Agler [1]. This also shows one reason why we are interested in K-spectral sets.

The main example of spectral sets given in [34] is provided by the celebrated von Neumann inequality for contractions.

Example 2.3. (Spectral sets for Hilbert space contractions) Suppose that $A \in \mathcal{B}(H)$, $\|A\| \leq 1$, is a Hilbert space contraction. Then the spectrum of A is included in the closed unit disk $\overline{\mathbb{D}}$. Let f be a rational function in $\mathcal{R}(\overline{\mathbb{D}})$. The von Neumann inequality, proved in the same paper [34], states that $\|f(A)\| \leq \|f\|_{\overline{\mathbb{D}}}$. This actually provides a functional calculus for functions in the disk algebra $A(\mathbb{D})$ for every given Hilbert space contraction. In particular, the closed unit disk $\overline{\mathbb{D}}$ is spectral for every $A \in \mathcal{B}(H)$ with $\|A\| \leq 1$.

Example 2.4. (Disks of the Riemann sphere) The above example generalizes to arbitrary closed disks: the closed disk $\overline{D}(a, R)$ of center a and radius R is spectral for $A \in \mathcal{B}(H)$ if and only if $\|A - aI\| \leq R$. Also, the exterior $D(a, r)^c = \{z : |z - a| \geq r\}$ of the open disk $D(a, r)$ is spectral for A if and only if $A - aI$ is invertible and $\|(A - aI)^{-1}\| \leq r^{-1}$. The right half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is spectral for $A \in \mathcal{B}(H)$ if and only if the numerical range $W(A)$ is included in $\mathbb{C}_+$. In conclusion, it is easy to recognize when disks of the Riemann sphere (interior/exterior of a disk or a half-plane) are spectral for a given operator.

Example 2.5. (The results of Berger and Ando/Okubo) It was proved by Okubo and Ando [22], and by Berger (see [9]) with a larger K-spectrality constant, that if $W(A) \subset \overline{\mathbb{D}}$, then $\overline{\mathbb{D}}$ is 2-spectral for A. Berger proved this fact using a "skew dilation" theorem: if $W(A) \subset \overline{\mathbb{D}}$, then A has a 2-dilation: there is a unitary operator U acting on a larger Hilbert space $K \supset H$ such that $A^n = 2P_H U^n \mid_H$ for every $n \geq 1$. Here P_H denotes the orthogonal projection onto H. Ando/Okubo proof uses a similarity to a contraction theorem: if $W(A) \subset \overline{\mathbb{D}}$ then there is an invertible operator L with $\|L\| \cdot \|L^{-1}\| \leq 2$ such that $\|L^{-1}AL\| \leq 1$. The von Neumann inequality for contractions implies then the 2-spectrality of $W(A)$. The famous counterexample to Halmos' similarity problem of Pisier [25] implies the existence of an operator A for which $\overline{\mathbb{D}}$ is K-spectral for some K, but A is not similar to a contraction.

Example 2.6. (The Sz.-Nagy dilation theorem) A geometric explanation of the validity of von Neumann's inequality is provided by the classical dilation theorem of Sz.-Nagy. This now classical and nice theorem says that for every Hilbert space contraction $A \in \mathcal{B}(H)$ there is a larger Hilbert space $K \supset H$ and a unitary operator U acting on K such that $A^n = P_H U^n \mid_H$ for $n \geq 0$. Here P_H denotes the orthogonal projection onto H.

Example 2.7. (Agler's theorem for the annulus) The relation between the von Neumann inequality and the Sz.-Nagy dilation theorem (for the unit disk) has a counterpart for annular domains. For $R > 1$, set $\mathbb{A}_R = \{z : \frac{1}{R} \leq |z| \leq R\}$. Agler [2] proved that if $A \in \mathcal{B}(H)$ has the annulus $\mathbb{A}_R$ as a spectral set, then A has a normal dilation U with $\sigma(U) \subset \partial\mathbb{A}_R$. Thus, for annuli, the analogue of the von Neumann inequality implies the analogue of Sz.-Nagy dilation theorem. According to counterexamples due to Dritschel-McCullough [16] and Agler-Harland-Raphael [3] the corresponding implication for some domains with at least two "holes" is false. See also [24].

Example 2.8. (In general, the intersection of two spectral sets is not a spectral set) We can write $\mathbb{A}_R = \overline{D}(0,R) \cap D(0, \frac{1}{R})^c$. Consider the invertible matrix

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad , \text{ with } A(t)^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix},$$

acting on $\mathbb{C}^2$ endowed with the Euclidean norm. For $t_0 = R - \frac{1}{R}$ we have $\|A(t_0)\| = \|A(t_0)^{-1}\| = R$. Therefore $\overline{D}(0,R)$ and $D(0, \frac{1}{R})^c$ are spectral sets for $A(t_0)$. However, $\mathbb{A}_R = \overline{D}(0,R) \cap D(0, \frac{1}{R})^c$ is not a spectral set for $A(t_0)$ if R is large. Indeed [19], the example of the function $f(z) = z - 1/z$ which verifies

$$\|f(A)\|/\|f\|_{\mathbb{A}_R} = 2\frac{R^2-1}{R^2+1}$$

shows that $\mathbb{A}_R$ is not a spectral set for $R > \sqrt{3}$.

The following sharper statement seems to be new. For the function $f(z) = g(z) - g(1/z)$, $g(z) = R\frac{z-1}{R^2-z}$, we get

$$\|f(A)\| = 2, \quad \|f\|_{\mathbb{A}_R} = \frac{1+R^2+2R}{1+R^2+R} < \frac{4}{3}.$$

Thus $\mathbb{A}_R$ is even not $\frac{3}{2}$-spectral for $A(t_0)$, for any $R > 1$.

It is an open problem to know if the intersection of two K-spectral sets is always a K'-spectral set for a suitable constant K'. The above example shows that we cannot always take $K' = K$.

3 The annulus as a K-spectral set

Shields [31] proved in 1974 the following result:

Theorem 3.1 (Shields). *Let $R > 1$. If $A \in \mathcal{B}(H)$ verifies $\|A\| \leq R$ and $\|A^{-1}\| \leq R$, then the annulus $\mathbb{A}_R$ is a $K(R)$-spectral set for A, with a constant $K(R)$ such that*

$$K(R) \leq 2 + \sqrt{\frac{R^2+1}{R^2-1}}. \tag{3.1}$$

Some comments are in order here. The bound for $K(R)$ provided by (3.1) goes to infinity when $R \to 1$. As mentioned in Example 2.8, it is an open problem to know if the intersection of two K-spectral sets is always a K'-spectral set.

Problem 3.2 (Shields [31]). *Is there a universal constant K such that the annulus $\mathbb{A}_R$ is a K-spectral set for A whenever $R > 1$, $\|A\| \leq R$ and $\|A^{-1}\| \leq R$?*

This question of Shields is a first explanation for the use of the word "asymptotic" in the title of this note. The limit case $R = 1$ means that A verifies $\|A\| \leq 1$ and $\|A^{-1}\| \leq 1$, which means that A is a unitary operator. In this case the unit circle is spectral for A. We should note however that this does not imply immediately an answer to Shields' problem.

We will discuss the positive answer to Shields' problem later on. For the moment, let us mention the following result proved by Stampfli [32], generalizing the above mentioned elementary characterization of unitary operators.

Theorem 3.3 (Stampfli [32]). *An invertible operator A is unitary if and only if $W(A^{\pm}) \subset \overline{\mathbb{D}}$, that is, $W(A) \subset \overline{\mathbb{D}}$ and $W(A^{-1}) \subset \overline{\mathbb{D}}$.*

4 The numerical range as a K-spectral set

The following result, proved in 1999 by Bernard and François Delyon [15], came quite as a surprise to the operator theory community.

Theorem 4.1 (Delyon brothers' theorem). *Let $A \in \mathcal{B}(H)$. Then $\overline{W(A)}$ is a K-spectral set for A, with*

$$K \leq 3 + \left(\frac{2\pi(\text{diameter}(W(A))^2}{\text{area}(W(A))}\right)^3. \tag{4.1}$$

Using this property of the rational functional calculus on the numerical range, a proof of a conjecture of Burkholder concerning almost everywhere convergence of products of conditional expectations was given in [15]. We refer the reader to [13] for other applications of Theorem 4.1.

It was remarked independently by Putinar-Sandberg [26] and Badea-Crouzeix-(B.) Delyon [8] that, as in Example 2.5, there is a similarity-dilation result behind Theorem 4.1. We follow here the version presented in [8]. In the next result, $I+P$, $P=P(\Omega)$, designates the C.Neumann's (or Poincaré-Neumann's) double layer potential operator (cf. [8, 26, 18, 30, 29]) associated with a non-empty convex set Ω, which is possibly unbounded. In the case when Ω is a bounded convex set, a modern proof of the invertibility of $I+P$ is given in [30, 18]. A description of P will be given in Section 6, together with a proof that Theorem 4.2 implies the skew dilation theorem of Berger.

Theorem 4.2 ([8])**.** *We assume that the convex domain Ω, possibly unbounded, is such that $I+P$ is an isomorphism of $\mathcal{C}(\partial\Omega)$ and that the operator $A \in \mathcal{B}(H)$ satisfies $\overline{W(A)} \subset \Omega$. Then there exists a larger Hilbert space K containing H, and a normal operator N acting on K with spectrum $\sigma(N) \subset \partial\Omega$, such that, for all rational functions r bounded in Ω,*

$$r(A) = P_H g(N) \,|_H \; .$$

Here P_H is the orthogonal projection from K onto H and $g = 2(I+P)^{-1}r$.

The upper bound for the constant of K-spectrality of the closure of the numerical range $W(A)$ provided by Equation (4.1) blows up when the area of the numerical range of A goes to zero. The limit case, when $\text{area}(W(A)) = 0$, is thus obtain for operators A such that $e^{i\theta}A$ is self-adjoint for a suitable real number θ. In this case $W(A)$ is even spectral for A. As in Shields' problem, the question of the existence of a universal constant arises, and this is the second appearance of the "asymptotic" aspect of K-spectral sets in this note. The existence of such a universal constant is a beautiful result due to Michel Crouzeix [12].

Theorem 4.3 (Crouzeix' theorem)**.** *Let $A \in \mathcal{B}(H)$. The closure of the numerical range $\overline{W(A)}$ is a 12-spectral set for A.*

It is a conjecture of Michel Crouzeix that the best possible constant for K-spectrality in the above theorem is 2. See [17, 10] for recent contributions about Crouzeix' conjecture.

5 Some asymptotic results

The following result [6] gives a positive answer to the problem of Shields.

Theorem 5.1 ([6]). *There is a universal constant K with $2 \leq K \leq 2+\frac{2}{\sqrt{3}}$, such that $\mathbb{A}_R$ is K-spectral for A whenever $A \in \mathcal{B}(H)$, $\|A\| \leq R$ and $\|A^{-1}\| \leq R$.*

Think again of the annulus $\mathbb{A}_R$ as the intersection of one closed disk and the exterior of an open disk. The following more general result looks at the intersection of several disks of the Riemann sphere as a K-spectral set, with a universal constant.

Theorem 5.2 ([6]). *Let $n \geq 2$ and let $A \in \mathcal{B}(H)$. Suppose that n disks D_j, $j = 1, \cdots, n$, of the Riemann sphere are spectral sets for A. Then $X = D_1 \cap \cdots D_n$ is K-spectral for A, with*

$$K \leq n + \frac{n(n-1)}{\sqrt{3}}.$$

This result provides a positive answer to a question of Michael Dritschel (private communication).

In relation to Stampfli's result given in Theorem 3.3, the following estimate was proved in [7]. See also [14].

Theorem 5.3 ([7]). *Let $\varepsilon > 0$. Suppose that $A \in \mathcal{B}(H)$ verifies $W(A^{\pm}) \subset (1+\varepsilon)\overline{\mathbb{D}}$. Then*

$$\inf\{\|A - U\| : U \text{ unitary operator } \} \leq C\varepsilon^{1/4}$$

for some constant $C > 0$. Moreover, the exponent $1/4$ is the best possible one.

Theorem 3.3 is obtained for $\varepsilon = 0$.

6 Some ingredients of, and comments about, the proofs

About the proof of Theorem 4.2.

We consider in what follows only the case when Ω is a bounded convex set in the complex plane and we refer to [8] for the general case. We start by recalling the definition of the operator P, following [30]. As Ω is a bounded convex set in the complex plane, its boundary $C = \partial\Omega$ is a rectifiable Jordan curve. Moreover, C is a curve of bounded rotation: that is, one-sided tangents exist at every point of C, and the angles which they make with a fixed direction are of bounded variation with respect to arc length. So except for a countable set, a tangent angle τ is defined and continuous with respect to arc length, and its discontinuities are jumps which can be assumed to be at most π in modulus. Represent C as a function of arc length by the equation $z = \zeta(s)$, $0 \leq s \leq L$, in such a way that $s = 0$ is a point of continuity of τ. The Poincaré-Neumann integral operator can be defined [30] as

$$P(f)(s) = \int_0^L f(t)\, d\psi(s,t),$$

where

$$\psi(s,t) = \begin{cases} \frac{1}{\pi} \arg\,(\zeta(t) - \zeta(s)) & for \;\; 0 \le t < s \le L\,; \\ \frac{1}{\pi} \arg\,((\zeta(t) - \zeta(s)) + 1 & for \;\; 0 \le s < t \le L\,; \\ \psi(s, s+0) & for \;\; 0 \le s = t < L\,; \\ \psi(s, s-0) & for \;\; s = t = L. \end{cases}$$

Notice that branches of the argument function are chosen so that for $s \neq t$ the function ψ is continuous and $\psi(s,t) = \psi(t,s)$. Since C is convex, $\psi(s,\cdot)$ is nondecreasing and $\int_0^L d\psi(s,t) = 1$ for all s. For points $\zeta(s) \in C$ at which the tangent angle τ is continuous, we have

$$(Tf)(s) = \int_0^L f(t) K(s,t)\, dt,$$

where $K(s,t)$ is the classical Poincaré-Neumann kernel of two-dimensional potential theory (cf. [29]).

The proof of Theorem 4.2 uses the classical Naimark's dilation theorem about the existence of a spectral measure dilating a certain regular positive measure. If $\Omega = \mathbb{D}$ is the unit disk and $r(z) = z^n$, with $n \ge 1$, we have (see [8, Remark 4.1]) $g = 2(I+P)^{-1} r = 2\,r$. In this case, Theorem 4.2 reduces to the skew dilation theorem of Berger mentioned above: every $A \in \mathcal{B}(H)$ with $W(A) \subset \overline{\mathbb{D}}$ satisfies

$$A^n = 2\, P_H U^n \,|_H, \qquad \forall\, n \ge 1,$$

for a suitable unitary operator U acting on K.

About the proof of Theorem 5.2.

Let $A \in \mathcal{B}(H)$, and consider the intersection $X = D_1 \cap D_2 \cap \cdots \cap D_n$ of n disks of the Riemann sphere $\overline{\mathbb{C}}$, each of them being spectral for A.

Convention. In what follows we will always suppose that for each spectral set D_j, the spectrum $\sigma(A)$ of A is included in the interior of D_j. The general case will then follow by a limit argument by slightly enlarging the disks. Note that each superset of a spectral set is spectral.

Decomposition of the Cauchy kernel. Consider a disk D among $D_1, \cdots, D_n$, which is centered in $\omega \in \mathbb{C}$ (if D is a half-plane we take $\omega = \infty$). We chose an arclength parametrization $s \mapsto \sigma = \sigma(s) \in \partial D$ of the boundary of D with orientation such that $\frac{1}{i}\frac{d\sigma}{ds}$ is the outward normal to D. Let $A \in \mathcal{B}(H)$ be a bounded operator with $\sigma(A) \subset \operatorname{int}(D)$. For $\sigma \in \partial D$, we consider the following Poisson kernel

$$\mu(\sigma, A, D) = \tfrac{1}{2\pi i}\left((\sigma - A)^{-1}\tfrac{d\sigma}{ds} - (\bar\sigma - A^*)^{-1}\tfrac{d\bar\sigma}{ds} - \tfrac{1}{\sigma-\omega}\tfrac{d\sigma}{ds}\right). \tag{6.1}$$

Notice that in case $\omega = \infty$ of a half-plane, the term involving ω on the right-hand side of (6.1) vanishes.

The first important step in the proof is the decomposition of the Cauchy kernel

$$\frac{1}{2\pi i}(\sigma - A)^{-1}d\sigma = \mu(\sigma, A, D)\, ds + \nu(\sigma, A, D)\, d\sigma$$

as the sum of the Poisson kernel and a residual kernel.

Decomposition of $f(A)$. For a rational function $f \in \mathcal{R}(X)$, the above decomposition of the Cauchy kernel leads to a decomposition of $f(A)$ as

$$f(A) = g_p(f) + g_r(f), \tag{6.2}$$

with

$$g_p(f) = \sum_{j=1}^{n}\int_{X\cap\partial D_j} f(\sigma)\,\mu(\sigma, A, D_j)\, ds, \quad g_r(f) = \sum_{j=1}^{n}\int_{X\cap\partial D_j} f(\sigma)\,\nu(\sigma, A, D_j)\, d\sigma.$$

Here p stands for "Poisson" and r for "residual". The given proof consists in showing that the norm of the map $f \mapsto g_p(f)$ is bounded by n, while the norm of the map $f \mapsto g_r(f)$ can be estimated by $n(n-1)/\sqrt{3}$. This is done using two basic lemmas on operator-valued integrals which leads to an estimation of the Poisson term. For the residual term, the decisive step is the invariance under Möbius maps (also called fractional linear transformations or homographic transformations) of our representation formula. We also use the fact that the variable $\overline{\sigma}$ which appears in $g_r(f)$ can be expressed in terms of σ due to the particular form of ∂X. Subsequently, a new path of integration is used in order to monitor the norm of the residual term. The new path of integration will be the circle of radius 1 in case of the annulus $\{R^{-1} \le |z| \le R\}$, and the positive real line in case of the sector $\{|\arg(z)| \le \theta\}$ for $\theta \in (0, \pi/2)$. We call these median lines.

The proof of our result is obtained by constructing a Voronoi-like tessellation of the Riemann sphere based on the reciprocal of the infinitesimal Carathéodory pseudodistance, sometimes also called infinitesimal Carathéodory metric. Here the new paths of integration (the median lines) are obtained using suitable edges of this tessellation, which requires some combinatorial considerations.

About the proof of Theorem 5.3.

Notice that it can be proved that

$$\inf\{\|A-U\| : U \text{ unitary operator }\} = \max\left(\|A\|-1, 1-\frac{1}{\|A^{-1}\|}\right).$$

The proof of the required estimate uses some techniques similar to ones from [28] and [4].

The proof that the exponent $1/4$ is the best possible one in Theorem 5.3 follows from the following construction. For each positive integer n of the form $n = 8k+4$ it is constructed in [7] a $n\times n$ matrix A_n verifying

$$W(A_n^{\pm}) \subset \frac{1}{\cos\frac{\pi}{n+1}}\overline{\mathbb{D}} \quad ; \quad \|A_n\| = 1+\frac{1}{8\sqrt{n}}.$$

The matrix A_n, defined for $n = 8k+4$, is given by $A_n = DBD$, where D is the diagonal matrix $D = \mathrm{diag}(e^{i\pi/2n}, \ldots, e^{(2\ell-1)i\pi/2n}, \ldots, e^{(2n-1)i\pi/2n})$ and $B = I + \frac{1}{2\,n^{3/2}}E$, where E is a matrix whose entries are defined by $e_{ij} = 1$ if $3k+2 \le |i-j| \le 5k+2$ and $e_{ij} = 0$ otherwise.

Taking

$$1+\varepsilon = \frac{1}{\cos\frac{\pi}{n}} = 1+\frac{\pi^2}{2n^2}+\mathcal{O}(\frac{1}{n^4}),$$

we see that the exponent $1/4$ cannot be improved.

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Sampling theorems associated with Stone-regular eigenvalue problems

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

We derive sampling representations for integral transforms whose kernels are Green's functions of Stone-regular eigenvalue problems multiplied by the characteristic determinant. Unlike the Birkhof-regular case, such sampling representations are generally speaking not convergent. We prove that the convergence can be achieved by adding a certain finite number of extra sampling points.

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1 Introduction

Sampling theory deals with the reconstruction of certain functions (signals) from their values (samples) at an appropiate sequence of points. Classical sampling theorem of Whittaker [35], Kotel'nikov [19] and Shannon [24] (the WKS theorem) states that any function of the form

$$F(\lambda) = \int_{-\pi}^{\pi} f(x) exp(-i\lambda x)\, dx, \quad f \in L_2(-\pi,\pi), \quad \lambda \in \mathbb{R}, \tag{1.1}$$

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can be reconstructed from its samples $F(k)$, $k \in \mathbb{Z}$, by the formula

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(k) \frac{\sin \pi(\lambda - k)}{\pi(\lambda - k)}, \tag{1.2}$$

where the series converges absolutely and uniformly on $\mathbb{R}$. Moreover, it can be written as a Lagrange interpolation series, since

$$\frac{\sin \pi(\lambda - k)}{\pi(\lambda - k)} = \frac{\Delta(\lambda)}{(\lambda - k)\Delta'(k)} \quad \text{with} \quad \triangle(\lambda) = \sin \lambda\pi = \pi\lambda \prod_{k=1}^{\infty} \Big(1 - \frac{\lambda^2}{k^2}\Big).$$

The WKS theorem deserved grate popularity by virtue of important applications in radio electronics and theory of signals.

Weiss [34] showed that there is a connection between sampling and expansions into series with respect to eigenfunctions of eigenvalue problems for certain ordinary differential operators. For example, formula (1.2) can be obtained with the help of the boundary value problem

$$iy'(x) = \lambda y(x), \quad -\pi < x < \pi, \quad y(-\pi) = y(\pi). \tag{1.3}$$

The kernel $\exp(-i\lambda x)$ of the integral transform (1.1) is a solution of the differential equation in (1.3) and the sampling points $\lambda_k = k$, $k \in \mathbb{Z}$, coincide with the eigenvalues of (1.3), which, in turn, coincide with the zeros of its characteristic function $\Delta(\lambda) = \sin \lambda\pi$. Moreover, the system of eigenfunctions $\exp(-ikx)$, $k \in \mathbb{Z}$, forms an orthogonal basis of the space $L_2(-\pi, \pi)$ and therefore any function $f \in L_2(-\pi, \pi)$ can be expanded into the L_2-convergent Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k \exp(ikx), \quad a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx = \frac{F(k)}{2\pi}. \tag{1.4}$$

Substituting (1.4) into (1.1) we arrive at (1.2). It can be shown that the series in (1.2) converges absolutely on $\mathbb{C}$ and uniformly on horizontal strips of $\mathbb{C}$.

Kramer [20] shaped this approach into the following abstract form known as Kramer's lemma. Let the function $F(\lambda)$, $\lambda \in \mathbb{R}$, has the form

$$F(\lambda) = \int_I f(x) K(x, \lambda) \, dx, \quad f \in L_2(I),$$

where the kernel $K(x, \lambda)$ possesses the following properties: $K(x, \lambda) \in L_2(I)$ for each $\lambda \in \mathbb{R}$ and there exists a sequence $\{\lambda_k\}$ such that $\{K(x, \lambda_k)\}$ is a complete orthogonal system in $L_2(I)$. Then the following representation holds:

$$F(\lambda) = \sum_k F(k) S_k(\lambda), \quad S_k(\lambda) = \Big(\int_I |K(x, \lambda_n)|^2 \, dx \Big)^{-1} \int_I K(x, \lambda) \overline{K(x, \lambda_k)} \, dx.$$

The proof is similar to the given above proof of (1.2). Kramer's Lemma allowed to derive sampling representations for a wide class of eigenvalue problems not necessarily selfadjoint. For the non-selfadjoint case there exists so-called biorthogonal form of Kramer's lemma (see, e.g., [15]). In this case eigenfunctions need to form a biorthogonal basis in L_2.

For obtaining sampled transforms one can also use the (compact) resolvent of a Hermitian operator [14]. For example, the kernel in (1.1) can be obtained as

$$\exp(-i\lambda x) = -2\Delta(\lambda)G(x,\pi,\lambda),$$

where

$$G(x,t,\lambda) = -\frac{1}{2\Delta(\lambda)}\begin{cases}\exp(i\lambda(t-x-\pi)),\, t>x,\\ \exp(i\lambda(t-x+\pi)),\, t<x,\end{cases}$$

is the Green function of (1.3). Instead of $\exp(-i\lambda x)$ one can use more general kernel $\varphi(x,\lambda) = \Delta(\lambda)G(x,t_0,\lambda)$, where $t_0 \in [-\pi,\pi]$ is fixed. However, this generalization obviously does not change the class (1.1) of functions $F(\lambda)$ being sampled. For the Sturm-Liouville differential operators using the Green function also does not enrich the variety of sampled transforms by virtue of existence of the transformation operator (see, e.g., [12]). Moreover, because of this reason for first and second orders it is usually sufficient to consider only simplest boundary value problems with zero coefficients when constructing sampled transforms.

In [1] the authors suggested another approach of deriving sampling representations. This approach uses the analytic nature of the Green function and seems to be more natural for boundary value problems. It does not require neither basisness of eigenfunctions, nor even their completeness. The only requirement was the Birkhoff-regularity of the boundary value problem. The corresponding sampling series are generally speaking of Hermite interpolation type because of a possibly multiple spectrum and appearance together with eigen- also of associated functions. In the present paper we generalize this approach for Stone-regular boundary value problems. For this purpose we summarize in the next section necessary information on Stone-regular problems. In section 3 we derive sampling representations for the corresponding integral transforms. It turns out that one has to add a finite number of extra sampling points.

2 Stone-regular problems

Let L be the differential operator generated by the differential expression

$$l(y) := i^n y^{(n)} + \sum_{j=0}^{n-2} p_j(x)y^{(j)} = \lambda y =: \rho^n y, \quad 0 \le x \le 1, \tag{2.1}$$

with complex-valued coefficients $p_j \in L(0,1)$ and n linearly independent boundary conditions

$$U_\nu(y) := U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \quad \nu = \overline{1,n}, \tag{2.2}$$

where

$$U_{\nu 0}(y) = a_\nu y^{(k_\nu)}(0) + \sum_{l=0}^{k_\nu - 1} \alpha_{\nu l} y^{(l)}(0), \;\; U_{\nu 1}(y) = b_\nu y^{(k_\nu)}(1) + \sum_{l=0}^{k_\nu - 1} \beta_{\nu l} y^{(l)}(1)$$

and $a_\nu,\ b_\nu,\ \alpha_{\nu l},\ \beta_{\nu l} \in \mathbb{C}$. Without loss of generality we assume that the boundary conditions are normalized, i.e. $n-1 \geq k_1 \geq k_2 \geq ... \geq k_n,\ k_\nu > k_{\nu+2}$, and $\mid a_\nu \mid + \mid b_\nu \mid > 0$ for $\nu = \overline{1,n}$. The number k_ν is called the *order* of the condition $U_\nu(y) = 0$ and $\kappa = k_1 + ... + k_n$ is the *total order* of (2.2), which we assume minimal among all equivalent boundary conditions. The operator $L : D(L) \to L_2(0,1)$, $y \mapsto l(y)$ has the domain of definition

$$D(L) = \{y \mid y^{(j)} \in AC[0,1],\, j = \overline{0, n-1},\, l(y) \in L_2(0,1),\, U_\nu(y) = 0,\, \nu = \overline{1,n}\}.$$

Let $y_1(\cdot, \lambda), \cdots, y_n(\cdot, \lambda)$ be the fundamental system of solutions of the differential equation (2.1) with $y_k^{(j-1)}(0,\lambda) = \delta_{jk},\ j,k = \overline{1,n}$. Then for all fixed $x,\ j,\ k$ the functions $y_k^{(j-1)}(x,\lambda)$ are entire in λ. The eigenvalues $\lambda_k,\ k \in \mathbb{N}$, of L coincide with the zeros of its characteristic determinant

$$\Delta(\lambda) = \det \|U_j(y_k)\|_{j,k=\overline{1,n}}, \tag{2.3}$$

which is also is entire function.

If λ is not an eigenvalue of L, then for any function $f \in L_2(0,1)$ the solution of $Ly = \lambda y + f$ exists and is given by the formula

$$y(x) = \int_0^1 G(x,\xi,\lambda) f(\xi)\, d\xi, \quad 0 \leq x \leq 1,$$

where $G(x,\xi,\lambda) = (\Delta(\lambda))^{-1} H(x,\xi,\lambda)$ is the Green function of L (see [23]) with

$$H(x,\xi,\lambda) = (-1)^n \begin{vmatrix} y_1(x,\lambda) & \dots & y_n(x,\lambda) & g(x,\xi,\lambda) \\ U_1(y_1) & \dots & U_1(y_n) & U_1(g) \\ \vdots & \vdots & \vdots & \vdots \\ U_1(y_n) & \dots & U_n(y_n) & U_n(g) \end{vmatrix},$$

$$g(x,\xi,\lambda) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{j=1}^n y_j(x,\lambda) z_j(\xi,\lambda), & \text{for } x > \xi, \\ -\dfrac{1}{2} \displaystyle\sum_{j=1}^n y_j(x,\lambda) z_j(\xi,\lambda), & \text{for } x < \xi, \end{cases}$$

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$$z_j(\xi,\lambda)=\frac{W_j(\xi,\lambda)}{W(\lambda)},\quad W(\lambda)=\det\|y_k^{(j-1)}(\xi,\lambda)\|_{j,k=\overline{1,n}},$$

where $W_j(\xi,\lambda)$, is the cofactor of $y_j^{(n-1)}(\xi,\lambda)$ in the determinant $W(\lambda)$. Thus, $G(x,\xi,\lambda)$ is meromorphic in λ (for fixed x, ξ) with the poles λ_k, $k\in\mathbb{N}$.

Denote by $m_g(\lambda_k)$ the *geometric multiplicity* of λ_k (i.e. the number of linearly independent eigenfunctions of L corresponding to λ_k). Let

$$y_{k,j,0},\ y_{k,j,1},\ \dots,\ y_{k,j,m_{kj}-1},\quad j=\overline{1,m_g(\lambda_k)},$$

be a complete system of eigen- and associated functions related to λ_k, i.e.

$$l(y_{k,j,\mu})=\lambda_k y_{k,j,\mu}+y_{k,j,\mu-1},\quad U_\nu(y_{k,j,\mu})=0,\ \nu=\overline{1,n},\quad \mu=\overline{0,m_{kj-1}},$$

where $y_{k,j,-1}=0$. The value m_{kj} is called the *multiplicity of the eigenfunction* $y_{k,j,0}$. Denote by $m_a(\lambda_k)$ the *algebraic multiplicity* of λ_k, i.e. the multiplicity of λ_k as a zero of the characteristic determinant $\Delta(\lambda)$. It is known (see [23]) that

$$m_a(\lambda_k)=\sum_{j=1}^{m_g(\lambda_k)} m_{kj}$$

The principal part of $G(x,\xi,\lambda)$ in a vicinity of λ_k has the form (see [23])

$$\operatorname*{Res}_{\zeta=\lambda_k}\frac{G(x,\xi,\zeta)}{\lambda-\zeta}=\sum_{j=1}^{m_g(\lambda_k)}\sum_{\nu=1}^{m_{kj}}\frac{1}{(\lambda-\lambda_k)^\nu}\sum_{l=1}^{m_{kj}+1-\nu}\overline{z_{k,j,l-1}(\xi)}y_{k,j,m_{kj}+1-\nu-l}(x),\tag{2.4}$$

where

$$z_{k,j,0},\ z_{k,j,1},\dots,\ z_{k,j,m_{kj}-1},\quad j=\overline{1,m_g(\lambda_k)},$$

is a system of eigen- and associated functions of the adjoint operator L^* corresponding to the eigenvalue $\overline{\lambda_k}$ that is appropriately normalized:

$$\int_0^1 y_{k,j,q}(x)\overline{z_{k_1,j_1,q_1}(x)}\,dx=-\delta_{k,k_1}\delta_{j,j_1}\delta_{q+q_1,m_{kj}-1}.$$

Put $\mathbb{C}_\delta=\mathbb{C}\setminus D_\delta,\ \delta>0$, with $D_\delta=\{\rho^n\in\mathbb{C}:|\rho-\rho_k|<\delta,\ \rho_k^n=\lambda_k,\ k\in\mathbb{N}\}$. Following [2], we call the problem (2.1), (2.2) and also the corresponding operator *Stone-regular* (*of order* $\alpha\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$) if here exists $M>0$ such that

$$|G(x,\xi,\lambda)|\le M|\lambda|^{\frac{1}{n}(\alpha+1-n)}\ \text{ for }\ \lambda\in\mathbb{C}_\delta.\tag{2.5}$$

Otherwise we call them *irregular*. We note that the set of Stone-regular problems of order $\alpha=0$ coincide with the set of Birkhoff-regular ones (see [23], [25], [26]).

Birkhoff [3], [4], Tamarkin [31], [32] and Stone [29] proved convergence, equiconvergence and summability results for eigenfunction expansions in eigenand associated functions of Birkhoff-regular problems; their results have later on been generalized in various directions (see the survey in [13]).

Stone [30] determined for the case $n = 2$ the class of boundary conditions that are not Birkhoff-regular (he called them irregular). He gave an exhaustive treatment of this narrowed topic, in particular he proved results on the summability of the corresponding eigenfunction expansions.

The work of Stone was extended by Khromov [16] and Benzinger [2] to a class of problems of the form (2.1), (2.2) for arbitrary $n \geq 2$. They defined a class of Stone-regular problems containing the Birkhoff-regular ones as a subclass. A detailed investigation of Stone-regular boundary value problems has been published recently by Locker [21]. We note that unlike Birkhoff-regularity, which depends only on boundary conditions, the Stone-regularity depends also on the differential equation (2.1). The definition of Stone-regularity (sometimes also called: almost regularity) can be easily extended to include more general boundary conditions (multipoint conditions, conditions including the eigenvalue parameter or general functionals) and also more general differential equations: pencils and systems (see [7], [8], [11], [22] [28], [33]).

Irregular eigenvalue problems, where Green's function grows exponentially, have properties completely different from those of Birkhoff- or Stone-regular problems and have been studied only by a few authors (see Eberhard [5], [6], Freiling [9], [10], Shkalikov [27], Khromov [17], [18]).

The purpose of the present paper is to show, how the sampling results of [1] for Birkhoff-regular problems can be modified for Stone-regular problems.

The eigenvalues of a Stone-regular problem (2.1), (2.2) (counted with multiplicities) form two sequences $\{\lambda'_k\}_{k\in\mathbb{N}}$ and $\{\lambda''_k\}_{k\in\mathbb{N}}$ with the asymptotics

$$\lambda'_k = (2\pi k)^n\Big(1+O\Big(\frac{\ln^s k}{k}\Big)\Big),\ \lambda''_k = (-2\pi k)^n\Big(1+O\Big(\frac{\ln^s k}{k}\Big)\Big),\ s = 1-\delta_{\alpha,0}. \quad (2.6)$$

Choose $m \in \mathbb{N}_0$ with $mn > \alpha - (n-1)$ and numbers $\mu_1, \ldots, \mu_m \in \mathbb{C}$, which we let for simplicity be distinct and lie outside $\{\lambda_k\}_{k\in\mathbb{N}}$. Put $D(\lambda) = \prod_{\nu=1}^{m}(\lambda-\mu_\nu)$,

$$G_D(x,\xi,\lambda) = \frac{1}{D(\lambda)}G(x,\xi,\lambda), \quad \lambda \notin \{\lambda_k\}_{k\in\mathbb{N}} \cup \{\mu_1, \cdots, \mu_m\} =: \sigma_\mu. \quad (2.7)$$

Lemma 1. *For $\lambda \notin \sigma_\mu$ the following representation holds:*

$$G_D(x,\xi,\lambda) = \sum_{\nu=1}^{m}\frac{Q_{0,\nu}(x,\xi)}{\lambda-\mu_\nu} + \sum_{k=1}^{\infty}\sum_{\nu=1}^{m_k}\frac{Q_{k,\nu}(x,\xi)}{(\lambda-\lambda_k)^\nu}, \quad (2.8)$$

where $m_k = \max_{1\leq j\leq m_g(\lambda_k)} m_{kj}$ and $Q_{k,\nu}(x,\xi)$ are continuous functions. The series and its derivatives with respect to λ converge uniformly for $x,\ \xi \in [0,1]$ and for λ on bounded subsets of $\mathbb{C}$. Moreover, $m_k \leq 2$ for large k and

$$|\, Q_{k,1}(x,\xi)\,| \leq C, \quad |\, Q_{k,2}(x,\xi)\,| \leq Ck^{n-1}. \quad (2.9)$$

Proof. Fix sufficiently small $\delta > 0$ and consider circles $\Gamma_N := \{\lambda : |\lambda| = R_N\}$, $N \in \mathbb{N}$, in $\mathbb{C}_\delta$ with $\max|\mu_\nu| < R_N < R_{N+1}$ and $\lim R_N = \infty$. Denote

$$I_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta}d\zeta, \quad \lambda \in int\Gamma_N. \tag{2.10}$$

Using (2.5) and the definition of m and $D(\lambda)$ for $|\lambda| \to \infty$, $\lambda \in \mathbb{C}_\delta$ we obtain $G_D(x,\xi,\lambda) = O(\rho^{-1})$, and hence (2.10) yields $I_N^{(\nu)}(\lambda) = o(1)$ as $N \to \infty$ uniformly in $x, \xi \in [0,1]$ and λ in bounded subsets of $\mathbb{C}$. On the other hand, we have

$$I_N(\lambda) = -G_D(x,\xi,\lambda) + \sum_{\nu=1}^{m} \operatorname*{Res}_{\zeta=\mu_\nu} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta} + \sum_{k=1}^{q_N} \operatorname*{Res}_{\zeta=\lambda_k} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta}$$

for certain $q_N \in \mathbb{N}$ with $\lim q_N = \infty$. Coming to the limit as $N \to \infty$ we get

$$G_D(x,\xi,\lambda) = \sum_{\nu=1}^{m} \operatorname*{Res}_{\zeta=\mu_\nu} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta} + \sum_{k=1}^{\infty} \operatorname*{Res}_{\zeta=\lambda_k} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta} \tag{2.11}$$

Since according to (2.4)

$$\operatorname*{Res}_{\zeta=\lambda_k} \frac{G(x,\xi,\zeta)}{\lambda-\zeta} = \sum_{\nu=1}^{m_k} \frac{R_{k,\nu}(x,\xi)}{(\lambda-\lambda_k)^\nu},$$

with certain continuous functions $R_{k,\nu}(x,\xi)$, formula (2.7) gives

$$\operatorname*{Res}_{\zeta=\mu_\nu} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta} = \frac{Q_{0,\nu}(x,\xi)}{\lambda-\mu_\nu}, \quad \operatorname*{Res}_{\zeta=\lambda_k} \frac{G_D(x,\xi,\zeta)}{\lambda-\zeta} = \sum_{\nu=1}^{m_k} \frac{Q_{k,\nu}(x,\xi)}{(\lambda-\lambda_k)^\nu}, \tag{2.12}$$

where

$$Q_{0,\nu}(x,\xi) = \frac{G(x,\xi,\mu_\nu)}{D'(\mu_\nu)}, \quad Q_{k,\nu}(x,\xi) = \sum_{l=\nu}^{m_k} R_{k,\nu}(x,\xi)\frac{1}{(l-\nu)!}\frac{d^{l-\nu}}{d\zeta^{l-\nu}}\frac{1}{D(\zeta)}\Big|_{\zeta=\lambda_k}.$$

Substituting (2.12) into (2.11) we arrive at (2.8). Further, according to (2.6) $m_k \le 2$ for sufficiently large k and as in the proof of Lemma 3.3 in [1] we get (2.9). $\square$

3 Sampling theorems

Fix $\xi_0 \in [0,1]$. Denote $\varphi(x,\lambda) := \Delta(\lambda)G(x,\xi_0,\lambda)$ and $N(\lambda) := D(\lambda)\Delta(\lambda)$. Consider the set $\mathbb{F}$ of integral transforms of the form

$$F(\lambda) = \int_0^1 f(x)\varphi(x,\lambda)\,dx, \quad f(x) \in L_2(0,1).$$

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Our goal is to derive a sampling representation for functions $F(\lambda) \in \mathbb{F}$. Recall that the WKS theorem corresponds to the case $n = 1$ and that for $n = 1$ each problem (2.1), (2.2) different from an initial value problem is Birkhoff-regular. For definiteness we assume that $n > 1$. At first we also assume that $m_a(\lambda_k) \le 2$ for all $k \in \mathbb{N}$, which according to (2.6) for sufficiently large k holds anyway. Denote

$$\mathbb{N}_1 = \{k \in \mathbb{N} : m_a(\lambda_k) = 1\}, \quad \mathbb{N}_2 = \{k \in \mathbb{N} : m_a(\lambda_k) = 2, m_g(\lambda_k) = 1\},$$

$$\mathbb{N}_3 = \{k \in \mathbb{N} : m_a(\lambda_k) = m_g(\lambda_k) = 2\}.$$

Under the preceding assumption we have $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$.

Theorem 1. *Let (2.1), (2.2) be Stone-regular and $m_a(\lambda_k) \le 2$, $k \in \mathbb{N}$. Then*

$$\begin{aligned} F(\lambda) =& \sum_{\nu=1}^{m} F(\mu_\nu) \frac{N(\lambda)}{(\lambda - \mu_\nu) N'(\mu_\nu)} + \sum_{k \in \mathbb{N}_1} F(\lambda_k) \frac{N(\lambda)}{(\lambda - \lambda_k) N'(\lambda_k)} + \\ &+ \sum_{k \in \mathbb{N}_2} \Big(F(\lambda_k) \Big(\frac{2N(\lambda)}{(\lambda - \lambda_k)^2 N''(\lambda_k)} - \frac{2N'''(\lambda_k) N(\lambda)}{3(\lambda - \lambda_k)(N''(\lambda_k))^2} \Big) + \\ &\qquad + F'(\lambda_k) \frac{2N(\lambda)}{(\lambda - \lambda_k) N''(\lambda_k)} \Big) + \sum_{k \in \mathbb{N}_3} F'(\lambda_k) \frac{2N(\lambda)}{(\lambda - \lambda_k) N''(\lambda_k)}. \end{aligned} \tag{3.1}$$

The series in the right-hand side of (3.1) and all its derivatives converge absolutely and uniformly on bounded subsets of $\mathbb{C}$. Moreover

$$\begin{cases} \Big| \dfrac{F(\lambda_k)}{N'(\lambda_k)} \Big| \le C, & k \in \mathbb{N}_1, \\ \Big| \dfrac{F(\lambda_k)}{N''(\lambda_k)} \Big| \le C k^{n-1}, \ \Big| \dfrac{F'(\lambda_k)}{N''(\lambda_k)} - \dfrac{F(\lambda_k) N'''(\lambda_k)}{3(N''(\lambda_k))^2} \Big| \le C, & k \in \mathbb{N}_2, \\ \Big| \dfrac{F'(\lambda_k)}{N''(\lambda_k)} \Big| \le C, & k \in \mathbb{N}_3. \end{cases} \tag{3.2}$$

The proof is similar to the proof of Theorem 4.2 in [1]. In the general case we get the following theorem, which is analogous to Theorem 4.3 in [1].

Theorem 2. *Let (2.1), (2.2) be Stone-regular. Then*

$$F(\lambda) = \sum_{\nu=1}^{m} F(\mu_\nu) \frac{N(\lambda)}{(\lambda - \mu_\nu) N'(\mu_\nu)} + \sum_{k=1}^{\infty} \sum_{\nu = m_a(\lambda_k) - m_k}^{m_a(\lambda_k) - 1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda), \tag{3.3}$$

where

$$S_{k,\nu}(\lambda) = \frac{1}{\nu!} \sum_{j=1}^{m_a(\lambda_k) - \nu} C_{k,j} \frac{N(\lambda)}{(\lambda - \lambda_k)^{m_a(\lambda_k) - \nu + 1 - j}},$$

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and the numbers $C_{k,j}$, $j = \overline{1, m_k}$, *can be found from the triangular non-singular system of linear algebraic equations*

$$\sum_{j=1}^{s} C_{k,j} \frac{N^{(m_a(\lambda_k)+s-j)}(\lambda_k)}{(m_a(\lambda_k)+s-j)!} = \delta_{s,1}, \quad s = \overline{1, m_k}.$$

The series in (3.3) *and all its termwise derivatives converge absolutely and uniformly on every bounded subset of* $\mathbb{C}$. *Moreover, the estimates* (3.2) *remain valid. In particular, for each fixed* $l \geq 0$ *they yield*

$$\frac{d^l}{d\lambda^l} \sum_{\nu=m_a(\lambda_k)-m_k}^{m_a(\lambda_k)-1} F^{(\nu)}(\lambda_k) S_{k,\nu}(\lambda) = O\Big(\frac{1}{k^n}\Big),$$

uniformly in λ *from bounded subsets of* $\mathbb{C}$.

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Volume of Support for Multivariate Continuous Refinable Functions

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Let $\{a(\alpha) : \alpha \in \mathbb{Z}^d\}$ be a finitely supported real sequence. The dilation equation associated with $\{a(\alpha)\}$ is defined by

$$\varphi(x) = \sum_{\alpha} a(\alpha)\varphi(2x - \alpha) \qquad \text{and} \qquad \sum_{\alpha} a(\alpha) = 2^d.$$

Assume that φ is continuous and nontrivial. We study in this paper the volume of the support for φ. We will show that if $d \in \{1, 2\}$ then the minimal volume among all those φ is $d + 1$. Under the restriction that φ is shift stable the above holds also for $d = 3$. Moreover, the minimal volume can be reached by some $\{a(\alpha)\}$.

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1 Introduction

Denote $\mathbb{Z}^d$ the d-dimensional integer lattice. Let $\{a(\alpha) : \alpha \in \mathbb{Z}^d\}$ be a finitely supported real sequence (mask) satisfying $\sum_{\alpha} a(\alpha) = 2^d$. It is known (see [3]) that under

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some suitable conditions the dilation equation

$$\varphi(x) = \sum_{\alpha} a(\alpha)\varphi(2x - \alpha) \tag{1.1}$$

has up to a constant a unique nontrivial solution – a refinable function. In this paper we focus on those refinable functions, which are continuous and have compact support. It is well known that such functions play an important role in wavelet theory for the construction of wavelets and in geometric modeling for fast generation of curves and surfaces. In our case the support of φ is the convex hull of $\Omega = \{\alpha : a(\alpha) \neq 0\}$ (see [3]). Denote $[\Omega]$ to be the convex hull of Ω and $\partial[\Omega]$ the boundary of $[\Omega]$. It is clear that $[\Omega]$ is a convex polytope, whose vertices are integers. Let vol$[\Omega]$ be the volume of $[\Omega]$ we are interested in the minimal value of vol$[\Omega]$ among all those Ω, i.e. $\Omega = \{\alpha : a(\alpha) \neq 0\}$ and the dilation equation (1.1) defined by $\{a(\alpha)\}$ has a nontrivial continuous solution. We note that for $d = 1$ this problem is trivial. Indeed, if the refinable function φ from (1.1) is continuous and nontrivial, then there is at least one element from Ω which is the inner point of $[\Omega]$ (see [3]). Thus, Ω has at least three elements, which gives vol$[\Omega] \geq 2$ (see also [4]). On the other hand, the hat function given by $N(x) = 1 - |x|$ if $|x| \leq 1$ and zero otherwise is continuous and satisfies

$$N(x) = \frac{1}{2}N(2x+1) + N(2x) + \frac{1}{2}N(2x-1)$$

with $\Omega = \{-1, 0, 1\}$. We obtain $[\Omega] = [-1, 1]$ and vol$[\Omega] = 2$. Therefore, for $d = 1$ the minimal value among all those Ω equals to 2. What is the analogue for $d \geq 2$? It is interesting to see that this problem is far from trivial and appears to be rather difficult. We will show in this paper

Theorem 1.1. *Let $d \in \{1, 2\}$. Then the support Ω of the continuous nontrivial solution φ of (1.1) associated with $\{a(\alpha)\}$ satisfies vol$[\Omega] \geq d + 1$.*

We do not know whether the above theorem is true for all $d \geq 1$. However, we have an analogue for $d = 3$. We need the concept of shift stability for functions (see [6]). A continuous function g in $\mathbb{R}^d$ is shift stable if there are two positive constants C_1 and C_2 so that for any finitely supported $\lambda = \{\lambda(\alpha)\} \in l_\infty$

$$C_1||\lambda||_\infty \leq ||\sum_{\alpha} \lambda(\alpha)g(\cdot - \alpha)||_C \leq C_2||\lambda||_\infty, \tag{1.2}$$

where $||\cdot||$ is the uniform norm of $C(\mathbb{R}^d)$. Our next result is

Theorem 1.2. *Let $d = 3$. If the nontrivial continuous solution of (1.1) is shift stable, then vol$[\Omega] \geq d + 1$.*

In the next section we first collect and prove some propositions concerning subdivision algorithms and cascade algorithms. Some results concerning polytopes will also be presented there. Finally, we give an example, which shows that the minimal volume can be reached by some continuous refinable function φ. Moreover, φ is shift stable and belongs to Lip 1. The two theorems will be verified in Section 3.

