Volume 3,Number 1 January 2008

ISSN:1559-1948 (PRINT), 1559-1956 (ONLINE)

EUDOXUS PRESS,LLC

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

SCOPE AND PRICES OF

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

A quartely international publication of **EUDOXUS PRESS,LLC**

ISSN:1559-1948(PRINT),1559-1956(ONLINE)

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Journal of Applied Functional Analysis(JAFA)
is published in January,April,July and October of each year by

EUDOXUS PRESS,LLC,
1424 Beaver Trail Drive,Cordova,TN38016,USA,
Tel.001-901-751-3553
anastassioug@yahoo.com
http://www.EudoxusPress.com visit also http://www.msci.memphis.edu/~ganastss/jafa.

Webmaster:Ray Clapsadle

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,11-32,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Bounds in spaces of Morrey under Cordes type conditions

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Abstract *In the study of boundary value problems for linear elliptic equations in nondivergence form with discontinuous coefficients we consider the class of discontinuity of Cordes type.*

In particular we state some local and non local a priori bounds for solutions of Dirichlet problem in unbounded domains.

The coefficients of lower terms in the differential operator belong to Morrey spaces and the principal coefficients are 'near' to functions satisfying a condition of Cordes type.

Our results are based on embedding theorems which allow us to require a summability lower than n for the coefficients of the operator L.

We introduce a modulus of continuity of the functions in Morrey spaces to obtain the dependence of the constants in the estimates. We state also a result about the multiplication operator from $W^1(\Omega)$ in $L^2(\Omega)$.

Mathematics Subject Classifications: 35J25, 46E35

Key words: elliptic equations, embedding theorems, a priori bounds, Morrey spaces, Cordes condition.

1. Introduction

Boundary value problems for linear elliptic equations in nondivergence form with discontinuous coefficients have been widely studied in bounded open sets. The paper of Miranda [24] represent a point of reference for many authors in the study of Dirichlet problem when coefficients have derivatives in the L^n spaces. Subsequent results were stated, for example, in [21, 23, 28].

Other results can be found in [2, 13, 15, 16] in wider classes of spaces while different classes of discontinuous operators were studied in [17, 18, 19, 20, 25].

When Ω is an unbounded open set, the problem was studied in more general spaces than L^n spaces in [26], in spaces of Morrey type in [7, 9, 10, 11] and in weighted spaces in [3, 4, 5, 6, 8, 12].

Basic tools for proving existence and, sometimes, uniqueness of solution of elliptic boundary value problems in Sobolev spaces are a priori bounds.

In this paper we state some a priori bounds for solutions of the problem

$$\begin{cases} Lu = f\,, \qquad f \in L^2(\Omega)\,, \\ u \in W^2(\Omega) \cap W_0^1(\Omega)\,, \end{cases} \tag{1.1}$$

where L is the operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u\,. \tag{1.2}$$

The coefficients a_i and a of the operator L belong to the class of Morrey type spaces $M^{p,\lambda}$ introduced in [27] which are larger than L^n spaces. We observe that, when Ω is a bounded open set, the spaces $M^{p,\lambda}(\Omega)$ are reduced to the classical Morrey space $L^{p,\lambda}(\Omega)$ (see [13], [14]) while, if $\Omega = R^n$, include $L^{p,\lambda}(R^n)$.

We require a lower summability for the coefficients of the operator L when we work with Morrey spaces with respect the other spaces. The reason is that our embedding theorems use some results stated by C.Fefferman [22], so we do not need to achieve n.

In this paper we consider the following class of discontinuity: the so-called Cordes condition introduced by H.O.Cordes in the study of Hölder continuity of solutions of elliptic equations. The requirement is that the eigenvalues of the matrix of the coefficients of the operator L do not scatter too much.

The interest of this type of conditions in the study of a priori bounds is due to the fact we get local estimates without the introduction of functions more regular close to coefficients a_{ij} and without further assumptions. The reason why we can apply embedding theorems also for $|x|$ large 'enough' is the kind of functions e_{ij} which approximate a_{ij}. Derivatives of such a functions are equal to zero and, so, we do not need further hypotheses on derivatives to use embedding results in the local a priori bounds.

A priori bounds (see Theorem 6.1 and Corollary 6.1 in Section 6) are obtained using embedding theorems (see Section 3) and local a priori bounds stated in Section 5.

In particular we prove that

$$\|u\|_{W^2(\Omega)} \leq c\left(\|Lu + \lambda\beta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega_o)}\right) ,$$

where $\lambda \geq 0$, $\beta : \Omega \to R_+$ and Ω_o is an bounded open subset of Ω.

We study also the dependence of the constants. This dependence turns out to be crucial to achieve some existence results.

To this aim it is necessary to introduce a kind of modulus of continuity of a function $g \in \tilde{M}^{p,\lambda}(\Omega)$ (see Section 2 for definitions) and to study the multiplication operator

$$u \longrightarrow gu$$

from $W^1(\Omega)$ in $L^2(\Omega)$ (see Lemma 3.1 and 3.2).

A recent paper [7] deals with problem (1.1) under conditions of Chicco type.

We remark that the two types of discontinuity require different hypotheses in the study of local bounds.

2. Notations and function spaces

Let E be a Lebesgue measurable subset of R^n and $\Sigma(E)$ the σ-algebra of Lebesgue measurable subsets of E.

We denote by $\mathcal{D}(A)$ the class of restrictions to A, $A \in \Sigma(E)$, of functions $\phi \in C_o^\infty(R^n)$ such that supp $\phi \cap \overline{A} \subset A$ and by $L^p_{loc}(A)$ the class of functions $f : A \to C$ such that

$\phi f \in L^p(A)$ for any $\phi \in \mathcal{D}(A)$. We set

$$|f|_{p,A} = \|f\|_{L^p(A)}\,, \qquad 1 \le p \le +\infty\,.$$

Let $B(x,r)$, $x \in R^n$, $r \in R_+$, be the open ball with center in x and radius r.

For $r \in \mathbf{R}_+$, we set $B_r = B(0,r)$ and denote by ζ_r a function of class $C_o^\infty(R^n)$ such that

$$\text{supp}\zeta_r \subset B_{2r}, \qquad 0 \le \zeta_r \le 1, \qquad \zeta_{r|B_r=1}, \qquad (\zeta_r)_x \le \frac{2}{r}\,.$$

Let Ω be an open subset of R^n. We set

$$\Omega(x,r) = \Omega \cap B(x,r) \qquad \forall\, x \in \Omega\,, \quad \forall\, r \in R_+\,.$$

Let us consider the spaces $M^{p,\lambda}(\Omega)$, $\tilde{M}^{p,\lambda}(\Omega)$, $M_o^{p,\lambda}(\Omega)$ defined in [27] (we refer also to [10] where we can find many properties of these spaces).

Let us define, for $1 \le p < +\infty$ and $0 \le \lambda < n$, $n \ge 2$,

$M^{p,\lambda}(\Omega)$ is the space of functions $g \in L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<\tau\le 1}} \tau^{-\lambda/p}\|g\|_{L^p(\Omega\cap B(x,\tau))} < +\infty\,, \tag{2.1}$$

equipped with the norm defined in (2.1);

$\tilde{M}^{p,\lambda}(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$;

$M_o^{p,\lambda}(\Omega)$ is the closure of $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

From the results in [27] we have the following characterizations of the spaces $\tilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$:

$\tilde{M}^{p,\lambda}(\Omega)$ is the subspace of $M^{p,\lambda}(\Omega)$ of the functions $g \in M^{p,\lambda}(\Omega)$ such that:

$$\forall \epsilon \in R_+ \ \exists \delta_\epsilon \in R_+ \text{ s.t. } (E \in \Sigma(\Omega), \ \sup_{x \in \Omega} |E \cap B(x,1)| \leq \delta_\epsilon \Rightarrow \|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \epsilon), \tag{2.2}$$

$M_o^{p,\lambda}(\Omega)$ is the subspace of $M^{p,\lambda}(\Omega)$ of the functions $g \in M^{p,\lambda}(\Omega)$ such that:

$$\forall \epsilon \in R_+ \ \exists h_\epsilon, k_\epsilon \in R_+ \text{ s.t. } (E \in \Sigma(\Omega), |E \cap B(0,k_\epsilon)| \leq h_\epsilon \ \Rightarrow \ \|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \epsilon). \tag{2.3}$$

Let us set:

$$M^p(\Omega) = M^{p,0}(\Omega), \quad \tilde{M}^p(\Omega) = \tilde{M}^{p,0}(\Omega), \quad M_o^p(\Omega) = M_o^{p,0}(\Omega).$$

The spaces $M^p(\Omega)$ and $M_o^p(\Omega)$ have been introduced and studied in [26].

It is useful to recall some results about Morrey type spaces introduced above.

We have the embedding:

$$M^{p_o,\lambda_0}(\Omega) \hookrightarrow M^{p,\lambda}(\Omega), \qquad p \leq p_o, \qquad \frac{\lambda - n}{p} \leq \frac{\lambda_0 - n}{p_o}$$

which implies in particular that:

$$L^\infty(\Omega) \hookrightarrow M^{p,\lambda}(\Omega).$$

The following inclusions hold:

$$L^n(\Omega) \subset M^{n,0}(\Omega) \subset M^{s,n-s}(\Omega), \qquad s \in]2,n[. \tag{2.4}$$

For example the constant functions belong to $M^{n,0}(\Omega)$ but do not belong to $L^n(\Omega)$. Furthermore the function $f(x) = \frac{1}{1+|x|^\alpha} \in M^{p,0}(\Omega)$ if $\alpha > 0$ while belongs to $L^p(\Omega)$ if $\alpha \in [0, \frac{n}{p}[$.

Remark 2.1 - We remark that if $g \in L^p_{loc}(\overline{\Omega})$, $1 \le p < +\infty$, and $\phi \in \mathcal{D}(\overline{\Omega})$, then $\phi\, g \in M_0^{p,\lambda}(\Omega)$ and as a consequence to the space $\tilde{M}^{p,\lambda}(\Omega)$ (see [7, Lemma 2.1 and Remark 2.1]). ∎

3. Embedding results

Embedding results due to C.Fefferman [22] (see also [14]) allow us to state the following lemma (see [27]).

Lemma 3.1 - *If Ω has the cone property and $g \in M^{s,n-s}(\Omega)$, $s \in]2, n]$, then for any $u \in W^1(\Omega)$ we get $gu \in L^2(\Omega)$ and*

$$|g\, u|_{2,\Omega} \le H\, \|g\|_{M^{s,n-s}(\Omega)}\, \|u\|_{W^1(\Omega)}\,, \tag{3.1}$$

where the constant H, independent of g and u, depends on n and s. ∎

Let us define the modulus of continuity of a function $g \in \tilde{M}^{p,\lambda}(\Omega)$ (see also [9]).

If $p \in [1, +\infty[$, $\lambda \in [0, n[$ and $g \in \tilde{M}^{p,\lambda}(\Omega)$, we set

$$\tau^p_\lambda[g](t) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_x |E\cap B(x,1)|\le t}} \|g\, \chi_E\|_{M^{p,\lambda}(\Omega)}\,, \qquad t \in R_+\,,$$

where χ_E is the characteristic function of E. From (2.2) it follows that that $g \in \tilde{M}^{p,\lambda}(\Omega)$ if and only if $g \in M^{p,\lambda}(\Omega)$ and

$$\lim_{t\to 0} \tau^p_\lambda[g](t) = 0\,.$$

We define the modulus of continuity of $g \in \tilde{M}^{p,\lambda}(\Omega)$ as a function $\tau[g] : R_+ \to R_+$ satisfying

$$\tau^p_\lambda[g](t) \le \tau[g](t)\,, \qquad \forall t \in R_+\,, \qquad \lim_{t\to 0} \tau\,[g](t) = 0\,.$$

In the case $g : \Omega \to R$, we put

$$A_r(g) = \{x \in \Omega : |g(x)| \geq r\} , \qquad r \in R_+ .$$

If $g \in L^p_{loc}(\overline{\Omega})$, $p \in [1, +\infty[$, we get

$$\lim_{r \to +\infty} |A_r(g) \cap B(x,1)| = 0 .$$

Let us denote, for all $k \in R_+$, by $r_k = r_k(g)$ a real number such that

$$|A_{r_k}(g) \cap B(x,1)| \leq \frac{1}{k+1} \tag{3.2}$$

and by $r[g]$ the function

$$r[g] : k \in R_+ \to r[g](k) = r_k \in R_+ . \tag{3.3}$$

Now we state the following lemma which we will use later. In [27] and in [10] we can find a similar inequality, but in this paper we emphasize the dependence of the constant in the final bound.

LEMMA 3.2 - *In the same hypotheses of Lemma 3.1 and if $g \in \tilde{M}^{s,n-s}(\Omega)$, $s \in]2, n]$, then for any $k \in R_+$ we have*

$$|g\, u|_{2,\Omega} \leq H\, \tau[g] \left(\frac{1}{k+1} \right) \|u\|_{W^1(\Omega)} + r[g](k)\, \|u\|_{L^2(\Omega)} \qquad \forall\, u \in W^1(\Omega) ,$$

where H is the constant in (3.1), $\tau[g]$ is the modulus of continuity of g in $\tilde{M}^{s,n-s}(\Omega)$ and $r[g]$ is the function defined by (3.3).

PROOF. Let

$$g_k = (1 - \chi_{A_{r_k}})\, g .$$

The function g_k so defined belongs to the space $L^\infty(\Omega)$. From Lemma 3.1 we get

$$|g\,u|_{2,\Omega} \leq |(g-g_k)\,u|_{2,\Omega} + |g_k\,u|_{2,\Omega} \leq$$

$$\leq H\|g-g_k\|_{M^{s,n-s}(\Omega)}\,\|u\|_{W^1(\Omega)} + |g_k\,u|_{2,\Omega} =$$

$$= H\|g\chi_{A_{r_k}}\|_{M^{s,n-s}(\Omega)}\,\|u\|_{W^1(\Omega)} + r[g](k)|u|_{2,\Omega}\,.$$

Taking in mind (3.2) and modulus of continuity we deduce the result. ■

4. Hypotheses

Let us set

$$B_+ = \{x \in B_1 : x_n > 0\}\,, \qquad B_o = \{x \in B_1 : x_n = 0\}\,,$$

and suppose that

h_1) there are a $d \in R_+$, an open cover $\{U_i\}_{i\in I}$ of $\partial\Omega$ and, for any $i \in I$, a C^2-diffeomorphism $\psi_i : \overline{U}_i \to \overline{B}_1$ such that:

- $\psi_i(U_i \cap \Omega) = B_+\,, \qquad \psi_i(U_i \cap \partial\Omega) = B_o\,;$
- the components of ψ_i and ψ_i^{-1} and of their first and second derivatives are bounded by a constant independent of i;
- for any $x \in \Omega_d$ there exists an $i \in I$ such that $B(x,d) \subset U_i$ and, for any $x \in \Omega \setminus \Omega_d$, we get $B(x,d) \subset \Omega$, where $\Omega_d = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < d\}$.

Remark 4.1 - It is easy to prove that h_1) holds when Ω has the uniform C^2 - regularity property defined in [1]. ■

Remark 4.2 - The condition $h_1)$ implies that there exists a number $\rho \in R_+$ such that, for any $x \in R^n$, $B(x,\rho) \cap \partial\Omega = \emptyset$ or $B(x,\rho) \cap \partial\Omega \neq \emptyset$ and $B(x,\rho) \subset U_i$ for some $i \in I$.

Let us consider in Ω the second order linear differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u \tag{4.1}$$

with the following conditions on the coefficients:

$h_2)$ $\qquad a_{ij} = a_{ji} \in L^\infty(\Omega)\,, \qquad i,j = 1,\dots,n\,,$

$h_3)$ $\qquad a_i \in \tilde{M}^{s,n-s}(\Omega)\,, \quad i = 1,\dots,n\,, \qquad a \in \tilde{M}^{t}(\Omega)\,,$

where

$$s \in]2,n]\,, \qquad t = 2 \quad \text{if} \quad n = 3\,, \qquad t > 2 \quad \text{if} \quad n = 4\,, \qquad t = \frac{n}{2} \quad \text{if} \quad n > 4\,.$$

$h_4)$ (*Cordes type condition*)

$$\operatorname*{ess\,inf}_{\Omega} \left(\sum_{i=1}^{n} a_{ii}\right)^2 \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{-1} > n-1\,.$$

Previous condition can be written in the following equivalent form

$$\operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} \Big(\delta_{ij} - g a_{ij}\Big)^2 < 1\,, \tag{4.2}$$

where $g = \dfrac{\sum_{i,j=1}^{n} \delta_{ij}\, a_{ij}}{\sum_{i,j=1}^{n} a_{ij}^2}$.

Let us set

$$u_x = \left(\sum_{i=1}^{n} u_{x_i}^2 \right)^{1/2}, \qquad u_{xx} = \left(\sum_{i,j=1}^{n} u_{x_i x_j}^2 \right)^{1/2}.$$

We consider a function $\beta : \Omega \to R_+$ such that the following hypothesis holds:

$$h_5) \qquad \beta \in \tilde{M}^t(\Omega) \quad \text{and} \quad \exists\, \delta \in \tilde{M}^{s,n-s}(\Omega) \quad \text{such that} \quad \beta_x \le \beta\, \delta\,.$$

For example, some functions which satisfy the hypothesis $h_5)$ are given by $\beta = 1$ or $\beta(x) = \frac{1}{(1+|x|^2)^\tau}$, $x \in \Omega$, $\tau > 0$.

Remark 4.3 - Let us note that hypothesis $h_4)$ implies that operator L defined in (4.1) is uniformly elliptic in Ω. ∎

Remark 4.4 - One can show that under hypotheses $h_1) - h_3)$ and $h_5)$ it follows that for any $s, \lambda \in R$ the operator

$$u \in W^2(\Omega) \;\to\; Lu + \lambda\, \beta\, u \in L^2(\Omega)$$

is bounded. ∎

5. Local a priori bounds

Let us set

$$L_o u = - \sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j},$$

and let us fix a bounded open subset V of R^n such that

$$V \subset \Omega \qquad \text{or} \qquad V \cap \partial\Omega \neq \emptyset \quad \text{and} \quad V \subset U_i \quad \text{for some } i \in I\,.$$

We can prove the following Lemma using as tool Lemma 3.2. We remark that Cordes conditions are sufficient to get estimates for $|x|$ large enough.

LEMMA 5.1 – *If the conditions* $h_1) - h_5)$, *hold and* λ_1 *is a real number, then there exists a constant* $c \in R_+$ *such that for any* $\lambda \in [\lambda_1, +\infty[$ *and for any function* v *satisfying*

$$v \in W^2(\Omega) \cap W_0^1(\Omega)\,, \qquad supp\, v \subset V\,.$$

we get

$$|v_{xx}|_{2,\Omega} \le c \Big(|Lv + \lambda\, g^{-1}\, \beta\, v|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} \Big)\,, \tag{5.1}$$

where c is a positive constant depending on $n,\ s,\ t,\ \|a_{ij}\|_\infty,\ \tau[\delta],\ \tau[\beta],\ \tau[a_i],\ \tau[a],\ r[\delta],$ $r[\beta],\ r[a_i],\ r[a]$.

PROOF. We start proving the inequality

$$|v_{xx}|^2_{2,\Omega} \le \left| -\sum_{i,j=1}^n \delta_{ij} v_{x_i x_j} + \lambda\, \beta\, v \right|^2_{2,\Omega} + |\eta\, v_x|^2_{2,\Omega}\,, \tag{5.2}$$

where $\eta = \sum_{i,j=1}^n \delta_{ij}\, \delta$ and $\lambda \ge 0$.

In fact we have for $\lambda \ge 0$

$$\begin{aligned} \int_\Omega \Big(-\sum_{i,j=1}^n \delta_{ij} v_{x_i x_j} + \lambda\, \beta\, v \Big)^2 dx \ge \int_\Omega \Big(-\sum_{i,j=1}^n \delta_{ij} v_{x_i x_j} \Big)^2 dx + \lambda^2 \int_\Omega \beta^2 v^2\, dx + \\ + 2\lambda \int_\Omega \beta\, v_x^2\, dx - 2\lambda \int_\Omega \beta\, \eta\, |v|\, v_x\, dx\,. \end{aligned} \tag{5.3}$$

Using the inequality

$$\int_\Omega \beta\, \eta\, |v|\, v_x\, dx \le \frac{\lambda}{2} \int_\Omega \beta^2 v^2\, dx + \frac{1}{2\lambda} \int_\Omega |\eta\, v_x|^2\, dx\,,$$

from (5.3) we get (5.2).

We proceed using hypothesis of Cordes type to get the result. Indeed if we set

$$h = \operatorname*{ess\,sup}_{\Omega} \left(\sum_{i,j=1}^{n} |\delta_{ij} - g a_{ij}|^2 \right)^{1/2} ,$$

from inequality (5.2) we get

$$\begin{aligned} |v_{xx}|_{2,\Omega} \leq & \left| - \sum_{i,j=1}^{n} (\delta_{ij} - g\, a_{ij}) v_{x_i x_j} + g\, L_o\, v + \lambda\, \beta\, v \right|_{2,\Omega} + |\eta\, v_x|_{2,\Omega} \leq \\ & \leq h\, |v_{xx}|_{2,\Omega} + \|g\|_\infty \big| L_o v + \lambda\, g^{-1} \beta\, v \big|_{2,\Omega} + |\eta\, v_x|_{2,\Omega} \, , \end{aligned}$$

from which we deduce the inequality

$$|v_{xx}|_{2,\Omega} \leq c_1 \bigg(|L_o v + \lambda\, g^{-1} \beta\, v|_{2,\Omega} + |\eta\, v_x|_{2,\Omega} \bigg) \, , \tag{5.4}$$

since $1 - h > 0$ from (4.2).

The function $\eta \in \tilde{M}^{s,n-s}(\Omega)$, then we can use Lemma 3.2 to estimate the last term in (5.4) to get

$$\begin{aligned} |v_{xx}|_{2,\Omega} \leq c_2 \bigg(& |L_o\, v + \lambda\, g^{-1}\, \beta\, v|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} + \\ & + H\, \tau[\eta] \left(\frac{1}{k+1} \right) |v_{xx}|_{2,\Omega} \bigg) \, . \end{aligned} \tag{5.5}$$

By definition of modulus of continuity given in Section 3 it follows that there exists $k_0 \in R_+$ such that from (5.5)

$$|v_{xx}|_{2,\Omega} \leq c_3 \left(|L_o\, v + \lambda\, g^{-1} \beta\, v|_{2,\Omega} + |v_x|_{2,\Omega} + |v|_{2,\Omega} \right) . \tag{5.6}$$

If $\lambda_1 < 0$, we fix $\lambda \in [\lambda_1, 0[$.

Using h_5) and applying to β Lemma 3.2 we get the bound

$$|\lambda\, g^{-1}\, \beta\, v|_{2,\Omega} \leq c_5 |\lambda_1| \, (\operatorname{ess\,inf} g)^{-1} \Big(|v_x|_{2,\Omega} + |v|_{2,\Omega} \Big) . \tag{5.7}$$

Now if we consider the inequality (5.6) with $\lambda = 0$, from (5.7) we easly deduce (5.1) with L_o instead of L.

Finally, applying Lemma 3.2 to the functions a_i and a verifying hypothesis h_3) we obtain the result. ∎

REMARK 5.1 - Lemma 5.1 can be proved in more general hypotheses, that is under Chicco type conditions (see [7] and, in weighted spaces, [3], [4]). In such a case the function η depends also on derivatives of functions which approximate a_{ij} and we can apply Lemma 3.2 for $|x|$ large enough introducing further assumptions. So the two types of discontinuity require different hypotheses in the study of local bounds. ∎

6. A priori bounds

We assume that the following further hypotheses hold:

h_6) there exists a function $\gamma : R_+ \to R_+$ such that

$$\operatorname*{ess\,sup}_{\Omega\setminus B_k} \sum_{i,j=1}^{n} |c_{ij} - g a_{ij}| \leq \gamma(k)\,, \qquad \forall k \in R_+\,, \qquad \lim_{k\to+\infty} \gamma(k) = 0\,,$$

where c_{ij}, for $i, j = 1, ..., n$, are constant functions satisfying

$$c_{ij} = c_{ji}\,, \quad i, j = 1, \ldots, n\,, \qquad \sum_{i,j=1}^{n} c_{ij}\,\xi_i\,\xi_j \geq \nu\,|\xi|^2 \qquad \forall\,\xi \in R^n\,, \quad \text{a.e. in } \Omega\,,$$

with ν positive constant independent of x and ξ ;

h_7)

$$a_i \in M_0^{s,n-s}(\Omega)\,, \qquad i = 1, ..., n\,, \qquad \operatorname*{ess\,inf}_{\Omega} a > 0\,.$$

Local a priori bound stated in Lemma 5.1 allows us to prove the following result.

THEOREM 6.1 – *If the hypotheses $h_1)-h_7)$ hold, then there exist a constant $c \in R_+$ and a bounded open set $\Omega_o \subset\subset \overline{\Omega}$ such that*

$$\|u\|_{W^2(\Omega)} \le c\Big(|Lu + \lambda\, g^{-1}\,\beta\, u|_{2,\Omega} + |u|_{2,\Omega_o}\Big) \tag{6.1}$$

$$\forall\, u \in W^2(\Omega) \cap W^1_0(\Omega)\,, \qquad \forall\, \lambda \ge 0\,,$$

where c is a positive constant depending on $\Omega,\ \nu,\ n,\ s,\ t,\ a_i,\ \|a_{ij}\|_\infty\,,\ c_{ij}\,,\ \tau[\delta],\ \tau[\beta],\ \tau[a],\ r[\delta],\ r[\beta],\ r[a]$.

PROOF. • STEP 1 (*Estimates at infinity).*

If the principal coefficients of L are suitable constants, we can use Corollary 5.2 in [10] to get the bound (6.2). Therefore if

$$\tilde{L}_o = -\sum_{i,j=1}^{n} c_{ij}\,\frac{\partial^2}{\partial x_i \partial x_j}\,,$$

and $g = \dfrac{\sum_{i,j=1}^n \delta_{ij}\, a_{ij}}{\sum_{i,j=1}^n a_{ij}^2}$, we have that

$$\|(1-\zeta_k)\,u\|_{W^2(\Omega)} \le c_1 \Big|\tilde{L}_o((1-\zeta_k)\,u) + (ga + \lambda\beta)(1-\zeta_k)\,u\Big|_{2,\Omega}\,, \tag{6.2}$$

from which

$$\begin{aligned}\|(1-\zeta_k)\,u\|_{W^2(\Omega)} &\le c_1\bigg(\Big|-\sum_{i,j=1}^{n}\big(c_{ij}-ga_{ij}\big)\big((1-\zeta_k)\,u\big)_{x_ix_j} + \\ &\quad - g\sum_{i,j=1}^{n} a_{ij}\big((1-\zeta_k)\,u\big)_{x_ix_j} + (ga+\lambda\,\beta)(1-\zeta_k)\,u\Big|_{2,\Omega}\bigg) \le \\ &\le c_1\bigg(\|g\|_\infty |L_o\big((1-\zeta_k)\,u\big) + (a+\lambda\, g^{-1}\,\beta)(1-\zeta_k)\,u|_{2,\Omega} + \\ &\quad + \gamma(k)|((1-\zeta_k)\,u)_{xx}|_{2,\Omega}\bigg)\,.\end{aligned} \tag{6.3}$$

Taking in mind h_6), by a suitable choice $k = k_0 \in R_+$ we get from (6.3)

$$\|(1-\zeta_{k_0})\,u\|_{W^2(\Omega)} \le c_2 \left| L_o\big((1-\zeta_{k_0})\,u\big) + (a+\lambda\, g^{-1}\,\beta)(1-\zeta_{k_0})\,u \right|_{2,\Omega} . \tag{6.4}$$

• STEP 2 (*Estimates on bounded sets*).

Let us consider a function $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that:

$$\varphi|_{B_{\frac{1}{2}}} = 1, \quad \operatorname{supp}\varphi \subset B_1\,, \quad \sup_{R^n} |\partial^\alpha \varphi| \le c_\alpha \quad \forall \alpha \in N_o^n\,.$$

Let us define for $x \in \Omega$,

$$\Phi = \Phi^x \;:\; y \in \mathbf{R}^n \to \varphi\left(\frac{x-y}{\tau}\right) .$$

We have

$$\Phi|_{B(x,\frac{\tau}{2})} = 1, \quad \operatorname{supp}\Phi \subset B_{(x,\tau)}\,, \quad \sup_{R^n} |\partial^\alpha \Phi| \le c'_\alpha \quad \forall \alpha \in N_0^n\,,$$

where $c'_\alpha = c_\alpha \tau^{-|\alpha|}$.

So, if $u \in W^2(\Omega) \cap W_0^1(\Omega)$, then the function $v = \Phi\, u \in W^2(\Omega) \cap W_0^1(\Omega)$ and either $\operatorname{supp} v \subset \Omega$ or $\operatorname{supp} v \cap \partial\Omega \neq \emptyset$ and $\operatorname{supp} v \subset U_i$ for some $i \in N$.

Let us fix $k \in R_+$ and set $w = \zeta_k u$. Then, we can apply Lemma 5.1 with $v = \Phi w$ and $L = L_o + a$ to get

$$|(\Phi w)_{xx}|_{2,\Omega} \le c_3 \Big(|L_o(\Phi w) + (a+\lambda g^{-1}\beta)\Phi w|_{2,\Omega} + |(\Phi w)_x|_{2,\Omega} + |\Phi w|_{2,\Omega} \Big) . \tag{6.5}$$

The first term of the right hand side in (6.5) can be bounded as it follows:

$$\begin{aligned} |L_o(\Phi w) + (a+\lambda g^{-1}\beta)\Phi w|_{2,\Omega} &\le |\Phi(L_o w + (a+\lambda g^{-1}\beta)w)|_{2,\Omega} + \\ &+ 2\sup_{i,j} \|a_{ij}\|_{L^\infty(\Omega)} |\Phi_x w_x|_{2,\Omega} + \sup_{i,j} \|a_{ij}\|_{L^\infty(\Omega)} |\Phi_{xx} w|_{2,\Omega} \le \\ &\le c_4 \Big(|L_o w + (a+\lambda g^{-1}\beta) w|_{2,\Omega(x,\tau)} + |w_x|_{2,\Omega(x,\tau)} + |\,w|_{2,\Omega(x,\tau)} \Big) . \end{aligned} \tag{6.6}$$

Hence from (6.5) and (6.6) we deduce the inequality

$$|w_{xx}|_{2,\Omega(x,\frac{\tau}{2})} \le c_5 \left(|L_o w + (a + \lambda g^{-1}\beta) w|_{2,\Omega(x,\tau)} + |w_x|_{2,\Omega(x,\tau)} + |w|_{2,\Omega(x,\tau)} \right).$$

Therefore, applying Lemma 1.1 in [10], we obtain

$$\begin{aligned} |(\zeta_k u)_{xx}|_{2,\Omega} \le c_6 \Big(& \big|(L_o(\psi_k u) + (a + \lambda g^{-1}\beta)\psi_k u\big|_{2,\Omega} + |(\zeta_k)_x\, u|_{2,\Omega} + \\ & + |\zeta_k\, u_x|_{2,\Omega} + |\zeta_k u|_{2,\Omega} \Big). \end{aligned} \tag{6.7}$$

Using the well known inequality (see [1])

$$|u_x|_{2,supp\zeta_k} \le K(\epsilon |u_{xx}|_{2,supp\zeta_k} + \epsilon^{-1}|u|_{2,supp\zeta_k}), \tag{6.8}$$

where $K = K(n,\Omega)$ and $0 < \epsilon < \epsilon_0$, $\epsilon_0 > 0$, by (6.7)

$$\|\zeta_k u\|_{W^2(\Omega)} \le c_7 \left(\big|(L_o(\zeta_k u) + (a + \lambda g^{-1}\beta)\zeta_k u\big|_{2,\Omega} + |\zeta_k u|_{2,\Omega} \right). \tag{6.9}$$

By (6.4) and (6.9) with $k = k_0$ and using again (6.8) we get

$$\|u\|_{W^2(\Omega)} \le c_8 \big(|L_o\, u + (a + \lambda\, g^{-1}\, \beta)\, u|_{2,\Omega} + |u|_{2,\Omega_o'} \big), \tag{6.10}$$

with $\Omega_o' = \operatorname{supp} \zeta_{k_0}$.

Moreover from Lemma 3.4 in [10] we have that for any $\epsilon \in R_+$ there exist $c(\epsilon) \in R_+$ and an open set $\Omega_\epsilon \subset\subset \Omega$ such that

$$\sum_{i=1}^n \|a_i u_{x_i}\|_{L^2_s(\Omega)} \le \epsilon\, \|u\|_{W^2_s(\Omega)} + c(\epsilon)\, |u|_{2,\Omega_\epsilon}. \tag{6.11}$$

From (6.10) and (6.11) we deduce the assertion with $\Omega_o = \Omega_o' \cup \Omega_\epsilon$. ∎

Remark 6.1 – We observe that in Theorem 6.1 we can suppose in place of the condition $\operatorname{ess\,inf}_\Omega a > 0$ in h_7)

$$a = a' + a'', \qquad a' \in M_0^t(\Omega), \qquad \operatorname*{ess\,inf}_\Omega a'' > 0. \ ∎$$

REMARK 6.2 – A different assumption in Theorem 6.1 could be the convergence of a_{ij} to more regular functions α_{ij} at infinity. For example functions such that $(\alpha_{ij})_{x_h}$ belong to the space $M_0^{s,n-s}(\Omega)$. Then we can modify the proof in Step 1 setting

$$\tilde{L}_o = -\sum_{i,j=}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

and using h_6) with α_{ij} in place of c_{ij}. ∎

From Theorem 6.1 it follows the following

COROLLARY 6.2 – *In the same hypotheses of Theorem 6.1 and if*

$$\beta^{-1} \in L^\infty_{loc}(\Omega) \tag{6.12}$$

then for any $s \in R$ there exist c, $\lambda_0 \in R_+$ such that

$$\|u\|_{W^2(\Omega)} \leq c \left| Lu + \lambda\, g^{-1}\, \beta\, u \right|_{2,\Omega} \tag{6.13}$$

$$\forall\, u \in W^2(\Omega) \cap W^1_0(\Omega)\,, \quad \forall\, \lambda \geq \lambda_0\,,$$

where c has the same dependence of the constant in Theorem 6.1.

PROOF. Using hypotheses (6.12) and taking in mind Remark 4.4 and Theorem 6.1 it follows that

$$\begin{aligned} \lambda |u|_{2,\Omega_o} \leq c_1 |\lambda \beta g^{-1} u|_{2,\Omega_o} \leq c_2 \Big(|Lu + \lambda \beta g^{-1} u|_{2,\Omega} + \|u\|_{W^2(\Omega)} \Big) \leq \\ \leq c_3 \Big(|Lu + \lambda \beta g^{-1} u|_{2,\Omega} + |u|_{2,\Omega_o} \Big) \end{aligned} \tag{6.14}$$

for any $u \in W^2(\Omega) \cap W^1_0(\Omega)$ and for any $\lambda \in R_+$, where Ω_o is the open set in Theorem 6.1. For λ large enough we deduce the result by (6.1) and (6.14). ∎

Remark 6.3 – Inequality (6.13) can be obtained under different assumptions if we suppose coefficients of the operator L more regular. We refer to the paper [10] where we can find some results. We remark that Theorem 6.1 allows us to obtain the result stated in Corollary 6.2 under hypotheses considerably weakened with respect to previous papers. ∎

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,33-50,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Applications of Randomly Pseudo-monotone Operators with Randomly Upper semicontinuity in Generalized Random Quasivariational Inequalities

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Abstract: Let (Ω, Σ) be a measurable space, E a topological vector space and X a nonempty subset of E. Let $S : X \times \Omega \to 2^X$ and $T : X \times \Omega \to E^*$ be two random mappings. Then the generalized random quasi-variational inequality (GRQVI) is to find for a measurable map $\hat{y} : \Omega \to X$ such that $\hat{y}(\omega) \in S(\omega, \hat{y}(\omega))$, $\hat{w} \in T(\omega, \hat{y}(\omega))$ and

$$Re\langle \hat{w}, \hat{y}(\omega) - x(\omega)\rangle \leq 0, \quad \forall\, x(\omega) \in S(\omega, \hat{y}(\omega)).$$

We use Chowdhury and Tan's [6] generalized version of Ky Fan's minimax inequality as a tool to obtain some general theorems on random solutions of the GRQVI on a paracompact set X in a Hausdorff locally convex space. The random multivalued operator T is either randomly strong psudo-monotone or randomly pseudo-monotone and is randomly upper semicontinuous from $Co(A)$ to the weak*-topology on E^* for each nonempty finite subset A of X.

Key words: Measurable space, generalized quasi-variational inequality, locally convex space, partition of unity, paracompact set, randomly lower semicontinuous, randomly upper semicontinuous, randomly strong pseudo-monotone, σ-algebra.

AMS Subject Classifications: 47H04, 47H05, 47H10, 49J35, 54C60

1. Introduction

The theory of variational inequalities provides a natural and elegant framework for the study of many seemingly unrelated free boundary value problems arising in various branches of engineering and mathematical sciences. Variational inequalities have many nice results related to nonlinear partial differential equations. Complementarity problem, which is closely related to variational inequality problem, plays an important role in general equilibrium theory, economics, management sciences and operations research.

An important and useful generalization of variational inequality is the quasi-variational inequality introduced and considered by Bensousson and Lions [2]. For further details, we refer to Baiocchi and Capelo [1]. On the other hand, Pang [17] has considered the quasi-complementarity problem. Karamardian [13] showed that if the set involved in a variational inequality problem is a convex cone, then variational inequality and complementarity problems have the same solution set. Pang [17] proved that the same relation is true for the quasi-complementarity problem and quasi-variational inequality problem.

The fundamental theory of random operators is an important branch of stochastic analysis and its development is required for the study of several classes of random operator equations. Almost half a century ago, the systematic study of random fixed point was initiated by the Prague school of probabilists. However, it received the attention it deserved only after the appearance of the survey paper by Bharucha-Reid [3] in 1976. Since then this discipline has been developed further in which many profound concepts and results were established with considerable generality, see for instance, the work of Shahzad [16], Xu [24], Itoh [12], Liu [15], Papageorgiou [18], Tan and Yuan [23], Yuan [25], Salahuddin [20], Khan and Salahuddin [11] etc.

The aim of this paper is to make further investigations in the same direction. We shall use Chowdhury and Tan's results [7,8] and Ky Fan's minimax inequality [10] as tools to obtain some general theorems on solutions of the GRQVI on a paracompact set X in a locally convex Hausdorff topological vector space, where the multivalued random operator T is randomly strong pseudo-monotone or randomly pseudo-monotone and is upper semicontinuous from $Co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$.

We shall use our following multivalued generalization of the classical *random pseudo-monotone operators*. The classical definition of a pseudo-monotone operator was introduced by Brezis, Nirenberg and Stampacchia in [4].

Let X be a set, 2^X the family of all nonempty subsets of X and $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. Let E be a topological vector space and E^* its continuous dual, $\langle w, x\rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$, and $\mathrm{Re}\langle w, x\rangle$ the real part of $\langle w, x\rangle$. If $X \subset E$, $S : X \to 2^X$ and $T : X \to E^*$, the quasi-variational inequality (QVI) is to find a point $y \in S(y)$ such that

$$\mathrm{Re}\langle T(y), y - x\rangle \ \leq\ 0, \quad \text{for all } \ x \in S(y),$$

which is introduced by Bensousson and Lions in 1973, see [2]. Again, we consider a multivalued mapping $T : X \to 2^{E^*}$, then the generalized quasi-variational inequality (GQVI) is to find a point $y \in S(y)$ and a point $w \in T(y)$ such that

$$\mathrm{Re}\langle w, y - x\rangle \ \leq\ 0, \quad \text{for all } \ x \in S(y),$$

which is introduced and studied by Chan and Pang [5] in 1982.

A measurable space (Ω, Σ) is a pair, where Ω is a set and Σ a σ-algebra of subsets of Ω. If X is a set, $A \subset X$ and $\mathcal{D}$ is nonempty family of subsets of X, we shall denote by $D \cap A$ the family $\{D \cap A : \ D \in \mathcal{D}\}$ and by $\sigma_X(\mathcal{D})$ the smallest σ-algebra on X generated by $\mathcal{D}$. If X is a topological space with topology τ_X, we shall use $\mathcal{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ-algebra on X if there is no ambiguity on the topology τ_X. Let X be a topological space and $F : (\Omega, \Sigma) \to 2^X$ be a correspondence, then F is said to be measurable (resp. weakly measurable) if $F^{-1}(B) = \{\omega \in \Omega : \ F(\omega) \cap B \neq \phi\} \in E$ for each closed (resp. open) subset B of X. The mapping F is said to have a measurable graph if

$$\operatorname{Graf} F \ = \ \{(\omega, y) \in \Omega \times X : \ y \in F(\omega)\} \in \Sigma \otimes B(X).$$

A function $F : \Omega \to X$ is a measurable selection of F if f is a measurable function such that $f(\omega) \in F(\omega)$, for all $\omega \in \Omega$.

Definition 1.1. Let (Ω, Σ) be a measurable space, E a topological vector space, X a nonempty subset of E and $T : \Omega \times X \to 2^{E^*}$. If $h : \Omega \times X \to \mathbb{R}$, then T is said to be

(i) *randomly h-pseudo-monotone* if for each fixed $\omega \in \Omega$, $y(\omega) \in X$ and every random net $\{y_\alpha(\omega)\}_{\alpha \in \Gamma}$ in X converging to $y(\omega)$ with

$$\limsup_{\alpha} [\inf_{u \in T(\omega, y_\alpha(\omega))} \operatorname{Re}\langle u, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))] \ \leq \ 0,$$

we have

$$\liminf_{\alpha} [\inf_{u \in T(\omega, y_\alpha(\omega))} \operatorname{Re}\langle u, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))]$$

$$\geq \ \inf_{w \in T(\omega, y(\omega))} \operatorname{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega)), \quad \text{for all } \ x(\omega) \in X,$$

(ii) *randomly pseudo-monotone if* T is randomly h-pseudo-monotone with $h \equiv 0$.

2. Generalized Random Quasi-variational Inequalities for Randomly Strong pseudo-monotone Operators

In this section, we shall introduce the notion of randomly pseudo-monotone operators and obtain some general theorem on solution of the GRQVI on paracompact sets in locally convex Hausdorff topological vector spaces.

We shall begin with the following:

Definition 2.1. Let (Ω, Σ) be a measurable space, E a topological vector space, X a nonempty subset of E, and $T : \Omega \times X \to 2^{E^*}$. If $h : \Omega \times X \to \mathbb{R}$, then T is said to be

(i) *randomly strong h-pseudo-monotone* if for each continuous function $\theta : \Omega \times X \to [0, 1]$, for fixed $\omega \in \Omega$, $y(\omega) \in X$ and every random net $\{y_\alpha(\omega)\}_{\alpha \in \Gamma}$ in X converging to $y(\omega)$ with

$$\limsup_{\alpha} [\theta(\omega, y_\alpha(\omega))\{\inf_{u \in T(\omega, y_\alpha(\omega))} \operatorname{Re}\langle u, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))\}] \ \leq \ 0,$$

we have

$$\limsup_{\alpha}[\theta(\omega, y_\alpha(\omega))\{\inf_{u\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle u, y_\alpha(\omega)-x(\omega)\rangle+h(\omega, y_\alpha(\omega))-h(\omega, x(\omega))\}]$$

$$\geq\ [\theta(\omega, y(\omega))\{\inf_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega)-x(\omega)\rangle+h(\omega, y(\omega))-h(\omega, x(\omega))\}],$$

for all $x(\omega)\in X$,

(ii) *randomly strong pseudo-monotone* if random operator T is randomly strong h-pseudo-monotone with $h\equiv 0$.

Remark 2.1. Every randomly strong pseudo-monotone operator is also a randomly pseudo-monotone operator.

Proposition 2.1. Let (Ω,Σ) be measurable space, X a nonempty subset of a topological vector space E. If $T:\Omega\times X\to E^*$ is randomly monotone and continuous from the relative weak topology on X to the weak*-topology on E^*, then random operator T is randomly strong pseudo-monotone.

Proof. Let $\theta:\Omega\times X\to[0,1]$ be any arbitrary continuous random functional. Suppose $\{y_\alpha(\omega)\}_{\alpha\in\Gamma}$ is a random net in X and for each $\omega\in\Omega$, $y(\omega)\in X$ with $y_\alpha(\omega)\to y(\omega)$ (and

$$\limsup_{\alpha}[\theta(\omega, y_\alpha(\omega))\{\mathrm{Re}\langle T(\omega, y_\alpha(\omega)), y_\alpha(\omega)-y(\omega)\rangle\}]\ \leq\ 0).$$

For any $x(\omega)\in X$, $\omega\in\Omega$ and $\epsilon>0$, there are $\beta_1,\beta_2\in\Gamma$ with

$$\mid \theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y_\alpha(\omega)-y(\omega)\rangle \mid<\frac{\epsilon}{2},\ \text{for all } \alpha\ \geq\ \beta_1$$

and

$$\mid \theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega))-T(\omega, y(\omega)), y(\omega)-x(\omega)\rangle \mid<\frac{\epsilon}{2},\ \text{for } \alpha\ \geq\ \beta_2.$$

Choose $\beta_0\in\Gamma$ with $\beta_0\geq\beta_1,\beta_2$. Thus

$$\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega)), y_\alpha(\omega)-x(\omega)\rangle$$

$$= \theta(\omega, y_\alpha(\omega))\langle T(\omega, y_\alpha(\omega)), y_\alpha(\omega)-y(\omega)\rangle+\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega)), y(\omega)-x(\omega)\rangle$$

$$\geq\ \theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y_\alpha(\omega)-y(\omega)\rangle+\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega)), y(\omega)-x(\omega)\rangle$$

$$=\ \theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y_\alpha(\omega)-y(\omega)\rangle+\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega)-T(\omega, y(\omega)), y(\omega)-x(\omega)\rangle$$

$$+\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y(\omega)-x(\omega)\rangle$$

$$>-\frac{\epsilon}{2}-\frac{\epsilon}{2}+\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y(\omega)-x(\omega)\rangle,\ \text{for all } \alpha\geq\beta_0,$$

so that

$$\inf_{\alpha\geq\beta_0}\theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y_\alpha(\omega)), y_\alpha(\omega)-x(\omega)\rangle$$

$$\geq -\epsilon + \inf_{\alpha \geq \beta_0} \theta(\omega, y_\alpha(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y(\omega) - x(\omega)\rangle.$$

It follows that

$$\begin{aligned}
&\limsup_{\beta} \theta(\omega, y_\beta(\omega))\mathrm{Re}\langle T(\omega, y_\beta(\omega)), y_\beta(\omega) - x(\omega)\rangle \\
\geq\ & \liminf_{\beta} \theta(\omega, y_\beta(\omega))\mathrm{Re}\langle T(\omega, y_\beta(\omega)), y_\beta(\omega) - x(\omega)\rangle \\
\geq\ & -\epsilon + \theta(\omega, y(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y(\omega) - x(\omega)\rangle.
\end{aligned}$$

As $\epsilon > 0$ is arbitrary,

$$\limsup_{\beta} \theta(\omega, y_\beta(\omega))\mathrm{Re}\langle T(\omega, y_\beta(\omega)), y_\beta(\omega) - x(\omega)\rangle \geq \theta(\omega, y(\omega))\mathrm{Re}\langle T(\omega, y(\omega)), y(\omega) - x(\omega)\rangle.$$

Hence random operator T is randomly pseudo-monotone.

Theorem 2.1. Let (Ω, Σ) be a measurable space, E a locally convex Hausdorff topological vector space, X a nonempty paracompact convex subset of E and $h : \Omega \times E \to \mathbb{R}$ be convex. Let $S : \Omega \times X \to 2^X$ be randomly upper semicontinuous such that for each fixed $\omega \in \Omega$, each $S(\omega, x(\omega))$ is compact convex and $T : \Omega \times X \to 2^{E^*}$ a randomly strong h-pseudo-monotone and randomly upper semicontinuous from $Co(A)$ to the weak*-topology on E^*, for each $A \in \mathcal{F}(X)$ and for each fixed $\omega \in \Omega$ such that $T(\omega, x(\omega))$ is weak*-compact convex. Suppose that the set

$$\begin{aligned}
\mathcal{A} = \{\omega \in \Omega, y(\omega) \in X : \sup_{x(\omega) \in S(\omega, y(\omega))} [\inf_{w \in T(\omega, y(\omega))} &\mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle \\
&+ h(\omega, y(\omega)) - h(\omega, x(\omega))] > 0\}
\end{aligned}$$

is open in X. Suppose further that there exists a nonempty compact subset K of X and a point $x_0(\omega) \in X$ for fixed $\omega \in \Omega$, such that $x_0(\omega) \in K \cap S(\omega, y(\omega))$ and

$$\inf_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega), x_0(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x_0(\omega)) > 0,$$

for all $y \in X \backslash K$ for $\omega \in \Omega$.

Then there exists a measurable map $\hat{y} : \Omega \to K$ such that

(i) $\hat{y} \in S(\omega, \hat{y}(\omega))$ and

(ii) there exists $\hat{\omega} \in T(\omega, \hat{y}(\omega))$ with

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega)\rangle \ \leq\ h(\omega, x(\omega)) - h(\omega, \hat{y}(\omega)), \text{ for all } \omega \in \Omega, \quad x(\omega) \in S(\omega, \hat{y}(\omega)).$$

Proof. We divide the proof into two steps:

Step 1. There exists a measurable map $\hat{y} : \Omega \to X$ such that $\hat{y}(\omega) \in S(\omega, \hat{y}(\omega))$ and

$$\sup_{x(\omega)\in S(\omega,\hat{y}(\omega))} [\inf_{w\in T(\omega,\hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))] \leq 0.$$

Suppose the contrary, for each fixed $\omega \in \Omega$ and $y(\omega) \in X$, either $y(\omega) \notin S(\omega, y(\omega))$ or there exists $x(\omega) \in S(\omega, y(\omega))$ such that

$$\inf_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega))\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega)) > 0,$$

i.e., $y(\omega) \notin S(\omega, y(\omega))$ or $y(\omega) \in \mathcal{A}$, for each fixed $\omega \in \Omega$. If for each $\omega \in \Omega$, $y(\omega) \notin S(\omega, y(\omega))$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that

$$\mathrm{Re}\langle p, y(\omega)\rangle - \sup_{x(\omega)\in S(\omega,y(\omega))} \langle p, x(\omega)\rangle > 0.$$

For each $\omega \in \Omega$, $y(\omega) \in X$, set

$$\gamma(y(\omega)) = \sup_{x(\omega)\in S(\omega,y(\omega))} [\inf_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))].$$

Let $V_0 = \{\omega \in \Omega, y(\omega) \in X \mid \gamma(y(\omega)) > 0\} = \mathcal{A}$ and for each $p \in E^*$, set

$$V_p = \{\omega \in \Omega, y(\omega) \in X : \mathrm{Re}\langle p, y(\omega)\rangle - \sup_{x(\omega)\in S(\omega,y(\omega))} \langle p, x(\omega)\rangle \geq 0\}.$$

Then $X = V_0 \cup \bigcup_{p\in E^*} V_p$. Since each V_p is open in X, by Lemma 1 in [21], and V_0 is open in X by hypothesis, then $\{V_0, V_p : p \in E^*\}$ is an open covering of X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see Theorem VIII. 4.2.[9]), i.e., for each $p \in E^*$, $\beta_p : X \to [0, 1]$ and $\beta_0 : X \to [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$, for all $y \in X\backslash V_p$ and $\beta_0(y) = 0$, for all $y \in X\backslash V_p$ and {support β_0, support $\beta_p : p \in E^*$} is locally finite and $\beta_0(y(\omega)) + \sum_{p\in E^*} \beta_p(y(\omega)) = 1$, for each $y(\omega) \in X$. Note that for each $A \in \mathcal{F}(X)$, h is randomly continuous on $Co(A)$, see [19, page 83]. Define $\phi : \Omega \times X \times X \to \mathbb{R}$ by

$$\phi(\omega, x(\omega), y(\omega)) = \beta_0(\omega, y(\omega))[\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle,$$

for each $x(\omega), y(\omega) \in X$, for fixed $\omega \in \Omega$.

Then we have the following.

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and for each fixed $x(\omega) \in Co(A)$, the random map

$$y(\omega) \to \min_{w \in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))$$

is randomly lower semicontinuous on $Co(A)$, by Lemma 3 in [6]; and the fact that h is randomly continuous on $Co(A)$, the random map

$$y(\omega) \to \beta_0(\omega, y(\omega))[\min_{w \in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))]$$

is randomly lower semicontinuous on $Co(A)$, by Lemma 3 in [22]. For $\omega \in \Omega$ and for each fixed $x(\omega) \in X$

$$y(\omega) \to \sum_{p \in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle$$

is randomly continuous on X. Hence for each $A \in \mathcal{F}(X)$ and for each $\omega \in \Omega$, $x(\omega) \in Co(A)$, the random map $y(\omega) \to \phi(\omega, x(\omega), y(\omega))$ is randomly lower semicontinuous on $Co(A)$.

(2) For fixed $\omega \in \Omega$ and for each $A \in \mathcal{F}(X)$, $y(\omega) \in Co(A)$,

$$\min_{x(\omega) \in A} \phi(\omega, x(\omega), y(\omega)) \ \leq \ 0.$$

Indeed, if this were false, then for some $A = \{x_1(\omega), \cdots, x_n(\omega)\} \in \mathcal{F}(X)$, for fixed $\omega \in \Omega$ and some $y(\omega) \in Co(A)$, (say $y(\omega) = \sum_{i=1}^{n} \lambda_i x_i(\omega)$, where $\lambda_1, \cdots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$), we have

$$\min_{1 \leq i \leq n} \phi(\omega, x_i(\omega), y(\omega)) > 0.$$

Then for each $i = 1, 2, \cdots, n$,

$$\beta_0(\omega, y(\omega))[\min_{w \in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x_i(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x_i(\omega))]$$

$$+ \sum_{p \in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x_i(\omega)\rangle > 0,$$

so that

$$0 = \phi(\omega, y(\omega), y(\omega)) = \beta_0(\omega, y(\omega))[\min_{w \in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - \sum_{i=1}^{n} \lambda_i x_i(\omega)\rangle$$

$$+h(\omega, y(\omega)) - h(\omega, \sum_{i=1}^{n} \lambda_i x_i(\omega))] + \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - \sum_{i=1}^{n} \lambda_i x_i(\omega)\rangle$$

$$\geq \sum_{i=1}^{n} \lambda_i(\beta_0(\omega, y(\omega))[\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x_i(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x_i(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x_i(\omega)\rangle) > 0,$$

which is a contradiction.

(3) For fixed $\omega \in \Omega$, suppose $A \in \mathcal{F}(X)$, $x(\omega), y(\omega) \in Co(A)$ and $\{y_\alpha(\omega)\}_{\alpha\in\Gamma}$ is a random net in X converging to $y(\omega)$ with $\phi(\omega, tx(\omega) + (1-t)y(\omega), y_\alpha(\omega)) \leq 0$, for all $\alpha \in \Gamma$ and $t \in [0,1]$. Then for $t = 0$, we have

$$\phi(\omega, y(\omega), y_\alpha(\omega)) \ \leq \ 0, \quad \text{for all} \quad \alpha \in \Gamma, \quad \text{i.e.,}$$

$$\beta_0(\omega, y_\alpha(\omega))[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle \leq 0, \quad \text{for all} \quad \alpha \in \Gamma.$$

Hence

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega)))]$$

$$+ \liminf_{\alpha}(\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle)$$

$$\leq \ \limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle)] \ \leq \ 0.$$

Therefore

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega)))] \ \leq \ 0.$$

Since random operator T is randomly strong h-pseudo-monotone, we have

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega)))]$$

$$\geq \ \beta_0(\omega, y(\omega))(\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))).$$

Thus

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega)))]$$

$$+\sum_{p\in E^*}\beta_p(\omega,y(\omega))\mathrm{Re}\langle p,y(\omega)-x(\omega)\rangle$$

$$\geq \beta_0(\omega,y(\omega))(\min_{w\in T(\omega,y(\omega))}\mathrm{Re}\langle w,y(\omega)-x(\omega)\rangle+h(\omega,y(\omega))-h(\omega,x(\omega)))$$

$$+\sum_{p\in E^*}\beta_p(\omega,y(\omega))\mathrm{Re}\langle p,y(\omega)-x(\omega)\rangle. \tag{2.1}$$

For $t=1$, we have

$$\phi(\omega,x(\omega),y_\alpha(\omega)) \leq 0, \quad \text{for all} \quad \alpha\in\Gamma, \quad \text{i.e.,}$$

$$\beta_0(\omega,y_\alpha(\omega))[\min_{w\in T(\omega,y_\alpha(\omega))}\mathrm{Re}\langle w,y_\alpha(\omega)-x(\omega)\rangle+h(\omega,y_\alpha(\omega))-h(\omega,x(\omega))$$

$$+\sum_{p\in E^*}\beta_p(\omega,y_\alpha(\omega))\mathrm{Re}\langle p,y_\alpha(\omega)-x(\omega)\rangle\leq 0, \quad \text{for all } \alpha\in\Gamma.$$

Therefore

$$\limsup_\alpha[\beta_0(\omega,y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))}\mathrm{Re}\langle w,y_\alpha(\omega)-x(\omega)\rangle+h(\omega,y_\alpha(\omega))-h(\omega,x(\omega)))]$$

$$+\liminf_\alpha[\sum_{p\in E^*}\beta_p(\omega,y_\alpha(\omega)\mathrm{Re}\langle p,y_\alpha(\omega)-x(\omega)\rangle]$$

$$\leq \limsup_\alpha[\beta_0(\omega,y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))}\mathrm{Re}\langle w,y_\alpha(\omega)-x(\omega)\rangle+h(\omega,y_\alpha(\omega))-h(\omega,x(\omega))$$

$$+\sum_{p\in E^*}\beta_p(\omega,y_\alpha(\omega))\mathrm{Re}\langle p,y_\alpha(\omega)-x(\omega)\rangle)]\leq 0.$$

Thus

$$\limsup_\alpha[\beta_0(\omega,y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))}\mathrm{Re}\langle w,y_\alpha(\omega)-x(\omega)\rangle+h(\omega,y_\alpha(\omega))-h(\omega,x(\omega)))]$$

$$+\sum_{p\in E^*}\beta_p(\omega,y(\omega))\mathrm{Re}\langle p,y(\omega)-x(\omega)\rangle\leq 0. \tag{2.2}$$

Hence by (2.1) and (2.2), we have

$$\phi(\omega,x(\omega),y(\omega)) \leq 0.$$

(4) By hypothesis, there exists a nonempty compact subset K of X and a point $x_0(\omega)\in X$ such that $x_0(\omega)\in K\cap S(\omega,y(\omega))$ and

$$\inf_{w\in T(\omega,y(\omega))}\mathrm{Re}\langle w,y(\omega)-x_0(\omega)\rangle+h(\omega,y(\omega))-h(\omega,x_0(\omega))>0, \text{ for each fixed } \omega\in\Omega,\ y\in X\backslash K.$$

Thus for each $y\in X\backslash K$,

$$\beta_0(\omega,y(\omega))[\inf_{w\in T(\omega,y(\omega))}\mathrm{Re}\langle w,y(\omega)-x_0(\omega)\rangle+h(\omega,y(\omega))-h(\omega,x_0(\omega))]>0,$$

whenever $\beta_0(\omega, y(\omega)) > 0$ and $\mathrm{Re}\langle p, y(\omega) - x_0(\omega)\rangle > 0$, for $p \in E^*$. Consequently,

$$\phi(\omega, x_0(\omega), y(\omega)) = \beta_0(\omega, y(\omega))[\inf_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x_0(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x_0(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x_0(\omega)\rangle > 0,$$

for all $y \in X\backslash K$, for fixed $\omega \in \Omega$.

Therefore ϕ satisfies all hypothesis of Theorem 2 in [6]. Hence by Theorem 2 in [6], there exists a measurable map $\hat{y} : \Omega \to K$ such that

$$\phi(\omega, x(\omega), \hat{y}(\omega)) \;\leq\; 0, \quad \text{for all} \;\; x(\omega) \in X \;\; \text{and for each} \;\; \omega \in \Omega, \quad \text{i.e.,}$$

$$\beta_0(\omega, \hat{y}(\omega))[\inf_{w\in T(\omega,\hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, \hat{y}(\omega))\mathrm{Re}\langle p, \hat{y}(\omega) - x(\omega)\rangle \;\leq\; 0, \tag{2.3}$$

for each fixed $\omega \in \Omega$.

If $\gamma(\hat{y}(\omega)) \geq 0$, choose any $\hat{x}(\omega) \in S(\omega, \hat{y}(\omega))$, such that

$$\inf_{w\in T(\omega,\hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - \hat{x}(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, \hat{x}(\omega)) \;\geq\; \frac{\gamma(\hat{y}(\omega))}{2} > 0.$$

If $\beta_0(\omega, \hat{y}(\omega)) > 0$, then $\hat{y}(\omega) \in V_0 \in \mathcal{A}$, so that $\gamma(\hat{y}(\omega)) > 0$. It follows that

$$\beta_0(\omega, \hat{y}(\omega))[\inf_{w\in T(\omega,\hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - \hat{x}(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, \hat{x}(\omega))] > 0.$$

If $\beta_p(\omega, \hat{y}(\omega)) > 0$, for some $p \in E^*$, then $\hat{y}(\omega) \in V_p$ and hence

$$\mathrm{Re}\langle p, \hat{y}(\omega)\rangle > \sup_{x(\omega)\in S(\omega,\hat{y}(\omega))} \mathrm{Re}\langle p, x(\omega)\rangle \;\geq\; \mathrm{Re}\langle p, \hat{x}(\omega)\rangle,$$

so that

$$\mathrm{Re}\langle p, \hat{y}(\omega) - \hat{x}(\omega)\rangle > 0.$$

Therefore, $\beta_p(\omega, \hat{y}(\omega))\mathrm{Re}\langle p, \hat{y}(\omega) - \hat{x}(\omega)\rangle > 0$, whenever $\beta_p(\omega, \hat{y}(\omega)) > 0$, for $p \in E^*$.

Since $\beta_0(\omega, \hat{y}(\omega)) > 0$ or $\beta_p(\omega, \hat{y}(\omega)) > 0$ for some $p \in E^*$, it follows that

$$\phi(\omega, \hat{x}(\omega), \hat{y}(\omega)) = \beta_0(\omega, \hat{y}(\omega))[\inf_{w\in T(\omega,\hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - \hat{x}(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, \hat{x}(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, \hat{y}(\omega))\mathrm{Re}\langle p, \hat{y}(\omega) - \hat{x}(\omega)\rangle > 0,$$

which contradicts (2.3). This contradiction proves step 1.

Step 2. There exists a measurable map $\hat{y} : \Omega \to X$ such that $\hat{w} \in T(\omega, \hat{y}(\omega))$ and

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega)) \leq 0,$$

for all $x(\omega) \in S(\omega, \hat{y}(\omega))$ and for fixed $\omega \in \Omega$.

Note that for each fixed $x(\omega) \in S(\omega, \hat{y}(\omega))$,

$$w \to \mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))$$

is convex and randomly continuous on $T(\omega, \hat{y}(\omega))$ and for each fixed $w \in T(\omega, \hat{y}(\omega))$

$$x(\omega) \to \mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))$$

is concave on $S(\omega, \hat{y}(\omega))$. Then by Kneser's Minimax Theorem in [14], we have

$$\begin{aligned}&\min_{w \in T(\omega,\hat{y}(\omega))} \max_{x(\omega)\in S(\omega,\hat{y}(\omega))} [\mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))]\\ &\quad = \max_{x(\omega)\in S(\omega,\hat{y}(\omega))} \min_{w \in T(\omega,\hat{y}(\omega))} [\mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))].\end{aligned}$$

Hence

$$\min_{w \in T(\omega,\hat{y}(\omega))} \max_{x(\omega)\in S(\omega,\hat{y}(\omega))} [\mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))] \ \leq \ 0,$$

by step 1. Since $T(\omega, \hat{y}(\omega))$ is compact, there exists a measurable map $\hat{y} : \Omega \to X$ with $\hat{w} \in T(\omega, \hat{y}(\omega))$ such that

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega)) \ \leq \ 0, \quad \text{for all} \ \ x(\omega) \in S(\omega, \hat{y}(\omega))$$

and for each fixed $\omega \in \Omega$. This completes the proof.

3. Generalized Random Quasi-Variational Inequalities for Randomly Pseudo-monotone Operators

In this section, we shall obtain some existence theorems of generalized random quasi-variational inequalities for randomly pseudo-monotone operators on para compact convex sets.

Theorem 3.1. Let (Ω, Σ) a measurable space, E a locally convex Hausdorff topological vector space, X a nonempty paracompact convex and bounded subset of E and $h : \Omega \times E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : \Omega \times X \to 2^X$ be randomly upper semicontinuous such that each $S(\omega, x(\omega))$, for each fixed $\omega \in \Omega$, is compact convex and $T : \Omega \times X \to 2^{E^*}$ the randomly h-pseudo-monotone and randomly upper semicontinuous from $Co(A)$ to the weak*-topology on E^*, for each $A \in \mathcal{F}(X)$ such that for fixed $\omega \in \Omega$, each $T(\omega, x(\omega))$ a weak*-compact convex on $T(X)$ is randomly bounded. Suppose that the set

$$\mathcal{A} = \{\omega \in \Omega, y(\omega) \in X : \sup_{x(\omega)\in S(\omega,y(\omega))} [\inf_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle$$

$$+h(\omega, y(\omega)) - h(\omega, x(\omega))] > 0\}$$

is open in X. Suppose further that there exists a nonempty compact subset K of X and a random point $x_0(\omega) \in X$ such that $x_0(\omega) \in K \cap S(\omega, y(\omega))$ and

$$\inf_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega) - x_0(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x_0(\omega)) > 0, \quad \text{for all } \; y \in X \backslash K,$$

for each fixed $\omega \in \Omega$. Then there exists a measurable map $\hat{y} : \Omega \to K$ such that

(i) $\hat{y}(\omega) \in S(\omega, \hat{y}(\omega))$ and

(ii) there exists $\hat{w} \in T(\omega, \hat{y}(\omega))$ with

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega)\rangle \;\leq\; h(\omega, x(\omega)) - h(\omega, \hat{y}(\omega)), \quad \text{for all } \; x \in S(\omega, \hat{y}(\omega))$$

and for each fixed $\omega \in \Omega$.

Proof. We divide the proof into two steps:

Step 1. There exists a measurable map $\hat{y} : \Omega \to X$ such that $\hat{y}(\omega) \in S(\omega, \hat{y}(\omega))$ and for each fixed $\omega \in \Omega$,

$$\sup_{x(\omega) \in S(\omega, \hat{y}(\omega))} [\inf_{w \in T(\omega, \hat{y}(\omega))} \mathrm{Re}\langle w, \hat{y}(\omega) - x(\omega)\rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega))] \;\leq\; 0.$$

Suppose the contrary, for each fixed $\omega \in \Omega$ and $y(\omega) \in X$, either $y(\omega) \notin S(\omega, y(\omega))$ or there exists $x(\omega) \in S(\omega, y(\omega))$ such that

$$\inf_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega)) > 0,$$

i.e., $y(\omega) \notin S(\omega, y(\omega))$ or $y(\omega) \in \mathcal{A}$. If $y(\omega) \notin S(\omega, y(\omega))$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that

$$\mathrm{Re}\langle p, y(\omega)\rangle - \sup_{x(\omega) \in S(\omega, y(\omega))} \mathrm{Re}\langle p, x(\omega)\rangle > 0.$$

For each fixed $\omega \in \Omega$, $y(\omega) \in X$, set

$$\gamma(y(\omega)) \;=\; \sup_{x(\omega) \in S(\omega, y(\omega))} [\inf_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))].$$

Let $V_0 = \{\omega \in \Omega, \; y(\omega) \in X \mid \gamma(y(\omega)) > 0\} = \mathcal{A}$ and for each $p \in E^*$, set

$$V_p \;=\; \{\omega \in \Omega, \; y(\omega) \in X : \; \mathrm{Re}\langle p, y(\omega)\rangle - \sup_{x(\omega) \in S(\omega, y(\omega))} \langle p, x(\omega)\rangle \geq 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 1 in [21] and V_0 is open in X by hypothesis, then $\{V_0, V_p : p \in E^*\}$ is an open covering of X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. For each $A \in \mathcal{F}(X)$, h is a randomly continuous on $Co(A)$, see [19, p.83].

Define $\phi : \Omega \times X \times X \to \mathbb{R}$ by

$$\phi(\omega, x(\omega), y(\omega)) = \beta_0(\omega, y(\omega))[\min_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p \in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle,$$

for each $x(\omega), y(\omega) \in X$ and for fixed $\omega \in \Omega$.

Then we have

(1) The same argument in proving (1) in the proof of Theorem 2.1, shows that for each $A \in \mathcal{F}(X)$ and for each fixed $x(\omega) \in Co(A)$, the random mapping $y(\omega) \to \phi(\omega, x(\omega), y(\omega))$, for fixed $\omega \in \Omega$, is lower semicontinuous on $Co(A)$.

(2) The same argument in proving (2) in the proof of Theorem 2.1, shows that for fixed $\omega \in \Omega$ and for each $A \in \mathcal{F}(X)$, $y(\omega) \in Co(A)$,

$$\min_{x(\omega) \in A} \phi(\omega, x(\omega), y(\omega)) \ \leq \ 0.$$

(3) Suppose $A \in \mathcal{F}(X)$, for fixed $\omega \in \Omega$, $x(\omega), y(\omega) \in Co(A)$, $\{y_\alpha(\omega)\}_{\alpha \in \Gamma}$ is a random net in X converging to $y(\omega)$ with

$$\phi(\omega, tx(\omega) + (1-t)y(\omega), y_\alpha(\omega)) \leq 0 \ \text{ for all } \ \alpha \in \Gamma \ \text{ and } \ t \in [0,1].$$

Case 1. $\beta_0(\omega, y(\omega)) = 0$. Note that $\beta_0(\omega, y_\alpha(\omega)) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(\omega, y_\alpha(\omega)) \to 0$. Since $T(X)$ is randomly strong bounded and $\{y_\alpha(\omega)\}_{\alpha \in \Gamma}$ a randomly bounded net, it follows that

$$\limsup_\alpha [\beta_0(\omega, y_\alpha(\omega))\{\min_{w \in T(\omega, y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}] = 0. \tag{3.1}$$

Also

$$\beta_0(\omega, y(\omega))[\min_{w \in T(\omega, y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))] = 0.$$

Thus

$$\limsup_\alpha [\beta_0(\omega, y_\alpha(\omega))\{\min_{w \in T(\omega, y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p \in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle$$

$$= \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle \quad \text{(by (3.1))}$$

$$= \beta_0(\omega, y(\omega))[\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle. \tag{3.2}$$

For $t = 1$, we have $\phi(\omega, x(\omega), y_\alpha(\omega)) \leq 0$, for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(\omega, y_\alpha(\omega))[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - x(\omega)\rangle \ \leq \ 0, \quad \text{for all} \quad \alpha \in \Gamma. \tag{3.3}$$

Therefore

$$\limsup_\alpha [\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}]$$

$$+ \liminf_\alpha [\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - x(\omega)\rangle]$$

$$\leq \ \limsup_\alpha [\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - x(\omega)\rangle] \ \leq \ 0 \quad \text{by (3.3).}$$

Thus

$$\limsup_\alpha [\beta_0(\omega, y_\alpha(\omega))(\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) + h(\omega, x(\omega)))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega) - x(\omega)\rangle \ \leq \ 0. \tag{3.4}$$

Hence by (3.2) and (3.4), we have

$$\phi(\omega, x(\omega), y(\omega)) \ \leq \ 0.$$

Case 2. $\beta_0(\omega, y(\omega)) > 0$. Since $\beta_0(\omega, y_\alpha(\omega)) \to \beta_0(\omega, y(\omega))$, there exists $\lambda \in \Gamma$ such that $\beta_0(\omega, y_\alpha(\omega)) > 0$ for all $\alpha \geq \lambda$ and for each fixed $\omega \in \Omega$. Then for $t = 0$, we have

$$\phi(\omega, y(\omega), y_\alpha(\omega)) \ \leq \ 0, \quad \text{for all} \quad \alpha \in \Gamma, \quad \text{i.e.,}$$

$$\beta_0(\omega, y_\alpha(\omega))[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))]$$

$$+ \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle \ \leq \ 0$$

for all $\alpha \in \Gamma$ and for each fixed $\omega \in \Omega$. Thus

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega)) + \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle] \leq 0. \tag{3.5}$$

Hence

$$\begin{aligned}
&\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))\}] \\
&\quad + \liminf_{\alpha}[\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle] \\
\leq\ & \limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega)) \\
&\quad + \sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle] \leq 0, \quad \text{by} \quad (3.5).
\end{aligned}$$

Since

$$\liminf_{\alpha}[\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega) - y(\omega)\rangle] = 0,$$

we have

$$\limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - y(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, y(\omega))\}] \leq 0. \tag{3.6}$$

Since $\beta_0(\omega, y_\alpha(\omega)) > 0$, for all $\alpha \geq \lambda$. It follows that

$$\begin{aligned}
&\beta_0(\omega, y(\omega)) \limsup_{\alpha}[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-y(\omega)\rangle+h(\omega, y_\alpha(\omega))-h(\omega, y(\omega))] \\
&= \limsup_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-y(\omega)\rangle+h(\omega, y_\alpha(\omega))-h(\omega, y(\omega))\}].
\end{aligned} \tag{3.7}$$

Since $\beta_0(\omega, y(\omega)) > 0$, by (3.6) and (3.7), we have

$$\limsup_{\alpha}[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-y(\omega)\rangle+h(\omega, y_\alpha(\omega))-h(\omega, y(\omega))] \leq 0.$$

Since T is randomly h-pseudomonotone, we have

$$\begin{aligned}
&\liminf_{\alpha}[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega) - x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))] \\
&\geq \min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega) - x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega)).
\end{aligned}$$

Since $\beta_0(\omega, y(\omega)) > 0$, we have

$$\beta_0(\omega, y(\omega))[\liminf_{\alpha}\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle+h(\omega, y_\alpha(\omega))-h(\omega, x(\omega))]$$

$$\geq \beta_0(\omega, y(\omega))[\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega)-x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))].$$

Thus,

$$\beta_0(\omega, y(\omega))[\liminf_{\alpha}\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}]$$

$$+\sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega)-x(\omega)\rangle$$

$$\geq \beta_0(\omega, y(\omega))[\min_{w\in T(\omega,y(\omega))} \mathrm{Re}\langle w, y(\omega)-x(\omega)\rangle + h(\omega, y(\omega)) - h(\omega, x(\omega))]$$

$$+\sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega)-x(\omega)\rangle. \tag{3.8}$$

For $t=0$, we also have

$$\phi(\omega, x(\omega), y_\alpha(\omega)) \leq 0, \quad \text{for all} \quad \alpha\in\Gamma, \quad \text{and} \quad \omega\in\Omega, \quad \text{i.e.,}$$

$$\beta_0(\omega, y_\alpha(\omega))[\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))]$$

$$+\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega)-x(\omega)\rangle \leq 0,$$

for all $\alpha\in\Gamma$, and for each fixed $\omega\in\Omega$. Therefore

$$0 \geq \liminf_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}$$

$$+\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega)-x(\omega)\rangle]$$

$$\geq \liminf_{\alpha}[\beta_0(\omega, y_\alpha(\omega))\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}]$$

$$+\liminf_{\alpha}[\sum_{p\in E^*} \beta_p(\omega, y_\alpha(\omega))\mathrm{Re}\langle p, y_\alpha(\omega)-x(\omega)\rangle]$$

$$= \beta_0(\omega, y(\omega))[\liminf_{\alpha}\{\min_{w\in T(\omega,y_\alpha(\omega))} \mathrm{Re}\langle w, y_\alpha(\omega)-x(\omega)\rangle + h(\omega, y_\alpha(\omega)) - h(\omega, x(\omega))\}]$$

$$+\sum_{p\in E^*} \beta_p(\omega, y(\omega))\mathrm{Re}\langle p, y(\omega)-x(\omega)\rangle. \tag{3.9}$$

Consequently, by (3.8) and (3.9), we have

$$\phi(\omega, x(\omega), y(\omega)) \leq 0.$$

Now, the remaining part of the proof of step 1 is similar to the proof in step 1 of Theorem 2.1.

Step 2. There exists $\hat{w} \in T(\omega, \hat{y}(\omega))$ such that for each $\omega \in \Omega$,

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega) \rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega)) \ \leq \ 0, \quad \text{for all } x(\omega) \in S(\omega, \hat{y}(\omega)).$$

Also the same proof in step 2 of Theorem 2.1, shows that there exists $\hat{w} \in T(\omega, \hat{y}(\omega))$ such that

$$\mathrm{Re}\langle \hat{w}, \hat{y}(\omega) - x(\omega) \rangle + h(\omega, \hat{y}(\omega)) - h(\omega, x(\omega)) \ \leq \ 0,$$

for all $x(\omega) \in S(\omega, \hat{y}(\omega))$, and for each fixed $\omega \in \Omega$. Hence result is completed.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,51-59,COPYRIGHT 2008 EUDOXUS PRESS, LLC

On Uniform Continuity and Lebesgue Property in Intuitionistic Fuzzy Metric Spaces

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Abstract

In this paper, we introduce the concepts of uniform continuity, $\mathbb{R}$-uniform continuity, equinormality and Lebesgue property in intuitionistic fuzzy metric spaces. We show that every continuous function on a compact intuitionistic fuzzy metric space is uniformly continuous. Thereafter, we prove every real valued continuous function is uniformly continuous in intuitionistic fuzzy metric spaces.

Key Words. Intuitionistic fuzzy metric space, uniformly continuous function, equinormal intuitionistic fuzzy metric, Lebesgue intuitionistic fuzzy metric.

M.S.C. (2000). 54A40, 54E35, 54E40

1. INTRODUCTION

Since the introduction of the concept of fuzzy set by Zadeh [19] in 1965, many authors have introduced the concept of fuzzy metric space in different ways [3, 4, 6, 9, 10, 13, 14]. George and Veeramani [6, 8] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [14] and defined a Hausdorff topology on this fuzzy metric space. They also showed that every metric induces a fuzzy metric. Gregori et al. [11] gave with the help of appropriate fuzzy notions of equinormality and Lebesgue property, several characterizations of those fuzzy metric spaces, in the sense of George and Veeramani [6, 8], for which every real valued continuous function is uniformly continuous was obtained.

Park [16] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani. Alaca et al. [1] defined the completions of intuitionistic fuzzy metric spaces. A complete intuitionistic fuzzy metric space Y is said to be an intuitionistic fuzzy completion of a given intuitionistic fuzzy metric space X if X is isometric to a dense subspace of Y. They gave an example of an intuitionistic fuzzy metric space that does not admit any intuitionistic fuzzy metric completion. However, they proved that every standard intuitionistic fuzzy metric space has an (up to isometry) unique intuitionistic fuzzy metric completion. They also showed that for each intuitionistic fuzzy metric space there is an (up to uniform isomorphism) unique complete intuitionistic fuzzy metric space that contains a dense subspace uniformly isomorphic to it. Many authors studied the concept of intuitionistic fuzzy metric space and its applications [12, 17].

The purpose of this paper is to introduce the concepts of uniform continuity, $\mathbb{R}$-uniform continuity, equinormality and Lebesgue property in intuitionistic fuzzy metric spaces. We show that every continuous function on a compact intuitionistic fuzzy metric space is uniformly continuous. Thereafter, we prove every real valued continuous function is uniformly continuous in intuitionistic fuzzy metric spaces. Also we give some relationships between equinormality and Lebesgue property in intuitionistic fuzzy metric spaces.

2. ON INTUITIONISTIC FUZZY METRIC SPACES

Definition 1 ([18]). *A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions: (i) $*$ is commutative and associative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for all $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.*

Definition 2 ([18]). *A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions: (i) $\Diamond$ is commutative and associative; (ii) $\Diamond$ is continuous; (iii) $a \Diamond 0 = a$ for all $a \in [0,1]$; (iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.*

Definition 3 ([16]). *A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions: for all x, y, $z \in X$, s, $t > 0$,*

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous;
(IFM-7) $N(x,y,t) \geq 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to [0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated [15], i.e., $x \Diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0, 1]$.*

Remark 2. *In intuitionistic fuzzy metric space X, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

Example 1. *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \; N_d(x, y, t) = \frac{md(x, y)}{ht^n + md(x, y)}$$

for all $h, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 3. *Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = m = n = 1$, we get*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \; N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 2. *Let $X = \mathbb{N} \backslash \{0\}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y, \\ \frac{y}{x} & \text{if } y \leq x, \end{cases}, \; N(x, y, t) = \begin{cases} \frac{y-x}{y} & \text{if } x \leq y, \\ \frac{x-y}{x} & \text{if } y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Remark 4. *Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric d on X satisfying*

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \; N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

where $M(x, y, t)$ and $N(x, y, t)$ are as defined in above example. Also note the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$.

Definition 4 ([6]). *Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$ is called the open ball with center x and radius r with respect to t.*

Definition 5 ([16]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set $B_{(M,N)}(x,r,t) = \{y \in X : M(x,y,t) > 1 - r,\ N(x,y,t) < r\}$ is called the open ball with center x and radius r with respect to t.*

Theorem 1 ([16]). *Every open ball $B_{(M,N)}(x,r,t)$ is an open set.*

Remark 5. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Define $\tau_{(M,N)} = \{A \subset X$:for each $x \in A$, there exist $t > 0$, $r \in (0,1)$ such that $B_{(M,N)}(x,r,t) \subset A\}$. Then $\tau_{(M,N)}$ is a topology on X.*

Lemma 1 ([17]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then $(X, \tau_{(M,N)})$ is a metrizable topological space.*

It was proved Lemma 1 for each $n \in \mathbb{N}$ and

$$U_n = \left\{(x,y) \in X \times X : M(x,y,\frac{1}{n}) > 1 - \frac{1}{n},\ N(x,y,\frac{1}{n}) < \frac{1}{n}\right\},$$

$\{U_n : n \in \mathbb{N}\}$ is a base for uniformity $\breve{U}_{(M,N)}$ on X whose induced topology coincides with $\tau_{(M,N)}$.

Let us recall that a uniformity $\breve{U}$ on a set X has the Lebesgue property provided that for each open cover G of X there is $U \in \breve{U}$ such that $\{U(x) : x \in X\}$ refines G, and $\breve{U}$ is said to be equinormal if for each pair of disjoint nonempty closed subsets A and B of X there is $U \in \breve{U}$ such that $U(A) \cap B = \varnothing$. A metric d on X has the Lebesgue property provided that the uniformity $\breve{U}_d$, induced by d, has the Lebesgue property and d is equinormal provided that $\breve{U}_d$ so is (see, for instance, [5]).

In this paper $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and positive integer numbers, respectively.

3. MAIN RESULTS

Gregori et al. [11] gave with the help of appropriate fuzzy notions of equinormality and Lebesgue property, several characterizations of those fuzzy metric spaces, in the sense of George and Veeramani [6, 8], for which every real valued continuous function is uniformly continuous were obtained. Now, we give fundamental definitions in intuitionistic fuzzy metric spaces as follows:

Definition 6. *A mapping f from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, M', N', *', \Diamond')$ is called uniformly continuous if for each $\varepsilon \in (0,1)$ and each $t > 0$, there exist $r \in (0,1)$ and $s > 0$ such that $M'(f(x), f(y), t) > 1 - \varepsilon$ and $N'(f(x), f(y), t) < \varepsilon$ whenever $M(x,y,s) > 1 - r$ and $N(x,y,s) < r$.*

In this paper by a compact intuitionistic fuzzy metric space we mean an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ such that $(X, \tau_{(M,N)})$ is a compact topological space.

Remark 6. *Every uniformly continuous mapping from the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to the intuitionistic fuzzy metric space $(Y, M', N', *', \Diamond')$ is continuous from $(X, \tau_{(M,N)})$ to $(Y, \tau_{(M',N')})$.*

Now, we define $\mathbb{R}$-uniform continuity in intuitionistic fuzzy metric spaces.

Definition 7. *A real valued function f on the intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is $\mathbb{R}$-uniformly continuous provided that for each $\varepsilon > 0$ there exist $r \in (0,1)$ and $s > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $M(x, y, s) > 1 - r$ and $N(x, y, s) < r$.*

Theorem 2. *Let f be a continuous mapping from the compact intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to the intuitionistic fuzzy metric space $(Y, M', N', *', \Diamond')$. Then f is uniformly continuous.*

Proof. We put $\varepsilon \in (0,1)$ and $t > 0$, then there exists $\delta > 0$ such that $(1-\delta) *' (1-\delta) > (1-\varepsilon)$ and $\delta \Diamond' \delta < \varepsilon$, by the continuity of $*'$and $\Diamond'$. So, for each $x \in X$ there exist $r_x, r'_x \in (0,1)$ and $s_x > 0$ such that

$$f\left(B_{(M,N)}(x, r'_x, s_x)\right) \subseteq B_{(M',N')}(f(x), \delta, t/2)$$

and $(1 - r_x) * (1 - r_x) > (1 - r'_x)$ and $r_x \Diamond r_x < r'_x$. Now, there exists a finite subset A of X such that $X = \cup_{x \in A} B_{(M,N)}(x, r_x, s_x/2)$. Put $r = \min\{r_x : x \in A\}$ and $s = \max\{s_x/2 : x \in A\}$. It is routine to show that $M'(f(x), f(y), t) > 1-\varepsilon$ and $N'(f(x), f(y), t) < \varepsilon$ whenever $M(x, y, s) > 1 - r$ and $N(x, y, s) < r$. So f is uniformly continuous. This completes the proof. $\square$

Now, we define equinormality in intuitionistic fuzzy metric spaces.

Definition 8. *An intuitionistic fuzzy metric (M, N) on a set X is called equinormal if for each pair of disjoint nonempty closed subsets A and B of $(X, \tau_{(M,N)})$ there is $s > 0$ such that $\sup\{M(a, b, s) : a \in A, b \in B\} < 1$ and $\inf\{N(a, b, s) : a \in A, b \in B\} > 0$.*

Now, we define Lebesgue property in intuitionistic fuzzy metric spaces.

Definition 9. *An intuitionistic fuzzy metric (M, N) on a set X has the Lebesgue property if for each open cover G of $(X, \tau_{(M,N)})$ there exist $r \in (0,1)$ and $s > 0$ such that $\{B_{(M,N)}(x, r, s) : x \in X\}$ refines G.*

Remark 7. *Notice that if (X, d) is a metric space, then the intuitionistic fuzzy metric (M_d, N_d) has the Lebesgue property (resp. is equinormal) if and only if d has the Lebesgue property (resp. is equinormal).*

Theorem 3. *Let $(X, M, N, *, \Diamond)$ and $(Y, M', N', *', \Diamond')$ are intuitionistic fuzzy metric spaces. Then, following are equivalent:*

(i) For each intuitionistic fuzzy metric space $(Y, M', N', *', \Diamond')$any continuous mapping from $(X, \tau_{(M,N)})$ to $(Y, \tau_{(M',N')})$ is uniformly continuous as a mapping from $(X, M, N, *, \Diamond)$ to $(Y, M', N', *', \Diamond')$.
(ii) Every real valued continuous function on $(X, \tau_{(M,N)})$ is $\mathbb{R}$-uniformly continuous on $(X, M, N, *, \Diamond)$.
(iii) Every real valued continuous function on $(X, \tau_{(M,N)})$ is uniformly continuous on $(X, \breve{U}_{(M,N)})$.
(iv) (M, N) is an equinormal intuitionistic fuzzy metric on X.
(v) $\breve{U}_{(M,N)}$ is an equinormal uniformity on X.
(vi) The uniformity $\breve{U}_{(M,N)}$ has the Lebesgue property.
(vii) The intuitionistic fuzzy metric (M, N) has the Lebesgue property.

Proof. (i) $\Rightarrow$ (ii). Let f be a real valued continuous function on $(X, \tau_{(M,N)})$ and $\varepsilon > 0$. We may assume without loss of generality that $\varepsilon \in (0, 1)$. Choose $n \in \mathbb{N}$ such that $1 - \varepsilon > \frac{1}{n}$. By assumption f is uniformly continuous as a mapping from $(X, M, N, *, \Diamond)$ to $(\mathbb{R}, M_d, N_d, *', \Diamond')$, where (M_d, N_d) is the Euclidean intuitionistic fuzzy metric on $\mathbb{R}$. Hence, there exist $r \in (0, 1)$ and $s > 0$ such that

$$\frac{\frac{1}{n}}{\frac{1}{n} + |f(x) - f(y)|} > 1 - \varepsilon \text{ and } \frac{|f(x) - f(y)|}{\frac{1}{n} + |f(x) - f(y)|} < \varepsilon$$

whenever $M(x, y, s) > 1 - r$ and $N(x, y, s) < r$. An easy computation shows that $|f(x) - f(y)| < \varepsilon$ whenever $M(x, y, s) > 1 - r$ and $N(x, y, s) < r$. We conclude that f is $\mathbb{R}$-uniformly continuous on $(X, M, N, *, \Diamond)$.

(ii) $\Rightarrow$ (iii) Let f a real valued continuous function on $(X, \tau_{(M,N)})$ and $\varepsilon > 0$. By assumption, there exist $r \in (0, 1)$ and $s > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $M(x, y, s) > 1-r$ and $N(x, y, s) < r$. Take $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \min\{r, s\}$. Then for all $x, y \in X$ such that $(x, y) \in U_n$, we obtain by Remark 2,

$$M(x, y, s) \geq M(x, y, \frac{1}{n}) > 1 - \frac{1}{n} \geq 1 - r$$

and

$$N(x, y, s) \leq N(x, y, \frac{1}{n}) < \frac{1}{n} \leq r,$$

so $|f(x) - f(y)| < \varepsilon$. We conclude that f is uniformly continuous on $(X, \breve{U}_{(M,N)})$.

(iii) $\Rightarrow$ (iv). Let A and B be two disjoint nonempty closed subsets of $(X, \tau_{(M,N)})$. There exists a continuous function $f : X \to [0, 1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$. Since by assumption f is uniformly continuous on $(X, \breve{U}_{(M,N)})$, for $\varepsilon = 1$ there is $n \in \mathbb{N}$ such that $|f(x) - f(y)| < 1$ whenever $M(x, y, \frac{1}{n}) > 1-\frac{1}{n}$ and $N(x, y, \frac{1}{n}) < \frac{1}{n}$.

Hence

$$M(a,b,\frac{1}{n}) \leq 1-\frac{1}{n} \text{ and } N(a,b,\frac{1}{n}) \geq \frac{1}{n}$$

for all $a \in A$ and $b \in B$. We conclude that (M,N) is equinormal intuitionistic fuzzy metric on X.

(iv) $\Rightarrow$ (v). Let A and B be two disjoint nonempty closed subsets of $(X,\tau_{(M,N)})$. By assumption, there exist $r \in (0,1)$ and $s > 0$ such that

$$\sup\{M(a,b,s) : a \in A, b \in B\} = 1-r$$

and

$$\inf\{N(a,b,s) : a \in A, b \in B\} = r.$$

Put

$$U = \{(x,y) \in X \times X : M(x,y,s) > 1-r,\ N(x,y,s) < s\}.$$

Then $U \in \breve{U}_{(M,N)}$ and $U(A) \cap B = \varnothing$. Hence $\breve{U}_{(M,N)}$ is an equinormal uniformity on X.

(v) $\Rightarrow$ (vi). Its clear from [2] Theorem 2.3.1.

(vi) $\Rightarrow$ (vii). Let G be an open cover of X. From our assumption it follows that there is an $n \in \mathbb{N}$ such that $\{B_{(M,N)}(x,\frac{1}{n},\frac{1}{n}) : x \in X\}$ refines G.Hence (M,N) is a Lebesgue intuitionistic fuzzy metric on X.

(vii) $\Rightarrow$ (i). Let $(Y,M',N',*',\Diamond')$ an intuitionistic fuzzy metric space and f a continuous mapping from $(X,\tau_{(M,N)})$ to $(Y,\tau_{(M',N')})$. Fix $\varepsilon \in (0,1)$ and $t > 0$. There is $\delta > 0$ such that $(1-\delta) *' (1-\delta) > 1-\varepsilon$ and $\delta\Diamond'\delta < \varepsilon$. Since f is continuous, for each $x \in X$ there is an open neighborhood V_x of x such that $f(V_x) \subseteq B_{(M',N')}(f(x),\delta,\frac{t}{2})$. By assumption there exist $r = r(t,\varepsilon) \in (0,1)$ and $s > 0$ such that $\{B_{(M,N)}(x,r,s) : x \in X\}$ refines $\{V_x : x \in X\}$.

Now if $M(x,y,s) > 1-r$ and $N(x,y,s) < s$ we have $y \in B_{(M,N)}(x,r,s)$, so $x,y \in V_z$ for some $z \in X$. Hence, $f(x)$ and $f(y)$ are in $B_{(M',N')}(f(z),\delta,\frac{t}{2})$. Thus,

$$M'(f(x),f(y),t) \geq M'(f(x),f(z),\frac{t}{2}) *' M'(f(z),f(y),\frac{t}{2}) > 1-\varepsilon$$

and

$$N'(f(x),f(y),t) \leq N'(f(x),f(z),\frac{t}{2})\Diamond' N'(f(z),f(y),\frac{t}{2}) < \varepsilon.$$

Then f is uniformly continuous from $(X,M,N,*,\Diamond)$ to $(Y,M',N',*',\Diamond')$. □

It is well known (see, for instance, [2]) that a metrizable topological space admits a metric with the Lebesgue property if and only if the set of nonisolated points is compact. From this result and the preceding theorem we deduce the following corollary.

Corollary 1. *A (intuitionistic fuzzy) metrizable topological space admits an intuitionistic fuzzy metric with the Lebesgue property if and only if the set of nonisolated points is compact.*

Remark 8. *Given a metrizable topological space X we denote by $\mathcal{F}\tilde{N}_{(M,N)}$ the supremum of all uniformities $\breve{U}_{(M,N)}$ induced by all compatible intuitionistic fuzzy metrics for X. It is easy to see that $\mathcal{F}\tilde{N}_{(M,N)}$ is exactly the fine uniformity of X. Hence, the classical theorem that if topological space X admits a metric d with the Lebesgue property, then the uniformity $\breve{U}_d$ coincides with the fine uniformity of X, can be reformulated as follows: If a topological space admits an intuitionistic fuzzy metric (M,N) with the Lebesgue property, then the uniformity $\breve{U}_{(M,N)}$ coincides with the uniformity $\mathcal{F}\tilde{N}_{(M,N)}$.*

We conclude the paper with an example which illustrates the obtained results.

Example 3. *Let X be the et of natural numbers and let $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm defined by $a * b = ab$ and $a\Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$. For each $x, y \in X$ and $t > 0$ let*

$$M(x,y,t) = \begin{cases} 1 & if \quad x = y, \\ \frac{1}{xy} & if \quad x \neq y, \end{cases} \quad and \; N(x,y,t) = \begin{cases} 0 & if \quad x = y, \\ \frac{xy-1}{xy} & if \quad x \neq y, \end{cases}$$

It is easy to check that $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. Note that there is no metric d on X for which (M,N) is the intuitionistic fuzzy metric induced by d. For each pair of disjoint nonempty subsets of X, A and B, we have

$$\sup\{M(a,b,s) : a \in A, b \in B\} \leq \frac{1}{2} \; and \; \inf\{N(a,b,s) : a \in A, b \in B\} \geq \frac{1}{2}$$

for all $t > 0$. From this fact it follows that (M,N) is an equinormal intuitionistic fuzzy metric on X (so $(X, M, N, *, \Diamond)$ satisfies conditions of our theorem), and that the $\tau_{(M,N)}$ discrete topology on X since for each $x \in X$,

$$\sup\{M(a,b,s) : a \in A, b \in B\} \leq \frac{1}{2} \; and \; \inf\{N(a,b,s) : a \in A, b \in B\} \geq \frac{1}{2}$$

and thus $B_{(M,N)}(x, \frac{1}{2}, \frac{1}{2}) = \{x\}$.

Acknowledgement. The authors would like to thank the referees for their help in the improvement of this paper.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,61-88,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Sharp estimates for solutions of parabolic equations with a lower order term

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Abstract

We give a comparison result for solutions of Cauchy-Dirichlet problems for parabolic equations by means of Schwarz symmetrization. The result takes into account the influence of the zero order term, on which any boundedness or sign assumption is assumed.

2000 Mathematics Subject Classification: 35B45, 35K15, 35K20.
Key words and phrases: Rearrangements, Comparison results, Schwarz symmetrization, Parabolic equations.

1 Introduction

Among the a priori estimates for solutions of elliptic boundary value problems, a particular role is assumed by those estimates, known as isoperimetric inequalities, that show interesting properties of geometric type. Indeed, the solution of the starting problem is compared with the solution of a suitable symmetric problem which is, in some sense, the "worst" one.

The pioneristic result in this direction is due to G.Talenti [22]. Afterwards, this result has been presented in several and different situations: for instance, we refer to the papers [3], [4], [6], [8], [13], [23], in which a particular emphasis has been reserved to the influence of the lower order terms. We restrict ourselves to describe briefly the case involving the zero order term. If Ω is an open bounded subset of $\mathbb{R}^N$, let $u \in H_0^1(\Omega)$ be the weak solution of the equation

$$-\Delta u + cu = f$$

where c is a nonnegative function. This condition guarantees that the solution u is nonnegative if f is nonnegative. Moreover, if $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same measure as Ω, let $v \in H_0^1(\Omega^\#)$ be the weak solution of the symmetrized problem

$$-\Delta v + c_\# v = f^\#,$$

where $c_\#$, $f^\#$ are the *increasing spherical rearrangement of* c and the *decreasing spherical rearrangement of* f. Roughly speaking, the functions $c_\#$ and $f^\#$ have some nice properties like simmetry, monotonicity and preserve the measures of the level sets of c and f respectively (we refer to section 2 for their precise definitions).

Then, it is known (see [3], [6], [8]) that u is dominated by v in the sense of rearrangements, i.e.

$$\int_0^s u^* \leq \int_0^s v^* \qquad \forall s \in [0, |\Omega|] \,, \tag{1.1}$$

where u^*, v^* are the *decreasing rearrangements* of u and v. Inequality (1.1) implies the following property (see [10], [8]):

$$\int_\Omega F\left(|u|\right) dx \leq \int_{\Omega^\#} F\left(|v|\right) dx$$

for any, increasing convex function F on $\mathbb{R}^+$, such that $F(0) = 0$. So that any Luxemburg norm of u can be estimated by the same norm of v (see [10]).

In the meantime, an analogous theory has been developed for parabolic operators (see for example [3], [7], [8], [14], [15], [20], [24], [25], [26]). Consider the problem

$$\begin{cases} u_t - \displaystyle\sum_{i,j=1}^N \left(a_{ij}\left(x,t\right) u_{x_i}\right)_{x_j} + cu = f & in \;\; \Omega \times (0,T) \\ \\ u = 0 & on \;\; \partial\Omega \times (0,T) \\ \\ u\left(x,0\right) = u_0\left(x\right) & x \in \Omega, \end{cases} \tag{1.2}$$

where we assume that the operator is uniformly parabolic, i.e.

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\,\xi_i\xi_j \geq |\xi|^2 \quad a.e.\ (x,t) \in \Omega \times (0,T)\,,\ \forall \xi \in \mathbb{R}^N, \tag{1.3}$$

and

$$c(x) \geq 0\ ;$$

then, if v is the solution of the "symmetrized" problem

$$\begin{cases} v_t - \Delta v = f^{\#} & in\ \ \Omega^{\#} \times (0,T) \\ \\ v = 0 & on\ \ \partial\Omega^{\#} \times (0,T) \\ \\ v(x,0) = u_0^{\#}(x) & x \in\ \Omega^{\#}, \end{cases} \tag{1.4}$$

for all $t \in [0,T]$ the following inequality holds

$$\int_0^s u^*(\sigma,t)\,d\sigma \leq \int_0^s v^*(\sigma,t)\,d\sigma,\ \forall s \in [0,|\Omega|]\,. \tag{1.5}$$

In problem (1.4) and in (1.5), the spherical rearrangement $f^{\#}$ and the decreasing rearrangements u^*, v^* are meant to be calculated with respect to x, for t fixed.

The main difficulty that appears in each of the above mentioned papers is linked to the presence of the time derivative term. This last one can be treated by two different methods. Following the approach contained in a paper of C. Bandle (see [7]), the crucial part consists in proving a delicate derivation formula with respect to the time variable for functions defined by integrals. In [7], such a formula is proved under strong regularity assumptions on the solutions. These hypotheses have been removed later in a paper of Mossino-Rakotoson (see [20]), where the formula is proved for fuctions $u \in H^1\left(0, T; L^2(\Omega)\right)$ by using the notion of relative rearrangement. Some generalizations of this result have been obtained in [5] or [16], where a formula concerning the second derivatives is also given.

Another approach uses the implicit time discretization scheme. In this way we replace the time derivative with a difference quotient, and by using a partition of the time interval $[0, T]$ of the form $0 = t_0 < t_1 < \ldots < t_n = T$, we are reduced to apply the above quoted comparison result to a sequence of elliptic problems with zero order term of the form

$$\begin{cases} -\left(a_{ij}^{(k)}(x)\, u_{x_i}^{(k)}\right)_{x_j} + \left(c + \dfrac{1}{t_k - t_{k-1}}\right) u^{(k)} = f^{(k)} + \dfrac{u^{(k-1)}}{t_k - t_{k-1}} \\ \\ u^{(k)} \in H_0^1(\Omega), \end{cases}$$

for $k = 1, \ldots, n$, where $u^{(k)} = u(x, t_k)$ and $f^{(k)}$, $a_{ij}^{(k)}$ are suitable discretization of the functions $f = f(x,t)$, $a_{ij} = a_{ij}(x,t)$. Then we reach the aim by passing to limit (see [3], [24], [25]). This method is described in more details in remark 2.

In all the results for parabolic operators mentioned above, the influence of the zero order term cu is always neglected, since this term is essentially omitted in (1.4) by using the sign condition $c(x) \geq 0$. Our aim is to find a comparison result of the type (1.5), between the solution u of the problem (1.2) and the solution v of a spherically symmetric problem which keeps in mind the zero order term. The candidate problem is the following

$$\begin{cases} v_t - \Delta v + c_{\#} v = f^{\#} & in \;\; \Omega^{\#} \times (0,T) \\ \\ v = 0 & on \;\; \partial\Omega^{\#} \times (0,T) \\ \\ v(x,0) = u_0^{\#}(x) & x \in \; \Omega^{\#}, \end{cases} \tag{1.6}$$

where

$$c_{\#} = \left(c^{+}\right)_{\#} - \left(c^{-}\right)^{\#},$$

c^{+} and c^{-} being the positive and the negative part of c and $\left(c^{+}\right)_{\#}$, $\left(c^{-}\right)^{\#}$ respectively their increasing and decreasing spherical rearrangements.

We consider weak solutions of the problem (1.2): namely we deal with functions $u \in L^2\left(0,T;H_0^1(\Omega)\right) \cap C\left([0,T];L^2(\Omega)\right)$ such that

$$\frac{\partial u}{\partial t} \in L^2\left(\Omega \times (0,T)\right)$$

and

$$\begin{cases} \displaystyle\int_\Omega \frac{\partial u}{\partial t}\varphi dx + \int_\Omega a_{ij} u_{x_i} \varphi_{x_j} dx + \int_\Omega c u \varphi dx = \int_\Omega f \varphi dx, \\ \\ u(0) = u_0, \end{cases} \tag{1.7}$$

for all $\varphi \in H_0^1(\Omega)$ and for a.e. $t \in [0,T]$. The existence of such a solution is guaranteed under suitable assumptions on the data.

The result is the following:

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, assume that the coefficients $a_{ij} \in L^\infty(\Omega \times (0,T))$ satisfy (1.3) and suppose*

$$c \in L^r(\Omega) \ \ with \ r > N/2 \ \ if \ \ N \geq 2, \ r \geq 1 \ \ if \ \ N = 1,$$

$f \in L^2(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$. Let u and v be the weak solutions of problems (1.2) and (1.6) respectively, then for all $t \in [0,T]$, (1.5) holds.

In section 2 we prove inequality (1.5) by assuming that c is bounded from below. Obviously, this assumption allows us to reduce the study to the case $c \geq 0$. In fact, if $c(x) \geq \lambda$ for a.e. $x \in \Omega$, the function $e^{\lambda t}u$ is the solution of a problem of type (1.2) in which the zero order coefficient is $(c - \lambda)$. This situation was already studied in [26]. We give a simpler proof that avoids to proceed by means of the approximation used in [26].

In section 3 we deal with the more general case, in which c is not bounded from below. The motivation of this study, besides its intrinsic interest, is also connected to some recent results obtained by various authors (see [9], [12], [17]), related to the existence of solutions of parabolic equations where the coefficient c has a singularity of the type

$$c(x) = -\frac{\lambda}{|x|^2}.$$

This situation can be classified as a limit case, in the sense that $c(x) \notin L^{N/2}$ $(N \geq 3)$, but belongs to the Lorentz space $L(N/2, \infty)$. Moreover the operator $-\Delta u - \left(\lambda/|x|^2\right) u$ is coercive if and only if $\lambda < \lambda_N := (N-2)^2/4$, where λ_N is the best constant in the classical Hardy inequality; it is positive when $\lambda = \lambda_N$: hence, the standard existence and regularity theories do not apply in this case. We will discuss this limit case in a forthcoming paper.

2 Proof of Theorem 1: the case $c(x) \geq \lambda$

Before going into a detailed proof of Theorem 1, we begin this section by recalling some definitions that are useful in the following. Let Ω be a bounded open subset of $\mathbb{R}^N$ and u be a real measurable function on Ω, we define the distribution function μ_u of u as

$$\mu_u(\theta) = |\{x \in \Omega : |u(x)| > \theta\}| \ , \theta \geq 0,$$

the decreasing and the increasing rearrangement of u as

$$\begin{aligned} &u^*(s) = \sup\{\theta \geq 0 : \mu_u(\theta) > s\} \ , \ s \in (0, |\Omega|) \ , \\ &u_*(s) = u^*(|\Omega| - s) \ , \quad s \in (0, |\Omega|) \,. \end{aligned}$$

Furthermore, if ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same measure as Ω, the functions

$$\begin{aligned} &u^\#(x) = u^*(\omega_N |x|^N) \ , \ x \in \Omega^\#, \\ &u_\#(x) = u_*(\omega_N |x|^N) \ , \ x \in \Omega^\#, \end{aligned}$$

are respectively the decreasing spherical rearrangement and the increasing spherical rearrangement of u. Here we just quote the well known Hardy-

Littlewood inequality (see [18]): if u, v are measurable functions on Ω, then

$$\int_0^{|\Omega|} u^*(s)\, v_*(s)\, ds \leq \int_\Omega |u(x)\, v(x)|\, dx \leq \int_0^{|\Omega|} u^*(s)\, v^*(s)\, ds\ . \qquad (2.1)$$

As we pointed out in the introduction, our aim is to obtain a comparison result for problems of the type (1.2). For this reason, in the following we will consider real functions u defined on the set $\Omega \times (0,T)$, where T is a real positive number, that are measurable with respect to the space variable x and denote by $\mu_u(\theta,t)$, $u^*(s,t)$, $u_*(s,t)$, $u^{\#}(x,t)$, $u_{\#}(x,t)$ the distribution function and the rearrangements of $u(x,t)$, with respect to x for t fixed. In other words, $u^{\#}(x,t)$ is the Steiner symmetrization of $u(x,t)$ with respect to the line $x = 0$.

As in [7], [26], problem (1.2) can be dealt by using a classic method introduced by Talenti in [22]. This method consists in choosing a suitable test function in (1.7), and it leads to the study of derivation formulas with respect to the variable t of integrals of the type

$$\int_{u(x,t)>u^*(s,t)} u(x,t)\, dx.$$

Such formula, obtained in [7] for regular functions (see also [5] and [16]), allows us to say that

$$\int_{u(x,t)>u^*(s,t)} \frac{\partial u}{\partial t}(x,t)\, dx = \int_0^s \frac{\partial u^*}{\partial t}(\sigma,t)\, d\sigma. \qquad (2.2)$$

In the paper [20], (2.2) has been obtained for less regular functions. More precisely, the following result is proved:

Lemma 2. *If u is a nonnegative function in $H^1\left(0,T;L^2\left(\Omega\right)\right)$, then u^* belongs to $H^1\left(0,T;L^2\left(0,|\Omega|\right)\right)$ and (2.2) holds if $\left|u\left(x,t\right)=u^*\left(s,t\right)\right|=0$, for a.e. $s\in\left(0,|\Omega|\right)$.*

Now we are able to prove Theorem 1 assuming that c is nonnegative.

In the first part of the proof we follow an approach similar to the one given in [26] (see also [3], [8], [20]) and we report it for completeness. It consists in deriving an integro-differential inequality for the decreasing rearrangement of u. In the second part we get the result by means of a maximum principle. This maximum principle *does not* make use of the method contained in [26], that consists in approximating the solution v of problem (1.6) with a sequence of solutions of suitable perturbated problems.

Since we will need that $\frac{\partial u}{\partial t}\in L^2\left(Q_T\right)$, we assume, for instance, the additional conditions (see [11])

$$\frac{\partial a_{ij}}{\partial t}\in C^0\left(\Omega\times[0,T]\right), i,j=1,\ldots,N$$

$$u_0\in H_0^1\left(\Omega\right).$$

Fixed $t\in[0,T]$, $h>0$ and $\theta\geq0$, we choose the function

$$\varphi_h\left(x\right)=\begin{cases}\operatorname{sign}\left(u\right) & \text{if} \quad |u|>\theta+h\\[2ex] \dfrac{|u|-\theta}{h}\operatorname{sign}\left(u\right) & \text{if} \quad \theta<|u|\leq\theta+h\\[2ex] 0 & \text{otherwise}\end{cases}$$

as a test function in (1.7). Letting h go to 0 and using (1.3) we obtain

$$-\frac{\partial}{\partial\theta}\int_{|u|>\theta}|\nabla u|^2\,dx\le\int_{|u|>\theta}\left[f(x,t)-c(x)\,u-\frac{\partial u}{\partial t}\right]\operatorname{sign}(u)\;dx. \quad (2.3)$$

The left hand side of (2.3) can be estimated from below using the following inequalities

$$N^2\omega_N^{2/N}\mu_u(\theta,t)^{2-(2/N)}\le\left(-\frac{\partial\mu_u}{\partial\theta}\right)\left(-\frac{\partial}{\partial\theta}\int_{|u|>\theta}|\nabla u|^2\,dx\right) \quad (2.4)$$

which are consequences of the isoperimetric inequality, the Fleming Rishel formula and the Schwartz inequality (we refer to [22] for more details). As regards to the term involving the derivative of u with respect to t, we notice that, since $u\in H_0^1(\Omega)$ for a.e. t, it follows $u^*\in C(]0,|\Omega|])$ (see [20]), therefore $|u=\theta|=0$ and $u^*(\mu_u(\theta,t))=\theta$, for a.e. θ, hence by lemma 1

$$\int_{|u|>\theta}\frac{\partial u}{\partial t}\operatorname{sign}(u)\;dx=\int_0^{\mu_u(\theta,t)}\frac{\partial u^*}{\partial t}ds\quad\text{for a.e. }\theta\ge 0. \quad (2.5)$$

The remaining terms of (2.3) can be treated by using (2.1) in the following way:

$$\begin{aligned}-\int_{|u|>\theta}c(x)\,u\operatorname{sign}(u)\;dx&\le-\int_0^{\mu_u(\theta,t)}c_*(s)\,u^*(s,t)\,ds\;,\\ \int_{|u|>\theta}f(x,t)\;\operatorname{sign}(u)\;dx&\le\int_0^{\mu_u(\theta,t)}f^*(s,t)\,ds.\end{aligned} \quad (2.6)$$

Collecting (2.4), (2.5) and (2.6), we have

$$N^2\omega_N^{2/N}\mu_u(\theta,t)^{2-(2/N)}\le\left(-\frac{\partial\mu_u}{\partial\theta}\right)\left(\int_0^{\mu_u(\theta,t)}\left\{f^*(s,t)-c_*(s)\,u^*(s,t)-\frac{\partial u^*}{\partial t}\right\}ds\right).$$

Making a change of variable (see [20]), we get

$$\begin{aligned}&\int_0^s \frac{\partial u^*}{\partial t}(\sigma,t)\,d\sigma - p(s)\,\frac{\partial u^*}{\partial s} + \int_0^s c_*(\sigma)\,u^*(\sigma,t)\,d\sigma \\ &\leq \int_0^s f^*(\sigma,t)\,d\sigma \quad \text{for a.e. } (s,t)\in Q_T^* := (0,|\Omega|)\times(0,T)\,,\end{aligned} \tag{2.7}$$

where $p(s) := N^2\omega_N^{2/N}s^{2-(2/N)}$. On the other hand, if we consider the solution v of problem (1.6), all the inequalities we used to get (2.7) become equalities and so we have

$$\begin{aligned}&\int_0^s \frac{\partial v^*}{\partial t}(\sigma,t)\,d\sigma - p(s)\,\frac{\partial v^*}{\partial s} + \int_0^s c_*(\sigma)\,v^*(\sigma,t)\,d\sigma \\ &= \int_0^s f^*(\sigma,t)\,d\sigma \quad \text{for a.e. } (s,t)\in Q_T^*.\end{aligned} \tag{2.8}$$

From (2.7)-(2.8) it follows that

$$\frac{\partial}{\partial t}\int_0^s w(\sigma,t)\,d\sigma - p(s)\,\frac{\partial w}{\partial s} + \int_0^s c_*(\sigma)\,w(\sigma,t)\,d\sigma \leq 0,$$

for a.e. $(s,t)\in Q_T^*$, where $w := u^* - v^*$. Then, if we set

$$\chi(s,t) := \int_0^s w(\sigma,t)\,d\sigma \text{ with } (s,t)\in\overline{Q_T^*},$$

by the boundary conditions of (1.2), (1.6), we have that χ satisfies

$$\begin{cases} \dfrac{\partial \chi}{\partial t} - p(s)\dfrac{\partial^2 \chi}{\partial s^2} + \displaystyle\int_0^s c_*(\sigma)\frac{\partial \chi}{\partial \sigma}(\sigma, t)\, d\sigma \leq 0 \quad a.e \text{ in } Q_T^* \\ \\ \chi(0,t) = \dfrac{\partial \chi}{\partial s}(|\Omega|, t) = 0 \quad \forall t \in [0,T] \\ \\ \chi(s,0) = 0 \quad \forall s \in [0, |\Omega|]. \end{cases} \tag{2.9}$$

Our aim is to show that

$$\chi \leq 0.$$

It can be easily proved that the function χ is continuous in $\overline{Q_T^*}$, so it has a maximum in $\overline{Q_T^*}$. We will prove that this maximum has to be zero. We argue by contradiction: let (s_0, t_0) be a maximum point of χ in $\overline{Q_T^*}$ such that $\chi(s_0, t_0) > 0$ and first assume that $(s_0, t_0) \in Q_T^*$. We begin observing that the term $\int_0^s c_*(\sigma)\frac{\partial \chi}{\partial \sigma}(\sigma, t)\, d\sigma$ in the differential inequality of problem (2.9) can be neglected in a suitable square neighbourhood of (s_0, t_0). Indeed, integrating by parts we obtain

$$\begin{aligned} \int_0^{s_0} c_*(\sigma)\frac{\partial \chi}{\partial \sigma}(\sigma, t_0)\, d\sigma &= \int_0^{s_0} c_*(\sigma)\, d\chi(\sigma, t_0) \\ &= \int_0^{s_0} [\chi(s_0, t_0) - \chi(\sigma, t_0)]\, dc_*(\sigma) > 0, \end{aligned} \tag{2.10}$$

then by continuity it is possible to find a suitable square neighbourhood $Q_\delta = Q_\delta(s_0, t_0)$ of (s_0, t_0) in which the function $\int_0^s c_*(\sigma)\frac{\partial \chi}{\partial \sigma}(\sigma, t)\, d\sigma$ is still positive. Therefore, provided to choose δ conveniently, from (2.9) we find

$$\frac{\partial \chi}{\partial t} - p\left(s\right)\frac{\partial^2 \chi}{\partial s^2} < 0 \ \text{ a.e. in } Q_\delta. \tag{2.11}$$

Hence we can reduce our study of problem (2.9) to the study of inequality (2.11) in the neighbourhood Q_δ. This allows us to proceed as in [20]. However, we cannot multiply directly both sides of (2.11) for χ^+, and integrate onto the interval $(s_0 - \delta, s_0 + \delta)$, because we don't know the values of χ on the parabolic boundary Γ_δ of Q_δ. So it is natural to consider, instead of χ^+, the function φ defined as

$$\varphi := \left(\chi_{/\overline{Q_\delta}} - \max_{\Gamma_\delta} \chi\right)^+ .$$

For our purpose we can suppose, without loss of generality, that

$$\chi\left(s,t\right) < \chi\left(s_0,t_0\right) \qquad \forall\left(s,t\right) \in \overline{Q_\delta} \backslash \left\{\left(s_0,t_0\right)\right\},$$

therefore $\varphi\left(s_{0,}t_0\right) > 0$.Multiplying the inequality (2.11) by $s^{(2/N)-2}\varphi$ we find

$$s^{(2/N)-2}\frac{\partial \chi}{\partial t}\varphi \leq N^2\omega_N^{2/N}\frac{\partial^2 \chi}{\partial s^2}\varphi \ \text{ a.e.in } Q_\delta \ . \tag{2.12}$$

We can prove as in [20] that

$$\chi \in W^{2,\infty}\left(s_0 - \delta, s_0 + \delta\right) \ ,$$

$$\varphi \in H^1\left(s_0 - \delta, s_0 + \delta\right) \ ,$$

so the integration by parts leads to :

$$\int_{s_0-\delta}^{s_0+\delta} \frac{\partial^2 \chi}{\partial s^2} \varphi ds = \left[\frac{\partial \chi}{\partial s} \varphi \right]_{s_0-\delta}^{s_0+\delta} - \int_{s_0-\delta}^{s_0+\delta} \left(\frac{\partial \varphi}{\partial s} \right)^2 ds \leq 0 \ .$$

Then, from (2.12) and by the definition of φ we have, for any $t \in (t_0 - \delta, t_0 + \delta)$,

$$0 \geq 2 \int_{t_0-\delta}^{t} \int_{s_0-\delta}^{s_0+\delta} s^{(2/N)-2} \frac{\partial \chi}{\partial t} \varphi ds d\tau = \int_{s_0-\delta}^{s_0+\delta} s^{(2/N-2)} \left[\int_{t_0-\delta}^{t} \frac{\partial}{\partial \tau} \left(\varphi^2 (s, \tau) \right) d\tau \right] ds$$
$$= \int_{s_0-\delta}^{s_0+\delta} s^{(2/N)-2} \varphi^2 (s, t) \, ds,$$

therefore $\varphi = 0$ in $Q_\delta = (s_0 - \delta, s_0 + \delta) \times (t_0 - \delta, t_0 + \delta)$ but this is a contradiction.

By similar arguments we come to the same conclusion if we suppose that (s_0, t_0) belongs either to the segment line $\{(s, T) : s \in (0, |\Omega|)\}$ or to the segment line $\{(|\Omega|, t) : t \in (0, T)\}$. □

Remark 1. Obviously the case in which c is unbounded from below could also be treated by truncating the coefficient c and passing to the limit.

3 Proof of Theorem 1: the general case

In this section we conclude the proof of theorem 1 considering the general case $c \in L^r(\Omega)$, with $r > N/2$ if $N \geq 2$, and $r \geq 1$ if $N = 1$.

The first part of the proof is exactly the same as the one of the previous case, the main difference consists in the proof of the maximum principle.

As before, we have that u satisfies

$$\int_0^s \frac{\partial u^*}{\partial t}\ (\sigma,t)\,d\sigma - p\,(s)\,\frac{\partial u^*}{\partial s} + \int_0^s \left[\left(c^+\right)_* - \left(c^-\right)^*\right] u^*\,(\sigma,t)\,d\sigma$$
$$\leq \int_0^s f^*\,(\sigma,t)\,d\sigma \quad \text{for a.e. } \ (s,t) \in Q_T^*$$

and this inequality holds as equality replacing u with the solution v of the symmetrized problem (1.6). However, if we set

$$\chi\,(s,t) = \int_0^s \left(u^* - v^*\right) d\sigma,$$

it is not possible to neglect the term $\int_0^s \chi\,(\sigma,t)\,\frac{d}{d\sigma}\left[\left(c^+\right)_* - \left(c^-\right)^*\right] d\sigma$ by means of the pointwise arguments we used to get (1.5), since c could not be bounded from below. In order to treat this term we proceed by approximation.

Let $v_\epsilon = v + \epsilon v_0$ be the solution of the following perturbated problem

$$\begin{cases} v_{\epsilon t} - \Delta v_\epsilon + \left[\left(c^+\right)_\# - \left(c^-\right)^\#\right] v_\epsilon = f^\# + \epsilon\delta & in \ \ \Omega^\# \times (0,T) \\ \\ v_\epsilon = 0 & on \ \ \partial\Omega^\# \times (0,T) \\ \\ v_\epsilon\,(x,0) = u_0^\#\,(x) + \epsilon v_0\,(x) & x \in \ \Omega^\#, \end{cases} \tag{3.1}$$

where $\epsilon > 0$, δ is the Dirac measure concentrated at the origin and v_0 is the unique solution vanishing on $\partial\Omega^\#$ of the equation

$$-\Delta v_0 + \left[\left(c^+\right)_\# - \left(c^-\right)^\#\right] v_0 = \delta \quad in\ \Omega^\#. \tag{3.2}$$

A suitable definition of weak solution to equation (3.2), together with some existence and regularity results, can be found in the monograph of G. Stampacchia [21]. Suppose for the sake of simplicity $N \geq 3:$ since $c \in L^r(\Omega)$ with $r > N/2$, by a regularity result contained in [2] to have that the solution v_0 of (3.2) is in the Lorentz space $L(N/(N-2), \infty)$.

Proceeding as in the proof of the case $c(x) \geq 0$, setting

$$\chi_\epsilon(s,t) := \int_0^s (u^* - v_\epsilon^*)\, d\sigma,$$

we get

$$\begin{cases} \dfrac{\partial \chi_\epsilon}{\partial t} - p(s)\dfrac{\partial^2 \chi_\epsilon}{\partial s^2} + \displaystyle\int_0^s \left[(c^+)_* - (c^-)^*\right] \dfrac{\partial \chi_\epsilon}{\partial \sigma}(\sigma,t)\, d\sigma \\ \\ \leq -\epsilon \ a.e \text{ in } Q_T^* \\ \\ \chi_\epsilon(0,t) = \dfrac{\partial \chi_\epsilon}{\partial s}(|\Omega|, t) = 0 \quad \forall t \in [0,T] \\ \\ \chi_\epsilon(s,0) = -\epsilon \displaystyle\int_0^s v_0^*(\sigma)\, d\sigma. \end{cases} \tag{3.3}$$

We want to prove that $\chi_\epsilon \leq 0$ in $\overline{Q_T^*}$, so that

$$\int_0^s u^*(\sigma,t)\, d\sigma \leq \int_0^s v_\epsilon^*(\sigma,t)\, d\sigma \ , \forall (s,t) \in \overline{Q_T^*},$$

and this obviously gives the desired result by letting ϵ goes to zero.

Let us suppose that $\chi_\epsilon > 0$ in a subset F of $\overline{Q_T^*}$ and let $\bar{t}$ be the minimum value of the projection of F on the t axis. We notice that

$$\chi_\epsilon (s,t) = \int_0^s (u^* - v^*)\, d\sigma - \epsilon \int_0^s v_0^* d\sigma$$

and then

$$\lim_{s\to 0} \left(c^-\right)^* (s)\, \chi_\epsilon (s,t) = 0.$$

Indeed, since v_0 is in $L\left(N/\left(N-2\right), \infty\right)$ we get

$$\begin{aligned}\left| \left(c^-\right)^* (s) \int_0^s v_0^* (\sigma)\, d\sigma \right| &\leq K \left(c^-\right)^* (s)\, s^{\frac{2}{N}} \leq K s^{\frac{2}{N}-1} \int_0^s \left(c^-\right)^* d\sigma \\ &\leq K s^{\frac{2}{N}-\frac{1}{r}} \left\| c^- \right\|_{L^r(\Omega)} ;\end{aligned}$$

obviously a similar estimate holds for the term $\left(c^-\right)^* (s) \int_0^s (u^* - v^*)\, d\sigma$, since $u^*, v^* \in L^{2^*}(0, |\Omega|)$. Hence integrating by parts we have that

$$\begin{aligned}\int_0^s \left[\left(c^+\right)_* - \left(c^-\right)^*\right] \frac{\partial \chi_\epsilon}{\partial \sigma} (\sigma, t)\, d\sigma &= \left[\left(c^+\right)_* - \left(c^-\right)^*\right] \chi_\epsilon (s,t) \\ &\quad - \int_0^s \chi_\epsilon (\sigma, t)\ d\left[\left(c^+\right)_* - \left(c^-\right)^*\right].\end{aligned}$$

By continuity arguments, we can choose $\tau > 0$ such that, for $t < \bar{t} + \tau$, it results

$$\int_0^s \chi_\epsilon (\sigma, t)\ d\left[\left(c^+\right)_* - \left(c^-\right)^*\right] - \epsilon \leq 0 \quad \forall s \in [0, |\Omega|],$$

so that by (3.3) we have

$$\begin{aligned}&\frac{\partial \chi_\epsilon}{\partial t} - p(s) \frac{\partial^2 \chi_\epsilon}{\partial s^2} + \left[\left(c^+\right)_* - \left(c^-\right)^*\right] \chi_\epsilon (s,t) \leq 0\ , \\ &a.e.\ in\ \ (0, |\Omega|) \times (0, \bar{t} + \tau).\end{aligned} \tag{3.4}$$

A similar computation can be done in the cases $N=1$ and $N=2$. Indeed, v_0 is bounded if $N=1$, while in the case $N=2$ we find that v_0 belongs to the Zygmund space $L_{\exp}(\Omega)$ (see [1]), namely

$$\sup_{s\in(0,|\Omega|)} \frac{v_0^{**}(s)}{\left(1+\log\frac{|\Omega|}{s}\right)} < \infty,$$

where

$$v_0^{**}(s) = \frac{1}{s}\int_0^s v_0^*(\sigma)\, d\sigma$$

is the maximal function of v_0^*.