2 Basic Properties

A subdivision scheme is defined by a fixed finitely supported mask $a = \{a(\alpha): \ \alpha \in \mathbb{Z}^d\}$. The Laurent polynomial

$$a(z) = \sum_{\alpha} a(\alpha) z^{\alpha}$$

is associated with this mask, where $z = (z_1, ..., z_d)^T \in \mathbb{R}^d$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ for $\alpha = (\alpha_1, ..., \alpha_d)^T$. Given an initial finite sequence of data values, $v^0 = \{v_{\alpha}^0\}$, a subdivision scheme with mask a defines recursively a new sequence of values v^k by applying the rule

$$v_{\alpha}^k = \sum_{\beta} v_{\beta}^{k-1} a(\alpha - 2\beta), \ \ k = 1, 2, ...$$

This scheme is said to be convergent if for each v^0 there exists a continuous function f_v such that

$$\lim_{k\to\infty} \sup_{\alpha} |f_v(\frac{\alpha}{2^k}) - v_{\alpha}^k| = 0$$

and $f_v \not\equiv 0$ for at least one v^0. If this is the case, the limit function f_v can be expressed as

$$f_v(x) = \sum_{\alpha} v_{\alpha}^0 \varphi(x - \alpha),$$

where φ is the refinable function given by (1.1). In fact, φ is the function obtained by subdivision from the initial data $v_{\alpha}^0 = \delta_{0,\alpha}$. On the other hand, φ can also be obtained by the so-called cascade algorithm. Thus, beginning with $\varphi_0(x) = N(x_1) \cdots N(x_d)$, where $N(y)$ is the hat function given in Section 1, one defines recursively

$$\varphi_k(x) = \sum_{\alpha} a(\alpha) \varphi_{k-1}(2x - \alpha), \ \ k = 1, 2, ...$$

It is known that the uniform convergence of φ_k to a nontrivial function is equivalent to the convergence of the corresponding subdivision scheme (see [3]). To describe the necessary and sufficient conditions of the convergence of the above present schemes we denote $a^k(\alpha) = \sum_{\beta} a^{k-1}(\beta) a(\alpha - 2\beta)$ with the understanding $a^1(\alpha) = a(\alpha)$. It is easy to check that $a^k(\alpha)$ are the coefficients of the Laurent polynomial $\prod_{l=0}^{k-1} a(z^{2^l})$ where $z^{\mu} = z_1^{\mu} \cdots z_d^{\mu}$, if $\mu \in \mathbb{R}$. Let us recall the following known result as (see e.g. [5, 6, 9])

Theorem 2.1. *A subdivision scheme associated with a finitely supported real mask $a = \{a(\alpha): \ \alpha \in \mathbb{Z}^d\}$ converges if and only if*

$$\sum_{\beta} a(\alpha + 2\beta) = 1, \quad \forall \ \alpha \in \mathbb{Z}^d \tag{2.1}$$

and for $E^d = \{0, 1\}^d$

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^d, \ e\in E^d} |a^k(\alpha) - a^k(\alpha - e)| = 0. \tag{2.2}$$

Furthermore, assume that φ satisfying (1.1) is shift stable, then (2.1) and (2.2) are valid.

Thus, by Theorem 2.1 if φ is shift stable, φ can be obtained by the subdivision algorithm with the initial data $v^0_\alpha = \delta_{0,\alpha}$ or by the cascade algorithm. For our goal we need also the following

Lemma 2.2. *Let $\{a(\alpha): \ \alpha \in \mathbb{Z}^d\}$ be a finite mask in $\mathbb{R}^d$ and $\Omega = \{\alpha: \ a(\alpha) \neq 0\}$. Assume that the subdivision scheme associated with $\{a(\alpha)\}$ converges to φ. If $a(\alpha') \neq 0$ for some α' and*

$$\{a(\alpha + 2\beta): \quad a(\alpha + 2\beta) \neq 0, \quad \beta \in \mathbb{Z}^d\} = \{a(\alpha')\}, \tag{2.3}$$

then $\alpha' \in \Omega \setminus \partial[\Omega]$ and $\varphi(\alpha') = 1$. Moreover, Let $\alpha \in \Omega$ be an extreme point of the polytope $[\Omega]$, then $|a(\alpha)| < 1$.

Proof. It follows from Theorem 2.1 (see (2.1)) that $a(\alpha') = 1$. On the one hand, we know that

$$\lim_{k\to\infty} |\varphi(\frac{\alpha}{2^k}) - \alpha^k(\alpha)| = 0$$

and $\varphi(x) = 0$ if $x \in \partial[\Omega] \cap [\Omega]$. On the other hand, one can easily see that

$$\begin{aligned} a^k((2^k-1)\alpha') &= \sum_\beta a^{k-1}(\beta) a((2^k-1)\alpha' - 2\beta) \\ &= a(\alpha') a^{k-1}(\alpha'(2^{k-1}-1)). \end{aligned}$$

Thus, $a^k((2^k-1)\alpha') = (a(\alpha'))^k = 1$. We conclude

$$\lim_{k\to\infty} \varphi(\frac{(2^k-1)\alpha'}{2^k}) = \varphi(\alpha') = 1.$$

Hence, α' must be an inner point of $[\Omega]$.

To verify the second assertion we observe that the Laurent polynomial associated with $\{a(\beta)\}$ is

$$a(z) = \sum_\beta a(\beta) z^\beta.$$

We know that $a^k(\beta)$ are the coefficients of Laurent polynomial

$$\begin{aligned} \prod_{l=0}^{k-1} a(z^{2^l}) &= \sum_{\beta_0,\ldots,\beta_{k-1}} a(\beta_0)\cdots a(\beta_{k-1}) z^{\beta_0 2^{l_0} + \ldots + \beta_{k-1} 2^{l_{k-1}}} \\ &= \sum_\beta a^k(\beta) z^\beta, \end{aligned} \tag{2.4}$$

where $\beta_0, ..., \beta_{k-1} \in \Omega$ and $(l_0, ..., l_{k-1})$ is a permutation of $(0, 1, ..., k-1)$. Clearly,

$$\frac{\beta_0 2^{l_0} + ... + \beta_{k-1} 2^{l_{k-1}}}{2^k - 1} \in [\Omega].$$

Since α is an extreme point of the convex polytope $[\Omega]$, one must have $\beta_0 2^{l_0} + ... + \beta_{k-1} 2^{l_{k-1}} = (2^k-1)\alpha$ if and only if $\beta_0 = ... = \beta_{k-1} = \alpha$. Hence, there is only one term in the first sum of (2.4), whose power is $(2^k-1)\alpha$. In other words, $a^k((2^k-1)\alpha) = (a(\alpha))^k$. We denote the corresponding refinable function by φ. As $\alpha' \in \partial[\Omega]$, we conclude

$$\lim_{k\to\infty} (a(\alpha))^k = \lim_{k\to\infty} a^k((2^k-1)\alpha) = \lim_{k\to\infty} \varphi(\frac{(2^k-1)\alpha}{2^k}) = \varphi(\alpha) = 0,$$

which implies $|a(\alpha)| < 1$. $\square$

Let $\mathcal{M}_d$ be the set of $d \times d$ unimodular matrices, namely,

$$\mathcal{M}_d = \{M: \ M \text{ is } d \times d \text{ matrix with integer entries and } |\det M| = 1\}.$$

Clearly, $\mathcal{M}_d$ is a group under the matrix production. In particular, $M \in \mathcal{M}_d$ implies $M^{-1} \in \mathcal{M}_d$. We have

Lemma 2.3. *Let $\{a(\alpha): \ \alpha \in \mathbb{Z}^d\}$ be a finite mask in $\mathbb{R}^d$ and satisfy $\sum_\alpha a(\alpha) = 2^d$. Let further $b(\alpha) = a(M\alpha)$ for any given $M \in \mathcal{M}_d$. Then, $\sum_\alpha b(\alpha) = 2^d$. Moreover, The existing nontrivial continuous solution of (1.1) associated with $\{a(\alpha)\}$ and $\{b(\alpha)\}$ respectively are the same.*

Proof. The first assertion is rather clear. To show the second one we write $\psi(x) = \varphi(Mx)$. Thus,

$$\begin{aligned}
\psi(x) = \varphi(Mx) &= \sum_\alpha a(\alpha)\varphi(2Mx - \alpha) \\
&= \sum_\alpha a(MM^{-1}\alpha)\varphi(2Mx - \alpha) \\
&= \sum_\alpha b(M^{-1}\alpha)\varphi(M(2x - M^{-1}\alpha)) \\
&= \sum_\beta b(\beta)\varphi(M(2x - \beta)) \\
&= \sum_\beta b(\beta)\psi(2x - \beta).
\end{aligned}$$

Clearly, if φ is continuous, so does ψ and vice versa. □

A d-polytope is defined to be a d-dimensional set, that is the convex hull of a finite number of vertices. A d-polytope is said to be simplical if each facet, i.e. $d-1$-face, is a simplex. Let us cite the following result concerning with the lower bound of the number of facets as (see [1]):

Lemma 2.4. *For a given simplical d-polytope Q, let f_{d-1} and f_0 be the numbers of facets and vertices, respectively. Then there holds*

$$f_{d-1} \geq (d-1)f_0 - (d+1)(d-2).$$

Finally, let us give an example, which shows that the minimum in Theorems 1.1 and 1.2 can be reached. We should formulate this example as

Lemma 2.5. *Let $d \geq 1$. There is a nonnegative mask $a(\alpha)$ in $\mathbb{R}^d$ such that vol$[\Omega] = d+1$. The equation (1.1) associated with this mask has a nontrivial continuous solution φ. Furthermore, φ is shift stable and $\varphi \in Lip1$.*

Proof. We choose $\{a(\alpha)\}$ to be the coefficients of

$$a(z) = \sum_\alpha a(\alpha) z^\alpha = \frac{1}{2}\prod_{l=1}^{d+1}(1 + z^{e_l}),$$

where e_l, $l = 1, ..., d$, is the coordinate vector of $\mathbb{R}^d$ and $e_{d+1} = e_1 + ... + e_d$. Thus, $[\Omega]$ is the convex hull given by $[0,1]^d \cup ([0,1]^d + e_{d+1})$. Moreover, $[\Omega]$ can be presented as

$$[\Omega] = \{e_1t_1 + e_2t_2 + ... + e_{d+1}t_{d+1} : \ 0 \leq t_l \leq 1, \ l = 1,..,d+1\}.$$

Thus, vol$[\Omega] = d+1$ as shown in [2]. The equation (1.1) associated with this mask has a continuous and nontrivial solution φ (see [8]). Moreover, one can easily verify that φ is shift stable. Next we show $\varphi \in Lip1$. We note that $\sum_\alpha a_{\alpha+2\beta} = 1$ for all $\alpha \in \mathbb{Z}^d$ and

$$\sum_\alpha a^n(\alpha) z^\alpha = \frac{1}{2^n} \prod_{j=0}^{n-1} \prod_{l=1}^{d+1} (1 + z^{2^j e_l}).$$

Obviously,

$$\prod_{j=0}^{n-1} (1 + z^{2^j e_l}) = \sum_{i=0}^{2^n - 1} z^{i e_l}.$$

On the other hand, let E_j and A be given by $E_j^\mu f(x) = f(x - \mu e_j)$ and $Af(x) = f(2x)$. Then (see [9]),

$$\varphi(\cdot) = A^n \prod_{l=0}^{n-1} a(E^{2^l}) \varphi(\cdot)$$

where $E = (E_1, ..., E_d)$. Now,

$$\begin{aligned} \varphi(x) - \varphi(x - 2^{-n} e_k) &= A^n \prod_{j=0}^{n-1} a(E^{2^j})(I - E_k)\varphi(x) \\ &= \frac{1}{2^n}(I - E_k) A^n \prod_{j=0}^{n-1} \prod_{\substack{l=1 \\ l \neq k}}^{d+1} (1 + E^{2^j e_l}) \varphi(x). \end{aligned}$$

We notice that, since

$$\sum_{\substack{l=1 \\ l \neq k}}^{d+1} i_l E_l = \sum_{\substack{l=1 \\ l \neq k}}^{d+1} j_l E_l$$

if and only if $i_l = j_l$, the coefficients of

$$\prod_{j=0}^{n-1} \prod_{\substack{l=1 \\ l \neq k}}^{d+1} (1 + E^{2^j e_l})$$

is 1. Consequently, for some $\alpha_i \in \mathbb{N}$

$$A^n \prod_{j=0}^{n-1} \prod_{\substack{l=1 \\ l \neq k}}^{d+1} (1 + E^{2^j e_l}) \varphi(x) = \sum_{i,j} \varphi(2^n x - \alpha_i e_j).$$

Therefore, as φ is compactly supported the number of terms of the sum, which are different to zero , is bounded. We conclude for some constant C, which does not depend on n,

$$|\varphi(x) - \varphi(x - 2^{-n} e_k)| \leq C 2^{-n},$$

i.e. $\varphi \in Lip1$. $\square$

3 Proof of Theorems 1.1 and 1.2

We are now in the position to prove Theorems 1.1 and 1.2. Let us first note (see [3, 4, 6]) that if (1.1) has a nontrivial continuous φ we can always suppose the Fourier-transformation of φ satisfying $\widehat{\varphi}(0) = 1$ and

$$\sum_{\alpha} \varphi(x - \alpha) = 1. \tag{3.1}$$

We notice also that if φ is nontrivial and continuous then there exists at least one integer $\alpha' \in [\Omega] \setminus \partial[\Omega]$ (see [3]). Thus, if $[\Omega] \setminus \partial[\Omega] = \{\alpha'\}$, then φ is interpolated, i.e. $\varphi(\alpha') = 1$ and $\varphi(\beta) = 0$ for all $\beta \in \mathbb{Z}^d \setminus \{\alpha'\}$. Obviously, in this case, φ is shift stable. In the following proof we will frequently use these facts.

Proof of Theorem 1.1. We already have this assertion for $d = 1$. So let $d = 2$. We know that $[\Omega]$ must have at least one inner point. If $[\Omega]$ has only one inner point, then, φ must be interpolated. Hence, φ is shift stable. It follows from Theorem 2.1 that the mask satisfies the sum rule (2.1). There are total four equations of (2.1), i.e., there are four sets of

$$\{a(\alpha + 2\beta) : \ \ a(\alpha + 2\beta) \neq 0, \ \beta \in \mathbb{Z}^2\}.$$

Only one set contains one element. Thus, by Lemma 2.2 the number of $[\Omega]$ is at least 7. According to Lemma 2.4 $[\Omega]$ can be divided into at least 6 simplexes with integer vertices. Let (x_1, x_2, x_3) be the vertices of any such simplex. Thus, $|\det(x_1 - x_3, x_2 - x_3)| \geq 1$ and the volume of (x_1, x_2, x_3) is at least 1/2. Therefore, if $[\Omega]$ contains at least 7 integers we must have $\mathrm{vol}[\Omega] \geq 3$.

Assume that $[\Omega]$ has two inner points. So if the integer number of $[\Omega]$ is at least 6, we conclude again $\mathrm{vol}[\Omega] \geq 3$, because $[\Omega]$ can also be divided into at least 6 simplexes with integer vertices.

Let the integer number of $[\Omega]$ be 5 with two inner points (say $x_0, x_1, ...x_4$). Clearly, $[\Omega]$ can be divided into 5 simplexes with integer vertices. The volume of each simplex is at least 1/2. Moreover, if one of simplexes has the volume greater than 1/2 then its volume must be at least 1. In this case we get again $\mathrm{vol}[\Omega] \geq 3$.

Next let us suppose that each those simplexes has the volume 1/2. We prove that there is no continuous and nontrivial φ. To see this, we note that there are two points x_i, x_j satisfying $x_i \equiv x_j \pmod 2$ and $(x_i + x_j)/2 \in ([\Omega] \setminus \partial[\Omega]) \cap \mathbb{Z}^2$. Moreover, one of x_i and x_j is an inner point. We may therefore suppose that $x_0 = 0$ and x_1 are inner points of $[\Omega]$ and $x_1 \equiv x_3 \pmod 2$. Let $(0, x_1, x_2)$ be a simplex of $[\Omega]$. Thus, $M^{-1} = (x_1, x_2) \in \mathcal{M}_2$ and $M\{0, x_1, x_2, x_3\} = \{0, e_1, e_2, -e_1\}$, where e_j is the coordinate unit vector. Consequently, $[M\Omega] = \{0, e_1, -e_1, e_2, 3e_1 - e_2\}$.

Let $M_1 = (e_1, e_1 + e_2) \in \mathcal{M}_2$. We obtain

$$[M_1 M\Omega] = [\{0, e_1, -e_1, e_1 + e_2, 2e_1 - e_2\}].$$

Lemma 2.3 allows us to change Ω in this way. We can write φ again to be the solution of (1.1) with the mask $b(M_1 M\alpha) = a(\alpha), \alpha \in [\Omega] \cap \mathbb{Z}^2$. We notice that $-e_1 \in M_1 M\Omega$. So $b(-e_1) \neq 0$. Moreover, $\varphi(0) \neq 0$ and $\varphi(e_1) \neq 0$. Otherwise φ is interpolated. As $\varphi(e_1) = b(e_1)\varphi(e_1)$ we conclude $b(e_1) = 1$. On the other hand, let $\varphi_2(y_2)$ be defined by

$$\varphi_2(y_2) = \int \varphi(y_1, y_2) dy_1.$$

Then, φ_2 has the mask

$$c_{-1} = \frac{b(2e_1 - e_2)}{2}, \quad c_0 = \frac{b(-e_1) + b(0) + b(e_1)}{2} \quad \text{and} \quad c_1 = \frac{b(e_1 + e_2)}{2}.$$

Clearly, φ_2 is nontrivial and continuous. Moreover, it follows from [7] that φ_2 is shift stable. Hence, by Theorem 2.1 there holds $c_0 = 1$ and $c_{-1} + c_1 = 1$. On the other hand, if we define $\varphi_1(y_1) = \int \varphi(y_1, y_2)dy_2$, then φ_1 has the mask

$$h_{-1} = \frac{b(-e_1)}{2}, \quad h_0 = \frac{b(0)}{2}, \quad h_1 = \frac{b(e_1 + e_2) + b(e_1)}{2} \quad \text{and} \quad h_2 = \frac{b(2e_1 - e_2)}{2}.$$

By the same reason φ_1 is shift stable. We conclude therefore $h_{-1} + h_1 = 1$ and $h_0 + h_2 = 1$. Comparing these two masks, we obtain in particular $b(0) = b(e_1 + e_2)$. In what follows we should again use M_1 to change $[M_1 M\Omega]$. So we get

$$M_1[M_1 M\Omega] = [\{0, e_1, -e_1, 2e_1 + e_2, e_1 - e_2\}].$$

Now repeating the above procedure we get $b(0) = b(2e_1 - e_2)$. Consequently, $b(0) = b(2e_1 - e_2) = b(e_1 + e_2) = 1$. As $b(e_1) = 1$, we obtain $b(-e_1) = 0$. This is a contradiction. So in this case φ cannot be nontrivial and continuous. Hence, we have always $\mathrm{vol}[\Omega] \geq 3$. $\square$

Proof of Theorem 1.2. The refinable function φ is now shift stable. Thus, by Theorem 2.1 the mask $\{a(\alpha)\}$ satisfies (2.1). In other words, there are $2^d = 8$ different relations of (2.1). Let us denote

$$B_\alpha = \{\alpha + 2\beta : \ a(\alpha + 2\beta) \neq 0, \quad \beta \in \mathbb{Z}^3\}.$$

Clearly, $B_\alpha \neq \emptyset$ and $B_\alpha \cap B_{\alpha'} = \emptyset$ if and only if $\alpha \not\equiv \alpha' \pmod 2$. Moreover $B_\alpha = B_{\alpha'}$ if and only if $\alpha \equiv \alpha' \pmod 2$. We observe the cases according to the numbers of relations, which contain only one nonzero element of $\{a(\alpha)\}$, i.e.

$$B_\alpha = \{\alpha + 2\beta : \ a(\alpha + 2\beta) \neq 0, \quad \beta \in \mathbb{Z}^3\} = \{\alpha'\}. \tag{3.2}$$

We remember (see Lemma 2.2) that if (3.2) is true then α' is an inner point of $[\Omega]$ and $\varphi(\alpha') = 1$. Let the number of relations (3.2) is $k \geq 0$, and denote $\alpha'_1, ..., \alpha'_k$ to be the corresponding elements of (3.2). Thus,

$$|B_\alpha| \geq 2, \qquad \forall \ \alpha \not\equiv \alpha'_j \pmod 2, \ \ j = 1, ..., k. \tag{3.3}$$

We notice that there must exist an inner integer point in $[\Omega]$ (see [3]). Consequently, if $k \geq 2$, then $[\Omega]$ has at least $k + 1$ inner integer points. To see this, we note that due to (3.1) there must exist an integer γ, which is different from α'_j and $\varphi(\gamma) \neq 0$. For otherwise, we would have $\sum_\alpha \varphi(\alpha) = k > 1$, which however contradicts to (3.1). We note also that if $[\Omega]$ has $j + 1$ inner integer points and $|\partial[\Omega] \cap \mathbb{Z}^d| = m$ then $[\Omega]$ can be triangulated into at least $2m - 4 + 3j$ simplexes, whose vertices are integers. Indeed, by Lemma 2.4 the polytope $[\Omega]$ has at least $2m - 4$ facets. Each such facet is a simplex in $\mathbb{R}^{d-1}$. Thus, each facet builds with a fixed inner point a simplex in $\mathbb{R}^d$. We obtain at least $2m - 4$ simplexes. If one of these simplexes contains another integer point we can again triangulate this into 4, 3 or 2 simplexes according to whether this point is an inner point, a point on facets or on edges, respectively. As $[\Omega]$ has $j + 1$

inner integer points, the number of simplexes for $[\Omega]$ is at least $2m-4+3j$. In the following discussion we will use this fact. Let us divide the proof into five cases.

Case 1. $k \in \{0,1\}$. We know that $[\Omega]$ must have an inner integer point. If $k=1$, α_1' is an inner point of $[\Omega]$. Thus, $|[\Omega]\cap\mathbb{Z}^3| \geq 15$. We may assume that $[\Omega]$ has only one inner integer point, say α_0. By Lemma 2.4 $[\Omega]$ has at least 24 facets. Clearly, each facet builds with α_0 a simplex. Hence, $[\Omega]$ can be triangulated into at least 24 simplexes, whose vertices are integers. Clearly, each simplex has a volume at least $1/3!$. We conclude $\mathrm{vol}[\Omega] \geq 24/3! = 4$. It is easy to see that if $[\Omega]$ has more than one inner point this inequality still holds. Thus $\mathrm{vol}[\Omega] \geq 24/3! = 4$ if $k=0,1$.

Case 2. $k \in \{2,3,4\}$ and there is an α such that $|B_\alpha| \geq 3$. We have already known that $[\Omega]$ has at least $k+1$ integer inner points. We may assume $|\partial[\Omega]\cap\mathbb{Z}^3| \geq 2(8-k)$. Thus, $[\Omega]$ can be triangulated into at least $2(16-2k)-4+3k \geq 24$ simplexes with integer vertices. We conclude again $\mathrm{vol}[\Omega] \geq 24/3! = 4$.

Case 3. $k \in \{2,3\}$ and there is no α such that $|B_\alpha| \geq 3$. We know that there exists an integer $\gamma \not\equiv \alpha_j' \pmod 2$ and $\varphi(\gamma) \neq 0$. If the number of such γ is one, then (1.1) tells us

$$\varphi(\gamma) = a(\gamma)\varphi(\gamma) + a(2\gamma-\alpha_1')\varphi(\alpha_1') + \ldots + a(2\gamma-\alpha_k')\varphi(\alpha_k').$$

Clearly, $a(2\gamma-\alpha_j') = 0$ because α_j' and $2\gamma-\alpha_j'$ belong to the same relation of B_α. However, the relation, that contains α_j', has only one element. Hence, $a(2\gamma-\alpha_j') = 0$. We conclude in this way $\varphi(\gamma) = a(\gamma)\varphi(\gamma)$. Thus, $a(\gamma) = 1$. But, the relation, that contains γ, has two elements and the sum of $a(\alpha)$ indexed with these elements is one. This gives a contradiction. Therefore, the number of such γ is at least two. We may without loss of the generality assume that $[\Omega]$ has $k+2$ inner integers and $|\partial[\Omega]\cap\mathbb{Z}^3| = 2(8-k)-2$. We conclude that the number of simplexes is at least $2(14-2k)-4+3k+3 \geq 24$. We obtain also $\mathrm{vol}[\Omega] \geq 24/3! = 4$.

Case 4. $k=4$ and there is no α such that $|B_\alpha| \geq 3$. According to Case 3 we may assume that $[\Omega]$ has 6 inner integer points and $|\partial[\Omega]\cap\mathbb{Z}^3| = 6$. If the number of the facets is more than 8, we have immediately $\mathrm{vol}[\Omega] \geq 4$. Let the number of the facets be 8. Hence, there are β_1 and β_2 in $\partial[\Omega]\cap\mathbb{Z}^3$ such that $\beta_1 \equiv \beta_2 \pmod 2$. Moreover, $(\beta_1+\beta_2)/2 \in [\Omega]\cap\mathbb{Z}^3$. Thus, β_1 and β_2 belong to different facets. It is not hard to see that there are two facets with the form $\{x_1,x_2,\beta_1\}$ and $\{x_1,x_2,\beta_2\}$, respectively. Let $\alpha \neq (\beta_1+\beta_2)/2$ be an inner integer of $[\Omega]$. So according to this α the polytope $[\Omega]$ can be triangulated into 8 simplexes. Two of them are $\{\alpha,x_1,x_2,\beta_1\}$ and $\{\alpha,x_1,x_2,\beta_2\}$. Clearly,

$$\begin{aligned}
[\{\alpha,x_1,x_2,\beta_1\}\cup\{\alpha,x_1,x_2,\beta_2\}] &= [\{\alpha,x_1,x_2,\beta_1\}]\cup[\{\alpha,x_1,x_2,\beta_2\}] \\
&= [\{\alpha,x_1,x_2,\beta_1,\beta_2\}].
\end{aligned}$$

Let us without loss of the generality suppose that beside $\alpha,x_1,x_2,\beta_1,\beta_2$ there is no integer on the facets of $[\{\alpha,x_1,x_2,\beta_1,\beta_2\}]$. Let $[\{\alpha,x_1,x_2,\beta_1,\beta_2\}]$ have $j+1$ inner integers. So for the rest 6 simplexes of $[\Omega]$ there are $k-j$ inner integers. Therefore, the total number of simplexes is at least $6+3(k-j)+6+3j = 24$ and $\mathrm{vol}[\Omega] \geq 24/3! = 4$.

Case 5. $k \geq 5$. Thus, we have $8-k$ sets B_α with $|B_\alpha| \geq 2$ provided $\alpha \not\equiv \alpha_j' \pmod 2$ for $j=1,\ldots,k$. We remember that $[\Omega]$ must have at least $k+1$ inner integer points.

Because $k \geq 5$ the set $\Omega \cap \partial[\Omega] \cap \mathbb{Z}^3$ contains at least four integers. Thus there are $\beta_1, \beta_2 \in B_\alpha$ for some α. If $|\partial\Omega| = 4$, then since $(\beta_1 + \beta_2)/2$ is an integer, the number of the facets are at least 6. We conclude that $[\Omega]$ can be triangulated into at least $6 + 3k$ simplexes. Thus, $\text{vol}[\Omega] \geq 4$ whenever $k \geq 6$. If the inner integer points are $k+2$, then the same discussion implies $\text{vol}[\Omega] \geq 4$ whenever $k \geq 5$. It remains to show the assertion for $k = 5$ and the number of inner integer points is $k+1$. If $[\Omega]$ has more than 8 facets, we have nothing more to do, since in this case $[\Omega]$ can be triangulated into at least $9 + 3k = 24$ simplexes and thus $\text{vol}[\Omega] \geq 4$. If $[\Omega]$ has exact 8 facets, we use the argument in Case 3. We obtain $|\partial\Omega \cap \mathbb{Z}^3| = 6$. Thus, $\partial\Omega$ contain two integers β_1, β_2 such that $\beta_1 \equiv \beta_1 \pmod 2$. As $(\beta_1 + \beta_2)/2$ is an integer, they cannot belong to the same facet. Now, like in Case 4 let $\alpha \neq (\beta_1 + \beta_2)/2$ be an inner integer of $[\Omega]$. So according to this α the polytope $[\Omega]$ can be triangulated into 8 simplexes. Two of them are $\{\alpha, x_1, x_2, \beta_1\}$ and $\{\alpha, x_1, x_2, \beta_2\}$. Clearly, $[\{\alpha, x_1, x_2, \beta_1\} \cup \{\alpha, x_1, x_2, \beta_2\}] = [\{\alpha, x_1, x_2, \beta_1\}] \cup [\{\alpha, x_1, x_2, \beta_2\}] = [\{\alpha, x_1, x_2, \beta_1, \beta_2\}]$. Let us without loss of the generality suppose that beside $\alpha, x_1, x_2, \beta_1, \beta_2$ there is no integer on the facets of $[\{\alpha, x_1, x_2, \beta_1, \beta_2\}]$. Let $[\{\alpha, x_1, x_2, \beta_1, \beta_2\}]$ have $j+1$ inner integers. So for the rest 6 simplexes there are $k - j$ inner integers. Therefore, the total number of simplexes is at least $6 + 3(k-j) + 6 + 3j = 24$ and $\text{vol}[\Omega] \geq 24/3! = 4$. $\square$

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On copositive approximation by bivariate polynomials on rectangular grids

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

In this paper, firstly we complete a result in the book [2] concerning quantitative estimates in the copositive approximation of smooth functions by bivariate polynomials on rectangular grids. Then, for bivariate functions which are only continuous, an error estimate in terms of a first order Ditzian-Totik bivariate modulus of continuity is obtained. The results are natural extensions of those well-known in the univariate case.

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1 Introduction

In the book [2], Subsection 2.6.3, pp. 186-194, some quantitative results in terms of the moduli of smoothness in copositive approximation by bivariate polynomials were obtained. For example, by using a result in unconstrained bivariate approximation in [1], one of them refers to the copositive approximation on a proper grid, of functions having continuous the mixed second order partial derivative. This result can be stated as follows.

Theorem 1 *(see Gal [2], Theorem 2.6.6, p. 187) If* $f : [0,1] \times [0,1] \to \mathbb{R}$ *has the partial derivative* $\frac{\partial^2 f}{\partial x \partial y}$ *continuous on* $[0,1] \times [0,1]$ *and changes its sign on the proper rectangular grid in* $(0,1) \times (0,1)$*, determined by the distinct segments*

$x = x_i, i \in \{1, ..., k\}, y = y_j, j \in \{1, ..., s\}$, *then for all* $n \geq n_0$ *and* $m \geq m_0$ *(with* n_0 *and* m_0 *depending only on* k, s, α, β, *where* $\alpha = min_{0\leq i\leq k}(x_{i+1} - x_i), \beta = min_{0\leq j\leq s}(y_{j+1} - y_j)$, $0 = x_0 = y_0, 1 = x_{k+1} = y_{s+1}$*), there exists a polynomial* $P_{n,m}(x, y)$ *of degrees* $\leq n$ *in* x *and* $\leq m$ *in* y*, which satisfies*

$$\|f - P_{n,m}\| \leq C \left[\frac{a}{n} + \frac{b}{m}\right],$$

and is copositive with f *on* $[0, 1]^2 \setminus \{A \cup B\}$*, where* $C = C(k, s, \alpha, \beta) > 0$,

$$a = \omega_2(\frac{\partial f}{\partial x}; \frac{1}{n}, 0) + \omega_2(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, 0),\ b = \omega_2(\frac{\partial f}{\partial y}; 0, \frac{1}{m}) + \omega_2(\frac{\partial^2 f}{\partial x \partial y}; 0, \frac{1}{m}),$$

$$\omega_2(f; \alpha, \beta) = \sup\{|f(x, y) - 2f(x + h, y + p) + f(x + 2h, y + 2p)|$$

$$; (x, y), (x + 2h, y + 2p) \in [0, 1] \times [0, 1], 0 \leq h \leq \alpha, 0 \leq p \leq \beta\},$$

$$A = \{(x, y) \in [0, 1]^2; x \in \cup_{i=1}^k [x_i - 1/n, x_i + 1/n], y \notin \cup_{j=1}^s [y_j - 1/m, y_j + 1/m],$$

$$\Pi_{j=1}^s (y - y_j) < 0\},$$

$$B = \{(x, y) \in [0, 1]^2; y \in \cup_{j=1}^s [y_j - 1/m, y_j + 1/m], x \notin \cup_{i=1}^k [x_i - 1/n, x_i + 1/n],$$

$$\Pi_{i=1}^k (x - x_i) < 0\}.$$

Remark 2 *As can easily be seen, unfortunately the above theorem states that the copositivity does not hold on the whole bidimensional interval* $[0, 1] \times [0, 1]$. *It is the first goal of this note to modify/complete the proof of the above theorem, by obtaining that the copositivity can hold on the whole bidimensional interval* $[0, 1] \times [0, 1]$, *in terms of the same quantitative estimate.*

Secondly, for functions which are only continuous, by using a result in bivariate comonotone approximation already proved in the same book [2], pp. 194-205, a quantitative result in the bivariate copositive polynomial approximation with the error in terms of the first order bivariate Ditzian-Totik modulus of continuity, $\omega_1^{\varphi}(f; 1/n, 1/m)$ *is obtained.*

2 Main Results

The first main result is the following.

Theorem 3 *If* $f : [0, 1] \times [0, 1] \to \mathbb{R}$ *has the partial derivative* $\frac{\partial^2 f}{\partial x \partial y}$ *continuous on* $[0, 1] \times [0, 1]$ *and changes its sign on the proper rectangular grid in* $(0, 1) \times (0, 1)$, *determined by the distinct segments* $x = x_i, i \in \{1, ..., k\}, y = y_j, j \in \{1, ..., s\}$, *then for all* $n \geq n_0$ *and* $m \geq m_0$ *(with* n_0 *and* m_0 *depending only on* k, s, α, β, *where* $\alpha = min_{0\leq i\leq k}(x_{i+1} - x_i), \beta = min_{0\leq j\leq s}(y_{j+1} - y_j)$, $0 = x_0 =$

$y_0, 1 = x_{k+1} = y_{s+1}$*), there exists a polynomial* $P_{n,m}(x, y)$ *of degrees* $\leq n$ *in* x *and* $\leq m$ *in* y*, which satisfies*

$$\|f - P_{n,m}\| \leq C \left[\frac{a}{n} + \frac{b}{m}\right],$$

and is copositive with f *on* $[0, 1] \times [0, 1]$*, where* $C = C(k, s, \alpha, \beta) > 0$ *and*

$$a = \omega_2(\frac{\partial f}{\partial x}; \frac{1}{n}, 0) + \omega_2(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, 0),\ b = \omega_2(\frac{\partial f}{\partial y}; 0, \frac{1}{m}) + \omega_2(\frac{\partial^2 f}{\partial x \partial y}; 0, \frac{1}{m}).$$

Here $\| \cdot \|$ *denotes the uniform norm in* $C([0, 1] \times [0, 1])$*.*

Proof. Keeping the notations, we construct the polynomials $Q_{n,m}(x, y)$ exactly as in the proof of Theorem 2.6.6, pp. 187-189 in [2], but we slightly modify the expression $E_{n,m}(x, y)$ as follows. Consider

$$E_{n,m}(x, y) = \varepsilon D_1 D_2 C \left[\frac{a}{n} + \frac{b}{m}\right] \prod_{i=1}^{k} q_n(x - x_i) \prod_{j=1}^{s} q_m(y - y_j),$$

where ε, C, $q_n(x)$, $q_m(y)$ are exactly as in the proof of Theorem 2.6.6, p. 188 in [2] and D_1, D_2 are strictly positive constants which will be determined later. It is known that $f(x, y)E_{n,m}(x, y) \geq 0$, for all $(x, y) \in [0, 1] \times [0, 1]$.

Define $P_{n,m}(x, y) = Q_{n,m}(x, y) + E_{n,m}(x, y), x, y \in [0, 1]$.

We distinguish three cases :

(i) $y \notin \cup_{j=1}^{s} [y_j - 1/m, y_j + 1/m]$;

(ii) $x \notin \cup_{i=1}^{k} [x_i - 1/n, x_i + 1/n]$;

(iii) there exists i such that $x \in [x_i - 1/n, x_i + 1/n]$ and there exists j such that $y \in [y_j - 1/m, y_j + 1/m]$.

Case (i) Consider fixed y and define the polynomial $\overline{P}_n(x) = Q_{n,m}(x, y) + h_n(x), x \in [0, 1]$, with $h_n(x) = \varepsilon_1 \varepsilon D_1 C \left[\frac{a}{n} + \frac{b}{m}\right] \prod_{i=1}^{k} q_n(x - x_i), x \in [0, 1]$, where $\varepsilon_1 = 1$ if $\prod_{j=1}^{k} q_m(y - y_j) \geq 0$ and $\varepsilon_1 = -1$ if $\prod_{j=1}^{k} q_m(y - y_j) < 0$. From here it follows that $f(x, y)h_n(x) \geq 0$, for all $x \in [0, 1]$.

Using now the same reasoning as in the proof of Theorem 1.5.4, (iv), p. 44-46 in [2] and choosing $D_1 \geq 2A^{-k}$, we obtain $f(x, y)\overline{P}_n(x) \geq 0$ $(\forall)$ $x \in [0, 1]$. Recall here that $A > 0$ is the constant in the statement of Lemma (A), p. 43, in [2] (see also [3]).

Now, let $x \in [0, 1]$ be arbitrary chosen. We have two subcases : $(i)_a$ $f(x, y)$ and $Q_{n,m}(x, y)$ have the same sign ; $(i)_b$ $f(x, y)$ and $Q_{n,m}(x, y)$ are of opposite signs.

Case $(i)_a$. Because $f(x, y)$ and $E_{n,m}(x, y)$ have the same sign, it immediately follows that $f(x, y)P_{n,m}(x, y) \geq 0$.

Case $(i)_b$. Because $f(x, y)\overline{P}_n(x) \geq 0$ and $f(x, y)h_n(x) \geq 0$, it necessarily follows that we have $|Q_{n,m}(x, y)| \leq |h_n(x)|$. Choosing $D_2 \geq A^{-s}$ and reasoning

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again as in the proof of Theorem 1.5.4, (iv), p. 44-46 in [2], it follows that $D_2 \left|\prod_{j=1}^{s} q_m(y-y_j)\right| \geq 1$, which easily implies that $|h_n(x)| \leq |E_{n,m}(x,y)|$.

In conclusion, we get that $|Q_{n,m}(x,y)| \leq |E_{n,m}(x,y)|$ and because $f(x,y)$ and $E_{n,m}(x,y)$ have the same sign, it is immediate that $f(x,y)P_{n,m}(x,y) \geq 0$, which finishes the proof of the case (i).

Case (ii). The proof uses the same type of reasoning as in the case (i).

Case (iii). The proof is identical with the proof of the case (iv) of Theorem 2.6.6 in [2], p. 189.

Finally, as in the proof of Theorem 2.6.6, p. 189 in [2], note that from its construction it follows in fact that $P_{n,m}(x,y)$ is of degrees $\leq 2kn$ in x and $\leq 2sm$ in y, but by a standard procedure we may reduce it to degrees $\leq n$ in x and $\leq m$ in y. ■

Remark 4 *The above theorem extends to bivariate case the result in univariate case in [3].*

The second main result one refers to the case of the absence of partial derivatives of f, namely to the case when f is only continuous.

Theorem 5 *If $f : [-1,1] \times [-1,1] \to \mathbb{R}$ is continuous on $[-1,1] \times [-1,1]$ (we write $f \in C([-1,1]\times[-1,1])$) and changes its sign on the proper rectangular grid in $(0,1) \times (0,1)$, determined by the distinct segments $x = x_i, i \in \{1, ..., k\}, y = y_j, j \in \{1, ..., k\}$, then for all $n \geq 1$ and $m \geq 1$, there exists a polynomial $Q_{n,m}(x,y)$ of degrees $\leq n$ in x and $\leq m$ in y, which satisfies*

$$\|f - Q_{n,m}\| \leq C(k) \cdot \omega_1^{\varphi}\left(f; \frac{1}{n}, \frac{1}{m}\right)$$

and is copositive with f on $[-1,1] \times [-1,1]$, where $C = C(k) > 0$ depends only on k and

$$\omega_1^{\varphi}(f; \delta_1, \delta_2) = \sup\{|\Delta_{h_1\varphi(x),h_2\varphi(y)} f(x,y)|; 0 \leq h_i \leq \delta_i, i = 1,2,\ x,y \in [-1,1]\},$$

with $\Delta_{h_1,h_2} f(x,y) = f(x+h_1/2, y+h_2/2) - f(x-h_1/2, y-h_2/2)$ if $(x \pm h_1/2, y \pm h_2/2) \in [-1,1]\times[-1,1]$, $\Delta_{h_1,h_2} f(x,y) = 0$ elsewhere and $\varphi(x) = \sqrt{1-x^2}$. Here $\|\cdot\|$ denotes the uniform norm in $C([-1,1] \times [-1,1])$.

Proof. Firstly, we prove that the polynomials $P_{n,m}$ in the statement of Theorem 2.6.10, p. 195 in [2], in addition to be upper bidimensional comonotone with f (that is satisfying $\frac{\partial^2 P_{n,m}}{\partial x \partial y}(x,y) \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y) \geq 0$, for all $x,y \in [-1,1]$) and to approximate f, they also satisfy the estimate

$$\left\|\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 P_{n,m}}{\partial x \partial y}\right\| \leq C(k) \cdot \omega_1^{\varphi}\left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, \frac{1}{m}\right). \tag{1}$$

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Indeed, looking into the proof of Theorem 2.6.10 at pages 204-206 in [2], we reason by mathematical induction. Thus, for $k = 0$ the above estimate (1) is exactly the second estimate in the statement of Theorem 2.6.12, p. 196 in [2]. Then, keeping the notations and the reasonings (by mathematical induction) from the page 205 in [2], taking into account the second estimate in the statement of Lemma 2.6.15, p. 199 in [2] obtained there for $\varepsilon \geq \omega_1^{\varphi}\left(\frac{\partial^2 f}{\partial x \partial y}; \frac{1}{n}, \frac{1}{m}\right)$, we immediately get exactly the above desired estimate.

Now, in order to get our result of copositive approximation, for given $f \in C([-1,1] \times [-1,1])$, let us define

$$F(x, y) = \int_{-1}^{x} \left[\int_{-1}^{y} f(t, s) ds\right] dt,\ x, y \in [-1, 1].$$

Clearly we have $\frac{\partial^2 F}{\partial x \partial y}(x, y) = f(x, y)$. Therefore, applying all the above considerations to F instead of f, we obtain the sequence of bivariate polynomials $Q_{n,m}(x, y) = \frac{\partial^2 P_{n,m}}{\partial x \partial y}(x, y)$, that clearly satisfy the requirements in the statement. ■

Remark 6 *As an open question would be interesting to study the general case in copositive bivariate approximation by polynomials, namely when the approximated function changes its sign along to some given algebraic curves (not necessarily segments) included in* $[0, 1] \times [0, 1]$ *or in* $[-1, 1] \times [-1, 1]$*, respectively.*

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From Bernstein polynomials to Bernstein copulas

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

In this paper we review Bernstein and checkerboard copulas for arbitrary dimensions and general grid resolutions in connection with discrete random vectors possessing uniform margins, and point out the relation to tensor product Bernstein operators. We further suggest a pragmatic and effective way to fit the dependence structure of multivariate data to Bernstein copulas via rook copulas, a subclass of checkerboard copulas, which is based on the multivariate empirical distribution.

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1 Introduction

In the history of approximation theory, univariate and multivariate Bernstein polynomials have played a central role since the beginning of the 20^{th} century, see, e.g., [11] for a survey of Bernstein polynomials in one variable and [1], chapters 8.4 and 18, for a short treatment of Bernstein polynomials in several variables. They have not only been used to provide a constructive proof of the famous Weierstraß approximation theorem for continuous functions on compact intervals, including explicit estimates for the rate of convergence, but also for more advanced applications in functional analysis and computer aided design, such as Bézier curves and surfaces, see, e.g., [7], [15] and [16]. Here, shape preserving and local smoothness properties of Bernstein polynomials are of central interest, in particular w.r.t. engineering applications. (It might be interesting to note here that Donald Knuth has used Bézier curves for the design of TEX-fonts.) Applications of Bernstein polynomials for modelling stochastic dependence in a nonparametric way have, in contrast, been considered much later.

The use of copulas for modelling and simulation purposes, for instance in risk management, is of increasing importance, see, e.g., [3], section 5.3, or [9], chapter 5, and the references given there. Let us recall that a (d-dimensional) copula C is the cumulative distribution function (cdf) of a random vector $\boldsymbol{U} = (U_1, \cdots, U_d)$ whose one-dimensional marginal distributions are uniform on the interval $[0,1]$. The following well-known theorem (see, e.g., [9], p. 186) deals with a key property of copulas.

Theorem of Sklar: Let F be the cdf of some random vector $\boldsymbol{X} = (X_1, \cdots, X_d)$, i.e., $F(x_1, \cdots, x_d) = P(X_1 \leq x_1, \cdots, X_d \leq x_d)$ with marginal cdfs $F_1, \cdots, F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that

$$F(x_1, \cdots, x_d) = C(F_1(x_1), \cdots, F_d(x_d)) \tag{1}$$

for all $x_1, \cdots, x_d \in \mathbb{R}$. If $F_1, \cdots, F_d$ are continuous, then C is uniquely determined. Vice versa: For a copula C and univariate cdfs $F_1, \cdots, F_d$ the assignment $F(x_1, \cdots, x_d) := C(F_1(x_1), \cdots, F_d(x_d))$ defines the cdf F of some d-variate random vector with marginal cdfs $F_1, \cdots, F_d$.

Thus, the theorem of Sklar states that the cdf F of any d-variate random vector can be written in terms of its marginal distribution functions $F_1, \cdots, F_d$ and a suitable copula C which thus describes the dependence structure of the vector components. Such a decomposition is often very useful in practice; for an exemplary application in the context of Bernstein copulas see Example 4.2. The definition of this specific copula type, constructed by means of Bernstein polynomials, is given in section 2.

The discussion of potential copula models has so far mostly focussed on other types, i.e., either the elliptical case (e.g., the Gaussian and t-copula) or the Archimedean case (e.g., Gumbel-, Clayton-, and Frank-copulas). It seems that the true impact of Bernstein polynomials on copula models has been discovered only more recently, first in the framework of approximation theory (see, e.g., [8], [10], [11]) and later in particular in connection with applications in finance (see, e.g., [2], [5], [6], [13], [14]). Bernstein copulas possess several benefits compared to the traditional approaches:

- Bernstein copulas allow for a very flexible, non-parametric and essentially non-symmetric description of dependence structures also in higher dimensions

- Bernstein copulas approximate any other given copula arbitrarily well
- Bernstein copula densities are given in an explicit form and can hence be easily used for Monte Carlo simulation studies.

In this paper, we review the construction of Bernstein copulas through discrete random vectors with uniform margins (called discrete skeletons), and point out their connection to checkerboard copulas, as discussed, e.g., in [8], [10] and [11], and to Bernstein tensor product operators (cf. the proof of Theorem 2.2). The explicit representation of Bernstein copulas in terms of tensor product Bernstein operators with a discrete skeleton has, to our knowledge, not been stated in the related literature before. This approach, amongst others, opens a pragmatic and storage saving approach to fit the dependence structure of observed data to Bernstein copulas via rook copulas, a special subcase of checkerboard copulas based on the multivariate empirical distribution. The tensor product representation might also be helpful in further studies on global smoothness preservation for copula approximation since it allows a direct transfer of results from multivariate approximation theory (as formulated, e.g., in [4] and [12]) into the copula context.

2 Some simple mathematical facts on Bernstein polynomials and Bernstein copulas

The assertions of the following lemma are well-known in the literature, but for convenience and better understanding in the copula context we give a short proof.