Dividing both sides of the inequality (3.4) by $p(s)$ we can rewrite it as

$$\begin{aligned} & p(s)^{-1}\frac{\partial\chi_\epsilon}{\partial t} - \frac{\partial^2\chi_\epsilon}{\partial s^2} + p(s)^{-1}\left[(c^+)_* - (c^-)^*\right]\chi_\epsilon \leq 0 \ , \\ & a.e.\ in\ \ (0,|\Omega|)\times(0,\bar{t}+\tau)\,. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by χ_ϵ^+ and integrating between 0 and $|\Omega|$,taking into account the boundary conditions in (3.3), we get

$$\begin{aligned} & \gamma_N\int_0^{|\Omega|} s^{(2/N)-2}\frac{\partial\chi_\epsilon}{\partial t}\chi_\epsilon^+ ds + \int_0^{|\Omega|}\left(\frac{\partial\chi_\epsilon^+}{\partial s}\right)^2 ds \\ & +\gamma_N\int_0^{|\Omega|}\left[(c^+)_* - (c^-)^*\right] s^{(2/N)-2}\left(\chi_\epsilon^+\right)^2 ds \leq 0, \end{aligned} \tag{3.6}$$

where $\gamma_N := N^{-2}\omega_N^{-2/N}$. We want to prove that if we replace the function χ_ϵ with the function $U_\epsilon := e^{-\lambda t}\chi_\epsilon$ (where $\lambda > 0$ is a suitable constant), the sum of those terms in (3.6) that don't contain the time derivative is non negative. This is essentially due to the fact that if $c \in L^r(\Omega)$ with $r > N/2$, the operator $-\Delta u + cu$ is coercive unless to multiply both sides of the equation by $e^{-\lambda t}$. Indeed, by (3.6) we have that U_ϵ satisfies

$$\begin{aligned}
&\gamma_N \int_0^{|\Omega|} s^{(2/N)-2} \frac{\partial U_\epsilon}{\partial t} U_\epsilon^+ ds + \int_0^{|\Omega|} \left(\frac{\partial U_\epsilon^+}{\partial s} \right)^2 ds \\
&+\gamma_N \int_0^{|\Omega|} \left[\lambda - (c^-)^* \right] s^{(2/N)-2} \left(U_\epsilon^+ \right)^2 ds \leq 0.
\end{aligned} \tag{3.7}$$

Now, if $B_\lambda := \{x : c^- (x) > \lambda\}$ and $B_\lambda^* := \left\{ s : (c^-)^* (s) > \lambda \right\} = [0, |B_\lambda|)$, we notice that

$$\begin{aligned}
&\int_0^{|\Omega|} \left(\frac{\partial U_\epsilon^+}{\partial s} \right)^2 ds + \gamma_N \int_0^{|\Omega|} \left[\lambda - \left(c^-\right)^* \right] s^{(2/N)-2} \left(U_\epsilon^+ \right)^2 ds \\
&\geq \int_0^{|\Omega|} \left(\frac{\partial U_\epsilon^+}{\partial s} \right)^2 ds - \gamma_N \int_{B_\lambda^*} \left[\left(c^-\right)^* - \lambda \right] s^{(2/N)-2} \left(U_\epsilon^+ \right)^2 ds.
\end{aligned} \tag{3.8}$$

If $s \in B_\lambda^*$, it results

$$\begin{aligned}
\left[\left(c^-\right)^* (s) - \lambda \right] s^{2/N} &\leq s^{(2/N)-1} \int_0^s \left[\left(c^-\right)^* (\sigma) - \lambda \right] d\sigma \\
&\leq \left(\int_0^s \left[\left(c^-\right)^* (\sigma) - \lambda \right]^{N/2} d\sigma \right)^{2/N} \\
&\leq \left(\int_0^{|B_\lambda|} \left[\left(c^-\right)^* (\sigma) - \lambda \right]^{N/2} d\sigma \right)^{2/N} = \left\| \left(c^-\right)^* - \lambda \right\|_{L^{N/2}\left(B_\lambda^*\right)}.
\end{aligned}$$

Using this last inequality and the one dimensional Hardy inequality, we obtain

$$\begin{aligned}\int_{B_\lambda^*}\left[\left(c^-\right)^*-\lambda\right]s^{(2/N)-2}\left(U_\epsilon^+\right)^2ds &\le \int_{B_\lambda^*}\left[\left(c^-\right)^*(\sigma)-\lambda\right]s^{2/N}\left(\frac{U_\epsilon^+}{s}\right)^2ds\\ &\le \left\|\left(c^-\right)^*-\lambda\right\|_{L^{N/2}(B_\lambda^*)}\int_{B_\lambda^*}\left(\frac{U_\epsilon^+}{s}\right)^2ds\\ &\le 4\left\|\left(c^-\right)^*-\lambda\right\|_{L^{N/2}(B_\lambda^*)}\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds,\end{aligned}$$

and by (3.8) we deduce

$$\begin{aligned}&\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds+\gamma_N\int_0^{|\Omega|}\left[\lambda-\left(c^-\right)^*\right]s^{(2/N)-2}\left(U_\epsilon^+\right)^2ds\\ &\ge\left[1-4\gamma_N\left\|\left(c^-\right)^*-\lambda\right\|_{L^{N/2}(B_\lambda^*)}\right]\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds.\end{aligned}$$

Then, provided we choose λ sufficiently large, we get

$$\alpha_N:=\left[1-\gamma_N\left\|\left(c^-\right)^*-\lambda\right\|_{L^{N/2}(B_\lambda^*)}\right]>0,$$

so we have

$$\begin{aligned}&\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds+\gamma_N\int_0^{|\Omega|}\left[\lambda-\left(c^-\right)^*\right]s^{(2/N)-2}\left(U_\epsilon^+\right)^2ds\\ &\ge\alpha_N\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds.\end{aligned}\tag{3.9}$$

We notice that we can obtain a similar estimate also in the cases $N=1,2:$ indeed, the case $N=1$ is much simpler, while in the case $N=2$ we find

$$\int_{B_\lambda^*}\left[\left(c^-\right)^*-\lambda\right]s^{-1}\left(U_\epsilon^+\right)^2ds\le 4C\left\|\left(c^-\right)^*-\lambda\right\|_{L^r(B_\lambda^*)}\int_0^{|\Omega|}\left(\frac{\partial U_\epsilon^+}{\partial s}\right)^2ds,$$

for some $r > 1$ and a suitable constant C.

By (3.7) it follows

$$\int_0^{|\Omega|} s^{(2/N)-2} \frac{\partial U_\epsilon}{\partial t} U_\epsilon^+ ds \leq 0 \tag{3.10}$$

for a.e. $t \in (0, \bar{t} + \tau)$. Therefore, by integrating (3.10) between 0 and t with $t \in (0, \bar{t} + \tau)$, and using again the boundary conditions of (3.3) we get

$$0 \geq 2 \int_0^t d\tau \int_0^{|\Omega|} s^{(2/N)-2} \frac{\partial U_\epsilon}{\partial \tau} U_\epsilon^+ ds = \int_0^{|\Omega|} s^{(2/N)-2} \left(\int_0^t \frac{\partial}{\partial \tau} \left(U_\epsilon^+ \right)^2 d\tau \right) ds$$
$$= \int_0^{|\Omega|} s^{(2/N)-2} \left(U_\epsilon^+ \right)^2 (s,t) \, ds$$

which implies $U_\epsilon^+ = 0$ in $[0, |\Omega|]$ for every $t \in (0, \bar{t} + \tau)$. This means that $U_\epsilon \leq 0$ in $[0, |\Omega|] \times [\bar{t}, \bar{t} + \tau)$, and then also $\chi_\epsilon \leq 0$ in $[0, |\Omega|] \times [\bar{t}, \bar{t} + \tau)$, but in the same rectangle the function χ_ϵ is positive. Then $\chi_\epsilon \leq 0$ in $\overline{Q_T^*}$. □

Remark 2. If $N > 2$, we could obtain the result of theorem 1 under the weaker assumption $c \in L^{N/2}(\Omega)$, by using the implicit time discretization scheme. We can take a partition of lenght $\tau = T/n$ $(n \in \mathbb{N})$ of the interval $(0, T)$ and we approximate the solutions u and v of problems (1.2)-(1.6) by the sequences

$$u_n(x,t) := u^{(k)}(x,t) \quad x \in \Omega, \; t \in [(k-1)\tau, k\tau]$$

$$v_n(x,t) := v^{(k)}(x,t) \quad x \in \Omega^{\#}, \; t \in [(k-1)\tau, k\tau]$$

where $u^{(k)}$ is the solution of the elliptic problem

$$\begin{cases} \dfrac{u^{(k)}-u^{(k-1)}}{\tau} - \left(a_{ij}^{(k)}(x)\,u_{x_i}^{(k)}\right)_{x_j} + cu^{(k)} = f^{(k)} & in \;\; \Omega \\ \\ u^{(k)} = 0 & on \;\; \partial\Omega, \end{cases} \tag{3.11}$$

with

$$a_{ij}^{(k)}(x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} a_{ij}(x,t)\,dt,$$

$$f^{(k)}(x) := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(x,t)\,dt$$

for any $k = 1, \ldots, n$ and $u^{(0)} := u_0$, while $v^{(k)}$ is the solution of the symmetrized problem

$$\begin{cases} \dfrac{v^{(k)}-v^{(k-1)}}{\tau} - \Delta v^{(k)} + c_{\#}v^{(k)} = f^{(k)\#} & in \;\; \Omega^{\#} \\ \\ v^{(k)} = 0 & on \;\; \partial\Omega^{\#}, \end{cases} \tag{3.12}$$

with $v^{(0)} := u_0^{\#}$. Using the results of [4] (*see* theorem 3.4) we can prove by induction that

$$\int_0^s u^{(k)*}(\sigma)\,d\sigma \leq \int_0^s v^{(k)*}(\sigma)\,d\sigma \tag{3.13}$$

for $k = 1, \ldots, n$. Actually the results of [4] can be applied in the case $c(x) + \frac{1}{\tau} \in L^{\infty}(\Omega)$, but they can be easily extended to the case $c(x) + \frac{1}{\tau} \in L^{N/2}(\Omega)$, since the operator

$$L^{(k)}u^{(k)} := -\left(a_{ij}^{(k)}(x)\,u_{x_i}^{(k)}\right)_{x_j} + \left(c(x) + \frac{1}{\tau}\right)u^{(k)}$$

is coercive (see [27]). Finally we pass to the limit and get (1.5).

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,89-111,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Integral and Differential Calculus in Riesz spaces and applications

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Abstract

In this paper we outline a new theory about integral and differential calculus for Riesz space-valued mappings defined on suitable Riesz spaces. In our abstract context, we prove some theorems similar to the classical ones, like for example the Fundamental Formula of Calculus and the theorem about exchanging order between limits and derivatives. As applications, we give some results about power series, a fixed point theorem, and some models of differential functional equations.

2000 AMS Mathematics Subject Classification: 28B15, 28B05, 28B10, 46G10.

Keywords: Riesz space, convergence, continuity, differentiability, Taylor formula, series, differential functional equations.

0 Introduction

In this paper a new theory is presented, concerning integral and differential calculus for functions defined in a suitable Riesz space and with values in another Riesz space, linked together with a "product" structure. This approach is, in a certain sense, a generalization of the one given in [1]. The concepts of uniform continuity, uniform differentiability, Riemann integrability are introduced and investigated, and some theorems like the corresponding classical ones are proved: among them we quote the Fundamental Formula of Calculus. Moreover a version of the Taylor formula is demonstrated: here we express the "remainder term" by means of our introduced abstract integral. One can find applications for example in the Itô formula, proved in [2], however we shall not deal with it here. Furthermore, a theory about exchanging order between limits and derivatives, power series and analyticity in our abstract context is given: a fixed point theorem, and some examples of differential and functional equations are then deduced. These equations too might have interesting formulations in the Stochastic Calculus, and some of them also in Theory of Fractals, though we chose not to treat them here.

1 Basic definitions and assumptions

A Riesz space R is said to be *Dedekind complete* if every nonempty subset $A \subset R$, bounded from above, has supremum in R.

From now on, we assume that R is a Dedekind complete Riesz space. Given a bounded sequence $(p_n)_n$ in R, we set:

$$\limsup_n p_n = \inf_{n \in I\!N} [\sup_{m \geq n} p_m]; \quad \liminf_n p_n = \sup_{n \in I\!N} [\inf_{m \geq n} p_m];$$

and we say that $\lim_n p_n = l \in R$ if $\limsup_n p_n = \liminf_n p_n = l$. This corresponds to the classical definition of order convergence or (o)-convergence (see also [5], [6]).

Assumptions 1.1 Let R_1, R_2, R be three Dedekind complete Riesz spaces. We say that (R_1, R_2, R) is a *product triple* if there exists a map $\cdot : R_1 \times R_2 \to R$, which we will call *product*, such that

1.1.1) $(r_1 + s_1) \cdot r_2 = r_1 \cdot r_2 + s_1 \cdot r_2, \quad r_1 \cdot (r_2 + s_2) = r_1 \cdot r_2 + r_1 \cdot s_2,$

1.1.2) $[r_1 \geq s_1,\, r_2 \geq 0] \Rightarrow [r_1 \cdot r_2 \geq s_1 \cdot r_2]$, $[r_1 \geq 0,\, r_2 \geq s_2] \Rightarrow [r_1 \cdot r_2 \geq r_1 \cdot s_2]$ for all $r_j, s_j \in R_j, j = 1, 2;$

1.1.3) if $(a_\lambda)_{\lambda \in \Lambda}$ is any family in R_1 with $a_\lambda \geq 0\ \forall\, \lambda$ and $\inf_\lambda a_\lambda = 0$, and $R_2 \ni b \geq 0$, then $\inf_\lambda (a_\lambda \cdot b) = 0$; if $(b_\lambda)_\lambda$ is any family in R_2 with $b_\lambda \geq 0\ \forall\, \lambda$ and $\inf_\lambda b_\lambda = 0$, and $R_1 \ni a \geq 0$, then $\inf_\lambda (a \cdot b_\lambda) = 0$.

A Dedekind complete Riesz space R is called an *algebra* if (R, R, R) is a product triple.

2 A Riemann-type integral in Riesz spaces

Let (R_1, R_2, R) be a *product triple* of Riesz spaces.

Given two elements a, $b \in R_1$, with $a \leq b$, we denote by $[a, b]$ and call *order interval* (or in short *interval*) the set of all elements $r \in R_1$, such that $a \leq r \leq b$. Given an order interval $[a, b] \subset R_1$, a *division* of $[a, b]$ is any finite set $T = \{x_0, x_1, \ldots, x_n\} \subset [a, b]$, such that $x_0 = a$, $x_n = b$ and $x_i \leq x_{i+1}$, $x_i \neq x_{i+1}$ for all $i = 0, \ldots, n-1$. The *mesh* of a division T is the quantity $\eta(T) = \sup_{i=1}^{n} (x_i - x_{i-1})$.

A *decomposition* of $[a, b]$ is a set $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\}$, where $\{x_0, x_1, \ldots, x_n\}$ is a division T of $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ $\forall\, i = 1, \ldots, n$. For such a decomposition E, we shall put $|E| = \eta(T)$.

We now introduce a Riemann-type integral in our setting, which will be useful in the sequel in order to prove our version of the Taylor formula. If $f : [a, b] \to R_2$ is a map and E is a decomposition of $[a, b]$, $E = \{([x_{i-1}, x_i], \xi_i)\colon\ i = 1, \ldots, n\}$, we denote by $S(f, E)$ and call *Riemann sum* associated with E the element of R given by $\displaystyle\sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f(\xi_i)$. A function $f : [a, b] \to R_2$ is said to be *Riemann integrable* (in short, *integrable*) in $[a,b]$ if there exists an element $Y \in R$ such that

$$\inf_{r \in R_1^+} (\sup\{|S(f,E) - Y| : |E| \leq r\}) = 0,$$

where R_1^+ is the set of all elements $r \in R_1$ such that $r \geq 0$ and $r \neq 0$. In this case we write $\int_a^b f(t)\,dt = Y$.

It is easy to see that such an element Y is uniquely determined.

The following results are easy to prove and will be useful in the sequel.

Proposition 2.1 *If f_1 and f_2 are integrable in $[a,b]$ and α_1, $\alpha_2 \in I\!R$, then $\alpha_1 f_1 + \alpha_2 f_2$ is integrable in $[a,b]$ too, and in this case we have*

$$\int_a^b (\alpha_1 f_1 + \alpha_2 f_2)(t)\,dt = \alpha_1 \int_a^b f_1(t)\,dt + \alpha_2 \int_a^b f_2(t)\,dt.$$

If f_1 and f_2 are integrable in $[a,b]$ and $f_1 \leq f_2$, then

$$\int_a^b f_1(t)\,dt \leq \int_a^b f_2(t)\,dt.$$

If f is integrable in an order interval $[a,b]$, then f is also integrable in any order interval $I \subset [a,b]$.

Thus it follows that for any integrable function $f : [a,b] \to R_2$ the *indefinite integral* is defined as an additive R-valued interval function on the family of all intervals in $[a,b]$. We shall denote it by $F(I) = \int_I f$, with the convention to define $F([b,a]) = -F([a,b])$ $(\forall\, a,b \in R_1,\ a \leq b)$, so that $F([a,a]) = 0\ \forall\, a \in R_1$, and we shall call *integral function* associated with f the map defined (with abuse of notation) as follows:

$$F(x) \equiv F([a,x]), \quad x \in [a,b].$$

Like in the classical case, uniform continuity implies integrability.

Definition 2.2 We say that a function $f : [a,b] \to R_2$ is *uniformly continuous* in $[a,b]$ if

$$\inf_{r \in R_1^+} [\sup\{|f(v) - f(u)| : u, v \in [a,b],\, u \leq v,\, v - u \leq r\}] = 0.$$

Proposition 2.3 *Every uniformly continuous function* $f : [a,b] \to R_2$ *is integrable in* $[a,b]$ *too.*

Proof. The proof is similar to the one in [4]. □

3 An abstract derivative in Riesz spaces

Throughout this section, we always assume that R_1 and R_2 are two Dedekind complete Riesz spaces and that (R_1, R_2, R_2) is a product triple; let now $[a,b] \subset R_1$ be an interval. We begin with the following:

Definition 3.1 We say that a function $f : [a,b] \to R_2$ is *uniformly differentiable* in $[a,b]$ if there exist a bounded function $f' : [a,b] \to R_2$ and an increasing family $(p_r)_{r \in R_1^+}$ such that $\inf_{r \in R_1^+} p_r = 0$ and

$$|f(v) - f(u) - (v-u)\, f'(x)| \leq (v-u)\, p_r \tag{1}$$

for every $r \in R_1^+$ and whenever $u, v, x \in [a,b]$, $u \leq x \leq v$, $v - x \leq r$ and $x - u \leq r$. In this case we say that f' is a *uniform derivative* of f or, when no confusion can arise, that f' is a *derivative* of f.

We observe that, in general, f' is not unique. Indeed, let R_1 and R_2 be the spaces of all bounded measurable real-valued functions, defined on $[0,1]$, vanishing on $[0,1/2]$ and $]1/2,1]$ respectively. For every $\psi_1 \in R_1$ and $\psi_2 \in R_2$, $\psi_1 \cdot \psi_2$ is identically zero (here, $\cdot$ is the usual product between functions): thus, it is not difficult to see that $(R_1, R_2, \{0\})$ is a product triple with respect to this product. Let $[a,b]$ be any arbitrary order interval of R_1, and $f : [a,b] \to R_2$ be any constant function: then clearly *every* function $f_1 : [a,b] \to R_2$ is a derivative of f.

This fact will not affect our results, and it will be clear from the context in which sense we deal with *derivatives.*

For instance, it is quite clear that every function $f : [a,b] \to R_2$, uniformly differentiable in $[a,b]$, is uniformly continuous in $[a,b]$.

Usual differentiation rules hold in our setting, for example:

Proposition 3.2 *Let* (R_1, R_2, R_2), (R_1, S_2, S_2), (R_2, S_2, T_1), (R_1, T_1, T_1) *be four product triples, and* $[a,b] \subset R_1$ *be an interval. If* $f : [a,b] \to R_2$, $g : [a,b] \to S_2$ *are two uniformly differentiable functions with derivatives* f', g' *respectively, then the map* $h = f \cdot g : [a,b] \to T_1$ *is uniformly* *differentiable too, with derivative h' given by $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$, $x \in [a, b]$.*

Therefore every "polynomial" function (in a commutative *algebra* R) is uniformly differentiable, and the usual differentiation rule is valid.

The following results are fundamental theorems of Integral Calculus, as in the classical case. We begin with the following version of the Torricelli-Barrow theorem: the proof is easy.

Theorem 3.3 *Let (R_1, R_2, R_2) be a product triple, and $f : [a, b] \to R_2$ be a uniformly continuous function (in $[a, b]$). Then its integral function F is uniformly differentiable in $[a, b]$ and $F'(x) = f(x)\ \forall\, x \in [a, b]$.*

We now turn to a version of the Fundamental Formula of Integral Calculus in an abstract setting.

Theorem 3.4 *Let (R_1, R_2, R_2) be a product triple, $[a, b] \subset R_1$ be an interval and $f : [a, b] \to R_2$ be a uniformly differentiable function, with derivative f'. Then, f' is integrable, and*

$$\int_a^b f'(t)\, dt = f(b) - f(a).$$

Proof. Choose arbitrarily $r \in R_1^+$ and take any decomposition $E = \{([x_{i-1}, x_i], \xi_i) : i = 1, \ldots, n\}$ of $[a, b]$, with $|E| \le r$. Let $(p_r)_{r \in R_1^+}$ be a family as in the definition of uniform differentiability. We get:

$$\begin{aligned} 0 &\leq \left| \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot f'(\xi_i) - [f(b) - f(a)] \right| \\ &\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) - (x_i - x_{i-1}) \cdot f'(\xi_i)| \\ &\leq \left(\sum_{i=1}^{n} (x_i - x_{i-1}) \right) \cdot p_r = (b-a)\, p_r. \end{aligned}$$

Thus the assertion follows. □

Remark 3.5 We can observe that Theorem 3.4 is true also if the end-points a and b are not comparable, provided they are contained in a larger interval $[A, B]$ where f is uniformly differentiable, and f' is its derivative. In fact, in case $A \leq a, b \leq B$, we can set $h = b - a$, and define

$$\int_a^b f'(t)dt = \int_a^{a+h} f'(t)dt = \int_a^{a+h^+} f'(t)dt - \int_{a+h}^{a+h^+} f'(t)dt : \qquad (2)$$

indeed, as $B - a \geq 0$, from $h = b - a \leq B - a$ it follows $h^+ \leq B - a$, and hence $[a, a + h^+] \subset [A, B]$; moreover, it follows also $[a + h, a + h^+] = [b, a + h^+] \subset [b, B]$. Thus, applying 3.4 to the last member of (2), it follows easily

$$\int_a^b f'(t)dt = f(a + h^+) - f(a) + f(a + h) - f(a + h^+) = f(b) - f(a).$$

4 The Taylor formula

We shall prove a version of the Taylor formula in our context. Besides the obvious applications in approximating functions, this formula has applications in stochastic integration (see [2]).

Definition 4.1 If a function $f : [a,b] \to R_2$ is uniformly differentiable and if its derivative f' is uniformly differentiable with derivative f'', we will say that f'' is a *uniform second derivative* or, when no confusion can arise, *second derivative* of f. By induction it is possible to introduce the (uniform) derivatives of order n for every $n \in I\!N$. If $f : [a,b] \to R_2$ is uniformly differentiable up to the order n, and if its n-th derivative $f^{(n)}$ is uniformly continuous, we say that f is *of class* $C^n([a,b])$. Furthermore, if $S \subset R_1$ contains at least an order interval, we say that $f : S \to R_2$ is *of class* $C^n(S)$ if it is of class $C^n([a,b])$ for every order interval $[a,b] \subset S$, and that $f : S \to R_2$ is *of class* $C^\infty(S)$ if it is of class $C^n(S)\ \forall\, n \in I\!N$.

Theorem 4.2 *Let R be an algebra, $[a,b] \subset R$ be an interval, and $f : [a,b] \to R$ have derivatives up to the order $n+1$: f', f'', $\ldots$, $f^{(n)}$, $f^{(n+1)}$. Fix arbitrarily $x_0 \in [a,b]$ and $h \in R$, such that $x_0 + h \in [a,b]$. Then we have:*

$$f(x_0 + h) = f(x_0) + \frac{h\, f'(x_0)}{1!} + \ldots + \frac{h^n\, f^{(n)}(x_0)}{n!} + B(x_0, h),$$

where $|B(x_0, h)| \leq \dfrac{|h|^{n+1}}{n!} \sup_{x \in [a,b]} |f^{(n+1)}(x)|.$

Proof. Fix x_0 and h as in the hypotheses, and define an auxiliary function $F : [a, b] \to R$ as follows:

$$F(t) = f(x_0 + h) - f(t) - \frac{(x_0 + h - t)\, f'(t)}{1!} - \ldots - \frac{(x_0 + h - t)^n\, f^{(n)}(t)}{n!}.$$

By hypothesis, F is uniformly differentiable and we have, $\forall\, t \in [a, b]$:

$$F'(t) \quad = \quad -\frac{(x_0 + h - t)^n\, f^{(n+1)}(t)}{n!},$$

and F' is bounded. Put $M = \sup_{x \in [a,b]} |f^{(n+1)}(x)|$. By Theorem 3.4 and Remark 3.5 we get:

$$F(x_0) = -\int_{x_0}^{x_0+h} F'(t)\, dt = \int_{x_0}^{x_0+h} \frac{(x_0 + h - t)^n}{n!}\, f^{(n+1)}(t)\, dt$$

$$= \int_{x_0}^{x_0+h^+} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t) dt - \int_{x_0+h}^{x_0+h^+} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t) dt,$$

and hence

$$\begin{aligned} |F(x_0)| \;&\leq\; M \left(\int_{x_0}^{x_0+h^+} \frac{|x_0 + h - t|^n}{n!} dt + \int_{x_0+h}^{x_0+h^+} \frac{|x_0 + h - t|^n}{n!} dt \right) \\ &\leq\; M\, \frac{h^+|h|^n + h^-|h|^n}{n!} = M \frac{|h|^{n+1}}{n!}, \end{aligned}$$

since $|x_0 + h - t| \leq |h|$. Thus the assertion follows. $\square$

5 Sequences of differentiable functions

In this section we give some conditions, under which it is possible to exchange the order between limits and derivatives. First of all we introduce the concept of uniform convergence for sequences of functions. We always suppose that $[a, b] \subset R_1$ is an order interval.

Definition 5.1 A sequence $(f_n : [a, b] \to R_2)_n$ is said to be *uniformly convergent* to $f : [a, b] \to R_2$ if $\lim_n \, [\sup_{t \in [a,\, b]} |f_n(t) - f(t)| \,] = 0$.

We now give two fundamental properties of uniform convergence, which will be useful in the sequel. The proofs are straightforward.

Theorem 5.2 *Let $(f_n : [a, b] \to R_2)_n$ be a sequence of integrable functions, uniformly convergent to a map $f : [a, b] \to R_2$. Then f is integrable and*

$$\lim_n \int_a^b f_n(t)\, dt = \int_a^b f(t)\, dt.$$

Theorem 5.3 *Let $(f_n)_n$ be a sequence of uniformly continuous functions $f_n : [a, b] \to R_2$, uniformly convergent to a mapping $f : [a, b] \to R_2$. Then f is uniformly continuous.*

Thanks to Theorems 3.4, 5.2, 5.3 and 3.3, it is possible to use a classical technique in order to prove the next result.

Theorem 5.4 *Let $(f_n : [a, b] \to R_2)_n$ be a sequence of uniformly differentiable functions, with derivatives f_n', $n \in I\!N$. Moreover, assume that the sequence $(f_n')_n$ is uniformly convergent in $[a, b]$ and that there* *exists* $\lim_n f_n(a)$ *in* R_2*. Then the sequence* $(f_n)_n$ *is uniformly convergent in* $[a,b]$ *to a uniformly differentiable function* $f : [a,b] \to R_2$*, and* $f' = \lim_n f'_n$ *in* $[a,b]$*.*

We recall that, analogously as in the classical case, it is possible to give the concept of series of elements of any Riesz space R and the ones of convergence and absolute convergence, and to deduce an analogue of the Cauchy criterion, together with its usual consequences. Thus, Theorem 5.4 implies the analogue of the classical result concerning differentiation term-by-term of a series of functions. We shall not write it down here, however we shall use it later.

6 Power series and applications

In this section we deal with power series: this will be the main tool in the subsequent applications.

Definition 6.1 Let R be any commutative algebra. We shall suppose that there exists a *multiplicative* unit in R, which will be denoted by $\underline{1}$. For every positive real number k, and for every positive element $r \in R$, we denote by $S_k(r)$ the following subset of R: $S_k(r) = \{x \in R : |x| \leq k \cdot r\}$; moreover, for each positive real number t, we set $U_t(r) = \bigcup_{0<k<t} S_k(r)$, $R_r = \bigcup_{t>0} U_t(r)$. In case $r = \underline{1}$, we shall simply write S_k and U_t rather than $S_k(\underline{1})$ and $U_t(\underline{1})$. A *power series* is a series of the type

$$\sum_{n=0}^{\infty} a_n x^n, \tag{3}$$

with $x \in R$, $a_n \in R \;\forall\, n$, and with the convention $x^0 = \underline{1} \;\forall\, x \in R$.

Proposition 6.2 *If the series (3) converges at some $x \in R$, then it converges uniformly and absolutely in every set $S_k(|x|)$ with $0 < k < 1$.*

Proof. From convergence of the series at x, it follows that the sequence $(|a_n x^n|)_n$ is bounded in R: let M denote any upper bound for that sequence. Now, for any real number $k \in]0,1[$ and every element $r \in S_k(|x|)$, we have $|a_n r^n| \leq |a_n| k^n |x^n| \leq M k^n$ for all positive integers n. This clearly implies the assertion. $\square$

The following results have many consequences: for example they show that some elements in R have an inverse, and give an expression for it.

Proposition 6.3 *The* geometric *series* $\sum_{n=0}^{\infty} x^n$ *absolutely converges in the set U_1; moreover, for every element $x \in U_1$ there exists the* inverse *of $\underline{1} - x$ in the algebra R: such inverse is the sum of the geometric series above.*

Proof. Let us fix $x \in U_1$, and choose any real number $\alpha \in]0,1[$ such that $|x| \leq \alpha \underline{1}$. Then we get $|x^n| \leq \alpha^n \underline{1}$: this clearly implies convergence of the geometric series at x. Moreover, for every positive integer n we have

$(\underline{1} - x)\sum_{j=0}^{n} x^j = \underline{1} - x^{n+1}$. From convergence of the series, we deduce that $\lim_n x^n = 0$, and finally $(\underline{1} - x)\sum_{n=0}^{\infty} x^n = \underline{1}$. $\square$

Proposition 6.4 *The* exponential *series* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges everywhere in the set* $R_{\underline{1}}$*.*

Proof. First of all, we observe that the exponential series obviously converges at the points $\nu\underline{1}$, for every positive integer ν. To conclude the proof, it will suffice to apply Proposition 6.2. $\square$

As usual, the sum of the exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ will be denoted by $\exp(x)$, whenever it exists. Moreover, usual techniques show that this function coincides with its derivative (see also Theorem 6.7 below), and obeys the usual algebraic rules of the exponential function, therefore $\exp(x)$ has always an inverse element, i.e. $\exp(-x)$.

Another consequence involves Taylor series:

Proposition 6.5 *Let* $f : U_t \to R$ *be a function of class* $C^\infty(U_t)$*. A sufficient condition for convergence to* f *of its Mc-Laurin series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{4}$$

is that there exists a positive element $M \in R$ *such that* $|f^{(n)}(x)| \leq M^n \underline{1}$ *holds, for every* $n \in I\!N$ *and every* $x \in U_t$*.*

Proof. The proof is an easy consequence of Theorem 4.2 and of convergence of the exponential series. $\square$

Now, given the power series (3), it is possible to associate to it its "derivative series"

$$\sum_{n=1}^{\infty} n\, a_n x^{n-1}, \quad x \in R. \tag{5}$$

We now prove the following:

Theorem 6.6 *Fix $t > 0$. The series* (3) *converges at every $x \in U_t$ if and only if the series* (5) *converges at every $x \in U_t$.*

Proof. We begin with the "if" part. Fix arbitrarily $x \in U_t$, and let $k \in]0, t[$ be any real number such that $x \in S_k$. By hypotheses and by Proposition 6.2, it follows that the series (5) absolutely converges at x. Now, from $|a_n\, x^n| = |x||a_n\, x^{n-1}| \leq |x|\, |n a_n x^{n-1}|$ it follows that the series (3) converges at x. Concerning the "only if" part, assume that the series (3) converges in U_t, and fix any element $x \in U_t$. Let k be any positive real number, $k < t$, such that $x \in S_k$. Set $k' := \dfrac{k+t}{2}$, $k'' := \dfrac{k'+k}{2}$, so that $k < k'' < k' < t$, and put $x_1 = k'\underline{1}, x_2 = k''\underline{1}$. Clearly, $x_1 \in U_t$ and therefore, thanks to Proposition 6.2, the series (3) converges absolutely at x_1; now, from $|n\, a_n\, x_2^{n-1}| = n\, |a_n|\, (k'')^{n-1}\underline{1} \leq |a_n|\, (k')^n \underline{1}$ (which holds at least definitely), we can deduce convergence of the series (5) at x_2. From Proposition 6.2 there follows convergence of (5) at x. $\square$

A consequence of Theorems 6.6 and 5.4 is the following:

Theorem 6.7 *Fix any positive real number* t. *If the series* (3) *converges* $\forall\, x \in U_t$ *and if* f *is its sum, that is*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \forall\, x \in U_t,$$

then we get

$$f'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \quad \forall\, x \in U_t.$$

Proof. Fix arbitrarily $x \in U_t$ and take an order interval $[-k\underline{1}, k\underline{1}]$ containing x and contained in U_t. Then, by Theorem 6.6, the series (5) converges (absolutely) at every $r \in U_t$, and by 6.2 the series (3) and (5) converge uniformly in $[-k\underline{1}, k\underline{1}]$. The assertion follows from this and term-by-term differentiation. $\square$

Of course, Theorem 6.7 implies that the sum of a power series $\sum_{n=0}^{\infty} a_n x^n$, convergent in U_t, is of class $C^{\infty}(U_t)$, and its Mc-Laurin series coincides with the initial power series.

A first consequence of the previous results is a fixed point theorem, of the type of Banach.

Theorem 6.8 *Let* $f : [a, b] \to [a, b]$ *be any mapping, satisfying*

$$|f(x_2) - f(x_1)| \leq K\ |x_2 - x_1|$$

for a suitable positive element $K \in U_1$, and all $x_1, x_2 \in [a, b]$. Then there exists a unique fixed point s for f. Moreover, s is the limit of every sequence $(s_n)_n$ defined by choosing s_0 arbitrarily in $[a, b]$ and requiring

$$s_{n+1} = f(s_n) \quad \forall\, n \in I\!N \cup \{0\}.$$

Proof. Let us fix arbitrarily $s_0 \in [a, b]$, and define the sequence $(s_n)_n$ as described, by iterations of f. For all $n, p \in I\!N$ we get

$$|s_{n+p} - s_n| \leq |s_1 - s_0| K^n (1 - K)^{-1}$$

by means of usual techniques. As $\lim_n K^n = 0$, the sequence $(s_n)_n$ is clearly convergent, and its limit s satisfies $f(s) = s$ thanks to continuity of f. Uniqueness can be proved by similar techniques. $\square$

As a remark, we can observe that a mapping $f : [a, b] \to [a, b]$ satisfies the contraction condition of 6.8 as soon as f is uniformly differentiable, and its derivative satisfies $|f'(x)| \leq K$ for all $x \in [a, b]$: this follows easily from Theorem 4.2.

Further applications allow us to obtain solutions of suitable functional equations, according with the following theorems. Though these equations are nothing but examples, from them one could find wide generalizations and also formulations in different contexts, for example in Stochastic Analysis, by specializing the underlying Riesz space.

Theorem 6.9 *Assume that* $\kappa(u) = \sum_{n=1}^{\infty} \gamma_n u^n$ *is a fixed convergent power series, with* $u \in R$ *and* $\gamma_n \in R \ \ \forall n \in I\!N$. *Then, for every element* $s \in U_1$, *there exist functions* $Y : R \to R$ *such that* $Y(u) - \kappa(u) = Y(su)$, *for all* $u \in R$.

Proof. We note that the given series converges absolutely at every $u \in R$ (see Proposition 6.2). Moreover, as $s \in U_1$, also $s^n \in U_1$ for every positive integer n, and $(\underline{1} - s^n)^{-1}$ exists, thanks to 6.3. Consider the following power series:

$$Y(u) = \sum_{n=1}^{\infty} \gamma_n (\underline{1} - s^n)^{-1} u^n.$$

From Proposition 6.3 we deduce:

$$|\gamma_n (\underline{1} - s^n)^{-1} u^n| \leq |\gamma_n u^n| \sum_{j=0}^{\infty} |s|^j = (\underline{1} - |s|)^{-1} |\gamma_n u^n| \quad \forall n \in I\!N.$$

By means of the Cauchy criterion, we then obtain (absolute) convergence of $Y(u)$ for every $u \in R$. Now,

$$Y(u) - Y(su) = \sum_{n=1}^{\infty} (\underline{1} - s^n) \gamma_n u^n (\underline{1} - s^n)^{-1} = \kappa(u)$$

for every $u \in R$, thus the equation is fulfilled. Clearly, if Y is any solution, also $Y + a$ is a solution, for every constant element $a \in R$. $\square$

A slightly different equation does not require analyticity of κ.

Existence and uniqueness of the solution could be deduced from some modifications of Theorem 6.8, but we have chosen a more direct approach.

Theorem 6.10 *Assume that $\kappa : R \to R$ is any bounded function, and fix two elements in R, α, β, with $\alpha \in U_1$. Then there is a unique bounded function $Y : R \to R$ satisfying $Y(u) - \kappa(u) = \alpha\, Y(\beta\, u)$, for all $u \in R$.*

Proof. Set $Y(u) = \sum_{n=0}^{\infty} \kappa(\beta^n u)\ \alpha^n$. Clearly Y is well-defined, because κ is bounded. Moreover, Y is bounded and satisfies $\alpha\, Y(\beta u) = Y(u) - \kappa(u)$, as required. Now, assume that another bounded function $Y_1 : R \to R$ exists, satisfying the same equation. Then we must have $\kappa(u) = Y_1(u) - \alpha Y_1(\beta u)$ and therefore

$$\begin{aligned} Y(u) &= \sum_{n=0}^{\infty} \kappa(\beta^n u)\alpha^n = \sum_{n=0}^{\infty} (Y_1(\beta^n u) - \alpha Y_1(\beta^{n+1} u))\alpha^n \qquad (6) \\ &= Y_1(u) - \sum_{n=1}^{\infty} \alpha^n Y_1(\beta^n u) + \sum_{n=1}^{\infty} \alpha^n Y_1(\beta^n u) \end{aligned}$$

for all $u \in R$ (absolute convergence of the series being ensured by boundedness of Y_1). The conclusion is now obvious. $\square$

We remark that the function Y here obtained is a generalization of the so-called *Weierstrass functions*, which are continuous but nowhere differentiable, and have self-similarity features. We also notice that uniqueness in the previous theorem is strictly related to boundedness of the function Y: if we drop such condition, there may exist many different solutions. For example, let us assume $\kappa = 0, \beta = 2\ \underline{1},\ \alpha = \frac{1}{2}\ \underline{1}$; the equation then reduces to $Y(2u) = 2Y(u)$: the solution given by Theorem 6.10 is identically 0, but every function of the type $Y_1(u) = ru$ clearly is a solution (though unbounded), for every constant $r \in R$.

Finally, we turn to some kind of differential functional equation. For the sake of simplicity, we shall deal with a very particular type of equation, as described in the following theorem.

Theorem 6.11 *Fix any positive element $s \in R_{\underline{1}}$. Then there exist non-trivial differentiable functions $Y : R_{\underline{1}} \to R$ satisfying the equation:*

$$Y'(u) = Y(s\,u) \tag{7}$$

for all $u \in R_{\underline{1}}$.

Proof. First of all, let us observe that the space of solutions is a linear one. Next, put

$$Y(u) = \sum_{n=0}^{\infty} \frac{s^{(n^2-n)/2}}{n!} u^n, \quad u \in R_{\underline{1}} :$$

it is not difficult to deduce convergence of the series in $R_{\underline{1}}$ and that the function Y obtained in this way is non-trivial and satisfies (7). $\square$

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.1,113-121,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Random fixed point theorems for uniformly Lipschitzain and asymptotically regular random operators*

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Abstract. Let (Ω, Σ) be a measurable space, X a Banach space, C a weakly compact convex subset of X and $T : \Omega \times C \to C$ a random operator. We prove the random version of a deterministic fixed point theorem when T is uniformly Lipschitzian random operators and satisfies property P such that $\sigma(T(\omega, \cdot)) \leq \sqrt{WCS(X)}$ for all $\omega \in \Omega$, and T is asymptotically regular on C. Our results also give stochastic version generalization of some results of Domínguez Benavides and Xu [A new geometrical coefficient for Banach spaces and its applications in fixed point theory, Nonlinear Anal. 25 No. 3 (1995), 311-325.].

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2000 Mathematics Subject Classification: 47H10, 47H09, 47H04.

Key words and phrases: random fixed point, uniformly Lipschitzian mapping, Lifschititz characteristic, random asymptotically regular.

*Supported by The Thailand Research Fund under grant BRG49800018/2549.

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1 Introduction

The study of random fixed point theorems was initiated by the Prague school of probability in the 1950s. Random operator theory is needed for the study of various classes of random equations (see [9] and references therein). Random fixed point theory has received much attention for the last two decades because of its importance in probabilistic functional analysis; the reader is referred to Beg and Shahzad [2], Shahzad and Latif [11] and Tan and Yaun [12]. Generalizations of the random fixed point theorems for continuous selfmaps to the case of non-selfmaps have been considered by many authors (see e.g. Beg et al. [2], and Shahzad and Latif [11]). On the other hand, the first fixed point theorem for uniformly Lipschitzian mapping in Banach spaces was given by Goebel and Kirk [8] who state a relationship between the existence of fixed point for uniformly Lipschitzian mappings and clarkson modulus of convexity.

In [3], Casini and Maluta prove the existence of fixed points of uniformly k-Lipschitzian mapping T with $k < \sqrt{N(X)}$ in a space X with uniform normal structure. (N(X) is the normal structure coefficient of X.) In 1995, Benavides and Xu [7] prove the existence of fixed point of a uniformly Lipschitzian mapping T such that the Lipschitz's constant $\sigma(T) < \sqrt{WCS(X)}$ and $WCSX > 1$. In 1996, Xu [14] gave the random version of Theorem 3.1 of Casini-Maluta [3] for uniformly Lipschitzian mappings.

The main goal of this paper is to establish some random fixed point theorems for Uniformly Lipschitzian and asymptotically regular operators. We will prove the random fixed point theorems for nonlinear uniformly Lipschitzian mappings in the frame work of a Banach space with $WCS(X) > 1$.

2 Preliminaries and notations

Through this paper we will consider a measurable spaces (Ω, Σ) (where Σ is a $\sigma-$algebra of subset of Ω) and (X, d) will be a metric spaces. We denote by $CL(X)(resp.CB(X), KC(X))$ the family of all nonempty closed (resp. closed bounded, compact) subset of X.

A set-valued operator $T : \Omega \to 2^X$ is call $(\Sigma)-$ measurable if, for any open subset B of X,

$$T^{-1}(B) := \{\omega \in \Omega : T(\omega) \cap B \neq \emptyset\}$$

belongs to Σ. A mapping $x : \Omega \to X$ is said to be a *measurable selector* of a measurable set-valued operator $T : \Omega \to 2^X$ if $x(\cdot)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. Let M be a nonempty closed subset of X. An operator $T : \Omega \times M \to 2^X$ is call a random operator if, for each fixed $x \in M$, the operator $T(\cdot, x) : \Omega \to 2^X$ is measurable. We will denote by $F(\omega)$ the fixed point set of $T(\omega, \cdot)$, i.e.,

$$F(\omega) := \{x \in M : x \in T(\omega, x)\} .$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, \cdot) : M \to 2^X, F(\omega)$ may be empty. A measurable operator $x : \Omega \to M$ is said to be a *random fixed point of a operator* $T : \Omega \times M \to 2^X$ if $x(\omega) \in T(\omega, x(\omega))$ for all $\omega \in \Omega$. Recall that $T : \Omega \times M \to 2^X$ is continuous if, for each fixed $\omega \in \Omega$, the operator $T : (\Omega, \cdot) \to 2^X$ is continuous.

Let C be a closed bounded convex subset of a Banach spaces X. A random operator $T : \Omega \times C \to C$ is said to be *nonexpansive* if, for fixed $\omega \in \Omega$ the map $T : (\omega, \cdot) \to C$ is nonexpansive. We will say that T is *uniformly Lipschitzian* if there exists a function $k : \Omega \to [1, +\infty)$ such that

$$\|T^n(\omega, x) - T^n(\omega, y)\| \le k(\omega)\|x - y\|$$

for all $x, y \in C$ and for each integer $n \ge 1$. Here $T^n(\omega, x)$ is the valued at x of the nth iterate of the map $T(\omega, \cdot)$. We will say that T is *asymptotically nonexpansive* if there exists a sequence of function $k_n : \Omega \to [1, +\infty)$ such that for each fixed $\omega \in \Omega, \lim_{n\to\infty} k_n(\omega) = 1$ and

$$\|T^n(\omega, x) - T^n(\omega, y)\| \le k_n(\omega)\|x - y\|$$

for all $x, y \in C$ and integer $n \ge 1$. The nonexpansive random map T is called *asymptotically regular* if for each $x \in K$,

$$\lim_{n\to\infty} \|T^{n+1}(\omega, x) - T^n(\omega, x)\| = 0$$

for each $\omega \in \Omega$.

Now recall the weakly convergent sequence coefficient $WCS(X)$ [7] of X is defined by

$$WCS(X) = \inf \left\{ \frac{A(\{x_n\})}{\inf_{y\in \bar{co}\{x_n\}} \limsup_{n\to\infty} \|x_n - y\|} : \{x_n\} \text{ is a weakly convergent sequence which is not norm-convergent} \right\},$$

where $A(\{x_n\}) = \limsup_{n\to\infty}\{\|x_i - x_j\| : i, j \ge n\}$ is the asymptotic diameter of $\{x_n\}$. We will use next relationship between the asymptotically center of a sequence and the characteristic of convexity of the space. Let C be a nonempty bounded closed subset of Banach spaces X

and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf \{r(x, \{x_n\}) : x \in C\}, \ where\{r(x, \{x_n\})\} = \limsup_n \|x_n - x\|, \\ A(C, \{x_n\}) &= \{x \in C : \{r(x, \{x_n\})\} = r(C, \{x_n\})\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Definition 2.1. *Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.*

We are going to list several result related to the concept of measurability which will be used repeatedly in next section.

Theorem 2.2. (cf. Wagner [13]). *Let (X, d) be a complete separable metric spaces and $F : \Omega \to CL(X)$ a measurable map. Then F has a measurable selector.*

Theorem 2.3. (cf. Tan and Yuan [12]). *Let X be a separable metric spaces and Y a metric spaces. If $f : \Omega \times X \to Y$ is a measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(\cdot, x(\cdot)) : \Omega \to Y$ is measurable.*

Follows form the separability of C and form Theorem 1.2 of Bharucha-Reid's book [**?**], we can easily prove the following proposition.

Proposition 2.4. *Let C be a closed convex separable subset of a Banach space X and (Ω, Σ) be a measurable space. Suppose $f : \Omega \to C$ is a function that is w-measurable, i.e., for each $x^* \in X^*$, the dual space of X, the numerically-valued function $x^*f : \Omega \to (-\infty, \infty)$ is measurable, then f is measurable.*

Theorem 2.5. (Benavidel, Lopez and Xu cf.[5]). *Suppose C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ a measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by*

$$r(\omega) := \inf_{x \in F(x)} f(\omega, x)$$

and the marginal map. $R : \Omega \to X$ defined by

$$R(\omega) := \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

Proposition 2.6. (Xu cf.[14]) *Let M be a separable metric space and $f : \Omega \times C \to \mathbb{R}$ be a Carathéodory map, i.e., for every $x \in M$, then the map $f(\cdot, x) : \Omega \to \mathbb{R}$ is measurable and for every $\omega \in \Omega$, the map $f(\omega, \cdot) : M \to \mathbb{R}$ is continuous. Then for any $s \in \mathbb{R}$, the map $F_s : \Omega \to M$ defined by*

$$F_s(\omega) = \{x \in M : f(\omega, x) < s, \quad \omega \in \Omega\}$$

is measurable.

Let M be a bounded convex sunset of a Banach space X. We recall that the Lifschitz characteristic for asymptotically regular mappings, is defined;

(i) A number $b \geq 0$ is said to have property (P_ω) with respect to M if there exists some $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| > r$ and each weakly convergent sequence $\{\xi_n\} \subset M$ for which $\limsup \|\xi_n - x\| \leq ar$ and $\limsup \|\xi_n - y\| \leq br$, there exists some $z \in M$ such that $\liminf \|\xi_n - z\| \leq r$;

(ii) $\kappa_\omega(M) = \sup\{b > 0 : b \text{ has property } (P_\omega) \text{ w.r.t. } M\}$;

(iii) $\kappa_\omega(X) = \inf\{\kappa_\omega(m) : M \text{ as above}\}$.

If S is a mapping from a set C into itself, then we use the symbol $|S|$ to denote the Lipscitz constant of S, i.e.

$$|S| = \sup\left\{\frac{\|Sx - Sy\|}{\|x - y\|} : x, y \in C, x \neq y\right\}.$$

For a mapping T on C, we set

$$\sigma(T) = \liminf_{n \to \infty} |T^n|.$$

A random operator $T(\omega, \cdot)$ on C has *property* (P) if there exists subsequence $\{T^{n_j}(\omega, \cdot)\}$ of $\{T^n(\omega, \cdot)\}$ converges uniformly to $\liminf_{n \to \infty} |T^n(\omega, \cdot)|$.

3 The Main results

In the framework of random nonexpansive operators Domínguez Benavides and Xu [7]. proved the following result:

Theorem 3.1. *Let C be a nonempty weakly compact convex separable subset of a Banach space with $WCS(X) > 1$ and $T : \Omega \times C \to C$ be a uniformly Lipschitzian random operator and $T(\omega, \cdot)$ has property (P), such that $\sigma(T(\omega, \cdot)) \leq \sqrt{WCS(X)}$ for all $\omega \in \Omega$. Suppose in addition that T is asymptotically regular on C. Then T has a random fixed point.*

Proof. It is easy to see that (cf. [7]),

$$WCS(X) = \sup\left\{M > 0 : M \cdot \limsup_{n\to\infty} \|x_n - x_\infty\| \leq D\{x_n\}\right\},$$

where the supremum is taken over all weakly (not strongly) convergent sequence $\{x_n\}$ in X and x_∞ is the weak limit of $\{x_n\}$ and $D\{x_n\} = \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\|$.

Fixed $x_0 \in C$, and consider the measurable function $x_0(\omega) = x_0$. Define a map $G_1 : \Omega \to CB(C)$ by

$$G_1(\omega) := w - cl\{T^n((\omega), x_0)\}, \quad \omega \in \Omega.$$

(Here $w - cl$ denote the closure under the weak topology of X.) Then $G_1 : \Omega \to CB(C)$ is w-measurable. By Lemma 2.2, G_1 has a w- measurable selector $x_1 : \Omega \to C$. Since C is separable $\{x_1\}$ is actually measurable by Proposition 2.4. By the definition of G_1, we note that $x_1(\omega)$ is a weak cluster point of $\{T^n((\omega), x_0)\}$ for each $\omega \in \Omega$. Hence, for a fix $\omega \in \Omega$, there exists a subsequence $\{n_{k(1)}\}$ of positive integer $\{n\}$ such that $\{T^{n_{k(1)}}((\omega), x_0)\}$ converging weakly to $x_1(\omega)$ and satisfies

$$\sigma(T(\omega, \cdot)) \leq \sigma(T(\omega, \cdot)) < \sqrt{WCS(X)}.$$

Next, define a map $G_2 : \Omega \to C$ by

$$G_2(\omega) := w - cl\{T^n((\omega), x_1)\}, \quad \omega \in \Omega.$$

Then $G_2 : \Omega \to C$ is w- measurable. By Lemma 2.2, G_2 has a w- measurable selector $x_2 : \Omega \to C$. Since C is separable $\{x_2\}$ is actually measurable by Proposition 2.4. Since for each $\omega \in \Omega$, $x_2(\omega)$ is a weak cluster point of $\{T^n((\omega), x_1)\}$. In fact, by definition of G_2, for a fix $\omega \in \Omega$, we have a subsequence $\{T^{n_{k(2)}}((\omega), x_1)\}$ of $\{T^n((\omega), x_1)\}$ converging weakly to $x_2(\omega)$.

By induction, for each $m \geq 1$ we construct $G_m(\omega) := w - cl\{T^n((\omega), x_{m-1})\}, \quad \omega \in \Omega$. Then $G_m : \Omega \to C$ is w- measurable. It follows again from Lemma 2.2, that G_m has a w-measurable selector x_m which measurable by Proposition 2.4. By definition of G_m, for a fix $\omega \in \Omega$, we have a subsequence $\{T^{n_{k(m)}}((\omega), x_{m-1}(\omega))\} \rightharpoonup x_m(\omega)$ for some $\{n_j\}$ of $\{n\}$. That is we can construct a sequence $\{x_m\}$ of measurable function $x_m : \Omega \to C$ such that for each

$\omega \in \Omega$ and integer $m \geq 1$,

$$x_m(\omega) = w - \lim_{j \to \infty} T^{n_j}(\omega, x_{m-1}(\omega)),$$

where $\{n_j\} := \{n_{k(m)}\}$, and for each $\omega \in \Omega$.
Note that the asymptotic regular of $T(\omega . \cdot)$ we have

$$x_m(\omega) = w - \lim_{j \to \infty} T^{n_j+p}(\omega, x_{m-1}(\omega)), \quad \forall p \geq 0.$$

We now show that $\{x_m(\omega)\}$ converges strongly to foxed point of T. Set for each integer $m \geq 1$,

$$B_m(\omega) = \limsup_j \|T^{n_j}(\omega, x_m(\omega)) - x_{m+1}(\omega)\|, \;\; L_m(\omega) = |T^{n_m}(\omega)|$$

for each $\omega \in \Omega$ and set

$$\alpha = \frac{[\sigma(T)]^2}{WCS(X)}.$$

Then $\alpha < 1$ and by the above definition of $WCS(X)$, for each $\omega \in \Omega$, we have

$$B_m(\omega) \leq \frac{1}{WCS(X)} D(\{T^{n_j}(\omega, x_m(\omega))\})$$

However, since $T(\omega, \cdot)$ have property P and from the w-lower semicontinuous of the norm of X, it follows that

$$\begin{aligned}
D(\{T^{n_j}(\omega, x_m(\omega))\}) &= \limsup_j \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))\| \\
&= \limsup_j \limsup_i \|T^{n_i+n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))\| \\
&\leq \limsup_j |T^{n_j}(\omega) \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_m(\omega)\| \\
&= \lim_{n_j} L_j(\omega) \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_m(\omega)\| \\
&\leq \sigma(T(\omega, \cdot)) \limsup_i (\liminf_j \|T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_{m-1}(\omega))\|) \\
&\leq \sigma(T(\omega, \cdot))(\limsup_i L_i(\omega)) \limsup_j \|x_m(\omega) - T^{n_j}(\omega, x_{m-1}(\omega))\| \\
&= [\sigma(T(\omega, \cdot))]^2 B_{m-1}(\omega).
\end{aligned}$$

We, therefore conclude that

$$B_m(\omega) \leq \frac{[\sigma(T(\omega, \cdot))]^2}{WCS(X)} B_{m-1}(\omega) \leq \alpha B_{m-1}(\omega).$$

Now using the w-lower semicontinuous of the norm of X again, we deduce that

$$\begin{aligned}
\|x_m(\omega) - x_{m+1}(\omega)\| &\leq \limsup_i \|x_m(\omega) - T^{n_i}(\omega, x_m(\omega))\| \\
&\quad + \limsup_i \|T^{n_i}(\omega, x_m(\omega)) - x_{m+1}(\omega)\| \\
&\leq \limsup_i \limsup_j \|T^{n_j}(\omega, x_{m-1}(\omega)) - T^{n_i}(\omega, x_m(\omega))\| + B_m(\omega) \\
&\leq \limsup_i |T^{n_j}| \limsup_j \|T^{n_j}(\omega, x_{m-1}(\omega)) - x_m(\omega)\| + B_m(\omega) \\
&= \sigma(T(\omega, \cdot)) B_{m-1}(\omega) + B_m(\omega).
\end{aligned}$$

This implies that $\{x_m(\omega)\}$ is Cauchy sequence for each $\omega \in \Omega$. For any $\omega \in \Omega$, let $x(\omega) = \lim_{m\to\infty} x_m(\omega)$. We will that $x(\omega)$ is a random fixed point of T. Indeed, for each $j \geq 1$ we have

$$\begin{aligned}
\|x(\omega) - T^{n_j}(\omega, x(\omega))\| &\leq \|x(\omega) - x_{m+1}(\omega))\| + \|x_{m+1}(\omega) - T^{n_j}(\omega, x_m(\omega))\| \\
&\quad + \|T^{n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x(\omega))\| \\
&\leq \|x(\omega) - x_{m+1}(\omega))\| + \|x_{m+1}(\omega) - T^{n_j}(\omega, x_m(\omega))\| \\
&\quad + |T^{n_j}|\|x_m(\omega) - x(\omega)\|.
\end{aligned}$$

Taking the upper limit as $j \to \infty$ yields

$$\limsup_j \|x(\omega) - T^{n_j}(\omega, x(\omega))\| \leq \|x(\omega) - x_{m+1}(\omega)\| + B_m(\omega) + \sigma(T(\omega, \cdot))\|x_m(\omega) - x(\omega)\|$$

which implies $T^{n_j}(\omega, x(\omega)) - x(\omega) \to 0$ as $m \to \infty$. Since $T(\omega, \cdot)$ is continuous and asymptotic regular, it follows that $x(\omega) = T(\omega, x(\omega))$. Observe that $x(\omega)$ is the limit of measurable mappings, so it is measurable. Hence $x(\omega)$ is a random fixed point of T. This completes the proof. □

Remark 3.2. *Theorem 3.1 is stochastic version of the Theorem 3.2 of Domínguez Benavides and Xu in [7]. for uniformly Lipschitzian and asymptotically regular mappings.*

Acknowledgement. The authors would like to thanks The Thailand Research Fund for financial support.

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Volume 3,Number 2 April 2008

ISSN:1559-1948 (PRINT), 1559-1956 (ONLINE)

EUDOXUS PRESS,LLC

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

SCOPE AND PRICES OF

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

A quartely international publication of **EUDOXUS PRESS,LLC**
ISSN:1559-1948(PRINT),1559-1956(ONLINE)

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Journal of Applied Functional Analysis(JAFA)
is published in January,April,July and October of each year by

EUDOXUS PRESS,LLC,
1424 Beaver Trail Drive,Cordova,TN38016,USA,
Tel.001-901-751-3553
anastassioug@yahoo.com
http://www.EudoxusPress.com visit also http://www.msci.memphis.edu/~ganastss/jafa.

Webmaster:Ray Clapsadle

Annual Subscription Current Prices:For USA and Canada,Institutional:Print $250,Electronic $220,Print and Electronic $310.Individual:Print $77,Electronic $60,Print &Electronic $110.For any other part of the world add $25 more to the above prices for Print.
Single article PDF file for individual $8.Single issue in PDF form for individual $25.

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JAFA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.2,139-162,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Multi-anisotropic Gevrey classes and ultradistributions

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Abstract

We consider a relevant generalization of the standard Gevrey classes, the so-called multi-anisotropic spaces, defined in terms of a given complete polyhedron. With respect to the previous literature on the subject, we concentrate here in the study of the topology. It is defined as inductive and projective limit of Banach spaces, in two equivalent ways, based on the estimates on the derivatives and on the Fourier transform, respectively. We consequently introduce the dual space, the class of the multi-anisotropic ultradistributions, of which we give different characterizations, study topological and algebraic properties and present some applications.
AMS Subject Classification (MSC 2000): 46F05, 46E10, 35A99.

Key Words: Generalized Gevrey functions, ultradistributions, inductive and projective limits of Banach spaces, complete polyhedra.

1 Introduction

The standard Gevrey classes play an important role in the study of partial differential equations as intermediate setting between the C^∞ and the analytic spaces. In particular, whenever the behavior of a differential operator (for instance local solvability and well-posedness of the Cauchy problem) is different in the C^∞ and analytic frame, it is natural to study the problem in Gevrey classes. Here, we deal with Roumieu type Gevrey classes, cf. [29, 30]; the classes of Beurling type, defined in [2] (cf. also [20]), have a parallel study.

Let us recall the definition of the standard Gevrey classes, in order to pass then to their generalizations. *Let Ω be an open subset of $\mathbb{R}^n$ and $s \in \mathbb{R}$, $s \geq 1$. We say that a function f belongs to the Gevrey class $G^s(\Omega)$ if $f \in C^\infty(\Omega)$ and for every compact subset K of Ω there is a constant $C > 0$ such that:*

$$\sup_{x \in K} |D^\alpha f(x)| \leq C^{|\alpha|+1} |\alpha|^{s|\alpha|}, \quad \forall \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n. \tag{1}$$

Observe that $G^1(\Omega)$ is the set of the analytic functions in Ω and that $G^s(\Omega) \subsetneq C^\infty(\Omega)$ for any $s \geq 1$. A useful characterization of the Gevrey functions with compact support $G^s_0(\mathbb{R}^n) = G^s(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n)$, for $s > 1$, is expressed by a decay estimate of the Fourier transform; namely the following result holds (cf. f.i. [28]).

A compactly supported distribution $f \in \mathcal{E}'(\mathbb{R}^n)$ belongs to $G^s_0(\mathbb{R}^n)$ if and only if there are two constants $C, \varepsilon > 0$ such that the Fourier transform of f, $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx$, satisfies

$$|\hat{f}(\xi)| \leq C \exp(-\varepsilon \langle \xi \rangle^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n, \tag{2}$$

where $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$.

The Gevrey classes $G^s(\Omega)$, $G^s_0(\Omega)$ can be endowed with natural locally convex topologies and the topological dual spaces, called *ultradistributions*, can be constructed, cf. for instance [20].

In literature, two kinds of generalizations of Gevrey classes were introduced, proceeding from the estimates on the derivatives (1) or the decay of the Fourier transform (2). The first approach consists of replacing $|\alpha|^{s|\alpha|}$ in (1) with a more general quantity M_α, depending on the multi-index α and obeying suitable conditions related to the properties of the corresponding function spaces. This case, usually referred to as *ultradifferentiable classes*, was analyzed by Roumieu in [30], under general hypotheses on the sequence $\{M_\alpha\}$. When M_α depends on α only through its length $|\alpha| := \alpha_1 + \cdots + \alpha_n$, the related Roumieu spaces coincide with the classes $\mathcal{E}^{\{M_p\}}(\Omega)$, $\mathcal{D}^{\{M_p\}}(\Omega)$, widely studied by many authors, among which Roumieu [29], Lions-Magenes [23], Mandelbrojt [24], Komatsu [20], Braun-Meise-Taylor [4], Rudin [31] and Matsumoto [25]. The second approach was followed by Björck [3], Liess-Rodino [22] and Calvo-Morando-Rodino [10], where the inhomogeneous Gevrey classes $G^{s,\lambda}(\Omega)$ are introduced by replacing in (2) the weight $\langle \xi \rangle$ with a function $\lambda(\xi)$ fulfilling suitable conditions.

The present paper is devoted to an important extension of the standard Gevrey classes and their dual spaces: the *multi-anisotropic Gevrey functions and ultradistributions*, defined by a generalization of the estimates (1) on the derivatives based on the notion of multi-quasi-elliptic or complete polyhedra introduced by Friberg [13], Pini [27] and Barozzi [1] in relation with hypoellipticity. They include, for particular choices of the polyhedron, the standard Gevrey classes and the anisotropic Gevrey classes studied, for instance, in [11], [14], [36]. An equivalent characterization via Fourier transform shows that the multianisotropic Gevrey functions and ultradistributions are a particular case of the inhomogeneous case of [3], [10], [22], for a suitable choice of the weight function. Comparing with the more general inhomogeneous Gevrey classes, our case allows a wider study, more direct proofs and more precise characterizations, as we can proceed with estimates on the derivatives.

We observe that, in general, the sequence M_α related to the multi-anisotropic Gevrey spaces cannot be written in terms of the length of α, and therefore they cannot be recovered by the theory of the classes $\mathcal{E}^{\{M_p\}}(\Omega), \mathcal{D}^{\{M_p\}}(\Omega)$; neither they fit in the frame of Roumieu [30]. In fact, the Roumieu ultradifferentiable classes (providing the test functions) are assumed to be rings of multiplication (cf. [30], Theorem 4), that is not true in our case (cf. Proposition 3.4), through very restrictive conditions on the sequence $\{M_\alpha\}$. However, the multi-anisotropic spaces keep the main features displayed by the Roumieu ultradistributions.

The multi-anisotropic Gevrey classes have many applications in the study of partial differential equations, among which we mention the works of Corli [12], Gindikin-Volevich [14], Hakobyan-Markaryan [16] and Kazharyan [18] regarding the hypoellipticity, Bouzar-Chaili [5, 6], Calvo-Hakobyan [9] and Zanghirati [35] for the iterates, Calvo [7, 8] for hyperbolic problems. The dual spaces, the multi-anisotropic ultradistributions, also have applications in the study of partial differential equations. In fact, all the results concerning hypoellipticity and iterates of a multi-quasi-elliptic operator ([1], [5], [6], [9], [12], [13], [14], [16], [18], [27], [35], [36]) can be reformulated by replacing the class of the Schwartz distributions with the space $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ corresponding to the Newton polyhedron $\mathcal{P}$ of the operator; moreover, we study in this frame the well-posedness of the Cauchy problem for weakly hyperbolic operators (cf. Theorem 4.3).

The paper is planned as follows.

In Section 2 we recall the notion of complete polyhedron in $\mathbb{R}^n$ and collect some related definitions and properties.

In Section 3 we introduce the multi-anisotropic Gevrey classes $G^{s,\mathcal{P}}(\Omega)$, $s \geq 1$, and $G_0^{s,\mathcal{P}}(\Omega)$, $s > 1$. In analogy with the standard Gevrey case, we can endow these spaces with locally convex topologies, defined as inductive and projective limits of Banach spaces. On the other hand, we can also characterize their topology in an equivalent way, through the behavior of the Fourier transform, that coincides with the one of the inhomogeneous Gevrey classes (cf. [10]). This allows also to see that the topology of the standard Gevrey classes (cf. f.i. [20] and [28]) can be equivalently formulated in terms of the Fourier transform, and therefore related to the topology of the inhomogeneous Gevrey classes. We then study the algebraic and topological properties of the operations.

In Section **??** we construct the topological dual of $G_0^{s,\mathcal{P}}(\Omega)$, the space of multi-anisotropic ultradistributions $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$, and of $G^{s,\mathcal{P}}(\Omega)$, the space of compactly supported multi-anisotropic ultradistributions

$\mathcal{E}'_{s,\mathcal{P}}(\Omega)$, of which we present different characterizations. In particular, the latter can be defined also by an exponential-type estimate on the Fourier transform or on the Fourier-Laplace transform (cf. Theorems 4.1 and 4.2); the latter gives a version of the Paley-Wiener-Schwartz theorem for distributions in our frame. We study the topological and algebraic properties of the multi-anisotropic ultradistributions, and show under which conditions we can set in these spaces a theory of linear partial differential operators with variable coefficients. As an application, we deal with the Cauchy problem, showing the well-posedness in the set of multi-anisotropic ultradistributions for a class of weakly hyperbolic operators (a generalization of the s-hyperbolic operators of Larsson [21]), called multi-quasi-hyperbolic and modeled in such a way to have well-posedness in the multi-anisotropic Gevrey classes (cf. [7]).

2 Complete polyhedra

To introduce our study of multi-anisotropic Gevrey functions and ultradistributions, we start by describing complete polyhedra and some related properties. Let $\mathcal{P}$ be a convex polyhedron in $\mathbb{R}^n$, then $\mathcal{P}$ can be obtained as convex hull of a finite set $\mathcal{V}(\mathcal{P}) \subset \mathbb{R}^n$ of convex-linearly-independent points, called the vertices of $\mathcal{P}$ and uniquely determined by $\mathcal{P}$. Moreover, if $\mathcal{P}$ has non-empty interior and the origin belongs to $\mathcal{P}$, there is a finite set $\mathcal{N}(\mathcal{P}) = \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$, with $|\nu| = 1 \ \forall \nu \in \mathcal{N}_0(\mathcal{P})$, such that $\mathcal{P} = \{z \in \mathbb{R}^n | \nu \cdot z \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P}), \nu \cdot z \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\}$ ($\mathcal{N}_1(\mathcal{P})$ is the set of the normal vectors to the faces of $\mathcal{P}$).

Definition 2.1. *A complete polyhedron is a convex polyhedron $\mathcal{P} \subset \mathbb{R}^n_+$ such that*

1. $\mathcal{V}(\mathcal{P}) \subset \mathbb{Q}^n$ *(i.e. all vertices have rational coordinates);*
2. *the origin $(0, 0, \ldots, 0)$ belongs to $\mathcal{P}$;*
3. $\mathcal{N}_0(\mathcal{P}) = \{e_1, e_2, \ldots, e_n\}$, *with $e_j = (0, \ldots, 0, 1_{j-th}, 0, \ldots, 0) \in \mathbb{R}^n$ for $j = 1, \ldots, n$;*
4. *every $\nu \in \mathcal{N}_1(\mathcal{P})$ has strictly positive components.*

Remark 1. *The condition 4 implies that for every $x \in \mathcal{P}$ the set $Q(x) = \{y \in \mathbb{R}^n | 0 \leq y \leq x\}$ is included in $\mathcal{P}$ and if s belongs to a face of $\mathcal{P}$ and $r > s$, then $r \notin \mathcal{P}$ (where for $x, y \in \mathbb{R}^n$, $y \leq x$ means that $y_i \leq x_i$, $i = 1, \ldots, n$; and $y < x$ means $y \leq x$, $y \neq x$).*

Let us now summarize some notations related to a complete polyhedron $\mathcal{P}$:
$k(s, \mathcal{P}) = \inf\{t > 0 : t^{-1}s \in \mathcal{P}\} = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu \cdot s, \quad \forall s \in \mathbb{R}^n_+$;
$\mu_j(\mathcal{P}) = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu_j^{-1}$;
$\mu = \mu(\mathcal{P}) = \max_{j=1,\ldots,n} \mu_j$ the formal order of $\mathcal{P}$;
$\mu^{(0)} = \mu^{(0)}(\mathcal{P}) = \min_{\gamma \in \mathcal{V}(\mathcal{P}) \setminus \{0\}} |\gamma|$ the minimum order of $\mathcal{P}$;
$\mu^{(1)} = \mu^{(1)}(\mathcal{P}) = \max_{\gamma \in \mathcal{V}(\mathcal{P})} |\gamma|$ the maximum order of $\mathcal{P}$.

Finally, we define the weight function associated to $\mathcal{P}$:

$$|\xi|_{\mathcal{P}} := \left(\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v| \right)^{\frac{1}{\mu}}, \quad \forall \xi \in \mathbb{R}^n. \tag{3}$$

It is a weight function according to the definition of Liess-Rodino [22].
The definition of the previous quantities is clarified by the following result (for the proof we refer to [7, 16]).

Proposition 2.1. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ with vertices $s^l = (s_1^l, \dots, s_n^l)$, for $l = 1, \dots, n(\mathcal{P})$. Then*

1. *for every $j = 1, 2, \dots, n$, there is a vertex s^{l_j} of $\mathcal{P}$ such that $s^{l_j} = s_j^{l_j} e_j$, $s_j^{l_j} = \max_{s \in \mathcal{P}} s_j =: m_j(\mathcal{P})$;*
2. *the boundary of $\mathcal{P}$ has at least one vertex lying outside the coordinate axes if and only if $\mu_j > m_j$, $\forall j = 1, \dots n$, that is equivalent to ask that the formal order $\mu(\mathcal{P})$ is greater than the maximum order $\mu^{(1)}(\mathcal{P})$;*
3. *if s belongs to $\mathcal{P}$, then $|\xi^s| \leq \sum_{l=1}^{n(\mathcal{P})} |\xi^{s^l}|$, $\forall \xi \in \mathbb{R}^n$, where $\xi^s = \prod_{j=1}^n \xi_j^{s_j}$ and $n(\mathcal{P})$ is the number of vertices of $\mathcal{P}$, including the origin.*

Proposition 2.2. *For any complete polyhedron $\mathcal{P}$ and any $s \in \mathbb{R}_+^n$, $k(s, \mathcal{P})$ is well defined and bounded as follows:*

$$\frac{|s|}{\mu^{(1)}} \leq k(s, \mathcal{P}) \leq \frac{|s|}{\mu^{(0)}}.$$

The associated weight function $|\xi|_{\mathcal{P}}$ satisfies for some constants $C_1, C_2 > 0$ and all $\xi \in \mathbb{R}^n$:

$$C_1 \langle \xi \rangle^{\frac{\mu^{(0)}}{\mu}} \leq |\xi|_{\mathcal{P}} \leq C_2 \langle \xi \rangle^{\frac{\mu^{(1)}}{\mu}}.$$

Considering a polynomial with complex coefficients, we can regard it as the symbol of a differential operator and associate a polyhedron to it.

Definition 2.2. *Let $P(D) = \sum_{|\alpha| \leq m} c_\alpha D^\alpha$, $c_\alpha \in \mathbb{C}$, be a differential operator in $\mathbb{R}^n$ with complex coefficients and $P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha, \xi \in \mathbb{R}^n$, its symbol. The Newton polyhedron or characteristic polyhedron associated to $P(D)$ (or $P(\xi)$) is the convex hull of the set $\{0\} \bigcup \{\alpha \in \mathbb{Z}_+^n : c_\alpha \neq 0\}$.*

The Newton polyhedron of an hypoelliptic operator is complete (cf. Friberg [13]), but the converse is not true in general (cf. Bouzar-Chaili [5, 6], Calvo-Hakobyan [9] and Zanghirati [35, 36]).
To clarify our treatment, we give now some examples of complete polyhedra (for more details cf. [7, 8]).

1. If $P(D)$ is an elliptic operator of order m, then its Newton polyhedron is the complete polyhedron of vertices $\{0, me_j,\ j = 1, \dots, n\}$. The set $\mathcal{N}_1(\mathcal{P})$ is reduced to the point $\nu = m^{-1} \sum_{j=1}^m e_j$, and $m_j(\mathcal{P}) = \mu_j(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \mu(\mathcal{P}) = m$, for all $j = 1, 2, \dots, n$; the weight function $|\xi|_{\mathcal{P}}$ associated to $\mathcal{P}$ is equivalent to $\langle \xi \rangle$.

2. If $P(D)$ is a quasi-elliptic operator of order m (cf. for instance [14], [28], [36]), its characteristic polyhedron $\mathcal{P}$ is complete and has vertices $\{0, m_j e_j,\ j = 1,\dots,n\}$, where $m_j = m_j(\mathcal{P})$ are fixed integers (the anisotropic case). The set $\mathcal{N}_1(\mathcal{P})$ is reduced to a point $\nu = \sum_{j=1}^n m_j^{-1} e_j$; then $\mu_j(\mathcal{P}) = m_j$, for all $j = 1,\dots,n$, $\mu^{(0)}(\mathcal{P}) = \min_{j=1,\dots,n} m_j$, $\mu(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \max_{j=1,\dots,n} m_j = m$. The weight function associated to $\mathcal{P}$ is $|\xi|_{\mathcal{P}} = (1 + |\xi_1|^{m_1} + \dots + |\xi_n|^{m_n})^{\frac{1}{m}}$.

3. If $\mathcal{P} \subset \mathbb{R}^2$ is the polyhedron of vertices $\mathcal{V}(\mathcal{P}) = \{(0,0),(0,3),(1,2),(2,0)\}$, then $\mathcal{P}$ is complete and $\mathcal{N}_1(\mathcal{P}) = \left\{\nu_1 = \left(\frac{1}{3},\frac{1}{3}\right),\ \nu_2 = \left(\frac{1}{2},\frac{1}{4}\right)\right\}$. We have $m_1(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = 2$, $m_2(\mathcal{P}) = m(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = 3$, $\mu(\mathcal{P}) = 4$. We observe that in this case the formal order $\mu(\mathcal{P})$ is bigger than the maximal order $m(\mathcal{P})$, as $\mathcal{P}$ has a vertex lying outside the coordinate axes (cf. Proposition 2.1). The weight function associated to $\mathcal{P}$ is $|\xi|_{\mathcal{P}} = (1 + |\xi_1|^2 + |\xi_2|^3 + |\xi_1\xi_2^2|)^{\frac{1}{4}}$.

Following Hakobyan-Markaryan [16], we define the complementary polyhedron associated to a complete polyhedron $\mathcal{P}$ and give an important property involving the corresponding weight function; the latter will be closely related to the behavior of the pointwise product, cf. Proposition 3.4.

Definition 2.3. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and let $\mu_j = \mu_j(\mathcal{P})$, $j = 1,\dots,n$. The complementary polyhedron associated to $\mathcal{P}$ is $\mathcal{P}^* = \{x \in \mathbb{R}^n_+ :\ x \cdot \lambda^0 \le 1\}$, for $\lambda^0 = \left(\frac{1}{\mu_1},\dots,\frac{1}{\mu_n}\right)$.*

Remark 2. *The formal orders of $\mathcal{P}$ and $\mathcal{P}^*$ coincide by definition. The complementary polyhedron $\mathcal{P}^*$ has only one face (besides the faces on the coordinate hyperplanes); $\mathcal{P}$ is included in $\mathcal{P}^*$ and they coincide only when $\mathcal{P}$ has only one face (the anisotropic and standard cases).*

Proposition 2.3. *Let $\mathcal{P}$ be a complete polyhedron and $\mathcal{P}^*$ its complementary polyhedron as in Definition 2.3, then the associated weight functions satisfy for some constant $C > 0$:*

$$|\xi + \eta|_{\mathcal{P}} \le C(|\xi|_{\mathcal{P}} + |\eta|_{\mathcal{P}^*}), \quad \forall \xi, \eta \in \mathbb{R}^n.$$

Moreover, for all polyhedra $\mathcal{P}'$ such that $\mu(\mathcal{P}') = \mu(\mathcal{P})$ and $\mathcal{P}^* \not\subseteq \mathcal{P}'$, there exist two sequences $\{\xi_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that

$$\lim_{k\to\infty} \frac{|\xi_k + \eta_k|_{\mathcal{P}}}{|\xi_k|_{\mathcal{P}} + |\eta_k|_{\mathcal{P}'}} = \infty.$$

It therefore follows that $|\xi|_{\mathcal{P}}$ satisfies the ring condition (cf. [3], [22]): $|\xi + \eta|_{\mathcal{P}} \le C(|\xi|_{\mathcal{P}} + |\eta|_{\mathcal{P}})$ for some $C > 0$ and all $\xi, \eta \in \mathbb{R}^n$, if and only if $\mathcal{P}$ has only one face (the anisotropic and standard cases).

Remark 3. *For every complete polyhedron $\mathcal{P}$, $\mathcal{P}^*$ is always included in the polyhedron of vertices $\{0, \mu e_j,\ j = 1,\dots,n\}$ defining the standard Gevrey classes, and therefore, $|\xi + \eta|_{\mathcal{P}} \le C(|\xi|_{\mathcal{P}} + \langle\eta\rangle)$, $\forall \xi, \eta \in \mathbb{R}^n$.*

Remark 4. *When $\mathcal{P}$ has more than one face, the inclusion $\mathcal{P} \subset \mathcal{P}^*$ is strict; therefore, from Proposition 2.3, there are two sequences $\{\xi_k\}_{k\in\mathbb{N}}, \{\eta_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that*

$$\lim_{k\to\infty} \frac{|\xi_k + \eta_k|_{\mathcal{P}}}{|\xi_k|_{\mathcal{P}} + |\eta_k|_{\mathcal{P}}} = \infty.$$

As an example, we can consider the polyhedron $\mathcal{P} \subset \mathbb{R}^2$ with vertices $\mathcal{V}(\mathcal{P}) = \{(0,0),(0,3),(2,2),(3,0)\}$, for which $|(\xi_1,\xi_2)|_{\mathcal{P}} = (1 + |\xi_1|^3 + \xi_1^2\xi_2^2 + |\xi_2|^3)^{\frac{1}{6}}$, and take the sequences $\xi_k = (k,0)$, $\eta_k = (0,k)$, $k \in \mathbb{N}$.

3 Multi-anisotropic Gevrey classes

We now introduce the multi-anisotropic Gevrey classes associated to a complete polyhedron $\mathcal{P}$. Firstly we define these spaces by suitable estimates on the growth of the derivatives, that generalize the condition (1) of the standard Gevrey classes. We then give some equivalent definitions; in particular the characterization by means of their Fourier transform is very useful for the applications to partial differential equations (cf. f.i. [7, 8]); it also shows that the multi-anisotropic Gevrey classes can be seen as a particular case of the inhomogeneous Gevrey classes $G^{s,\lambda}(\Omega)$ (cf. [10]), for $\lambda(\xi) = |\xi|_{\mathcal{P}}$. We finally present their topological and algebraic properties.

Definition 3.1. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$, Ω an open set in $\mathbb{R}^n$ and $s \in \mathbb{R}$, $s \geq 1$. We denote by $G^{s,\mathcal{P}}(\Omega)$ the multi-anisotropic Gevrey class of order s associated to $\mathcal{P}$, that is the set of all $f \in C^\infty(\Omega)$ such that for any compact subset K of Ω there is a constant $C > 0$ such that*

$$\sup_{x \in K} |D^\alpha f(x)| \leq C^{|\alpha|+1} (\mu k(\alpha, \mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall \alpha \in \mathbb{N}^n. \tag{4}$$

For $s > 1$, the multi-anisotropic Gevrey classes with compact support are $G_0^{s,\mathcal{P}}(\Omega) = G^{s,\mathcal{P}}(\Omega) \cap C_0^\infty(\Omega)$.

Remark 5. *If $\mathcal{P}$ is the Newton polyhedron of an elliptic operator (cf. Example 1 of Section 2), then $G^{s,\mathcal{P}}(\Omega)$ coincides with $G^s(\Omega)$, the set of the standard Gevrey functions in Ω. If $\mathcal{P}$ is the Newton polyhedron of a quasi-elliptic operator (cf. Example 2 of Section 2), then $G^{s,\mathcal{P}}(\Omega) = G^{s,q}(\Omega)$ is the set of the anisotropic Gevrey functions, cf. [14], [28], [36].*

Remark 6. *We have the following inclusions:*

$$G^s(\Omega) \subseteq G^{s\frac{\mu}{\mu(1)}}(\Omega) \subseteq G^{s,\mathcal{P}}(\Omega) \subseteq G^{s\frac{\mu}{\mu(0)}}(\Omega), \tag{5}$$

that show that the multi-anisotropic Gevrey classes include the standard Gevrey classes of the same order.

For the characterization via Fourier transform of the multi-anisotropic Gevrey classes and, in the next section, of the multi-anisotropic ultradistributions (cf. Theorem 4.1), it is useful to introduce some spaces of multi-anisotropic Gevrey classes in which the Fourier transform is an automorphism. At this aim we define the space $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$, in analogy with [3], [29], [30].

Definition 3.2. *We say that a function $f \in L^1(\mathbb{R}^n)$ belongs to $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$ if $f, \hat{f} \in C^\infty(\mathbb{R}^n)$ and there is a constant $\varepsilon > 0$ such that for all $\alpha \in \mathbb{N}^n$ it holds:*

$$\begin{aligned} p_{\alpha,\varepsilon}(f) &= \sup_{x \in \mathbb{R}^n} \exp(\varepsilon |x|_{\mathcal{P}}^{\frac{1}{s}}) |D^\alpha f(x)| < \infty \\ \pi_{\alpha,\varepsilon}(f) &= \sup_{\xi \in \mathbb{R}^n} \exp(\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}) |D^\alpha \hat{f}(\xi)| < \infty. \end{aligned} \tag{6}$$

We easily see that $G_0^{s,\mathcal{P}}(\mathbb{R}^n) \subset \mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n) \subset G^{s,\mathcal{P}}(\mathbb{R}^n)$ and $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$ is included in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In analogy with [3], from (6) we can see that the Fourier transform is an automorphism in $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$.

¿From now on $\mathcal{P}$ is a complete polyhedron and the Gevrey order s is strictly bigger than 1.

Theorem 3.1. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$, then f belongs to $G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ if and only if there are two positive constants C, ε such that $|\hat{f}(\xi)| \leq C \exp(-\varepsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}})$, $\forall \xi \in \mathbb{R}^n$.*

Remark 7. *Theorem 3.1 can be reformulated in the setting $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$ as follows.*
Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then f belongs to $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$ if and only if there are two positive constants C, ε such that $|\hat{f}(\xi)| \leq C \exp(-\varepsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}})$, $\forall \xi \in \mathbb{R}^n$.

Then the following version of Paley-Wiener-Schwartz Theorem can be established for multi-anisotropic Gevrey classes with compact support.

Theorem 3.2. *Let K be a convex compact subset of $\mathbb{R}^n$. If f is a function in $G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ with support contained in K then there exist two positive constants C, ε such that the Fourier-Laplace transform F of f satisfies:*

$$|F(\zeta)| \leq C|K| \exp(H_K(\Im\zeta) - \varepsilon|\zeta|_{\mathcal{P}}^{\frac{1}{s}}), \qquad \forall \zeta \in \mathbb{C}^n, \tag{7}$$

where $|K|$ is the Lebesgue measure of K, $H_K(t) := \sup_{x \in K} t \cdot x$, $\forall t \in \mathbb{R}^n$ is the so-called support function of K and the weight function $|\cdot|_{\mathcal{P}}$ is defined by naturally extending formula (3) to $\mathbb{C}^n$. Conversely, every entire analytic function F in $\mathbb{C}^n$ satisfying (7) is the Fourier-Laplace transform of a function in $G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ with support contained in K.

For the proof of Theorem 3.1 we refer to Calvo [7], Theorem 3.2 can be proved analogously.
A natural topological structure is defined in the multi-anisotropic Gevrey spaces analogously to the standard Gevrey and ultradifferentiable functions (cf. f.i. [20], [28], [30]).
Let K be a compact subset of $\mathbb{R}^n$. We denote by $C^\infty(K)$ the space of all the C^∞-Whitney jets on K (cf. [34]). We also write $C_0^\infty(K)$ for the space of all the functions of class C^∞ in $\mathbb{R}^n$ with support (possibly empty) contained in K; and, in analogy with [20], we proceed to construct the topology of the multi-anisotropic Gevrey classes.

Definition 3.3. *For any compact subset K of Ω and any positive constant C, the space $G^{s,\mathcal{P}}(K, C)$ is the set of all $f \in C^\infty(K)$ such that:*

$$\|f\|_{G^{s,\mathcal{P}}(K,C)} := \sup_{\alpha \in \mathbb{N}^n} C^{-|\alpha|}(\mu k(\alpha, \mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} \sup_{x \in K} |D^\alpha f(x)| < \infty. \tag{8}$$

For $s > 1$, we also set $G_0^{s,\mathcal{P}}(K, C) := G^{s,\mathcal{P}}(K, C) \cap C_0^\infty(K)$.

The spaces $G^{s,\mathcal{P}}(K, C)$ and $G_0^{s,\mathcal{P}}(K, C)$ are Banach spaces with respect to the norm (8).
Let us begin to describe the topology of $G_0^{s,\mathcal{P}}(\Omega)$. Since $G_0^{s,\mathcal{P}}(\Omega) = \bigcup_{K \subset\subset \Omega, C>0} G_0^{s,\mathcal{P}}(K, C)$, then $G_0^{s,\mathcal{P}}(\Omega)$ is endowed with the inductive limit topology of the Banach spaces $G_0^{s,\mathcal{P}}(K, C)$, as K ranges over the family of all compact subsets of Ω and C over $\mathbb{R}_+$; therefore we write

$$G_0^{s,\mathcal{P}}(\Omega) = \mathrm{indlim}_{K \subset\subset \Omega, C>0} G_0^{s,\mathcal{P}}(K, C). \tag{9}$$

For a detailed study of the inductive limit topology in an abstract topological setting, we refer to [19], [32].
In a similar way, for any compact subset K of $\mathbb{R}^n$, we endow the space $G_0^{s,\mathcal{P}}(K) := G^{s,\mathcal{P}}(\mathbb{R}^n) \cap C_0^\infty(K)$ with

the inductive limit topology of the Banach spaces $G_0^{s,\mathcal{P}}(K,C)$ and write $G_0^{s,\mathcal{P}}(K) = \mathrm{indlim}_{C>0} G_0^{s,\mathcal{P}}(K,C)$. Analogously to [20], we have the following result.

Proposition 3.1. *For any $0 < C_1 < C_2$ and any compact subset K of Ω, the inclusion mappings*

$$G^{s,\mathcal{P}}(K,C_1) \hookrightarrow G^{s,\mathcal{P}}(K,C_2), \quad G_0^{s,\mathcal{P}}(K,C_1) \hookrightarrow G_0^{s,\mathcal{P}}(K,C_2)$$

are compact operators.

It turns out that for any compact subset K of $\mathbb{R}^n$, $G_0^{s,\mathcal{P}}(K)$ and $G_0^{s,\mathcal{P}}(\Omega)$ are (DFS)-spaces according to [19, 20]; in particular they are separable complete bornologic Montel and Schwartz spaces. By the characterization of the Gevrey multi-anisotropic functions via Fourier transform (cf. Theorem 3.1), we can construct an equivalent topology on $G_0^{s,\mathcal{P}}(\Omega)$, that can be seen as a particular case of the topology of the inhomogeneous Gevrey classes (cf. [10]).

Definition 3.4. *Let K be a compact subset of $\mathbb{R}^n$ and ε a positive constant. We define the space*

$$G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon) := \{f \in \mathcal{E}'(\mathbb{R}^n) : \operatorname{supp} f \subseteq K,\ \|f\|_{G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)} := \sup_{\xi\in\mathbb{R}^n} \exp(\varepsilon|\xi|_\mathcal{P}^{\frac{1}{s}})|\hat{f}(\xi)| < \infty\}.$$

We observe that we can also take f in $\mathcal{E}'(\Omega)$, considering its null extension in $\mathbb{R}^n$ as a distribution in $\mathcal{E}'(\mathbb{R}^n)$. $G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)$ is a Banach space with norm $\|\cdot\|_{G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)}$. The following result holds (cf. [10]).

Lemma 3.1. *Let $K_1 \subseteq K_2$ be two compact subsets of Ω and $0 < \varepsilon_2 < \varepsilon_1$. Then the inclusion map $G_0^{s,|\cdot|_\mathcal{P}}(K_1,\varepsilon_1) \hookrightarrow G_0^{s,|\cdot|_\mathcal{P}}(K_2,\varepsilon_2)$ is a compact operator.*

¿From Theorem 3.1 and Definition 3.4, it follows that the space $G_0^{s,\mathcal{P}}(\Omega)$ can be described as $G_0^{s,\mathcal{P}}(\Omega) = \bigcup_{K\subset\subset\Omega,\varepsilon>0} G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)$. Then $G_0^{s,\mathcal{P}}(\Omega)$ is endowed with the inductive limit topology of the Banach spaces $G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)$, as K ranges over the family of the compact subsets of Ω and ε on $\mathbb{R}_+$. Analogously to (9), we can write

$$G_0^{s,\mathcal{P}}(\Omega) = \mathrm{indlim}_{K\subset\subset\Omega,\varepsilon>0} G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon). \tag{10}$$

For any compact subset K of $\mathbb{R}^n$ we write also $G_0^{s,|\cdot|_\mathcal{P}}(K) := \mathrm{indlim}_{\varepsilon>0} G_0^{s,|\cdot|_\mathcal{P}}(K,\varepsilon)$.

Proposition 3.2. *Let τ_0 be the inductive limit topology on $G_0^{s,\mathcal{P}}(\Omega)$ defined by (9) and τ_1 the topology defined in $G_0^{s,\mathcal{P}}(\Omega)$ by (10), then τ_0 and τ_1 are equivalent.*

Proof. In view of the Open Mapping Theorem for (LF)-spaces (cf. [26], Theorem 8.4.11), it suffices to prove that the identity map $id : (G_0^{s,\mathcal{P}}(\Omega),\tau_0) \to (G_0^{s,\mathcal{P}}(\Omega),\tau_1)$ is continuous. Therefore, we have to prove that for any compact subset K of Ω and any constant $C_1 > 0$ the restriction of the operator id to $G_0^{s,\mathcal{P}}(K,C_1) \to (G_0^{s,\mathcal{P}}(\Omega),\tau_1)$ is continuous. If f belongs to $G_0^{s,\mathcal{P}}(K,C_1)$ then, analogously to the proof of Theorem 3.1 (cf. [7]), its Fourier transform satisfies

$$|\hat{f}(\xi)|^{\frac{1}{\mu s}} \le \|f\|_{G^{s,\mathcal{P}}(K,C_1)}^{\frac{1}{\mu s}} C_2^{\frac{1}{\mu s}} (n(\mathcal{P}))^{\frac{1}{\mu s}} |K|^{\frac{1}{\mu s}} \frac{(2^{n(\mathcal{P})} C^{\frac{1}{\mu}})^{\frac{N}{s}} e^N N!}{(|\xi|_\mathcal{P}^{\frac{1}{s}})^N}, \quad \forall \xi \in \mathbb{R}^n,$$

for $N = 0, 1, 2, \ldots$, where $n(\mathcal{P})$ is the number of vertices of $\mathcal{P}$, C_2 is a suitable positive constant independent of f and $C = \max(\mu^{\mu s} C_1^{\mu}, C_2)$. Summing over $N = 0, 1, 2, \ldots$, we obtain for a suitable $\varepsilon > 0$

$$|\hat{f}(\xi)| \leq C_2 \|f\|_{G^{s,\mathcal{P}}(K,C_1)} n(\mathcal{P}) |K| \exp(-\varepsilon |\xi|_{\mathcal{P}}^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n.$$

Therefore, for any $C_1 > 0$ there is an $\varepsilon > 0$ such that $\|f\|_{G_0^{s,|\cdot|_{\mathcal{P}}}(K,\varepsilon)} \leq C_2 n(\mathcal{P}) |K| \|f\|_{G^{s,\mathcal{P}}(K,C_1)}$ for all $f \in G_0^{s,\mathcal{P}}(K, C_1)$; thus for a suitable positive ε the inclusion $G_0^{s,\mathcal{P}}(K, C_1) \hookrightarrow G_0^{s,|\cdot|_{\mathcal{P}}}(K, \varepsilon)$ is a continuous embedding of Banach spaces. Since the inclusion map $G_0^{s,|\cdot|_{\mathcal{P}}}(K, \varepsilon) \hookrightarrow (G_0^{s,\mathcal{P}}(\Omega), \tau_1)$ is continuous, by the definition of the inductive topology τ_1, the result is proved. □

We treat now the topology of the multi-anisotropic Gevrey functions with arbitrary support. First we define for any compact subset K of Ω: $G^{s,\mathcal{P}}(K) := \bigcup_{C>0} G^{s,\mathcal{P}}(K, C)$ and we endow $G^{s,\mathcal{P}}(K)$ with the inductive limit topology of Banach spaces $G^{s,\mathcal{P}}(K, C)$, writing $G^{s,\mathcal{P}}(K) = \mathrm{indlim}_{C>0} G^{s,\mathcal{P}}(K, C)$. ¿From Proposition 3.1, it turns out that $G^{s,\mathcal{P}}(K)$ are (DFS)-spaces; in particular they are reflexive and complete Hausdorff, Montel and Schwartz spaces. Let $\{K_j\}_{j\in\mathbb{N}}$ be an exhaustive sequence of compact subsets of Ω, i.e. for every $j = 1, 2, \ldots$ $K_{j-1} \subset \overset{\circ}{K}_j$ (with $K_0 := \emptyset$) and $\bigcup_{j\in\mathbb{N}} K_j = \Omega$. For any pair of indices l, j with $l > j$ we denote by ρ_j^l the restriction operator $\rho_j^l : G^{s,\mathcal{P}}(K_l) \to G^{s,\mathcal{P}}(K_j)$ defined by $\rho_j^l(f) = f_{|K_j}$ for all $f \in G_0^{s,\mathcal{P}}(K_l)$. Of course, the mappings ρ_j^l are linear and continuous and, for any $j = 1, 2, \ldots$, they satisfy $\rho_j^{j+1} \rho_{j+1}^{j+2} = \rho_j^{j+2}$. Therefore, following [32] (cf. also [19]), the projective limit of the sequence $\{G_0^{s,\mathcal{P}}(K_j)\}_{j\in\mathbb{N}}$ with respect to the mappings $\{\rho_j^l\}_{l>j}$ is defined as the space of all sequences $\{f_j\}_{j\in\mathbb{N}}$ of C^∞-jets $f_j \in G^{s,\mathcal{P}}(K_j)$ such that

$$\rho_j^{j+1}(f_{j+1}) = f_j, \quad j = 1, 2, \ldots. \tag{11}$$

This projective space is isomorphic to $G^{s,\mathcal{P}}(\Omega)$, if we identify any sequence $\{f_j\}_{j\in\mathbb{N}}$ fulfilling (11) with the function f in $G^{s,\mathcal{P}}(\Omega)$ defined by setting $f(x) = f_j(x)$ for $x \in K_j$, $j = 1, 2, \ldots$. Then $G^{s,\mathcal{P}}(\Omega)$ is endowed with the projective topology of the sequence $\{G^{s,\mathcal{P}}(K_j)\}_{j\in\mathbb{N}}$ with respect to the mappings $\{\rho_j^l\}_{l>j}$ (cf. [15], [19]); this procedure does not depend on the sequence $\{K_j\}_{j\in\mathbb{N}}$ and makes $G^{s,\mathcal{P}}(\Omega)$ a complete Schwartz space (cf. [15], [19]). Therefore we will write

$$G^{s,\mathcal{P}}(\Omega) = \mathrm{projlim}_{K\subset\subset\Omega} G^{s,\mathcal{P}}(K) = \mathrm{projlim}_{K\subset\subset\Omega}(\mathrm{indlim}_{C\to\infty} G^{s,\mathcal{P}}(K, C)).$$

Concerning the topology in the spaces $\mathcal{S}^{s,\mathcal{P}}$, we just observe that the semi-norms $p_{\alpha,\varepsilon}$, $\pi_{\alpha,\varepsilon}$ in (6) endow $\mathcal{S}^{s,\mathcal{P}}$ with a locally convex topology.

Remark 8. *The inclusions* (5) *of Gevrey classes are continuous with respect to the previously defined topology.*

We now consider the algebraic properties of multi-anisotropic Gevrey classes. It is clear that if $u, v \in G^{s,\mathcal{P}}(\Omega)$, $k \in \mathbb{C}$, then $u + v \in G^{s,\mathcal{P}}(\Omega)$, $ku \in G^{s,\mathcal{P}}(\Omega)$.

Proposition 3.3. *If $\alpha \in \mathbb{N}^n$, then for any $0 < C < C'$ and any compact subset K of Ω, the derivative $D^\alpha : G^{s,\mathcal{P}}(K,C) \to G^{s,\mathcal{P}}(K,C')$ is continuous and it satisfies $\|D^\alpha f\|_{G^{s,\mathcal{P}}(K,C')} \leq M\|f\|_{G^{s,\mathcal{P}}(K,C)}$ for every $f \in G^{s,\mathcal{P}}(K,C)$ and some constant $M > 0$ depending only on α, C and C'. Therefore, the derivative D^α defines a continuous linear operator from each of the spaces $G^{s,\mathcal{P}}(\Omega)$ and $G_0^{s,\mathcal{P}}(\Omega)$ into itself.*

The proof is immediate by direct computation, or after observing that the sequence $(\mu k(\alpha,\mathcal{P}))^{\mu s k(\alpha,\mathcal{P})}$, $\alpha \in \mathbb{N}^n$, fulfills the differentiation property stated by Roumieu (cf. [30], inequality (7)). The problem of the pointwise multiplication for multi-anisotropic Gevrey functions is solved by the following result.

Proposition 3.4. *Let Ω be an open subset of $\mathbb{R}^n$, $\mathcal{P}$ be a complete polyhedron and $\mathcal{P}^*$ its complementary polyhedron given by Definition 2.3; then for any $s > 1$ we have: $G^{s,\mathcal{P}}(\Omega) \cdot G^{s,\mathcal{P}^*}(\Omega) \subseteq G^{s,\mathcal{P}}(\Omega)$ and for any complete polyhedron $\mathcal{P}'$ such that $\mu(P') = \mu(\mathcal{P})$ and $\mathcal{P}^* \nsubseteq \mathcal{P}'$: $G^{s,\mathcal{P}}(\Omega) \cdot G^{s,\mathcal{P}'}(\Omega) \nsubseteq G^{s,\mathcal{P}}(\Omega)$.*

For the proof and counterexamples we refer to Calvo [8] and Hakobyan-Markaryan [16].

Remark 9. *In view of the inclusion $\mathcal{P} \subseteq \mathcal{P}^*$, we have $G^{s,\mathcal{P}^*}(\Omega) \subseteq G^{s,\mathcal{P}}(\Omega)$ and they coincide only in the standard or the anisotropic case (cf. [8], Proposition 2.1). Moreover, from Proposition 3.4, $G^{s,\mathcal{P}}(\Omega)$ is not an algebra under the pointwise multiplication, except the standard and anisotropic cases.*

Remark 10. *As the inclusion $G^s(\Omega) \subseteq G^{s,\mathcal{P}^*}(\Omega)$ holds for any complete polyhedron $\mathcal{P}$, then $G^s(\Omega) \cdot G^{s,\mathcal{P}}(\Omega) \subseteq G^{s,\mathcal{P}}(\Omega)$.*

¿From the topological point of view, the following results can be proved.

Proposition 3.5. *Let $\mathcal{P}$ be a complete polyhedron, $\mathcal{P}^*$ its complementary polyhedron, $C_1, C_2 > 0$ and K a compact subset of Ω. Then $\|fg\|_{G^{s,\mathcal{P}}(K,C_1+C_2)} \leq \|f\|_{G^{s,\mathcal{P}}(K,C_1)}\|g\|_{G^{s,\mathcal{P}^*}(K,C_2)}$, for every $f \in G^{s,\mathcal{P}}(K,C_1)$ and $g \in G^{s,\mathcal{P}^*}(K,C_2)$.*

Proposition 3.6. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and $\mathcal{P}^*$ its complementary polyhedron. Then, for any open subset Ω of $\mathbb{R}^n$, any compact subset K of Ω and $s > 1$:*

1. *$G^{s,\mathcal{P}}(\Omega)$ is a topological $G^{s,\mathcal{P}^*}(\Omega)$ -module;*
2. *$G_0^{s,\mathcal{P}}(\Omega)$ is a $G^{s,\mathcal{P}^*}(\Omega)$ -module in which the pointwise multiplication is hypocontinuous.*

By the properties of the multiplication of multi-anisotropic Gevrey functions (cf. Remark 10, Proposition 3.5), we can give an equivalent topological description of the space $G^{s,\mathcal{P}}(\Omega)$ in analogy with the inhomogeneous Gevrey case.

Corollary 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $\{K_j\}_{j\in\mathbb{N}}$ an exhaustive sequence of compact subsets of Ω; chosen any $\chi_j \in G_0^s(K_j)$ such that $0 \leq \chi_j \leq 1$ and $\chi_j|_{K_{j-1}} \equiv 1$ then $G^{s,\mathcal{P}}(\Omega) = \mathrm{projlim}_{j\to\infty} G_0^{s,|\cdot|_\mathcal{P}}(K_j)$, where the projective limit in the right-hand side is taken with respect to the mappings $\varphi_j : G_0^{s,|\cdot|_\mathcal{P}}(K_{j+1}) \to G_0^{s,|\cdot|_\mathcal{P}}(K_j)$, $f \mapsto \chi_j f$.*

Corollary 3.2. *If Ω is an open subset of $\mathbb{R}^n$ and K_1, K_2 are compact subsets of Ω such that $K_1 \subseteq K_2$, then the inclusion maps: $G_0^{s,\mathcal{P}}(K_1) \hookrightarrow G^{s,\mathcal{P}}(K_2)$, $G_0^{s,\mathcal{P}}(K_1) \hookrightarrow G^{s,\mathcal{P}}(\Omega)$ are topological homomorphisms.*

The proof is analogous to that in [4], following from Propositions 3.5 and 3.6.
Combining Propositions 3.3, 3.4, 3.6, we get the following result.

Proposition 3.7. *Let $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ be a linear partial differential operator with coefficients $a_\alpha \in G^{s,\mathcal{P}^*}(\Omega)$. Then the following are linear continuous operators:*

$$P(x,D) : G^{s,\mathcal{P}}(\Omega) \to G^{s,\mathcal{P}}(\Omega), \qquad P(x,D) : G_0^{s,\mathcal{P}}(\Omega) \to G_0^{s,\mathcal{P}}(\Omega).$$

Concerning the convolution product of multi-anisotropic Gevrey functions, the following result can be established.

Proposition 3.8. *Let $f \in G^{s,\mathcal{P}}(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{comp}}(\mathbb{R}^n)$. Then the convolution product $f * g$ belongs to $G^{s,\mathcal{P}}(\mathbb{R}^n)$; more precisely for any compact subset K of $\mathbb{R}^n$ there is a constant $C > 0$ such that $\|f * g\|_{G^{s,\mathcal{P}}(K,C)} \le \|f\|_{G^{s,\mathcal{P}}(K-H,C)} \|g\|_{L^1(H)}$, where $H = \operatorname{supp} g$. Analogously, if $f \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ then $f * g \in G^{s,\mathcal{P}}(\mathbb{R}^n)$; in particular for any compact subset K of $\mathbb{R}^n$ there exists a constant $C > 0$ such that $\|f * g\|_{G^{s,\mathcal{P}}(K,C)} \le \|f\|_{G^{s,\mathcal{P}}(H,C)} \|g\|_{L^1(K-H)}$, where $H = \operatorname{supp} f$.*

Corollary 3.3. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbb{R}^n$ and set $S(f,g) := f * g$; then the following are hypocontinuous bilinear maps:*

$$S : G^{s,\mathcal{P}}(\mathbb{R}^n) \times L^1_{\mathrm{comp}}(\mathbb{R}^n) \to G^{s,\mathcal{P}}(\mathbb{R}^n), \qquad S : G_0^{s,\mathcal{P}}(\mathbb{R}^n) \times L^1_{\mathrm{loc}}(\mathbb{R}^n) \to G^{s,\mathcal{P}}(\mathbb{R}^n).$$

Remark 11. *In view of Proposition 3.8, analogously to the standard Gevrey case we can prove that $G_0^{s,\mathcal{P}}(\Omega)$ is dense in $G^{s,\mathcal{P}}(\Omega)$, $G_0^{s,\mathcal{P}}(\Omega)$ is dense in $C_0^\infty(\Omega)$ and $G^{s,\mathcal{P}}(\Omega)$ is dense in $C^\infty(\Omega)$ with continuous inclusions. The proof is based on the convolution with functions $\varphi_\varepsilon \in G^s(\Omega) \subseteq G^{s,\mathcal{P}}(\Omega)$, $\varphi_\varepsilon \to \delta$ (cf. [4], Lemma 3.8 and [30], Corollary 3).*

4 Multi-anisotropic ultradistributions

Now we present the topological dual spaces of the multi-anisotropic Gevrey classes: the multi-anisotropic ultradistributions. They admit equivalent characterizatons according to the topology defined in the multi-anisotropic Gevrey spaces; in particular we study the properties of the Fourier transform and prove a version of the Paley-Wiener-Schwartz Theorem for multi-anisotropic ultradistributions with compact support (cf. Theorems 4.1, 4.2). We then examine the algebraic and topological properties and give an application concerning the well posedness of the Cauchy problem for a certain class of operators with constant coefficients.

Definition 4.1. *For any $s > 1$ and any open set Ω of $\mathbb{R}^n$, the space of multi-anisotropic ultradistributions $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ is the topological dual space of the multi-anisotropic Gevrey class $G_0^{s,\mathcal{P}}(\Omega)$, endowed with the strong dual topology.*

Proposition 4.1. *The following conditions are equivalent:*

1. *u belongs to* $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$*;*
2. *u is a linear form on* $G_0^{s,\mathcal{P}}(\Omega)$ *such that* $u(f_j) \to 0$ *for any sequence* $\{f_j\}_{j\in\mathbb{N}} \subset G_0^{s,\mathcal{P}}(\Omega)$ *converging to* 0 *in* $G_0^{s,\mathcal{P}}(\Omega)$*;*
3. *for any compact subset* K *of* Ω *and any* $\varepsilon > 0$*, there is a constant* $C_\varepsilon > 0$ *such that for all* $f \in G_0^{s,\mathcal{P}}(\Omega)$*, with* $\operatorname{supp} f \subseteq K$*, it holds* $|u(f)| \le C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|}(\mu k(\alpha,\mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} \sup_{x\in K} |D^\alpha f(x)|$.

The proof follows from standard topological arguments (cf. [19], [32], [33]).

Remark 12. *For any complete polyhedron* $\mathcal{P}$ *in* $\mathbb{R}^n$*, any open subset* Ω *of* $\mathbb{R}^n$ *and* $t > s > 1$*, the inclusions* $\mathcal{D}'(\Omega) \subset \mathcal{D}'_{t,\mathcal{P}}(\Omega) \subset \mathcal{D}'_{s,\mathcal{P}}(\Omega)$ *hold continuously (as a consequence of the continuous inclusions* $G_0^{s,\mathcal{P}}(\Omega) \subset G_0^{t,\mathcal{P}}(\Omega) \subset C_0^\infty(\Omega)$*, cf. Remark 11). ¿From Remark 6, we derive also the following inclusions of multi-anisotropic and standard ultradistributions:* $\mathcal{D}'_{s\frac{\mu}{\mu(0)}}(\Omega) \subseteq \mathcal{D}'_{s,\mathcal{P}}(\Omega) \subseteq \mathcal{D}'_{s\frac{\mu}{\mu(1)}}(\Omega) \subseteq \mathcal{D}'_s(\Omega)$.

Let us mention some particular cases of multi-anisotropic ultradistributions.

1. If $\mathcal{P}$ is the Newton polyhedron of an elliptic operator, then $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ coincides with $\mathcal{D}'_s(\Omega)$, the space of the ultradistributions associated to the standard Gevrey class $G_0^s(\Omega)$;
2. If $\mathcal{P}$ is the Newton polyhedron of a quasi-elliptic operator, then $\mathcal{D}'_{s,\mathcal{P}}(\Omega) = \mathcal{D}'_{s,q}(\Omega)$ is the set of the anisotropic ultradistributions associated to the anisotropic Gevrey class $G_0^{s,q}(\Omega)$. From Proposition 4.1, $\mathcal{D}'_{s,q}(\Omega)$ is the set of the linear forms on $G_0^{s,q}(\Omega)$ such that for any compact subset K of Ω and any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ for which the inequality $|u(f)| \le C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|}(\alpha \cdot q)^{-s\alpha\cdot q} \sup_{x\in K} |D^\alpha f(x)|$ holds for all $f \in G_0^{s,q}(\Omega)$ with $\operatorname{supp} f \subseteq K$.

Definition 4.2. *For every* $s > 1$*, we denote by* $\mathcal{E}'_{s,\mathcal{P}}(\Omega)$ *the topological dual space of the multi-anisotropic Gevrey class* $G^{s,\mathcal{P}}(\Omega)$*, endowed with the strong dual topology.*

As in the space of Schwartz distributions, we can define the support of a multi-anisotropic ultradistribution $u \in \mathcal{D}'_{s,\mathcal{P}}(\Omega)$ as the intersection of all closed subsets of Ω in whose complement u vanishes; then, in analogy with Proposition 4.1, we have the following result.

Proposition 4.2. *The following conditions are equivalent:*

1. *u belongs to* $\mathcal{E}'_{s,\mathcal{P}}(\Omega)$*;*
2. *u is a linear form on* $G^{s,\mathcal{P}}(\Omega)$ *such that* $u(f_j) \to 0$ *for any sequence* $\{f_j\}_{j\in\mathbb{N}} \subset G^{s,\mathcal{P}}(\Omega)$ *converging to* 0 *in* $G^{s,\mathcal{P}}(\Omega)$*;*
3. *u is a linear form on* $G^{s,\mathcal{P}}(\Omega)$ *and there is a compact subset* K *of* Ω *such that for any* $\varepsilon > 0$ *there is a constant* $C_\varepsilon > 0$ *such that for all* $f \in G^{s,\mathcal{P}}(\Omega)$ *it holds:*

$$|u(f)| \le C_\varepsilon \sup_{\alpha\in\mathbb{N}} \varepsilon^{|\alpha|}(\mu k(\alpha,\mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} \sup_{x\in K} |D^\alpha f(x)|. \tag{12}$$

4. u belongs to $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ and has compact support in Ω.

We can equivalently define the multi-anisotropic ultradistributions with compact support by means of the Fourier transform and the Fourier-Laplace transform, in analogy with the standard ultradistributions (cf. [20], [28]). It follows that the multi-anisotropic ultradistributions can be seen as a particular case of the inhomogeneous ultradistributions treated in [10].
If u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$, then its Fourier transform can be defined by setting $\hat{u}(\xi) = u_x(e^{-ix\cdot\xi})$, $\forall \xi \in \mathbb{R}^n$, that is well defined as $e^{-ix\cdot\xi}$ belongs to $G^s(\mathbb{R}^n) \subset G^{s,\mathcal{P}}(\mathbb{R}^n)$. The rules of derivation and multiplication of the Fourier transform in $\mathcal{E}'(\mathbb{R}^n)$ are still satisfied in $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$.

Theorem 4.1. *If u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$, then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$|\hat{u}(\xi)| \leq C_\varepsilon \exp(\varepsilon|\xi|_{\mathcal{P}}^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n. \tag{13}$$

Proof. If u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$, then $\hat{u}(\xi) = u_x(e^{-ix\cdot\xi})$ is a function, since u has compact support. From formula (12), taking $f(x) = e^{-ix\cdot\xi}$ we get:

$$\begin{aligned}|\hat{u}(\xi)| = |u_x(e^{-ix\cdot\xi})| &\leq C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|}(\mu k(\alpha,\mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} \sup_{x\in H} |D_x^\alpha e^{-ix\cdot\xi}| \\ &\leq C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|}(\mu k(\alpha,\mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} |\xi|_{\mathcal{P}}^{\mu k(\alpha,\mathcal{P})}.\end{aligned}$$

Applying the inequality $t^d \leq e^t d^d$ for $t = \frac{\varepsilon_1}{s}|\xi|_{\mathcal{P}}^{\frac{1}{s}}$ and $d = \mu k(\alpha,\mathcal{P})$ with an arbitrary $\varepsilon_1 > 0$, we can estimate

$$\begin{aligned}|\hat{u}(\xi)| &\leq C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|}(\mu k(\alpha,\mathcal{P}))^{-s\mu k(\alpha,\mathcal{P})} \exp\left(\varepsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right) (\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})} \left(\frac{s}{\varepsilon_1}\right)^{s\mu k(\alpha,\mathcal{P})} \\ &\leq C_\varepsilon \sup_{\alpha\in\mathbb{N}^n} \varepsilon^{|\alpha|} \left(\frac{s}{\varepsilon_1}\right)^{s\frac{\mu}{\mu_0}|\alpha|} \exp\left(\varepsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{s}}\right).\end{aligned}$$

Now taking $\varepsilon = \left(\frac{s}{\varepsilon_1}\right)^{-s\frac{\mu}{\mu(0)}}$ we obtain $|\hat{u}(\xi)| \leq C_\varepsilon \exp(\varepsilon_1|\xi|_{\mathcal{P}}^{\frac{1}{s}})$, that gives the result. $\square$

Definition 4.3. *The space of multi-anisotropic ultradistribution $\mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$.*

We have obviously the inclusions $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n) \subset \mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n) \subset \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$, where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions, dual space of $\mathcal{S}(\mathbb{R}^n)$.
In analogy with the spaces $\mathcal{S}'(\mathbb{R}^n)$, it is possible to define the Fourier transform also for $u \in \mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$, by the use of Parseval's formula: $\hat{u}(f) = u(\hat{f})$, for all $f \in \mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$ (as also $\hat{f} \in \mathcal{S}^{s,\mathcal{P}}(\mathbb{R}^n)$). For $u \in \mathcal{E}'_{s,\mathcal{P}}$, this definition obviously coincides with the previous one.

Proposition 4.3. *The Fourier transform is an automorphism of $\mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$.*

Corollary 4.1. *Any function $U(\xi)$ satisfying for all $\varepsilon > 0$ the condition (13) in $\mathbb{R}^n$ belongs also to $\mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$, and therefore is the Fourier transform of an element u of $\mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n)$.*

It follows from the inclusion $\mathcal{S}'_{s,\mathcal{P}}(\mathbb{R}^n) \subset \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$ that if $U(\xi)$ satisfies (13), then it is the Fourier transform of an element of $\mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$ (as we will need in the proof of Theorem 4.3).
Finally, we prove a version of Paley-Wiener-Schwartz Theorem for multi-anisotropic ultradistributions.

Theorem 4.2. *If u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ and $\operatorname{supp} u \subseteq K$, where K is a convex compact subset of $\mathbb{R}^n$, then for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that the Fourier-Laplace transform U of u satisfies*

$$|U(\zeta)| \leq C_\varepsilon \exp\left(H_K(\Im\zeta) + \varepsilon|\zeta|_{\mathcal{P}}^{\frac{1}{s}}\right), \quad \forall\zeta \in \mathbb{C}^n, \tag{14}$$

where $H_K(t) := \sup_{x\in K} x \cdot t$, for $t \in \mathbb{R}^n$. If an entire analytic function $U(\zeta)$ satisfies (14)*, then U is the Fourier-Laplace transform of an ultradistribution $u \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq K$.*

Proof. If u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ and $\operatorname{supp} u \subseteq K$, then (14) holds in analogy with Theorem 4.1.
Conversely, let $U(\zeta)$ satisfy (14). Then, as in the standard Gevrey case (cf. [4]), the linear form:

$$u(\varphi) := (2\pi)^{-n} \int U(-\xi)\hat{\varphi}(\xi)d\xi, \quad \forall\varphi \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$$

defines a multi-anisotropic ultradistribution $u \in \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq K$, whose Fourier-Laplace transform coincides with U. Therefore, in view of *4.* of Proposition 4.2, u belongs to $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$. □

Remark 13. *The inequality* (14) *is an estension to the complex space $\mathbb{C}^n$ of* (13)*.*

We now study the topological properties of the multi-anisotropic ultradistributions $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ and $\mathcal{E}'_{s,\mathcal{P}}(\Omega)$. In view of Definition 4.1 and according to [32], Theorem 4, the strong dual topology makes $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ a (FS)-space; in particular it is a complete bornologic Montel and Schwartz space. An explicit fundamental system of continuous semi-norms is given by setting for any $u \in \mathcal{D}'_{s,\mathcal{P}}(\Omega)$: $p_{B^j_m}(u) := \sup_{f\in B^j_m} |u(f)|$, $j, m = 1, 2, \ldots$, where, for any pair of positive integers (j, m), $B^j_m := \{f \in G_0^{s,\mathcal{P}}(K_j, C_j) : \|f\|_{G^{s,\mathcal{P}}(K_j,C_j)} \leq m\}$, $\{K_j\}_{j\in\mathbb{N}}$ is an exhaustive sequence of compact subsets of Ω and $\{C_j\}_{j\in\mathbb{N}}$ is an increasing sequence of positive numbers diverging to $+\infty$. Following the arguments of Komatsu ([20], Proposition 5.11 and Theorem 5.12), $\mathcal{E}'_{s,\mathcal{P}}(\Omega)$ are complete bornologic Montel and Schwartz spaces.
In the space of multi-anisotropic ultradistributions, we can define the usual elementary operations.
Let $\mathcal{P}$ be a complete polyhedron and $\mathcal{P}^*$ its complementary polyhedron. For any $u \in \mathcal{D}'_{s,\mathcal{P}}(\Omega)$, $f \in G^{s,\mathcal{P}^*}(\Omega)$ and $\alpha \in \mathbb{N}^n$ the following operations are well defined:

- the product $fu(\varphi) := u(f\varphi)$, $\forall\varphi \in G_0^{s,\mathcal{P}}(\Omega)$;
- the derivative $D^\alpha u(\varphi) := (-1)^{-|\alpha|}u(D^\alpha\varphi)$, $\forall\varphi \in G_0^{s,\mathcal{P}}(\Omega)$.

¿From Propositions 3.3 and 3.4, the derivative is continuous in $\mathcal{D}'_{s,\mathcal{P}}(\Omega)$ and $\mathcal{E}'_{s,\mathcal{P}}(\Omega)$ and setting $M(f, u) := fu$ the following maps are bilinear and hypocontinuous:

$$M : G^{s,\mathcal{P}^*}(\Omega) \times \mathcal{D}'_{s,\mathcal{P}}(\Omega) \to \mathcal{D}'_{s,\mathcal{P}}(\Omega),$$

$$M : G^{s,\mathcal{P}^*}(\Omega) \times \mathcal{E}'_{s,\mathcal{P}}(\Omega) \to \mathcal{E}'_{s,\mathcal{P}}(\Omega),$$

$$M : G_0^{s,\mathcal{P}^*}(\Omega) \times \mathcal{D}'_{s,\mathcal{P}}(\Omega) \to \mathcal{E}'_{s,\mathcal{P}}(\Omega).$$

Corollary 4.2. *Let $\mathcal{P}$ be a complete polyhedron and $\mathcal{P}^*$ its complementary polyhedron. Let $P(x,D) := \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ be a linear partial differential operator with coefficients $a_\alpha \in G^{s,\mathcal{P}^*}(\Omega)$. Then the following are continuous linear operators:*

$$P(x,D) : \mathcal{D}'_{s,\mathcal{P}}(\Omega) \to \mathcal{D}'_{s,\mathcal{P}}(\Omega), \qquad P(x,D) : \mathcal{E}'_{s,\mathcal{P}}(\Omega) \to \mathcal{E}'_{s,\mathcal{P}}(\Omega).$$

If $u \in \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$ and $f \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$, or $u \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ and $f \in G^{s,\mathcal{P}}(\mathbb{R}^n)$, the convolution product $u * f$ is defined by $(u*f)(x) = u_y(f(x-y))$, where u_y acts on $f(x-y)$ as an ultradistribution in y.

Proposition 4.4. *Let $f \in G^{s,\mathcal{P}}(\mathbb{R}^n)$ and $u \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ (or $f \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$ and $u \in \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$); then the convolution product $u*f$ belongs to $G^{s,\mathcal{P}}(\mathbb{R}^n)$.*

Proof. We prove the proposition for $f \in G^{s,\mathcal{P}}(\mathbb{R}^n)$ and $u \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$; the other case is similar and we omit it. For all $\alpha \in \mathbb{N}^n$, $D_x^\alpha(u*f)(x) = u_y(D_x^\alpha f(x-y))$ is well defined, since $D_x^\alpha f(x-y) \in G^{s,\mathcal{P}}(\mathbb{R}^n_x)$ for all $y \in \mathbb{R}^n$ and u is linear and continuous in $G^{s,\mathcal{P}}(\mathbb{R}^n)$; therefore $u*f$ belongs to $C^\infty(\mathbb{R}^n)$; now we prove that it is also in $G^{s,\mathcal{P}}(\mathbb{R}^n)$. From the definitions of $G^{s,\mathcal{P}}(\mathbb{R}^n)$ and $\mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ (cf. formulas (4) and (12), respectively), for any compact subset K of $\mathbb{R}^n$ there is a $C>0$ such that for any $\alpha \in \mathbb{N}^n$ it holds:

$$\begin{aligned}
&\sup_{x\in K} |D_x^\alpha(u*f)(x)| = \sup_{x\in K} |u_y(D_x^\alpha f(x-y))| \\
&\le C_\varepsilon \sup_{\beta\in\mathbb{N}^n} \varepsilon^{|\beta|}(\mu k(\beta,\mathcal{P}))^{-s\mu k(\beta,\mathcal{P})} \sup_{x\in K}\sup_{y\in H} |D_x^\alpha D_y^\beta f(x-y)| \\
&\le C_\varepsilon \sup_{\beta\in\mathbb{N}^n} \varepsilon^{|\beta|}(\mu k(\beta,\mathcal{P}))^{-s\mu k(\beta,\mathcal{P})} \sup_{z\in H-K} |D^{\alpha+\beta} f(z)| \\
&\le C_\varepsilon \sup_{\beta\in\mathbb{N}^n} \varepsilon^{|\beta|}(\mu k(\beta,\mathcal{P}))^{-s\mu k(\beta,\mathcal{P})} C^{|\alpha|+|\beta|+1} (\mu k(\alpha+\beta,\mathcal{P}))^{s\mu k(\alpha+\beta,\mathcal{P})} \\
&\le C_\varepsilon \sup_{\beta\in\mathbb{N}^n} \varepsilon^{|\beta|}(\mu k(\beta,\mathcal{P}))^{-s\mu k(\beta,\mathcal{P})} C^{|\alpha|+|\beta|+1} (\mu k(\alpha,\mathcal{P}) + \mu k(\beta,\mathcal{P}))^{s(\mu k(\alpha,\mathcal{P})+\mu k(\beta,\mathcal{P}))},
\end{aligned}$$

where in the last step we have applied

$$k(\alpha+\beta,\mathcal{P}) = \max_{\nu\in\mathcal{V}(\mathcal{P})} (\alpha+\beta)\cdot\nu \le \max_{\nu\in\mathcal{V}(\mathcal{P})} \alpha\cdot\nu + \max_{\nu\in\mathcal{V}(\mathcal{P})} \beta\cdot\nu = k(\alpha,\mathcal{P}) + k(\beta,\mathcal{P}).$$

¿From the elementary inequality $t^d \le d^d e^{t-d}$ we obtain:

$$\begin{aligned}
\sup_{x\in K} |D_x^\alpha(u*f)(x)| &\le C_\varepsilon \sup_{\beta\in\mathbb{N}^n} C^{|\alpha|+|\beta|+1} \varepsilon^{|\beta|} (\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})} e^{s\mu k(\beta,\mathcal{P})} e^{s\mu|\alpha|} \\
&\le CC_\varepsilon (Ce^{s\mu})^{|\alpha|} (\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})} \sup_{\beta\in\mathbb{N}^n} \varepsilon^{|\beta|} e^{s\mu k(\beta,\mathcal{P})} C^{|\beta|}.
\end{aligned}$$

So, taking $\varepsilon = (Ce^{s\mu})^{-1}$ and $C_1 = Ce^{s\mu}, C_2 = CC_\varepsilon$ for the previous choice of ε, we get the estimate

$$\sup_{x\in K} |D_x^\alpha(u*f)(x)| \le C_2 C_1^{|\alpha|} (\mu k(\alpha,\mathcal{P}))^{s\mu k(\alpha,\mathcal{P})}, \quad \forall \alpha \in \mathbb{N}^n,$$

that implies that $u*f$ belongs to $G^{s,\mathcal{P}}(\mathbb{R}^n)$. □

The convolution product satisfies the usual properties (analogously to the case of Schwartz distributions); in particular, for any $f \in G^{s,\mathcal{P}}(\mathbb{R}^n)$, $u \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$ (or $f \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$, $u \in \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$) and $g \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$, it holds $(u * f) * g = u * (f * g)$ and $\mathrm{supp}(u * f) \subseteq \mathrm{supp}\, u + \mathrm{supp}\, f$. Therefore, we can define the product by convolution of two ultradistributions $u, v \in \mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$, one of which has compact support, by setting $(u * v) * g = u * (v * g)$ for all $g \in G_0^{s,\mathcal{P}}(\mathbb{R}^n)$; this is well defined in view of Proposition 4.4 and $u * v$ belongs to $\mathcal{D}'_{s,\mathcal{P}}(\mathbb{R}^n)$.

To conclude our treatment, we show some applications of the multi-anisotropic ultradistributions to the study of partial differential equations.

We first point out that the multi-anisotropic ultradistributions are the suitable setting for the problems (for instance hypoelliptcity, local solvability and iterates of operators) in which existence or regularity results are proved in the frame of multi-anisotropic Gevrey classes, cf. for instance [1], [5], [6], [9], [12], [13], [14], [16], [18], [27], [35], [36]. In fact, all these results can be reformulated by replacing the class of Schwartz distributions with the corresponding multi-anisotropic ultadistributions.

As further application, we study the Cauchy problem in multi-anisotropic ultradistributions for weakly hyperbolic operators. The well-posedness will be proved for the so-called multi-quasi-hyperbolic operators introduced by Calvo [7] and generalizing the s-hyperbolic operators of Larsson [21]; they are shaped in such a way that the well-posedness of the Cauchy problem holds for multi-anisotropic Gevrey classes. Let us recall the definition; for more precise characterizations and properties we refer to [7].