Lemma 2.1. Let $B(m,k,z) = \binom{m}{k} z^k (1-z)^{m-k}$, $0 \le z \le 1$, $k = 0, \cdots, m \in \mathbb{N}$. Then we have

$$\int_0^1 m\, B(m-1,k,z)\, dz = 1 \text{ for } k = 0, \cdots, m-1.$$

Further,

$$\frac{d}{dz} B(m,k,z) = m\left[B(m-1,k-1,z) - B(m-1,k,z)\right] \text{ for } k = 0, \cdots, m$$

with the convention $B(m-1,-1,z) = B(m-1,m,z) = 0$. For the Bernstein operator $\mathcal{B}_m$ defined by $\mathcal{B}_m f : z \mapsto \sum_{k=0}^m f\left(\frac{k}{m}\right) B(m,k,z)$ for real-valued functions f on $[0,1]$ and $z \in [0,1]$, this yields

$$\frac{d}{dz} \mathcal{B}_m f(z) = m \sum_{k=0}^{m-1} \Delta_m f\left(\frac{k}{m}\right) B(m-1,k,z)$$

where $\Delta_m f(z) := f\left(z + \frac{1}{m}\right) - f(z)$ for $z \in [0,1]$ denotes the forward difference operator.

Proof. Let $\mathrm{B}(x,y) := \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$ for $x, y > 0$ denote the Beta function and Γ the Gamma function, as usual. Then

$$\begin{aligned}\int_0^1 m\,B(m-1,k,z)\,dz &= m\binom{m-1}{k}\mathrm{B}(k+1,m-k)\\ &= m\binom{m-1}{k}\frac{\Gamma(k+1)\Gamma(m-k)}{\Gamma(m+1)}\\ &= \frac{m\,(m-1)!}{k!(m-k-1)!}\times\frac{k!(m-k-1)!}{m!}\\ &= 1.\end{aligned}$$

Further, for $0 < k < m$,

$$\begin{aligned}\frac{d}{dz}B(m,k,z) &= k\binom{m}{k}z^{k-1}(1-z)^{m-k} - (m-k)\binom{m}{k}z^k(1-z)^{m-k-1}\\ &= m\binom{m-1}{k-1}z^{k-1}(1-z)^{(m-1)-(k-1)}\\ &\quad - m\binom{m-1}{k}z^k(1-z)^{m-1-k}\\ &= m\left[B(m-1,k-1,z) - B(m-1,k,z)\right]\end{aligned}$$

which, by the above convention, also holds for $k \in \{0, m\}$. The remaining statement follows easily from this. $\square$

Theorem 2.1 and Definition. For $d \in \mathbb{N}$ let $\boldsymbol{U} = (U_1, \cdots, U_d)$ be a random vector whose marginal component U_i follows a discrete uniform distribution over $T_i := \{0, 1, \cdots, m_i - 1\}$ with $m_i \in \mathbb{N}$, $i = 1, \cdots, d$. Let further

$$p\left(k_1, \cdots, k_d\right) := P\left(\bigcap_{i=1}^{d}\{U_i = k_i\}\right) \text{ for all } \left(k_1, \cdots, k_d\right) \in \bigtimes_{i=1}^{d} T_i.$$

Then

$$c_B^{\boldsymbol{U}}\left(u_1, \cdots, u_d\right) := \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} p\left(k_1, \cdots, k_d\right)\prod_{i=1}^{d} m_i B\left(m_i - 1, k_i, u_i\right),$$

$(u_1, \cdots, u_d) \in [0,1]^d$, defines the density of a d-dimensional copula $C_B^{\boldsymbol{U}}$, called *Bernstein copula*. We call $c_B^{\boldsymbol{U}}$ the Bernstein copula density induced by $\boldsymbol{U}$. The vector $\boldsymbol{U}$ is also called the discrete skeleton of the Bernstein copula.

Proof. For fixed $1 \le j \le d$ we obtain, according to Lemma 2.1 above,

$$\begin{aligned}&\int_0^1 c_B^{\boldsymbol{U}}\left(u_1, \cdots, u_d\right)\,du_j\\ &= \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} p\left(k_1, \cdots, k_d\right)\prod_{\substack{i=1\\ i\neq j}}^{d} m_i B\left(m_i - 1, k_i, u_i\right)\int_0^1 m_j B\left(m_j - 1, k_j, u_j\right)\,du_j\end{aligned}$$

4

$$
\begin{aligned}
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p\left(k_1, \cdots, k_d\right) \prod_{\substack{i=1 \\ i \neq j}}^{d} m_i B\left(m_i-1, k_i, u_i\right) \\
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_{j-1}=0}^{m_{j-1}-1} \sum_{k_{j+1}=0}^{m_{j+1}-1} \cdots \sum_{k_d=0}^{m_d-1} \left(\sum_{k_j=0}^{m_j-1} p\left(k_1, \cdots, k_d\right) \right) \prod_{\substack{i=1 \\ i \neq j}}^{d} m_i B\left(m_i-1, k_i, u_i\right) \\
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_{j-1}=0}^{m_{j-1}-1} \sum_{k_{j+1}=0}^{m_{j+1}-1} \cdots \sum_{k_d=0}^{m_d-1} P\left(\bigcap_{\substack{i=1 \\ i \neq j}}^{d} \left\{U_i = k_i\right\} \right) \prod_{\substack{i=1 \\ i \neq j}}^{d} m_i B\left(m_i-1, k_i, u_i\right) \\
&= c_B^{\boldsymbol{U}^{\setminus j}}\left(u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_d\right)
\end{aligned}
$$

for $\left(u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_d\right) \in [0,1]^{d-1}$, where $\boldsymbol{U}^{\setminus j} = \left(U_1, \cdots, U_{j-1}, U_{j+1}, \cdots, U_d\right)$ (note that for $j = 1$, the symbol $\boldsymbol{U}^{\setminus j}$reads $\left(U_2, \cdots, U_d\right)$, likewise for $j = d$). We thus obtain another Bernstein copula density, but of dimension $d-1$ instead of d. Continuing integration according to the remaining variables except for the variable u_r for fixed $1 \leq r \leq d$, we end up with

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 c\left(u_1, \cdots, u_d\right)\, du_1 \cdots du_{r-1}\, du_{r+1} \cdots du_d \\
&\quad = \sum_{k_r=0}^{m_r-1} P\left(U_r = k_r\right)\, m_r B\left(m_r-1, k_r, u_r\right) \\
&\quad = \sum_{k_r=0}^{m_r-1} \frac{1}{m_r} m_r B\left(m_r-1, k_r, u_r\right) = \sum_{k_r=0}^{m_r-1} B\left(m_r-1, k_r, u_r\right) \\
&\quad = \sum_{k=0}^{m_r-1} \binom{m_r-1}{k} u_r^k \left(1-u_r\right)^{m_r-k-1} = 1
\end{aligned}
$$

for all $u_r \in [0,1]$ which proves that the r-th marginal density of $c_B^{\boldsymbol{U}}$ is that of a continuous uniform distribution over $[0,1]$, for every $1 \leq r \leq d$. □

Remark 2.1. Note that the line of proof above shows that if $\boldsymbol{U} = \left(U_1, \cdots, U_d\right)$ is a random vector with joint Bernstein copula density $c_B^{\boldsymbol{U}}$ as above, then also any partial random vector $\boldsymbol{V} = \left(U_{i_1}, \cdots, U_{i_n}\right)$ with $n < d$ and $1 \leq i_1 < \cdots < i_n \leq d$ possesses a Bernstein copula density $c_B^{\boldsymbol{V}}$ given by

$$
\begin{aligned}
&c_B^{\boldsymbol{V}}\left(u_{i_1}, \cdots, u_{i_n}\right) \\
&\quad = \sum_{k_{i_1}=0}^{m_{i_1}-1} \cdots \sum_{k_{i_n}=0}^{m_{i_n}-1} P\left(\bigcap_{\ell=1}^{n} \left\{U_{i_\ell} = k_{i_\ell}\right\} \right) \prod_{\ell=1}^{n} m_{i_\ell} B\left(m_{i_\ell}-1, k_{i_\ell}, u_{i_\ell}\right),
\end{aligned}
$$

$\left(u_{i_1}, \cdots, u_{i_n}\right) \in [0,1]^n$.

Theorem 2.2. Under the conditions of Theorem 2.1, the Bernstein copula $C_B^{\boldsymbol{U}}$

induced by $\boldsymbol{U}$ is explicitly given by

$$\begin{aligned} C_B^{\boldsymbol{U}}\left(x_1,\cdots,x_d\right) &:= \int_0^{x_d}\cdots\int_0^{x_1} c_B^{\boldsymbol{U}}\left(u_1,\cdots,u_d\right)\,du_1\cdots du_d \\ &= \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} P\left(\bigcap_{i=1}^{d}\left\{U_i < k_i\right\}\right)\prod_{i=1}^{d} B\left(m_i,k_i,x_i\right) \end{aligned}$$

for $\left(x_1,\cdots,x_d\right)\in\left[0,1\right]^d$.

Proof. Let $F_{\boldsymbol{U}}$ denote the cdf of $\boldsymbol{U}$, i.e. $F_{\boldsymbol{U}}\left(x_1,\cdots,x_d\right) = P\left(\bigcap_{i=1}^{d}\left\{U_i\le x_i\right\}\right)$ for $\left(x_1,\cdots,x_d\right)\in\mathbb{R}^d$, and let $\boldsymbol{Z} = \left(Z_1,\cdots,Z_d\right)$ be given by $Z_i := \frac{U_i+1}{m_i}$ for $i = 1,\cdots,d$. Then for the cdf of $\boldsymbol{Z}$, we obtain

$$\begin{aligned} F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right) &= P\left(\bigcap_{i=1}^{d}\left\{U_i\le k_i-1\right\}\right) = P\left(\bigcap_{i=1}^{d}\left\{U_i < k_i\right\}\right) \\ &= F_{\boldsymbol{U}}\left(k_1-1,\cdots,k_d-1\right) \end{aligned}$$

for $\left(k_1,\cdots,k_d\right)\in\bigtimes_{i=1}^{d} T_i$. By applying Lemma 2.1 consecutively d times, it follows that

$$\begin{aligned} c_B^{\boldsymbol{U}}\left(u_1,\cdots,u_d\right) &= \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} P\left(\bigcap_{i=1}^{d}\left\{U_i = k_i\right\}\right)\prod_{i=1}^{d} m_i B\left(m_i-1,k_i,u_i\right) \\ &= \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} P\left(\bigcap_{i=1}^{d}\left\{U_i\in\left(k_i-1,k_i\right]\right\}\right)\prod_{i=1}^{d} m_i B\left(m_i-1,k_i,u_i\right) \\ &= \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} \Delta_{m_1,\cdots,m_d} F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)\prod_{i=1}^{d} m_i B\left(m_i-1,k_i,u_i\right) \\ &= \frac{\partial^d}{\partial x_1\cdots\partial x_d}\mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d} F_{\boldsymbol{Z}}\left(u_1,\cdots,u_d\right) \end{aligned}$$

for $\left(u_1,\cdots,u_d\right)\in\left[0,1\right]^d$ where $\Delta_{m_1,\cdots,m_d} := \Delta_{m_1}\circ\cdots\circ\Delta_{m_d}$ is the tensor product of the forward difference operators $\Delta_{m_1},\cdots,\Delta_{m_d}$ from Lemma 2.1 and $\mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d}$ is the tensor product of the Bernstein operators $\mathcal{B}_{m_1},\cdots,\mathcal{B}_{m_d}$ in the sense of [1], section 8.4 (i.e., roughly speaking, the operator with index m_i is applied with the i-th of the d components as a variable and all other components remaining fixed). By integration, we thus obtain

$$\begin{aligned} C_B^{\boldsymbol{U}}\left(x_1,\cdots,x_d\right) &= \int_0^{x_d}\cdots\int_0^{x_1} c\left(u_1,\cdots,u_d\right)\,du_1\cdots du_d \\ &= \mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d} F_{\boldsymbol{Z}}\left(x_1,\cdots,x_d\right) \\ &= \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)\prod_{i=1}^{d} B\left(m_i,k_i,x_i\right) \\ &= \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} P\left(\bigcap_{i=1}^{d}\left\{U_i < k_i\right\}\right)\prod_{i=1}^{d} B\left(m_i,k_i,x_i\right) \end{aligned}$$

for $\left(x_1,\cdots,x_d\right)\in\left[0,1\right]^d$, as stated. $\square$

Remark 2.2. Note that the term $\Delta_{m_1,\cdots,m_d} F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)$ in the proof above corresponds – up to an index shift – to the d-th order difference of the d-increasing cdf $F_{\boldsymbol{Z}}$, see, e.g., [17], chapter 6, or [8], Proposition 4.2. For instance, for $d=2$, we obtain

$$\begin{aligned}\Delta_{m_1,m_2} F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\frac{k_2}{m_2}\right) = F_{\boldsymbol{Z}}\left(\frac{k_1+1}{m_1},\frac{k_2+1}{m_2}\right) - F_{\boldsymbol{Z}}\left(\frac{k_1+1}{m_1},\frac{k_2}{m_2}\right)\\ - F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\frac{k_2+1}{m_2}\right) + F_{\boldsymbol{Z}}\left(\frac{k_1}{m_1},\frac{k_2}{m_2}\right).\end{aligned}$$

Remark 2.3. From a probabilistic point of view, in the light of Lemma 2.1, Bernstein copula densities $c_B^{\boldsymbol{U}}(u_1,\cdots,u_d)$ can also be considered as mixtures of densities of random vectors $\boldsymbol{Y}(k_1,m_1,\cdots,k_d,m_d) = \left(Y_{(k_1,m_1)},\cdots,Y_{(k_d,m_d)}\right)$ with independent components which follow beta distributions with parameters k_j+1 and m_j-k_j and density

$$\begin{aligned}f_{Y_{(k_j,m_j)}}(z) &= m_j\binom{m_j-1}{k_j} z^{k_j}(1-z)^{m_j-1-k_j}\\ &= \frac{1}{\mathrm{B}(k_j+1,m_j-k_j)} z^{k_j}(1-z)^{m_j-1-k_j}\end{aligned}$$

for $j=1,\cdots,d$ and $z\in[0,1]$. Here $\boldsymbol{U}$ is the mixing random vector. From an algorithmic point of view, this representation is particularly useful for Monte Carlo simulations with Bernstein copulas.

3 Bernstein and checkerboard copulas

There is also a natural relationship between Bernstein and checkerboard copulas as discussed in [2] , [5] and [6]. We refer to a slightly more general setup here.

Theorem 3.1 and Definition. Under the assumptions of Theorem 2.1 define the intervals $I_{k_1,\cdots,k_d} := \bigtimes_{j=1}^{d}\left(\frac{k_j}{m_j},\frac{k_j+1}{m_j}\right]$ for all possible choices $(k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d} T_i$. Then the function

$$c_{CB}^{\boldsymbol{U}} := \prod_{i=1}^{d} m_i \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} p(k_1,\cdots,k_d)\ \mathbb{1}_{I_{k_1,\cdots,k_d}}$$

is the density of a d-dimensional copula $C_{CB}^{\boldsymbol{U}}$, called *checkerboard copula* (induced by $\boldsymbol{U}$). Similarly as before, $\boldsymbol{U}$ is called the discrete skeleton of the checkerboard copula. Here $\mathbb{1}_A$ denotes the indicator random variable of the set A, as usual.

Proof. The assertion is a direct consequence of the fact that a random vector $\boldsymbol{W}=(W_1,\cdots,W_d)$ follows a checkerboard copula iff the conditional distribution of $\boldsymbol{W}$ given $\boldsymbol{U}$ fulfills the conditions

$$P^{\boldsymbol{W}}(\bullet\,|\,\boldsymbol{U}=(k_1,\cdots,k_d)) = \mathcal{U}(I_{k_1,\cdots,k_d})\ \text{ for all }\ (k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d} T_i,$$

where $\mathcal{U}(I_{k_1,\cdots,k_d})$ denotes the continuous uniform distribution over $I_{k_1,\cdots,k_d}$ and

$$\boldsymbol{U} = (k_1, \cdots, k_d) \;\; \Leftrightarrow \;\; \boldsymbol{W} \in I_{k_1,\cdots,k_d} \text{ for all } (k_1, \cdots, k_d) \in \bigtimes_{i=1}^{d} T_i$$

(i.e., $\boldsymbol{U}$ denotes in some sense the "coordinates" of $\boldsymbol{W}$ w.r.t. the grid induced by $I_{k_1,\cdots,k_d}$). □

Remark 3.1. The Bernstein copula induced by $\boldsymbol{U}$ can be regarded as a naturally smoothed version of the checkerboard copula induced by $\boldsymbol{U}$, replacing the discontinuous indicator functions

$$\mathbb{1}_{I_{k_1,\cdots,k_d}}(u_1, \cdots, u_d) = \prod_{i=1}^{d} \mathbb{1}_{\left(\frac{k_i}{m_i}, \frac{k_i+1}{m_i}\right]}(u_i)$$

by the continuous polynomials

$$\prod_{i=1}^{d} B(m_i - 1, k_i, u_i), \;\; (u_1, \cdots, u_d) \in [0,1]^d.$$

Theorem 3.2 (Approximation Theorem)**.** Every copula C in d dimensions can be uniformly approximated by a sequence $\left\{C_{CB,r}^{\boldsymbol{U}_r}\right\}_{r\in\mathbb{N}}$ of checkerboard copulas with grid constants $m_{r1}, \ldots, m_{rd} \in \mathbb{N}$, if $\min_{1\le k\le d}\{m_{rk}\}$ tends to infinity when r tends to infinity. If C is the cdf of the random vector $\boldsymbol{Z} = (Z_1, \cdots, Z_d)$ an admissible choice of the discrete skeletons $\boldsymbol{U}_r$, $r \in \mathbb{N}$ is given by the random vectors $\boldsymbol{U}_r = (U_{r1}, \cdots, U_{rd})$ with $U_{rj} := \lceil m_{rj} \cdot Z_j - 1\rceil$ for $j = 1, \cdots, d$ where $\lceil z\rceil := \min\{k \in \mathbb{Z} \,|\, z \le k\}$ for $z \in \mathbb{R}$ (rounding upwards). In this case,

$$\begin{aligned} p_r(k_1, \cdots, k_d) &= P\left(\bigcap_{i=1}^{d} \{U_{ri} = k_i\}\right) = P\left(\bigcap_{j=1}^{d} \left\{\frac{k_j}{m_{rj}} < Z_j \le \frac{k_j + 1}{m_{rj}}\right\}\right) \\ &= P\left(\boldsymbol{Z} \in I_{k_1,\cdots,k_d}\right) \end{aligned}$$

for all $(k_1, \cdots, k_d) \in \bigtimes_{i=1}^{d} T_{ri}$.

Proof. The statement Theorem 3.2 as well as the following Corollary 3.1 follows from a straight-forward extension of the two-dimensional case discussed in [8], section 5. □

Corollary 3.1. Every copula C in d dimensions can be uniformly approximated by a sequence $\left\{C_{B,r}^{\boldsymbol{U}_r}\right\}_{r\in\mathbb{N}}$ of Bernstein copulas with discrete skeletons and grid constants $m_{r1}, \ldots, m_{rd} \in \mathbb{N}$, if $\min_{1\le k\le d}\{m_{rk}\}$ tends to infinity when r tends to infinity. The discrete skeletons may be chosen identically as in the checkerboard copula approximation.

The practical importance of Theorem 3.2 lies in the fact that the Monte Carlo simulation of – especially high dimensional – copulas is generally difficult, while a simulation of checkerboard copulas is comparatively easy.

4 Bernstein and rook copulas

In most practical applications, e.g., when modelling financial portfolios containing different stocks and derivatives or insurance portfolios with different types of risk, the stochastic dependence structure of the various model variables is not explicitly known, see, e.g., [9], [13] and [14] for numerous examples. In such situations, assumptions on the class of corresponding (parametric) copula families are sometimes made on the basis of statistical tests. Alternatively, a non-parametric approach could be chosen, for instance identifying the discrete skeleton of a checkerboard or Bernstein copula directly via the observed data. A major problem here is to find a suitable contingency table since the marginal distributions must be discretely uniform, which means that a set of side conditions has to be fulfilled. Also, this approach becomes ineffective for higher dimensions d, since in general $\prod_{i=1}^{d} m_i$ real numbers have to be stored in order to describe the distribution of the discrete skeleton completely. Such problems are completely avoided if so-called rook copulas are used for modelling the discrete skeleton.

A rook copula is a particular checkerboard copula with the same grid size in each dimension that distributes probability mass according to the placement of rooks on a checkerboard without mutual threatening. It can in general be constructed in d dimensions as follows. Let

$$M := \begin{bmatrix} \sigma_{01} & \sigma_{02} & \cdots & \sigma_{0,d-1} & \sigma_{0d} \\ \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1,d-1} & \sigma_{1d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{m-2,1} & \sigma_{m-2,2} & \cdots & \sigma_{m-2,d-1} & \sigma_{m-2,d} \\ \sigma_{m-1,1} & \sigma_{m-1,2} & \cdots & \sigma_{m-1,d-1} & \sigma_{m-1,d} \end{bmatrix}$$

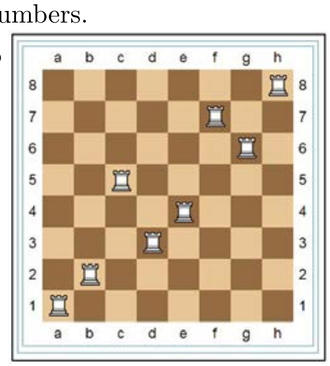
denote a matrix of permutations in column vector notation, i.e. each column $(\sigma_{0k}, \sigma_{1k}, \cdots, \sigma_{m-1,k})$ is a permutation of the set $T := \{0, 1, \cdots, m-1\}$ for $k = 1, \cdots, d$. A checkerboard copula C is a rook copula iff there holds

$$p_m(k_1, \cdots, k_d) = P\left(\bigcap_{i=1}^{d} \{U_i = k_i\}\right) = \frac{1}{m}$$
$$\Leftrightarrow (k_1, \cdots, k_d) = (\sigma_{t1}, \sigma_{t2}, \cdots, \sigma_{t,d}) \text{ for some } t \in T.$$

The distribution of the discrete skeleton of a rook copula can thus be completely described by storing just $m \cdot d$ instead of m^d real numbers.

Example 4.1. The rook copula corresponding to the picture on the right is given by the matrix

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 4 & 2 & 3 & 6 & 5 & 7 \end{bmatrix}^T.$$



In practical applications, in the case of continuous distributions, the permutation matrix pertaining to a rook copula can directly be extracted from the ranks of the observed random vectors according to the following procedure. Given a matrix $\boldsymbol{x} = [x_{ij}]$ of data, where $i = 1, \cdots, n$ is the i-th out of n independent d-dimensional observation row vectors and $j = 1, \cdots, d$ is the corresponding component (dimension) index:

- For each j, calculate the rank r_{ij} of the observation x_{ij} among $x_{1j}, \cdots, x_{nj}$ for $i = 1, \cdots, n$.
- Form the matrix $M := [(r_{ij} - 1)]$ of permutations for the empirical rook copula.

W.r.t. Monte Carlo simulations, it is extremely easy to generate samples that follow either a rook copula or a Bernstein copula with the same discrete skeleton. For simplicity, we explain the procedure by means of the following example only.

Example 4.2. The following table contains some original data (x_{i1}, x_{i2}), $i = 1, \cdots, 20$ from an insurance portfolio of storm and flooding losses, observed over a period of 20 years, their ranks and the permutation matrix M.

i	x_{i1}	x_{i2}	r_{i1}	r_{i2}	M	
1	0.468	0.966	4	9	3	8
2	9.951	2.679	20	20	19	19
3	0.866	0.897	8	4	7	3
4	6.731	2.249	19	19	18	18
5	1.421	0.956	13	8	12	9
6	2.040	1.141	17	15	16	14
7	2.967	1.707	18	18	17	17
8	1.200	1.008	11	10	10	9
9	0.426	1.065	3	12	2	11
10	1.946	1.162	15	16	14	15
11	0.676	0.918	5	6	4	5
12	1.184	1.336	10	17	9	16
13	0.960	0.933	9	7	8	6
14	1.972	1.077	16	13	15	12
15	1.549	1.041	14	11	13	10
16	0.819	0.899	6	5	5	4
17	0.063	0.710	1	1	0	0
18	1.280	1.118	12	14	11	13
19	0.824	0.894	7	3	6	2
20	0.227	0.837	2	2	1	1

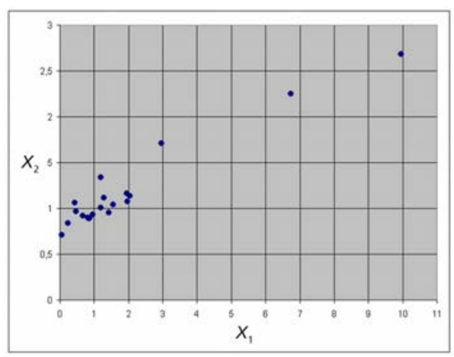


Figure 1: Scatterplot of observed risks x_{i1} and x_{i2} (in million euros)

In the first step, we draw a pair $(\sigma_{i1}, \sigma_{i2})$ out of M with equal probability $\frac{1}{m} = \frac{1}{20}$ w.r.t. the index $i \in \{0, \cdots, m-1\} = \{0, \cdots, 19\}$. In the second step, we either draw a sample $\boldsymbol{Z} = (Z_1, Z_2)$ from a continuous uniform distribution over the rectangle $I_{\sigma_{i1}, \sigma_{i2}} = \left[\frac{\sigma_{i1}}{m}, \frac{\sigma_{i1}+1}{m}\right] \times \left[\frac{\sigma_{i2}}{m}, \frac{\sigma_{i2}+1}{m}\right]$ for the rook copula, or a sample $\boldsymbol{Z} = (Z_1, Z_2)$ with independent components where Z_j follows a beta distribution with parameters $\sigma_{ij} + 1$ and $m - \sigma_{ij}$, $j \in \{1, 2\}$.

A generalization of the procedure to arbitrary dimensions, replacing the rectangle $I_{\sigma_{i1}, \sigma_{i2}}$ by a general cube, is obvious.

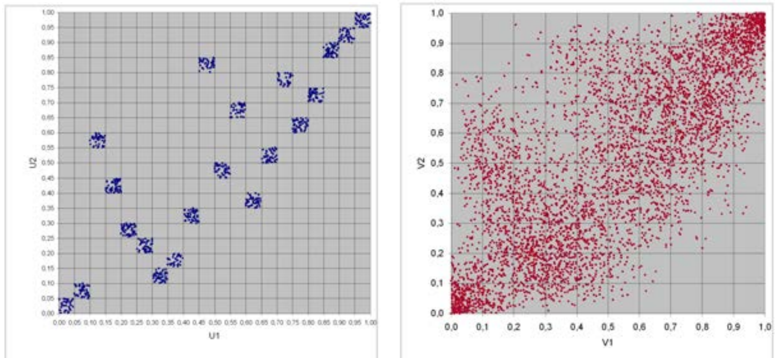


Figure 2: 5000 simulated random vectors following the rook copula (left) and the Bernstein copula (right)

Note that according to a fundamental theorem in statistics, the empirical distribution function of a multivariate observation converges uniformly to the true cdf when the sample size increases. Likewise, the empirical copula based on the extracted marginal ranks converges uniformly to the true underlying copula. This implies that with an increasing number of observed data, the rook copulas as well as the Bernstein copulas with the discrete skeletons derived from the marginal ranks converge to the true underlying copula as well, since in both cases the grid constant m corresponds to the sample size.

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Blended Fejer-type Approximation

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Dedicated to Prof Dr. H. H. Gonska on the occasion of his 65th birthday

Babuska introduced the concept of periodic Hilbert spaces in studying optimal approximation of linear functionals. We used these spaces to study the approximation properties of trigonometric interpolation and periodic spline interpolation [1,4,8] .
We will continue the investigation of approximation by generalized Fourier partial sums constructed by Boolean methods [3,6] . We will consider the construction of bivariate periodic Hilbert spaces . In these spaces we will consider bivariate Fejer opertors . In particular we will introduce approximately blended Fejer operators and study their approximation order.

AMS 2010 subject classification : 42A10, 42A24, 41A35
Key words and phrases : Boolean sum, Fejer operator, periodic Hilbert space, approximation order

1 Periodic Hilbert spaces

We denote by $l_\infty(\mathbf{Z}^n)$ the linear space of bounded discrete complex-valued functions on $\mathbf{Z}^n$ with norm

$$\|F\|_\infty = \sup_{k\in\mathbf{Z}^n} |F_k| < \infty.$$

The linear subspace of summable discrete functions is denoted by $l_1(\mathbf{Z}^n)$ with norm

$$\|F\|_1 = \sum_{k\in\mathbf{Z}^n} |F_k| < \infty$$

Any $F \in l_1(\mathbf{Z}^n)$ is related to an absolutely convergent Fourier series:

$$f(x) = \sum_{k\in\mathbf{Z}^n} F_k e^{ixk}.$$

The associated function f is an element of the algebra $C(\mathbf{T}^n)$ of continuous periodic function with maximum norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbf{T}^n\}.$$

The *Wiener algebra* $A(\mathbf{T}^n)$ is the linear subspace of $C(\mathbf{T}^n)$ of those functions with absolutely convergent Fourier series. The norm for $f \in A(\mathbf{T}^n)$ is defined by

$$\|f\|_a = \sum_{k\in\mathbf{Z}^n} |F_k| \ .$$

Here the *finite Fourier transform* recovers F from f :

$$F_k = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} f(x) e^{-ixk} dx \ .$$

Note the continuous imbeddings of spaces

$$A(\mathbf{T}^n) \subset C(\mathbf{T}^n) \subset L_2(\mathbf{T}^n) \ : \ \|f\|_2 \le \|f\|_\infty \le \|f\|_a \, .$$

are true where the norm for $f \in L_2(\mathbf{T}^n)$ is given by

$$\|f\|_2 = \left(\frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} |f(x)|^2 dx \right)^{\frac{1}{2}} = \left(\sum_{k\in\mathbf{Z}^n} |F_k|^2 \right)^{\frac{1}{2}} .$$

Periodc Hilbert spaces are subspaces of the Wiener algebra $A(\mathbf{T}^n)$. The *defining positive summable discrete function* $D \in l_1(\mathbf{Z}^n)$ describes the smoothness of the functions of the *periodic Hilbert space* :

$$H_D(\mathbf{T}^n) := \{f \in L_2(\mathbf{T}^n) : \sum_{k\in\mathbf{Z}^n} |F_k|^2 / D_k < \infty\} \ .$$

Note that the associated *generating function*

$$d(x) = \sum_{k\in\mathbf{Z}^n} D_k e^{ixk}$$

and its translates $d(\cdot - c)$ are elements from $H_D(\mathbf{T}^n)$.

The univariate periodic *Sobolev space* $W^q(\mathbf{T}) = H_D(\mathbf{T})$ is a simple example with defining function

$$D_k = \frac{1}{k^{2q}}, \quad D_0 = 1, \quad q \ge 1 \ .$$

The function d is given by

$$d(x) = 1 + (-1)^q B_{2q}(x)$$

2

where

$$B_{2q}(x) = \sum_{|k|>0} (ik)^{-2q} e^{ixk}, q \geq 1,$$

is the periodically extended *Bernoulli polynomial* .

The construction of *periodic tensor product Hilbert space*

$$H_{D\otimes D}(\mathbf{T}^2) = H_D(\mathbf{T}) \otimes H_D(\mathbf{T})$$

is based on the defining discrete function

$$(D \otimes D)_{(k_1,k_2)} = D_{k_1} D_{k_2}$$

The associated generating function is of tensor product-type:

$$d_{D\otimes D}(x_1, x_2) = (d \otimes d)(x_1, x_2) = d(x_1)d(x_2) \quad .$$

In particular for the special choice $D_k = \frac{1}{k^{2q}}, \quad D_0 = 1, \quad q \geq 1$, we obtain the tensor product Sobolev space $W^q(\mathbf{T}) \otimes W^q(\mathbf{T})$.

2 Generalized Fourier sums

We will investigate approximation properties of *generalized Fourier sum* (*Fourier means*) of $f \in L_2(\mathbf{T}^n)$ defined by

$$S_\psi(f)(x) = \sum_{k\in\mathbf{Z}^n} \psi_k F_k e^{ixk} \ , \ \psi \in l_\infty(\mathbf{Z}^n, [0,1]) \ .$$

Theorem 1 *S_ψ is a bounded linear operator from $H_D(\boldsymbol{T}^n)$ into $L_2(\boldsymbol{T}^n)$:*

$$\|S_\psi(f)\|_2 \leq \|f\|_D \left\|\sqrt{D}\psi\right\|_\infty .$$

Proof: We apply Parseval's equality :

$$\|S_\psi(f)\|_2^2 = \sum_{k\in\mathbf{Z}^n} |F_k\psi_k|^2 = \sum_{k\in\mathbf{Z}^n} \frac{|F_k|^2}{D_k}|\psi_k|^2 D_k$$

$$\leq \sum_{k\in\mathbf{Z}^n} \frac{|F_k|^2}{D_k} \cdot \sup_l |\psi_l^2 D_l| = \|f\|_D^2 \left\|\psi^2 D\right\|_\infty \leq \|f\|_D^2 \left\|\sqrt{D}\psi\right\|_\infty^2 .$$

The preceeding result is used to obtain error bounds. Replacing ψ by $1-\psi$ we obtain

Theorem 2 *Assume $f \in H_D(\mathbf{T^n})$. Then*

$$\|f - S_\psi(f)\|_2 \leq \|f\|_D \left\|\sqrt{D}(1-\psi)\right\|_\infty \quad .$$

3

The quantity $\left\|\sqrt{D}(1-\psi)\right\|_\infty$ describes the *approximation error in the mean square norm* . The classical example is given by the *Fourier partial sum* related to

$$\psi_k = \left[1 - \left|b^{-1}k\right|\right]_+^0 .$$

In this case we have

$$S_\psi(f)(x) = \sum_{k\in\mathbf{Z}} \left[1 - \left|b^{-1}k\right|\right]_+^0 F_k e^{ixk} = \sum_{|k|\le b} F_k e^{ixk} =: S_b(f)(x).$$

The approximation error in the mean square norm is determined by

$$\left\|\sqrt{D}(1-\psi)\right\|_\infty = \sup_{|k|>b} \sqrt{D_k} = \sup_{|k|>b} |k|^{-q} \le b^{-q}.$$

Theorem 3 *For $f \in W^q(\boldsymbol{T})$ we have*

$$\|f - S_b(f)\|_2 \le \|f\|_D \, b^{-q}, \;\; q \ge 1.$$

The next classical example is given by the *Fejer sum* related to

$$\psi_k = \left[1 - \left|b^{-1}k\right|\right]_+^1 .$$

In this case we have

$$S_\psi(f)(x) = \sum_{|k|<b} [1 - |b^{-1}k|] F_k e^{ixk} =: F_b(f)(x).$$

Again, the mean square error for Fejer sum F_b is determined by

$$\left\|\sqrt{D}(1-\psi)\right\|_\infty = \sup\{\sup_{0<|k|<b} \left|b^{-1}k\right| |k|^{-q}, \; \sup_{|k|\ge b} |k|^{-q}\} \le b^{-1}, \;\; q \ge 1 \; .$$

Theorem 4 *For $f \in W^q(\boldsymbol{T})$ we have*

$$\|f - F_b(f)\|_2 \le \|f\|_D \, b^{-1}, \quad q \ge 1 \; .$$

To compare the operators S_b and F_b we note that the error for S_b decreases by enlarging both b, q while the error for F_b decreases only by enlarging b while the order of approximation of F_b is 1 and does not depend on the degree of smoothness q .

On the other hand the Fejer sum operator F_b is bounded on $C(\mathbf{T})$ in contrast to the Fourier partial sum operator S_b . As a consequence as is well known for any $f \in C(\mathbf{T})$ $F_b(f)$ converges uniformly to f as b tends to infinity .

To improve the approximation order of F_b we use Boolean sum techniques ([3],[6], [7]) .

3 Boolean constructions

The set of discrete functions $l_\infty(\mathbf{Z}^n,[0,1]) = \{\psi \in l_\infty(\mathbf{Z}^n) : 0 \le \psi_k \le 1\}$ which is essential in the definition of S_ψ possesses algebraic properties. Given $\psi_1, \psi_2 \in l_\infty(\mathbf{Z}^n,[0,1])$ we have

$$\psi_1 \cdot \psi_2 \in l_\infty(\mathbf{Z}^n,[0,1]) \ ,$$

$$\psi_1 \oplus \psi_2 := \psi_1 + \psi_2 - \psi_1 \cdot \psi_2 \in l_\infty(\mathbf{Z}^n,[0,1])$$

in view of

$$0 \le \psi_1 + (1-\psi_1)\psi_2 \le 1.$$

The related operators are given by

$$S_{\psi_1} S_{\psi_2} = S_{\psi_1\psi_2} \quad , \quad S_{\psi_1} \oplus S_{\psi_2} := S_{\psi_1} + S_{\psi_2} - S_{\psi_1} S_{\psi_2} = S_{\psi_1 \oplus \psi_2}.$$

$S_{\psi_1\psi_2}$ is the called the *product operator* while $S_{\psi_1\oplus\psi_2}$ is the called the *Boolean sum operator* . Since $I = S_1$ is the identity operator the *remainder operator* of S_ψ is given by

$$I - S_\psi = S_{1-\psi}.$$

Theorem 5 *The mean square error of the product operator $S_{\psi_1} S_{\psi_2}$ is described by*

$$\|f - S_{\psi_1} S_{\psi_2}(f)\|_2 \le \|f\|_D \left(\left\| \sqrt{D}(1-\psi_1) \right\|_\infty + \left\| \sqrt{D}(1-\psi_2) \right\|_\infty \right).$$

Proof: An application of Theorem 2 yields

$$\|f - S_{\psi_1} S_{\psi_2}(f)\|_2 \le \|f\|_D \left\| \sqrt{D}(1-\psi_1\psi_2) \right\|_\infty \quad .$$

Since

$$1 - \psi_1 \cdot \psi_2 = (1-\psi_1)\,\psi_2 + (1-\psi_2) \ .$$

we can conclude

$$\left\| \sqrt{D}(1-\psi_1\psi_2) \right\|_\infty = \left\| \sqrt{D}((1-\psi_1)\,\psi_2 + (1-\psi_2)) \right\|_\infty$$

$$\le \left\| \sqrt{D}(1-\psi_1)\psi_2 \right\|_\infty + \left\| \sqrt{D}(1-\psi_2) \right\|_\infty \le \left\| \sqrt{D}(1-\psi_1) \right\|_\infty + \left\| \sqrt{D}(1-\psi_2) \right\|_\infty$$

which completes the proof .

Theorem 6 *The mean square error of the Boolean sum operator $S_{\psi_1} \oplus S_{\psi_2}$ is described by*

$$\|f - S_{\psi_1} \oplus S_{\psi_2}(f)\|_2 \le \|f\|_D \left\| \sqrt{D}(1-\psi_1)(1-\psi_2) \right\|_\infty .$$

In particular for $\psi_1 = \psi_2 = \psi$ the estimate

$$\|f - S_\psi \oplus S_\psi(f)\|_2 \le \|f\|_D \left\| \sqrt{D}(1-\psi)^2 \right\|_\infty$$

holds .

Proof: In this case we have

$$1-\psi_1\oplus\psi_2 = 1-\psi_1-\psi_2+\psi_1\psi_2 = (1-\psi_1)(1-\psi_2).$$

By Theorem 2 we obtain

$$\|f - S_{\psi_1}\oplus S_{\psi_2}(f)\|_2 \le \|f\|_D \left\|\sqrt{D}(1-\psi_1)(1-\psi_2)\right\|_\infty \ .$$

As an example we consider the *Boolean Fejer sum* . We have

$$\psi(k)\oplus\psi(k) = 2\psi(k)-\psi(k)^2$$

$$= 2\left[1-\left|b^{-1}k\right|\right]_+^1 - \left[1-\left|b^{-1}k\right|\right]_+^2 = \left[1-\left|b^{-2}k^2\right|\right]_+^1 .$$

Thus we have

$$[F_b\oplus F_b](f)(x) = \sum_{|k|<b}\left[1-\left|b^{-2}k^2\right|\right] F_k e^{ixk} = 2F_b(f) - F_b^2(f))$$

which is a *Riesz means* [2] . The improvement of the approximation order by $F_b\oplus F_b$ compared with that of F_b is described in the following theorem.

Theorem 7 *For* $f\in W^q(\boldsymbol{T})$ *we have*

$$\|f-[F_b\oplus F_b](f)\|_2 \le \|f\|_D\, b^{-2}, \quad q\ge 2 \ .$$

Proof: The mean square error for Boolean Fejer sum $F_b\oplus F_b$ is determined by

$$\left\|\sqrt{D}(1-\psi)^2\right\|_\infty = \sup\{\sup_{0<|k|<b}\left|b^{-2}k^2\right||k|^{-q},\ \sup_{|k|\ge b}|k|^{-q}\}\le b^{-2},\ q\ge 2\ .$$

While the Boolean sum operator for the Fejer operator improves the approximation order in the case of higher smoothnes this procedure does not work for the Fourier partial sum operator since $S_b\oplus S_b = S_b$.

As the Fejer operator itself the Boolean sum operator $F_b\oplus F_b = 2F_b - F_b^2$ is bounded on $C(\mathbf{T})$. As a consequence for any $f\in C(\mathbf{T})$ $F_b\oplus F_b(f)$ converges uniformly to f as b tends to infinity .

4 Bivariate Fejer approximation

Blending approximation is one tool to extend univariate approximation methods to the bivariate situation. As the periodic Hilbert space of bivariate functions we choose the *periodic tensor product Hilbert space*

$$H_{D\otimes D}(\mathbf{T}^2) = H_D(\mathbf{T})\otimes H_D(\mathbf{T}).$$

The defining discrete function is given by

$$(D \otimes D)_{(k_1,k_2)} = D_{k_1} D_{k_2} \ .$$

We construct the defining $\psi \in l_\infty(\mathbf{Z}^2, [0,1])$ of S_ψ on $L_2(\mathbf{T}^2)$ by using *parametric extensions* of $\varphi, \zeta \in l_\infty(\mathbf{Z}, [0,1])$ wich are defined by

$$(\varphi \otimes 1)_{(k_1,k_2)} = \varphi_{k_1}, \quad (1 \otimes \zeta)_{(k_1,k_2)} = \zeta_{k_2}.$$

Recall that the *tensor product* of $\varphi, \zeta \in l_\infty(\mathbf{Z})$ is defined by

$$(\varphi \otimes \zeta)_{(k_1,k_2)} = \varphi_{k_1} \zeta_{k_2}.$$

Note that $\varphi, \zeta \in l_\infty(\mathbf{Z}, [0,1])$ yields

$$\varphi \otimes \zeta \ = \ (\varphi \otimes 1)(1 \otimes \zeta) \in l_\infty(\mathbf{Z}^2, [0,1]) \ .$$

The associated bivariate generalized Fourier partial sum is given by

$$S_{\varphi \otimes \zeta}(f)(x_1, x_2) = \sum_{k_1, k_2 \in \mathbf{Z}} \varphi_{k_1} \zeta_{k_2} F_{(k_1,k_2)} e^{i(k_1 x_1 + k_2 x_2)} \ .$$

We introduce the *tensor product operator* notation :

$$S_\varphi \otimes S_\zeta = S_{\varphi \otimes \zeta} = S_{\varphi \otimes 1} S_{1 \otimes \zeta} \ .$$

Note that $S_\varphi \otimes I = S_{\varphi \otimes 1}, \quad I \otimes S_\zeta = S_{1 \otimes \zeta}, \quad I \otimes I = I$.

Theorem 8 *Assume*

$$f \in H_D(\boldsymbol{T}) \otimes H_D(\boldsymbol{T}) = H_{D \otimes D}(\boldsymbol{T}^2) \ .$$

Then the mean square error estimate for the tensor product operator is described by

$$\|f - S_\varphi \otimes S_\varphi(f)\|_2 \leq \|f\|_{D \otimes D} \|\sqrt{D}\|_\infty 2 \|(1-\varphi)\sqrt{D}\|_\infty.$$

Proof Since

$$S_{\varphi \otimes \varphi} = S_{\varphi \otimes 1} S_{1 \otimes \varphi}$$

we apply Theoerem 5 :

$$\|f - S_{\varphi \otimes 1} S_{1 \otimes \varphi}(f)\|_2$$

$$\leq \|f\|_{D \otimes D} \left(\left\| \sqrt{D \otimes D}(1 - \varphi \otimes 1) \right\|_\infty + \left\| \sqrt{D \otimes D}(1 - 1 \otimes \varphi) \right\|_\infty \right).$$

Now we have ($1 \otimes 1 = 1$)

$$\left\| \sqrt{D \otimes D}(1 - \varphi \otimes 1) \right\|_\infty = \left\| \sqrt{D}(1 - \varphi)) \right\|_\infty \left\| \sqrt{D} \right\|_\infty \ ,$$

$$\left\|\sqrt{D\otimes D}(1-1\otimes\varphi)\right\|_\infty = \left\|\sqrt{D}(1-\varphi))\right\|_\infty \left\|\sqrt{D}\right\|_\infty \ ,$$

which completes the proof.

Next we consider the Boolean sum of $\varphi\otimes 1,\ 1\otimes\varphi$ is given by

$$(\varphi\otimes 1)\oplus(1\otimes\varphi) = \varphi\otimes 1 \ + \ 1\otimes\varphi \ - \ \varphi\otimes\varphi \ \in l_\infty(\mathbf{Z}^2,[0,1]) \ .$$

In this case the associated generalized Fourier partial sum operator

$$S_{(\varphi\otimes 1)\oplus(1\otimes\varphi)} = S_{\varphi\otimes 1} + S_{1\otimes\varphi} - S_{\varphi\otimes\varphi} = S_\varphi\otimes I + I\otimes S_\varphi - S_\varphi\otimes S_\varphi \ .$$

is called the *Blending operator* . We also use the Boolean sum notation for the Blending operator :

$$(S_\varphi\otimes I)\oplus(I\otimes S_\varphi) := S_\varphi\otimes I + I\otimes S_\varphi - S_\varphi\otimes S_\varphi .$$

The Fourier series representation is given by the bivariate series

$$\begin{aligned}&(S_\varphi\otimes I)\oplus(I\otimes S_\varphi)(f)(x_1,x_2)\\ &= \sum_{k_1,k_2\in\mathbf{Z}} (\varphi_{k_1}+\varphi_{k_2}-\varphi_{k_1}\varphi_{k_2})F_{(k_1,k_2)}e^{i(k_1x_1+k_2x_2)}\end{aligned}$$

Theorem 9 *Assume*

$$f\in H_D(\boldsymbol{T})\otimes H_D(\boldsymbol{T}) = H_{D\otimes D}(\boldsymbol{T}^2) \ .$$

Then the mean square error estimate of the Blending operator is described by

$$\|f-(S_\varphi\otimes I)\oplus(I\otimes S_\varphi)(f)\|_2 \le \|f\|_{D\otimes D}\|(1-\varphi)\sqrt{D}\|_\infty^2 .$$

Proof: Since

$$(S_\varphi\otimes I)\oplus(I\otimes S_\varphi) = S_{\varphi\otimes 1}\oplus S_{1\otimes\varphi}$$

we apply Theorem 6 :

$$\begin{aligned}&\|f-S_{\varphi\otimes 1}\oplus S_{1\otimes\varphi}(f)\|_2\\ &\le \|f\|_{D\otimes D}\left\|\sqrt{D\otimes D}(1-\varphi\otimes 1)(1-1\otimes\varphi)\right\|_\infty .\end{aligned}$$

Again we have ($1\otimes 1 = 1$)

$$\begin{aligned}&\left\|\sqrt{D\otimes D}(1-\varphi\otimes 1)(1-1\otimes\varphi)\right\|_\infty\\ &= \left\|\sqrt{D\otimes D}(1-\varphi)\otimes(1-\varphi)\right\|_\infty = \left\|\sqrt{D}(1-\varphi)\right\|_\infty^2 \ .\end{aligned}$$

We next consider a finite-dimensional version of the *transfinite* Blending operator $(S_\varphi \otimes I) \oplus (I \otimes S_\varphi)$. The *approximate Blending operator* is defined by

$$(S_\varphi \otimes S_{\varphi'}) \oplus' (S_{\varphi'} \otimes S_\varphi) := S_\varphi \otimes S_{\varphi'} + S_{\varphi'} \otimes S_\varphi - S_\varphi \otimes S_\varphi$$

assuming the relations $0 \leq \varphi \leq \varphi' \leq 1$. We have

$$(S_\varphi \otimes S_{\varphi'}) \oplus' (S_{\varphi'} \otimes S_\varphi) = S_{\varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi} .$$

Note that

$$0 \leq \varphi \otimes \varphi \leq \varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi \leq \varphi \otimes 1 + 1 \otimes \varphi - \varphi \otimes \varphi \leq 1 .$$

The Fourier series representation of the approximate Blending operator is given by

$$\begin{aligned} &(S_\varphi \otimes S_{\varphi'}) \oplus' (S_{\varphi'} \otimes S_\varphi)(f)(x_1, x_2) \\ &= \sum_{k_1, k_2 \in \mathbf{Z}} [\varphi_{k_1} \varphi'_{k_2} + \varphi'_{k_1} \varphi_{k_2} - \varphi_{k_1} \varphi_{k_2}] F_{(k_1, k_2)} e^{i(k_1 x_1 + k_2 x_2)} . \end{aligned}$$

Choosing $\varphi,\ \varphi'$ with *bounded support* we obtain a *bivariate trigonometric polynomial* .