Definition 4.4. *We say that a differential operator with constant coefficients in $\mathbb{R}_t \times \mathbb{R}^n_x$*

$$P(D) = P(D_t, D_x) = D_t^m + \sum_{|\nu|+j \leq m, j \neq m} a_{\nu j} D_x^\nu D_t^j \tag{15}$$

is multi-quasi-hyperbolic with respect to $\mathcal{P}$ of order $s > 1$ (for short $(s, \mathcal{P})$-hyperbolic) if there exists a constant $C > 0$ such that for any $(\tau, \xi) \in \mathbb{C} \times \mathbb{R}^n$ the symbol of $P(D)$ satisfies the condition:

$$P(\tau, \xi) = \tau^m + \sum_{|\nu|+j \leq m, j \neq m} a_{\nu j} \xi^\nu \tau^j = 0 \quad \Longrightarrow \quad |\Im \tau| \leq C |\xi|_{\mathcal{P}}^{\frac{1}{s}}.$$

Obviously, if $P(D)$ is $(s, \mathcal{P})$-hyperbolic then it is also weakly hyperbolic, i.e. all the roots τ of the characteristic equation $P_m(\tau, \xi) = \tau^m + \sum_{|\nu|+j=m, j \neq m} \xi^\nu \tau^j = 0$ are real. In the opposite direction, if $P_m(\tau, \xi)$ satisfies the weakly hyperbolic assumption, in order to have the $(s, \mathcal{P})$-hyperbolicity we need to ask some Levi-type conditions on the lower order terms, modeled on $\mathcal{P}$, cf. [7]. We just point out the following result, in order to give an idea of the multi-quasi-hyperbolicity conditions.

Proposition 4.5. *Let $P(D)$ in (15) be weakly hyperbolic and suppose that the multiplicity of the roots of its principal symbol $P_m(\tau, \xi)$ is equal to $M \leq m$ and the lower order terms satisfy for some $k < M$ and a constant $C > 0$:*

$$|a_{\nu j} \xi^\nu| \leq C |\xi|_{\mathcal{P}}^k \langle \xi \rangle^{m-M-j} \qquad \text{for } \; |\nu| + j \leq m - 1. \tag{16}$$

Then $P(D)$ is $(\frac{M}{k}, \mathcal{P})$-hyperbolic.

In [7] it was proved that, under the hypothesis of $(s, \mathcal{P})$-hyperbolicity, the Cauchy problem (17) admits a unique solution $u \in C^\infty(\mathbb{R}, G^{s,\mathcal{P}}(\mathbb{R}^n))$ for any data $u_k \in G^{r,\mathcal{P}}(\mathbb{R}^n)$ if $r < s$.

Theorem 4.3. *Let $P(D)$ be an $(s,\mathcal{P})$-hyperbolic differential operator in $\mathbb{R}_t \times \mathbb{R}^n_x$. Let $1 < r < s$ and $u_k \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n_x)$ $(k = 0, 1, \ldots, m-1)$. Then the Cauchy problem*

$$\begin{cases} P(D)u = D_t^m u + \sum_{|\nu|+j\le m,\ j\ne m} a_{\nu j} D_x^\nu D_t^j u = 0 \\ D_t^k u(0,x) = u_k(x), \qquad k = 0, 1, \ldots, m-1 \end{cases} \tag{17}$$

admits a unique solution $u \in C^\infty(\mathbb{R}, \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n_x))$.

Proof. For the finite speed of propagation of weakly hyperbolic operators with constant coefficients, it is not restrictive to take the data with compact support $u_k \in \mathcal{E}'_{s,\mathcal{P}}(\mathbb{R}^n)$, $k = 0, \ldots, m-1$. Therefore, after performing the partial Fourier transform (with respect to the x variable and taking t as parameter) of the Cauchy problem (17), the unique solution of the equivalent Cauchy problem (an ordinary Cauchy problem in t, with ξ as parameter) is given by:

$$\hat{u}(t,\xi) = \sum_{j=0}^{m-1} \hat{u}_j(\xi) F_j(t,\xi), \tag{18}$$

where $\hat{u}_j$ $(j = 0, \ldots, m-1)$ are the Fourier transforms of the data of (17) and F_j are the unique solutions of the Cauchy problems

$$\begin{cases} P(D_t,\xi)F_j = 0 \\ D_t^k F_j(0,\xi) = \delta_{jk}, \quad k = 0, \ldots, m-1, \end{cases}$$

for $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Let us fix now an arbitrary $T > 0$. The F_j are estimated by Lemma 12.7.7 of Hörmander [17] thanks to the hypothesis of weak hyperbolicity of $P(D)$, and therefore for suitable constants $c_1, C_1, c_2, C_2, C' > 0$ and all $\xi \in \mathbb{R}^n, t \in [-T,T]$ we have:

$$|F_j(t,\xi)| \le (C_1\langle\xi\rangle)^{m+1} c_1 \exp(C'|t||\xi|_{\mathcal{P}}^{\frac{1}{s}}) \le c_2 \exp(C_2(1+|t|)|\xi|_{\mathcal{P}}^{\frac{1}{s}}).$$

Then we can estimate $\hat{u}(t,\xi)$ given by (18) as follows: for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|\hat{u}(t,\xi)| \le \sum_{j=0}^{m-1} c_2 C_\varepsilon \exp(\varepsilon|\xi|_{\mathcal{P}}^{\frac{1}{r}}) \exp(C_2(1+|t|)|\xi|_{\mathcal{P}}^{\frac{1}{s}}).$$

As $r < s$, for $t \in [-T,T]$ and for all $\varepsilon' > 0$, taking ε sufficiently small (depending on ε', r, s, T), there is a constant $C_{\varepsilon'} > 0$ such that it holds $|\hat{u}(t,\xi)| \le C_{\varepsilon'} \exp(\varepsilon'|\xi|_{\mathcal{P}}^{\frac{1}{r}})$, $\forall \xi \in \mathbb{R}^n$; this proves that for every fixed $t \in [-T,T]$ we have $u \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$. Similar estimates for $\partial_t^j \hat{u}(t,\xi)$ (cf. [17], Lemma 12.7.7) show that $\partial_t^j u \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$ for all $t \in [-T,T]$; since T is arbitrarily fixed, we conclude that $u \in C^\infty(\mathbb{R}, \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$. □

We finally illustrate the notion of multi-quasi-hyperbolicity and the previous result by some examples.

1. Let us consider the case when $\mathcal{P}$ is the Newton polyhedron of an elliptic operator, i.e. $|\xi|_{\mathcal{P}} = \langle\xi\rangle$. From Proposition 4.5, any differential operator $P(D) = P_m(D) + Q(D)$ in $\mathbb{R}^{n+1}$, such that its principal part $P_m(D)$ is hyperbolic and $Q(D)$ has order $q < m$, is $\frac{m}{q}$-hyperbolic; therefore, for any $r < \frac{m}{q}$ and any data $u_k \in \mathcal{D}'_r(\mathbb{R}^n)$, $k = 0, \ldots, m-1$, there is a unique solution $u \in C^\infty(\mathbb{R}, \mathcal{D}'_r(\mathbb{R}^n))$

of the corresponding Cauchy problem.

If $P_m(D)$ is hyperbolic, with multiplicity of the characteristics equal to M and $q = m - M + k$ for a given k, $0 < k < M$, then $P(D)$ is $\frac{M}{k}$-hyperbolic. When $k = M - 1$, (16) is always satisfied and we obtain the well-known result of G^s well-posedness for $s < \frac{M}{M-1}$ (cf. for instance [21]).

2. Let $\mathcal{P}$ be the complete polyhedron in $\mathbb{R}^2$ with vertices $\mathcal{V}(\mathcal{P}) = \{(0,0),\ (0,2),\ (1,0)\}$; then the associated weight function is $|(\xi,\eta)|_{\mathcal{P}} = \left(1 + |\xi| + |\eta^2|\right)^{\frac{1}{2}}$. Consider the operator of order 3:

$$P(D_x, D_y, D_t) = P_3(D_x, D_y, D_t) + P_2(D_x, D_y, D_t) + P_1(D_x, D_y, D_t) + c, \tag{19}$$

such that the principal part $P_3(D_x, D_y, D_t)$ is hyperbolic, $P_2(D_x, D_y, D_t) = c_1 D_y^2$ and $P_1(D_x, D_y, D_t)$ is any operator of order 1, with $c_1, c \in \mathbb{C}$. It is multi-quasi-hyperbolic of order $\frac{3}{2}$ with respect to $\mathcal{P}$ (cf. Proposition 4.5) and therefore, from Theorem 4.3, for any $r < \frac{3}{2}$ and any data $u_k \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$, $k = 0, 1, 2$, there is a unique solution $u \in C^\infty(\mathbb{R}, \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n))$ of the corresponding Cauchy problem.

3. Let $\mathcal{P}$ be the same polyhedron as in Example 2; then if we ask the conditions of Proposition 4.5 with $M = 2$, $k = 1$ to the operator (19), we have that $P_3(D_x, D_y, D_t)$ is hyperbolic with multiplicity of the characteristics equal to 2 and the lower order terms must be of the following kind: $P_2(D_x, D_y, D_t) = c_1 D_y^2 + c_2 D_x D_y + c_3 D_y D_t$ and $P_1(D_x, D_y, D_t)$ is any operator of order 1, with $c_1, c_2, c_3 \in \mathbb{C}$. This implies that $P(D)$ is multi-quasi-hyperbolic of order 2 with respect to $\mathcal{P}$ and for any $r < 2$ and any data $u_k \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$, $k = 0, 1, 2$, there is a unique solution $u \in C^\infty(\mathbb{R}, \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n))$ of the corresponding Cauchy problem. Observe that the lower order terms here are more general than in Example 2.

5 Acknowledgements

The authors are grateful to Prof. O. Liess, University of Bologna, and Prof. L. Rodino, University of Torino, for useful ideas and valuable remarks during the writing of this paper.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.2,163-173,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Periodic Solutions for p-Laplacian Duffing Equations with a Deviating Argument

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Abstract

In this paper, by means of Mawhin's continuation theorem, the existence of periodic solutions for a p-Laplacian Duffing equation with deviating argument is obtained.

Keywords: periodic solution, Mawhin's continuation theorem, deviating argument.

2000 AMS Subject Classification: 34K13, 34L30

1. INTRODUCTION

Consider the p-Laplacian Duffing equation with a deviating argument

$$(\varphi_p(x'(t)))' + g(x(t-\tau(t))) = e(t), \tag{1.1}$$

where $p > 1$ is a constant, $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$ is a one-dimensional p-Laplacian, e, τ are periodic functions with period $T > 0$, and $g, e, \tau \in C(\mathbb{R}, \mathbb{R})$.

There has been a great deal of research works on such an equation which is used to describe fluid mechanical and nonlinear elastic mechanical phenomena. For example, when $p = 2$ and $\tau(t) \equiv 0$, the existence of $T-$periodic solutions to Eq.(1.1) was extensively studied in [1-3]. In [4-6, 9, 13], by using the time maps and the phase plane analysis, the authors discussed the existence of periodic solutions to Eq.(1.1) for $p \neq 2$ and $\tau(t) \equiv 0$. On the other hand, for $p = 2$ and $\tau(t) \not\equiv 0$, the existence of $T-$periodic solutions to several second order scalar differential equations were also studied in [8, 10-12]. In [8], X. Huang and Z. Xiang studied the following type of Duffing equation with a single constant deviating argument

$$x''(t) + g(x(t-\tau)) = p(t). \tag{1.2}$$

Under a one-sided boundedness condition imposed on $g(x)$ such as

$$|g(x)| < R_0 \text{ for } x > M, \tag{1.3}$$

where $M > 0, R_0 > 0$ are constants, and a signal condition $xg(x) > 0$ for $|x| > M$, the authors obtained a periodic solution for Eq.(1.2). In [12], S. Ma, Z. Wang and J. Yu studied delay Duffing equations of the type

$$x''(t) + m^2 x(t) + g(x(t-\tau)) = p(t). \tag{1.4}$$

They established several criteria to guarantee the existence of periodic solutions of Eq.(1.4) by assuming

$$\sup_{x \in R} |g(x)| < \infty. \tag{1.5}$$

Recently, S. Lu and W. Ge in [10] discussed the existence of periodic solutions for the second order differential equation with multiple deviating arguments

$$x''(t) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t) g(x(t-\gamma_j(t))) = p(t). \tag{1.6}$$

In their work, some linear growth condition imposed on $g(x)$ such as

$$\lim_{|x| \to +\infty} \frac{|g(x)|}{|x|} = r \in [0, +\infty). \tag{1.7}$$

was needed.

The main technique of these works [8, 10-12] is to convert the problem into the abstract form $Lx = Nx$, with L being a non-invertible linear operator. Thus the existence of solutions of the problem can be given by the Mawhin's continuation theorem [7]. But as far as we are aware of, the corresponding problem of Eq.(1.1) with $p \neq 2$ and $\tau(t) \not\equiv 0$ has never been studied. This is mainly due to the fact that the Mawhin's continuation theorem is not applicable directly since the p-Laplacian $\varphi_p(u) = |u|^{p-2}u$ is not linear with respect to u except when $p = 2$.

In this paper, we translate equation (1.1) into a two-dimension system to ensure Mawhin's continuation theorem can be applied. This method can also be used to solve problems for other equations with p-Laplacian. Moreover, the one-side growth condition we impose on $g(x)$ in order to obtain *a priori* bound of periodic solutions for Eq.(1.1) is weaker than the corresponding ones in (1.3), (1.5) and (1.7).

2. MAIN RESULT

First, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach Spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = Ker\ L \oplus X_1$ and $Y = Im\ L \oplus Y_1$ and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Now, let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be $L-$ compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to X$ is compact.

MAWHIN'S CONTINUATION THEOREM [7] Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is $L-$ compact on $\overline{\Omega}$. If

(1)$Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

(2)$Nx \notin Im\ L, \forall x \in \partial\Omega \cap Ker\ L$; and

(3)$deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$, where $J : Im\ Q \to Ker\ L$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \bigcap D(L)$.

In order to use Mawhin's continuation theorem to study the existence of $T-$periodic solutions for Eq.(1.1), we rewrite Eq.(1.1) in the following form

$$\begin{cases} x_1'(t) &= \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -g(x_1(t-\tau(t))) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a $T-$periodic solution to Eqs.(2.1), then $x_1(t)$ must be a $T-$periodic solution to Eq.(1.1). Thus, the problem of finding a $T-$periodic solution for Eq. (1.1) reduces to finding one for Eq. (2.1).

Now, we set $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) :\ \phi(t+T) \equiv \phi(t)\}$ with norm $|\phi|_0 = \max_{t\in[0,T]} |\phi(t)|$, $X = Y = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) :\ x(t) \equiv x(t+T)\}$ with norm $||x|| = \max\{|x_1|_0, |x_2|_0\}$.

Clearly, X and Y are Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$N : X \to Y, \quad Nx = \begin{pmatrix} \varphi_q(x_2) \\ -g(x_1(t-\tau(t))) + e(t). \end{pmatrix}$$

It is easy to see that $Ker\ L = \mathbb{R}^2$, $Im\ L = \{y \in Y : \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds; \quad Qy = \frac{1}{T}\int_0^T y(s)ds,$$

and let K denote the inverse of $L|_{KerP \cap D(L)}$. Obviously, $KerL = ImQ = R^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds. \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{s}{T}, & 0 \le s < t \le T. \\ \dfrac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

From (2.2), one can easily see that N is $L-$compact on $\overline{\Omega}$, where Ω is an open, bounded subset of X.

THEOREM 1. Suppose the following conditions are satisfied,

[A1] $\int_0^T e(s)ds = 0$ and $e(t) \not\equiv 0$;

[A2] there exists a constant $d > 0$ such that $ug(u) > 0$ for $|u| > d$ or $ug(u) < 0$ for $|u| > d$;

[A3] there is a constant $r_0 \ge 0$ such that $\lim\limits_{u\to-\infty} \frac{|g(u)|}{|u|^{p-1}} = r_0$,

then Eq.(1.1) has at least one $T-$periodic solutions if $2r_0T^p < 1$.

PROOF. Considering the following operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1). \tag{2.3}$$

Let $\Omega_1 \in \{x : x \in X, Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from (2.3), we see

$$\begin{cases} x_1'(t) = \lambda\varphi_q(x_2(t)) = \lambda|x_2(t)|^{q-2}x_2(t) \\ x_2'(t) = -\lambda g(x_1(t-\tau(t))) + \lambda e(t) \end{cases} \tag{2.4}$$

From the first equation of (2.4), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$. Hence by the second equation of (2.4),

$$[\varphi_p(\frac{1}{\lambda}x_1'(t))]' + \lambda g(x_1(t-\tau(t))) = \lambda e(t),$$

i.e.,

$$[\varphi_p(x_1'(t))]' + \lambda^p g(x_1(t-\tau(t))) = \lambda^p e(t). \tag{2.5}$$

Integrating the two sides of (2.5) on $[0, T]$, we get

$$\int_0^T g(x_1(t-\tau(t)))dt = 0 \tag{2.6}$$

So there is a constant $\xi \in [0, T]$ such that

$$g(x_1(\xi - \tau(\xi))) = 0.$$

From assumption $[A2]$, we see that $|x_1(\xi - \tau(\xi))| \leq d$. Write $\xi - \tau(\xi) = kT + t_0$, where $k \in \mathbb{Z}$ and $t_0 \in [0, T)$. Then

$$|x_1(t_0)| = |x_1(\xi - \tau(\xi))| \leq d,$$

which implies

$$|x_1|_0 \leq d + \int_0^T |x_1'(s)|ds. \tag{2.7}$$

On the other hand, taking absolute values and integrating both sides of Eq.(2.5) on $[0, T]$, we have

$$\begin{aligned} \textstyle\int_0^T |[\varphi_p(x_1'(t))]'|dt &\leq \lambda^p[\textstyle\int_0^T |g(x_1(t-\tau(t)))|dt + \int_0^T |e(t)|dt] \\ &< \textstyle\int_0^T |g(x_1(t-\tau(t)))|dt + \int_0^T |e(t)|dt. \end{aligned} \tag{2.8}$$

In view of $r_0 T^p < 1$, it is easy to see that there is a constant $\varepsilon > 0$ (independent of λ) such that

$$(r_0 + \varepsilon)T^p < 1. \tag{2.9}$$

For such a constant $\varepsilon > 0$, we have from assumption $[A3]$ that there is a constant $\rho > d$ (independent of λ) such that

$$|g(u)| \leq (r_0 + \varepsilon)|u|^{p-1}, \quad \text{for} \quad u < -\rho. \tag{2.10}$$

Let

$$\begin{aligned} E_1 &= \{t \in [0, T] : \ |x_1(t-\tau(t))| \leq \rho\}, \\ E_2 &= \{t \in [0, T] : \ x_1(t-\tau(t)) > \rho\}, \\ E_2 &= \{t \in [0, T] : \ x_1(t-\tau(t)) < -\rho\}. \end{aligned}$$

From (2.6), we know that

$$(\int_{E_1} + \int_{E_2} + \int_{E_3})g(x_1(t-\tau(t)))dt = 0.$$

It follows from [A2] that

$$\int_{E_2} |g(x_1(t-\tau(t)))|dt = |\int_{E_2} g(x_1(t-\tau(t)))dt| \leq \int_{E_1} |g(x_1(t-\tau(t)))|dt + \int_{E_3} |g(x_1(t-\tau(t)))|dt$$

and then by (2.10), we have

$$\begin{aligned} \textstyle\int_0^T |g(x_1(t-\tau(t)))|dt &= (\textstyle\int_{E_1} + \int_{E_2} + \int_{E_3})|g(x_1(t-\tau(t)))|dt \\ &\leq 2\textstyle\int_{E_1} |g(x_1(t-\tau(t)))|dt + 2\int_{E_3} |g(x_1(t-\tau(t)))|dt \\ &\leq 2g_\rho T + 2(r_0 + \varepsilon)T|x_1|_0^{p-1}, \end{aligned} \tag{2.11}$$

where $g_\rho = \max_{|u|\le\rho} |g(u)|$. Substituting (2.11) into (2.8), we get

$$\int_0^T |[\varphi_p(x_1'(t))]'|dt < 2g_\rho T + |e|_1 + 2(r_0+\varepsilon)T|x_1|_0^{p-1}, \tag{2.12}$$

where $|e|_1 = \int_0^T |e(s)|ds$.

From $\varphi_p(x_1'(0)) = \varphi_p(x_1'(T))$, we know that there is $\eta \in [0,T]$ such that

$$\varphi_p(x_1'(\eta)) = 0.$$

So

$$\begin{aligned} |\varphi_p(x_1')|_0 &\le \int_0^T |\varphi_p(x_1'(t))|'dt \\ &\le 2g_\rho T + |e|_1 + 2(r_0+\varepsilon)T|x_1|_0^{p-1}. \end{aligned}$$

In view of $|\varphi_p(x_1')|_0 = |x_1'|_0^{p-1}$, we get

$$|x_1'|_0^{p-1} \le 2(r_0+\varepsilon)T|x_1|_0^{p-1} + 2g_\rho T + |e|_1.$$

By (2.7),

$$|x_1'|_0^{p-1} < 2(r_0+\varepsilon)T(d + \int_0^T |x_1'(s)|ds)^{p-1} + 2g_\rho T + |e|_1. \tag{2.13}$$

Next we prove that there exists a constant $R > 0$ such that

$$|x_1'|_0 \le R. \tag{2.14}$$

Case 1. If $1 < p \le 2$, i.e., $0 < p-1 \le 1$, then from (2.13) we know that

$$\begin{aligned} |x_1'|_0^{p-1} &\le 2(r_0+\varepsilon)Td^{p-1} + 2(r_0+\varepsilon)T(\int_0^T |x_1'(s)|ds)^{p-1} + 2g_\rho T + |e|_1 \\ &\le 2(r_0+\varepsilon)Td^{p-1} + 2(r_0+\varepsilon)T^p|x_1'|_0^{p-1} + 2g_\rho T + |e|_1. \end{aligned}$$

By (2.9), we have

$$|x_1'|_0^{p-1} < \frac{2(r_0+\varepsilon)Td^{p-1} + 2g_\rho T + |e|_1}{1 - 2(r_0+\varepsilon)T^p},$$

i.e.,

$$|x_1'|_0 < \left[\frac{2(r_0+\varepsilon)Td^{p-1} + 2g_\rho T + |e|_1}{1 - 2(r_0+\varepsilon)T^p}\right]^{\frac{1}{p-1}} := R_1 \tag{2.15}$$

Case 2. If $p > 2$, then $p - 1 > 1$. We know from [A1] that $x_1(t)$ is not constant and then $\int_0^T |x_1'(s)|ds > 0$. In fact, if $x_1(t)$ is a constant, then by (2.5), $e(t) = c$ and this contradicts assumption [A1]. Now it is elementary to check that there is a constant $h > 0$ (independent of λ) such that

$$(1+u)^{p-1} < 1 + pu, \qquad \forall u \in (0,h]. \tag{2.16}$$

(i) If $\frac{d}{T|x_1'|_0} \ge h$, then

$$|x_1'|_0 \le \frac{d}{Th}. \tag{2.17}$$

(ii) If $\frac{d}{T|x_1'|_0} < h$, then from (2.13) and (2.16),

$$\begin{aligned} |x_1'|_0^{p-1} &< 2(r_0+\varepsilon)T(d + T|x_1'|_0)^{p-1} + 2g_\rho T + |e|_1 \\ &= 2(r_0+\varepsilon)T \cdot T^{p-1}|x_1'|_0^{p-1}[1 + \tfrac{d}{T|x_1'|_0}]^{p-1} + 2g_\rho T + |e|_1 \\ &\le 2(r_0+\varepsilon)T^p|x_1'|_0^{p-1}[1 + \tfrac{pd}{T|x_1'|_0}] + 2g_\rho T + |e|_1 \\ &= 2(r_0+\varepsilon)T^p|x_1'|_0^{p-1} + 2pd(r_0+\varepsilon)T^{p-1}|x_1'|_0^{p-2} + 2g_\rho T + |e|_1, \end{aligned}$$

and then

$$[1-2(r_0+\varepsilon)T^p]|x_1'|_0^{p-1} < 2pd(r_0+\varepsilon)T^{p-1}|x_1'|_0^{p-2} + 2g_\rho T + |e|_1.$$

From (2.9) and $p-1 > p-2$, it follows that there exists $R_2 > 0$ such that

$$|x_1'|_0 \leq R_2. \tag{2.18}$$

Let $R = \max\{R_1, R_2, \frac{d}{Th}\}$. It is easy to see from (2.15), (2.17) and (2.18) that (2.14) holds in any case. Thus by (2.6),

$$|x_1|_0 \leq d + T|x_1'|_0 \leq d + TR := M_1. \tag{2.19}$$

By the first equation of (2.4), we have

$$\int_0^T |x_2(s)|^{q-2}x_2(s)ds = 0,$$

which implies that there is a constant $t_2 \in [0,T]$ such that $x_2(t_2) = 0$. So

$$|x_2|_0 \leq \int_0^T |x_2'(s)|ds. \tag{2.20}$$

On the other hand, by using the second equation of (2.4), we obtain

$$\int_0^T |x_2'(s)|ds \leq \lambda(g_{M_1}T + |e|_1) < g_{M_1}T + |e|_1,$$

where $g_{M_1} = \max\limits_{|u|\leq M_1} |g(u)|$. So from (2.20), we have

$$|x_2|_0 \leq g_{M_1}T + |e|_1 := M_2. \tag{2.21}$$

Let $\Omega_2 = \{x \in Ker\ L : Nx \in Im\ L\}$. If $x \in \Omega_2$, then $x \in Ker\ L$ and $QNx = 0$. From assumption [A1] we see that

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ g(x_1) = 0. \end{cases} \tag{2.22}$$

So

$$|x_1| \leq d \leq M_1, \qquad x_2 = 0 \leq M_2. \tag{2.23}$$

Let $\Omega = \{x = (x_1, x_2)^\top \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where N_1 and N_2 are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > dg_d$, where $g_d = \max\limits_{|u|\leq d} |g(u)|$. Then $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$. From (2.19), (2.21) and (2.23), it is easy to see that conditions (1) and (2) of Mawhin's Continuation Theorem are satisfied.

Next, we claim that condition (3) of Mawhin's Continuation Theorem is also satisfied. For this, define the isomorphism $J : Im\ Q \to Ker\ L$ by

$$J(x_1, x_2) = \begin{cases} (x_2, x_1), & \text{if } \ ug(u) < 0 \text{ for } |u| > d, \\ (-x_2, x_1), & \text{if } \ ug(u) > 0 \text{ for } |u| > d, \end{cases}$$

and let $H(v, \mu) := \mu v + \frac{1-\mu}{T} JQNv, \quad (v, \mu) \in \Omega \times [0,1]$. By simple calculation, we obtain, for $(x, \mu) \in \partial(\Omega \cap KerL) \times [0,1]$,

$$x^\top H(x,\mu) = \begin{cases} \mu(x_1^2 + x_2^2) + \frac{1-\mu}{T}(-x_1 g(x_1) + |x_2|^q) > 0, & \text{if } \ ug(u) < 0 \text{ for } |u| > d, \\ \mu(x_1^2 + x_2^2) + \frac{1-\mu}{T}(x_1 g(x_1) + |x_2|^q) > 0, & \text{if } \ ug(u) > 0 \text{ for } |u| > d. \end{cases}$$

Hence

$$\begin{aligned} & deg\{JQN, \Omega \cap KerL, 0\} = deg\{H(x,0), \Omega \cap KerL, 0\} \\ = \ & deg\{H(x,1), \Omega \cap KerL, 0\} = deg\{I, \Omega \cap KerL, 0\} \\ \neq \ & 0, \end{aligned}$$

and so condition (3) of Mawhin's Continuation Theorem is also satisfied.

Therefore, by Mawhin's Continuation Theorem, we conclude that equation

$$Lx = Nx$$

has a solution $x(t) = (x_1(t), x_2(t))^\top$ on $\overline{\Omega}$, i.e., Eq.(1.1) has a $T-$periodic solution $x_1(t)$ with $|x_1|_0 \leq M_2$.

THEOREM 2 Suppose assumptions [A1] and [A2] in Theorem 1 hold, and [A3] is replaced by

[A3]$'$ there is a constant $r_1 \geq 0$ such that $\lim\limits_{u \to +\infty} \frac{|g(u)|}{u^{p-1}} = r_1$,

then Eq. (1.1) has at least one $T-$periodic solution if $2r_1T^p < 1$.

In fact, if [A3]$'$ holds, then (2.10) can be replaced by

$$|g(u)| \leq |r_1 + \varepsilon| u^{p-1}, \quad \text{for} \;\; u > \rho.$$

So it follows from

$$\left(\int_{E_1} + \int_{E_2} + \int_{E_3}\right) g(x_1(t - \tau(t)))dt = 0$$

that

$$\int_{E_3} |g(x_1(t - \tau(t)))|dt = |\int_{E_1} g(x_1(t - \tau(t)))dt| \leq \int_{E_1} |g(x_1(t - \tau(t)))|dt + \int_{E_2} |g(x_1(t - \tau(t)))|dt$$

and so (2.12) is satisfied. The rest of the proof is analogous to the proof of Theorem 1.

REMARK 1. It is obviously that the one-side linear growth condition [A3] (or [A3]$'$) is weaker than the corresponding ones in (1.3), (1.5) and (1.7).

Acknowledgements

Research of the first author was partially supported by the Research Grants Council of the Hong Kong SAR, China (Project No. HKU7040/03P)

Research of the second author was partially supported by the National Natural Science Foundation, China (Project No. 10371006)

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BOSBACH AND RIEČAN STATES ON RESIDUATED LATTICES

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Abstract. Residuated lattices were introduced firstly as generalization of ideal lattices of rings and they served as algebraic structures for substructural logics. The notion of a state is an analogue to probability measure and it has been studied for different types of non-commutative fuzzy structures such as pseudo MV-algebras, pseudo BL-algebras and bounded non-commutative Rℓ-monoids. In this paper we investigate the states on residuated lattices and we show that the extension of Georgescu's original problem from pseudo BL-algebras has negative solution for good residuated lattices.

1. Introduction

The notion of a state is an analogue to probability measure and it has a very important role in the theory of quantum structures ([10]). The state on MV-algebras was introduced firstly by F.Kôpka and F.Chovanec ([21]) and the state on BL-algebras was introduced by B.Riečan ([23]). In the case of non-commutative fuzzy structures, the states were introduced by A. Dvurečenskij ([7]) for pseudo MV-algebras, by G. Georgescu ([13]) for pseudo BL-algebras and by A.Dvurečenskij and J.Rachůnek ([11]) for bounded non-commutative Rℓ-monoids.
In the case of a pseudo MV-algebra M, A.Dvurečenskij proved in [6] that there is an ℓ-group (G, u) with strong unit u such that M is isomorphic to $\Gamma(G, u) = \{g \in G/0 \leq g \leq u\}$. This allowed him to define a partial addition $+$, that is $x + y$ is defined if $x \leq y^- = u - y$ and the state is a mapping $s : M \to [0, 1]$ which preserves the partial addition $+$ and $s(1) = 1$. We recall that the elements a and b are orthogonal if $a + b$ is defined in M.
The other non-commutative structures don't have such a group representation and it was more difficult to define the notion of states for these structures.
We recall that a state on MV-algebras always exists in contrast to pseudo MV-algebras ([7]), on the other hand, in [9] it was solved the existence of states for linear pseudo BL-algebras (see also [8]).
In the case of pseudo BL-algebras G.Georgescu defined in [13] the Bosbach state and this definition was generalized by A. Dvurečenskij and J. Rachůnek ([11]) for non-commutative Rℓ-monoids.
For a good pseudo BL-algebra G.Georgescu proved that any Bosbach state is also a Riečan state, but he formulated as open problem to find an example of Riečan state on a good pseudo BL-algebra which is not a Bosbach state.
Inspired by the above mentioned results, in this paper we extend the notion of states to residuated lattices and the final results consist of proving that any Bosbach state on a good residuated lattice is a Riečan state, but conversely it turns out not to be true.

Date: 2006.03.15.
Key words and phrases. Residuated lattice, Bosbach state, Riečan state, orthogonal elements, normal filter.
2000 *Mathematics Subject Classification 06B05, 03G25, 28E15.*

The paper is organized as follows.
In Section 2 we give the definition of a residuated lattice and we prove the basic properties of this structure which are also valid for other structures such as pseudo BL-algebras, weak pseudo BL-algebras and bounded non-commutative Rℓ-monoids. The distance functions defined in this section are very important for the main results in the next section.
In Section 3 we define the Bosbach and Riečan states on a residuated lattice and we investigate their basic properties, proving that any Bosbach state is a Riečan state.
As an answer to Georgescu's open problem, we give an example of a Riečan state on a good residuated lattice which is not a Bosbach state.
We refer to [3], [16] and [22] for general notions on lattices theory and for unexplained notions and results on residuated lattices.

2. Residuated lattices and their basic properties

Definition 2.1. A *residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:
(L_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice ;
(L_2) $(L, \odot, 1)$ is a monoid ;
(L_3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in L$.

In the sequel we agree that the operations $\wedge, \vee, \odot$ have higher priority than the operations $\rightarrow, \rightsquigarrow$.

Examples 2.2. Let's consider $L = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	a	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

.

Then $(L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a residuated lattice.

Examples 2.3. [4],[5] Let's consider a pseudo MV-algebra $\mathbf{M} = (M, \oplus, ^-, ^\sim, 0, 1)$ with the additional operation $x \odot y = (y^- \oplus x^-)^\sim$.
The order on $\mathbf{M}$ is defined by $x \leq y$ iff $x^- \oplus y = 1$ (iff $y \oplus x^\sim = 1$).
Defining $x \wedge y = x \odot (x^- \oplus y)$ and $x \vee y = x \oplus x^\sim \odot y$, according to [14], Prop.1.13, $(M, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.
Applying [14], Prop.1.7, $(M, \odot, 1)$ is a non-commutative monoid.
If we define $x \rightarrow y = y \oplus x^\sim$ and $x \rightsquigarrow y = x^- \oplus y$, then according to [14], Prop.1.12 we have $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$.
Thus, $\mathbf{M} = (M, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded residuated lattice.

Remark 2.4. (1) If additionally, for any $x, y \in L$ the structure $\mathbf{L}$ satisfies the conditions:
(L_4) $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$,
(L_5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$
then $\mathbf{L}$ is a *pseudo BL-algebra*.
(2) If $\mathbf{L}$ satisfies the conditions (L_1), (L_2), (L_3) and (L_5), then it is *a weak pseudo BL-algebra* (or *pseudo MTL-algebra*).

(3) If $\mathbf{L}$ satisfies the conditions (L_1), (L_2), (L_3) and (L_4), then it is a *bounded Rℓ-monoid* (*divisible residuated lattice*) ([11]).

One can easily prove that the structure $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ from Example 2.2 is a pseudo MTL-algebra, but it is not a pseudo BL-algebra because $(b \rightarrow a) \odot b \neq b \odot (b \rightsquigarrow a)$, so (L_4) does not hold.
There are residuated lattices that are not pseudo MTL-algebras. Indeed, put $L' = L \oplus (L_{2\times 2} \oplus L_3)$, where $L_{2\times 2} = L_2 \times L_2$ and $\oplus$ represents the ordinal sum of structures (see [19]). Note that $L_{2\times 2} \oplus L_3$ does not satisfy (L_5), therefore L' doesn't satisfy it either. Also, L' doesn't satisfy (L_4) because L doesn't. Thus, L' is a residuated lattice that is neither pseudo MTL-algebra, nor Rℓ-monoid.

Definition 2.5. A residuated lattice is called *commutative* if $x \odot y = y \odot x$.

Proposition 2.6. *A residuated lattice is commutative iff $x \rightarrow y = x \rightsquigarrow y$.*

Proof. " $\Rightarrow$ ": For any $x, y \in L$ we have the following equivalences:

$$x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z \Leftrightarrow x \leq y \rightsquigarrow z,$$

hence, $y \rightarrow z = y \rightsquigarrow z$.
" $\Leftarrow$ ": For any $z \in L$, $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow x \leq y \rightsquigarrow z \Leftrightarrow y \odot x \leq z$.
Thus, $x \odot y = y \odot x$ (indeed, for $z = y \odot x$ we have $x \odot y \leq y \odot x$ and for $z = x \odot y$ we get $y \odot x \leq x \odot y$) □

In a residuated lattice $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ we define two negations for all $x \in L : x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$.

Proposition 2.7. ([1],[20]) *In any residuated lattice the following properties hold:*
(1) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$;
(2) $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(3) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$;
(4) $x \rightarrow x = x \rightsquigarrow x = 1$;
(5) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
(6) $0 \rightarrow x = 0 \rightsquigarrow x = 1$;
(7) $x \odot 0 = 0 \odot x = 0$;
(8) $(x \rightarrow y) \odot x \leq y$ *and* $x \odot (x \rightsquigarrow y) \leq y$;
(9) $x \leq y \rightarrow (x \odot y)$ *and* $x \leq y \rightsquigarrow (y \odot x)$;
(10) $x \leq y$ *implies* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$ *for any* $z \in L$;
(11) $(x \rightarrow y) \odot x \leq x \wedge y$ *and* $x \odot (x \rightsquigarrow y) \leq x \wedge y$;
(12) $(x \rightarrow y) \odot x \leq x \leq y \rightarrow (x \odot y)$ *and* $(x \rightarrow y) \odot x \leq y \leq x \rightarrow (y \odot x)$ *;*
(13) $x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y)$ *and* $x \odot (x \rightsquigarrow y) \leq x \leq y \rightsquigarrow (y \odot x)$;
(14) *If* $x \leq y$ *then* $z \rightarrow x \leq z \rightarrow y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(15) *If* $x \leq y$ *then* $y \rightarrow z \leq x \rightarrow z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(16) $1 \rightarrow x = x$ *and* $1 \rightsquigarrow x = x$;
(17) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$;
(18) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$ *;*
(19) $x \rightarrow y = x \rightarrow (x \wedge y)$;
(20) $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$;
(21) $y \leq x \rightarrow y$ *and* $y \leq x \rightsquigarrow y$;
(22) *If* $x \leq y$ *then* $x \leq z \rightarrow y$ *and* $x \leq z \rightsquigarrow y$ *;*
(23) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$;

(24) $x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$;
(25) $1^- = 1^\sim = 0$ *and* $0^- = 0^\sim = 1$;
(26) $x^- \odot x = 0$ *and* $x \odot x^\sim = 0$;
(27) $x \leq y^-$ *iff* $x \odot y = 0$ *and* $x \leq y^\sim = 0$ *iff* $y \odot x = 0$;
(28) $x \leq x^{-\sim}$ *and* $x \leq x^{\sim -}$;
(29) $x \to y^- = (x \odot y)^-$ *and* $x \rightsquigarrow y^\sim = (y \odot x)^\sim$;
(30) $x \leq y^-$ *iff* $y \leq x^\sim$;
(31) *If* $x \leq y$, *then* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;
(32) $x \leq x^\sim \to y$ *and* $x \leq x^- \rightsquigarrow y$;
(33) $x \to y \leq y^- \rightsquigarrow x^-$ *and* $x \rightsquigarrow y \leq\rightsquigarrow y^\sim \to x^\sim$;
(34) $x \to y^\sim = y \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^- = y \to x^\sim$;
(35) $x^{\sim-\sim} = x^\sim$ *and* $x^{-\sim-} = x^-$;
(36) $x \to x^\sim = x \rightsquigarrow x^-$;
(37) $x \odot (\vee_{i\in I} y_i) = \vee_{i\in I}(x \odot y_i)$;
(38) $(\vee_{i\in I} y_i) \odot x = \vee_{i\in I}(y_i \odot x)$;
(39) $(\vee_{i\in I} x_i) \rightsquigarrow y = \wedge_{i\in I}(x_i \rightsquigarrow y)$;
(40) $(\vee_{i\in I} x_i) \to y = \wedge_{i\in I}(x_i \to y)$;
(41) $y \rightsquigarrow (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \rightsquigarrow x_i)$;
(42) $y \to (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \to x_i)$;
(43) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(44) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(45) $(x \vee y) \to (x \wedge y) = (x \to y) \wedge (y \to x)$;
(46) $(x \vee y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$;
(47) $(x \vee y)^- = x^- \wedge y^-$ *and* $(x \vee y)^\sim = x^\sim \wedge y^\sim$;
(48) $(x \wedge y)^- \geq x^- \vee y^-$ *and* $(x \wedge y)^- \geq x^\sim \vee y^\sim$;
(49) $(x \vee y)^{-\sim} \geq x^{-\sim} \vee y^{-\sim}$ *and* $(x \vee y)^{\sim-} \geq x^{\sim-} \vee y^{\sim-}$;
(50) $y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim} = x \to y^{-\sim}$;
(51) $y^\sim \to x^\sim = x^{\sim-} \rightsquigarrow y^{\sim-} = x \rightsquigarrow y^{\sim-}$.

If a residuated lattice is a chain, then it is a weak pseudo BL-algebra.

Definition 2.8. ([4], [5]) A residuated lattice L is *good* if $x^{-\sim} = x^{\sim-}$ for any $x \in L$.

Proposition 2.9. *In any good residuated lattice we have* $(x^\sim \odot y^\sim)^- = (x^- \odot y^-)^\sim$.

Proof. Applying Proposition 2.7(29),(50),(51) we have:

$$\begin{aligned}(x^\sim \odot y^\sim)^- &= x^\sim \to y^{\sim-} = x^\sim \to y^{-\sim} = y^{-\sim-} \rightsquigarrow x^{\sim-}\\ &= y^- \rightsquigarrow x^{\sim-} = y^- \rightsquigarrow x^{-\sim} = (x^- \odot y^-)^\sim.\end{aligned}$$

(In the last equality we also applied Proposition 2.7(29)). □

Proposition 2.10. *In any good residuated lattice we have* $x^{-\sim} \odot y^{-\sim} \leq (x \odot y)^{-\sim}$.

Proof. Because the residuated lattice is good and by Proposition 2.7(8), we have:

$$\begin{aligned}(x \odot y)^{-\sim} &= (x \odot y)^{\sim-} \geq (x \odot y)^{\sim-} \wedge x^{\sim-} \geq x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{\sim-})\\ &= x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{-\sim}) = x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \to y^-))^\sim.\end{aligned}$$

But, applying Proposition 2.7(29) and Proposition 2.7(2) we have:

$$\begin{aligned}x^{\sim-} \rightsquigarrow (x \to y^-)^\sim &= x^{\sim-} \rightsquigarrow ((x \to y^-) \rightsquigarrow 0) = (x^{\sim-} \to y^-) \odot x^{\sim-} \rightsquigarrow 0\\ &= ((x^{\sim-} \to y^-) \odot x^{\sim-})^\sim \geq (x^{\sim-} \wedge y^-)^\sim \geq x^{\sim-\sim} \vee y^{-\sim} = x^\sim \vee y^{-\sim}.\end{aligned}$$

(By Proposition 2.7(29) we have $(x^{\sim-} \to y^-) \odot x^{\sim-} \leq (x^{\sim-} \wedge y^-)$, so $((x^{\sim-} \to y^-) \odot x^{\sim-})^\sim \geq (x^{\sim-} \wedge y^-)^\sim)$.
It follows that

$$\begin{aligned}(x \odot y)^{-\sim} &\geq x^{\sim-} \odot (x^\sim \vee y^{-\sim}) = (x^{\sim-} \odot x^\sim) \vee (x^{\sim-} \odot y^{-\sim}) \\ &= 0 \vee (x^{\sim-} \odot y^{-\sim}) = x^{\sim-} \odot y^{\sim-} = x^{-\sim} \odot y^{-\sim}.\end{aligned}$$

(We applied Proposition 2.7(37-38) and Proposition 2.7(26)). □

Proposition 2.11. *Let L be a good residuated lattice. We define a total binary operation $\oplus$ on L by $x \oplus y := (y^\sim \odot x^\sim)^-, x, y \in L$. Then for all $x, y, z \in L$ we have:*
(1) $x \oplus y := (y^- \odot x^-)^\sim$;
(2) $\oplus$ *is associative;*
(3) $x, y \leq x \oplus y$;
(4) $x \oplus 0 = x^{-\sim} = 0 \oplus x$;
(5) $x \oplus 1 = 1 = 1 \oplus x$;
(6) $x \oplus y = x^- \rightsquigarrow y^{-\sim} = y^\sim \to x^{-\sim}$.

Proof. (1) follows from Proposition 2.9.
(2) We have :

$$\begin{aligned}(x \oplus y) \oplus z &= (y^\sim \odot x^\sim)^- \odot z = (z^\sim \odot (y^\sim \odot x^\sim)^{-\sim})^- \\ &= z^\sim \to (y^\sim \odot x^\sim)^{-\sim-} = z^\sim \to (y^\sim \odot x^\sim)^- = z^\sim \to (y^\sim \to x^{\sim-})\end{aligned}$$

(we applied Proposition 2.7(29),(35));

$$\begin{aligned}x \oplus (y \oplus z) &= x \oplus (z^\sim \odot y^\sim)^- = ((z^\sim \odot y^\sim)^{-\sim} \odot x^\sim)^- \\ &= (z^\sim \odot y^\sim)^{-\sim} \to x^{\sim-} = (z^\sim \odot y^\sim)^{-\sim} \to x^{-\sim} \\ &= (z^\sim \odot y^\sim) \to x^{-\sim} = z^\sim \to (y^\sim \to x^{-\sim}) = z^\sim \to (y^\sim \to x^{\sim-}) = (x \oplus y) \oplus z.\end{aligned}$$

(we applied Proposition 2.7(50),(1));
(3) By Proposition 2.7(29),(32) we have:

$$x \oplus y = (y^- \odot x^-)^\sim = x^- \rightsquigarrow y^{-\sim} \geq x$$

$$x \oplus y = (y^\sim \odot x^\sim)^- = y^\sim \to x^{\sim-} \geq y;$$

(4) Applying Proposition 2.7(26),(16) we get:
$x \oplus 0 = (0^\sim \odot x^\sim)^- = 0^\sim \to x^{-\sim} = 1 \to x^{-\sim} = x^{\sim-} = x^{-\sim}$;
$0 \oplus x = (x^- \odot 0^-)^\sim = x^- \rightsquigarrow 0^{-\sim} = x^- \rightsquigarrow 0 = x^{-\sim}$.
(5) $x \oplus 1 = (1^- \odot x^-)^\sim = x^- \rightsquigarrow 1^{-\sim} = x^- \rightsquigarrow 1 = 1$ (Proposition 2.7(29),(5);
$1 \oplus x = (x^- \odot 1^-)^\sim = 1^- \rightsquigarrow x^{-\sim} = 0 \rightsquigarrow x^{-\sim} = 1$ (Proposition 2.7(29),(6)).
(6) $x \oplus y = (y^\sim \odot x^\sim)^- = y^\sim \to x^{\sim-} = y^\sim \to x^{-\sim}$.
$x \oplus y = (y^- \odot x^-)^\sim = x^- \rightsquigarrow y^{-\sim}$.
It follows that $x \oplus y = x^- \rightsquigarrow y^{-\sim} = y^\sim \to x^{-\sim}$. □

In a residuated lattice we can define two *distance functions*:

$$d_1(x, y) = (x \to y) \wedge (y \to x) = x \vee y \to x \wedge y$$

$$d_2(x, y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) = x \vee y \rightsquigarrow x \wedge y.$$

(later equalities hold according to Proposition 2.7(45-46)).

Proposition 2.12. *The two distance functions fulfill the following properties:*
(1) $d_1(x,y) = d_1(y,x)$ *and* $d_2(x,y) = d_2(y,x)$;
(2) $d_1(x,y) = 1$ *iff* $x = y$ *iff* $d_2(x,y) = 1$;
(3) $d_1(x,0) = x^-$ *and* $d_2(x,0) = x^\sim$;
(4) $d_1(x,1) = x = d_2(x,1)$;
(5) $d_1(x,y) \le d_2(x^-,y^-)$;
(6) $d_2(x,y) \le d_1(x^\sim,y^\sim)$;
(7) $d_1(x,y) \le d_1(x^{\sim-},y^{\sim-})$;
(8) $d_2(x,y) \le d_2(x^{-\sim},y^{-\sim})$;
(9) $d_2(x^-,y^-) = d_1(x^{-\sim},y^{-\sim})$;
(10) $d_1(x^\sim,y^\sim) = d_2(x^{\sim-},y^{\sim-})$.

Proof. (1) Obvious.
(2) $d_1(x,y) = 1 \Leftrightarrow x \to y = 1$ and $y \to x = 1 \Leftrightarrow x \le y$ and $y \le x \Leftrightarrow x = y$;
Similarly, $d_2(x,y) = 1 \Leftrightarrow x = y$.
(3) $d_1(x,0) = (x \to 0) \wedge (0 \to x) = x^- \wedge 1 = x^-$;
$d_2(x,0) = (x \rightsquigarrow 0) \wedge (0 \rightsquigarrow x) = x^\sim \wedge 1 = x^\sim$;
(4) $d_1(x,1) = (x \to 1) \wedge (1 \to x) = 1 \wedge x = x$;
$d_2(x,1) = (x \rightsquigarrow 1) \wedge (1 \rightsquigarrow x) = 1 \wedge x = x$.
(5) By Proposition 2.7(33) we have:
$d_1(x,y) = (x \to y) \wedge (y \to x) \le (y^- \rightsquigarrow x^-) \wedge (x^- \rightsquigarrow y^-) = d_2(x^-,y^-)$.
(6) By Proposition 2.7(33) we have:
$d_2(x,y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \le (y^\sim \to x^\sim) \wedge (x^\sim \to y^\sim) = d_1(x^\sim,y^\sim)$.
(7) By (5) and (6) we get: $d_2(x,y) \le d_1(x^\sim,y^\sim) \le d_2(x^{\sim-},y^{\sim-})$.
(8) Similarly, $d_1(x,y) \le d_2(x^-,y^-) \le d_1(x^{-\sim},y^{-\sim})$.
(9) By above properties we get:
$d_2(x^-,y^-) \le d_1(x^{-\sim},y^{-\sim}) \le d_2(x^{-\sim-},y^{-\sim-}) = d_2(x^-,y^-)$, hence $d_2(x^-,y^-) = d_1(x^{-\sim},y^{-\sim})$.
(10) Similarly:
$d_1(x^\sim,y^\sim) \le d_2(x^{\sim-},y^{\sim-}) \le d_1(x^{\sim-\sim},y^{\sim-\sim}) = d_1(x^\sim,y^\sim)$, hence $d_1(x^\sim,y^\sim) = d_2(x^{\sim-},y^{\sim-})$. $\square$

The following result is inspired by [13].

Proposition 2.13. *On the residuated lattice L let $s : L \longrightarrow [0,1]$ be a function such that $s(1) = 1$. Then the following are equivalent:*
(i) $1 + s(x \wedge y) = s(x \vee y) + s(d_1(x,y))$ *for all* $x,y \in L$;
(ii) $1 + s(x \wedge y) = s(x) + s(x \to y)$ *for all* $x,y \in L$;
(iii) $s(x) + s(x \to y) = s(y) + s(y \to x)$ *for all* $x,y \in L$, *where the + is the usual addition of real numbers.*

Proof. (i)⇒(ii) If $a \le b$, then $a \wedge b = a$, $a \vee b = b$, $a \to b = 1$ and

$$d_1(a,b) = (a \to b) \wedge (b \to a) = 1 \wedge (b \to a) = b \to a,$$

hence by hypothesis, $1 + s(a) = s(b) + s(b \to a)$.
Letting $a = x \wedge y$ and $b = y$ it follows that

$$1 + s(x \wedge y) = s(y) + s(y \to x \wedge y) = s(y) + s(y \to x) \text{ (we applied Proposition 2.7(19).)}$$

(ii)⇒(iii) $s(x) + s(x \to y) = 1 + s(x \wedge y) = 1 + s(y \wedge x) = s(y) + s(y \to x)$.
(iii)⇒(i) We have that $d_1(x,y) = x \vee y \to x \wedge y$, hence, applying the hypothesis we get

that:

$$\begin{aligned} s(x \vee y) + s(d_1(x,y)) &= s(x \vee y) + s(x \vee y \to x \wedge y) = \\ &= s(x \wedge y) + s(x \wedge y \to x \vee y) = s(x \wedge y) + s(1) = \\ &= 1 + s(x \wedge y) \\ (x \wedge y \leq x \vee y &\Rightarrow x \wedge y \to x \vee y = 1). \end{aligned}$$

□

Proposition 2.14. *On the residuated lattice L let $s : L \longrightarrow [0,1]$ be a function such that $s(1) = 1$. Then the following are equivalent:*
(i) $1 + s(x \wedge y) = s(x \vee y) + s(d_2(x,y))$ *for all* $x, y \in L$*;*
(ii) $1 + s(x \wedge y) = s(x) + s(x \rightsquigarrow y)$ *for all* $x, y \in L$*;*
(iii) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$ *for all* $x, y \in L$*.*

Proof. Similarly. □

3. Bosbach state and Riečan state

Inspired by [13], [11], [12], we introduce the notion of Bosbach state on a residuated lattice L.

Definition 3.1. A *Bosbach state* on a residuated lattice L is a function $s : L \to [0,1]$ such that the following conditions hold for any $x, y \in L$:
(B_1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(B_2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(B_3) $s(0) = 0$ and $s(1) = 1$.

Examples 3.2. Let's consider the set $L = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the residuated lattice $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ with $0 < a < b < c < 1$ where the operations $\odot, \to, \rightsquigarrow$ are given in the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

The function $s : L \to [0,1]$ defined by: $s(0) = 0, s(a) = 1, s(b) = 1, s(c) = 1, s(1) = 1$ is the unique Bosbach state on $\mathbf{L}$.
$\mathbf{L}$ is actually even a good pseudo-MTL algebra.

Proposition 3.3. *Let s be a Bosbach state on L. Then for all $x, y \in L$ the following properties hold:*
(1) $s(x \to y) = s(s \rightsquigarrow y)$;
(2) $s(d_1(x,y)) = s(d_2(x,y))$;
(3) $s(x^-) = s(x^\sim) = 1 - s(x)$;
(4) $s(x^{-\sim}) = s(x^{\sim -}) = s(x^{--}) = s(x^{\sim\sim}) = s(x)$;
(5) $x \leq y$ *implies* $1 + s(x) = s(y) + s(y \to x) = s(y) + s(y \rightsquigarrow x)$;
(6) $x \leq y$ *implies* $s(x) \leq s(y)$;
(7) $s(x \odot y) = 1 - s(x \to y^-)$ *and* $s(y \odot x) = 1 - s(x \rightsquigarrow y^\sim)$;

(8) $s(x)+s(y)=s(x\odot y)+s(y^{-}\to x)$;
(9) $s(x)+s(y)=s(y\odot x)+s(y^{\sim}\rightsquigarrow x)$;
(10) $s(x^{-}\to y^{-})=s(y^{-\sim}\to x^{-\sim})$;
(11) $s(x^{\sim}\to y^{\sim})=s(y^{\sim-}\to x^{\sim-})$

Proof. (1) By Proposition 2.13 and Proposition 2.14 it follows that:
$s(x)+s(x\to y)=1+s(x\wedge y)=s(x)+s(x\rightsquigarrow y)$, hence
$s(x\to y)=s(x\rightsquigarrow y)$;
(2) By Proposition 2.13 and Proposition 2.14, $s(d_1(x,y))=1+s(x\wedge y)-s(s\vee y)=s(d_2(x,y))$.
(3) $s(x)+s(x^{-})=s(x)+s(x\to 0)=s(0)+s(0\to x)=s(0)+s(1)=1$, hence
$s(x^{-})=1-s(x)$. By (1) we have $s(x^{\sim})=s(x^{-})$, hence $s(x^{-})=s(x^{\sim})=1-s(x)$;
(4) Applying (3) twice;
(5) Since $x\leq y$ we have $x\to y=x\rightsquigarrow y=1$.
Applying (B_1) and (B_3) we get that:
$s(y)+s(y\to x)=s(x)+s(x\to y)=s(x)+1$.
Similarly, by (B_2) and (B_3) we obtain $s(y)+s(y\rightsquigarrow x)=s(x)+1$;
(6) By (5) and (3) we get that $s(y)-s(x)=1-s(y\to x)=s((y\to x)^{-})\geq 0$;
(7) By (3) $s((x\odot y)^{-})=1-s(x\odot y)$.
But $(x\odot y)^{-}=x\to y^{-}$, so $s(x\odot y)=1-s(x\to y^{-})$.
Similarly, $s(y\odot x)=1-s((y\odot x)^{\sim})=1-s(x\rightsquigarrow y^{\sim})$;
(8) Applying (B_1) and (7) we have:

$$\begin{aligned} s(x\odot y)+s(y^{-}\to x) &= s(x\odot y)+s(x)-s(x\to y^{-})-s(y^{-})\\ &= s(x\odot y)+s(x)-(1-s(x\odot y))-1+s(y)=s(x)+s(y); \end{aligned}$$

(9) Applying (B_2) and (7) we have:

$$\begin{aligned} s(y\odot x)+s(y^{\sim}\rightsquigarrow x) &= s(y\odot x)+s(x)+s(x\rightsquigarrow y^{\sim})-s(y^{\sim})\\ &= s(y\odot x)+s(x)+1-s(y\odot x)-1+s(y)=s(x)+s(y); \end{aligned}$$

(10) By Proposition 2.7(33) we have $x^{-}\rightsquigarrow y^{-}\leq y^{-\sim}\to x^{-\sim}$, so,by (1) and (6) it follows that:

$$\begin{aligned} s(x^{-}\to y^{-}) &= s(x^{-}\rightsquigarrow y^{-})\leq s(y^{-\sim}\to x^{-\sim})\leq s(x^{-\sim-}\rightsquigarrow y^{-\sim-})\\ &= s(x^{-}\rightsquigarrow y^{-})=s(x^{-}\to y^{-}). \end{aligned}$$

Thus, $s(x^{-}\to y^{-})=s(y^{-\sim}\to x^{-\sim})$;
(11) Similarly. □

According to [13], [11], [23] we introduce the following notion.

Definition 3.4. Let L be a good residuated lattice. The elements $x,y\in L$ are called *orthogonal*, denoted by $x\perp y$ iff $y^{-\sim}\leq x^{-}$.
If the elements $x,y\in L$ are orthogonal, we define a partial operation $+$ on L by $x+y:=x\oplus y$.

Proposition 3.5. *In a good residuated lattice the following are equivalent:*
(i) $x\perp y$;
(ii) $x^{-\sim}\leq y^{\sim}$;
(iii) $y^{-\sim}\odot x^{-\sim}=0$.

Proof. (*i*) $\Leftrightarrow$ (*iii*) $x \perp y \Leftrightarrow y^{-\sim} \leq x^{-} = x^{-\sim-} \Leftrightarrow y^{-\sim} \odot x^{-\sim} = 0$ (by Proposition 2.7(27)).
(*ii*) $\Leftrightarrow$ (*iii*) $x^{-\sim} \leq y^{\sim} = y^{\sim-\sim} = y^{-\sim-} \Leftrightarrow y^{-\sim} \odot x^{-\sim} = 0$ (by Proposition 2.7(29)). □

Proposition 3.6. *In a good residuated lattice the following hold:*
(1) $x^{\sim} \perp x$ *and* $x \perp x^{-}$;
(2) *If* $x \leq y$ *then* $x \perp y^{-}$ *and* $y^{\sim} \perp x$;
(3) *If* L *is commutative, then* $x \perp y$ *iff* $y \perp x$.

Proof. (1) $x^{-\sim} = x^{\sim-} \Rightarrow x^{\sim} \perp x; x^{-\sim-} = x^{--\sim} = x^{-} \Rightarrow x \perp x^{-}$.
(2) $x \leq y \Rightarrow y^{-} \leq x^{-} \Rightarrow y^{-\sim-} \leq x^{-} \Rightarrow y^{--\sim} \leq x^{-} \Rightarrow x \perp y^{-}$.
$x \leq y \Rightarrow x^{-\sim} \leq y^{-\sim} = y^{\sim-} \Rightarrow y^{\sim} \perp x$.
(3) $x \perp y \Leftrightarrow y^{-\sim} \odot x^{-\sim} = 0 \Leftrightarrow x^{-\sim} \odot y^{-\sim} = 0 \Leftrightarrow y \perp x$. □

The following notion of a state was firstly defined for BL-algebras in [23], in [13] for pseudo BL-algebras and in [11] for another more general structure.

Definition 3.7. Let L be a good residuated lattice. A *Riečan state* on L is a function $s : L \to [0,1]$ such that the following conditions hold for all $x, y \in L$:
(R_1) If $x \perp y$, then $s(x+y) = s(x) + s(y)$;
(R_2) $s(1) = 1$.

Example 3.8. Let's consider again the residuated lattice in Example 3.2.
One can easily check that $x^{-\sim} = x^{\sim-}$ for any $x \in L$, so L is a good residuated lattice. We claim that the Bosbach state $s : L \to [0,1]$ defined by: $s(0) = 0, s(a) = 1, s(b) = 1, s(c) = 1, s(1) = 1$ is also Riečan state on $\mathbf{L}$.
Indeed, the orthogonal elements of $\mathbf{L}$ are the pairs (x, y) of the following table:

x	y	x^{-}	$y^{-\sim}$	$x \oplus y$
0	0	1	0	0
0	a	1	1	1
0	b	1	1	1
0	c	1	1	1
0	1	1	1	1
a	0	0	0	1
b	0	0	0	1
c	0	0	0	1
1	0	0	0	1

One cas easy check that s is a Riečan state.

Proposition 3.9. *Let s be a Riečan state on the good residuated lattice L. Then the following properties hold for all $x, y \in L$:*
(1) $s(x^{-}) = s(x^{\sim}) = 1 - s(x)$;
(2) $s(0) = 0$;
(3) $s(x^{-\sim}) = s(x^{\sim-}) = s(x^{--}) = s(x^{\sim\sim}) = s(x)$;
(4) *If* $x \leq y$ *then* $s(y) - s(x) = 1 - s(x \oplus y^{-}) = 1 - s(y^{\sim} \oplus x)$;
(5) *If* $x \leq y$ *then* $s(x) \leq s(y)$.

Proof. (1) By Proposition 3.6(1) we have $x \perp x^{-}$ and $x^{\sim} \perp x$. By (R_1) and Proposition 2.7(26) it follows that:
$s(x) + s(x^{-}) = s(x \oplus x^{-}) = s(x^{-\sim} \odot x^{\sim})^{-} = s(x^{\sim-} \odot x^{\sim})^{-}) = s(0^{-}) = s(1) = 1$, hence

$s(x^-) = 1 - s(x)$.
Similarly, $s(x^\sim) = 1 - s(x)$;
(2) $s(0) = s(1^-) = 1 - s(1) = 0$;
(3) $s(x^{-\sim}) = 1 - s(x^-) = 1 - 1 + s(x) = s(x)$. Similarly the others;
(4) By Proposition 3.6(2),(1) we have :

$$s(x \oplus y^-) = s(x) + s(y^-) = s(x) + 1 - s(y)$$

$$s(y^\sim \oplus x) = s(y^\sim) + s(x) = 1 - s(y) + s(x);$$

(5) By (4) $s(y) - s(x) = 1 - s(x \oplus y^-) \geq 0$. □

Theorem 3.10. *Let L be a good residuated lattice. Any Bosbach state on L is a Riečan state on L.*

Proof. Let s be a Bosbach state on L. Assume $x \perp y$, i.e. $y^{-\sim} \leq x^-$.
By Proposition 3.6(3),(5) we have:
$1 + s(y) = 1 + s(y^{-\sim}) = s(x^-) + s(x^- \to y^{-\sim}) = 1 - s(x) + s(x^- \to y^{-\sim})$, hence
$s(x^- \to y^{-\sim}) = s(x) + s(y)$.
On the other hand, $x \oplus y = (y^\sim \odot x^\sim)^- = (y^- \odot x^-)^\sim = x^- \rightsquigarrow y^{-\sim}$, hence
$s(x \oplus y) = s(x^- \rightsquigarrow y^{-\sim}) = s(x^- \to y^{-\sim}) = s(x) + s(y)$.
Therefore, $s(x \oplus y) = s(x) + s(y)$ and by hypothesis $s(1) = 1$.
It follows then that s is a Riečan state on L. □

By the next example we show that there exists a Riečan state which is not Bosbach state.

Example 3.11. Let's consider again the residuated lattice in Example 2.2,
$\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ where $L = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$. It is easy to check that $\mathbf{L}$ is a good residuated lattice.
The function $s : L \longrightarrow [0, 1]$ defined by $s(0) = 0, s(a) = 1/2, s(b) = 1/2, s(c) = 1, s(1) = 1$ is a Riečan state.
Indeed, the orthogonal elements of $\mathbf{L}$ are the pairs (x, y) of the following table:

x	y	x^-	$y^{-\sim}$	$x \oplus y$
0	0	1	0	0
0	a	1	b	b
0	b	1	b	b
0	c	1	1	1
0	1	1	1	1
a	0	b	0	b
a	a	b	b	1
a	b	b	b	1
b	0	b	0	b
b	a	b	b	1
b	b	b	b	1
c	0	0	0	1
1	0	0	0	1

One can easily check that s is a Riečan state.
But the function s above defined is not a Bosbach state.
Indeed, trying to check the conditions (B_2) of Bosbach state definition we obtain:

$s(a) + s(a \rightsquigarrow b) = s(a) + s(1) = 1/2 + 1 = 3/2$
$s(b) + s(b \rightsquigarrow a) = s(b) + s(b) = 1/2 + 1/2 = 1,$
so the condition (B_2) doesn't hold.
We conclude that s is not a Bosbach state.

Remarks 3.12. (1) In case of good pseudo-BL algebras, in [13] G.Georgescu left as an open problem to find an example of Riečan state which is not a Bosbach state ;
(2) In case of good Rℓ-monoids, A.Dvurečenskij and J.Rachůnek ([11]) proved that any Riečan state is also a Bosbach state, hence it is true for pseudo BL-algebras ;
(3) By the above example we proved that in case of good residuated lattices, Riečan states need not be Bosbach state. Moreover, as the above structure is actually a pseudo MTL-algebra, we can see that this is also valid for the class of pseudo MTL-algebras.

In the sequel we shall write state instead of Bosbach state.

Definition 3.13. Let L be a residuated lattice. A nonempty set F of L is called *filter* of L if the following conditions hold:
(F_1) If $x, y \in F$, then $x \odot y \in F$;
(F_2) If $x \in F, y \in L, x \leq y$ then $y \in F$.

Proposition 3.14. [12] *If F is a filter of L then:*
(F_3) $1 \in F$;
(F_4) *If $x \in F, y \in L$, then $y \rightarrow x \in F, y \rightsquigarrow x \in F$;*
(F_5) *If $x, y \in F$ then $x \wedge y \in F$.*

Proposition 3.15. [12] *For a subset F of L the following are equivalent:*
(1) *F is a filter;*
(2) *$1 \in F$ and if $x, x \rightarrow y \in F$, then $y \in F$;*
(3) *$1 \in F$ and if $x, x \rightsquigarrow y \in F$, then $y \in F$.*
A set F that fulfills (2) and (3) is called deductive system.

Definition 3.16. A filter H of L is called *normal* if for any $x, y \in L$

$$x \rightarrow y \in H \text{ iff } \mathrm{x} \rightsquigarrow \mathrm{y} \in \mathrm{H}.$$

Example 3.17. Let's consider the good residuated lattice from Example 2.2,
$\mathbf{L} = \{L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1\}$ where $L = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$.
It is easy to check that $H = \{c, 1\}$ is a normal filter of L.

With any normal filter H of L we associate a binary relation $\equiv_H$ on L by defining

$$x \equiv_H y \text{ iff } \mathrm{d_1(x,y) \in H} \text{ iff } \mathrm{d_2(x,y) \in H}.$$

Remark 3.18. $x \equiv_H y$ iff $x \rightarrow y, y \rightarrow x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$.

Proposition 3.19. *For a given normal filter H of L the relation $x \equiv_H y$ is an equivalent relation on L.*

Proof. Reflexivity: $x \equiv_H x$ because $x \rightarrow x = x \rightsquigarrow x = 1 \in F$.
Symmetry: Obviously $x \equiv_H y \Rightarrow y \equiv_H x$ according to the above remark.
Transitivity: Let's consider $x \equiv_H y$ and $y \equiv_H z$.
We apply Proposition 2.7(17-18):

$$x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$$
$$x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z).$$

Because $x \rightsquigarrow y \in H$ it follows that $(y \rightsquigarrow z) \to (x \rightsquigarrow z) \in H$.
Because $y \rightsquigarrow z \in H$ and $(y \rightsquigarrow z) \to (x \rightsquigarrow z) \in H$, we get $z \rightsquigarrow x \in H$.
Also by Proposition 2.7(17-18) changing x and z we have:

$$z \to y \leq (y \to x) \rightsquigarrow (z \to x)$$
$$z \rightsquigarrow y \leq (y \rightsquigarrow x) \to (z \rightsquigarrow x),$$

so we get $z \rightsquigarrow x \in H$.
Similarly, $x \to z \in H$ and $z \to x \in H$.
Thus, $x \equiv_H z$ and the transitivity is proved. □

For any $x \in L$, let x/L be the equivalence class $x/\equiv_H$.
L/H becomes a residuated lattice with the natural operation induced from those of L.
If $x, y \in L$, then $x/H \leq y/H$ iff $x \to y \in H$ iff $x \rightsquigarrow y \in H$.

Definition 3.20. If $s : L \to [0, 1]$ is a state on L, we define the *kernel Ker(s)* by

$$Ker(s) = \{x \in L \mid s(x) = 1\}.$$

Proposition 3.21. *If s is a state on residuated lattice L, then $Ker(s)$ is a normal filter on L.*

Proof. Obviously $1 \in Ker(s)$.
Assume $x, x \to y \in Ker(s)$, that is $s(x) = s(x \to y) = 1$.
Then, by Proposition 2.13(iii), $s(y) + s(y \to x) = s(x) + s(x \to y) = 2$, hence
$s(y) = s(y \to x) = 1$. It follows that $y \in Ker(s)$. Thus, $Ker(s)$ is a filter of L.
By Propositions 2.13 and 2.14 we have
$s(x) + s(x \to y) = 1 + s(x \wedge y) = s(x) + s(x \rightsquigarrow y)$, hence $s(x \to y) = s(x \rightsquigarrow y)$.
It follows that $x \to y \in Ker(s)$ iff $x \rightsquigarrow y \in Ker(s)$.
Thus, $Ker(s)$ is a normal filter of L. □

Proposition 3.22. *If s is a state on the good residuated lattice L, the following properties hold:*
(1) $x/Ker(s) = y/Ker(s)$ *iff* $s(x \wedge y) = s(x \vee y)$;
(2) *If* $s(x \wedge y) = s(x \vee y)$, *then* $s(x) = s(y) = s(x \wedge y)$.

Proof. (1) $x/Ker(s) = y/Ker(s)$ iff $d_1(x, y) \in Ker(s)$ iff $s(d_1(x, y)) = 1$ iff $s(x \wedge y) = s(x \vee y)$ (by Proposition 2.13(i));
(2) By Proposition 3.9(5) we have $s(x \wedge y) \leq s(x), s(y) \leq s(x \vee y)$ and by hypothesis it follows that $s(x) = s(y) = s(x \wedge y)$. □

Theorem 3.23. *Let L be a good residuated lattice. If s is a Riečan state on L, then the function $\hat{s} : L/Ker(s) \to [0, 1]$ defined by $\hat{s}(x/Ker(s)) = s(x)$ is a Riečan state on $L/Ker(s)$.*

Proof. First we prove that $\hat{s}$ is well-defined.
Indeed, if $x/Ker(s) = y/Ker(s)$, then by Proposition 3.22 it follows that $s(x \wedge y) = s(x \vee y)$. Then by Proposition 2.13(i)we have $s(d_1(x, y)) = 1$.
It follows that $d_1(x, y) \in Ker(s)$ and similarly, $d_2(x, y) \in Ker(s)$.
Thus, $x \equiv_{Ker(s)} y$.
Moreover, if $x \equiv_{Ker(s)} y$, then $s(x) = s(y)$.
Indeed, $x \equiv_{Ker(s)} y$ is equivalent to $s(x \to y) = s(y \to x) = 1$ and by Proposition 2.13(iii) it follows that $s(x) = s(y)$.
Applying the method used in [11] we prove now that $\hat{s}$ is a Riečan state on $L/Ker(s)$.

First we recall that if $\hat{x} \leq \hat{y}$, then there is an element $x_1 \in \hat{x}$ such that $x_1 \leq y$. Indeed, it is sufficient to take $x_1 = x \wedge y$.
Assume that $\hat{x} \perp \hat{y}$, that is $\hat{y}^{-\sim} \leq \hat{x}^{-}$, hence $\hat{x}^{-\sim} \leq \hat{y}^{-\sim\sim} = \hat{y}^{\sim-\sim} = \hat{y}^{\sim}$, so $\hat{x}^{-\sim} \leq \hat{y}^{\sim}$. Let's take $x_1 \in \hat{x}^{-\sim}$ such that $x_1 \leq y^{\sim}$. Hence, $x_1^{-\sim} \leq y^{\sim}$ and by Proposition 3.5(5) it follows that $x_1 \perp y$.
Therefore,

$$\begin{aligned}\hat{s}(\hat{x} + \hat{y}) &= \hat{s}(\hat{y}^{\sim} \odot \hat{x}^{\sim})^{-} = \hat{s}((y^{\sim} \odot x^{\sim})^{-})\hat{} = s(y^{\sim} \odot x^{\sim})^{-} = s(x \oplus y) \\ &= s(x^{-\sim} \oplus y^{-\sim}) = \hat{s}(\hat{x_1} + \hat{y}^{-\sim}) = s(x_1 + y^{-\sim}) = s(x_1) + s(y^{-\sim}) = s(x) + s(y) \\ &= \hat{s}(\hat{x}) + \hat{s}(\hat{y})\end{aligned}$$

(We took in consideration that $x_1 \equiv_{Ker(s)} x$ implies $s(x_1) = s(x)$). $\square$

Remarks 3.24. Let's consider the set $L = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the residuated lattice $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ with $0 < a < b < c < 1$, where the operations $\odot, \rightarrow, \rightsquigarrow$ are given in the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

.

One can easily check that L is a not good residuated lattice ($a^{-\sim} = a$, but $a^{\sim-} = b$).
(1) G.Georgescu proved that for pseudo BL-algebras the existence of a state is equivalent with the existence of a maximal filter which is normal ([13]). In the above example, $H = \{1\}$ is a maximal and normal filter of L, but there are no states on L.
Indeed, assume that A admits a Bosbach state s such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = \gamma$, $s(1) = 1$. From $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$, taking $x = a, y = 0$, $x = b, y = 0$ and respectively $x = c, y = 0$ we get $\alpha = 1/2$, $\beta = 1/2$, $\gamma = 1/2$.
On the other hand, taking $x = b, y = a$ we get $\beta + \gamma = \alpha + 1$, so $1 = 3/2$ which is a contradiction. Hence, L does not admit a Bosbach state.
(2) A.Dvurečenskij proved in [8] that every linear pseudo BL-algebra admits a state. The above example shows that there exist linear residuated lattices having no states.
(3) In the case of a BL-algebra A, it is proved that a filter H is normal if and only if $x \odot H = H \odot x$ for any $x \in A$ ([5],Proposition 1.3). This equality doesn't hold in the case of residuated lattices. Indeed, let's consider the normal filter $H = \{a, b, c, 1\}$ of the residuated lattice L in Example 3.2. In this case we have $c \odot H = \{a, c\}$ and $H \odot c = \{a, b, c\}$, so $c \odot H \neq H \odot c$.

Acknowledgment

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Computing VaR and AVaR of Skewed-T Distribution

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Abstract

We consider the skewed-T distribution defined as a normal mixture with inverse gamma distribution. Analytical formulas for its value-at-risk, VaR quantile, and average value-at-risk, AVaR conditional mean are derived. High-accuracy approximations are developed and numerically tested.

Keywords: *skewed-T distribution, asymmetric, value-at-risk, VaR, AVaR*

1 Introduction

The skewed-T distribution is a popular choice for modeling financial time series of asset returns. The VaR quantile and the average VaR quantile, i.e. AVaR, of

*Rachev gratefully acknowledges research support by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst.

those returns are usually estimated from a large sample of observations. If such large sample is not available, as in a case when only short history of returns is present, then we need a reliable way for assessing the magnitude of the VaR and AVaR risk measures. Analytical formulas might help in this case and let thorough analysis been performed on the risk measures by varying the distribution parameters and assessing different risk levels.

We note that asymmetric distributions can be defined in different ways from their symmetric counterparts. For one such case of skewed-T distribution we see analytical formulas for VaR and AVaR derived in [1] (AVaR is denoted CVaR in [1]). In this case the skewed Student-t density function was proposed by Hansen (1994) in [2]. The density is an extension of the conventional symmetric Student-t distribution. The asymmetry is introduced by weighting differently, multiplying by different weights, the negative and the positive values of the symmetric Student-t distribution.

We consider skewed-T distribution defined as a normal mixture with inverse gamma distribution (e.g. see [3] for details). Such skewed-T random variable, X, is defined by

$$X = \mu + \gamma W + Z\sqrt{W},$$

where $W \sim Ig(\nu/2, \nu/2)$, i.e. W is inverse gamma random variable, Z is multivariate normal random variable $Z \sim N_d(0, \Sigma)$, and W, Z are independent. In the paper we present the analytical formulas for α-level VaR(X) and AVaR(X) risk measures. We denote the d-dimensional distribution by $X \sim t_d(\nu, \mu, \Sigma, \gamma)$ where ν stands for degrees of freedom, $\nu \geq 4$, μ is a location parameter, and Σ is d-by-d covariance matrix. Finally, the sign of γ controls the distribution asymmetry: positive for skewed to the right, having fat right tail of asset returns, and vice-versa, negative for skewed to the left; except for the case $\nu = 5$.

In the paper we specifically consider the cases for $\gamma \neq 0$, that is, the cases with "true" asymmetry in X. Nevertheless, we note that for small γ's (of order 10^{-6}) our formulas numerically converge to the symmetric, conventional Student-t, AVaR formula

$$AVaR_\alpha(X) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\sqrt{\nu}}{(\nu-1)\alpha\sqrt{\pi}} \left(1 + \frac{(VaR_\alpha(X))^2}{\nu}\right)^{\frac{1-\nu}{2}}$$

for $\nu > 1$.

The paper is organized as follows: in next Section 2 we state the classical definitions of VaR and AVaR. Then we elaborate on the form of the skewed-T pdf needed for development of the analytical formulas. An integral representation of the Bessel function (involved in the pdf) is utilized. The analytical formulas for VaR and AVaR are stated in two theorems. Their properties are discussed and one related proposition is stated. The proofs of the theorems and the proposition are presented in the appendix with thorough details. Two corollaries state approximation versions of the theorems statements. The corollaries results are very useful in numerical implementations. Theire proofs are in the appendix as well. In Section 3 we discuss some issues arising in numerical implementations of the developed formulas. The issues are resolved via the Bessel function asymptotic forms and via locating the spike-like unimodal peak of the quadrature integrand function. The paper is concluded in Section 4.

2 VaR and AVaR for skewed-T distribution

The definition of VaR calls for a confidence level $\alpha \in (0,1)$. Then the VaR of portfolio return at confidence level α is defined as the smallest number x_0 such that the probability that the loss X exceeds x_0 is not larger than $(1-\alpha)$. That is, in general

$$\begin{aligned} VaR_\alpha(X) &= \ \inf\ \{x_0 : P(X > x_0) \le 1-\alpha\} \\ &= \ \inf\ \{x_0 : F_X(x_0) \ge \alpha\} \\ &= F_X^{-1}(\alpha) \end{aligned}$$

where $F_X(\cdot)$ is the cdf (cumulative distribution function) of X, F_X^{-1} is the inverse function of F_X provided one exists, and the last equality holds for continuous distributions. In probabilistic terms VaR is the α-quantile of the loss distribution. If we consider the random variable X for modeling the asset returns then $-X$ models the asset losses. Here we will not distinguish between the two, but we will derive analytical formulas for both tails of the skewed-T distribution, that is, formulas for smaller and larger α values for the VaR quantile.

If we let the random variable X denote portfolio loss then the definition of the α level AVaR is given by the following conditional expectation

$$\begin{aligned} AVaR(X) &= E\left[X|X \ge VaR_\alpha(X)\right] \\ &= \frac{1}{1-\alpha}\int_{X\ge x_0} xf(x)dx \end{aligned}$$

that is, the α level average value-at-risk $AVaR(X)$ is the average loss larger than the α level quantile loss $VaR_\alpha(X)$. Similarly to the VaR case, we will derive AVaR results for both distribution tails combined with the two cases for the sign of the asymmetry parameter γ which controls the fatness of the distribution tails.

For our analytical results we need the probability density function, $f(x)$, of the skewed-T random variable X, which is given by

$$f(x) = \frac{2^{1-(\nu+d)/2}}{\Gamma(\frac{\nu}{2})(\pi\nu)^{d/2}\left|\Sigma\right|^{1/2}} * \frac{\exp((x-\mu)'\Sigma^{-1}\gamma)}{\left(1+\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu}\right)^{(\nu+d)/2}}$$

$$* \frac{K_{(\nu+d)/2}\left(\sqrt{(\nu+(x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma}\right)}{\left(\sqrt{(\nu+(x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma}\right)^{-(\nu+d)/2}}$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ (also know as modified Hankel function or Macdonald function). For details see, for example, [4] and [5]. The skewed-T distribution is in the class of generalized hyperbolic distributions. When we model asset returns with this distribution then the resulting portfolio return can be seen as a random variable which is a linear combination of skewed-T returns. Because of a property of the generalized hyperbolic distributions such linear combination has a generalized hyperbolic distribution as well (see Corollary 2.2.4 in [8]).

For assessing the VaR and AVaR of a single asset we consider its return as univariate, $d = 1$, skewed-T random variable. The corresponding univariate probability density (pdf) function is

$$f(x) = \frac{\nu^\lambda \gamma^{2\lambda}}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}2^{2\lambda}} \int_0^\infty t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dt$$

where, for convenience, we set $\mu = 0$, $\Sigma \equiv \sigma^2 = 1$, and $\lambda = (\nu + 1)/2$. That is, we consider X normalized by the standard transformation

$$\frac{X - \mu}{\sigma} = \frac{\gamma}{\sigma} W + N(0,1)\sqrt{W}.$$

where $N(0, 1)$ stands for a random variable from the standard normal distribution. For numerical implementations we note that the skewness controlling parameter γ of the normalized skewed-T random variable $\frac{X-\mu}{\sigma}$ actually becomes $\frac{\gamma}{\sigma}$ when the above normalized pdf $f(x)$ is utilized.

In the last form of the pdf, $f(x)$, we applied the following integral representation of the Bessel function (with $y = \sqrt{(\nu + x^2)\gamma^2}$)

$$K_\lambda(y) = \frac{1}{2}\left(\frac{y}{2}\right)^\lambda \int_0^\infty t^{-\lambda-1} e^{-t-\frac{y^2}{4t}} dt.$$

Among other places this representation can be seen in [6]. We note that the skewed-T distribution is also popular under the name asymmetric Laplace distribution, for example, in statistical applications in medical research: see [7] where the inverse gamma random variable W is replaced with a special case of its reciprocal values.

2.1 VaR formula for skewed-T distribution

In this section we develop a method for computing $VaR(X)$, i.e., the $1 - \alpha$ quantile of a skewed-T random variable X. We look for a formula and/or numerical procedure yielding $x_0 = \text{VaR}_\alpha$ which is such that

$$1 - \alpha = \int_{x_0}^\infty f(x)dx$$

where $f(x)$ is the univariate density which is also normalized with the notations we introduced.

For a given skewed-T random variable we know the sign of the skewness controlling parameter γ yielding heavier right or left distribution tail. Similarly, we have to know the sign of x_0, the α level $VaR(X)$, in order to distinguish between the two distribution tails (that is, we have to know whether we look for the VaR quantile for a "smaller" or for a "larger" α value). We so, first, compute the above integral on the interval $[0, \infty]$, and we set

$$1 - \alpha_0 = \int_0^\infty f(x)dx.$$

Hence, if the given VaR level α is such that $\alpha < \alpha_0$ then we look for negative x_0, otherwise, we look for positive x_0. This naturally leads to two cases in the $x_0 = \mathrm{VaR}_\alpha$ calculation depending on whether the specified VaR level α is greater than or less than the α_0 value we set. These two cases will have to be combined with the other two cases coming from the sign of the skewness controlling parameter γ. This naturally leads to total of four cases which we describe and study in this section and in the next section with respect to the VaR and AVaR formulas.

The formula for computing α_0 comes as a corollary of the following theorem.

Theorem 1. *The VaR formula for skewed-T random variable, that is, the value of $x_0 = VaR_\alpha$ is coming as the unique zero of the following equation (provided $\gamma > 0$)*

$$g(x_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0.$$

For negative skewness the value of $x_0 = VaR_\alpha$ is coming as the unique zero of the next equation (i.e., provided $\gamma < 0$)

$$g(x_0) = 1 - \alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0,$$

In both cases the zero, x_0 of g, is sought on the interval $(-\infty, 0]$ provided $\alpha < \alpha_0$ or, on the interval $[0, +\infty)$ provided $\alpha > \alpha_0$.

Proof. For the case $\gamma > 0$ see the Appendix. $\square$

In the theorem statement the $\Phi(\cdot)$ stands for the standard normal cdf, the constant

$$C = \frac{\nu^\lambda \gamma^{2\lambda}}{\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2}) 2^{2\lambda}}$$

depends on the skewed-T distribution parameters: degrees of freedom ν, $\lambda = (\nu+1)/2$, and γ is the skewness controlling parameter.

We note that $g(\cdot)$ is an increasing function of x_0 from $(-\alpha)$ to $1-\alpha$ as x_0 ranges from minus to plus infinity. Hence, the unique zero of g can be found by any numerical routine, for example, by one like a bi-sectional search.

The integrals on infinite intervals in the theorem have to be computed numerically. Also the search for the zero x_0 of g has to be performed on infinite intervals. We so derive approximations of the theorem statement where the infinite intervals are replaced with finite intervals. We prove that the accumulated numerical error is bounded by 10^{-9} when the infinite intervals are replaced by finite intervals.

Corollary 1. *The integral on infinite interval in Theorem 1 can be replaced with the following integral on a finite interval*

$$\int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

where

$$t_0(x) = \left(\frac{3\sqrt{2} + \sqrt{18 + 2\gamma x}}{2}\right)^2$$

The approximation error $R(t_0(x_0))$ *brought in*

$$g(x_0) = -\alpha + \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt + R(t_0(x_0))$$

for $\gamma > 0$*, and for* $\gamma < 0$ *as well, is less than* 10^{-9}*.*

Furthermore, the search for the zero x_0 *of* g *is performed on the following intervals*

$$\begin{aligned} [-9/\gamma, 0] &\quad \textit{if} \quad \alpha < \alpha_0, \gamma > 0, \\ [0, +\infty) &\quad \textit{if} \quad \alpha > \alpha_0, \gamma > 0, \\ (-\infty, 0] &\quad \textit{if} \quad \alpha < \alpha_0, \gamma < 0, \\ [0, -9/\gamma] &\quad \textit{if} \quad \alpha > \alpha_0, \gamma < 0, \end{aligned}$$

Proof. For the case $\gamma > 0$ see the Appendix. □

Getting rid of the error term $R(t_0(x_0))$ still preserves the increasing nature of $g(x_0)$. Hence, the search for the unique zero x_0 is fine with the approximation we propose in Corollary 1. The value of α_0 which specifies the sign of x_0 comes as corollary from the above approximation after substituting the later with zero.

Corollary 2. *The formula for* α_0 *is*

$$\begin{aligned} \alpha_0 &= \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(-\sqrt{2t}) dt, && \textit{if } \gamma > 0, \\ \alpha_0 &= 1 + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(-\sqrt{2t}) dt, && \textit{if } \gamma < 0, \end{aligned}$$

Proof. By substitution $x_0 = 0$. □

The proofs of the theorem and the first corollary in the appendix are presented for the case $\gamma > 0$. The formulas for $g(x_0)$ when $\gamma < 0$ are derived from their $\gamma > 0$ counterparts by substituting X with $-X$, γ with $-\gamma$, and the VaR level α of X with $1 - \alpha$ which is the VaR level α for $-X$.

In both cases, for positive and negative skewness, when the zero x_0 of $g(\cdot)$ has to be sought on infinite interval

$$\begin{aligned} x_0 &\in (-\infty, 0] \text{ provided } \alpha < \alpha_0 \text{ and } \gamma < 0, \\ x_0 &\in [0, +\infty) \text{ provided } \alpha > \alpha_0 \text{ and } \gamma > 0, \end{aligned}$$

we perform some additional analysis which let us do the zero search on a finite interval. Note that these are the cases of the heavy tail in the skewed-T distribution. For the case $x_0 \in [0, +\infty)$ let us assume for example that we are looking for Value-at-Risk, $x_0 = \mathrm{VaR}_\alpha$, at confidence level α less than, say, 99.99%. Hence, $1 - \alpha \geq 0.0001$, or in general $1 - \alpha \geq \epsilon$ where the ϵ is a small positive number which we can specify in advance. That is, if we choose, for example, $\epsilon = 0.0002$ then we will be able to do the search for the zero x_0 of g on a finite interval but, the user specified VaR confidence level must not be greater than 99.98%. Thus, for a chosen small positive number ϵ, the $-/+$ infinity in the above two intervals can be replaced with $-/+$ "big" number M depending on ϵ.

Proposition 1. *The infinite intervals for the search of the unique α level $x_0 = VaR_\alpha$ can be replaced by finite intervals such that the $-/+$ infinity in Corollary 1 is replaced with $-/+M$ given by*

$$M = \frac{2d + 3\sqrt{8d}}{\mp\gamma}$$

where

$$d = \frac{\nu\gamma^2}{4}\left[\epsilon\Gamma\left(\frac{\nu+2}{2}\right)\right]^{-2/\nu}$$

and ϵ is an arbitrary small positive number specified in advance.

Proof. See the Appendix □

Here ν and γ are the skewed-T distribution parameters, and $\Gamma(\cdot)$ is the Gamma function. Technically, in the case $x_0 \in [0, +\infty)$, we have that $g(0) = -\alpha + \alpha_0 < 0$, while $g(M) \geq 1 - \alpha - \epsilon > 0$ and $g(+\infty) = 1 - \alpha > 0$. So, the interval $[0, +\infty)$ for the zero x_0 is replaced with $[0, M]$ provided that ϵ is chosen such that $\epsilon < 1 - \alpha$. Note, M depends on ϵ.

Thorough proof for the M formula in the first case, $x_0 \in (-\infty, 0]$, is presented in the Appendix.

2.2 AVaR formula for skewed-T distribution

The AVaR (average VaR, also known as conditional value-at-risk CVaR, or ETL, i.e. expected tail loss) is defined as the expectation of a distribution tail. As we already discussed it with respect to the VaR formula, for the skewed-T distribution we distinguish four cases depending on whether we are interested in computing the expectation of the left or the right distribution tail and, on the sign of the skewness controlling parameter γ. As it is also discussed, the interest in computing the conditional expectation of the left or the right distribution tail might depend on whether the distribution is utilized for modeling asset returns or asset losses. We so study all possible four cases for the conditional tail expectations.

Case 1: $x_0 \geq 0, \gamma > 0$,

$$AVaR_1 = E[X|X > x_0] = \frac{1}{P(X > x_0)}\int_{x_0}^{\infty} xf(x)dx$$

Case 2: $x_0 \leq 0, \gamma > 0$,

$$AVaR_2 = E[X|X < x_0] = \frac{1}{P(X < x_0)}\int_{-\infty}^{x_0} xf(x)dx$$

Case 3: $x_0 \geq 0, \gamma < 0$,

$$AVaR_3 = E[X|X > x_0] = \frac{1}{P(X > x_0)}\int_{x_0}^{\infty} xf(x)dx$$

Case 4: $x_0 \leq 0, \gamma < 0$,

$$AVaR_4 = E[X|X < x_0] = \frac{1}{P(X < x_0)}\int_{-\infty}^{x_0} xf(x)dx$$

The alpha level AVaR depends on x_0 which stands for the alpha level VaR, that is, $x_0 = \mathrm{VaR}_\alpha$ is the $1-\alpha$ quantile of the distribution. The x_0 is such that

$$1-\alpha = \int_{x_0}^{\infty} f(x)dx$$

The VaR_α is studied in the previous section.

Along with the notations we have so far, like C and $t_0(x)$, here we introduce two additional notations

$$\beta = \sqrt{\nu\gamma^2 + (\gamma x_0)^2} \quad \text{and} \quad h_0 = \frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}$$

and, an expression which repeatedly appears in the final AVaR formulas

$$KI = \left[K_{\lambda-1}(\beta) \left(\frac{2}{\beta}\right)^{\lambda-1} e^{\gamma x_0} - \sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt \right]$$

which is the difference between the modified Bessel function of the third kind with index $(\lambda-1)$ and a finite integral. Moreover,

$$KI = K_{\lambda-1}(\beta) \left(\frac{2}{\beta}\right)^{\lambda-1} e^{\gamma x_0}, \quad \text{if} \quad 9 + \gamma x_0 < 0$$

that is, the integral vanishes (has a value of order 10^{-9}) for γ and x_0 such that their values satisfy the last inequality.

Theorem 2. *The AVaR formula for skewed-T random variable, in each one of the four cases we describe, is*

$$\begin{aligned}
AVaR_1 &= \frac{\gamma\nu}{(1-\alpha)(\nu-2)} + \frac{4C}{(1-\alpha)\gamma^2} KI \\
AVaR_2 &= \frac{-4C}{\alpha\gamma^2} KI \\
AVaR_3 &= \frac{4C}{(1-\alpha)\gamma^2} KI \\
AVaR_4 &= \frac{\gamma\nu}{\alpha(\nu-2)} + \frac{-4C}{\alpha\gamma^2} KI
\end{aligned}$$

Proof. See the Appendix for all details about the proof of the $AVaR_1$ formula. The $AVaR_2$ formula is derived as a complement for $AVaR_1$ to the mean, $E(X) = (\gamma\nu)/(\nu-2)$, of the normalized skewed-T random variable X. The other two formulas, $AVaR_3$ and $AVaR_4$, come from the first two after the substitution of X with $-X$, the substitution of the skewness parameter γ with $-\gamma$, and the substitution of $\alpha = \alpha_X$ with $(1-\alpha) = \alpha_{(-X)}$. □

We note that the AVaR calculation for the skewed-T distribution requires numerical integration on infinite interval. Similarly to the previous section we approximate it with integration on a finite interval. Next we state this result.

Corollary 3. *The integral on infinite interval in the KI expression in Theorem 2 can be replaced with the following integral on a finite interval*

$$\int_0^{t_0(x_0)} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt$$

The approximation error is lees than

$$\frac{\nu\gamma}{(1-\alpha)(\nu-2)} 10^{-9}.$$

Proof. See the Appendix for all details about the proof of the $AVaR_1$ approximation. □

Theorems 1 and 2 provide the analytical formulas for VaR and AVaR of the skewed-T distribution defined as a normal mixture with inverse Gamma distribution. The approximation versions of the formulas presented in Corollaries 1 and 3 allow to carry on numerical tests of those formulas. The results are presented in the next section.

3 About issues in numerical implementations

In this section we present our findings when the analytical VaR and AVaR formulas are tested in numerical experiments. We, first, generate ten million variates from the skewed-T distribution. This let us achieve good sample estimates for VaR and AVaR quantities at different confidence levels α. We vary the confidence level from one to ninety nine percent. Then we compare the estimated quantities with their analytical counterparts. The percent relative error $100 * |(sampleEstimate - analyticalResult)/sampleEstimate|$ stays below one percent. But, when we begin vary the distribution parameters significantly then we discover that some additional theoretical work must be done.

Two important issues deserve our attention. First, the accuracy of the numerical integration, and second, the asymptotic behavior of the modified Bessel function of the third kind. The later is well studied in the scientific literature. We so only point out the way we utilize this asymptotic behavior for achieving high numerical accuracy.

About the first issue, the accuracy in the numerical integration, after extensive numerical testing we note that the integrand function, that is, the function defining the quadratures has a spike-like shape which can easily destroy the accuracy in the numerical integration. Finding the exact point at which the spike appears requires additional amount of numerical calculations or theoretical analysis. So, we basically determine that the spike must appear near the point $\gamma^2/2$. This fact is stated and proved in the next proposition.

Proposition 2. *The integrand function in the quadratures in Theorems 1 and 2, respectively, in Corollaries 1 and 3*

$$u(t) = t^{-\theta} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right)$$

where $\theta = (\nu+2)/2$, or $\theta = \lambda - 1/2 = \nu/2$, has maximum for $t > 0$ in a neighborhood of $t = \gamma^2/2$.

Proof. We use the notation h_0 introduced in the previous section. The first derivative of $u(t)$ becomes

$$u'(t) = t^{-\theta-2} e^{-\frac{\nu\gamma^2}{4t}} \left[\left(\frac{\nu\gamma^2}{4} - \theta t \right) \Phi(h_0) - (\gamma x_0 + 2t) \frac{\sqrt{t}}{2\sqrt{2}} \Phi'(h_0) \right]$$

The first expression multiplying the cdf $\Phi(h_0)$ changes sign at $t_1 = \nu\gamma^2/(4\theta)$ being positive for $t < t_1$ and negative for $t > t_1$. The second expression multiplying the pdf $\Phi'(h_0)$ changes sign at $t_2 = -\gamma x_0/2$ only if γ and $x_0 = VaR_\alpha$ are such that $\gamma x_0 < 0$ (note, we are interested in partitioning the quadratures for $t \in [0, t_0(x_0)]$, that is for positive t's, on two intervals which union covers the quadratures interval).

If $\gamma x_0 > 0$ then the second expression in $u'(t)$ does not change sign. Hence, the derivative has a unique zero, respectively the integrand function in the quadratures has a unique maximum, for t "near" t_1, that is, for $t < t_1$ because the second expression in $u'(t)$ takes on negative values which shift the t_1 zero to the left. We note that in both cases for θ, i.e. for $(\nu + 2)/2$ and $\nu/2$, the t_1 value is close to the $\gamma^2/2$ approximation which we suggest in this proposition, and which is numerically tested.

If $\gamma x_0 < 0$ then the derivative $u'(t)$ can change sign (eventually more than once) for t between $t_1 \approx \gamma^2/2$ and $t_2 = -\gamma x_0/2$. Otherwise, for $t < \min(t_1, t_2)$ the derivative $u'(t)$ takes on positive values, and for $t > \max(t_1, t_2)$ the derivative has negative values. Therefore there is at least one maximum for the integrand function $u(t)$ between t_1 and t_2, that is, for t belonging to the interval $[\min(t_1, t_2), \max(t_1, t_2)]$.

We combine the conclusions from the above two cases, and we approximate the location of the maximum with $t_1 \approx \gamma^2/2$ for all cases.

□

Based on the proposition and the numerical experiments, we conclude that every case of numerical integration in the formulas for VaR and AVaR must be partitioned in two quadratures at $\gamma^2/2$. This is especially important for near symmetric skewed-T distributions, that is when γ goes to zero. This completely resolves the numerical issues arising in the analytical VaR calculation. However, in the analytical AVaR calculation we have to deal with the Bessel function evaluation involved in our formula.

We use the following two asymptotic properties of the modified Bessel function of the third kind

$$K_\lambda(x) \longrightarrow \sqrt{\frac{\pi}{2x}} e^{-x}$$

for large $x >> |\lambda^2 - 1/4|$, and

$$K_\lambda(x) \to \frac{\Gamma(\lambda)}{2} \left(\frac{2}{x} \right)^\lambda$$

for small positive $x << \sqrt{\lambda + 1}$. The first asymptotic is especially important for large γ. For such γ's we have that β goes to plus infinity and β has the order of z_0. In this case we see that the Beseel part

$$K_{\lambda-1}(\beta) \left(\frac{2}{\beta} \right)^{\lambda-1} e^{\gamma x_0}$$

in the KI expression tends to zero which significantly helps in the numerical calculations because, otherwise, one might have to deal with an undefined numeric expression looking like zero times infinity. On the other side when evaluating the above expression for small γ then the second asymptotic significantly improves the accuracy in the analytical AVaR calculation.

4 Conclusions

We develop analytical formulas for computing the α level VaR and AVaR for a random variable X having the asymmetric Student-t distribution, also known as the skewed-T distribution. It is defined as a normal mixture with inverse Gamma distribution. The distribution pdf and the AVaR formula require the Bessel function calculation.

The analytical formulas are tested and they appear to be accurate for different confidence levels α. The parameters of the skewed-T distribution $X \sim t_d(\nu, \mu, \sigma, \gamma)$ are also varied. For the normalized random variable $(X-\mu)/\sigma$ it is important to vary the degrees of freedom parameter ν and the ratio γ/σ when the achieved numerical accuracy is tested against very large sample estimates. We tested the analytical formulas for ν in the range $[4, 400]$ where the results for $\nu > 300$ are approximated very well with $\nu = 300$. The test range for the ratio γ/σ is $[10^{-4}, 10^2]$. In all tested cases the analytical formulas yield results which differ from ten million sample size estimates by less than one percent.

The derived analytical formulas are very useful if only a small number of sample observations is available. In this case we find that the sample estimates tend to underestimate the heavy tail extreme quantiles. On the other side, for large sample size we find that the sample estimates tend to overestimate the light (short) tail extreme quantiles. Further qualitative and quantitative analysis could be performed on the derived formulas in a future research concerning their applications in modeling financial time series.

Appendix

The VaR formula for $\gamma > 0$ (Proof of Theorem 1)

The result for the α level Value-at-Risk, $x_0 = VaR_\alpha$ of a skewed-T random variable, derived in the main text, says that x_0 is the unique zero of the following equation

$$g(x_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0, \text{ if } \gamma > 0,$$

where the zero, x_0 of g, is sought in the interval $[-9/\gamma, 0]$ provided $\alpha < \alpha_0$, or in the interval $[0, +\infty]$ provided $\alpha > \alpha_0$. Furthermore, $g(\cdot)$ is an increasing function of x_0 which actually implies the uniqueness of the zero.

Here we prove this fact. We begin with the VaR definition

$$1 - \alpha = \int_{x_0}^\infty f(x)dx$$

which is rewritten in

$$g(x_0) = 1 - \alpha - \int_{x_0}^{\infty} f(x)dx.$$

Next, we simplify the integral

$$\begin{aligned}\int_{x_0}^{\infty} f(x)dx &= \int_{x_0}^{\infty} C \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dtdx \\ &= \frac{C}{\gamma} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{x_0}^{\infty} e^{\gamma x - \frac{(\gamma x)^2}{4t}} d(\gamma x)dt.\end{aligned}$$

The change of variables $z = \gamma x$, $z_0 = \gamma x_0$, in the inner integral yields

$$\begin{aligned}\int_{x_0}^{\infty} f(x)dx &= \frac{C}{\gamma} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{z_0}^{\infty} e^{z - \frac{z^2}{4t}} dzdt \\ &= \frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-\lambda-1/2} e^{-\frac{\nu\gamma^2}{4t}} \left[1 - \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right)\right] dt.\end{aligned}$$

We now use the following identity

$$\frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-\lambda-1/2} e^{-\frac{\nu\gamma^2}{4t}} dt = 1$$

(note, the above right-hand side, 1, must be replaced with -1 if $\gamma < 0$). We substitute back $\lambda = (\nu+1)/2$, and we obtain

$$\int_{x_0}^{\infty} f(x)dx = 1 - \frac{2C\sqrt{\pi}}{\gamma} \int_0^{\infty} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt$$

which completes the proof.

The approximation VaR formula for $\gamma > 0$ (Proof of Corollary 1)

We note that for t such that

$$\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t} < -6.$$

the tail of the last integral (in the above proof) becomes infinitely small. This is so because the standard normal cdf satisfies $\Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) < \Phi(-6) < 10^{-9}$, and the integral from the remaining integrand function is bounded by one over the tail of the integral (this statement is rigorously proved below). The tail is on the interval $t \in [t_0(x_0), +\infty]$ where $t_0(x_0)$ satisfies the above inequality. The last inequality is equivalent to a quadratic inequality with respect to $\sqrt{t}$

$$2t - 6\sqrt{2t} - \gamma x_0 > 0$$

which is true for

$$t > t_0(x_0) \equiv \left(\frac{3\sqrt{2} + \sqrt{18 + 2\gamma x_0}}{2}\right)^2$$

(which coincides with the notation $t_0(x)$ in the main text). We partition the last integral on $[0,\infty]$ as integral on $[0,t_0(x_0)]$ plus integral on $[t_0(x_0),\infty]$. For the latter we argue above that it is infinitely small (basically, it is the error term $R(t_0(x_0))$). This is so because we have the following inequality

$$\begin{aligned}
R(t_0(x_0)) &\equiv \frac{2C\sqrt{\pi}}{\gamma}\int_{t_0(x_0)}^{\infty} t^{-(\nu+2)/2}e^{-\frac{\nu\gamma^2}{4t}}\Phi\left(\frac{\gamma x_0}{\sqrt{2t}}-\sqrt{2t}\right)dt \\
&\leq \frac{2C\sqrt{\pi}}{\gamma}\Phi(-6)\int_{t_0(x_0)}^{\infty} t^{-(\nu+2)/2}e^{-\frac{\nu\gamma^2}{4t}}dt
\end{aligned}$$

The last expression simplifies to, and is bounded by $\Phi(-6)<10^{-9}$

$$\Phi(-6)\frac{\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)};\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\leq\Phi(-6)$$

where $\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)};\frac{\nu}{2}\right)$ is the lower incomplete Gamma function. Therefore

$$\int_{x_0}^{\infty} f(x)dx \approx 1-\frac{2C\sqrt{\pi}}{\gamma}\int_0^{t_0(x_0)} t^{-(\nu+2)/2}e^{-\frac{\nu\gamma^2}{4t}}\Phi\left(\frac{\gamma x_0}{\sqrt{2t}}-\sqrt{2t}\right)dt$$

which completes the proof of the approximation formula for $g(x_0)$. Rigorously we have

$$g(x_0) = -\alpha+\frac{2C\sqrt{\pi}}{\gamma}\int_0^{t_0(x_0)} t^{-(\nu+2)/2}e^{-\frac{\nu\gamma^2}{4t}}\Phi\left(\frac{\gamma x_0}{\sqrt{2t}}-\sqrt{2t}\right)dt + R(t_0(x_0))$$

and,

$$g(x_0) \geq -\alpha+\frac{2C\sqrt{\pi}}{\gamma}\int_0^{t_0(x_0)} t^{-(\nu+2)/2}e^{-\frac{\nu\gamma^2}{4t}}\Phi\left(\frac{\gamma x_0}{\sqrt{2t}}-\sqrt{2t}\right)dt$$

where we chose the lower bound of $g(x_0)$ as its approximation.

Furthermore, (recall, we are in the case $\gamma>0$), we note that if $\alpha>\alpha_0$, i.e., $x_0>0$, then the quadratic inequality is true for $t>t_0(x_0)$. And, if $\alpha<\alpha_0$, i.e., $x_0<0$, then the quadratic inequality is true for $t>t_0(0)$ provided $18+2\gamma x_0>0$. Otherwise, if $18+2\gamma x_0<0$ then the quadratic inequality is true for any t. Hence, the search for negative x_0 should be performed only for $x_0>-9/\gamma$. Finally, the approximation for $g(x_0)$ increases in x_0 because from the expression we have for $g(x_0)$ we see that $\Phi(\cdot)$ increases in x_0 and, the integral in the $g(x_0)$ expression is an integral from nonnegative function on the interval $[0,t_0(x_0)]$ where the right end $t_0(x_0)$ of the interval also increases with respect to x_0.

The $AVaR_1$ formula for skewed-T (Proof of Theorem 2)

The proof for the following formula

$$AVaR_1 = \frac{\gamma\nu}{(1-\alpha)(\nu-2)}+\frac{4C}{(1-\alpha)\gamma^2}KI$$

is presented below.

$$
\begin{aligned}
AVaR_1 &= E[X|X > x_0] \\
&= \frac{1}{P(X > x_0)} \int_{x_0}^{\infty} x f(x) dx \\
&= \frac{1}{(1-\alpha)} \int_{x_0}^{\infty} x \frac{\nu^{\lambda}\gamma^{2\lambda}}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}2^{2\lambda}} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} e^{\gamma x - \frac{(\gamma x)^2}{4t}} dt dx \\
&= \frac{C}{(1-\alpha)\gamma^2} \int_0^{\infty} t^{-\lambda-1} e^{-t-\frac{\nu\gamma^2}{4t}} \int_{x_0}^{\infty} (\gamma x) e^{\gamma x - \frac{(\gamma x)^2}{4t}} d(\gamma x) dt
\end{aligned}
$$

We set $z = \gamma x$, $z_0 = \gamma x_0$, and simplify the inner integral, first, with integration by parts

$$
\begin{aligned}
\int_{z_0}^{\infty} z e^{z-\frac{z^2}{4t}} dz &= -2t \int_{z_0}^{\infty} e^z d\left(e^{\frac{-z^2}{4t}}\right) \\
&= 2t \left(e^{z_0 - \frac{z_0^2}{4t}} + \int_{z_0}^{\infty} e^{z-\frac{z^2}{4t}} dz\right)
\end{aligned}
$$

and, second, with change of variables technique $h = \frac{z}{\sqrt{2t}} - \sqrt{2t}$, $h_0 = \frac{z_0}{\sqrt{2t}} - \sqrt{2t}$ leading to closed-form result with standard normal cdf

$$
\begin{aligned}
\int_{z_0}^{\infty} z e^{z-\frac{z^2}{4t}} dz &= 2t \left(e^{z_0 - \frac{z_0^2}{4t}} + e^t \sqrt{2t} \int_{h_0}^{\infty} e^{-\frac{h^2}{2}} dh\right) \\
&= 2t \left(e^{z_0 - \frac{z_0^2}{4t}} + e^t \sqrt{2t}\sqrt{2\pi}(1 - \Phi(h_0))\right)
\end{aligned}
$$

Hence, the conditional expectation of the right tail of the skewed-T distribution becomes

$$
\begin{aligned}
AVaR_1 &= \frac{2C}{(1-\alpha)\gamma^2} \int_0^{\infty} t^{-\lambda} e^{-t-\frac{\nu\gamma^2}{4t}} \left(e^{z_0-\frac{z_0^2}{4t}} + e^t\sqrt{2t}\sqrt{2\pi}(1-\Phi(h_0))\right) dt \\
&= \frac{2C}{(1-\alpha)\gamma^2}(E_1 + E_2 - E_3),
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= e^{z_0} \int_0^{\infty} t^{-\lambda} e^{-t-\frac{\nu\gamma^2}{4t}-\frac{z_0^2}{4t}} dt \\
E_2 &= 2\sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} dt \\
E_3 &= 2\sqrt{\pi} \int_0^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt
\end{aligned}
$$

We combine the integral representation of the Bessel function with one earlier notation $\beta = \sqrt{\nu\gamma^2 + (\gamma x_0)^2}$ (recall $z_0 = \gamma x_0$) which simplifies E_1 to a closed-form result

$$E_1 = e^{z_0} \int_0^\infty t^{-\lambda} e^{-t-\frac{\beta^2}{4t}} dt$$
$$= 2e^{z_0} \left(\frac{2}{\beta}\right)^{\lambda-1} K_{\lambda-1}(\beta)$$

We note that the result for E_1 (along with the $\frac{2C}{(1-\alpha)\gamma^2}$ multiplier) corresponds to the Bessel function term in KI

$$\frac{4C}{(1-\alpha)\gamma^2} K_{\lambda-1}(\beta) \left(\frac{2}{\beta}\right)^{\lambda-1} e^{\gamma x_0}$$

in the formula for $AVaR_1$.

Next, we show that E_2 simplifies to the first term in the $AVaR_1$ formula. We apply the Gamma function definition $\Gamma(u) = \int_0^\infty v^{u-1} e^{-v} dv$ for argument $u = \lambda - 3/2$ with change of variables $v = \frac{\nu\gamma^2}{4t}$. Hence

$$E_2 = 2\sqrt{\pi} \int_0^\infty t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} dt$$
$$= 2\sqrt{\pi} \left(\frac{4}{\nu\gamma^2}\right)^{\lambda-3/2} \Gamma(\lambda - 3/2)$$
$$= \sqrt{\pi}\ \frac{2^{2\lambda-2}\Gamma(\lambda - 1/2)}{\nu^{(\lambda-3/2)}\gamma^{2\lambda-3}(\lambda - 3/2)}$$
$$= \frac{\gamma^3 \nu}{2(\nu - 2)C}$$

where the second to the last equality comes from the following property $\Gamma(u) = (u-1)\Gamma(u-1)$. For the last equality we use the definition of the notation C and recall that $\lambda = (\nu+1)/2$. We note that adjusting the last result for E_2 with the multiplier $\frac{2C}{(1-\alpha)\gamma^2}$ yields the first term in the $AVaR_1$ formula.

Finally, we deal with the E_3 expression

$$\frac{E_3}{2\sqrt{\pi}} = \int_0^\infty t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt$$

which is an integral on infinite interval. We note that the result for E_3 (along with the $\frac{2C}{(1-\alpha)\gamma^2}$ multiplier) corresponds to the third term in the $AVaR_1$ formula, that is, to the integral expression in KI. This completes the proof of the theorem.

The approximation formula for $AVaR_1$ (Proof of Corollary 3)

We keep work on the E_3 expression from the end of the previous proof. We will approximate the integral with one on a finite interval. We note that the argument, h_0, of the standard normal cdf tends to minus infinity as t goes to plus infinity (recall $h_0(t) = \frac{z_0}{\sqrt{2t}} - \sqrt{2t}$). Hence, for sufficiently large t we have

$\Phi(h_0)$ going to zero. Technically, sufficiently large t can be determined (as in the proof of Theorem 1) from $\Phi(-6) < 10^{-9}$. The inequality $h_0 < -6$ is true for $t > t_0(x_0)$ where the last notation $t_0(x_0)$ is already specified in the main text and in the proof of Theorem 1. We so obtain

$$\frac{E_3}{2\sqrt{\pi}} = \int_0^{t_0(x_0)} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt + \int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt$$

The first integral corresponds to the third term in the $AVaR_1$ formula, that is, to the integral expression in KI (*"here we complete the proof of this formula"*). The second integral "vanishes" because it simplifies similarly to the proof of Theorem 1. We have that $h_0(t) \leq h_0(t_0(x_0)) = -6$ for $t \geq t_0(x_0)$, hence the integral is bounded by

$$\int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi(h_0) dt \leq \Phi(-6) * \int_{t_0(x_0)}^{\infty} t^{-\lambda+1/2} e^{-\frac{\nu\gamma^2}{4t}} dt$$

As it is already done earlier in this proof, the last expression must be adjusted by the multipliers $\frac{2C}{(1-\alpha)\gamma^2}$ and $2\sqrt{\pi}$. Hence, after simplification (similar to the one in the proof of Theorem 1) we obtain that the second integral is bounded by

$$\Phi(-6) * \frac{\nu\gamma}{(1-\alpha)(\nu-2)} * \frac{\gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2}-1\right)}{\Gamma\left(\frac{\nu}{2}-1\right)}$$

where the last multiplier is bounded by 1, the second multiplier is a constant for given distribution parameters and AVaR level α, and the first multiplier is bounded by 10^{-9}. Therefore, we assume that the error made in the transition from integral on infinite interval to integral on finite interval is infinitely small.

The "big" M formula (Proof of Proposition 1)

The proof for the following formula is presented below

$$M = \frac{2d + 3\sqrt{8d}}{(-\gamma)}$$

where

$$d = \frac{\nu\gamma^2}{4}\left[\epsilon\Gamma\left(\frac{\nu+2}{2}\right)\right]^{-2/\nu}$$

in the case $\alpha < \alpha_0$ and $\gamma < 0$.

In this case we look for the unique zero x_0 of

$$g(x_0) = 1 - \alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{t_0(x_0)} t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \Phi\left(\frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}\right) dt = 0,$$

in $(-\infty, 0]$, and we prove that for a given small positive number ϵ such that $\alpha > \epsilon$ the zero search can be performed on $[-M, 0]$ rather than on $(-\infty, 0]$. We note that we are in the very left end of the heavy left tail of the skewed-T distribution, that is, the small ϵ represent the probability for being in $(-\infty, -M]$. In the main text we said that $g(\cdot)$ is an increasing function from $g(-\infty) = -\alpha < 0$ to $g(0) = -\alpha + \alpha_0 > 0$. So, we now prove that

$$g(-M) \leq -\alpha + \epsilon,$$

that is, $g(-M) < 0$ provided that ϵ is chosen such that $(\epsilon < \alpha)$ it is less than the user specified confidence level α for VaR_α.

We utilize the notations

$$F(t,k) = t^{-(\nu+2)/2} e^{-\frac{\nu\gamma^2}{4t}} \frac{e^{-k^2/2}}{\sqrt{2\pi}}$$
$$h_0(t) = \frac{\gamma x_0}{\sqrt{2t}} - \sqrt{2t}$$

and the definition of the standard normal cdf $\Phi(\cdot)$ to rewrite the integral in the $g(x_0)$ expression as follows

$$\int_0^{t_0(x_0)} \int_{-\infty}^{h_0(t)} F(t,k)dkdt.$$

Next, we change the order of integration, and present the integral as a sum of two integrals

$$\int_{-\infty}^{-6} \int_0^{t_0(x_0)} F(t,k)dtdk + \int_{-6}^{\infty} \int_0^{h_0^{-1}(k)} F(t,k)dtdk$$

where $t = h_0^{-1}(k)$ is the inverse function of $k = h_0(t)$. The two functions are defined on the intervals $k \in [-6, +\infty)$ and $t \in [0, t_0(x_0)]$ respectively (note, the inverse function exists because $h_0(\cdot)$ is a monotone function). The first double integral is bounded by

$$\frac{\Gamma\left(\frac{\nu\gamma^2}{4t_0(x_0)}; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \Phi(-6)$$

which is a product of (positive) multiplier less than one (i.e. the ratio of upper incomplete Gamma function and the Gamma function) and, technically, the zero value $\Phi(-6) \approx 10^{-9}$. We so focus only on the second double integral in the last expression for $g(x_0)$. Hence,

$$\begin{aligned} g(x_0) &= 1 - \alpha - \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{\infty} \left[\Gamma\left(\frac{\nu}{2}\right) - \gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(k)}; \frac{\nu}{2}\right)\right] \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \\ &= -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{\infty} \gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(k)}; \frac{\nu}{2}\right) \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \end{aligned}$$

where $\gamma\left(\cdot; \frac{\nu}{2}\right)$ is the lower incomplete Gamma function, i.e., $\Gamma\left(\frac{\nu}{2}\right) = \gamma\left(\cdot; \frac{\nu}{2}\right) + \Gamma\left(\cdot; \frac{\nu}{2}\right)$. Note, for the pair of double integrals in $g(x_0)$ we utilized the identity

$$\frac{2C\sqrt{\pi}}{\gamma} \left(\frac{\nu\gamma^2}{4}\right)^{-\nu/2} \Gamma\left(\frac{\nu}{2}\right) = -1$$

for $\gamma < 0$ (otherwise, the above expression simplifies to plus one for $\gamma > 0$). Next we have

$$g(x_0) = -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{6} \gamma\left(\cdot; \frac{\nu}{2}\right) \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk + \int_{6}^{\infty} \frac{\gamma\left(\cdot; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk$$

where the second integral is technically equal zero for reasons described earlier with respect to the first double integral in $g(x_0)$. Therefore, we focus on bounding the integral on finite interval $[-6, 6]$. We note that the lower incomplete gamma function in this integral, $\gamma\left(\cdot; \frac{\nu}{2}\right)$, is an increasing function of k in its argument $\frac{\nu\gamma^2}{4h_0^{-1}(k)}$. Hence,

$$\begin{aligned} g(x_0) &\leq -\alpha + \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \int_{-6}^{6} \gamma\left(\cdot; \frac{\nu}{2}\right)\Big|_{k=6} \frac{e^{-k^2/2}}{\sqrt{2\pi}} dk \\ &= -\alpha + \frac{\gamma\left(\frac{\nu\gamma^2}{4h_0^{-1}(6)}; \frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \end{aligned}$$

Next we bound the lower incomplete gamma function $\gamma(u; a)$ by

$$\gamma(u; a) = \int_0^u t^{a-1} e^{-t} dt \leq u^a / a.$$

Hence,

$$g(x_0) \leq -\alpha + \frac{\left(\frac{\nu\gamma^2}{4h_0^{-1}(6)}\right)^{\nu/2}}{\frac{\nu}{2}\Gamma\left(\frac{\nu}{2}\right)} \equiv l(x_0)$$

where

$$\begin{aligned} h_0^{-1}(6) &= \left.\frac{(\sqrt{k^2 + 4\gamma x_0} - k)^2}{8}\right|_{k=6} \\ &= \frac{(\sqrt{9 + \gamma x_0} - 3)^2}{2} \end{aligned}$$

Here we may observe that

$$h_0^{-1}(6)\big|_{x_0=-M} = d$$

where d is specified in the M definition.

Finally, some tedious algebraic manipulations show that $l(-M) = -\alpha + \epsilon$ which completes the proof for the "big" M formula.

Remark: If we decide that we do not have to ignore (we do not want to ignore) the two infinite small terms (integrals) in the proof (involving a constant times $\Phi(-6)$ where the absolute value of that constant is less than one) then we may choose ϵ such that $\epsilon < \alpha - 2 * 10^{-9}$ for some specified in advance VaR level α.

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Ball Closure Property In Fuzzy Metric Spaces

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Abstract

The aim of this paper to introduce the ball closure property in fuzzy metric spaces. We also give some theorems on relationship between ball closure property and density and also compactness.

Key Words. Fuzzy metric space, closed ball, compact subset.
M.S.C. (2000). 54A40, 54E35.

1. INTRODUCTION

Since the concept of fuzzy set was introduced by Zadeh [10] in 1965, many authors have introduced the concept of fuzzy metric space in different ways [1-5]. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] and defined a Hausdorff topology on this fuzzy metric space. Malik and Toma [7] introduced the ball closure property in metric spaces.

In this paper we study on ball closure property in fuzzy metric spaces in the sense of George and Veeramani. We also give some relations between ball closure property and density and also compactness.

2. PRELIMINARIES

Definition 1 ([8]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t-norm if* $*$ *is satisfying the following conditions:*

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 ([3]). *A triple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* x, y, $z \in X$, s, $t > 0$,

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Remark 1. *In fuzzy metric space* X, $M(x, y, .)$ *is non-decreasing for all* $x, y \in X$.

Example 1. *Let* (X, d) *be a metric space. Denote* $a * b = ab$ *for all* $a, b \in [0, 1]$ *and let* M_d *be a fuzzy set on* $X^2 \times (0, \infty)$ *defined as follows:*

$$M_d(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all $k, m, n \in \mathbb{R}^+$. *Then* $(X, M_d, *)$ *is a fuzzy metric space.*

Remark 2. *Note the above example holds even with the t-norm* $a * b = \min\{a, b\}$ *and hence* M *is a fuzzy metric with respect to any continuous t-norm. In the above example by taking* $k = m = n = 1$, *we get*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

We call this fuzzy metric induced by a metric d *the standard fuzzy metric.*

Definition 3 ([3]). *Let* $(X, M, *)$ *be a fuzzy metric space and let* $r \in (0, 1)$, $t > 0$ *and* $x \in X$. *The set*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

is called the open ball with center x *and radius* r *with respect to* t.

Theorem 1 ([3]). *Every open ball* $B(x, r, t)$ *is an open set.*

Definition 4 ([3]). *Let* $(X, M, *)$ *be a fuzzy metric space and let* $r \in (0, 1)$, $t > 0$ *and* $x \in X$. *The set*

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$$

is called the closed ball with center x *and radius* r *with respect to* t.

Theorem 2 ([3]). *Every closed ball* $B[x, r, t]$ *is a closed set.*

Remark 3. *Let* $(X, M, *)$ *be a fuzzy metric space. Define* $\tau = \{A \subset X :$ *for each* $x \in A$, *there exist* $t > 0$, $r \in (0, 1)$ *such that* $B(x, r, t) \subset A\}$. *Then* τ *is a topology on* X.

Remark 4.

(i) Since $\{B(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, ...\}$ is a local base at x, the topology τ is first countable.
(ii) Every fuzzy metric space is Hausdorff.
(iii) Let $(X, M, *)$ be an fuzzy metric space and τ be the topology on X induced by the fuzzy metric. Then for a sequence $(x_n)_n$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$ and for all $t > 0$.
(iv) In a fuzzy metric space every compact set is closed and bounded.

Definition 5 ([9]). *Let* $(X, M, *)$ *be a fuzzy metric space,* $x \in X$ *and* $A \subset X$. *The degree of closeness* x *to* A *is defined by*

$$M(A, x, t) = \sup\{M(y, x, t) : y \in A\}$$

for all $t > 0$.

3. MAIN RESULTS

Let $(X, M, *)$ be a fuzzy metric space and A be a subset of X. We denote

$$\begin{aligned} B(A, r, t) &= \{x \in X : M(A, x, t) > 1 - r\} \\ B[A, r, t] &= \{x \in X : M(A, x, t) \geq 1 - r\} \end{aligned}$$

for all $r \in (0, 1)$ and for all $t > 0$.

Definition 6. *Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to have ball closure property iff for all $r \in (0, 1)$ and $t > 0$ we have $\overline{B(A, r, t)} = B[A, r, t]$.*

If each subset A of X has this property we say that the fuzzy metric space $(X, M, *)$ has the ball closure property.

Example 2. *Let $X = [0, 1] \cup [2, 3]$. Define $a * b = ab$ and*

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $t > 0$. Let $A = \{1\}$ and $r = 1 - \frac{t}{t+1}$. Then

$$\begin{aligned} B\left[\{1\}, 1 - \frac{t}{t+1}, t\right] &= \left\{x \in X : M(\{1\}, x, t) \geq \frac{t}{t+1}\right\} \\ &= [0, 1] \cup \{2\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \overline{B\left(\{1\}, 1 - \frac{t}{t+1}, t\right)} &= \overline{\left\{x \in X : M(\{1\}, x, t) > \frac{t}{t+1}\right\}} \\ &= \overline{(0, 1)} = [0, 1]. \end{aligned}$$

Hence $(X, M, *)$ does not have the ball closure property.

Lemma 1. *Let $(X, M, *)$ be a fuzzy metric space and A be a subset of X. Then, $B(A, r, t) = B(\overline{A}, r, t)$ for all $r \in (0, 1)$ and $t > 0$.*

Proof. Since $A \subset \overline{A}$, then $B(A, r, t) \subset B(\overline{A}, r, t)$ for all $r \in (0, 1)$ and $t > 0$.

Let x be an arbitrary element of $B\left(\overline{A}, r, t\right)$. Then $M\left(\overline{A}, x, t\right) > 1 - r$ and there exists a point $y \in \overline{A}$ such that $M(y, x, t) > 1 - r$ for $r \in (0, 1)$ and $t > 0$. Therefore, there exists a t_0, $0 < t_0 < t$ such that $M(y, x, t_0) > 1 - r$. Let $1 - r_0 = M(y, x, t_0)$. Since $1 - r_0 > 1 - r$, there exists $s \in (0, 1)$ such that $1 - r_0 > 1 - s > 1 - r$. Now for given r_0 and s such that $1 - r_0 > 1 - s$ we can find r_1, $0 < r_1 < 1$ such that $(1 - r_0) * (1 - r_1) \geq 1 - s$. Now consider the ball $B(y, r_1, t - t_0) \subset A$. Let $z \in B(y, r_1, t - t_0)$, then $M(y, z, t - t_0) > 1 - r_1$. Therefore,

$$\begin{aligned} M(x, z, t) &\geq M(y, x, t_0) * M(y, z, t - t_0) \\ &\geq (1 - r_0) * (1 - r_1) \geq 1 - s \\ &> 1 - r \end{aligned}$$

which implies $x \in B(A, r, t)$. Then, $B(\overline{A}, r, t) \subset B(A, r, t)$ which completes the proof. $\square$

Theorem 3. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent:*

(i) $(X, M, *)$ has the ball closure property.
(ii) Every closed subset of X has the ball closure property.
(iii) Every countable subset of X has the ball closure property.
(iv) For every point $x \in X$, $r \in (0, 1)$, $t > 0$ and every sequence $(x_n)_n$ in X, with $M(x_n, x, t) \longrightarrow 1 - r$, there exists a sequence $(y_n)_n$ in $B((x_n)_n, r, t)$ such that $y_n \longrightarrow x$.