Theorem 10 *Assume*

$$f \in H_D(\boldsymbol{T}) \otimes H_D(\boldsymbol{T}) = H_{D \otimes D}(\boldsymbol{T}^2) .$$

Then the mean square error estimate of the approximate Blending operator is described by

$$\begin{aligned} &\|f - (S_\varphi \otimes S_{\varphi'}) \oplus' (S_{\varphi'} \otimes S_\varphi)(f)\|_2 \\ &\leq \|f\|_{D \otimes D} [\|(1-\varphi)\sqrt{D}\|_\infty^2 + \|\sqrt{D}\|_\infty 2 \|(1-\varphi')\sqrt{D}\|_\infty] . \end{aligned}$$

Proof: By Theorem 2 we have

$$\begin{aligned} &\|f - (S_\varphi \otimes S_{\varphi'}) \oplus' (S_{\varphi'} \otimes S_\varphi)(f)\|_2 \\ &\leq \|f\|_{D \otimes D} \|\sqrt{D \otimes D}[1 - (\varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi)]\|_\infty . \end{aligned}$$

Since

$$\begin{aligned} &\varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi \\ &= \varphi \otimes 1 + 1 \otimes \varphi - \varphi \otimes \varphi - \varphi \otimes (1-\varphi') - (1-\varphi') \otimes \varphi \end{aligned}$$

implies

$$\begin{aligned} &1 - [\varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi] \\ &= (1-\varphi) \otimes (1-\varphi) + \varphi \otimes (1-\varphi') + (1-\varphi') \otimes \varphi. \end{aligned}$$

we can conclude

$$\|\sqrt{D \otimes D}[1 - (\varphi \otimes \varphi' + \varphi' \otimes \varphi - \varphi \otimes \varphi)]\|_\infty$$

$$\le \|\sqrt{D\otimes D}[(1-\varphi)\otimes(1-\varphi)]\|_\infty$$
$$+\|\sqrt{D\otimes D}[\varphi\otimes(1-\varphi')]\|_\infty+\|\sqrt{D\otimes D}[(1-\varphi')\otimes\varphi]\|_\infty$$
$$\le \|\sqrt{D}(1-\varphi)\|_\infty^2+2\|\sqrt{D}(1-\varphi')\|_\infty\|\sqrt{D}\varphi\|_\infty$$

which completes the proof .

We appy the preceeding results to the classical *Fejer sum* :

$$F_b(f)(x)=\sum_{|k|<b}[1-|b^{-1}k|]F_k e^{ixk} \ .$$

The *tensor product Fejer sum* is given by

$$F_b\otimes F_b(f)(x_1,x_2)=$$
$$\sum_{|k_1|<b}\sum_{|k_2|<b}[1-|b^{-1}k_1|]_+[1-|b^{-1}k_2|]_+F_{(k_1,k_2)}e^{i(x_1k_1+x_2k_2)} \ .$$

while the *blended Fejer sum* is given by the bivariate series

$$(F_b\otimes I)\oplus(I\otimes F_b)(f)(x_1,x_2)$$
$$=\sum_{k_1,k_2\in\mathbf{Z}}\left([1-|b^{-1}k_1|]_+ + [1-|b^{-1}k_2|]_+\right)F_{(k_1,k_2)}e_{k_1}(x_1)e_{k_2}(x_2)$$
$$-\sum_{k_1,k_2\in\mathbf{Z}}[1-|b^{-1}k_1|]_+[1-|b^{-1}k_2|]_+F_{(k_1,k_2)}e_{k_1}(x_1)e_{k_2}(x_2)$$

An application of Theorem 8 and Theorem 9 yields

Theorem 11 *Assume*
$$f\in W^q(\boldsymbol{T})\otimes W^q(\boldsymbol{T}) \ .$$
The mean square error for the tensor product Fejer sum is described by

$$\|f-F_b\otimes F_b(f)\|_2\le 2\|f\|_{D\otimes D}\ b^{-1}$$

while the mean square error for the blended Fejer sum is described by

$$\|f-(F_b\otimes I)\oplus(I\otimes F_b)(f)\|_2\le \|f\|_{D\otimes D}\ b^{-2}$$

with $D_k=\frac{1}{k^{2q}}, \quad D_0=1, \quad q\ge 1$.

The *approximately blended Fejer sum* is given by the bivariate trigonometric polynomial

$$(F_b\otimes F_{b^2})\oplus'(F_{b^2}\otimes F_b)(f)(x_1,x_2)$$
$$=\sum_{|k_1|<b}\sum_{|k_2|<b^2}[1-|b^{-1}k_1|][1-|b^{-2}k_2|]F_{(k_1,k_2)}e^{i(x_1k_1+x_2k_2)}$$

$$+\sum_{|k_1|<b^2}\sum_{|k_2|<b}[1-|b^{-2}k_1|][1-|b^{-1}k_2|]F_{(k_1,k_2)}e^{i(x_1k_1+x_2k_2)}$$
$$-\sum_{|k_1|<b}\sum_{|k_2|<b}[1-|b^{-1}k_1|][1-|b^{-1}k_2|]F_{(k_1,k_2)}e^{i(x_1k_1+x_2k_2)}\ .$$

An application of Theorem 10 now yields

Theorem 12 *Assume*

$$f \in W^q(\boldsymbol{T}) \otimes W^q(\boldsymbol{T})\ .$$

The mean square error for the approximately blended Fejer sum is described by

$$\|f-(F_b \otimes F_{b^2}) \oplus' (F_{b^2} \otimes F_b)(f)\|_2 \leq 3\|f\|_{D\otimes D}\ b^{-2}$$

with $D_k = \frac{1}{k^{2q}}$, $D_0 = 1$, $q \geq 1$.

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An answer to a conjecture on positive linear operators

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

We answer and generalize an open problem of Tachev on the bounds of the remainder term in the Bernstein approximation and give a positive answer to a conjecture of Cao, Gonska and Kacsó concerning positive linear operators.

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1 Introduction

Let $f\colon[0,1]\to\mathbb{R}$. For any $n\in\mathbb{N}$ the Bernstein operator is defined by

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),\qquad x\in[0,1].$$

In what follows we shall denote by g the function $g\colon[0,1]\to\mathbb{R}$,

$$g(x)=\begin{cases}0, & x=0,\\ x\,\ln x+(1-x)\,\ln(1-x), & x\in(0,1),\\ 0, & x=1.\end{cases}$$

The function g appears in numerous problems in approximation theory (see, e.g., [1], [2], [3], [8], [7], [11], [12]).

From [1, Lemma 3, Eq. (18)] Berens and Lorentz obtained the following estimate for the uniform approximation of g by B_ng:

$$\|B_ng-g\|\le\frac{7}{n},\quad n=1,2,\dots \tag{1}$$

1

Let us consider the remainder term in the Bernstein approximation of the function g,

$$R_n(g,x) = B_n(g,x) - g(x), \qquad x \in [0,1].$$

In 2002, Tachev proposed the following open problem:

Open Problem 1 (Tachev [11]) *Find the best constants $k_1, k_2 > 0$, α_1, α_2, a_1, a_2 and β, b, such that*

$$k_1 \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n^\beta} \leq R_n(g,x) \leq k_2 \frac{x^{a_1}(1-x)^{a_2}}{n^b}, \tag{2}$$

for all $x \in [0,1]$.

In 2002, Lupaş gave an answer to the Open Problem 1 in a particular case:

Theorem 2 (Lupaş [7]) *For all $x \in [0,1]$, the following inequalities hold true.*

$$\frac{x(1-x)}{2n} \leq R_n(g,x) \leq \sqrt{2}\sqrt{\frac{x(1-x)}{n}}.$$

In 2012, Tachev showed that if $\beta = 1$ and $b = \frac{1}{2}$, the best possible constants $\alpha_1, \alpha_2, a_1, a_2$ in (2) are

$$\alpha_1 = \alpha_2 = 1, \quad \text{and} \quad a_1 = a_2 = \frac{1}{2}.$$

More precisely, he proves the following result.

Theorem 3 (Tachev [12]) *It is not possible to find*

$$\alpha_1 < 1 \quad \text{or} \quad \alpha_2 < 1 \quad \text{or} \quad a_1 > \frac{1}{2} \quad \text{or} \quad a_2 > \frac{1}{2}$$

and $k_1,\ k_2 > 0$ such that

$$k_1 \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n} \leq R_n(g,x) \leq k_2 \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}$$

hold true for all $x \in [0,1]$ and $n \in \mathbb{N}$.

The previous results are in closer connection with the following conjecture.

Conjecture 4 (Cao, Gonska and Kacsó [2, 3]) *Let $T_n\colon C[0,1] \to C[0,1]$ be a sequence of linear operators, $\varepsilon_n > 0$, with $\lim_{n\to\infty} \varepsilon_n = 0$, $\varphi(x) = \sqrt{x(1-x)}$, and $0 \leq \beta < \lambda \leq 1$. If, for every $f \in C[0,1]$, one has*

$$|T_n(f,x) - f(x)| \leq C(f)\, \omega_2^{\varphi^\lambda}(f; \varepsilon_n \varphi^{1-\lambda}(x)),$$

where $C(f)$ is a constant dependent only on f, then the lower pointwise estimates

$$c(f)\, \omega_2^{\varphi^\beta}(f; \varepsilon_n \varphi^{1-\lambda}(x)) \leq |T_n(f;x) - f(x)|, \qquad f \in C[0,1],$$

do not hold in general.

Let us recall the definition of the generalized modulus $\omega_2^{\varphi^\lambda}$ (see, e.g., [6, Chap. 2])

$$\omega_2^{\varphi^\lambda}(f;\delta) = \sup_{0\le h\le\delta} \|\Delta^2_{h\varphi^\lambda} f\|,$$

where

$$\Delta^2_{h\varphi^\lambda} f(x) = \begin{cases} f(x - h\varphi^\lambda(x)) - 2f(x) + f(x + h\varphi^\lambda(x)), \\ \qquad\qquad\qquad\qquad 0 \le x - h\varphi^\lambda(x) \le x + h\varphi^\lambda(x) \le 1, \\ 0, \qquad\qquad\qquad\quad \text{otherwise.} \end{cases}$$

By using Theorem 3, Tachev [12] gives a positive answer to Conjecture 4 in the particular case when T_n is the Bernstein operator B_n.

The main tasks of this paper are:

- to establish a relation of the type (2) for any linear positive operator preserving linear functions $T_n\colon C[0,1] \to C[0,1]$;
- to give a solution to the Tachev Open Problem 1;
- to answer the Cao, Gonska and Kacsó Conjecture 4 for all positive linear operators preserving linear functions $T_n\colon C[0,1] \to C[0,1]$.

2 Auxiliary Results

In the following we shall use the following notations:

$I = [0,1]$,

$C(I)$, the Banach space of all continuous functions $f\colon I \to \mathbb{R}$ endowed with the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$,

$e^j(t) = t^j, \quad t \in [0,1], \quad j \in \mathbb{N}$,

$[x_1, \ldots, x_m; f]$, the divided difference of the function f at the specified distinct knots $x_1, \ldots, x_m$, defined by

$$[x_1, \ldots, x_m; f] = \sum_{k=1}^{m} \frac{f(x_k)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_m)}.$$

A function $f\colon I \to \mathbb{R}$ is said to be s-convex (s-concave) on I if for every system $x_1, \ldots, x_{s+2}$ distinct points in I we have $[x_1, \ldots, x_{s+2}; f] > 0$ (< 0).

Following [8], we denote by $K_s(I)$ $(\overline{K}_s(I))$ the cone of all s-convex (s-concave) functions on I.

In what follows we shall need the following results:

Theorem 5 ([8]) *Let $A: C[0,1] \to \mathbb{R}$ be a positive linear functional with $A(e_0) = 1$. If $f \in K_3(I)$, then:*

$$A(f) - f(\bar{a}) \geq \frac{A(e_2) - \bar{a}^2}{2} f''(\bar{z}). \tag{3}$$

where $\bar{a} = A(e_1)$ *and* $\bar{z} = \bar{a} + \dfrac{A\left((e_1 - \bar{a})^3\right)}{3A\left((e_1 - \bar{a})^2\right)}$.

An application of the Jensen inequality to the concave logarithmic function yields

$$\alpha\, x_1 + (1-\alpha)x_2 \geq x_1^{\alpha} x_2^{1-\alpha}, \qquad x_1, x_2 > 0, \quad \alpha \in (0,1),$$

or

$$x_1 + x_2 \geq \frac{x_1^{\alpha}}{\alpha^{\alpha}} \frac{x_2^{1-\alpha}}{(1-\alpha)^{1-\alpha}}, \qquad x_1, x_2 > 0, \quad \alpha \in (0,1). \tag{4}$$

We will also make use of the following result.

Lemma 6 *Let $A: C[0,1] \to \mathbb{R}$ be a positive linear functional with $A(e_0) = 1$. Define the functional $B: C[0,1] \to \mathbb{R}$ by*

$$B(f) := A\left(f(e_0 - e_1)\right).$$

Then, with $\bar{b} := B(e_1)$, *the following equalities hold true.*

$$B(e_0) = 1, \quad B(e_1) = 1 - \bar{a}, \quad B\left((e_1 - \bar{b})^2\right) = A\left((e_1 - \bar{a})^2\right).$$

The proof reduces fairly easily to explicit computations.

3 Main results

The following theorem gives an answer to and generalizes the Open Problem 1 raised by Tachev.

Theorem 7 *Let $\alpha, x \in (0,1)$ and $T_n: C[0,1] \to C[0,1]$ be a positive linear operator preserving the linear functions. Then, the following inequalities hold true.*

$$2\left(T_n(e_2;x) - x^2\right) \leq T_n(g;x) - g(x) \leq \frac{k(\alpha)\left(T_n(e_2;x) - x^2\right)^{\frac{1+\alpha}{2}}}{2^{\alpha-1} x^{\alpha}(1-x)^{\alpha}}, \tag{5}$$

where $k(\alpha) = \alpha^{\frac{\alpha}{2}}(1-\alpha)^{\frac{1-\alpha}{2}}$.

Proof. It is known that $g \in K_3(I)$. Define the linear functional $A(f) := T_n(f;x)$.

If $T_n((e_1 - x)^2; x) = 0$, then $T_n(f;x) = f(x)$ for every $f \in C[0,1]$. and inequality (5) is trivial. Let us suppose that $T_n((e_1 - x)^2; x) > 0$.

Let us prove that $\bar{z} = x + \dfrac{T_n\left((e_1 - x)^3; x\right)}{3T_n\left((e_1 - x)^2; x\right)} \in (0,1)$.

4

First, let us note that $\{e_0,\ e_1,\ e_3\}$ and $\{e_0,\ e_0-e_1,\ (e_0-e_1)^3\}$ are Chebyshev systems on I.

The inequality $\bar{z} > 0$ is equivalent to $T_n(e_3;x) > x^3$. Since e_3 is convex, by Jessen inequality for positive linear functionals, we deduce that

$$T_n(e_3;x) \geq x^3.$$

If $T_n(e_3;x) = x^3$, then the positive linear functional $A\colon[0,1] \to \mathbb{R}$, $A(f) = T_n(f;x)$, satisfies the relations:

$$A(e_0) = 1, \quad A(e_1) = x, \quad A(e_3) = x^3.$$

By Šaškin's ([10]) theorem, we deduce that $A(f) = f(x)$, for all $f \in C[0,1]$, so $T_n((e_1-x)^2;x) = 0$. This is in contradiction with the initial assumption.

The inequality $\bar{z} < 1$ is equivalent to

$$(1-x)^3 < T_n((1-e_1)^3;x).$$

The function $(1-e_1)^3$ is convex. Next, we just follow the same procedure as above and obtain that $T_n((e_1-x)^2;x) = 0$, which contradicts the initial assumption.

Now let us properly begin the proof of inequalities (5). By (3), we get

$$A(g) - g(x) \geq \frac{A(e_2) - x^2}{2} g''(\bar{z}). \tag{6}$$

But $g''(\bar{z}) \geq 4$, and so, from (6) we obtain

$$T_n(g;x) - g(x) \geq 2(T_n(e_2;x) - x^2).$$

The function $g_1\colon[0,1] \to \mathbb{R}$, $g_1(x) = x\ln x$, $x \in (0,1]$, $g(0) = 0$ belongs to $K_1(I) \cap \overline{K}_2(I)$.

In what follows, we need the following inequality (see [8, Corollary 3.2]).

$$A(g_1) - g_1(x) \leq (A(e_2) - x^2)\frac{1}{\sqrt{T_n(e_2;x)}}. \tag{7}$$

By Lemma 6 and (7) we get

$$B(g_1) - g_1(1-x) \leq (A(e_2) - x^2)\frac{1}{\sqrt{(1-x)^2 + T_n(e_2;x) - x^2}} \tag{8}$$

But

$$A(g) - g(x) = A(g_1) - g_1(x) + B(g_1) - g_1(1-x) \tag{9}$$

From (7), (8) and (9) we obtain

$$\begin{aligned} & T_n(g;x) - g(x) && (10)\\ \leq\ & (T_n(e_2;x) - x^2)\left(\frac{1}{\sqrt{x^2 + T_n(e_2;x) - x^2}} + \frac{1}{\sqrt{(1-x)^2 + T_n(e_2;x) - x^2}}\right) \end{aligned}$$

From (4) we obtain

$$x^2 + T_n(e_2;x) - x^2 \geq \frac{x^{2\alpha}}{\alpha^\alpha} \frac{\left(T_n(e_2;x) - x^2\right)^{1-\alpha}}{(1-\alpha)^{1-\alpha}} \tag{11}$$

$$(1-x)^2 + T_n(e_2;x) - x^2 \geq \frac{(1-x)^{2\alpha}}{\alpha^\alpha} \frac{\left(T_n(e_2;x) - x^2\right)^{1-\alpha}}{(1-\alpha)^{1-\alpha}} \tag{12}$$

From (10), (11) and (12), we get

$$T_n(g;x) - g(x) \leq \frac{\left(T_n(e_2;x) - x^2\right)^{\frac{1+\alpha}{2}}}{x^\alpha (1-x)^\alpha} \alpha^{\frac{\alpha}{2}} (1-\alpha)^{\frac{1-\alpha}{2}} \left(x^\alpha + (1-x)^\alpha\right)$$

By using the inequality

$$x^\alpha + (1-x)^\alpha \leq 2^{1-\alpha}, \qquad x \in [0,1],$$

the proof is completed. □

In the case of the Bernstein operator, we deduce the following results.

Corollary 8 *For all $x \in [0,1]$ the following inequalities are satisfied.*

$$2\frac{x(1-x)}{n} \leq B_n(g;x) - g(x) \leq \frac{\alpha^{\frac{\alpha}{2}} (1-\alpha)^{\frac{1-\alpha}{2}}}{2^{\alpha-1}} \frac{x^{\frac{1-\alpha}{2}} (1-x)^{\frac{1-\alpha}{2}}}{n^{\frac{1+\alpha}{2}}}, \tag{13}$$

and

$$2\frac{x(1-x)}{n} \leq B_n(g;x) - g(x) \leq \frac{1}{n}. \tag{14}$$

Proof. Relation (13) follows from (5) because $B_n(e_2;x) - x^2 = x(1-x)/n$. From (13), for $\alpha \nearrow 1$, we obtain (14). □

Remark 9 *We note that (14), already proved in [9, Lemma 3.2], is an improvement of Berens and Lorentz's inequality (1).*

Corollary 10 *The best constant β in (2) is $\beta = 1$.*

Corollary 11 *Let $b \in \left(\frac{1}{2}, 1\right)$. Then*

$$2\frac{x(1-x)}{n} \leq B_n(g;x) - g(x) \leq (2b-1)^{\frac{2b-1}{2}} (1-b)^{1-b} 2^{3(1-b)} \frac{x^{1-b}(1-x)^{1-b}}{n^b}, \tag{15}$$

and $a_1 = a_2 = 1-b$ are the best constants in (2).

Proof. Inequalities (15) follows from (13) for $\alpha = 2b-1$.

Since $g \leq 0$, we have:

$$\begin{aligned} & B_n(g;x) \\ \geq\ & \binom{n}{1} x^1 (1-x)^{n-1} g\left(\frac{1}{n}\right) \geq \binom{n}{1} x\, g\left(\frac{1}{n}\right) \\ =\ & x\left(-\ln n + (n-1)\ln\left(1 - \frac{1}{n}\right)\right) \\ \geq\ & x\left(-\ln n - 1\right), \qquad x \in [0,1]. \end{aligned} \tag{16}$$

Suppose that there exist some constants $a_1 > 1 - b$ and a_2 such that

$$B_n(g;x) - g(x) \le K \frac{x^{a_1}(1-x)^{a_2}}{n^b}, \tag{17}$$

where K is a constant independent of x and n.

Since $g(x) \le x \ln x$, from (16) and (17), we obtain

$$x\,(-\ln n - \ln x - 1) \le B_n(g;x) - g(x) \le K \frac{x^{a_1}(1-x)^{a_2}}{n^b}.$$

For $x = \frac{e^{-a}}{n}$, $a > 1$, the previous inequalities yield

$$(a-1)\, e^{-a}\, n^{b-1+a_1} \le K\, e^{-a\,a_1} \left(1 - \frac{e^{-a}}{n}\right)^{a_2}, \qquad \text{for all } n \in \mathbb{N},\ n > 1,$$

which is not true. $\square$

In what follows, we give a positive answer to Conjecture 4 in the case when T_n are positive linear operators preserving linear functions. To this end we need the following result.

Lemma 12 *Let T_n, $n \in \mathbb{N}$, be positive linear operators preserving the linear functions. If there exist some constants a_1, a_2 and $C > 0$ such that*

$$T_n(g;x) - g(x) \ge C\,\varepsilon_n^2\, x^{a_1}(1-x)^{a_2}, \qquad \text{forall } x \in [0,1] \tag{18}$$

then $a_1, a_2 \ge 1$.

Proof. Let us suppose that inequality (18) is satisfied for some $a_1 < 1$. Since $g(x) \le 0$, we obtain

$$T_n(g;x) - g(x) \le -g(x). \tag{19}$$

From (18) and (19), we get

$$C\,\varepsilon_n^2\, x^{a_1}(1-x)^{a_2} \le -x \ln x - (1-x)\ln(1-x),$$

or

$$C\,\varepsilon_n^2 (1-x)^{a_2} \le -x^{1-a_1} \ln x - (1-x)\frac{\ln(1-x)}{x^{a_1}},$$

which, for $x \searrow 0$, gives $C\,\varepsilon_n^2 \le 0$, which is false. Similarly, for $x \nearrow 1$, we obtain that $a_2 \ge 1$. $\square$

Now we are in position to answer Conjecture 4.

Let us suppose that there exists $C(f) > 0$, $\beta \in [0,1]$ and $\lambda \in (0,1)$ such that

$$|T_n(f;x) - f(x)| \ge C(f)\omega_2^{\varphi^\beta}(f; \varepsilon_n \varphi^{1-\lambda}(x))$$

for all $x \in [0,1]$ and $f \in C[0,1]$. In particular, we have

$$T_n(e_2;x) - x^2 \ge C(e_2)\omega_2^{\varphi^\beta}(e_2; \varepsilon_n \varphi^{1-\lambda}(x)) = C(e_2)\, \frac{x^{1-\lambda}(1-x)^{1-\lambda}}{2^{2\beta-1}}\, \varepsilon_n^2 \tag{20}$$

On the other hand, from (5) we have $T_n(g;x) - g(x) \ge 2(T_n(e_2;x) - x^2)$, and so $T_n(g;x) - g(x) \ge C(e_2)\, \frac{x^{1-\lambda}(1-x)^{1-\lambda}}{2^{2\beta-2}}\, \varepsilon_n^2$, which contradicts Lemma 12.

7

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Approximation by Szász-Mirakyan-Baskakov operators

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

In the present article we find the hypergeometric representation of the Szász-Baskakov operators and obtain the moments using confluent Hypergeometric functions, which can be related to Laguerre polynomials. We study approximation by linear combinations of Szász-Baskakov operators and establish a Voronovskaja-type theorem. The case of weighted approximation is also considered.

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1 Introduction

There are several integral modifications of the well known Szász-Mirakyan operators [13] available in the literature which include the most common modifications due to Kantorovich and Durrmeyer. In the year 1983 Prasad et al. [12] introduced the modification of the Szász-Mirakyan operator by taking the weights of Baskakov basis functions as

$$S_n(f,x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^{\infty} b_{n,k}(t)f(t)dt, x \in \mathbb{R}_+ \equiv [0,\infty) \tag{1.1}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $b_{n,k}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}} = \frac{(n)_k}{k!}\frac{t^k}{(1+t)^{n+k}}$ and $(n)_k$ represents the Pochhammer symbol given by

$$(n)_k = n(n+1)(n+2)(n+3)....(n+k-1).$$

Some approximation properties on such operators were later studied by several researchers Gupta [6] improved the estimates of [12]. In further studies Gupta-Gupta ([9],[10]) considered the simultaneous approximation and they obtained asymptotic formula, error estimation and inverse theorem for these operators. Recently Tuncer et al. [1] estimated the rate of convergence for functions having derivatives of bounded variation. The q analogue of these operators was discussed in [7] and the Stancu variant of q operators was studied in [11]. Deo [4] also claimed to study the inverse theorem for the operators S_n, but he has copied some parts without giving proper citations for example Lemma 2.3 of [4] was copied and repeated the same misprint as in [10].

Alternatively (1.1) with $k! = (1)_k$, can be written as

$$\begin{aligned} S_n(f(t),x) &= (n-1)\int_0^{\infty}\frac{f(t)e^{-nx}}{(1+t)^n}\sum_{k=0}^{\infty}\left(\frac{nxt}{1+t}\right)^k\frac{(n)_k}{(1)_k k!}dt \\ &= (n-1)\int_0^{\infty}\frac{e^{-nx}f(t)}{(1+t)^n}\ {}_1F_1\left(n;1;\frac{nxt}{1+t}\right)dt, \end{aligned}$$

where the function ${}_1F_1$ is known as the Pochhammer-Berens confluent hypergeometric function defined as

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{x^k}{k!}.$$

2 Moments

In this section we establish the moments of Szász-Baskakov operators.

Lemma 1 *For $n > 0$ and $r \geq 0$, we have*

$$S_n(t^r,x) = \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ {}_1F_1(-r;1;-nx). \tag{2.1}$$

Further, we have

$$S_n(t^r,x) = \frac{\Gamma(n-r+1)\Gamma(r+1)}{\Gamma(n-1)}\ L_r(-nx),$$

where $L_r(-nx)$ is the Laguerre polynomials.

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Proof. Substituting $f(t) = t^r$ in (1.1) and using the definition Beta integral, we have

$$S_n(t^r, x) = (n-1)\sum_{k=0}^{\infty} \frac{e^{-nx}(nx)^k}{k!}\frac{(n)_k}{k!}\int_0^{\infty} \frac{t^{k+r}}{(1+t)^{n+k}}dt$$
$$= (n-1)\sum_{k=0}^{\infty} \frac{e^{-nx}(nx)^k}{k!}\frac{(n)_k}{k!}\frac{\Gamma(k+r+1)\Gamma(n-r-1)}{\Gamma(n+k)}.$$

Using $k! = (1)_k$ and $\Gamma(k+r+1) = \Gamma(r+1)(r+1)_k$, we have

$$S_n(t^r, x) = (n-1)e^{-nx}\Gamma(n-r-1)\sum_{k=0}^{\infty} \frac{(nx)^k}{k!}\frac{(n)_k}{(1)_k}\frac{\Gamma(r+1)(r+1)_k}{(n)_k\Gamma(n)}$$
$$= \frac{e^{-nx}\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\sum_{k=0}^{\infty}\frac{(r+1)_k}{(1)_k}\frac{(nx)^k}{k!}$$
$$= \frac{e^{-nx}\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ {}_1F_1(r+1;1;nx),$$

which on using ${}_1F_1(a;b;x) = e^x\ {}_1F_1(b-a;b;-x)$, leads us to (2.1).

It is obvious that the confluent Hypergeometric functions can be related with the generalized Laguerre polynomials $L_n^m(x)$ with the relation

$$L_n^m(x) = \frac{(m+n)!}{m!n!}\ {}_1F_1(-n; m+1; x).$$

Thus

$$S_n(t^r, x) = \frac{\Gamma(n-r-1)\Gamma(r+1)}{\Gamma(n-1)}\ L_r(-nx),$$

where $L_r(-nx) = L_r^0(-nx)$ is the simple Laguerre polynomials. ∎

Remark 2 *By definition of the operators $S_n(1, x) = 1$, using Lemma 1, we have*

$$S_n(t, x) = \frac{1+nx}{n-2}, S_n(t^2, x) = \frac{n^2x^2+4nx+2}{(n-2)(n-3)}.$$

The higher order moments can be obtained easily by Lemma 1. For fixed $x \in I \equiv [0, \infty)$, define the function ψ_x by $\psi_x(t) = t - x$. The central moments for the operators S_n are given by

$$(S_n\psi_x^0)(x) = 1, (S_n\psi_x^1)(x) = \frac{1+2x}{n-2}, (S_n\psi_x^2)(x) = \frac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)}.$$

Moreover, let $x \in I$ be fixed. For $r = 0, 1, 2, \ldots$ and $n \in N$, the central moments for the operators S_n satisfy

$$(S_n\psi_x^r)(x) = O(n^{-[(r+1)/2]}).$$

In view of above, an application of the Schwarz inequality, for $r = 0, 1, 2, \ldots,$ yields

$$(S_n|\psi_x^r|)(x) \le \sqrt{(S_n\psi_x^{2r})(x)} = O(n^{-r/2}). \tag{2.2}$$

3 Better Approximation

Very recently Bhardwaj and Deo [5] studied these operators again and they constructed the following operators as

$$\widehat{S}_n(f,x) = (n-1)\sum_{k=0}^{\infty} s_{n,k}(r_n(x)) \int_0^{\infty} b_{n,k}(t)f(t)dt, \tag{3.1}$$

with $r_n(x) = \frac{(n-2)x-1}{n}$. We may remark here that these operators are defined for $n \geq 3$ and on the compact interval $x \in [1,\infty)$. Also it is observed that for the form (3.1), the Remark 21.6 of [5] should be:

$$\begin{aligned} |\widehat{S}_n(f,x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{(n+6)x^2+2(n+3)x+2}{(n-2)(n-3)} + \frac{(1+2x)^2}{(n-2)^2}}\right) \\ +\omega\left(f, \frac{1+2x}{n-2}\right). \end{aligned} \tag{3.2}$$

It follows that the degree of approximation is at most $O(n^{-1})$. To increase the degree of approximation, in the next section we consider linear combinations of the Szász-Baskakov operators. The more general case of the above estimate in the q setting is already available in the literature. We refer the readers to Theorem 1 of [7].

4 Linear combinations of Szász-Baskakov operators

It is known that the simple generalized Laguerre polynomials $L_k(x)$ have the following representation

$$L_k(x) = \sum_{j=0}^{k}(-1)^j \binom{k}{k-j} \frac{x^j}{j!}. \tag{4.1}$$

Therefore the coefficient of the leading term in (4.1) is $(-1)^k \cdot \frac{1}{k!}$. Hence

$$\begin{aligned} S_n(t^k, x) &= \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \cdot L_k(-nx) \qquad (4.2) \\ &= \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \cdot \sum_{j=0}^{k} \binom{k}{k-j} \frac{n^j x^j}{j!} \\ &= \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \cdot \left[\frac{n^k x^k}{k!} + \sum_{j=0}^{k-1} \binom{k}{k-j} \frac{n^j x^j}{j!} \right] \\ &= \frac{\Gamma(n-k-1)}{\Gamma(n-1)} \cdot n^k x^k + \frac{\Gamma(n-k-1)k!}{\Gamma(n-1)} \sum_{j=0}^{k-1} \binom{k}{k-j} \frac{n^j x^j}{j!}. \end{aligned}$$

Following the ideas from [15] we will consider the following linear combinations

$$S_{n,r} = \sum_{i=0}^{r} \alpha_i(n) \cdot S_{n_i}, \qquad (4.3)$$

where n_i, $i = 0, 1, \ldots, r$-are different positive numbers. Determine $\alpha_i(n)$ such that $S_{n,r}p = p$ for all $p \in \mathbf{P}_r$. This seems to be natural as the operators S_n don't preserve linear functions. The requirement that each polynomial of degree at most r should be reproduced leads to a linear system of equations:

$$S_{n,r}(t^k, x) = x^k,\ 0 \le k \le r. \qquad (4.4)$$

Therefore (4.2) and (4.3) imply the system

$$\begin{array}{l} \alpha_0 + \alpha_1 + \cdots + \alpha_r = 1 \\ \sum_{i=0}^{r} \alpha_i \cdot \frac{\Gamma(n_i-k-1)}{\Gamma(n_i-1)} \cdot n_i^k = 1,\ 1 \le k \le r. \end{array} \qquad (4.5)$$

The unique solution of this system is

$$\alpha_i = \frac{\Gamma(n_i-1)}{\Gamma(n_i-r-1)} \cdot \prod_{\substack{j=0 \\ j \ne i}}^{r} \frac{1}{(n_i-n_j)},\ 0 \le i \le r. \qquad (4.6)$$

To verify this, let us set firstly $k = r$ in the second equation in (4.5). By using (4.6) the left side of second equation of (4.5) is equal to

$$\sum_{i=0}^{r} n_i^r \prod_{\substack{j=0 \\ j \ne i}}^{r} \frac{1}{(n_i-n_j)} = f[n_0, n_1, \ldots, n_r],\ f(t) = t^r, \qquad (4.7)$$

where in (4.7), we have used the well known formula for the representation of the divided difference $f[n_0, n_1, \ldots, n_r]$ over the knots $n_0, \ldots, n_r$ for $f(t) = t^r$. But the latter is equal to the leading coefficient in the Lagrange interpolation

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polynomial of degree r over the knots $n_0, \dots, n_r$, which obviously is equal to 1. Further if $1 \le k < r$ in (4.5) then we should verify that

$$\sum_{i=0}^{r} \frac{\Gamma(n_i - k - 1)}{\Gamma(n_i - r - 1)} \cdot n_i^k \cdot \prod_{\substack{j=0 \\ j \neq i}}^{r} \frac{1}{(n_i - n_j)} = 1.$$

Consequently

$$(n_i - r - 1)(n_i - r) \cdot (n_i - k - 2) \cdot n_i^k = h(n_i) \in \mathbf{P}_r,$$

with leading coefficient 1. So in a similar way as above we see that the second equation in (4.5) holds true for all $1 \le k \le r$. To verify the first equation in (4.5) we only need to observe that $\alpha_0 + \cdots + \alpha_r$ equals to the divided difference $f[n_0, \cdots, n_r]$ for

$$f(t) = (t - r - 1)(t - r) \dots (t - 2) = t^r + \dots$$

and the latter is equal to 1. According to (4.2) by the same method we observe that

$$\sum_{i=0}^{r} \alpha_i \frac{\Gamma(n_i - k - 1)k!}{\Gamma(n_i - 1)} \binom{k}{k-j} \frac{n_i^j x^j}{j!} = 0$$

for $0 \le j \le k - 1$ and $1 \le k \le r$. To obtain a direct estimate for approximation by linear combinations $S_{n,r}$ one needs two additional assumptions:

$$n = n_0 < n_1 < \cdots < n_r \le A \cdot n, (A = A(r)), \tag{4.8}$$

$$\sum_{i=0}^{r} |\alpha_i(n)| \le C. \tag{4.9}$$

The first of these conditions guarantees that

$$\left(S_{n,r}|\psi_x^{r+1}|\right)(x) = O\left(n^{-\frac{r+1}{2}}\right), \, n \to \infty, \tag{4.10}$$

which follows from (2.2). The second condition is that the sum of the absolute values of the coefficients should be bounded independent of n. This is due to the fact that the linear combinations are no longer positive operators.We end this section with the following example:

Example 3 *Let $n_0 = n, n_1 = 2n, n_2 = 3n$. Then by simple calculations we verify that*

$$\alpha_0 = \frac{(n-2)(n-3)}{(-n)(-2n)} = \frac{1}{2} - \frac{5}{2} \cdot \frac{1}{n} + \frac{3}{n^2},$$

$$\alpha_1 = -4 + 10\frac{1}{n} - 6 \cdot \frac{1}{n^2},$$

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$$\alpha_2 = \frac{9}{2} - \frac{15}{2} \cdot \frac{1}{n} + \frac{3}{n^2}.$$

The two assumptions on α_i are fulfilled.

But if we choose $\alpha_0 = n, \alpha_1 = n+1, \alpha_2 = n+2$ it is easy to verify that the condition (4.9) is not satisfied. So we should be careful with the choice of the coefficients of the linear combination.

5 Direct estimate for linear combinations $S_{n,r}$

The main result in this section is

Theorem 4 *Let $f \in C_B[0,\infty)$. Then for every $x \in [0,\infty)$ and for $C > 0, n > r$ we have*

$$|(S_{n,r}f)(x) - f(x)| \leq C \cdot \omega_{r+1}\left(f, \frac{1}{\sqrt{n}}\right),$$

where $C_B[0,\infty)$ be the space of all real valued continuous bounded functions f defined on $[0,\infty)$.

Corollary 5 *If $f^{(r+1)} \in C_B[0,\infty)$ then*

$$|(S_{n,r}f)(x) - f(x)| \leq C \cdot \left(\frac{1}{\sqrt{n}}\right)^{r+1} \cdot \|f^{(r+1)}\|_{C_B[0,\infty)}.$$

Proof. The classical Peetre's K_r-functional for $f \in C_B[0,\infty)$ is defined by

$$K_r(f,\delta^r) = \inf\{\|f-g\| + \delta^r \cdot \|g^{(r)}\| : g \in W_\infty^r\},\ \delta > 0, \tag{5.1}$$

where $W_\infty^r = \{g \in C_B[0,\infty), g^{(r)} \in C_B[0,\infty)\}$. From the classical book of DeVore-Lorentz [3] there exists a positive constant C such that

$$K_r(f,\delta^r) \leq C\omega_r(f,\delta). \tag{5.2}$$

Let $g \in W_\infty^r$. By Taylor's expansion of g we get

$$g(t) = g(x) + \sum_{i=1}^{r} \frac{(t-x)^i}{i!} g^{(i)} + \frac{(t-x)^{r+1}}{(r+1)!} \cdot g^{(r+1)}(\xi_{t,x}). \tag{5.3}$$

We apply the operator to the both sides of (5.3). Now (4.4) implies

$$S_{n,r}(g,x) - g(x) = S_{n,r}\left(\frac{(t-x)^{r+1}}{(r+1)!} \cdot g^{(r+1)}(\xi_{t,x}); x\right)$$

Therefore

$$|S_{n,r}(g,x) - g(x)| \leq \sum_{i=0}^{r} |\alpha_i| \cdot S_{n_i}\left(\frac{|t-x|^{r+1}}{(r+1)!}; x\right) \cdot \|g^{(r+1)}\|_{C_B[0,\infty)}$$

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From (4.9) and (4.10) it follows

$$|S_{n,r}(g,x) - g(x)| \leq C(r) \cdot n^{-\frac{r+1}{2}} \cdot \|g^{(r+1)}\|.$$

Consequently

$$|S_{n,r}(f,x) - f(x)| \leq |S_{n,r}(f-g,x) - (f-g)(x)| + |S_{n,r}(g,x) - g(x)| \leq$$

$$\leq 2\|f-g\| + C(r) \cdot n^{-\frac{r+1}{2}} \cdot \|g^{(r+1)}\|.$$

Taking the infimum on the right side over all $g \in W_\infty^{r+1}$ and using (5.1),(5.2) we get the required result. ■

Remark 6 *If $r = 1$ then linear combinations of only two Szász-Baskakov operators*

$$S_{n,r} = \alpha_0 \cdot S_{n_0} + \alpha_1 \cdot S_{n_1}$$

guarantees, that the linear functions will be preserved. Then Theorem 4 implies direct estimate in term of second-order moduli of smoothness, which improves the estimate (3.2) from [5].

6 Generalized Voronovskaja-type Theorem

The following Voronovskaja-type estimate was proved in [5]-(see Remark 21.8 there:)

$$\lim_{n\to\infty} n[S_n(f,x) - f(x)] = (2x+1)f'(x) + (x^2+1)f''(x).$$

The aim of this section is to generalized this estimate for linear combinations of S_n.

Theorem 7 *Let $f, f', \ldots, f^{(r+2)} \in C_B[0,\infty)$. Then, if $r = 2k+1, k = 0,1,2,\ldots$ for $x \in [0,\infty)$ it follows*

$$\lim_{n\to\infty} n^{k+1} \cdot [S_{n,2k+1}(f,x) - f(x)] = P_{2k+2}(x) \cdot f^{(2k+2)}(x),$$

where

$$P_{2k+2}(x) = \lim_{n\to\infty} \left(S_{n,2k+1}(\psi_x^{2k+2}(t),x)n^{k+1}\right)$$

.

Proof. By Taylor's expansion of f we obtain

$$f(t) = f(x) + \sum_{i=1}^{2k+2} \frac{(t-x)^i}{i!} \cdot f^{(i)}(x) + \frac{(t-x)^{2k+2}}{(2k+2)!} \cdot R(t,x), \qquad (6.1)$$

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where $R(t,x)$ is a bounded function for all $t, x \in [0,\infty)$ and $\lim_{t\to x} R(t,x) = 0$. We apply $S_{n,2k+1}$ to the both sides of (6.1) to get

$$S_{n,2k+1}(f,x) - f(x) = \frac{f^{(2k+2)}(x)}{(2k+2)!} \cdot S_{n,2k+1}(\psi_x^{2k+2}, x) + I, \tag{6.2}$$

where

$$I = \frac{1}{(2k+2)!} \cdot S_{n,2k+1}\left((t-x)^{2k+2} \cdot R(t,x); x\right).$$

From (2.2), (4.9) and (4.10) we get

$$|S_{n,2k+1}(\psi_x^{2k+2}, x)| = O\left(n^{-[\frac{2k+3}{2}]}\right) = O\left(n^{-(k+1)}\right). \tag{6.3}$$

Let $\varepsilon > 0$ be given. Since $\xi(t,x) \to 0$ as $t \to x$, then there exists $\delta > 0$ such that when $|t-x| < \delta$ we have $|\xi(t,x)| < \varepsilon$ and when $|t-x| \geq \delta$ we write

$$|\xi(t,x)| \leq C < C \cdot \frac{(t-x)^2}{\delta^2}.$$

Thus for all $t, x \in [0,\infty)$

$$|\xi(t,x)| \leq \varepsilon + C \cdot \frac{(t-x)^2}{\delta^2}$$

and

$$|I| \leq C\varepsilon \cdot n^{-(k+1)} + \frac{C}{\delta^2} \cdot \left|S_{n,2k+1}((t-x)^{2k+4}, x)\right| \leq$$
$$\leq C\varepsilon \cdot n^{-(k+1)} + \frac{C}{\delta^2} \cdot n^{-(k+2)}.$$

Hence

$$n^{k+1} \cdot |I| \leq C\varepsilon + \frac{C}{\delta^2}\frac{1}{n}.$$

So

$$\lim_{n\to\infty} n^{k+1} \cdot |I| = 0.$$

Combining the estimates (6.1), (6.2) and (6.3), we get the desired result. This completes the proof of the theorem. ■.

7 Weighted approximation by Szász-Baskakov operators

First we point out that our statements in previous sections are formulated for $f \in C_B[0,\infty)$. On the other hand, the Szász-Baskakov operator S_n is well-defined for much larger class of functions, satisfying certain polynomial growth at infinity, that is $|f(t)| \leq M(1+t)^m$, for some $M > 0, m > 0$. In this case we consider the weight

$$\rho(x) = (1+x)^{-m},\ x \in I = [0,\infty).$$

The polynomial weighted space associated to this weight is defined by

$$C_\rho(I) = \{f \in C(I) : \|f\|_\rho < \infty\|,$$

where

$$\|f\|_\rho = \sup_{x\geq 0} \rho(x)|f(x)|.$$

Recently the case of weighted approximation by a broad class of linear positive operators, satisfying some conditions was considered in [2]. For $a \in \mathbb{N}_0, b > 0, c \geq 0$ we denote

$$\varphi(x) = \sqrt{(1+ax)(bx+c)}.$$

For $\lambda \in [0,1], r = 1, f \in C_\rho(I)$ we consider the K-functional

$$K_{1,\varphi^\lambda}(f,t)_\rho = \inf\{\|f-g\|_\rho + t\|\varphi^\lambda g'\|_\rho,\ g \in W^\infty_{1,\lambda}(\varphi)\},$$

where $W^\infty_{1,\lambda}(\varphi)$ consists of all functions $g \in C_\rho[0,\infty)$ such that $\|\varphi^\lambda g'\|_\rho < \infty$. According to the second moment of Szász-Baskakov operator we set $a = 2, b = 1, c = 0$. Hence $\varphi(x) = \sqrt{x(1+2x)}$. One of the main statements in [2] is Theorem 1, which we cite here as:

Theorem A *Let $L_n :\to C_\rho(I)$ be a sequence of positive linear operators satisfying the following conditions:*
(i) $L_n(e_0) = e_0$.
(ii) There exists a constant C_1 and a sequence $\{\alpha_n\}$ such that

$$L_n((t-x)^2, x) \leq C_1\alpha_n \cdot \varphi^2(x).$$

(iii) There exists a constant $C_2 = C_2(m)$ such that for each $n \in \mathbb{N}$,

$$L_n((1+t)^m, x) \leq C_2(1+x)^m,\ x \geq 0.$$

(iv) There exists a constant $C_3 = C_3(m)$ such that for every $m \in \mathbb{N}$,

$$\rho(x)L_n\left(\frac{(t-x)^2}{\rho(t)}, x\right) \leq C_3\alpha_n \cdot \varphi^2(x),\ x \geq 0.$$

Then for $\lambda \in [0,1]$, there exists a constant $C_4 = C_4(m,\lambda)$ such that for any $f \in C_\rho(I), x \in I, n \in \mathbb{N}$, one has

$$\rho(x)|f(x) - L_n(f,x)| \leq C_4 K_{1,\varphi^\lambda}(f, \sqrt{\alpha_n} \cdot \varphi^{1-\lambda}(x))_\rho, x \geq 0.$$

It is easy to verify that all 4 conditions in Theorem A are satisfied for $L_n = S_n$ with $\alpha_n = \frac{1}{n}$. The condition (i) is obvious. The conditions (ii), (iii) follow from the representation of the second moment and (4.2). To verify (iv) we apply again the Schwarz inequality and estimate for the fourth moment of S_n. So we arrive at the proof of the following statement for a weighted approximation by S_n:

Theorem 8 *By the conditions of Theorem A the following holds true*

$$\rho(x)|f(x) - S_n(f,x)| \leq C_4 K_{1,\varphi^\lambda}(f, \sqrt{\alpha_n} \cdot \varphi^{1-\lambda}(x))_\rho, x \geq 0.$$

If $\lambda = 1$ we get the estimate in term of the first Ditzian-Totik modulus of smoothness. For $\lambda = 0$ we get the estimate in term of the first modulus of continuity. The both estimates are in a point-wise form.

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k-th order Kantorovich type modification of the operators U_n^ρ

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Dedicated to Heiner Gonska on the occasion of his 65th birthday

Abstract

H. Gonska and R. Păltănea introduced and investigated a remarkable class of positive linear operators constituting a link between the classical Bernstein operators and the genuine Bernstein-Durrmeyer operators. In this paper we study the k-th order Kantorovich type modification of these operators. For the modified operators we establish explicit formulas as well as recursion formulas for the images of the monomials and the moments of arbitrary order. Applications are given to asymptotic relations. In particular, this unifies several known results for classical sequences of positive linear operators.

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1 Introduction

We consider sequences $Q_n^{k,\rho}$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\rho \in \mathbb{R}_+$, of positive linear operators which can be considered as the k-th order Kantorovich modification of the operators U_n^ρ. For a function $f \in C[0,1]$ the operators U_n^ρ are defined by

$$\begin{aligned} U_n^\rho(f,x) &:= \sum_{j=0}^{n} F_{n,j}^\rho(f) p_{n,j}(x) \\ &= f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \sum_{j=1}^{n-1} p_{n,j}(x) \int_0^1 \mu_{n,j}^\rho(t) f(t)dt \end{aligned}$$

where $p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}$, $0 \le j \le n$, $x \in [0,1]$.

Moreover, for $1 \le j \le n-1$, $\mu_{n,j}^\rho(t) = \dfrac{t^{j\rho-1}(1-t)^{(n-j)\rho-1}}{B(j\rho,(n-j)\rho)}$ with Euler's Beta function $B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$.