Proof. (i)$\Longleftrightarrow$(ii) is clear from Lemma 1. (i)$\Longrightarrow$(iii) is obvious. We will prove (iii)$\Longrightarrow$(iv)$\Longrightarrow$(i).

(iii)$\Longrightarrow$(iv): Let the assumptions of (iv) be satisfied.

Then $M((x_n)_n, x, t) \geq 1 - r$ for $r \in (0, 1)$ and $t > 0$, hence $x \in B[(x_n)_n, r, t]$. Therefore $x \in \overline{B((x_n)_n, r, t)}$.

(iv)$\Longrightarrow$(i): Let A be an arbitrary subset of X and let $x \in B[A, r, t]$ for $r \in (0, 1)$ and $t > 0$. If $M(A, x, t) > 1 - r$, then $x \in B(A, r, t) \subset \overline{B(A, r, t)}$ for all $r \in (0, 1)$ and $t > 0$. If $M(A, x, t) = 1 - r$, then there exists a sequence $(x_n)_n \subset A$ such that $M(x_n, x, t) \longrightarrow 1 - r$. From (iv) we have $x \in \overline{B(A, r, t)}$. This implies $B[A, r, t] \subset \overline{B(A, r, t)}$. The $\overline{B(A, r, t)} \subset B[A, r, t]$ is a consequence of the continuity of the function $x \longrightarrow M(A, x, t)$. $\square$

Theorem 4. *Let $(X, M, *)$ be a fuzzy metric space satisfying the ball closure property and let U be an open set in X. Then $(U, M, *)$ has the ball closure property.*

Proof. It is sufficient to show that $(U, M, *)$ satisfies the condition (iv) of Theorem 3. Let $x \in X$, $r \in (0, 1)$ and $(x_n)_n$ satisfy the assumptions of (iv). Then there exists a sequence $(y_n)_n$ in $B((x_n)_n, r, t)$ such that $y_n \longrightarrow x$, $x \in U$. Since U is open, we can suppose without loss of generality that $(y_n)_n \in B((x_n)_n, r, t) \cap U$ such that $y_n \longrightarrow x$ which completes the proof. $\square$

Theorem 5. *Let $(X, M, *)$ be a fuzzy metric space satisfying the ball closure property and D be a dense subset of X. Then $(D, M, *)$ has the ball closure property.*

Proof. We again make use of the assertion (iv) of Theorem 3. Let $x \in D$, $r \in (0, 1)$, $t > 0$ and $(x_n)_n \subset D$ satisfy $M(x_n, x, t) \longrightarrow 1 - r$. Then there exists a sequence $(y_n)_n$ in $B((x_n)_n, r, t)$ such that $y_n \longrightarrow x$. Since the D is dense in X there exists a sequence $(z_n)_n$ in D such that $M(y_n, z_n, t) > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and $t > 0$. Obviously $z_n \longrightarrow x$. $\square$

Theorem 6. *Let $(X, M, *)$ be a fuzzy metric space. Then the following are equivalent.*

(i) Every compact subset of X has the ball closure property.
(ii) Every singleton has the ball closure property.
(iii) For each $x, y \in X$, $r \in (0,1)$ and $t > 0$, there exists $z \in X$ such that

$$M(x,z,t) > 1-r \text{ and } M(y,z,t) > M(x,y,t).$$

Proof. (i)$\Longrightarrow$(ii) is clear. Let us prove (ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

(ii)$\Longrightarrow$(iii): From (ii) we have

$$\overline{B(y,1-M(x,y,t),t)} = B[y,1-M(x,y,t),t]$$

and $x \in B[y,1-M(x,y,t),t]$. Since $x \in \overline{B(y,1-M(x,y,t),t)}$, then

$$B(x,r,t) \cap B(y,1-M(x,y,t),t) \neq \varnothing$$

for all $r \in (0,1)$ and $t > 0$ which is precisely the assertion of (iii).

(iii)$\Longrightarrow$(i): For every compact set $A \subset X$, since $M(A,x,t)$ is a continuous function, then $\overline{B(A,r,t)} \subset B[A,r,t]$. We will show the opposite inclusion. Let x be an arbitrary element of $B[A,r,t]$. If $M(A,x,t) > 1-r$, then $x \in B(A,r,t) \subset \overline{B(A,r,t)}$, for all $r \in (0,1)$ and $t > 0$. If $M(A,x,t) = 1-r$, then there exists a sequence $(x_n)_n$ in A such that $M(x_n,x,t) \longrightarrow 1-r$, for all $r \in (0,1)$ and $t > 0$. Since A is compact, we can choose a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $x_{n_k} \longrightarrow x_0$, $x_0 \in A$. Making use of continuity the metric, we have $M(x,x_0,t) = 1-r$, for $r \in (0,1)$ and $t > 0$. Hence $x \in B[x_0,r,t]$ i.e., $M(x,x_0,t) \geq 1-r$ for $r \in (0,1)$ and $t > 0$. On the other hand from (iii) there exists $z \in X$ such that

$$M(x,z,t) > 1-r \text{ and } M(x_0,z,t) > M(x,x_0,t).$$

Then, $M(x,z,t) > 1-r$ and $M(x_0,z,t) > 1-r$ which implies that

$$B(x,r,t) \cap B(x_0,r,t) \neq \varnothing$$

for all $r \in (0,1)$ and $t > 0$. Hence $x \in \overline{B(x_0,r,t)}$ which shows $B[x_0,r,t] \subset \overline{B(x_0,r,t)}$. Since $\overline{B(x_0,r,t)} \subset \overline{B(A,r,t)}$, then the inclusion $B[A,r,t] \subset \overline{B(A,r,t)}$ is proved. □

Theorem 7. *Let $(X,M,*)$ be a complete fuzzy metric space. Then the following are equivalent.*

(i) Every totally bounded subset of X has the ball closure property.
(ii) Every compact subset of X has the ball closure property.
(iii) Every singleton in X has the ball closure property.
(iv) For each $x, y \in X$, $r \in (0,1)$ and $t > 0$, there exists $z \in X$ such that

$$M(x,z,t) > 1-r \text{ and } M(y,z,t) > M(x,y,t).$$

Proof. Applying Lemma 1 and Theorem 6 we obtain the equivalence of the assertions (i) -(iv) as an immediate consequence of the fact that in a complete fuzzy metric space the closure of each totally bounded set is a compact set. □

Theorem 8. *Let* $(X_1, M_1, *)$ *and* $(X_2, M_2, *)$ *are fuzzy metric spaces in which every compact subset has the ball closure property. Then also the product space* $(X_1 \times X_2, M, *)$ *has this property where* M *is defined by the*

$$M((x_1, x_2), (y_1, y_2), t) = \min\{M_1(x_1, y_1, t), M_2(x_2, y_2, t)\}.$$

Proof. We use the (iii) of Theorem 6. For $r \in (0, 1)$ and $t > 0$, there exists $(z_1, z_2) \in X_1 \times X_2$ such that

$$M_1(x_1, z_1, t) > 1 - r,\ M_2(x_2, z_2, t) > 1 - r$$

and consequently

$$M((x_1, x_2), (z_1, z_2), t) > 1 - r.$$

Since

$$M_1(y_1, z_1, t) > M_1(x_1, y_1, t) \text{ and } M_2(y_2, z_2, t) > M_2(x_2, y_2, t),$$

therefore

$$M((y_1, y_2), (z_1, z_2), t) > M((x_1, x_2), (y_1, y_2), t)$$

which completes the proof. □

Acknowledgement. The author would like to thank the referees for their help in the improvement of this paper.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.2,215-232,COPYRIGHT 2008 EUDOXUS PRESS, LLC

The Canonical Coherent States Associated With Quotients of the Affine Weyl-Heisenberg Group*

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Abstract

This paper is concerned with uncertainty principles in the context of the affine-Weyl-Heisenberg group in one and two dimensions. As the representation of this group fails to be square integrable, we explore various admissible sections of this group that appear in the context of coorbit and α-modulation spaces, and calculate the resulting uncertainty principles as well as its minimizers with respect to these sections. In special cases, we also consider the relationships of minimzing states with admissibility conditions.

Keywords: Affine Weyl-Heisenberg group, representations via quotients, uncertainty principles, minimizing states, coorbit spaces.
AMS Subject Classification: 22D10, 47B25

1 Introduction

In this paper, we are concerned with the construction of uncertainty principles and the computation of associated canonical coherent states. This topic of uncertainty based localization is of interest in abstract as well as in applied areas and has been therefore extensively considered in an abundant number of papers and books, see, e.g., [1, 2, 4, 5, 7, 8, 9, 12, 15, 16, 19]; even when no uncertainty principle applies, attempts have been made to derive some sort of optimally localized coherent states, see [14]. The motivation of the present paper is the interest in very special coherent states of the affine Weyl-Heisenberg group which was introduced by B. Torresani, see [21, 22]. He was the first who constructed irreducible representation and discussed concepts of square-integrability. Square integrability is always an important aspect in practical applications of group representations since it ensures the invertibility of the associated integral transform. It is shown in [21] that the representations of the affine Weyl-Heisenberg group unfortunately fail to be square integrable, i.e., there does not exist any admissible vector. Possible remedies are to factor out a suitable closed subgroup and to work with quotients and/or to weaken the concept of admissibility. These techniques for treating the affine Weyl-Heisenberg group can be nicely used in their full glory for the construction of mixed smoothness spaces and Banach frames for them. These mixed smoothness spaces are specific coorbit spaces that lie in between Besov and modulation spaces and coincide with the α-modulation spaces, see [6]. Once this abstract theory is developed, it is quite natural to ask for uncertainty principle and suitable minimizing states for these special group representations modulo quotients. Suitable means here admissible, although we want to emphazise that uncertainty relations and minimizing functions are clearly of interest by themselves. To these topics we dedicate this paper.

The most famous uncertainty relation is associated with the Short Time Fourier Transform (STFT). The STFT or so-called Gabor transform, see [10], is obtained by applying the action of the Weyl-Heisenberg group to a suitable window function and taking the inner product with the function under consideration. As a well-known result, the Gaussian function minimizes the associated Heisenberg uncertainty relation and therefore gives rise

*This work has been supported through the European Union's Human Potential Programme, under contract HPRN–CT–2002–00285 (HASSIP), and through DFG, Grants Da 360/4–3, MA 1657/15–1, TE 354/1–2.

to canonical coherent states of the Weyl-Heisenberg group. More recent studies considered the uncertainty principles which are related to the affine group in one dimension and the similitude group as well as the affine group in two dimensions [1, 5, 16]. For the one dimensional affine group it was possible to find an analytical solution of the form:

$$\psi(x) = c(x-\eta)^{-\frac{1}{2}-i\eta\mu_2+i\mu_1}, \tag{1}$$

where c is some constant, η is purely imaginary and $\mu_1, \mu_2 \in \mathbb{R}$. However, for the multi-dimensional case, it was not possible to find solutions which simultaneously minimize all appearing uncertainties with respect to all the parameters involved, and therefore solutions that accounted for various subgroups were employed, see, e.g., [6, 16, 19, 20, 21, 22].

In this paper, we follow similar lines, and we are especially interested in the case of the affine Weyl-Heisenberg group and in the associated mixed smoothness spaces that lie in between Besov and modulation spaces. However, we are not further interested in the spaces themselves but rather in the special representations that pop up in the construction process. For them we consider in detail the computation of canonical coherent states and their admissibility properties. The issue of admissibility is discussed in greater detail in Sections 3.2 and 4.2. In Section 3.2, we consider an example where switching to quasi-coherent states allows an easy verificaton of admissibility, and it turns out that minimal uncertainty and admissibility fit together quite nicely. Moreover, in Section 4.2 we are able to give explicit and sufficient conditions for admissibility but which can, unfortunately, not be fulfilled by the derived minimizing states. As a suitable alternative, the structure of the coherent states (having exponential decay in Fourier space) suggests the application of a smooth cut off operator that provides us then with admissible and 'near' minimizing coherent states.

The paper is organized as follows: In Section 2, we discuss some basic results and summarize related work. We calculate the minimizers for the one dimensional affine Weyl-Heisenberg group and address the issue of admissibility in Sections 3. Finally, in Section 4, we continue by analyzing the two-dimensional affine Weyl-Heisenberg group and explore some possible subgroups for obtaining valid minimizers.

2 Background and Related Work

A general theorem which is well-known in quantum mechanics and harmonic analysis [8] relates an uncertainty principle to any two self-adjoint operators and provides a mechanism for deriving a minimizing function for the uncertainty relation. Before repeating this well-known result on uncertainties, let us fix some notation. Let A, B be two self-adjoint operators. Their *commutator* is defined by

$$[A,B] := AB - BA,$$

the *expectation* of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\mu(A) := \mu_A := \langle A\psi, \psi\rangle$$

and, finally, the *variance* of A with respect to some state $\psi \in \mathrm{dom}(A)$ by

$$\Delta A_\psi := \mu((A-\mu(A))^2).$$

Theorem 1 *Let two self-adjoint infinitesimal generators A and B of a unitary representation of some Lie group be given, then for all $\psi \in \mathrm{dom}([A,B]) \cap \mathrm{dom}(A) \cap \mathrm{dom}(B)$ they obey the uncertainty relation:*

$$\Delta A_\psi \Delta B_\psi \;\geq \frac{1}{4}|\langle [A,B]\psi, \psi\rangle|^2. \tag{2}$$

A state ψ is said to have minimal uncertainty *if the above inequality turns into an equality. This happens iff there exists an $\eta \in i\mathbb{R}$ such that*

$$(A-\mu_A)\psi = \eta(B-\mu_B)\psi. \tag{3}$$

For an extensive discussion on the latter theorem concerning the definiteness of ψ, we refer the reader to [4, 15].

The Weyl-Heisenberg group as well as the affine group are both related to well-known transforms in signal processing: the STFT and the wavelet transform, respectively. Both can be derived from square integrable representations of these groups. The windowed Fourier transform is related to the Weyl-Heisenberg group and the wavelet transform is related to the affine group. The linearized operation of the group at the identity element can be described by the infinitesimal generators of the related Lie algebra. If the group representation is unitary, then the infinitesimal generators can transformed to be self-adjoint operators. Thus, the general uncertainty theorem stated above provides a tool for obtaining uncertainty principles using these infinitesimal generators. In the case of the Weyl-Heisenberg group, the canonical functions that minimize the corresponding uncertainty relation are Gaussian functions.

The canonical functions that minimize the uncertainty relations for the affine group in one dimension and for the similitude group in two dimensions, were the subject of previous studies [1, 5]. In these studies, it was shown that there is no non-trivial canonical function that minimizes the uncertainty relation associated with the similitude group of $\mathbb{R}^2$, $SIM(2)$. Thus, there is no non-zero solution for the set of differential equations obtained for these group generators. Rather than using the original generators of the $SIM(2)$ group, a different set of operators was used in [5] that includes elements of the enveloping algebra, i.e., polynomials in the generators of the algebra, in order to obtain the $2D$ isotropic Mexican hat as a minimizer. Further results were achieved in [1] where a symmetry in the set of commutators was obtained for the $SIM(2)$ group and a possible minimizer in the frequency domain for some fixed direction was computed. This solution is a real valued wavelet which is confined to some convex cone in the positive half-plane of the frequency space with an exponential decay inside the cone.

The extension of these studies to the affine group in two dimensions resulted in two possible solutions [16]. The first accounted for the overall scaling and rotation and utilizes the results of [1, 5]. The second solution was obtained by exploiting a symmetry in the group of commutators which led to

$$\psi(x, y) = (\eta + x)^{-\frac{1}{2} - i\mu_{11} + i\eta\mu_{b_x}} e^{i\mu_{b_y} y}. \tag{4}$$

It belongs to L_2 with respect to the variable x if we select $\mathrm{Re}(\eta)\mu_{b_x} \geq 0$, but it belongs not to L_2 in terms of the variable y, although it is periodic.

The affine Weyl-Heisenberg (AWH) group has already been addressed in this context in the early $90's$. The paper [21] considered wavelets associated with representations of the AWH group. It shows that the canonical representation of the AWH group is not square integrable, but can be regularized with some density function. This work was later extended to N-dimensional AWH wavelets [22]. In [17] a scaling was introduced in the Heisenberg group with an intertwining operator. More recently, [19] proposed a mechanism to construct generalized uncertainty principles and their minimizing wavelets in anisotropic Sobolev spaces. A new set of uncertainty principles was introduced in this paper by weakening the two operator relations and by introducing a multi-dimensional operator setting. Recently, a study [6] has considered generalizations of the coorbit space theory based on group representations modulo quotients. This is based on applying the general theory to the AWH group and obtaining families of smoothness spaces that can be identified with the α-modulation spaces.

3 The $1D$ Affine Weyl-Heisenberg Group

The affine Weyl-Heisenberg group is generated by time translations $b \in \mathbb{R}$, frequency translations $\omega \in \mathbb{R}$, spatial dilations $a \in \mathbb{R}_+$, and a toral component $\phi \in \mathbb{R}$, and is equipped with the group law

$$(b, \omega, a, \phi) \circ (b', \omega', a', \phi') = (b + ab', \omega + \omega'/a, aa', \phi + \phi' + \omega b'a).$$

The AWH group can be viewed as the extension of the affine group, incorporating frequency translations or, alternatively, as the extension of the Weyl-Heisenberg group incorporating dilations. The Stone-von-Neumann representation of G_{AWH} on $L_2(\mathbb{R})$ is given by:

$$[U(b, \omega, a, \phi)\psi](x) = a^{-\frac{1}{2}} e^{i\omega(x-b)} e^{i\phi} \psi(\frac{x-b}{a}). \tag{5}$$

This representation, however, fails to be square integrable, see [20]. The AWH group raises a special interest as it "contains" both, the affine group as well as the Weyl-Heisenberg group: If we consider cases where $a = 1$, we are in the Weyl-Heisenberg framework, and if we consider cases where $\omega = 0$ we are in the affine framework. Two independent studies have regarded these attributes, and suggested a specific section of the AWH [6, 20], where the scale is represented as a function of the frequency. It was proven that this section is admissible.

In addition, this paper introduces a mechanism that starts at the Weyl-Heisenberg case and allows a smooth transition towards the affine case.

In what follows, we consider a section where the scale is a function of the frequency and calculate the appropriate minimizing functions with respect to the uncertainty principle related to it. Then, we study a section where the frequency is regarded as a function of the scale, consider its admissibility and calculate the appropriate minimizers.

3.1 The Sections Where the Scale is a Function of the Frequency

As suggested in [21, 22] and considered in the context of α-modulation spaces, see [6], we treat the affine Weyl-Heisenberg group and its representation by factoring out a closed subgroup and work with the quotients.

We will consider G_{AWH}/H with

$$H := (0, 0, a, \phi) \in G_{AWH}$$

and Borel sections that do not depend on b, namely:

$$\sigma(b, \omega) = (b, \omega, \beta(\omega), 0).$$

Further, it was shown in [6] that the specific section

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha}$$

is admissible.

Next, we consider the effect of varying the value of $\alpha \in [0, 1]$. If $\alpha = 0$ then we obtain:

$$\beta(\omega) = \eta_0(\omega)^{-1} = (1 + |\omega|)^0 = 1.$$

Thus, there are practically no dilations and we obtain Gabor analysis. For $\alpha \to 1$ we obtain:

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha} \overset{|\alpha| \to 1}{\longrightarrow} \frac{1}{1 + |\omega|}.$$

Thus, the frequency translations and modulations are inversely proportional which is close to wavelet analysis. The intermediate case for which $\alpha = \frac{1}{2}$ is known as the Fourier-Bros-Iagolnitzer transform, see [3].

The representation for the quotient as a function of α is then given by

$$[U(b, \omega, \eta_\alpha^{-1}(\omega))\psi](x) = (1 + |\omega|)^{\frac{\alpha}{2}} e^{i\omega(x-b)} \psi\left((1 + |\omega|)^\alpha (x - b)\right).$$

As can be seen, this representation is not C^1 for $\omega = 0$. Nevertheless, when calculating the infinitesimal generators, we may take the one-sided derivatives.

Lemma 2 *The infinitesimal operators T_b, T_ω associated with the one dimensional G_{AWH} are given by*

$$(T_b\psi)(x) = -i\frac{\partial}{\partial x}\psi(x), \quad \textit{and} \quad (T_\omega\psi)(x) = (i\frac{\alpha}{2} - x)\psi(x) + i\alpha x \frac{\partial}{\partial x}\psi(x). \tag{6}$$

The state ψ which is the minimizer of the associated uncertainty is of the form

$$\psi(x) = e^{\frac{-ix}{\alpha}} (\alpha\lambda x + 1)^{-\frac{1}{2} - \frac{i\mu_\omega}{\alpha} + \frac{i\mu_b}{\alpha\lambda} + \frac{i}{\alpha^2\lambda}}, \tag{7}$$

where $\lambda \in i\mathbb{R}$.

Proof; Taking the (one-sided) derivatives with respect to ω and b and evaluating them at $b = 0, \omega = 0$ leads to

$$\frac{\partial}{\partial b}U(b, \omega, \eta_\alpha(\omega), 0)\psi|_{b=0,\omega=0}(x) = -\frac{\partial}{\partial x}\psi(x), \qquad \frac{\partial}{\partial \omega}U(b, \omega, \eta_\alpha(\omega), 0)\psi|_{b=0,\omega=0}(x) = (\frac{\alpha}{2} + ix)\psi(x) + \alpha x \frac{\partial}{\partial x}\psi(x).$$

By its construction, these operators are only skew symmetric and not self-adjoint, but a multiplication with the imaginary unit i assures self-adjointness. We therefore consider operators $T_b = i\,\frac{\partial}{\partial b}U$ and $T_\omega = i\,\frac{\partial}{\partial \omega}U$. This proves (6).

The commutator between these two operators is non-zero. This implies that we cannot exactly measure the mean values of the spatial frequency and the position simultaneously. By means of Theorem 1, we may calculate those states that minimize the corresponding uncertainty principle. Indeed, eq. (3) provides us with the differential equation

$$-i\frac{\partial}{\partial x}\psi(x) - \mu_b\psi(x) = \lambda\left((\frac{i\alpha}{2} - x)\psi(x) + i\alpha x\frac{\partial}{\partial x}\psi(x) - \mu_\omega\psi(x)\right),$$

i.e.,

$$\frac{\partial}{\partial x}\psi(x) = i\psi(x)\left(\frac{-\lambda x + \frac{i\lambda\alpha}{2} - \lambda\mu_\omega + \mu_b}{\alpha\lambda x + 1}\right). \tag{8}$$

Now (8) can be solved by separation of variables which leads us to (7). □

Let $\lambda = i\gamma, \gamma \in \mathbb{R}$. If $\gamma < 0$ and if $\gamma > 0$ respectively, then the solution is contained in L_2 if $\mu_b > -\frac{1}{\alpha}$ and if $\mu_b < -\frac{1}{\alpha}$ respectively. In Figure 1 we have plotted ψ, which is the minimizer for this section of the AWH group in $1D$.

3.2 The Sections where the Frequency is a Function of the Scale

In the previous section we have considered the section

$$\beta(\omega) = \eta_\alpha(\omega)^{-1} = (1 + |\omega|)^{-\alpha}.$$

We note that it is not possible to obtain the affine framework using this approach. Hence, let us explore the inverse relationship

$$\omega = \zeta_\alpha(a) = a^{-\frac{1}{\alpha}} - 1 \ ,$$

which determines the frequency as a function of the scale a.

Let us denote $\kappa = \frac{1}{\alpha}$, and restrict the discussion to values of κ ranging between 0 and 1 (corresponding to values of α ranging between 1 and ∞). Thus we obtain $\omega = \zeta(a) = a^{-\kappa} - 1$. If κ is selected to be zero, we then obtain no frequency modulation as then $\omega = 0$, and thus we are in the affine case. If κ is selected to be one, we again observe reciprocal relations between scale and frequency of the form: $|\omega| = a^{-1} - 1$, which is the same as $a = \frac{1}{1+|\omega|}$, and corresponds to the case of Gabor-like wavelets.

This concept has again an interpretation in the group theoretical setting. Once more we are working with the affine Weyl-Heisenberg group G_{AWH}, but this time we consider the subgroup

$$H := (0, \omega, 1, \phi) \in G_{AWH}$$

and the associated homogeneous space $X = G_{AWH}/H$. In order to make this setting well-defined, first of all it is necessary to establish square-integrability, see, e.g., [1] for a detailed discussion. In general, let a quasi-invariant measure μ on X and a section σ be given. Then a unitary representation U of G on a Hilbert space $\mathcal{H}$ is called *square-integrable modulo* $(H;\sigma)$ if there exists a function $\psi \in \mathcal{H}$ such that the self-adjoint operator $A_\sigma : \mathcal{H} \to \mathcal{H}$ (depending on σ and ψ) weakly defined by

$$A_\sigma f := \int_X \langle f, U(\sigma(h))\psi\rangle_{\mathcal{H}} U(\sigma(x))\psi d\mu(h) \tag{9}$$

is bounded and has a bounded inverse. The function ψ is then called *admissible*. If A_σ is a multiple of the identity then we are in the strictly admissible case. We consider the case of the affine group in the following lemma.

Lemma 3 *Let $\psi \in L_2(\mathbb{R})$. The operator A_σ in (9) for the affine Weyl-Heisenberg group, i.e.*

$$A_\sigma f(x) = \int\int \langle f, \psi_{a,b}\rangle\psi_{a,b}(x)\ db\ \frac{da}{a} \tag{10}$$

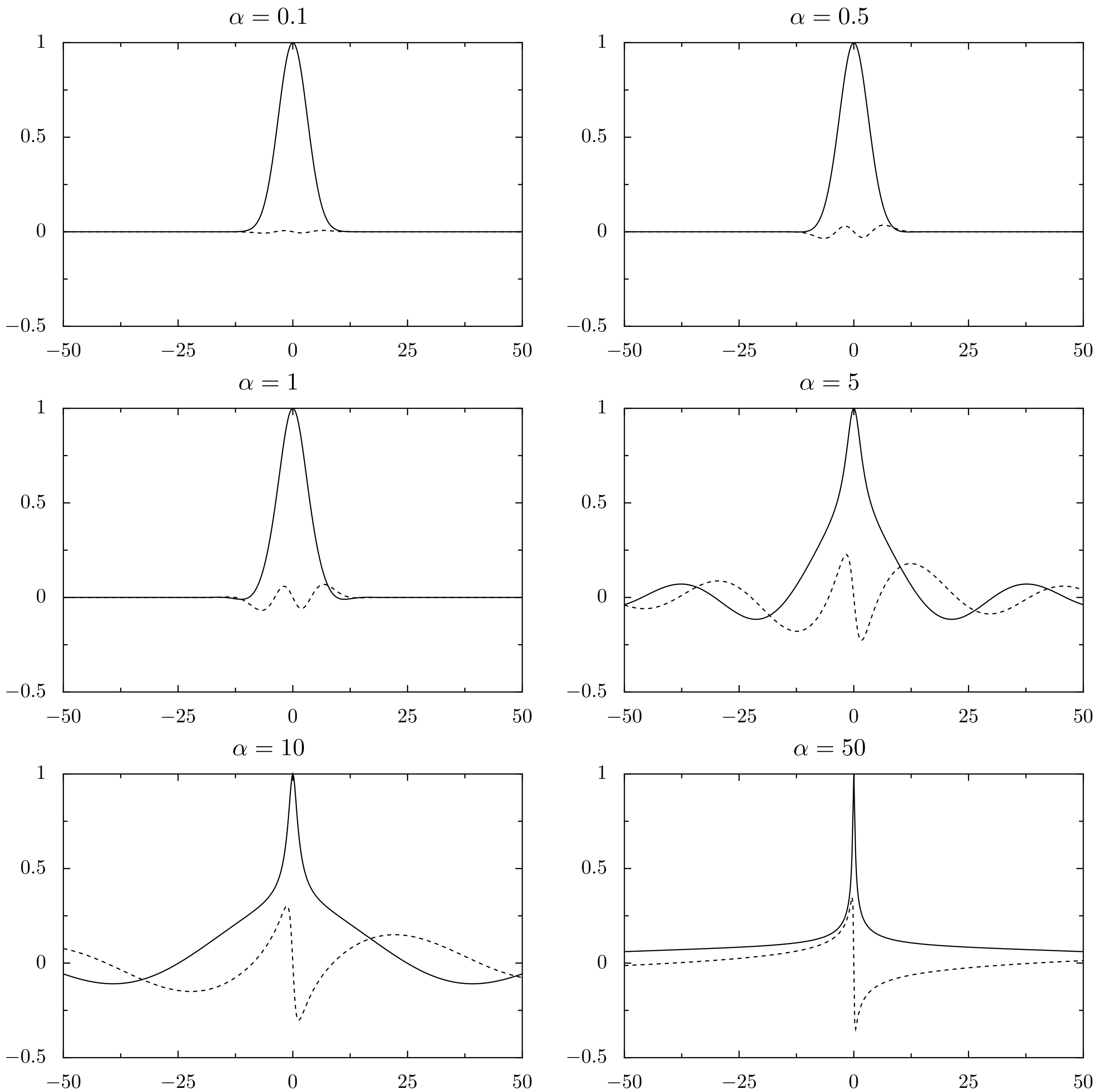

Figure 1: The minimizers of the AWH uncertainty for $\lambda = 0.1i$, $\mu_b = \mu_\omega = 0$ and different values of α. The real part is plotted solid, the imaginary part is dashed. Note the transition from a Gaussion for small α (Weyl-Heisenberg case) to the Cauchy wavelet for large α (affine case).

with the section $\sigma(a,b) = (b, \zeta(a), a, 0)$, *i.e.,*

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} e^{2\pi i \zeta(a)(x-b)} \psi \left(\frac{x-b}{a}\right), \tag{11}$$

can be written as a Fourier multiplier operator:

$$\widehat{A_\sigma f} = m_\zeta \hat{f} \tag{12}$$

with the symbol

$$m_\zeta(\gamma) := \int_{\mathbb{R}_+} |\hat{\psi}(a(\gamma - \zeta(a)))|^2 da. \tag{13}$$

Proof. We follow the approach of [13]. For the sake of this calculation we use the approximation

$$A_\sigma^T f(x) = \int \int \langle f, \psi_{a,b} \rangle \psi_{a,b}(x) \ \chi_{[-\frac{T}{2}, \frac{T}{2}]}(b) \ db \frac{da}{a}. \tag{14}$$

In order to compute $\widehat{A_\sigma^T f(\gamma)}$, we first derive the Fourier transform of $\psi_{a,b}$:

$$\hat{\psi}_{a,b}(\gamma) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{2\pi i \zeta(a)(x-b)} e^{-2\pi i \gamma x} \psi \left(\frac{x-b}{a}\right) dx. \tag{15}$$

If we apply the change of variables $y = \frac{x-b}{a}$ we obtain

$$\hat{\psi}_{a,b}(\gamma) \quad = \quad \sqrt{a} e^{-2\pi i b \gamma} \hat{\psi}(a(\gamma - \zeta(a))). \tag{16}$$

With the help of Plancherel's theorem we further obtain

$$\langle f, \psi_{a,b} \rangle = \langle \hat{f}, \hat{\psi}_{a,b} \rangle = \int \sqrt{a} \hat{f}(\omega) e^{2\pi i b \omega} \bar{\hat{\psi}}(a(\omega - \zeta(a))) d\omega \tag{17}$$

and thus

$$\begin{aligned} \widehat{A_\sigma^T f(\gamma)} &= \int \int \hat{f}(\omega) \bar{\hat{\psi}}(a(\omega - \zeta(a))) \hat{\psi}(a(\gamma - \zeta(a))) \int e^{-2\pi i b(\gamma - \omega)} \chi_{[-\frac{T}{2}, \frac{T}{2}]}(b) \ dbdad\omega \\ &= \int \hat{\psi}(a(\gamma - \zeta(a))) \int \hat{f}(\omega) \bar{\hat{\psi}}(a(\omega - \zeta(a))) T \frac{\sin(\pi(\gamma - \omega)T)}{\pi(\gamma - \omega)T} d\omega da. \end{aligned}$$

The term

$$T \frac{\sin(\pi(\gamma - \omega)T)}{\pi(\gamma - \omega)T} \tag{18}$$

can be seen as an approximation of a δ-function when T approaches infinity. Thus, we obtain by standard arguments

$$\widehat{A_\sigma(f)}(\gamma) = \hat{f}(\gamma) \ \int |\hat{\psi}(a(\gamma - \zeta(a)))|^2 da = \hat{f}(\gamma) m_\zeta(\gamma). \tag{19}$$

□

In order to verify the boundedness of A_σ and its inverse one has to check

$$C_1 \le m_\zeta(\gamma) \le C_2 \tag{20}$$

almost everywhere for constants $0 < C_1, C_2 < \infty$. Unfortunately, for the current particular choice of H and β condition (20) cannot be verified. This can already be seen in the simple situation $\kappa = 1$, i. e. $\zeta(a) = \frac{1}{a} - 1$, then

$$\begin{aligned} m_\zeta(\gamma) &= \int_0^\infty |\hat{\psi}(a(\gamma - \frac{1}{a} + 1))|^2 da \\ &= \int_0^\infty |\hat{\psi}(a(\gamma + 1) - 1))|^2 da \\ &= \frac{1}{|\gamma + 1|} \int_0^\infty |\hat{\psi}(x)|^2 dx \qquad \stackrel{|\gamma| \to \infty}{\longrightarrow} 0. \end{aligned}$$

Nevertheless, a possible remedy for the non-admissibility of this section is given in the weaker framework of quasi-coherent states, see [2]. In this case, we might consider (19) with respect to some positive density $\iota(a,\gamma)$. For the special situation $\kappa = 1$ we might choose $\iota(a,\gamma) = 1/a$, then we obtain

$$\tilde{m}_\zeta(\gamma) \;=\; \int_0^\infty |\hat{\psi}(a(\gamma - \frac{1}{a} + 1))|^2 \iota(a,\gamma)\; da \;=\; \int_0^\infty \frac{|\hat{\psi}(x-1)|^2}{x}\; dx \;. \tag{21}$$

This leads to quasi-coherent states that have the standard properties of the covariant coherent states: overcompleteness, resolution of a positive operator A_σ and having a reproducing kernel.

We now turn to calculating the infinitesimal operators and the coherent states minimizing the corresponding uncertainty relation. The choice of the section $\omega = \zeta(a)$ leads to the representation

$$[U(b,\zeta(a),a,0)\;\psi](x) = \frac{1}{\sqrt{a}} e^{i(a^{-\kappa}-1)(x-b)} \psi\left(\frac{x-b}{a}\right)$$

for which the two infinitesimal generators and the minimizing coherent state are derived in the following lemma.

Lemma 4 *The infinitesimal operators with respect to the representation above are*

$$T_a\psi(x) = (\kappa x - \frac{i}{2})\psi(x) - ix\psi_x(x) \quad \text{and} \quad T_b\psi(x) = -i\psi_x(x).$$

The minimizing states are then given by

$$\psi(x) = (1-\rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{\kappa}{\rho}} e^{-i\kappa x}, \tag{22}$$

where $\rho = ir$ with $r \in \mathbb{R}$. Then, the minimizing states belong to L_2 if $r < 0$ and $\mu_b < -\kappa$ or if $r > 0$ and $\mu_b > -\kappa$.

Proof: The proof can be performed by following the lines of the proof of Lemma 2. This time, the corresponding differential equation is given by

$$\frac{d}{dx}\psi(x) = \frac{\left(\mu_b - \frac{i\rho}{2} + \rho\kappa x - \rho\mu_a\right)}{i(\rho x - 1)}\psi(x), \tag{23}$$

where ρ is purely imaginary. Eq. (23) can again be solved by separation of variables which leads to (22). $\square$

The next lemma verifies the admissibility (19) for the restriction to quasi-coherent states with $\iota(a,\gamma) = 1/a$ and $\kappa = 1$. Therefore, at least for $\kappa = 1$, minimal uncertainty and admissibility fit together quite nicely.

Lemma 5 *Assume $0 < r \in \mathbb{R}$ and $\mu_b + 1 > r/2 > 0$. Then the minimizing elements (22) are admissible quasi-coherent states, i.e. the integral (21) is positive and bounded.*

Proof: For simplifying the notation, we define $\eta := i\mu_a - 1/2 - \mu_b/r - 1/r$ which yields

$$\psi(x) = (-\rho)^\eta (x + i/r)^\eta e^{-ix}.$$

As long as $r > 0$ and $\Re(\eta) < 0$ the Fourier transform of ψ is given by

$$\hat{\psi}(\omega) = \begin{cases} c(\omega+1)^{-\eta-1} e^{-1(\omega+1)/r} & \text{for } \omega \geq -1 \\ 0 & \text{elsewhere} \end{cases} \tag{24}$$

with $c = (-\rho)^\eta \sqrt{2\pi}(-i)^\eta \Gamma(-\eta)^i$. Thus the admissibility condition becomes with $\eta' = \Re\eta$

$$|c|^2 \int_0^\infty \frac{\omega^{-2(\eta'+1)} e^{-2\omega/r}}{\omega}\; d\omega = |c|^2 (r/2)^{-2(\eta'+2)} \Gamma(-2(\eta'+1))$$

provided that $\mu_b + 1 > r/2$. $\square$

Note that the case $\kappa = 0$ reproduces the classical pure wavelet conditions. For $\kappa = 1$ the current solution reduces to

$$\psi(x) = (1 - \rho x)^{i\mu_a - \frac{1}{2} - i\frac{\mu_b}{\rho} - i\frac{1}{\rho}} e^{-ix}. \tag{25}$$

It is interesting to compare this solution to the one obtained for the previously used section $a = \beta(\omega)$ for $\alpha = 1$, which yields:

$$\psi(x) = (1 + \lambda x)^{i\frac{\mu_b}{\lambda} - \frac{1}{2} - i\mu_\omega + \frac{i}{\lambda}} e^{-ix} \quad . \tag{26}$$

The constraints for the two solutions to agree for $\alpha = 1$ are derived as follows

$$\lambda = -\rho, \quad \mu_a = -\mu_\omega \quad .$$

In Figure 2 we have plotted the minimizing ψ, for this choice of a section for the AWH group in $1D$.

4 The $2D$ Affine Weyl-Heisenberg Group

In this section we are interested in finding uncertainty minimizers for the two-dimensional affine Weyl-Heisenberg group with a generating element (A, ω, b, ϕ), $\omega, b \in \mathbb{R}^2, A \in Gl(2, \mathbb{R}), \phi \in \mathbb{R}$ and group law

$$(b, \omega, A, \phi) \circ (b', \omega', A', \phi') = (b + Ab', \omega + A^{-1}\omega', AA', \phi + \phi' + \omega^T Ab').$$

The Stone-von-Neumann representation of the AWH group in two dimensions is given by:

$$[U(b, \omega, A, \phi)\psi](x, y) = \frac{1}{|\det(A)|} e^{i(\omega_x(x-b_x)+\omega_y(y-b_y)+\phi)} \psi\left(A(x - b_x, y - b_y)\right) \tag{27}$$

with the unimodular Haar measure

$$\frac{1}{|\det(A)|^2} db d\omega dm(A) d\phi,$$

where $dm(A)$ denotes a usual measure when parametrizing the matrix A. In our discussion we explore various subgroups of the full $2D$ AWH group, starting from the $SIM(2)$ group, and moving to the full group structure.

The admissibility of the derived minimizing coherent states is in the cases discussed below either difficult to check or not confirmable. For instance in Section 4.2 we are able to give explicit and sufficient conditions (see Theorem 7 stating that a function which has a compactly supported Fourier transform is admissible) for admissibility but which can, however, not be fulfilled by the derived minimizing coherent states. Nevertheless, the Fourier transform of the minimizing states have its support (up to shifting etc.) in the positive Euclidian half space in which they decay exponentially. Hence, chosing the support size large enough and applying a smooth cut off operator provides us with admissible and 'near' minimizing coherent states.

4.1 The $2D$ Similitude Weyl-Heisenberg Subgroup

We start our discussion by considering the similitude group, which only allows rotations and scalings $A = A(a, \theta)$. The general representation of the $2D$ similitude Weyl-Heisenberg subgroup is given by

$$[U(b, \omega, (a, \theta), 0)\psi](x, y) = \frac{1}{a} e^{i(\omega_x(x-b_x)+\omega_y(y-b_y))} \psi\left(\tau_\theta\left(\frac{x - b_x}{a}, \frac{y - b_y}{a}\right)\right), \tag{28}$$

where $\tau_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. This representation fails to be square integrable [22], therefore, we are faced with the interesting question of selecting an appropriate section. To this end, let $H = (0, 0, (a, 0), \phi)$, consider $G_{AWH} \setminus H$ and take the section

$$\sigma(b, \omega, \theta) = (b, \omega, (\Phi(\omega), \theta), 0).$$

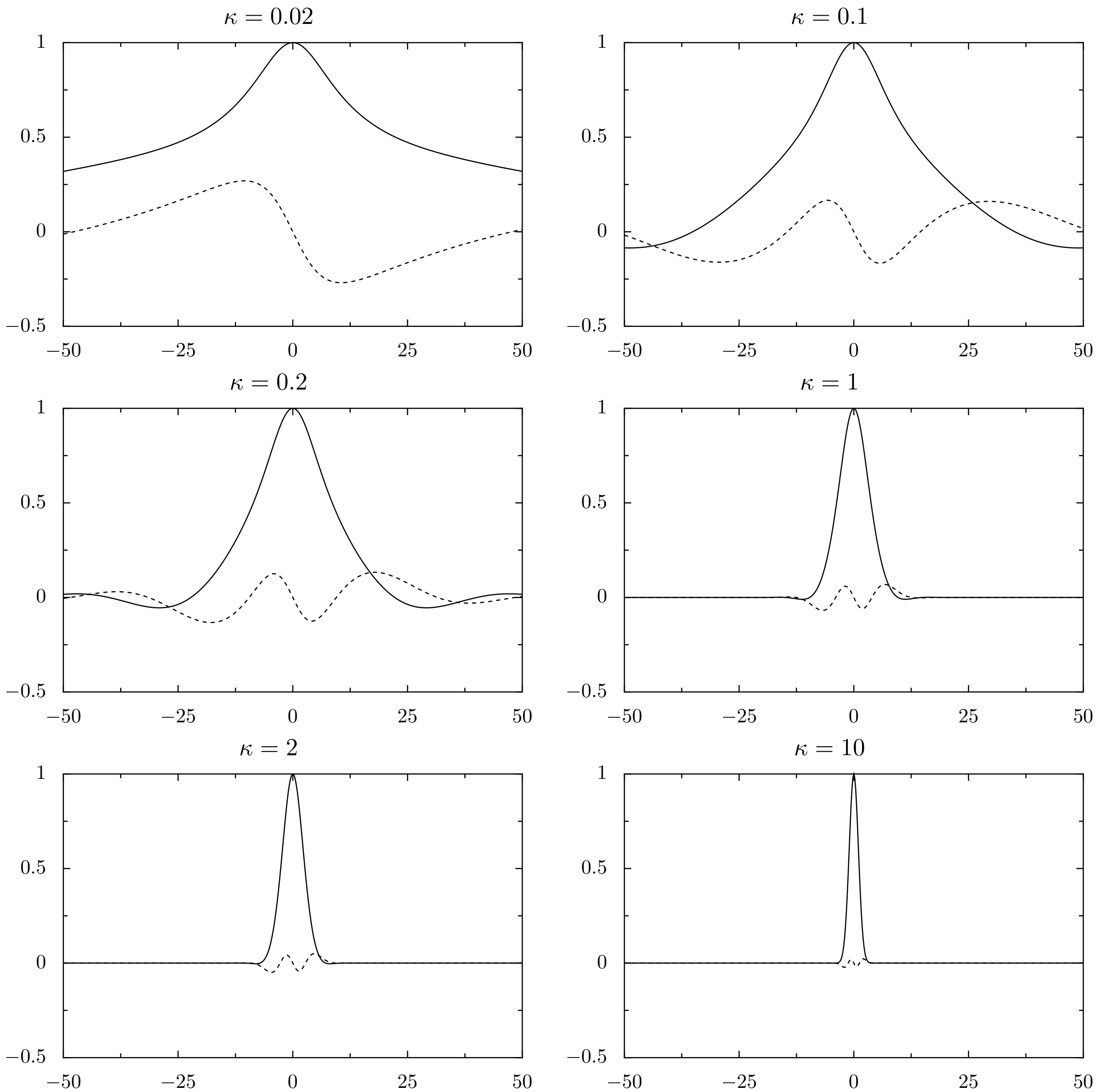

Figure 2: The minimizers of the AWH uncertainty for $\rho = 0.1i$, $\mu_a = \mu_b = 0$ and different values of κ. The real part is plotted solid, the imaginary part is dashed. Note the transition from a Cauchy wavelet for small κ (affine case) to a Gaussian for large κ (Weyl-Heisenberg case).

We consider a coupling between the frequency ω and the scaling a by $a = \Phi(\omega)$. More specifically, in the spirit of the α-modulation spaces framework, we assume that the function Φ depends only on the p-norm of the frequency vector ω

$$\Phi(\omega) = \frac{1}{(1 + \|\omega\|_p)^\alpha}. \tag{29}$$

As before, we would like to obtain the infinitesimal generators of this group by calculating the appropriate derivatives of the representation of this group at the identity element. Depending on the choice of the section $\Phi(\omega)$, we will obtain different infinitesimal operators from the partial derivatives with respect to ω_k:

$$\tilde{T}_{\omega_k} = (T_{\omega_k} + \Phi'_{\omega_k}(0)T_a), \tag{30}$$

for $k \in \{x, y\}$. Therefore, we have to estimate the derivative of Φ at $\omega = 0$. Our particular choice of Φ yields

$$\begin{aligned}\Phi'_{\omega_k}(\omega) &= -\alpha(1 + \|\omega\|_p)^{-\alpha-1}\frac{1}{p}\|\omega\|_p^{1-p} p\omega_k^{p-1}\text{sign}(\omega_k) \\ &= \frac{-\alpha}{(1 + \|\omega\|_p)^{\alpha+1}}\left[\frac{\omega_k^p}{\sum_j |\omega_j|^p}\right]^{\frac{p-1}{p}} \text{sign}(\omega_k).\end{aligned} \tag{31}$$

Next, we have to evaluate this expression at $\omega_k = 0$ for all k. In contrast to the $1D$ situation, the resulting infinitesimal operators and the corresponding commutation relations strongly depend on the choice of p. We start by selecting the L_1-norm, as this allows a straight forward calculation of infinitesimal generators.

4.1.1 AWH Minimizers Using the L_1-Norm

In this case: $a = \Phi(\omega) = \frac{1}{(1+|\omega_x|+|\omega_y|)^\alpha}$, thus the representation becomes:

$$[U(b, \omega, \Phi(\omega), \theta, 0)\psi](x, y) = (1+|\omega_x|+|\omega_y|)^\alpha e^{i((x-b_x)\omega_x+(y-b_y)\omega_y)}\psi\left((1 + |\omega_x| + |\omega_y|)^\alpha \tau_\theta(x - b_x, y - b_y)\right). \tag{32}$$

As before, we would like to obtain the infinitesimal generators of this group by calculating the appropriate derivatives of the representation of this group at the identity element. Depending on the choice of the section Φ, we obtain the infinitesimal genrators with respect to ω_k from the directional derivative of Φ at 0 in direction $k \in x, y$:

$$\tilde{\tilde{T}}_{\omega_k} = (\tilde{T}_{\omega_k} + \Phi_{\omega_k}(0)\tilde{T}_a). \tag{33}$$

Our particular choice of Φ yields

$$\Phi_{\omega_k}(0) = -\alpha. \tag{34}$$

Then the self-adjoint infinitesimal operators are given by:

$$\begin{aligned}T_{\omega_x}\psi(x, y) &= (i\alpha - x)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)), \\ T_{\omega_y}\psi(x, y) &= (i\alpha - y)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)), \\ T_{b_x}\psi(x, y) &= -i\psi_x(x, y), \\ T_{b_y}\psi(x, y) &= -i\psi_y(x, y), \\ T_\theta\psi(x, y) &= i(y\psi_x(x, y) - x\psi_y(x, y)).\end{aligned} \tag{35}$$

Out of the ten commutation relations, three vanish, $[T_{\omega_x}, T_{b_y}] = 0$, $[T_{\omega_y}, T_{b_x}] = 0$, $[T_{b_x}, T_{b_y}] = 0$, and we are left with seven partial differential equations.

1.

$$(i\alpha-x)\psi(x, y)+i\alpha(x\psi_x(x, y)+y\psi_y(x, y))-\mu_{\omega_x}\psi(x, y) = \lambda_1\left((i\alpha - y)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)) - \mu_{\omega_y}\psi(x, y)\right)$$

2.

$$(i\alpha - x)\psi(x, y) + i\alpha(x\psi_x(x, y) + y\psi_y(x, y)) - \mu_{\omega_x}\psi(x, y) = \lambda_2\left(-i\psi_x(x, y) - \mu_{b_x}\psi(x, y)\right)$$

3.

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = \lambda_3\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y)\right)$$

4.

$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi(x,y) = \lambda_4\left(-i\psi_y(x,y) - \mu_{b_y}\psi(x,y)\right)$$

5.

$$(i\alpha - y)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_y}\psi(x,y) = \lambda_5\left(i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y)\right)$$

6.

$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y) = \lambda_6\left(-i\psi_x(x,y) - \mu_{b_x}\psi(x,y)\right)$$

7.

$$i(y\psi_x(x,y) - x\psi_y(x,y)) - \mu_\theta\psi(x,y) = \lambda_7\left(-i\psi_y(x,y) - \mu_{b_y}\psi(x,y)\right)$$

The only simultaneous solution to these equations is the trivial one, $\psi = 0$ everywhere. Therefore, we aim at finding a partial solution to this set of equations, which involves operators from the enveloping algebra. First of all, we impose rotational invariance.

Suppose that the minimizer is of the form $g(r)$ where $r = \sqrt{x^2 + y^2}$. Then, we consider the following infinitesimal operators with respect to $g(r)$:

$$\begin{aligned} T_\theta g(r) &= 0, \\ T_b g(r) &= (T_{b_x}^2 + T_{b_y}^2)g(r) = -\frac{d^2 g}{dr^2}(r) - \frac{1}{r}\frac{dg}{dr}(r). \end{aligned}$$

Moreover, the operators $T_{\omega_x}, T_{\omega_y}$ are commuting with respect to $g(r)$,i.e., $[T_{\omega_x}, T_{\omega_y}]g(r) = 0$. These observations lead to two possible solutions: the first involves defining a new operator: $T_\omega = T_{\omega_x}T_{\omega_y} - T_{\omega_y}T_{\omega_x}$ and considering its commutator relations with T_θ and T_b. Then, any function $g(r)$ that is rotation invariant is a valid minimizer of the uncertainties related to these operators.

Another option is to consider T_{ω_x} and T_{ω_y} with respect to $g(r)$. The commutators of these operators with T_b are not equal to zero and we obtain the differential equation

$$g_{rr}(r) + \frac{1}{r}g_r(r) + \mu_b g(r) = 0 \tag{36}$$

whose solution is given by Bessel functions of the first and second kind:

$$\psi(r) = c_1 J_0(\sqrt{\mu_b}r) + c_2 Y_0(\sqrt{\mu_b}r). \tag{37}$$

Nevertheless, this solution does not belong to L_2.

Another interesting effort is to find a solution for a single differential equation and thus obtain a selective minimal uncertainty with respect to two operators only. For example, let us consider equation 2 only:

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = \lambda_2\left(-i\psi_x(x,y) - \mu_{b_x}\psi(x,y)\right) \quad .$$

Choosing $\lambda_2 = 0$ yields

$$(i\alpha - x)\psi(x,y) + i\alpha(x\psi_x(x,y) + y\psi_y(x,y)) - \mu_{\omega_x}\psi(x,y) = 0.$$

A possible solution is given by the expression:

$$\psi(x,y) = y^{-i\frac{\mu_{\omega_x}}{\alpha} - 1} e^{-i\frac{x}{\alpha}} \tau\left(\frac{x}{y}\right), \tag{38}$$

where τ is an arbitrary function of the variable $\frac{x}{y}$. This function, however, is not even a member of L_2. Therefore we are faced here with an example where minimal uncertainty and admissibility do not really fit together. Nevertheless, let us again emphasize that the minimizing states are always of interest by themselves. A particular solution is depicted in Figure 3.

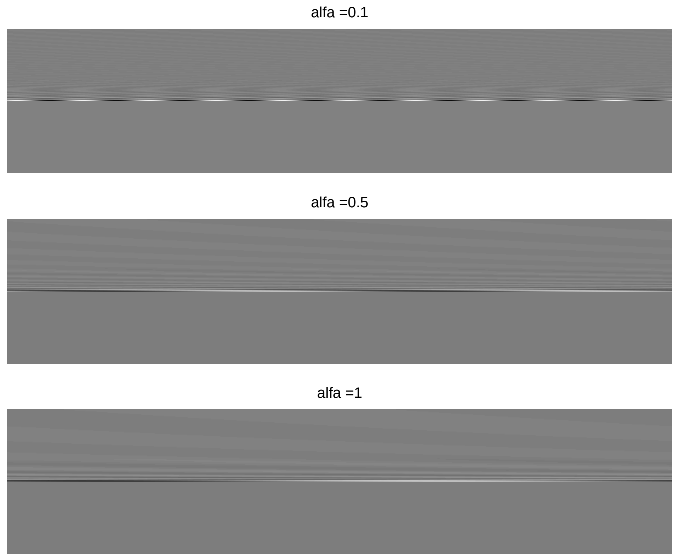

Figure 3: A possible minimizing function for $\tau\left(\frac{x}{y}\right) = 1$.

4.1.2 AWH Minimizers Using the L_2-Norm

As we have seen in the previous section, the choice $a = \Phi(\omega) = \left(1 + \sqrt{\omega_x^2 + \omega_y^2}\right)^{-\alpha}$ proves to be futile. It is interesting to explore another solution to the problem mentioned in Section 4.1, where the relationship between the scale and frequency is given by

$$a = \Phi(\omega) = \left(1 + \omega_x^2 + \omega_y^2\right)^{-\alpha} \quad .$$

The unitary representation induced by this section of the similitude Weyl-Heisenberg group is then given by

$$[U(b, \omega, (\Phi(\omega), \theta), 0)\psi](x, y) = (1 + \omega_x^2 + \omega_y^2)^{\alpha} e^{i((x-b_x)\omega_x + (y-b_y)\omega_y)} \psi\left((1 + \omega_x^2 + \omega_y^2)^{\alpha} \tau_\theta(x - b_x, y - b_y)\right), \quad (39)$$

and τ_θ is the same as already defined. The infinitesimal generators are then given by:

$$\begin{aligned} T_{\omega_x}\psi(x, y) &= -x\psi(x, y), \\ T_{\omega_y}\psi(x, y) &= -y\psi(x, y), \\ T_{b_x}\psi(x, y) &= -i\psi_x(x, y), \\ T_{b_y}\psi(x, y) &= -i\psi_y(x, y), \\ T_\theta\psi(x, y) &= i(y\psi_x(x, y) - x\psi_y(x, y)). \end{aligned} \quad (40)$$

It is interesting to note that the dependency on the parameter α has disappeared. This means that selecting this type of section may provide a solution regardless of the smoothness space we are dealing with.

The differential equations resulting from the non-commuting operators are:

$$\begin{aligned} -x\psi(x, y) - \mu_{\omega_x}\psi(x, y) &= \lambda_1(-i\psi_x(x, y) - \mu_{b_x}\psi(x, y)), \\ -x\psi(x, y) - \mu_{\omega_x}\psi(x, y) &= \lambda_2(i(y\psi_x(x, y) - x\psi_y(x, y)) - \mu_\theta\psi(x, y)), \\ -y\psi(x, y) - \mu_{\omega_y}\psi(x, y) &= \lambda_3(-i\psi_y(x, y) - \mu_{b_y}\psi(x, y)), \\ -y\psi(x, y) - \mu_{\omega_y}\psi(x, y) &= \lambda_4(i(y\psi_x(x, y) - x\psi_y(x, y)) - \mu_\theta\psi(x, y)), \\ -i\psi_x(x, y) - \mu_{b_x}\psi(x, y) &= \lambda_5(i(y\psi_x(x, y) - x\psi_y(x, y)) - \mu_\theta\psi(x, y)), \\ -i\psi_y(x, y) - \mu_{b_y}\psi(x, y) &= \lambda_6(i(y\psi_x(x, y) - x\psi_y(x, y)) - \mu_\theta\psi(x, y)). \end{aligned} \quad (41)$$

If, again, we search for a solution which is rotational invariant, i.e. $\psi(x, y) = g(r)$, we may satisfy all equations that involve the operator T_θ. Moreover, applying restrictions to our parameters, e.g. $\lambda_1 = \lambda_3 =$

$\lambda, \mu_{b_x} = \mu_{b_y}, \omega_x = \omega_y$, we may obtain a rotationally invariant solution to the first and third equation as well, which is a Gaussian function

$$\psi(x,y) = e^{-\frac{i}{\lambda}\left(\frac{x^2+y^2}{2}\right)}, \quad \text{with} \quad \lambda \in i\mathbb{R}. \tag{42}$$

4.2 AWH Minimizers with Anisotropic Scaling

In the previous treatment of the two-dimensional case, we regard the frequency as a vector, but treat the scale as a scalar argument. Moreover, we use the $SIM(2)$ group rather than the full affine group. We are interested to add more degrees of freedom to our setting, and as a first step we observe a relationship where the scale is also a two dimensional vector and not a scalar

$$a_x = \beta_1(\omega_x), \qquad a_y = \beta_2(\omega_y).$$

We explore a generalization to two dimensions of the one-dimensional affine Weyl-Heisenberg group, and ignore for now rotation and shear. We thus consider the group with a generic element $g = (b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi,)$ where $b_x, b_y, \omega_x, \omega_y, \phi \in \mathbb{R}$ and $a_x, a_y \in \mathbb{R}_+$ equipped with a group law

$$(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi) \circ (b'_x, b'_y, \omega'_x, \omega'_y, a'_x, a'_y, \phi') =$$

$$(b_x + a_x b'_x, b_y + a_y b'_y, \omega_x + a_x^{-1}\omega'_x, \omega_y + a_y^{-1}\omega'_y, a_x a'_x, a_y a'_y, \phi + \phi' + \omega_x a_x b'_x + \omega_y a_y b'_y).$$

This is a subgroup of the $2D$ AWH group. The inverse element of $g \in G$ is given by

$$g^{-1} = (-a_x^{-1}b_x, -a_y^{-1}b_y, -a_x\omega_x, -a_y\omega_y, a_x^{-1}, a_y^{-1}, -\phi + b_x\omega_x + b_y\omega_y). \tag{43}$$

Let us look at the following representation

$$[U(b_x, b_y, \omega_x, \omega_y, a_x, a_y, \phi)\psi](x,y) = \frac{1}{\sqrt{a_x a_y}} e^{i((x-b_x)\omega_x + (y-b_y)\omega_y)+\phi} \psi\left(\frac{x-b_x}{a_x}, \frac{y-b_y}{a_y}\right) \tag{44}$$

which is the $2D$ extension of the Stone-von-Neumann representation of the $1D$ AWH group. This representation fails to be square integrable, and therefore we restrict ourselves to the homogeneous space $G_{AWH}\backslash H$ with

$$H := (0,0,0,0,a_x,a_y,\phi) \in G_{AWH}. \tag{45}$$

Next, we consider the section $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0)$, and would like to prove that this section is admissible.

We define a self-adjoint operator $A_\sigma f$

$$\begin{aligned} (A_\sigma f)(x,y) &:= \int_X \langle f, U(\sigma(h))\psi\rangle U(\sigma(h))\psi d\mu(h) \\ &= \int\int\int\int \langle f, \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y}\rangle \psi_{\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),b_x,b_y}(x,y) db_x db_y d\omega_x d\omega_y. \end{aligned} \tag{46}$$

It can be written as a Fourier multiplier operator

$$\widehat{(A_\sigma f)} = m_{\beta_x,\beta_y}\hat{f} \tag{47}$$

where

$$m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int\int |\hat{\psi}(\beta_x(\omega_x)(\gamma_x - \omega_x), \beta_y(\omega_y)(\gamma_y - \omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y. \tag{48}$$

Next, we follow the lines of [6] to show that m_{β_x,β_y} is bounded from above and below, i.e.

$$C_1 \le m_{\beta_x,\beta_y} \le C_2 \tag{49}$$

for constants $0 < C_1 < C_2 < \infty$. We start with the following lemma which is a straight forward generalization of Lemma 5.1 in [6] to the $2D$-case. Therefore we omit the details.

Lemma 6 *Consider the specific section σ given by the functions*

$$\beta_x(\omega_x) = \beta_{x,\alpha_x}(\omega_x) = (1+|\omega_x|)^{-\alpha_x},$$
$$\beta_y(\omega_x) = \beta_{y,\alpha_y}(\omega_y) = (1+|\omega_y|)^{-\alpha_y}.$$

Let us define

$$r_{\gamma_x}(\omega_x) := \beta_x(\omega_x)(\gamma_x - \omega_x) = (1+|\omega_x|)^{-\alpha_x}(\gamma_x-\omega_x),$$
$$r_{\gamma_y}(\omega_y) := \beta_y(\omega_y)(\gamma_y - \omega_y) = (1+|\omega_y|)^{-\alpha_y}(\gamma_y-\omega_y).$$

Then, for any fixed $A > 0$, there exist $\gamma_{x,A}, \gamma_{y,A} > 0$ such that for all $\gamma_x \geq \gamma_{x,A}, \gamma_y \geq \gamma_{y,A}$ the functions $r_{\gamma_x}, r_{\gamma_y}$ are invertible on

$$A_{\omega_x} = \{\omega_x : r_{\gamma_x}(\omega_x) \in [-A, A]\} \quad \textit{and} \quad A_{\omega_y} = \{\omega_y : r_{\gamma_y}(\omega_y) \in [-A, A]\}$$

respectively. The inverse functions $r^{-1}_{\gamma_x}, r^{-1}_{\gamma_y}$ of $r_{\gamma_x}, r_{\gamma_y}$ on $[-A, A]$ have the form

$$r^{-1}_{\gamma_x} = -xg_1(\gamma_x, x) + \gamma_x, \qquad r^{-1}_{\gamma_y} = -yg_2(\gamma_y, y) + \gamma_y,$$

with some functions $g_1(\gamma_x, x), g_2(\gamma_y, y)$ satisfying

$$xg_1(\gamma_x, x) + g_1(\gamma_x, x)^{\frac{1}{\alpha_x}} = 1 + \gamma_x \quad \textit{and} \quad yg_2(\gamma_y, y) + g_2(\gamma_y, y)^{\frac{1}{\alpha_y}} = 1 + \gamma_y.$$

Furthermore, g_1, g_2 fulfill

$$\lim_{\gamma_x\to\infty} \gamma_x^{-\alpha_x} g_1(\gamma_x, x) = 1, \qquad \lim_{\gamma_y\to\infty} \gamma_y^{-\alpha_y} g_2(\gamma_y, y) = 1$$

uniformly for $x, y \in [-A, A]$.

Theorem 7 *Let the Borel section σ be given by $\sigma(b_x, b_y, \omega_x, \omega_y) = (b_x, b_y, \omega_x, \omega_y, \beta_x(\omega_x), \beta_y(\omega_y), 0)$ with $\beta_x(\omega_x) = (1 + |\omega_x|)^{-\alpha_x}, \beta_y(\omega_y) = (1 + |\omega_y|)^{-\alpha_y}$. Let ψ be a non zero L_2 function whose Fourier transform is compactly supported. Then, ψ is admissible, i.e., the condition*

$$C_1 \leq m_{\beta_x,\beta_y}(\gamma_x, \gamma_y) \leq C_2$$

is satisfied for $0 < C_1 \leq C_2 < \infty$.

Proof. The proof can be performed by following the lines of the proof of Theorem 5.2 in [6]. For reader's convenience, we briefly sketch the arguments. We consider the case where either γ_x or γ_y tend to $+\infty$. Let us assume that $\text{supp}(\hat{\psi}) \subset [-A, A] \times [-A, A]$. We substitute $x = r_{\gamma_x}(\omega_x), y = r_{\gamma_y}(\omega_y)$ for $\gamma_x \geq \gamma_{x,A} > 0, \gamma_y \geq \gamma_{y,A} > 0$ in the expression for $m_{\beta_x,\beta_y}(\gamma_x, \gamma_y)$ to obtain

$$\begin{aligned} m_{\beta_x,\beta_y}(\gamma_x, \gamma_y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{\psi}(r_{\gamma_x}(\omega_x), r_{\gamma_y}(\omega_y))|^2 \beta_x(\omega_x)\beta_y(\omega_y) d\omega_x d\omega_y \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\hat{\psi}(x, y)|^2 \beta_x(r^{-1}_{\gamma_x}(x))\beta_y(r^{-1}_{\gamma_y}(y))(r^{-1}_{\gamma_x})'(r^{-1}_{\gamma_y})' dxdy. \end{aligned} \tag{50}$$

Next, we calculate the values of the derivatives of the inverse functions $r^{-1}_{\gamma_x}, r^{-1}_{\gamma_y}$ using

$$r'_{\gamma_x}(\omega_x) = \beta'_x(\omega_x)(\gamma_x - \omega_x) - \beta_x(\omega_x) = -\beta_x(\omega_x)\left(\alpha_x \frac{\gamma_x - \omega_x}{1 + \omega_x} + 1\right),$$
$$r'_{\gamma_y}(\omega_y) = \beta'_y(\omega_y)(\gamma_y - \omega_y) - \beta_y(\omega_y) = -\beta_y(\omega_y)\left(\alpha_y \frac{\gamma_y - \omega_y}{1 + \omega_y} + 1\right)$$

to obtain

$$(r^{-1}_{\gamma_x})'(x) = \frac{1}{r'_{\gamma_x}(r^{-1}_{\gamma_x}(x))} = -\frac{1}{\beta_x(r^{-1}_{\gamma_x}(x)\left(1 + \alpha_x \frac{\gamma_x - r^{-1}_{\gamma_x}(x)}{1 + r^{-1}_{\gamma_x}(x)}\right)},$$
$$(r^{-1}_{\gamma_y})'(y) = \frac{1}{r'_{\gamma_y}(r^{-1}_{\gamma_y}(y))} = -\frac{1}{\beta_y(r^{-1}_{\gamma_y}(y)\left(1 + \alpha_y \frac{\gamma_y - r^{-1}_{\gamma_y}(y)}{1 + r^{-1}_{\gamma_y}(y)}\right)}.$$

Thus, for values of $\gamma_x \geq \gamma_{x,A} > 0, \gamma_y \geq \gamma_{y,A} > 0$ we have

$$m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int_{-A}^{A}\int_{-A}^{A} |\hat{\psi}(x,y)|^2 G(\gamma_x,\gamma_y,x,y)dxdy \tag{51}$$

where

$$G(\gamma_x,\gamma_y,x,y) = \frac{1}{\left(1+\alpha_x \frac{\gamma_x - r_{\gamma_x}^{-1}(x)}{1+r_{\gamma_x}^{-1}(x)}\right)} \frac{1}{\left(1+\alpha_y \frac{\gamma_y - r_{\gamma_y}^{-1}(y)}{1+r_{\gamma_y}^{-1}(y)}\right)} = \frac{1}{1+\alpha_x x g_1(\gamma_x,x)^{1-\frac{1}{\alpha_x}}} \frac{1}{1+\alpha_y y g_2(\gamma_y,y)^{1-\frac{1}{\alpha_y}}},$$

where we have used the definitions in the previous lemma. According to this lemma, we may substitute $\gamma_x^{\alpha_x}$ for $g_1(\gamma_x,x)$ when γ_x goes to infinity, and the same for $g_2(\gamma_y,y)$ when $\gamma_y \to \infty$

$$\begin{aligned}
\lim_{\gamma_x\to\infty,\gamma_y\to\infty} G(\gamma_x,\gamma_y,x,y) &= \lim_{\gamma_x\to\infty,\gamma_y\to\infty} \frac{1}{1+\alpha_x x g_1(\gamma_x,x)^{1-\frac{1}{\alpha_x}}} \frac{1}{1+\alpha_y y g_2(\gamma_y,y)^{1-\frac{1}{\alpha_y}}} \\
&= \lim_{\gamma_x\to\infty,\gamma_y\to\infty} \frac{1}{1+\alpha_x x \gamma_x^{\alpha_x(1-\frac{1}{\alpha_x})}} \frac{1}{1+\alpha_y y \gamma_y^{\alpha_y(1-\frac{1}{\alpha_y})}} \\
&= 1,
\end{aligned}$$

and therefore we finally have

$$\lim_{\gamma_x\to\infty,\gamma_y\to\infty} m_{\beta_x,\beta_y}(\gamma_x,\gamma_y) = \int_{-A}^{A}\int_{-A}^{A} |\hat{\psi}(x,y)|^2 dxdy \tag{52}$$

for any L_2-function with compact support in the Fourier domain, and thus we obtain that m_{β_x,β_y} is bounded from below and above. $\square$

Now, that this section is proven to be admissible, we would like to explore the uncertainty principle minimizers associated with this representation. We assume that it should be a two-dimensional extension of the one-dimensional solution obtained earlier. The representation for the quotient as a function of α_x, α_y is then given by:

$$\begin{gathered}
[U(b_x,b_y,\omega_x,\omega_y,\beta_x(\omega_x),\beta_y(\omega_y),0)\psi](x,y) = \\
(1+|\omega_x|)^{\frac{\alpha_x}{2}}(1+|\omega_y|)^{\frac{\alpha_y}{2}} e^{i(\omega_x(x-b_x)+\omega_y(y-b_y))}\psi\left((1+|\omega_x|)^{\alpha_x}(x-b_x),(1+|\omega_y|)^{\alpha_y}(y-b_y)\right).
\end{gathered}$$

From this representation we may see that the x and y axes are not correlated, and thus we obtain the following infinitesimal generators

$$\begin{aligned}
(T_{b_x}\psi)(x,y) &= -\frac{\partial}{\partial x}\psi(x,y), \\
(T_{b_y}\psi)(x,y) &= -\frac{\partial}{\partial y}\psi(x,y), \\
(T_{\omega_x}\psi)(x,y) &= (\frac{\alpha_x}{2}+i)\psi(x,y) + \alpha_x x\frac{\partial}{\partial x}\psi(x,y), \\
(T_{\omega_y}\psi)(x,y) &= (\frac{\alpha_y}{2}+i)\psi(x,y) + \alpha_y y\frac{\partial}{\partial y}\psi(x,y).
\end{aligned} \tag{53}$$

In order to make these operators self-adjoint, we multiply them by i. The commutators between the x and y operators vanish, and we have to solve two independent one-dimensional problems, with the following solutions

$$\psi(x,y) = (\alpha_x\lambda_x x+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_x}}{\alpha_x}+\frac{i\mu_{b_x}}{\alpha_x\lambda_x}+\frac{i}{\alpha_x^2\lambda_x}} e^{\frac{-ix}{\alpha_x}} (\alpha_y\lambda_y y+1)^{-\frac{1}{2}-\frac{i\mu_{\omega_y}}{\alpha_y}+\frac{i\mu_{b_y}}{\alpha_y\lambda_y}+\frac{i}{\alpha_y^2\lambda_y}} e^{\frac{-iy}{\alpha_y}}. \tag{54}$$

In order for this solution to be a member of L_2, the following should be met: we denote $\lambda_x = i\varpi_x, \lambda_y = i\varpi_y$ where $\varpi_x, \varpi_y \in \mathbb{R}$. Then, if $\varpi_x, \varpi_y < 0$, then $\mu_{b_x} > -\frac{1}{\alpha_x}, \mu_{b_y} > -\frac{1}{\alpha_y}$. If $\varpi_x, \varpi_y > 0$, then $\mu_{b_x} < -\frac{1}{\alpha_x}, \mu_{b_y} < -\frac{1}{\alpha_y}$. Unfortunately, the Fourier transform of this minimizing state is not compactly supported. However, it has at least exponential decay. Therefore, by choosing a sufficiently large interval and using a smooth cut off function we obtain admissible and 'almost' minimizing states.

Acknowledgements: The authors want to thank Y. Y. Zeevi and N. A. Sochen for many fruitful discussions.

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BOUNDARY VALUE PROBLEMS IN PLAN SECTOR WITH CORNERS FOR A CLASS OF SOBOLEV SPACES OF DOUBLE WEIGHT

H. BENSERIDI & M. DILMI

ABSTRACT. In this note, we study some boundary value problems of elasticity in plan domain with corners for a class of *Sobolev* spaces of double weight. We give a generalization of some results of existence and unicity of the solution obteined by *P. Grisvard* [6] in classical spaces of nul weight, by *D. Teniou* [8] in *Sobolev* spaces of simple weight and by *M. Dauge* [4] for the system of *Stokes*.

AMS Classification : *Mathematics Subject Classification,35B65.*
Keywords : *Elasticity, Lame, Regularity, Singularity, Sobolev.*

1. Notations

Let Ω be a sector in $\mathbb{R}^2$ of sides Γ_0, Γ_ω, angle ω and the vertexe S. Let L be the *Lamé* operator :

$$L = \mu\Delta + (\lambda + \mu)\,\nabla div,$$

λ and μ are *Lamé* coefficients ($\lambda \geq 0, \mu > 0$) (u_i), $(-f_i)$, $i = 1, 2$ designate respectively the components of the displacement vector and the density of external powers. $\sum = (\sigma_{ij}), i = 1, 2 \quad , j = 1, 2$ designate the stress tensor. The stress tensor and the displacement vector are related via *Hooke's* law :

$\sigma_{ij} = 2\mu\varepsilon_{ij}(u) + \lambda\, div(u)\delta_{ij}$, where $\varepsilon_{ij}(u) = \dfrac{1}{2}(\partial_j u_i + \partial_i u_j)$, $\varepsilon_{ij}(u)$ is called tensor of linear deformation associated to u.

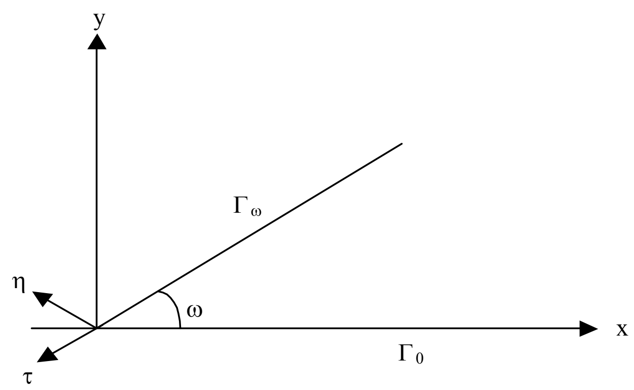

FIGURE 1

Lamé operator being invariant by translation and rotation; we may then suppose, without lass of generality, that S is the origine O and that the first side of Ω is on the positive x axis. We define B by : $B=\mathbb{R}\times]0\,,\,\omega[$. Let θ_0,θ_∞ be two reals such that: $\theta_0\leq\theta_\infty$, we put $\theta_0=\eta_0-1$, $\theta_\infty=\eta_\infty-1$.

Definition 1.1. *For $m\in\mathbb{N}$, we define the spaces :*
$H^m_{\theta_0,\theta_\infty}(\Omega)=\left\{u\in L^2_{loc}(\Omega):\ r^{\theta_0-m+|\alpha|}(1+r)^{\theta_\infty-\theta_0}D^\alpha u(x,y)\in L^2(\Omega),\forall\alpha\in\mathbb{N}^2,\ |\alpha|\leq m\right\}.$
equiped with the inner product :

$$\begin{aligned}\langle u,v\rangle &= \sum_{|\alpha|\leq m}\iint_\Omega r^{2(\theta_0-m+|\alpha|)}(1+r)^{2(\theta_\infty-\theta_0)}D^\alpha u(x,y)D^\alpha v(x,y)dxdy.\\ H^m_{\theta_0,\theta_\infty}(B) &= \left\{u\in L^2_{loc}(B)\ /\ e^{\theta_0\,t}\ (1+e^t)^{\theta_\infty-\theta_0}u(t,\theta)\in H^m(B)\right\}.\end{aligned}$$

equiped with the inner product :

$$\langle u,v\rangle=\sum_{|\alpha|\leq m}\iint_B D^\alpha\left(e^{\theta_0\,t}\ (1+e^t)^{\theta_\infty-\theta_0}u(t,\theta)\right)D^\alpha\left(e^{\theta_0\,t}\ (1+e^t)^{\theta_\infty-\theta_0}v(t,\theta)\right)dtd\theta.$$

Lemma 1.1.(cf. [4]). *Let θ_1,θ_2 be two reals, we assume that $\theta_1\leq\theta_2$. Let k be a positive integer then: $f\in H^k_{\theta_1,\theta_2}(\Omega)$ if and only if $f\in H^k_{\theta_1,\theta_1}(\Omega)\cap H^k_{\theta_2,\theta_2}(\Omega)$ and we have*

$$\|f\|_{H^k_{\theta_1,\theta_2}(\Omega)}\leq c\left[\|f\|_{H^k_{\theta_1,\theta_1}(\Omega)}+\|f\|_{H^k_{\theta_2,\theta_2}(\Omega)}\right],$$

c benig a constant which depends only on θ_1,θ_2.

Definition 1.2. *Let $m\in\mathbb{N}$, we define the space $V^m(B)$ by :*
$V^m(B)=\left\{u\in L^2(B)\ /\ (1+\xi^2)^{\frac{k}{2}}u\in L^2(\mathbb{R}\ ,H^{m-k}(]0,\omega[)\ ),\ pour\ k=0,1,...m\right\}.$

$V^m(B)$ is a *Hilbert* space with the inner product given by :

$$\langle u,v\rangle=\sum_{k=0}^m\int\int_B(1+\xi^2)^k\left|D^{m-k}_\theta u\right|\left|D^{m-k}_\theta v\right|d\theta d\xi.$$

We define by F the *Fourier* transform with repect to the first variable in B.
The application: $F:\ H^m(B)\longrightarrow V^m(B)$ is an isomorphism.

Proposition 1.1 (cf. [4, 8]). *$k\in\mathbb{N}$, $\eta_1\leq\eta_2$. Let C_{η_1,η_2} be the band of the complexe C defined by*

$$C_{\eta_1,\eta_2}=\{\ \lambda\in C\ /\ \ Im\lambda\in[\eta_1,\eta_2]\}.$$

Let $f\in H^k_{\eta_1,\eta_2}(B)$, then
(1) *The Fourier transform exists for all $\lambda\in C_{\eta_1,\eta_2}$.*
(2) *The application*

$$\begin{aligned}&[\eta_1,\eta_2]\longrightarrow L^2\,(B)\\ &\eta\longmapsto\widehat{f}\,(\cdot+i\eta,\cdot)\end{aligned}$$

is continuous

(3) *The application*

$$\begin{array}{c} C_{\eta_1,\eta_2} \longrightarrow H^k\left(]0,\omega[\right) \\ \lambda \longmapsto \widehat{f}\left(\lambda,\cdot\right) \end{array}$$

is analytical

4) *There exists a constant c which depends only η_1,η_2 such that*

$$\left\| \widehat{f}\left(\cdot+i\eta,\cdot\right)\right\|_{V^k(B)} \leq c\left\|f\right\|_{H^k_{\eta_1,\eta_2}(B)}.$$

2. Position of the problem $P^k_{\theta_0,\theta_\infty}(\Omega)$

We take into consideration ten types of problems governed by *Lamé* operator. If $f\in L^2_{\theta_0,\theta_\infty}(\Omega)^2$, we look for u if possible in $H^2_{\theta_0,\theta_\infty}(\Omega)^2$ as a solution for the following problem :

$$P^k_{\theta_0,\theta_\infty}(\Omega):\left\{\begin{array}{l} Lu=f \quad \text{in } \Omega, \\ B_0^K u=0 \text{ on } \Gamma_0, \\ B_\omega^K u=0 \text{ on } \Gamma_\omega, \end{array}\right. \quad , k=1,2,...,10.$$

The operators of trace

$$\begin{array}{rclcrl}
B_0^1 u &=& u, & & B_\omega^1 u = & u \\
B_0^2 u &=& \sum(u).\eta\ , & & B_\omega^2 u = & \sum(u).\eta \\
B_0^3 u &=& u\,, & & B_\omega^3\, u = & \sum(u).\eta \\
B_0^4 u &=& \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\tau \end{array}\right. & , & B_\omega^4 = & \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\tau \end{array}\right. \\
B_0^5 u &=& u\,, & & B_\omega^5 u = & \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\tau) \end{array}\right. \\
B_0^6 u &=& \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\tau) \end{array}\right. & , & B_\omega^6 u = & \sum(u).\eta \\
B_0^7 u &=& u\,, & & B_\omega^7 = & \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\eta \end{array}\right. \\
B_0^8 u &=& \left\{\begin{array}{c} u.\tau \\ (\sum(u).\eta).\eta \end{array}\right. & , & B_\omega^8 u = & \sum(u).\eta \\
B_0^9 u &=& \left\{\begin{array}{c} u.\tau \\ \sum(u).\eta).\eta \end{array}\right. & , & B_\omega^9 u = & \left\{\begin{array}{c} u.\tau \\ (\sum(u).\eta).\eta \end{array}\right. \\
B_0^{10} u &=& \left\{\begin{array}{c} u.\tau \\ (\sum(u).\eta).\eta \end{array}\right. & , & B_\omega^{10} u = & \left\{\begin{array}{c} u.\eta \\ (\sum(u).\eta).\tau \end{array}\right.
\end{array}$$

3. **Usage of polar coordinates**

We put $x = r\cos\theta\,,\ y = r\sin\theta\,$, with $r = e^t$.
After the multiplication by (e^{2t}), the system $Lu = f$ be cames

$$(E)\left\{\begin{array}{c} \left[\partial_t^2 + \partial_\theta^2 + \nu_0(\sin\theta.\partial_t + \cos\theta.\partial)\right]\widetilde{u}_1 + \\ \nu_0\left[\cos\theta.\partial_t - \sin\theta.\partial_\theta\right]\left[\sin\theta.\partial_t + \cos\theta.\partial_\theta\right]\widetilde{u}_2 = \widetilde{g}_1 \\ \\ \left[\partial_t^2 + \partial_\theta^2 + \nu_0(\sin\theta.\partial_t + \cos\theta.\partial_\theta)\right]\widetilde{u}_2 + \\ \nu_0\left[\sin\theta.\partial_t + \cos\theta.\partial_\theta\right]\left[\cos\theta.\partial_t - \sin\theta.\partial_\theta\right]\widetilde{u}_1 = \widetilde{g}_2 \end{array}\right.$$

where

$$\begin{array}{rcl} \widetilde{g}_i(t,\theta) & = & e^{2t} f_{\,i}(\,e^t\cos\theta, e^t\sin\theta\,), \\ \widetilde{u}_i(t,\theta) & = & u_i(e^t\cos\theta, e^t\sin\theta), \\ \nu_0 & = & (1-2\nu\,)^{-1}. \end{array}$$

with ν is the coefficient of *Poisson* defined by $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ we have

$$\left.\begin{array}{l} u \in H^2_{\theta_0,\theta_\infty}(\Omega)^{\,2} \\ f\ \in H^2_{\theta_0,\theta_\infty}(\Omega)^{\,2} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \widetilde{u} \in H^2_{\eta_0,\eta_\infty}(B)^2 \\ \widetilde{g} \in L^2_{\eta_0,\eta_\infty}(B)^2 \end{array}\right.$$

Let I_1^K be the isomorphisme of $H^k_{\theta_0,\theta_\infty}(\Omega)^2$ on $H^k_{\eta_0,\eta_\infty}(B\,)^2$ defined by

$$I_1^k(u_1,u_2) = (\widetilde{u}_1,\widetilde{u}_2).$$

Propriety 3.1. *The application I_2^k defined by*

$$\begin{array}{rcl} I_2^k & : & H^k_{\eta_0,\eta_\infty}(B)^2 \longrightarrow H^k_{\eta_0,\eta_\infty}(B)^2 \\ (\widetilde{u}_1,\widetilde{u}_2) & \longrightarrow & V = (V_1,V_2) = (\cos\theta.\widetilde{u}_1 + \sin\theta.\widetilde{u}_2, -\sin\theta.\widetilde{u}_1 + \cos\theta.\widetilde{u}_2) \end{array}$$

is an isomorphisme.

We put $g = (g_1,g_2)$ with $(g_1,g_2) = \dfrac{1}{\lambda+\mu} I_2^2(\widetilde{g}_1,\widetilde{g}_2)$.

With these changes, (E) becames

$$P^k_{\eta_0,\eta_\infty}(B\,)\left\{\begin{array}{l} 2(1-\nu)(\frac{\partial^2 V_1}{\partial t^2} - V_1) + \dfrac{\partial^2 V_2}{\partial t\partial\theta} + (1-2\nu)\dfrac{\partial^2 V_1}{\partial\theta^2} - (3-4\nu)\dfrac{\partial V_2}{\partial\theta} = g_1 \\ (1-2\nu)(\dfrac{\partial^2 V_2}{\partial t^2} - V_2) + \dfrac{\partial^2 V_1}{\partial t\partial\theta} + 2(1-\nu)\dfrac{\partial^2 V_2}{\partial\theta^2} - (3-4\nu\,)\dfrac{\partial V_1}{\partial\theta} = g_2 \\ \widetilde{B}_0^k V = 0, \text{on } \mathbb{R}\times\{0\} \\ \widetilde{B}_\omega^k V = 0,\ \text{on } \mathbb{R}\times\{\omega\} \end{array}\right.$$

where $\widetilde{B}_0^k$, $\widetilde{B}_\omega^k$ are boundary operators of B_0^k, B_ω^k, after the *Fourier* transform. The application I_2^k is an isomorphisme of the space $H^k_{\eta_0,\eta_\infty}(B)$ in itself (propriety 3.1). Thus : $V \in H^2_{\eta_0,\eta_\infty}(B)^2$ and $g \in L^2_{\eta_0,\eta_\infty}(B)^2$.

Propriety 3.2. *The problems* $P^k_{\theta_0,\theta_\infty}(\Omega)$, $P^k_{\eta_0,\eta_\infty}(B)$ *are equivalents.*
This result, from propriety 3.1.

4. Fourier transforme of $\mathbf{P}^k_{\theta_0,\theta_\infty}(\Omega)$

We have found on what procedes that $V \in H^2_{\eta_0,\eta_\infty}(B)^2$. Thus after the proposition (1.1), V and its derivatives of order ≤ 2 admits a *Fourier* transform with repect to the first variable t in the band C_{η_0,η_∞} defined by

$$C_{\eta_0,\eta_\infty} = \{\lambda \in C \ \ / \ \eta_0 \leq \ Im\lambda \ \ \leq \eta_\infty\}\,.$$

We have also: $g \in L^2_{\eta_0,\eta\infty}(B)^2$, then g admits a *Fourier* transform in the same band.
For simplicity, we write: $\partial_\theta X = X'$ for all function $X(\lambda,\theta)$.

Transforming the problem $P^k_{\eta_0,\eta_\infty}(B)$ by *Fourier*, we obtain

$$P^k_\lambda(B)\begin{cases} (1-2\nu)\widehat{V}_1'' - 2(1-\nu)(1+\lambda^2)\widehat{V}_1 - (3-4\nu-i\lambda)\widehat{V}_2' = \widehat{g}_1 \\ 2(1-\nu)\widehat{V}_2'' - (1-2\nu)(1+\lambda^2)\widehat{V}_2 + (3-4\nu+i\lambda)\widehat{V}_1' = \widehat{g}_2 \\ C_0^k\widehat{V} = 0, \quad \text{for } \theta = 0 \\ C_\omega^k\widehat{V} = 0, \quad \text{for } \theta = \omega \end{cases}$$

where C_0^k , C_ω^k are boundary operators of $\widetilde{B}_0^k$, $\widetilde{B}_\omega^k$, after the *Fourier* transform.
More precisely we have

$$\begin{aligned}
C_0^1\widehat{V} &= \widehat{V}, & C_\omega^1\widehat{V} &= \widehat{V} \\
C_0^2\widehat{V} &= \begin{cases} (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \\ (i\lambda\nu+1-\nu)\widehat{V}_1 + (1-\nu)\widehat{V}_2' \end{cases}, & C_\omega^2\widehat{V} &= \begin{cases} (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \\ (i\lambda\nu+1-\nu)\widehat{V}_1 + (1-\nu)\widehat{V}_2' \end{cases} \\
C_0^3\widehat{V} &= \widehat{V}, & C_\omega^3\widehat{V} &= \begin{cases} (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \\ (i\lambda\nu+1-\nu)\widehat{V}_1 + (1-\nu)\widehat{V}_2' \end{cases} \\
C_0^4\widehat{V} &= \begin{cases} \widehat{V}_2 \\ (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \end{cases}, & C_\omega^4\widehat{V} &= \begin{cases} \widehat{V}_2 \\ (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \end{cases} \\
C_0^5\widehat{V} &= \widehat{V}, & C_\omega^5\widehat{V} &= \begin{cases} \widehat{V}_2 \\ (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \end{cases} \\
C_0^6\widehat{V} &= \begin{cases} \widehat{V}_2 \\ (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \end{cases}, & C_\omega^6\widehat{V} &= \begin{cases} (i\lambda-1)\widehat{V}_2 + \widehat{V}_1' \\ (i\lambda\nu+1-\nu)\widehat{V}_1 + (1-\nu)\widehat{V}_2' \end{cases} \\
C_0^7\widehat{V} &= \widehat{V}, & C_\omega^7\widehat{V} &= \begin{cases} \widehat{V}_1 \\ (i\lambda\nu+1-\nu)\widehat{V}_1 + (1-\nu)\widehat{V}_2' \end{cases}
\end{aligned}$$

$$C_0^8\widehat{V} = \left\{ \begin{array}{c} \widehat{V}_1 \\ (i\lambda\nu+1-\nu)\widehat{V}_1+(1-\nu)\widehat{V}_2' \end{array} \right. , C_\omega^8\widehat{V} = \left\{ \begin{array}{c} (i\lambda-1)\widehat{V}_2+\widehat{V}_1' \\ (i\lambda\nu+1-\nu)\widehat{V}_1+(1-\nu)\widehat{V}_2' \end{array} \right.$$

$$C_0^9\widehat{V} = \left\{ \begin{array}{c} \widehat{V}_1 \\ (i\lambda\nu+1-\nu)\widehat{V}_1+(1-\nu)\widehat{V}_2' \end{array} \right. , C_\omega^9\widehat{V} = \left\{ \begin{array}{c} \widehat{V}_1 \\ (i\lambda\nu+1-\nu)\widehat{V}_1+(1-\nu)\widehat{V}_2' \end{array} \right.$$

$$C_0^{10}\widehat{V} = \left\{ \begin{array}{c} \widehat{V}_1 \\ (i\lambda\nu+1-\nu)\widehat{V}_1+(1-\nu)\widehat{V}_2' \end{array} \right. , C_0^{10}\widehat{V} = \left\{ \begin{array}{c} \widehat{V}_2 \\ (i\lambda-1)\widehat{V}_2+\widehat{V}_1' \end{array} \right. .$$

Finally, we have to study the following problem :

λ is a fixed in the band C_{η_0,η_∞}, $\widehat{g}$ being in $L^2(]0,\omega[)^2$, we look for $\widehat{V}$ if possible in $H^2(]0,\omega[)^2$ solution for the problem $P_\lambda^k(B)$. The study of the homogeneous problem corresponding to $P_\lambda^k(B)$ gives the following results: (cf. *Benseridi* and *Merouani* [2]).

Result 4.1. *Let D^k be the determinant of Cramer system given by the conditions: $(C_0^K\widehat{V}=0, C_\omega^K\widehat{V}=0)$, then: $D^k = c_k\Phi_k,\ k=1,2,...,10$.*

The (c_k) are constants, and (Φ_k) are the following functions :

$$\begin{aligned}
\Phi_1(\lambda) &= \frac{1}{\lambda^2}\left[(3-4\nu)^2 sh^2(\lambda\omega)-\lambda^2\sin^2(\omega)\right] \\
\Phi_2(\lambda) &= \lambda^2\left[sh^2(\lambda\omega)-\lambda^2\sin^2(\omega)\right] \\
\Phi_3(\lambda) &= \left[4(1-\nu)^2+\lambda^2\sin^2(\omega)+(3-4\nu)sh^2(\lambda\omega)\right] \\
\Phi_4(\lambda) &= sh(\lambda-i)\omega sh(\lambda+i)\omega \\
\Phi_5(\lambda) &= \frac{1}{\lambda}\left[(3-4\nu)sh(2\lambda\omega)+\lambda\sin(2\omega)\right] \\
\Phi_6(\lambda) &= \lambda(sh(2\lambda\omega)+\lambda\sin(2\omega)) \\
\Phi_7(\lambda) &= \frac{1}{\lambda}\left[(3-4\nu)sh(2\lambda\omega)+\lambda\sin(2\omega)\right] \\
\Phi_8(\lambda) &= \lambda(sh(2\lambda\omega)-\lambda\sin(2\omega)) \\
\Phi_9(\lambda) &= sh(\lambda-i)\omega sh(\lambda+i)\omega \\
\Phi_{10}(\lambda) &= ch(\lambda-i)\omega ch(\lambda+i)\omega
\end{aligned}$$

Result 4.2. *Let F_k be the set of zeros of $\Phi_k(\lambda)$, then the homogeneous problem corresponding to $P_\lambda^k(B)$ admits a unique solution (the trivial solution) if and only if $\lambda \notin F_k$.*

Proposition 4.1 (cf. [4]). *For all* $\lambda \in \mathbb{C}/F_k$, *for all* $\widehat{g} \in L^2(]0,\omega[)^2$, *there exists one and only one* $\widehat{V}_\lambda \in H^2(]0,\omega[)^2$ *solution for the problem* $P_\lambda^k(B)$. *In addition, the resolved of* $P_\lambda^k(B)$,

$$\begin{array}{rcl} R_\lambda & : & L^2(]0,\omega[)^2 \longrightarrow H^2(]0,\omega[)^2 \\ \widehat{g} & \longmapsto & R_\lambda(\widehat{g}) = \widehat{V}_\lambda \end{array}$$

such that the application

$$\begin{array}{rcl} \mathbb{C}/F_k & \longrightarrow & L\left(L^2(]0,\omega[)^2, H^2(]0,\omega[)^2\right), \\ \lambda & \longmapsto & R_\lambda \end{array}$$

is analytical.

5. Existence and uniqueness of $\eta_$solutions

In order to the study the existence and uniqueness of the solution for the problem $P_{\theta_0,\theta_\infty}^k(\Omega)$, it is important to introduce the following definition :

Defnition 5.1. *Let* $\eta \in [\eta_0,\eta_\infty]$, *we call* $\eta-$*solution for the problem* $P_{\theta_0,\theta_\infty}^k(\Omega)$, *all elements* u *of* $H_{\eta+1,\eta+1}^2(\Omega)^2$, *verifying*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ B_0^k u = 0 & \text{on } \Gamma_0, \\ B_\omega^k u = 0 & \text{on } \Gamma_\omega. \end{cases}$$

The following propriety is a direct result of lemma (1.1).

Propriety 5.1. u *is a solution for the problem* $P_{\theta_0,\theta_\infty}^k(\Omega)$ *iff* u *is a* η_0-*solution of the* $P_{\theta_0,\theta_\infty}^k(\Omega)$ *and* u *is a* $\eta_\infty-$*solution of* $P_{\theta_0,\theta_\infty}^k(\Omega)$.

Proposition 5.1. *If* Φ_k *have no zero of imaginary part* η, *the problem* $P_{\theta_0,\theta_\infty}^k(\Omega)$ *has a unique* $\eta-$ *solution, in addition there exists a positive constant* c *such as*

$$\|u\|_{H_{\eta+1,\eta+1}^2(\Omega)^2} \leq c\,\|f\|_{L_{\theta_0,\theta_\infty}^2(\Omega)^2}.$$

Remark 5.1. *For the demonstration of the proposition 5.1, (cf. Teniou* [8]).

6. Comparaison of the $\eta-$solutions

Let $\eta_1,\eta_2 \in [\eta_0,\eta_\infty]$, $\eta_1 \leq \eta_2$. We know that if Φ_k have no zero of imaginary part η_1(resp η_2), the problem $P_{\theta_0,\theta_\infty}^k(\Omega)$ admits a unique η_1-solution (resp η_2-solution) that we not by u_{η_1} (resp u_{η_2}).

We put $V_{\eta_1} = I_2^2 \circ I_1^2(u_{\eta_1})$, $V_{\eta_2} = I_2^2 \circ I_1^2(u_{\eta_2})$, with this notation , we have the following proposition :

Proposition 6.1. *We assume that Φ_k have no zero of imaginary part η_1 or η_2, then*

$$V_{\eta_1} - V_{\eta_2} = i \sum_{\lambda_0 \in F_k \cap \{\eta_1 \leq Im\lambda \leq \eta_2\}} Res(e^{i\,t\lambda} R_\lambda(\widehat{g}))\,|_{\lambda=\lambda_0} \,.$$

Proof. The proof of this proposition is very classical. We note first that the sum has a meaning because the set $F_k \cap \{\eta_1 \leq Im\lambda \leq \eta_2\}$ is finite and the residuals are well defined (these are the functions on $(t, \theta) \in B$).

Let γ be the contour made by the straight line $\mathbb{R} + i\eta_1, \mathbb{R} + i\eta_2$. We know that R_λ is analytical on $\mathbb{C}/F_k$, hence

$$\int_\gamma e^{it\lambda} R_\lambda(\widehat{g}) d\lambda = 2\pi i \sum_{\lambda_0 \in F_k \cap \{\eta_1 \leq Im\lambda \leq \eta_2\}} Res(e^{i\,t\lambda} R_\lambda(\widehat{g}))\,|_{\lambda=\lambda_0}$$

and

$$\int_\gamma e^{it\lambda} R_\lambda(\widehat{g}) d\lambda = \int_{[-\varepsilon+i\eta_1, \varepsilon+i\eta_1]} e^{it\lambda} R_\lambda(\widehat{g}) d\lambda + \int_{[\varepsilon+i\eta_1, \varepsilon+i\eta_2]} e^{it\lambda} R_\lambda(\widehat{g}) d\lambda$$

$$+ \int_{[\varepsilon+i\eta_2, -\varepsilon+i\eta_2]} e^{it\lambda} R_\lambda(\widehat{g}) d\lambda + \int_{[-\varepsilon+i\eta_2, -\varepsilon+i\eta_1]} e^{it\lambda} R_\lambda(\widehat{g}) d\lambda$$

with the limit when $\varepsilon \to \infty$, we obtain

$$\lim_{\varepsilon\to\infty} \int_\gamma e^{it\lambda} R_\lambda(\widehat{g}) d\lambda = \int_{-\infty}^{+\infty} e^{it(\xi+i\eta_1)} R_{\xi+i\eta_1}(\widehat{g}) d\xi - \int_{-\infty}^{+\infty} e^{it(\xi+i\eta_2)} R_{\xi+i\eta_2}(\widehat{g}) d\xi$$

the integrals

$$\int_{[\varepsilon+i\eta_1, \varepsilon+i\eta_2]} e^{it\lambda}\ R_\lambda(\widehat{g}) d\lambda\ , \quad \int_{[-\varepsilon+i\eta_2, -\varepsilon+i\eta_1]} e^{it\lambda}\ R_\lambda(\widehat{g}) d\lambda,$$

tends to zero, thus

$$\tfrac{e^{-\eta_1 t}}{2\pi} \int_{-\infty}^{+\infty} e^{it\xi} R_{\xi+i\eta_1}(\widehat{g}) - \tfrac{e^{-\eta_2 t}}{2\pi} \int_{-\infty}^{+\infty} e^{it\xi} R_{\xi+i\eta_2}(\widehat{g}) = i \sum_{\lambda_0 \in F_k \cap \{\eta_1 \leq Im\lambda \leq \eta_2\}} Res(e^{i\,t\lambda} R_\lambda(\widehat{g}))\,|_{\lambda=\lambda_0}$$

but

$$V_{\eta_1} = \tfrac{e^{-\eta_1 t}}{2\pi} \int_{-\infty}^{+\infty} e^{it\xi} R_{\xi+i\eta_1}(\widehat{g}) d\xi \ ,$$

$$V_{\eta_2} = \tfrac{e^{-\eta_2 t}}{2\pi} \int_{-\infty}^{+\infty} e^{it\xi} R_{\xi+i\eta_2}(\widehat{g}) d\xi.$$

Which ends the proof.

7. Existence and uniqueness of the solution for the problem $P^k_{\theta_0,\theta_\infty}(\Omega)$

Our objective now is to prove a theorem of existence and uniqueness and regularity of the solution of the problem $P^k_{\theta_0,\theta_\infty}(\Omega)$.

Theorem 7.1. *Let θ_0, θ_∞ be two reals such that $\theta_0 \leq \theta_\infty$.*
We assume that Φ_k have no zero in the band C_{η_0,η_∞}, thus for all $f \in L^2_{\theta_0,\theta_\infty}(\Omega)^2$, there exists one and only one $u \in H^2_{\theta_0,\theta_\infty}(\Omega)^2$ solution for the problem $P^k_{\theta_0,\theta_\infty}(\Omega)$,and we have

$$\|u\|_{H^2_{\theta_0,\theta_\infty}(\Omega)^2} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega)^2}.$$

Proof.

(1) *Existence*. The hypothesis that Φ_k has no zero on the band C_{η_0,η_∞} ensures the existence of η_0-solution and the $\eta_\infty-$solution of $P^k_{\theta_0,\theta_\infty}(\Omega)$, that we note u_{η_0}, u_{η_∞} and in addition: $F_k \cap \{\eta_0 \leq Im\lambda \leq \eta_\infty\} = \emptyset$.

We put $V_{\eta_0} = I_2^2 \circ I_1^2(\ u_{\eta_0}), V_{\eta_\infty} = I_2^2 \circ I_1^2(\ u_{\eta_\infty})$, proposition (6.1) implies that $V_{\eta_0} - V_{\eta_\infty} = i \sum\limits_{\lambda_0 \in F_k \cap \{\eta_0 \leq Im\lambda \leq \eta_\infty\}} Res(e^{i\,t\lambda} R_\lambda(\widehat{g}))\,|_{\lambda=\lambda_0}$. As a result $V_{\eta_0} = V_{\eta_\infty}$, this shows that $u_{\eta_0} = u_{\eta_\infty}$. Now, we put now $u = u_{\eta_0}$, it is clear that $u \in H^2_{\theta_0,\theta_0}(\Omega)^2$ and $u \in H^2_{\theta_\infty,\theta_\infty}(\Omega)^2$. The lemma(1.1) shows that $u \in H^2_{\theta_0,\theta_\infty}(\Omega)^2$. Thus u is a solution of $P^k_{\theta_0,\theta_\infty}(\Omega)$ by construction.

(2) *Uniqueness*. We assume that there exist two solutions u^1, $u^2 \in H^2_{\theta_0,\theta_\infty}(\Omega)^2$. thus u^1, u^2 are η_0-solution and $\eta_\infty-$solution (propriety 5.1). Thus from the uniqueness of $\eta-$solutions $u^1 = u^2$.

(3) *Continuity with repect to the data*. We deduce from the proposition 5.1 that

$$\|u\|_{H^2_{\theta_0,\theta_0}(\Omega)^2} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega)^2} \text{ and } \|u\|_{H^2_{\theta_\infty,\theta_\infty}(\Omega)^2} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega)^2},$$

and from the lemma(1.1): $\|u\|_{H^2_{\theta_0,\theta_\infty}(\Omega)^2} \leq c\left[\|u\|_{H^2_{\theta_0,\theta_0}(\Omega)^2} + \|u\|_{H^2_{\theta_\infty,\theta_\infty}(\Omega)^2}\right]$
thus

$$\|u\|_{H^2_{\theta_0,\theta_\infty}(\Omega)^2} \leq c\,\|f\|_{L^2_{\theta_0,\theta_\infty}(\Omega)^2}.$$

Remark 7.1.

- We can verify that the functions Φ_1, Φ_2 have no zeros in the band $\{-1 \leq Im\lambda < 0\}$ if $\omega \leq \pi$. This ensures the regularity of the solution u in $H^2_{\theta_0,\theta_\infty}(\Omega)^2$ if $\theta_0 \geq 0$, $\theta_\infty < 1$ and the vertexe is convexe of *Dirichlet* type or *Neumann* type.

- When $\theta_0 = \theta_\infty$, we reproduce the results of *Teniou* [8].

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.2,243-251,COPYRIGHT 2008 EUDOXUS PRESS, LLC

On the Product of Laplace and Logistic Random Variables

by

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Abstract: The exact distribution of the product $\mid XY \mid$ is derived when X and Y are Laplace and logistic random variables distributed independently of each other. Tabulations of the associated percentage points are given.

1 Introduction

For given random variables X and Y, the distribution of the product $\mid XY \mid$ is of interest in problems in biological and physical sciences, econometrics, and classification. The exact distribution of $\mid XY \mid$ has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results).

However, there is relatively little work of the above kind when X and Y belong to different families. In the applications, it is quite possible that X and Y could arise from different but similar distributions. In this note, we study the exact distribution of $\mid XY \mid$ when X and Y are independent random variables having the Laplace and logistic distributions specified by the probability density functions (pdfs)

$$f_X(x) \quad = \quad \frac{\lambda}{2} \exp\left(-\lambda \mid x \mid\right) \tag{1}$$

and

$$f_Y(y) \quad = \quad \frac{\mu \exp(-\mu y)}{\left\{1 + \exp(-\mu y)\right\}^2}, \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\lambda > 0$ and $\mu > 0$. Tabulations of the associated percentage points are also provided.

Laplace and logistic distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. Both are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. In many of the application areas, one would be interested in products of Laplace and logistic random variables. Some examples are:

1. in communication theory, X and Y could represent the random noise corresponding to two different signals.
2. in ocean engineering, X and Y could represent distributions of navigation errors.
3. in finance, X and Y could represent distributions of log-returns of two different commodities.
4. in image and speech recognition, X and Y could represent "input" distributions.

For further discussion of applications, the reader is referred to Balakrishnan (1992) and Kotz *et al.* (2001).

The results of this note are organized as follows: exact expressions for the pdf and the cumulative distribution function (cdf) of $\mid XY \mid$ are given in Section 2; moment properties of $\mid XY \mid$ including its characteristic function and moments are considered in Section 3; finally, tabulations of the percentile points of $\mid XY \mid$ obtained by inverting the derived cdf are provided in Section 4.

The calculations of this note involve the modified Bessel function of the first kind defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k,$$

and the modified Bessel function of the third kind defined by

$$K_\nu(x) = \frac{\pi \{I_{-\nu}(x) - I_\nu(x)\}}{2\sin(\nu\pi)}.$$

The calculations also require the following representation of (2):

$$f_{|Y|}(y) = 2\sum_{k=0}^{\infty} (-1)^k \mu(k+1) \exp\{-\mu(k+1)y\} \tag{3}$$

for $y > 0$ (the series does not hold for $y = 0$). The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 PDF and CDF of $\mid XY \mid$

Theorem 1 derives explicit expressions for the pdf and the cdf of $\mid XY \mid$ in terms of the modified Bessel function of the third kind.

Theorem 1 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. The cdf of $Z = \mid XY \mid$ can be expressed as*

$$F(z) = 1 - 4\sqrt{\lambda\mu z} \sum_{k=0}^{\infty} (-1)^k \sqrt{k+1} K_1\left(2\sqrt{\lambda\mu(k+1)z}\right) \tag{4}$$

for $z > 0$. The corresponding pdf is:

$$
\begin{aligned}
f(z) &= 4\lambda\mu\sum_{k=0}^{\infty}(-1)^k(k+1)\left\{K_0\left(2\sqrt{\lambda\mu(k+1)z}\right)+\frac{K_1\left(2\sqrt{\lambda\mu(k+1)z}\right)}{2\sqrt{\lambda\mu(k+1)z}}\right\} \\
&\quad -2\sqrt{\frac{\lambda\mu}{z}}\sum_{k=0}^{\infty}(-1)^k\sqrt{k+1}K_1\left(2\sqrt{\lambda\mu(k+1)z}\right) \qquad (5)
\end{aligned}
$$

for $z > 0$.

Proof: Using the relationship (3), one can write

$$
\begin{aligned}
\Pr\left(\mid XY \mid > z\right) &= \int_0^{\infty}\Pr\left(\mid X \mid > \frac{z}{y}\right)f_{|Y|}(y)dy \\
&= 2\int_0^{\infty}\exp\left(-\frac{\lambda z}{y}\right)\left[\sum_{k=0}^{\infty}(-1)^k\mu(k+1)\exp\left\{-\mu(k+1)y\right\}\right]dy \\
&= 2\sum_{k=0}^{\infty}(-1)^k\mu(k+1)\int_0^{\infty}\exp\left\{-\frac{\lambda z}{y}-\mu(k+1)y\right\}dy \\
&= 4\sum_{k=0}^{\infty}(-1)^k\mu(k+1)\sqrt{\frac{\lambda z}{(k+1)\mu}}K_1\left(2\sqrt{\lambda\mu(k+1)z}\right),
\end{aligned}
$$

where the last step follows by direct application of equation (3.471.9) in Gradshteyn and Ryzhik (2000). The switching of the integral with the sum can be justified by the dominated convergence theorem:

$$
\begin{aligned}
& 2\int_0^{\infty}\exp\left(-\frac{\lambda z}{y}\right)\left|\sum_{k=0}^{n}(-1)^k\mu(k+1)\exp\left\{-\mu(k+1)y\right\}\right|dy \\
\leq\ & 2\int_0^{\infty}\exp\left(-\frac{\lambda z}{y}\right)\sum_{k=0}^{n}\mu(k+1)\exp\left\{-\mu(k+1)y\right\}dy \\
<\ & 2\int_0^{\infty}\exp\left(-\frac{\lambda z}{y}\right)\sum_{k=0}^{\infty}\mu(k+1)\exp\left\{-\mu(k+1)y\right\}dy \\
=\ & 2\int_0^{\infty}\exp\left(-\frac{\lambda z}{y}\right)\frac{\mu\exp(-\mu y)}{\left\{1-\exp(-\mu y)\right\}^2}dy \\
<\ & \infty. \qquad (6)
\end{aligned}
$$

The result of (5) follows by using the property $K_\nu'(z) = -K_{\nu-1}(z) - (\nu/z)K_\nu(z)$, see http: // functions.wolfram.com / BesselAiryStruveFunctions/BesselK/20/01/02/0001/. The term by term differentiation of (4) can also be justified by the dominated convergence theorem. ■

Figure 1 illustrates possible shapes of the pdf (5) for $\lambda = 1$ and a range of values of μ. Note that the shapes are unimodal and that the value of μ largely dictates the behavior of the pdf near $z = 0$.

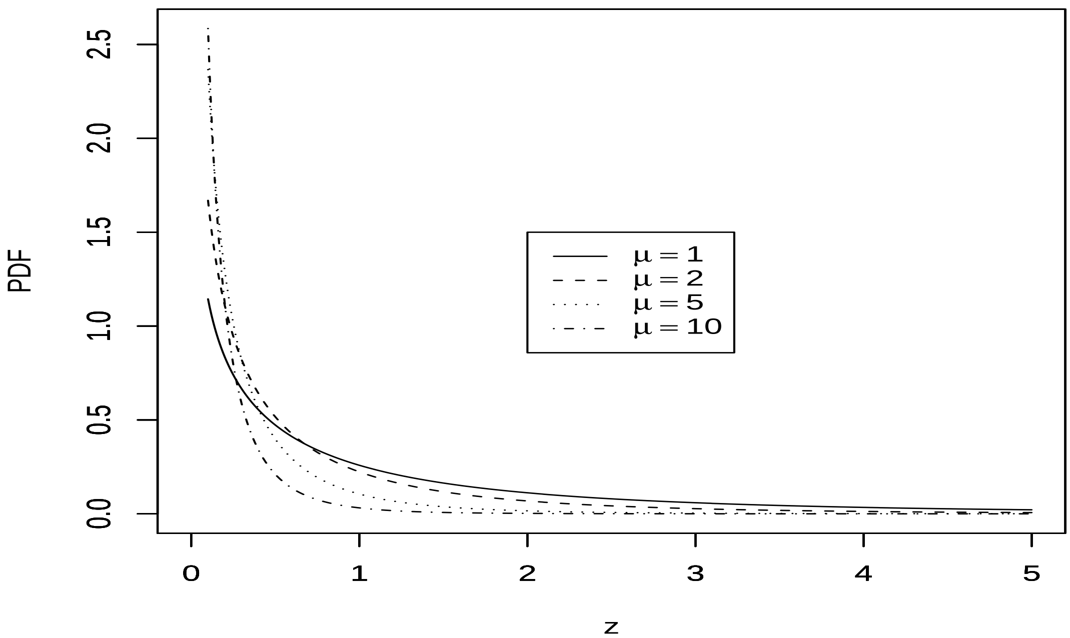

Figure 1. Plots of the pdf (5) for $\mu = 1, 2, 5, 10$ and $\lambda = 1$.

3 Moment Properties of $\mid XY \mid$

The moment properties of $\mid XY \mid$ can be derived by knowing the same for X and Y. It is well known (see, for example, Johnson *et al.* (1994, 1995)) that

$$E\left(\mid X \mid^n\right) = \frac{n!}{\lambda^n}$$

and

$$E\left(\mid Y \mid^n\right) = 2\mu^{-n} n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+k)^n}.$$

Thus, the nth moment of $Z = \mid XY \mid$ is

$$E\left(Z^n\right) = \frac{2\,(n!)^2}{(\lambda\mu)^n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+k)^n}.$$

In particular,

$$E(Z) = \frac{2}{\lambda\mu} \sum_{k=0}^{\infty} \frac{(-1)^k}{1+k},$$

which can be reduced to

$$E(Z) = \frac{2\log 2}{\lambda\mu},$$

and

$$E\left(Z^2\right) = \frac{8}{(\lambda\mu)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^2},$$

which can be reduced to

$$E\left(Z^2\right) = \frac{2\pi^2}{3(\lambda\mu)^2}.$$

Using the fact that the characteristic function (chf) of $\mid X \mid$ is

$$E\left[\exp(it \mid X \mid)\right] = \frac{\lambda}{\lambda - it},$$

where $i = \sqrt{-1}$ denotes the complex unit, the chf of $\mid XY \mid$ can be derived as

$$\begin{aligned}
& E\left[\exp(it \mid XY \mid)\right] \\
= \; & 2\lambda\mu \int_0^{\infty} \frac{1}{\lambda - ity} \frac{\exp(-\mu y)}{\{1+\exp(-\mu y)\}^2} dy \\
= \; & 2\lambda\mu \int_0^{\infty} \frac{\lambda + ity}{\lambda^2 + t^2 y^2} \frac{\exp(-\mu y)}{\{1+\exp(-\mu y)\}^2} dy \\
= \; & 2\lambda\mu \int_0^{\infty} \frac{\lambda + ity}{\lambda^2 + t^2 y^2} \left[\sum_{k=0}^{\infty}(-1)^k \mu(k+1)\exp\{-\mu(k+1)y\}\right] dy \\
= \; & 2\lambda^2\mu^2 \sum_{k=0}^{\infty}(-1)^k (k+1) \int_0^{\infty} \frac{\exp\{-\mu(k+1)y\}}{\lambda^2 + t^2 y^2} dy \\
& +2it\lambda\mu^2 \sum_{k=0}^{\infty}(-1)^k (k+1) \int_0^{\infty} \frac{y\exp\{-\mu(k+1)y\}}{\lambda^2 + t^2 y^2} dy \\
= \; & \frac{2\lambda\mu^2}{t} \sum_{k=0}^{\infty}(-1)^k (k+1) \left\{\sin\frac{\lambda\mu(k+1)}{t}\mathrm{ci}\frac{\lambda\mu(k+1)}{t} - \cos\frac{\lambda\mu(k+1)}{t}\mathrm{si}\frac{\lambda\mu(k+1)}{t}\right\} \\
& +2i\mu^2 \sum_{k=0}^{\infty}(-1)^{k+1} (k+1) \frac{\partial}{\partial p}\left\{\sin\frac{p\lambda}{t}\mathrm{ci}\frac{p\lambda}{t} - \cos\frac{p\lambda}{t}\mathrm{si}\frac{p\lambda}{t}\right\}\Bigg|_{p=\mu(k+1)},
\end{aligned}$$

where the last step follows by equation (2.3.7.11) in Prudnikov *et al.* (1986, volume 1). The $\mathrm{si}(\cdot)$ and $\mathrm{ci}(\cdot)$ are the sine and cosine integrals defined by

$$\mathrm{si}(x) = -\int_x^{\infty}\frac{\sin t}{t}dt$$

and

$$\mathrm{ci}(x) = -\int_x^{\infty}\frac{\cos t}{t}dt,$$

respectively. The switching of the integral with the sum above can be justified as in the proof of Theorem 1.

4 Percentiles of $\mid XY \mid$

In this section, we provide tabulations of percentage points z_p associated with the cdf (4). These values are obtained by numerically solving the equation

$$4\sqrt{\lambda \mu z} \sum_{k=0}^{\infty} (-1)^k \sqrt{k+1} K_1 \left(2\sqrt{\lambda \mu (k+1) z} \right) \quad = \quad 1-p.$$

Evidently, this involves computation of the modified Bessel function of the third kind and routines for this are widely available. We used the function BesselK $(\cdot)$ in the algebraic manipulation package, MAPLE. Table 1 provides the numerical values of z_p for $\mu = 0.1, 0.2, \ldots, 10$ and $\lambda = 1$.

Table 1. Percentage points of $Z = \mid XY \mid$ for $\lambda = 1$.

μ	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	0.030	0.209	0.513	35.507	52.939	103.517
0.2	0.015	0.104	0.257	17.754	26.470	51.759
0.3	0.010	0.070	0.171	11.836	17.646	34.506
0.4	0.007	0.052	0.128	8.877	13.235	25.879
0.5	0.006	0.042	0.103	7.101	10.588	20.703
0.6	0.005	0.035	0.086	5.918	8.823	17.253
0.7	0.004	0.030	0.073	5.073	7.563	14.788
0.8	0.004	0.026	0.064	4.438	6.617	12.940
0.9	0.003	0.023	0.057	3.945	5.882	11.502
1	0.003	0.021	0.051	3.551	5.294	10.352
1.1	0.003	0.019	0.047	3.228	4.813	9.411
1.2	0.002	0.017	0.043	2.959	4.412	8.626
1.3	0.002	0.016	0.039	2.731	4.072	7.963
1.4	0.002	0.015	0.037	2.536	3.781	7.394
1.5	0.002	0.014	0.034	2.367	3.529	6.901
1.6	0.002	0.013	0.032	2.219	3.309	6.470
1.7	0.002	0.012	0.030	2.089	3.114	6.089
1.8	0.002	0.012	0.028	1.973	2.941	5.751
1.9	0.002	0.011	0.027	1.869	2.786	5.448
2	0.001	0.010	0.026	1.775	2.647	5.176
2.1	0.001	0.010	0.024	1.691	2.521	4.929
2.2	0.001	0.010	0.023	1.614	2.406	4.705
2.3	0.001	0.009	0.022	1.544	2.302	4.501
2.4	0.001	0.009	0.021	1.479	2.206	4.313
2.5	0.001	0.008	0.020	1.420	2.118	4.141
2.6	0.001	0.008	0.020	1.366	2.036	3.981
2.7	0.001	0.008	0.019	1.315	1.961	3.834
2.8	0.001	0.007	0.018	1.268	1.891	3.697
2.9	0.001	0.007	0.018	1.224	1.826	3.570
3	0.001	0.007	0.017	1.184	1.765	3.451
3.1	0.001	0.007	0.017	1.145	1.708	3.339
3.2	0.001	0.007	0.016	1.110	1.654	3.235
3.3	0.001	0.006	0.016	1.076	1.604	3.137

3.4	0.001	0.006	0.015	1.044	1.557	3.045
3.5	0.001	0.006	0.015	1.014	1.513	2.958
3.6	0.001	0.006	0.014	0.986	1.471	2.875
3.7	0.001	0.006	0.014	0.960	1.431	2.798
3.8	0.001	0.006	0.014	0.934	1.393	2.724
3.9	0.001	0.005	0.013	0.910	1.357	2.654
4	0.001	0.005	0.013	0.888	1.323	2.588
4.1	0.001	0.005	0.013	0.866	1.291	2.525
4.2	0.001	0.005	0.012	0.845	1.260	2.465
4.3	0.001	0.005	0.012	0.826	1.231	2.407
4.4	0.001	0.005	0.012	0.807	1.203	2.353
4.5	0.001	0.005	0.011	0.789	1.176	2.300
4.6	0.001	0.005	0.011	0.772	1.151	2.250
4.7	0.001	0.004	0.011	0.756	1.126	2.203
4.8	0.001	0.004	0.011	0.740	1.103	2.157
4.9	0.001	0.004	0.011	0.725	1.080	2.113
5	0.001	0.004	0.010	0.710	1.059	2.070
5.1	0.001	0.004	0.010	0.696	1.038	2.030
5.2	0.001	0.004	0.010	0.683	1.018	1.991
5.3	0.001	0.004	0.010	0.670	0.999	1.953
5.4	0.001	0.004	0.010	0.658	0.980	1.917
5.5	0.001	0.004	0.009	0.646	0.963	1.882
5.6	0.001	0.004	0.009	0.634	0.945	1.849
5.7	0.001	0.004	0.009	0.623	0.929	1.816
5.8	0.001	0.004	0.009	0.612	0.913	1.785
5.9	0.000	0.004	0.009	0.602	0.897	1.755
6	0.000	0.004	0.009	0.592	0.882	1.725
6.1	0.000	0.003	0.008	0.582	0.868	1.697
6.2	0.000	0.003	0.008	0.573	0.854	1.670
6.3	0.000	0.003	0.008	0.564	0.840	1.643
6.4	0.000	0.003	0.008	0.555	0.827	1.617
6.5	0.000	0.003	0.008	0.546	0.814	1.593
6.6	0.000	0.003	0.008	0.538	0.802	1.568
6.7	0.000	0.003	0.008	0.530	0.790	1.545
6.8	0.000	0.003	0.008	0.522	0.779	1.522
6.9	0.000	0.003	0.007	0.515	0.767	1.500
7	0.000	0.003	0.007	0.507	0.756	1.479
7.1	0.000	0.003	0.007	0.500	0.746	1.458
7.2	0.000	0.003	0.007	0.493	0.735	1.438
7.3	0.000	0.003	0.007	0.486	0.725	1.418
7.4	0.000	0.003	0.007	0.480	0.715	1.399
7.5	0.000	0.003	0.007	0.473	0.706	1.380
7.6	0.000	0.003	0.007	0.467	0.697	1.362
7.7	0.000	0.003	0.007	0.461	0.688	1.344
7.8	0.000	0.003	0.007	0.455	0.679	1.327
7.9	0.000	0.003	0.006	0.449	0.670	1.310
8	0.000	0.003	0.006	0.444	0.662	1.294
8.1	0.000	0.003	0.006	0.438	0.654	1.278

8.2	0.000	0.003	0.006	0.433	0.646	1.262
8.3	0.000	0.003	0.006	0.428	0.638	1.247
8.4	0.000	0.002	0.006	0.423	0.630	1.232
8.5	0.000	0.002	0.006	0.418	0.623	1.218
8.6	0.000	0.002	0.006	0.413	0.616	1.204
8.7	0.000	0.002	0.006	0.408	0.608	1.190
8.8	0.000	0.002	0.006	0.404	0.602	1.176
8.9	0.000	0.002	0.006	0.399	0.595	1.163
9	0.000	0.002	0.006	0.395	0.588	1.150
9.1	0.000	0.002	0.006	0.390	0.582	1.138
9.2	0.000	0.002	0.006	0.386	0.575	1.125
9.3	0.000	0.002	0.006	0.382	0.569	1.113
9.4	0.000	0.002	0.005	0.378	0.563	1.101
9.5	0.000	0.002	0.005	0.374	0.557	1.090
9.6	0.000	0.002	0.005	0.370	0.551	1.078
9.7	0.000	0.002	0.005	0.366	0.546	1.067
9.8	0.000	0.002	0.005	0.362	0.540	1.056
9.9	0.000	0.002	0.005	0.359	0.535	1.046
10	0.000	0.002	0.005	0.355	0.529	1.035

We hope these numbers will be of use to the practitioners mentioned in Section 1. Similar tabulations could be easily derived for other values of p, μ and λ by using the BesselK $(\cdot)$ function in MAPLE. A sample program is shown in the Appendix below.

Appendix

The following procedure in MAPLE can be used to generate tables similar to that presented in Section 3.

```
percent:=proc(lambda,mu,p)
local ff,pp,k,z;
ff:=sum((-1)**k*sqrt(k+1)*BesselK(1,2*sqrt(lambda*mu*(k+1)*z)),k=0..infinity);
ff:=1-4*sqrt(lambda*mu*z)*ff;
pp:=evalf(fsolve(ff=p,z=0..200)):
end proc;
```

Acknowledgments

The author would like to thank the Editor and the referee for carefully reading the paper and for their comments which greatly improved the paper.

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Volume 3,Number 3 July 2008

ISSN:1559-1948 (PRINT), 1559-1956 (ONLINE)

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A quartely international publication of **EUDOXUS PRESS,LLC**

ISSN:1559-1948(PRINT),1559-1956(ONLINE)

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,269-283,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Some Applications of Intuitionistic Fuzzy Metric Spaces

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January 4, 2007

Abstract. In this paper, we prove that every intuitionistic fuzzy metric space has a basis that is countably locally finite. We use this result to conclude that every intuitionistic fuzzy metric space is metrizable. We also introduce notions of orbitally complete intuitionistic fuzzy metric spaces and generalized contraction mappings, and prove fixed point theorems which are extensions of the classical Banach's fixed point principle and fixed point theorems of Edelstein [4] and Grabiec [7]. Our results also extend, generalize and fuzzify several fixed point theorems on metric spaces, Menger probabilistic metric spaces, uniform spaces and fuzzy metric spaces.

Keywords. Generalized contraction; Uniformly convergence; Intuitionistic fuzzy metric space; Orbitally complete intuitionistic fuzzy metric space.

AMS [2000]. 46S40, 47H10, 54H25.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [16]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [8], and George and Veeramani [5,6] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently, many authors have proved fixed point theorems involving fuzzy sets [2-4,7,9,10,12,14].

As a generalization of fuzzy sets, Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [15] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [5], and showed that every metric induces an intuitionistic fuzzy metric, every fuzzy metric space is an intuitionistic fuzzy metric space and found a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete.

In this paper, we observe that an intuitionistic fuzzy metric space is normal and prove that in an intuitionistic fuzzy metric space X every open cover admits a countably locally finite refinement which covers X. Using this result, we prove that every intuitionistic fuzzy metric space has a countably locally finite basis. We introduce notions of orbitally complete intuitionistic fuzzy metric spaces and generalized contraction mappings, and prove fixed point theorems. Our results also extend, generalize and fuzzify several fixed point theorems on metric spaces, Menger probabilistic metric spaces, uniform spaces and fuzzy metric spaces.

2 Preliminaries

Definition 1 *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *satisfies the following conditions:*

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if* $\Diamond$ *satisfies the following conditions:*

(a) $\Diamond$ is commutative and associative;
(b) $\Diamond$ is continuous;
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Remark 1 *The concepts of triangular norms (shortly t-norms) and triangular conorms (shortly t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and fuzzy unions, respectively. These concepts were originally introduced by Menger [11]. Several examples for these concepts were proposed by many authors (see [9-11,15]).*

Lemma 1 ([15]) *If* $*$ *is a continuous t-norm,* $\Diamond$ *is a continuous t-conorm and* $r_i \in (0,1)$, $1 \leq i \leq 7$, *then*

(a) If $r_1 > r_2$, there are $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_2 \Diamond r_4 \leq r_1$.
(b) If $r_5 \in (0,1)$, there are $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \Diamond r_7 \leq r_5$.

Definition 3 ([15]) *A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space (shortly IFM-space) if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x,\ y,\ z \in X,\ s,\ t > 0$,*

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$;
(IFM-2) $M(x, y, t) > 0$;
(IFM-3) $M(x, y, t) = 1$ if and only if $x = y$;
(IFM-4) $M(x, y, t) = M(y, x, t)$;
(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(IFM-6) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous;
(IFM-7) $N(x, y, t) > 0$;
(IFM-8) $N(x, y, t) = 0$ if and only if $x = y$;
(IFM-9) $N(x, y, t) = N(y, x, t)$;
(IFM-10) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$;
(IFM-11) $N(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y,\ t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2 *Every fuzzy metric space $(X, M, *)$ is an IFM-space of the form $(X, M,\ 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, i.e. $x \Diamond y = 1 - ((1 - x) * (1 - y))$ for all $x, y \in [0, 1]$. But the converse is not true.*

Remark 3 ([15]) *In an IFM-space $(X, M, N, *, \Diamond)$, $M(x, y, .)$ is nondecreasing and $N(x,\ y, .)$ is nonincreasing for all $x, y \in X$.*

Example 1 (Induced intuitionistic fuzzy metric [15]) *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},\ \ N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Remark 4 *Note that the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$ and hence (M_d, N_d) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm.*

Definition 4 ([15]) *Let $(X, M, N, *, \Diamond)$ be an IFM-space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r,\ N(x, y, t) < r\}$. Let $\tau_{(M,N)}$ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $r \in (0, 1)$ such that $B(x, r, t) \subset A$. Then $\tau_{(M,N)}$ is a topology on X (induced by the intuitionistic fuzzy metric*

(M, N)). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x in X if and only if $M(x_n, x, t)$ tends to 1 and $N(x_n, x, t)$ tends to 0 as n tends to ∞, for each $t > 0$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$ and $N(x_n, x_m, \varepsilon) < \lambda$ for all $n, m \geq n_0$. An IFM-space is called complete if every Cauchy sequence is convergent.

Remark 5 *Since $*$ and $\diamondsuit$ are continuous, the limit is uniquely determined from (IFM-5) and (IFM-10).*

3 Main Results

Theorem 2 *Every intuitionistic fuzzy metric space is normal.*

Proof. Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space, and S, K be two disjoint closed sets in X. Let $x \in S$ then $x \in K^c$. Since K^c is open, there exist $r_x \in (0, 1)$ and $t_x > 0$ such that $B(x, r_x, t_x) \cap K = \varnothing$ for all $x \in S$. Similarly, there exist $r_y \in (0, 1)$ and $t_y > 0$ such that $B(x, r_y, t_y) \cap S = \varnothing$ for all $x \in K$. Let $s = \min\{r_x, t_x, r_y, t_y\}$. Then we can find a $p_0 \in (0, p)$ such that $(1 - p_0) * (1 - p_0) > 1 - p$ and $p_0 \diamondsuit p_0 < p$. Define $U = \cup_{x \in S} B(x, p_0, p/2)$ and $V = \cup_{y \in K} B(y, p_0, p/2)$. Clearly U and V are open sets such that $S \subset U$ and $K \subset V$. Now, we claim that $U \cap V = \varnothing$. Let $z \in U \cap V$. Then there exist $x \in S$ and $y \in K$ such that $z \in B(x, p_0, p/2)$ and $z \in B(y, p_0, p/2)$. Therefore, we have

$$\begin{aligned} M(x, y, p) &\geq M(x, z, p/2) * M(y, z, p/2) \\ &\geq (1 - p_0) * (1 - p_0) > 1 - p \end{aligned}$$

and

$$\begin{aligned} N(x, y, p) &\leq N(x, z, p/2) \diamondsuit N(y, z, p/2) \\ &\leq p_0 \diamondsuit p_0 < p. \end{aligned}$$

Hence, $y \in B(x, p, p)$. But $B(x, p, p) \subset B(x, r_x, t_x)$, since $s < t_x, r_x$. Thus, $B(x, r_x, t_x) \cap K$ is nonempty which is a contradiction. Therefore, $U \cap V = \varnothing$. Hence, X is normal. ■

Remark 6 *From the above theorem, we can easily deduce that every metrizable space is normal. Since every intuitionistic fuzzy metric space is normal, Urysohn's lemma and Tietze extension theorem are true in the case of intuitionistic fuzzy metric spaces.*

Lemma 3 *Let $(X, M, N, *, \diamondsuit)$ be an IFM-space. If A is an open covering of X, then there is an open covering B of X such that B is a countably locally finite refinement of A.*

Proof. Since A is an open covering of X, by well ordering theorem, we can choose a well ordering $<$ for the collection A. Choose $n \in \mathbb{N}$ and for $U \in A$, define $S_n(U) = \{x \in X : B(x, 1/n, 1/n) \subset U\}$ and $R_n(U) = S_n(U) - \cup_{V<U} V$. If $V, W \in A$ with $V < W$ and if $x \in R_n(V), y \in R_n(W)$ then we claim that $M(x, y, 1/n) \leq 1 - 1/n$ and $N(x, y, 1/n) \geq 1/n$. Since $x \in R_n(V)$, we have $x \in S_n(V)$ and since $y \in R_n(W)$ and $V < W$, clearly y is not in V and hence $M(x, y, 1/n) \leq 1-1/n$ and $N(x, y, 1/n) \geq 1/n$. We can find a $s \in (0, 1/n)$ such that $(1 - s) * (1 - s) * (1 - s) > 1 - 1/n$ and $s \Diamond s \Diamond s < 1/n$.

Now, let $E_n(U) = \cup\{B(x, s, 1/3n) : x \in R_n(U)\}$. Clearly $E_n(U)$'s are open and we claim that $E_n(U)$'s are disjoint. Let $V, W \in A$ with $V < W$ and if $x \in E_n(V), y \in E_n(W)$ then we claim that $M(x, y, 1/3n) \leq 1-s$ and $N(x, y, 1/3n) \geq s$. In fact, assume that $M(x, y, 1/3n) \geq 1 - s$ and $N(x, y, 1/3n) \leq s$. Since $x \in E_n(V)$ and $y \in E_n(W)$, there exist $x_0 \in R_n(V)$ and $y_0 \in R_n(W)$ such that $M(x, x_0, 1/3n) \leq 1 - s$, $M(y_0, y, 1/3n) \leq 1 - s$, $N(x, x_0, 1/3n) \geq s$ and $N(y_0, y, 1/3n) \geq s$. Since $V < W$, we have $M(x_0, y_0 1/n) \leq 1 - 1/n$ and $N(x_0, y_0 1/n) \geq 1/n$. But

$$
\begin{aligned}
1 - 1/n &\geq M(x_0, y_0 1/n) \geq M(x, x_0, 1/3n) * M(x, y, 1/3n) * M(y, y_0, 1/3n) \\
&\geq (1 - s) * (1 - s) * (1 - s) \\
&> 1 - 1/n
\end{aligned}
$$

and

$$
\begin{aligned}
1/n &\leq N(x_0, y_0 1/n) \leq N(x, x_0, 1/3n) \Diamond N(x, y, 1/3n) \Diamond N(y, y_0, 1/3n) \\
&\leq s \Diamond s \Diamond s \\
&< 1/n
\end{aligned}
$$

which are contradictions and hence $M(x, y, 1/3n) \leq 1-s$ and $N(x, y, 1/3n) \geq s$.

Define $E_n = \{E_n(U) : U \in A\}$. If $y \in E_n(U)$, then there exists x in $R_n(U)$ such that $y \in B(x, s, 1/3n)$. But $s < 1/n$ and hence we have $y \in B(x, s, 1/3n) \subset y \in B(x, 1/n, 1/n) \subset U$. Since $E_n(U) \subset U$ for all $U \in A$, E_n refines A. We claim that E_n is locally finite. Since $s \in (0, 1)$, we can find a $r_0 \in (0, 1)$ such that $(1-r_0)*(1-r_0) > 1-s$ and $r_0 \Diamond r_0 < s$. For each $x \in X$, $B(x, r_0, 1/6n)$ intersects almost one element of E_n. If $B(x, r_0, 1/6n)$ intersect $E_n(U)$ and $E_n(V)$ with $U < V$, then there exist $y \in E_n(U)$ and $z \in E_n(V)$ such that $M(x, y, 1/6n) > 1-r_0$, $M(x, z, 1/6n) > 1-r_0$, $N(x, y, 1/6n) < r_0$ and $N(x, z, 1/6n) < r_0$. Since $U < V$, we have $M(y, z, 1/3n) > 1 - s$ and $N(y, z, 1/3n) < s$. Therefore, we have

$$
\begin{aligned}
M(y, z, 1/3n) &\geq M(x, y, 1/6n) * M(x, z, 1/6n) \\
&\geq (1 - r_0) * (1 - r_0) > 1 - s
\end{aligned}
$$

and

$$
\begin{aligned}
N(y, z, 1/3n) &\leq N(x, y, 1/6n) \Diamond N(x, z, 1/6n) \\
&\leq r_0 \Diamond r_0 < s
\end{aligned}
$$

which are contradictions and hence E_n is locally finite.

Now consider the family $B = \cup_{n\in\mathbb{N}} E_n$ and let $x \in X$. Since A is a covering of X, we can find a $U \in A$ such that U is the first element of A that contains x. Since U is open, we can choose $n \in \mathbb{N}$ such that $B(x, 1/n, 1/n) \subset U$. Hence $x \in S_n(U)$, but U is the first element of A that contains x, $x \in R_n(U)$ and hence $x \in E_n$. Thus, we get a family B of sets satisfying the required conditions. ■

Theorem 4 *Every intuitionistic fuzzy metric space has a basis that is countably locally finite.*

Proof. For a given $m \in \mathbb{N}$, define $A_m = \{B(x, 1/m, 1/m) : x \in X\}$, then A_m covers X for each m. By Lemma 3, we can find an open covering D_m of X which is a countably locally finite refinement of A_m. Let $D = \cup_{m\in\mathbb{N}} D_m$, then D is countably locally finite. We claim that D is a basis for X. Let $x \in X$. For given $r \in (0, 1)$ and $t > 0$, we can choose $m \in \mathbb{N}$ such that $(1-1/m)*(1-1/m) > 1-r$, $1/m\Diamond 1/m < r$ and $1/m < t/2$. If B is the element of D_m which contains x and since D_m refines A_m, then we can find a $x_0 \in X$ such that $B \subset B(x_0, 1/m, 1/m)$. Therefore, for any $y \in B$, we have

$$\begin{aligned} M(x, y, t) &> M(x, y, 2/m) \geq M(x, x_0, 1/m) * M(y, x_0, 1/m) \\ &\geq (1 - 1/m) * (1 - 1/m) > 1 - r \end{aligned}$$

and

$$\begin{aligned} N(x, y, t) &< N(x, y, 2/m) \leq N(x, x_0, 1/m)\Diamond N(y, x_0, 1/m) \\ &\leq 1/m\Diamond 1/m < r. \end{aligned}$$

Thus, $y \in B(x, y, t)$ and hence $B \subset B(x, y, t)$. ■

Remark 7 *Since the topology induced by a metric and the corresponding intuitionistic fuzzy metric are same, by using above theorem, we can deduce that every metric space has a basis that is countably locally finite.*

Corollary 5 *Every intuitionistic fuzzy metric space is metrizable.*

Since every intuitionistic fuzzy metric space is regular (Theorem 2) and since every intuitionistic fuzzy metric space has a basis that is countably locally finite, the result follows from the Nagata-Smirnov theorem [13].

Definition 5 *A sequence of maps $T_i : X \to X$ on an IFM-space $(X, M, N, *, \Diamond)$ converges uniformly to a map $T : X \to X$ if and only if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $k = k(\varepsilon, \lambda)$ such that*

$$M(Tx, T_i x, \varepsilon) > 1 - \lambda \text{ and } N(Tx, T_i x, \varepsilon) < \lambda$$

for every $x \in X$ and $i \geq k$.

Corollary 6 *Let X be any nonempty set and (Y, d) be a metric space. Let $(Y, M, N, *, \Diamond)$ be the induced intuitionistic fuzzy metric space. Then a sequence of functions $\{f_n\}$ from X to Y converges uniformly to a function f from X to Y with respect to the metric d if and only if $\{f_n\}$ converges uniformly to f with respect to the intuitionistic fuzzy metric (M, N).*

Proof. Assume that $\{f_n\}$ converges uniformly to f with respect to the intuitionistic fuzzy metric (M, N). Then, for given $r \in (0,1)$ and $t > 0$, there exists $k \in \mathbb{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ and $N(f_n(x), f(x), t) < r$ for all $n \geq k$. Let $\varepsilon > 0$ be given. Taking $r = \varepsilon/(t+\varepsilon)$, we have $M(f_n(x), f(x), t) > 1 - \varepsilon/(t+\varepsilon)$ and $N(f_n(x), f(x), t) < \varepsilon/(t+\varepsilon)$, and hence $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq k$. Therefore, $\{f_n\}$ converges uniformly to f with respect to the metric d. The converse part can also be proved in the same way. ■

Definition 6 *An orbit of T at a point $x_0 \in X$ is a sequence $\{x_n\}$ given by*

$$O(T, x_0) = \{x_n : x_n \in Tx_{n-1}, n = 1, 2, 3, ...\}.$$

An IFM-space $(X, M, N, *, \Diamond)$ is called T-orbitally complete if every Cauchy sequence which is a subsequence of an orbit T at each $x \in X$ converges to a point of X.

Definition 7 *A mapping T on IFM-space $(X, M, N, *, \Diamond)$ will be called a generalized contraction if and only if there exists a constant $q \in (0,1)$ such that for every $x, y \in X$,*

$$\begin{aligned} M(Tx, Ty, qt) \geq \; & \min\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), \\ & M(x, Ty, 2t), M(y, Tx, 2t)\}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} N(Tx, Ty, qt) \leq \; & \max\{N(x, y, t), N(x, Tx, t), N(y, Ty, t), \\ & N(x, Ty, 2t), N(y, Tx, 2t)\} \end{aligned} \tag{2.2}$$

for all $t > 0$.

Throughout this paper, $(X, M, N, *, \Diamond)$ will denote the IFM-space with the following conditions: for all $x, y \in X$ and $t > 0$

(IFM-7') $N(x, y, t) < 1$,
(IFM-11') $N(x, y, .) : (0, \infty) \to [0, 1)$ is continuous,
(IFM-12) $\lim_{t\to\infty} M(x, y, t) = 1$,
(IFM-13) $\lim_{t\to\infty} N(x, y, t) = 0$.

Lemma 7 *Let $\{x_n\}$ be a sequence in an IFM-space $(X, M, N, *, \Diamond)$ with the conditions (IFM-12) and (IFM-13). If there exists a constant $q \in (0,1)$ such that*

$$M(x_n, x_{n+1}, qt) \geq M(x_{n-1}, x_n, t) \tag{2.3}$$

and

$$N(x_n, x_{n+1}, qt) \leq N(x_{n-1}, x_n, t) \tag{2.4}$$

for all $t > 0$ and $n = 1, 2, 3, ...$ then $\{x_n\}$ is a Cauchy sequence in X.

Proof. To prove that $\{x_n\}$ is a Cauchy sequence, we prove (2.5) and (2.6) are true for all $n \geq n_0$ and for every $m \in \mathbb{N}$,

$$M(x_n, x_{n+m}, t) > 1 - \lambda \tag{2.5}$$

and

$$N(x_n, x_{n+m}, t) < \lambda. \tag{2.6}$$

Here we use induction method. From (2.3) and (2.4), we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, t/q) \\ &\geq M(x_{n-2}, x_{n-1}, t/q^2) \\ &\geq ... \geq M(x_0, x_1, t/q^n) \to 1 \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+1}, t) &\leq N(x_{n-1}, x_n, t/q) \\ &\leq N(x_{n-2}, x_{n-1}, t/q^2) \\ &\leq ... \leq N(x_0, x_1, t/q^n) \to 0 \end{aligned}$$

as $n \to \infty$, i.e. for $t > 0$ and $\lambda \in (0, 1)$, we can choose $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, t) > 1 - \lambda \text{ and } N(x_n, x_{n+1}, t) < \lambda.$$

Thus, (2.5) and (2.6) are true for $m = 1$. Suppose that (2.5) and (2.6) are true for m then we shall show that they are also true for $m + 1$.

Using the definition of intuitionistic fuzzy metric space, (2.3) with (2.5) and (2.4) with (2.6), we have

$$M(x_n, x_{n+m+1}, t) \geq \min\{M(x_n, x_{n+m}, t/2), M(x_{n+m}, x_{n+m+1}, t/2)\} > 1 - \lambda$$

and

$$N(x_n, x_{n+m+1}, t) \leq \max\{N(x_n, x_{n+m}, t/2), N(x_{n+m}, x_{n+m+1}, t/2)\} < \lambda.$$

Hence (2.5) and (2.6) are true for $m + 1$. This completes the proof. ■

Lemma 8 *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) \geq M(x, y, t)$ and $N(x, y, qt) \leq N(x, y, t)$ for $x, y \in X$. Then $x = y$.*

Proof. Since $M(x, y, qt) \geq M(x, y, t)$ and $N(x, y, qt) \leq N(x, y, t)$, then $M(x, y, t) \geq M(x, y, q^{-1}t)$ and $N(x, y, t) \leq N(x, y, q^{-1}t)$. By repeated application of above inequalities, we have

$$\begin{aligned} M(x, y, t) &\geq M(x, y, q^{-1}t) \geq M(x, y, q^{-2}t) \geq ... \geq M(x, y, q^{-n}t) \geq ...\text{and} \\ N(x, y, t) &\leq N(x, y, q^{-1}t) \leq N(x, y, q^{-2}t) \leq ... \leq N(x, y, q^{-n}t) \leq ..., \ n \in \mathbb{N} \end{aligned}$$

which $\to 1$ and $\to 0$ as $n \to \infty$, respectively. Thus $M(x, y, t) = 1$ and $N(x, y, t) = 0$ for all $t > 0$ and we get $x = y$. ■

Theorem 9 *Let $(X, M, N, *, \Diamond)$ be an IFM-space where $t*t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. If $T : X \to X$ is a generalized contraction on X and X is T-orbitally complete, then T has a unique fixed point $y \in X$ and $\lim_{n\to\infty} T^n x = y$ for every $x \in X$.*

Proof. Let x be an arbitrary point of X. Then we can construct a sequence $\{x_n\}$ in X as follows:

$$x_0 = x, x_1 = Tx_0, x_2 = Tx_1, ..., x_n = Tx_{n-1}, ... \tag{2.7}$$

We will show that the sequence is fundamental in X, i.e., for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there is an integer n_0 such that $n, m \geq n_0$ imply $M(x_n, x_m, \varepsilon) > 1-\lambda$ and $N(x_n, x_m, \varepsilon) < \lambda$.

Observe that, by (IFM-3) and (IFM-8), $x \neq y$ implies

$$M(x, y, qt) < M(x, y, t) \text{ and } N(x, y, qt) > N(x, y, t) \tag{2.8}$$

for some $t > 0$. Also (d) in Definition 1 and $t * t \geq t$ imply

$$M(x, z, t+s) \geq \min\{M(x, y, t), M(y, z, s)\} \tag{2.9}$$

and (d) in Definition 2 and $(1-t)\Diamond(1-t) \leq (1-t)$ imply

$$N(x, z, t+s) \leq \max\{N(x, y, t), N(y, z, s)\} \tag{2.10}$$

for all $x, y, z \in X$ and $s, t > 0$.

Suppose that in the sequence (2.7) $x_{n-1} \neq x_n$ for every integer n, since $x_{n-1} = x_n = Tx_{n-1}$ for some integer n implies immediately that (2.7) is fundamental. Then, for $x_{n-1}, x_n \in X$, by (2.1)

$$\begin{aligned} M(x_n, x_{n+1}, qt) &= M(Tx_{n-1}, Tx_n, qt) \geq \min\{M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), \\ &\quad M(x_n, x_{n+1}, t), M(x_{n-1}, x_{n+1}, 2t), M(x_n, x_n, 2t)\} \\ &= \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t), M(x_{n-1}, x_{n+1}, 2t)\} \end{aligned}$$

and since $M(x_{n-1}, x_{n+1}, 2t) \geq \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}$ it follows that

$$M(x_n, x_{n+1}, qt) \geq \min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}, \tag{2.11}$$

by (2.2)

$$\begin{aligned} N(x_n, x_{n+1}, qt) &= N(Tx_{n-1}, Tx_n, qt) \leq \max\{N(x_{n-1}, x_n, t), N(x_{n-1}, x_n, t), \\ &\quad N(x_n, x_{n+1}, t), N(x_{n-1}, x_{n+1}, 2t), N(x_n, x_n, 2t)\} \\ &= \max\{N(x_{n-1}, x_n, t), N(x_n, x_{n+1}, t), N(x_{n-1}, x_{n+1}, 2t)\}. \end{aligned}$$

and since $N(x_{n-1}, x_{n+1}, 2t) \leq \max\{N(x_{n-1}, x_n, t), N(x_n, x_{n+1}, t)\}$ it follows that

$$N(x_n, x_{n+1}, qt) \leq \max\{N(x_{n-1}, x_n, t), N(x_n, x_{n+1}, t)\} \tag{2.12}$$

for all $t > 0$.

Since we assume that $x_n \neq x_{n+1}$ for each integer n, (2.8) implies that

$$\begin{aligned} M(x_n, x_{n+1}, qt) &\geq M(x_n, x_{n+1}, t), \\ N(x_n, x_{n+1}, qt) &\leq N(x_n, x_{n+1}, t) \end{aligned}$$

which are impossible. Then,for each integer n, we have

$$M(x_n, x_{n+1}, qt) \geq M(x_{n-1}, x_n, t), \tag{2.13}$$
$$N(x_n, x_{n+1}, qt) \leq N(x_{n-1}, x_n, t). \tag{2.14}$$

By (2.13) and (2.14), for an arbitrary integer n, we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_{n-1}, x_n, t/q) \geq ... \geq M(x_0, x_1, t/q^n), \\ N(x_n, x_{n+1}, t) &\leq N(x_{n-1}, x_n, t/q) \leq ... \leq N(x_0, x_1, t/q^n). \end{aligned}$$

By noting that $M(x_0, x_1, t/q^n) \to 1$ and $N(x_0, x_1, t/q^n) \to 0$ as $n \to \infty$, using inequalities (2.11) and (2.12), we have

$$\begin{aligned} M(x_n, x_{n+1}, qt) &\geq M(x_{n-1}, x_n, t), \\ N(x_n, x_{n+1}, qt) &\leq N(x_{n-1}, x_n, t) \end{aligned}$$

for each integer n, $q \in (0, 1)$ and $t > 0$. Hence, by Lemma 7, $\{x_n\}$ is a Cauchy sequence in X. Since (2.7) is an orbit of T at $x \in X$ and X is T-orbitally complete, there is a point $y \in X$ such that $y = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x$.

Now, we prove that

$$Ty = \lim_{n\to\infty} x_{n+1} = y. \tag{2.15}$$

Let $B(Ty, \lambda, \varepsilon)$ be any neighborhood of Ty. Since $y = \lim_{n\to\infty} x_n$, there exists an integer k such that $n \geq k$ implies

$$M\left(x_n, y, \left(\frac{1-q}{2q}\right)\varepsilon\right) > 1-\lambda,\ N\left(x_n, y, \left(\frac{1-q}{2q}\varepsilon\right)\right) < \lambda, \tag{2.16}$$
$$M\left(x_n, x_{n+1}, \left(\frac{1-q}{2q}\varepsilon\right)\right) > 1-\lambda,\ N\left(x_n, x_{n+1}, \left(\frac{1-q}{2q}\varepsilon\right)\right) < \lambda. \tag{2.17}$$

Then, by (2.1) and (2.2), we have

$$\begin{aligned} M(x_{n+1}, Ty, \varepsilon) &= M(Tx_n, Ty, q\varepsilon/\varepsilon) \geq \min\{M(x_n, y, \varepsilon/q), M(x_n, x_{n+1}, \varepsilon/q), \\ &\quad M(y, Ty, \varepsilon/q), M(x_n, Ty, 2\varepsilon/q), M(y, x_{n+1}, 2\varepsilon/q)\}, \end{aligned}$$

$$\begin{aligned} N(x_{n+1}, Ty, \varepsilon) &= N(Tx_n, Ty, q\varepsilon/\varepsilon) \leq \max\{N(x_n, y, \varepsilon/q), N(x_n, x_{n+1}, \varepsilon/q), \\ &\quad N(y, Ty, \varepsilon/q), N(x_n, Ty, 2\varepsilon/q), N(y, x_{n+1}, 2\varepsilon/q)\} \end{aligned}$$

for all $t > 0$. Since by (2.9)

$$\begin{aligned} M\left(y, Ty, \frac{\varepsilon}{q}\right) &= M\left(y, Ty, \left(\frac{1-q}{2q} + \frac{1+q}{2q}\right)\varepsilon\right) \\ &\geq \min\left\{M\left(y, x_{n+1}, \left(\frac{1-q}{2q}\right)\varepsilon\right), M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right)\right\} \end{aligned}$$

and

$$M\left(x_n, Ty, \frac{2\varepsilon}{q}\right) \geq \min\left\{M\left(x_n, x_{n+1}, \frac{\varepsilon}{q}\right), M\left(x_{n+1}, Ty, \frac{\varepsilon}{q}\right)\right\},$$

we obtain, as M is nondecreasing, that

$$M(x_{n+1}, Ty, \varepsilon) \geq \min\left\{\begin{array}{c} M\left(x_n, y, \left(\frac{1-q}{2q}\right)\varepsilon\right), M\left(x_n, x_{n+1}, \left(\frac{1-q}{2q}\right)\varepsilon\right), \\ M\left(x_{n+1}, y, \left(\frac{1-q}{2q}\right)\varepsilon\right), M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right) \end{array}\right\}, \tag{2.18}$$

and also since by (2.10)

$$\begin{aligned} N\left(y, Ty, \frac{\varepsilon}{q}\right) &= N\left(y, Ty, \left(\frac{1-q}{2q} + \frac{1+q}{2q}\right)\varepsilon\right) \\ &\leq \max\left\{N\left(y, x_{n+1}, \left(\frac{1-q}{2q}\right)\varepsilon\right), N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right)\right\} \end{aligned}$$

and

$$N\left(x_n, Ty, \frac{2\varepsilon}{q}\right) \leq \max\left\{N\left(x_n, x_{n+1}, \frac{\varepsilon}{q}\right), N\left(x_{n+1}, Ty, \frac{\varepsilon}{q}\right)\right\},$$

we obtain, as N is nonincreasing, that

$$N(x_{n+1}, Ty, \varepsilon) \leq \max\left\{\begin{array}{c} N\left(x_n, y, \left(\frac{1-q}{2q}\right)\varepsilon\right), N\left(x_n, x_{n+1}, \left(\frac{1-q}{2q}\right)\varepsilon\right), \\ N\left(x_{n+1}, y, \left(\frac{1-q}{2q}\right)\varepsilon\right), N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right) \end{array}\right\} \tag{2.19}$$

Hence for all $n \geq k$, by (2.16) and (2.17), we have

$$M(x_{n+1}, Ty, \varepsilon) > 1 - \lambda \text{ and } N(x_{n+1}, Ty, \varepsilon) < \lambda \text{ or} \tag{2.20}$$

$$\begin{aligned} M(x_{n+1}, Ty, \varepsilon) &= M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right) \text{ and} \\ N(x_{n+1}, Ty, \varepsilon) &= N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right). \end{aligned} \tag{2.21}$$

Thus, we proved (2.15) if (2.20) is valued. If (2.20) were false, then substitutings in (2.18) and (2.19) ε by $\varepsilon_1 = \frac{1+q}{2q}\varepsilon > \varepsilon$, it would follow

$$\begin{aligned} M(x_{n+1}, Ty, \varepsilon_1) &= M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon_1\right) = M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^2\varepsilon\right), \\ N(x_{n+1}, Ty, \varepsilon_1) &= N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon_1\right) = N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^2\varepsilon\right). \end{aligned}$$

Hence

$$\begin{aligned} M(x_{n+1}, Ty, \varepsilon) &= M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right) = M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^2\varepsilon\right), \\ N(x_{n+1}, Ty, \varepsilon) &= N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right) = N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^2\varepsilon\right). \end{aligned}$$

Proceeding in this direction, we would obtain that

$$\begin{aligned} 1-\lambda &> M(x_{n+1}, Ty, \varepsilon) = ... = M\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^k\varepsilon\right) \to 1, \\ \lambda &< N(x_{n+1}, Ty, \varepsilon) = ... = N\left(x_{n+1}, Ty, \left(\frac{1+q}{2q}\right)^k\varepsilon\right) \to 0 \end{aligned}$$

as $k \to \infty$ which are contradictions. Therefore the inequalities (2.20) is correct, which imply (2.15). So, we conclude, there exists a fixed point for T.

To prove the uniqueness of the fixed point y in (2.15), suppose that $w \neq y$ and $Tw = w$. Then, by (2.1) and (2.2), we have

$$\begin{aligned} M(y,w,qt) &= M(Ty,Tw,qt) \geq \min\left\{ \begin{array}{c} M(y,w,t), M(y,Ty,t), M(w,Tw,t), \\ M(y,Tw,2t), M(w,Ty,2t) \end{array} \right\} \\ &= \min\{M(y,w,t), 1, 1, M(y,w,2t), M(w,y,2t)\} \\ &= M(y,w,t), \end{aligned}$$

$$\begin{aligned} N(y,w,qt) &= N(Ty,Tw,qt) \leq \max\left\{ \begin{array}{c} N(y,w,t), N(y,Ty,t), N(w,Tw,t), \\ N(y,Tw,2t), N(w,Ty,2t) \end{array} \right\} \\ &= \max\{N(y,w,t), 0, 0, N(y,w,2t), N(w,y,2t)\} \\ &= N(y,w,t) \end{aligned}$$

hence, by Lemma 8, we have $w = y$. This completes the proof. ■

Next, we prove Theorem 9 in T-orbitally complete metric space:

Corollary 10 *Let (X,d) be a metric space and let $T : X \to X$ be a mapping. If*

$$d(Tx,Ty) \leq q \max\left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \\ \left[d(x,Ty) + d(y,Tx)\right]/2 \end{array} \right\}$$

for some $q < 1$, all $x, y \in X$ and X is T-orbitally complete, then T has a unique fixed point $z \in X$ and $\lim_{n\to\infty} T^n x = z$.

Proof. The proof follows from Theorem 9 considering the induced IFM-space $(X, M, N, *, \Diamond)$, where $a*b = ab$, $a\Diamond b = \min\{1, a+b\}$, $M(x,y,t) = t/t+d(x,y)$ and $N(x,y,t) = d(x,y)/t+d(x,y)$. ■

Corollary 11 *Let $(X, M, N, *, \Diamond)$ be a complete IFM-space where $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. If $T : X \to X$ is a contraction on X, i.e., if for each $x, y \in X$*

$$M(Tx, Ty, qt) \geq M(x, y, t), N(Tx, Ty, qt) \leq N(x, y, t)$$

then there is a unique $z \in X$ such that $z = Tz$. Moreover, $T^n x \to z$ for every $z \in X$.

Proof. The proof is easy from Theorem 9. ■

Next, we prove Corollary 11 in complete fuzzy metric space:

Corollary 12 ([7]) *Let $(X, M, *,)$ be a complete fuzzy metric space where $t*t \geq t$ for all $t \in [0,1]$. If $T : X \to X$ is a contraction on X, i.e., if for each $x, y \in X$*

$$M(Tx, Ty, qt) \geq M(x, y, t)$$

then there is a unique $z \in X$ such that $z = Tz$.

Proof. In the view of Remark 2, the proof follows from Corollary 11. ■

We will denote subset I of X defined by

$$I = \{x \in X : \text{there is some } T_i \text{ such that } T_i x = x\}.$$

Theorem 13 *Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of maps on an IFM-space $(X, M, N, *, \Diamond)$ where $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. Let $T : X \to X$ be generalized contraction on X and X be T-orbitally complete. If each T_i has at least one fixed point y_i and if the sequence $\{T_i\}_{i\in\mathbb{N}}$ on the subset I converges uniformly to T, then the sequence $\{x_i\}_{i\in\mathbb{N}}$ converges to a unique fixed point y of T.*

Proof. By Theorem 9, the mapping T has a unique fixed point y. To show that $y = \lim_{i\to\infty} y_i$, let $B(y, \lambda, \varepsilon)$ be an arbitrary neighborhood of y. We will show that

$$M(y_i, y, \varepsilon) > 1 - \lambda \text{ and } N(y_i, y, \varepsilon) < \lambda$$

for almost all $i \in \mathbb{N}$. Since $\{T_i\}$ converges uniformly to T, there exists $k \in \mathbb{N}$ such that

$$M(Tx, T_i x, (1-q)\varepsilon/2) > 1 - \lambda \text{ and } N(Tx, T_i x, (1-q)\varepsilon/2) < \lambda \tag{2.22}$$

for $i \geq k$. For arbitrary $y_i \in X$ for which $T_i y_i = y_i$ we have by (2.9) and (2.10)

$$\begin{aligned} M(y_i, y, \varepsilon) &= M\left(y_i, y, \left(\frac{1-q}{2} + \frac{1+q}{2}\right)\varepsilon\right) \\ &\geq \min\left\{M\left(T_i y_i, T y_i, \left(\frac{1-q}{2}\right)\varepsilon\right), M\left(T y_i, y, \left(\frac{1+q}{2}\right)\varepsilon\right)\right\} \end{aligned} \tag{2.23}$$

Since T is a generalized contraction, $y = Ty, y_i = Ty_i, M$ is nondecreasing and N is nonincreasing, we have

$$M\left(Ty_i, y, \left(\frac{1+q}{2}\right)\varepsilon\right) = M\left(Ty_i, Ty, \left(q\frac{1+q}{2q}\right)\varepsilon\right)$$

$$\geq \min\left\{\begin{array}{c} M\left(y_i, y, \left(\frac{1+q}{2q}\right)\varepsilon\right), M\left(y_i, Ty_i, \left(\frac{1+q}{2q}\right)\varepsilon\right), M\left(y, Ty, \left(\frac{1+q}{2q}\right)\varepsilon\right), \\ M\left(y_i, Ty, \left(\frac{1+q}{q}\right)\varepsilon\right), M\left(y, Ty_i, \left(\frac{1+q}{q}\right)\varepsilon\right) \end{array}\right\}$$

$$\geq \min\left\{\begin{array}{c} M\left(y_i, y, \left(\frac{1+q}{2q}\right)\varepsilon\right), M\left(T_iy_i, Ty_i, \left(\frac{1+q}{2q}\right)\varepsilon\right), \\ M\left(y, Ty_i, \left(\frac{1+q}{q}\right)\varepsilon\right) \end{array}\right\}.$$

Using that

$$M\left(y, Ty_i, (1+q)/q\varepsilon\right) \geq \min\left\{M\left(y_i, y, \left(\frac{1+q}{2q}\right)\varepsilon\right), M\left(T_iy_i, Ty_i, \left(\frac{1+q}{2q}\right)\varepsilon\right)\right\}$$

we have

$$\begin{aligned} M\left(y, Ty_i, (1+q)/2\varepsilon\right) &\geq \min\left\{M\left(y_i, y, (1+q)/2q\varepsilon\right), M\left(T_iy_i, Ty_i, (1+q)/2q\varepsilon\right)\right\} \\ &\geq \min\left\{M\left(y_i, y, (1+q)/2q\varepsilon\right), M\left(T_iy_i, Ty_i, (1-q)/2q\varepsilon\right)\right\}. \end{aligned}$$

Then (2.23) results in

$$M(y_i, y, \varepsilon) \geq \min\left\{M\left(T_iy_i, Ty_i, (1-q)/2\varepsilon\right), M\left(Ty_i, y, (1+q)/2\varepsilon\right)\right\}.$$

Hence, by (2.22) we have

$$M(y_i, y, \varepsilon) > 1-\lambda \text{ and } N(y_i, y, \varepsilon) < \lambda \text{ or} \tag{2.24}$$

$$M(y_i, y, \varepsilon) = M\left(y_i, y, (1+q)/2q\varepsilon\right) \text{ and } N(y_i, y, \varepsilon) = N\left(y_i, y, (1+q)/2q\varepsilon\right) \tag{2.25}$$

for all $i \geq k$. The assertion of Theorem follows if (2.24) is valid. Since $M(y_i, y, \varepsilon) \leq 1-\lambda$ and $N(y_i, y, \varepsilon) \geq \lambda$ imply (as in the proof of Theorem 3)

$$\begin{aligned} M(y_i, y, \varepsilon) &= M\left(y_i, y, (1+q)/2q\varepsilon\right) \to 1 \text{ and} \\ N(y_i, y, \varepsilon) &= N\left(y_i, y, (1+q)/2q\varepsilon\right) \to 0 \end{aligned}$$

as $n \to \infty$, which is a contradiction, we see that (2.24) is correct. This completes the proof. ∎

Corollary 14 *Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of maps on an IFM-space $(X, M, N, *, \Diamond)$ such that $x_i = Tx_i$ for some $x_i \in X$ and let T_0 be a contraction mapping on X with a fixed point $x_0 \in X$. If $\{T_i\}$ converges uniformly to T_0, then the sequence $\{x_i\}$ converges to x_0.*

Proof. The proof follows easily from Theorem 13. ∎

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,285-297,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Some Theorems in Intuitionistic Fuzzy Metric Spaces

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September 17, 2006

Abstract

In this paper, we prove that the topology induced by any complete intuitionistic fuzzy metric space is completely metrizable. We also introduce the notions of t-uniformly continuity, t-equicontinuity and t-equinormality, and state new version of the Ascoli-Arzela theorem for intuitionistic fuzzy metric spaces.

Keywords. Topology, intuitionistic fuzzy metric space, t-uniform continuity,t-equicontinuity, t-equinormality, intuitionistic fuzzy contractive map, compact.

M.S.C. (2000). 54A40; 03E72; 54H25

1 Introduction

One of the main problems in the theory of fuzzy topological spaces is to obtain a consistent notion of a fuzzy metric space. Many authors have investigated this question and several different notions of a fuzzy metric space have been defined. In particular, George and Veeramani [5,6] introduced and studied an interesting notion of fuzzy metric space. Recently, using the idea of intuitionistic fuzzy sets introduced by Atanassov [1], Park [16] defined the notion of intuitionistic fuzzy metric space as a natural generalization of fuzzy metric space due to George and Veeramani [5], and proved that every intuitionistic fuzzy metric space generates a Hausdorff first countable topology. Saadati and Park [17] obtained a stronger result. In fact, they introduced a uniform structure on intuitionistic fuzzy metric space in the sense of Park [16] and proved (Lemma 1) that the topology generated by any intuitionistic fuzzy metric space is metrizable.

In this paper, we prove that if intuitionistic fuzzy metric space is complete, then the generated topology is completely metrizable. We introduce the concept of t-uniform continuity and uniform continuity, and discuss several properties of them. We observe that every t-uniformly continuous mapping is uniformly continuous but converse does not hold in general. Moreover, we show that every continuous mapping on a compact intuitionistic fuzzy metric space is t-uniformly continuous and, for each intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ such that every real valued continuous function is uniformly continuous, there is an intuitionistic fuzzy metric on X compatible with the topology generated by (M, N) for which every real valued continuous function is t-uniformly continuous. We also introduce the concept of t-equinormality to characterize intuitionistic fuzzy metric spaces for which every real valued continuous function is t-uniform continuous. We observe that a compact intuitionistic fuzzy metric space is separable and introduce the concept of t-equicontinuity. Using these results, we extend the Ascoli-Arzela theorem on intuitionistic fuzzy metric space. We also prove that every intuitionistic fuzzy metric space is normal and as a result of that Urysohn's lemma and Tietze extension theorem are true in the case of intuitionistic fuzzy metric space in the sense of Park [16].

2 Preliminaries

Definition 1 *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is continuous t-norm if* $*$ *is satisfying the following conditions: for all* $a, b, c, d \in [0,1]$

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Definition 2 *A binary operation* $\lozenge : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm if* $\lozenge$ *is satisfying the following conditions: for all* $a, b, c, d \in [0,1]$

(a) $\lozenge$ is commutative and associative;

(b) $\lozenge$ is continuous;

(c) $a \lozenge 0 = a$ for all $a \in [0,1]$;

(d) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$.

Remark 1 *The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [15] in his study of statistical metric spaces. Several examples for these concepts were proposed by many authors (see[8,10-14,16,18]).*

Remark 2 ([16])

(a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \Diamond r_2$.

(b) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_5 \geq r_7 \Diamond r_7$.

Definition 3 ([16]) *The 5-tuble $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,*

(a) $M(x, y, t) + N(x, y, t) \leq 1$;

(b) $M(x, y, t) > 0$;

(c) $M(x, y, t) = 1$ iff $x = y$;

(d) $M(x, y, t) = M(y, x, t)$;

(e) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;

(f) $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous;

(g) $N(x, y, t) > 0$;

(h) $N(x, y, t) = 0$ iff $x = y$;

(i) $N(x, y, t) = N(y, x, t)$;

(j) $N(x, z, t + s) \leq N(x, y, t) \Diamond N(y, z, s)$ for all $t, s > 0$;

(k) $N(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of nonnearness between x and y with respect to t, respectively.

Until now, $(X, M, N, *, \Diamond)$ denotes an intuitionistic fuzzy metric space with the following condition:

(l) $N(x, y, t) < 1$ for all $x, y \in X$ and $t > 0$.

Remark 3 *In a metric space (X, d) if we define $a*b = ab$, $a \Diamond b = \min\{1, a + b\}$, $M_d(x, y, t) = t/[t+d(x, y)]$ and $N_d(x, y, t) = d(x, y)/[t+d(x, y)]$ then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space [16]. We call this $(M_d,\ N_d)$ as the standard intuitionistic fuzzy metric induced by d. In the case that d is the Euclidean metric on $\mathbb{R}$, the induced intuitionistic fuzzy metric will be denoted by $(M_{|.|}, N_{|.|})$ and the corresponding intuitionistic fuzzy metric space, denoted $(\mathbb{R}, M_{|.|}, N_{|.|}, *, \Diamond)$, will be called the Euclidean intuitionistic fuzzy metric space. In [16], it is proved that every intuitionistic fuzzy metric (M, N) on X generates* *a topology $\tau_{(M,N)}$ on X which has as a base the family of open sets of the form $\{B(x,r,t) : x \in X, r \in (0,1), t > 0\}$, where $B(x,r,t) = \{y \in X : M(x,y,t) > 1-r, N(x,y,t) < r\}$ for every r, with $r \in (0,1)$, and $t > 0$. Clearly, this topology is Hausdorff and first countable. Moreover, if (X,d) is a metric space, then the topology generated by d coincides with the topology $\tau_{(M_d,N_d)}$ generated by the intuitionistic fuzzy metric (M_d, N_d).*

3 The Results

Definition 4 *A topological space (X,τ) is said to be admit a compatible intuitionistic fuzzy metric if there is an intuitionistic fuzzy metric (M,N) on X such that $\tau_{(M,N)} = \tau$.*

Lemma 1 ([17]) *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Then $(X, \tau_{(M,N)})$ is a metrizable topological space.*

Remark 4 *By results of Park [16] mentioned above, every metrizable topological space admits a compatible intuitionistic fuzzy metric. By results of Saadati and Park [17], it is easy to see that if $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space then $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity compatible with $\tau_{(M,N)}$, where $U_n = \{(x,y) \in X \times X : M(x,y,1/n) > 1-(1/n), N(x,y,1/n) < 1/n\}$ for all $n \in \mathbb{N}$. Thus, if $(X, M, N, *, \diamondsuit)$ is an intuitionistic fuzzy metric space then the topological space $(X, \tau_{(M,N)})$ is metrizable.*

Corollary 2 *A topological space is metrizable if and only if it admits a compatible intuitionistic fuzzy metric.*

Proof. Let (X,τ) be a metrizable topological space and d be a metric on X compatible with τ. Then, the intuitionistic fuzzy metric (M_d, N_d), induced by d, is compatible with τ [16]. The converse follows immediately from Lemma 1. ■

Corollary 3 ([16]) *Every separable intuitionistic fuzzy metric space is second countable.*

Proof. Let $(X, M, N, *, \diamondsuit)$ be a separable intuitionistic fuzzy metric space. By Lemma 1, $(X, \tau_{(M,N)})$ is a separable metrizable space. So, it is second countable. ■

Definition 5 ([16]) *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. A sequence $\{x_n\}$ in X converges to x in X if and only if $M(x_n, x, t)$ tends to 1 and $N(x_n, x, t)$ tends to 0 as n tends to ∞, for each $t > 0$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $\varepsilon, 0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$. An intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent. It is called compact, if every sequence in X contains a convergent subsequence.*

Let us recall that a metrizable topological space (X, τ) is said to be completely metrizable if it admits a complete metric [3]. On the other hand, an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is called complete [16] if every Cauchy sequence is convergent. If $(X, M, N, *, \Diamond)$ is a complete intuitionistic fuzzy metric space, we will say that (M, N) is a complete intuitionistic fuzzy metric on X.

Theorem 4 *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Then, $(X, \tau_{(M,N)})$ is completely metrizable.*

Proof. It follows from the proof of Lemma 1 that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X compatible with $\tau_{(M,N)}$, where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - (1/n), N(x, y, 1/n) < 1/n\}$ for every $n \in \mathbb{N}$. Then, there exists a metric d on X whose induced uniformity coincides with $\mathcal{U}$. To show that d is complete, we will prove that for given a Cauchy sequence (x_n) in (X, d), (x_n) is Cauchy sequence in $(X, M, N, *, \Diamond)$. For given $r \in (0, 1)$ and $t > 0$, choose a $k \in \mathbb{N}$ such that $1/k \leq \min\{r, t\}$. Then, there exists a $n_0 \in \mathbb{N}$ such that $(x_n, x_m) \in U_k$ for every $n, m \geq n_0$. Consequently,

$$M(x_n, x_m, t) \geq M(x_n, x_m, 1/k) > 1 - (1/k) \geq 1 - r$$

and

$$N(x_n, x_m, t) \leq N(x_n, x_m, 1/k) < (1/k) \leq r$$

for each $n, m \geq n_0$.

Hence, we have shown that (x_n) is Cauchy sequence in complete intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, so it is convergent with the respect to $\tau_{(M,N)}$. Thus, d is a complete metric on X. We conclude that $(X, \tau_{(M,N)})$ is completely metrizable. ■

Corollary 5 *A topological space is completely metrizable if and only if it admits a compatible complete intuitionistic fuzzy metric.*

Proof. Let (X, τ) be a completely metrizable space and d be a complete metric on X compatible with τ. Then, the intuitionistic fuzzy metric (M_d, N_d) induced by d is complete [16], and it is compatible with τ. The converse follows immediately from Theorem 4. ■

Since every completely metrizable space is a Baire space [3], we deduce from Theorem 4 the following.

Corollary 6 ([16]) *Every complete intuitionistic fuzzy metric space is a Baire space.*

Definition 6 *A mapping f from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ is said to be uniformly continuous if for each $\varepsilon \in (0, 1)$, there exist $r \in (0, 1)$ and $s > 0$ such that $M(x, y, s) > 1 - r$ and $N(x, y, s) < r$ imply $O(f(x), f(y), t) > 1 - \varepsilon$ and $P(f(x), f(y), t) < \varepsilon$, for all $x, y \in X$ and $t > 0$.*

Definition 7 *A mapping f from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ is said to be t - uniformly continuous if for each $\varepsilon \in (0,1)$, there exists $r \in (0,1)$ such that $M(x,y,t) > 1-r$ and $N(x,y,t) < r$ imply $O(f(x), f(y), t) > 1-\varepsilon$ and $P(f(x), f(y), t) < \varepsilon$, for all $x, y \in X$ and $t > 0$.*

Remark 5 *It is easy to see that every t-uniformly continuous mapping is uniformly continuous. In example below, we will show that the converse does not hold. It is also clear that every t-uniformly continuous mapping from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ is continuous from $(X, \tau_{(M,N)})$ to $(Y, \tau_{(O,P)})$. By a compact intuitionistic fuzzy metric space, we mean an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ such that $(X, \tau_{(M,N)})$ is a compact topological space. In [7-9], these notions were introduced and studied in fuzzy metric spaces.*

Example 1 *Let $X = \{1\} \cup \left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$, $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0,1]$. For each $x, y \in X$ and $t > 0$, define*

$$M(x,y,t) = \begin{cases} 1, & \text{if } x = y \\ txy, & \text{if } x \neq y \text{ and } t < 1 \\ xy, & \text{if } x \neq y \text{ and } t \geq 1 \end{cases}$$

and

$$N(x,y,t) = \begin{cases} 0, & \text{if } x = y \\ 1 - txy, & \text{if } x \neq y \text{ and } t < 1 \\ 1 - xy, & \text{if } x \neq y \text{ and } t \geq 1 \end{cases}$$

It is easy to check that $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space. Moreover, for each $x \in X$, $B(x, 1/2, 1/2) = \{x\}$, so $\tau_{(M,N)}$ is the discrete topology on X.

Now, let f be any continuous real valued function on X. Given $\varepsilon \in (0,1)$ and $t > 0$, choose $r = s = 1/2$. Then, $M(x,y,s) > 1-r$ and $N(x,y,s) < r$ if and only if $x = y$. Therefore, f is uniformly continuous from $(X, M, N, *, \Diamond)$ to $(\mathbb{R}, M_{|.|}, N_{|.|}, *, \Diamond)$.

Finally, let $f(1) = 1$ and $f(x) = 0$ for all $x \in X \backslash \{1\}$. Given $\varepsilon = 1/2$ and $t = 1$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} < r$ for each $r \in (0,1)$, so $M(1, 1 - \frac{1}{n+1}, t) > 1 - r$ and $N(1, 1 - \frac{1}{n+1}, t) < r$, but $M_{|.|}(f(1), f(1 - \frac{1}{n+1}), t) = M_{|.|}(1,0,1) = 1/2 = 1 - \varepsilon$ and $N_{|.|}(f(1), f(1 - \frac{1}{n+1}), t) = N_{|.|}(1,0,1) = 1/2 = \varepsilon$. We conclude that f is not t-uniformly continuous from $(X, M, N, *, \Diamond)$ to $(\mathbb{R}, M_{|.|}, N_{|.|}, *, \Diamond)$.

Proposition 7 *Every continuous mapping from a compact intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ is t-uniformly continuous.*

Proof. Suppose that there is a continuous mapping f from the compact intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ which is not t-uniformly continuous. Then there exist $\varepsilon \in (0, 1)$, and two sequences (x_n) and (y_n) in X such that $M(x_n, y_n, t) > 1 - 1/n$ and $N(x_n, y_n, t) < 1/n$ but $O(f(x_n), f(y_n), t) \leq 1-\varepsilon$ and $P(f(x_n), f(y_n), t) \geq \varepsilon$ for $t > 0$ and all $n \in \mathbb{N}$.

By compactness of $(X, \tau_{(M,N)})$, there are subsequences (x_{n_k}) and (y_{n_k}) of (x_n) and (y_n) respectively, and points $x, y \in X$ such that $x_{n_k} \to x$ and $y_{n_k} \to y$ in $(X, \tau_{(M,N)})$. Since

$$M(x, y, 3t) \geq M(x, x_{n_k}, t) * M(x_{n_k}, y_{n_k}, t) * M(y_{n_k}, y, t)$$

and

$$N(x, y, 3t) \leq N(x, x_{n_k}, t) \Diamond N(x_{n_k}, y_{n_k}, t) \Diamond N(y_{n_k}, y, t),$$

those immediately follow that $M(x, y, 3t) = 1$ and $N(x, y, 3t) = 0$, so $x = y$. Hence, by continuity of f, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(y)$ in $(Y, \tau_{(O,P)})$. Choose $\delta > 0$ such that $(1-\delta) \star (1-\delta) > 1-\varepsilon$ and $\delta \triangle \delta < \varepsilon$. Then, there is $k_0 \in \mathbb{N}$ such that $O(f(x), f(x_{n_k}), t/2) > 1-\delta$, $O(f(x), f(y_{n_k}), t/2) > 1-\delta$, $P(f(x), f(x_{n_k}), t/2) < \delta$ and $P(f(x), f(y_{n_k}), t/2) < \delta$ for all $k \geq k_0$. So

$$\begin{aligned} O(f(x_{n_k}), f(y_{n_k}), t) &\geq O(f(x_{n_k}), f(x), t/2) \star O(f(x), f(y_{n_k}), t/2) \\ &\geq (1-\delta) \star (1-\delta) > 1-\varepsilon \end{aligned}$$

and

$$\begin{aligned} P(f(x_{n_k}), f(y_{n_k}), t) &\leq P(f(x_{n_k}), f(x), t/2) \triangle P(f(x), f(y_{n_k}), t/2) \\ &\leq \delta \triangle \delta < \varepsilon \end{aligned}$$

for all $k \geq k_0$, which are contradictions. We conclude that f is t-uniformly continuous. ■

Remark 6 *Let f be a t-uniformly continuous mapping from the intuitionistic fuzzy metric space X to the intuitionistic fuzzy metric space Y. If (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is also a Cauchy sequence in Y.*

Next, we will characterize intuitionistic fuzzy metric spaces for which real valued continuous functions are t-uniformly continuous.

Definition 8 *An intuitionistic fuzzy metric (M, N) on a set X is said to be t-equinormal if for each pair of disjoint nonempty closed subsets A and B of $(X, \tau_{(M,N)})$ and each $t > 0$,* $\sup\{M(a, b, t) : a \in A, b \in B\} < 1$ *and* $\inf\{N(a, b, t) : a \in A, b \in B\} > 0$.

Theorem 8 *For an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, the followings are equivalent.*

(a) For each intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$, any continuous mapping from $(X, \tau_{(M,N)})$ to $(Y, \tau_{(O,P)})$ is t-uniformly continuous as a mapping from $(X, M, N, *, \Diamond)$ to $(Y, O, P, \star, \triangle)$.

(b) Any real valued continuous function on $(X, \tau_{(M,N)})$ is t-uniformly continuous from $(X, M, N, *, \Diamond)$ to the Euclidean intuitionistic fuzzy metric space $(\mathbb{R}, M_{|.|}, N_{|.|}, *, \Diamond)$.

(c) The intuitionistic fuzzy metric (M, N) is t-equinormal.

Proof. (a) $\Longrightarrow$ (b). Obvious.

(b) $\Longrightarrow$ (c). Let A and B be two disjoint nonempty closed subsets of $(X, \tau_{(M,N)})$ and fix $t > 0$. Let $f : X \to [0, 1]$ be a continuous function such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$. Put $\varepsilon = 1/(t + 1)$. By assumption, there is $r \in (0, 1)$ such that

$$\frac{t}{t + |f(x) - f(y)|} > 1 - \varepsilon \text{ and } \frac{|f(x) - f(y)|}{t + |f(x) - f(y)|} < \varepsilon$$

whenever $M(x, y, t) > 1-r$ and $N(x, y, t) < r$. Since we have $t/(t+|f(a) - f(b)|) = t/(t + 1) = 1 - \varepsilon$ and $|f(a) - f(b)| / (t + |f(a) - f(b)|) = 1/(t + 1) = \varepsilon$, it follow that $M(a, b, t) \leq 1 - r$ and $N(a, b, t) \geq r$ for all $a \in A$ and $b \in B$. We conclude that (M, N) is a t-equinormal intuitionistic fuzzy metric on X.

(c) $\Longrightarrow$ (a). Suppose that there is an intuitionistic fuzzy metric space $(Y, O, P, \star, \triangle)$ and a continuous mapping f from $(X, \tau_{(M,N)})$ to $(Y, \tau_{(O,P)})$ which is not t-uniformly continuous. Then, there exist $\varepsilon \in (0, 1)$, and two sequences (a_n) and (b_n) such that $M(a_n, b_n, t) > 1 - 1/n$ and $N(a_n, b_n, t) < 1/n$ but $O(f(a_n), f(b_n), t) \leq 1 - \varepsilon$ and $P(f(a_n), f(b_n), t) \geq \varepsilon$ for $t > 0$ and all $n \in \mathbb{N}$. We distinguish two cases:

Case I. The sequence (a_n) has a convergent subsequence (a_{n_k}) with limit $a \in X$. If the sequence (b_{n_k}) has a cluster point $b \in X$, then as in the proof of Proposition 7, we obtain that $M(a, b, 3t) = 1$ and $N(a, b, 3t) = 0$, so $a = b$, and by continuity of f, $O(f(a_{n_k}), f(b_{n_k}), t) > 1-\varepsilon$ and $P(f(a_{n_k}), f(b_{n_k}), t) < \varepsilon$ which provide contradictions. Otherwise, without loss of generality, we may suppose that $\{a\} \cup \{a_{n_k} : k \in \mathbb{N}\}$ and $\{b_{n_k} : k \in \mathbb{N}\}$ are disjoint closed subsets of X. But $\sup \{M(a_{n_k}, b_{n_k}, t) : k \in \mathbb{N}\} = 1$ and $\inf \{N(a_{n_k}, b_{n_k}, t) : k \in \mathbb{N}\} = 0$ which provide contradictions.

Case II. The sequence (a_n) has no convergent subsequence. Then, without loss of generality, we may suppose that $\{a_{n_k} : k \in \mathbb{N}\}$ and $cl \{b_{n_k} : k \in \mathbb{N}\}$ are disjoint closed subsets of X. Thus, we obtain a contradiction again. This completes the proof. ■

Let us recall that a metric d on a set X is equinormal ([4]) if for each pair of disjoint nonempty closed subsets A and B of X, $d(A, B) > 0$.

Proposition 9 *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space for which every real valued continuous function is uniformly continuous from $(X, M, N, *, \Diamond)$ to the Euclidean intuitionistic fuzzy metric space $(\mathbb{R}, M_{|.|}, N_{|.|}, *, \Diamond)$. Then, there is an intuitionistic fuzzy metric (O, P) on X compatible with $\tau_{(M,N)}$ for which every real valued continuous function on $(X, \tau_{(O,P)})$ is t-uniformly continuous from $(X, O, P, \star, \triangle)$ to the Euclidean intuitionistic fuzzy metric space.*

Proof. The set $\{U_n : n \in \mathbb{N}\}$ is a countable base for a uniformity $\mathcal{U}$ on X compatible with $\tau_{(M,N)}$, where $U_n = \{(x, y) : M(x, y, 1/n) > 1 - (1/n), N(x, y, 1/n) < 1/n\}$ for all $n \in \mathbb{N}$. Let d be any metric on X whose induced uniformity coincides with $\mathcal{U}$. Obviously, the topology τ_d generated by d coincides with $\tau_{(M,N)}$. Since every real valued continuous function on the metric space (X, d) is uniformly continuous, d is an equinormal metric on X (see [2,4]). Now, let (M_d, N_d) be the intuitionistic fuzzy metric induced by d, where $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$. It is clear that (M_d, N_d) is compatible with $\tau_{(M,N)}$. Now, let A and B be two disjoint nonempty closed subsets of X and $t > 0$. By equinormality of d, there exists $\delta > 0$ such that $d(a, b) \geq \delta$ for all $a \in A$ and $b \in B$. Therefore, for each $a \in A$ and $b \in B$, we have $M_d(a, b, t) = t/(t + d(a, b)) \leq t/(t + \delta)$ and $N_d(a, b, t) = d(a, b)/(t + d(a, b)) \geq \delta/(t + \delta)$. Hence, $\sup\{M_d(a, b, t) : a \in A, b \in B\} < 1$ and $\inf\{N_d(a, b, t) : a \in A, b \in B\} > 0$. So, by Theorem 8, every real valued continuous function on $(X, \tau_{(M_d,N_d)})$ is t-uniformly continuous from $(X, M_d, N_d, *, \Diamond)$ to the Euclidean intuitionistic fuzzy metric space. ■

Proposition 10 *Every compact intuitionistic fuzzy metric space is separable.*

Proof. Let $(X, M, N, *, \Diamond)$ be any compact intuitionistic fuzzy metric space. Then, for given $r \in (0, 1)$ and $t > 0$, we can find $x_1, x_2, ..., x_n$ in X such that $X = \cup_{i=1}^{n} B(x_i, r, t)$. In particular, for each $n \in \mathbb{N}$, we can find a finite set A_n such that $X = \cup_{n \in A_n} B(a, 1/n, 1/n)$. Let $A = \cup_{i=1}^{\infty} A_n$, then A is countable. We claim that $X \subset A$. Let $x \in X$, then for each n, there exists $a_n \in A_n$ such that $x \in B(a_n, 1/n, 1/n)$. Thus, a_n converges to x but $a_n \in A$ for all n and hence $x \in \overline{A}$. Therefore, A is dense in X and X is separable. ■

Definition 9 ([16]) *Let X be any nonempty set and $(Y, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is called converge uniformly to a function f from X to Y if given $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ and $N(f_n(x), f(x), t) < r$ for all $n \geq n_0$ and $x \in X$.*

Definition 10 *A family of mappings F from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to an intuitionistic fuzzy metric space $(X, O, P, \star, \triangle)$ is said to be t-equicontinuous if for each $r \in (0, 1)$ and $t > 0$, there exists $r_0 \in (0, 1)$ such that $M(x, y, t) > 1 - r_0$ and $N(x, y, t) < r_0$ imply $O(f(x), f(y), t) > 1 - r$ and $P(f(x), f(y), t) < r$ for all $f \in F$.*

Lemma 11 *Let $\{f_n\}$ be an t-equicontinuous sequence of mappings from an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ to a complete intuitionistic fuzzy metric space $(X, O, P, \star, \triangle)$. If $\{f_n\}$ converges for each point of a dense subset A of X, then $\{f_n\}$ converges at each point of X and the limit function is continuous.*

Proof. Let $s \in (0, 1)$ and $t > 0$. Then, we can find a $r \in (0, 1)$ such that $(1 - r) \star (1 - r) \star (1 - r) > 1 - s$ and $r \triangle r \triangle r < s$. Since $F = \{f_n\}$ is a t-equicontinuous family, for $r \in (0, 1)$ and $t > 0$, we can find $r_1 \in (0, 1)$ such that

$M(x,y,t/3) > 1-r_1$ and $N(x,y,t/3) < r_1$ imply $O(f_n(x), f_n(y), t/3) > 1-r$ and $P(f_n(x), f_n(y), t/3) < r$ for all $f_n \in F$. Since A is dense in X, there exists $y \in B(x, r_1, t) \cap A$ and $(f_n(y))$ converges for that y. This $(f_n(y))$ being a Cauchy sequence, for $r \in (0,1)$ and $t > 0$, we can find a $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $O(f_n(y), f_m(y), t/3) > 1-r$ and $P(f_n(y), f_m(y), t/3) < r$. Therefore, for any $x \in X$, we have

$$\begin{aligned} O(f_n(x), f_m(x), t) &\geq O(f_n(x), f_n(y), t/3) \star O(f_n(y), f_m(y), t/3) \\ &\quad \star O(f_m(x), f_m(y), t/3) \\ &\geq (1-r) \star (1-r) \star (1-r) \\ &> 1-s \end{aligned}$$

and

$$\begin{aligned} P(f_n(x), f_m(x), t) &\leq P(f_n(x), f_n(y), t/3) \vartriangle P(f_n(y), f_m(y), t/3) \\ &\vartriangle P(f_m(x), f_m(y), t/3) \\ &\leq r \vartriangle r \vartriangle r \\ &< s. \end{aligned}$$

Thus, $(f_n(x))$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges. Define $f(x) = \lim_{n\to\infty} f_n(x)$. We claim that f is continuous. Let $s_0 \in (0,1)$ be given. Then, we can find a $r_0 \in (0,1)$ such that $(1-r_0) \star (1-r_0) \star (1-r_0) > 1-s_0$ and $r_0 \vartriangle r_0 \vartriangle r_0 < s_0$. Since F is t-equicontinuous, for $r_0 \in (0,1)$ we can find $r_1 \in (0,1)$ such that $M(x,y,t/3) > 1-r_1$ and $N(x,y,t/3) < r_1$ imply $O(f_n(x), f_n(y), t/3) > 1-r_0$ and $P(f_n(x), f_n(y), t/3) < r_0$ for all $f_n \in F$. Since $f_n(x)$ converges to $f(x)$, for $r_0 \in (0,1)$ we can find an $i \in \mathbb{N}$ such that $O(f_n(x), f(x), t/3) > 1-r_0$ and $P(f_n(x), f(x), t/3) < r_0$ for all $n \geq i$. Since $f_n(y)$ converges to $f(y)$, for $r_0 \in (0,1)$ we can find a $j \in \mathbb{N}$ such that $O(f_n(x), f(y), t/3) > 1-r_0$ and $P(f_n(x), f(y), t/3) < r_0$ for all $n \geq j$. Hence, for all $n \geq \max\{i,j\}$, we have

$$\begin{aligned} O(f(x), f(y), t) &\geq O(f(x), f_n(x), t/3) \star O(f_n(x), f_n(y), t/3) \\ &\quad \star O(f_n(y), f(y), t/3) \\ &\geq (1-r_0) \star (1-r_0) \star (1-r_0) \\ &> 1-s_0 \end{aligned}$$

and

$$\begin{aligned} P(f(x), f(y), t) &\leq P(f(x), f_n(x), t/3) \vartriangle P(f_n(x), f_n(y), t/3) \\ &\vartriangle P(f_n(y), f(y), t/3) \\ &\leq r_0 \vartriangle r_0 \vartriangle r_0 \\ &< s_0. \end{aligned}$$

Hence, f is continuous. ■

Theorem 12 (Intuitionistic fuzzy Ascoli-Arzela theorem) *Let $(X, M, N, *, \Diamond)$ be a compact intuitionistic fuzzy metric space and $(Y, O, P, \star, \triangle)$ be a complete intuitionistic fuzzy metric space. Let F be a t-equicontinuous family of functions from X to Y. Let (f_n) be a sequence in F such that $cl\{f_n(x) : n = 1, 2, ...\}$ is a compact subset of Y for each $x \in X$. Then, there exists a continuous function f from X to Y and a subsequence (g_n) of (f_n) such that g_n converges uniformly to f on X.*

Proof. Since X is a compact intuitionistic fuzzy metric space, by Proposition 10, X is separable. Let $A = \{x_1, x_2, ...\}$ be a countable dense subset of X. Hence, for each i, $cl\,\{f_n(x_i) : n = 1, 2, ...\}$ is a compact subset of Y. Since every intuitionistic fuzzy metric space is first countable [16], every compact subset of Y is sequentially compact. Therefore, we get a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i, i = 1, 2, ...$. By Lemma 11, we get a continuous function f from X to Y such that $g_n(x)$ converges to $f(x)$ for all x in X. We claim that g_n converges uniformly for f on X. Let $s \in (0, 1)$ and $t > 0$ be given, then we can find a $r \in (0, 1)$ such that $(1-r) \star (1-r) \star (1-r) > 1-s$ and $r \triangle r \triangle r < s$. Since F is t-equicontinuous, we can find $r_1 \in (0, 1)$ such that $M(x, y, t/3) > 1 - r_1$ and $N(x, y, t/3) < r_1$ imply $O(g_n(x), g_n(y), t/3) > 1 - r$ and $P(g_n(x), g_n(y), t/3) < r$ for all n. Since X is compact, by Proposition 7, f is t-uniformly continuous. Hence, for $r \in (0, 1)$, we can find $r_2 \in (0, 1)$ such that $M(x, y, t/3) > 1-r_2$ and $N(x, y, t/3) < r_2$ imply $O(f(x), f(y), t/3) > 1-r$ and $P(f(x), f(y), t/3) < r$. Let $r_0 = \min\{r_1, r_2\}$. Since X is compact and A is dense in X, $X = \cup_{i=1}^{k} B(x_i, r_0, t/3)$ for some finite k. If $x \in X$, then we can find $i, 1 \leq i \leq k$, such that $M(x, x_i, t/3) > 1 - r_0$ and $N(x, x_i, t/3) < r_0$. Since $r_0 = \min\{r_1, r_2\}$, by the t-equicontinuity of F, we have $O(g_n(x), g_n(x_i), t/3) > 1-r$ and $P(g_n(x), g_n(x_i), t/3) < r$, and by the uniform continuity of f, we have $O(f(x_i), f(x), t/3) > 1 - r$ and $P(f(x_i), f(x), t/3) < r$. Since $g_n(x_j)$ converges to $f(x_j)$, we can find a $n_0 \in \mathbb{N}$ such that $O(g_n(x_j), f(x_j), t/3) > 1 - r$ and $P(g_n(x_j), f(x_j), t/3) < r$ for all $n \geq n_0$ and $1 \leq j \leq k$. Thus, for all $x \in X$, we have

$$\begin{aligned} O(g_n(x), f(x), t) &\geq O(g_n(x), g_n(x_i), t/3) \star O(g_n(x_i), f(x_i), t/3) \\ &\quad \star O(f(x_i), f(x), t/3) \\ &\geq (1-r) \star (1-r) \star (1-r) \\ &> 1-s \end{aligned}$$

and

$$\begin{aligned} P(g_n(x), f(x), t) &\leq P(g_n(x), g_n(x_i), t/3) \triangle P(g_n(x_i), f(x_i), t/3) \\ &\triangle \; P(f(x_i), f(x), t/3) \\ &\leq r \triangle r \triangle r \\ &< s. \end{aligned}$$

Hence, g_n converges uniformly to f on X. ■

Theorem 13 *Every intuitionistic fuzzy metric space is normal.*

Proof. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space, and S, K be two disjoint closed sets in X. Let $x \in S$ then $x \in K^c$. Since K^c is open, there exist $r_x \in (0, 1)$ and $t_x > 0$ such that $B(x, r_x, t_x) \cap K = \varnothing$ for all $x \in S$. Similarly, there exist $r_y \in (0, 1)$ and $t_y > 0$ such that $B(x, r_y, t_y) \cap S = \varnothing$ for all $x \in K$. Let $s = \min\{r_x, t_x, r_y, t_y\}$. Then we can find a $p_0 \in (0, p)$ such that $(1 - p_0) * (1 - p_0) > 1 - p$ and $p_0 \Diamond p_0 < p$. Define $U = \cup_{x \in S} B(x, p_0, p/2)$ and $V = \cup_{y \in K} B(y, p_0, p/2)$. Clearly U and V are open sets such that $S \subset U$ and $K \subset V$. Now, we claim that $U \cap V = \varnothing$. Let $z \in U \cap V$. Then there exist $x \in S$ and $y \in K$ such that $z \in B(x, p_0, p/2)$ and $z \in B(y, p_0, p/2)$. Therefore, we have

$$\begin{aligned} M(x, y, p) &\geq M(x, z, p/2) * M(y, z, p/2) \geq (1 - p_0) * (1 - p_0) \\ &> 1 - p \end{aligned}$$

and

$$\begin{aligned} N(x, y, p) &\leq N(x, z, p/2) \Diamond N(y, z, p/2) \leq p_0 \Diamond p_0 \\ &< p. \end{aligned}$$

Hence, $y \in B(x, p, p)$. But $B(x, p, p) \subset B(x, r_x, t_x)$, since $s < t_x, r_x$. Thus, $B(x, r_x, t_x) \cap K$ is nonempty which is a contradiction. Therefore, $U \cap V = \varnothing$. Hence, X is normal. ■

Remark 7 *From the above theorem, we can easily deduce that every metrizable space is normal. Since every intuitionistic fuzzy metric space is normal, Urysohn's lemma and Tietze extension theorem are true in the case of intuitionistic fuzzy metric spaces.*

4 Conclusions

In this work, we have shown that the topology generated by any complete intuitionistic fuzzy metric space is completely metrizable and if the intuitionistic fuzzy metric space is complete then the generated topology is completely metrizable. We also introduce the concepts of t-uniform continuity, uniform continuity, t-equinormality and t-equicontinuity, and discuss their properties to extend and Ascoli-Arzela theorem on intuitionistic fuzzy metric spaces. During this work, we think some arised natural questions: firstly, if every open cover in an intuitionistic fuzzy metric space X admits a countably locally finite refinement which covers X, then ,in other way, one could show that every intuitionistic fuzzy metric space has a countably locally finite basis which implies every intuitionistic fuzzy metric space is also metrizable. Secondly, in [17] it is proved that if a compact structure on an intuitionistic fuzzy metric space implies its precompactedness and completedness, then one could show that the Niemytzki-Tychonoff theorem is also true in the case of intuitionistic fuzzy metric spaces.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,299-313,COPYRIGHT 2008 EUDOXUS PRESS, LLC

A general sampling theory in the functional Hilbert space induced by a Hilbert space valued kernel

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Abstract

Let $\mathbb{H}$ be a separable Hilbert space and Ω a fixed subset of $\mathbb{R}$. Consider an $\mathbb{H}$-valued function $K : \Omega \longrightarrow \mathbb{H}$ and $x \in \mathbb{H}$. Then, the function $f_x : \Omega \longrightarrow \mathbb{C}$ given by $f_x(t) := \langle x, K(t) \rangle_{\mathbb{H}}$ is well-defined. Denote by $\mathcal{H}_K$ the set of functions obtained in this way. Although a variety of sampling results for $\mathcal{H}_K$ is known in the literature, there exist some simple examples where they do not apply because the implicit interpolation condition that appears does not adjust the former pattern we need. The main aim of this paper is to obtain a more general sampling result including most of these special cases. In this way, the concept of interpolation condition is redefined and we study how to combine two of them in order to obtain a new sampling result. Some examples are obtained in this new framework.

Keywords: Reproducing kernel Hilbert spaces; Sampling formulas; Biorthonormal Riesz bases.
AMS: 94A20; 44A05; 46E22.

1 Statement of the problem

For the past few years a significant mathematical literature on the topic of sampling theorems associated with differential or difference problems has flourished [1, 4, 5, 6, 7, 8]. See also [15] and the references therein. In turn, we might consider the Weiss-Kramer sampling theorem as the *leitmotiv* of all these results [11, 13]. Roughly speaking, the common situation for these sampling problems is the following:

Let f be a function defined on $\mathbb{C}$ by $f(t) = \int_I F(x)\, K(x,t)\, dx$, $F \in L^2(I)$, (or $f(t) = \sum_n F(n)\, K(n,t)$, $F \in \ell^2$). The kernel K, which belongs to $L^2(I)$ (or ℓ^2) for each fixed $t \in \mathbb{C}$, satisfies the differential (difference) equation appearing in a differential (difference) problem (P) which has the sequence of eigenvalues $\{t_n\}$. Moreover, whenever we substitute in K the spectral parameter t by $\{t_n\}$ we obtain the sequence of orthogonal eigenfunctions associated with (P) which constitutes an orthogonal basis for $L^2(I)$ (ℓ^2). Under these circumstances,

f is an entire function which can be recovered from its samples $\{f(t_n)\}$ by means of a sampling formula $f(t) = \sum_n f(t_n)\, S_n(t)$, where the sampling functions $\{S_n\}$ are given by $S_n(t) = \|K(\cdot, t_n)\|^{-2} \langle K(\cdot, t), K(\cdot, t_n)\rangle$ (the inner product in $L^2(I)$ or ℓ^2).

All the results above can be formulated in an abstract way following the approach in Saitoh's book [12]. Namely, let $\mathbb{H}$ be a separable Hilbert space, and Ω a fixed subset of $\mathbb{R}$. Given an $\mathbb{H}$-valued function $K : \Omega \to \mathbb{H}$, for $x \in \mathbb{H}$, the function $f(t) := \langle x, K(t)\rangle_{\mathbb{H}}$ is well-defined as a function $f : \Omega \to \mathbb{C}$. We denote by $\mathcal{H}_K$ the set of functions obtained in this way and by T the linear transform

$$\begin{array}{rccc} T: & \mathbb{H} & \longrightarrow & \mathcal{H}_K \\ & x & \longmapsto & f \end{array} \tag{1}$$

Hereafter we refer the function K as the *kernel* of the transform T. Note that the continuity of the kernel K implies that the functions in $\mathcal{H}_K$ are continuous in Ω, a natural setting for sampling purposes. If we define in $\mathcal{H}_K$ the norm $\|f\|_{\mathcal{H}_K} = \inf\{\|x\|_{\mathbb{H}} : f = T(x)\}$ we obtain a reproducing kernel Hilbert space (RKHS hereafter) whose reproducing kernel is given by $k(t, s) := \langle K(s), K(t)\rangle_{\mathbb{H}}$ i.e., for each $s \in \Omega$ the function k_s defined as $k_s(t) := k(t, s)$ belongs to $\mathcal{H}_K$, and the reproducing property

$$f(s) = \langle f, k_s\rangle_{\mathcal{H}_K} = \langle f, k(\cdot, s)\rangle_{\mathcal{H}_K}\,, \quad s \in \Omega\,, \quad f \in \mathcal{H}_K \tag{2}$$

holds. Recall that the Moore-Aronszajn procedure [2] leads to the same RKHS via the *positive definite (or positive matrix)* function k. Under these circumstances it is known that the linear operator T is one-to-one if and only if T is an isometry between $\mathbb{H}$ and $\mathcal{H}_K$, or, equivalently, if and only if the set of functions $\{K(t)\}_{t\in\Omega}$ is complete in $\mathbb{H}$ [12]. An important property of $\mathcal{H}_K$ is that convergence in its norm implies pointwise convergence. In fact, by the reproducing property, we have that

$$|f(t) - f_n(t)| = |\big\langle f - f_n, k(\cdot, t)\big\rangle| \le \|f - f_n\|_{\mathcal{H}_K} \|K(t)\|_{\mathbb{H}}\,.$$

Notice that the last inequality also implies uniform convergence in subsets of Ω where the function $k(t, t) = \|K(t)\|^2$ is bounded. The RKHS $\mathcal{H}_K$ has been largely studied in the mathematical literature (see the superb monograph [12] and references therein).

A sampling result for $\mathcal{H}_K$ can be easily established (see [9] or [10]). Namely, let $\{x_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty$ be a pair of biorthonormal Riesz bases for a Hilbert space $\mathbb{H}$. Assume that, for each fixed $t \in \Omega$, $K(t)$ can be written as $K(t) = \sum_{n=1}^\infty \overline{S_n(t)}\, x_n^*$, where the functions $S_n \in \mathcal{H}_K$ satisfy, for some fixed sequence $\{t_n\}_{n=1}^\infty$ in Ω, the interpolation property: $S_n(t_m) = a_n \delta_{n,m}$ for some constants $\{a_n\}_{n=1}^\infty \subset \mathbb{C} \setminus \{0\}$. Then, any function $f \in \mathcal{H}_K$ can be expanded as

$$f(t) = \sum_{n=1}^\infty f(t_n) \frac{S_n(t)}{a_n}\,, \quad t \in \Omega\,, \tag{3}$$

where the convergence of the series is absolute and uniform on subsets of Ω where the function $\|K(t)\|$ is bounded.

Recall that a Riesz basis $\{w_n\}_{n=1}^\infty$ for $\mathbb{H}$ is the image of an orthonormal basis by means of a bounded invertible operator on $\mathbb{H}$. Any Riesz basis $\{w_n\}_{n=1}^\infty$ has a unique biorthonormal (dual) Riesz basis $\{w_n^*\}_{n=1}^\infty$, i.e., such that $\langle w_n, w_m^*\rangle_{\mathbb{H}} = \delta_{n,m}$, and the expansions

$$x = \sum_{n=1}^\infty \langle x, w_n^*\rangle_{\mathbb{H}} w_n = \sum_{n=1}^\infty \langle x, w_n\rangle_{\mathbb{H}} w_n^*$$

hold for every $x \in \mathbb{H}$ (see [3] or [14] for more details and proofs).

Perhaps the most important examples of RKHS $\mathcal{H}_K$ that verify the mentioned result are the classical Paley-Wiener spaces of bandlimited functions, i.e., square integrable functions in $\mathbb{R}$ such that their Fourier transforms are zero outside a bounded set in $\mathbb{R}$. For instance, any function of the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x) e^{itx} dx \,, \quad t \in \mathbb{R} \,,$$

where $F \in L^2[-\pi, \pi]$, can be expanded as the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \operatorname{sinc}(t - n) \,, \quad t \in \mathbb{R} \,,$$

where sinc stands for cardinal sine function (or sinc function) defined as $\operatorname{sinc} t = \sin \pi t / \pi t$ for $t \neq 0$ and $\operatorname{sinc} 0 = 1$.

The sampling series (3) might also contain samples from functions which are related to f in some sense. Thus, this sampling result can be generalized in the following way: Let $\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty} \cup \{y_n^*\}_{n=1}^{\infty}$ be a pair of biorthonormal Riesz bases for the Hilbert space $\mathbb{H}$. For each fixed $t \in \Omega$, $K(t)$ can be written as

$$K(t) = \sum_{n=1}^{\infty} \left[\overline{S_n(t)} x_n^* + \overline{T_n(t)} y_n^* \right] ,$$

where $S_n(t)$ and $T_n(t)$ denote the evaluation at $t \in \Omega$ of the functions $S_n = T(x_n) \in \mathcal{H}_K$ and $T_n = T(y_n) \in \mathcal{H}_K$ obtained by means of the linear transform (1) .

Assume that there exist two kernels $K_1, K_2 : \Omega \longrightarrow \mathbb{H}$ each defining a function in the way K does, i.e., $f_j(t) := \langle x, K_j(t) \rangle_{\mathbb{H}}$, $j = 1, 2$. Let T_1 and T_2 be the corresponding linear transforms. These kernels can be written as

$$K_j(t) = \sum_{n=1}^{\infty} \left[\overline{S_n^j(t)} x_n^* + \overline{T_n^j(t)} y_n^* \right] , \quad j = 1, 2 \,,$$

where $S_n^j(t) = T_j(x_n)[t]$ and $T_n^j(t) = T_j(y_n)[t]$ for $t \in \Omega$, $j = 1, 2$. Suppose there exist two sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ in Ω such that the functions $\{S_n^j\}_{n=1}^{\infty}$ and $\{T_n^j\}_{n=1}^{\infty}$, $j = 1, 2$, satisfy the interpolation conditions

$$\begin{array}{lll} S_n^1(s_m) = a_n \delta_{n,m}; & T_n^1(s_m) = b_n \delta_{n,m}, & \{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C} \\ S_n^2(t_m) = c_n \delta_{n,m}; & T_n^2(t_m) = d_n \delta_{n,m}, & \{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty} \subset \mathbb{C} \,, \end{array}$$

where $\Delta_n := a_n d_n - b_n c_n \neq 0$ for all $n \in \mathbb{N}$. Suppose that f and the functions f_1 and f_2 are related in the sense that $\ker T \subseteq \ker T_1 \cap \ker T_2$. This implies that $\ker T = \{0\}$ so that $\mathcal{H}_K$ becomes a RKHS under the inner product $\langle f, g \rangle_{\mathcal{H}_K} := \langle x, y \rangle_{\mathbb{H}}$ where $Tx = f$ and $Ty = g$. Under these conditions, the technique used in [9] gives the following result:

Theorem 1 *The sequence $\{S_n\}_{n=1}^{\infty} \cup \{T_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathcal{H}_K$ and, expansions with respect to this basis allow to recover any function f in $\mathcal{H}_K$ from the samples $\{f_1(s_n)\}_{n=1}^{\infty}$*

and $\{f_2(t_n)\}_{n=1}^\infty$ *of* f_1 *and* f_2, *by means of the sampling formula*

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty}\Big[\frac{d_n f_1(s_n) - b_n f_2(t_n)}{\Delta_n} S_n(t) + \frac{a_n f_2(t_n) - c_n f_1(s_n)}{\Delta_n} T_n(t)\Big] \\ &= \sum_{n=1}^{\infty}\Big[f_1(s_n)\frac{d_n S_n(t) - c_n T_n(t)}{\Delta_n} + f_2(t_n)\frac{a_n T_n(t) - b_n S_n(t)}{\Delta_n}\Big], \quad t \in \Omega\,. \end{aligned} \tag{4}$$

The convergence of the series is absolute and uniform on subsets of Ω *where the function* $\|K(t)\|$ *is bounded.*

Using a matrix notation, formula (4) can be written as

$$f(t) = \sum_{n=1}^{\infty} \begin{pmatrix} f_1(s_n) & f_2(t_n) \end{pmatrix} \begin{pmatrix} a_n & c_n \\ b_n & d_n \end{pmatrix}^{-1} \begin{pmatrix} S_n(t) \\ T_n(t) \end{pmatrix}, \tag{5}$$

from which it is not difficult to derive a more general result involving the samples of N functions related to f.

Multi-channel sampling in Paley-Wiener spaces can be easily derived from this approach [9]. As a different example of Theorem 1 we can obtain a *Hermite-type interpolation series* for $\mathcal{H}_K$, i.e., a sampling series which involves samples of any function $f \in \mathcal{H}_K$ and its first derivative, and in addition, the sampling functions generalize the classical Hermite interpolation polynomials. Indeed, let $\{t_n\}_{n=1}^\infty$ be a sequence of distinct nonzero real numbers such that $\sum_{n=1}^\infty |t_n|^{-2} < \infty$. There exists an entire function $P(t)$ with simple zeros at the sequence $\{t_n\}_{n=1}^\infty$. Specifically, the function $P(t)$ is given by the canonical product

$$P(t) = \begin{cases} \displaystyle\prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n}\Big)\exp(t/t_n) & \text{if } \displaystyle\sum_{n=1}^{\infty}|t_n|^{-1} = \infty \\ \displaystyle\prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n}\Big) & \text{if } \displaystyle\sum_{n=1}^{\infty}|t_n|^{-1} < \infty\,. \end{cases}$$

Consider $\{x_n\}_{n=1}^\infty \cup \{y_n\}_{n=1}^\infty$ and $\{x_n^*\}_{n=1}^\infty \cup \{y_n^*\}_{n=1}^\infty$ a pair of biorthonormal Riesz bases for $\mathbb{H}$ and $Q(t) := P(t)^2$, which has double zeros at $\{t_n\}_{n=1}^\infty$. Take the functions

$$S_n(t) = \frac{Q(t)}{(t-t_n)^2} \quad \text{and} \quad T_n(t) = \frac{Q(t)}{t-t_n}$$

and define the kernels $K(t), K_1(t)$ and $K_2(t)$, $t \in \mathbb{R}$, by

$$K(t) = \sum_{n=1}^{\infty}\Big[\frac{Q(t)}{(t-t_n)^2} x_n^* + \frac{Q(t)}{t-t_n} y_n^*\Big],$$

$K_1(t) = K(t)$ and $K_2(t) = K'(t)$. It is easy to check the interpolation conditions:

$$S_n(t_m) = \frac{Q''(t_n)}{2}\delta_{n,m}\,; \quad T_n(t_m) = 0$$

$$S_n'(t_m) = \frac{Q'''(t_n)}{6}\delta_{n,m}\,; \quad T_n'(t_m) = \frac{Q''(t_n)}{2}\delta_{n,m}\,.$$

Taking into account that $Q''(t_n) \neq 0$ for all $n \in \mathbb{N}$, any function $f \in \mathcal{H}_K$ can be expanded as the series

$$f(t) = \sum_{n=1}^{\infty} \Big[f(t_n)\Big(1 - \frac{Q'''(t_n)}{3Q''(t_n)}(t-t_n)\Big) + f'(t_n)(t-t_n)\Big] \frac{2Q(t)}{Q''(t_n)(t-t_n)^2}\,, \quad t \in \mathbb{R}\,.$$

1.1 An easy anomalous example

Although Theorem 1 is a quite general sampling result, the following example exhibits a situation where it does not work. Indeed, consider the usual orthonormal basis for $L^2[-\pi,\pi]$ given by

$$\Big\{\frac{1}{\sqrt{2\pi}}\Big\} \cup \Big\{\frac{1}{\sqrt{\pi}}\cos nx\Big\}_{n=1}^{\infty} \cup \Big\{\frac{1}{\sqrt{\pi}}\sin nx\Big\}_{n=1}^{\infty}$$

and the kernels

$$K(t) = K_1(t) = \cos tx + \sin tx \;\text{ and } K_2(t) = \cos tx\,,$$

from which we define, for each $\phi \in L^2[-\pi,\pi]$, the transforms

$$f(t) = T\phi(t) = \big\langle \phi, K(t) \big\rangle \;\text{ and }\; f_i(t) = T_i\phi(t) = \big\langle \phi, K_i(t)\big\rangle\,, \quad i \in \{1,2\}\,.$$

Notice that, if $\phi \in L^2[-\pi,\pi]$, we obtain that $\big\langle \phi, \cos(tx)\big\rangle$ is an even function of t and that $\big\langle \phi, \sin(tx)\big\rangle$ is an odd one. Consequently, $\phi \in \ker T$ implies that $T_2\phi(t) = \big\langle \phi, \cos(tx)\big\rangle = -\big\langle \phi, \sin(tx)\big\rangle$ is both an odd and an even function of t. Therefore, $\phi = 0$ so that T is one-to-one.

The corresponding sampling functions are given by

$$\begin{aligned}
S_0(t) &= \Big\langle \frac{1}{\sqrt{2\pi}}, K(t)\Big\rangle = \sqrt{2\pi}\,\operatorname{sinc} t \\
S_n(t) &= \Big\langle \frac{1}{\sqrt{\pi}}\cos nx, K(t)\Big\rangle = \frac{2t(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N}) \\
T_n(t) &= \Big\langle \frac{1}{\sqrt{\pi}}\sin nx, K(t)\Big\rangle = \frac{2n(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N})\,.
\end{aligned}$$

In this case, $S_0^1 = S_0^2 = S_0$, $S_n^1 = S_n^2 = S_n$, $T_n^1 = T_n$ and $T_n^2 = 0$ for $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$, the following interpolation condition holds:

$$\begin{pmatrix} S_n^1(m) & T_n^1(m) \\ S_n^2(m) & T_n^2(m) \end{pmatrix} = \delta_{m,n} \begin{pmatrix} \sqrt{\pi} & \sqrt{\pi} \\ \sqrt{\pi} & 0 \end{pmatrix}\,.$$

However, what is happening with S_0^1 and S_0^2? Even if we decide to define $T_0^1 = T_0^2 := 0$, the interpolation matrix that we used in (5) will be singular and we will not be able to apply Theorem 1.

This example gives us a suitable generalization that, in practice, can be very useful. As we only need one coefficient for S_0 (T_0 is not defined), we can consider the matrix

$$\begin{pmatrix} S_0^1(m) \\ S_0^2(m) \end{pmatrix} = \delta_{m,0} \begin{pmatrix} \sqrt{2\pi} \\ \sqrt{2\pi} \end{pmatrix} \tag{6}$$

where $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The coefficient of $S_0(t)$ can be chosen in two ways for the sampling formula, $f_1(0)/\sqrt{2\pi}$ or $f_2(0)/\sqrt{2\pi}$, the choice being at our disposal; we might choose the simplest, or perhaps the only one available, etc.

2 A general sampling result in $\mathcal{H}_K$

The aim in this section is to prove a new sampling result which also applies for anomalous examples. To this end, we need to introduce new interpolation conditions which will be combined in an appropriate way to obtain the desired sampling result. These topics are the subject of the next three subsections:

2.1 Interpolation condition of type (L, M)

Consider $L, M \in \mathbb{N}$ such that $L \geq M$ and define $\mathsf{M} := \{1, 2, \ldots, M\}$ and $\mathsf{L} := \{1, 2, \ldots, L\}$.

Consider a linear independent system for $\mathbb{H}$ written as

$$\{x_{1,n}\}_{n\in\mathbb{I}} \cup \{x_{2,n}\}_{n\in\mathbb{I}} \cup \cdots \cup \{x_{M,n}\}_{n\in\mathbb{I}}$$

where $\mathbb{I} \subseteq \mathbb{N}$ can be a finite set. Suppose that we have L linear transforms $T_1, T_2, \ldots, T_L$ with associate kernels $K_1(t), K_2(t), \ldots, K_L(t)$ and defined as follows

$$f_\ell(t) = T_\ell(x)[t] = \big\langle x, K_\ell(t)\big\rangle\,, \quad \ell \in \mathsf{L}\,.$$

Denote

$$S^\ell_{m,n}(t) = T_\ell(x_{m,n})[t]\,, \quad t \in \Omega\,,$$

where $\ell \in \mathsf{L}$, $m \in \mathsf{M}$ and $n \in \mathbb{I}$.

Definition 1 *We say that an interpolation condition of type (L, M) is satisfied by these elements if there exist L sets of points $\{t^\ell_n\}_{n\in\mathbb{I}}$ in Ω, $\ell \in \mathsf{L}$, such that, for any fixed $\ell \in \mathsf{L}$ and $m \in \mathsf{M}$,*

$$S^\ell_{m,n}(t^\ell_k) = a^n_{\ell,m}\delta_{n,k}\,, \qquad n, k \in \mathbb{I}\,,$$

holds, and the coefficients $a^n_{\ell,m} \in \mathbb{C}$ verify that the rank of the matrices

$$A^n := \begin{pmatrix} a^n_{1,1} & a^n_{1,2} & \cdots & a^n_{1,M} \\ a^n_{2,1} & a^n_{2,2} & \cdots & a^n_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{L,1} & a^n_{L,2} & \cdots & a^n_{L,M} \end{pmatrix}$$

is just M for all $n \in \mathbb{I}$.

Definition 2 *Denote by ψ any increasing function from M into L, which we shall call a* choice function. *For any matrix B with L rows $b(1), b(2), \ldots, b(L)$, we define the choice of M rows of B by means of ψ as*

$$B_\psi := \begin{pmatrix} b[\psi(1)] & b[\psi(2)] & \cdots & b[\psi(M)] \end{pmatrix}^\top\,.$$

As we can see, B_ψ is a matrix obtained from B by choosing the rows of B given by $\psi(1), \psi(2), \ldots, \psi(M)$.

In these terms, the rank of A^n is M if and only if there exists a choice function ψ_n such that $A^n_{\psi_n}$ is not singular.

Definition 3 *We shall call such a ψ_n, an A^n-*regular choice function.

Any A^n-regular choice function ψ is related to A^n in such a way that A^n_ψ is regular. However, this choice function can be applied to any matrix with L rows and any number of columns.

If we have a vector $v = \begin{pmatrix} v_1 & v_2 & \cdots & v_L \end{pmatrix}^\top$, then we can choose the same rows from both A^n and v. The product of the matrix and the vector obtained in such a way is given by

$$A^n_\psi v_\psi = \begin{pmatrix} a^n_{\psi(1),1} & a^n_{\psi(1),2} & \cdots & a^n_{\psi(1),M} \\ a^n_{\psi(2),1} & a^n_{\psi(2),2} & \cdots & a^n_{\psi(2),M} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{\psi(M),1} & a^n_{\psi(M),2} & \cdots & a^n_{\psi(M),M} \end{pmatrix} \begin{pmatrix} v_{\psi(1)} \\ v_{\psi(2)} \\ \vdots \\ v_{\psi(M)} \end{pmatrix} .$$

2.2 Compatibility

We began this article by showing an example in which two interpolation conditions were used. For the first one, $\mathbb{I}_1 = \{0\}$, the linear independent system was just one vector and we had only one transform, i.e., it was an interpolation condition of type $(1,1)$. In the second one, $\mathbb{I}_2 = \mathbb{N}$, the partition of the linear independent system had two elements and there were two transforms, so it was an interpolation condition of type $(2,2)$. The goal of this section is to establish which properties must be verified by two interpolation conditions in order for a sampling theorem to be possible. It is the topic *compatible interpolation conditions.*

The fact of working with two interpolation conditions at least makes the notation used hard. For the sake of clarity, hereafter, an index k_j denotes that the indexed element corresponds to the j-th interpolation condition of those we are using.

Consider two interpolation conditions of types (L_1, M_1) and (L_2, M_2), respectively. This means that we have two linear independent systems in $\mathbb{H}$:

$$\begin{aligned} \mathfrak{S}_1 :=& \{x^1_{1,n_1}\}_{n_1\in\mathbb{I}_1} \cup \{x^1_{2,n_1}\}_{n_1\in\mathbb{I}_1} \cup \cdots \cup \{x^1_{M_1,n_1}\}_{n_1\in\mathbb{I}_1} \\ \mathfrak{S}_2 :=& \{x^2_{1,n_2}\}_{n_2\in\mathbb{I}_2} \cup \{x^2_{2,n_2}\}_{n_2\in\mathbb{I}_2} \cup \cdots \cup \{x^2_{M_2,n_2}\}_{n_2\in\mathbb{I}_2} \end{aligned}$$

where $\mathbb{I}_1, \mathbb{I}_2 \subseteq \mathbb{N}$ (possibly finite) and $M_1, M_2 \in \mathbb{N}$. Moreover, we suppose the j-th interpolation condition, $j \in \{1,2\}$, has $L_j \in \mathbb{N}$ transforms T_{ℓ_j}, $\ell_j \in \mathsf{L}_j$ and $M_j \leq L_j$, defined from each kernel $K_{\ell_j}(t)$, $\ell_j \in \mathsf{L}_j$, by

$$f_{\ell_j}(t) = T_{\ell_j}(x)[t] = \langle x, K_{\ell_j}(t) \rangle , \qquad \ell_j \in \mathsf{L}_j .$$

Denote

$$S^{\ell_i}_{m_j,n_j} = T_{\ell_i}(x^j_{m_j,n_j})$$

for $\ell_i \in \mathsf{L}_i$, $m_j \in \mathsf{M}_j$, $n_j \in \mathbb{I}_j$ and $i, j \in \{1,2\}$, i.e., the image of $\mathfrak{S}_1 \cup \mathfrak{S}_2$ by each of the transforms of both the interpolation conditions is calculated.

As they are interpolation conditions, there exist sequences $\{t^{\ell_j}_{k_j}\}_{k_j\in\mathbb{I}_j}$, where $\ell_j \in \mathsf{L}_j$ and $j \in \{1,2\}$, such that

$$S^{\ell_j}_{m_j,n_j}(t^{\ell_j}_{k_j}) = a^{n_j}_{\ell_j,m_j}\delta_{n_j,k_j}$$

holds for $m_j \in \mathsf{M}_j$, $\ell_j \in \mathsf{L}_j$, $n_j, k_j \in \mathbb{I}_j$ and $j \in \{1,2\}$ in such a way the coefficients $a^{n_j}_{\ell_j,m_j} \in \mathbb{C}$ satisfy that the rank of the matrix

$$A^{n_j} := \begin{pmatrix} a^{n_j}_{1,1} & a^{n_j}_{1,2} & \cdots & a^{n_j}_{1,M_j} \\ a^{n_j}_{2,1} & a^{n_j}_{2,2} & \cdots & a^{n_j}_{2,M_j} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n_j}_{L_j,1} & a^{n_j}_{L_j,2} & \cdots & a^{n_j}_{L_j,M_j} \end{pmatrix}$$

is just M_j for all $n_j \in \mathbb{I}_j$, $j \in \{1,2\}$.

Definition 4 *Two interpolation conditions are compatible if the following* compatibility condition *holds:*

$$S^{\ell_2}_{m_1,n_1}(t^{\ell_2}_{k_2}) = 0 \quad , \quad S^{\ell_1}_{m_2,n_2}(t^{\ell_1}_{k_1}) = 0\,,$$

for $\ell_j \in \mathsf{L}_j$, $m_j \in \mathsf{M}_j$, $n_j, k_j \in \mathbb{I}_j$, $j \in \{1,2\}$.

Observe that this condition implies that $\mathfrak{S}_1 \cup \mathfrak{S}_2$ is a linear independent system. Indeed, suppose there exists $x^2_{m_2,n_2}$ in $\mathfrak{S}_2$ such that

$$x^2_{m_2,n_2} = \sum_{m_1=1}^{M_1} \alpha_{m_1} x^1_{m_1,n_1} + \sum_{k=1}^{N} \beta_k x_k$$

where $x_k \in \mathfrak{S}_1 \cup \mathfrak{S}_2$ is not in $\{x^2_{m_2,n_2}, x^1_{1,n_1}, x^1_{2,n_1}, \ldots, x^1_{M_1,n_1}\}$ and there exists an index $k_1 \in \mathsf{M}_1 = \{1,2,\ldots,M_1\}$ such that $\alpha_{k_1} \neq 0$. For all $\ell_1 \in \mathsf{L}_1$, we obtain that

$$0 = [T_{\ell_1} x^2_{m_2,n_2}](t^{\ell_1}_{n_1}) = \sum_{m_1=1}^{M_1} \alpha_{m_1} a^{n_1}_{\ell_1,m_1}\,,$$

i.e., A^{n_1} has a zero column or it has at least two linear dependent columns so that its rank cannot be M_1.

2.3 The sampling result

Consider the pair of dual Riesz bases for $\mathbb{H}$ given by

$$\bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{x_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \qquad \text{and} \qquad \bigcup_{r=1}^{R} \bigcup_{m_r=1}^{M_r} \{x^*_{m_r,n_r}\}_{n_r \in \mathbb{I}_r} \tag{7}$$

and suppose that $R \in \mathbb{N}$ interpolation conditions are satisfied, and each is compatible with every other (see Definition 1 and Definition 4).

Suppose that the following condition is satisfied, as well:

$$\ker T \subseteq \bigcap_{r=1}^{R} \bigcap_{\ell_r=1}^{L_r} \ker T_{\ell_r}\,. \tag{8}$$

We show the transform T is one-to-one. First, observe that the kernel K can be written as

$$K(t)=\sum_{r=1}^{R}\sum_{m_r=1}^{M_r}\sum_{n_r\in\mathbb{I}_r}\overline{S_{m_r,n_r}(t)}x^*_{m_r,n_r}$$

where $S_{m_r,n_r}(t)=\langle x_{m_r,n_r},K(t)\rangle$. Analogously, for $\ell_q\in\mathsf{L}_q$ and $1\le q\le R$, the kernel K_{ℓ_q} can be expanded as

$$K_{\ell_q}(t)=\sum_{r=1}^{R}\sum_{m_r=1}^{M_r}\sum_{n_r\in\mathbb{I}_r}\overline{S^{\ell_q}_{m_r,n_r}(t)}x^*_{m_r,n_r}\,.$$

Suppose that $x\in\mathbb{H}$ verifies $Tx=0$. Then, for all $q\in\{1,2,\dots,R\}$ and $\ell_q\in\mathsf{L}_q$, we have that

$$f_{\ell_q}=T_{\ell_q}x=0$$

and therefore, for $p_q\in\mathbb{I}_q$,

$$0=f_{\ell_q}(t^{\ell_q}_{p_q})=\Big\langle x,K_{\ell_q}(t^{\ell_q}_{p_q})\Big\rangle=\Big\langle x,\sum_{r=1}^{R}\sum_{m_r=1}^{M_r}\sum_{n_r\in\mathbb{I}_r}\overline{S^{\ell_q}_{m_r,n_r}(t^{\ell_q}_{p_q})}x^*_{m_r,n_r}\Big\rangle\,.$$

Thus, compatibility implies

$$0=\Big\langle x,\sum_{m_q=1}^{M_q}\sum_{n_q\in\mathbb{I}_q}\overline{S^{\ell_q}_{m_q,n_q}(t^{\ell_q}_{p_q})}x^*_{m_q,n_q}\Big\rangle\,,$$

and by using the q-th interpolation condition, we have

$$0=\Big\langle x,\sum_{m_q=1}^{M_q}\overline{a^{p_q}_{\ell_q,m_q}}x^*_{m_q,p_q}\Big\rangle=\sum_{m_q=1}^{M_q}a^{p_q}_{\ell_q,m_q}\big\langle x,x^*_{m_q,p_q}\big\rangle\,,\qquad 1\le\ell_q\le L_q\,.$$

Thus, we have a homogeneous linear system with L_q equations and M_q unknowns whose unique solution is the trivial one. Consequently, we obtain that $\langle x,x^*_{m_q,p_q}\rangle=0$ for all $q\in\{1,2,\dots R\}$, $m_q\in\mathsf{M}_q$ and $p_q\in\mathbb{I}_q$. Since $\{x^*_{m_r,n_r}\,:\,1\le r\le R,\ m_r\in\mathsf{M}_r,\ n_r\in\mathbb{I}_r\}$ is a Riesz basis, $x=0$. Observe that the following sequences

$$\bigcup_{r=1}^{R}\bigcup_{m_r=1}^{M_r}\{S_{m_r,n_r}\}_{n_r\in\mathbb{I}_r}\qquad\text{and}\qquad\bigcup_{r=1}^{R}\bigcup_{m_r=1}^{M_r}\{S^*_{m_r,n_r}\}_{n_r\in\mathbb{I}_r}\tag{9}$$

are dual Riesz bases of $\mathcal{H}_K$.

If we do the same for any $x\in\mathbb{H}$ such that $f=Tx$, we obtain the consistent linear system

$$f_{\ell_q}(t^{\ell_q}_{p_q})=\sum_{m_q=1}^{M_q}a^{p_q}_{\ell_q,m_q}\big\langle x,x^*_{m_q,p_q}\big\rangle\,,\qquad 1\le\ell_q\le L_q\,,$$

which has a unique solution. For each $p_q\in\mathbb{I}_q$ we can find a choice function ψ_{p_q} such that $A^{p_q}_{\psi_{p_q}}$ is regular. Thus, we can write the coefficients $\langle x,x^*_{m_q,p_q}\rangle$ with respect to the samples $f_{\ell_q}(t^{\ell_q}_{p_q})$ by means of

$$\Big(\big\langle x,x^*_{1,p_q}\big\rangle\quad\big\langle x,x^*_{2,p_q}\big\rangle\quad\cdots\quad\big\langle x,x^*_{M_q,p_q}\big\rangle\Big)^\top=\big(A^{p_q}_{\psi_{p_q}}\big)^{-1}F^{p_q}_{\psi_{p_q}}\,,$$

where $F^{p_q}_{\psi_{p_q}}$ is obtained from the vector

$$F^{p_q} := \Big(f_1(t^1_{p_q}) \quad f_2(t^2_{p_q}) \quad \cdots \quad f_{L_q}(t^{L_q}_{p_q}) \Big)^\top \tag{10}$$

by using the choice function ψ_{p_q}.

Now, expanding f by using the Riesz basis given by $\{S_{m_r,n_r} : n_r \in \mathbb{I}_r\}_{m_r=1}^{M_r}$ and taking into account that T is an isometry, we have that

$$\begin{aligned} f(t) &= \sum_{r=1}^{R} \sum_{m_r=1}^{M_r} \sum_{n_r \in \mathbb{I}_r} \langle f, S^*_{m_r,n_r} \rangle_{\mathcal{H}_K} S_{m_r,n_r}(t) \\ &= \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \sum_{m_r=1}^{M_r} \langle x, x^*_{m_r,n_r} \rangle_{\mathbb{H}} S_{m_r,n_r}(t) \\ &= \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \big(F^{n_r}_{\psi_{n_r}} \big)^\top \Big[\big(A^{n_r}_{\psi_{n_r}} \big)^{-1} \Big]^\top \mathbb{S}_{n_r}(t) \,, \end{aligned}$$

where F^{n_r} is given by (10) and

$$\mathbb{S}_{n_r}(t) = \big(S_{1,n_r}(t) \quad S_{2,n_r}(t) \quad \cdots \quad S_{M_r,n_r}(t) \big)^\top \,, \tag{11}$$

which is just the sampling formula we are looking for. Convergence is, as we know, in the $\mathcal{H}_K$-norm sense and, also, absolute in Ω and uniform in those subsets of Ω where $\|K(t)\|$ is bounded. Consequently, we have proved the next result:

Theorem 2 *Consider the dual Riesz bases given by* (7)*. Suppose we have $R \in \mathbb{N}$ interpolation conditions of types (L_r, M_r), where $1 \le r \le R$, each two of them being compatible (in the sense of Definition 4). Assume condition* (8) *is satisfied. Then, the sets in* (9) *are dual Riesz bases for $\mathcal{H}_K$ and for each set of A^{n_r}-regular choice functions $\big\{\psi_{n_r} : \mathsf{M}_r \longrightarrow \mathsf{L}_r \mid n_r \in \mathbb{I}_r \,,\ 1 \le r \le R\big\}$, we have that any function $f \in \mathcal{H}_K$ can be recovered from its samples*

$$\big\{ f_{\ell_r}(t^{\ell_r}_{n_r}) : n_r \in \mathbb{I}_r \,, \quad \ell_r \in \mathsf{L}_r \big\}_{r=1}^{R}$$

by the following sampling formula

$$f(t) = \sum_{r=1}^{R} \sum_{n_r \in \mathbb{I}_r} \big(F^{n_r}_{\psi_{n_r}} \big)^\top \Big[\big(A^{n_r}_{\psi_{n_r}} \big)^{-1} \Big]^\top \mathbb{S}_{n_r}(t) \,, \quad t \in \Omega \,,$$

where F^{n_r} and $\mathbb{S}_{n_r}$ are given by (10) *and* (11)*, respectively. Convergence is absolute and, also, uniform in those subsets of Ω where $\|K(t)\|$ is bounded.*

Notice that the number of transforms we have for each interpolation condition of type (L, M) is M at least. However, it is not important how many of them we have at most. In fact, only the possibility of finding a regular choice of rows of A^n is needed. Thus, if L is a finite or infinite set of indices, we have an interpolation condition of type $(\operatorname{card}\mathsf{L}, M)$ if there exist some points $\{t^\ell_n : n \in \mathbb{I}\}_{\ell \in \mathsf{L}}$ such that, for any fixed $\ell \in \mathsf{L}$ and $m \in \mathsf{M}$,

$$S^\ell_{m,n}(t^\ell_k) = a^n_{\ell,m} \delta_{n,k} \,, \quad n, k \in \mathbb{I} \,,$$

holds, and for each $n \in \mathbb{I}$ there exists an A^n-regular choice function ψ_n where

$$A^n = \begin{pmatrix} a^n_{\ell,1} & a^n_{\ell,2} & \cdots & a^n_{\ell,M} \end{pmatrix}_{\ell \in \mathsf{L}}$$

represents a function from L into $\mathbb{C}^M$ for each $n \in \mathbb{I}$. If these more general interpolation conditions are used, Theorem 2 still remains valid.

3 Sampling by using linear operators in $\mathbb{H}$

Let $\mathbb{H}$ be a separable Hilbert space. Consider two dual Riesz bases of $\mathbb{H}$ written as

$$\{x_{1,n}\}_{n=1}^{\infty} \cup \{x_{2,n}\}_{n=1}^{\infty} \cup \cdots \cup \{x_{M,n}\}_{n=1}^{\infty}\,,$$
$$\{x^*_{1,n}\}_{n=1}^{\infty} \cup \{x^*_{2,n}\}_{n=1}^{\infty} \cup \cdots \cup \{x^*_{M,n}\}_{n=1}^{\infty}\,,$$

where $M \in \mathbb{N}$. Given a kernel $K : \Omega \subset \mathbb{R} \longrightarrow \mathbb{H}$, we define the linear transform T as in (1). Suppose we have a family of bounded linear operators $\{L_\lambda : \mathbb{H} \longrightarrow \mathbb{H}\}_{\lambda \in \Lambda}$. Related to $f = Tx \in \mathcal{H}_K$ we define the functions $f_\lambda(t) := \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}}$, where $\lambda \in \Lambda$. For any fixed $n \in \mathbb{N}$ and $1 \le m \le M$, we denote $S^n_{\lambda,m}(t) := \big\langle L_\lambda x_{m,n}, K(t) \big\rangle_{\mathbb{H}}$ and, for any $n \in \mathbb{N}$, we write $S_{m,n}(t) := \big\langle x_{m,n}, K(t) \big\rangle_{\mathbb{H}}$.

In the sequel, we assume that $x \in \ker T$ implies $L_\lambda x \in \ker T$ for all $\lambda \in \Lambda$ (which occurs when T commutes with L_λ for all $\lambda \in \Lambda$), and that there exists a family of sequences $\big\{\{t^\lambda_k\}_{k \in \mathbb{N}} : \lambda \in \Lambda\big\} \subset \Omega$ such that $S^n_{\lambda,m}(t^\lambda_k) = a^n_{\lambda,m}\delta_{n,k}$ for $n, k \in \mathbb{N}$, $1 \le m \le M$ and $\lambda \in \Lambda$. For each $n \in \mathbb{N}$, define the function

$$\begin{array}{rccl} A^n : & \Lambda & \longrightarrow & \mathbb{C}^M \\ & \lambda & \longmapsto & (a^n_{\lambda,1}, a^n_{\lambda,2}, \ldots, a^n_{\lambda,M}) \end{array}$$

and suppose that there exists a sequence $\{\psi_n : \mathsf{M} \longrightarrow \Lambda\}_{n \in \mathbb{N}}$ of A^n-regular choice functions. Then, the following result holds:

Theorem 3 *Under the hypotheses as above, any $f \in \mathcal{H}_K$ can be recovered from its samples $\big\{\{f_\lambda(t^\lambda_n)\}_{n \in \mathbb{N}} : \lambda \in \Lambda\big\}$ by means of the sampling formula*

$$f(t) = \sum_{n \in \mathbb{N}} \big(F^n_{\psi_n}\big)^\top \Big[\big(A^n_{\psi_n}\big)^{-1}\Big]^\top \mathbb{S}_n(t)\,, \quad t \in \Omega\,,$$

where $F^n(\lambda) := f_\lambda(t^\lambda_n)$ and $\mathbb{S}_n(t) = \begin{pmatrix} S_{1,n}(t) & S_{2,n}(t) & \cdots & S_{M,n}(t) \end{pmatrix}^\top$. Convergence is absolute and uniform in subsets of Ω where $\|K(t)\|$ is bounded.

Proof: Defining $K_\lambda(t) := L^*_\lambda[K(t)]$ for $\lambda \in \Lambda$, where L^*_λ denotes the adjoint operator of L_λ, we have

$$f_\lambda(t) := T_\lambda(x)[t] = \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}} = \big\langle x, K_\lambda(t) \big\rangle_{\mathbb{H}}\,, \quad t \in \Omega\,.$$

If $x \in \ker T$ then, by assumption, $L_\lambda x \in \ker T$ for all $\lambda \in \Lambda$, i.e., $0 = \big\langle L_\lambda x, K(t) \big\rangle_{\mathbb{H}} = \big\langle x, K_\lambda(t) \big\rangle_{\mathbb{H}}$ for all $\lambda \in \Lambda$. As a consequence,

$$\ker T \subseteq \bigcap_{\lambda \in \Lambda} \ker T_\lambda\,,$$

and Theorem 2 implies the desired result. ∎

4 Some illustrative examples

In this section we go back to our anomalous example in order to handle it into the new setting. We also give another example which involves two interpolation conditions of type $(2,2)$.

4.1 The introductory example revisited

Consider the orthonormal basis of $L^2[-\pi,\pi]$ given by

$$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos nx\right\}_{n=1}^{\infty} \cup \left\{\frac{1}{\sqrt{\pi}}\sin nx\right\}_{n=1}^{\infty}$$

and the kernels $K_1(t) = K(t) = \cos tx + \sin tx,\ K_2(t) = \cos tx - \sin tx$ and $K_3(t) = \cos tx$. Notice that, if $\phi \in L^2[-\pi,\pi]$, we obtain that $\langle \phi, \cos(tx)\rangle$ is an even function of t and that $\langle \phi, \sin(tx)\rangle$ is an odd one. Consequently, $\phi \in \ker T$ implies that $T_3\phi(t) = \langle \phi, \cos(tx)\rangle = -\langle \phi, \sin(tx)\rangle$ is both an odd and an even function of t. Thus, T is injective and trivially $\{0\} = \ker T \subseteq \ker T_1 \cap \ker T_2 \cap \ker T_3$.

Sampling functions are given by

$$\begin{aligned}
S_0(t) &= \left\langle \frac{1}{\sqrt{2\pi}}, K(t)\right\rangle = \sqrt{2\pi}\,\mathrm{sinc}\, t \\
S_n(t) &= \left\langle \frac{1}{\sqrt{\pi}}\cos nx, K(t)\right\rangle = \frac{2t(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N}) \\
T_n(t) &= \left\langle \frac{1}{\sqrt{\pi}}\sin nx, K(t)\right\rangle = \frac{2n(-1)^n \sin \pi t}{\sqrt{\pi}(t^2-n^2)} \quad (n \in \mathbb{N})
\end{aligned}$$

Easy calculations show that $S_n^1 = S_n^2 = S_n^3 = S_n$, that $T_n^3 = 0$ and that $-T_n^2 = T_n^1 = T_n$ for $n \in \mathbb{N}$ and $S_0^1 = S_0^2 = S_0^3 = S_0$. Thus, we have two interpolation conditions of types $(3,1)$ and $(3,2)$, respectively. The first one verifies that:

$$\begin{pmatrix} S_0^1(m) \\ S_0^2(m) \\ S_0^3(m) \end{pmatrix} = \begin{pmatrix} \sqrt{2\pi} \\ \sqrt{2\pi} \\ \sqrt{2\pi} \end{pmatrix} \delta_{0,m} \quad m \in \mathbb{N} \cup \{0\}\,,$$

and the second one, that:

$$\begin{pmatrix} S_n^1(m) & T_n^1(m) \\ S_n^2(m) & T_n^2(m) \\ S_n^3(m) & T_n^3(m) \end{pmatrix} = \begin{pmatrix} \sqrt{\pi} & \sqrt{\pi} \\ \sqrt{\pi} & -\sqrt{\pi} \\ \sqrt{\pi} & 0 \end{pmatrix} \delta_{n,m} \quad m \in \mathbb{N} \cup \{0\}\,,$$

so we have a couple of compatible interpolation conditions.

We define $f(t) := \langle F, K(t)\rangle$ and $f_k(t) := \langle F, K_k(t)\rangle$ for $k = 1,2,3$ and $F \in L^2[-\pi,\pi]$. Finally, Theorem 2 yields the following sampling result:

Corollary 4 *Any function f defined as*

$$f(t) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} F(x)[\cos tx + \sin tx]dx\,, \quad t \in \mathbb{R}\,,$$

where $F \in L^2[-\pi,\pi]$, can be recovered from the samples $f(0)$ and $\{f_k(n)\}_{n=1}^\infty$, $k = 1,2,3$, of its related functions f_1, f_2, f_3 by means of the following sampling formula:

$$f(t) = f(0)\,\mathrm{sinc}\, t + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \big[\mathcal{G}_1(n) S_n(t) + \mathcal{G}_2(n) T_n(t)\big]\,, \quad t \in \mathbb{R}\,,$$

where $\big(\mathcal{G}_1(n)\ \mathcal{G}_2(n)\big)$ is any row of the matrix

$$\begin{pmatrix} f_3(n) & f_3(n) - f_2(n) \\ \\ f_3(n) & f_1(n) - f_3(n) \\ \\ \frac{f_2(n)+f_1(n)}{2} & \frac{f_1(n)-f_2(n)}{2} \end{pmatrix}. \tag{12}$$

Convergence is absolute and uniform in subsets of Ω where $\|K(t)\|$ is bounded.

This result allows us to recover any classical band-limited function to $[-\pi,\pi]$ by means of the samples of the related functions f_1, f_2, f_3 since band-limited functions can be written as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(x)[\cos tx + \sin tx]dx\,, \quad t \in \mathbb{R}\,,$$

where $F \in L^2[-\pi,\pi]$. Notice that the above integral representation involves the Hartley transform of F (see [16]).

4.2 Another example

In this example, we use two compatible interpolation conditions of type $(2,2)$. To this end, consider $\{x_n, y_n\}_{n=1}^\infty \cup \{\widetilde{x}_n, \widetilde{y}_n\}_{n=1}^\infty$, a Riesz basis for $\mathbb{H}$ whose dual Riesz basis is given by $\{x_n^*, y_n^*\}_{n=1}^\infty \cup \{\widetilde{x}_n^*, \widetilde{y}_n^*\}_{n=1}^\infty$. Suppose we have five kernels $K, K_1, K_2, \widetilde{K}_1, \widetilde{K}_2$ each of them defining a transform, as usual: $f(t) = (Tx)[t] := \big\langle x, K(t)\big\rangle$, $f_j(t) = (T_j x)[t] := \big\langle x, K_j(t)\big\rangle$, and $\widetilde{f}_j(t) = (\widetilde{T}_j x)[t] := \big\langle x, \widetilde{K}_j(t)\big\rangle$ where $t \in \Omega$, $x \in \mathbb{H}$ and $j \in \{1,2\}$, denoting a tilde over an element that it is related to the second interpolation condition. Assume that the following condition holds:

$$\ker T \subseteq \ker T_1 \cap \ker T_2 \cap \ker \widetilde{T}_1 \cap \ker \widetilde{T}_2\,.$$

Denote

$$\begin{aligned} S_n &:= T x_n\,, \quad T_n := T y_n\,, \\ \widetilde{S}_n &:= T \widetilde{x}_n\,, \quad \widetilde{T}_n := T \widetilde{y}_n\,, \end{aligned}$$

and suppose there exist sequences $\{s_n\}_{n=1}^\infty$, $\{t_n\}_{n=1}^\infty$, $\{\widetilde{s}_n\}_{n=1}^\infty$ and $\{\widetilde{t}_n\}_{n=1}^\infty$ such that

$$\begin{pmatrix} (T_1(x_n))(s_m) & (T_1(y_n))(s_m) \\ (T_2(x_n))(t_m) & (T_2(y_n))(t_m) \end{pmatrix} = \begin{pmatrix} a_{1,1}^n & a_{1,2}^n \\ a_{2,1}^n & a_{2,2}^n \end{pmatrix} \delta_{m,n} =: A^n \delta_{m,n}$$

for the first interpolation condition, and

$$\begin{pmatrix} (\widetilde{T}_1(\widetilde{x}_n))(s_m) & (\widetilde{T}_1(\widetilde{y}_n))(s_m) \\ (\widetilde{T}_2(\widetilde{x}_n))(t_m) & (\widetilde{T}_2(\widetilde{y}_n))(t_m) \end{pmatrix} = \begin{pmatrix} \widetilde{a}_{1,1}^n & \widetilde{a}_{1,2}^n \\ \widetilde{a}_{2,1}^n & \widetilde{a}_{2,2}^n \end{pmatrix} \delta_{m,n} =: \widetilde{A}^n \delta_{m,n}$$

for the second one, being the matrices A^n and $\widetilde{A}^n$ invertible. For these interpolation conditions the compatibility condition reads:

$$\begin{pmatrix} (\widetilde{T}_1(x_n))(\widetilde{s}_m) & (\widetilde{T}_1(y_n))(\widetilde{s}_m) \\ (\widetilde{T}_2(x_n))(\widetilde{t}_m) & (\widetilde{T}_2(y_n))(\widetilde{t}_m) \end{pmatrix} = \begin{pmatrix} (T_1(\widetilde{x}_n))(s_m) & (T_1(\widetilde{y}_n))(s_m) \\ (T_2(\widetilde{x}_n))(t_m) & (T_2(\widetilde{y}_n))(t_m) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, as a consequence of Theorem 2 we deduce the following sampling result for the preceding compatible interpolation conditions:

Corollary 5 *Any function f in the Hilbert space $\mathcal{H}_K$ can be recovered from the sequences of samples $\{f_1(s_n)\}_{n=1}^{\infty}$, $\{f_2(t_n)\}_{n=1}^{\infty}$, $\{\widetilde{f}_1(\widetilde{s}_n)\}_{n=1}^{\infty}$ and $\{\widetilde{f}_2(\widetilde{t}_n)\}_{n=1}^{\infty}$ by means of the following sampling formula*

$$\begin{aligned} f(t) = \sum_{n=1}^{\infty}\Big[f_1(s_n)\frac{a_{2,2}^n S_n(t) - a_{2,1}^n T_n(t)}{a_{1,1}^n a_{2,2}^n - a_{1,2}^n a_{2,1}^n} + f_2(t_n)\frac{a_{1,1}^n T_n(t) - a_{1,2}^n S_n(t)}{a_{1,1}^n a_{2,2}^n - a_{1,2}^n a_{2,1}^n}\Big] + \\ + \sum_{n=1}^{\infty}\Big[\widetilde{f}_1(s_n)\frac{\widetilde{a}_{2,2}^n \widetilde{S}_n(t) - \widetilde{a}_{2,1}^n \widetilde{T}_n(t)}{\widetilde{a}_{1,1}^n \widetilde{a}_{2,2}^n - \widetilde{a}_{1,2}^n \widetilde{a}_{2,1}^n} + \widetilde{f}_2(t_n)\frac{\widetilde{a}_{1,1}^n \widetilde{T}_n(t) - \widetilde{a}_{1,2}^n \widetilde{S}_n(t)}{\widetilde{a}_{1,1}^n \widetilde{a}_{2,2}^n - \widetilde{a}_{1,2}^n \widetilde{a}_{2,1}^n}\Big]. \end{aligned}$$

The convergence of the series above is absolute and uniform in subsets of Ω where $\|K(t)\|$ is bounded.

4.3 A comment on the choice of the samples

Theorem 2 allows us to combine several interpolation conditions whose types are not necessarily equal. Corollary 4 gives a sampling formula for the example of subsection 1.1 which shows some advantages of our approach. Indeed, for each $n \in \mathbb{N}$, we can choose any row of the matrix in (12). Thus, if the samples of f_2 are lost for $n \in \mathbf{N} \subseteq \mathbb{N}$, we can still recover f by using the second row of that matrix for $n \in \mathbf{N}$.

On the other hand, fixed $n \in \mathbb{N}$, the rows of (12) are obtained as solutions of a consistent linear system with three equations and two unknowns, so we can choose any two of them in order to solve the system. This means that every row of (12) is equal to every other. As a consequence, we can choose any element of the first column and any element (not necessarily in the same row) of the second one. This remark allows us to avoid cancellation errors by choosing the appropriate elements of (12). For instance, suppose we have that $f_1(n_0)f_2(n_0) > 0$ and that $f_1(n_0)f_3(n_0) < 0$ for $n_0 \in \mathbb{N}$. Then, we can choose the third element of the first column, $\frac{1}{2}[f_1(n_0) + f_2(n_0)]$, and the second element of the second one, $f_1(n) - f_3(n)$.

Acknowledgments: This work has been supported by the grant BFM2003–01034 from the D.G.I. of the Spanish *Ministerio de Ciencia y Tecnología.*

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,315-332,COPYRIGHT 2008 EUDOXUS PRESS, LLC

PERTURBATIONS OF OPERATORS ON TENSOR PRODUCTS AND SPECTRUM LOCALIZATION OF MATRIX DIFFERENTIAL OPERATORS*

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Abstract

We investigate spectrum perturbations of a class of operators on the tensor product of a separable Hilbert space and a finite dimensional space. The abstract results are applied to the higher order matrix nonselfadjoint differential operators on finite and infinite real segments. Besides, bounds for the spectra and estimates for the norm of the resolvents of the differential operators are derived. We also investigate conditions, under which the considered operators generate analytic and stable semigroups. Our main tool is a combined use of some properties of operators on tensor products of Hilbert spaces and the recent estimates for norms of resolvents of matrices.

Key words: nonselfadjoint differential operators, higher order operators resolvent, spectrum, tensor product

AMS (MOS) subject classification: 34L15, 47E05

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* This research was supported by the Kamea Fund of the Israel

1 Introduction and notation

The spectrum of ordinary differential operators was considered in a lot of papers and books, cf. [1, 7, 8, 10] and references therein. However, the bounds for the spectrum of the non-selfadjoint higher order differential operators are not enough investigated. The paper [11] should be mentioned. In that paper, the author studies the problem of localization of the spectrum for a class of a scalar differential operator on a finite segment.

In the present paper we establish bounds for the spectrum of a class of the higher order matrix nonselfadjoint differential operators on finite and infinite real segments. In addition, we derive estimates for the norm of the resolvents and investigate the conditions, under which the considered operators generate analytic and stable semigroups. We consider the differential operators as operators on the tensor product of a separable Hilbert space and a finite dimensional space. Our main tool is a combined use of some properties of operators on tensor product of Hilbert spaces and the recent estimates for norms of resolvents of matrices.

A few words about the contents. The paper consists of 11 sections. In Section 2 we formulate the main result of the paper-Theorem 2.1 on spectrum localization of abstract operators. In Section 3, the proof of Theorem 2.1 is presented. In Sections 4 and 5 we establish some auxiliary results which are used in the following sections. Section 6 deals with sectorial operators. Note that all the differential operators considered here are sectorial. In Section 7 we derive the conditions, under which the considered operators generate stable semigroups. Sections 8, 9 and 10 are devoted to differential operators. In Section 8, the periodic boundary conditions are imposed. The Dirichlet problem on a finite interval is explored in Section 9. Section 10 is devoted to operators on the whole real line. In Section 11 we give an example.

Let $\mathbf{C}^n$ be the complex Euclidean space with the scalar product $\subset .,. _n$, the norm $\|.\|_n = \sqrt{< .,. >_n}$ and the unit operator I_n. Let J be a (finite or infinite) segment of the real axis. $L^2(J) = L^2(J, \mathbf{C}^n)$ denotes the complex Hilbert space of functions defined on J with values in $\mathbf{C}^n$, the scalar product

$$(f,h)_{L^2} := \int_J \subset f(x), h(x)\ _n dt \ \ (f, h \in L^2(J, \mathbf{C}^n))$$

and the norm $|.|_{L^2}$. For a linear operator A, $R_\lambda(A)$ is the resolvent, $Dom\ (A)$ is the domain and $\sigma(A)$ is the spectrum. Moreover, $\beta(A) = \inf\ Re\ \sigma(A)$ and $\theta(A) = \inf\ |\sigma(A)|$.