The operators U_n^ρ were introduced in [8] and studied in [2, 3]. They constitute a non-trivial link between the Bernstein operators B_n and their genuine Bernstein-Durrmeyer variant U_n^1 and inherit many useful properties of these classical operators. In [2] it is proved that

$$\lim_{\rho\to\infty} U_n^\rho f = B_n f = U_n^\infty f \text{ uniformly on } [0,1] \text{ for any } f \in C[0,1].$$

In [4] Gonska and the authors of this paper investigated the k-th order Kantorovich modification of the Bernstein operators, i.e.,

$$D^k \circ B_n \circ I_k \tag{1}$$

where D^k denotes the k-th order ordinary differential operator and

$$I_k f = f, \text{ if } k = 0, \text{ and } I_k(f,x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt, \text{ if } k \in \mathbb{N}.$$

In the investigation a crucial role was played by the moments of the new operators, and by the images of monomials under them.

In this paper we generalize this concept to the operators U_n^ρ and consider

$$Q_n^{k,\rho} = D^k \circ U_n^\rho \circ I_k,\ n \in \mathbb{N}, k \in \mathbb{N}_0, \rho \in \mathbb{R}_+.$$

This general definition contains many known operators as special cases. For $\rho \to \infty$ the operators $Q_n^{k,\infty}$ are the k-th order Kantorovich modification of the Bernstein operators (1) which include the Bernstein operators $Q_n^{0,\infty}$, the Kantorovich operators $Q_n^{1,\infty}$ and the Kantorovich operators of second order $Q_n^{2,\infty}$ considered by Nagel in [7]. For $\rho = 1$ we get the genuine Bernstein-Durrmeyer operators $Q_n^{0,1}$, the Bernstein-Durrmeyer operators $Q_n^{1,1}$ and the auxiliary operators $Q_n^{k,1}$ considered in [6, (3.5)].

In this paper we establish recurrence relations and explicit representations for the moments and the images of the monomials for the general operators $Q_n^{k,\rho}$. This enables us to get asymptotic relations for the operators $Q_n^{k,\rho}$. In particular, we establish a Voronovskaja type formula for $Q_n^{k,\rho}$, which unifies the corresponding formulas for the special cases mentioned above.

Throughout this paper we use the notations $a^{\underline{k}} := \prod_{l=0}^{k-1}(a-l)$, $a^{\overline{k}} := \prod_{l=0}^{k-1}(a+l)$, $k \in \mathbb{N}$, $a^{\underline{0}} = a^{\overline{0}} := 1$ and for abbreviation we define $X := x(1-x)$, $X' := 1-2x$, $Y := y(1-y)$, $Y' := 1-2y$.

2 Monomials and moments – explicit formulas

In this section we prove general explicit formulas for the moments and the images of the monomials of the operators $Q_n^{k,\rho}$. In what follows we denote by $e_\nu(t) = t^\nu$, $\nu \in \mathbb{N}_0$, the monomials and by

$$\vec{\Delta}_h^l f(x) = \sum_{\kappa=0}^{l} (-1)^{l-\kappa} \binom{l}{\kappa} f(x+\kappa h) \tag{2}$$

the l-th order forward difference of a function f with step h and define

$$p_\nu^\rho(\xi) := \prod_{l=1}^{\nu-1} \left(\xi + \frac{l}{\rho}\right), \ \nu \in \mathbb{N}.$$

We first consider the images of the monomials for the case $k = 0$.

Theorem 1 *Let $n \in \mathbb{N}$, $\rho \in \mathbb{R}_+$, $\nu \in \mathbb{N}_0$, $\nu \le n$. Then*

$$(Q_n^{0,\rho} e_0)(x) = 1, \tag{3}$$

$$(Q_n^{0,\rho} e_\nu)(x) = \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \sum_{j=1}^{\nu} \binom{n}{j} j \left(\vec{\Delta}_1^{j-1} p_\nu^\rho(1)\right) x^j, \ \nu \in \mathbb{N}. \tag{4}$$

Proof. For (3) see [8, (2.7)].
In order to prove (4) we take into account that

$$\frac{\int_0^1 t^\nu t^{i\rho-1}(1-t)^{(n-i)\rho-1} dt}{\int_0^1 t^{i\rho-1}(1-t)^{(n-i)\rho-1} dt} = \frac{\Gamma(i\rho+\nu)\Gamma(n\rho)}{\Gamma(i\rho)\Gamma(n\rho+\nu)} = \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \prod_{l=0}^{\nu-1} \left(i + \frac{l}{\rho}\right).$$

Thus we get for $\nu \ge 1$

$$\begin{aligned}(Q_n^{0,\rho} e_\nu)(x) &= x^n + \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \sum_{i=1}^{n-1} p_{n,i}(x) \prod_{l=0}^{\nu-1} \left(i + \frac{l}{\rho}\right) \\ &= \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} nx \sum_{i=1}^{n} p_{n-1,i-1}(x) \prod_{l=1}^{\nu-1} \left(i + \frac{l}{\rho}\right).\end{aligned}$$

We have

$$p_\nu(i) = \prod_{l=1}^{\nu-1} \left(i + \frac{l}{\rho}\right) = \sum_{j=0}^{\nu-1} \frac{\vec{\Delta}_1^j p_\nu^\rho(1)}{j!} \prod_{l=1}^{j} (i-l),$$

3

which can be derived by the using the Newton representation of the interpolation polynomial of p_ν^ρ for the equidistant knots $1, 2, \ldots, \nu$, evaluated for $\xi = i$. Then

$$\begin{aligned}(Q_n^{0,\rho} e_\nu)(x) &= \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} nx \sum_{i=1}^{n} p_{n-1,i-1}(x) \sum_{j=0}^{\nu-1} \frac{\vec{\Delta}_1^j p_\nu^\rho(1)}{j!} \prod_{l=1}^{j} (i-l) \\ &= \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} nx \sum_{j=0}^{\nu-1} \frac{\vec{\Delta}_1^j p_\nu^\rho(1)}{j!} \sum_{i=j+1}^{n} p_{n-j-1,i-j-1}(x) \frac{(n-1)!}{(n-j-1)!} x^j \\ &= \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \sum_{j=1}^{\nu} \binom{n}{j} j \left(\vec{\Delta}_1^{j-1} p_\nu^\rho(1) \right) x^j.\end{aligned}$$

□

Remark 1 *Using (2), the representation (4) can be rewritten into*

$$(Q_n^{0,\rho} e_\nu)(x) = \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \sum_{j=1}^{\nu} \binom{n}{j} j x^j \sum_{\kappa=0}^{j-1} (-1)^{j-1-\kappa} \binom{j-1}{\kappa} p_\nu^\rho(1+\kappa).$$

Now we look at the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: Then $\dfrac{1}{(n)^{\overline{\nu}}} = \dfrac{(n-1)!}{(n+\nu-1)!}$, and with [5, (3.48)]

$$\vec{\Delta}_1^{j-1} p_\nu^1(1) = \sum_{\kappa=0}^{j-1} (-1)^{j-1-\kappa} \binom{j-1}{\kappa} \frac{(\kappa+\nu)!}{(\kappa+1)!} = (\nu-1)! \binom{\nu}{j}.$$

Thus

$$(Q_n^{0,1} e_\nu)(x) = \frac{n!(\nu-1)!}{(n+\nu-1)!} \sum_{j=1}^{\nu} \binom{n-1}{j-1} \binom{\nu}{j} x^j,$$

which coincides with the formula given in [11, Lemma 1.11].
$\underline{\rho \to \infty}$: Then $\dfrac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \to \dfrac{1}{n^\nu}$, and

$$\begin{aligned}\vec{\Delta}_1^{j-1} p_\nu^\infty(1) &= \sum_{\kappa=0}^{j-1} (-1)^{j-1-\kappa} \binom{j-1}{\kappa} (k+1)^{\nu-1} \\ &= \frac{1}{j} \sum_{\kappa=1}^{j} (-1)^{j+\kappa} \binom{j}{\kappa} \kappa^\nu = (j-1)! \sigma_\nu^j,\end{aligned}$$

where σ_ν^j denote the Stirling numbers of second kind. Thus

$$(Q_n^{0,\infty} e_\nu)(x) = \frac{1}{n^\nu} n! \sum_{j=1}^{\nu} \frac{1}{(n-j)!} \sigma_\nu^j x^j,$$

which coincides with the known result for the Bernstein operators (see [4, p. 720]).

Next we consider the images of the monomials for the case $k \in \mathbb{N}$.

Theorem 2 *Let $n \in \mathbb{N}$, $\rho \in \mathbb{R}_+$, $\nu \in \mathbb{N}_0$, $\nu + k \leq n$. Then*

$$
\begin{aligned}
&(Q_n^{k,\rho} e_\nu)(x) \\
&= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=0}^{\nu} \frac{n!}{(n-j-k)!j!} (j+k) \left(\vec{\Delta}_1^{j+k-1} p_{\nu+k}^{\rho}(1) \right) x^j .
\end{aligned} \tag{5}
$$

Proof. By using $Q_n^{k,\rho} e_\nu = \frac{\nu!}{(\nu+k)!} D^k Q_n^{0,\rho} e_{\nu+k}$ we get from (4) for $k \in \mathbb{N}$

$$
\begin{aligned}
&(Q_n^{k,\rho} e_\nu)(x) \\
&= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=1}^{\nu+k} \binom{n}{j} j \left(\vec{\Delta}_1^{j-1} p_{\nu+k}^{\rho}(1) \right) (x^j)^{(k)} \\
&= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=k}^{\nu+k} \binom{n}{j} j \left(\vec{\Delta}_1^{j-1} p_{\nu+k}^{\rho}(1) \right) \frac{j!}{(j-k)!} x^{j-k} \\
&= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=0}^{\nu} \frac{n!}{(n-j-k)!j!} (j+k) \left(\vec{\Delta}_1^{j+k-1} p_{\nu+k}^{\rho}(1) \right) x^j .
\end{aligned}
$$

□

Remark 2 *Using again (2), the representation (5) can be rewritten into*

$$
\begin{aligned}
&(Q_n^{k,\rho} e_\nu)(x) \\
&= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=0}^{\nu} \frac{n!(j+k)!}{(n-j-k)!j!} x^j \\
&\quad \times \sum_{\kappa=0}^{j+k-1} (-1)^{j+k-1-\kappa} \frac{1}{\kappa!(j+k-1-\kappa)!} p_{\nu+k}^{\rho}(1+\kappa).
\end{aligned}
$$

Again we consider the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: Then $\dfrac{1}{(n)^{\overline{\nu+k}}} = \dfrac{(n-1)!}{(n+\nu+k-1)!}$, and again with [5, (3.48)]

$$
\begin{aligned}
\vec{\Delta}_1^{j+k-1} p_{\nu+k}^{1}(1) &= \sum_{\kappa=0}^{j+k-1} (-1)^{j+k-1-\kappa} \binom{j+k-1}{\kappa} \frac{(\kappa+\nu+k)!}{(\kappa+1)!} \\
&= (\nu+k-1)! \sum_{\kappa=0}^{j+k-1} (-1)^{j+k-1-\kappa} \binom{j+k-1}{\kappa} \binom{\kappa+\nu+k}{\kappa+1} \\
&= (\nu+k-1)! \binom{\nu+k}{j+k}.
\end{aligned}
$$

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Thus

$$(Q_n^{k,1}e_\nu)(x)$$
$$= \frac{\nu!}{(\nu+k)!}\frac{(n-1)!}{(n+\nu+k-1)!}\sum_{j=0}^{\nu}\frac{n!(j+k)}{(n-j-k)!j!}x^j(\nu+k-1)!\binom{\nu+k}{j+k}$$
$$= \frac{n!(\nu+k-1)!}{(n+\nu+k-1)!}\sum_{j=0}^{\nu}\binom{n-1}{n-j-k}\binom{\nu}{j}x^j.$$

This coincides with the corresponding result in [6, Satz 4.2] for the auxiliary operators with the notation $Q_n^{k,1} = M_{n-1,k-1}$ there.

$\underline{\rho\to\infty}$: Then $\dfrac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \to \dfrac{1}{n^{\nu+k}}$ and

$$\vec{\Delta}_1^{j+k-1}p_{\nu+k}^{\infty}(1) = \frac{1}{(j+k)}\sum_{\kappa=1}^{j+k}(-1)^{j+k+\kappa}\binom{j+k}{\kappa}\kappa^{\nu+k} = (j+k-1)!\sigma_{\nu+k}^{j+k}.$$

Thus

$$(Q_n^{k,\infty}e_\nu)(x) = \frac{\nu!}{(\nu+k)!}\frac{1}{n^{\nu+k}}\sum_{j=0}^{\nu}\frac{n!(j+k)!}{(n-j-k)!j!}\sigma_{\nu+k}^{j+k}x^j$$

which coincides with [4, (2)]).

For the evaluation of $Q_n^{k,\rho}e_\nu$, $k\in\mathbb{N}$, for special values of ν, we use the representation

$$p_{\nu+k}^{\rho}(\xi) = \sum_{l=0}^{\nu+k-1}\rho^{-l}\sigma_l(1,2,\ldots,\nu+k-1)\xi^{\nu+k-1-l},$$

with the notation $\sigma_j(x_0,x_1,\ldots,x_n)$, $j\in\mathbb{N}$, for the symmetric function which is the sum of all products of j distinct values from the set $\{x_0,x_1,\ldots x_n\}$ and $\sigma_0(x_0,x_1,\ldots,x_n) := 1$.

For the monomial e_m it is known (see e.g. [9, Theorem 1.2.1]) that

$$\vec{\Delta}_1^{j+k-1}e_m(1) = \begin{cases} 0, & m < j+k-1,\\ (j+k-1)!\tau_{m-(j+k-1)}(1,2,\ldots,j+k), & 0\le j+k-1\le m,\end{cases}$$

with the complete symmetric function $\tau_j(x_0,x_1,\ldots,x_n)$ which is the sum of all products of $x_0,x_1,\ldots,x_n$ of total degree j, $j\in\mathbb{N}$, and $\tau_0(x_0,x_1,\ldots,x_n) := 1$.

Thus we can rewrite $(Q_n^{k,\rho}e_\nu)$ into

$$(Q_n^{k,\rho}e_\nu)(x) = \frac{\nu!}{(\nu+k)!}\frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}}\sum_{j=0}^{\nu}\frac{n!(j+k)!}{(n-j-k)!j!}x^j \tag{6}$$
$$\times\sum_{l=0}^{\nu-j}\rho^{-l}\sigma_l(1,2,\ldots,\nu+k-1)\tau_{\nu-l-j}(1,2,\ldots,j+k).$$

As a corollary we present the results for $\nu = 0$, 1, 2.

6

Corollary 1 *For $k \in \mathbb{N}_0$ the images for the first monomials are given by*

$$\begin{aligned}
(Q_n^{k,\rho}e_0)(x) &= \frac{\rho^k}{(n\rho)^{\overline{k}}} \cdot n^{\underline{k}}, \\
(Q_n^{k,\rho}e_1)(x) &= \frac{\rho^{k+1}}{(n\rho)^{\overline{k+1}}} \cdot n^{\underline{k}} \left[\frac{1}{2}k\left(1+\frac{1}{\rho}\right) + (n-k)x\right], \\
(Q_n^{k,\rho}e_2)(x) &= \frac{\rho^{k+2}}{(n\rho)^{\overline{k+2}}} \cdot n^{\underline{k}} \left[\frac{1}{2}k\left(\frac{3k+1}{6} + \frac{k+1}{\rho} + \frac{3k+5}{6\rho^2}\right)\right. \\
&\qquad \left. +(n-k)\left((k+1)\left(1+\frac{1}{\rho}\right)x + (n-k-1)x^2\right)\right].
\end{aligned}$$

Proof. For $k = 0$ the identities follow from Theorem 1. For $k \in \mathbb{N}$ we derive the proposition by using the representation (6) and the fact that for $m \in \mathbb{N}$

$$\begin{aligned}
\sigma_0(1,\ldots,m) = \tau_0(1,\ldots,m) &= 1, \\
\sigma_1(1,\ldots,m) = \tau_1(1,\ldots,m) &= \frac{1}{2}m(m+1), \\
\sigma_2(1,\ldots,m) &= \frac{1}{24}(m-1)m(m+1)(3m+2), \\
\tau_2(1,\ldots,m) &= \frac{1}{24}m(m+1)(m+2)(3m+1).
\end{aligned}$$

□

Next we consider the moments of $Q_n^{k,\rho}$. For abbreviation we use the notation

$$M_{n,m}^{k,\rho}(x) = [Q_n^{k,\rho}(e_1 - xe_0)^m](x),\ m \in \mathbb{N}_0,\ x \in [0,1] \tag{7}$$

and use the fact that $M_{n,m}^{k,\rho}(x) = \sum_{\nu=0}^{m} \binom{m}{\nu}(-x)^{m-\nu}(Q_n^{k,\rho}e_\nu)(x)$.
Again we first treat the case $k = 0$.

Theorem 3 *Let $n \in \mathbb{N}$, $\rho \in \mathbb{R}_+$, $m \in \mathbb{N}_0$, $m \le n$. Then*

$$M_{n,0}^{0,\rho}(x) = 1, \tag{8}$$

$$M_{n,1}^{0,\rho}(x) = 0, \tag{9}$$

$$\begin{aligned}
M_{n,m}^{0,\rho}(x) = (-x)^m + \sum_{j=1}^{m}(-x)^j \sum_{\nu=1}^{j} \frac{\rho^{\nu+m-j}}{(n\rho)^{\overline{\nu+m-j}}}(-1)^\nu \binom{m}{j-\nu} \\
\times \binom{n}{\nu}\nu\vec{\Delta}_1^{\nu-1}p_{\nu+m-j}^{\rho}(1),\ m \ge 2.
\end{aligned} \tag{10}$$

Proof. For (8) and (9) see [8, (2.7)].
In order to prove (10) we apply Theorem 1. With the indextransform $j \to j - m + \nu$, changing the order of summation and applying the indextransform

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$\nu \to \nu + m - j$, we derive

$$
\begin{aligned}
&M_{n,m}^{0,\rho}(x) \\
&= (-x)^m + \sum_{\nu=1}^{m} \binom{m}{\nu} (-x)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \sum_{j=1}^{\nu} \binom{n}{j} j \left(\vec{\Delta}_1^{j-1} p_\nu^\rho(1) \right) x^j \\
&= (-x)^m + \sum_{\nu=1}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \\
&\qquad \times \sum_{j=m-\nu+1}^{m} \binom{n}{j-m+\nu} (j-m+\nu) \left(\vec{\Delta}_1^{j-m+\nu-1} p_\nu^\rho(1) \right) x^j \\
&= (-x)^m + \sum_{j=1}^{m} x^j \sum_{\nu=m+1-j}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{\overline{\nu}}} \\
&\qquad \times \binom{n}{j-m+\nu} (j-m+\nu) \left(\vec{\Delta}_1^{j-m+\nu-1} p_\nu^\rho(1) \right) \\
&= (-x)^m + \sum_{j=1}^{m} x^j \sum_{\nu=1}^{j} \binom{m}{\nu+m-j} (-1)^{j-\nu} \frac{\rho^{\nu+m-j}}{(n\rho)^{\overline{\nu+m-j}}} \\
&\qquad\qquad \times \binom{n}{\nu} \nu \left(\vec{\Delta}_1^{\nu-1} p_{\nu+m-j}^\rho(1) \right) \\
&= (-x)^m + \sum_{j=1}^{m} (-x)^j \sum_{\nu=1}^{j} \frac{\rho^{\nu+m-j}}{(n\rho)^{\overline{\nu+m-j}}} (-1)^\nu \binom{m}{j-\nu} \binom{n}{\nu} \nu \vec{\Delta}_1^{\nu-1} p_{\nu+m-j}^\rho(1).
\end{aligned}
$$

□

Remark 3 *Analogously as for the images of monomials, (10) can be rewritten into*

$$
\begin{aligned}
M_{n,m}^{0,\rho}(x) = (-x)^m &+ \sum_{j=1}^{m} (-x)^j \sum_{\nu=1}^{j} \frac{\rho^{\nu+m-j}}{(n\rho)^{\overline{\nu+m-j}}} \frac{n!}{(n-\nu)!} \binom{m}{j-\nu} \\
&\times \sum_{\kappa=0}^{\nu-1} (-1)^{\kappa+1} \frac{1}{\kappa!(\nu-1-\kappa)!} p_{\nu+m-j}^\rho(1+\kappa).
\end{aligned}
$$

Next we consider the special cases $\rho = 1$ and $\rho \to \infty$.

<u>$\rho = 1$</u>: Then $\dfrac{1}{(n)^{\overline{\nu+m-j}}} = \dfrac{(n-1)!}{(n+\nu+m-j-1)!}$, and with [5, (3.48)]

$$
\begin{aligned}
\vec{\Delta}_1^{\nu-1} p_{\nu+m-j}^1(1) &= \sum_{\kappa=0}^{\nu-1} (-1)^{\nu-1-\kappa} \binom{\nu-1}{\kappa} \frac{(\kappa+\nu+m-j)!}{(\kappa+1)!} \\
&= (\nu+m-j-1)! \binom{\nu+m-j}{\nu}.
\end{aligned}
$$

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Thus

$$M_{n,m}^{0,1}(x) = (-x)^m + \sum_{j=1}^{m} (-x)^j \sum_{\nu=1}^{j} (-1)^\nu \frac{(n-1)!}{(n+\nu+m-j-1)!} \times \frac{n!}{(n-\nu)!} \binom{m}{j-\nu} \binom{j}{\nu} \frac{(\nu+m-j-1)!}{(\nu-1)!},$$

which coincides with the result in [11, Korollar 1.12].

$\underline{\rho \to \infty}$: Then $\dfrac{\rho^{\nu+m-j}}{(n\rho)^{\overline{\nu+m-j}}} \to \dfrac{1}{n^{\nu+m-j}}$ and

$$\begin{aligned}\vec{\Delta}_1^{\nu-1} p_{\nu+m-j}^{\infty}(1) &= \frac{1}{\nu} \sum_{\kappa=1}^{\nu} (-1)^{\nu+\kappa} \binom{\nu}{\kappa} \kappa^{\nu+m-j} \\ &= (\nu-1)! \sigma_{\nu+m-j}^{\nu}.\end{aligned}$$

Thus

$$M_{n,m}^{0,\infty}(x) = (-x)^m + \sum_{j=1}^{m} (-x)^j \sum_{\nu=1}^{j} \frac{1}{n^{\nu+m-j}} \frac{n!}{(n-\nu)!} \binom{m}{j-\nu} \sigma_{\nu+m-j}^{\nu}.$$

In our next theorem we evaluate the moments for the case $k \in \mathbb{N}$.

Theorem 4 *Let $n \in \mathbb{N}$, $\rho \in \mathbb{R}_+$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, $m+k \le n$. Then*

$$M_{n,m}^{k,\rho}(x) = \sum_{j=0}^{m} (-x)^j \sum_{\nu=0}^{j} \frac{\rho^{\nu+m-j+k}}{(n\rho)^{\overline{\nu+m-j+k}}} (-1)^\nu \binom{m}{j-\nu} \times \frac{(\nu+m-j)!}{(\nu+m-j+k)!} \frac{n!(\nu+k)}{(n-\nu-k)!\nu!} \vec{\Delta}_1^{\nu+k-1} p_{\nu+m-j+k}^{\rho}(1). \tag{11}$$

Proof. We now use Theorem 2 and carry out the same steps as in the proof of Theorem 3 to derive

$$\begin{aligned}&M_{n,m}^{k,\rho}(x) \\ &= \sum_{\nu=0}^{m} \binom{m}{\nu} (-x)^{m-\nu} \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \sum_{j=0}^{\nu} \frac{n!(j+k)}{(n-j-k)!j!} \left(\vec{\Delta}_1^{j+k-1} p_{\nu+k}^{\rho}(1)\right) x^j \\ &= \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \\ &\quad \times \sum_{j=m-\nu}^{m} \frac{n!(j-m+\nu+k)}{(n-j+m-\nu-k)!(j-m+\nu)!} \left(\vec{\Delta}_1^{j-m+\nu+k-1} p_{\nu+k}^{\rho}(1)\right) x^j\end{aligned}$$

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$$= \sum_{j=0}^{m} x^j \sum_{\nu=m-j}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{\overline{\nu+k}}} \times \frac{n!(j-m+\nu+k)}{(n-j+m-\nu-k)!(j-m+\nu)!} \vec{\Delta}_1^{j-m+\nu+k-1} p_{\nu+k}^{\rho}(1)$$

$$= \sum_{j=0}^{m} x^j \sum_{\nu=0}^{j} \binom{m}{\nu+m-j} (-1)^{j-\nu} \frac{(\nu+m-j)!}{(\nu+m-j+k)!} \frac{\rho^{\nu+m-j+k}}{(n\rho)^{\overline{\nu+m-j+k}}} \times \frac{n!(\nu+k)}{(n-\nu-k)!\nu!} \vec{\Delta}_1^{\nu+k-1} p_{\nu+m-j+k}^{\rho}(1)$$

$$= \sum_{j=0}^{m} (-x)^j \sum_{\nu=0}^{j} \frac{\rho^{\nu+m-j+k}}{(n\rho)^{\overline{\nu+m-j+k}}} (-1)^{\nu} \binom{m}{j-\nu} \times \frac{(\nu+m-j)!}{(\nu+m-j+k)!} \frac{n!(\nu+k)}{(n-\nu-k)!\nu!} \vec{\Delta}_1^{\nu+k-1} p_{\nu+m-j+k}^{\rho}(1).$$

□

Remark 4 *With (2), we can rewrite the representation (11) into*

$$M_{n,m}^{k,\rho}(x) = \sum_{j=0}^{m} (-x)^j \sum_{\nu=0}^{j} \frac{\rho^{\nu+m-j+k}}{(n\rho)^{\overline{\nu+m-j+k}}} \binom{m}{j-\nu} \frac{(\nu+m-j)!}{(\nu+m-j+k)!} \times \frac{n!(\nu+k)!}{(n-\nu-k)!\nu!} \sum_{\kappa=0}^{\nu+k-1} (-1)^{k+1+\kappa} \frac{1}{\kappa!(\nu+k-1-\kappa)!} p_{\nu+m-j+k}^{\rho}(1+\kappa).$$

From Theorem 4 we derive the following identity for the special case $\rho = 1$.
$\underline{\rho = 1}$: Then $\frac{1}{(n)^{\overline{\nu+m-j+k}}} = \frac{(n-1)!}{(n+\nu+m-j+k-1)!}$, and with [5, (3.48)]

$$\vec{\Delta}_1^{\nu+k-1} p_{\nu+m-j+k}^{1}(1) = \sum_{\kappa=0}^{\nu+k-1} (-1)^{\nu+k-1-\kappa} \binom{\nu+k-1}{\kappa} \frac{(\kappa+\nu+m-j+k)!}{(\kappa+1)!} = (\nu+m-j+k-1)! \binom{\nu+m-j+k}{\nu+k}.$$

Thus

$$M_{n,m}^{k,1}(x) = \sum_{j=0}^{m} (-x)^j \sum_{\nu=0}^{j} (-1)^{\nu} \frac{(n-1)!}{(n+\nu+m-j+k-1)!} \times \frac{n!}{(n-\nu-k)!} \binom{m}{j-\nu} \binom{j}{\nu} \frac{(\nu+m-j+k-1)!}{(\nu+k-1)!}.$$

This coincides with the result [6, Korollar 4.4] for the moments of the auxiliary operators named $M_{n-1,k-1}$ there.

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With the same notations and arguments used for Corollary 1, the moments (10) and (11) can be computed by using

$$\begin{aligned}&\vec{\Delta}_1^{\nu+k-1} p^{\rho}_{\nu+m-j+k}(1)\\ =&(\nu+k-1)!\sum_{l=0}^{m-j}\rho^{-l}\sigma_l(1,2,\ldots,\nu+m-j+k-1)\tau_{m-j-l}(1,2,\ldots,\nu+k).\end{aligned}$$

Corollary 2 *For $k\in\mathbb{N}_0$ the first moments are given by*

$$\begin{aligned}M^{k,\rho}_{n,0}(x)&=\frac{\rho^k}{(n\rho)^{\overline{k}}}n^{\underline{k}},\quad M^{k,\rho}_{n,1}(x)\ =\ \frac{\rho^{k+1}}{(n\rho)^{\overline{k+1}}}n^{\underline{k}}\frac{1}{2}k\left(1+\frac{1}{\rho}\right)X',\\ M^{k,\rho}_{n,2}(x)&=\frac{\rho^{k+2}}{(n\rho)^{\overline{k+2}}}n^{\underline{k}}\left(1+\frac{1}{\rho}\right)\left\{\left[n-\left(1+\frac{1}{\rho}\right)k(k+1)\right]X\right.\\ &\qquad\left.+\frac{k}{12}\left[(3k+1)\left(1+\frac{1}{\rho}\right)+\frac{4}{\rho}\right]\right\}.\end{aligned}$$

3 Monomials and moments – recursion formulas

First of all, we establish recurrence formulas for the functions

$$R^{k,\rho,y}_{n,m}(x):=\left[Q^{k,\rho}_n(e_1-ye_0)^m\right](x),\tag{12}$$

where $y\in[0,1]$ is a parameter.

Theorem 5 *The following relations hold for $k\in\mathbb{N}_0$:*

$$R^{k,\rho,y}_{n,0}(x)=\frac{\rho^k}{(n\rho)^{\overline{k}}}n^{\underline{k}},\tag{13}$$

$$R^{k,\rho,y}_{n,1}(x)=\frac{\rho^{k+1}}{(n\rho)^{\overline{k+1}}}n^{\underline{k}}\left[\frac{1}{2}k\left(1+\frac{1}{\rho}\right)+n(x-y)-k\left(x+\frac{y}{\rho}\right)\right],\tag{14}$$

$$\begin{aligned}&\frac{(m+k+1)(n\rho+m+k)}{m+1}R^{k,\rho,y}_{n,m+1}(x)\\ &=\rho X\frac{d}{dx}R^{k,\rho,y}_{n,m}(x)+\left(\rho kX'+\rho n(x-y)+(m+k)Y'\right)R^{k,\rho,y}_{n,m}(x)\\ &\quad+mYR^{k,\rho,y}_{n,m-1}(x)+\frac{\rho k(n-k+1)}{m+1}R^{k-1,\rho,y}_{n,m+1}(x),\quad m\in\mathbb{N}.\end{aligned}\tag{15}$$

Proof. (13) and (14) follow immediately from Corollary 1. To prove (15) we first note that in case $k=0$ the last term on the right hand side vanishes. Furthermore, for $k=0$ (15) is Theorem 3.1 in [2]. Thus we have

$$\begin{aligned}&(n\rho+m+k)R^{0,\rho,y}_{n,m+k+1}(x)=\rho X\frac{d}{dx}R^{0,\rho,y}_{n,m+k}(x)\\ &+\left((m+k)Y'+n\rho(x-y)\right)R^{0,\rho,y}_{n,m+k}(x)+(m+k)YR^{0,\rho,y}_{n,m+k-1}(x).\end{aligned}\tag{16}$$

On the other hand,

$$D^k R^{0,\rho,y}_{n,m+k}(x) = D^k U^\rho_n (e_1 - ye_0)^{m+k}(x) = \frac{(m+k)!}{m!} D^k U^\rho_n I_k (e_1 - ye_0)^m(x),$$

so that

$$D^k R^{0,\rho,y}_{n,m+k}(x) = \frac{(m+k)!}{m!} R^{k,\rho,y}_{n,m}(x). \tag{17}$$

Applying D^k to both sides of (16), and then using (17), we obtain (15). □

Let us remark that, according to (7) and (12),

$$M^{k,\rho}_{n,m}(x) = R^{k,\rho,x}_{n,m}(x). \tag{18}$$

So, we shall replace y by x in Theorem 5, but first we need

Lemma 1 *For each $m \in \mathbb{N}_0$,*

$$\left(\frac{d}{dx} R^{k,\rho,y}_{n,m}(x) \right)_{y=x} = \frac{d}{dx} M^{k,\rho}_{n,m}(x) + m M^{k,\rho}_{n,m-1}(x). \tag{19}$$

Proof. According to the definition of U^ρ_n, we have:

$$\begin{aligned}
\frac{d}{dx} M^{k,\rho}_{n,m}(x) &= \frac{d}{dx} D^k U^\rho_n I_k (e_1 - xe_0)^m(x) \\
&= \frac{m!}{(m+k)!} \frac{d}{dx} D^k U^\rho_n (e_1 - xe_0)^{m+k}(x) \\
&= \frac{m!}{(m+k)!} \frac{d}{dx} \left[D^k \left(\sum_{j=0}^n p_{n,j}(x) F^\rho_{n,j} (e_1 - xe_0)^{m+k} \right) \right](x) \\
&= \frac{m!}{(m+k)!} \frac{d}{dx} \sum_{j=0}^n p^{(k)}_{n,j}(x) F^\rho_{n,j} (e_1 - xe_0)^{m+k} \\
&= \frac{m!}{(m+k)!} \left(\sum_{j=0}^n p^{(k+1)}_{n,j}(x) F^\rho_{n,j} (e_1 - xe_0)^{m+k} \right. \\
&\qquad \left. - \sum_{j=0}^n p^{(k)}_{n,j}(x)(m+k) F^\rho_{n,j} (e_1 - xe_0)^{m+k-1} \right) \\
&= \left(\frac{d}{dx} R^{k,y,\rho}_{n,m}(x) \right)_{y=x} - m M^{k,\rho}_{n,m-1}(x).
\end{aligned}$$

□

Now, as a consequence of Theorem 5 and Lemma 1, we have

Theorem 6 *The following relations hold for $k \in \mathbb{N}_0$:*

$$M^{k,\rho}_{n,0}(x) = \frac{\rho^k}{(n\rho)^{\overline{k}}} n^{\underline{k}}, \tag{20}$$

$$M^{k,\rho}_{n,1}(x) = \frac{\rho^{k+1}}{(n\rho)^{\overline{k+1}}} n^{\underline{k}} \frac{1}{2} k \left(1 + \frac{1}{\rho} \right) X', \tag{21}$$

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$$
\begin{aligned}
&\frac{(m+k+1)(n\rho+m+k)}{m+1}M_{n,m+1}^{k,\rho}(x) \qquad (22)\\
&=\rho X\frac{d}{dx}M_{n,m}^{k,\rho}(x)+(k(\rho+1)+m)\,X'M_{n,m}^{k,\rho}(x)\\
&\quad+m(\rho+1)XM_{n,m-1}^{k,\rho}(x)+\frac{\rho k(n-k+1)}{m+1}M_{n,m+1}^{k-1,\rho}(x), \quad m\in\mathbb{N}.
\end{aligned}
$$

Obviously $R_{n,0}^{k,\rho,0}(x)=Q_n^{k,\rho}e_\nu(x)$; thus, setting $y=0$ in Theorem 5, we get

Theorem 7 *The following relations hold for* $k\in\mathbb{N}_0$ *:*

$$
Q_n^{k,\rho}e_0(x)=\frac{\rho^k}{(n\rho)^{\overline{k}}}n^{\underline{k}}, \qquad (23)
$$

$$
Q_n^{k,\rho}e_1(x)=\frac{\rho^{k+1}}{(n\rho)^{\overline{k+1}}}n^{\underline{k}}\left[\frac{1}{2}k\left(1+\frac{1}{\rho}\right)+(n-k)x\right], \qquad (24)
$$

$$
\begin{aligned}
&\frac{(\nu+k+1)(n\rho+\nu+k)}{\nu+1}Q_n^{k,\rho}e_{\nu+1}(x) \qquad (25)\\
&=\rho X\frac{d}{dx}Q_n^{k,\rho}e_\nu(x)+(\rho kX'+\rho nx+\nu+k)\,Q_n^{k,\rho}e_\nu(x)\\
&\quad+\frac{\rho k(n-k+1)}{\nu+1}Q_n^{k-1,\rho}e_{\nu+1}(x), \quad \nu\in\mathbb{N}.
\end{aligned}
$$

As in the case of the moments, from these recurrence formulas we can compute the images of the monomials under $Q_n^{k,\rho}$.

The following special cases deserve to be mentioned separately, since they are related to the operators investigated in [4]. From Theorem 6 we infer

$$
\begin{aligned}
&\frac{(r+k+1)n}{r+1}M_{n,r+1}^{k,\infty}(x) \qquad (26)\\
&=X\frac{d}{dx}M_{n,r}^{k,\infty}(x)+kX'M_{n,r}^{k,\infty}(x)\\
&\quad+rXM_{n,r-1}^{k,\infty}(x)+\frac{k(n-k+1)}{r+1}M_{n,r+1}^{k-1,\infty}(x).
\end{aligned}
$$

Moreover, Theorem 7 leads to

$$
\begin{aligned}
&\frac{(r+k+1)n}{r+1}Q_n^{k,\infty}e_{r+1}(x) \qquad (27)\\
&=X\frac{d}{dx}Q_n^{k,\infty}e_r(x)+(kX'+nx)\,Q_n^{k,\infty}e_r(x)\\
&\quad+\frac{k(n-k+1)}{r+1}Q_n^{k-1,\infty}e_{r+1}(x).
\end{aligned}
$$

4 Asymptotic relations

The preceding results allow us to establish asymptotic relations for the operators $Q_n^{k,\rho}$. To this aim, an essential tool is the following theorem.

Theorem 8 *For each $m \in \mathbb{N}_0$ we have, as $n \to \infty$,*

$$M_{n,m}^{k,\rho}(x) = \mathcal{O}\left(n^{-\left[\frac{m+1}{2}\right]}\right), \tag{28}$$

uniformly with respect to $x \in [0,1]$.

Proof. For the moments $M_{n,j}^{0,\rho}(x)$ of U_n^ρ, $j \in \mathbb{N}_0$, we know from [2, Corollary 3.2] and Theorem 6 for $k = 0$ that

$$M_{n,0}^{0,\rho}(x) = 1,\ M_{n,1}^{0,\rho}(x) = 0,$$

and for $j \in \mathbb{N}$

$$(n\rho + j)M_{n,j+1}^{0,\rho}(x) = \rho X \frac{d}{dx} M_{n,j}^{0,\rho}(x) + jX' M_{n,j}^{0,\rho}(x) + j(\rho+1)XM_{n,j-1}^{0,\rho}(x).$$

Now it is easy to verify by induction on j that for $n \to \infty$

$$M_{n,j}^{0,\rho}(x) = \mathcal{O}\left(n^{-\left[\frac{j+1}{2}\right]}\right), \tag{29}$$

uniformly with respect to $x \in [0,1]$. On the other hand, from [1, Lemma 1] we deduce

$$M_{n,m}^{k,\rho}(x) = \sum_{j=m}^{m+k} \binom{k}{j-m} \frac{m!}{j!} \frac{d^{j-m}}{dx^{j-m}} M_{n,j}^{0,\rho}(x). \tag{30}$$

Comparing (29) and (30) we conclude the proof. □

Let us remark that (28) is exactly what is needed in order to apply Sikkema's result [10]. Consequently, we have

Theorem 9 *Let $f \in C[0,1]$ be $2q$ times differentiable at $x \in [0,1]$. Then*

$$Q_n^{k,\rho} f(x) = \sum_{m=0}^{2q} \frac{1}{m!} f^{(m)}(x) M_{n,m}^{k,\rho}(x) + o\left(n^{-q}\right). \tag{31}$$

If $f \in C^{2q}[0,1]$, then (31) holds uniformly with respect to $x \in [0,1]$.

Using the known values of the moments $M_{n,m}^{k,\rho}(x)$, $m = 0,1,2$, (see Corollary 2) it is easy to derive from Theorem 9 the Voronovskaja type formula for the operators $Q_n^{k,\rho}$. More precisely, we have:

Theorem 10 *Let $f \in C[0,1]$ be two times differentiable at $x \in [0,1]$. Then*

$$\begin{aligned}&\lim_{n\to\infty} n\left(Q_n^{k,\rho} f(x) - f(x)\right) &(32)\\ &= \frac{\rho+1}{2\rho}\left(Xf''(x) + kX'f'(x) - (k-1)kf(x)\right).\end{aligned}$$

If $f \in C^2[0,1]$, then (32) holds uniformly with respect to $x \in [0,1]$.

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Remark 5 *With the notation $I_0 f = f$, $I_{-1} f = f'$ and $I_{-2} f = f''$ (32) can be rewritten as*

$$\lim_{n\to\infty} n\left(Q_n^{k,\rho} f(x) - f(x)\right) = \frac{\rho+1}{2\rho}\left(XI_{k-2} f(x)\right)^{(k)}.$$

For special values of the parameters k and ρ, from (32) we obtain the Voronovskaja type formulas for several classical operators.

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On the class of operators U_n^ϱ linking the Bernstein and the genuine Bernstein-Durrmeyer operators

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Dedicated to Prof. Dr. dr.h.c. Heiner Gonska
on the occasion of his 65th birthday

Abstract

We consider the class of operators U_n^ϱ introduced and investigated by Păltănea and Gonska and study further properties, such as variation diminution and global smoothness preservation. We also establish upper and lower estimates for iterates of these operators. The results we provide here come as a natural extension of the known results for both the Bernstein and the genuine Bernstein-Durrmeyer operators.

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1 Basic properties

Definition 1 *Let $\varrho > 0$ and $n \in \mathbb{N}$. The operators $U_n^\varrho : C[0,1] \to \Pi_n$ are defined by*

$$\begin{aligned} U_n^\varrho(f,x) &:= \sum_{k=0}^{n} F_{n,k}^\varrho(f) p_{n,k}(x) \\ &:= \sum_{k=1}^{n-1} \left(\int_0^1 \frac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)dt \right) p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \end{aligned}$$

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for $f \in C[0,1], x \in [0,1]$, *where* $B(\cdot,\cdot)$ *is Euler's Beta function. The fundamental functions* $p_{n,k}$ *are given by*

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le k \le n, \quad k,n \in \mathbb{N}_0, \quad x \in [0,1].$$

For $\varrho = 1$ and $f \in C[0,1]$, one has

$$U_n^1(f,x) = U_n(f,x) = (n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x)$$
$$+(1-x)^n f(0) + x^n f(1),$$

where U_n are the genuine Bernstein-Durrmeyer operators (see [5]), while for $\varrho \to \infty$, for each $f \in C[0,1]$ the sequence $U_n^\varrho(f,x)$ converges uniformly to the Bernstein polynomial $B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x)$. The operators U_n^ϱ were introduced in [17] by Păltănea and further investigated by Păltănea and Gonska in [10] and [11], where also the latter convergence was shown, and many interesting results and historical remarks can be found.

Moreover, it was noted in [18, (2.1)] that $U_n^\varrho = B_n \circ \mathcal{B}_{n\varrho}$, where $\mathcal{B}_{n\varrho}$ are Beta-type operators (see [14, p. 63] and [15]) given by

$$\mathcal{B}_r(f,x) := \begin{cases} f(x), \quad x = 0,1; \\ \dfrac{\int_0^1 t^{rx-1}(1-t)^{r-rx-1}f(t)dt}{B(rx, r-rx)}, \ 0 < x < 1, \end{cases}$$

for $r > 0, f \in C[0,1], x \in [0,1]$.

U_n^ϱ share many properties common for the well–known operators $B_n, U_n, \mathcal{B}_n$, such as being positive linear operators preserving linear functions. Furthermore,

Lemma 2 *If* f *is convex on* $C[0,1]$*, then*

$$U_n^\varrho(f,x) \ge U_{n+1}^\varrho(f,x) \ge f(x), \ 0 < x < 1. \tag{1}$$

The inequalities are strict when f *is strictly convex on* $[0,1]$.

Proof. This result is a consequence of ([1], Th.1) and ([4], Corollary 4.2).
Let $f \in C[0,1]$ convex. If $s > r > 0$, then

$$\mathcal{B}_r(f,x) \ge \mathcal{B}_s(f,x).$$

We choose $s = (n+1)\varrho$ and $r = n\varrho$ in the inequality above and we compose to the left with the $(n+1)$-st Bernstein operator. We get then

$$(B_{n+1} \circ \mathcal{B}_{n\varrho})(f,x) \ge (B_{n+1} \circ \mathcal{B}_{(n+1)\varrho})(f,x) = U_{n+1}^\varrho(f,x). \tag{2}$$

Next in the inequality bellow we compose to the right with $\mathcal{B}_{n\varrho}(f,x)$

$$B_n(f,x) \geq B_{n+1}(f,x)$$

and get

$$U_n^{\varrho}(f,x) = (B_n \circ \mathcal{B}_{n\varrho})(f,x) \geq (B_{n+1} \circ \mathcal{B}_{n\varrho})(f,x). \tag{3}$$

Combining (2) and (3) we get (1). ∎

Lemma 3 *If f is convex on $C[0,1]$, then*

$$U_n^{\varrho}(f,x) \geq B_n(f,x), 0 < x < 1. \tag{4}$$

The inequality is strict if f is strictly convex on $[0,1]$.

Proof. In [18] it is shown that for $f \in C[0,1]$ convex and $0 < \varrho < \sigma$,

$$U_n^{\varrho}(f,x) \geq U_n^{\sigma}(f,x).$$

Letting $\sigma \to \infty$ in the inequality above we get (4). ∎

2 Variation diminution

Shape preservation properties of an approximation method are considered to be of great importance in both Approximation Theory and Computer Aided Geometric Design. Among them, we discuss first the variation diminution.
As it is known that both B_n and U_n satisfy this property (see [19] and [5], respectively), it is natural to consider the question whether the operators U_n^{ϱ} are also variation–diminishing. To that end, we refer to [8], which contains historical remarks clarifying the various meanings of "variation–diminishing" employed in the past as well as an approach to prove this property.

Let K be any interval on the real line, and let $f : K \to \mathbb{R}$ be an arbitrary function. For an ordered sequence $x_0 < x_1 < ... < x_n$ of points in K, let $S[f(x_k)]$ denote the number of sign changes in the finite sequence of ordinates $f(x_k)$, where zeros are disregarded. The number of sign changes of f in the interval K is defined by

$$S_K[f] = \sup S[f(x_k)],$$

where the supremum is taken over all ordered finite sets $\{x_k\}$.
Let I and J be two intervals, let U be a subspace of $C(I)$, and suppose that $L : U \to C(J)$ is a linear operator reproducing constant functions.
The operator L is said to be (strongly) variation-diminishing (as an operator from U into $C(J)$) if

$$S_J[Lf] \leq S_I[f], \text{ for all } f \in U.$$

The main result presented in [8] (see Theorem 1 there) reads as follows.

Theorem 4 *Let $I = (a, b)$ or $I = (a, \infty)$ with $a \geq 0$, let $w : I \to \mathbb{R}_+$ be a strictly positive continuous weight function, and $[\alpha, \beta] \subset [0, \infty)$. Consider a linear and positive definite functional $A : C(I) \to \mathbb{R}$ having the following property: there exists a subspace $C_w^{[\alpha,\beta]}(I) \subset C(I)$ such that for $f \in C_w^{[\alpha,\beta]}(I)$ the function $Lf : (\alpha, \beta) \to \mathbb{R}$ given by $(Lf)(x) := A_t[t^x \cdot w(t) \cdot f(t)]$ is well–defined. If the function Lf has one-sided limits at the endpoints, then*

$$S_{[\alpha,\beta]}[Lf] \;\leq\; S_I[f], \qquad \forall f \in C_w^{[\alpha,\beta]}(I),$$

where, for $x \in \{\alpha, \beta\}$, one understands by $sgn(Lf)(x)$ the sign of the corresponding one-sided limit.

Theorem 5 *The operators U_n^ϱ have the (strong) variation-diminishing property, that is,*

$$S_{[0,1]}[U_n^\varrho f] \leq S_{[0,1]}[f] \text{ for all } f \in C[0,1].$$

Proof. We use the fact that $U_n^\varrho = B_n(\mathcal{B}_{n\varrho})$ and that the Bernstein operators B_n are (strongly) variation–diminishing. Thus we have

$$S_{[0,1]}[U_n^\varrho f] \leq S_{[0,1]}[\mathcal{B}_{n\varrho} f] = S_{[0,1]}\left[\int_0^1 t^{n\varrho x-1}(1-t)^{n\varrho-n\varrho x-1} f(t)\, dt\right].$$

Substituting $\left(\dfrac{t}{1-t}\right)^{n\varrho} = u$ the above integral becomes

$$\frac{1}{n\varrho}\int_0^\infty u^x \cdot \frac{1}{u(u^{\frac{1}{n\varrho}}+1)^{n\varrho}} \cdot f\left(\frac{u^{\frac{1}{n\varrho}}}{u^{\frac{1}{n\varrho}}+1}\right) du.$$

Obviously, the number of sign changes of $f(t), t \in [0,1]$ equals the number of sign changes of the function $g(u) = f\left(\dfrac{u^{\frac{1}{n\varrho}}}{u^{\frac{1}{n\varrho}}+1}\right)$, $u \in [0, \infty)$. Applying Theorem 4 for the functional $A(g) = \displaystyle\int_0^\infty g(u)\, du$ with $w(u) = \dfrac{1}{u(u^{\frac{1}{n\varrho}}+1)^{n\varrho}}$ we get that the operators U_n^ϱ have the (strong) variation–diminishing property on $C[0,1]$. ■

Remark 6 *As degree $U_n^\varrho e_i = i,\ i = 0, 1, \ldots, n$ (with $e_i(x) = x^i$, see [10, Lemma 3.5]) and U_n^ϱ have the (strong) variation–diminishing property, it follows from Theorem 7 in [8] that $U_n^\varrho, n \in \mathbb{N}$ preserve the convexity of order i, for $i = 0, 1, \ldots, n$ (i.e., $U_n^\varrho f$ is convex of order i, provided that f is convex of order i). This preservation of convexity by U_n^ϱ was proved first by Gonska and Păltănea (see [10, Th. 4.1], where also more details about the terminology and historical references can be found) and recently in [18], both using different methods.*

3 Global smoothness preservation

Over the last decades there has been considerable interest in the preservation of global smoothness in various contexts. This intensive research culminated in the book by Anastassiou and Gal [2].
The results in this section generalize the corresponding statements available in the literature for both Bernstein (see [3]) and genuine Bernstein–Durrmeyer operators (see [12, S.3.3.2]) and they supplement results on the behavior of the operators U_n^ϱ with respect to Lipschitz classes very recently given in [18].
To that end, we use first the following result given earlier by Cottin and Gonska [3, Theorem 2.2].

Lemma 7 *Let $k \geq 0$ and $s \geq 1$ be integers, and let $I = [a,b]$ and $I' = [c,d] \subset [a,b]$ be compact intervals with non-empty interior. Furthermore, let $L : C^k(I) \to C^k(I')$ be a linear operator having the following properties:*

(i) L is almost convex of orders $k-1$ and $k+s-1$,

(ii) L maps $C^{k+s}(I)$ into $C^{k+s}(I')$,

(iii) $L(\Pi_{k-1}) \subseteq \Pi_{k-1}$ and $L(\Pi_{k+s-1}) \subseteq \Pi_{k+s-1}$

(iv) $L(C^k(I)) \not\subset \Pi_{k-1}$

Then for all $f \in C^k(I)$ and all $\delta \geq 0$ we have

$$K_s(D^k Lf;\delta)_{I'} \leq \frac{1}{k!}||D^k Le_k|| \cdot K_s\left(f^{(k)}; \frac{1}{(k+s)^{\underline{s}}}\frac{||D^{k+s}Le_{k+s}||}{||D^k Le_k||}\delta\right) \qquad (5)$$

with the following notations for the rising and falling factorial:

$$x^{\overline{k}} = x(x+1)\cdot \ldots \cdot (x+k-1),$$
$$x^{\underline{k}} = x(x-1)\cdot \ldots \cdot (x-k+1).$$

First we provide the corresponding quantitative statement regarding the smoothing effect of the operators U_n^ϱ.

Theorem 8 *Let $k \geq 0$ and $s \geq 1$ be fixed integers. Then for all $n \geq k+s$, all $f \in C^k[0,1]$ and all $\delta \geq 0$ the following inequality holds*

$$K_s(D^k U_n^\varrho f;\delta)_{[0,1]} \leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} K_s\left(f^{(k)}; \varrho^s \frac{(n-k)^{\underline{s}}}{(n\varrho+k)^{\overline{s}}}\delta\right)_{[0,1]}. \qquad (6)$$

Proof. It can be easily verified that the assumptions of Lemma 7 are satisfied by the operators U_n^ϱ. Using its assertion and the fact that

$$D^m U_n^\varrho e_m = m!\varrho^m \frac{n^{\underline{m}}}{(n\varrho)^{\overline{m}}}, \quad m \in \{k, k+s\}. \qquad (7)$$

(easily deduced from [10, Lemma 5.1]) we immediately get the statement of our theorem. ■

We now consider two special cases of $s \geq 1$ which are of particular interest. The first is the case $s = 1$ leading to

Proposition 9 *Let $k \geq 0$ be a fixed integer. Then for all $n \geq k+1, f \in C^k[0,1]$ and $\delta \geq 0$ we have*

$$\omega_1(D^k U_n^\varrho f;\delta) \leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \widetilde{\omega}_1\left(f^{(k)}; \frac{\varrho(n-k)}{n\varrho+k}\delta\right) \\ \leq 1 \cdot \widetilde{\omega}_1(f^{(k)};\delta) \leq 2 \cdot \omega_1(f^{(k)};\delta).$$

where $\widetilde{\omega}_1(f,\cdot)$ denotes the least concave majorant of $\omega_1(f,\cdot)$ and is given by

$$\widetilde{\omega}_1(f,t) := \begin{cases} \sup\limits_{\substack{0\leq x\leq t\leq y\leq 1\\ x\neq y}} \dfrac{(t-x)\omega_1(f,y)+(y-t)\omega_1(f,x)}{y-x}, & \text{for } 0 \leq t \leq 1,\\ \omega_1(f,t), & \text{for } t > 1. \end{cases}$$

The leftmost inequality is best possible in the sense that for e_{k+1} both sides are equal and do not vanish.

Proof. Theorem 8 gives in this particular case

$$K_1(D^k U_n^\varrho f;\delta)_{[0,1]} \leq \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} K_1\left(f^{(k)}; \frac{\varrho(n-k)}{(n\varrho+k)}\delta\right)_{[0,1]}.$$

For the K-functional K_1 it is known from Brudnyi's representation theorem (see, e.g. [16], p.1258) that $K_1(f,\delta) = \frac{1}{2}\widetilde{\omega}_1(f,2\delta)$. Using this representation on both sides of the inequality involving K_1 and the fact that $\omega_1(f,t) \leq \widetilde{\omega}_1(f,t) \leq 2\omega_1(f,t)$ leads to our first assertion.
Furthermore, for the function $e_{k+1}(x) = x^{k+1}$ it can be easily verified that, for $n \geq k+1$ and $\delta > 0$, both sides in the leftmost inequality above equal

$$(k+1)! \cdot \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \cdot \frac{\varrho(n-k)}{n\varrho+k} \cdot \delta > 0.$$

■

Thus it follows

Corollary 10 *For a fixed integer $k \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(k)} \in Lip_M(\tau;[0,1])$ for some $M \geq 0$ and some $0 < \tau \leq 1$, then $D^k U_n^\varrho f$ is in the same Lipschitz class.*

The second case we discuss in more detail is $s = 2$. Here we get

Proposition 11 *Let $k \geq 0$ be a fixed integer. Then for all $n \geq k+2, f \in C^k[0,1]$ and $\delta \geq 0$ we have*

$$\omega_2(D^k U_n^\varrho f;\delta) \leq 3 \cdot \varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}} \left[1 + \varrho^2 \frac{(n-k)(n-k-1)}{2(n\varrho+k)(n\varrho+k+1)}\right] \omega_2\left(f^{(k)};\delta\right) \\ \leq \frac{9}{2}\omega_2(f^{(k)};\delta).$$

Proof. Instead of using the statement of Theorem 8 and the equivalence between the K–functional K_2 and the modulus ω_2, which would deteriorate the constants, we start from the definition of ω_2 and employ the function $Z_\delta(f)$ from Žuk's paper [20] (see Lemma 1 there).

First recall the identity

$$K_2(f;\delta) = K(f;\delta;C[0,1],C^2[0,1]) = K(f;\delta;C[0,1],W_{2,\infty}[0,1]),$$

where $W_{2,\infty}[0,1] := \{f \in C[0,1] : f' \text{ absolutely continuous}, \ ||f''||_{L_\infty} < \infty\}$, and $||f''||_{L_\infty} = vrai \sup\limits_{x\in[0,1]} |f''(x)|$.

Let now $f \in C^k[0,1], 0 < \delta < \frac{1}{2}$ be arbitrary given, and let $0 < h \le \delta$. Then for a typical difference figuring in the definition of $\omega_2(D^k U_n^\varrho f;\delta)$ we have

$$\begin{aligned}&|D^kU_n^\varrho f(x-h) - 2D^kU_n^\varrho f(x) + D^kU_n^\varrho f(x+h)| = \\ &\qquad |\{D^kU_n^\varrho(f-g;x-h) - 2D^kU_n^\varrho(f-g;x) + D^kU_n^\varrho(f-g;x+h)\}+ \\ &\qquad\quad \{D^kU_n^\varrho(g;x-h) - 2D^kU_n^\varrho(g;x) + D^kU_n^\varrho(g;x+h)\}|\end{aligned}$$

where $g \in C^k[0,1]$ with $g^{(k)} \in W_{2,\infty}[0,1]$ arbitrarily chosen.
Taking into account that the operator U_n^ϱ preserves convexity and using equality (7), the absolute value of the first term in braces can be estimated from above by

$$4||D^kU_n^\varrho(f-g)||_\infty \le 4\varrho^k \frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}}||(f-g)^{(k)}||_\infty.$$

For the modulus of the second expression in braces we have

$$\begin{aligned}&|D^kU_n^\varrho(g;x-h) - 2D^kU_n^\varrho(g;x) + D^kU_n^\varrho(g;x+h)| \\ &= |D^{k+2}U_n^\varrho(g;\xi)|\cdot h^2 \text{ (for some } \xi \text{ between } x-h \text{ and } x+h) \\ &\le |D^{k+2}U_n^\varrho g|\cdot h^2 \le \varrho^{k+2}\frac{n^{\underline{k+2}}}{(n\varrho)^{\overline{k+2}}}\cdot h^2\cdot ||g^{(k+2)}||_{L_\infty}.\end{aligned}$$

We substitute now the function $g^{(k)} \in W_{2,\infty}[0,1]$ by Žuk's function $Z_h(f^{(k)})$, yielding

$$||(f-g)^{(k)}|| = ||f^{(k)} - Z_h(f)|| \le \frac{3}{4}\cdot\omega_2(f^{(k)};h)$$

and

$$||g^{(k+2)}||_{L_\infty} = ||Z_h''(f)||_{L_\infty} \le \frac{3}{2}\cdot\frac{1}{h^2}\cdot\omega_2(f^{(k)};h).$$

Combining these estimates and taking into account the preceding steps we obtain

$$\begin{aligned}\omega_2(D^kU_n^\varrho f;\delta) &\le 4\cdot\frac{3}{4}\varrho^k\frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}}\omega_2(f^{(k)};\delta) + \frac{3}{2}\cdot\varrho^{k+2}\frac{n^{\underline{k+2}}}{(n\varrho)^{\overline{k+2}}}\cdot\omega_2(f^{(k)};h) \\ &= 3\cdot\varrho^k\frac{n^{\underline{k}}}{(n\varrho)^{\overline{k}}}\left[1 + \varrho^2\frac{(n-k)(n-k-1)}{2(n\varrho+k)(n\varrho+k+1)}\right]\omega_2\left(f^{(k)};\delta\right) \\ &\le \frac{9}{2}\omega_2(f^{(k)};\delta).\end{aligned}$$

■

Defining Lipschitz classes with respect to the second order modulus by

$$\mathrm{Lip}_M^*(\tau,[0,1]) := \left\{ f \in C[0,1] : \omega_2(f;\delta) \leq M \cdot \delta^\tau,\ 0 \leq \delta \leq \frac{1}{2} \right\},\ 0 < \tau \leq 2,$$

we get

Corollary 12 *For a fixed integer $k \geq 0$ the following assertion holds for all $n \in \mathbb{N}$. If $f^{(k)} \in Lip_M^*(\tau;[0,1])$ for some $M \geq 0$ and some $0 < \tau \leq 2$, then*

$$D^k U_n^\varrho f \in Lip_{4.5M}^*(\tau;[0,1]).$$

4 Upper and lower estimates for iterates of U_n^ϱ

The operators U_n^ϱ are of the form given in [13] for certain general positive linear operators preserving linear functions, so that we can apply the general results provided there for iterates of such operators. We have namely

$$U_n^\varrho(e_2;x) = \left(1 - \frac{\varrho+1}{n\varrho+1}\right) x^2 + \frac{\varrho+1}{n\varrho+1} x,$$

Hence an application of Theorem 6 as well as of Corollaries 7, 8 and 10 in [13] (with the coefficient of x^2 in the above $a_n = 1 - \frac{\varrho+1}{n\varrho+1}$) yields the following statements.

Corollary 13 *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\Phi : [0,1] \to \mathbb{R}$ be a function such that Φ^2 is concave. Then for $n, k \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$ the following pointwise estimate holds for the iterates of U_n^ϱ*

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq 2 \cdot K_2^\Phi \left(f; \frac{\varphi^2(x)}{\Phi^2(x)} \cdot \frac{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}{2} \right).$$

Corollary 14 *Let $\Phi : [0,1] \to \mathbb{R}$ be an admissible step–weight function of the Ditzian–Totik modulus and such that Φ^2 is concave. Then for all $n, k \in \mathbb{N},\ f \in C[0,1]$ and $x \in [0,1]$, we have*

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq c \cdot \omega_2^\Phi \left(f; \frac{\varphi(x)}{\Phi(x)} \cdot \sqrt{\frac{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}{2}} \right),$$

where the constant c depends only on the function Φ.
In particular, for $\Phi = \varphi^\lambda,\ \lambda \in [0,1],\ x \in [0,1]$ we get

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq c \cdot \omega_2^{\varphi^\lambda} \left(f; \varphi^{1-\lambda}(x) \cdot \sqrt{\frac{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}{2}} \right).$$

In terms of the classical modulus of smoothness we have

Corollary 15 *For all $f \in C[0,1]$, $n,k \in \mathbb{N}$, $x \in [0,1]$, and each $h > 0$ we have the following pointwise estimate*

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq \left[1 + \frac{1}{2h^2} \cdot \left(1 - (1 - \frac{\varrho+1}{n\varrho+1})^k\right) \cdot x(1-x)\right] \cdot \omega_2(f;h).$$

Taking, in particular, $h = \sqrt{\left(1 - (1 - \frac{\varrho+1}{n\varrho+1})^k\right) \cdot x(1-x)}$, and $h = \sqrt{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}$, yields

$$|[U_n^\varrho]^k(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2\left(f; \sqrt{\left(1 - (1 - \frac{\varrho+1}{n\varrho+1})^k\right) \cdot x(1-x)}\right), \quad \textit{and}$$

$$\|[U_n^\varrho]^k f - f\| \leq \frac{9}{8} \cdot \omega_2\left(f; \sqrt{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}\right),$$

respectively.

Furthermore, in terms of the second order Ditzian–Totik modulus we get

Corollary 16 *For all $f \in C[0,1]$, $n,k \in \mathbb{N}$, and $h \in \left(0, \frac{1}{2}\right]$ there holds the uniform estimate*

$$\|[U_n^\varrho]^k f - f\| \leq \left[1 + \frac{3}{2h^2} \cdot \left(1 - (1 - \frac{\varrho+1}{n\varrho+1})^k\right)\right] \cdot \omega_2^\varphi(f;h).$$

For the particular choice $h = \sqrt{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}$, this gives

$$\|[U_n^\varrho]^k f - f\| \leq \frac{5}{2} \cdot \omega_2^\varphi\left(f; \sqrt{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}\right).$$

Remark 17 *Note that for $n \in \mathbb{N}$, and $0 < \varrho < \infty$, one has $0 \leq 1 - \dfrac{\varrho+1}{n\varrho+1} < 1$ and $1 - \dfrac{\varrho+1}{n\varrho+1} \to 1$, for $n \to \infty$, so, for k fixed the results in the above imply uniform convergence as $n \to \infty$. For n fixed and $k \to \infty$, one has $[U_n^\varrho]^k \to B_1 f$ (see [11]).*

Applying the general results given above for $k = 1$ (no iterates) we get the following direct estimates, which supplement the corresponding results given by Păltănea [17, Th. 2.3].

Corollary 18 *Let $\varphi(x) = \sqrt{x(1-x)}$ and let $\Phi : [0,1] \to \mathbb{R}$ be an admissible step–weight function of the Ditzian–Totik modulus such that Φ^2 is concave. Then for $f \in C[0,1]$ and $x \in [0,1]$ the following estimates hold for U_n^ϱ:*

$$|U_n^\varrho(f;x) - f(x)| \leq 2 \cdot K_2^\Phi\left(f; \frac{\varphi^2(x)}{\Phi^2(x)} \cdot \frac{\varrho+1}{2(n\varrho+1)}\right)$$

and

$$|U_n^\varrho(f;x) - f(x)| \leq c \cdot \omega_2^\Phi \left(f; \frac{\varphi(x)}{\Phi(x)} \cdot \sqrt{\frac{\varrho+1}{2(n\varrho+1)}} \right),$$

where the constant c depends only on the function Φ.
In particular, for $\Phi = \varphi^\lambda,\ \lambda \in [0,1],\ x \in [0,1]$ we get

$$|U_n^\varrho(f;x) - f(x)| \leq c \cdot \omega_2^{\varphi^\lambda} \left(f; \varphi^{1-\lambda}(x) \cdot \sqrt{\frac{\varrho+1}{2(n\varrho+1)}} \right).$$

Furthermore, in terms of the second order Ditzian–Totik modulus with $h = \sqrt{\frac{\varrho+1}{n\varrho+1}}$ respectively $h = \sqrt{\frac{1}{n\varrho+1}}$, one has the uniform estimates

$$||U_n^\varrho(f;x) - f(x)|| \leq \frac{5}{2} \cdot \omega_2^{\varphi^\lambda} \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}} \right),$$

$$||U_n^\varrho(f;x) - f(x)|| \leq \frac{5+3\varrho}{2} \cdot \omega_2^{\varphi^\lambda} \left(f; \sqrt{\frac{1}{n\varrho+1}} \right).$$

In terms of the classical modulus of smoothness we get for the particular choices $h = \sqrt{\frac{\varrho+1}{n\varrho+1}x(1-x)}$ respectively $h = \sqrt{\frac{x(1-x)}{n\varrho+1}}$ the local estimates

$$|U_n^\varrho(f;x) - f(x)| \leq \frac{3}{2} \cdot \omega_2 \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}x(1-x)} \right),$$

$$|U_n^\varrho(f;x) - f(x)| \leq \frac{3+\varrho}{2} \cdot \omega_2 \left(f; \sqrt{\frac{x(1-x)}{n\varrho+1}} \right),$$

and for $h = \sqrt{\frac{\varrho+1}{n\varrho+1}}$ respectively $h = \sqrt{\frac{1}{n\varrho+1}}$ the global estimates

$$|U_n^\varrho(f;x) - f(x)| \leq \frac{9}{8} \cdot \omega_2 \left(f; \sqrt{\frac{\varrho+1}{n\varrho+1}} \right), \tag{8}$$

$$|U_n^\varrho(f;x) - f(x)| \leq \frac{9+\varrho}{8} \cdot \omega_2 \left(f; \sqrt{\frac{1}{n\varrho+1}} \right). \tag{9}$$

Remark 19 *The reason we choose two different representations for h is that one case is better suited for the instance when $\varrho \to \infty$, and the other for small values of ϱ. However this are not the only choices available. One can manipulate h in order to get the best possible estimate.*

Concerning the magnitude of the constants appearing in inequalities like (9), we show

Theorem 20 *For all $f \in C[0,1]$, the best possible constant c in the uniform estimate*

$$|U_n^\varrho(f;x) - f(x)| \le c \cdot \omega_2\left(f; \sqrt{\frac{1}{n\varrho+1}}\right). \tag{10}$$

cannot be smaller than 1 for $1 \le \varrho < \infty$.

Proof. Recall that for convex functions $f \in C[0,1]$ one has $U_n^\varrho f \ge f$ and $B_n f \ge f$. Moreover, according to Lemma 3, it holds $U_n^\varrho f \ge B_n f$, thus

$$0 \le B_n f(x) - f(x) \le U_n^\varrho f(x) - f(x),\ x \in [0,1],$$

implying

$$||B_n f - f|| \le ||U_n^\varrho f - f||.$$

Let now n and ϱ be fixed, $0 < \varepsilon < \frac{1}{n\varrho}$, and consider the convex function

$$f_\varepsilon(x) = \begin{cases} 0, & 0 \le x \le 1-\varepsilon, \\ \frac{1}{\varepsilon}x + 1 - \frac{1}{\varepsilon}, & 1-\varepsilon < x \le \varepsilon. \end{cases}$$

We have

$$B_n f_\varepsilon(x) = \sum_{k=0}^{n-1} p_{n,k}(x) f_\varepsilon\left(\frac{k}{n}\right) + x^n \cdot f_\varepsilon(1) = x^n,$$

thus

$$||B_n f_\varepsilon - f_\varepsilon|| = \max_{x\in[0,1]} (B_n f_\varepsilon(x) - f_\varepsilon(x)) = B_n f_\varepsilon(1-\varepsilon) - f_\varepsilon(1-\varepsilon) = (1-\varepsilon)^n.$$

Next we compute

$$\omega_2\left(f_\varepsilon; \frac{1}{\sqrt{n\varrho+1}}\right) = \sup_{\substack{|h|\le \frac{1}{\sqrt{n\varrho+1}} \\ x\pm h\in[0,1]}} |f_\varepsilon(x-h) - 2f_\varepsilon(x) + f_\varepsilon(x+h)| = f_\varepsilon(1) = 1,$$

since the largest possible value for the second order difference is obtained for $x = 1-\varepsilon, h = \varepsilon (< \frac{1}{n\varrho} \le \frac{1}{\sqrt{n\varrho+1}})$. Hence, there holds

$$\frac{||B_n f_\varepsilon - f_\varepsilon||}{\omega_2\left(f_\varepsilon; \frac{1}{\sqrt{n\varrho+1}}\right)} = (1-\varepsilon)^n.$$

Assume that there exists a constant a such that

$$\frac{||B_n f - f||}{\omega_2\left(f; \frac{1}{\sqrt{n\varrho+1}}\right)} \le a < 1, \quad \text{for each } f \in [0,1].$$

Then, since $\lim_{\varepsilon\to 0}(1-\varepsilon)^n = 1$, for fixed n, we can choose $\varepsilon > 0$, such that $(1-\varepsilon)^n > a$. Taking the function $f = f_\varepsilon$, we arrive from above to a contradiction.
Thus

$$\|B_n f_\varepsilon - f_\varepsilon\| \le c_1 \cdot \omega_2\left(f_\varepsilon; \sqrt{\frac{1}{n\varrho+1}}\right) \text{ with } c_1 \ge 1.$$

Since
$$||B_n f_\varepsilon - f_\varepsilon|| \le ||U_n^\varrho f_\varepsilon - f_\varepsilon||,$$
it follows that
$$|U_n^\varrho f_\varepsilon - f_\varepsilon| \le c \cdot \omega_2 \left(f_\varepsilon; \sqrt{\frac{1}{n\varrho + 1}} \right), \text{ with } c \ge 1,$$
so for the best constant in (10) it also holds $c \ge 1$. ■

We apply now the results of Corollaries $13 - 16$ in [13] for iterates of U_n^ϱ. This shows that lower inequalities in terms of the classical moduli, corresponding to the upper ones in the above, are not possible. More precisely, for $k \in \mathbb{N}$ fixed, we have

Corollary 21 *Lower inequalities of the form*
$$C(f)\omega_2 \left(f; \sqrt{1 - (1 - \frac{\varrho + 1}{n\varrho + 1})^k} \right) \le \|[U_n^\varrho]^k(f) - f\| \quad \textit{for all} \;\; f \in C[0,1]$$
do not hold.

Corollary 22 *The lower pointwise estimates*
$$C(f)\omega_2 \left(f; \sqrt{\left(1 - (1 - \frac{\varrho + 1}{n\varrho + 1})^k\right) x(1 - x)} \right) \le |[U_n^\varrho]^k(f; x) - f(x)| \;\; \textit{for} \;\; f \in C[0,1]$$
do not hold.

Corollary 23 *Let $0 < \lambda \le 1$ be fixed. The lower pointwise estimates*
$$C(f)\omega_2 \left(f; \varphi^{1-\lambda}(x) \sqrt{\frac{1 - (1 - \frac{\varrho+1}{n\varrho+1})^k}{2}} \right) \le |[U_n^\varrho]^k(f; x) - f(x)|, \; f \in C[0,1],$$
do not hold.

Moreover, we have

Corollary 24 *For $l \ge 3$ it is not possible to have an inequality of the type*
$$C(f) \cdot \omega_l \left(f; \sqrt[l]{1 - (1 - \frac{\varrho + 1}{n\varrho + 1})^k} \right) \le \|[U_n^\varrho]^k(f) - f\|$$
for all $f \in C[0,1]$ and all $n \in \mathbb{N}$.

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On Zermelo's navigation problem with *Mathematica*

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Abstract

Zermelo's navigation problem requires determination of the optimal trajectory and the associated guidance of a boat (ship, aircraft) traveling between two given points so that the transit time is minimized. In the present paper we study this minimum time optimal control problem mainly numerically by the power of *Mathematica*. By the Manipulate command we show the families of trajectories of the navigation problem for certain values of parameters.

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1 Introduction

According to [5, p. 150], Zermelo was the first to formulate and solve in [16] and [17] a problem that now is called the navigation problem of Zermelo. The problem came to Zermelo's mind when the airship Graf Zeppelin circumnavigated the Earth in August 1929. He considered a vector field given in the Euclidean plane that describes the distribution of winds as depending on place and time and treats the question how an airship or plane, moving at a constant speed against the surrounding air, has to fly in order to reach a given point B from a given point A in the shortest time possible. With

(i) $x = x(t)$ and $y = y(t)$ the Cartesian coordinates of the airship at time t,

(ii) $u = u(t, x, y)$ and $v = v(t, x, y)$ the corresponding components of the vector field representing the speed of the wind (water) in respect to the Cartesian system,

(iii) $\beta = \beta(t, x, y)$ the angle between the momentary speed (u_0, v_0) of the airship against the surrounding air and the x-axis, and normalizing to $|(u_0, v_0)| = 1$, one has the system of differential equations that describes the problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u + \cos\beta \ \text{ and } \ \frac{\mathrm{d}y}{\mathrm{d}t} = v + \sin\beta.$$

Using the calculus of variations, Zermelo obtained the following differential equation for the heading angle β

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \sin^2\beta\,\frac{\partial v}{\partial x} + \sin\beta\cos\beta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \cos^2\beta\,\frac{\partial u}{\partial y}.$$

The previous differential equation called the Zermelo's differential equation, is a necessary condition for β to be the optimal guidance function.

Other historical remarks on the contribution of Zermelo and others of his time to this problem may be found in [5, pp. 150–152]. We also note the interesting papers [11] and [10]. In all these papers as well as in [2, pp. 17–22], the investigation was based on the calculus of variations. Later on the main tool of investigation became the maximum principle of Pontryagin, [15], [3, pp. 77–79], [4, pp. 228–231], and [13]. We emphasize the interest on the navigation problem for routing and guidance of airplanes (boats) as a part of flight planning [6], [9], [14], [1], and [8]. Since the navigation problem is obviously nonlinear, up to the author's knowledge, there is no solution in closed form and its approaches make intensively use of numerical methods [7], [6], [9], [14], [3, pp. 77–79], [1], and [8]. Hereafter we will use the power of *Mathematica* to study numerically the navigation problem of Zermelo.

2 A planar form of the navigation problem

Zermelo's navigation problem is a minimum-time paths through a region of position-dependent-time vector velocity. Under a general form the problem supposes that a boat travels through a zone of currents. The magnitude and direction of the currents are given by the functions of time and position

$$u = u(t, x, y) \ \text{ and } \ v = v(t, x, y),$$

where (x, y) are Cartesian coordinates giving the position of the boat at time t and (u, v) are the velocity components of the boat at the current point (x, y) at time t in the x and y directions, respectively. The speed of the boat relative to the water is supposed to be a constant $V > 0$.

The problem requires to steer the boat in such a way to minimize the time necessary to travel from a given point $A = (x_1, y_1)$ at instant a to another given point $B = (x_2, y_2)$ at instant b.

The equations of the motion are

$$\begin{cases} x'(t) = V\cos\beta(t) + u(t, x(t), y(t)), \\ y'(t) = V\sin\beta(t) + v(t, x(t), y(t)), \quad t \in [a, b], \end{cases} \tag{2.1}$$

where β is the heading angle of the boat's axis relative to a fixed coordinate axis, let it be the horizontal axis, and is the control function.

In a more compact form the navigation problem can be stated as

$$\begin{array}{lll} x'(t) = V\cos\beta(t) + u(t, x(t), y(t)), & & \\ y'(t) = V\sin\beta(t) + v(t, x(t), y(t)), & \text{dynamics} & \\ x(a) = x_1, \quad y(a) = y_1, & \text{initial conditions,} & (2.2) \\ x(b) = x_2, \quad y(b) = y_2, & \text{final conditions,} & (2.3) \\ [a, b], & \text{finite horizon,} & \\ \beta \in C([a, b], \mathbb{R}), & \text{control function,} & \\ u, v : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, & \text{components of the velocity of water,} & \\ V, & \text{relative speed of the boat,} & \\ g(b) = b \to \min, & \text{cost functional.} & \end{array}$$

We suppose that the functions u and v are continuous in the first variable and of class C^1 in the second and third variables.

By the maximum principle of Pontryagin under the form in [3, §2.4], we have that

$$\lambda_1' = -\frac{\partial H}{\partial x} = -\lambda_1 \frac{\partial u}{\partial x} - \lambda_2 \frac{\partial v}{\partial x}, \tag{2.4}$$

$$\lambda_2' = -\frac{\partial H}{\partial y} = -\lambda_1 \frac{\partial u}{\partial y} - \lambda_2 \frac{\partial v}{\partial y}, \tag{2.5}$$

$$0 = \frac{\partial H}{\partial \beta} = V(-\lambda_1 \sin\beta + \lambda_2 \cos\beta) \implies \lambda_1 \sin\beta = \lambda_2 \cos\beta, \tag{2.6}$$

where the Hamiltonian of the system is

$$H(t, x, y, \beta, \lambda_1, \lambda_2) = \lambda_1(V\cos\beta + u(t, x, y)) + \lambda_2(V\sin\beta + v(t, x, y)) + 1. \tag{2.7}$$

If we solve the system of differential equations (2.1) and (2.4)–(2.6), then we find $x, y, \beta, \lambda_1, \lambda_2$. By (2.6) and the initial and final conditions (2.2)–(2.3) we find the solutions that solve the navigation problem. By [10] or [11] we have that there exists a solution.

Remark 1 If the functions u and v do not depend explicitly upon t, that is,

$$\begin{cases} x'(t) = V\cos\beta(t) + u(x(t), y(t)), \\ y'(t) = V\sin\beta(t) + v(x(t), y(t)), \end{cases} \tag{2.8}$$

then the problem is autonomous and therefore we take $a = 0$ initial instant and $(0, 0)$ is the final point. $\triangle$

From now on we suppose that the functions u and v do not depend on t, i.e., the equations (2.8) hold. Then the Hamiltonian does not explicitly depend on

t, and then $H = constant$ is a prime integral. Because we minimize time, this constant has to be 0. Then from (2.7) we have that $H = 0$. We invoke (2.6) and get that

$$\lambda_1 = \frac{-\cos\beta}{V + u\cos\beta + v\sin\beta} \quad \text{and} \quad \lambda_2 = \frac{-\sin\beta}{V + u\cos\beta + v\sin\beta}\,. \tag{2.9}$$

Substituting (2.9) in (2.4) and (2.5) (or asking for consistency between (2.6) and $\mathrm{d}(\partial H/\partial\beta)/\,\mathrm{d}t = 0$) it follows the Zermelo's navigation formula

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \sin^2\beta\,\frac{\partial v}{\partial x} + \sin\beta\cos\beta\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \cos^2\beta\,\frac{\partial u}{\partial y}\,. \tag{2.10}$$

Now the nonlinear equations (2.8) and (2.10) give the general solution for our navigation problem. If we take into account the initial and final conditions, we get the concrete solution if the data are consistent.

We now study a special case considering for the current of water the following function

$$u(x,y) = -(V/h)y, \quad v(x,y) = 0, \tag{2.11}$$

where h is a nonzero real number. Now we express the data of the problem as functions depending on the angle β. From (2.10) we write

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{V}{h}\cos^2\beta \implies \frac{\mathrm{d}\beta}{\cos^2\beta} = \frac{V}{h}\,\mathrm{d}t \implies \tan\beta = \tan\beta_f + \frac{V}{h}(t - t_f), \tag{2.12}$$

where t_f is the final time and β_f is the final angle, both still unknown.
From the second equation in (2.8) we have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = V\sin\beta \implies \frac{\mathrm{d}y}{\mathrm{d}\beta} = V\sin\beta\,\frac{h}{V}\frac{1}{\cos^2\beta} = h\frac{\sin\beta}{\cos^2\beta},$$
$$\implies y = y(\beta) = h(\sec\beta - \sec\beta_f).$$

Now we take into account the first equation in (2.8), that is,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = V\cos\beta - \frac{V}{h}\,y \implies \mathrm{d}x = h(\sec\beta - \sec^3\beta + \sec\beta_f\sec^2\beta)\,\mathrm{d}\beta.$$

From the last equation by integration we find that

$$x = x(\beta) = -\frac{h}{2}\left[\ln\frac{\sec\beta_f + \tan\beta_f}{\sec\beta + \tan\beta} + (\tan\beta_f - \tan\beta)\sec\beta_f \right.$$
$$\left. -\ (\sec\beta_f - \sec\beta)\tan\beta_f\right].$$

The angular limits of the navigation problem, the initial angle β_0 and the final one β_f, can be obtained asking from the following system of nonlinear equation

$$x(\beta_0) = x_1 \ \text{and}\ y(\beta_0) = y_1. \tag{2.13}$$

Now all the elements of the trajectory are determined and we pass to the numerical approach.

3 Numerical approach to the navigation problem

We introduce now a particular case of (2.11) such that $V = 2$ and $h = 2$. Clearly the initial time is $a = 0$. We choose the initial position at $(x(0), y(0)) = (7.32, -3.727)$. The final position is at the origin $(0, 0)$. Then by (2.13) we have that the initial angle is $105\,^\circ$ whereas the final angle is $240.004\,^\circ$. The minimum time to steer the boat from the initial point to the origin by formula (2.12) is 5.46439. By Figure 1 the trajectory of this problem is given in blue whereas the heading direction vectors appear in red. The numerical results in the picture are consistent to [3, pp. 77–79].

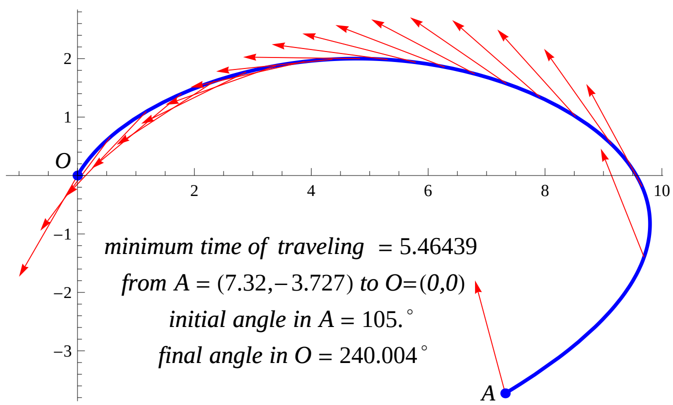

Figure 1: A navigation problem

Figure 2 exhibits a case of dynamical navigation problem when the initial data belong to some intervals. The black (horizontal) vector at A represents the direction and magnitude of the current vector at the initial point. The blue (oblique) vector at A represents the tangent to the trajectory. The figure is self-explanatory.

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```
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 {{h, 2, "h"}, 0.5, 5, 0.5, Appearance → "Labeled"},
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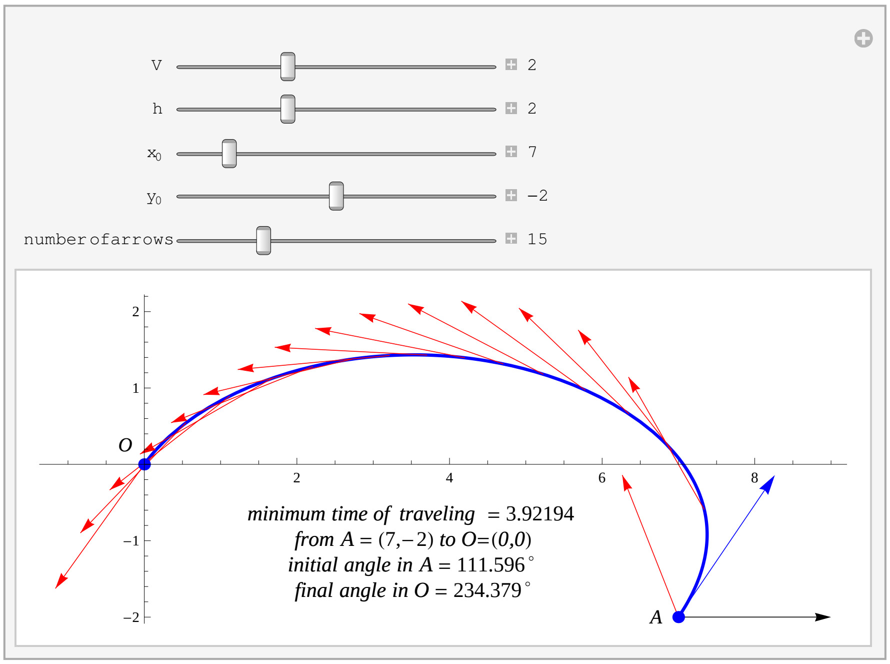

Figure 2: A dynamical navigation problem

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Simultaneous Approximation by a Class of Szász-Mirakjan Operators

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

We study the shape preserving property and the simultaneous approximation by a sequences of Durrmeyer type modification of Szász-Mirakjan operators with a parameter. These operators preserve linear functions and make a link between the Phillips operators and the classical Szász-Mirakjan operators.

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Key Words and Phrases: Szász-Mirakjan type operators, Durrmeyer type operators, Phillips operators, shape preserving property, simultaneous approximation.

1 Introduction

Several Durrmeyer type modification of Szász-Mirakjan operators are known. A not exhaustive list of them is given in References. In [10] a new Durrmeyer modification of Szász-Mirakjan operators is given, using two parameters $\alpha > 0,\ \rho > 0$, in the following way:

$$L_\alpha^\rho(f,x) = e^{-\alpha x} f(0) + \sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t) f(t)\, dt,\ x \in [0,\infty), \tag{1.1}$$

where

$$s_{\alpha,k}(x) = e^{-\alpha x} \cdot \frac{(\alpha x)^k}{k!}, \quad \Theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)} \cdot e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}, \tag{1.2}$$

and where $f : [0,\infty) \to \mathbb{R}$ is an integrable function for which the integrals and the series above are convergent.

Operators L_α^ρ preserve linear functions, as can be easily verified. For $\rho = 1$, when $\Theta_{\alpha,k}^\rho$ is equal to $\alpha s_{\alpha,k-1}$, L_α^ρ becomes the Phillips operators, [11]. On the other hand we shall prove that the limit of operators L_α^ρ for $\rho \to \infty$ are the Szász-Mirakjan operators, [12], with a continuous parameter $\alpha > 0$, given by

$$S_\alpha(f,x) = \sum_{k=0}^{\infty} s_{\alpha,k}(x) f\left(\frac{k}{\alpha}\right), \quad x \geq 0. \tag{1.3}$$

Also we shall prove that operators L_α^ρ preserve convexity of higher order and they have the property of simultaneous approximation on compact sets.

The link given by operators L_α^ρ between the Phillips operators and the Szász-Mirakjan operators is the analogous with the link, shown in [2], between the "genuine" Durrmeyer operators and the Bernstein operators, given by a class of Durrmeyer type operators.

2 Auxiliary results

We use the following notation $I = [0,\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $e_m(x) = x^m$, $(x \in I,\ m = 0,1,\ldots)$. Denote $W = \{f : I \to \mathbb{R},\ f$ integrable and $\exists M > 0,\ \exists q \geq 0 :\ |f(t)| \leq Me^{qt},\ (t \geq 0)\}$. We denote by W_α^ρ the set of functions $f \in W$, satisfying the condition given in the definition of W with $q < \alpha\rho$. In [10] it is given the following simple result.

Lemma A *$L_\alpha^\rho(f)$ exists for any $f \in W_\alpha^\rho$, when $\alpha > 0$ and $\rho > 0$.*

For fixed $\alpha > 0$ and $\rho > 0$ set $T_m(x) = L_\alpha^\rho(e_m, x)$, for $m \in \mathbb{N}_0$, and $x \geq 0$.

Lemma 1 *Let $x \in [0,\infty)$ and $m \in \mathbb{N}$. We have*

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = x^2 + \frac{\rho+1}{\alpha\rho} \cdot x. \tag{2.1}$$

$$T_m(x) = \left(x + \frac{m-1}{\alpha\rho}\right) T_{m-1}(x) + \frac{x}{\alpha} \cdot T'_{m-1}(x). \tag{2.2}$$

Consequently operators L_α^ρ transform any polynomial into a polynomial of the same degree.

Proof. $T_0(x)$ and $T_1(x)$ can be obtained immediately by a direct computation. It follows relation (2.2) for $m = 1$. Let now $m \geq 2$. Then we have successively

$$\begin{aligned} T_m(x) &= \sum_{k=1}^{\infty} s_{\alpha,k}(x)\frac{1}{\Gamma(k\rho)(\alpha\rho)^m}\int_0^{\infty} e^{-s}s^{k\rho+m-1}ds \\ &= \sum_{k=1}^{\infty} s_{\alpha,k}(x)\frac{(k\rho)(k\rho+1)\dots(k\rho+m-1)}{(\alpha\rho)^m} \\ &= \sum_{k=1}^{\infty}\Big(\Big(\frac{m-1}{\alpha\rho}+x\Big)s_{\alpha,k}(x)+\frac{x}{\alpha}\cdot s'_{\alpha,k}(x)\Big)\frac{(k\rho)(k\rho+1)\dots(k\rho+m-2)}{(\alpha\rho)^{m-1}} \\ &= \left(x+\frac{m-1}{\alpha\rho}\right)T_{m-1}(x)+\frac{x}{\alpha}\cdot T'_{m-1}(x). \end{aligned}$$

From relation (2.2), we deduce $T_2(x)$ and the last part of Lemma 1. ■

Corollary 2 *For $m \in \mathbb{N}$, $x \geq 0$, we have*

$$\begin{aligned} T_m(x) = x^m &+ \frac{\rho+1}{2\alpha\rho}\cdot m(m-1)x^{m-1} \\ &+ \frac{\rho+1}{24(\alpha\rho)^2}\cdot m(m-1)(m-2)[(3m-5)\rho+3m-1]x^{m-2}+\dots \end{aligned} \tag{2.3}$$

Proof. Formula (2.3) can be obtained by induction using relation (2.2). ■

Lemma 3 *Let $\alpha > 0$, $\rho > 0$, $k \in \mathbb{N}$ and $m \geq 0$.*

i) If $\alpha\rho > m$, then

$$\int_0^{\infty} \Theta^{\rho}_{\alpha,k}(t)e^{mt}dt = \Big(\frac{\alpha\rho}{\alpha\rho-m}\Big)^{k\rho}. \tag{2.4}$$

ii) If $\alpha\rho > 2m$, and $c > 0$, then

$$\int_c^{\infty} \Theta^{\rho}_{\alpha,k}(t)e^{mt}dt \leq \sqrt{\frac{k\rho}{2\pi}}\cdot\frac{2^{k\rho}}{\nu_m(\rho,c)e^{\nu_m(\rho,c)}},\ \text{where } \nu_m(\rho,c)=\frac{c}{2}\cdot(\alpha\rho-2m). \tag{2.5}$$

Proof. i) It follows by a direct computation. See also [10] - Lemma 1.

ii) Using the change of variable $u = \alpha\rho t$, then taking into account that $u = 2k\rho$ is the point of maximum for the function $u \mapsto u^{k\rho}e^{-\frac{u}{2}}$ and finally using the well-known formula $\Gamma(x) \geq \sqrt{\frac{2\pi}{x}}\cdot\left(\frac{x}{e}\right)^x$, for $x > 0$, we obtain successively

$$\int_c^\infty \Theta^\rho_{\alpha,k}(t)e^{mt}dt = \int_{\alpha\rho c}^\infty \frac{u^{k\rho-1}e^{-u}e^{\frac{mu}{\alpha\rho}}}{\Gamma(k\rho)}\,du$$
$$\leq \frac{(2k\rho)^{k\rho}e^{-k\rho}}{\alpha\rho c\Gamma(k\rho)}\int_{\alpha\rho c}^\infty e^{-\frac{u}{2}}e^{\frac{mu}{\alpha\rho}}du$$
$$= \frac{(2k\rho)^{k\rho}e^{-k\rho}}{\Gamma(k\rho)\nu_m(\rho,c)e^{\nu_m(\rho,c)}}$$
$$\leq \sqrt{\frac{k\rho}{2\pi}}\cdot\frac{2^{k\rho}}{\nu_m(\rho,c)e^{\nu_m(\rho,c)}}.$$

■

3 The limit of operators L^ρ_α when $\rho\to\infty$

In [10] there is proved that for each function f which belongs to the closure in sup-norm of the space of polynomials, and for each constant $\alpha>0$, $L^\rho_\alpha(f)$ converges on compacts to $S_\alpha(f)$. Here we extend this result in the following way.