Let Q be a constant $n \times n$-matrix. Let $\lambda_k(Q)$ $(k = 1, ..., n)$ be the eigenvalues of Q with their multiplicities. The following quantity plays a key role hereafter:

$$g(Q) = (N^2(Q) - \sum_{k=1}^{n} |\lambda_k(Q)|^2)^{1/2}, \tag{1.1}$$

where $N(Q)$ is the Hilbert-Schmidt (Frobenius) norm of Q, i.e. $N^2(Q) = Trace(QQ^*)$. The asterisk means the adjoint. If Q is a normal matrix: $QQ^* = Q^*Q$, then $g(Q) = 0$. The following relations are true:

$$g(Qe^{i\tau} + zI) = g(Q) \ \ (z \in \mathbf{C}, \tau \in \mathbf{R}), \tag{1.2}$$

$$g^2(Q) \leq N^2(Q^* e^{-i\tau} - Qe^{i\tau})/2 \ \ (\tau \in \mathbf{R}) \tag{1.3}$$

and

$$g^2(Q) \leq N^2(Q) - |Trace Q^2|.$$

If matrices Q and Q_1 commute, then $g(Q + Q_1) \leq g(Q) + g(Q_1)$. For the proofs see [4, Section 2.1].

2 Statement of the the main result

Let E be a separable Hilbert space with a scalar product $< .,. >_E$, and the norm $\|.\|_E = \sqrt{< .,. >_E}$. Let $H = E \otimes \mathbf{C}^n$ be the tensor product of E and $\mathbf{C}^n$. I_H, I_E and I_n are the unit operators in H, E and $\mathbf{C}^n$, respectively. The scalar product in H is defined by

$$< y \otimes h, y_1 \otimes h_1 >_H = < y, y_1 >_E < h, h_1 >_n \quad (y, y_1 \in E, h, h_1 \in \mathbf{C}^n)$$

and the cross norm is $\|.\|_H = \sqrt{< .,. >_H}$. From the theory of tensor products we only need the basic definition and elementary facts which can be found in [1].

Everywhere below S is an invertible normal operator in E and $S_0 = S \otimes I_n$; $c_k, \ k = 0, ..., m$ are constant $n \times n$-matrix matrices and c_m is invertible. Consider the operator

$$W(S) = \sum_{k=0}^{m} c_k \otimes S^k \ \ (Dom\ (W(S)) = Dom\ (S_0^m)).$$

Let $E(s)$ be the orthogonal resolution of the identity defined on $\sigma(S)$, such that

$$S = \int_{\sigma(S)} s dE(s).$$

Then

$$W(S) = \int_{\sigma(S)} W(s) \otimes dE(s),$$

where

$$W(s) = \sum_{k=0}^{m} c_k s^k \ \ (s \in \sigma(S)).$$

Note that according to (1.1).

$$g(W(s)) = \sqrt{N^2(W(s)) - \sum_{k=1}^{n} |\lambda_k(W(s))|^2}, \tag{2.1}$$

where $\lambda_k(W(s))$ are the eigenvalues of $W(s)$ with a fixed s.

Furthermore, let T be a linear operator in H having the properties $Dom\ (T) = Dom\ (S_0^m)$ and

$$q := \|(W(S) - T)S_0^{-\nu}\|_H < \infty \tag{2.2}$$

for some $\nu \in [0, m]$. Here $S_0^0 = I_H$.

Now we are in a position to formulate the main result of the paper.

Theorem 2.1 *Assume (2.2) and there is a constant v_0, such that*

$$g(W(s)) \le v_0 |s|^\nu \quad (s \in \sigma(S)). \tag{2.3}$$

Then the spectrum of T lies in the set

$$\cup_{s\in\sigma(S);j=1,\dots,n}\{\lambda \in \mathbf{C} : |\lambda - \lambda_j(W(s))| \le |s|^\nu y(q, v_0)\}, \tag{2.4}$$

where $y(q, v_0)$ is the extreme right-hand root of the algebraic equation

$$y^n = q \sum_{k=0}^{n-1} \frac{v_0^k y^{n-k-1}}{\sqrt{k!}}. \tag{2.5}$$

The proof of this theorem is presented in the next section. Put

$$w_n = \sum_{j=0}^{n-1} \frac{1}{\sqrt{k!}}.$$

By setting $z = \frac{v_0}{z}$, it follows from (2.5) that

$$z^n = \frac{q}{v_0} \sum_{k=0}^{n-1} \frac{z^{n-k-1}}{\sqrt{k!}}.$$

Due to Lemma 1.11.1 from [3], $y(q, v_0) \le \delta(q, v_0)$, where

$$\delta(q, v_0) = \begin{cases} qw_n & \text{if } qw_n \ge v_0, \\ (qv_0^{n-1} w_n)^{1/n} & \text{if } qw_n \le v_0 \end{cases}. \tag{2.6}$$

This result and Theorem 2.1 imply

Corollary 2.2 *Under conditions (2.2) and (2.3), the spectrum of T lies in the set*

$$\cup_{s\in\sigma(S);j=1,\dots,n}\{\lambda \in \mathbf{C} : |\lambda - \lambda_j(W(s))| \le |s|^\nu \delta(q, v_0)\}. \tag{2.7}$$

Note that *if all the matrices* c_k *are selfadjoint, then* $g(W(s)) = 0$ *and* $y(q, v_0) = q$. *Since* $\nu \le m$, *everywhere below, we can take* m *instead of* ν. But in appropriate situations some $\nu < m$ allows us to derive sharper results.

Take $H = L^2(J, \mathbf{C}^n) = L^2(J) \otimes \mathbf{C}^n$. Consider in $L^2(J, \mathbf{C}^n)$ the operator B defined by

$$Bu = \sum_{k=0}^{m} a_k(x) S_0^k u \quad (x \in J,\ u \in Dom\ (B) = Dom\ (S^m)), \tag{2.8}$$

where $a_k(x),\ \ k = 0, ..., m$ are $n \times n$-matrix-valued functions measurable and bounded on J. In addition $a_m(x)$ is invertible. It is assumed that for a nonnegative integer $\nu \le m$,

$$a_k(x) = c_k + b_k(x) \text{ for } k = 0, ..., \nu \text{ and } a_k(x) \equiv c_k \text{ for } k = \nu + 1, ..., m, \tag{2.9}$$

where $b_k(x)$ are variable $n \times n$-matrices and c_k are constant $n \times n$-matrices, again. Put $\|b_k\|_C = \sup_x \|b_k(x)\|_n$.

Corollary 2.3 *Under conditions (2.9) and (2.3), the spectrum of operator* B *lies in the set (2.4) (and therefore, in the set (2.7)) with* $q = q_0$, *where*

$$q_0 := \sum_{j=0}^{\nu} \|b_j\|_C\ \theta^{j-\nu}(S). \tag{2.10}$$

In other words, for any $\lambda \in \sigma(B)$, *there is a point* $s \in \sigma(S)$ *and an eigenvalue* $\lambda_j(W(s))$ *of the matrix* $W(s)$, *such that*

$$|\lambda - \lambda_j(W(s))| \le |s|^\nu y(q_0, v_0) \le |s|^\nu \delta(q_0, v_0).$$

Indeed, since,

$$B - W(S) = \sum_{j=0}^{\nu} b_j(x) S_0^j,$$

we have

$$\|(W(S) - B)S_0^{-\nu}\|_H \le \sum_{k=0}^{\nu} \|b_k\|_C \|S^{k-\nu}\|_E \le q_0.$$

Now the required result is due to Theorem 2.1 and Corollary 2.2 .

3 Proof of Theorem 2.1

For numbers $s \in \sigma(S)$ and $\lambda \in \mathbf{C}$, put

$$\rho(W(s), \lambda) = min_{j=1,...,n} |\lambda_j(W(s)) - \lambda|.$$

Lemma 3.1 *For a $\lambda \in \mathbf{C}$, let*

$$p(\lambda,\nu) := \sup_{s\in\sigma(S)} |s|^{\nu} \sum_{k=0}^{n-1} \frac{g^k(W(s))}{\sqrt{k!}\rho^{k+1}(W(s),\lambda)} < \infty. \tag{3.1}$$

Then the operator $S_0^{\nu}(W(S) - I_H\lambda)^{-1}$ is bounded and

$$\|S_0^{\nu}(W(S) - I_H\lambda)^{-1}\| \leq p(\lambda,\nu).$$

Proof: Let Q be a linear operator in $\mathbf{C}^n$, and $\rho(Q,\lambda)$ the distance between $\sigma(Q)$ of Q and a complex point λ. Then

$$\|(Q - I_n\lambda)^{-1}\|_n \leq \sum_{k=0}^{n-1} \frac{g^k(Q)}{\sqrt{k!}\rho^{k+1}(Q,\lambda)} \ (\lambda \notin \sigma(Q))$$

where $\rho(Q,\lambda) = min_{j=1,\ldots,n}|\lambda_j(Q) - \lambda|$. For the proof see [4, Section 2.1]. Replacing in this estimate Q by $W(s)$, we have

$$\|(W(s) - I_n\lambda)^{-1}\|_n \leq \sum_{k=0}^{n-1} \frac{g^k(W(s))}{\sqrt{k!}\rho^{k+1}(W(s),\lambda)}. \tag{3.2}$$

But

$$(W(S) - I_H\lambda)^{-1} = \int_{\sigma(S)} (W(s) - I_n\lambda)^{-1} \otimes dE(s),$$

and

$$S_0^{\nu}(W(S) - I_H\lambda)^{-1} = \int_{\sigma(S)} s^{\nu}(W(s) - I_n\lambda)^{-1} \otimes dE(s).$$

Consequently

$$\|S_0^{\nu}(W(S) - I_H\lambda)^{-1}\|_H = \sup_{s\in\sigma(S)} |s|^{\nu}\|(W(s) - I_n\lambda)^{-1}\|_n. \tag{3.3}$$

In particular,

$$\|(W(S) - I_H\lambda)^{-1}\|_H = \sup_{s\in\sigma(S)} \|(W(s) - I_n\lambda)^{-1}\|_n.$$

This and (3.2) prove the lemma. $\square$

Lemma 3.2 *Let the conditions (2.2) and*

$$qp(\lambda,\nu) < 1 \tag{3.4}$$

hold. Then λ is a regular point for T and

$$\|(T - I_H\lambda)^{-1}\|_H \leq p(\lambda,0)(1 - qp(\lambda,\nu))^{-1}. \tag{3.5}$$

Proof: Since

$$(W(S)-I_H\lambda)^{-1}-(T-I_H\lambda)^{-1}=(T-I_H\lambda)^{-1}(T-W(S))(W(S)-I_H\lambda)^{-1}=$$

$$(T-I_H\lambda)^{-1}(T-W(S))S_0^{-\nu}S_0^{\nu}(W(S)-I_H\lambda)^{-1},$$

the condition

$$\theta_0(\lambda):=q\|S_0^{\nu}(W(S)-I_H\lambda)^{-1}\|_H<1 \tag{3.6}$$

implies

$$\|(T-\lambda)^{-1}\|_H\le\|(W(S)-I_H\lambda)^{-1}\|_H(1-\theta_0(\lambda))^{-1}. \tag{3.7}$$

Now the previous lemma proves the required result. □

Proof of Theorem 2.1: Let $\lambda\in\sigma(T)$. Due to the previous lemma, for some $s\in\sigma(S)$ and j,

$$q|s|^{\nu}\sum_{k=0}^{n-1}\frac{g^k(W(s))}{\sqrt{k!}|\lambda_j(W(s))-\lambda|^{k+1}}\ge 1.$$

Hence due to (2.3),

$$q|s|^{\nu}\sum_{k=0}^{n-1}\frac{(v_0|s|^{\nu})^k}{\sqrt{k!}|\lambda_j(W(s))-\lambda|^{k+1}}\ge 1.$$

Consequently, $|\lambda_j(W(s))-\lambda|\le z(s)$, where $z(s)$ is the extreme right-hand root of the equation

$$q|s|^{\nu}\sum_{k=0}^{n-1}\frac{(v_0|s|^{\nu})^k}{\sqrt{k!}z^{k+1}}=1. \tag{3.8}$$

Put in this equation $z=|s|^{\nu}y$. Then we have equation (2.5). Hence

$$z(s)\le|s|^{\nu}y(q,v_0).$$

This finishes the proof. □

4 Auxiliary inequalities

In the present section we suggest the inequalities, which allow us to check condition (2.3). Let

$$S=S^*. \tag{4.1}$$

Then due to (2.1) and (1.3) with $\tau=0$, we have

$$\sqrt{2}g(W(s))\le\sum_{k=0}^{m}N(c_k-c_k^*)|s|^k. \tag{4.2}$$

That is, condition (2.3) holds with $\nu = m$ and

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{m} N(c_k - c_k^*) \theta^{k-m}(S). \tag{4.3}$$

If, additionally,

$$c_k = c_k^*, \ k = \nu + 1, ..., m,$$

then under (4.1) condition (2.3) holds with

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{\nu} N(c_k - c_k^*) \theta^{k-\nu}(S).$$

The case

$$S = -S^* \tag{4.4}$$

can be reduced to (4.1), if we take iS instead of S. Besides,

$$\sqrt{2} g(W(s)) \le \sum_{k=0}^{m} N(c_k + c_k^*) |s|^k.$$

That is, condition (2.3) holds with $\nu = m$ and

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{m} N(c_k + c_k^*) \theta^{k-m}(S).$$

Let, additionally, $c_k = -c_k^*, \ k = \nu + 1, ..., m$. Then condition (2.3) holds with

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{\nu} N(c_k + c_k^*) \theta^{k-\nu}(S).$$

Consider now the operator

$$B = S_0^m + a_0(x), \ x \in J \tag{4.5}$$

with $a_0(x) = b_0(x) + c_0$ and $\nu = 0$. Take $W(S) = S_0^m + c_0$. According to (1.2) $g(W(s)) = g(c_0) = v_0$. Thanks to (2.7), taking into account that in the considered case $\nu = 0$ we can assert that

$$\sigma(B) \subset \cup_{s \in \sigma(S); j=1,...,n} \{\lambda \in \mathbf{C} : |\lambda - s^m - \lambda_j(c_0)| \le \delta(q, v_0)\}. \tag{4.6}$$

5 Additional estimates for resolvents

In this section we derive additional estimates for the resolvent of the operator B defined by (2.8), which in appropriate situations, are more useful than Lemma 3.2.

Put

$$\zeta_n = (n-1)^{-(n-1)/2}$$

and

$$w(\lambda, s) := \frac{\zeta_n [N^2(W(s)) - 2Re\ (\overline{\lambda}\ Trace\ (W(s)))\ + n|\lambda|^2]^{(n-1)/2}}{|det\ (\lambda I - W(s))|}$$

for $s \in \sigma(S), \lambda \in \mathbf{C}$. It should be noted that

$$w(\lambda, s) \leq \tilde{w}(\lambda, s), \tag{5.1}$$

where

$$\tilde{w}(\lambda, s) := \frac{\zeta_n [N(W(s)) + |\lambda|\sqrt{n}]^{n-1}}{|det\ (\lambda I - W(s))|}.$$

So in the next lemma $w(\lambda, s)$ can be replaced by the simple function $\tilde{w}(\lambda, s)$.

Lemma 5.1 *Let*

$$\psi(\lambda, \nu) := \sup_{\sigma(S)} |s|^{\nu} w(\lambda, s) < \infty.$$

Then the operator $S_0^{\nu}(W(S) - I_H\lambda)^{-1}$ *is bounded and*

$$\|S_0^{\nu}(W(S) - I_H\lambda)^{-1}\|_H \leq \psi(\lambda, \nu).$$

Proof: We need the following result: let Q be a constant $n \times n$-matrix. Then for all $\lambda \notin \sigma(Q)$,

$$\|(I\lambda - Q)^{-1}\ det\ (\lambda I - Q)\|_n \leq \zeta_n [N^2(Q) - 2Re\ (\overline{\lambda}\ Trace\ (Q))\ + n|\lambda|^2]^{(n-1)/2}. \tag{5.2}$$

For the proof see [4, p. 28]. Hence,

$$\|(I\lambda - W(s))^{-1}\| \leq w(s, \lambda) \leq \tilde{w}(s, \lambda)\ \ (\lambda \notin \sigma(W(s))).$$

Taking into account (3.3), we get the required result. $\square$

The latter lemma and relations (3.6), (3.7) yield

Theorem 5.2 *Let an operator* T *satisfy the conditions (2.2) and*

$$q\psi(\lambda, \nu) < 1 \tag{5.3}$$

hold. Then λ *is a regular point of operator* T *and*

$$\|(T - I_H\lambda)^{-1}\|_H \leq \psi(\lambda, 0)(1 - q\psi(\lambda, \nu))^{-1}. \tag{5.4}$$

Recall that B and q_0 are defined by (2.8) and (2.10), respectively.

Corollary 5.3 *Let the conditions (2.9) and (5.3) hold with* $q = q_0$. *Then* λ *is a regular point of operator* B *and*

$$\|(B - I_H\lambda)^{-1}\|_H \leq \psi(\lambda, 0)(1 - q_0\psi(\lambda, \nu))^{-1}. \tag{5.5}$$

Therefore, under conditions (2.9), for any $\lambda \in \sigma(B)$, *there is an* $s \in \sigma(S)$, *such that*

$$q_0\zeta_n|s|^{\nu}\ \frac{[N(W(s)) + \sqrt{n}|\lambda|]^{n-1}}{|det\ (\lambda I - W(s))|} \geq 1.$$

6 Sectorial operators

For a natural number $\mu \geq 1$, let

$$m = 2\mu, \tag{6.1}$$

and the following conditions hold:

$$a_m(x) = a_m^*(x) \ (x \in J) \text{ is a strongly positive definite matrix and } S^\mu \text{ is selfadjoint .} \tag{6.2}$$

We use the notion of a sectorial operator, which can be found, for instance, in [5].

Theorem 6.1 *Under conditions (6.1) and (6.2), let S be a normal operator defined by the expression*

$$Su = du/dx + w_0 u \ \ (u \in D(S))$$

with a constant $w_0 \in \mathbf{C}$ and

$$D(S) = \{u \in L^2(J, \mathbf{C}^n) : du/dx \in L^2(J, \mathbf{C}^n) + \textit{selfafjoint boundary conditions}\}.$$

In addition, let

$$a_m(x) \textit{ have bounded derivatives on } J \textit{ up to order } \mu. \tag{6.3}$$

Then the operator B defined by (2.8) is sectorial.

Proof: Firstly, let $w_0 = 0$. Take $A_0 = S_0^\mu a_m(.) S_0^\mu$. Clearly,

$$A_0 - a_m(x) S_0^m = \sum_{k=0}^{\mu-1} \binom{\mu}{k} a_k^{(\mu-k)}(x) S_0^{k+\mu}$$

where $\binom{n}{k} = n!/(n-k)!k!$ are the binomial coefficients. So $(A_0 - a_m(x)S_0^m)S_0^{-m+1}$ is a bounded operator and

$$(S_0^\mu a_m(x) S_0^\mu u, u) \leq \beta_0 \ (S_0^m u, u) \ \ (u \in D(B)).$$

where $\beta_0 = \sup_x \|a_m(x)\|_n$. Hence it follows that $A_0 \leq \beta_0 S_0^m$ and therefore

$$S_0^{-m\tau} = S_0^{-m+1} \leq \beta_0^\tau A_0^{-\tau} \ \ (\tau = (m-1)/m)).$$

Thus $(A_0 - a_m(x)S_0^m)A_0^{-\tau}$ is a bounded operator. Since the operator

$$(B - a_m(x)S_0^m)S_0^{-m+1}$$

is bounded, we can assert that $(A_0 - B)A_0^{-\tau}$ is a bounded operator. But A_0 is selfadjoint positive definite. So due to Theorem 1.4.6 from [5] operator B is sectorial. The case $w_0 \neq 0$ follows by a bounded perturbation argument. □

Similarly the following theorem can be proved.

Theorem 6.2 *Under conditions (6.1) and (6.2), let S be a normal operator defined by the expression*

$$Su = d^2u/dx^2 + w_0 u \ \ (u \in D(S))$$

with a constant $w_0 \in \mathbf{C}$ and

$$D(S) = \{u \in L^2(J, \mathbf{C}^n) : d^2u/dx^2 \in L^2(J, \mathbf{C}^n) + \textit{selfafjoint boundary conditions}\}.$$

In addition, let

$$a_m(x) \textit{ have bounded derivatives on } J \textit{ up to order } 2\mu. \tag{6.4}$$

Then the operator B defined by (2.8) is sectorial.

Under the hypothesis of Theorem 6.1 or 6.2, thanks to Theorem 1.3.4 [5], the semigroup e^{-Bt} generated by $-B$ is an analytic. Moreover, for any $a < \beta(B)$, where

$$\beta(B) := \inf \ Re\ \sigma(B)$$

there is a constant γ_a, such that

$$\|e^{-Bt}\|_H \leq \gamma_a e^{-at} \ \ (t \geq 0) \tag{6.5}$$

cf. [5, p. 21].

7 Stable operators

We will say that operator B is stable if $\beta(B) = \inf \ Re\ \sigma(B) > 0$. Under the hypothesis of Theorems 6.1 and 6.2, according to (6.5) this means that the semigroup e^{-Bt} is exponentially stable. The aim of this section is to establish explicit stability conditions for the considered operators.

Due to (2.4) and (2.7), for any $\lambda \in \sigma(B)$, there are $s \in \sigma(S)$ and $j = 1, ..., n$, such that

$$Re\ \lambda_j(W(s)) - Re\ \lambda \leq |s|^\nu y(q_0, v_0) \leq |s|^\nu \delta(q_0, v_0)$$

where q_0 is defined by (2.10). Hence

$$\beta(B) = \inf \ Re\ \sigma(B) \geq \beta(W, \nu),$$

where

$$\beta(W, \nu) := \inf \{Re\ \lambda_j(W(s)) - |s|^\nu y(q_0, v_0) : s \in \sigma(S); j = 1, ..., n\}.$$

For a matrix Q, put

$$\beta_R(Q) = \min_k \lambda_k(Q + Q^*)/2 \text{ and } \alpha_R(Q) = \max_k \lambda_k(Q + Q^*)/2,$$

and consider the case
$$S = S^* > 0. \tag{7.1}$$
Take into account that for any matrix Q and an eigenvalue $\lambda_j(Q)$ we have
$$< (Q + Q^*)h, h >_n = 2Re\ \lambda_j(Q),$$
where h is the normed eigenvector. Then
$$\beta_R(W(s)) \geq \sum_{k=0}^{m} \beta_R(c_k)s^k.$$
Hence
$$\beta(B) \geq \inf \{\sum_{k=0}^{m} \beta_R(c_k)s^k - s^\nu y(q_0, v_0) : s \in \sigma(S)\}.$$
We thus get

Theorem 7.1 *Let the conditions (7.1) and*
$$\inf \{\sum_{k=0}^{m} \beta_R(c_k)s^k - s^\nu y(q_0, v_0) : s \in \sigma(S)\} > 0$$
hold. Then the operator B defined by (2.8) is stable.

For instance, if
$$y(q_0, v_0) < \sum_{k=\nu}^{m} \beta_R(c_k)\theta^{k-\nu}(S),$$
then under (7.1) we get
$$\beta(B) \geq \sum_{k=0}^{m} \beta_R(c_k)\theta^k(S) - \theta^\nu(S)y(q_0, v_0) > 0$$
and thus B is stable.

8 Periodic boundary conditions

Consider the operator
$$A = \sum_{k=0}^{m} \tilde{a}_k(x)\frac{d^k}{dx^k} \quad (x \in (0, 1)) \tag{8.1}$$
with bounded matrix-valued coefficients $\tilde{a}_k(x)$. Put $H = L^2([0, 1], \mathbf{C}^n)$, $E = L^2[0, 1]$ and
$$Dom\ (A) = \{u \in H = L^2([0, 1], \mathbf{C}^n) : u^{(k)} \in H : \ k = 1, ..., m;$$

$$u^{(j)}(0) = u^{(j)}(1);\ j = 0, ..., m-1\}. \tag{8.2}$$

Take $\nu = m$ and

$$Su = -iu' - \pi;\ u \in Dom\ (S) = \{v \in L^2[0,1] : v' \in L^2[0,1];\ v(0) = v(1)\}.$$

Clearly, S is selfadjoint and invertible,

$$\sigma(S) = \{2\pi k - \pi : k = 0, \pm 1, \pm 2, ...\} \text{ and } \theta(S) = \pi. \tag{8.3}$$

Substitute $u' = i(S+\pi)u$ into (8.1). Then we have operator B defined by (2.8) with

$$a_j(x) = \sum_{k=j}^{m} \tilde{a}_k i^k \tbinom{k}{j} \pi^{k-j} \quad (j = 0, ..., m). \tag{8.4}$$

So $a_m(x) = \tilde{a}_k i^m$. Under (2.9) due to (4.3), condition (2.3) holds with

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{m} N(c_k - c_k^*) \pi^{k-m}. \tag{8.5}$$

In the considered case, according to (2.10)

$$q_0 = \sum_{j=0}^{m} \|b_j\|_C \pi^{j-m}. \tag{8.6}$$

Recall that $W(s), y(q, v_0)$ and $\delta(q, v_0)$ are defined in Section 2. For brevity, put

$$s_j = \pi(2j-1).$$

Thanks to Corollary 2.3, we get

Corollary 8.1 *Let A be defined by (8.1), (8.2). Then, under (2.9), for any $\lambda \in \sigma(A)$ there are an integer $k = 0, \pm 1, ...$ and an eigenvalue $\lambda_j(W(s_k))$ of matrix $W(s_k)$, such that*

$$|s_k|^{-m} |\lambda - \lambda_j(W(s_k))| \le y(q_0, v_0) \le \delta(q_0, v_0).$$

Moreover, for a $\lambda \in \mathbf{C}$, let

$$p_1(\lambda, m) := \sup_{j=0,\pm 1,...} \sum_{k=0}^{n-1} \frac{v_0^k |s_j|^{(k+1)m}}{\sqrt{k!}\rho^{k+1}(W(s_j), \lambda)} < \frac{1}{q_0}, \tag{8.7}$$

where q_0 and v_0 are is defined by (8.5) and (8.6), respectively. Lemma 3.2 implies

Corollary 8.2 *Let A be defined by (8.1), (8.2). In addition, let conditions (2.9) and (8.7) hold. Then λ is a regular point for A and*

$$\|(A - I_H \lambda)^{-1}\| \le p_1(\lambda, 0)(1 - q_0 p_1(\lambda, m))^{-1}.$$

where

$$p_1(\lambda, 0) := \sup_{j=0,\pm 1,...} \sum_{k=0}^{n-1} \frac{v_0^k |s_j|^{km}}{\sqrt{k!}\rho^{k+1}(W(s_j), \lambda)}.$$

Now let conditions (6.1)-(6.3) hold with $J = [0,1]$ and (8.4) taken into account. Then due to Theorem 6.1the operator, defined by (8.1), (8.2) generates an analytic semigroup.

9 Dirichlet conditions

Consider the operator

$$A = \sum_{k=0}^{m} a_k(x)(-1)^k \frac{d^{2k}}{dx^{2k}} \quad (x \in (0,1)), \tag{9.1}$$

where $a_k(x)$ are matrix-valued functions bounded on $[0,1]$. Let H and E be the same as in the previous section. Put

$$Dom\ (A) = \{u \in H = L^2([0,1], \mathbf{C}^n) : u^{(k)} \in H :\ k = 1, ..., 2m,$$
$$u^{(j)}(0) = u^{(j)}(1) = 0,\ j = 0, ..., m-1\}. \tag{9.2}$$

That is the boundary conditions

$$u^{(j)}(0) = u^{(j)}(1) = 0;\ j = 0, ..., m-1$$

hold. Take $\nu = m$ and $Su = u''$ with

$$Dom\ (S) = \{u \in E = L^2[0,1] : u^{''} \in E;\ \ u(0) = u(1) = 0\}.$$

We have $\sigma(S) = \{(\pi k)^2 : k = 1, 2, ...\}$. So S is invertible and $\theta(S) = \pi^2$. Due to (4.3), condition (2.3) holds with

$$v_0 = \frac{1}{\sqrt{2}} \sum_{k=0}^{m} N(c_k - c_k^*)\pi^{2(k-m)}. \tag{9.3}$$

Under consideration, (2.10) implies

$$q_0 = \sum_{j=0}^{m} \|b_j\|_C \pi^{2(j-m)}. \tag{9.4}$$

Now Corollary 2.3 implies

Corollary 9.1 *Let the operator A be defined by (9.1) and (9.2) and condition (2.9) hold. Then for any $\lambda \in \sigma(A)$, there are an integer $k = 1, 2, ...$ and an eigenvalue $\lambda_j(W((k\pi)^2)$ of matrix $W((k\pi)^2)$, such that*

$$(\pi k)^{-2m}|\lambda - \lambda_j(W((k\pi)^2)| \le y(q_0, v_0) \le \delta(q_0, v_0).$$

Now let conditions (6.1), (6.2) and (6.4) hold with $J = [0,1]$. Then the operator, defined by (9.1), (9.2) generates an analytic semigroup. Moreover, Theorem 7.1 yields the following result.

Corollary 9.2 *Let the conditions (2.9), (6.1),*

$$\pi^2\beta_R(c_{2k}) > \alpha_R(c_{2k-1})\ (k < m/2)\ \textit{and}\ \pi^2\beta_R(c_m) > \pi^2 y(q_0, v_0) + \alpha_R(c_{m-1})$$

hold. Then the operator A defined by (9.1), (9.2) satisfies the inequality

$$\beta(A) > \sum_{k=0}^{\mu} \pi^{4k}\beta_R(c_{2k}) - \sum_{k=0}^{\mu-1} \alpha_R(c_{2k+1})\pi^{4k+2} - \pi^{2m} y(q_0, v_0) > 0.$$

That is, A is stable.

10 Operators on the whole line

Put $H = L^2(R^1, \mathbf{C}^n)$, $E = L^2(R^1)$ and consider the operator

$$A = \sum_{k=0}^{m} \tilde{a}_k(x) \frac{d^k}{dx^k} \quad (x \in \mathbf{R}) \tag{10.1}$$

with matrix coefficients bounded on $\mathbf{R}$ and the domain

$$Dom\ (A) = \{u \in H = L^2(R^1, \mathbf{C}^n) : u^{(k)} \in H;\ k = 1, ..., m\}, \tag{10.2}$$

Take $\nu = m$ and

$$Su = u' - u, u \in Dom\ (S) = \{y \in L^2(R^1) : y' \in L^2(R^1)\}.$$

Then S is normal and invertible. Moreover, $\sigma(S) = \{it - 1 : t \in R^1\}$ and $\theta(S) = 1$.

Substituting $u' = Su + u$ in (10.1), we have the operator B defined by (2.8) with

$$a_j = \sum_{k=j}^{m} \binom{k}{j} \tilde{a}_k$$

Due to (1.3),

$$\sqrt{2} g(W(s)) \le \sum_{k=0}^{m} N(s^k c_k + \overline{s}^k c_k^*) \le 2 \sum_{k=0}^{m} |s|^k N(c_k)$$

Thus (2.3) holds with $\nu = m$ and

$$v_0 = \sqrt{2} \sum_{k=0}^{m} N(c_k). \tag{10.3}$$

According to (2.10),

$$q_0 := \sum_{j=0}^{m} \|b_j\|_C. \tag{10.4}$$

Thanks to Corollary 2.3, we get the following result.

Corollary 10.1 *Let A be defined by (10.1) and (10.2), and condition (2.9) hold. Then for any $\lambda \in \sigma(A)$ there are a $t \in \mathbf{R}$ and an eigenvalue $\lambda_j(W(it-1))$ of matrix $W(it-1)$, such that*

$$(t^2 + 1)^{-m/2} |\lambda - \lambda_j(W(it-1))| \le y(q_0, v_0) \le \delta(q_0, v_0),$$

where v_0 and q_0 are defined by (10.3) and (10.4).

11 Example

In space $H = L^2([0,1], \mathbf{C}^n)$ consider the operator

$$(Au)(x) = \tilde{a}_m(x)\frac{d^m u(x)}{dx^m} + \tilde{a}_0(x)u(x) \ \ (u \in D(A); \ x \in (0,1)) \tag{11.1}$$

with bounded matrix-valued coefficients $\tilde{a}_m(x), \tilde{a}_0(x)$. and

$$Dom\ (A) = \{u \in H = L^2([0,1], \mathbf{C}^n) : u^{(k)} \in H : \ k = 1, ..., m;$$

$$u^{(j)}(0) = u^{(j)}(1); \ j = 0, ..., m-1\}. \tag{11.2}$$

Take $\nu = m$ and

$$Su = -iu' - \pi; \ u \in Dom\ (S) = \{v \in L^2[0,1] : v' \in L^2[0,1]; \ v(0) = v(1)\}.$$

So $\sigma(S)$ is defined by (8.3). Substitute $u' = i(S + \pi)u$ into (11.1). Then we have operator B defined by (2.8) with

$$a_j(x) = \tilde{a}_m(x) i^m \tbinom{m}{j} \pi^{m-j} \ \ (j = 1, ..., m)$$

and

$$a_0(x) = \tilde{a}_0(x) + \tilde{a}_m(x)\pi^m i^m.$$

Put

$$c_j = a_j(0), \ b_j(x) = a_j(x) - a_j(0) \ \ (j = 0, ..., m).$$

Then v_0 and q_0 are defined by (8.5) and (8.6), respectively. Recall also that

$$w_n = \sum_{j=0}^{n-1} \frac{1}{\sqrt{k!}}.$$

Assume that

$$q w_n \le v_0. \tag{11.3}$$

Then according to (2.6)

$$\delta(q_0, v_0) = (q_0 v_0^{n-1} w_n)^{1/n}.$$

Thanks to Corollary 8.1, we can assert that for the operator A defined by (11.1), (11.2) under condition (11.3), for any $\lambda \in \sigma(A)$ there are an integer $l = 0, \pm 1, ...$ and an eigenvalue $\lambda_j(W(s_l))$ of the matrix

$$W(s_l) = \sum_{k=0}^{m} c_k s_l^k,$$

where $s_l = \pi(2l - 1)$, such that

$$|s_l|^{-m} |\lambda - \lambda_j(W(s_l))| \le (q_0 v_0^{n-1} w_n)^{1/n}.$$

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,333-340,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Composition Followed by Differentiation Between Weighted Bergman Spaces and Bloch Type Spaces

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Abstract: The boundedness of products of differentiation operators and composition operators between the weighted Bergman space and the Bloch type space are discussed in this paper.

MSC 2000: 47B38, 30H05.

Keywords: differentiation operator, composition operator, Bloch type space, weighted Bergman space.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all functions analytic on $\mathbb{D}$. Let dA denote the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α is the set of all functions f analytic on $\mathbb{D}$ satisfying

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$.

An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space, or β-Bloch space $\mathcal{B}^\beta (\beta > 0)$ if

$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\beta |f'(z)| < \infty.$$

Under the norm $\|\cdot\|_{\mathcal{B}^\beta}$, $\mathcal{B}^\beta$ is a Banach space. When $\beta = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well known Bloch space. Let $\mathcal{B}^\beta_0$ denote the subspace of $\mathcal{B}^\beta$ consisting of those $f \in \mathcal{B}^\beta$ for which

$$(1 - |z|^2)^\beta |f'(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

This space is called the little β-Bloch space(see [9, 10]).

Let φ be an analytic self map of $\mathbb{D}$. Associated with φ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{D})$. It is a well known consequence of the Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy and Bergman spaces. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces (see, for example, [2] and [10]).

Let D be the differentiation operator. The composition operator is one of the typical bounded operators, while the differentiation operator is typically unbounded on many analytic function spaces. The products of the composition operator and differentiation operator are defined by

$$DC_\varphi(f) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D})$$

and

$$C_\varphi D(f) = f'(\varphi), \quad f \in H(\mathbb{D})$$

respectively. The operator DC_φ was first studied by Hibschweiler and Portnoy in [4], where the boundedness of DC_φ between Hardy space and Bergman space are investigated. In [5], the boundedness and the compactness of DC_φ on Bloch type spaces are studied.

In this paper, we study DC_φ between weighted Bergman spaces and Bloch type spaces. Sufficient and necessary conditions for the boundedness of the operator DC_φ are given. As a byproduct, we obtain the characterization of the boundedness of $C_\varphi D$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2 Main results and proofs

In this section we give our main results. In order to prove the main results of this paper, the following auxiliary lemmas are needed.

Lemma 1. *Assume that $0 < p < \infty$ and $\alpha > -1$. If $f \in A^p_\alpha$, then*

$$|f'(z)| \leq C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha+p)/p}}. \tag{1}$$

Proof. Let $\beta(z, w)$ denote the Bergman metric between two points z and w in $\mathbb{D}$. It is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For $a \in \mathbb{D}$ and $r > 0$, the set $\mathbb{D}(a, r) = \{z \in \mathbb{D} : \beta(a, z) < r\}$ is a Bergman metric disk with center a and radius r. It is well known that (see, for example, [10])

$$\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} \asymp \frac{1}{(1-|z|^2)^2} \asymp \frac{1}{(1-|a|^2)^2} \asymp \frac{1}{|\mathbb{D}(a,r)|},$$

when $z \in \mathbb{D}(a,r)$, where $|\mathbb{D}(a,r)|$ is the hyperbolic area of the disk $\mathbb{D}(a,r)$. For $0<r<1$ and $z \in \mathbb{D}$, by the subharmonicity of $|f'(z)|^p$ and the well known asymptotic formula (see, for example, [6, 7])

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) \asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{\alpha+p} dA(z),$$

we obtain

$$\begin{aligned} |f'(z)|^p &\le \frac{C}{(1-|z|^2)^2} \int_{\mathbb{D}(z,r)} |f'(w)|^p dA(w) \\ &\le \frac{C}{(1-|z|^2)^{2+\alpha+p}} \int_{\mathbb{D}(z,r)} (1-|w|^2)^{\alpha+p} |f'(w)|^p dA(w) \\ &\le \frac{C\|f\|^p_{A^p_\alpha}}{(1-|z|^2)^{2+\alpha+p}}, \end{aligned}$$

from which we obtain the desired result.

By a similar argument and using the following asymptotic formula ([6, 7])

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) \asymp |f(0)|^p + |f'(0)|^p + \int_{\mathbb{D}} |f''(z)|^p (1-|z|^2)^{\alpha+2p} dA(z),$$

we obtain the following lemma.

Lemma 2. *Assume that $p>0$, $\alpha>-1$ and $f \in A^p_\alpha$. Then there is a positive constant C independent of f such that*

$$|f''(z)| \le C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha+2p)/p}}. \tag{2}$$

Let $0<p<\infty$, μ be a positive Borel measure on $\mathbb{D}$ and

$$D_p(\mu) = \Big\{ f \in H(\mathbb{D}) \mid \|f\|^p_{D_p(\mu)} = \int_{\mathbb{D}} |f'(z)|^p d\mu(z) < \infty \Big\}.$$

Lemma 3. *Let μ be a positive measure on $\mathbb{D}$ and $0<p,\beta<\infty$. Then the following statements are equivalent:*
(a) $i : \mathcal{B}^\beta \longmapsto D_p(\mu)$ is bounded;
(b) $i : \mathcal{B}^\beta_0 \longmapsto D_p(\mu)$ is bounded;
(c)

$$\int_{\mathbb{D}} \frac{d\mu(z)}{(1-|z|^2)^{\beta p}} < \infty.$$

Remark. The above lemma was obtained by Zhao when $0<\beta\le 1$ (see [8]). In fact, his proof implies that the result also holds for $\beta>1$. Partial results can also be found in [1] when $\beta=1$.

Now we are in a position to formulate and prove the main results of this paper.

Theorem 1. *Suppose that $0 < p, \beta < \infty, -1 < \alpha < \infty$ and φ is an analytic self-map of the unit disk. Then $DC_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if the following conditions are satisfied:*

(a)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{(2+\alpha+2p)/p}} < \infty; \tag{3}$$

(b)

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+p)/p}} < \infty. \tag{4}$$

Proof. Suppose that conditions (3) and (4) hold. Then, for arbitrary $z \in \mathbb{D}$ and $f \in A^p_\alpha$, (1) and (2) imply

$$\begin{aligned}
& (1-|z|^2)^\beta |(DC_\varphi f)'(z)| \\
\le\ & (1-|z|^2)^\beta |(f'(\varphi)\varphi')'(z)| \\
\le\ & (1-|z|^2)^\beta |\varphi'(z)|^2 |f''(\varphi(z))| + (1-|z|^2)^\beta |\varphi''(z)||f'(\varphi(z))| \\
\le\ & C\frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+2}} \|f\|_{A^p_\alpha} + C\frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} \|f\|_{A^p_\alpha}.
\end{aligned} \tag{5}$$

In addition, from Lemma 1 we see that

$$|(DC_\varphi f)(0)| \le \frac{C|\varphi'(0)|\|f\|_{A^p_\alpha}}{(1-|\varphi(0)|^2)^{\frac{2+\alpha}{p}+1}}.$$

From this, by taking the supremum in the inequality (5) over $\mathbb{D}$, then employing conditions (3) and (4), we obtain that $DC_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded.

Conversely, suppose that $DC_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ is bounded, i.e. there exists a constant C such that $\|DC_\varphi f\|_{\mathcal{B}^\beta} \le C\|f\|_{A^p_\alpha}$ for all $f \in A^p_\alpha$. Then, taking $f(z) = z$ and $f(z) = z^2$, we obtain that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi''(z)| < \infty \tag{6}$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(\varphi'(z))^2 + \varphi''(z)\varphi(z)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$, we have that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)|^2 < \infty. \tag{7}$$

For fixed $w \in \mathbb{D}$, we define the test function

$$f_w(z) = \frac{(1-|w|^2)^{(\alpha+2)/p}}{(1-\overline{w}z)^{(4+2\alpha)/p}}.$$

It is easy to check that $f_w \in A^p_\alpha$ and $\sup_{w\in\mathbb{D}} \|f_w\|_{A^p_\alpha} \le C$(or see [3]). Moreover

$$|f_w'(z)| = \frac{4+2\alpha}{p} \frac{(1-|w|^2)^{(2+\alpha)/p}|w|}{|1-\overline{w}z|^{(4+2\alpha+p)/p}};$$

$$|f_w''(z)| = \frac{4+2\alpha}{p}\frac{4+2\alpha+p}{p}\frac{(1-|w|^2)^{(2+\alpha)/p}|w|^2}{|1-\overline{w}z|^{(4+2\alpha+2p)/p}}.$$

Hence, we have

$$\begin{aligned} C\|DC_\varphi\|_{A^p_\alpha\to\mathcal{B}^\beta} &\geq \|DC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}^\beta} \\ &\geq -\frac{4+2\alpha}{p}\frac{4+2\alpha+p}{p}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}} \\ &\quad +\frac{4+2\alpha}{p}\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}} \end{aligned}$$

for $\lambda\in\mathbb{D}$. Therefore, we obtain

$$\begin{aligned} \frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}} &\leq \frac{4+2\alpha+p}{p}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}} \\ &\quad +C\|DC_\varphi\|_{A^p_\alpha\to\mathcal{B}^\beta}. \end{aligned} \tag{8}$$

Next, set

$$g_w(z) = \frac{4+2\alpha+p}{4+2\alpha}\frac{(1-|w|^2)^{(\alpha+2)/p}}{(1-\overline{w}z)^{(4+2\alpha)/p}} - \frac{1-|w|^2}{1-\overline{w}z}\frac{(1-|w|^2)^{(\alpha+2)/p}}{(1-\overline{w}z)^{(4+2\alpha)/p}}, \quad w\in\mathbb{D}.$$

Then, since

$$g_w(z) = \frac{1-|w|^2}{1-\overline{w}z}\in H^\infty,$$

we see that $g_w\in A^p_\alpha$, moreover $\sup_{w\in\mathbb{D}}\|g_w\|_{A^p_\alpha}\leq C$. Also, we have $g'_{\varphi(\lambda)}(\varphi(\lambda))=0$ and

$$|g''_{\varphi(\lambda)}(\varphi(\lambda))| = \frac{4+2\alpha+p}{p}\frac{|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}}.$$

Hence, we obtain

$$\infty > C\|DC_\varphi\|_{A^p_\alpha\to\mathcal{B}^\beta}\geq\|DC_\varphi g_{\varphi(\lambda)}\|_{\mathcal{B}^\beta}\geq\frac{4+2\alpha+p}{p}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}}.$$

Thus

$$\begin{aligned} \sup_{|\varphi(\lambda)|>\frac12}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}} &\leq \sup_{|\varphi(\lambda)|>\frac12}4\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}} \\ &\leq \sup_{|\varphi(\lambda)|>\frac12}C\|DC_\varphi\|_{A^p_\alpha\to\mathcal{B}^\beta}<\infty. \end{aligned} \tag{9}$$

By (7), we see that

$$\begin{aligned} \sup_{|\varphi(\lambda)|\leq\frac12}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}} &\leq \frac{4^{(2+\alpha+2p)/p}}{3^{(2+\alpha+2p)/p}}\sup_{|\varphi(\lambda)|\leq\frac12}(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2 \\ &< \infty. \end{aligned} \tag{10}$$

Therefore, from (9) and (10) we see that

$$\sup_{\lambda\in\mathbb{D}}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+2p)/p}}<\infty.$$

From this and (8), we obtain

$$\sup_{\lambda\in\mathbb{D}}\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}}<\infty. \tag{11}$$

From (11) and (6), we have that

$$\sup_{|\varphi(\lambda)|>\frac12}\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}}\le 2\sup_{|\varphi(\lambda)|>\frac12}\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}} \tag{12}$$

and

$$\sup_{|\varphi(\lambda)|\le\frac12}\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{(2+\alpha+p)/p}}\le\frac{4^{\frac{2+\alpha+p}{p}}}{3^{\frac{2+\alpha+p}{p}}}\sup_{|\varphi(\lambda)|\le\frac12}(1-|\lambda|^2)^\beta|\varphi''(\lambda)|<\infty. \tag{13}$$

Combining (12) and (13), we obtain (4). The proof is completed.

Theorem 2. *Suppose that $0<p,\beta<\infty,-1<\alpha<\infty$ and φ is an analytic self-map of the unit disk. Then, $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta_0$ is bounded if and only if $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta$ is bounded,*

$$\lim_{|z|\to1}(1-|z|^2)^\beta|\varphi'(z)|^2=0; \tag{14}$$

and

$$\lim_{|z|\to1}(1-|z|^2)^\beta|\varphi''(z)|=0. \tag{15}$$

Proof. First assume that $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta_0$ is bounded. Then, it is clear that $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta$ is bounded. Taking the functions $f(z)=z$ and $f(z)=z^2$ respectively, we obtain (14) and (15).

Conversely, assume that $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta$ is bounded, and that (14) and (15) hold. Then, for each polynomial $p(z)$, we have that

$$\begin{aligned}&(1-|z|^2)^\beta|(DC_\varphi p)'(z)|\\ \le\;&(1-|z|^2)^\beta|\varphi'(z)|^2|p''(\varphi(z))|+(1-|z|^2)^\beta|\varphi''(z)p'(\varphi(z))|.\end{aligned} \tag{16}$$

Since

$$\sup_{w\in\mathbb{D}}|p''(w)|<\infty\quad\text{and}\quad\sup_{w\in\mathbb{D}}|p'(w)|<\infty,$$

from (14)-(16) it follows that $DC_\varphi p\in\mathcal{B}^\beta_0$. Since the set of all polynomials is dense in A^p_α (see [3]), we have that for every $f\in A^p_\alpha$ there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|f-p_n\|_{A^p_\alpha}\to0$ as $n\to\infty$. Hence

$$\|DC_\varphi f-DC_\varphi p_n\|_{\mathcal{B}^\beta}\le\|DC_\varphi\|_{A^p_\alpha\to\mathcal{B}^\beta}\|f-p_n\|_{A^p_\alpha}\to0$$

as $n\to\infty$, by using the boundedness of the operator $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta$. Since $\mathcal{B}^\beta_0$ is a closed subset of $\mathcal{B}^\beta$, we obtain $DC_\varphi(A^p_\alpha)\subset\mathcal{B}^\beta_0$. Therefore $DC_\varphi: A^p_\alpha\to\mathcal{B}^\beta_0$ is bounded.

Theorem 3. *Suppose that φ is an analytic self-map of the unit disk. Assume that $p, \beta > 0$ and $\alpha > -1$. Then the following statements are equivalent:*

(a) $DC_\varphi : \mathcal{B}^\beta \to A^p_\alpha$ is bounded;

(b) $DC_\varphi : \mathcal{B}^\beta_0 \to A^p_\alpha$ is bounded;

(c)

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^{\beta p}} dA_\alpha(z) < \infty.$$

Proof. Let $f \in A^p_\alpha$. We have

$$\begin{aligned} \|DC_\varphi f\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |(DC_\varphi f)(z)|^p dA_\alpha(z) \\ &= \int_{\mathbb{D}} |f'(\varphi(z))|^p |\varphi'(z)|^p dA_\alpha(z) \\ &= \int_{\mathbb{D}} |f'(\varphi(z))|^p d\mu(z) = \int_{\mathbb{D}} |f'(z)|^p d\mu \circ \varphi^{-1}(z), \end{aligned}$$

where

$$d\mu(z) = |\varphi'(z)|^p dA_\alpha(z).$$

By Lemma 3, we know that $DC_\varphi : \mathcal{B}^\beta (\text{or } \mathcal{B}^\beta_0) \to A^p_\alpha$ is bounded if and only if

$$\infty > \int_{\mathbb{D}} \frac{d\mu \circ \varphi^{-1}}{(1-|z|^2)^{\beta p}} = \int_{\mathbb{D}} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^{\beta p}} dA_\alpha(z),$$

finishing the proof of the theorem.

Remark 1. Note that by modifying the proofs of Theorems 1-3, we can prove the following results. We omit the details.

Theorem 4. *Suppose that $0 < p, \beta < \infty, -1 < \alpha < \infty$ and φ is an analytic self-map of $\mathbb{D}$. Then*

(a) $C_\varphi D : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{(2+\alpha+2p)/p}} < \infty;$$

(b) $C_\varphi D : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $\varphi \in \mathcal{B}^\beta_0$;

(c) $C_\varphi D : \mathcal{B}^\beta \to A^p_\alpha$ is bounded if and only if $C_\varphi D : \mathcal{B}^\beta_0 \to A^p_\alpha$ is bounded if and only if

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\beta p}} dA(z) < \infty.$$

Acknowledgments. The first author of this paper is supported in part by the NNSF China (No.10671115), PHD Foundation (No. 20060560002) and NSF of Guangdong Province (No. 06105648).

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,341-346,COPYRIGHT 2008 EUDOXUS PRESS, LLC

A NOTE ON INTEGRAL INEQUALITIES INVOLVING THE PRODUCT OF TWO FUNCTIONS ON TIME SCALES

A.TUNA, B. I. YAŞAR AND S. KUTUKCU

Abstract. In this article we study new integral inequalities involving two functions and their derivatives on time scales.

1. Introduction

The theory of dynamic equations on time scales(aka measure chains) was introduced by Hilger [3] with the motivation of providing a unified approach to continuous and discrete analysis. The generalized derivative or Hilger derivative $f^{\triangle}(t)$ of a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a so-called "time scale" (an arbitrary closed nonempty subset of $\mathbb{R}$) becomes the usual derivative when $\mathbb{T} = \mathbb{R}$, that is $f^{\triangle}(t) = f'(t)$. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t)$ reduces to the usual forward difference, that is $f^{\triangle}(t) = \triangle f(t)$. This theory not only brought equations leading to new applications. Also, this theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems [1, 2].

In this paper, we establish new integral inequalities involving two functions and their derivatives on time scales.

Now, first we mention without proof several fundamental definitions and result from the calculus on time scales in an excellent introductory text by Bohner and Peterson [2].

2. General Definitions

Definition 1. *A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.*

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It also assumed throughout that in $\mathbb{T}$ the interval $[a, b]$ means the set $\{t \in \mathbb{T} : \ s < t\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2. *The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$ are called the jump operators.*

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in the following way:

Definition 3. *A nonmaximal element $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$,left-scattered if $\rho(t) < t$.*

In the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$.

2000 *Mathematics Subject Classification.* Primary 26D15, 39A10,

Key words and phrases. Integral inequalities; Product of two functions; Time scales.

Definition 4. *The mapping* $\mu : \mathbb{T} \to \mathbb{R}^+$ *defined by* $\mu(t) = \sigma(t) - t$ *is called the graininess function.*

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 5. *Let* $f : \mathbb{T} \to \mathbb{R}$. f *is called differentiable at* $t \in \mathbb{T}^k$, *with (delta) derivative* $f^{\triangle}(t) \in \mathbb{R}$ *if given* $\varepsilon > 0$ *there exists a neighborhood* U *of* t *such that, for all* $s \in U$,

$\left\| f^\sigma(t) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right\| \leq \varepsilon \left\| \sigma(t) - s \right\|,$

where $f^\sigma = f \circ \sigma$.

If $\mathbb{T} = \mathbb{R}$, *then* $f^{\triangle}(t) = \frac{df(t)}{dt}$, *and if* $\mathbb{T} = \mathbb{R}$, *then* $f^{\triangle}(t) = f(t+1) - f(t)$.

Some basic properties of delta derivatives are the following [2].

Theorem 1. *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *and let* $t \in \mathbb{T}^k$.

(i) If f *is differentiable at* t, *then* f *is continuous at* t.

(ii) If f *is differentiable at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^{\triangle}(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

(iii) If f *is differentiable at* t *and* t *is right-dense, then*

$$f^{\triangle}(t) = \lim_{t \to s} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f *is differentiable at* t, *then*

$$f^\sigma(t) = f(t) + \mu(t) f^{\triangle}(t)$$

Example 1. *(i) If* $f : \mathbb{T} \to \mathbb{R}$ *is defined by* $f(t) = \alpha$ *for all* $t \in \mathbb{T}$, *where* $\alpha \in \mathbb{R}$ *is constant, then* $f^{\triangle}(t) \equiv 0$.

(ii) If $f : \mathbb{T} \to \mathbb{R}$ *is defined by* $f(t) = t$ *for all* $t \in \mathbb{T}$, *then* $f^{\triangle}(t) \equiv 1$

Definition 6. *The function* $f : \mathbb{T} \to \mathbb{R}$ *is said to be rd-continuous (denote* $f \in C_{rd}(\mathbb{T},\mathbb{R})$*) if, at all* $t \in \mathbb{T}$,

(i) f *is continuous at every right-dense point* $t \in \mathbb{T}$,

(ii) $lim_{s \to t^-} f(s)$ *exists and is finite at every left-dense point* $t \in \mathbb{T}$.

Definition 7. *Let* $f \in C_{rd}(\mathbb{T},\mathbb{R})$. *Then* $g : \mathbb{T} \to \mathbb{R}$ *is called the antiderivative of* f *on* $\mathbb{T}$ *if it is differentiable on* $\mathbb{T}$ *and satisfies* $g^{\triangle}(t) = f(t)$ *for any* $t \in \mathbb{T}^k$. *In this case, we defined*

$$\int_a^t f(s) \triangle s = g(t) - g(a), \qquad t \in \mathbb{T}.$$

Theorem 2. *If* f *is* $\triangle$*-integrable on* $[a, b]$, *then so is* $|f|$*,and*

$$\left| \int_a^b f(t) \triangle t \right| \leq \int_a^b |f(t)| \triangle t.$$

We assume that $\mathbb{T} = [a, b]$ is an arbitrary interval on time scale. Our main results are given in the following theorem.

3. Main Result

Theorem 3. *Let $f, g \in C^1_{rd}(\mathbb{T},\mathbb{R})$. Then*

$$\begin{aligned} &\left| f(t)g(t) - \frac{1}{2}\left[g(t)F + f(t)G\right]\right| \\ \leq\ & \frac{1}{4}\left[|g(t)| \int_a^b \left|f^{\triangle}(\tau)\right| \triangle \tau + |f(t)| \int_a^b \left|g^{\triangle}(\tau)\right| \triangle \tau\right] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &|f(t)g(t) - [g(t)F + f(t)G] + FG| \\ \leq\ & \frac{1}{4}\left(\int_a^b \left|f^{\triangle}(\tau)\right| \triangle \tau\right)\left(\int_a^b \left|g^{\triangle}(\tau)\right| \triangle \tau\right) \end{aligned} \tag{3.2}$$

for all $t \in \mathbb{T}$, where

$$F = \frac{f(a) + f(b)}{2},\ G = \frac{g(a) + g(b)}{2}. \tag{3.3}$$

The constant $\frac{1}{4}$ in (3.1) and (3.2) is sharp.

Proof. From the hypotheses of Theorem 3. we have the following identities

$$f(t) - F = \frac{1}{2}\left[\int_a^t f^{\triangle}(\tau) \triangle \tau - \int_t^b f^{\triangle}(\tau) \triangle \tau\right], \tag{3.4}$$

$$g(t) - G = \frac{1}{2}\left[\int_a^t g^{\triangle}(\tau) \triangle \tau - \int_t^b g^{\triangle}(\tau) \triangle \tau\right]. \tag{3.5}$$

Multiplying both sides of (3.4) and (3.5) by $g(t)$ and $f(t)$ respectively, adding the resulting identities and rewriting we have

$$\begin{aligned} &f(t)g(t) - \frac{1}{2}\left[g(t)F + f(t)G\right] \\ =\ & \frac{1}{4}\left[g(t)\left[\int_a^t f^{\triangle}(\tau) \triangle \tau - \int_t^b f^{\triangle}(\tau) \triangle \tau\right]\right. \\ &\left. + f(t)\left[\int_a^t g^{\triangle}(\tau) \triangle \tau - \int_t^b g^{\triangle}(\tau) \triangle \tau\right]\right] \end{aligned} \tag{3.6}$$

From (3.6) and using the properties of modulus we have

$$\left| f(t)g(t) - \frac{1}{2}\left[g(t)F + f(t)G\right]\right| \leq \frac{1}{4}\left[|g(t)| \int_a^b \left|f^{\triangle}(\tau)\right| \triangle \tau + |f(t)| \int_a^b \left|g^{\triangle}(\tau)\right| \triangle \tau\right].$$

This is the required inequality in (3.1).

Multiplying the left sides and right sides of (3.4) and (3.5) we get

$$
\begin{aligned}
(3.7) \qquad & f(t)g(t) - [g(t)F + f(t)G] + FG \\
= \; & \frac{1}{4}\left[\int_a^t f^{\triangle}(\tau)\triangle\tau - \int_t^b f^{\triangle}(\tau)\triangle\tau\right]\left[\int_a^t g^{\triangle}(\tau)\triangle\tau - \int_t^b g^{\triangle}(\tau)\triangle\tau\right]
\end{aligned}
$$

From (3.7) and using the properties of modulus we have

$$
|f(t)g(t) - [g(t)F + f(t)G] + FG| \leq \frac{1}{4}\left(\int_a^b \left|f^{\triangle}(\tau)\right|\triangle\tau\right)\left(\int_a^b \left|g^{\triangle}(t)\right|\triangle\tau\right).
$$

This proves the inequality in (3.2).

To prove the sharpness of the constant $\frac{1}{4}$ in (3.1) and (3.2), assume that the inequalities (3.1) and (3.2) hold with constant $c > 0$ and $k > 0$ respectively.

That is,

$$
\begin{aligned}
(3.8) \qquad & \left|f(t)g(t) - \frac{1}{2}[g(t)F + f(t)G]\right| \\
\leq \; & c\left[|g(t)|\int_a^b \left|f^{\triangle}(\tau)\right|\triangle\tau + |f(t)|\int_a^b \left|g^{\triangle}(\tau)\right|\triangle\tau\right],
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9) \qquad & |f(t)g(t) - [g(t)F + f(t)G] + FG| \\
\leq \; & k\left(\int_a^b \left|f^{\triangle}(\tau)\right|\triangle\tau\right)\left(\int_a^b \left|g^{\triangle}(\tau)\right|\triangle\tau\right),
\end{aligned}
$$

for $t \in \mathbb{T}$. In (3.8) and (3.9), choose $f(t) = g(t) = t$ and hence $f^{\triangle}(t) = g^{\triangle}(t) = 1$ by Example 1(ii), $F = G = \frac{a+b}{2}$. Then by simple computation, we get

$$
(3.10) \qquad \left|t - \frac{1}{2}(a+b)\right| \leq 2c(b-a),
$$

and

$$
(3.11) \qquad \left|t(t-(a+b)) + (\frac{a+b}{2})^2\right| \leq k(b-a)^2.
$$

By taking $t = b$, from (3.10) we observe that $c \geq \frac{1}{4}$ and from (3.11) it is easy to observe that $k \geq \frac{1}{4}$, which proves the sharpness of the constants in (3.1) and (3.2). The proof is complete. □

Remark 1. *The results of Theorem 3. holds for an arbitrary time scale. Thus for some peculiar time scales, by Theorem 3., we immediately obtain the following two corollaries.*

Corollary 1. *Let* $\mathbb{T}=\mathbb{R}$ *and* $f,g\in C^1([a,b],\mathbb{R})$, $[a,b]\subset\mathbb{R}$, $a<b$. *Then*

$$\begin{aligned} &\left| f(t)g(t)-\frac{1}{2}\left[g(t)F+f(t)G\right]\right| \\ \leq\ & \frac{1}{4}\left[|g(t)|\int_a^b |f'(\tau)|\,d\tau+|f(t)|\int_a^b |g'(\tau)|\,d\tau\right], \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &|f(t)g(t)-[g(t)F+f(t)G]+FG| \\ \leq\ & \frac{1}{4}\left(\int_a^b |f'(\tau)|\,d\tau\right)\left(\int_a^b |g'(\tau)|\,d\tau\right) \end{aligned} \tag{3.13}$$

for all $t\in[a,b]$, *where*

$$F=\frac{f(a)+f(b)}{2},\ G=\frac{g(a)+g(b)}{2}.$$

The constant $\frac{1}{4}$ *in (3.12) and (3.13) is sharp.*

Corollary 2. *Let* $\mathbb{T}=\mathbb{Z}$ *and* $\{u_i\},\{v_i\}$ *for* $i=0,1,2,...,n$, $n\in\mathbb{N}$ *be sequences of real numbers. Then*

$$\begin{aligned} &\left| u_iv_i-\frac{1}{2}\left[v_iU+u_iV\right]\right| \\ \leq\ & \frac{1}{4}\left[|v_i|\sum_{j=0}^{n-1}|\triangle u_j|+|u_i|\sum_{j=0}^{n-1}|\triangle v_j|\right], \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &|u_iv_i-[v_iU+u_iV]+UV| \\ \leq\ & \frac{1}{4}\left(\sum_{j=0}^{n-1}|\triangle u_j|\right)\left(\sum_{j=0}^{n-1}|\triangle v_j|\right), \end{aligned} \tag{3.15}$$

for $i=0,1,2,...,n$, *where*

$$U=\frac{u_0+u_n}{2},\ V=\frac{v_0+v_n}{2},$$

and $\triangle$ *is the forward difference operator. The constant* $\frac{1}{4}$ *in (3.14) and (3.15) is sharp.*

Remark 2. *If we take* $g(t)=1$ *and hence* $g^{\triangle}(t)=0$ *in (3.1), then by simple calculation we get the inequality*

$$|f(t)-F|\leq\frac{1}{2}\int_a^b\left|f^{\triangle}(\tau)\right|\triangle\tau, \tag{3.16}$$

Remark 3. *Dividing both sides of (3.6) and (3.7) by $(b-a)$, then integrating both sides with respect to t over $\mathbb{T}$ and closely looking at the proof of Theorem 3 we get*

$$\begin{aligned}
(3.17)\qquad & \left| \frac{1}{b-a}\int_a^b f(t)g(t)\triangle t \right. \\
& \left. -\frac{1}{2(b-a)}\left[F\int_a^b g(t)\triangle t + G\int_a^b f(t)\triangle t\right]\right| \\
\leq\ & \frac{1}{4(b-a)}\left[\left(\int_a^b |g(t)|\triangle t\right)\left(\int_a^b \left|f^{\triangle}(t)\right|\triangle t\right)\right. \\
& \left. +\left(\int_a^b |f(t)|\triangle t\right)\left(\int_a^b \left|g^{\triangle}(t)\right|\triangle t\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
(3.18)\qquad & \left|\frac{1}{b-a}\int_a^b f(t)g(t)\triangle t - \frac{1}{(b-a)}\left[F\int_a^b g(t)\triangle t + G\int_a^b f(t)\triangle t - FG\right]\right| \\
\leq\ & \frac{1}{4}\left(\int_a^b \left|f^{\triangle}(t)\right|\triangle t\right)\left(\int_a^b \left|g^{\triangle}(t)\right|\triangle t\right).
\end{aligned}$$

Conclusion 1. *If we take $\mathbb{T}=\mathbb{R}$, we note the inequalities (4.5) and (4.6) are similar to those of the well known inequalities due to Grüss and Čebyeš, see* [4, 5]

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SOME NEW INTEGRAL INEQUALITIES FOR RETARDED VOLTERRA EQUATIONS

B. I. YAŞAR AND A.TUNA

ABSTRACT. In this article we study some new integral inequalities for retarded Volterra equations.

1. INTRODUCTION

Integral inequalities are very useful in the qualitative analysis of differential and integral equations. Starting with [6], several recent investigations, see [7, 11, 12, 13, 14], were devoted to retarded integral inequalities. Next [5] has considered case of retarded Volterra integral equations. There, bounds on the solutions and, by means of examples, established also it is shown the usefulness of results in investigating the asymptotic behavior of the solutions.

In this article we study some new integral inequalities for retarded Volterra equations.

First we mention several fundamental Theorems without proof by Olivia Lipovan [5].

2. PRELIMINARIES

2.1. The linear case.

Theorem 1. *Let* $k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $a \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *with* $(t,s) \to \partial_t a(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. *Assume in addition that* α *is nondecreasing and* $\alpha(t) \le t$ *for* $t \ge 0$. *If* $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfies*

$$u(t) \le k(t) + \int_0^{\alpha(t)} a(t,s)u(s)ds, \quad t \ge 0, \tag{2.1}$$

then

$$u(t) \le k(t) + e^{\int_0^{\alpha(t)} a(t,s)ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r,s)ds} \partial_r \left(\int_0^{\alpha(r)} a(r,s)k(s)ds \right) dr, \quad t \ge 0. \tag{2.2}$$

Corollary 1. *Assume* α, a *are as in Theorem 1. and* $k(t) \equiv k > 0$. *If* $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfies (2.1), then*

$$u(t) \le k e^{\int_0^{\alpha(t)} a(t,s)ds}, \quad t \ge 0. \tag{2.3}$$

2000 *Mathematics Subject Classification.* Primary 26D15, 39A10,

Key words and phrases. Integral inequalities; Volterra equations; Retarded Volterra inequalities.

Remark 1. *We note that for $\partial_t a(t,s) \equiv 0$ in Corollary 1. we get an inequality obtained in* [6]. *If, in addition, $\alpha(t) = t$, the inequality given by Corollary 1. reduces to Gronwall's inequality* [4].

Theorem 2. *Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that α is nondecreasing with $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds, \quad t \geq 0, \tag{2.4}$$

then

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r)dr, \quad t \geq 0. \tag{2.5}$$

Remark 2. *Considering $\alpha(t) = t$ in Theorem 2., we obtain Morro's inequality* [4].

2.2. **The nonlinear case.**

Theorem 3. *Let a, α be as in Theorem 1. Assume $k, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $k(0) > 0$, $\omega(t) > 0$ for $t > 0$ and $\int_1^{\infty} \frac{dt}{\omega(t)} = \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)\omega(u(s))ds, \quad t \geq 0,$$

then

$$u(t) \leq G^{-1}\left(G(k(t)) + \int_0^{\alpha(t)} a(t,s)ds\right), \quad t \geq 0. \tag{2.6}$$

where $G(t) = \int_1^t \frac{ds}{\omega(s)}, \quad t \geq 0.$

Theorem 4. *Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that a, k, α are nondecreasing functions with $\alpha(t) \leq t$ for $t \geq 0$. Let also $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function such that $\omega(t) > 0$ for $t > 0$ and $\int_1^{\infty} \frac{dt}{\omega(t)} = \infty$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)\omega(u(s))ds, \quad t \geq 0, \tag{2.7}$$

then

$$u(t) \leq G^{-1}\left(G(k(t)) + a(t) \int_0^{\alpha(t)} b(s)ds\right), \quad t \geq 0. \tag{2.8}$$

where $G(t) = \int\limits_{1}^{t} \frac{ds}{\omega(s)}, \quad t \geq 0.$

3. Main Results

Theorem 5. *Let $a, c, g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$. Assume in addition that α is nondecreasing and $\alpha(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$(u(t))^p \leq a(t) + c(t) \int\limits_{0}^{\alpha(t)} [g(s)u(s) + h(s)]\, ds, \quad t \geq 0, \tag{3.1}$$

then

$$u(t) \leq \left\{ a(t) + c(t) \left[e^{\int\limits_{0}^{\alpha(t)} \frac{g(s)c(s)}{p} ds} \int\limits_{0}^{t} e^{-\int\limits_{0}^{\alpha(r)} \frac{g(s)c(s)}{p} ds} \partial_r \left(\int\limits_{0}^{\alpha(r)} F(s) ds \right) dr \right] \right\}^{\frac{1}{p}}, \quad t \geq 0. \tag{3.2}$$

where

$$F(t) = g(t) \left(\frac{p-1}{p} + \frac{a(t)}{p} \right) + h(t). \tag{3.3}$$

Proof. Obviously, if $\alpha(t) = 0$, then the inequality (3.2) holds. Thus, in the next proof, we always assume that $\alpha(t) \leq t$ with $t > 0$,

Define a function $z(t)$ by

$$z(t) = \int\limits_{0}^{\alpha(t)} [g(s)u(s) + h(s)]\, ds. \tag{3.4}$$

Then (3.1) can be restated as

$$(u(t))^p \leq a(t) + c(t)z(t). \tag{3.5}$$

Using the elementary inequality (See [8,p,30])

$$x^{\frac{1}{p}} y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q},$$

where $x \geq 0$, $y \geq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, we observe that

$$\begin{aligned} u(t) &\leq [a(t) + c(t)z(t)]^{\frac{1}{p}} \\ &\leq \frac{p-1}{p} + \frac{a(t)}{p} + \frac{c(t)}{p} z(t). \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we have

$$\begin{aligned} z'(t) &= [g(\alpha(t))u(\alpha(t)) + h(\alpha(t))]\, \alpha'(t) + \int\limits_{0}^{\alpha(t)} [g(s)u(s) + h(s)]\, ds, \quad t \geq 0, \\ &\leq \left[g(\alpha(t)) \left[\tfrac{p-1}{p} + \tfrac{a(\alpha(t))}{p} + \tfrac{c(\alpha(t))}{p} z(\alpha(t)) \right] + h(\alpha(t)) \right] \alpha'(t) \\ &+ \int\limits_{0}^{\alpha(t)} \left[g(s) \left(\tfrac{p-1}{p} + \tfrac{a(s)}{p} + \tfrac{c(s)}{p} z(s) \right) + h(s) \right] ds \end{aligned}$$

$$= \left[g(\alpha(t))\left[\tfrac{p-1}{p}+\tfrac{a(\alpha(t))}{p}\right]+h(\alpha(t))\right]\alpha'(t)+\int_0^{\alpha(t)}\left[g(s)\left(\tfrac{p-1}{p}+\tfrac{a(s)}{p}\right)+h(s)\right]ds$$
$$+\left[\tfrac{g(\alpha(t))c(\alpha(t))}{p}\alpha'(t)+\int_0^{\alpha(t)}\tfrac{g(s)c(s)}{p}ds\right]z(t).$$

or, equivalently,

$$z'(t)-z(t)\tfrac{d}{dt}\left(\int_0^{\alpha(t)}\tfrac{g(s)c(s)}{p}ds\right)\le \tfrac{d}{dt}\left(\int_0^{\alpha(t)}F(s)ds\right)$$

where

$$F(t)=g(s)\left(\tfrac{p-1}{p}+\tfrac{a(s)}{p}\right)+h(s)$$

Multiplying the above inequality by $e^{-\int_0^{\alpha(t)}\frac{g(s)c(s)}{p}ds}$, we get

$$\tfrac{d}{dt}\left(z(t)e^{-\int_0^{\alpha(t)}\frac{g(s)c(s)}{p}ds}\right)\le e^{-\int_0^{\alpha(t)}\frac{g(s)c(s)}{p}ds}\tfrac{d}{dt}\left(\int_0^{\alpha(t)}F(s)ds\right)$$

Consider now the integral on the interval $[0,t]$ to obtain

$$z(t)\le e^{\int_0^{\alpha(t)}\frac{g(s)c(s)}{p}ds}\int_0^t e^{-\int_0^{\alpha(r)}\frac{g(s)c(s)}{p}ds}\partial_r\left(\int_0^{\alpha(r)}F(s)ds\right)dr,\quad t\ge 0. \tag{3.7}$$

Combine the above inequality with $(u(t))^p\le a(t)+c(t)z(t)$ to get (3.2) and, with this, the proof is complete. □

Corollary 2. *If we take $p=1$, $h(t)\equiv 0$, it is clear that we can have the same Theorem 1.2 and Remark 1.2. in* [5].

Theorem 6. *Assume that $u,a,c,g,h,m\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\alpha\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $\omega\in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$ with $(t,s)\to\frac{\partial}{\partial t}\omega(t,s)\in C(\mathbb{R}_+\times\mathbb{R}_+,\mathbb{R}_+)$. Assume in addition that α is nondecreasing and $\alpha(t)\le t$ for $t\ge 0$,*

$$(u(t))^p\le a(t)+c(t)\int_0^{\alpha(t)}\omega(t,s)\left[g(s)(u(s))^p+h(s)u(s)+m(s)\right]ds,\quad t\ge 0, \tag{3.8}$$

implies

(3.9)

$$u(t)\le \left\{a(t)+c(t)\left(e^{\int_0^{\alpha(t)}\omega(t,s)A(s)ds}\int_0^t e^{-\int_0^{\alpha(r)}\omega(r,s)A(s)ds}\partial_r\left(\int_0^{\alpha(r)}\omega(r,s)B(s)ds\right)dr\right)\right\}^{\frac{1}{p}}$$

for $t\ge 0$. Where

$$A(t)=c(t)\left(g(t)+\frac{h(t)}{p}\right) \tag{3.10}$$

and

$$B(t) = a(t)g(t) + h(t)\left(\frac{p-1}{p} + \frac{a(t)}{p}\right) + m(t). \tag{3.11}$$

Proof. Obviously, if $\alpha(t) = 0$, then the inequality (3.9) holds. Thus, in the next proof, we always assume that $\alpha(t) \leq t$ with $t > 0$,

Define a function $z(t)$ by

$$z(t) = \int_0^{\alpha(t)} \omega(t,s)\left[g(s)\left(u(s)\right)^p + h(s)u(s) + m(s)\right]ds \tag{3.12}$$

As in the proof of Theorem 5., we easily obtain (3.5) and (3.6). Combining (3.12), (3.5) and (3.6), we have

$$\begin{aligned}
z'(t) &= \omega(t,\alpha(t))\left[g(\alpha(t))\left(u(\alpha(t))\right)^p + h(\alpha(t))u(\alpha(t)) + m(\alpha(t))\right]\alpha'(t) \\
&\quad + \int_0^{\alpha(t)} \tfrac{\partial}{\partial t}\omega(t,s)\left[g(s)\left(u(s)\right)^p + h(s)u(s) + m(s)\right]ds, \quad t \geq 0, \\
&\leq \left\{\omega(t,\alpha(t))\left[g(\alpha(t))a(\alpha(t)) + c(\alpha(t))g(\alpha(t))z(t)\right]\right. \\
&\qquad \left. + h(\alpha(t))\left[\tfrac{p-1}{p} + \tfrac{a(\alpha(t))}{p} + \tfrac{c(\alpha(t))}{p}z(t)\right] + m(\alpha(t))\right\}\alpha'(t) \\
&\quad + \int_0^{\alpha(t)} \left[\tfrac{\partial}{\partial t}\omega(t,s)\left[g(s)a(s) + c(s)g(s)z(s)\right]\right. \\
&\qquad \left. + h(s)\left(\tfrac{p-1}{p} + \tfrac{a(s)}{p} + \tfrac{c(s)}{p}z(s)\right) + m(s)\right]ds \\
&\leq \left[\omega(t,\alpha(t))c(\alpha(t))\left(g(\alpha(t)) + \tfrac{h(\alpha(t))}{p}\right)\alpha'(t)\right. \\
&\quad \left. + \int_0^{\alpha(t)} \tfrac{\partial}{\partial t}\omega(t,s)c(s)\left(g(s) + \tfrac{h(s)}{p}\right)ds\right] z(t) \\
&\quad + \omega(t,\alpha(t))\left[a(\alpha(t))g(\alpha(t)) + h(\alpha(t))\left(\tfrac{p-1}{p} + \tfrac{a(\alpha(t))}{p}\right) + m(\alpha(t))\right]\alpha'(t) \\
&\quad + \int_0^{\alpha(t)} \tfrac{\partial}{\partial t}\omega(t,s)\left[a(s)g(s) + h(s)\left(\tfrac{p-1}{p} + \tfrac{a(s)}{p}\right) + m(s)\right]ds
\end{aligned}$$

or, equivalently,

$$z'(t) - z(t)\tfrac{d}{dt}\left(\int_0^{\alpha(t)} \omega(t,s)A(s)ds\right) \leq \tfrac{d}{dt}\left(\int_0^{\alpha(t)} \omega(t,s)B(s)ds\right)$$

where

$$A(t) = c(t)\left(g(t) + \tfrac{h(t)}{p}\right)$$

and

$$B(t) = a(t)g(t) + h(t)\left(\tfrac{p-1}{p} + \tfrac{a(t)}{p}\right) + m(t)$$

Multiplying the above inequality by $e^{-\int_0^{\alpha(t)} \omega(t,s)A(s)ds}$, we get

$$\frac{d}{dt}\left(z(t)e^{-\int\limits_0^{\alpha(t)}\omega(t,s)A(s)ds}\right)\leq e^{-\int\limits_0^{\alpha(t)}\omega(t,s)A(s)ds}\frac{d}{dt}\left(\int\limits_0^{\alpha(t)}\omega(t,s)B(s)ds\right)$$

Consider now the integral on the interval $[0,t]$ to obtain

$$z(t)\leq e^{\int\limits_0^{\alpha(t)}\omega(t,s)A(s)ds}\int\limits_0^t e^{-\int\limits_0^{\alpha(r)}\omega(r,s)A(s)ds}\partial_r\left(\int\limits_0^{\alpha(r)}\omega(r,s)B(s)ds\right)dr,\quad t\geq 0.$$

Combine the above inequality with $(u(t))^p\leq a(t)+c(t)z(t)$ to get (3.9) and, with this, the proof is complete. □

Corollary 3. *In Theorem 6. if we take* $p=1$, $c(t)\equiv 1$, $g(t)\equiv 0$, $h(t)\equiv 1$, $m(t)\equiv 0$, *we get Theorem 1.1.,Corollary1.1. and Remark 1.1.in* [5].

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.3,353-381,COPYRIGHT 2008 EUDOXUS PRESS, LLC

The Dirichlet-Neumann Problem for the 2-D Laplace Equation in an Exterior Cracked Domain with Neumann Condition on Cracks.

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Abstract

The mixed Dirichlet-Neumann problem for the Laplace equation in an unbounded plane domain with cuts (cracks) is studied. The Dirichlet condition is given on closed curves making up the boundary of the domain, while the Neumann condition is specified on the cuts. The existence of a classical solution is proved by potential theory and a boundary integral equation method. The integral representation for a solution is obtained in the form of potentials. The density of the potentials satisfies a uniquely solvable Fredholm integral equation of the second kind and index zero. Singularities of the gradient of the solution at the tips of the cuts are investigated.

AMS Subject Classification : 35J05, 35J25.

Key words and Phrases: *Laplace equation, Dirichlet–Neumann problem, boundary integral equation method.*

1. Introduction.

The Boundary of a 2-D cracked domain consists of both closed curves and open arcs (cuts). Open arcs model cracks in solids and screens or wings in fluids. Different physical processes in cracked domains can be described by boundary value problems for the Laplace equation, for example, distribution of stationary heat and electric fields in cracked solids, electric flow in cracked semiconductors, flow of an ideal fluid over several obstacles and wings, etc. Appropriate boundary conditions must be specified on the total boundary, i.e. on both closed curves and open arcs (cracks). The Neumann boundary condition reflects the nonflow (of fluid, electric current, etc.) through the boundary. The Dirichlet boundary condition corresponds to the given temperature in heat theory, fluid pressure in hydrodynamics, electric potential in electrostatics, etc.

Boundary value problems with mixed boundary conditions were not treated in cracked domains by rigorous mathematical methods before. Even in the case of Laplace and Helmholtz equations the problems in domains bounded by closed curves [2], [13–17] and problems in the exterior of cuts (cracks) [14, 16], [18–20] were treated separately, because different methods were used in their analysis. Previously the Neumann problem in the exterior of a cut was reduced to a hypersingular integral equation [14, 16, 18, 19] or to an infinite algebraic system of equations [20], while the Dirichlet problem in domains bounded by closed curves was reduced to the Fredholm equation of the second kind [13–17]. The combination of these methods in case of domains bounded by closed curves and cuts leads to an integral equation, which is algebraic or hypersingular on cuts, and it is an equation of the second kind with compact integral operators on the closed curves. The integral equation on the whole boundary is rather complicated to be effectively studied by standard methods. The approach suggested in the present paper enables us to reduce the mixed Dirichlet–Neumann problem in a cracked domain to the Fredholm integral equation of the second kind and index zero on the whole boundary with the help of a nonclassical angular potential. It is shown that the Fredholm integral equation is uniquely solvable, therefore the integral equation can be computed by a standard code by discretization and inversion of the matrix. So our approach is constructive, because it gives the way for finding the numerical solution for mixed boundary value problem with complicated boundary in an exterior domain. Our approach is based on [5–6], where the problems in the exterior of cuts were reduced to the Fredholm integral equations using the angular potential. At first these problems were reduced to the Cauchy singular integral equation with additional conditions. Next, the singular integral equation was reduced to the Fredholm integral equation of the 2nd kind and index zero by regularization. In [7–10] our approach has been applied to the Dirichlet and Neumann problems for the Laplace and Helmholtz equation in cracked domains. Some nonlinear problems of fluid dynamics were studied in [4]. Using an integral representation for a solution of our problem in the form of potentials, obtained in the present paper, we derive explicit asymptotic formulas for singularities of the gradient of the solution at the tips of the cuts (cracks).

2. Formulation of the problem.

By a simple open curve we mean a non-closed smooth arc of finite length without self-intersections [16].

In the plane $x = (x_1, x_2) \in R^2$ we consider the exterior multiply connected domain bounded by simple open curves $\Gamma_1^1, ..., \Gamma_{N_1}^1 \in C^{2,\lambda}$ and simple closed curves

$\Gamma_1^2, ..., \Gamma_{N_2}^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, so that the curves have no points in common. We put

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1, \ \ \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2, \ \ \Gamma = \Gamma^1 \cup \Gamma^2.$$

The exterior connected domain bounded by Γ^2 will be called $\mathcal{D}$. We assume that each curve Γ_n^k is parametrized by the arc length $s: \Gamma_n^k = \{x: \ x = x(s) = (x_1(s), x_2(s)), \ s \in [a_n^k, b_n^k]\}$, $n = 1, ..., N_k$, $k = 1, 2$, so that $a_1^1 < b_1^1 < ... < a_{N_1}^1 < b_{N_1}^1 < a_1^2 < b_1^2 < ... < a_{N_2}^2 < b_{N_2}^2$ and the domain $\mathcal{D}$ is on the right when the parameter s increases on Γ_n^2. Therefore points $x \in \Gamma$ and values of the parameter s are in one-to-one correspondence except a_n^2, b_n^2, which correspond to the same point x for $n = 1, ..., N_2$. Below the sets of the intervals on the Os axis

$$\bigcup_{n=1}^{N_1} [a_n^1, b_n^1], \ \ \bigcup_{n=1}^{N_2} [a_n^2, b_n^2], \ \ \bigcup_{k=1}^{2} \bigcup_{n=1}^{N_k} [a_n^k, b_n^k]$$

will be denoted by the same symbols as the corresponding sets of curves, that is, by Γ^1, Γ^2 and Γ respectively.

We put $C^0(\Gamma_n^2) = \{\mathcal{F}(s): \ \mathcal{F}(s) \in C^0[a_n^2, b_n^2], \ \mathcal{F}(a_n^2) = \mathcal{F}(b_n^2)\}$, and

$$C^0(\Gamma^2) = \bigcap_{n=1}^{N_2} C^0(\Gamma_n^2).$$

By $\mathcal{D}_n$ we denote the interior domain bounded by the curve Γ_n^2, if $n = 1, ..., N_2$.

The tangent vector to Γ at the point $x(s)$ we denote by $\tau_x = (\cos\alpha(s), \ \sin\alpha(s))$, where $\cos\alpha(s) = x_1'(s)$, $\sin\alpha(s) = x_2'(s)$. Let $\mathbf{n}_x = (\sin\alpha(s), \ -\cos\alpha(s))$ be the normal vector to Γ at $x(s)$. The direction of $\mathbf{n}_x$ is chosen such that it will coincide with the direction of τ_x if $\mathbf{n}_x$ is rotated anticlockwise through an angle of $\pi/2$. Therefore $\mathbf{n}_x$ is the inward normal for $\mathcal{D}$ on Γ^2.

We consider the curves Γ^1 as a set of cuts. The side of Γ^1 which is on the left, when the parameter s increases, will be denoted by $(\Gamma^1)^+$, and the opposite side will be denoted by $(\Gamma^1)^-$.

We say, that the function $u(x)$ belongs to the smoothness class $\mathbf{K}$ if

1) $u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}) \cap C^2(\mathcal{D}\backslash\Gamma^1)$,

2) $\nabla u \in C^0(\overline{\mathcal{D}\backslash\Gamma^1}\backslash\Gamma^2\backslash X)$, where X is a point-set, consisting of the end-points of Γ^1 :

$$X = \bigcup_{n=1}^{N_1} \left(x(a_n^1) \cup x(b_n^1) \right),$$

3) in the neighbourhood of any point $x(d) \in X$ for some constants $\mathcal{C} > 0$, $\epsilon > -1$ the inequality holds

$$|\nabla u| \leq \mathcal{C}\, |x - x(d)|^{\epsilon}, \tag{1}$$

where $x \to x(d)$ and $d = a_n^1$ or $d = b_n^1$, $n = 1, ..., N_1$.

Remark. In the definition of the class $\mathbf{K}$ we consider Γ^1 as a set of cuts. In particular, by $C^0(\overline{\mathcal{D}\backslash\Gamma^1})$ we denote a class of functions, which are continuously extended on the cuts Γ^1 from the left and right and are continuous at the tips of cuts Γ^1. However values of these functions on Γ^1 from the left and right can be different everywhere except at the tips, so that the functions may have a jump on Γ^1.

Let us formulate the mixed Dirichlet-Neumann problem for the Laplace equation in the domain $\mathcal{D}\backslash\Gamma^1$.

Problem $\mathbf{U}$. Find a function $u(x)$ of class $\mathbf{K}$ so that $u(x)$ satisfies the Laplace equation

$$u_{x_1x_1}(x) + u_{x_2x_2}(x) = 0, \qquad x \in \mathcal{D}\backslash\Gamma^1,$$

the boundary conditions

$$\left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^+} = F^+(s), \qquad \left.\frac{\partial u(x)}{\partial \mathbf{n}_x}\right|_{x(s)\in(\Gamma^1)^-} = F^-(s), \tag{2a}$$

$$u(x(s))|_{\Gamma^2} = F(s),$$

and the following conditions as $|x| = \sqrt{x_1^2 + x_2^2} \to \infty$

$$|u(x)| \le \text{const}, \qquad |\nabla u| = o(|x|^{-1}). \tag{2b}$$

All conditions of the problem $\mathbf{U}$ must be satisfied in the classical sense.

The edge condition (1) ensures the absence of point sources at the ends of Γ^1. It is assumed that $N_2 \ge 1$. If $N_1 = 0$ and the cuts Γ^1 are absent, then the problem $\mathbf{U}$ transforms to the classical Dirichlet problem in an exterior domain $\mathcal{D}$ without cuts.

Using the energy equalities we can prove the following assertion.

Theorem 1. *The problem* $\mathbf{U}$ *has at most one solution.*

By $\int_{\Gamma^k} ...\, d\sigma$ we mean

$$\sum_{n=1}^{N_k} \int_{a_n^k}^{b_n^k} ...\, d\sigma.$$

Proof. Consider the homogeneous problem $\mathbf{U}$ and assume that $u_0(x)$ is a solution of the homogeneous problem (with $F^{\pm}(s) \equiv 0,\ F(s) \equiv 0$). Our aims is to show that $u_0(x) \equiv 0$. According to [3], smoothness of a solution on the part of the boundary with the Dirichlet condition is the least value among smoothness of the boundary data and smoothness of the boundary. Therefore, $u_0(x) \in C^{1,\lambda}(\overline{\mathcal{D}} \setminus \Gamma^1)$. Combining this result with the smoothness ensured for $u_0(x)$ by the class $\mathbf{K}$, we have $\nabla u_0(x) \in C^0(\overline{\mathcal{D} \setminus \Gamma^1} \setminus X)$, and inequality (1) holds at the tips of Γ^1. We envelope each cut Γ_n^1 $(n = 1, ..., N_1)$ by a closed contour so that all contours lie in $\mathcal{D} \setminus \Gamma^1$. Next we write the energy equalities for a domain, bounded by our auxiliary contours, Γ^2 and the circle of a large enough radius r. We allow the auxiliary contours shrink to Γ^1 and let r tend to infinity. Using the conditions at infinity (2b) and the smoothness of $u_0(x)$ established above, we obtain

$$\|\nabla u_0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = \int_{\Gamma^1} \left[u_0^+ \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^+ - u_0^- \left(\frac{\partial u_0}{\partial \mathbf{n}_x}\right)^- \right] ds - \int_{\Gamma^2} u_0 \frac{\partial u_0}{\partial \mathbf{n}_x} ds.$$

Taking into account the homogeneous boundary conditions (2a), we have

$$\|\nabla u_0\|^2_{L_2(\mathcal{D}\setminus\Gamma^1)} = 0.$$

Hence $u_0(x) \equiv \text{const}$ and $\text{const} = 0$ due to homogeneous Dirichlet boundary condition on Γ^2. Therefore $u_0(x) \equiv 0$, and the theorem is proved thanks to the linearity of the problem $\mathbf{U}$.

3. Integral equations at the boundary.

Below we assume that

(3) $$F^+(s), F^-(s) \in C^{0,\lambda}(\Gamma^1), \quad F(s) \in C^0(\Gamma^2), \quad \lambda \in (0,1].$$

Note that the Hölder exponent λ in the description of smoothness of these functions and in the description of smoothness of the boundary Γ is the same. If the exponents are different in practice, then by λ we denote the least.

If $\mathcal{B}_1(\Gamma^1)$, $\mathcal{B}_2(\Gamma^2)$ are Banach spaces of functions given on Γ^1 and Γ^2, then for functions given on Γ we introduce the Banach space $\mathcal{B}_1(\Gamma^1) \cap \mathcal{B}_2(\Gamma^2)$ with the norm

$$\|\cdot\|_{\mathcal{B}_1(\Gamma^1)\cap\mathcal{B}_2(\Gamma^2)} = \|\cdot\|_{\mathcal{B}_1(\Gamma^1)} + \|\cdot\|_{\mathcal{B}_2(\Gamma^2)}.$$

An example of such a Banach space is $C^0(\Gamma) = C^0(\Gamma^1) \cap C^0(\Gamma^2)$.

We shall construct the solution of the problem $\mathbf{U}$ from the smoothness class $\mathbf{K}$ with the help of potential theory for harmonic functions.

We consider an angular potential [1,5], [12, Appendix] for the Laplace equation:

(4) $$w_1[\mu](x) = -\frac{1}{2\pi}\int_{\Gamma^1} \mu(\sigma) V(x, y(\sigma))\, d\sigma.$$

The kernel $V(x, y(\sigma))$ is defined (up to indeterminacy $2\pi m$, $m = \pm 1, \pm 2, \ldots$) by the formulae

$$\cos V(x, y(\sigma)) = \frac{x_1 - y_1(\sigma)}{|x - y(\sigma)|}, \qquad \sin V(x, y(\sigma)) = \frac{x_2 - y_2(\sigma)}{|x - y(\sigma)|},$$

where

$$y = y(\sigma) = (y_1(\sigma), y_2(\sigma)) \in \Gamma^1, \;\; |x - y(\sigma)| = \sqrt{(x_1 - y_1(\sigma))^2 + (x_2 - y_2(\sigma))^2}.$$

One can see, that $V(x,\sigma)$ is the angle between the vector $\overrightarrow{y(\sigma)x}$ and the direction of the Ox_1 axis. More precisely, $V(x, y(\sigma))$ is a many-valued harmonic function conjugate to $\ln|x-y(\sigma)|$.

Below by $V(x, y(\sigma))$ we denote an arbitrary fixed branch of this function, which varies continuously with σ along each curve Γ_n^1 $(n = 1, ..., N_1)$ for given fixed $x \notin \Gamma^1$.

Under this definition of $V(x, y(\sigma))$, the potential $w_1[\mu](x)$ is a many-valued function. In order that the potential $w_1[\mu](x)$ be single-valued, it is necessary to impose the following additional conditions

$$\int_{a_n^1}^{b_n^1} \mu(\sigma)\, d\sigma = 0, \;\; n = 1, ..., N_1. \tag{5}$$

Below we suppose that the density $\mu(\sigma)$ belongs to the Banach space $C_q^{\omega}(\Gamma^1)$, $\omega \in (0,1]$, $q \in [0,1)$ and satisfies conditions (5).

We say, that $\mu(s) \in C_q^{\omega}(\Gamma^1)$ if

$$\mu(s) \prod_{n=1}^{N_1} \left|s - a_n^1\right|^q \left|s - b_n^1\right|^q \in C^{0,\omega}(\Gamma^1),$$

where $C^{0,\omega}(\Gamma^1)$ is a Hölder space with exponent ω and

$$\|\mu(s)\|_{C_q^{\omega}(\Gamma^1)} = \left\| \mu(s) \prod_{n=1}^{N_1} \left|s - a_n^1\right|^q \left|s - b_n^1\right|^q \right\|_{C^{0,\omega}(\Gamma^1)}.$$

As shown in [1], [5], [12, Appendix], for such $\mu(\sigma)$ the angular potential $w_1[\mu](x)$ belongs to the class $\mathbf{K}$. In particular, the inequality (1) holds with $\epsilon = -q$, if $q \in (0,1)$. Moreover, integrating $w_1[\mu](x)$ by parts and using (5), we express the angular potential in terms of a double layer potential

$$w_1[\mu](x) = \frac{1}{2\pi} \int_{\Gamma^1} \rho(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)|\, d\sigma,$$

with the density

$$\rho(\sigma) = \int_{a_n^1}^{\sigma} \mu(\xi)\, d\xi, \quad \sigma \in [a_n^1, b_n^1], \quad n = 1, ..., N_1.$$

Consequently, $w_1[\mu](x)$ satisfies the Laplace equation outside Γ^1 and the conditions at infinity (2b).

Let us construct a solution of the problem **U** . We seek a solution of the problem in the following form

$$u[\nu,\mu](x) = v_1[\nu](x) + w[\mu](x) + h[\nu,\mu](x) \quad , \tag{6}$$

where

$$w[\mu](x) = w_1[\mu](x) + w_2[\mu](x) \quad , \tag{7a}$$

$$v_1[\nu](x) = -\frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma) \ln|x - y(\sigma)|\, d\sigma,$$

$$w_2[\mu](x) = -\frac{1}{2\pi}\int_{\Gamma^2} \mu(\sigma)\frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)|\, d\sigma,$$

and $w_1[\mu](x)$ is given by (4). By $h[\nu,\mu](x)$ we denote the sum of point sources placed at the fixed points Y_k lying inside Γ_k^2 $(k = 1, ..., N_2)$ and a constant:

$$h[\nu,\mu](x) = -\frac{1}{2\pi}\sum_{k=2}^{N_2}\int_{\Gamma_k^2} \mu(\sigma) d\sigma \, \ln|x - Y_k| +$$

$$+\frac{1}{2\pi}\left[\int_{\Gamma^2} \mu(\sigma) d\sigma + \int_{\Gamma^1} \nu(\sigma) d\sigma - \int_{\Gamma_1^2} \mu(\sigma) d\sigma\right] \ln|x - Y_1| + \int_{\Gamma^2} \mu(\sigma) d\sigma =$$

$$= \frac{1}{2\pi}\int_{\Gamma^1} \nu(\sigma) d\sigma \ln|x - Y_1| + h_2[\mu](x).$$

Here

$$h_2[\mu](x) = -\frac{1}{2\pi}\sum_{k=2}^{N_2}\int_{\Gamma_k^2} \mu(\sigma) d\sigma \, \ln|x - Y_k| + \tag{7b}$$

$$+\frac{1}{2\pi}\left[\int\limits_{\Gamma^2}\mu(\sigma)d\sigma-\int\limits_{\Gamma_1^2}\mu(\sigma)d\sigma\right]\ln|x-Y_1|+\int\limits_{\Gamma^2}\mu(\sigma)d\sigma\ ;\quad Y_k\in\mathcal{D}_k\ ,\quad k=1,...,N_2\ .$$

Clearly, $h[\nu,\mu](x)$ obeys the Laplace equation in $R^2\backslash\bigcup\limits_{k=1}^{N_2}Y_k$ and belongs to

$$C^\infty\left(R^2\backslash\bigcup_{k=1}^{N_2}Y_k\right).$$

Besides, if $x(s)\in\Gamma$, then $h[\nu,\mu](x(s))\in C^{1,\lambda}(\Gamma)$ in s. We need the function $h[\nu,\mu](x)$ to construct a uniquely solvable integral equation. Moreover, $h[\nu,\mu](x)$ is taken in such a way that $u[\nu,\mu](x)$ in (6) satisfies conditions (2b) at infinity.

We will look for the density $\nu(\sigma)$ in the space $C^{0,\lambda}(\Gamma^1)$.

We will seek $\mu(s)$ in the Banach space $C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)$, $\omega\in(0,1]$, $q\in[0,1)$ with the norm $\|\cdot\|_{C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)}=\|\cdot\|_{C_q^\omega(\Gamma^1)}+\|\cdot\|_{C^0(\Gamma^2)}$. Besides $\mu(s)$ must satisfy conditions (5).

It follows from [1,5,17], [12, Appendix] that for such $\mu(s)$, $\nu(s)$ the function (6) belongs to the class **K** and satisfies all conditions of the problem **U** except the boundary conditions (2a).

To satisfy the boundary conditions, we insert (6) in (2a), use limit formulas for the normal derivative of the angular potential [1,5], [12, Appendix] and arrive at the system of integral equations for the densities $\mu(s)$, $\nu(s)$

$$(8a)\qquad \pm\frac{1}{2}\nu(s)+\frac{1}{2\pi}\int\limits_{\Gamma^1}\nu(\sigma)\frac{\cos\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma-$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|}d\sigma-$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_x}\frac{\partial}{\partial\mathbf{n}_y}\ln|x(s)-y(\sigma)|\,d\sigma+\frac{\partial}{\partial\mathbf{n}_x}h[\nu,\mu](x(s))=F^{\pm}(s),\quad s\in\Gamma^1,$$

$$(8b)\qquad -\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)V\left(x(s),y(\sigma)\right)d\sigma-\frac{1}{2\pi}\int\limits_{\Gamma^1}\nu(\sigma)\ln|x(s)-y(\sigma)|\,d\sigma+$$

$$+\frac{1}{2}\mu(s)-\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_y}\ln|x(s)-y(\sigma)|\,d\sigma+h[\nu,\mu](x(s))=F(s),\quad s\in\Gamma^2.$$

By $\varphi_0(x,y)$ we denote the angle between the vector $\overrightarrow{xy}$ and the direction of the normal $\mathbf{n}_x$. The angle $\varphi_0(x,y)$ is taken to be positive if it is measured anticlockwise from $\mathbf{n}_x$ and negative if it is measured clockwise from $\mathbf{n}_x$. Besides, $\varphi_0(x,y)$ is continuous in $x,y\in\Gamma$ if $x\neq y$. Note, that for $x(s),y(\sigma)\in\Gamma$ and $x\neq y$ we have the relationships

$$\frac{\partial}{\partial\mathbf{n}_x}\ln|x(s)-y(\sigma)|=\frac{\partial}{\partial\tau_x}V(x(s),y(\sigma))=\frac{\partial}{\partial s}V(x(s),y(\sigma))=$$

$$=-\frac{\cos\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|}=-\frac{\sin\left(V(x(s),y(\sigma))-\alpha(s)\right)}{|x(s)-y(\sigma)|},$$

$$\frac{\partial}{\partial\mathbf{n}_x}V(x(s),y(\sigma))=-\frac{\partial}{\partial\tau_x}\ln|x(s)-y(\sigma)|=-\frac{\partial}{\partial s}\ln|x(s)-y(\sigma)|=$$

$$=\frac{\sin\varphi_0\left(x(s),y(\sigma)\right)}{|x(s)-y(\sigma)|}=-\frac{\cos\left(V(x(s),y(\sigma))-\alpha(s)\right)}{|x(s)-y(\sigma)|},$$

where $\alpha(s)$ is the inclination of the tangent τ_x to the Ox_1 axis, and $V(x,y(\sigma))$ is the kernel of the angular potential (4).

The 2nd integral term in (8a) is a Cauchy singular integral. The kernel of the third integral term in (8b) has a weak singularity as $s=\sigma$.

Equation (8a) is obtained as $x\to x(s)\in(\Gamma^1)^\pm$ and comprises two integral equations. The upper sign denotes the integral equation on $(\Gamma^1)^+$, the lower sign denotes the integral equation on $(\Gamma^1)^-$.

In addition to the integral equations written above we have conditions (5).

Subtracting the integral equations (8a), we find

(9) $$\nu(s)=\left(F^+(s)-F^-(s)\right)\in C^{0,\lambda}(\Gamma^1).$$

We note that $\nu(s)$ is found completely and satisfies all required conditions. Hence, the potential $v_1[\nu](x)$ is found completely as well. Additionally,

$$h[\nu,\mu](x) = \frac{1}{2\pi}\int\limits_{\Gamma^1}(F^+(\sigma) - F^-(\sigma))d\sigma \ln|x - Y_1| + h_2[\mu](x),$$

where $h_2[\mu](x)$ is given by (7b).

We introduce the function $f(s)$ on Γ by the formulas

$$f(s) = \frac{1}{2}\left(F^+(s) + F^-(s)\right) - \tag{10a}$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^1}\left(F^+(\sigma) - F^-(\sigma)\right)\frac{\cos\varphi_0\left(x(s), y(\sigma)\right)}{|x(s) - y(\sigma)|}d\sigma -$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^1}(F^+(\sigma) - F^-(\sigma))d\sigma\frac{\partial}{\partial \mathbf{n}_x}\ln|x(s) - Y_1|, \quad s \in \Gamma^1,$$

and

$$f(s) = F(s) + \tag{10b}$$

$$+\frac{1}{2\pi}\int\limits_{\Gamma^1}\left(F^+(\sigma) - F^-(\sigma)\right)\ln|x(s) - y(\sigma)|d\sigma -$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^1}(F^+(\sigma) - F^-(\sigma))d\sigma \ln|x(s) - Y_1|, \quad s \in \Gamma^2,$$

where $F^{\pm}(s)$ and $F(s)$ are specified in (2a) and satisfy conditions (3). As shown in [6], if $s \in \Gamma^1$, then $f(s) \in C^{0,\lambda}(\Gamma^1)$. Consequently,

$$f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2). \tag{10c}$$

Adding the integral equations (8a) we obtain the integral equation for $\mu(s)$ on Γ^1

$$-\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)\frac{\sin\varphi_0(x(s), y(\sigma))}{|x(s) - y(\sigma)|}d\sigma - \tag{11a}$$

$$-\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial \mathbf{n}_x}\frac{\partial}{\partial \mathbf{n}_y}\ln|x - y(\sigma)|\,d\sigma + \frac{\partial}{\partial \mathbf{n}_x}h_2[\mu](x(s)) = f(s), \quad s \in \Gamma^1,$$

where $f(s)$ is given by (10a).

Equation (8b) on Γ^2 takes the form

$$(11b)\qquad -\frac{1}{2\pi}\int\limits_{\Gamma^1}\mu(\sigma)V(x(s),y(\sigma))d\sigma+$$

$$+\frac{1}{2}\mu(s)-\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu(\sigma)\frac{\partial}{\partial\mathbf{n}_y}\ln|x(s)-y(\sigma)|\,d\sigma+h_2[\mu](x(s))=f(s),\quad s\in\Gamma^2,$$

where $f(s)$ is given in (10b).

Thus, if $\mu(s)$ is a solution of equations (11), (5) from the space $C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)$ with $\omega\in(0,1]$, $q\in[0,1)$, then the potential (6) with $\nu(s)$ from (9) satisfies all conditions of the problem $\mathbf{U}$ and belongs to the class $\mathbf{K}$.

The following theorem holds.

Theorem 2. *Let $\Gamma^1\in C^{2,\lambda}$, $\Gamma^2\in C^{1,\lambda}$ and conditions (3) hold. If equations (11), (5) have a solution $\mu(s)$ from the Banach space $C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)$ for some $\omega\in(0,1]$ and $q\in[0,1)$, then the solution of the problem $\mathbf{U}$ exists, belongs to the class $\mathbf{K}$ and is given by (6), where $\nu(s)$ is defined in (9).*

If $s\in\Gamma^2$, then (11b) is an equation of the second kind with a weak singularity in the kernel. If $s\in\Gamma^1$, then (11a) is a Cauchy singular integral equation of the first kind [16].

Our further treatment will be aimed to the proof of the solvability of the system (11), (5) in the Banach space $C_q^\omega(\Gamma^1)\cap C^0(\Gamma^2)$. Moreover, we reduce the system (11), (5) to a Fredholm equation of the second kind and index zero, which can be easily computed by classical methods.

Equation (11b) on Γ^2 can be rewritten in the form

$$(12)\qquad \mu(s)+\int\limits_{\Gamma}\mu(\sigma)A_2(s,\sigma)d\sigma=2f(s),\quad s\in\Gamma^2,$$

where

$$A_2(s,\sigma)=\Big\{-\frac{1}{\pi}\Big(1-\delta(\Gamma^2,\sigma)\Big)V(x(s),y(\sigma))-$$

$$-\frac{1}{\pi}\delta(\Gamma^2,\sigma)\frac{\partial}{\partial \mathbf{n}_y}\ln|x(s)-y(\sigma)| - \frac{1}{\pi}\sum_{k=2}^{N_2}\delta(\Gamma_k^2,\sigma)\ln|x(s)-Y_k| +$$

$$+\frac{1}{\pi}\Big(\delta(\Gamma^2,\sigma)-\delta(\Gamma_1^2,\sigma)\Big)\ln|x(s)-Y_1| + 2\delta(\Gamma^2,\sigma)\Big\}.$$

By $\delta(\gamma,\sigma)$ we denote the characteristic function of the set γ:

$$\delta(\gamma,\sigma) = \begin{cases} 0, & \text{if } \sigma \notin \gamma \\ 1, & \text{if } \sigma \in \gamma \end{cases}$$

The kernel $A_2(s,\sigma)$ has a weak singularity if $s=\sigma\in\Gamma^2$, and $A_2(s,\sigma)$ is continuous if $s\neq\sigma$ $(s\in\Gamma^2,\ \sigma\in\Gamma)$.

Remark. Evidently, $f(a_n^2)=f(b_n^2)$ and $A_2(a_n^2,\sigma)=A_2(b_n^2,\sigma)$ for $\sigma\in\Gamma$, $\sigma\neq a_n^2,b_n^2$ $(n=1,...,N_2)$. Hence, if $\mu(s)$ is a solution of equation (12) from $C^0\left(\bigcup_{n=1}^{N_2}[a_n^2,b_n^2]\right)$, then, according to the equality (12), $\mu(s)$ automatically satisfies the matching conditions $\mu(a_n^2)=\mu(b_n^2)$ for $n=1,...,N_2$ and, therefore, belongs to $C^0(\Gamma^2)$. This observation can be helpful in finding numerical solutions, since we may discard the matching conditions $\mu(a_n^2)=\mu(b_n^2)$, $(n=1,...,N_2)$, which are automatically fulfilled.

It can be easily proved that

$$-\frac{\partial}{\partial s}\ln\frac{|x(s)-y(\sigma)|}{|s-\sigma|} = \frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma-s} \in C^{0,\lambda}(\Gamma^1\times\Gamma^1)$$

(see [5], [6] for details). Therefore we can rewrite (11a) in the form

(13)
$$\frac{1}{\pi}\int_{\Gamma^1}\mu(\sigma)\frac{d\sigma}{\sigma-s} + \int_{\Gamma}\mu(\sigma)M(s,\sigma)d\sigma =$$

$$= -2f(s), \quad s\in\Gamma^1,$$

where

$$M(s,\sigma) = \frac{1}{\pi}\left\{\Big(1-\delta(\Gamma^2,\sigma)\Big)\left(\frac{\sin\varphi_0(x(s),y(\sigma))}{|x(s)-y(\sigma)|} - \frac{1}{\sigma-s}\right) +\right.$$

$$+\,\delta(\Gamma^2,\sigma)\frac{\partial}{\partial \mathbf{n}_x}\left[\frac{\partial}{\partial \mathbf{n}_y}\ln|x(s)-y(\sigma)|+\sum_{k=2}^{N_2}\delta(\Gamma_k^2,\sigma)\ln|x(s)-Y_k|\;-\right.$$
$$\left.-(1-\delta(\Gamma_1^2,\sigma))\ln|x(s)-Y_1|\right]\Big\}$$

and $M(s,\sigma)\in C^{0,\lambda}(\Gamma^1\times\Gamma)$.

4. The Fredholm integral equation and the solution of the problem.

Inverting the singular integral operator in (13), we arrive at the following integral equation of the second kind [5,6]:

(14)
$$\mu(s)+\frac{1}{Q_1(s)}\int_{\Gamma}\mu(\sigma)A_0(s,\sigma)d\sigma+\frac{1}{Q_1(s)}\sum_{n=0}^{N_1-1}G_n s^n=$$
$$=\frac{1}{Q_1(s)}\Phi_0(s),\quad s\in\Gamma^1,$$

where

$$A_0(s,\sigma)=-\frac{1}{\pi}\int_{\Gamma^1}\frac{M(\xi,\sigma)}{\xi-s}Q_1(\xi)d\xi,$$

$$Q_1(s)=\prod_{n=1}^{N_1}\left|\sqrt{s-a_n^1}\sqrt{b_n^1-s}\right|\mathrm{sign}(s-a_n^1)\ ,$$

$$\Phi_0(s)=\frac{1}{\pi}\int_{\Gamma^1}\frac{2Q_1(\sigma)f(\sigma)}{\sigma-s}d\sigma,$$

and $G_0,...,G_{N_1-1}$ are arbitrary constants.

To derive equations for $G_0,...,G_{N_1-1}$, we substitute $\mu(s)$ from (14) in the conditions (5), then we obtain

(15)
$$\int_{\Gamma}\mu(\sigma)l_n(\sigma)d\sigma+\sum_{m=0}^{N_1-1}B_{nm}G_m=H_n,\quad n=1,...,N_1\ ,$$

where

$$l_n(\sigma)=-\int_{\Gamma_n^1}Q_1^{-1}(s)A_0(s,\sigma)ds,$$

$$B_{nm} = -\int\limits_{\Gamma_n^1} Q_1^{-1}(s) s^m ds, \tag{16}$$

$$H_n = -\int\limits_{\Gamma_n^1} Q_1^{-1}(s)\Phi_0(s) ds.$$

By B we denote the $N_1 \times N_1$ matrix with the elements B_{nm} from (16). As shown in [6, lemma 7], [11] the matrix B is invertible. The elements of the inverse matrix will be called $(B^{-1})_{nm}$. Inverting the matrix B in (15), we express the constants $G_0, ..., G_{N_1-1}$ in terms of $\mu(s)$ as

$$G_n = \sum_{m=1}^{N_1} (B^{-1})_{nm} \left[H_m - \int\limits_{\Gamma} \mu(\sigma) l_m(\sigma) d\sigma \right].$$

We substitute G_n in (14) and obtain the following integral equation for $\mu(s)$ on Γ^1

$$\mu(s) + \frac{1}{Q_1(s)} \int\limits_{\Gamma} \mu(\sigma) A_1(s,\sigma) d\sigma = \frac{1}{Q_1(s)} \Phi_1(s), \quad s \in \Gamma^1, \tag{17}$$

where

$$A_1(s,\sigma) = A_0(s,\sigma) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} l_m(\sigma),$$

$$\Phi_1(s) = \Phi_0(s) - \sum_{n=0}^{N_1-1} s^n \sum_{m=1}^{N_1} (B^{-1})_{nm} H_m \, .$$

It can be verified directly that any solution of (17) in the required space satisfies conditions (5) automatically. It can be shown using the properties of singular integrals [2], [16], that $\Phi_0(s)$, $A_0(s,\sigma)$ are Hölder continuous functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Therefore, $\Phi_1(s)$, $A_1(s,\sigma)$ are also Hölder continuous functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. Consequently, any solution of (17) belongs to $C^{\omega}_{1/2}(\Gamma^1)$, and below we look for $\mu(s)$ on Γ^1 in this space.

We put

$$Q(s) = \Big(1 - \delta(\Gamma^2, s)\Big) Q_1(s) + \delta(\Gamma^2, s), \quad s \in \Gamma.$$

Instead of $\mu(s) \in C^{\omega}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ we introduce the new unknown function $\mu_*(s) = \mu(s)Q(s) \in C^{0,\omega}(\Gamma^1) \cap C^0(\Gamma^2)$ and rewrite (12), (17) in the form of one equation

$$\mu_*(s) + \int\limits_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma = \Phi(s), \quad s \in \Gamma, \tag{18}$$

where

$$A(s,\sigma) = \Big(1 - \delta(\Gamma^2, s)\Big)\, A_1(s,\sigma) + \delta(\Gamma^2, s)A_2(s,\sigma),$$

$$\Phi(s) = \Big(1 - \delta(\Gamma^2, s)\Big)\, \Phi_1(s) + 2\delta(\Gamma^2, s)f(s).$$

Thus, the system of equations (5), (11) for $\mu(s)$ has been reduced to the equation (18) for the function $\mu_*(s)$. It is clear from our consideration that any solution of (18) gives a solution of system (5), (11) and conversely.

As noted above, $\Phi_1(s)$ and $A_1(s,\sigma)$ are Hölder continuous functions if $s \in \Gamma^1$, $\sigma \in \Gamma$. More precisely (see [6]), $A_1(s,\sigma)$ belongs to $C^{0,p}(\Gamma^1)$ in s uniformly with respect to $\sigma \in \Gamma$, where $p = \min\{1/2, \lambda\}$. Besides, taking into account (10c) we have $\Phi_1(s) \in C^{0,p}(\Gamma^1)$. Consequently, from equation (18) we can conclude the following assertion.

Lemma. *Let $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, and $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$, where $p = \min\{\lambda, 1/2\}$. If $\mu_*(s)$ from $C^0(\Gamma)$ satisfies the equation (18), then $\mu_*(s)$ belongs to $C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$.*

The condition $\Phi(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ holds if conditions (3) hold.

Hence below we will seek $\mu_*(s)$ from $C^0(\Gamma)$.

Consider equation (18). The integral operator

$$\int\limits_{\Gamma} \mu_*(\sigma)Q^{-1}(\sigma)A_2(s,\sigma)d\sigma = \int\limits_{\Gamma^1} \mu_*(\sigma)Q_1^{-1}(\sigma)A_2(s,\sigma)d\sigma + \int\limits_{\Gamma^2} \mu_*(\sigma)A_2(s,\sigma)d\sigma$$

is compact from $C^0(\Gamma)$ into $C^0(\Gamma^2)$. Indeed, using Arzela theorem one can verify that the 1st term is a compact operator from $C^0(\Gamma^1)$ into $C^0(\Gamma^2)$, because $A_2(s,\sigma) \in C^0(\Gamma^2 \times \Gamma^1)$.

The 2nd term is a compact operator from $C^0(\Gamma^2)$ into $C^0(\Gamma^2)$, because $A_2(s,\sigma)$ is a polar kernel [17], i.e. it has a weak singularity as $s=\sigma\in\Gamma^2$ and it is continuous if $s\neq\sigma$ $(s,\sigma\in\Gamma^2)$. Furthermore, using Arzela theorem one can show that the integral operator $\int_\Gamma \mu_*(\sigma)Q^{-1}(\sigma)A_1(s,\sigma)d\sigma$ is compact from $C^0(\Gamma)$ into $C^0(\Gamma^1)$ since $A_1(s,\sigma)\in C^0(\Gamma^1\times\Gamma)$. Therefore the integral operator from (18):

$$\mathbf{A}\mu_*(s)=\int\limits_\Gamma \mu_*(\sigma)Q^{-1}(\sigma)A(s,\sigma)d\sigma$$

is a compact operator mapping $C^0(\Gamma)$ into itself. Therefore, (18) is a Fredholm equation of the second kind and index zero in the Banach space $C^0(\Gamma)$.

Let us show that if $\mu_*^0(s)$ is a solution of the homogeneous equation (18) from $C^0(\Gamma)$, then it is the trivial solution, i.e. $\mu_*^0(s)\equiv 0$. Let $\mu_*^0(s)\in C^0(\Gamma)$ be a solution of the homogeneous equation (18). According to the Lemma, $\mu_*^0(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, $\quad p=\min\{\lambda,1/2\}$. Therefore the function $\mu^0(s)=\mu_*^0(s)Q^{-1}(s)\in C^p_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ converts the homogeneous equations (12), (17) into identities. Using the homogeneous identity (17), we check, that $\mu^0(s)$ satisfies conditions (5). Besides, acting on the homogeneous identity (17) with a singular operator with the kernel $(s-t)^{-1}$, we find that $\mu^0(s)$ satisfies the homogeneous equation (13). Consequently, $\mu^0(s)$ satisfies the homogeneous equations (11). On the basis of Theorem 2, the function $u[0,\mu^0](x)=w[\mu^0](x)+h_2[\mu^0](x)$ given by (6), (7) is a solution of the homogeneous problem $\mathbf{U}$. According to Theorem 1, $\Big(w[\mu^0](x)+h_2[\mu^0](x)\Big)\equiv 0,\ x\in\mathcal{D}\backslash\Gamma^1$. Using the limit formulas for tangential derivatives of an angular potential [1,5], [12, Appendix], we obtain

$$\lim_{x\to x(s)\in(\Gamma^1)^+}\frac{\partial}{\partial\tau_x}\Big(w[\mu^0](x)+h_2[\mu^0](x)\Big)-$$
$$-\lim_{x\to x(s)\in(\Gamma^1)^-}\frac{\partial}{\partial\tau_x}\Big(w[\mu^0](x)+h_2[\mu^0](x)\Big)=\mu^0(s)\equiv 0,\quad s\in\Gamma^1.$$

Hence, $\Big(w[\mu^0](x)+h_2[\mu^0](x)\Big)=\Big(w_2[\mu^0](x)+h_2[\mu^0](x)\Big)\equiv 0,\;\; x\in\mathcal{D}$, and $\mu^0(s)$ satisfies (11b), which takes the form

$$\frac{1}{2}\mu^0(s)-\frac{1}{2\pi}\int\limits_{\Gamma^2}\mu^0(\sigma)\frac{\partial}{\partial\mathbf{n}_y}\ln|x(s)-y(\sigma)|\,d\sigma+ \tag{19}$$

$$+h_2[\mu^0](x(s))=0,\;\; s\in\Gamma^2,$$

where $h_2[\mu](x)$ is specified in (7b). The Fredholm equation (19) arises when solving the homogeneous Dirichlet problem for harmonic functions in the exterior domain $\mathcal{D}$ by the double layer potential with the sum of point sources placed inside the curves $\Gamma_1^2,...,\Gamma_{N_2}^2$. The equation (19) has only the trivial solution $\mu^0(s)\equiv 0$ in $C^0(\Gamma^2)$. This is shown in the appendix.

Consequently, if $s\in\Gamma$, then $\mu^0(s)\equiv 0,\quad \mu_*^0(s)=\mu^0(s)Q^{-1}(s)\equiv 0$. Thus, the homogeneous Fredholm equation (18) has only the trivial solution in $C^0(\Gamma)$.

We have proved the following assertion.

Theorem 3. *If* $\Gamma^1\in C^{2,\lambda},\;\Gamma^2\in C^{1,\lambda},\;\lambda\in(0,1]$, *then (18) is a Fredholm equation of the second kind and index zero in the space* $C^0(\Gamma)$. *Moreover, equation (18) has a unique solution* $\mu_*(s)\in C^0(\Gamma)$ *for any* $\Phi(s)\in C^0(\Gamma)$.

As a consequence of Theorem 3 and the Lemma we obtain

Corollary. *If* $\Gamma^1\in C^{2,\lambda}$, $\Gamma^2\in C^{1,\lambda}$, $\lambda\in(0,1]$, *then equation (18) has a unique solution* $\mu_*(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, *for any* $\Phi(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, *where* $p=\min\{\lambda,1/2\}$.

We recall that $\Phi(s)$ belongs to the class of smoothness required in the Corollary if $f(s)\in C^{0,\lambda}(\Gamma^1)\cap C^0(\Gamma^2)$. As mentioned above, if $\mu_*(s)\in C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$ is a solution of (18), then $\mu(s)=\mu_*(s)Q^{-1}(s)\in C^p_{1/2}(\Gamma^1)\cap C^0(\Gamma^2)$ is a solution of system (5), (11). We obtain the following statement.

Theorem 4. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, *then the system of equations (5), (11) has a solution* $\mu(s) \in C^{p}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$, *for any* $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$. *Moreover, this solution is expressed by the formula* $\mu(s) = \mu_*(s) Q^{-1}(s)$, *where* $\mu_*(s) \in C^{0,p}(\Gamma^1) \cap C^0(\Gamma^2)$ *is the unique solution of the Fredholm equation (18) in* $C^0(\Gamma)$.

Remark. The solution of the system (5), (11) ensured by Theorem 4, is unique in the space $C^{p_o}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ for any $p_o \in (0,p]$. More precisely, the system (5), (11) has at most one solution in the space $C^{\omega}_{q}(\Gamma^1) \cap C^0(\Gamma^2)$ for any $\omega \in (0,1]$ and $q \in [0,1)$. The proof of this fact almost coincides with the proof of Theorem 3.

According to (10), $f(s)$ belongs to $C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$ if (3) holds. Therefore, the condition $f(s) \in C^{0,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$ in Theorem 4 can be replaced by the condition that (3) holds. On the basis of Theorem 2 and Theorem 4 we arrive at the final result.

Theorem 5. *If* $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$ *and condition (3) holds, then the solution of the problem* **U** *exists, belongs to the class* **K** *and is given by (6), where* $\nu(s)$ *is defined in (9) and* $\mu(s)$ *is a solution of system (5), (11) from* $C^{p}_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$, $p = \min\{1/2, \lambda\}$, *ensured by Theorem 4.*

It can be checked directly that the solution of the problem **U** constructed in Theorem 5 satisfies condition (1) with $\epsilon = -1/2$. Explicit expressions for the singularities of the solution gradient at the end-points of the open curves will be presented in the next section.

Theorem 5 ensures the existence of a classical solution of the problem **U** when $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{1,\lambda}$, and condition (3) holds. The uniqueness of the classical solution follows from Theorem 1. On the basis of our consideration we suggest the following scheme for solving the problem **U**. At first, we find the unique solution $\mu_*(s)$ of the Fredholm equation

(18) from $C^0(\Gamma)$. This solution automatically belongs to $C^{0,p}(\Gamma^1)\cap C^0(\Gamma^2)$, $p = \min\{\lambda, 1/2\}$. Secondly, we construct the solution of equations (5), (11) from $C^p_{1/2}(\Gamma^1) \cap C^0(\Gamma^2)$ by the formula $\mu(s) = \mu_*(s)Q^{-1}(s)$. Finally, substituting $\nu(s)$ from (9) and $\mu(s)$ in (6), we obtain the solution of the problem **U**.

5. The behaviour of the gradient of the solution at the tips of the cuts Γ^1.

In the present section by $u(x) = u[\nu, \mu](x)$ we denote the solution of problem **U** constructed in the previous section. The integral representation for $u(x)$ obtained in Theorem 5 enables us to derive explicit formulas for the singularities of ∇u at the tips of the cuts Γ^1. It follows from the definition of the class **K** that the gradient of the solution of the problem **U** might be unbounded at the end-points of Γ^1, where the estimate (1) holds with $\epsilon = -1/2$. Our aim now is to investigate in detail the behaviour of $\nabla u(x)$ at the end-points of Γ^1. Let $x(d)$ be one of these points ($d = a^1_n$ or $d = b^1_n$, where $n = 1, ..., N_1$). In the neighbourhoods of $x(d)$ we introduce the polar system of coordinates

$$x_1 = x_1(d) + |x - x(d)| \cos\varphi, \quad x_2 = x_2(d) + |x - x(d)| \sin\varphi.$$

We will assume that $\varphi \in (\alpha(d), \alpha(d) + 2\pi)$, if $d = a^1_n$, and $\varphi \in (\alpha(d) - \pi, \alpha(d) + \pi)$, if $d = b^1_n$. Recall that $\alpha(s)$ is the angle between the Ox_1 axis and the tangent vector τ_x drawn at the point $x(s) \in \Gamma$. Hence, $\alpha(d) = \alpha(a^1_n + 0)$, if $d = a^1_n$, and $\alpha(d) = \alpha(b^1_n - 0)$ if $d = b^1_n$. Thus, the angle φ varies continuously in a neighbourhood of $x(d)$ cut along the contour Γ^1.

Let $\mu_1(s) = \mu(s)|s - d|^{1/2} = Q^{-1}(s)\mu_*(s)|s - d|^{1/2}$ and put $\mu_1(d) = \mu_1(a^1_n) = \mu_1(a^1_n + 0)$, if $d = a^1_n$, $\mu_1(d) = \mu_1(b^1_n) = \mu_1(b^1_n - 0)$ if $d = b^1_n$.

Recall that X is the set of end-points of Γ^1. The following theorem is easily proved using the results obtained in [5] and using the properties of Cauchy type integrals near the end-points of the integration line given in [16, section 22], [2, section 8].

Theorem 7. *Let $x \to x(d) \in X$. Then in the neighbourhood of the point $x(d)$, the derivatives of the solution of the problem* **U** *satisfy the relations*

$$\frac{\partial}{\partial x_1}u(x) = -(-1)^m \frac{\mu_1(d)}{2|x-x(d)|^{1/2}}\sin\gamma -$$

$$-(-1)^m \frac{\nu(d)}{2\pi}[\ln|x-x(d)|\cos\alpha(d) + \varphi\sin\alpha(d)] + O(1),$$

$$\frac{\partial}{\partial x_2}u(x) = (-1)^m \frac{\mu_1(d)}{2|x-x(d)|^{1/2}}\cos\gamma +$$

$$+(-1)^m \frac{\nu(d)}{2\pi}[-\ln|x-x(d)|\sin\alpha(d) + \varphi\cos\alpha(d)] + O(1),$$

where $m = 0$, $\gamma = [\varphi + \alpha(d) - \pi]/2$ if $d = a_n^1$, and $m = 1$, $\gamma = [\varphi + \alpha(d)]/2$ if $d = b_n^1$. Besides, $O(1)$ denotes functions which are continuous at the point $x(d)$. Furthermore, the functions denoted by $O(1)$ are continuous in the neighbourhood of the point $x(d)$, cut along the contour Γ^1.