Theorem 4 *For any $\alpha>0$, any $f\in W$ and any $b>0$ there is $\rho_0>0$, such that $L^\rho_\alpha(f)$ exists for all $\rho\geq\rho_0$ and we have*

$$\lim_{\rho\to\infty} L^\rho_\alpha(f,x) = S_\alpha(f,x),\ \textit{uniformly for}\ x\in[0,b]. \tag{3.1}$$

Proof. Let $M>0$ and $q>0$, such that $f(t)\leq Me^{qt}$, for $t\geq0$. Choose $\varepsilon>0$ arbitrarily. Take $\rho_0>\frac{q}{\alpha}$. Then $f\in W^\rho_\alpha$, for $\rho\geq\rho_0$ and hence, from Lemma A, $L^\rho_\alpha(f)$ exists. Since the function $\rho\mapsto\left(\frac{\alpha\rho}{\alpha\rho-q}\right)^\rho$ is decreasing on interval $[\rho_0,\infty)$, we obtain from Lemma 3- i), for $x\in[0,b]$ and $\rho\geq\rho_0$:

$$\sum_{k=1}^\infty s_{\alpha,k}(x)\int_0^\infty \Theta^\rho_{\alpha,k}(t)|f(t)|dt \leq M\sum_{k=1}^\infty \frac{1}{k!}\Big(\alpha b\Big(\frac{\alpha\rho_0}{\alpha\rho_0-q}\Big)^{\rho_0}\Big)^k.$$

Also we have
$$\sum_{k=1}^\infty s_{\alpha,k}(x)\Big|f\Big(\frac{k}{\alpha}\Big)\Big| \leq M\sum_{k=1}^\infty\frac{1}{k!}\cdot\Big(\alpha b e^{\frac{q}{\alpha}}\Big)^k.$$

From these two absolute and uniformly convergent series we deduce that there is $N\in\mathbb{N}$, such that, for all $x\in[0,b]$ and $\rho\geq\rho_0$ we have

$$\Big|\sum_{k=N+1}^\infty s_{\alpha,k}(x)\int_0^\infty\Theta^\rho_{\alpha,k}(t)f(t)dt\Big| < \frac{\varepsilon}{6},\quad \Big|\sum_{k=N+1}^\infty s_{\alpha,k}(x)f\Big(\frac{k}{\alpha}\Big)\Big| < \frac{\varepsilon}{6}.$$

4

Hence we obtain, for all $x \in [0, b]$ and $\rho \geq \rho_0$,

$$|L_\alpha^\rho(f, x) - S_\alpha(f, x)| \leq \sum_{k=1}^{N} s_{\alpha,k}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t) \left| f(t) - f\left(\frac{k}{\alpha}\right) \right| dt + \frac{\varepsilon}{3}. \qquad (3.2)$$

Set $M_1 = \max_{1 \leq k \leq N} \left| f\left(\frac{k}{\alpha}\right) \right|$. Fix a number $c > \frac{N \ln 4}{\alpha}$. Then

$$\int_c^\infty \Theta_{\alpha,k}^\rho(t) \left| f(t) - f\left(\frac{k}{\alpha}\right) \right| dt \leq \int_c^\infty \Theta_{\alpha,k}^\rho(t)(Me^{qt} + M_1)dt, \ 1 \leq k \leq N.$$

From the choice of c and using the notation $\nu_q(\rho, c) = \frac{c}{2}(\alpha\rho - 2q)$ it follows

$$\lim_{\rho\to\infty} \sqrt{\frac{k\rho}{2\pi}} \cdot \frac{2^{k\rho}}{\nu_q(\rho, c)e^{\nu_q(\rho,c)}} = 0.$$

This is also true, if instead of q we have 0. Then, by Lemma 3 - ii), for the choices $m = q$ and $m = 0$ we obtain

$$\lim_{\rho\to\infty} \int_c^\infty \Theta_{\alpha,k}^\rho(t)(Me^{qt} + M_1)dt = 0, \text{ for } 1 \leq k \leq N.$$

Since $\sum_{k=1}^{N} s_{\alpha,k}(x) \leq 1$, for $x \geq 0$ it follows that there is $\rho_\varepsilon^1 \geq \rho_0$, such that, for $\rho > \rho_\varepsilon^1$:

$$\sum_{k=1}^{N} s_{\alpha,k}(x) \int_c^\infty \Theta_{\alpha,k}^\rho(t) \left| f(t) - f\left(\frac{k}{\alpha}\right) \right| dt < \frac{\varepsilon}{6}.$$

By combining with relation (3.2) we obtain for all $x \in [0, b]$ and $\rho \geq \rho_\varepsilon^1$

$$|L_\alpha^\rho(f, x) - S_\alpha(f, x)| \leq \sum_{k=1}^{N} s_{\alpha,k}(x) \int_0^c \Theta_{\alpha,k}^\rho(t) \left| f(t) - f\left(\frac{k}{\alpha}\right) \right| dt + \frac{\varepsilon}{2}. \qquad (3.3)$$

From the choice of c we have $\frac{k}{\alpha} \in (0, c)$, for $1 \leq k \leq N$. From the continuity of function f there is $\delta > 0$, such that $\delta < \min\left\{\frac{1}{\alpha}, c - \frac{N}{\alpha}\right\}$ and $\left| f(t) - f\left(\frac{k}{\alpha}\right) \right| < \frac{\varepsilon}{6N}$, if $\left| t - \frac{k}{\alpha} \right| \leq \delta$. It follows,

$$\sum_{k=1}^{N} s_{\alpha,k}(x) \int_{\frac{k}{\alpha}-\delta}^{\frac{k}{\alpha}+\delta} \Theta_{\alpha,k}^\rho(t) \left| f(t) - f\left(\frac{k}{\alpha}\right) \right| dt < \frac{\varepsilon}{6}. \qquad (3.4)$$

Denote $M_2 = \max_{t\in[0,c]} |f(t)|$. Take $\rho > \max\left\{\frac{1}{\delta\alpha}, 1\right\}$. For $1 \leq k \leq N$, denote $y_k(\rho) = k\rho - \delta\alpha\rho$ and $z_k(\rho) = k\rho + \delta\alpha\rho$. Since $y_k(\rho) < k\rho - 1 < z_k(\rho)$ function $u \mapsto u^{k\rho-1}e^{-u}$ is increasing on $[0, y_k(\rho)]$ and decreasing on $[z_k(\rho), \alpha\rho c]$. Using the change of variable $u = \alpha\rho t$ and the inequality $\Gamma(x) \geq \sqrt{\frac{2\pi}{x}} \cdot \left(\frac{x}{e}\right)^x$, for $x > 0$, we obtain successively

$$\begin{aligned}\int_0^{\frac{k}{\alpha}-\delta}\Theta^{\rho}_{\alpha,k}(t)\Big|f(t)-f\Big(\frac{k}{\alpha}\Big)\Big|dt &\le 2M_2\int_0^{y_k(\rho)}\frac{u^{k\rho-1}e^{-u}}{\Gamma(k\rho)}du\\ &\le 2M_2\frac{(y_k(\rho))^{k\rho}e^{-y_k(\rho)}}{\Gamma(k\rho)}\\ &\le\sqrt{\frac{2}{\pi}}\cdot M_2\sqrt{k\rho}\left[\Big(1-\frac{\alpha\delta}{k}\Big)^k e^{\alpha\delta}\right]^{\rho}.\end{aligned}$$

But $(1-\frac{\alpha\delta}{k}\Big)^k e^{\alpha\delta}<1$ and hence there is $\rho^2_\varepsilon>\max\left\{\rho^1_\varepsilon,\frac{1}{\delta\alpha},1\right\}$, such that for all $\rho>\rho^2_\varepsilon$ and $1\le k\le N$ to have

$$\int_0^{\frac{k}{\alpha}-\delta}\Theta^{\rho}_{\alpha,k}(t)\Big|f(t)-f\Big(\frac{k}{\alpha}\Big)\Big|dt<\frac{\varepsilon}{6N}. \tag{3.5}$$

In a similar way we obtain

$$\begin{aligned}\int_{\frac{k}{\alpha}+\delta}^{c}\Theta^{\rho}_{\alpha,k}(t)\Big|f(t)-f\Big(\frac{k}{\alpha}\Big)\Big|dt &\le 2M_2\int_{z_k(\rho)}^{\alpha\rho c}\frac{u^{k\rho-1}e^{-u}}{\Gamma(k\rho)}du\\ &\le 2M_2(\alpha\rho c-z_k(\rho))\frac{(z_k(\rho))^{k\rho-1}e^{-z_k(\rho)}}{\Gamma(k\rho)}\\ &\le\sqrt{\frac{2}{\pi}}\cdot M_2\frac{c\alpha-k-\alpha\delta}{k+\alpha\delta}\sqrt{k\rho}\left[\Big(1+\frac{\alpha\delta}{k}\Big)^k e^{-\alpha\delta}\right]^{\rho}.\end{aligned}$$

Since $(1+\frac{\alpha\delta}{k}\Big)^k e^{-\alpha\delta}<1$ there is $\rho^3_\varepsilon>\rho^2_\varepsilon$, such that for all $\rho>\rho^3_\varepsilon$ and $1\le k\le N$ to have

$$\int_{\frac{k}{\alpha}+\delta}^{c}\Theta^{\rho}_{\alpha,k}(t)\Big|f(t)-f\Big(\frac{k}{\alpha}\Big)\Big|dt<\frac{\varepsilon}{6N}. \tag{3.6}$$

From relations (3.3), (3.4), (3.5) and (3.6) we arrive to

$$|L^{\rho}_{\alpha}(f,x)-S_{\alpha}(f,x)|<\varepsilon,\ \text{for } \rho>\rho^3_\varepsilon.$$

■

4 Convexity of operators $L^{\rho}_{\alpha}(f)$

We extend the definition of functions $s_{\alpha,k}$, for all $k\in\mathbb{Z}$ by $s_{\alpha,k}=0$, if $k<0$.

Lemma 5 *Let $\alpha>0$, $\rho>0$, $r\in\mathbb{N}_0$.*

i) For $k\in\mathbb{Z}$ and $x\in I$ we have

$$s^{(r)}_{\alpha,k}(x)=\alpha^r\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}s_{\alpha,k-j}(x). \tag{4.1}$$

ii) Let $f \in W_\alpha^\rho \cap C^r(I)$. Then for any $x \in I$ we have:

$$(L_\alpha^\rho(f,x))^{(r)} = s_{\alpha,0}^{(r)}(x)f(0) + \sum_{k=1}^{\infty} s_{\alpha,k}^{(r)}(x)\int_0^\infty \Theta_{\alpha,k}^\rho(t)f(t)\,dt. \tag{4.2}$$

Proof. i) It follows by induction by relation: $s'_{\alpha,k}(x) = \alpha(s_{\alpha,k-1}(x) - s_{\alpha,k}(x))$.

ii) It suffices to show that, for any points $0 \le a < b$ and any $r \in \mathbb{N}$ we have

$$T_{a,b}^r := \max_{x\in[a,b]} |s_{\alpha,0}^{(r)}(x)f(0)| + \sum_{k=1}^{\infty} \max_{x\in[a,b]} \left| s_{\alpha,k}^{(r)}(x)\int_0^\infty \Theta_{\alpha,k}^\rho(t)f(t)\,dt\right| < \infty.$$

If these conditions are true, then relation (4.2) can be proved by induction. Indeed, let denote by $S_r(x)$ the series given in the right side of relation (4.2). Suppose that $(L_\alpha^\rho(f))^{(r)}(x_0) = S_{r-1}(x_0)$, for $x_0 \in I$ and $r \ge 1$. Choose a compact neighbourhood of x_0 of the form $[a,b] \subset I$. Using condition $T_{a,b}^r < \infty$, from Weierstrass's theorem it follows that series $S_r(x)$ is uniformly convergent for $x \in [a,b]$. Then, series $S_{r-1}(x)$ can be differentiated term-by-term on $[a,b]$. We obtain relation (4.2) for $x = x_0$. Since x_0 was chosen arbitrarily we obtain relation (4.2) for all $x \in I$. We pass to prove that $T_{a,b}^r < \infty$ for any $0 \le a < b$. Let $M > 0$, $0 \le q < \alpha\rho$ be such that $|f(t)| \le Me^{qt}$, for $t \ge 0$. From Lemma 3 we have

$$\begin{aligned}
T_{a,b}^r &= M\sum_{k=0}^{\infty} \max_{x\in[a,b]} \left| s_{\alpha,k}^{(r)}(x)\left(\frac{\alpha\rho}{\alpha\rho - q}\right)^{k\rho}\right| \\
&\le M\sum_{k=0}^{\infty} \max_{x\in[a,b]} \alpha^r \sum_{j=0}^{r} \binom{r}{j} s_{\alpha,k-j}(x)\left(\frac{\alpha\rho}{\alpha\rho - q}\right)^{k\rho} \\
&\le M\alpha^r \sum_{k=0}^{\infty} \left(\frac{\alpha\rho}{\alpha\rho - q}\right)^{k\rho} \sum_{j=0}^{\min\{r,k\}} \binom{r}{j}\frac{(\alpha b)^{k-j}}{(k-j)!} \\
&\le M\alpha^r \sum_{k=0}^{\infty} \left(\left(\frac{\alpha\rho}{\alpha\rho - q}\right)^{\rho}\alpha b\right)^k \frac{k^r}{k!} \sum_{j=0}^{\min\{r,k\}} \binom{r}{j}\frac{1}{(\alpha b)^j} \\
&< \infty.
\end{aligned}$$

■

The main result in this section is the following.

Theorem 6 *For all $\alpha > 0$ and $\rho > 0$ and $r \in \mathbb{N}$, if $f \in W_\alpha^\rho \cap C^r(I)$ satisfies condition $f^{(r)} \ge 0$ on I then $(L_\alpha^\rho)^{(r)}(f) \ge 0$ on I.*

Proof. For $r = 0$ this fact reduces to the positivity of operator L_α^ρ. So that we take $r \ge 1$ and $f \in W_\alpha^\rho \cap C^r(I)$ with $f^{(r)} \ge 0$ on I. We have

$$f(t) = \sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!}\cdot t^j + \int_0^t \frac{(t-u)^{r-1}}{(r-1)!}\cdot f^{(r)}(u)du, \quad t \in I.$$

Since $f \in W_\alpha^\rho$, it follows that the function $f - \sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!} \cdot e_j$ belongs to W_α^ρ and consequently, the function $t \mapsto \int_0^t \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)du$, $t \in I$, belongs to W_α^ρ. From Lemma 1 it follows that $L_\alpha^\rho(\Pi_{r-1}) \subset \Pi_{r-1}$, where Π_{r-1} is the set of polynomials of degree at most $r-1$. Then $(L_\alpha^\rho)^{(r)}\Big(\sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!} \cdot e_j\Big) = 0$. From Lemma 5, for $x \in I$, we obtain

$$\begin{aligned}(L_\alpha^\rho)^{(r)}(f,x) &= \sum_{k=1}^{\infty} s_{\alpha,k}^{(r)}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t)\Big(\int_0^t \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)du\Big)dt \\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{r} s_{\alpha,k-j}(x)U_{k,j},\end{aligned}$$

where

$$U_{k,j} = (-1)^{r-j}\alpha^r \binom{r}{j} \int_0^\infty \Theta_{\alpha,k}^\rho(t)\Big(\int_0^t \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)du\Big)dt.$$

We have

$$\sum_{k=1}^{\infty}\sum_{j=0}^{r} s_{\alpha,k-j}(x)|U_{k,j}| \le \alpha^r \sum_{k=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j} s_{\alpha,k-j}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t)\left| f(t) - \sum_{i=0}^{r-1}\frac{f^{(i)}(0)}{i!}\cdot t^i\right| dt.$$

The following inequalities

$$\alpha^r \sum_{k=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j} s_{\alpha,k-j}(x)\int_0^\infty \Theta_{\alpha,k}^\rho(t)|f(t)|dt < \infty,$$

$$\alpha^r \sum_{i=0}^{r-1}\frac{|f^{(i)}(0)|}{i!}\sum_{k=1}^{\infty}\sum_{j=0}^{r}\binom{r}{j} s_{\alpha,k-j}(x)\int_0^\infty \Theta_{\alpha,k}^\rho(t)t^i dt < \infty$$

can be proved similarly as the inequality $T_{a,b}^r < \infty$ in Lemma 5. We extend conventionally the definition of $\Theta_{\alpha,k}^\rho$, for $k = 0$, by $\Theta_{\alpha,0}^\rho = 0$. Then $U_{0,j} = 0$, $j \ge 0$. Now, the inequality

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\min\{r,k\}} s_{\alpha,k-j}(x)|U_{k,j}| < \infty$$

allows us to rewrite

$$(L_\alpha^\rho)^{(r)}(f,x) = \sum_{m=0}^{\infty} s_{\alpha,m}(x)\sum_{j=0}^{r} U_{m+j,j}.$$

In order to prove $(L_\alpha^\rho)^{(r)}(f,x) \ge 0$ it suffices to show that

$$\sum_{j=0}^{r} U_{m+j,j} \ge 0, \text{ for all } m \ge 0.$$

Since $f^{(r)} \geq 0$ we can rewrite

$$U_{m+j,j} = (-1)^{r-j}\alpha^r \binom{r}{j} \int_0^\infty f^{(r)}(u)\Big(\int_u^\infty \Theta^\rho_{\alpha,m+j}(t)\frac{(t-u)^{r-1}}{(r-1)!}dt\Big)du.$$

Consequently, in order to prove $(L^\rho_\alpha)^{(r)}(f) \geq 0$ it is sufficient to prove

$$\Psi^r_m(u) \geq 0, \quad \text{for all } m \in \mathbb{N}_0,\ u \geq 0,\ r \in \mathbb{N}. \tag{4.3}$$

where

$$\Psi^r_m(u) = \sum_{j=0}^r (-1)^{r+j}\binom{r}{j}\int_u^\infty \Theta^\rho_{\alpha,m+j}(t)\frac{(t-u)^{r-1}}{(r-1)!}\,dt. \tag{4.4}$$

We find by induction, for $0 \leq i \leq r-1$, that

$$\frac{\mathrm{d}^i}{\mathrm{d}u^i}\Psi^r_m(u) = \sum_{j=0}^r (-1)^{r+j+i}\binom{r}{j}\int_u^\infty \Theta^\rho_{\alpha,m+j}(t)\frac{(t-u)^{r-i-1}}{(r-i-1)!}\,dt. \tag{4.5}$$

For any $q \in \mathbb{N}$ we obtain

$$\int_0^\infty \Theta^\rho_{\alpha,m+j}(t)\frac{t^q}{q!}\,dt = \frac{((m+j)\rho)((m+j)\rho+1)\ldots((m+j)\rho+q-1)}{q!(\alpha\rho)^q}.$$

This shows that the above integral is a polynomial of degree q in variable j. Since $\frac{\mathrm{d}^i}{\mathrm{d}u^i}\Psi^r_m(0)$ is the finite difference of order r of a polynomial of degree at most $r-1$ we deduce

$$\frac{\mathrm{d}^i}{\mathrm{d}u^i}\Psi^r_m(0) = 0,\ m \in \mathbb{N}_0,\ r \in \mathbb{N},\ 0 \leq i \leq r-1. \tag{4.6}$$

Note that this relation holds true also in the case $m = 0$, when, by convention, $\Theta^\rho_{\alpha,0} = 0$.

Also, clearly, we have

$$\lim_{u\to\infty}\frac{\mathrm{d}^i}{\mathrm{d}u^i}\Psi^r_m(u) = 0,\ m \in \mathbb{N}_0,\ r \in \mathbb{N},\ 0 \leq i \leq r-1. \tag{4.7}$$

From (4.5) we obtain also

$$\begin{aligned}\frac{\mathrm{d}^r}{\mathrm{d}u^r}\Psi^r_m(u) &= \sum_{j=0}^r (-1)^j\binom{r}{j}\Theta^\rho_{\alpha,m+j}(u)\\ &= \frac{(\alpha\rho)^{(m+j)\rho}e^{-\alpha\rho u}u^{m\rho-1}}{\Gamma((m+j)\rho)}\sum_{j=0}^r(-1)^j\binom{r}{j}(u^\rho)^j.\end{aligned}$$

From Descartes' rule of signs, the polynomial $P(s) = \sum_{j=0}^r(-1)^j\binom{r}{j}s^j$ has at most r positive roots. Then function $\frac{\mathrm{d}^r}{\mathrm{d}u^r}\Psi^r_m(u)$ has at most r roots on interval I.

Now it follows by induction that, for $0 \le i \le r$ function $(\mathrm{d}^i/\mathrm{d}u^i)\Psi_m^r$ has at most i roots on interval $(0,\infty)$. Indeed, for $i \ge 1$, suppose that the function $(\mathrm{d}^i/\mathrm{d}u^i)\Psi_m^r$ has at most i roots on $(0,\infty)$. If we suppose also that function $(\mathrm{d}^{i-1}/\mathrm{d}u^{i-1})\Psi_m^r(u)$ has at least i roots on $(0,\infty)$, we obtain a contradiction, by taking into account relations (4.6) and (4.7). Finally it follows that Ψ_m^r has no roots on interval $(0,\infty)$.

Applying r-times l'Hôpital's rule we obtain, for $0 \le j < r$

$$\begin{aligned}
&\lim_{u\to\infty} \int_u^\infty \Theta^\rho_{\alpha,m+j}(t)\frac{(t-u)^{r-1}}{(r-1)!}\,dt \Big/ \int_u^\infty \Theta^\rho_{\alpha,m+r}(t)\frac{(t-u)^{r-1}}{(r-1)!}\,dt \\
&= \lim_{u\to\infty} \frac{(-1)^r(\alpha\rho)^{(m+j)\rho}u^{(m+j)\rho-1}}{(-1)^r(\alpha\rho)^{(m+r)\rho}u^{(m+r)\rho-1}} \cdot \frac{\Gamma((m+r)\rho)}{\Gamma((m+j)\rho)} \\
&= 0.
\end{aligned}$$

It follows that, for sufficiently large u, the sign of $\Psi_m^r(u)$ is equal to the sign of the term of index $j = r$, from the sum which define function Ψ_m^r in (4.4), namely $\int_u^\infty \Theta^\rho_{\alpha,m+r}(t)\frac{(t-u)^{r-1}}{(r-1)!}\,dt$. Consequently, we obtain relation (4.3). The proof is finished. ■

Remark 7 From Theorem 6 it follows, more general, that any function $f \in W_\alpha^\rho$ which is r convex of I, $r \ge -1$, i.e. f satisfies condition $[f; x_1, \ldots, x_{r+1}] \ge 0$, for any distinct points $x_1, \ldots x_{r+1} \in I$, where $[f; x_1, \ldots, x_{r+1}]$ denotes the divided difference, is transformed by operator L_α^ρ into a function which is also convex of order r. Since we do not use this more general property we do not give the details of the proof.

5 Simultaneous approximation

We need of the following Lemma.

Lemma 8 *Let $\alpha > 0$, $\rho > 0$, $q > 0$, $b > 0$ and $r, j \in \mathbb{N}_0$. Let $c > \frac{2}{\rho}(2^\rho - 1)b$. Then we have*

$$\lim_{\alpha\to\infty} \alpha^r \sum_{k=0}^\infty s_{\alpha,k-j}(x) \int_c^\infty \Theta^\rho_{\alpha,k}(t)e^{qt}dt = 0, \quad \textit{uniformly for } x \in [0,b]. \tag{5.1}$$

Proof. Take α such that $\frac{c}{2}(\alpha\rho - 2q) > 1$. Denote $C = \sqrt{\frac{\rho}{2\pi}} \cdot e^{cq}2^{j\rho}$. By Lemma 3 - ii):

$$\begin{aligned}
&\alpha^r \sum_{k=0}^\infty s_{\alpha,k-j}(x) \int_c^\infty \Theta^\rho_{\alpha,k}(t)e^{qt}dt \alpha^r \sum_{k=0}^\infty s_{\alpha,k}(x) \int_c^\infty \Theta^\rho_{\alpha,k+j}(t)e^{qt}dt \\
&\le C\alpha^r e^{-\left(x+\frac{c}{2}\rho\right)\alpha} \sum_{k=0}^\infty \frac{(2^\rho\alpha x)^k}{k!} \cdot \sqrt{k+j} \le C\alpha^r e^{-\left(x+\frac{c}{2}\rho\right)\alpha} \sum_{k=0}^\infty \frac{(2^\rho\alpha x)^k}{k!} \cdot (k+j) \\
&\le C\alpha^r(2^\rho\alpha b + j)e^{\left[(2^\rho-1)b-\frac{c}{2}\rho\right]\alpha}.
\end{aligned}$$

From the condition imposed to c it follows (5.1). ■

Our main result is the following

Theorem 9 *For any $f \in W \cap C^r(I)$, $r \in \mathbb{N}$ and $\rho > 0$ there is $\alpha_0 > 0$ such that $L_\alpha^\rho(f)$ exists for all $\alpha \geq \alpha_0$ and for any interval $[0,b] \subset [0,\infty)$ we have*

$$\lim_{\alpha \to \infty} (L_\alpha^\rho)^{(r)}(f) = f^{(r)}, \text{ uniformly on interval } [0,b]. \tag{5.2}$$

Proof. Let $f \in W \cap C^r(I)$, such that $|f(t)| \leq Me^{qt}$, $t \in I$, for certain constants $M > 0$ and $q > 0$. We can choose α_0 any number such that $\alpha_0 \rho > q$. Then, for $\alpha \geq \alpha_0$, $L_\alpha^\rho(f)$ exists, by Lemma A. Choose an interval $[0,b] \subset [0,\infty)$.

Denote by D^r the operator of differentiation of order r and consider operator $J^r : C(I) \to C^r(I)$, given by

$$J^r(g,x) = \int_0^x \frac{(x-u)^{r-1}}{(r-1)!} \cdot g(u)du, \; g \in C(I), \; x \in I.$$

Then consider the Kantorovich modification of operator L_α^ρ, $K_{\alpha,\rho}^r := D^r \circ L_\alpha^\rho \circ J^r$. Consider the domain of $K_{\alpha,\rho}^r$ to be the space of function $g \in C(I)$ with property that $J^r(g) \in W_\alpha^\rho$. Since $J^r(f^{(r)}) = f - \sum_{j=0}^{r-1} \frac{f^{(j)}(0)}{j!} \cdot e_j$, $f \in W_\alpha^\rho$ and any polynomial belongs to space W_α^ρ, it follows $J^r(f^{(r)}) \in W_\alpha^\rho$. Since $L_{\alpha,\rho}^r(\Pi_{r-1}) \subset \Pi_{r-1}$ it follows $D^r(L_{\alpha,\rho}^r)(f) = K_{\alpha,\rho}^r(f^{(r)})$. Hence the theorem which we want to prove is equivalent with: $\lim_{\alpha\to\infty} K_{\alpha,\rho}^r(f^{(r)}) = f^{(r)}$, uniformly on $[0,b]$.

Fix $c > \frac{2}{\rho}(2^\rho - 1)b$. Denote by χ_A the characteristic function of a set $A \subset \mathbb{R}$. Consider operator $U_{\alpha,\rho}^r : C[0,c] \to C(I)$, given by $U_{\alpha,\rho}^r(g) = K_{\alpha,\rho}^r(\tilde{g} \cdot \chi_{[0,c]})$, $g \in C[0,c]$, where $\tilde{g} \in C(I)$ is an arbitrary extension of g. We consider function $\tilde{g}$ only for the correctness of the notation. Also, on the subspace of $C[c,\infty)$ of functions g satisfying condition $|g(t)| \leq Me^{qt}$, $t \geq c$, with some $M > 0$ and $0 \leq q < \alpha\rho$, consider operator $V_{\alpha,\rho}^r$ given by $V_{\alpha,\rho}^r(g) = K_{\alpha,\rho}^r(\tilde{g} \cdot \chi_{[c,\infty)})$, where $\tilde{g} \in C(I)$ is an arbitrary extension of g. From Theorem 6 it follows that operator $K_{\alpha,\rho}^r$ is positive. Consequently, operators $U_{\alpha,\rho}^r$ and $V_{\alpha,\rho}^r$ are also positive.

Then we have

$$K_{\alpha,\rho}^r(f^{(r)}) = U_{\alpha,\rho}^r(f^{(r)}|_{[0,c]}) + V_{\alpha,\rho}^r(f^{(r)}|_{[c,\infty)}).$$

Since

$$\begin{aligned} J^r(f^{(r)}\chi_{[c,\infty)}, t) &= \int_c^{\max\{c,t\}} \frac{(t-u)^{r-1}}{(r-1)!} \cdot f^{(r)}(u)du \\ &= \Big[f(t) - \sum_{j=0}^{r-1} \frac{f^{j)}(c)}{j!} \cdot (t-c)^j\Big]\chi_{[c,\infty)}(t), \end{aligned}$$

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from Lemma 5 we obtain, for $x \in [0, b]$

$$V^r_{\alpha,\rho}(f^{(r)}|_{[c,\infty)}, x) = \sum_{k=1}^{\infty}(s_{\alpha,k}(x))^{(r)} \int_c^{\infty} \Theta^{\rho}_{\alpha,k}(t)\Big[f(t) - \sum_{j=0}^{r-1}\frac{f^{j)}(c)}{j!}\cdot(t-c)^j\Big]dt$$
$$= \sum_{j=0}^{r}\alpha^r\binom{r}{j}(-1)^{r-j}\sum_{k=1}^{\infty}s_{\alpha,k-j}(x)\int_c^{\infty}\Theta^{\rho}_{\alpha,k}(t)\Big[f(t)-\sum_{j=0}^{r-1}\frac{f^{j)}(c)}{j!}\cdot(t-c)^j\Big]dt.$$

Then using Lemma 8 we obtain

$$\lim_{\alpha\to\infty} V^r_{\alpha,\rho}(f^{(r)}|_{[c,\infty)}, x) = 0, \text{ uniformly with regard to } x \in [0, b]. \tag{5.3}$$

Now, we shall prove

$$\lim_{\alpha\to\infty} V^r_{\alpha,\rho}(g) = g, \text{ uniformly on } [0, b], \text{ for all } g \in C[0, c]. \tag{5.4}$$

By Popoviciu-Bohmann-Korovkin's theorem in order to prove (5.4) it suffices to prove it only for the test functions e_j, $j = 0, 1, 2$, restricted to interval $[0, c]$. We can write $V^r_{\alpha,\rho}(e_j) = (D^r \circ L^{\rho}_{\alpha} \circ J^r)(e_j \cdot \chi_{[0,c]})$. Denote $\epsilon_j = J^r(e_j \cdot \chi_{[0,c]})$. We have

$$\epsilon_j(t) = \int_0^{\min\{c,t\}} \frac{(t-u)^{r-1}}{(r-1)!}\cdot u^j\,du.$$

For computing ϵ_1 we use decomposition $u = u - t + t$ and for computing ϵ_2 we use decomposition $u^2 = (u - t)^2 + 2t(u - t) + t^2$. We obtain

$$\epsilon_0(t) = \frac{1}{r!}\cdot t^r - \frac{(t-c)^r}{r!}\chi_{[c,\infty)}(t),$$
$$\epsilon_1(t) = \frac{1}{(r+1)!}\cdot t^{r+1} + \frac{(t-c)^r}{(r-1)!}\Big[\frac{t-c}{r+1} - \frac{t}{r}\Big]\chi_{[c,\infty)}(t),$$
$$\epsilon_2(t) = \frac{2}{(r+2)!}\cdot t^{r+2} + \frac{(t-c)^r}{(r-1)!}\Big[-\frac{(t-c)^2}{r+2} + 2\frac{t(t-c)}{r+1} - \frac{t^2}{r}\Big]\chi_{[c,\infty)}(t).$$

Summarizing, $\epsilon_j(t) = \frac{j!}{(r+j)!}\cdot t^{r+j} + h_{r+j}(t)\chi_{[c,\infty)}(t)$, where h_{r+j} is a polynomial of degree $r+j$. Hence $V^r_{\alpha,\rho}(e_j) = \frac{j!}{(r+j)!}D^r(L^{\rho}_{\alpha}(e_{r+j}))+D^r(L^{\rho}_{\alpha}(h_{r+j}\cdot\chi_{[c,\infty)}))$, $j = 0, 1, 2$. Using Lemmas 5 and 8 we obtain

$$\lim_{\alpha\to\infty} D^r(L^{\rho}_{\alpha}(h_{r+j}\cdot\chi_{[c,\infty)})) = 0, \text{ uniformly on interval } [0, b].$$

Using Corollary 2 we obtain

$$V^r_{\alpha,\rho}(e_0) = e_0,$$
$$V^r_{\alpha,\rho}(e_1) = e_1 + \frac{(\rho+1)r}{2\alpha\rho}\cdot e_0,$$
$$V^r_{\alpha,\rho}(e_2) = e_2 + \frac{(\rho+1)(r+1)}{\alpha\rho}\cdot e_1 + \frac{(\rho+1)r}{12(\alpha\rho)^2}\cdot(\rho(3r+1)+3r+5)e_0.$$

Thus $\lim_{\alpha\to\infty} V^r_{\alpha,\rho}(e_j) = e_j$, uniformly on $[0,b]$, for $j = 0,1,2$. Consequently

$$\lim_{\alpha\to\infty} V^r_{\alpha,\rho}(f^{(r)}\chi_{[0,c]}) = 0, \text{ uniformly on interval } [0,b].$$

The theorem is proved. ■

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On some operators linking the Bernstein and the genuine Bernstein-Durrmeyer operators

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Dedicated to Prof. Dr. dr.h.c. Heiner Gonska
on the occasion of his 65th birthday

Abstract

H. Gonska and R. Păltănea investigated in 2010 a class of operators linking the Bernstein and the genuine Bernstein-Durrmeyer ones. In this paper we give new proofs of some results of these authors, and investigate new properties of the operators.

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1 Introduction

In this paper we are concerned with the operators U_n^{ϱ} introduced by R. Păltănea [15] and further investigated by H. Gonska and R. Păltănea in [7] and [8]. They are defined as follows. Let $n \geq 1$ and $\varrho > 0$. For $0 \leq k \leq n$ consider the functionals $F_{n,k}^{\varrho} : C[0,1] \to \mathbb{R}$,

$$F_{n,k}^{\varrho}(f) = \begin{cases} f(0), \text{ if } k = 0, \\ \int\limits_0^1 \dfrac{t^{k\varrho-1}(1-t)^{(n-k)\varrho-1}}{B(k\varrho,(n-k)\varrho)} f(t)dt, \text{ if } 1 \leq k \leq n-1, \\ f(1), \text{ if } k = n, \end{cases} \tag{1}$$

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here $B(\cdot,\cdot)$ is Euler's Beta function. Now define $U_n^\varrho : C[0,1] \to \Pi_n$ by

$$U_n^\varrho f(x) = \sum_{k=0}^{n} F_{n,k}^\varrho(f) p_{n,k}(x), f \in C[0,1], x \in [0,1], \tag{2}$$

where Π_n is the space of polynomials of degree at most n, and the fundamental polynomials $p_{n,k}(x)$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \le k \le n, \quad k, n \in \mathbb{N}_0, \quad x \in [0,1]. \tag{3}$$

In particular, $U_n := U_n^1, n \ge 1$, are the well-known genuine Bernstein-Durrmeyer operators. The basic properties of U_n^ϱ were studied in [7] and [8].
In this paper we give new proofs of some theorems from [7], [8], and investigate other properties of the operators U_n^ϱ. The difference $U_n^\varrho - U_n^\sigma$ is estimated, as well as two commutators of operators from this class. The behavior of U_n^ϱ with respect to Lipschitz classes is also studied. Throughout the paper we use the notation $e_i(t) = t^i, i = 0, 1, ...; t \in [0,1]$.

2 The difference $U_n^\varrho - U_n^\sigma$

A first approach in order to study this difference is based on a method presented in [5]. We need the following result from that paper :

Theorem 1 *Let $A, B : C[0,1] \to C[0,1]$ be positive linear operators such that*

$$(A - B)((e_1 - x)^i)(x) = 0 \text{ for } i = 0, 1, \ldots, n \text{ and } x \in [0,1],$$

also satisfying $Ae_0 = Be_0 = e_0$. Then for all $f \in C[0,1], x \in [0,1]$ we have

$$|(A - B)(f)(x)| \le c_1 \cdot \omega_{n+1}\left(f; \sqrt{\frac{1}{2}(A+B)(|e_1 - x|^{n+1})(x)}\right).$$

Here c_1 is an absolute constant independent of f, x and A and B, and $\omega_{n+1}(f,\cdot)$ denotes the $(n+1)$-st order modulus of smoothness.

We choose $A = U_n^\varrho$ and $B = U_n^\sigma$. Both operators reproduce linear functions so we have

$$(U_n^\varrho - U_n^\sigma)((e_1 - x)^i)(x) = 0 \text{ for } i = 0, 1, x \in [0,1].$$

According to [7], Corollary 3.3, the second moments for U_n^ϱ and U_n^σ are given by

$$M_{n,2}^t(x) = \frac{(t+1)x(1-x)}{nt+1}$$

where $t = \varrho, \sigma$. As a consequence of Theorem 1 the following statement holds:

Proposition 2

$$\begin{aligned}|(U_n^\varrho - U_n^\sigma)(f)(x)| &\le c_1 \cdot \omega_2 \left(f; \sqrt{\frac{1}{2}(U_n^\varrho + U_n^\sigma)(|e_1 - x|^2)(x)} \right) \\ &\le c_1 \cdot \omega_2 \left(f; \sqrt{\frac{1}{2}\frac{2n\varrho\sigma + (n+1)(\varrho+\sigma)+2}{(n\varrho+1)(n\sigma+1)} x(1-x)} \right).\end{aligned}$$

Another approach is described in the sequel. Consider the Beta operator, introduced independently by A. Lupaş [12] and Mühlbach [14]:

$$\mathcal{B}_r f(x) := \begin{cases} f(x), \quad x = 0, 1; \\ \dfrac{\int\limits_0^1 t^{rx-1}(1-t)^{r-rx-1} f(t)dt}{B(rx, r-rx)}, 0 < x < 1, \end{cases} \tag{4}$$

for $r > 0, f \in C[0,1], x \in [0,1]$. It is not difficult to see that

$$U_n^\varrho = B_n \circ \mathcal{B}_{n\varrho}, \tag{5}$$

where $B_n : C[0,1] \to \Pi_n$ is the classical Bernstein operator:

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad f \in C[0,1], \ x \in [0,1]. \tag{6}$$

The following result is a consequence of ([1], Th.1).

Theorem 3 *If $g \in C[0,1]$ is convex, and $s > r > 0$, then*

$$\mathcal{B}_r g(x) \ge \mathcal{B}_s g(x). \tag{7}$$

Now we are in a position to state

Theorem 4 *Let $f \in C[0,1], n \ge 1, \varrho > 0, \sigma > 0$. Then*

$$|(U_n^\varrho - U_n^\sigma) f(x)| \le \frac{9}{4}\omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma - \varrho|}{(n\varrho+1)(n\sigma+1)} x(1-x)} \right), \tag{8}$$

where ω_2 is the second order modulus of smoothness.

Proof. Suppose that $0 < \varrho < \sigma$ and set $r := n\varrho, s := n\sigma$. According to (7), we have for each convex function $g \in C[0,1]$,

$$\mathcal{B}_{n\varrho} g \ge \mathcal{B}_{n\sigma} g.$$

This entails

$$B_n(\mathcal{B}_{n\varrho} g) \ge B_n(\mathcal{B}_{n\sigma} g). \tag{9}$$

Now (5) and (9) yield

$$U_n^{\varrho} g \geq U_n^{\sigma} g, \quad g \in C[0,1] \text{ convex.} \tag{10}$$

Let $x \in [0,1]$ be fixed. Consider the functional $\Phi : C[0,1] \to \mathbb{R}$,

$$\Phi(f) := U_n^{\varrho} f(x) - U_n^{\sigma} f(x), \; f \in C[0,1].$$

The linear functional Φ is bounded on $C[0,1]$ endowed with the uniform norm; moreover, Φ is different from 0, and according to (10),

$$\Phi(g) \geq 0, \quad g \in C[0,1] \text{ convex.} \tag{11}$$

By a result of T. Popoviciu [16] (see also [17]) it follows that for each $f \in C[0,1]$ there exist distinct points t_0, t_1, t_2 in $[0,1]$ such that

$$\Phi(f) = \Phi(e_2)[t_0, t_1, t_2; f], \tag{12}$$

where $[t_0, t_1, t_2; f]$ is the divided difference of the function f on the nodes t_0, t_1, t_2. According to [7],

$$U_n^{\varrho} e_2(x) = x^2 + \frac{\varrho + 1}{n\varrho + 1} x(1-x),$$

so that

$$\Phi(e_2) = U_n^{\varrho} e_2(x) - U_n^{\sigma} e_2(x) = \frac{(n-1)(\sigma - \varrho)}{(n\varrho + 1)(n\sigma + 1)} x(1-x).$$

On the other hand, if $g \in C^2[0,1]$, then

$$[t_0, t_1, t_2; g] = \frac{1}{2} g''(\xi)$$

for some $\xi \in [0,1]$. Thus (12) leads to

$$U_n^{\varrho} g(x) - U_n^{\sigma} g(x) = \frac{(n-1)(\sigma - \varrho)}{(n\varrho + 1)(n\sigma + 1)} x(1-x) \frac{g''(\xi)}{2}, \; g \in C^2[0,1].$$

This entails

$$|U_n^{\varrho} g(x) - U_n^{\sigma} g(x)| \leq \frac{(n-1)(\sigma - \varrho)}{2(n\varrho + 1)(n\sigma + 1)} x(1-x) ||g''||_{\infty}, \; g \in C^2[0,1]. \tag{13}$$

As a consequence of Theorem 4.2 and Corollary 4.3 in [4],
for $h^2 = \dfrac{(n-1)(\sigma - \varrho)}{(n\varrho + 1)(n\sigma + 1)} x(1-x)$, $\alpha = 2$ and $\beta_2 = \dfrac{h^2}{2}$ we obtain

$$|(U_n^{\varrho} - U_n^{\sigma})(f)(x)| \leq \left(2 \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{2}\right) \omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma - \varrho|}{(n\varrho + 1)(n\sigma + 1)} x(1-x)}\right)$$

$$\leq \frac{9}{4} \omega_2 \left(f; \sqrt{\frac{(n-1)|\sigma - \varrho|}{(n\varrho + 1)(n\sigma + 1)} x(1-x)}\right).$$

■

4

3 The commutators $[U_n^\varrho; U_n^\sigma]$ and $[U_m^\varrho; U_n^\varrho]$

The problem of studying the commutator $[A;B] := AB - BA$ of two positive linear operators A and B was raised by A. Lupaş in [13]. Some answers to Lupaş's problem can be found in [6]. Here we shall study the commutators $[U_n^\varrho; U_n^\sigma]$ and $[U_m^\varrho; U_n^\varrho]$. First of all, we need information about the moments of the investigated operators.
Let $M_{n,j}^\varrho(x) := U_n^\varrho(e_1 - xe_0)^j(x)$, be the j-th moment of U_n^ϱ. Then, according to [7],

$$M_{n,0}^\varrho(x) = 1, M_{n,1}^\varrho(x) = 0, \tag{14}$$

$$\begin{aligned} M_{n,j+1}^\varrho(x) = \frac{1}{n\varrho + j}\Big(\varrho x(1-x)\frac{d}{dx}M_{n,j}^\varrho(x) + j(1-2x)M_{n,j}^\varrho(x) + \\ + j(\varrho+1)x(1-x)M_{n,j-1}^\varrho(x)\Big), j \geq 1. \end{aligned} \tag{15}$$

By using (14) and (15) it is not difficult to prove by induction on j that

$$M_{n,j}^\varrho(x) = \mathcal{O}\left(n^{-[\frac{j+1}{2}]}\right) \tag{16}$$

uniformly with respect to $x \in [0,1]$. Now let

$$M_{n,r}^{\varrho,\sigma}(x) := U_n^\varrho U_n^\sigma (e_1 - xe_0)^r(x) \tag{17}$$

be the r-th moment of $U_n^\varrho U_n^\sigma$. According to [9], Theorem 4,

$$M_{n,r}^{\varrho,\sigma}(x) = \sum_{\substack{i,k\geq 0\\ i+k=r}} \sum_{j=k}^{r} \binom{r}{k} \frac{1}{(j-k)!} M_{n,j}^\varrho(x)(M_{n,i}^\sigma(x))^{(j-k)}. \tag{18}$$

Combining (16) and (18) (see also [9], Corollary 1), we get

$$M_{n,r}^{\varrho,\sigma}(x) = \mathcal{O}\left(n^{-[\frac{r+1}{2}]}\right) \tag{19}$$

uniformly with respect to $x \in [0,1]$. Now by a result of Sikkema [18] we have

$$U_n^\varrho U_n^\sigma f(x) = \sum_{r=0}^{6} \frac{f^{(r)}(x)}{r!} M_{n,r}^{\varrho,\sigma}(x) + o(n^{-3}) \tag{20}$$

uniformly with respect to $x \in [0,1]$, for each $f \in C^6[0,1]$. It follows that for $f \in C^6[0,1]$,

$$(U_n^\varrho U_n^\sigma - U_n^\sigma U_n^\varrho)f(x) = \sum_{r=0}^{6} \frac{f^{(r)}(x)}{r!}(M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x)) + o(n^{-3}). \tag{21}$$

A combination of hand calculations and MAPLE shows that

$$M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x) = 0, r = 0,1,2,3, \tag{22}$$

5

$$\lim_{n\to\infty} n^3(M_{n,4}^{\varrho,\sigma}(x) - M_{n,4}^{\sigma,\varrho}(x)) = \frac{(\sigma-\varrho)(\varrho+1)(\sigma+1)}{\varrho^2\sigma^2}x(1-x), \tag{23}$$

$$\lim_{n\to\infty} n^3(M_{n,r}^{\varrho,\sigma}(x) - M_{n,r}^{\sigma,\varrho}(x)) = 0, r = 5, 6, \tag{24}$$

uniformly with respect to $x \in [0,1]$. From (21)-(24) we derive

Theorem 5 *For each $f \in C^6[0,1]$ one has*

$$\lim_{n\to\infty} n^3(U_n^\varrho U_n^\sigma - U_n^\sigma U_n^\varrho)f(x) = \frac{(\sigma-\varrho)(\varrho+1)(\sigma+1)}{\varrho^2\sigma^2}x(1-x)f^{(4)}(x), \tag{25}$$

uniformly with respect to $x \in [0,1]$.

In particular, we see that U_n^ϱ and U_n^σ do not commute if $\varrho \neq \sigma$. On the other hand it is well known (see [10]) that U_n^1 and U_m^1 (i.e., the genuine Bernstein-Durrmeyer operators) do commute. A combination of hand calculations and MAPLE shows that

$$(U_m^\varrho U_n^\varrho - U_n^\varrho U_m^\varrho)e_r(x) = 0, r = 0, 1, 2, 3, \tag{26}$$

and

$$(U_m^\varrho U_n^\varrho - U_n^\varrho U_m^\varrho)e_4(x) = \frac{\varrho^3(\varrho-1)(\varrho+1)^2(m-1)(n-1)(n-m)}{(m\varrho+1)(m\varrho+2)(m\varrho+3)(n\varrho+1)(n\varrho+2)(n\varrho+3)}. \tag{27}$$

We see that U_m^ϱ and U_n^ϱ do not commute if $\varrho \neq 1, m \neq 1, n \neq 1$ and $m \neq n$.

4 New proofs of some theorems from [7] and [8]

In [7] the authors proved that for each $n \geq 1$ and $f \in C[0,1]$,

$$\lim_{\varrho\to\infty} U_n^\varrho f = B_n f, \text{ uniformly on } [0,1]. \tag{28}$$

Thus, for n fixed and $\varrho \in [1,\infty)$, the operators U_n^ϱ constitute a link between the genuine Bernstein-Durrmeyer operators U_n and the Bernstein operators B_n. The authors of [8] proved that for $n \geq 1$ and $f \in C[0,1]$,

$$\lim_{\varrho\to 0^+} U_n^\varrho f = B_1 f, \text{ uniformly on } [0,1]. \tag{29}$$

Moreover, they proved

Theorem 6 *For $U_n^\varrho, 0 < \varrho < \infty, n \geq 1,$ we have*

$$|U_n^\varrho f(x) - B_1 f(x)| \leq \frac{9}{4}\omega_2\left(f; \sqrt{\frac{n\varrho-\varrho}{n\varrho+1}x(1-x)}\right).$$

In what follows, we give a different proof of (29). First of all, we have

$$F^{\varrho}_{n,k}(e_j) = \frac{k\varrho(k\varrho+1)\cdot ... \cdot(k\varrho+j-1)}{n\varrho(n\varrho+1)\cdot ... \cdot(n\varrho+j-1)},\ j\geq 0,\ 0\leq k\leq n,$$

and consequently,

$$\lim_{\varrho\to 0^+} F^{\varrho}_{n,k}(e_0) = 1 \tag{30}$$

and

$$\lim_{\varrho\to 0^+} F^{\varrho}_{n,k}(e_j) = \frac{k}{n},\quad j=1,2,.... \tag{31}$$

Now let $p\in\Pi, p = a_0e_0 + a_1e_1 + ... + a_me_m$ for some $a_0, a_1, ..., a_m \in \mathbb{R}$. Then, according to (30) and (31),

$$\lim_{\varrho\to 0^+} F^{\varrho}_{n,k}(p) = a_0 + (a_1 + ... + a_m)\frac{k}{n} = p(0) + (p(1) - p(0))\frac{k}{n}.$$

This leads to

$$\lim_{\varrho\to 0^+} U^{\varrho}_{n}p = \sum_{k=0}^{n}\left(p(0) + (p(1)-p(0))\frac{k}{n}\right)p_{n,k} = p(0)e_0 + (p(1)-p(0))e_1,$$

and so

$$\lim_{\varrho\to 0^+} U^{\varrho}_{n}p = B_1p,\quad p\in\Pi. \tag{32}$$

Since Π is dense in $C[0,1]$, and $||U^{\varrho}_{n}|| = ||B_1|| = 1$, (29) is a consequence on (32).

In the sequel we shall be concerned with shape preserving properties of the operators U^{ϱ}_{n}. In [7], Theorem 4.1, the authors proved that for $n\geq 1$ and $\varrho > 0$, the operators U^{ϱ}_{n} transform k-convex functions into k-convex functions. Basically this means that if $f^{(k)}\geq 0$, then $(U^{\varrho}_{n})^{(k)}f\geq 0, k\geq 0$; see [7] for the complete terminology. Here we shall present briefly another proof of this theorem.

First, let $\alpha\geq -1, \beta\geq -1$ be real numbers. For $r>0$ consider the kernel

$$(x,y)\in[0,1]\times]0,1[\to K^{\alpha,\beta}_{r}(x,y) := \frac{y^{rx+\alpha}(1-y)^{r(1-x)+\beta}}{B(rx+\alpha+1, r(1-x)+\beta+1)},$$

and the operator

$$\mathcal{B}^{\alpha,\beta}_{r}f(x) := \int_0^1 K^{\alpha,\beta}_{r}(x,y)f(y)dy,\ f\in C[0,1],\ x\in[0,1].$$

In particular, $\mathcal{B}^{-1,-1}_{r}$ is the operator $\mathcal{B}_r$ discussed in Section 2. Let us remark that the kernel $K^{\alpha,\beta}_{r}$ can be represented also as

$$K^{\alpha,\beta}_{r}(x,y) = \frac{e^{\alpha\log y + (r+\beta)\log(1-y)}\cdot e^{rx(\log y - \log(1-y))}}{B(rx+\alpha+1, r(1-x)+\beta+1)}.$$

According to [11], Theorem 1.1, part (a), p. 99, and (1.5), p. 100, $K_r^{\alpha,\beta}$ is a totally positive kernel. Moreover, a direct computation yields

$$\mathcal{B}_r^{\alpha,\beta} e_k(x) = \frac{(rx+\alpha+1)(rx+\alpha+2)\cdot \ldots \cdot (rx+\alpha+k)}{(r+\alpha+\beta+2)\cdot \ldots \cdot (r+\alpha+\beta+k+1)}.$$

Thus, for any $k \geq 0, \mathcal{B}_r^{\alpha,\beta} e_k$ is a polynomial of degree k with leading coefficient

$$a_{r,k}^{\alpha,\beta} := \frac{r^k}{(r+\alpha+\beta+2)\cdot \ldots \cdot (r+\alpha+\beta+k+1)}.$$

By [3] Theorem 2.3 and Remark 2.5, $\mathcal{B}_r^{\alpha,\beta}$ transforms k-convex functions into k-convex functions, $k \geq 0$. Since the Bernstein operator B_n does the same, we conclude that $B_n \circ \mathcal{B}_r^{\alpha,\beta}$ preserves k-convexity. In particular, $U_n = B_n \circ \mathcal{B}_r^{-1,-1}$ preserves k-convexity, and this is the content of [7], Theorem 4.1.