This theorem establishes the following curious fact. In the general case, the derivatives of the solution of the problem **U** near the end-point $x(d)$ of the contour Γ^1 behave as $O(|x-x(d)|^{-1/2}) + O(\ln|x-x(d)|^{-1})$. However, if $\mu_1(d) = \nu(d) = 0$, then $\nabla u(x)$ will be bounded and even continuous at the end-point $x(d)$ of Γ^1. This effect of disappearence of singularities happens for certain functions $F^{\pm}(s)$, $F(s)$ given in the boundary condition (2a), since the condition $\mu_1(d) = \nu(d) = 0$ specifies restrictions on these functions.

Appendix.

Here we prove the following assertion.

Proposition A. *If* $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, *then there exists only the trivial solution of the homogeneous Fredholm equation (19) in $C^0(\Gamma^2)$.*

Proof. Let $\mu^0(s) \in C^0(\Gamma^2)$ be a non-trivial solution of the homogeneous equation (19). The kernel of the integral term in (19) has a weak singularity. It can be shown with the help of [16, Sec.51], that the integral term in (19) belongs to $C^{0,\lambda/4}(\Gamma^2)$ in s, therefore $\mu^0(s) \in C^{0,\lambda/4}(\Gamma^2)$. Now we consider the function

$$g[\mu^0](x) = w_2[\mu^0](x) + h_2[\mu^0](x), \tag{A1}$$

where $w_2[\mu^0](x)$ and $h_2[\mu^0](x)$ were introduced in (7a), (7b). The function $g[\mu^0](x)$ belongs to $C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ and satisfies the following homogeneous Dirichlet problem for the Laplace equation

$$\Delta g = 0 \ \text{ in } \ \mathcal{D}, \quad g|_{\Gamma^2} = 0, \quad |g| \le const \quad \text{in } \ \overline{\mathcal{D}}.$$

Indeed, substituting $g[\mu^0](x)$ in the boundary condition, we get the identity (19). According to the uniqueness theorem for the Dirichlet problem, we have

$$g[\mu^0](x) \equiv 0, \quad x \in \overline{\mathcal{D}}. \tag{A2}$$

Therefore, letting $|x| \to \infty$ in the expression for $g[\mu^0](x)$, we obtain

$$\int_{\Gamma^2} \mu^0(\sigma) d\sigma = 0. \tag{A3}$$

We consider the function

$$g^*[\mu^0](x) = -\frac{1}{2\pi}\left[-\int_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \sigma} \ln|x - y(\sigma)| d\sigma + \right. \tag{A4}$$

$$+ \sum_{k=2}^{N_2} \int_{\Gamma_k^2} \mu^0(\sigma) d\sigma \, V(x, Y_k) -$$

$$\left. - \left(\int_{\Gamma^2} \mu^0(\sigma) d\sigma - \int_{\Gamma_1^2} \mu^0(\sigma) d\sigma \right) V(x, Y_1) \right],$$

where $V(x,y)$ is the kernel of the angular potential from (4). The function $g^*[\mu^0](x)$ is harmonic conjugate to $g[\mu^0](x)$, i.e. the Cauchy-Riemann relations $\partial_{x_1} g = \partial_{x_2} g^*$, $\partial_{x_2} g = -\partial_{x_1} g^*$ hold. Consequently, $g^*[\mu^0](x) \equiv Const$ in $\mathcal{D}$. It is clear from (A4), that $g^*[\mu^0](x)$ is a many-valued function, because $V(x, Y_k)$ are many-valued functions $(k = 1, ..., N_2)$. Indeed, when passing around the point Y_k the value of the function $V(x, Y_k)$ changes for 2π. Evidently, $g^*[\mu^0](x)$ can be constant in $\mathcal{D}$ only if $g^*[\mu^0](x)$ is single-valued. In order for $g^*[\mu^0](x)$ be single-valued, the following N_2 conditions must hold

$$\int\limits_{\Gamma_k^2} \mu^0(\sigma) d\sigma = 0, \quad k = 2, ..., N_2\,,$$

$$\int\limits_{\Gamma^2} \mu^0(\sigma) d\sigma - \int\limits_{\Gamma_1^2} \mu^0(\sigma) d\sigma = 0.$$

Along with (A3) we obtain

$$(A5) \qquad \int\limits_{\Gamma_k^2} \mu^0(\sigma) d\sigma = 0, \qquad k = 1, ..., N_2.$$

Under these conditions, $g^*[\mu^0](x)$ takes the form of the modified single-layer potential [16]

$$(A6) \qquad g^*[\mu^0](x) \equiv w_2^*[\mu^0](x) = \frac{1}{2\pi} \int\limits_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \sigma} \ln|x - y(\sigma)| d\sigma,$$

and $g[\mu^0](x)$ transforms to the ordinary double-layer potential

$$(A7) \qquad g[\mu^0](x) \equiv w_2[\mu^0](x) = -\frac{1}{2\pi} \int\limits_{\Gamma^2} \mu^0(\sigma) \frac{\partial}{\partial \mathbf{n}_y} \ln|x - y(\sigma)| d\sigma \in$$

$$\in C^0(\overline{R^2 \backslash \Gamma^2}) \cap C^2(R^2 \backslash \Gamma^2).$$

The potentials (A6) and (A7) are connected by the Cauchy-Riemann relations in $R^2 \backslash \Gamma^2$. Because of $\mu^0(s) \in C^{0,\lambda/4}(\Gamma^2)$, the potential (A6) is a harmonic function, which belongs to $C^0(R^2) \cap C^2(R^2 \backslash \Gamma^2)$ (see [16] for details). Note, that (A6) is continuous when passing through Γ^2 and is represented on Γ^2 by a singular integral (for this we have stressed that $\mu^0(s)$ is a Hölder continuous function).

As stated above, $w_2^*[\mu^0](x)$ in $\overline{\mathcal{D}}$ is equal to a constant, which is equal to zero due to the behaviour of this potential at infinity, so that $w_2^*[\mu^0](x) \equiv 0$ in $\overline{\mathcal{D}}$.

We consider the internal domain $\mathcal{D}_k$ bounded by Γ_k^2 $(k = 1, ..., N_2)$. In this domain the potential (A6) satisfies the Dirichlet problem

$$\Delta w_2^* = 0 \ \text{ in } \ \mathcal{D}_k, \quad w_2^*|_{\Gamma_k^2} = 0,$$

which has the unique solution

$$w_2^*[\mu^0](x) \equiv 0, \ \ x \in \overline{\mathcal{D}_k}, \ \ (k = 1, ...N_2).$$

It follows from the Cauchy-Riemann relations and the smoothness of the double-layer potential that

$$w_2[\mu^0](x) \equiv c_k, \ \ x \in \overline{\mathcal{D}_k}, \ \ k = 1, ...N_2,$$

where $c_1, ..., c_{N_2}$ are constants. Using (A2) and the jump relation for the double-layer potential $w_2[\mu^0](x)$ on Γ^2, we get

$$\mu^0(s)|_{\Gamma_k^2} \equiv -c_k, \ \ k = 1, ...N_2.$$

According to (A5), $c_k = 0, \ \ k = 1, ..., N_2$, and therefore

$$\mu^0(s)|_{\Gamma_k^2} \equiv 0, \ \ k = 1, ...N_2.$$

Consequently,

$$\mu^0(s) \equiv 0 \ \text{ on } \ \Gamma^2.$$

Hence, the homogeneous equation (19) has only the trivial solution. The proof is completed.

Because of (19) is a Fredholm equation of the second kind, the following Corollary holds.

Corollary A. *If* $\Gamma^2 \in C^{1,\lambda}$, $\lambda \in (0,1]$, *then the inhomogeneous Fredholm equation (19) is uniquely solvable in* $C^0(\Gamma^2)$ *for any right-hand side from* $C^0(\Gamma^2)$.

The inhomogeneous equation (19) is a particular case of (11b) if the exterior domain $\mathcal{D}$ does not contain cuts.

Acknowledgment.

This research was supported by the RFBR grants No. 06-01-00001, 05-01-00050.

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Journal of Applied Functional Analysis

A quartely international publication of Eudoxus Press, LLC of TN.

Editor in Chief: George Anastassiou

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Volume 3,Number 4 October 2008

ISSN:1559-1948 (PRINT), 1559-1956 (ONLINE)

EUDOXUS PRESS,LLC

JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS

A quartely international publication of **EUDOXUS PRESS,LLC**

ISSN:1559-1948(PRINT),1559-1956(ONLINE)

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Journal of Applied Functional Analysis(JAFA)
is published in January,April,July and October of each year by

EUDOXUS PRESS,LLC,
1424 Beaver Trail Drive,Cordova,TN38016,USA,
Tel.001-901-751-3553
anastassioug@yahoo.com
http://www.EudoxusPress.com visit also http://www.msci.memphis.edu/~ganastss/jafa.

Webmaster:Ray Clapsadle

Annual Subscription Current Prices:For USA and Canada,Institutional:Print $250,Electronic $220,Print and Electronic $310.Individual:Print $77,Electronic $60,Print &Electronic $110.For any other part of the world add $25 more to the above prices for Print.
Single article PDF file for individual $8.Single issue in PDF form for individual $25.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.4,399-417,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Neumann and Mixed Boundary Value Problems

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Abstract

The Neumann boundary value problem is investigated for the inhomogeneous polyanalytic equation. The mixed k-Neumann and $n-k$ Schwarz boundary value problem has also been studied.

AMS(2000): 32A30, 30G20, 35J55, 31A10

Keywords and phrases: Neumann problem, Schwarz problem, Gauss theorem, Cauchy-Pompeiu representation, inhomogeneous polyanalytic equation

1. Introduction

The Neumann problem is well studied for harmonic functions and solved under certain conditions via the Neumann function. Extending the concept of Neumann functions for the Laplacian to Neumann functions for powers of the Laplacian, an explicit representation of the solution to the n-Neumann problem for $\Delta^n u = f$ has been given in [7]. Integral representations for solutions to higher order differential equations can be obtained by iterating those representation formulas for first order equations. This procedure has been applied in [3, 4, 5, 7, 8] to get explicit representation formula for the solution of higher order partial differential equations.
Neumann boundary conditions are given via outer normal derviations ∂_ν. For the unit disc $\mathbb{D}$ this is

$$\partial_\nu = z\partial_z + \bar{z}\partial_{\bar{z}}$$

In [2,6], the following results are proved.

Theorem 1. The Neumann problem $w_{\bar{z}} = f$ in $\mathbb{D}$, $\partial_\nu w = \gamma$ on $\partial\mathbb{D}$, $w(0) = c$ for $f \in C^\alpha(\overline{\mathbb{D}}, \mathbb{C})$, $0 < \alpha < 1$, $\gamma \in C(\partial\mathbb{D}, \mathbb{C})$, $c \in \mathbb{C}$ is solvable if and only if for $|z| = 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\frac{d\zeta}{(1-\bar{z}\zeta)\zeta} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{f(\zeta)d\bar{\zeta}}{1-\bar{z}\zeta} + \frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)d\xi d\eta}{(1-\bar{z}\zeta)^2} = 0 \tag{1}$$

The solution then is given as

$$\begin{aligned} w(z) = c &- \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma(\zeta)\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} f(\zeta)\log(1-z\bar{\zeta})d\bar{\zeta} \\ &-\frac{z}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)d\xi d\eta}{\zeta(\zeta-z)} \end{aligned} \tag{2}$$

□

Theorem 2. The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc

$\partial_{\bar{z}}^m w = f$ in $\mathbb{D}$, Re $\partial_{\bar{z}}^s w = \beta_s$ on $\partial\mathbb{D}$, Im $\partial_{\bar{z}}^s w(0) = b_s$, $0 \le s \le m-1$,

is uniquely solvable for $f \in L_1(\mathbb{D}, \mathbb{C})$, $\beta_s \in C(\partial\mathbb{D}, \mathbb{R})$, $b_s \in \mathbb{R}$, $0 \le s \le m-1$. The solution is given by

$$\begin{aligned} w(z) &= i\sum_{s=0}^{m-1}\frac{b_s}{s!}(z+\bar{z})^s + \sum_{s=0}^{m-1}\frac{(-1)^s}{2\pi i s!}\int\limits_{|\zeta|=1} \beta_s(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta - z + \overline{\zeta - z})^s\frac{d\zeta}{\zeta} \\ &+\frac{(-1)^m}{2\pi(m-1)!}\int\limits_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)(\zeta - z + \overline{\zeta - z})^{m-1}d\xi d\eta \end{aligned} \tag{3}$$

□

In [5], the Dirichlet problem and the half-Neumann problem for the inhomogeneous polyanalytic equation have been studied. In the following section the Neumann problem for the inhomogenous polyanalytic equation is considered. The third section deals with the mixed problem of n-Neumann and m-Schwarz boundary conditions. Our methods are based on repeated applications of Gauss theorem, Cauchy Pompeiu formula and Cauchy Pompeiu operators for the differential operators $\partial_z^m \partial_{\bar{z}}^n$ [1, 3]. For boundary value problems of polyharmonic functions, see [9, 10].

2. Neumann problem

Let D be a regular domain i.e. a bounded domain in the complex plane with a smooth boundary. Often D will be chosen to be unit disc $\mathbb{D}$ in order to receive explicit formulas. The Neumann problem is improperly formulated for the first order equation because the differential operator of the equation becomes involved into boundary condition. For this reason a half-Neumann problem is considered in [5]. However, in this section we consider the full Neumann problem. The integrals in the following lemma can be computed using Cauchy-Pompeiu formula [1].

Lemma 1. For $r \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, $|z| < 1$ and $|\tilde{\zeta}| < 1$, we have

(i)
$$L_r(z, \tilde{\zeta}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \bar{\zeta}^r \log(1 - z\bar{\zeta}) \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\zeta}{\zeta^2}$$
$$- \frac{z}{\pi} \int\limits_{|\zeta|<1} \bar{\zeta}^r \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\xi d\eta}{\zeta(\zeta - z)}$$
$$= -\frac{1}{r+1} (\bar{\tilde{\zeta}}^{r+1} - \bar{z}^{r+1}) \log(1 - z\bar{\tilde{\zeta}})$$

(ii)
$$N_r(\tilde{\zeta}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \bar{\zeta}^r \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\zeta}{\zeta^2} = -\frac{\bar{\tilde{\zeta}}^{r+1}}{r+1}$$ □

Theorem 3. The Neumann problem for the inhomogeneous polyanalytic equation in the unit disc $\partial_{\bar{z}}^n w = f$ in $\mathbb{D}$, $\partial_\nu(\partial_{\bar{z}}^r w) = \alpha_r$ on $\partial\mathbb{D}$, $\partial_{\bar{z}}^r w(0) = a_r$ for $0 \le r \le n-1$ is uniquely solvable for $f \in C^\alpha(\overline{\mathbb{D}}, \mathbb{C})$, $0 < \alpha < 1$, $\alpha_r \in C(\partial\mathbb{D}, \mathbb{C}), a_r \in \mathbb{C}$, $0 \le r \le n-1$, if and only if for $1 \le k \le n-1$ and $|z| = 1$

$$\sum_{r=k}^{n-1} a_r \frac{\bar{z}^{r-k+1}}{(r-k)!} + \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\alpha_{k-1}(\zeta)}{\bar{\zeta} - \bar{z}} d\bar{\zeta}$$

$$- \sum_{r=k}^{n-1} \frac{(-1)^{r-k+1}}{(r-k+1)!} \frac{1}{2\pi i} \int_{|\zeta|=1} \alpha_r(\zeta)(\bar{\zeta} - \bar{z})^{r-k} [\bar{\zeta} - (r-k+1)\bar{z} \log(1 - z\bar{\zeta})] \frac{d\zeta}{\zeta}$$

$$+ \frac{(-1)^{n-k}}{(n-k)!} \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta)(\bar{\zeta} - \bar{z})^{n-k-1} [\bar{\zeta} - (n-k)\bar{z} \log(1 - z\bar{\zeta})] \frac{d\zeta}{\zeta^2}$$

$$- \frac{(-1)^{n-k}}{(n-k)!} \frac{\bar{z}}{\pi} \int_{|\zeta|<1} f(\zeta)(\bar{\zeta} - \bar{z})^{n-k-1} \left[\frac{(\bar{\zeta} - \bar{z})}{(1 - \bar{z}\zeta)^2} + \frac{n-k}{\zeta(1 - \bar{z}\zeta)} \right] d\xi d\eta = 0 \tag{4}$$

and

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \alpha_{n-1}(\zeta) \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} - \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta) \frac{\bar{\zeta} d\bar{\zeta}}{\bar{\zeta} - \bar{z}} - \frac{\bar{z}}{\pi} \int_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{(1 - \bar{z}\zeta)^2} = 0 \tag{5}$$

The solution then is given by

$$w(z) = \sum_{r=0}^{n-1} a_r \frac{\bar{z}^r}{r!} - \sum_{r=0}^{n-1} \frac{(-1)^r}{r!} \frac{1}{2\pi i} \int_{|\zeta|=1} \alpha_r(\zeta)(\bar{\zeta} - \bar{z})^r \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta}$$

$$+ \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta)(\bar{\zeta} - \bar{z})^{n-1} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2}$$

$$- \frac{(-1)^{n-1}}{(n-1)!} \frac{z}{\pi} \int_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \frac{(\bar{\zeta} - \bar{z})^{n-1}}{\zeta - z} d\xi d\eta \tag{6}$$

Proof: The conditions (5) coincide with (1) and (6) with (2) for $n = 1$. Assuming Theorem 3 is proved for $n-1$ rather than n the problem is decomposed into the system

$$\partial_{\bar{z}}^{n-1} w = \omega \text{ in } \mathbb{D}, \partial_\nu (\partial_{\bar{z}}^r w) = \alpha_r \text{ on } \partial\mathbb{D}, \partial_{\bar{z}}^r w(0) = a_r \text{ for } 0 \le r \le n-2 \tag{7}$$

$$\partial_{\bar{z}} \omega = f \text{ in } \mathbb{D}, \partial_\nu \omega = \alpha_{n-1} \text{ on } \partial\mathbb{D}, \omega(0) = a_{n-1} \tag{8}$$

and the solution (6) for $n-1$ instead of n and ω instead of f where

$$\omega(z) = a_{n-1} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\zeta \alpha_{n-1}(\zeta) - f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2}$$

$$- \frac{z}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta(\zeta - z)} d\xi d\eta \tag{9}$$

The last two terms of (6) with $n-1$ instead of n and ω instead of f can be written as

$$a_{n-1}J_1(z) - \frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1} \frac{\alpha_{n-1}(\tilde{\zeta})}{\tilde{\zeta}} J_2(z,\tilde{\zeta})d\tilde{\zeta}$$

$$+\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1} \frac{f(\tilde{\zeta})}{\tilde{\zeta}^2} J_2(z,\tilde{\zeta})d\tilde{\zeta} - \frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1} \frac{f(\tilde{\zeta})}{\tilde{\zeta}} J_3(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} \tag{10}$$

where

$$J_1(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\bar{\zeta}-\bar{z})^{n-2}\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta^2} - \frac{z}{\pi}\int\limits_{|\zeta|<1} \frac{(\bar{\zeta}-\bar{z})^{n-2}}{\zeta(\zeta-z)}d\xi d\eta$$

$$= \frac{(-1)^n \bar{z}^{n-1}}{n-1}, \tag{11}$$

$$J_2(z,\tilde{\zeta}) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\bar{\zeta}-\bar{z})^{n-2}\log(1-z\bar{\zeta})\log(1-\zeta\bar{\tilde{\zeta}})\frac{d\zeta}{\zeta^2}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1} (\bar{\zeta}-\bar{z})^{n-2}\log(1-\zeta\bar{\tilde{\zeta}})\frac{d\xi d\eta}{\zeta(\zeta-z)}$$

Using binomial expansion and expressing $J_2(z,\tilde{\zeta})$ in terms of $L_r(z,\tilde{\zeta})$, we obtain

$$J_2(z,\tilde{\zeta}) = -\frac{1}{n-1}(\bar{\tilde{\zeta}}-\bar{z})^{n-1}\log(1-z\bar{\tilde{\zeta}}) \tag{12}$$

Using Cauchy-Pompeiu formula and computing the boundary integrals, we have

$$\begin{aligned} J_3(z,\tilde{\zeta}) &= -\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \zeta(\bar{\zeta}-\bar{z})^{n-2}\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta^2(\zeta-\tilde{\zeta})} \\ &\quad +\frac{z}{\pi}\int\limits_{|\zeta|<1} \zeta(\bar{\zeta}-\bar{z})^{n-2}\frac{d\xi d\eta}{\zeta(\zeta-z)(\zeta-\tilde{\zeta})} \\ &= -\frac{1}{(n-1)}\frac{z}{\tilde{\zeta}-z}(\bar{\tilde{\zeta}}-\bar{z})^{n-1} \end{aligned} \tag{13}$$

Substituting (11), (12), (13) in (10) and then subsequently in (6) with $n-1$ instead of n, we obtain (6).

Using (9), the boundary integral in (5) with ω instead of f can be written as

$$a_{n-1}\bar{z} - \frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\left(\frac{\alpha_{n-1}(\tilde{\zeta})}{\tilde{\zeta}} - \frac{f(\tilde{\zeta})}{\tilde{\zeta}^2}\right)\Phi_1(z,\tilde{\zeta})d\tilde{\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}\Phi_2(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta} \tag{14}$$

where

$$\Phi_1(z,\tilde{\zeta}) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\zeta\bar{\tilde{\zeta}})\frac{d\zeta}{\zeta^2(1-\bar{z}\zeta)} = -\bar{\tilde{\zeta}}$$

and $\Phi_2(z,\tilde{\zeta}) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\zeta-\tilde{\zeta})\zeta(1-\bar{z}\zeta)} = \frac{\bar{z}}{1-\bar{z}\tilde{\zeta}}$

Using Gauss theorem, the area integral in (5) with ω instead of f can be written as

$$-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\omega(\zeta)}{\zeta-z}\frac{d\zeta}{\zeta} - \frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{(\bar{\zeta}-\bar{z})}{(1-\bar{z}\zeta)^2}d\xi d\eta$$

Substituting ω from (8) and computing the boundary integrals involved we can write the above expression as

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\alpha_{n-1}(\zeta)}{\zeta}\bar{z}\log(1-z\bar{\zeta})d\zeta - \frac{\bar{z}}{2\pi i}\int\limits_{|\zeta|=1}\frac{f(\zeta)}{\zeta^2}\log(1-z\bar{\zeta})d\zeta$$

$$-\frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)\frac{(\bar{\zeta}-\bar{z})}{(1-\bar{z}\zeta)^2}d\xi d\eta \tag{15}$$

Substituting (14) and (15) in (5) with $n-1$ instead of n and ω instead of f we get (4) for $k = n-1$.

For $1 \le k \le n-2$, the last two integrals of (4) with $n-1$ instead of n and ω instead of f can be written as

$$-\frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\left(\frac{\alpha_{n-1}(\tilde{\zeta})}{\tilde{\zeta}} - \frac{f(\tilde{\zeta})}{\tilde{\zeta}^2}\right)\psi_1(z,\tilde{\zeta})d\tilde{\zeta} - \frac{1}{\pi}\int\limits_{|\tilde{\zeta}|<1}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}\psi_2(z,\tilde{\zeta})d\tilde{\xi}d\tilde{\eta}$$

$$+\bar{z}\psi_3(z) - (n-k-1)\psi_4(z) \tag{16}$$

where

$$\psi_1(z,\tilde{\zeta}) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\bar{\zeta}-\bar{z})^{n-k-1}\log(1-\zeta\bar{\tilde{\zeta}})\frac{d\zeta}{\zeta^2}$$

Using binomial expansion, $\psi_1(z,\tilde{\zeta})$ can be expressed as linear combination of $\bar{z}^r N_r(\tilde{\zeta})$. Similar using Lemma 1(ii) it follows that

$$\psi_1(z,\tilde{\zeta}) = -\frac{1}{n-k}((\bar{\tilde{\zeta}}-\bar{z})^{n-k} - (-\bar{z})^{n-k}) \tag{17}$$

$$\psi_2(z,\tilde{\zeta}) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\bar{\zeta}-\bar{z})^{n-k-1}\frac{\zeta\, d\zeta}{\zeta^2(\tilde{\zeta}-\zeta)} = 0 \tag{18}$$

$$\psi_3(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \omega(\zeta)(\bar{\zeta}-\bar{z})^{n-k-2}\frac{d\zeta}{\zeta^2}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\frac{(\bar{\zeta}-\bar{z})^{n-k-1}}{(1-\bar{z}\zeta)^2}d\xi d\eta$$

Applying Gauss theorem on the area integral and the fact that $\dfrac{\partial\omega}{\partial\bar{z}} = f$ in $\mathbb{D}$, computations similar to that in $\psi_1(z,\tilde{\zeta})$ yield

$$\bar{z}\psi_3(z) = \frac{1}{n-k}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{\alpha_{n-1}(\zeta)}{\zeta} - \frac{f(\zeta)}{\zeta^2}\right)(\bar{z}(\bar{\zeta}-\bar{z})^{n-k-1} + (-\bar{z})^{n-k})d\zeta$$

$$+\frac{\bar{z}}{n-k}\frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{(\bar{\zeta}-\bar{z})^{n-k}}{(1-\bar{z}\zeta)^2}d\xi d\eta$$

Lastly

$$\psi_4(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \omega(\zeta)(\bar{\zeta}-\bar{z})^{n-k-2}\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta^2}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)(\bar{\zeta}-\bar{z})^{n-k-2}\frac{d\xi d\eta}{\zeta(1-\bar{z}\zeta)} \tag{19}$$

Computations similar to those involved in $J_1(z)$, $J_2(z,\tilde{\zeta})$ and $J_3(z,\tilde{\zeta})$ yield,

$$\psi_4(z)=\frac{(-1)^{n-k}a_{n-1}\bar{z}^{n-k-1}}{(n-k-1)}+\frac{1}{n-k-1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{\alpha_{n-1}(\zeta)}{\zeta}-\frac{f(\zeta)}{\zeta^2}\right)(\bar{\zeta}-\bar{z})^{n-k-1}$$

$$\log(1-z\bar{\zeta})d\zeta-\frac{1}{n-k-1}\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\frac{(\bar{\zeta}-\bar{z})^{n-k-1}}{1-\bar{z}\zeta}d\xi d\eta \tag{20}$$

Substituting (15), (16), (17) and (18), (14) can be simplified to

$$(-1)^{n-k+1}a_{n-1}\bar{z}^{n-k}+\frac{1}{n-k}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{\alpha_{n-1}(\zeta)}{\zeta}-\frac{f(\zeta)}{\zeta^2}\right)$$

$$(\bar{\zeta}-\bar{z})^{n-k-1}[\bar{\zeta}-(n-k)\bar{z}\log(1-z\bar{\zeta})]d\zeta$$

$$+\frac{1}{n-k}\frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}f(\zeta)(\bar{\zeta}-\bar{z})^{n-k-1}\left[\frac{\bar{\zeta}-\bar{z}}{(1-\bar{z}\zeta)^2}+\frac{n-k}{\zeta(1-\bar{z}\zeta)}\right]d\xi d\eta \tag{21}$$

Inserting (21) in (4) with $n-1$ instead of n and ω instead of f and simplifying terms, we obtain (3) for $1\le k\le n-2$. Applying the solvability condition (1) for (8), (5) follows.
Using the differentiability of the operators $T_{m,n}$ [3], it follows that w given by (6) is indeed a solution. $\square$

3. Mixed Neumann-Schwarz Problem

In this section, we investigate mixed boundary value problem arising from n-Neumann and m-Schwarz boundary conditions. Since some of the computations are lengthy, the details have been avoided. All these computations can be made using Cauchy-Pompeiu formula, Gauss theorem, binomial and multinomials.

For p, q, $r\in\mathbb{N}_0$, we denote $\dfrac{r!}{p!q!(r-p-q)!}$ by $N(p,q,r)$ and the subset $\{(p,q): p+q\le r\}$ of $\mathbb{N}_0\times\mathbb{N}_0$ by $A(r)$.

Lemma 2. For $r,s\in\mathbb{N}_0$, $n\in\mathbb{N}$ and $|\tilde{\zeta}|$, $|z|<1$, we have

$$C_{r,n}(z,\tilde{\zeta})=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta^2}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}(\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\xi d\eta}{\zeta(\zeta-z)}$$

$$= \sum_{j=0}^{r} \binom{r}{j} (-1)^j (\tilde{\zeta} + \bar{\tilde{\zeta}})^{r-j} B_{j,n}(z) \tag{22}$$

where

$$\begin{aligned} B_{r,n}(z) &= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\zeta + \bar{\zeta})^r (\bar{\zeta} - \bar{z})^{n-1} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2} \\ &\quad - \frac{z}{\pi} \int\limits_{|\zeta|<1} (\zeta + \bar{\zeta})^r (\bar{\zeta} - \bar{z})^{n-1} \frac{d\xi d\eta}{\zeta(\zeta - z)} \\ &= \sum_{(p,q)\in A(r)} N(p,q,r) \bar{z}^{r-p-q} A_{p,q+n}(z) \end{aligned} \tag{23}$$

and

$$A_{r,n}(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \zeta^r (\bar{\zeta} - \bar{z})^{n-1} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2} - \frac{z}{\pi} \int\limits_{|\zeta|<1} \zeta^r (\bar{\zeta} - \bar{z})^{n-1} \frac{d\xi d\eta}{\zeta(\zeta - z)}$$

$$= \frac{(-1)^{n+1}}{n} \left[\bar{z}^n z^r - \sum_{s=0}^{r-2} \binom{n}{s+1} (-1)^s \frac{r}{r-s-1} \bar{z}^{n-(s+1)} z^{r-(s+1)} \right]$$

Proof: To express $B_{r,n}$ in terms of $A_{p,q+n}$, we write $(\zeta + \bar{\zeta})^r = (\bar{z} + \zeta + \bar{\zeta} - \bar{z})^r$

$$= \sum_{(p,q)\in A(r)} N(p,q,r) \bar{z}^{r-p-q} \zeta^p (\bar{\zeta} - \bar{z})^q$$

Remaining area integrals and boundary integrals can be computed by using Cauchy Pompeiu formula and Gauss theorem. □

Using similar arguments as in the above lemma, we obtain the following

Lemma 3. For $r \in \mathbb{N}_0$, $n \in \mathbb{N}, |\tilde{\zeta}|, |z| < 1$ we have

(i) If $$F_{r,n}(z, \tilde{\zeta}) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\tilde{\zeta} + \bar{\tilde{\zeta}} - (\zeta + \bar{\zeta}))^r (\bar{\zeta} - \bar{z})^{n-1} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2(\zeta - \tilde{\zeta})}$$

$$- \frac{z}{\pi} \int\limits_{|\zeta|<1} (\tilde{\zeta} + \bar{\tilde{\zeta}} - (\zeta + \bar{\zeta}))^r (\bar{\zeta} - \bar{z})^{n-1} \frac{d\xi d\eta}{\zeta(\zeta - z)(\zeta - \tilde{\zeta})}$$

then $F_{r,n}(z, \tilde{\zeta})$ are given as in (22) with $B_{j,n}(z)$ replaced by $E_{j,n}(z, \tilde{\zeta})$ and $E_{r,n}(z, \tilde{\zeta})$ are as in (23) with $A_{p,q+n}(z)$ replaced by $D_{p,q+n}(z, \tilde{\zeta})$. These $D_{r,n}(z, \tilde{\zeta})$ are given by

$$D_{r,n}(z, \tilde{\zeta}) = \frac{1}{2\pi i} \int\limits_{|\zeta|<1} \zeta^r (\bar{\zeta} - \bar{z})^{n-1} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta^2(\zeta - z)}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}\zeta^r(\bar\zeta-\bar z)^{n-1}\frac{d\xi d\eta}{\zeta(\zeta-z)(\zeta-\tilde\zeta)}$$

$$=\frac{\tilde\zeta^r}{n}\left[\frac{(-\bar z)^n}{\tilde\zeta}+\frac{z}{\bar{\tilde\zeta}}\frac{(\bar{\tilde\zeta}-\bar z)^n}{\tilde\zeta-z}\right]+\sum_{j=0}^{r-1}\tilde\zeta^{r-1-j}A_{j,n}(z)$$

(ii) If $H_{r,n}(z,\tilde\zeta)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r(\bar\zeta-\bar z)^{n-1}\log(1-z\bar\zeta)\frac{d\zeta}{\zeta^2(1-\zeta\bar\zeta)}$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r(\bar\zeta-\bar z)^{n-1}\frac{d\xi d\eta}{\zeta(\zeta-z)(1-\zeta\bar{\tilde\zeta})}$$

then $H_{r,n}(z,\tilde\zeta)$ are given as in (22) with $B_{j,n}(z)$ replaced by $G_{j,n}(z,\tilde\zeta)$ and $G_{j,n}(z,\tilde\zeta)$ are given as in (23) with $A_{p,q+n}(z)$ replaced by $I_{p,q+n}(z,\tilde\zeta)$ where

$$I_{r,n}(z,\bar\zeta)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta^r(\bar\zeta-\bar z)^{n-1}\log(1-z\bar\zeta)\frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde\zeta})}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}\zeta^r(\bar\zeta-\bar z)^{n-1}\frac{d\xi d\eta}{\zeta(\zeta-z)(1-\zeta\bar{\tilde\zeta})}$$

$$=(-1)^{n-1}\sum_{s=0}^{n-1}\binom{n-1}{s}(-1)^s\bar z^{n-1-s}K_{r,s}(z,\tilde\zeta)$$

with

$$K_{r,s}(z,\tilde\zeta)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta^r\bar\zeta^s\log(1-z\bar\zeta)\frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde\zeta})}$$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}\zeta^r\bar\zeta^s\frac{d\xi d\eta}{\zeta(\zeta-z)(1-\zeta\bar{\tilde\zeta})}$$

$$=\begin{cases}\bar{\tilde\zeta}^{s+1-r}\log(1-z\bar{\tilde\zeta})-\dfrac{1}{s+1}\dfrac{1}{1-z\bar{\tilde\zeta}}[z\bar{\tilde\zeta}^{s+2-r}-\bar z^{s+1}z^r] \\ \qquad\qquad\text{if } s\ \geq\ r-1 \text{ or } r\ =\ 2 \text{ and } s\ =\ 0 \\ -z^{r-s-1}\sum\limits_{k=0}^{\infty}\dfrac{1}{r-s-1+k}(z\bar{\tilde\zeta})^k-\dfrac{1}{(s+1)(1-z\bar{\tilde\zeta})}[z^{r-s-1}-z^r\bar z^{s+1}] \\ \qquad\qquad\text{if } r\ \geq\ 3,\ 0\ \leq\ s\ \leq\ r-2\end{cases}$$

(iii) If $N_{r,n}(z,\tilde{\zeta}) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1} (\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2}$

then $N_{r,n}(z,\tilde{\zeta})$ are given as in (22) with $B_{j,\,n}(z)$ replaced by $M_{j,\,n}(z)$ and $M_{r,\,n}(z)$ are given as in (23) with $A_{p,q+n}(z)$ replaced by $L_{p,q+n}(z)$. These $L_{r,n}(z)$ can be expressed as $(-1)^{n-r}\binom{n-1}{r-1}\bar{z}^{n-r}$.

(iv) If $Q_{r,n}(z,\tilde{\zeta}) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1} (\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2(\zeta-\tilde{\zeta})}$

then $Q_{r,n}\ (z,\tilde{\zeta})$ are given as in (22) with $B_{j,n}(z)$ replaced by $P_{j,n}(z,\tilde{\zeta})$ and $P_{j,n}(z,\tilde{\zeta})$ are given as in (23) with $A_{p,q+n}(z)$ replaced by $O_{p,q+n}(z,\tilde{\zeta})$. $O_{p,q+n}(z,\tilde{\zeta})$ can be expressed as $O_{r,n}(z,\tilde{\zeta}) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}\zeta^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2(\zeta-\tilde{\zeta})}$ $= (-1)^{n-1}\sum\limits_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\bar{z}^{n-1-j}\tilde{\zeta}^{r-j-2}\ \chi_A(r,s),\ A=\{(r,s)\}:r+s\geq 2\}$, χ_A being the characteristic function of A

(v) If $T_{r,n}(z,\tilde{\zeta}) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1} (\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde{\zeta}})}$

then $T_{r,n}\ (z,\tilde{\zeta})$ are given as in (22) with $B_{j,n}(z)$ replaced by $S_{j,n}(z,\tilde{\zeta})$ and $S_{j,n}(z,\tilde{\zeta})$ are given as in (23) with $A_{p,q+n}(z)$ replaced by $R_{p,q+n}(z,\tilde{\zeta})$ where $R_{r,n}$ can be expressed as

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\zeta^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde{\zeta}})}$$

$$=(-1)^{n-1}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\bar{z}^{n-1-j}\bar{\tilde{\zeta}}^{j-r+1}\chi_A(j,s)$$

where A is as above in (iv).

(vi) If $q_{r,n}(z,\tilde{\zeta}) = \dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1} (\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r(\bar{\zeta}-\bar{z})^{n-1}\frac{d\zeta}{\zeta^2}$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}(\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^r\frac{(\bar{\zeta}-\bar{z})^n}{(1-\bar{z}\zeta)^2}d\xi d\eta,$$

then $q_{r,n}(z,\tilde{\zeta})$ are given as in (22) with $B_{j,n}(z)$ replaced $p_{j,n}(z)$ where $p_{r,n}(z)$ can be expressed as

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\zeta+\bar\zeta)^r(\bar\zeta-\bar z)^{n-1}\frac{d\zeta}{\zeta^2}-\frac{1}{\pi}\int\limits_{|\zeta|<1}(\zeta+\bar\zeta)^2\frac{(\bar\zeta-\bar z)^n}{(1-\bar z\zeta)^2}d\xi d\eta$$

$$=\sum_{(p,q)\in A(r)}(-1)^{q+n-p}N(p,\ q,\ r)\binom{q+n-1}{p-1}\frac{q+n}{q+n+1}\bar z^{q+n-p}$$

(vii) If $U_{r,n}(z,\tilde\zeta)=\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r(\bar\zeta-\bar z)^{n-1}\frac{d\zeta}{\zeta^2(\zeta-\tilde\zeta)}$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\frac{(\bar\zeta-\bar z)^{n-1}}{(1-\bar z\zeta)^2}\frac{d\xi d\eta}{\zeta-\tilde\zeta},$$

then $U_{r,n}(z,\tilde\zeta)$ are given as in (22) with $B_{j,n}(z)$ replaced by $T_{j,n}(z,\tilde\zeta)$. $T_{j,n}(z,\tilde\zeta)$ are given in (23) with $S_{p,q+n}(z,\tilde\zeta)$ instead of $A_{p,q+n}(z)$. These $S_{r,n}(z,\tilde\zeta)$ can be expressed as

$$\frac{n}{n+1}O_{r,n-1}(z,\tilde\zeta)+\frac{1}{n+1}\tilde\zeta^r\frac{(\bar{\tilde\zeta}-\bar z)^{n+1}}{(1-\bar z\tilde\zeta)^2}$$

$O_{r,n}(z,\tilde\zeta)$ are as in (iv) above.

(viii) If $u_{r,n}(z,\tilde\zeta)=\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\ (\bar\zeta-\bar z)^{n-1}\ \frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde\zeta})}$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\frac{(\bar\zeta-\bar z)^{n-1}}{(1-\bar z\zeta)^2}\frac{d\xi d\eta}{1-\zeta\bar{\tilde\zeta}},$$

then $u_{r,n}(z,\tilde\zeta)$ are given as (22) with $B_{j,n}(z)$ replaced by $t_{j,n}(z,\tilde\zeta)$ where $t_{j,n}(z,\tilde\zeta)$ are given in (23) with $s_{p,q+n}(z,\tilde\zeta)$ instead of $A_{p,q+n}(z)$. These $s_{r,n}(z,\tilde\zeta)$ can be expressed as $\dfrac{1}{2\pi i}\displaystyle\int_{|\zeta|=1}\zeta^r(\bar\zeta-\bar z)^{n-1}\frac{d\zeta}{\zeta^2(1-\zeta\bar{\tilde\zeta})}-\frac{1}{\pi}\int_{|\zeta|<1}\frac{\zeta^r(\bar\zeta-\bar z)^n}{(1-\bar z\zeta)^2}\frac{d\xi d\eta}{1-\zeta\bar{\tilde\zeta}}=$ $\dfrac{n}{n+1}\ O_{r,n}(z,\tilde\zeta)$ □

Decomposing the term $\dfrac{1}{\zeta^k(\zeta-z)^2}$ and using again Gauss theorem we obtain the following

Lemma 4. For $r\in\mathbb{N}_0$, $n\in\mathbb{N}$, $|\tilde\zeta|$, $|z|<1$, we have

(i) If $n_r(z,\tilde\zeta)=-\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\frac{\bar\zeta d\bar\zeta}{\bar\zeta-\bar z}$

$$-\frac{z}{\pi}\int\limits_{|\zeta|<1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\ \frac{d\xi d\eta}{(1-\bar z\zeta)^2}$$

then $n_r(z,\tilde\zeta)$ is equal to

$$\sum_{k=0}^{r}\sum_{j=0}^{[(k+1)/2]}\binom{r}{k}\binom{k}{j+1}(\tilde\zeta+\bar{\tilde\zeta})^{r-k}\,\bar z^{k+1-2j},$$

$[k]$ being the greatest integer less than or equal to k.

(ii) If $c_r(z,\tilde\zeta) = -\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\frac{\bar\zeta d\bar\zeta}{(\bar\zeta-\bar z)(\zeta-\tilde\zeta)}$

$$-\frac{\bar z}{\pi}\int\limits_{|\zeta|<1}(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^r\frac{d\xi d\eta}{(1-\bar z\zeta)^2(\zeta-\tilde\zeta)}$$

then $c_r(z,\tilde\zeta)$ are expressible as in (22) with $B_{j,n}(z)$ replaced by $b_j(z,\tilde\zeta)$. The $b_r(z,\tilde\zeta)$ are given by $\sum\limits_{j=0}^{r}\binom{r}{j}\,a_{r-j,j}(z,\tilde\zeta)$ where

$$a_{r,s}(z,\tilde\zeta) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta^r\bar\zeta^{s+1}d\bar\zeta}{(\bar\zeta-\bar z)(\zeta-\tilde\zeta)} - \frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\frac{\zeta^r\bar\zeta^s d\xi d\eta}{(1-\bar z\zeta)^2(\zeta-\tilde\zeta)}$$

$$= \frac{(r-1)\bar z^{s+2-r} - r\bar z^{s+3-r}\tilde\zeta + \bar z\tilde\zeta^r\bar{\tilde\zeta}^{s+1}}{(s+1)(1-\bar z\tilde\zeta)^2} + \chi_B(r,s)\sum_{k=r-1}^{s+2}\frac{k\tilde\zeta^{k-r+1}}{\bar z^{k-(s+1)}},$$

where $B = \{(r,s) : s+2<r\}$.

(ii) If $f_r(z,\tilde\zeta) = -\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\zeta|=1}\frac{(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^s\bar\zeta d\bar\zeta}{(\bar\zeta-\bar z)(1-\zeta\bar{\tilde\zeta})} - \frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\frac{(\tilde\zeta+\bar{\tilde\zeta}-\zeta-\bar\zeta)^s d\xi d\eta}{(1-\bar z\zeta)^2(1-\zeta\bar{\tilde\zeta})}$

then $f_r(z,\tilde\zeta)$ are expressible as in (22) with $B_{j,n}(z)$ replaced by $e_j(z,\tilde\zeta)$. These $e_s(z,\tilde\zeta)$ are given as $\sum\limits_{j=[s/2]}^{s}\binom{s}{j}\,d_{s-j,j}(z,\tilde\zeta)$ where

$$d_{r,s}(z,\tilde\zeta) = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta^r\bar\zeta^{s+1}d\zeta}{(\bar\zeta-\bar z)(1-\zeta\bar{\tilde\zeta})} - \frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\zeta^r\bar\zeta^s\frac{d\xi d\eta}{(1-\bar z\zeta)^2(1-\zeta\bar{\tilde\zeta})}$$

$$= \left(\frac{1}{\bar{\tilde\zeta}-\bar z}\left[\bar{\tilde\zeta}^{s+2-r} - \left(\frac{2s+3-r}{s+1}\right)\bar z^{s+2-r}\right]\right.$$

$$-\frac{\bar{z}}{(s+1)(\bar{\bar{\zeta}}-\bar{z})^2}\left\{\bar{\bar{\zeta}}^{s+2-r}-\bar{z}^{s+2-r}\right\}\Bigg)\chi_C(r,s)$$

where $C=\{(r,s): s+2>r\}$. $\square$

Theorem 4. For $n,m\geq 1$, the mixed n-Neumann and m-Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc $\partial_{\bar{z}}^{n+m}w = f$ in $\mathbb{D}$, $\partial_\nu(\partial_{\bar{z}}^r w) = \alpha_r, \mathrm{Re}(\partial_{\bar{z}}^{n+s}w) = \beta_s$ on $\partial\mathbb{D}$, $\partial_{\bar{z}}^r w(0) = a_r$ and Im $\partial_{\bar{z}}^{n+s}w(0) = b_s$, where $f\in C^\alpha(\bar{\mathbb{D}},\mathbb{C})$, $0<\alpha<1$, $\alpha_r\in C(\partial\mathbb{D},\mathbb{C})$, $\beta_s\in C(\partial\mathbb{D},\mathbb{R})$, and $a_r\in\mathbb{C}$, $b_s\in\mathbb{R}$ for $0\leq r\leq n-1$, $0\leq s\leq m-1$ is uniquely solvable if and only if for $1\leq k\leq m-1$ and $|z|=1$,

$$\sum_{r=k}^{n-1} a_r\frac{\bar{z}^{r-k+1}}{(r-k)!}+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left[\frac{\alpha_{k-1}(\zeta)}{\zeta(\bar{z}\zeta-1)}+\sum_{r=k}^{n-1}\frac{(-1)^{r-k}}{(r-k+1)!}\ \frac{\alpha_r(\zeta)}{\zeta}(\bar{\zeta}-\bar{z})^{r-k}\right.$$
$$\left.(\bar{\zeta}-(r-k+1)\bar{z}\log(1-z\bar{\zeta}))\right]d\zeta$$
$$+\frac{(-1)^{n-k}}{(n-k)!}i\sum_{s=0}^{m-1}\frac{b_s}{s!}[M_{s,n-k+1}(z)+\bar{z}p_{s,n-k}(z)-(n-k)\bar{z}B_{s,n-k}(z)]$$
$$-\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\ \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\beta_s(\zeta)}}{\zeta}\ [N_{s,n-k+1}(z,\zeta)+\bar{z}q_{s,n-k}(z,\zeta)-(n-k)\bar{z}\ c_{s,n-k}(z,\zeta)$$
$$+2\zeta\{Q_{s,n-k+1}(z,\zeta)+\bar{z}U_{s,n-k}(z,\zeta)-(n-k)\bar{z}F_{s,n-k}(z,\zeta)\}]d\zeta$$
$$+\frac{(-1)^m}{(m-1)!}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta}\phi(z,\zeta)+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\psi(z,\zeta)\right]d\xi d\eta=0 \tag{24}$$

where

$$\phi(z,\zeta)=[(-N_{m-1,n-k+1}-2\zeta Q_{m-1,n-k+1})-\bar{z}(q_{m-1,n-k}-2\zeta U_{m-1,n-k})$$
$$+(n-k)\ \bar{z}(C_{m-1,n-k}+2\zeta F_{m-1,n-k})](z,\zeta)$$
$$\psi(z,\zeta)=[(2T_{m-1,n-k+1}-Q_{m-1,n-k+1})+\bar{z}(2u_{m-1,n-k}-U_{m-1,n-k})$$
$$-(n-k)\ \bar{z}(2H_{m-1,n-k}-F_{m-1,n-k})]\ (z,\zeta)$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left[\frac{\alpha_{n-1}(\zeta)}{\zeta(\bar{z}\zeta-1)}-\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\ \frac{\beta_s(\zeta)}{\zeta}(n_s+2\zeta c_s)(z,\zeta)\right]d\zeta+$$

$$i\sum_{s=0}^{m-1}\sum_{j=0}^{[(s+1)/2]}\binom{s}{j+1}\frac{b_s}{s!}\bar{z}^{s+1-2j}+\frac{(-1)^m}{2\pi(m-1)!}\int\limits_{|\zeta|<1}[\frac{f(\zeta)}{\zeta}\ (-n_{m-1}-2\zeta c_{m-1})(z,\zeta)$$

$$+\frac{\overline{f(\zeta)}}{\bar{\zeta}}(2f_{m-1}-n_{m-1})\ (z,\zeta)]\,d\xi d\eta=0 \tag{25}$$

The solution is given as

$$w(z)=\sum_{r=0}^{n-1}a_r\frac{\bar{z}^r}{r!}-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Big[\sum_{r=0}^{n-1}\frac{(-1)^r}{r!}\ \frac{\alpha_r(\zeta)}{\zeta}\ (\bar{\zeta}-\bar{z})^r\log(1-z\bar{\zeta})+$$

$$\frac{(-1)^{n-1}}{(n-1)!}\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\ \frac{\beta_s(\zeta)}{\zeta}\ (C_{s,n}+2\zeta F_{s,n})\ (z,\zeta)]\,d\zeta$$

$$+i\frac{(-1)^{n-1}}{(n-1)!}\sum_{s=0}^{m-1}\frac{b_s}{s!}B_{s,n}(z)+\frac{(-1)^m}{(m-1)!}\frac{1}{2\pi}\int\limits_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta}\ (-C_{m-1,n}-2\zeta F_{m-1,n})(z,\zeta)\right.$$

$$\left.+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\ (-C_{m-1,n}+2H_{m-1,n})\ (z,\zeta)\right]d\xi d\eta \tag{26}$$

Proof: The problem is decomposed into the system

$$\partial_{\bar{z}}^n w=\omega \text{ in } \mathbb{D},\ \partial_\nu(\partial_{\bar{z}}^r w)=\alpha_r \text{ on } \partial\mathbb{D},\ \partial_{\bar{z}}^r w(0)=a_r \text{ for } 0\le r\le n-1 \tag{27}$$

and $\partial_{\bar{z}}^m\omega=f$ in $\mathbb{D}$, Re $(\partial_{\bar{z}}^s\omega)=\beta_s$ on $\partial\mathbb{D}$, Im $\partial_{\bar{z}}^s\omega(0)=b_s$ for $0\le s\le m-1$.(28)

Using Theorem 3, the problem (27) is uniquely solvable if and only if (4) and (5) hold with ω instead of f and w is given by (6) with ω instead of f.

The problem (28) is uniquely solvable and ω is given as in (3). Substituting this value of ω in (6) with ω instead of f, the last two integrals of the expression can be written as

$$i\sum_{s=0}^{m-1}(-1)^s\,\frac{b_s}{s!}\left[\frac{1}{2\pi i}\int\limits_{|\zeta|=1}W(\zeta,0,s)\log(1-z\bar{\zeta})\ \frac{d\zeta}{\zeta^2}-\frac{z}{\pi}\int\limits_{|\zeta|<1}W(\zeta,0,s)\frac{d\xi d\eta}{\zeta(\zeta-z)}\right]$$

$$+\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\ \frac{1}{2\pi i}\int\limits_{|\tilde{\zeta}|=1}\frac{B_s(\tilde{\zeta})}{\tilde{\zeta}}\left[\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\ W(\zeta,\tilde{\zeta},s)\ \log(1-z\bar{\zeta})\ \frac{d\zeta}{\zeta^2}\right.$$

$$\left.-\frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\ \frac{W(\zeta,\tilde{\zeta},s)}{\zeta(\zeta-z)}\ d\xi d\eta\right]d\bar{\zeta}$$

$$+\frac{(-1)^m}{(m-1)!}\frac{1}{2\pi}\int\left[\frac{f(\tilde{\zeta})}{\tilde{\zeta}}\left[\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}W(\zeta,\tilde{\zeta},m-1)\log(1-z\tilde{\zeta})\frac{d\zeta}{\zeta^2}\right.\right.$$

$$\left.-\frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{\tilde{\zeta}+\zeta}{\tilde{\zeta}-\zeta}\frac{W(\zeta,\tilde{\zeta},m-1)}{\zeta(\zeta-z)}d\xi\, d\eta\right]+\frac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}}\ [\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{1+\zeta\bar{\tilde{\zeta}}}{1-\zeta\bar{\tilde{\zeta}}}\ W(\zeta,\tilde{\zeta},m-1)$$

$$\left.\log(1-z\bar{\zeta})\ \frac{d\zeta}{\zeta^2}-\frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{1+\zeta\bar{\tilde{\zeta}}}{1-\zeta\bar{\tilde{\zeta}}}\frac{W(\zeta,\tilde{\zeta},m-1)}{\zeta(\zeta-z)}d\xi\, d\eta]\right]d\tilde{\xi}\, d\tilde{\eta} \tag{29}$$

where $W(\zeta,\tilde{\zeta},s)=(\tilde{\zeta}+\bar{\tilde{\zeta}}-\zeta-\bar{\zeta})^s\ (\bar{\zeta}-\bar{z})^{n-1}$

The integral in the first sum is equal to $B_{(s,n)}(z)$ (Lemma 2). The integral in the second summand can be expressed as $-C_{s,n}\ (z,\tilde{\zeta})\ -2\tilde{\zeta}F_{s,n}(z,\tilde{\zeta})$ (Lemma 2, 3). The last integrals in (29) with $\dfrac{\overline{f(\tilde{\zeta})}}{\bar{\tilde{\zeta}}}$ is equal to $-C_{m-1,n}\ (z,\tilde{\zeta})$ $+2H_{m-1,n}\ (z,\tilde{\zeta})$ (Lemma 2, 3). Substituting these values in (6) with ω instead of f, we obtain (26).

The last two integrals of (4) with ω instead of f are expressible as $L_1+\bar{z}L_2$ $-(n-k)\bar{z}L_3$.

Using Lemma 3 (iii), (v) L_1 can be written as

$$L_1=i\sum_{s=0}^{m-1}\frac{b_s}{s!}\ M_{s,n-k+1}(z)$$

$$-\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\ \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\beta_s(\zeta)}{\zeta}\ [N_{s,n-k+1}+2\zeta Q_{s,n+k+1}]\ (z,\zeta)d\zeta$$

$$+\frac{(-1)^{m-1}}{2\pi(m-1)!}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\ [(N_{m-1,n-k+1}\ +2\zeta Q_{m-1,n-k+1})(z,\zeta)$$

$$+\frac{\overline{f(\zeta)}}{\bar{\zeta}}\ (2T_{m-1,n-k+1}\ -N_{m-1,n-k+1})\ (z,\zeta)\ ]\ d\xi d\eta$$

Using Lemma 3 (vi), (vii), L_2 can be expressed as

$$L_2 = i\sum_{s=0}^{m-1}\frac{b_s}{s!}p_{s,n-k}(z) - \sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\beta_s(\zeta)}{\zeta}\left(q_{s,n-k}+2\zeta\, U_{s,n-k}\right)(z,\zeta)d\zeta$$
$$+\frac{(-1)^m}{2\pi(m-1)!}\int\limits_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta}\left(-q_{m-1,n-k}-2\zeta U_{m-1,n-k}\right)(z,\zeta)\right.$$
$$\left.+\frac{\overline{f(\zeta)}}{\bar\zeta}\left(2u_{m-1,n-k}-q_{m-1,n-k}\right)(z,\zeta)\right]d\xi d\eta$$

Lemma 2, Lemma 3 (i), (ii) enable us to write

$$L_3 = i\sum_{s=0}^{m-1}\frac{b_s}{s!}\ B_{s,n-k}(z)$$
$$-\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\beta_s(\zeta)}{\zeta}[C_{s,n-k}+2\zeta F_{s,n-k}]\ (z,\zeta)d\zeta$$
$$+\frac{(-1)^{m+1}}{(m-1)!}\ \frac{1}{2\pi}\int\limits_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta}(C_{m-1,n-k}+2\zeta F_{m-1,n-k})\ (z,\zeta)\right.$$
$$\left.+\frac{\overline{f(\zeta)}}{\bar\zeta}\ (C_{m-1,n-k}-2H_{m-1,n-k})\ (z,\zeta)\right]d\xi d\eta$$

Substituting $L_1+\bar zL_2-(n-k)\bar zL_3$ in (6) with ω instead of f, we obtain (24).

Applying Lemma 2, the last two integrals of (5) with ω instead of f can be written as

$$i\sum_{s=0}^{m-1}\frac{b_s}{s!}\sum_{j=0}^{[(s+1)/2]}\binom{s}{j+1}\ \bar z^{s+1-2j}\ -\sum_{s=0}^{m-1}\frac{(-1)^s}{s!}\int\limits_{|\zeta|=1}\frac{\beta_s(\zeta)}{\zeta}\ (n_s+2\zeta c_s)\ (z,\zeta)\ d\zeta$$
$$+\frac{(-1)^{m+1}}{2\pi(m-1)!}\int\limits_{|\zeta|<1}\left[\frac{f(\zeta)}{\zeta}\ (n_{m-1}+2\zeta C_{m-1})\ (z,\zeta)\right.$$
$$\left.+\frac{\overline{f(\zeta)}}{\bar\zeta}\ (n_{m-1}-2f_{m-1})(z,\zeta)\ \right]\ d\xi d\eta$$

Substituting this value in (5) with ω instead of f, we obtain (25).

Remark 3.5: The mixed boundary value problem arising with first n-Schwarz and last m-Neumann boundary conditions can be solved in a similar way. This has been avoided here due to lack of space. □

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The Construction of a Kind of Quadrature Formulas[1]

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Abstract
In this paper, a kind of quadrature formulas are constructed by using the Euler-Maclaurin summation formula. It has the same nodes as composite trapezoidal rule. But its convergence order is very high. Numerical results are presented which also show that they are simple and efficient numerical integration rules.
Keywords: Numerical integration; Convergence order; Absolute error.

1. Introduction

Numerical integrations are often encountered in practice, for example, in wavelet-Galerkin methods for integral equations, we need to calculate a lot of numerical integrations(see [1,3,4]). The composite trapezoidal rule and Simpson's rule are simple, and have recursive relations. But their convergence order is very low. Gaussian rule has high algebraic accuracy ,but has no recurrence relations. In paper[5], a method of construction of quadrature formulas for the calculation of inner products of smooth function and scaling functions is presented. In this paper, we use the Euler-Maclaurin summation formula to construct a kind of quadrature formulae, we call it modified composite trapezoidal rule. It has the same nodes as composite trapezoidal rule, and has recurrence relations. Furthermore, its convergence order is very high.

2. The Construction of the Quadrature formulas

Suppose $f(x) \in C^{2k+3}[a,b]$, where $k = 1, 2, \cdots$, and let $h = \frac{b-a}{n}, n \geq 2k$, and B_{2j} be Bernoulli numbers, i.e. $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \cdots$. Set

$$p_{2k+3}(x) = (-1)^k \sum_{n=1}^{\infty} \frac{2\sin 2\pi nx}{(2\pi n)^{2k+3}},$$

then, we have the Euler-Maclaurin summation formula[2]:

$$\frac{h}{2}[f(a)+f(b)+2\sum_{i=1}^{n-1} f(a+ih)] = \int_a^b f(x)dx + \sum_{i=1}^{k+1} \frac{B_{2i}}{(2i)!} h^{2i}[f^{(2i-1)}(b) - f^{(2i-1)}(a)]$$

$$+h^{2k+3} \int_a^b p_{2k+3}(n\frac{x-a}{b-a}) f^{(2k+3)}(x)dx, \tag{1}$$

[1]This paper is supported by the National Natural Science Foundation of China (60372071).

and have the estimation:

$$\Big|\int_a^b p_{2k+3}(n\frac{x-a}{b-a})f^{(2k+3)}(x)dx\Big| \le M_{2k+3}2^{-2k-2}(\pi)^{-2k-3}(b-a)\sum_{j=1}^{\infty}\frac{1}{j^{2k+3}}, \qquad (2)$$

where $M_{2k+3} = \max\limits_{a\le x\le b}|f^{(2k+3)}(x)|$, $\sum\limits_{j=1}^{\infty} 1/(j^{2k+3})$ is Riemann series. Set

$$T_n = \frac{h}{2}[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a+ih)],$$

$$P_k = \sum_{i=1}^{k}\frac{B_{2i}}{(2i)!}h^{2i}[f^{(2i-1)}(a) - f^{(2i-1)}(b)], \qquad (3)$$

and

$$E_k = \frac{B_{2k+2}}{(2k+2)!}h^{2k+2}[f^{(2k+1)}(a) - f^{(2k+1)}(b)] - h^{2k+3}\int_a^b p_{2k+3}(n\frac{x-a}{b-a})f^{(2k+3)}(x)dx,$$

then, (1) can be rewritten as

$$\int_a^b f(x)dx = T_n + P_k + E_k. \qquad (4)$$

From E_k, we know that the convergence order of $T_n + P_n$ is $O(h^{2k+2})$, Consequently, Eq.(4)can be rewritten as

$$\int_a^b f(x)dx = T_n + P_k + O(h^{2k+2}). \qquad (5)$$

The idea is to generate an approximation to derivative in P_k by the vales of the function at the given nodes, and to ensure that the convergence order is also $O(h^{2k+2})$.

For $f(x) \in C^{2k+1}[a,b]$, using the Taylor formula, we have

$$f(x+ih) - f(x) = \sum_{j=1}^{2k}\frac{f^{(j)}(x)}{j!}(ih)^j + \frac{f^{(2k+1)}(\xi_i(x))}{(2k+1)!}(ih)^{2k+1}, \qquad (6)$$

where $i = 1, 2, \cdots, 2k,\ x \le \xi_i(x) \le x + ih$.

Let

$$\varepsilon_i(x) = -\frac{f^{(2k+1)}(\xi_i(x))i^{2k+1}}{(2k+1)!}, \qquad E = (\varepsilon_1(x), \varepsilon_2(x), \cdots, \varepsilon_{2k}(x))^\top,$$

$$y_i = f^{(i)}(x)h^i, \quad Y = (y_1, y_2, \cdots, y_{2k})^\top, \quad \tilde{f}_i = f(x+ih) - f(x),$$

$$F = (\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_{2k})^\top, \quad A = (a_{ij}), \quad a_{ij} = \frac{i^j}{j!}, \quad i, j = 1, 2, \cdots, 2k,$$

then (6) can be rewritten as

$$AY = F + h^{2k+1}E. \qquad (7)$$

Proposition 1. *The matrix A in (7) is nonsingular.*

Proof. Let the ith row of A be divided by $i, (i = 1, 2, \cdots, 2k)$, the jth column of A be divided by $j!, (j = 1, 2, \cdots, 2k)$, then we get a matrix whose determinant is a Vandermode determinant. Therefor, the matrix A is nonsingular.

By(7), we have

$$Y = A^{-1}F + h^{2k+1}A^{-1}E. \tag{8}$$

Let $A^{-1} = (b_{ij})$, then (8)can be rewritten as

$$f^{(i)}(x)h^i = \sum_{j=1}^{2k} b_{ij}[f(x+jh) - f(x)] + h^{2k+1}\sum_{j=1}^{2k} b_{ij}\varepsilon_j(x),\ \ i = 1, 2, \cdots, 2k. \tag{9}$$

Let $x = a$, we have

$$f^{(i)}(a)h^i = \sum_{j=1}^{2k} b_{ij}[f(a+jh) - f(a)] + h^{2k+1}\sum_{j=1}^{2k} b_{ij}\varepsilon_j(a). \tag{10}$$

Let $x = b$, and replace h by $-h$,(now $b - ih \leq \xi_i(b) \leq b$), we have

$$f^{(i)}(b)(-h)^i = \sum_{j=1}^{2k} b_{ij}[f(b-jh) - f(b)] - h^{2k+1}\sum_{j=1}^{2k} b_{ij}\varepsilon_j(b). \tag{11}$$

Substituting (10) and (11)into P_k in (3) gives

$$P_k = h\sum_{i=1}^{k}\frac{B_{2i}}{(2i)!}\Big\{\sum_{j=1}^{2k} b_{ij}\Big[f(a+jh) - f(a) + f(b-jh) - f(b) + (\varepsilon_j(a) - \varepsilon_j(b))h^{2k+1}\Big]\Big\}. \tag{12}$$

Let

$$\tilde{P}_k = h\sum_{i=1}^{k}\frac{B_{2i}}{(2i)!}\Big\{\sum_{j=1}^{2k} b_{ij}[f(a+jh) - f(a) + f(b-jh) - f(b)]\Big\}, \tag{13}$$

$$\tilde{E}_k = h^{2k+2}\sum_{i=1}^{k}\frac{B_{2i}}{(2i)!}\Big\{\sum_{j=1}^{2k} b_{ij}[\varepsilon_j(a) - \varepsilon_j(b)]\Big\}. \tag{14}$$

and

$$T_n^{(k)} = T_n + \tilde{P}_k. \tag{15}$$

then (4) can be rewritten as

$$\int_a^b f(x)dx = T_n^{(k)} + \tilde{E}_k + E_k. \tag{16}$$

where $T_n^{(k)}$ is a modified composite trapezoidal rule which we want to construct, and $\tilde{E}_k = P_k - \tilde{P}_k$. Now, we estimate the $\tilde{E}_k + E_k$ in (16):

$$\begin{aligned} |\tilde{E}_k| &\leq 2h^{2k+2}M_{2k+1}\sum_{i=1}^{k}\frac{|B_{2i}|}{(2i)!}\Big(\sum_{j=1}^{2k}|b_{ij}|\Big)\frac{i^{2k+1}}{(2k+1)!} \\ &= h^{2k+2}\frac{2M_{2k+1}}{(2k+1)!}\sum_{i=1}^{k}\frac{|B_{2i}|i^{2k+1}}{(2i)!}\Big(\sum_{j=1}^{2k}|b_{ij}|\Big), \end{aligned} \tag{17}$$

where $\quad M_{2k+1} = \max\limits_{a\leq x\leq b}|f^{(2k+1)}(x)|$.

Consequently,

$$|\tilde{E}_k + E_k| \leq h^{2k+2}\Big[\frac{2M_{2k+1}|B_{2k+2}|}{(2k+2)!} + \frac{2M_{2k+1}}{(2k+1)!}\sum_{i=1}^{k}\frac{|B_{2i}|i^{2k+1}}{(2i)!}\Big(\sum_{j=1}^{2k}|b_{ij}|\Big)\Big]$$

$$+h^{2k+3}M_{2k+3}2^{-2k-2}(\pi)^{-2k-3}(b-a)\sum_{j=1}^{\infty}\frac{1}{j^{2k+3}}. \tag{18}$$

Then we have

Proposition 2. *If $f(x) \in C^{2k+3}[a,b]$, then the convergence order of the modified composite trapezoidal rule $T_n^{(k)}$ is $O(h^{2k+2})$. That is*

$$\int_a^b f(x)dx = T_n^{(k)} + O(h^{2k+2}). \tag{19}$$

3. Some examples of the modified composite trapezoidal rule

Suppose $f(x) \in C^{2k+3}[a,b], k = 1,2,\cdots, h = \frac{b-a}{n}, n \geq 2k$. Let $f_i = f(a+ih), f_{-i} = f(b-ih), i = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, n$. Notice that $f_0 = f(a) = f_{-n}, f_{-0} = f(b) = f_n$. Let $T_n = \frac{h}{2}(f_0 + f_{-0} + 2\sum_{i=1}^{n-1} f_i)$. Now, we use the method introduced above to constructed some formulas of $T_n^{(k)}$ for different k:

k=1:

$$T_n^{(1)} = T_n + \frac{h}{24}[-3(f_0+f_{-0}) + 4(f_1+f_{-1}) - (f_2+f_{-2})]. \tag{20}$$

The formula of $T_n^{(1)}$ has the following properties:

(i) When $n=2$, $T_2^{(1)}$ is the Simpson's rule; when $n=3$, $T_3^{(1)}$ is the Newton-Cotes formula with $n=3$(see [2]).

(ii) When $n \geq 6$

$$T_n^{(1)} = h\sum_{i=3}^{n-3} + h\Big[\frac{3}{8}(f_0+f_{-0}) + \frac{7}{6}(f_1+f_{-1}) + \frac{23}{24}(f_2+f_{-2})\Big]. \tag{21}$$

(iii) By(21), we have a recurrence relations of $T_{2n}^{(1)}$ and $T_n^{(1)}$:

$$T_{2n}^{(1)} = \frac{1}{2}T_n^{(1)} + \frac{h}{2}\sum_{i=1}^{n-2} f_{i+\frac{1}{2}} + \frac{h}{48}\Big[(f_2+f_{-2}) - 5(f_1+f_{-1}) + 28(f_{\frac{1}{2}} + f_{-\frac{1}{2}})\Big]. \tag{22}$$

k=2:

$$T_n^{(2)} = T_n - \frac{h}{1440}[245(f_0+f_{-0}) - 462(f_1+f_{-1})$$
$$+336(f_2+f_{-2}) - 146(f_3+f_{-3}) + 27(f_4+f_{-4})]. \tag{23}$$

The formula of $T_n^{(2)}$ has the following properties:

(i) $T_4^{(2)}$ and $T_5^{(2)}$ are Newton-Cotes formulas with $n=4$ and $n=5$(see [2]), respectively.

(ii) When $n \geq 10$, we have

$$T_n^{(2)} = h\sum_{i=5}^{n-5} f_i + h\Big[\frac{95}{288}(f_0+f_{-0}) + \frac{317}{240}(f_1+f_{-1}) + \frac{23}{30}(f_2+f_{-2})$$
$$+\frac{793}{720}(f_3+f_{-3}) + \frac{157}{160}(f_4+f_{-4})\Big]. \tag{24}$$

(iii) $T_{2n}^{(2)}$ and $T_n^{(2)}$ have a recurrence relations

$$T_{2n}^{(2)} = \frac{1}{2}T_n^{(2)} + \frac{h}{2}\sum_{i=2}^{n-3} f_{i+\frac{1}{2}} + \frac{h}{2}\Big[\frac{317}{240}(f_{\frac{1}{2}} + f_{-\frac{1}{2}}) - \frac{133}{240}(f_1 + f_{-1}) + \frac{793}{720}(f_{\frac{3}{2}}$$

$$+f_{-\frac{3}{2}}) + \frac{103}{480}(f_2 + f_{-2}) - \frac{73}{720}(f_3 + f_{-3}) + \frac{3}{160}(f_4 + f_{-4})\Big]. \tag{25}$$

k=3:

$$T_n^{(3)} = T_n - \frac{h}{362880}[71043(f_0 + f_{-0}) - 167064(f_1 + f_{-1})$$

$$+198327(f_2 + f_{-2}) - 171072(f_3 + f_{-3}) + 94569(f_4 + f_{-4})$$

$$-29928(f_5 + f_{-5}) + 4125(f_6 + f_{-6})]. \tag{26}$$

The formula of $T_n^{(3)}$ has the following properties:

(i) $T_6^{(3)}$ and $T_7^{(3)}$ are Newton-Cotes formulas with $n = 6$ and $n = 7$(see [2]),respectively.

(ii) When $n \geq 14$, we have

$$T_n^{(3)} = h\sum_{i=7}^{n-7} f_i + h\Big[\frac{5257}{17280}(f_0 + f_{-0}) + \frac{22081}{15120}(f_1 + f_{-1})$$

$$+\frac{54851}{120960}(f_2 + f_{-2}) + \frac{103}{70}(f_3 + f_{-3}) + \frac{89437}{120960}(f_4 + f_{-4})$$

$$+\frac{16367}{15120}(f_5 + f_{-5}) + \frac{23917}{24192}(f_6 + f_{-6})\Big]. \tag{27}$$

(iii) $T_{2n}^{(3)}$ and $T_n^{(3)}$ have a recurrence relations

$$T_{2n}^{(3)} = \frac{1}{2}T_n^{(3)} + \frac{h}{2}\sum_{i=3}^{n-4} f_{i+\frac{1}{2}}$$

$$+\frac{h}{2}\Big[\frac{22081}{15120}(f_{\frac{1}{2}} + f_{-\frac{1}{2}}) - \frac{121797}{120960}(f_1 + f_{-1}) + \frac{103}{70}(f_{\frac{3}{2}} + f_{-\frac{3}{2}})$$

$$+\frac{34586}{120960}(f_2 + f_{-2}) + \frac{16367}{15120}(f_{\frac{5}{2}} + f_{-\frac{5}{2}}) - \frac{408793}{846720}(f_3 + f_{-3})$$

$$+\frac{31523}{120960}(f_4 + f_{-4}) - \frac{1247}{15120}(f_5 + f_{-5}) + \frac{275}{24192}(f_6 + f_{-6})\Big]. \tag{28}$$

k=4:

$$T_n^{(4)} = T_n - h[\frac{19087}{89600}(f_0 + f_{-0}) - \frac{427487}{725760}(f_1 + f_{-1}) + \frac{3498217}{3628800}(f_2 + f_{-2})$$

$$-\frac{500327}{403200}(f_3 + f_{-3}) + \frac{6467}{5670}(f_4 + f_{-4}) - \frac{2616161}{3628800}(f_5 + f_{-5})$$

$$+\frac{24019}{80640}(f_6 + f_{-6}) - \frac{263077}{3628800}(f_7 + f_{-7}) + \frac{8183}{1036800}(f_8 + f_{-8})]. \tag{29}$$

The formula of $T_n^{(4)}$ has the following properties:

(i) When $n \geq 18$, we have

$$T_n^{(4)} = h\sum_{i=9}^{n-9} f_i + h\Big[\frac{25713}{89600}(f_0 + f_{-0}) + \frac{1153247}{725760}(f_1 + f_{-1}) + \frac{130583}{3628800}(f_2 + f_{-2})$$
$$+\frac{903527}{403200}(f_3 + f_{-3}) - \frac{797}{5670}(f_4 + f_{-4}) + \frac{6244961}{3628800}(f_5 + f_{-5})$$
$$+\frac{56621}{80640}(f_6 + f_{-6}) + \frac{3891877}{3628800}(f_7 + f_{-7}) + \frac{1028617}{1036800}(f_8 + f_{-8})\Big]. \tag{30}$$

(ii) Let $h \leq 1$, by(18), we can obtain

$$\Big|\int_a^b f(x)dx - T_n^{(4)}\Big| \leq \frac{h^9}{100}\max_{a\leq x\leq b}|f^{(9)}(x)|$$

(iii) $T_{2n}^{(4)}$ and $T_n^{(4)}$ have a recurrence relations

$$T_{2n}^{(4)} = \frac{1}{2}T_n^{(4)} + \frac{h}{2}\sum_{i=4}^{n-5} f_{i+\frac{1}{2}}$$
$$+\frac{h}{2}\Big[\frac{1153247}{725760}(f_{\frac{1}{2}} + f_{-\frac{1}{2}}) - \frac{1408913}{907200}(f_1 + f_{-1}) + \frac{903527}{403200}(f_{\frac{3}{2}} + f_{-\frac{3}{2}})$$
$$+\frac{1821587}{7257600}(f_2 + f_{-2}) + \frac{6244961}{3628800}(f_{\frac{5}{2}} + f_{-\frac{5}{2}}) - \frac{310211}{201600}(f_3 + f_{-3})$$
$$+\frac{3891877}{3628800}(f_{\frac{7}{2}} + f_{-\frac{7}{2}}) + \frac{8220479}{7257600}(f_4 + f_{-4}) - \frac{2616161}{3628800}(f_5 + f_{-5})$$
$$+\frac{24019}{80640}(f_6 + f_{-6}) - \frac{263077}{3628800}(f_7 + f_{-7}) + \frac{8183}{1036800}(f_8 + f_{-8})\Big]. \tag{31}$$

4. Numerical examples

Now we use the modified composite trapezoidal rule $T_n^{(k)}, k = 1,2,3,4$, composite trapezoidal rule T_n, composite Simpson's rule S_n, and composite two-point Gaussian rule G_n, i.e. using two-point Gaussian rule on each subinterval, to approximate integrals. $T_n^{(k)}$ and T_n as before, S_n and G_n as following:

$$\int_a^b f(x)dx \approx S_n := \frac{h}{3}\Big[f(x_0) + f(x_{2n}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i})\Big], \tag{32}$$

where $h = \frac{b-a}{2n}, x_i = a + ih, i = 0, 1, \cdots, n.$

$$\int_a^b f(x)dx \approx G_n := \sum_{i=0}^{n-1}\frac{h}{2}[(f(\frac{h}{2}t_1 + \frac{x_i + x_{i+1}}{2}) + f(\frac{h}{2}t_2 + \frac{x_i + x_{i+1}}{2})], \tag{33}$$

where $h = \frac{b-a}{n}, x_i = a + ih, i = 0, 1, \cdots, n, t_1 = 0.577350269189628, t_2 = -t_1.$

Example 1. $\int_0^\pi \sin x dx = 2.$

Example 2. $\int_0^1 \frac{x}{1+e^x} = 0.1705573495024382\cdots.$

Example 3. $\int_0^1 \frac{1}{1+10x^2} = 0.3998760050557660\cdots.$

Example 4. $\int_0^{\frac{3}{2}\pi} \cos 15x dx = \frac{1}{15}$.
Example 5. $\int_0^{2\pi} x \cos x \sin 30x dx = -0.2096724796611 \cdots$.
Example 6. $\int_0^1 \frac{1}{1-0.98x^2} dx = 2.67096531488 \cdots$.

The numerical results are shown in Table 1. The first column are the ordinal numbers of these examples, the second are numbers of nodes, and the others are absolute errors, which correspond to quadrature Formulas.

Table 1.

example	n	$T_n^{(1)}$	$T_n^{(2)}$	$T_n^{(3)}$	$T_n^{(4)}$	T_n	S_n	G_n
1	10	4.9E-04	2.3E-05	1.1E-06	5.2E-08	1.6E-02	1.1E-04	4.5E-06
1	20	3.2E-05	4.1E-07	6.3E-09	1.0E-10	4.1E-03	6.8E-06	2.8E-07
2	10	6.9E-07	5.6E-09	5.4E-11	1.6E-12	3.5E-04	1.7E-07	7.1E-09
2	20	4.7E-08	1.1E-10	3.6E-13	2.2E-15	8.9E-05	1.1E-08	4.5E-10
3	80	1.6E-09	7.8E-10	6.1E-11	6.1E-12	2.1E-06	2.0E-10	8.3E-12
3	160	5.4E-10	6.4E-12	1.4E-13	4.5E-15	5.4E-07	1.3E-11	5.2E-13
4	160	3.9E-05	3.2E-07	4.6E-07	1.2E-07	1.0E-03	1.4E-05	5.9E-07
4	320	3.4E-06	6.2E-08	9.8E-10	2.7E-12	2.4E-04	8.9E-07	3.7E-08
5	640	4.0E-05	1.6E-06	7.2E-08	2.7E-09	1.5E-03	8.9E-06	3.7E-07
5	1280	2.6E-06	2.9E-08	3.9E-10	5.6E-12	3.8E-04	5.5E-07	2.3E-08
6	1280	2.2E-06	8.8E-08	7.1E-09	8.9E-10	2.5E-04	5.8E-07	2.5E-08
6	2560	1.6E-07	1.9E-09	6.0E-11	9.2E-12	6.2E-05	3.7E-08	1.5E-09

From Table 1, we can see that the modified composite trapezoidal rule $T_n^{(4)}$ is significantly better than the others.

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Stability of a numerical algorithm for non-stationary transport equation

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Abstract. The initial-boundary value problem for one-dimensional linear transport equation with a source term is considered. This is rewritten as a Cauchy problem: $dw/dt + Aw = F,\ w\big|_{t=0} = w_0$, where w represents a suitable subset of a Hilbert space, whose elements are pairs of real-valued functions depending on three variables: a space variable $z\in[0, H]$, an angle variable ν, with $\mu = \cos\nu \in [-1, 1]$ and a time variable $t\in[0, T]$. A is a linear strictly positive operator. A difference scheme is given in order to approximate the space derivatives appearing in A. Then, the operator A is decomposed as $A = A_1 + A_2$ (where both A_1 and A_2 are positive operators) and another difference scheme is given to approximate the time derivatives. Finally, the numerical integration with respect to μ is carried out. One obtains an algorithm, which is stable and approximates the exact solution with an accuracy of second order in time step τ and in space step h.

Key words: transport equation, difference scheme, Krank-Nicholson scheme, bicycle splitting-up method.
MSC : 35J99, 65N99.

1 Introduction

The main problem in the nuclear physics is to find the neutrons distribution in the reactor, hence its density, φ. This is a scalar function, which is studied in a plan-parallel geometry and depends on the next variables: the position of the neutron on the z – axis, the neutron speed, v_c, which makes an angle ν with Oz and the time, t. The density is the solution of an integral – differential equation, named the neutron transport equation.

Many authors paid attention to this problem, [1],[2],[4]–[7], but their papers are theoretical studies. In this paper we present a numerical algorithm in order to find the solution of a boundary value and initial value problem for the non – stationary transport equation. We prove that the iterative process is stable and the approximation of the solution with respect to time step, τ, is of the τ^2 order.

2 Problem Formulation

Let us consider a transport equation in a plan – parallel geometry:

$$\frac{1}{v_c}\cdot\frac{\partial\varphi}{\partial t} + \mu\frac{\partial\varphi}{\partial z} + \sigma\cdot\varphi = \frac{\sigma_s}{2}\int_{-1}^{1}\varphi d\mu + f(z,\mu,t) \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} &\varphi = 0 \text{ if } z = 0, \quad \mu > 0 \\ &\varphi = 0 \text{ if } z = H, \quad \mu < 0 \end{aligned} \tag{2}$$

and the initial condition: $\varphi = \varphi_0$ if $t = 0$. (3)

In the right-hand side of (1), f is the radioactive source function, the functions σ, σ_s are continuous in the interval [0, H] and satisfy the conditions:

$$0 < \sigma_0 \le \sigma \le \sigma_1 < \infty; \quad 0 \le \sigma_s \le \sigma_s' < \infty; \quad 0 < \sigma_{c_0} \le \sigma_c = \sigma - \sigma_s \tag{4}$$

Further on, we consider for simplicity, $v_c = 1$. Using the notations:

$$\varphi^+ = \varphi(z, \mu, t)\ ; \ \varphi^- = \varphi(z, -\mu, t), \text{ where } \mu > 0, \tag{5}$$

the equation (1) can be written in the form:

$$\begin{aligned} \frac{\partial \varphi^+}{\partial t} + \mu \frac{\partial \varphi^+}{\partial z} + \sigma \cdot \varphi^+ &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-) d\mu + f^+ \\ \frac{\partial \varphi^-}{\partial t} - \mu \frac{\partial \varphi^-}{\partial z} + \sigma \cdot \varphi^- &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-) d\mu + f^- \end{aligned} \tag{6}$$

Substituting: $\mu' = -\mu > 0$, we get:

$$\int_{-1}^{0} \varphi(z,\mu,t) d\mu = -\int_{1}^{0} \varphi(z,-\mu',t) d\mu' = \int_{0}^{1} \varphi(z,-\mu',t) d\mu' = \int_{0}^{1} \varphi^- d\mu .$$

The boundary value problem becomes:

$$\varphi^+(0,\mu,t) = 0; \quad \varphi^-(H,\mu,t) = 0, \quad \forall \mu \in [0,1], \forall t \in [0,T] \tag{7}$$

Adding and subtracting the equations (6) and introducing the notations:

$$\begin{aligned} u &= \frac{1}{2}(\varphi^+ + \varphi^-) \qquad g = \frac{1}{2}(f^+ + f^-) \\ v &= \frac{1}{2}(\varphi^+ - \varphi^-) \qquad r = \frac{1}{2}(f^+ + f^-) \end{aligned} \tag{8}$$

we obtain the following system:

$$\begin{aligned} \frac{\partial u}{\partial t} + \mu \cdot \frac{\partial v}{\partial z} + \sigma \cdot u &= \sigma_s \int_0^1 u\, d\mu' + g \\ \frac{\partial v}{\partial t} + \mu \cdot \frac{\partial u}{\partial z} + \sigma \cdot v &= r. \end{aligned} \tag{9}$$

The boundary- initial conditions are:

$$\begin{aligned} u + v = 0 \ \text{ for } \ z = 0 \\ u - v = 0 \ \text{ for } \ z = H \end{aligned} \tag{10}$$

and respectively: $$u = u^0, \ v = v^0 \ \text{ for } \ t = 0. \tag{11}$$

Now we rewrite the problem (9)-(11) in a operator form. For this purpose, we introduce the vector functions having two scalar components:

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \quad F = \begin{pmatrix} g \\ r \end{pmatrix}. \tag{12}$$

and the operator

$$A = \begin{pmatrix} \sigma - \sigma_s \int_0^1 d\mu' & \mu \dfrac{\partial}{\partial z} \\ \mu \dfrac{\partial}{\partial z} & \sigma \end{pmatrix} \tag{13}$$

Let us define in the measurable set $D = [0, H]\times[0, 1]$, a Hilbert space $L_2(D)$, (the functions quadratically integrable), with the scalar product:

$$(\alpha(t), \beta(t)) = \sum_{i=1}^{2} \int_0^1 d\mu \int_0^H \alpha^i(z,\mu,t)\beta^i(z,\mu,t)dz \tag{14}$$

where α^i, β^i are the components of the vector functions α, β. Here, the scalar product is a function of time. For solving the problem (9)-(11), we consider that the vector function w is defined on the set $[0,T]$ and has the values in the Hilbert space $L_2(D)$. The notation $w(t)$ defines an element of $L_2(D)$, which corresponds to a function $(z, \mu) \to w(z,\mu,t)$ with t fixed.

Let us consider Φ, the set of functions $w(t)$, which have the components u, v and $\dfrac{\partial u}{\partial z}, \dfrac{\partial v}{\partial z}$ continuous on D. Then, the domain of definition $\mathcal{D}(A) = \Phi_0$ is the subset of Φ with the elements w that verify conditions (10) and have $\dfrac{\partial w}{\partial t}$ continuous on D.

Let us now define the operator

$$L = \frac{\partial}{\partial t} + A \tag{15}$$

with the domain $\mathcal{D}(L) = \Phi_0$. Consequently, the problem (9)-(11) becomes:

$$\frac{\partial w}{\partial t} + Aw = F, \quad (z,\mu,t) \in D\times[0,T], \quad D = [0,H]\times[0,1] \tag{16}$$

$$w\big|_{t=0} = w^0, \qquad \forall(z,\mu) \in D \tag{17}$$

where

$$F \in L_2(D\times[0,T]), \quad w^0 \in \Phi, \quad w(t) \in \Phi_0.$$

We prove that A is a positive operator, namely, $(Aw, w) > 0$ for each $w \neq 0$, $w \in \Phi_0$. We have

$$(Aw, w) = \int_0^1 d\mu \int_0^H \left[\sigma u^2 - \sigma_s u \int_0^1 u d\mu' + \mu u \frac{\partial v}{\partial z} + \mu v \frac{\partial u}{\partial z} + \sigma v^2 \right] dz \tag{18}$$

Using the Hölder inequality we obtain

$$\left(\int_0^1 1 \cdot u d\mu \right) \le \left(\int_0^1 1^2 d\mu \right)\left(\int_0^1 u^2 d\mu \right) = \int_0^1 u^2 d\mu$$

Finally, for $\sigma_s \le \sigma$ we get

$$(Aw,w)\geq \sigma\int_0^H\left(\left(\int_0^1 u\,d\mu\right)^2-\left(\int_0^1 u\,d\mu\right)^2\right)dz+\int_0^1 d\mu\int_0^H\left(\sigma v^2+\mu\frac{\partial}{\partial z}(uv)\right)dz\geq$$
$$\geq\frac{1}{4}\int_0^1\mu\left((\varphi^+)^2-(\varphi^-)^2\right)\Big|_0^H d\mu=\frac{1}{4}\int_0^1\mu\left((\varphi^+)^2+(\varphi^-)^2\right)d\mu>0 \tag{19}$$

according with (7). If the operator L is positive, the equation:

$$Lw = F$$

has only one solution. Indeed, let w_1 be an element such that

$$Lw_1 = F$$

Hence, $L(w - w_1) = 0 \Rightarrow L\overline{w}=0 \Rightarrow (L\overline{w},\overline{w})=0$. But the operator L is positive definite, such that $\overline{w}=0 \Rightarrow w = w_1$.

In order to get a solution of the problem (16)-(17), we go through three stages.

First, a difference scheme is given in order to approximate the space derivatives which appearing in A. We consider on z – axis two points systems:

- a principal system, $\{z_k\}_k$, $k\in\{0,1,...,N\}$ with $z_0 = 0$ and $z_N = H$;
- a secondary system, $\{z_{k+1/2}\}_k$, $k\in\{0,1,...,N-1\}$, which verifies the inequality: $z_{k-1/2} < z_k < z_{k+1/2}$.

Integrating the first equation (9) on the intervals: $(z_0, z_{1/2})$, $(z_{k-1/2}, z_{k+1/2})$, $k\in\{1, 2,\dots, N-1\}$, $(z_{N-1/2}, z_N)$ and the second equation on (z_{k-1}, z_k), $k\in\{1, 2,\dots, N-1\}$, the system can be written in the form:

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{z_0}^{z_{1/2}}u\,dz+\mu\int_{z_0}^{z_{1/2}}\frac{\partial v}{\partial z}dz+\int_{z_0}^{z_{1/2}}\sigma u\,dz=\int_{z_0}^{z_{1/2}}\sigma_s\,dz\int_0^1 u\,d\mu'+\int_{z_0}^{z_{1/2}}g\,dz\\
&\frac{\partial}{\partial t}\int_{z_0}^{z_1}v\,dz+\mu\int_{z_0}^{z_1}\frac{\partial u}{\partial z}dz+\int_{z_0}^{z_1}\sigma v\,dz=\int_{z_0}^{z_1}r\,dz\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial}{\partial t}\int_{z_{k-1/2}}^{z_{k+1/2}}u\,dz+\mu\int_{z_{k-1/2}}^{z_{k+1/2}}\frac{\partial v}{\partial z}dz+\int_{z_{k-1/2}}^{z_{k+1/2}}\sigma u\,dz=\int_{z_{k-1/2}}^{z_{k+1/2}}\sigma_s\,dz\int_0^1 u\,d\mu'+\int_{z_{k-1/2}}^{z_{k+1/2}}g\,dz\\
&\frac{\partial}{\partial t}\int_{z_k}^{z_{k+1}}v\,dz+\mu\int_{z_k}^{z_{k+1}}\frac{\partial u}{\partial z}dz+\int_{z_k}^{z_{k+1}}\sigma v\,dz=\int_{z_k}^{z_{k+1}}r\,dz\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\partial}{\partial t}\int_{z_{N-1/2}}^{z_N}u\,dz+\mu\int_{z_{N-1/2}}^{z_N}\frac{\partial v}{\partial z}dz+\int_{z_{N-1/2}}^{z_N}\sigma u\,dz=\int_{z_{N-1/2}}^{z_N}\sigma_s\,dz\int_0^1 u\,d\mu'+\int_{z_{N-1/2}}^{z_N}g\,dz\\
&\frac{\partial}{\partial t}\int_{z_{N-1}}^{z_N}v\,dz+\mu\int_{z_{N-1}}^{z_N}\frac{\partial u}{\partial z}dz+\int_{z_{N-1}}^{z_N}\sigma v\,dz=\int_{z_{N-1}}^{z_N}r\,dz
\end{aligned} \tag{20}$$

With the following notations:

$$\begin{aligned}
&\Delta z_0=z_{1/2}-z_0,\quad \Delta z_k=z_{k+1/2}-z_{k-1/2},\quad k=1,2,..,N-1\\
&\Delta z_N=z_N-z_{N-1/2},\quad \Delta z_{k-1/2}=z_k-z_{k-1},\quad k=1,2,...,N
\end{aligned} \tag{21}$$

we define the mean values:

$$\sigma_k = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma\, dz, \quad \sigma_{k+1/2} = \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma\, dz, \quad \sigma_{s_k} = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma_s\, dz$$

$$g_k = \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} g\, dz, \qquad r_{k+1/2} = \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma\, dz \tag{22}$$

Let $h = \max\limits_{0 \le k \le N-1} (\Delta z_0, \Delta z_N, \Delta z_k, \Delta z_{k+1/2})$. Dividing the first equation (20) by Δz_0 and the second by $\Delta z_{1/2}$ we obtain:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} u\, dz\right] + \mu \frac{1}{\Delta z_0} [v]_{z_0}^{z_{1/2}} + \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} \sigma(z)\, u(z,\mu,t)\, dz = \\ &\qquad = \int_0^1 \left[\frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} \sigma_s u\, dz\right] d\mu' + \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} g\, dz \\ &\frac{\partial}{\partial t}\left[\frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} v\, dz\right] + \mu \frac{1}{\Delta z_{1/2}} [u]_{z_0}^{z_1} + \frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} \sigma v\, dz = \frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} r\, dz \end{aligned} \tag{23}$$

In accordance with the boundary conditions:

$$v_0 = v\big|_{z=0} = \frac{1}{2}\left(\varphi^+ - \varphi^-\right)\big|_{z=0} = -\frac{1}{2}\varphi^- \tag{24}$$

$$u_0 = u\big|_{z=0} = \frac{1}{2}\left(\varphi^+ + \varphi^-\right)\big|_{z=0} = \frac{1}{2}\varphi^- \tag{25}$$

we have: $v_0 = - u_0$ and the relations (23) can be rewritten in the form:

$$\begin{aligned} &\frac{\partial u_0}{\partial t} + \mu \frac{v_{1/2} + u_0}{\Delta z_0} + \sigma_0 u_0 = \sigma_{s_0} \int_0^1 u_0 d\mu' + g_0,\ g_0 + \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} g dz \\ &\frac{\partial v_{1/2}}{\partial t} + \mu \frac{u_1 - u_0}{\Delta z_{1/2}} + \sigma_{1/2} v_{1/2} = r_{1/2} \end{aligned} \tag{26}$$

where the functions u, v are replaced by their values in the points: $z = 0$, $z = 1/2$, $z = 1$. Similarly, we get

$$\begin{aligned} &\frac{\partial u_k}{\partial t} + \mu \frac{v_{k+1/2} - v_{k-1/2}}{\Delta z_k} + \sigma_k u_k = \sigma_{s_k} \int_0^1 u_k d\mu' + g_k \\ &\frac{\partial v_{k+1/2}}{\partial t} + \mu \frac{u_{k+1} - u_k}{\Delta z_{k+1/2}} + \sigma_{k+1/2} v_{k+1/2} = r_{k+1/2}, \quad k = 1,2,\dots,N-1 \end{aligned} \tag{27}$$

$$\frac{\partial v_{N-1/2}}{\partial t} + \mu \frac{u_N - u_{N-1}}{\Delta z_{N-1/2}} + \sigma_{N-1/2} v_{N-1/2} = r_{N-1/2} \tag{28}$$

$$\frac{\partial u_N}{\partial t} + \mu \frac{u_N - v_{N-1/2}}{\Delta z_N} + \sigma_N u_N = \sigma_{s_N} \int_0^1 u_N d\mu' + g_N,\ g_N = \frac{1}{\Delta z_N} \int_{z_{N-1/2}}^{z_N} g dz$$

where $u_N = v_N$.

Let us consider $M(0, 2N)$, the Hilbert space of the vector functions $\alpha = (\alpha_0, \alpha_{1/2}, \alpha_1, \dots \alpha_N)$ with the scalar product:

$$(\alpha, \beta) = \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2} \alpha_{i/2} \beta_{i/2} \, d\mu \tag{29}$$

and the norm: $$\|\alpha\| = \sqrt{(\alpha, \alpha)}, \ \alpha \in M_h(0, 2N) \,. \tag{30}$$

We define the vector functions:

$$\begin{aligned} \varphi &= (u_0, v_{1/2}, u_1, \dots, u_{N-1}, v_{N-1/2}, u_N) \\ f &= (g_0, r_{1/2}, g_1, \dots, g_{N-1}, r_{N-1/2}, g_N) \\ \varphi^0 &= (u_0^0, v_{1/2}^0, u_1^0, \dots, u_{N-1}^0, v_{N-1/2}^0, u_N^0) \end{aligned} \tag{31}$$

and the operator: $A = L - S$, where

$$L = \begin{bmatrix} \dfrac{\mu}{\Delta z_0} + \sigma_0 & \dfrac{\mu}{\sigma_0} & 0 & 0 & \cdots & 0 & 0 \\ -\dfrac{\mu}{\Delta z_{1/2}} & \sigma_{1/2} & \dfrac{\mu}{\Delta z_{1/2}} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{N-1/2} & \dfrac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & 0 & \cdots & -\dfrac{\mu}{\Delta z_N} & \dfrac{\mu}{\Delta z_N} + \sigma_N \end{bmatrix} \tag{32}$$

$$S = diag\left(\sigma_{s_{i/2}} \int_0^1 \gamma_{i/2} \, d\mu \right), i = 0,1,\dots,2N; \ \gamma_{i/2} = \begin{cases} 1 & \text{if } \ i/2 \ \text{ is entier number} \\ 0 & \text{if } \ i/2 \ \text{ is rational number.} \end{cases} \tag{33}$$

Then, the system (26) – (28) has the form:

$$\begin{aligned} &\frac{d\varphi}{dt} + A\varphi = f, \quad t \in [0, T] \\ &\varphi(z, \mu, 0) = \varphi^0 \end{aligned} \tag{34}$$

where φ, f, φ^0 were defined by (31). In spite of the fact that we use, for simplicity of the writing, the same notation as (1) – (3), here φ is the numerical solution for our problem.

It is seen that the operator S is positive and we shall prove that L and A are positive operators. Indeed, let $w \in M$ and then

$$\begin{aligned} (Lw, w) &= \int_0^1 \Delta z_0 \left(\mu \frac{w_{1/2} + w_0}{\Delta z_0} + \sigma_0 w_0 \right) w_0 d\mu + \int_0^1 \Delta z_{1/2} \left(\mu \frac{w_1 - w_0}{\Delta z_{1/2}} + \sigma_0 w_0 \right) w_{1/2} d\mu + \\ &+ \sum_{i=1}^{N-1} \int_0^1 \Delta z_i \left(\mu \frac{w_{i+1/2} + w_{i-1/2}}{\Delta z_i} + \sigma_i w_i \right) w_i d\mu + \sum_{i=1}^{N-1} \int_0^1 \Delta z_{i+1/2} \left(\mu \frac{w_{i+1} - w_i}{\Delta z_{i+1/2}} + \sigma_{i+1/2} w_{i+1/2} \right) w_{i+1/2} d\mu + \\ &+ \int_0^1 \Delta z_N \left(\mu \frac{w_N + w_{N-1}}{\Delta z_N} + \sigma_N w_N \right) w_N d\mu = \int_0^1 \mu (w_0^2 + w_N^2) d\mu + \sum_{i=1}^{2N} \int_0^1 \Delta z_{i/2} \sigma_{i/2} w_{1/2}^2 \, d\mu > \\ &> \sigma_0 \sum_{i=1}^{2N} \int_0^1 \Delta z_{i/2} w_{i/2}^2 d\mu = \sigma_0 \| w \|^2 > 0 \end{aligned}$$

because continuous functions on $[0,H], \sigma_i$, are bounded on $[0,H]$ and by hypothesis $\sigma \geq \sigma_0 > 0$. Using above results, we obtain

$$(Aw, w) = \int_0^1 \mu(w_0^2 + w_N^2)d\mu + \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2}(\sigma_{i/2}w_{i/2}^2 - \sigma_{s_i/2}w_{i/2}\int_0^1 w_{i/2}\gamma_{i/2}d\mu')d\mu \geq$$
$$\geq \int_0^1 \mu(w_0^2 + w_N^2)d\mu + \sum_{i=0}^{2N} \int_0^1 (\sigma_{i/2} - \sigma_{s_i/2})\Delta z_{i/2}w_{i/2}^2 d\mu \geq \sigma_{c_0}\|w\|^2$$

In the second stage a difference scheme is given to approximate the time derivatives. This is used together with the bicycle splitting-up method, which writes the operator A as a sum:

$$A = A_1 + A_2 \tag{35}$$

where

$$A_1 = \begin{bmatrix} \frac{\mu}{\Delta z_0} & \frac{\mu}{\Delta z_0} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\mu}{\Delta z_{1/2}} & 0 & \frac{\mu}{\Delta z_{1/2}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-\mu}{\Delta z_{N-1/2}} & 0 & \frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-\mu}{\Delta z_N} & \frac{\mu}{\Delta z_N} \end{bmatrix}$$

and

$$A_2 = diag\left(\sigma_{i/2} - \sigma_{s_i/2}\int_0^1 \gamma_{i/2}d\mu\right).$$

In the following, we shall prove that A_1 is a positive operator. If $\alpha = (\alpha_0, \alpha_{1/2}, \alpha_1, \ldots \alpha_N) \in M_h$, we have

$$\left(A_1\alpha, \alpha\right) = \sum_{i=0}^{2N}\int_0^1 \Delta z_{i/2}(A_1\alpha)_{i/2}\alpha_{i/2}d\mu = \int_0^1 \Delta z_0\left(\frac{\mu}{\Delta z_0}\alpha_0 + \frac{\mu}{\Delta z_0}\alpha_{1/2}\right)\alpha_0 d\mu +$$
$$+ \int_0^1 \Delta z_{1/2}\left(\frac{-\mu}{\Delta z_{1/2}}\alpha_0 + \frac{\mu}{\Delta z_{1/2}}\alpha_1\right)\alpha_{1/2}d\mu + \cdots$$
$$\cdots + \int_0^1 \Delta z_{N-1/2}\left(\frac{-\mu}{\Delta z_{N-1/2}}\alpha_{N-1} + \frac{\mu}{\Delta z_{N-1/2}}\alpha_N\right)\alpha_{N-1/2}d\mu +$$
$$+ \int_0^1 \Delta z_N\left(\frac{-\mu}{\Delta z_N}\alpha_{N-1/2} + \frac{\mu}{\Delta z_N}\alpha_N\right)\alpha_N d\mu = \int_0^1 \mu(\alpha_0^2 + \alpha_N^2)d\mu > 0$$

We shall also prove that A_2 is a positive definite operator. Indeed,

$$\int_0^1 (\sigma_{i/2}\alpha_{i/2}^2 - \sigma_{s_{i/2}}\alpha_{i/2}\int_0^1 \alpha_{i/2}\gamma_{i/2}d\mu')d\mu \geq \int_0^1 (\sigma_{i/2} - \sigma_{s_{i/2}})\alpha_{i/2}^2 d\mu \geq \sigma_{c_0}\int_0^1 \alpha_{i/2}^2 d\mu > 0$$

In these conditions we can apply the bicycle splitting-up method. Let us divide the close interval [0, T] into n subintervals by choosing points: $t_0 = 0, t_1, \ldots, t_n = T$. Next, we take an arbitrary subinterval: $[t_{j-1}, t_{j+1}] = [t_{j-1}, t_{j-1/2}] \cup [t_{j-1/2}, t_j] \cup [t_j, t_{j+1/2}] \cup [t_{j+1/2}, t_{j+1}]$, which has the length equal to 4τ, where τ is the time step. Approximating the operators A_1, A_2 on this the subinterval by: $\Lambda_k^j = A_k(t_j), k = 1,2$, we shall obtain from (34) a difference system using the Krank-Nicholson scheme, [5]:

$$\frac{\varphi^{j-1/2} - \varphi^{j-1}}{\tau} + \Lambda_1^j \frac{\varphi^{j-1/2} + \varphi^{j-1}}{2} = 0 \tag{36}$$

$$\frac{\varphi^{j} - \varphi^{j-1/2}}{\tau} + \Lambda_2^j \frac{\varphi^{j} + \varphi^{j-1/2}}{2} = f^j + \frac{\tau}{2}\Lambda_2^j f^j \tag{37}$$

$$\frac{\varphi^{j+1/2} - \varphi^{j}}{\tau} + \Lambda_2^j \frac{\varphi^{j+1/2} + \varphi^{j}}{2} = f^j - \frac{\tau}{2}\Lambda_2^j f^j \tag{38}$$

$$\frac{\varphi^{j+1} - \varphi^{j+1/2}}{\tau} + \Lambda_1^j \frac{\varphi^{j+1} + \varphi^{j+1/2}}{2} = 0 \tag{39}$$

with $f^j = f(t_j)$. Adding (37) and (38) we get

$$\frac{\varphi^{j+1/2} - \varphi^{j-1/2}}{2\tau} + \Lambda_2^j \frac{\varphi^j + \dfrac{\varphi^{j-1/2} + \varphi^{j+1/2}}{2}}{2} = f^j \tag{40}$$

and the system (36)-(39) can be rewritten:

$$\begin{aligned}
\left(E + \frac{\tau}{2}\Lambda_1^j\right)\varphi^{j-1/2} &= \left(E - \frac{\tau}{2}\Lambda_1^j\right)\varphi^{j-1} \\
\left(E + \frac{\tau}{2}\Lambda_2^j\right)(\varphi^j - \tau f^j) &= \left(E - \frac{\tau}{2}\Lambda_2^j\right)\varphi^{j-1/2} \\
\left(E + \frac{\tau}{2}\Lambda_2^j\right)\varphi^{j+1/2} &= \left(E - \frac{\tau}{2}\Lambda_2^j\right)(\varphi^j + f^j\tau) \\
\left(E + \frac{\tau}{2}\Lambda_1^j\right)\varphi^{j+1} &= \left(E - \frac{\tau}{2}\Lambda_1^j\right)\varphi^{j+1/2}
\end{aligned} \tag{41}$$

where E is unit matrix. Let us define:

$$T_k^j = (E + \frac{\tau}{2}\Lambda_k^j)^{-1}(E - \frac{\tau}{2}\Lambda_k^j) \tag{42}$$

and from (41) we have:

$$\begin{aligned}
\varphi^{j-1/2} &= T_1^j \varphi^{j-1} \\
\varphi^{j} &= \tau f^j + T_2^j \varphi^{j-1/2} \\
\varphi^{j+1/2} &= T_2^j \varphi^j + T_2^j \tau f^j \\
\varphi^{j+1} &= T_1^j \varphi^{j+1/2} = T_1^j T_2^j \varphi^j + T_1^j T_2^j f^j \tau = \\
&= T_1^j T_2^j f^j \tau + T_1^j T_2^j T_2^j T_1^j \varphi^{j-1} + T_1^j T_2^j f^j \tau = T^j \varphi^{j-1} + 2\tau T_1^j T_2^j f^j
\end{aligned} \tag{43}$$

where $T^j = T_1^j T_2^j T_2^j T_1^j$.

Now we prove that the algorithm is stable and leads to a numerical solution, which approximates the exact solution with an accuracy of second order in time step τ.

Approximation

For the estimation of approximation order, we shall expand with respect to the power of small τ, the expression

$$T_k^j = \left(E + \frac{\tau}{2}\Lambda_k^j\right)^{-1}\left(E - \frac{\tau}{2}\Lambda_k^j\right) = E - \tau\Lambda_k^j + \frac{\tau^2}{2}(\Lambda_k^j)^2 \ldots$$

when $\frac{\tau}{2}\left\| \Lambda_k^j \right\| < 1,\ k = 1,2.$ Then

$$T_1^j T_2^j = E - \tau\Lambda^j + \frac{\tau^2}{2}\left[\left(\Lambda_1^j + \Lambda_2^j\right)^2 + (\Lambda_1^j\Lambda_2^j - \Lambda_2^j\Lambda_1^j)\right] + o(\tau^3)$$

where $\Lambda^j = (\Lambda_1^j + \Lambda_2^j)/2$. If $\Lambda_1^j\Lambda_2^j = \Lambda_2^j\Lambda_1^j$, we get

$$T_1^j T_2^j = E - \tau\Lambda^j + \frac{\tau^2}{2}(\Lambda^j)^2 + o(\tau^3)$$

When the operators are non-commutative, the approximation with the splitting-up algorithm is of the first order with respect to τ. Let us now consider

$$T^j = \prod_{k=1}^{2} T_k^j \prod_{k=2}^{1} T_k^j = T_1^j T_2^j T_2^j T_1^j = E - 2\tau\Lambda^j + \frac{(2\tau)^2}{2}(\Lambda^j)^2 + o(\tau^3)$$

Hence, the following estimation is valid in the interval $[t_{j-1}, t_{j+1}]$:

$$\varphi^{j+1} = \left[E - 2\tau\Lambda^j + \frac{(2\tau)^2}{2}(\Lambda^j)^2\right]\varphi^{j-1} + 2\tau(E - \tau\Lambda^j)f^j + o(\tau^3)$$

and

$$\frac{\varphi^{j+1} - \varphi^{j-1}}{2\tau} + \Lambda^j(E - \tau\Lambda^j)\varphi^{j-1} = (E - \tau\Lambda^j)f^j + O_1(\tau^2) \tag{44}$$

Using the Taylor series expansion of the solution φ in the neighborhood of the point t_{j-1} and substituting t_j for t, we can write:

$$\varphi^j = \varphi(z, \mu, t_j) = \varphi^{j-1} + \left(\frac{\partial\varphi}{\partial t}\right)^{j-1}\tau + o(\tau^2) \tag{45}$$

Then, we eliminate $\left(\frac{\partial\varphi}{\partial t}\right)^{j-1}$, writing the transport equation (35) in the point t_{j-1} in the form:

$$\left(\frac{\partial\varphi}{\partial t}\right)^{j-1} = -\Lambda^j\varphi^{j-1} + f^j + o_2(\tau)$$

and (45) becomes:

$$\varphi^j = \varphi^{j-1}(E - \tau\Lambda^j) + \tau f^j + o(\tau^2)\,.$$

Finally, we get

$$\frac{\varphi^{j+1} - \varphi^j}{2\tau} + \Lambda^j\varphi^j = f^j + o(\tau^2) \tag{46}$$

This relation is an approximation with the accuracy of second order in time step τ of the initial equation (35) on the interval $[t_{j-1,}\, t_{j+1}]$.

Stability

The algorithm: $\varphi^{j+1} = T^j \varphi^j + \tau f^j$ on $D \times [0,T]$, $\varphi(z,\mu,0) = \varphi^0$ on D, is stable, if for any step h, which characterizes the first difference scheme, and for any $j \le T/\tau$, τ - the time step, we have the inequality:

$$\left\| \varphi^j \right\|_{\Phi_h} \le C_1 \left\| \varphi^0 \right\|_{\Phi_h} + C_2 \left\| f \right\|_{\Phi_{h,\tau}} \tag{47}$$

where C_1, C_2 are the independent constants of h, τ, φ^0, f and $\Phi_h, \Phi_{h,\tau}$ are the systems of these reticular functions.

Turning to the formula (43) we get:

$$\varphi^{j+1} = T_1^j T_2^j \varphi^j + \tau T_1^j T_2^j f^j \quad \text{for } t \in [t_j, t_{j+1}] \tag{48}$$

First, we prove that:

$$T_k^j = (E + \frac{\tau}{2}\Lambda_k^j)^{-1}(E - \frac{\tau}{2}\Lambda_k^j) = (E - \frac{\tau}{2}\Lambda_k^j)(E + \frac{\tau}{2}\Lambda_k^j)^{-1} \tag{49}$$

using the identity: $(E + \frac{\tau}{2}\Lambda_k^j)^{-1}(E + \frac{\tau}{2}\Lambda_k^j) = E$. Indeed, multiplying to the left with $(E + \frac{\tau}{2}\Lambda_k^j)^{-1}(E - \frac{\tau}{2}\Lambda_k^j)$ and using the next commutative property:

$$(E - \frac{\tau}{2}\Lambda_k^j)(E + \frac{\tau}{2}\Lambda_k^j) = (E + \frac{\tau}{2}\Lambda_k^j)(E - \frac{\tau}{2}\Lambda_k^j) \tag{50}$$

we obtain (49). Then, we have

$$\begin{aligned} T_1^j T_2^j &= (E + \frac{\tau}{2}\Lambda_1^j)^{-1}(E - \frac{\tau}{2}\Lambda_1^j)(E + \frac{\tau}{2}\Lambda_2^j)^{-1}(E - \frac{\tau}{2}\Lambda_2^j) = \\ &= (E - \frac{\tau}{2}\Lambda_1^j)(E + \frac{\tau}{2}\Lambda_1^j)^{-1}(E - \frac{\tau}{2}\Lambda_2^j)(E + \frac{\tau}{2}\Lambda_2^j)^{-1} \end{aligned} \tag{51}$$

Now, we make use of the Kellog theorem,[5]: *if* Λ_k^j *is a positive definite operator and* $\tau/2 \ge 0$, *then we have:*

$$\left\| (E - \frac{\tau}{2}\Lambda_k^j)(E + \frac{\tau}{2}\Lambda_k^j)^{-1} \right\| \le 1 \tag{52}$$

In accordance with (51) and (52) we get:

$$\begin{aligned} \| T_1 T_2 \| &= \left\| (E + \frac{\tau}{2}\Lambda_1^j)^{-1}(E - \frac{\tau}{2}\Lambda_1^j)(E + \frac{\tau}{2}\Lambda_2^j)^{-1}(E - \frac{\tau}{2}\Lambda_2^j) \right\| \le \\ &\le \left\| (E - \frac{\tau}{2}\Lambda_1^j)(E + \frac{\tau}{2}\Lambda_1^j)^{-1} \right\| \cdot \left\| (E - \frac{\tau}{2}\Lambda_2^j)(E + \frac{\tau}{2}\Lambda_2^j)^{-1} \right\| \le 1 \cdot 1 = 1 \end{aligned} \tag{53}$$

Then, the equality (48) becomes:

$$\left\| \varphi^{j+1} \right\| \le \left\| T_1^j T_2^j \varphi^j \right\| + \tau \left\| T_1^j T_2^j f^j \right\| \le \left\| T_1^j \right\| \cdot \left\| T_2^j \right\| \left(\left\| \varphi^j \right\| + \tau \left\| f^j \right\| \right) \le \left\| \varphi^j \right\| + \tau \left\| f^j \right\| \tag{54}$$

Using the recurrence formula (54) we obtain:

$$\left\| \varphi^{j+1} \right\| \le \left\| \varphi^j \right\| + \tau \left\| f^j \right\| \le \left\| \varphi^{j-1} \right\| + 2\tau \left\| f^j \right\| \le .... \le \| g \| + T \cdot \| f \| \tag{55}$$

where $\| f \| = \max\limits_j \left\| f^j \right\|$ and T is the total length of the time interval. Hence, a sufficient condition that the algorithm to be stable is the following:

$$\left\| T_k^j \right\| \le 1, \ \forall j = 1, 2, ..., n \tag{56}$$

which is always true when we can applied the Kellog theorem.
We shall now summarize the above results in the following theorem.

Theorem

Suppose the following conditions hold:

(1) the solution φ of the problem (34) has bounded derivatives with respect to t up to the second order;
(2) the operators A_1 and A_2 are positive definite;
(3) $\frac{\tau}{2} \left\| \Lambda_k^j \right\| < 1$, where τ is time step.

Then the numerical algorithm (41) is stable and approximates the exact solution with an accuracy of second order in τ.

In the practical application, we use instead of the system (36) – (39) on the interval $[t_{j-1}, t_{j+1}]$,the following splitting-up method:

$$\begin{aligned}
&\frac{\varphi^{j-2/3} - \varphi^{j-1}}{\tau} + \Lambda_1^j \frac{\varphi^{j-1} + \varphi^{j-2/3}}{2} = 0 \\
&\frac{\varphi^{j-1/3} - \varphi^{j-2/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j-2/3} + \varphi^{j-1/3}}{2} = 0 \\
&\frac{\varphi^{j-2/3} - \varphi^{j-1}}{2\tau} = f^j \\
&\frac{\varphi^{j+2/3} - \varphi^{j+1/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j+2/3} + \varphi^{j+1/3}}{2} = 0 \\
&\frac{\varphi^{j+1} - \varphi^{j+2/3}}{\tau} + \Lambda_1^j \frac{\varphi^{j+1} + \varphi^{j+2/3}}{2} = 0
\end{aligned} \tag{57}$$

or in other form:

$$\begin{aligned}
&\varphi^{j-2/3} = \left(E + \frac{\tau}{2} \Lambda_1^j \right)^{-1} \left(E - \frac{\tau}{2} \Lambda_1^j \right) = T_1^j \varphi^{j-1} \\
&\varphi^{j-1/3} = T_2^j \varphi^{j-2/3} \\
&\varphi^{j+1/3} = \varphi^{j-1/3} + 2\tau f^j \\
&\varphi^{j+2/3} = T_2^j \varphi^{j+1/3} \\
&\varphi^{j+1} = T_1^j \varphi^{j+2/3}
\end{aligned} \tag{58}$$

Similarly, we obtain for $\Lambda_1^j = A_1(t_j) \geq 0, \Lambda_2^j = A_2(t_j) \geq 0$ and the recurrence formula

$$\varphi^{j+1} = T_1^j T_2^j T_2^j T_1^j \varphi^{j-1} + 2\tau T_1^j T_2^j f^j$$

that the schema (57) is stable. At the beginning, for determining the solution of the system (57), we consider the first equation, for a fixed μ, and the operators (32), (33). We get:

$$\begin{bmatrix} 1+\frac{\tau}{2}\frac{\mu}{\Delta z_0} & \frac{\tau}{2}\frac{\mu}{\Delta z_0} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & 1 & \frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\tau}{2}\frac{\mu}{\Delta z_1} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} & 1 & \frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\tau}{2}\frac{\mu}{\Delta z_N} & 1+\frac{\tau}{2}\frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \vdots \\ v_{N-1/2}^{j-2/3} \\ u_N^{j-2/3} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\frac{\tau}{2}\frac{\mu}{\Delta z_0} & -\frac{\tau}{2}\frac{\mu}{\Delta z_0} & 0 & \cdots & 0 & 0 & 0 \\ \frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & 1 & -\frac{\tau}{2}\frac{\mu}{\Delta z_{1/2}} & \cdots & 0 & 0 & 0 \\ 0 & \frac{\tau}{2}\frac{\mu}{\Delta z_1} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} & 1 & -\frac{\tau}{2}\frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\tau}{2}\frac{\mu}{\Delta z_N} & 1-\frac{\tau}{2}\frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-1} \\ v_{1/2}^{j-1} \\ u_1^{j-1} \\ \vdots \\ v_{N-1/2}^{j-1} \\ u_N^{j-1} \end{bmatrix}.$$

We obtain an analogous relation from the last equation. For the second and the fourth equation we have:

$$\begin{bmatrix} 1+\frac{\tau}{2}\left(\sigma_0 - \sigma_{s0}\int_0^1 d\mu\right) & 0 & 0 & \cdots & 0 \\ 0 & 1+\frac{\tau}{2}\sigma_{1/2} & 0 & \cdots & 0 \\ 0 & 0 & 1+\frac{\tau}{2}\left(\sigma_1 - \sigma_{s1}\int_0^1 d\mu\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\frac{\tau}{2}\left(\sigma_N - \sigma_{sN}\int_0^1 d\mu\right) \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-1/3} \\ v_{1/2}^{j-1/3} \\ u_1^{j-1/3} \\ \vdots \\ u_N^{j-1/3} \end{bmatrix} =$$

$$= \begin{bmatrix} 1-\frac{\tau}{2}\left(\sigma_0-\sigma_{s0}\int\limits_0^1 d\mu\right) & 0 & 0 & \cdots & 0 \\ 0 & 1-\frac{\tau}{2}\sigma_{1/2} & 0 & \cdots & 0 \\ 0 & 0 & 1-\frac{\tau}{2}\left(\sigma_1-\sigma_{s1}\int\limits_0^1 d\mu\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\frac{\tau}{2}\left(\sigma_N-\sigma_{sN}\int\limits_0^1 d\mu\right) \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \vdots \\ u_N^{j-2/3} \end{bmatrix}$$

We obtain the following relations for the numerical solution:

$$\varphi^{j-1/3} = \left(E+\frac{\tau}{2}A_2^j\right)^{-1}\left(E-\frac{\tau}{2}A_2^j\right)\varphi^{j-2/3}$$

Elements of the product are:

$$\varphi_i^{j-1/3} = \frac{1}{1+\frac{\tau\sigma_{ci}}{2}}\left[\left(1-\frac{\tau\sigma_i}{2}\right)\varphi_i^{j-2/3}+\frac{\tau\sigma_{si}}{2}\int\limits_0^1 \varphi_i^{j-2/3}d\mu\right], i=0,1,...,N \tag{59}$$

$$\varphi_{i-1/2}^{j-1/3} = \frac{1-\sigma_{i-1/2}\tau/2}{1+\sigma_{i-1/2}\tau/2}\cdot\varphi_i^{j-2/3}, \quad i=1,2,...,N \tag{60}$$

Analogously, we have

$$\varphi_i^{j+2/3} = \frac{1}{1+\frac{\tau\sigma_{ci}}{2}}\left[\left(1-\frac{\tau\sigma_i}{2}\right)\varphi_i^{j+1/3}+\frac{\tau\sigma_{si}}{2}\int\limits_0^1 \varphi_i^{j+1/3}d\mu\right], i=0,1,...,N \tag{61}$$

$$\varphi_{i-1/2}^{j+2/3} = \frac{1-\sigma_{i-1/2}\tau/2}{1+\sigma_{i-1/2}\tau/2}\cdot\varphi_{i-1/2}^{j+1/3}, \quad i=1,2,...,N \tag{62}$$

At the third stage, we consider the points: $\mu_0 = 0$, μ_1, ..., $\mu_m = 1$, in the interval [0, 1] and compute the integrals with respect to μ, using a numerical integration (trapezoidal approximation):

$$\int\limits_0^1 \psi(\mu)d\mu \approx \sum_{l=1}^{m} S_l\psi_l, \quad \psi_l = \psi(\mu_l) \tag{63}$$

Then, the system (57) can be written in the form:

$$
\begin{aligned}
&(E+\frac{\tau}{2}A_{1,l}^{j})\varphi_l^{j-2/3}=(E-\frac{\tau}{2}A_{1,l}^{j})\varphi_l^{j-1}\\
&(E+\frac{\tau}{2}A_{2,l}^{j})\varphi_l^{j-1/3}=(E-\frac{\tau}{2}A_{2,l}^{j})\varphi_l^{j-2/3}\\
&\varphi_l^{j+1/3}=\varphi_l^{j-1/3}+2\tau f_l^{j}\\
&(E+\frac{\tau}{2}A_{2,l}^{j})\varphi_l^{j+2/3}=(E-\frac{\tau}{2}A_{2,l}^{j})\varphi_l^{j+1/3}\\
&(E+\frac{\tau}{2}A_{1,l}^{j})\varphi_l^{j+1}=(E-\frac{\tau}{2}A_{1,l}^{j})\varphi_l^{j+2/3}
\end{aligned}
\tag{64}
$$

In this choice of the steps, which correspond to the variables z, t , we use the condition:

$$\tau \le \min_i(\Delta z_{i/2}) \tag{65}$$

3 Numerical Example

We wish to find the solution of the problem:

$$\frac{d\varphi(z,\mu,t)}{dt}+A\varphi(z,\mu,t)=f(z,\mu,t),\quad (z,\mu,t)\in[0,4]\times[0,1]\times[0,2]$$

(66)

$$\varphi(z,\mu,0)=\varphi^0(z,\mu) \tag{67}$$

Considering the partition of [0,4] into four subintervals of equal length by points:

$$z_0=0<z_{1/2}<z_1<z_{3/2}<z_2=4$$

with:

$\Delta z_0=z_{1/2}-z_0=1$; $\Delta z_{1/2}=z_1-z_0=2$; $\Delta z_1=z_{3/2}-z_{1/2}=2$;
$\Delta z_{3/2}=z_2-z_1=2$; $\Delta z_2=z_2-z_{3/2}=1$.

The partition of the interval [0, 1] is: $\mu_0=0<\mu_1=1/2<\mu_2=1$.

For the variable t, we consider the regular partition of the interval [0,2] by the points: $t_0=0<t_{1/3}<t_{2/3}<t_1<t_{4/3}<t_{5/3}<t_2=2$. The initial value problem is defined by:

$$\varphi^0=\left(u_0^0,v_{1/2}^0,u_1^0,v_{3/2}^0,u_2^0\right)=\left(1,1,1,1,1\right) \tag{68}$$

The functions $\sigma(z)$, $\sigma_s(z)$ and f, which here depends only of μ are defined with the help of the table 1. The values of $\varphi_i, i\in\{0,1/2,1,3/2,2\}$ with respect to μ_l and t_j are presented in table 2.

Table 1

z	0	1	1/2	3/2	2
$\sigma(z)$	1	1.4	1.65	1.76	1.9
$\sigma_s(z)$	0.9	0.95	1	0.8	0.7

From the relations (8) and using the mean values for $u_{1/2}$, v_1, $u_{3/2}$ we obtain the density, φ^+, for $\mu>0$ and the density, φ^-, for $\mu<0$ for each value of z_i and t_j :

$$\varphi^{+}(1/2,\mu,t) = v_{1/2} + \frac{u_0 + 2\cdot u_1}{3} \qquad \varphi^{-}(0,-\mu,t) = 2u_0,\ \varphi^{-}(2,-\mu,t) = 0$$

$$\varphi^{+}(1,\mu,t) = u_1 + \frac{v_{1/2} + v_{3/2}}{2} \qquad \varphi^{-}(1/2,\mu,t) = \frac{u_0 + 2\cdot u_1}{3} - v_{1/2}$$

$$\varphi^{+}(3/2,\mu,t) = v_{3/2} + \frac{2\cdot u_1 + u_2}{3} \qquad \varphi^{-}(1,\mu,t) = u_1 - \frac{v_{1/2} + v_{3/2}}{2}$$

$$\varphi^{+}(2,\mu,t) = 2u_N,\quad \varphi^{+}(0,\mu,t) = 0 \qquad \varphi^{-}(3/2,\mu,t) = \frac{2\cdot u_1 + u_2}{3} - v_{3/2}$$

Table 2

φ	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	μ = 1/2	$\mu=0$	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$
u_0	1	0.69	0.44	0.92	0.67	0.46	1.25	0.97	0.69	1.17	0.94	0.71	1.17	0.73	0.38
$v_{1/2}$	1	0.98	0.95	0.62	0.62	0.59	0.95	0.9	0.82	0.59	0.56	0.5	0.59	0.54	0.43
u_1	1	0.99	0.99	0.87	0.85	0.87	1.2	1.17	1.09	1.05	1.03	0.98	1.05	1.04	0.98
$v_{3/2}$	1	1.	1.	0.56	0.56	0.56	0.89	0.86	0.79	0.5	0.48	0.44	0.5	0.5	0.5
u_2	1	1.	1.	0.69	0.69	0.69	0.99	0.99	0.92	0.68	0.68	0.64	0.68	0.65	0.59

Table 3

φ +	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1/2	2	1.88	1.76	1.52	1.4	1.32	2.17	2	1.77	1.7	1.56	1.35	1.7	1.47	1.2
1	2	1.99	1.97	1.5	1.44	1.44	2.12	2.05	1.9	1.6	1.55	1.45	1.6	1.6	1.45
3/2	2	2	2	1.37	1.36	1.37	2	1.97	1.83	1.4	1.39	1.31	1.4	1.4	1.35
2	2	2	2	1.38	1.38	1.38	1.98	1.98	1.84	1.36	1.36	1.3	1.36	1.3	1.2

Table 4

φ -	t = 1/3			t = 2/3			t = 4/3			t = 5/3			t = 2		
	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$	$\mu=0$	μ = 1/2	$\mu=1$
0	2	1.4	0.87	1.85	1.34	0.92	2.5	1.94	1.38	2.34	1.88	1.42	2.34	1.46	1.96
1/2	0	0	0	0.27	0.18	0.14	0.27	0.2	0.14	0.5	0.44	0.39	0.5	0.29	0.35
1	0	0.4	0.02	0.28	0.26	0.29	0.3	0.28	0.29	0.51	0.51	0.51	0.51	0.52	0.52
3/2	0	0.06	0	0.25	0.24	0.24	0.24	0.25	0.25	0.41	0.43	0.43	0.42	0.4	0.35
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

We mention that, the numerical values of φ + and φ^{-} were obtained for the neutron speed, v_c =1. Also, in accordance with [3], we multiply with a factor 2 the number of the particles, which pass through a surface with unitary area.

Finally, to obtain the true values of φ + and φ^{-}, we will multiply these with $2v_c$.

The results of this numerical example prove its practical importance: how depends the density in a point z at the time t for different values of angle ν.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.4,443-460,COPYRIGHT 2008 EUDOXUS PRESS, LLC

Asymptotic distribution of the sample average value-at-risk in the case of heavy-tailed returns

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*Rachev gratefully acknowledges research support by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst.

Abstract

In this paper, we provide a stable limit theorem for the asymptotic distribution of the sample average value-at-risk when the distribution of the underlying random variable X describing portfolio returns is heavy-tailed. We illustrate the convergence rate in the limit theorem assuming that X has a stable Paretian distribution and Student's t distribution.