5 The behavior of U_n^ϱ with respect to Lipschitz classes

Fix an integer $m \geq 0$ and $M > 0$. We say that a function $f \in C[0,1]$ belongs to the Lipschitz class $Lip_m(M)$ if

$$|\Delta_h^m f(x)| \leq M h^m$$

for all $x \in [0,1]$ and $h > 0$ such that $x + mh \in [0,1]; \Delta_h^m f(x)$ stands for the m-th order difference of f with step h at x. According to [3], Proposition 2.1, $f \in Lip_m(M)$ if and only if $\frac{M}{m!} e_m \pm f$ are m- convex functions.

Theorem 7 *If $f \in Lip_m(M)$, then for all $n \geq 1, \varrho > 0$,*

$$U_n^\varrho f \in Lip_m\left(\frac{M\varrho^m n(n-1)\cdot \ldots \cdot (n-m+1)}{m!(n\varrho)(n\varrho+1)\cdot \ldots \cdot (n\varrho+m-1)}\right).$$

Proof. Let $f \in Lip_m(M)$. Then $\frac{M}{m!} e_m \pm f$ are m-convex functions, so that

$$\frac{M}{m!} U_n^\varrho e_m \pm U_n^\varrho f$$

are m-convex functions. Since

$$U_n^\varrho e_m(x) = \frac{\varrho^m n(n-1)\cdot \ldots \cdot (n-m+1)}{(n\varrho)(n\varrho+1)\cdot \ldots \cdot (n\varrho+m-1)} x^m + \text{terms of lower degree},$$

we deduce that

$$\frac{M}{m!} \cdot \frac{\varrho^m n(n-1)\cdot \ldots \cdot (n-m+1)}{(n\varrho)(n\varrho+1)\cdot \ldots \cdot (n\varrho+m-1)} e_m \pm U_n^\varrho f$$

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are m-convex functions.
This means that $U_n^\varrho f$ belongs to the class

$$Lip_m\left(\frac{M\varrho^m n(n-1)\cdot\ldots\cdot(n-m+1)}{m!(n\varrho)(n\varrho+1)\cdot\ldots\cdot(n\varrho+m-1)}\right).$$

■

Let now $M>0$ and $0<\gamma\leq 1$. Define

$$Lip(\gamma,M):=\{f\in C[0,1]:|f(x)-f(y)|\leq M|x-y|^\gamma, x,y\in[0,1]\},$$

and remark that $Lip(1,M)=Lip_1(M)$. Let ω be the usual modulus of continuity.

Theorem 8 *For all $n\geq 1$ and $\varrho>0$,*

a) $\omega(U_n^\varrho f,\delta)\leq 2\omega(f,\delta), f\in C[0,1], \delta>0$;

b) $U_n^\varrho(Lip(\gamma,M))\subset Lip(\gamma,M)$.

Proof. According to Theorem 7, $U_n^\varrho(Lip_1(M))\subset Lip_1(M)$, hence $U_n^\varrho(Lip(1,M))\subset Lip(1,M)$. Now the statements a) and b) follow from [2], Corollary 6 and Corollary 7. ■

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Approximation by Positive Linear Operators on Variable $L^{p(\cdot)}$ Spaces

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

Approximation of functions by Bernstein type positive linear operators on L^p spaces is a classical topic in approximation theory. In this paper we consider the approximation on an open set Ω by positive linear operators on a variable space $L^{p(\cdot)}$ associated with a general exponent function $p : \Omega \to [1, \infty)$. The topic is motivated by applications in electrorheological fluids and learning theory. Under an assumption of log-Hölder continuity of the exponent function p, we provide quantitative estimates for the approximation when the approximated function lies in a variable Sobolev space. The uniform boundedness of the Kantorovich operators and the Durrmeyer operators on the variable spaces is proved when the exponent function p is Lipschitz α with $0 < \alpha \leq 1$, which yields rates of approximation. The technical difficulty arising from the uniform boundedness is overcome by the Lipschiz continuity of the exponent function and localization of Bernstein type positive linear operators.

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1 Introduction and Motivations

Approximation of functions by positive linear operators is a classical topic in the field of approximation theory [15]. It was motivated by the Weierstrass approximation theorem verifying the denseness of polynomials in the space $C[0, 1]$

of continuous functions on the interval $[0,1]$ and started with the investigation of approximation of continuous functions by the classical *Bernstein operators* defined [4] as

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \qquad x \in [0,1],\ f \in C[0,1], \tag{1.1}$$

where $\{b_{n,k}\}_{k=0}^{n}$ is the Bernstein basis given by

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{1.2}$$

To approximate discontinuous functions, one often replaces the point evaluation functionals in (1.1) by some integrals and considers the corresponding Bernstein type positive linear operators on $L^p[0,1]$ spaces with $1 \le p < \infty$ where $L^p[0,1]$ is the Banach space consisting of all integrable functions f on $[0,1]$ with the L^p-norm

$$\|f\|_{L^p} := \left(\int_0^1 |f(x)|^p dx\right)^{1/p} \tag{1.3}$$

finite. Examples of such positive linear operators on $L^p[0,1]$ include the *Kantorovich operators* [13] defined by

$$K_n(f,x) = \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t) dt b_{n,k}(x), \qquad x \in [0,1] \tag{1.4}$$

and the *Durrmeyer operators* [11] by

$$D_n(f,x) = \sum_{k=0}^{n} (n+1) \int_0^1 b_{n,k}(t) f(t) dt b_{n,k}(x), \qquad x \in [0,1]. \tag{1.5}$$

Quantitative behaviors of the approximation by the above mentioned positive linear operators have been well understood due to a large literature (e.g. [6, 5]), which can be found in the book [10] and references therein.

In this paper we study the approximation of functions by positive linear operators on variable L^p spaces. Note that (1.4) and (1.5) may be regarded as operators on the space $L^p(0,1)$. So the functions for approximation considered in this paper are defined on a connected open subset Ω of $\mathbb{R}$ such as $\Omega = (0,1), (0,\infty)$ and $(-\infty,\infty)$. The variable L^p space, $L^{p(\cdot)}$, is associated with a measurable function $p : \Omega \to [1,\infty)$ called the *exponent function.* The space $L^{p(\cdot)}$ consists of all measurable function f on Ω such that $\int_\Omega \left(|f(x)|/\lambda\right)^{p(x)} dx \le 1$ for some $\lambda > 0$. Its norm cannot be defined through replacing the constant p in (1.3) by the exponent function $p(x)$. It is defined by scaling as

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1 \right\}. \tag{1.6}$$

With this definition, $L^{p(\cdot)}$ becomes a Banach space.

The idea of variable $L^{p(\cdot)}$ spaces was introduced by Orlicz [16] who considered necessary and sufficient conditions on a sequence $\{y_k\}_{k\in\mathbb{N}}$ for the series $\sum_k x_k y_k$ to converge, given that $\{p_k > 1\}_k$ and $\{x_k\}_k$ are real sequences with the series $\sum_k x_k^{p_k}$ convergent. Mathematical analysis [14] of variable spaces $L^{p(\cdot)}$ was motivated by connections between these function spaces and variational integrals with non-standard growth related to modeling of electrorheological fluids, which can be found in [1] and references therein. Important analysis topics include boundedness of various maximal operators [9, 7], continuity of translates, and denseness of smooth functions [14].

The purpose of this paper is to raise the issue of approximation on the variable spaces $L^{p(\cdot)}$ by positive linear operators.

Definition 1 *We say that a linear operator L_n on $L^{p(\cdot)}$ is positive if it maps $\left(L^{p(\cdot)}\right)_+$ into itself, where $\left(L^{p(\cdot)}\right)_+$ denotes the positive cone of $L^{p(\cdot)}$ consisting of all functions f in $L^{p(\cdot)}$ such that $f(x) \geq 0$ almost everywhere.*

We aim at providing some quantitative estimates for the approximation, and then demonstrating that the uniform boundedness of the linear operators including (1.4) and (1.5) is essentially different from that in the classical L^p spaces.

To illustrate the technical difficulty arising from the variety of the exponent function $p(\cdot)$, we choose the Durrmeyer operator (1.5) and consider the standard trick [12] for bounding

$$\int_0^1 |D_n(f,x)|^{p(x)}\, dx \leq \int_0^1 \sum_{k=0}^n (n+1) \int_0^1 b_{n,k}(t)\, |f(t)|^{p(x)}\, dt b_{n,k}(x) dx.$$

From this expression, we see that the term $|f(t)|^{p(x)}$ appears and is different from $|f(t)|^{p(t)}$, a quantity in the definition of the norm $\|f\|_{L^{p(\cdot)}}$. This difference is essential, which motivates the introduction of various regularity concepts for the exponent function towards approximation analysis stated in the next section.

Our investigation of approximation on variable spaces $L^{p(\cdot)}$ is also motivated by its applications in learning theory [8]. In [20] a Durrmeyer operator associated with a general probability measure ρ_X on $\Omega = (0,1)$ was introduced as

$$\widetilde{D}_n(f,x) = \sum_{k=0}^n \frac{1}{\int_\Omega b_{n,k}(t) d\rho_X(t)} \int_\Omega b_{n,k}(t) f(t) d\rho_X(t) b_{n,k}(x), \qquad x \in \Omega$$

and was applied to error analysis of support vector machine algorithms for classification. Further mathematical analysis can be found in the literature (e.g., [3]). In learning theory, the least squares loss $\int_{\Omega\times Y}(f(x)-y)^2 d\rho$ with an output space $Y \subseteq \mathbb{R}$ and a probability measure ρ on $\Omega \times Y$ is the most commonly used tool to measure errors, which leads to learning algorithms for predicting conditional means [17, 18] affected by outliers. To reduce the affect of outliers, the ℓ^1 loss $\int_{\Omega\times Y} |f(x)-y|d\rho$ is often used to produce learning algorithms predicting conditional medians [19]. One may use the variable loss $\int_{\Omega\times Y} |f(x)-y|^{p(x)}d\rho$ to learn conditional means for some events and conditional medians for some others. To this end, a variable space involving a general probability measure ρ_X on Ω might be defined by means of scaled integrals of type $\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} d\rho_X$. Further discussion on this point is out of the scope of this paper.

2 Main Results

The first main result of this paper is error analysis for the approximation by positive linear operators on the variable space $L^{p(\cdot)}$. Here regularity of the approximated functions is needed. We use the variable Sobolev space, $W^{1,p(\cdot)}$, defined to be the subspace of $L^{p(\cdot)}$ consisting of functions f such that its distributional gradient or derivative ∇f exists almost everywhere and satisfies $\nabla f \in L^{p(\cdot)}$. Its norm is given by

$$\|f\|_{1,p(\cdot)} = \|f\|_{L^{p(\cdot)}} + \|\nabla f\|_{L^{p(\cdot)}}. \tag{2.1}$$

To give quantitative estimates for the approximation on the variable space $L^{p(\cdot)}$, we need to assume that the exponent function p is log-Hölder continuous [9, 7].

Definition 2 *We say that the exponent function $p:\Omega\to[1,\infty)$ is log-Hölder continuous if there exists a positive constant $C_p>0$ such that*

$$|p(x)-p(y)| \le \frac{C_p}{-\log|x-y|}, \qquad x,y\in\Omega, |x-y|<\frac{1}{2}. \tag{2.2}$$

We say that p is log-Hölder continuous at infinity (when Ω is unbounded) if there holds

$$|p(x)-p(y)| \le \frac{C_p}{\log(e+|x|)}, \qquad x,y\in\Omega, |y|\ge|x|. \tag{2.3}$$

Note that the condition (2.3) for the log-Hölder continuity at infinity is automatically satisfied if Ω is bounded. When Ω is unbounded, (2.3) tells us that the limit $\lim_{x\to\infty} p(x)$ exists.

Denote

$$p_- = \inf_{x\in\Omega} p(x), \quad p_+ = \sup_{x\in\Omega} p(x).$$

Our quantitative estimate for the approximation on the variable space $L^{p(\cdot)}$ of Sobolev functions by positive linear operators can be stated as follows. It will be proved in Section 3.

Theorem 3 *Assume that the exponent function $p : \Omega \to [1, \infty)$ satisfies $1 < p_- \le p_+ < \infty$. If p is log-Hölder continuous satisfying (2.2) and (2.3), then for any positive linear operator L_n on $L^{p(\cdot)}$ and $f \in W^{1,p(\cdot)}$, we have*

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \le (A_p + 1)\, \|f\|_{1,p(\cdot)} \Delta_n, \tag{2.4}$$

where with the constant 1 function $\mathbf{1}$, Δ_n is the quantity defined by

$$\Delta_n = \sup_{x \in \Omega} \left\{ L_n(|\cdot - x|, x) + |L_n(\mathbf{1}, x) - 1| \right\}, \tag{2.5}$$

and A_p is a constant depending only on p.

For general functions, we can measure the regularity by the K-functional $K(f, t)_{p(\cdot)}$ between $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ defined for $f \in L^{p(\cdot)}$ as

$$K(f, t)_{p(\cdot)} = \inf \left\{ \|f - g\|_{L^{p(\cdot)}} + t \|g\|_{1,p(\cdot)} : \ g \in W^{1,p(\cdot)} \right\}, \qquad t > 0. \tag{2.6}$$

Then the following corollary is a standard consequence of Theorem 3, though getting reasonable bounds for the operator norm $\|L_n\|$ is non-trivial.

Corollary 4 *Under the assumption of Theorem 3, there holds*

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \le (\|L_n\| + 1)\, K\left(f, \frac{A_p + 1}{\|L_n\| + 1} \Delta_n\right)_{p(\cdot)}, \qquad \forall f \in L^{p(\cdot)}. \tag{2.7}$$

At a first glance, one might expect that the approximation theorems on the variable space $L^{p(\cdot)}$ can be stated in a straight forward way from those on the classical L^p spaces. However, this is not trivial at all. Even the uniform boundedness of the linear operators (1.4) and (1.5) is not clear and the operator norm $\|L_n\|$ is hard to obtain when a general exponent function p is involved.

The second main result of this paper, to be proved in Section 4, is a uniform bound for the Kantorovich operator (1.4) and the Durrmeyer operator (1.5) when the general exponent function p is Lipschitz α with $0 < \alpha \le 1$.

Theorem 5 *Let $\Omega = (0, 1)$ and $0 < \alpha \le 1$. If $p_- > 1$, and the exponent function $p : \Omega \to [1, \infty)$ is Lipschitz α satisfying*

$$|p(x) - p(y)| \le C_\alpha |x - y|^\alpha, \qquad x, y \in \Omega \tag{2.8}$$

for some positive constant C_α, then there exists a positive constant $A_{\alpha,p}$ depending only on α and p such that for $L_n = K_n$ or D_n, we have

$$\|L_n\| \le A_{\alpha,p}, \qquad \forall n \in \mathbb{N}. \tag{2.9}$$

It would be interesting to obtain the uniform boundedness of the Kantorovich operator or the Durrmeyer operator on $L^{p(\cdot)}$ associated with a general exponent function p. In particular, we conjecture that the uniform boundedness holds true when p is log-Hölder continuous satisfying (2.2).

Theorem 5 can be extended to positive linear operators on $L^{p(\cdot)}$ defined on unbounded domains such as integral versions of the Szász-Mirakjan operators and the Baskakov operators [2] on $\Omega = (0, \infty)$. Some of our analysis can be extended to the multidimensional case where Ω is an open domain in $\mathbb{R}^d$ with $d > 1$.

3 Proof of Approximation Estimates

We need the boundedness of the Hardy-Littlewood maximal operator on variable spaces $L^{p(\cdot)}$ which can be found in [9, 7]. The Hardy-Littlewood maximal operator M is defined for locally integrable functions f on Ω as

$$M(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_{B \cap \Omega} |f(y)| dy, \qquad x \in \Omega, \tag{3.1}$$

where the supremum is taken over all balls B which contains x and for which $|B \cap \Omega| > 0$. Here $|E|$ denotes the Lebesgue measure of a subset E of Ω. It was shown in [9, 7] that when the exponent function p is log-Hölder continuous, the Hardy-Littlewood maximal operator M is bounded on the variable space $L^{p(\cdot)}$.

Lemma 6 *If the exponent function $p : \Omega \to [1, \infty)$ satisfies $1 < p_- \le p_+ < \infty$ and the log-Hölder continuity conditions (2.2) and (2.3), then there exists a constant A_p depending only on p such that*

$$\|M(f)\|_{L^{p(\cdot)}} \le A_p \|f\|_{L^{p(\cdot)}}, \qquad \forall f \in L^{p(\cdot)}. \tag{3.2}$$

Now we can prove our first main result on quantitative estimates for the approximation by positive linear operators on the variable space $L^{p(\cdot)}$ when the approximated function lies in the Sobolev space $W^{1,p(\cdot)}$.

Proof of Theorem 3. Let $f \in W^{1,p(\cdot)}$. Express the difference function $L_n(f, x) - f(x)$ as

$$L_n(f, x) - f(x) = L_n(f(\cdot) - f(x), x) + L_n(f(x), x) - f(x), \qquad x \in \Omega.$$

By the linearity, $L_n(f(x), x) = f(x) L_n(\mathbf{1}, x)$.

For $x, t \in \Omega$, we have

$$|f(t) - f(x)| = \left| \int_x^t \nabla f(u) du \right| \le M(\nabla f)(x) |t - x|.$$

Since L_n is a positive linear operator, the trivial relation $-|f(t) - f(x)| \leq f(t) - f(x) \leq |f(t) - f(x)|$ yields $-L_n(|f(\cdot) - f(x)|, x) \leq L_n(f(\cdot) - f(x), x) \leq L_n(|f(\cdot) - f(x)|, x)$. It follows that

$$\begin{aligned}|L_n(f(\cdot) - f(x), x)| &\leq L_n(|f(\cdot) - f(x)|, x) \leq L_n(M(\nabla f)(x)|\cdot - x|, x) \\ &\leq M(\nabla f)(x) L_n(|\cdot - x|, x).\end{aligned}$$

Therefore, for any $x \in \Omega$, there holds

$$\begin{aligned}|L_n(f, x) - f(x)| &\leq M(\nabla f)(x) L_n(|\cdot - x|, x) + |f(x)| \, |L_n(\mathbf{1}, x) - 1| \\ &\leq \Delta_n \left(M(\nabla f)(x) + |f(x)| \right).\end{aligned}$$

The above bound gives

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \leq \Delta_n \left(\|M(\nabla f)\|_{L^{p(\cdot)}} + \|f\|_{L^{p(\cdot)}} \right).$$

Now we apply Lemma 6 and find $\|M(\nabla f)\|_{L^{p(\cdot)}} \leq A_p \|\nabla f\|_{L^{p(\cdot)}}$. Hence

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \leq \Delta_n \left(A_p \|\nabla f\|_{L^{p(\cdot)}} + \|f\|_{L^{p(\cdot)}} \right) \leq \Delta_n \left(A_p + 1 \right) \|f\|_{1,p(\cdot)}.$$

The proof of Theorem 3 is complete.

The proof for the approximation estimates for general functions on the variable space $L^{p(\cdot)}$ is standard.

Proof of Corollary 4. The triangle inequality tells us that for any $g \in W^{1,p(\cdot)}$, there holds

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \leq \|L_n(f - g)\|_{L^{p(\cdot)}} + \|L_n(g) - g\|_{L^{p(\cdot)}} + \|g - f\|_{L^{p(\cdot)}}.$$

Then we apply Theorem 3 and see from the definition of the operator norm $\|L_n\|$ that

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \leq \left(\|L_n\| + 1 \right) \|f - g\|_{L^{p(\cdot)}} + \left(A_p + 1 \right) \|g\|_{1,p(\cdot)} \Delta_n.$$

Taking the infimum over $g \in W^{1,p(\cdot)}$ in the above bound yields

$$\|L_n(f) - f\|_{L^{p(\cdot)}} \leq \left(\|L_n\| + 1 \right) K \left(f, \frac{A_p + 1}{\|L_n\| + 1} \Delta_n \right)_{p(\cdot)}.$$

The proof of Corollary 4 is complete.

4 Uniform Boundedness on Variable Spaces

In this section, we prove our second main result. The technical difficulty arising from the uniform boundedness is overcome by the Lipschiz continuity of the exponent function and localization of Bernstein type positive linear operators. Here the Lipschitz continuity condition (2.8) and the lower bound $p_- > 1$ play a crucial role, and they imply the finiteness of the upper bound p_+. Note that here $\Omega = (0,1)$.

Proof of Theorem 5. To give the proof in a unified way for both the Kantorovich operator and Durrmeyer operator, we use the notation χ_E for the characteristic function of a set E and define kernels $\{q_{n,k}\}_{k=0}^n$ as

$$q_{n,k}(t) = \begin{cases} (n+1)\chi_{(k/(n+1),(k+1)/(n+1))}(t), & \text{for } L_n = K_n, \\ (n+1)b_{n,k}(t), & \text{for } L_n = D_n. \end{cases} \tag{4.1}$$

Note that $0 \leq q_{n,k}(t) \leq n+1$ and $\int_\Omega q_{n,k}(t)dt = 1$. Then

$$L_n(f,x) = \sum_{k=0}^n \int_\Omega q_{n,k}(t)f(t)dtb_{n,k}(x), \qquad x \in \Omega = (0,1).$$

Let $f \in L^{p(\cdot)}$ have norm 1. That means $\int_\Omega |f(x)|^{p(x)}dx \leq 1$.

Let $x \in \Omega$, $n \in \mathbb{N}$, and $\beta = \frac{2}{p_- - 1} > 0$. Define a subset Ω_n of Ω as

$$\Omega_n = \left\{t \in \Omega : |f(t)| > n^\beta\right\},$$

and

$$L_n^*(f,x) = \sum_{k=0}^n \int_{\Omega_n} q_{n,k}(t)f(t)dtb_{n,k}(x).$$

By the definition of p_-, we have $p(t) \geq p_- > 1$ and $|f(t)| > n^\beta$ for every $t \in \Omega_n$, which implies

$$|f(t)| \leq |f(t)|^{p(t)} \left(n^\beta\right)^{1-p(t)} \leq |f(t)|^{p(t)} n^{\beta(1-p_-)} = n^{-2}|f(t)|^{p(t)}.$$

It follows that

$$\begin{aligned} |L_n^*(f,x)| &\leq n^{-2} \sum_{k=0}^n \int_{\Omega_n} q_{n,k}(t)|f(t)|^{p(t)}dtb_{n,k}(x) \\ &\leq n^{-2} \sum_{k=0}^n \int_{\Omega_n} (n+1)|f(t)|^{p(t)}dtb_{n,k}(x) \\ &= n^{-2} \int_{\Omega_n} (n+1)|f(t)|^{p(t)}dt \leq \frac{2}{n}. \end{aligned}$$

Here we have used the assumption that $\int_\Omega |f(x)|^{p(x)}dx \leq 1$.

Define

$$L_n^{**}(f,x) = \sum_{k \notin J_{n,x}} \int_{\Omega \setminus \Omega_n} q_{n,k}(t)f(t)dtb_{n,k}(x),$$

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where $J_{n,x}$ is the index set defined by

$$J_{n,x} = \left\{ k \in \{0, \dots, n\} : |k - nx| \leq n^{\frac{3}{4}} \right\} = \left\{ k \in \{0, \dots, n\} : \left| \frac{k}{n} - x \right| \leq n^{-\frac{1}{4}} \right\}. \tag{4.2}$$

We use the bound $|f(t)| \leq n^\beta$ for $t \in \Omega \setminus \Omega_n$ and

$$n^{\frac{1}{4}} \left| \frac{k}{n} - x \right| > 1, \qquad \text{for } k \notin J_{n,x},$$

and see that

$$\begin{aligned} |L_n^{**}(f,x)| &\leq \sum_{k \notin J_{n,x}} \int_{\Omega \setminus \Omega_n} (n+1) n^\beta dt b_{n,k}(x) \\ &\leq (n+1) n^\beta \sum_{k \notin J_{n,x}} b_{n,k}(x) \\ &\leq (n+1) n^\beta \sum_{k \notin J_{n,x}} \left(n^{\frac{1}{4}} \left| \frac{k}{n} - x \right| \right)^{2r} b_{n,k}(x), \end{aligned}$$

where r is the integer part of $2\beta + 5$. Then $2\beta + 4 < r \leq 2\beta + 5$.

It is well known in the classical approximation theory (see e.g., [15, 10]) that there exists a positive constant M_r depending only on $r \in \mathbb{N}$ such that

$$B_n \left((\cdot - x)^{2r}, x \right) \leq M_r n^{-r}, \qquad \forall x \in (0,1). \tag{4.3}$$

Applying this bound to continue our estimates gives

$$\begin{aligned} |L_n^{**}(f,x)| &\leq (n+1) n^\beta n^{\frac{r}{2}} \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^{2r} b_{n,k}(x) \\ &= (n+1) n^\beta n^{\frac{r}{2}} B_n \left((\cdot - x)^{2r}, x \right) \\ &\leq (n+1) n^\beta n^{\frac{r}{2}} M_r n^{-r} \leq 2 M_r n^{1+\beta-\frac{r}{2}} \leq \frac{2M_r}{n}. \end{aligned}$$

Let us turn to the key part defined by

$$\begin{aligned} I_n &:= \int_\Omega |L_n(f,x) - L_n^*(f,x) - L_n^{**}(f,x)|^{p(x)} dx \\ &= \int_\Omega \left| \sum_{k \in J_{n,x}} \int_{\Omega \setminus \Omega_n} q_{n,k}(t) f(t) dt b_{n,k}(x) \right|^{p(x)} dx. \end{aligned}$$

Applying the Hölder inequality and the relation $\sum_{k=0}^n b_{n,k}(x) = 1$ yields

$$I_n \leq \int_\Omega \sum_{k \in J_{n,x}} \left(\int_{\Omega \setminus \Omega_n} q_{n,k}(t) |f(t)| dt \right)^{p(x)} b_{n,k}(x) dx.$$

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Since $q_{n,k}(t) \geq 0$ and $\int_\Omega q_{n,k}(t)dt = 1$, we apply the Hölder inequality again and find

$$I_n \leq \int_\Omega \left\{ \sum_{k \in J_{n,x}} \int_{\Omega \setminus \Omega_n} q_{n,k}(t)|f(t)|^{p(x)} dt \right\} b_{n,k}(x) dx$$
$$\leq \int_\Omega \left\{ \int_{\Omega \setminus \Omega_n} |f(t)|^{p(t)} I_{n,x,t} dt \right\} dx, \tag{4.4}$$

where $I_{n,x,t}$ is the quantity defined by

$$I_{n,x,t} = \sum_{k \in J_{n,x}} q_{n,k}(t)|f(t)|^{p(x)-p(t)} b_{n,k}(x), \qquad t \in \Omega \setminus \Omega_n. \tag{4.5}$$

Recall the index set (4.2). It tells us that for $k \in J_{n,x}$, we have $\left|\frac{k}{n} - x\right| \leq n^{-\frac{1}{4}}$.

For the Kantorovich operator $L_n = K_n$, we have $q_{n,k}(t) \neq 0$ only when $t \in (k/(n+1), (k+1)/(n+1))$. In this case there hold $|x - t| \leq 2n^{-\frac{1}{4}}$ for $k \in J_{n,x}$, and $|p(x) - p(t)| \leq 2C_\alpha n^{-\frac{\alpha}{4}}$ by the Lipschitz condition (2.8). It follows that

$$I_{n,x,t} \leq \sum_{k \in J_{n,x}} q_{n,k}(t) \left(n^{\beta 2 C_\alpha n^{-\frac{\alpha}{4}}} + |f(t)|^{-p(t)} \right) b_{n,k}(x)$$
$$\leq \left(M_{p,\alpha} + |f(t)|^{-p(t)} \right) \sum_{k=0}^{n} q_{n,k}(t) b_{n,k}(x), \tag{4.6}$$

where the number $M_{p,\alpha}$ defined by

$$M_{p,\alpha} = \sup_{n \in \mathbb{N}} n^{\beta 2 C_\alpha n^{-\frac{\alpha}{4}}}$$

is finite because taking logarithms tells us that

$$\log \left(n^{\beta 2 C_\alpha n^{-\frac{\alpha}{4}}} \right) = \beta 2 C_\alpha n^{-\frac{\alpha}{4}} \log n \to 0 \quad (n \to \infty).$$

The argument for the Durrmeyer operator is more complicated. Recall

$$J_{n,t} = \left\{ k \in \{0, 1, \ldots, n\} : \left| \frac{k}{n} - t \right| \leq n^{-\frac{1}{4}} \right\}.$$

We need to separate the summation $\sum_{k \in J_{n,x}}$ into $\sum_{k \in J_{n,x} \cap J_{n,t}} + \sum_{k \in J_{n,x} \setminus J_{n,t}}$. For $k \in J_{n,x} \cap J_{n,t}$ we have again $|x - t| \leq 2n^{-\frac{1}{4}}$ and $|p(x) - p(t)| \leq 2C_\alpha n^{-\frac{\alpha}{4}}$.

Hence

$$\begin{aligned}
&\sum_{k\in J_{n,x}\cap J_{n,t}} q_{n,k}(t)|f(t)|^{p(x)-p(t)} b_{n,k}(x)\\
&\leq \sum_{k\in J_{n,x}\cap J_{n,t}} q_{n,k}(t)\left(n^{\beta 2C_\alpha n^{-\frac{\alpha}{4}}} + |f(t)|^{-p(t)}\right) b_{n,k}(x)\\
&\leq \left(M_{p,\alpha} + |f(t)|^{-p(t)}\right)\sum_{k=0}^{n} q_{n,k}(t) b_{n,k}(x).
\end{aligned}$$

For the second summation term, we apply the moment estimate (4.3) to the integer part r' of the positive number $3+2\beta p_+$ and get from the upper bound $p_+ = \sup_{x\in\Omega} p(x)$ of p that

$$\begin{aligned}
&\sum_{k\in J_{n,x}\setminus J_{n,t}} q_{n,k}(t)|f(t)|^{p(x)-p(t)} b_{n,k}(x)\\
&\leq \sum_{k\in J_{n,x}\setminus J_{n,t}} \left(n^{\frac{1}{4}}\left|\frac{k}{n}-t\right|\right)^{2r'} (n+1) b_{n,k}(t)\left(n^{\beta p_+} + |f(t)|^{-p(t)}\right) b_{n,k}(x)\\
&\leq n^{\frac{r'}{2}} M_{r'} n^{-r'}(n+1)\left(n^{\beta p_+} + |f(t)|^{-p(t)}\right)\\
&\leq 2M_{r'} n^{1-\frac{r'}{2}}\left(n^{\beta p_+} + |f(t)|^{-p(t)}\right) \leq 2M_{r'} + 2M_{r'}|f(t)|^{-p(t)}.
\end{aligned}$$

Putting the above bounds for the two terms into (4.5), we know that

$$I_{n,x,t} \leq \left(M_{p,\alpha} + |f(t)|^{-p(t)}\right)\sum_{k=0}^{n} q_{n,k}(t) b_{n,k}(x) + 2M_{r'} + 2M_{r'}|f(t)|^{-p(t)}.$$

Combining this with the bound (4.6) for the Kantorovich operator and (4.4), we see by $\int_\Omega b_{n,k}(t)dt = \frac{1}{n+1}$ and $\sum_{k=0}^{n} b_{n,k}(t) = 1$ that

$$\begin{aligned}
I_n &\leq \int_\Omega \left\{ \int_{\Omega\setminus\Omega_n} |f(t)|^{p(t)}\left(M_{p,\alpha}\sum_{k=0}^{n} q_{n,k}(t) b_{n,k}(x) + 2M_{r'}\right)\right.\\
&\qquad \left. + \sum_{k=0}^{n} q_{n,k}(t) b_{n,k}(x) + 2M_{r'} dt \right\} dx\\
&\leq (M_{p,\alpha} + 2M_{r'})\int_\Omega |f(t)|^{p(t)} dt + 1 + 2M_{r'} \leq M_{p,\alpha} + 1 + 4M_{r'}.
\end{aligned}$$

Based on the above estimates for the three terms L_n^*, L_n^{**} and $L_n - L_n^* - L_n^{**}$, we conclude that

$$\begin{aligned}
\int_\Omega |L_n(f,x)|^{p(x)}\, dx &\leq \int_\Omega 3^{p(x)}\Big\{ |L_n^*(f,x)|^{p(x)} + |L_n^{**}(f,x)|^{p(x)}\\
&\qquad + |L_n(f,x) - L_n^*(f,x) - L_n^{**}(f,x)|^{p(x)}\Big\} dx\\
&\leq 3^{p_+}\left\{ \left(\frac{2}{n}\right)^{p_+} + \left(\frac{2}{n}\right)^{p_-} + \left(\frac{2M_r}{n}\right)^{p_+} + \left(\frac{2M_r}{n}\right)^{p_-} + M_{p,\alpha} + 1 + 4M_{r'}\right\}.
\end{aligned}$$

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Thus, from $p(x) \geq p_-$, we find

$$\int_\Omega \left(\frac{|L_n(f,x)|}{\lambda} \right)^{p(x)} dx \leq \left(\frac{1}{\lambda} \right)^{p_-} \int_\Omega |L_n(f,x)|^{p(x)} dx \leq 1,$$

if we choose

$$\lambda = A_{\alpha,p} := 3^{1+C_\alpha} \left\{ 2 + 2^{1+C_\alpha} + 2M_r + (2M_r)^{1+C_\alpha} + M_{p,\alpha} + 1 + 4M_{r'} \right\},$$

since the Lipschitz condition (2.8) gives $|p(x) - p(y)| \leq C_\alpha |x-y|^\alpha \leq C_\alpha$, and thereby the relations $p_+ \leq p_- + C_\alpha$ and $\frac{p_+}{p_-} \leq 1 + \frac{C_\alpha}{p_-} \leq 1 + C_\alpha$. Therefore, we have $\|L_n(f)\|_{L^{p(\cdot)}} \leq A_{\alpha,p}$. This bound is true for any $f \in L^{p(\cdot)}$ with $\|f\|_{L^{p(\cdot)}} = 1$. So $\|L_n\| \leq A_{\alpha,p}$. This completes the proof of Theorem 5.

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Further interpretation of some fractional Ostrowski and Grüss type inequalities

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Abstract

We further interpret and simplify earlier produced fractional Ostrowski and Grüss type inequalities involving several functions.

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1 Background

Let $\nu \geq 0$; the operator I_{a+}^{ν}, defined for $f \in L_1[(a,b)]$ is given by

$$I_{a+}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\, dt, \tag{1}$$

for $a \leq x \leq b$, is called the left Riemann-Liouville fractional integral operator of order ν. For $\nu = 0$, we set $I_{a+}^{0} := I$, the identity operator, see [1], p. 392, also [7].

Let $\nu \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^n([a,b])$ (it means $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions).

Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt = \left(I_{a+}^{n-\nu} f^{(n)}\right)(x), \tag{2}$$

and it exists almost everywhere for $x \in [a,b]$.

Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator ([2], [8], [9]) of order α is denoted by

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \quad \forall x \in [a,b]. \tag{3}$$

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We set $I_{b-}^{0} := I$, the identity operator.

Let now $f \in AC^{m}([a,b])$, $m \in \mathbb{N}$, with $m := \lceil \alpha \rceil$.

We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^{m} I_{b-}^{m-\alpha} f^{(m)}(x), \tag{4}$$

we set $D_{b-}^{0} f := f$, that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)} \int_{x}^{b} (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{5}$$

We need

Proposition 1 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \geq x_0$. Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.*

Proposition 2 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \leq x_0$. Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.*

We also mention

Theorem 3 *([4], p. 362) Let $f \in C^{m}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

Convention 4 *([4], p. 360) We suppose that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{6}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{7}$$

for all $x, x_0 \in [a,b]$.

2 Motivation

We mention some Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 5 *([6]) Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_i \in AC^{m}([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$, with $f_i^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $i = 1, ..., r$. Assume that $\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1, ..., r$. Denote by*

$$\theta(f_1, ..., f_r)(x_0) := r \int_{a}^{b} \left(\prod_{k=1}^{r} f_k(x) \right) dx - \sum_{i=1}^{r} \left[f_i(x_0) \int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right]. \tag{8}$$

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Then

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{9}$$

$$\left. + \left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Inequality (9) is sharp, infact it is attained.

Theorem 6 *([6]) Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$; $x_0 \in [a,b]$ and $D_{x_0-}^{\alpha}f_i \in L_1([a,x_0])$, $D_{*x_0}^{\alpha}f_i \in L_1([x_0,b])$, for all $i = 1,...,r$. Then*

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{10}$$

$$\left. + \left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Theorem 7 *([6]) Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $x_0 \in [a,b]$; $i = 1,...,r$. Assume $D_{x_0-}^{\alpha}f_i \in L_q([a,x_0])$, and $D_{*x_0}^{\alpha}f_i \in L_q([x_0,b])$, $i = 1,...,r$. Then*

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right| \leq$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\alpha\right)} \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])} I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{11}$$

$$\left. + \left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])} I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

Next we mention some Caputo fractional Grüss type inequalities for several functions.

Theorem 8 *([6]) Let* $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_i \in AC([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{x_0-} f_i \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{*x_0} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,...,r$. *Denote by*

$$\Delta(f_1,...,f_r) := r(b-a)\int_a^b \left(\prod_{k=1}^r f_k(x)\right) dx - \tag{12}$$

$$\sum_{i=1}^r \left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^r f_j(x)\right) dx\right)\right].$$

Then

$$|\Delta(f_1,...,f_r)| \leq (b-a)\cdot$$

$$\sum_{i=1}^r \left[\left[\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{x_0-} f_i\right\|_{\infty,[a,x_0]} \sup_{x_0\in[a,b]} I^{\alpha+1}_{a+}\left(\prod_{\substack{j=1\\ j\neq i}}^r |f_j(x_0)|\right)\right] + \right.$$

$$\left.\left[\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{*x_0} f_i\right\|_{\infty,[x_0,b]} \sup_{x_0\in[a,b]} I^{\alpha+1}_{b-}\left(\prod_{\substack{j=1\\ j\neq i}}^r |f_j(x_0)|\right)\right]\right]. \tag{13}$$

Theorem 9 *([6]) Let* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, *and* $f_i \in AC([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$, $x_0 \in [a,b]$. *Assume that* $\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{x_0-} f_i\right\|_{L_q([a,x_0])}$, *and* $\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{*x_0} f_i\right\|_{L_q([x_0,b])} < \infty$, $i = 1,...,r$. *Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{(b-a)\,\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\cdot$$

$$\sum_{i=1}^r \left[\left[\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{x_0-} f_i\right\|_{L_q([a,x_0])} \sup_{x_0\in[a,b]} I^{\alpha+\frac{1}{p}}_{a+}\left(\prod_{\substack{j=1\\ j\neq i}}^r |f_j(x_0)|\right)\right] + \right.$$

$$\left.\left[\sup_{x_0\in[a,b]} \left\|D^{\alpha}_{*x_0} f_i\right\|_{L_q([x_0,b])} \sup_{x_0\in[a,b]} I^{\alpha+\frac{1}{p}}_{b-}\left(\prod_{\substack{j=1\\ j\neq i}}^r |f_j(x_0)|\right)\right]\right]. \tag{14}$$

3 Main Results

We make

Remark 10 *Let $g \in C([a,b])$, $\alpha > 0$, $x_0 \in [a,b] \subset \mathbb{R}$. Notice that*

$$I_{a+}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} g(z)\, dz. \tag{15}$$

Hence

$$\left| I_{a+}^{\alpha+1}(g)(x_0) \right| \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} |g(z)|\, dz \leq$$

$$\frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha}\, dz = \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \frac{(x_0 - a)^{\alpha+1}}{(\alpha+1)}$$

$$= \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{16}$$

That is

$$\left| I_{a+}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{17}$$

Similarly we have

$$I_{b-}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha} g(z)\, dz, \tag{18}$$

and

$$\left| I_{b-}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha}\, dz$$

$$= \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \frac{(b - x_0)^{\alpha+1}}{(\alpha+1)} = \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{19}$$

That is

$$\left| I_{b-}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{20}$$

Consequently we derive

$$I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \overset{(17)}{\leq} \frac{\left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}, \tag{21}$$

and

$$I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\overset{(20)}{\leq}\frac{\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}}{\Gamma\left(\alpha+2\right)}\left(b-x_0\right)^{\alpha+1}. \tag{22}$$

Therefore it holds

$$\left|\theta\left(f_1,...,f_r\right)\left(x_0\right)\right|\overset{(9)}{\leq}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right]\overset{((21),(22))}{\leq}$$

$$\frac{1}{\Gamma\left(\alpha+2\right)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right]\left(x_0-a\right)^{\alpha+1}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right]\left(b-x_0\right)^{\alpha+1}\right]=:\left(\xi_1\right). \tag{24}$$

Call

$$M_1\left(f_1,...,f_r\right)\left(x_0\right):=\max_{i=1,...,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]},\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\right\}. \tag{25}$$

Then

$$\left(\xi_1\right)\leq\frac{M_1\left(f_1,...,f_r\right)\left(x_0\right)}{\Gamma\left(\alpha+2\right)}\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\left(x_0-a\right)^{\alpha+1}+\right.$$

$$\left.\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\left(b-x_0\right)^{\alpha+1}\right]=:\left(\xi_2\right). \tag{26}$$

Call

$$\psi_1\left(f_1,...,f_r\right)\left(x_0\right):=\max\left\{\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]},\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \tag{27}$$

So that

$$(\xi_2) \leq \frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{28}$$

We have proved simpler interpretations of Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 11 *Here all as in Theorem 5, $M_1(f_1,...,f_r)(x_0)$ as in (25) and $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq \tag{29}$$
$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{30}$$

We make

Remark 12 *Let $g \in C([a,b])$, $\alpha \geq 1$, $x_0 \in [a,b] \subset \mathbb{R}$. We have that*

$$\left|I_{a+}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{31}$$

and

$$\left|I_{b-}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{32}$$

Consequently we derive

$$I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right) \overset{(31)}{\leq} \frac{\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{33}$$

$$I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right) \overset{(32)}{\leq} \frac{\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{34}$$

Therefore it holds

$$|\theta(f_1,...,f_r)(x_0)| \overset{(10)}{\leq} \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right] \overset{((33),(34))}{\leq}$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right](x_0-a)^{\alpha}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right](b-x_0)^{\alpha}\right]=:(\eta). \tag{35}$$

Call

$$M_2(f_1,...,f_r)(x_0) := \max_{i=1,...,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])}, \left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\right\}. \tag{36}$$

Then

$$(\eta) \leq \frac{M_2(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\cdot$$

$$\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}(x_0-a)^{\alpha}+\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}(b-x_0)^{\alpha}\right] \tag{37}$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\left[(b-x_0)^{\alpha}+(x_0-a)^{\alpha}\right] \tag{38}$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \tag{39}$$

We have proved

Theorem 13 *Let all as in Theorem 6, $M_2(f_1,...,f_r)(x_0)$ as in (36) and $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}\left[(b-x_0)^{\alpha}+(x_0-a)^{\alpha}\right] \tag{40}$$

$$\leq \frac{M_2(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \tag{41}$$

Similarly we obtain

Theorem 14 *Let all as in Theorem 7. Call*

$$M_3(f_1,...,f_r)(x_0) := \max_{i=1,...,r}\left\{\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\}. \tag{42}$$

Here $\psi_1(f_1,...,f_r)(x_0)$ *as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\left[(b-x_0)^{\alpha+\frac{1}{p}}+(x_0-a)^{\alpha+\frac{1}{p}}\right] \leq \tag{43}$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}}. \tag{44}$$

Finally we give a simpler interpretation of Caputo fractional Grüss type inequalities (13), (14).

Theorem 15 *All as in Theorem 8. We define*

$$M_4(f_1,...,f_r) := \max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{\infty,[a,x_0]}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{\infty,[x_0,b]}\right\} \tag{45}$$

and

$$\psi_2(f_1,...,f_r)(x_0) :=$$

$$\max\left\{\sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}, \sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \tag{46}$$

Then

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_4(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\Gamma(\alpha+2)}(b-a)^{\alpha+2}. \tag{47}$$

Theorem 16 *All as in Theorem 9. We define*

$$M_5(f_1,...,f_r) := \max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\} \tag{48}$$

Here ψ_2 *is as in (46). Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_5(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}+1}. \tag{49}$$

We finish with applications.

4 Applications

We apply above theory for $r=2$. In that case

$$\theta(f_1,f_2)(x_0)=2\int_a^b f_1(x)\,f_2(x)\,dx-f_1(x_0)\int_a^b f_2(x)\,dx-f_2(x_0)\int_a^b f_1(x)\,dx, \tag{50}$$

$x_0\in[a,b]$,

$$M_1(f_1,f_2)(x_0)=$$
$$\max\left\{\left\|D^{\alpha}_{x_0-}f_1\right\|_{\infty,[a,x_0]},\left\|D^{\alpha}_{x_0-}f_2\right\|_{\infty,[a,x_0]},\left\|D^{\alpha}_{*x_0}f_1\right\|_{\infty,[x_0,b]},\left\|D^{\alpha}_{*x_0}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{51}$$

$$\psi_1(f_1,f_2)(x_0)=\max\left\{\|f_1\|_{\infty,[a,x_0]}+\|f_2\|_{\infty,[a,x_0]},\|f_1\|_{\infty,[x_0,b]}+\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{52}$$

$$M_2(f_1,f_2)(x_0)=\max\left\{\left\|D^{\alpha}_{x_0-}f_1\right\|_{L_1([a,x_0])},\right.$$
$$\left.\left\|D^{\alpha}_{x_0-}f_2\right\|_{L_1([a,x_0])},\left\|D^{\alpha}_{*x_0}f_1\right\|_{L_1([x_0,b])},\left\|D^{\alpha}_{*x_0}f_2\right\|_{L_1([x_0,b])}\right\}, \tag{53}$$

$$M_3(f_1,f_2)(x_0):=\max\left\{\left\|D^{\alpha}_{x_0-}f_1\right\|_{L_q([a,x_0])},\right.$$
$$\left.\left\|D^{\alpha}_{x_0-}f_2\right\|_{L_q([a,x_0])},\left\|D^{\alpha}_{*x_0}f_1\right\|_{L_q([x_0,b])},\left\|D^{\alpha}_{*x_0}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{54}$$

$$\Delta(f_1,f_2)=2\left[(b-a)\int_a^b f_1(x)\,f_2(x)\,dx-\left(\int_a^b f_1(x)\,dx\right)\left(\int_a^b f_2(x)\,dx\right)\right], \tag{55}$$

$$M_4(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_1\right\|_{\infty,[a,x_0]},\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_2\right\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_1\right\|_{\infty,[x_0,b]},\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{56}$$

$$\psi_2(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[x_0,b]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{57}$$

and

$$M_5(f_1,f_2)=\max\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_1\right\|_{L_q([a,x_0])},\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_2\right\|_{L_q([a,x_0])},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_1\right\|_{L_q([x_0,b])},\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{58}$$

above $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.

Proposition 17 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_1, f_2 \in AC^m([a,b])$, with $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Assume that $\left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} \left[(b-x_0)^{\alpha+1} + (x_0-a)^{\alpha+1} \right] \tag{59}$$

$$\leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} (b-a)^{\alpha+1}. \tag{60}$$

Proof. By Theorem 11. ∎

Proposition 18 *Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f_1, f_2 \in AC^m([a,b])$. Suppose that $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$; $x_0 \in [a,b]$ and $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_1([x_0,b])$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} \left[(b-x_0)^{\alpha} + (x_0-a)^{\alpha} \right] \tag{61}$$

$$\leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} (b-a)^{\alpha}. \tag{62}$$

Proof. By Theorem 13. ∎

Proposition 19 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $f_1, f_2 \in AC^m([a,b])$. Suppose that $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$. Assume $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_q([a,x_0])$, and $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_q([x_0,b])$. Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \left[(b-x_0)^{\alpha+\frac{1}{p}} + (x_0-a)^{\alpha+\frac{1}{p}} \right] \tag{63}$$

$$\leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} (b-a)^{\alpha+\frac{1}{p}}. \tag{64}$$

Proof. By Theorem 14. ∎

Proposition 20 *Let $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_1, f_2 \in AC([a,b])$. Assume that $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. Then*

$$|\Delta(f_1,f_2)| \leq \frac{2M_4(f_1,f_2)\,\psi_2(f_1,f_2)}{\Gamma(\alpha+2)} (b-a)^{\alpha+2}. \tag{65}$$

Proof. By Theorem 15. ∎

Proposition 21 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, *and* $f_1, f_2 \in AC([a,b])$, $x_0 \in [a,b]$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_q([a,x_0])}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_q([x_0,b])} < \infty$, $i = 1, 2$. *Then*

$$|\Delta(f_1, f_2)| \leq \frac{2M_5(f_1, f_2)\,\psi_2(f_1, f_2)}{\left(\alpha + \frac{1}{p}\right)(p(\alpha - 1) + 1)^{\frac{1}{p}}\,\Gamma(\alpha)} (b-a)^{\alpha + \frac{1}{p} + 1}. \tag{66}$$

Proof. By Theorem 16. ■

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