Keywords average value-at-risk, risk measures, heavy-tails, asymptotic distribution, Monte Carlo

1 Introduction

The average value-at-risk (AVaR) risk measure has been proposed in the literature as a coherent alternative to the industry standard Value-at-Risk (VaR), see Artzner et al. (1998) and Pflug (2000). It has been demonstrated that it has better properties than VaR for the purposes of risk management and, being a downside risk-measure, it is superior to the classical standard deviation and can be adopted in a portfolio optimization framework, see Rachev et al. (2006), Stoyanov et al. (2007), Biglova et al. (2004), and Rachev et al. (2008).

The AVaR of a random variable X at tail probability ϵ is defined as

$$AVaR_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F^{-1}(p)dp.$$

where $F^{-1}(x)$ is the inverse of the cumulative distribution function (c.d.f.) of the random variable X. The random variable may describe the return of stock, for example. A practical problem of computing portfolio AVaR is that usually we do not know explicitly the c.d.f. of portfolio returns. In order to solve this practical problem, the Monte Carlo method is employed. The returns of the portfolio constituents are simulated and then the returns of the portfolio are calculated. In effect, we have a sample from the portfolio return distribution which we can use to estimate AVaR. The sample AVaR equals,

$$\widehat{AVaR}_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F_n^{-1}(p)dp.$$

where $F_n^{-1}(p)$ denotes the inverse of the sample c.d.f. $F_n(x) = \frac{1}{n}\sum_{i=1}^n I\{X_i \leq x\}$ in which $I\{A\}$ denotes the indicator function of the event A, and $X_1, \ldots, X_n$

is a sample of independent, identically distributed (i.i.d.) copies of a random variable X.

Under a very general regularity condition, the larger the sample, the closer the estimate to the true value. Suppose that $E\max(-X,0) < \infty$. Then, it is easy to demonstrate that the following relation holds,

$$E\max(-X,0) < \infty \qquad \Longleftrightarrow \qquad AVaR_\epsilon(X) < \infty.$$

Thus, by the strong law of large numbers, the condition $E\max(-X,0) < \infty$ is necessary and sufficient for the almost sure convergence of the sample AVaR to the true one,

$$\widehat{AVaR}_\epsilon(X) \xrightarrow{a.s.} AVaR_\epsilon(X) \quad \text{as} \quad n \to \infty. \tag{1}$$

However, with any finite sample, the sample AVaR will fluctuate about the true value and, having only a sample estimate, we have to know the probability distribution of the sample AVaR in order to build a confidence interval for the true value. The problem of computing the distribution of the sample AVaR is a complicated one even if we know the distribution of X. From a practical viewpoint, X describes portfolio return which can be a complicated function of the joint distribution of the risk drivers. Therefore, we can only rely on large sample theory to gain insight into the fluctuations of sample AVaR. That is, for a large n, we can use a limiting distribution to calculate a confidence interval. In this respect, a limit theorem for the distribution of the sample AVaR can be regarded as a way to describe the speed of convergence in (1).

Concerning the finite sample properties, the estimator $\widehat{AVaR}_\epsilon(X)$ has a negative bias,

$$\widehat{AVaR}_\epsilon(X) \leq AVaR_\epsilon(X).$$

The asymptotic bias is of order $O(n^{-1})$ and we consider it negligible for the purposes of our study. For further details, see Trindade et al. (2007).

In this paper, we discuss the asymptotic distribution of the sample AVaR assuming that the random variable X can be heavy-tailed and may have an infinite second moment. In such a case, we cannot take advantage of the classical Central Limit Theorem (CLT) to establish a limit theorem. For this reason, we resort to the Generalized CLT and the characterization of the domains of attraction of stable distributions which appear as limiting distribution in it.

Stable distributions are introduced by their characteristic functions. The random variable Z is said to have a stable distribution if its characteristic function $\varphi(t) = Ee^{itZ}$ has the form

$$\varphi(t) = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) + i\mu t\}, & \alpha \neq 1 \\ \exp\{-\sigma |t| (1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \ln(|t|)) + i\mu t\}, & \alpha = 1 \end{cases} \tag{2}$$

and is denoted by $Z \in S_\alpha(\sigma, \beta, \mu)$. The parameter $\alpha \in (0, 2]$ is called the tail index and governs the tail behavior and the kurtosis of the distribution. Smaller α indicates heavier tails and higher kurtosis. If $\alpha < 2$, then Z has infinite variance. If $1 < \alpha \leq 2$, then Z has finite mean and the AVaR of Z can be calculated. The Gaussian distribution appears as a stable distribution with $\alpha = 2$. The stable distributions with $\alpha < 2$ are referred to as *stable Paretian distributions*. The parameter $\beta \in [-1, 1]$ is a skewness parameter. If $\beta = 0$, the distribution is symmetric with respect to μ. Positive β indicates that the distribution is skewed to the right and negative β indicates that the distribution is skewed to the left. The parameter $\sigma > 0$ is a scale parameter and $\mu \in \mathbf{R}$ is a location parameter.

The notion of slowly varying functions is extensively used in the paper. A positive function $L(x)$ is said to be slowly varying at infinity if the following limit relation is satisfied,

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad \forall t > 0. \tag{3}$$

The main result concerning the domains of attraction of stable distributions is given in the following theorem.

Theorem 1. *Let $X_1, \ldots, X_n$ be i.i.d. with c.d.f. $F(x)$. There exist $a_n > 0$, $b_n \in \mathbf{R}$, $n = 1, 2, \ldots$, such that the distribution of*

$$a_n^{-1}[(X_1 + \ldots + X_n) - b_n]$$

converges as $n \to \infty$ to $S_\alpha(1, \beta, 0)$ if and only if both

(i) $x^\alpha[1 - F(x) + F(-x)] = L(x)$ is slowly varying at infinity.

(ii) $\dfrac{1 - F(x) - F(-x)}{1 - F(x) + F(-x)} \to \beta$ as $x \to \infty$

The a_n must satisfy

$$\lim_{n \to \infty} \frac{nL(a_n)}{a_n^\alpha} = \begin{cases} (\Gamma(1-\alpha)\cos(\pi\alpha/2))^{-1} & \text{if} \quad 0 < \alpha < 1, \\ 2/\pi & \text{if} \quad \alpha = 1, \\ \left(\frac{\Gamma(2-\alpha)}{\alpha - 1} |\cos\frac{\pi\alpha}{2}|\right)^{-1} & \text{if} \quad 1 < \alpha < 2. \end{cases} \tag{4}$$

The b_n may be chosen as follows:

$$b_n = \begin{cases} 0 & \textit{if} \quad 0 < \alpha < 1, \\ na_n \displaystyle\int_{-\infty}^{\infty} \sin(x/a_n) dF(x) & \textit{if} \quad \alpha = 1, \\ n \displaystyle\int_{-\infty}^{\infty} x dF(x) & \textit{if} \quad 1 < \alpha < 2. \end{cases} \tag{5}$$

In all cases, $a_n = n^{1/\alpha} L_0(n)$ *where* $L_0(n)$ *is slowly varying at infinity.*

For further information about stable distributions and their properties, see Samorodnitsky and Taqqu (1994).

The result in Theorem 1 characterizes the domains of attraction of stable Paretian laws. If the index α characterizing the tails of the c.d.f. $F(x)$ in condition *(i)* satisfies $\alpha \geq 2$, then the tail index of the limiting distribution equals $\alpha^* = 2$. Thus, the relationship between the tail index of the limiting distribution, which we denote by α^*, and the tail index in condition *(i)* can be generalized as $\alpha^* = \min(\alpha, 2)$. If $\alpha > 2$, then $EX_1^2 < \infty$ and we are in the setting of the classical CLT. The centering and normalization can be done $b_n = nEX_1$ and $a_n = n^{1/2}\sigma_{X_1}$, where σ_{X_1} denotes the standard deviation of X_1. The case $\alpha = 2$ is more special because the variance of X_1 is infinite and a_n cannot be chosen in this fashion. Moreover, the proper normalization cannot be obtained by computing the limit $\alpha \to 2$ in equation (4). Under the more simple assumptions that the function $L(x)$ in condition *(i)* equals a constant A, Zolotarev and Uchaikin (1999) provide the formula $a_n = (n \log n)^{1/2} A^{1/2}$.

The paper is organized in the following way. Section 2 provides a stable limit theorem for the asymptotic distribution of the sample AVaR. In Section 3, we apply the theorem assuming that the random variable X has a stable Paretian distribution and also Student's t distribution. Under these assumptions, we study the effect of skewness and heavy tails on the convergence rate in the limit theorem.

2 A stable limit theorem

In order to develop the limit theorem, we need a few additional facts related to building a linear approximation to AVaR and estimating the rate of improvement of the linear approximation. They are collected in the following proposition.

Proposition 1. *Suppose* X *is a r.v. with c.d.f.* F *which satisfies the condition* $E\max(-X, 0) < \infty$ *and* F *is differentiable at the* ϵ*-quantile of* X*. Denote by* F_n *the sample c.d.f. of* $X_1, \ldots, X_n$ *which is a sample of i.i.d.*

copies of X. There exists a linear functional Δ defined on the difference $G - F$ where the functions G and F are c.d.f.s, such that

$$|\phi(F_n) - \phi(F) - \Delta(F_n - F)| = o(\rho(F_n, F)) \tag{6}$$

where $\rho(F_n, F) = \sup_x |F_n(x) - F(x)|$ stands for the Kolmogorov metric and

$$\phi(G) = -\frac{1}{\epsilon}\int_0^{\epsilon} G^{-1}(p)dp$$

in which G^{-1} is the inverse of the c.d.f. G. The linear functional Δ has the form

$$\Delta(F_n - F) = \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} (q_\epsilon - x)d(F_n(x) - F(x)). \tag{7}$$

where q_ϵ is the ϵ-quantile of X.

Proof. The condition $E\max(-X, 0) < \infty$ guarantees $\phi(F) < \infty$. Note that $\phi(F_n)$ is convergent with any finite sample.

Consider the difference $\phi(F_n) - \phi(F)$.

$$\begin{aligned}
\phi(F_n) - \phi(F) &= -\frac{1}{\epsilon}\int_0^{\epsilon} F_n^{-1}(p)dp + \frac{1}{\epsilon}\int_0^{\epsilon} F^{-1}(p)dp \\
&= -\frac{1}{\epsilon}\int_{-\infty}^{F_n^{-1}(\epsilon)} pdF_n(p) + \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} pdF(p) \\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} pdF_n(p) - \frac{1}{\epsilon}\int_{q_\epsilon}^{F_n^{-1}(\epsilon)} pdF_n(p) + \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} pdF(p) \\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} pd(F_n(p) - F(p)) - \frac{C_n}{\epsilon}(F(q_\epsilon) - F_n(q_\epsilon)) \\
&= -\frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} pd(F_n(p) - F(p)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon))
\end{aligned}$$

where, by the mean-value theorem, the constant C_n is between q_ϵ and $F_n^{-1}(\epsilon)$. For example if we assume, for the sake of being particular, that $q_\epsilon \leq F_n^{-1}(\epsilon)$, then $q_\epsilon \leq C_n \leq F_n^{-1}(\epsilon)$. Due to the assumption that F is differentiable at q_ϵ, $F_n^{-1}(\epsilon) \to q_\epsilon$ in almost sure sense as n increases indefinitely. As a result, $C_n \to q_\epsilon$ in almost sure sense.

Choose the linear functional $\Delta(F_n - F)$ as in equation (7). The fact that it is linear with respect to the difference of the c.d.f.s is a property of the integral. Consider the left-had side of (7), which we denote by LHS, having in mind the expression derived above. We obtain

$$
\begin{aligned}
LHS &= \left| -\frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} x d(F_n(x) - F(x)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) - L(F_n - F) \right| \\
&= \left| -\frac{1}{\epsilon} \int_{-\infty}^{q_\epsilon} q_\epsilon d(F_n(x) - F(x)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) \right| \\
&= \left| -\frac{q_\epsilon}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) + \frac{C_n}{\epsilon}(F_n(q_\epsilon) - F(q_\epsilon)) \right| \\
&= \frac{|C_n - q_\epsilon|}{\epsilon} |F_n(q_\epsilon) - F(q_\epsilon)| \\
&\leq \frac{|C_n - q_\epsilon|}{\epsilon} \sup_x |F_n(x) - F(x)| \\
&= \frac{|C_n - q_\epsilon|}{\epsilon} \rho(F_n, F)
\end{aligned}
$$

As a result,

$$
\frac{|\phi(F_n) - \phi(F) - L(F_n - F)|}{\rho(F_n, F)} \to 0, \text{ as } n \to \infty
$$

in almost sure sense. As a result we obtain the asymptotic relation in equation (6). □

Corollary 1. *Under the assumptions in the proposition,*

$$
|\phi(F_n) - \phi(F) - \Delta(F_n - F)| = o(n^{-1/2}). \tag{8}
$$

Proof. By the Kolmogorov theorem, the metric $\rho(F_n, F)$ approaches zero at a rate equal to $n^{-1/2}$ which indicates the rate of improvement of the linear approximation $\Delta(F_n - F)$. □

The main result is given in the theorem below. The idea is to use the linear approximation $\Delta(F_n - F)$ of the AVaR functional in order to obtain an asymptotic distribution as $n \to \infty$.

Theorem 2. *Suppose that X is random variable with c.d.f. $F(x)$ which satisfies the following conditions*

a) $x^\alpha F(-x) = L(x)$ *is slowly varying at infinity*

b) $\int_{-\infty}^{0} x dF(x) < \infty$

c) $F(x)$ *is differentiable at* $x = q_\epsilon$, *where* q_ϵ *is the* ϵ*-quantile of* X.

Then, there exist $c_n > 0$, $n = 1, 2 \ldots$, such that for any $0 < \epsilon < 1$,

$$c_n^{-1}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right) \xrightarrow{w} S_{\alpha^*}(1, 1, 0), \tag{9}$$

in which $\xrightarrow{w}$ denotes weak limit, $1 < \alpha^* = \min(\alpha, 2)$, and $c_n = n^{1/\alpha^*-1}L_0(n)/\epsilon$ where L_0 is slowly varying at infinity. Furthermore, the c_n are representable as $c_n = a_n/n\epsilon$ where a_n stands for the normalizing sequence in Theorem 1 and must satisfy the condition in equation (4).

Proof. By the result in Proposition 1,

$$\phi(F_n) - \phi(F) = \Delta(F_n - F) + o(n^{-1/2}) \tag{10}$$

where ϕ is the AVaR functional and $\Delta(F_n - F)$ is given in (6). Simplifying the expression for $\Delta(F_n - F)$, we obtain

$$\phi(F_n) - \phi(F) = \frac{1}{n\epsilon}\sum_{i=1}^{n}[(q_\epsilon - X_i)_+ - E(q_\epsilon - X_i)_+] + o(n^{-1/2}) \tag{11}$$

It remains to apply the domains of attraction characterization in Theorem 1 to the right-hand side of equation (11). For this purpose, consider the expression

$$\sum_{i=1}^{n} Y_i - nEY_1 \tag{12}$$

where $Y_i = (q_\epsilon - X_i)_+$ are i.i.d. random variables. Denote by $F_Y(x)$ the c.d.f. of Y. The left-tail behavior of X assumed in a) implies $x^\alpha(1 - F_Y(x)) = L(x)$ as $x \to \infty$ where $L(x)$ is the slowly varying function assumed in a). This is demonstrated by

$$\begin{aligned} x^\alpha(1 - F_Y(x)) &= x^\alpha P(\max(q_\epsilon - X, 0) > x) \\ &= x^\alpha P(X < q_\epsilon - x) \\ &\sim x^\alpha P(X < -x) \end{aligned} \tag{13}$$

Furthermore, the asymptotic behavior of the left tail of Y is $F_Y(-x) = 0$ which holds for any $x \geq -q_\epsilon$. As a result, condition (i) from Theorem 1 holds.

Condition b) implies that the tail exponent α in a) must satisfy the inequality $\alpha > 1$. Therefore, subtracting nEY_1 in (12) is a proper centering of the sum as suggested in (5) in Theorem 1. Note that if $\alpha \geq 2$, then Y is in the domain of attraction of the normal distribution and the same choice of centering is appropriate. Thus, the tail index of the limiting distribution satisfies $1 < \alpha^* = \min(\alpha, 2)$.

Finally, computing condition (ii) in Theorem 1 from the tail behavior of Y yields $\beta = 1$. Essentially, this follows because $F_Y(-x) = 0$ if $x \geq -q_\epsilon$.

Therefore, all conditions in Theorem 1 are satisfied and as, a result, there exists a sequence of normalizing constants a_n satisfying (4), such that

$$a_n^{-1}\left(\sum_{i=1}^{n} Y_i - nEY_1\right) \xrightarrow{w} S_{\alpha^*}(1,1,0). \tag{14}$$

as $n \to \infty$. In order to apply this result to sample AVaR, we need (14) reformulated for the average rather than the sum of Y_i. Thus, a more suitable form is

$$n\epsilon a_n^{-1}\left(\frac{1}{n\epsilon}\sum_{i=1}^{n}(Y_i - EY_i)\right) \xrightarrow{w} S_{\alpha^*}(1,1,0). \tag{15}$$

as $n \to \infty$.

As a final step, we apply the limit result in (15) to equation (11). Multiplying both sides of (11) by $n\epsilon a_n^{-1}$ yields the limit

$$n\epsilon a_n^{-1}(\phi(F_n) - \phi(F)) \xrightarrow{w} S_{\alpha^*}(1,1,0) \tag{16}$$

as $n \to \infty$. It remains only to verify if the normalization does not lead to explosion of the residual. Indeed,

$$n\epsilon a_n^{-1} o(n^{-1/2}) = \frac{n^{1/2}}{a_n} o(1) = o(1),$$

because the factor $n^{1/2}/a_n$ approaches zero by the asymptotic behavior of a_n given in the domains of attraction characterization in Theorem 1.

□

A number of comments are collected in the following remarks.

Remark 1. By definition, the AVaR is the negative of the average of the quantiles of X beyond a reference quantile q_ϵ. For this reason, it is only the behavior of the left tail of X which matters and the assumptions a) and b) in Theorem 2 concern the left tail only. Condition c) is technical and allows the calculation of the influence function of AVaR.

Remark 2. If $\alpha > 2$ in condition a), then $\int_{-\infty}^{0} x^2 dF(x) < \infty$ and the limiting distribution is the standard normal distribution. In this case, the normalizing sequence c_n should be calculated using $\sigma_\epsilon^2 = D(q_\epsilon - X)_+$,

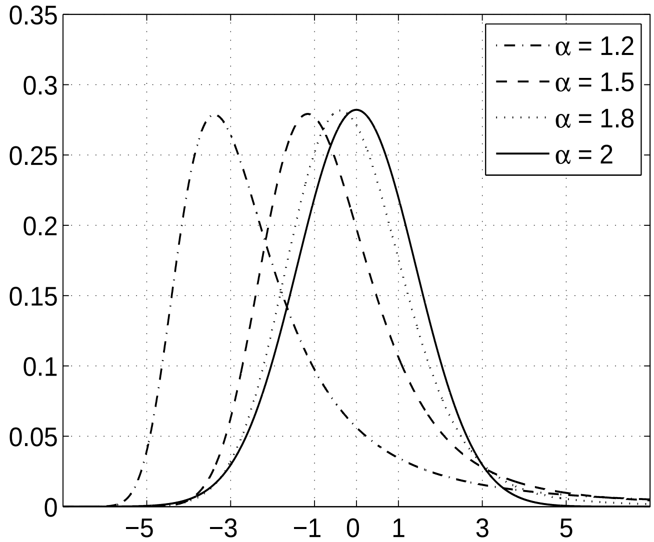

Figure 1: Densities of the limiting stable distribution corresponding to different tail behavior.

$$c_n = n^{-1/2}\sigma_\epsilon/\epsilon.$$

The case $\alpha > 2$ is considered in detail in Stoyanov and Rachev (2007).

Remark 3. The limiting stable distribution is totally skewed to the right, $\beta = 1$. However, the observed skewness in the shape of the distribution decreases as $\alpha \to 2$, see Figure 1. At the limit, when $\alpha = 2$, the limiting distribution is Gaussian and is symmetric irrespective of the value of β. Therefore, the degree of the observed skewness in the limiting distribution is essentially determined by the tail behavior of X, or by the value of α, and is not influenced by any other characteristic.

Remark 4. When $\epsilon \to 1$, then AVaR approaches the mean of X (or the sample average if we consider the sample AVaR),

$$\lim_{\epsilon \to 1} AVaR_\epsilon(X) = EX.$$

Unfortunately, there is no such continuity in equation (9) unless X has finite variance. That is, generally it is not true that the weak limit in equation (9) holds for the sample average letting $\epsilon \to 1$. The reason is that if $\epsilon = 1$, then both tails of the distribution of X matter and the limiting stable distribution can have any $\beta \in [-1, 1]$. The condition $DX < \infty$ is sufficient to guarantee that the limiting distribution is normal for any $\epsilon \in (0, 1]$ and in this case there is continuity in equation (9) as $\epsilon \to 1$.

As an illustration of the singularity at $\epsilon = 1$, consider the following example. Suppose that the right tail of X is heavier than the left tail and as a consequence,

$$\int_{-\infty}^{q_\epsilon} x^2 dF(x) < \infty, \quad \text{for any} \quad \epsilon < 1,$$

but $EX^2 = \infty$. Under this assumption, the limiting distribution of the sample AVaR is normal for any $\epsilon < 1$. If $\epsilon = 1$, then the limiting distribution becomes stable non-Gaussian due to the heavier right tail. Thus, there is a change in the limiting distribution of the sample AVaR with $\epsilon < 1$ and the sample average.

3 Examples

The result in Theorem 2 provides the limiting distribution but does not provide any insight on the rate of convergence. That is, it does not give an answer to the question how many observations are needed in order for the distribution of the left-had side in equation (9) to be sufficiently close to the distribution of the right-hand side in terms of a selected probability metric. In this section, we provide illustrations of the stable limit theorem and the rate of convergence assuming particular distributions of X.

3.1 Stable Paretian Distributions

We remarked that stable Paretian distributions are stable distributions with tail index $\alpha < 2$. This distinction is made since their properties are very different from the properties of the normal distribution which appears as a stable distribution with $\alpha = 2$. For example, in contrast to the normal distribution, stable Paretian distributions have heavy tails exhibiting power decay. In the field of finance, stable Paretian distribution were proposed as a model for stock returns and other financial variables, see Rachev and Mittnik (2000).

Denote by X the random variable describing the return of a given stock. In this section, we assume that $X \in S_\alpha(\sigma, \beta, \mu)$ with $1 < \alpha < 2$, $\beta \neq 1$, and our goal is to apply the result in Theorem 2 which provides a tool of computing the confidence interval of the sample AVaR of X on condition that the Monte Carlo method is used with a large number of scenarios. Since by

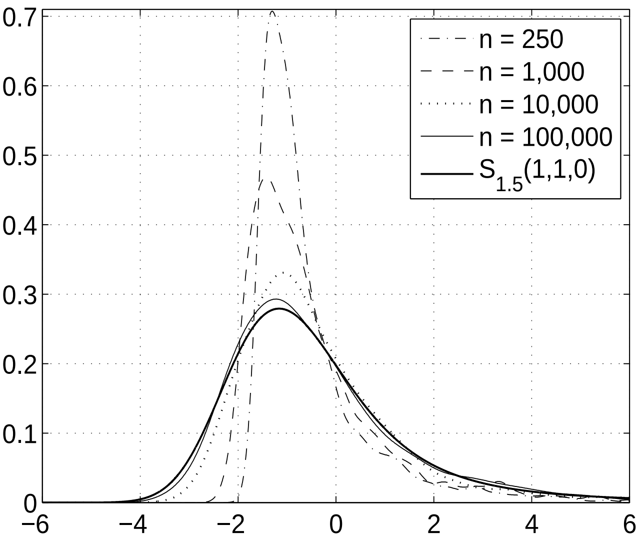

Figure 2: The density of the sample AVaR as n increases with $\beta = 0.7$ and $\epsilon = 0.01$.

assumption $\alpha > 1$, which guarantees convergence of the sample AVaR to the theoretical AVaR in almost sure sense. In the case of stable distributions, the quantity $AVaR_\epsilon(X)$ can be calculated using a semi-analytic expression given in Stoyanov et al. (2006).

In order to apply the result in Theorem 2, first we have to check if the conditions are satisfied and then choose the scaling constants c_n. For this purpose, we use the following property of stable Paretian distributions, see Samorodnitsky and Taqqu (1994).

Property 1. *Let* $X \in S_\alpha(\sigma, \beta, \mu)$ $\;0 < \alpha < 2$. *Then*

$$\lim_{\lambda\to\infty} \lambda^\alpha P(X > \lambda) = C_\alpha \frac{1+\beta}{2}\sigma^\alpha$$
$$\lim_{\lambda\to\infty} \lambda^\alpha P(X < -\lambda) = C_\alpha \frac{1-\beta}{2}\sigma^\alpha$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin(x)dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}, & \alpha \neq 1 \\ 2/\pi, & \alpha = 1 \end{cases}$$

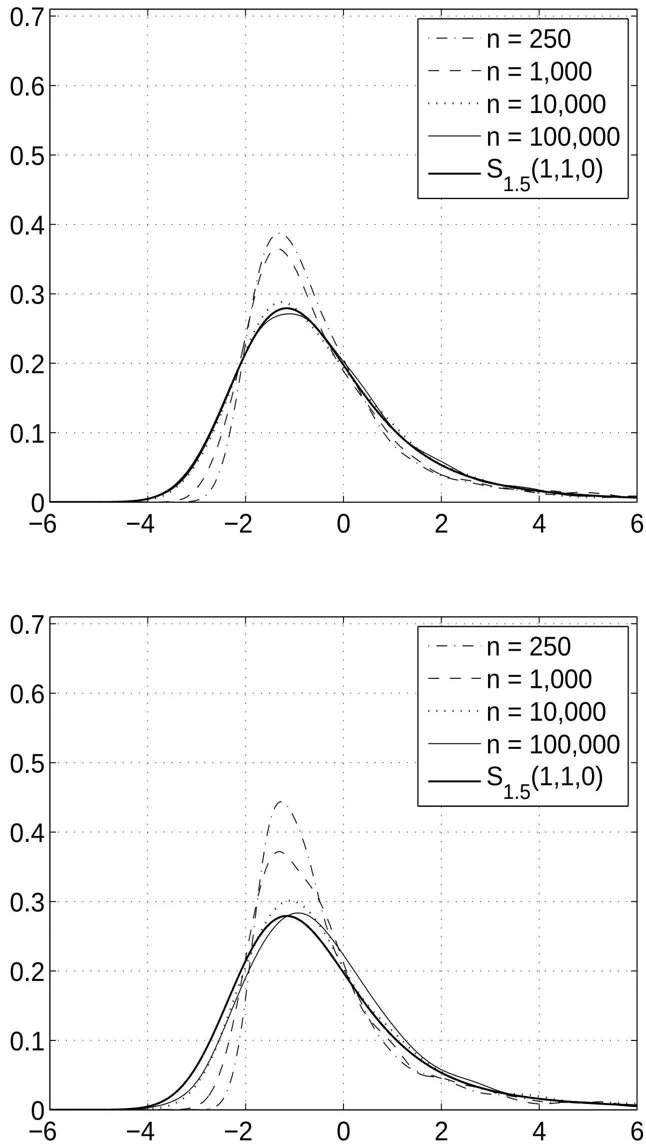

Figure 3: The density of the sample AVaR as n increases with $\beta = 0.7$ (top) and $\beta = -0.7$ (bottom) and $\epsilon = 0.05$.

This property provides the asymptotic behavior of the left tail of the distribution. We further assume that $\beta \neq 1$ since in this case the asymptotic behavior of the left tail is different, see Samorodnitsky and Taqqu (1994). Condition b) is satisfied because of the assumption $1 < \alpha < 2$ and, finally, condition c) is satisfied for any choice of $0 < \epsilon < 1$ since all stable distributions have densities. Therefore, all assumptions are satisfied and the result in Theorem 2 holds with $\alpha^* = \alpha$ and the scaling constants c_n should be chosen in the following way,

$$c_n = n^{1/\alpha - 1} \left(\frac{1-\beta}{2} \right)^{1/\alpha} \frac{\sigma}{\epsilon}.$$

Note that in this case, the skewness in the distribution of X translates into a different scaling of the normalizing constants. If X is negatively skewed ($\beta < 0$), the scaling factor is larger than if X is skewed positively ($\beta > 0$).

We carry out a Monte Carlo study assuming $X \in S_{1.5}(\beta, 1, 0)$ where $\beta = \pm 0.7$ and two choices of the tail probability $\epsilon = 0.01$ and $\epsilon = 0.05$. We generate 2,000 samples from the corresponding distribution the size of which equals $n = 250, 1,000, 10,000,$ and $100,000$.

Figure 2 illustrates the convergence rate for the case $\epsilon = 0.01$ as the number of observations increases. While from the plot it seems that $n = 100,000$ results in a density which is very close to that of the limiting distribution, but the Kolmogorov test fails. The convergence rate is much slower in the heavy-tailed case than in the setting of the classical CLT. Stoyanov and Rachev (2007) suggest that about 5,000 simulations are sufficient for the purposes of confidence bounds estimation when the distribution has bounded support. Apparently, much more observations are needed in this heavy-tailed case.

The plots in Figure 3 indicate that as the tail probability ϵ increases, the behavior of the sample AVaR distribution improves. Furthermore, the the behavior improves when X turns from being negatively to positively skewed.

3.2 Student's *t* distribution

Student's t distribution is a widely used model for a stock return distribution. X has Student's t distribution, $X \in t(\nu)$, with $\nu > 0$ degrees of freedom if the density of X equals,

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \ x \in \mathbb{R}.$$

A few simple properties of Student's t distribution are collected in the next proposition.

Proposition 2. *Suppose that $X \in t(\nu)$ and denote the c.d.f. of X by $F(x)$. Then, $x^\nu F(-x) = L(x)$ where $L(x)$ is a slowly varying function at infinity and also*

$$\lim_{x\to\infty} x^\nu F(-x) = \nu^{\nu/2-1}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)\sqrt{\pi}}. \tag{17}$$

Proof. The fact that $L(x)$ is a slowly varying function is checked directly applying the definition and the limit in (17) is obtained by applying l'Hospital's rule. $\square$

The result in this proposition and Theorem 2 imply that for $\nu > 2$, the limiting distribution of the sample AVaR is the Gaussian distribution. If $1 < \nu \leq 2$, then the limiting distribution is stable with $\alpha^* = \nu$. If $\nu \leq 1$, then the AVaR of X diverges. The scaling constants c_n should be chosen in a different way depending on the value of ν,

$$c_n = \begin{cases} n^{-1/2}\sigma_\epsilon/\epsilon, & \text{if } \nu > 2 \\ n^{1/\nu-1}A_\nu/\epsilon, & \text{if } 1 < \nu < 2 \end{cases} \tag{18}$$

where $\sigma_\epsilon^2 = D(q_\epsilon - X)_+$ and

$$A_\nu^\nu = \nu^{\nu/2-1}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu/2)\sqrt{\pi}}\frac{\Gamma(2-\nu)}{\nu-1}|\cos(\pi\nu/2)|.$$

The value of the constant A_ν is obtained by taking into account the limit in (17) and the condition in equation (4). Stoyanov and Rachev (2007) consider in detail the case $\nu > 2$ and provide the formula for σ_ϵ. This case is in the classical setting of the CLT as the variance of X is finite.

We carry out a Monte Carlo experiment in order to study the convergence rate of the sample AVaR distribution to the limiting distribution. We fix the degrees of freedom, the number of simulations to 100,000, and $\epsilon = 0.05$. Next we generate 2,000 samples from which the sample AVaR is estimated. Thus we obtain 2,000 estimates of $AVaR_\epsilon(X)$, $X \in t(\nu)$. Finally, we calculate the Kolmogorov distance

$$\rho(G_\nu, G) = \sup_x |G_\nu(x) - G(x)|$$

where G_ν is the c.d.f. of the sample AVaR approximated by the sample c.d.f. obtained with the 2,000 estimates, and G is the c.d.f. of the limiting distribution $S_{\alpha^*}(1, 1, 0)$ where $\alpha^* = \min(\nu, 2)$.

Figure 4 shows the values of $\rho(G_\nu, G)$ as ν varies from 1.05 to 3. The horizontal line shows the critical value of the Kolmogorov statistic: if the

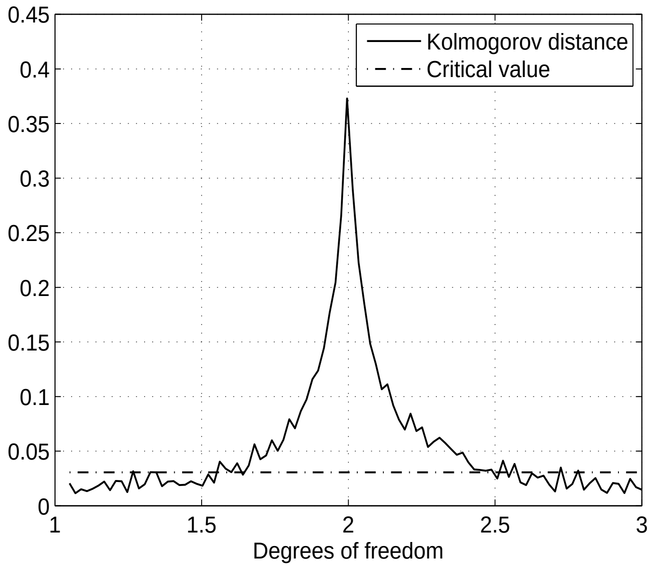

Figure 4: The Kolmogorov distance between the sample AVaR distribution of $X \in t(\nu)$ obtained with 100,000 simulations and the limiting distribution.

calculated $\rho(G_\nu, G)$ is below the critical value, we accept the hypothesis that the sample AVaR distribution is the same as the limiting distribution, otherwise we reject it. Since we use a sample c.d.f. to approximate $G_\nu(x)$, the solid line fluctuates a little but we notice that for $\nu \leq 1.5$ and $\nu \geq 2.5$ it seems that 100,000 observations are enough in order to accept the limiting distribution as a model. For the middle values, larger samples are needed. This observation indicates that the rate of convergence of the sample AVaR distribution to the limiting distribution deteriorates as ν approaches 2 and is slowest for $\nu = 2$. This finding can be summarized in the following way by considering all possible cases for ν:

- $\nu > 2$. As ν decreases from larger values to 2, the tail thickness increases which results in higher absolute moments becoming divergent, $E|X|^\delta = \infty$, $\delta \geq \nu$. The limiting distribution is the Gaussian distribution but the tails becoming thicker results in deterioration of the convergence rate to the Gaussian distribution.

- $\nu = 2$. The limiting distribution is the Gaussian distribution even though the variance of X is infinite. This case is not covered by the limit theory behind the classical CLT.

- $1 < \nu < 2$. We continue decreasing ν and the tails become so thick that they start influencing the limiting distribution which is stable Paretian, $S_\nu(1, 1, 0)$, and depends on ν. However, the convergence rate starts improving.

- $0 < \nu \leq 1$. The tails of X become so heavy that $AVaR_\epsilon(X) = \infty$.

4 Conclusion

In the paper, we study the asymptotic distribution of the sample AVaR. We provide a stable limit theorem describing all possible asymptotic laws depending on the behavior of the left tail of the random variable X. If we assume that X describes the return distribution of a stock, then the left tail describes losses. Intuitively, the asymptotic distribution of the sample AVaR is determined by the behavior of extreme losses.

Furthermore, in order to adopt the asymptotic law and draw conclusions based on it, we need insight on the rate of convergence in the stable limit theorem. We illustrate the rate of convergence by Monte Carlo experiments assuming a stable distribution and Student's t distribution for X. In summary, the convergence rate deteriorates as the tail exponent $\alpha \to 2$ and it improves as the distribution of X becomes more positively skewed. Generally, the skewness of X does not influence the asymptotic law.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.4,461-483,COPYRIGHT 2008 EUDOXUS PRESS, LLC

TRANSPORT PROCESSES IN NETWORKS WITH SCATTERING RAMIFICATION NODES

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Dedicated to Rainer Nagel on the occasion of his 65th birthday.

Abstract. We investigate the streaming of particles with different velocities in a network. In the vertices of the network the particles are scattered, i.e. they change their velocity but obey a Kirchhoff law. This situation will be formulated as an abstract Cauchy problem for an operator $(A, D(A))$ on a suitable Banach space X. Then the problem is studied using semigroup methods. The main emphasis is on the asymptotic behaviour.

keywords: transport processes, networks, semigroups, positivity, spectral theory

1. Statement of the problem

We consider a transport process with absorption and scattering as described by the classical linear Boltzmann equation, see [7], [11], [14]. As many authors before, e.g. [26], [27], [13], [28], [29], [18], we use the theory of strongly continuous operator semigroups, see [8], [10], [22], in particular the theory of positive semigroups on Banach lattices, see [20] to show wellposedness and to discuss the asymptotic behaviour of the solutions. However, while the problem is usually considered on a domain in $\mathbb{R}^n$, we study the transport process in a network. This seems to be physically relevant, and it is mathematically interesting to discuss how the network structure influences the process. Moreover, we assume that absorption and scattering takes place only in the ramification nodes of the network and that a Kirchhoff law holds in each node. As predecessors we mention papers studying transport equations in slab geometry as e.g. [4], [5], [6] and [16]. Closer to our setting is [2] who concentrates on the wellposedness of a similar problem and discusses some applications to physics. Our paper is mainly inspired by [15] and [17]. These authors assume that all particles move with the same speed in the network. However, in doing so they developed the semigroup techniques we will use.

Our network is represented by a simple, directed and weighted graph $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of vertices (or nodes) and $E = \{e_1, \ldots, e_m\}$ is the set of edges (or arcs). If two vertices are connected by an edge, then the particles can move between the vertices in the direction given by the edge. The velocity of each particle is constant during its motion along an edge, however, for different particles this velocity can vary between a minimal speed $v_{min} > 0$ and a maximal speed $v_{max} > v_{min}$. By the assumption on the minimal speed, each particle will reach a vertex after a finite time. In these vertices the particles are scattered, i.e. they change their velocity, or will be absorbed. Thereafter, they are distributed to the outgoing edges of the vertex according to the (positive) weight of the outgoing edge. We consider only the case that each vertex has at least one incoming and one outgoing edge.

This physical situation will now be modelled in mathematical terms. The edges $e_j, j = 1, \dots, m$, are parameterised over the intervals $[0, l_j]$ where $e_j(0)$ is the tail of the edge e_j and $e_j(l_j)$ is the head of the edge e_j: $e_j(0) \xrightarrow{e_j} e_j(l_j)$.

If edge e_j is an outgoing edge of vertex v_i, then ω_{ij} gives the weight of the edge e_j. In each vertex v_i the weights of the outgoing edges shall sum up to 1, i.e.

$$\sum_{j=1}^{m} \omega_{ij} = 1, \tag{1}$$

for each $i \in \{1, \dots, n\}$.

Our transport process is then described by the equations

$$(F) \begin{cases} \dfrac{\partial}{\partial t} u_j(x,v,t) = -v \dfrac{\partial}{\partial x} u_j(x,v,t), \quad x \in (0,l_j),\ v \in [v_{min}, v_{max}],\ t \geq 0, \\ \quad u_j(x,v,0) = f_j(x,v), \quad x \in (0,l_j),\ v \in [v_{min}, v_{max}], \qquad (IC) \\ \phi^-_{ij} u_j(0,\cdot,t) = \omega_{ij} J \displaystyle\sum_{k=1}^{m} \phi^+_{ik} u_k(l_k,\cdot,t), \quad t \geq 0, \qquad (BC) \end{cases}$$

where $j = 1, \dots, m,\ i = 1, \dots, n$.

Here, u_j gives the density of the particles on edge e_j depending on the position x, the velocity v and the time t. The first equation is the well-known one-dimensional transport equation without scattering and absorption effects, while (IC) is the usual initial condition for $t = 0$. The equation (BC) is a condition in the vertices of the graph and models the scattering, absorption, and redistribution of particles in the vertices. The operator J appearing in (BC) is called *scattering operator*. It converts, in each vertex v_i, the incoming velocity profile $\sum_{k=1}^{m} \phi^+_{ik} u_k(l_k,\cdot,t)$ into an outgoing velocity profile. Then the ω^{th}_{ij} part of this velocity profile is leaving vertex v_i into edge e_j. For the scattering operator J we assume the following.

General Assumption 1.1. The operator J is a positive contraction from $Y := L^1[v_{min}, v_{max}]$ to Y.

Since $\|f\|_1 = \int_{v_{min}}^{v_{max}} f(v)\, dv$ gives the total number of the particles for $f \in Y_+$, where

$$Y_+ = \{f \in Y : f(v) \geq 0 \text{ for almost all } v \in [v_{min}, v_{max}]\},$$

this assumption means that no particles can enter the system.

The properties of J will play an important role for the asymptotics of the process and we will later make additional assumptions on J, see Sections 3, 5 and 6, with interesting consequences on the spectral properties and the asymptotic behaviour of the corresponding semigroup.

The coefficients ϕ^-_{ij} and ϕ^+_{ik} in (BC) arise from matrices coding the structure of the graph and are defined below. In this way, the equations in (BC) relate the one-dimensional particle transport to the underlying network.

To describe the graph we use the following matrices, see also [15].

Definition 1.2. (1) The *outgoing incidence matrix* $\Phi^- = (\phi^-_{ij})_{n\times m}$ is defined by

$$\phi^-_{ij} := \begin{cases} 1, & v_i = e_j(0), \text{ i.e. } v_i \xrightarrow{e_j}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The *weighted outgoing incidence matrix* $\Phi^-_w = (\phi^-_{ij,w})_{n\times m}$ is defined by

$$\phi^-_{ij,w} := \begin{cases} \omega_{ij}, & v_i = e_j(0), \text{ i.e. } v_i \xrightarrow[\omega_{ij}]{e_j}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) The *incoming incidence matrix* $\Phi^+ = (\phi^+_{ij})_{n\times m}$ is defined by

$$\phi^+_{ij} := \begin{cases} 1, & v_i = e_j(l_j), \text{ i.e. } \xrightarrow{e_j} v_i, \\ 0, & \text{otherwise.} \end{cases}$$

(4) The *weighted transposed adjacency matrix* $\mathbb{A} = (\alpha_{ij})_{n\times n}$ is defined by $\mathbb{A} := \Phi^+(\Phi^-_w)^T$, i.e.

$$\alpha_{ij} = \begin{cases} \omega_{jk}, & \text{if } v_j = e_k(0) \text{ and } v_i = e_k(l_k), \text{ i.e. } v_j \xrightarrow{e_k} v_i, \\ 0, & \text{otherwise.} \end{cases}$$

(5) The *weighted transposed adjacency matrix* $\mathbb{B} = (\beta_{ij})_{m\times m}$ of the line graph is defined by $\mathbb{B} := (\Phi^-_w)^T\Phi^+$, i.e.

$$\beta_{ij} = \begin{cases} \omega_{ki}, & \text{if } e_i(0) = e_j(l_j) = v_k, \text{ i.e. } \xrightarrow{e_j} v_k \xrightarrow[\omega_{ki}]{e_i}, \\ 0, & \text{otherwise.} \end{cases}$$

These matrices determine the structure of the graph completely, see [3] and [9]. However, we need the following operator version of the above defined (scalar) matrices.

Definition 1.3. Let Id_Y denote the identity operator on Y. We introduce the following operator matrices.

(1) $\tilde{\Phi}^- := (\phi^-_{ij} Id_Y)_{n\times m}$,
(2) $\tilde{\Phi}^-_w := (\phi^-_{ij,w} Id_Y)_{n\times m}$,
(3) $\tilde{\Phi}^+ := (\phi^+_{ij} Id_Y)_{n\times m}$,
(4) $\Phi^+_J := (\phi^+_{ij} J)_{n\times m}$,
(5) $\tilde{\mathbb{A}} := (\alpha_{ij} Id_Y)_{n\times n}$,
(6) $\tilde{\mathbb{B}} := (\beta_{ij} Id_Y)_{m\times m}$,
(7) $\mathbb{B}_J := (\beta_{ij} J)_{m\times m}$.

These operator matrices define operators in the canonical way on products of the space Y.

We close this section with a useful observation, see [15, Sect. 2].

Remark 1.4. In each column of Φ^-, Φ^-_w, $\tilde{\Phi}^-$, $\tilde{\Phi}^-_w$, Φ^+, $\tilde{\Phi}^+$ and Φ^+_J there is exactly one non-zero entry. Furthermore, an easy computation using the condition (1) yields

$$\Phi^-(\Phi^-_w)^T = Id_{\mathbb{C}^n}$$

and

$$\tilde{\Phi}^-(\tilde{\Phi}^-_w)^T = Id_{Y^n}.$$

Moreover, it follows that $\mathbb{A}$ and $\mathbb{B}$ are column stochastic matrices.

2. Equation (F) as an abstract Cauchy problem

Since we want to treat the problem (F) with semigroup methods, we rewrite it as an abstract Cauchy problem on a suitable state space X. As the state space for our problem we choose

$$X := L^1([0,l_1],Y) \times \cdots \times L^1([0,l_m],Y)$$

which is isomorphic to

$$L^1([0,l_1] \times [v_{min}, v_{max}]) \times \cdots \times L^1([0,l_m] \times [v_{min}, v_{max}]).$$

If all arc lengths are equal to l, then

$$X \cong L^1([0,l],Y^m) \cong (L^1([0,l],Y))^m \cong (L^1([0,l] \times [v_{min}, v_{max}]))^m.$$

The space X is endowed with the norm

$$\|\cdot\|_1 : X \to \mathbb{R}, \qquad \|u\|_1 := \sum_{j=1}^m \int_0^{l_j} \int_{v_{min}}^{v_{max}} |u_j(x,v)|\, dv\, dx,$$

where $u = (u_j)_{1 \le j \le m} \in X$. In the spirit of [15] we choose an abstract "boundary space" as

$$\partial X := Y^n,$$

endowed with the norm

$$\|\cdot\|_1 : \partial X \to \mathbb{R}, \qquad \|f\|_1 := \sum_{i=1}^n \int_{v_{min}}^{v_{max}} |f_i(v)|\, dv,$$

where $f = (f_i)_{1 \le i \le n} \in \partial X$.

Furthermore, we define

$$W := W^{1,1}([0,l_1],Y) \times \cdots \times W^{1,1}([0,l_m],Y)$$

which is a Banach space for the norm

$$\|\cdot\|_W : W \to \mathbb{R}, \qquad u \mapsto \|u\|_W := \|u\|_1 + \|\tfrac{\partial}{\partial x} u\|_1.$$

The trace operators

$$\Gamma_0, \Gamma_l : W \to Y^m$$

are defined by

$$\Gamma_0 u := (u_j(0))_{1 \le j \le m},$$

and

$$\Gamma_l u := (u_j(l_j))_{1 \le j \le m},$$

respectively, where $u = (u_j)_{1 \le j \le m} \in W$, and give the velocity profiles at the endpoints of the edges. Both operators are continuous on $(W, \|\cdot\|_W)$.

To formulate (F) as an abstract Cauchy problem we proceed as in [15], and start from the following "maximal" operator on X.

Definition 2.1. The operator $(A_w, D(A_w))$ is defined by

$$D(A_w) := \{u \in W : \Gamma_0 u \in \operatorname{rg}(\tilde{\Phi}_w^-)^T\},$$

$$(A_w u)_j(x,v) := -v \tfrac{\partial}{\partial x} u_j(x,v), \quad x \in [0,l_j],\ v \in [v_{min}, v_{max}],\ j = 1, \dots, m.$$

The condition $\Gamma_0 u \in \operatorname{rg}(\tilde{\Phi}_w^-)^T$ means that the proportion of the mass leaving vertex v_i over edge e_j is determined by the weight ω_{ij}. However, this does not contain the complete boundary condition (BC) from (F). To formulate a condition equivalent to (BC) we introduce the following continuous operators on $(W, \|\cdot\|_W)$.

Definition 2.2. The *outgoing boundary operator* L is defined by

$$L : W \to \partial X, \quad u \mapsto \tilde{\Phi}^{-}\Gamma_0 u,$$

while for the *incoming boundary operator* M_J we take

$$M_J : W \to \partial X, \quad u \mapsto \Phi_J^{+}\Gamma_l u.$$

Note that $(\tilde{\Phi}^{+}\Gamma_l u)_i$ gives the velocity profile coming into vertex v_i. Then, $(M_J u)_i$ gives the velocity profile in vertex v_i after the scattering and $(Lu)_i$ gives the velocity profile leaving vertex v_i. Thus, the condition

$$Lu = M_J u \tag{2}$$

expresses the Kirchhoff law.

The operator corresponding to our original problem (F) is now given as follows.

Definition 2.3. The operator $(A, D(A))$ is defined by

$$\begin{aligned} D(A) &:= \{u \in D(A_w) : Lu = M_J u\}, \\ Au &:= A_w u. \end{aligned}$$

To show the equivalence fix t in (BC). Then $(u_j(0,\cdot,t))_{1\le j\le m} \in \operatorname{rg}(\tilde{\Phi}_w^{-})^T$. Taking the sum over j in (BC) yields the Kirchhoff law.

On the other hand, let us require that $Lv = M_J v$ and $\Gamma_0 v \in \operatorname{rg}(\tilde{\Phi}_w^{-})^T$ for $v \in D(A_w)$. Then there exists $d = (d_i)_{1\le i\le n} \in Y^n$ such that $\Gamma_0 v = (\tilde{\Phi}_w^{-})^T d$. Since in each row of $(\tilde{\Phi}_w^{-})^T$ there is exactly one non-zero entry, it follows from the condition $\Gamma_0 v \in \operatorname{rg}(\tilde{\Phi}_w^{-})^T$ that for every $j \in \{1,\dots,m\}$ there exists exactly one $i \in \{1,\dots,n\}$ such that

$$v_j(0,\cdot) = \omega_{ij} d_i. \tag{3}$$

With that we compute for $i = 1,\dots,n$

$$J\sum_{j=1}^{m} \phi_{ij}^{+} v_j(l_j,\cdot) \overset{M_J v = Lv}{=} \sum_{j=1}^{m} \phi_{ij}^{-} v_j(0,\cdot) \overset{(3)}{=} \sum_{j=1}^{m} \phi_{ij}^{-}\omega_{ij} d_i = \sum_{j=1}^{m} \omega_{ij} d_i \overset{(1)}{=} d_i. \tag{4}$$

Combining (4) and (3) yields

$$v_j(0,\cdot) = \omega_{ij} J \sum_{i=1}^{m} \phi_{ij}^{+} v_j(l_j,\cdot).$$

If we multiply both sides by ϕ_{ij}^{-} and remember that $\omega_{ij} \neq 0$ if and only if $\phi_{ij}^{-} \neq 0$, we see that (BC) is fulfilled.

Thus, (F) can equivalently be formulated as the abstract Cauchy problem

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), & t \ge 0, \\ u(0) = u_0, \end{cases}$$

for the operator $(A, D(A))$ in the Banach space X and the initial value $u_0 = (f_j)_{1\le j\le m}$.

Proposition 2.4. *The operator $(A, D(A))$ is closed and densely defined.*

Proof. Consider the norm $\|\cdot\|_{\mathcal{G}}$ on W given by

$$\|\cdot\|_{\mathcal{G}} : W \to \mathbb{R}, \quad u \mapsto \|u\|_{\mathcal{G}} := \|u\|_1 + \sum_{j=1}^{m} \int_0^{l_j} \int_{v_{min}}^{v_{max}} v \left|\tfrac{\partial}{\partial x} u_j(x,v)\right| \, dv \, dx.$$

Since $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{G}}$ are equivalent, also $(W, \|\cdot\|_{\mathcal{G}})$ is a Banach space.

To prove the closedness of A we have to show that $(D(A), \|\cdot\|_{\mathcal{G}})$ is a Banach space. Therefore, suppose that $(u^{(n)})_{n\in\mathbb{N}} \subseteq D(A)$ is a Cauchy sequence converging to $u \in W$. Then for all $n \in \mathbb{N}$ there exists an $f_n \in \partial X$ such that

$$\Gamma_0 u^{(n)} = (\tilde{\Phi}_w^-)^T f_n.$$

By the continuity of Γ_0 on $(W, \|\cdot\|_{\mathcal{G}})$ it follows that

$$f_n = \tilde{\Phi}^-(\tilde{\Phi}_w^-)^T f_n = \tilde{\Phi}^-\Gamma_0 u^{(n)} \longrightarrow \tilde{\Phi}^-\Gamma_0 u =: f.$$

Hence,

$$\Gamma_0 u = \lim_{n\to\infty} \Gamma_0 u^{(n)} = \lim_{n\to\infty} (\tilde{\Phi}_w^-)^T f_n = (\tilde{\Phi}_w^-)^T f,$$

i.e.

$$\Gamma_0 u \in \operatorname{rg}(\tilde{\Phi}_w^-)^T.$$

Since L and M_J are continuous operators on $(W, \|\cdot\|_{\mathcal{G}})$, also the condition $Lu = M_J u$ is fulfilled and therefore $u \in D(A)$.

An easy computation shows that the set

$$K := \{u \in W : \Gamma_0 u = \Gamma_l u = 0\}$$

is dense in W with respect to the norm $\|\cdot\|_1$. Since $K \subseteq D(A) \subseteq W \subseteq X$ and W is dense in X, also $D(A)$ is dense in X. □

This is the basis to prove the generator property of A in Section 4. Before doing so we investigate its spectral properties.

3. Spectral properties

In this section we apply the method from [15] to determine the spectrum $\sigma(A)$ of A.

To do so we try to characterise $\sigma(A)$ by a characteristic equation in the boundary space ∂X. We use operator matrix techniques developed by R. Nagel and A. Rhandi, see [19] and [23] and refer to [15] where this has been done for a finite dimensional boundary space ∂X.

We start with the decomposition of $D(A_w)$ as in [12] for which it is essential that $L|_{D(A_w)}$ is surjective.

Proposition 3.1. *The operator L is surjective from $D(A_w)$ to ∂X.*

Proof. Let $f \in \partial X$. Then $g = (g_j)_{1\le j\le m} := (\tilde{\Phi}_w^-)^T f \in Y^m$. Now, consider the element $u = (u_j)_{1\le j\le m} \in X$ where

$$u_j : [0, l_j] \to Y, \quad x \mapsto g_j$$

is a constant function for $1 \le j \le m$. Clearly, we have $u \in D(A_w)$. Applying L to u yields

$$Lu = \tilde{\Phi}^-\Gamma_0 u = \tilde{\Phi}^-(u_j(0))_{1\le j\le m} = \tilde{\Phi}^- g = \tilde{\Phi}^-(\tilde{\Phi}_w^-)^T f = f.$$

□

Next, we consider the operator A_w with homogeneous boundary conditions.

Definition 3.2. The operator $(A_0, D(A_0))$ is defined by

$$\begin{aligned} D(A_0) &:= \{u \in D(A_w) : Lu = 0\}, \\ A_0 u &:= A_w u. \end{aligned}$$

Lemma 3.3. *The domain $D(A_0)$ of A_0 coincides with $K := \{u \in W : \Gamma_0 u = 0\}$.*

Proof. The inclusion $K \subseteq D(A_0)$ is clear.

To show the other inclusion, suppose that $u \in D(A_0)$. Then, by the condition $\Gamma_0 u \in \operatorname{rg}(\tilde{\Phi}_w^-)^T$, there exists $f \in \partial X$ such that

$$\Gamma_0 u = (\tilde{\Phi}_w^-)^T f.$$

Therefore, and since $Lu = 0$, we obtain

$$0 = Lu = \tilde{\Phi}^- \Gamma_0 u = \tilde{\Phi}^- (\tilde{\Phi}_w^-)^T f = f,$$

hence

$$\Gamma_0 u = (\tilde{\Phi}_w^-)^T f = (\tilde{\Phi}_w^-)^T 0 = 0.$$

□

Hence, it is clear that A_0 can be written as an $m \times m$ operator matrix whose entries in the off-diagonal are 0 and with the same operator in each entry in the diagonal. Its domain is given by the product of the domain of the operator in the diagonal. Each of the diagonal entries is the generator of a strongly continuous semigroup, see [24, Sect. 3.1], and the semigroup $(T_0(t))_{t\geq 0}$ generated by A_0 is just the direct sum of these semigroups. More precisely, it is given by

$$(T_0(t)u)_j(x,v) := \chi_j(x,v,t) u_j(x - vt, v),$$

where

$$\chi_j(x,v,t) := \begin{cases} 1, & \text{if } 0 \leq x - vt \leq l_j, \\ 0, & \text{otherwise,} \end{cases}$$

$j = 1, \ldots, m$. Similarly, the resolvent of A_0 is obtained as

$$(R(\lambda, A_0)u)_j(x,v) = \int_0^x \tfrac{1}{v} e^{-\lambda \frac{x-r}{v}} u_j(r,v)\, dr,$$

$j = 1, \ldots, m$. From this representation one can easily see that $T_0(t)$ and $R(\lambda, A_0)$ are positive for $t \geq 0$ and $\lambda \in \mathbb{R}$, respectively. It is also clear that the semigroup $(T_0(t))_{t\geq 0}$ is nilpotent. This implies that the spectrum of A_0 is empty. Hence, by [12, Lemma 1.2], we can decompose the domain of A_w for any $\lambda \in \mathbb{C}$ as

$$D(A_w) = D(A_0) \oplus \ker(\lambda - A_w). \tag{5}$$

By Prop. 3.1 the operator L is surjective. Therefore, the restriction of L to $\ker(\lambda - A_w)$ is bijective. By the open mapping theorem, its inverse D_λ is bounded for every $\lambda \in \mathbb{C}$. Before we give the explicit form of D_λ we first introduce the following notation.

Definition 3.4. The operator $\epsilon_\lambda \in \mathcal{L}(Y^m, X), \lambda \in \mathbb{C}$, is defined by

$$\epsilon_\lambda : Y^m \to X, \quad (\epsilon_\lambda f)_j(x,v) := e^{-\frac{\lambda}{v}x} f_j(v),$$

where $f = (f_j)_{1\leq j\leq m} \in Y^m, \quad x \in [0, l_j], \quad v \in [v_{min}, v_{max}]$.

We now define an operator which turns out to be the inverse of $L|_{\ker(\lambda - A_w)}$.

Definition 3.5. For $\lambda \in \mathbb{C}$ the operator

$$D_\lambda : \partial X \to \ker(\lambda - A_w)$$

is defined by

$$f \mapsto D_\lambda f := \epsilon_\lambda (\tilde{\Phi}_w^-)^T f.$$

It is clear that D_λ maps to $\ker(\lambda - A_w)$. So it suffices to check that D_λ is the inverse of $L|_{\ker(\lambda - A_w)}$.

Proposition 3.6. *For $\lambda \in \mathbb{C}$ we have*

$$LD_\lambda = Id_{\partial X} \tag{6}$$

and

$$D_\lambda L = Id_{\ker(\lambda - A_w)}. \tag{7}$$

Proof. Let $f \in \partial X$ and recall that $\tilde{\Phi}^-(\tilde{\Phi}_w^-)^T = Id_{\partial X}$, see Remark 1.4. Thus,

$$LD_\lambda f = \tilde{\Phi}^- \Gamma_0 \epsilon_\lambda \tilde{\Phi}_w^- f = \tilde{\Phi}^-(\tilde{\Phi}_w^-)^T f = f,$$

and (6) is satisfied. To show (7) take an element $u = (u_j)_{1 \le j \le m} \in \ker(\lambda - A_w)$. The functions $w = (w_j)_{1 \le j \le m} \in W$ of the form

$$w_j(x, v) = f_j(v) e^{-\frac{\lambda}{v} x},$$

where $x \in [0, l_j], v \in [v_{min}, v_{max}], f_j \in Y$, and $\Gamma_0 w \in \mathrm{rg}(\tilde{\Phi}_w^-)^T$ compose the kernel of $\lambda - A_w$. Therefore, there exists $d \in \partial X$ such that $\Gamma_0 u = (\tilde{\Phi}_w^-)^T d$. Thus, u can be written as $u = \epsilon_\lambda (\tilde{\Phi}_w^-)^T d$. Hence,

$$D_\lambda L u = \epsilon_\lambda (\tilde{\Phi}_w^-)^T \tilde{\Phi}^- \Gamma_0 u = \epsilon_\lambda (\tilde{\Phi}_w^-)^T \tilde{\Phi}^- (\tilde{\Phi}_w^-)^T d = \epsilon_\lambda (\tilde{\Phi}_w^-)^T d = u.$$

□

To prove a characteristic equation for the spectrum of A we work on the product space $X \times \partial X$ and extend the given operators, see also [15, Sect. 3].

Definition 3.7. (1) $\mathcal{X} := X \times \partial X$.

(2) $\mathcal{A}_0 := \begin{pmatrix} A_w & 0 \\ -L & 0 \end{pmatrix}, \quad D(\mathcal{A}_0) := D(A_w) \times \{0\}^n.$

(3) $\mathcal{X}_0 := X \times \{0\}^n = \overline{D(A_w) \times \{0\}^n} = \overline{D(\mathcal{A}_0)}$.

(4) $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ M_J & 0 \end{pmatrix}, \quad D(\mathcal{B}) := W \times \partial X.$

(5) $\mathcal{A} := \mathcal{A}_0 + \mathcal{B} = \begin{pmatrix} A_w & 0 \\ M_J - L & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A_w) \times \{0\}^n.$

Remark 3.8. (1) An easy computation shows that the resolvent of $\mathcal{A}_0$ is given by

$$R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, A_0) & D_\lambda \\ 0 & 0 \end{pmatrix},$$

for each $\lambda \in \mathbb{C}$.

(2) The part $\mathcal{A}|_{\mathcal{X}_0}$ of $\mathcal{A}$ in $\mathcal{X}_0$ is given by

$$D(\mathcal{A}|_{\mathcal{X}_0}) = D(A) \times \{0\}^n, \quad \mathcal{A}|_{\mathcal{X}_0} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{A}|_{\mathcal{X}_0}$ can be identified with the operator $(A, D(A))$.

The next proposition shows that the spectrum of A is characterised by the spectrum of an operator on the boundary space ∂X. Moreover, an explicit form of the resolvent of A is given.

Proposition 3.9. *For $\lambda \in \mathbb{C}$ we have*

$$\lambda \in \sigma(\mathcal{A}) \quad \Longleftrightarrow \quad \lambda \in \sigma(A) \quad \Longleftrightarrow \quad 1 \in \sigma(M_J D_\lambda). \tag{CE}$$

For $\lambda \in \rho(A) = \rho(\mathcal{A})$ the resolvent operators are

$$R(\lambda, A) = (Id_X + D_\lambda(Id_{\partial X} - M_J D_\lambda)^{-1} M_J) R(\lambda, A_0),$$

and

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & D_\lambda(Id - M_J D_\lambda)^{-1} \\ 0 & 0 \end{pmatrix},$$

respectively.

Proof. To show the equivalence

$$\lambda \in \sigma(\mathcal{A}) \quad \Leftrightarrow \quad 1 \in \sigma(M_J D_\lambda), \tag{8}$$

we proceed as [15, Prop. 3.3]. First, we decompose

$$\lambda - \mathcal{A} = \lambda - \mathcal{A}_0 - \mathcal{B} = (\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))(\lambda - \mathcal{A}_0).$$

Note that $\rho(\mathcal{A}_0) = \mathbb{C}$. From this we see that $\lambda - \mathcal{A}$ is invertible if and only if $\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0)$ is invertible. Since

$$\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0) = \begin{pmatrix} Id_X & 0 \\ -M_J R(\lambda, A_0) & Id_{\partial X} - M_J D_\lambda \end{pmatrix},$$

one can easily see that the invertibility of $\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0)$ is equivalent to $1 \notin \sigma(M_J D_\lambda)$ and (8) is shown. The inverse is then given by

$$(\mathcal{I} - \mathcal{B}R(\lambda, \mathcal{A}_0))^{-1} = \begin{pmatrix} Id_X & 0 \\ (Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0) & (Id_{\partial X} - M_J D_\lambda)^{-1} \end{pmatrix}$$

and the resolvent of $\mathcal{A}$ is

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda) & D_\lambda(Id_{\partial X} - M_J D_\lambda)^{-1} \\ 0 & 0 \end{pmatrix},$$

where $R(\lambda) = (Id_Y + D_\lambda(Id_{\partial X} - M_J D_\lambda)^{-1} M_J) R(\lambda, A_0)$.

If $\lambda > 0$, then by Proposition 3.10 (1) $\|M_J D_\lambda\| < 1$ and therefore $1 \notin \sigma(M_J D_\lambda)$. Hence, the resolvent set of $\mathcal{A}$ is non-empty. Furthermore, A can be identified with the part $\mathcal{A}|_{\mathcal{X}_0}$ of $\mathcal{A}$ in $\mathcal{X}_0$. So we can apply [8, Prop. IV.2.17] to prove that $\sigma(\mathcal{A}) = \sigma(A)$.

Since

$$\left(\begin{smallmatrix} R(\lambda) & 0 \\ 0 & 0 \end{smallmatrix}\right) = R(\lambda, \mathcal{A})|_{\mathcal{X}_0} = R(\lambda, \mathcal{A}|_{\mathcal{X}_0}),$$

it follows that

$$R(\lambda, A) = R(\lambda)$$

for $\lambda \in \rho(A)$.

□

The condition $1 \in \sigma(M_J D_\lambda)$ will be called "characteristic equation". Indeed, it is a condition in ∂X, hence in a space much smaller than the state space X. To use it we compute $M_J D_\lambda$ as

$$M_J D_\lambda = \Phi_J^+ \begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l_m}} \end{pmatrix} (\tilde{\Phi}_w^-)^T.$$

Here, and in the following Q_g denotes the multiplication by a function $g \in L^\infty[v_{min}, v_{max}]$, i.e.

$$Q_g : Y \to Y, \quad f \mapsto Q_g f := gf.$$

This form of $M_J D_\lambda$ (and Proposition 3.9) immediately allows the following conclusions.

Proposition 3.10. (1) *Let $\lambda \in \mathbb{C}$. If $\Re\lambda > 0$ then $\|M_J D_\lambda\| < 1$. Thus, the spectral bound of A satisfies* $\mathrm{s}(A) \le 0$.

(2) *If $\|Jf\|_1 = \|f\|_1$ holds for all $f \ge 0$, then* $\mathrm{s}(A) = 0$.

(3) *The resolvent fulfills $R(\lambda, A) \ge 0$ for all $\lambda > 0$.*

Proof. (1), (2) First, using that J is a contraction, we estimate the norm of $M_J D_\lambda$ as

$$\begin{aligned} \|M_J D_\lambda\| &= \|\Phi_J^+ \begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l_m}} \end{pmatrix} (\tilde{\Phi}_w^-)^T\| \\ &\le \|\Phi_J^+\| \left\| \begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l_m}} \end{pmatrix} \right\| \|(\tilde{\Phi}_w^-)^T\| \\ &= \|J\| \max_{1 \le j \le m} \|Q_{e^{-\frac{\lambda}{\cdot} l_j}}\| \le \max_{1 \le j \le m} \|Q_{e^{-\frac{\lambda}{\cdot} l_j}}\|. \end{aligned}$$

Suppose now that $\Re\lambda > 0$. Then

$$\|M_J D_\lambda\| \le e^{-\frac{\Re\lambda}{v_{max}} \min_{1 \le j \le m} l_j} < 1,$$

and therefore $1 \notin \sigma(M_J D_\lambda)$ which is equivalent to $\lambda \notin \sigma(A)$ by (CE). Moreover, if $\lambda = 0$ then

$$\begin{aligned} \sigma(M_J D_0) &= \sigma(\Phi_J^+ \begin{pmatrix} Q_{e^{-\frac{0}{\cdot} l_1}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{0}{\cdot} l_m}} \end{pmatrix} (\tilde{\Phi}_w^-)^T) \\ &= \sigma(\Phi_J^+ (\tilde{\Phi}_w^-)^T) = \sigma((\alpha_{ij} J)_{n \times n}), \end{aligned}$$

where $\mathbb{A} = (\alpha_{ij})_{n \times n}$. By [21, Sect. 4], this can be further decomposed into

$$\sigma(M_J D_0) = \sigma(\mathbb{A})\sigma(J).$$

By the assumption in (2) on J, we have that $\mathrm{r}(J) = 1$ and from the positivity of J we know that $\mathrm{r}(J) \in \sigma(J)$, see [25, Prop. V.4.1]. Since $\mathbb{A}$ is a column stochastic matrix, $1 \in \sigma(\mathbb{A})$ and again by (CE) it follows that $0 \in \sigma(A)$. So we conclude that $\mathrm{s}(A) = 0$.

(3) If $\lambda > 0$ then $R(\lambda, A_0)$, D_λ, M_J, and $M_J D_\lambda$ are positive operators. Since $\|M_J D_\lambda\| < 1$, the inverse of $Id - M_J D_\lambda$ is given by the Neumann series, i.e.

$$(Id - M_J D_\lambda)^{-1} = \sum_{n=0}^{\infty} (M_J D_\lambda)^n.$$

From this representation we see that it is also a positive operator. So $R(\lambda, A) = R(\lambda, A_0) + D_\lambda(1 - M_J D_\lambda)^{-1} M_J R(\lambda, A_0)$ consists only of positive operators and is therefore also positive. $\square$

Note that assertion (3) in the above proposition also follows from Theorem 4.6. In order to use (CE) we investigate $\sigma(M_J D_\lambda)$ in more detail.

Lemma 3.11. *For $\lambda \in \mathbb{C}$ the following holds.*

(1)

$$\begin{aligned}
\sigma(M_J D_\lambda) \setminus \{0\} &= \sigma(\begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l_1}} J & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l_m}} J \end{pmatrix} \tilde{\mathbb{B}}) \setminus \{0\} \\
&= \sigma(\tilde{\mathbb{B}} \begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l_1}} J & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l_m}} J \end{pmatrix}) \setminus \{0\}.
\end{aligned}$$

(2) *If all arc lengths are equal to l, then*

$$\sigma(M_J D_\lambda) = \sigma(\mathbb{A}) \sigma(J Q_{e^{-\frac{\lambda}{\cdot} l}}).$$

Proof. (1) The first assertion follows from the fact that

$$\sigma(EF) \setminus \{0\} = \sigma(FE) \setminus \{0\} \text{ for } E \in \mathcal{L}(X_1, X_2) \text{ and } F \in \mathcal{L}(X_2, X_1), \tag{9}$$

where X_1 and X_2 are arbitrary Banach spaces.

(2) If all arc lengths are equal to l, then we have

$$\begin{aligned}
M_J D_\lambda &= \Phi_J^+ \begin{pmatrix} Q_{e^{-\frac{\lambda}{\cdot} l}} & & 0 \\ & \ddots & \\ 0 & & Q_{e^{-\frac{\lambda}{\cdot} l}} \end{pmatrix} (\tilde{\Phi}_w^-)^T \\
&= \begin{pmatrix} J Q_{e^{-\frac{\lambda}{\cdot} l}} & & 0 \\ & \ddots & \\ 0 & & J Q_{e^{-\frac{\lambda}{\cdot} l}} \end{pmatrix} \tilde{\Phi}^+ (\tilde{\Phi}_w^-)^T \\
&= \begin{pmatrix} J Q_{e^{-\frac{\lambda}{\cdot} l}} & & 0 \\ & \ddots & \\ 0 & & J Q_{e^{-\frac{\lambda}{\cdot} l}} \end{pmatrix} \tilde{\mathbb{A}} \\
&= \tilde{\mathbb{A}} \begin{pmatrix} J Q_{e^{-\frac{\lambda}{\cdot} l}} & & 0 \\ & \ddots & \\ 0 & & J Q_{e^{-\frac{\lambda}{\cdot} l}} \end{pmatrix} \\
&= (\alpha_{ij} J Q_{e^{-\frac{\lambda}{\cdot} l}})_{n \times n},
\end{aligned}$$

where $\mathbb{A} = (\alpha_{ij})_{n\times n}$. The spectrum of operator matrices of this special form is given by

$$\sigma(M_J D_\lambda) = \sigma(\mathbb{A})\sigma(JQ_{e^{-\frac{\lambda}{\cdot} l}}),$$

see [21, Sect. 4]. □

We now make additional assumptions on J and discuss the spectrum of A with the help of the characteristic equation (CE) and the above lemma. First, we consider the case that the operator J is compact.

Proposition 3.12. *If J is a compact operator, then*

$$\sigma_p(\mathcal{A}) = \sigma_p(A) = \sigma(A).$$

Proof. Clearly, the point spectra of $\mathcal{A}$ and A coincide, i.e.

$$\sigma_p(\mathcal{A}) = \sigma_p(A). \tag{10}$$

Let now $\lambda \in \sigma(A)$. By (CE), this means $1 \in \sigma(M_J D_\lambda)$. Since J is a compact operator, also $M_J D_\lambda$ is compact and therefore $\sigma(M_J D_\lambda) = \sigma_p(M_J D_\lambda) \cup \{0\}$. So $Id_{\partial X} - M_J D_\lambda$ is not injective, i.e. there exists $f_\lambda \in \partial X, f_\lambda \neq 0$, such that

$$(Id_{\partial X} - M_J D_\lambda) f_\lambda = 0.$$

Using that $D_\lambda f_\lambda \in \ker(\lambda - A_w)$ and that $LD_\lambda = Id_{\partial X}$ we obtain

$$\begin{aligned}(\lambda - \mathcal{A})\begin{pmatrix} D_\lambda f_\lambda \\ 0 \end{pmatrix} &= \begin{pmatrix} \lambda - A_w & 0 \\ L - M_J & \lambda \end{pmatrix}\begin{pmatrix} D_\lambda f_\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda - A_w) D_\lambda f_\lambda \\ LD_\lambda f_\lambda - M_J D_\lambda f_\lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ f_\lambda - M_J D_\lambda f_\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

So $\lambda - \mathcal{A}$ is not injective, and therefore $\lambda \in \sigma_p(\mathcal{A})$. Combining this with (10) we obtain

$$\sigma(A) \subseteq \sigma_p(\mathcal{A}) = \sigma_p(A) \subseteq \sigma(A)$$

as claimed. □

A physically realistic assumption is that the scattering operator J is a compact integral operator with a strictly positive kernel. More precisely, we assume that $J \in \mathcal{L}(Y)$ is given by

$$Jf := \int_{v_{min}}^{v_{max}} k(\cdot, w) f(w)\, dw, \quad f \in Y.$$

The measurable kernel

$$k : [v_{min}, v_{max}] \times [v_{min}, v_{max}] \to \mathbb{R}$$

fulfills $k(v, w) > 0$ for almost all $v, w \in [v_{min}, v_{max}]$. Moreover, we assume that

$$\int_{v_{min}}^{v_{max}} k(v, w)\, dv = 1 \text{ for all } w \in [v_{min}, v_{max}] \tag{11}$$

so that our General Assumption 1.1 is satisfied. Moreover, these assumptions imply the irreducibility of J, see [25, Example V.6.4] and Definition 5.1 below.

Under these assumptions we can show that 0 is the only spectral value of A on the imaginary axis.

Theorem 3.13. *Suppose that all the arc lengths are equal to l and suppose that the scattering operator J is as above. Then*

$$\sigma(A) \cap i\mathbb{R} = \{0\}.$$

Proof. By the proof of Proposition 3.10 we already know that $0 \in \sigma(A)$.

From assumption (11) follows that

$$\|Jf\|_1 = \|f\|_1 \text{ for all } f \geq 0. \tag{12}$$

Hence, the adjoint operator $J' \in \mathcal{L}(Y')$ where $Y' \cong L^\infty[v_{min}, v_{max}]$ satisfies

$$J'\mathbf{1} = \mathbf{1},$$

where $\mathbf{1}$ denotes the constant one function. By the irreducibility of J and [25, Thm. V.5.2] we then obtain that there exists $g \in Y_+$ such that $Jg = g$ and $g(v) > 0$ for almost all $v \in [v_{min}, v_{max}]$. Consider the Banach space $\tilde{Y} := L^1([v_{min}, v_{max}], g(v)dv)$. The positive operator $\tilde{J} := Q_{g^{-1}} J Q_g \in \mathcal{L}(\tilde{Y})$ is similar to J and satisfies

$$\tilde{J}\mathbf{1} = \mathbf{1}. \tag{13}$$

Since J is irreducible, the same holds for $\tilde{J}$, and also

$$\|\tilde{J}f\|_{\tilde{Y}} = \|f\|_{\tilde{Y}}$$

remains true for $f \in \tilde{Y}_+$. This again implies for the adjoint operator $\tilde{J}' \in \mathcal{L}(Y')$ of $\tilde{J}$ that

$$\tilde{J}'\mathbf{1} = \mathbf{1}. \tag{14}$$

Suppose now that there is a spectral value $\lambda \in i\mathbb{R} \setminus \{0\}$ of A. Define the operator $\tilde{J}_\lambda := Q_{g^{-1}} J Q_{e^{-\frac{\lambda}{\cdot} l}} Q_g \in \mathcal{L}(\tilde{Y})$. Note that $\tilde{J}_\lambda$ is similar to $JQ_{e^{-\frac{\lambda}{\cdot} l}} \in \mathcal{L}(Y)$. Therefore, their spectra coincide. We know from the characteristic equation (CE) with the help of Lemma 3.11 (2) that there must exist an $\alpha \in \sigma(\tilde{J}_\lambda)$ such that $|\alpha| = 1$. Since J is compact, $\alpha \in \sigma_p(\tilde{J}_\lambda)$. So there exists $f \in \tilde{Y}, f \neq 0$, such that

$$\tilde{J}_\lambda f = \alpha f.$$

Since

$$|f| = |\alpha f| = |\tilde{J}_\lambda f| \leq |\tilde{J}_\lambda||f| = \tilde{J}|f|,$$

we have

$$|\tilde{J}|f| - |f|| = \tilde{J}|f| - |f|.$$

From

$$\langle \mathbf{1}, |\tilde{J}|f| - |f|| \rangle = \langle \mathbf{1}, \tilde{J}|f| \rangle - \langle \mathbf{1}, |f| \rangle = \langle \tilde{J}'\mathbf{1}, |f| \rangle - \langle \mathbf{1}, |f| \rangle \overset{(14)}{=} 0,$$

it follows that $\tilde{J}|f| = |f|$. By [25, Thm. V.5.2] the fixed space of $\tilde{J}$ is one-dimensional and by (13) we conclude that it is spanned by $\mathbf{1}$. Therefore, we can assume that $|f| = \mathbf{1}$. Thus, f is a unimodular eigenfunction of $\tilde{J}_\lambda$.

If we take $h \in L^\infty[v_{min}, v_{max}] \subseteq \tilde{Y}$, then

$$0 \leq |\tilde{J}_\lambda h| \leq |\tilde{J}_\lambda||h| = \tilde{J}|h| \leq \tilde{J}(\|h\|_\infty \mathbf{1}) = \|h\|_\infty \tilde{J}\mathbf{1} = \|h\|_\infty \mathbf{1}.$$

Therefore, $\tilde{J}_\lambda(L^\infty[v_{min}, v_{max}]) \subseteq L^\infty[v_{min}, v_{max}]$.

By Gelfand's theorem

$$L^\infty[v_{min}, v_{max}] \cong C(K)$$

holds for a suitable compact space K. So far, we have shown that all the assumptions of [25, Prop. V.7.4] are fulfilled. Hence,

$$\tilde{J}_\lambda|_{L^\infty[v_{min}, v_{max}]} = \alpha Q_f \tilde{J} Q_{f^{-1}}|_{L^\infty[v_{min}, v_{max}]}.$$

This implies

$$k(v,w)e^{-\frac{\lambda}{w}l} = \alpha f(v)k(v,w)\overline{f(w)}$$

for almost all $v, w \in [v_{min}, v_{max}]$. Since k is strictly positive, this means that

$$\overline{f(v)} = \alpha e^{\frac{\lambda}{w}l}\overline{f(w)}$$

has to be fulfilled for almost all $v, w \in [v_{min}, v_{max}]$. Evidently, this is not possible, hence there is no spectral value $\lambda \neq 0$ on the imaginary axis. □

4. Wellposedness

In this section we show the generator property of A and hence the wellposedness of (F). We first renorm the space X and then check that A fulfills all the conditions in the Phillips generation theorem, see [20, Thm. C-II 1.2]. Therefore, A is the generator of a contraction semigroup on X for this norm.

Since J is contractive on Y_+ also $\mathbb{B}_J$ is contractive on Y_+^m as is shown in the following lemma.

Lemma 4.1. *If $f \in Y_+^m$, then*

$$\|\mathbb{B}_J f\|_1 - \|f\|_1 \leq 0.$$

Proof. Let $f \in Y_+^m$. Then the following computation shows the assertion.

$$\begin{aligned}
\|\mathbb{B}_J f\|_1 - \|f\|_1 &= \sum_{j=1}^m \int_{v_{min}}^{v_{max}} (\mathbb{B}_J f - f)_j(v)\, dv \\
&= \sum_{j=1}^m \int_{v_{min}}^{v_{max}} [J(\mathbb{B}f)_j - f_j](v)\, dv \\
&= \sum_{j=1}^m \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^m b_{jk} f_k\right) - f_j \right](v)\, dv \\
&= \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^m f_k \sum_{j=1}^m b_{jk}\right) - \sum_{j=1}^m f_j \right](v)\, dv \\
&\overset{\mathbb{B}\text{ column stochastic}}{=} \int_{v_{min}}^{v_{max}} \left[J\left(\sum_{k=1}^m f_k\right) - \sum_{j=1}^m f_j \right](v)\, dv \\
&= \sum_{j=1}^m (\|Jf_j\|_1 - \|f_j\|_1) \\
&\overset{\text{Gen. Ass. 1.1}}{\leq} 0.
\end{aligned}$$

□

There is an alternative way of writing the domain of A which uses the operator matrix $\mathbb{B}_J$.

Proposition 4.2. *The domain of A is given by*

$$D(A) = \{u \in W : \Gamma_0 u = \mathbb{B}_J \Gamma_l u\}.$$

Proof. If $u \in D(A)$ then $\tilde{\Phi}^- \Gamma_0 u = \Phi_J^+ \Gamma_l u$ and there exists $f \in \partial X$ such that $\Gamma_0 u = (\tilde{\Phi}_w^-)^T f$. With that we compute

$$\Phi_J^+ \Gamma_l u = \tilde{\Phi}^- \Gamma_0 u = \tilde{\Phi}^- (\tilde{\Phi}_w^-)^T f = f.$$

This implies

$$\Gamma_0 u = (\tilde{\Phi}_w^-)^T f = (\tilde{\Phi}_w^-)^T \Phi_J^+ \Gamma_l u = \mathbb{B}_J \Gamma_l u.$$

On the other hand, if for $u \in W$ the condition $\Gamma_0 u = \mathbb{B}_J \Gamma_l u$ is fulfilled, then $\Gamma_0 u \in \operatorname{rg}(\tilde{\Phi}_w^-)^T$ holds since $\mathbb{B}_J = (\tilde{\Phi}_w^-)^T \Phi_J^+$ Moreover,

$$Lu = \tilde{\Phi}^- \Gamma_0 u = \tilde{\Phi}^- \mathbb{B}_J \Gamma_l u = \tilde{\Phi}^- (\tilde{\Phi}_w^-)^T \Phi_J^+ \Gamma_l u = \Phi_J^+ \Gamma_l u = M_J u.$$

□

This representation of $D(A)$ is needed in Lemma 4.5 to show the dispersivity of A where dispersive means the following.

Definition 4.3. An operator $(B, D(B))$ on a Banach lattice Z is called *dispersive* if for every $z \in D(B)$ one has $\Re < Bz, \psi > \leq 0$ for some $\psi \in Z'_+$ such that $\|\psi\| \leq 1$ and $< z, \psi >= \|z^+\|$.

If X is endowed with the following norm, then A is dispersive.

Definition 4.4. The norm $\|\cdot\|_{1,v}$ on X is

$$\|\cdot\|_{1,v} : X \to \mathbb{R}, \quad u = (u_j)_{1 \leq j \leq m} \mapsto \|u\|_{1,v} := \sum_{j=1}^{m} \int_0^{l_j} \int_{v_{min}}^{v_{max}} \frac{1}{v} |u_j(x,v)| \, dv \, dx.$$

Since

$$\tfrac{1}{v_{max}} \|\cdot\|_1 \ \leq \ \|\cdot\|_{1,v} \ \leq \ \tfrac{1}{v_{min}} \|\cdot\|_1,$$

the norm $\|\cdot\|_{1,v}$ is equivalent to the original norm $\|\cdot\|_1$ on X.

Now we check the dispersivity of A.

Lemma 4.5. *The operator $(A, D(A))$ is dispersive on the Banach lattice $(X, \|\cdot\|_{1,v})$.*

Proof. The dual space of X is

$$\begin{aligned} X' &\cong L^\infty([0,l_1], Y') \times \cdots \times L^\infty([0,l_m], Y') \\ &\cong L^\infty([0,l_1] \times [v_{min}, v_{max}]) \times \cdots \times L^\infty([0,l_m] \times [v_{min}, v_{max}]) \end{aligned}$$

where $Y' = L^\infty[v_{min}, v_{max}]$. Let $u \in D(A)$ and let $\Psi = (\Psi_k)_{1 \leq k \leq m} \in X'$ be defined by

$$\Psi_k(x, \cdot) = \begin{cases} v \mapsto \frac{1}{v}, & u_k(x,\cdot) = u_k^+(x,\cdot), \\ v \mapsto 0, & \text{else}, \end{cases}$$

where $x \in [0, l_k]$. Clearly, $\|\Psi\| \leq 1$ for $\Psi \in (X, \|\cdot\|_{1,v})'$.

Next, we compute

$$\begin{aligned}
\langle u, \Psi\rangle &= \sum_{k=1}^{m} \int_0^{l_k} \langle u_k(x,\cdot), \Psi_k(x,\cdot)\rangle \, dx \\
&= \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} u_k(x,v)\Psi_k(x,v) \, dv \, dx \\
&= \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} u_k(x,v)\tfrac{1}{v}\chi_k(x) \, dv \, dx \\
&= \|u^+\|_{1,v},
\end{aligned}$$

where

$$\chi_k(x) = \begin{cases} 1, & \text{if } u_k(x,\cdot) = u_k^+(x,\cdot), \\ 0, & \text{else.} \end{cases}$$

We then obtain

$$\begin{aligned}
&\langle Au, \Psi\rangle \\
= \quad & \sum_{k=1}^{m} \int_0^{l_k} \int_{v_{min}}^{v_{max}} \tfrac{1}{v}(-v)\tfrac{\partial}{\partial x} u_k^+(x,v) \, dv \, dx \\
= \quad & -\sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \int_0^{l_k} \tfrac{\partial}{\partial x} u_k^+(x,v) \, dx \, dv \\
= \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left(u_k^+(0,v) - u_k^+(l_k,v)\right) \, dv \\
\overset{\text{Prop. 4.2}}{=} \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left((\mathbb{B}_J(\Gamma_l u))_k^+ - u_k^+(l_k,\cdot)\right)(v) \, dv \\
\leq \quad & \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} \left((\mathbb{B}_J(\Gamma_l u)^+)_k(v) - u_k^+(l_k,v)\right) \, dv \\
= \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{k=1}^{m} J\left(\sum_{j=1}^{m} \beta_{kj} u_j^+(l_j,\cdot)\right)\right)(v) \, dv - \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v) \, dv \\
= \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{j=1}^{m} J\left[\left(\sum_{k=1}^{m} \beta_{kj}\right) u_j^+(l_j,\cdot)\right]\right)(v) \, dv - \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v) \, dv \\
\overset{\mathbb{B}\text{ column stochastic}}{=} \quad & \int_{v_{min}}^{v_{max}} \left(\sum_{j=1}^{m} J u_j^+(l_j,\cdot)\right)(v) \, dv - \sum_{k=1}^{m} \int_{v_{min}}^{v_{max}} u_k^+(l_k,v) \, dv \\
= \quad & \sum_{j=1}^{m} \|J u_j^+(l_j,\cdot)\|_1 - \sum_{k=1}^{m} \|u_k^+(l_k,\cdot)\|_1 \\
\overset{\text{Gen. Ass. 1.1}}{\leq} \quad & 0.
\end{aligned}$$

This shows that all the conditions of Definition 4.3 are fulfilled, hence A is dispersive. □

We now obtain the generator property of A.

Theorem 4.6. *The operator $(A, D(A))$ on X is the generator of a positive and bounded strongly continuous semigroup $(T(t))_{t\geq 0}$ with bound $\frac{v_{max}}{v_{min}}$.*

Proof. By Proposition 2.4 and Lemma 4.5 it follows that A is a densely defined, dispersive operator and by Proposition 3.10 the operator $\lambda - A$ is surjective for $\lambda > 0$. Therefore, the Phillips theorem, see [20, Thm. C-II 1.2], implies that A is the generator of a positive contraction semigroup on $(X, \|\cdot\|_{1,v})$. Returning to our original norm $\|\cdot\|_1$ on X we obtain that the semigroup is bounded by $\frac{v_{max}}{v_{min}}$. □

5. Irreducibility of the semigroup

Irreducibility of the semigroup is an important property. It turns out that we need both a condition on the structure of the graph and on the scattering operator J in the vertices to obtain irreducibility. In Section 6, this will lead to a precise description of the asymptotic behaviour of the semigroup. We briefly recall the basic definitions, see [20] and [25].

Definition 5.1. (1) A positive linear operator S on a Banach lattice E is called *irreducible* if there is no closed ideal in E which is invariant under S apart from $\{0\}$ and E.

(2) A positive semigroup $(S(t))_{t\geq 0}$ on a Banach lattice E is called *irreducible* if there is no closed ideal in E which is invariant under $(S(t))_{t\geq 0}$ apart from $\{0\}$ and E.

The irreducibility of our semigroup $(T(t))_{t\geq 0}$ on the Banach lattice X can be characterised in the following way, cf. [20, Def. C-III 3.1].

Proposition 5.2. *The following assertions are equivalent.*

(1) *The semigroup $(T(t))_{t\geq 0}$ on X is irreducible.*
(2) *If $u \in X$ and $u \gneq 0$, then $R(\lambda, A)u \gg 0$ for all $\lambda > 0$.*

Here, $u \gneq 0$ means that $u \neq 0$ and u is positive ($u \geq 0$), i.e. $u_j(x, v) \geq 0$ for almost all $x \in [0, l_j]$ and $v \in [v_{min}, v_{max}]$ where $j = 1, \dots, m$. $R(\lambda, A)u \gg 0$ means that $R(\lambda, A)u$ is strictly positive, i.e. $(R(\lambda, A)u)_j(x, v) > 0$ for almost all $x \in [0, l_j]$ and $v \in [v_{min}, v_{max}]$ where $j = 1, \dots, m$. The same notation will be used to indicate positivity and strict positivity for functions in Y.

To show irreducibility for our semigroup we need the following concept from graph theory.

Definition 5.3. A directed graph is called *strongly connected* if for any two vertices v, w of the graph there exists a path from v to w and from w to v.

We obtain irreducibility of our semigroup combining two assumptions on the graph G and the scattering operator J.

Proposition 5.4. *Let G be strongly connected and suppose that*

$$Jf \gg 0 \text{ if } f \gneq 0. \tag{15}$$

Then the semigroup $(T(t))_{t\geq 0}$ generated by A is irreducible.

Proof. Suppose that $\lambda > 0$ and let $u \gneq 0$. Then also $R(\lambda, A_0)u \gneq 0$ and $M_J R(\lambda, A_0)u \gneq 0$. The inverse of $Id_{\partial X} - M_J D_\lambda$ is given by the Neumann series

$$(Id_{\partial X} - M_J D_\lambda)^{-1} = \sum_{n=0}^{\infty} (M_J D_\lambda)^n.$$

The operator $M_J D_\lambda$ has the same zero pattern as the adjacency matrix $\mathbb{A}$. Observe that $\mathbb{A}^k$ has a non-zero entry at position ij if there is a path from vertex v_j to vertex v_i of length k. Since G is assumed to be strongly connected, for every pair i, j there exists $k \in \mathbb{N}$ such that the entry ij of $\mathbb{A}^k$ and thus of $(M_J D_\lambda)^k$ is nonzero. This entry can be written as the composition of J with an operator composed of J and multiplictions by strictly positive functions. By assumption (15) we conclude that

$$(Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0)u \gg 0$$

and therefore by the special form of D_λ also

$$D_\lambda (Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0)u \gg 0.$$

This implies

$$R(\lambda, A)u \gg 0,$$

which is by Proposition 5.2 equivalent to the irreducibility of the semigroup. □

In the following examples we show that only the combination of the two assumptions in Proposition 5.4 leads to irreducibility.

Example 5.5. If we drop the assumption of the strong connectivity of the graph, then the semigroup need not be irreducible.

To prove this we decompose the graph into its strongly connected components. Assuming the graph to be not strongly connected, there exists a strongly connected component $C = (V', E'), V' \subseteq V, E' \subseteq E$ such that there is no edge $e \in E \setminus E'$ that is an incoming edge for a vertex $v \in V'$. Without loss of generality we can assume that $V' = \{v_r, \dots, v_n\}$ for some $2 < r < n$, and $E' = \{e_s, \dots, e_m\}$ for some $1 < s < m - 1$. The incidence matrices have the form

$$\tilde{\Phi}_w^- = \begin{pmatrix} \tilde{\Phi}_{11}^- & 0 \\ \tilde{\Phi}_{21}^- & \tilde{\Phi}_{22}^- \end{pmatrix} \text{ and } \tilde{\Phi}^+ = \begin{pmatrix} \tilde{\Phi}_{11}^+ & \tilde{\Phi}_{12}^+ \\ 0 & \tilde{\Phi}_{22}^+ \end{pmatrix} \text{ respectively,}$$

where $\tilde{\Phi}_{11}^-$ and $\tilde{\Phi}_{11}^+$ are $(r-1) \times (s-1)-$, $\tilde{\Phi}_{12}^-$ and $\tilde{\Phi}_{12}^+$ are $(r-1) \times (m-s+1)-$, $\tilde{\Phi}_{21}^-$ and $\tilde{\Phi}_{21}^+$ are $(n-r+1) \times (s-1)-$ and $\tilde{\Phi}_{22}^-$ and $\tilde{\Phi}_{22}^+$ are $(n-r+1) \times (m-s+1)-$ operator matrices. Moreover, since there is no path from a vertex $v_i, 1 \le i \le r-1$, leading into the subgraph C, we have

$$(M_J D_\lambda)^n = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}, n \in \mathbb{N},$$

where M_{11} is an $(r-1) \times (r-1)-$, M_{12} is an $(r-1) \times (n-r+1)-$, and M_{22} is an $(n-r+1) \times (n-r+1)-$ operator matrix. Take an element $u \in X$ such that $u = (u_j)_{1 \le j \le m} \gneq 0$ and $u_j = 0$ for $j \in \{r, \dots, n\}$. Then, it follows from the special form of the operators appearing in the resolvent of A that $(R(\lambda, A)u)_j = 0$ for $\lambda > 0$ and $j \in \{r, \dots, n\}$. By Proposition 5.2, the semigroup is not irreducible.

Note that in (15) we do not only require the irreducibility of J but a stronger condition. If G is strongly connected and J is only assumed to be irreducible, then the semigroup generated by A is not necessarily irreducible as the following example shows.

Example 5.6. Define

$$k : [v_{min}, v_{max}] \times [v_{min}, v_{max}] \to \mathbb{R},$$

$$k(v,w) := \begin{cases} c & \text{if } v \in [v_{min}, v'] \text{ and } w \in [v', v_{max}] \\ & \text{or if } v \in (v', v_{max}] \text{ and } w \in [v_{min}, v'], \\ 0 & \text{else,} \end{cases}$$

where $0 \neq c \in \mathbb{R}$ and $v_{min} < v' < v_{max}$. Then the integral operator

$$J : Y \to Y,$$

$$(Jf)(v) := \int_{v_{min}}^{v_{max}} k(v,w) f(w)\, dw = \begin{cases} c \displaystyle\int_{v'}^{v_{max}} f(w)\, dw \text{ if } v_{min} \leq v \leq v', \\ c \displaystyle\int_{v_{min}}^{v'} f(w)\, dw \text{ if } v' < v \leq v_{max}, \end{cases}$$

is irreducible which can be shown by an easy computation.

Consider a graph with the incidence matrices $\tilde{\Phi}_w^- = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\tilde{\Phi}^+ = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and suppose that both arcs have length l. Let $u = (u_1, u_2) \in X$ such that $u_1 \gneq 0, u_1(x,v) = 0$ if $0 \leq x \leq l$ and $v_{min} \leq v \leq v'$ and $u_2 \equiv 0$. Then also $(R(\lambda, A_0)u)_2 = 0$ and

$$(f_1, f_2) := M_J R(\lambda, A_0) u = (0, J(\int_0^l \tfrac{1}{\cdot} e^{-\lambda \frac{l-r}{\cdot}} u_1(r, \cdot)\, dr)).$$

Observe that if $f \in Y$ with $f|_{[v_{min}, v']} = 0$, then

$$(J^{2k+1} f)|_{[v', v_{max}]} = 0 \tag{16}$$

and

$$(J^{2k} f)|_{[v_{min}, v']} = 0 \text{ for } k \in \mathbb{N}. \tag{17}$$

Therefore, $f_2|_{[v', v_{max}]} = 0$ holds. Suppose that $\lambda > 0$. Then the inverse of $Id_{\partial X} - M_J D_\lambda$ is given by the Neumann series

$$\begin{aligned}(Id_{\partial X} - M_J D_\lambda)^{-1} &= \sum_{k=0}^{\infty} (M_J D_\lambda)^k \\ &= \begin{pmatrix} \sum_{k=0}^{\infty} (JQ_{e^{-\frac{\lambda}{\cdot} l}})^{2k} & \sum_{k=0}^{\infty} (JQ_{e^{-\frac{\lambda}{\cdot} l}})^{2k+1} \\ \sum_{k=0}^{\infty} (JQ_{e^{-\frac{\lambda}{\cdot} l}})^{2k+1} & \sum_{k=0}^{\infty} (JQ_{e^{-\frac{\lambda}{\cdot} l}})^{2k} \end{pmatrix}.\end{aligned}$$

For $f \in Y$ the function $Q_{e^{-\frac{\lambda}{\cdot} l}} f$ vanishes on the same set as f. since these operators are multiplications by positive functions. Therefore we conclude, using (16) and (17), that

$$((Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0) u)_1|_{[v', v_{max}]} = 0$$

and, by the definition of D_λ, also

$$(D_\lambda (Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0) u)_1|_{[v', v_{max}]} = 0$$

holds. With these considerations it follows that

$$\begin{aligned}&(R(\lambda, A)u)_1|_{[v', v_{max}]} \\ =&(R(\lambda, A_0)u)_1 + D_\lambda (Id_{\partial X} - M_J D_\lambda)^{-1} M_J R(\lambda, A_0) u)_1|_{[v', v_{max}]} = 0.\end{aligned}$$

By Proposition 5.2 the semigroup $(T(t))_{t \geq 0}$ is not irreducible.

6. Asymptotic behaviour

As our final result we describe the asymptotic behaviour of the solutions using the theory of positive and irreducible semigroups from [20].

We use the following notation. We denote by $[-u, u]$ for $u = (u_j)_{1\leq j\leq m} \in X$ order intervals, i.e.

$$\begin{aligned}[-u, u] \quad = \quad & \{w = (w_j)_{1\leq j\leq m} \in X : -u_j(x) \leq w_j(x) \leq u_j(x) \text{ for almost all} \\ & x \in [0, l_j], j = 1, \ldots, m\},\end{aligned}$$

and the absolute value of u is $|u| = (|u_j|)_{1\leq j\leq m}$ where $|u_j|(x, v) = |u_j(x, v)|$ for $x \in [0, l_j]$ and $v \in [v_{min}, v_{max}]$. A lattice norm $\|\cdot\|_X$ on X is called *strictly monotone* if $|u| < |w|$ implies $\|u\|_X < \|w\|_X$ for all $u, w \in X$. The notation is taken from [20] and also the definitions can be found in this book. Moreover, the fixed space of the semigroup $(T(t))_{t\geq 0}$ is

$$\operatorname{fix}(T(t))_{t\geq 0} = \bigcap_{t\geq 0} \operatorname{fix}(T(t)) = \{u \in X : T(t)u = u \text{ for all } t \geq 0\}.$$

By [8, Cor. IV.3.8 (i)] the equality

$$\operatorname{fix}(T(t))_{t\geq 0} = \ker A$$

holds.

To treat the asymptotic behaviour of the semigroup the following compactness property of the semigroup is important.

Lemma 6.1. *Let $0 \in \sigma_p(A)$ and suppose that the semigroup $(T(t))_{t\geq 0}$ is irreducible. Then $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively compact for the weak operator topology, hence it is mean ergodic, i.e.*

$$\lim_{r\to\infty} \tfrac{1}{r} \int_0^r T(s)u \, ds,$$

exists for all $u \in X$, see [8, Def. V.4.3].

Proof. Since $0 \in \sigma_p(A)$ we know from [8, Cor. IV.3.8] that there exists $0 \neq u \in \operatorname{fix}(T(t))_{t\geq 0}$. Then, from the positivity of the semigroup, the inequality

$$|u| = |T(t)u| \leq T(t)|u| \tag{18}$$

follows for $t \geq 0$. Suppose that $|u| < T(t)|u|$. Since $(T(t))_{t\geq 0}$ is a contraction semigroup with respect to the strictly monotone norm $\|\cdot\|_{1,v}$ from Definition 4.4 on X, we obtain

$$\|u\|_{1,v} < \|T(t)|u|\|_{1,v} \leq \|u\|_{1,v}$$

which is a contradiction. Thus, in (18) we have equality and we can assume in the following without loss of generality that $u \gneq 0$. Since the semigroup is irreducible we obtain from [20, Prop. C-III 3.5 (a)] that u is a quasi-interior point of X which implies that

$$X_u := \bigcup_{n\geq 1} [-nu, nu]$$

is dense in X.

Let $n \in \mathbb{N}$ and take $w \in [-nu, nu]$, i.e. $-nu \leq w \leq nu$. Then

$$-nu = -nT(t)u \leq T(t)w \leq nT(t)u = nu,$$

for all $t \geq 0$. Since the order interval $[-nu, nu]$ is weakly compact in X, see [25, p. 92], the orbit $\{T(t)w : t \geq 0\}$ is relatively weakly compact in X. So far, we have shown that the orbits of elements w from the dense subset X_u of X are relatively weakly compact. Since the semigroup $(T(t))_{t\geq 0}$ is bounded, this suffices to prove that $\{T(t) : t \geq 0\} \subseteq \mathcal{L}(X)$ is relatively weakly compact, see [8, Lem. V.2.7].

The mean ergodicity of $(T(t))_{t\geq 0}$ follows from [8, Lem. V.4.6]. □

The mean ergodicity of the semigroup allows a decomposition of X into the direct sum of $\ker A$ and $\overline{\operatorname{rg} A}$. If the semigroup is irreducible, then $\ker A$ is one-dimensional. If in addition the scattering operator is as in Theorem 3.13, then the semigroup converges strongly to the one-dimensional projection onto $\ker A$. This is shown in the next theorem.

Theorem 6.2. *Let G be strongly connected. Then, under the assumptions of Theorem 3.13, the space X can be decomposed into the direct sum*

$$X = X_1 \oplus X_2$$

where $X_1 = \operatorname{fix}(T(t))_{t\geq 0} = \ker A$ is one-dimensional and spanned by a strictly positive eigenvector $u \in \ker A$ of A, $u \gg 0$, and $(T(t)|_{X_2})_{t\geq 0}$ is strongly stable.

Proof. Observe first that all the assumptions of Proposition 5.4 are fulfilled and hence $(T(t))_{t\geq 0}$ is irreducible. Since $(T(t))_{t\geq 0}$ is mean ergodic by Lemma 6.1, the space X can be decomposed into

$$X = \ker A \oplus \overline{\operatorname{rg}(A)} =: X_1 \oplus X_2,$$

where $\ker A = \operatorname{fix}(T(t))_{t\geq 0}$, see [8, Lem. V.4.4]. From Proposition 3.12 it is clear that $0 \in \sigma_p(A)$. As in the proof of Lemma 6.1 we can show that there exists $u \in \ker A$ such that $u \gneq 0$. Moreover, we find by the same construction as in the proof of [8, Lem. V.2.20 (i)] $\phi \in X'$ such that $\phi \gneq 0$ and $A'\phi = 0$. Thus, by [20, Prop. C-III 3.5] we obtain that

$$\dim \ker A = 1,$$

and that u is strictly positive, i.e. $u \gg 0$.

Both spaces X_1 and X_2 are invariant under $(T(t))_{t\geq 0}$. Consider now the restricted semigroup $(T_2(t))_{t\geq 0}$ where $T_2(t) := T(t)|_{X_2}$. Its generator $(A_2, D(A_2))$ is given by

$$\begin{aligned} D(A_2) &= D(A) \cap X_2, \\ A_2 v &= Av. \end{aligned}$$

In the next step we show that $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. From [8, Prop. IV.1.12] we have that

$$\sigma_p(A_2') = \sigma_r(A_2),$$

where $\sigma_r(A_2) = \{\lambda \in \mathbb{C} : \operatorname{rg}(\lambda - A_2) \text{ is not dense in } X_2\}$ denotes the residual spectrum. Since $\sigma_r(A_2) \subseteq \sigma(A)$ and $\sigma(A) \cap i\mathbb{R} = \{0\}$ by Theorem 3.13, we only have to prove that $0 \notin \sigma_p(A_2') = \sigma_r(A_2)$. Clearly, $(T_2(t))_{t\geq 0}$ is a mean ergodic bounded semigroup on X_2. So, by [8, Thm. V.4.5], $\ker A_2$ separates $\ker A_2'$. But $\ker A_2 = \{0\}$ and thus $\ker A_2' = \{0\}$. Hence, it follows that $\sigma_p(A_2') \cap i\mathbb{R} = \emptyset$. Now we can apply the Arendt-Batty-Lyubich-Vũ Theorem, see [1, Thm. 5.5.5], to show the strong stability of $(T_2(t))_{t\geq 0}$. □

We reformulate the above theorem as our final result.

Corollary 6.3. *Under the conditions of the above theorem the semigroup* $(T(t))_{t\geq 0}$ *converges to the one-dimensional projection* $P \in \mathcal{L}(X)$ *onto* $\operatorname{fix}(T(t))_{t\geq 0}$*, i.e.*

$$\lim_{t\to\infty} \|T(t)w - Pw\| = 0 \text{ for all } w \in X.$$

Here, $P = u \otimes \phi$ *where* u *and* $0 \ll \phi \in X'$ *are as in (the proof of) Theorem 6.2 and* $\langle u, \phi\rangle = 1$.

7. Acknowledgements

The author thanks Rainer Nagel sincerely for many helpful discussions and suggestions.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.4,485-495,COPYRIGHT 2008 EUDOXUS PRESS, LLC

$(\alpha, \beta) - L^p$ 2 -Norm Orthogonality and Characterizations of 2 - Inner Product Spaces

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ABSTRACT

In the present paper we have characterised $(\alpha, \beta) - L^p$ orthogonality in a 2-normed linear space. In some way the results proved in this paper generalize some of the similar characterization of generalized L^p- orthogonality derived earlier by Zheng Liu[8].

Mathematics Subject Classification 2000: Primary 46C15.

Key words and Phrases: $(\alpha, \beta) - L^p$ orthogonality, Birkhoff orthogonality, 2-normed linear spaces, 2-inner product spaces, Homogeniety.

INTRODUCTION

Recently there has been special interest to deal with certain analytic functional aspects in 2-normed spaces of finite or infinite dimensional type.

Usually orthogonality is dealt in inner product spaces but there is a concept like orthogonality in normal linear spaces ([2], [3], [5], [6],[7] and [8]). As has been noted earlier (for example see reference [7]) Birkhoff orthogonality plays a typical role in a normed linear space. In some analytic consideration also Birkhoff orthogonality is important.

In the present paper we have introduced $(\alpha, \beta) - L^p \Updownarrow$-orthogonality for a pair (x,z) and (y,z) in 2-normed spaces. We have also developed certain properties in the line of those given earlier by Liu [8] as was given in the normed spaces to be carried over in the setting of 2-normed space and 2-inner product spaces.

PRELIMINARIES AND NOTATIONS

DEFINITION 1. Let $p > 1$ be a fixed real number. If $(x, z) \in X \times X$, then we say that (x,z) is L^p-orthogonal and we denote $(x, z) \perp_{L^p} (y, z)$ provided

$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p$ is called left L^p orthogonality. In a similar way$(x, y) \perp_{L^p} (x, z)$ provided $\| x, y + z \|^p = \| x, y \|^p + \| x, z \|^p$ is called right L^p -orthogonality in 2-normed spaces.

DEFINITION 2. Let $p \geq 1$ and $\alpha, \beta \neq 1$ be fixed real numbers. If $(x, z) \in X \times X$ then (x,z) is 2-norm $(\alpha, \beta) - L^p-$ orthogonal to (y,z) denote by$(x, z) \perp_{L^p} (y, z)(\alpha, \beta)$ provided that

$$\| x + y, z \|^p + \| \alpha x + \beta y, z \|^p = \| \alpha x + y, z \|^p + \| x + \beta y, z \|^p$$

and $z \notin V(x, y)$ (where V(x,y) is the linear span of $x, y \in X$. Similarly we say (x,z) is $(\alpha, \beta) - L^p-$ orthogonal to (y,z) denoted by $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ provided that

$$\| x, y + z \|^p + \| x, \alpha y + \beta z \|^p = \| x, \alpha y + z \|^p + \| x, y + \beta z \|^p .$$

LEMMA 1. For all $(x, z), (y, z) \in X \times X, \alpha, \beta \neq 1, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only $((y, z) \perp_{L^p} (x, z))(\alpha, \beta)$.

The following theorem and corollary demonstrate that the concept of $(\alpha, \beta) - L^p-$ orthogonality is non-vacuous.

THEOREM 1. Let $p > 1$ and $\alpha, \beta \neq 1$ be a fixed real numbers. If $(x, z) \neq 0, (y, z) \in X \times X$ then there exists a real number such that $((x, z) \perp_{L^p} (ax + y, z))(\alpha, \beta)$.

PROOF. Set

$$f(t) = \| x+tx+y, z \|^p + \| ax+\beta(tx+y), z \|^p - \| ax+tx+y, z \|^p - \| x+\beta(tx+y), z \|^p .$$

Clearly, f is a continuous function on $-\infty < t < \infty$, and we have, for $t \neq 0$

$$f(t) = |t|^p[(\| x+\frac{1}{t}(x+y), z \|^p - \| x, z \|)^p - (\| \beta x+\frac{1}{t}(\alpha x+\beta y), z \|^p - \| \beta x, z \|)^p$$

$$-(\| x+\frac{1}{t}(\alpha x+y), z \|^p - \| x, z \|)^p - (\| \beta x+\frac{1}{t}(x+\beta y), z \|^p - \| \beta x, z \|)^p].$$

Then for $t \neq 0$

$$\frac{f(t)}{|t|^{p-1} sgnt} = \frac{\| x + \frac{1}{t}(x + y), z \|^p - \| x, z \|^p}{\frac{1}{t}} + \frac{\| \beta x + \frac{1}{t}(\alpha x + \beta y), z \| - \| \beta x, z \|^p}{\frac{1}{t}}$$

$$- \frac{\| x + \frac{1}{t}(\alpha x + y), z \|^2 - \| x, z \|^p}{\frac{1}{t}} - \frac{\| \beta x + \frac{1}{t}(x + \beta y), z \| - \| \beta x, z \|^p}{\frac{1}{t}},$$

and hence

$$\lim_{t \to \overset{+}{-}\infty} \frac{f(t)}{|t|^{p-1} sgnt} = p \| x, z \|^{p-1} J_{\underset{-}{+}}(x, z)(x+y) + p \| \beta x, z \|^{p-1} J_{\underset{-}{+}}(\beta x, z)(\alpha x+\beta y)$$

$$-p \| x \|^{p-1} J_{\underset{-}{+}}(x, z)(\alpha x+y) - p \| \beta x, z \|^{p-1} J_{\underset{-}{+}}(\beta x, z)(x+\beta y),$$

where $J_+((x, z)(y, z))$ and $J_-((x, z)(y, z))$ are respectively the right and left Gateaux derivative of the norm at (x,z), keeping second co-ordinate as fixed in the direction of (y,z). By James [5] we see that

$$J_{\underset{-}{+}}(x, z)(rx + sy) = r \| x, z \| + s J_{\underset{-}{+}}(x, z)(y, z)$$

for some $s \geq 0$ and r, therefore,

$$lim \frac{f(t)}{p-1} = p \| x, z \|^p (1 + \alpha\beta^{p-1} - \alpha - \beta^{p-1})$$

$$p \| x, z \|^p (1 - \alpha)(1 - \beta^{p-1}).$$

Thus for any fixed real number $\alpha, \beta \neq 1$ we have either $f(t) \to \infty$ as $t \to +\infty$ or $f(t) \to -\infty$. Hence there is a real number a such that $f(a) = 0$, which was to be proved.

COROLLARY 1. Let $p > 1$ and $\alpha, \beta \neq 1$ be the fixed real numbers. If $x \neq 0, z \neq 0, (y, z) \in X$, then there exist a real number a such that $((ax + yz) \perp_{L^p} (x, z)(\alpha, \beta))$.

PROOF. The result follows from Theorem 1 and Lemma 1.

Lemma 2. Let (x,z) or $(y, z) \in X \times X$ and $\alpha, \beta \neq 1$
(i) $\alpha, \beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_{L^p} (\beta y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,
(ii) if $\beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta))$ if and only if $((x, z) \perp_{L^p} \beta((y, z))(\alpha, \frac{1}{\beta})$,
(iii)if $\alpha \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_{L^p} (y, z))(\frac{1}{\alpha}, \beta)$.
Homogenity, symmetry and left and right additivity of $(\alpha, \beta) - L^p-$ orthogonality are defined in usual way, i.e. 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous provided for all $x, y, z, \in X$ and real numbers $a, b, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $((ax, z) \perp_{L^p} (by, z))(\alpha, \beta)$; 2-norm $(\alpha, \beta) - L^p - 2$ norm orthogonality is symmetric provided for all $x, y, z \in X, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $(y, z) \perp_{L^p} (x, z))(\alpha, \beta)$; 2 -norm $(\alpha, \beta) - L^p-$ orthogonality is left additive if and only if for all $x, y, w \in X, ((x, z) \perp_{L^p} (w, z))(\alpha, \beta)$ and $((y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ implies $((x + y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ and 2-norm $(\alpha, \beta) - L^p-$ orthogonality is right additive if and only if for all $(x, z), (y, z), (w, z) \in X \times X$, $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ and if $((y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ imply $((x, z) \perp_{L^p} (y + w, z))(\alpha, \beta)$.

The following two corollaries are immediate consequences of the definition of homogeneity and Lemma 1 and Lemma 2.

COROLLARY 2. For all $\alpha, \beta \neq 1$,2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous if and only if 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous.

COROLLARY 3. Suppose 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous .
(i) If $\alpha, \beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_{L^p} (y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,
(ii) if $\beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_{L^p} ((y, z))(\alpha, \frac{1}{\beta})$,

(iii)if $\alpha \neq 0, ((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$ if and only if $((x,z) \perp_{L^p} (y,z))(\frac{1}{\alpha},\beta)$.
Now let us further study some consequences of homogeneity.

LEMMA 3. If $\alpha \neq -1$ and $(\alpha,\beta) - L^p - 2-$norm orthogonality is homogeneous, then $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$ implies

$$\| x+y,z \|^p = (1-|\beta|^p) \| y,z \|^p + \| x+\beta y,z \|^p .$$

PROOF. From Corollary 3, it suffices to assume $|\alpha| < 1$. Suppose $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$. Then $\| x+y,z \|^p + \| \alpha x+\beta y,z \|^p = \| \alpha x+y,z \|^p$ $+ \| x+\beta y,z \|^p$ keeping second co-ordinate as fixed. Since the result is immediate for $\alpha = 0$. We may assume that $\alpha \neq 0$. We are denoting the statement by $P(n)$ i.e.

$$P(n) : \| x+y,z \|^p + \| \alpha^n x+\beta y,z \|^p = \| \alpha^n x+y,z \|^p + \| x+\beta y,z \|^p .$$

Clearly $P(1)$ is true and if $P(n)$ is true for some positive integer n. Since 2-norm $(\alpha,\beta) - L^p-$ orthogonality is homogeneous, $((\alpha^n x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, we have

$$\| \alpha^n x+y,z \|^p + \| \alpha^{n+1} x+\beta y,z \|^p = \| \alpha^{n+1} x+y,z \|^p + \| \alpha^n x+\beta y,z \|^p .$$

Adding this to $P(n)$ we obtain

$$\| x+y,z \|^p + \| \alpha^{n+1} x+\beta y,z \|^p = \| \alpha^{n+1} x+y,z \|^p + \| x+\beta y,z \|^p,$$

which is $P(n+1)$. Thus $P(n)$ is true for all positive integer n, but

$$\lim_{t\to\infty^+} \alpha^n = 0,$$

so in the limit, by continuity of the norm, we have

$$\| x+y,z \|^p + |\beta|^p \| y,z \|^p = \| y,z \|^p + \| x+\beta y,z \|^p,$$

and the conclusion of the lemma follows.

THEOREM 2. If $\alpha,\beta \neq -1$ and 2-norm $(\alpha,\beta) - L^p-$ orthogonality is homogeneous, then $((x,y) \perp_{L^p} (y,z))(\alpha,\beta)$ implies

$$\| x+y,z \|^p = \| x,z \|^p + \| y,z \|^p$$

i.e. 2 - norm $(\alpha, \beta) - L^p-$ orthogonality implies L_p orthogonality.

PROOF. By Corollary 6, we may assume $|\beta| < 1$. Suppose $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ and let $Q(n)$ denote the statement

$$Q(n) :\| x + y, z \|^p = (1 - |\beta^n|^p) \| y, z \|^p + \| x + \beta^n y, z \|^p .$$

The statement $Q(1)$ is Lemma 3. If we assume $Q(n)$ is true for some positive integer n, since $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ by homogeneity, we have by Lemma3,

$$\| x + \beta^n y, z \|^p = (1 - |\beta|^p) \| \beta^n y, z \|^p + \| x + \beta^{n+1} y, z \|^p$$

Substituting this in $Q(n)$ we obtain

$$\| x + y, z \|^p = (1 - |\beta^{n+1}|^p) \| y, z \|^p + \| x + \beta^{n+1} y, z \|^p$$

or $Q(n + 1)$.
Hence $Q(n)$ holds for all positive integer n. Since $\beta^n \to 0$ as $n \to +\infty$ by taking limit in $Q(n)$ we obtain

$$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p .$$

THEOREM 3. If 2 - norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous, then $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $\| x - y, z \| = \| x + y, z \|$ i.e. 2 - norm $(\alpha, \beta) - L^p-$ orthogonality implies 2 - norm isosceles orthogonality.

PROOF. If $(\alpha, \beta) = -1$ the result is follows. Otherwise by Lemma 1 we may assume without loss of generality that $\alpha \neq -1$. If $\beta = -1$ the result is immediate from Lemma 7. If $\beta \neq -1$, by Theorem 4, we have
$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p$. But by homogeneity $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ holds, so $\| x - y, z \|^p = \| x, z \|^p + \| -y, z \|^p$ and the result follows.

LEMMA 4. For all $\alpha, \beta \neq 1$, each one of the following:
(i) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is symmetric and left additive,
(ii) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is symmetric and right additive,
(iii) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is left and right additive,
implies that $(\alpha, \beta) - L^p$ 2 - norm orthogonality is homogeneous.

PROOF. Suppose (i) holds and $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, where x , y and z are arbitrary elements. Since the result is obvious for x = 0 or y = 0. We will assume $x \neq 0$ and $y \neq 0$. By Corollary 1, there exist a real number a such that $((ay-x,z) \perp_{L^p} (y,z))(\alpha,\beta)$. Left additive then gives $((ay,z) \perp_{L^p} (y,z))(\alpha,\beta)$ and hence $a=0$. Thus $((-x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, using left additive and symmetry, we find it now follows that $((nx,z) \perp_{L^p} (my,z))(\alpha,\beta)$ for all integers m and n, i.e.

$$\| nx+my,z \|^p + \| \alpha nx+\beta my,z \|^p = \| \alpha nx+my,z \|^p + \| nx+\beta my,z \|^p$$

or

$$\| x+\frac{m}{n}y,z \|^p + \| \alpha x+\beta\frac{m}{n}y,z \|^p = \| \alpha x+\frac{m}{n}y,z \|^p + \| x+\beta\frac{m}{n}y,z \|^p .$$

From the continuity of the norm it follows that

$$\| x+ky,z \|^p + \| \alpha x+\beta ky,z \|^p = \| \alpha x+ky,z \|^p + \| x+\beta ky,z \|^p$$

for all real numbers k or $((x,z) \perp_{L^p} (ky,z))(\alpha,\beta)$ for all k. So $(\alpha,\beta)-L^p$- 2 - norm orthogonality is homogeneous. By similar reasoning we can also get that (ii) and (iii) imply $(\alpha,\beta)-L^p-$ 2 - norm orthogonality is homogeneous. By similar reasoning, we can also get that (ii) and (iii) imply $(\alpha,\beta)-L^p-$ 2 - norm orthogonality is homogeneous.

The result may be summarized as follows:

THEOREM 4. Let $p>1$ and $\alpha,\beta \neq 1$. The following are equivalent.

(i) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is homogeneous,

(ii) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is symmetric and left additive,

(iii)$(\alpha,\beta)-L^p$ 2 - norm orthogonality is symmetric and right additive,

(iv) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is left and right additive.

Finally we give two characterizations of inner product spaces based on the relation between $(\alpha,\beta)-L^p$, 2 - norm orthogonality and Birkhoff 2 - norm orthogonality.

DEFINITION 3. If $(x,z),(y,z) \in X \times X$, we say (x,z) is Birkhoff orthogonal to (y,z), denoted $(x,z) \perp_\beta (y,z)$ provided $\| x+ky,z \| \geq \| x,z \|$ for all real numbers k.

THEOREM 5. Let $1 < p \leq 2$ and $0 < \alpha, \beta < 1$ be fixed real numbers. Then Birkhoff 2 - norm orthogonality implies $(\alpha, \beta) - L^p$ 2 - norm orthogonality in X if and only if X is an 2 - inner product space and p = 2.

PROOF. Let $(x, z) \perp_\beta (y, z)$. By assumption and homogeneity of Birkhoff 2 - norm orthogonality we get,

$$\| x + y, z \|^p = \| \alpha x + y, z \|^p + \| x + \beta y, z \|^p - \| \alpha x + \beta y, z \|^p$$

$$= (\| \alpha^2 x + y, z \|^p + \| \alpha x + \beta y, z \|^p - \| \alpha^2 x + \beta y, z \|^p)$$

$$+(\| \alpha x + \beta y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha x + \beta y, z \|^p) - \| \alpha x + \beta y, z \|^p$$

$$= (\| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p) - \| \alpha x + \beta^2 y, z \|^p + \| \alpha x + \beta y, z \|^p$$

$$= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p - \| \alpha x + \beta^2 y, z \|^p$$

$$+(\| \alpha^2 x + \beta y, z \|^p + \| \alpha x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta^2 y, z \|^p)$$

$$= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta^2 y, z \|^p .$$

Thus by induction we see that $(x, z) \perp_\beta (y, z)$. implies

$$\| x + y, z \|^p = \| \alpha^n x + y, z \|^p + \| x + \beta^n y, z \|^p - \| \alpha^n x + \beta^n y, z \|^p$$

for $n \geq 1$. In the limit this yields $(x, z) \perp_p (y, z)$. implies

$$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p \qquad (A)$$

If for $p = 2$ then (A)yields

$$\| x + y, z \|^2 = \| x, z \|^2 + \| y, z \|^2 \qquad (B)$$

From the definition of 2 - inner product space, we have

$$\| x + y, z \|^2 = (x + y, x + y/z)$$

and

$$\| x, z \|^2 = (x, x/z)$$

$$\| y, z \|^2 = (y, y/z)$$

From (B) $(x+y, x/z)+(x+y, y/z)=(x, x/z)+(y, y/z)$
i.e. $\quad (x, x/z)+y, x/z)+(x, y/z)(y, y/z)$
$=(x, x/z)+(y, y/z)+2(x, y/z)=0 \Rightarrow (x, y/z)=0$
Although in 1 - norm space the proof that identity (A) implies

$$\mu_p(X)=\sup_{(x,z)\perp_B(y,z)} \frac{(\| x,z \|^p +(\| y,z \|^p)^{\frac{1}{p}}}{(\| x+y,z \|^p} = 1$$

which in turn implies that X is an inner product space by the technique of Amir[1]. But one has to explore whether the same proof will work for $1<p\leq 2$ in the context of 2 - norm spaces.
The other part is obvious.

THEOREM 6. Let $1<p\leq 2$ and $0<\alpha,\beta<1$, be fixed real numbers. Then $(\alpha,\beta)-L^p$ 2 - norm orthogonality implies Birkhoff 2 - norm orthogonality in X if and only if X is an inner product space and p = 2.

PROOF. We first prove that if $(\alpha,\beta)-L^p$ 2 - norm orthogonality implies Birkhoff orthogonality then X is strict convex. If not then we can choose $x\neq y$ as extreme points of the units ball of X such that $\| x,z \| = \| y,z \|$
$=\| \frac{x+y}{z}, z \| = 1$. Then

$$\| \frac{x+y}{z} + y, z \|^p + \| \alpha\frac{x+y}{z} + \beta y, z \|^p \neq \| \alpha\frac{x+y}{z} + y, z \|^p + \| \frac{x+y}{z} + \beta y, z \|^p .$$

For otherwise $2^p+(\alpha+\beta)^p=(\alpha+1)^p+(\beta+1)^p$ which requires $\alpha=1$ or $\beta=1$, i.e. $(\frac{x+y}{z}, z)$ is not $(\alpha,\beta)-L^p$ 2 - norm orthogonal to (y,z). Without loss of generality we assume $\alpha\geq\beta$. By Theorem 2 we can choose $a\neq 0$ such that $((\frac{x+y}{z}, z) \perp_{L^p} (\frac{x+y}{z}+y, z))(\alpha+\beta)$. Hence $(\frac{x+y}{z}, z) \perp_{L^p} a\frac{x+y}{z}+y, z)$ i.e.$\| \frac{x+y}{z} + k(a\frac{x+y}{z}+y), z \| \geq \| \frac{x+y}{z}, z \| = 1$ for all real numbers k. Putting $k=-1/2$ yields $|\alpha|\leq 1$, and then $k=-\frac{1}{a+2}$ yields $|a+2|\leq 1$. Thus $a=-1$. But then $(\frac{x+y}{z}, z \perp_{L^p} a\frac{y-x}{z}, z)(\alpha,\beta)$ and then it gives

$$1+ \| \frac{\alpha-\beta}{z}x + \frac{\alpha+\beta}{z}y, z \|^p = \| \frac{\alpha-1}{z}x + \frac{\alpha+1}{z}y, z \|^p + \| \frac{1-\beta}{z}x + \frac{1+\beta}{z}y, z \|^p .$$

So we have

$$\alpha^p = \alpha^p \| \frac{\alpha-\beta}{2\alpha}x + \frac{\alpha+\beta}{2\alpha}y, z \|^p = \| \frac{\alpha-1}{z}x + \frac{\alpha+1}{z}y, z \|^p$$

and it follows $\| \frac{\alpha-1}{2\alpha}x + \frac{\alpha+1}{2\alpha}y, z \|= 1$.
Writing $(y, z) = \frac{1-\alpha}{1+\alpha}(x) + (1 - \frac{\alpha-1}{\alpha+1})(\frac{\alpha-1}{2\alpha}x + \frac{\alpha+1}{2\alpha}y)$.

We see that y is a convex combination of two points of the unit sphere which is false since y was taken to be an extrem point of the unit ball X. Thus X must be strictly convex.

Now we prove that if $(\alpha, \beta) - L^p$ 2 - norm orthogonality implies $(\alpha, \beta) - L^p$ 2 - norm orthogonality. If not, then there exists $(x, z)(y, z) \in X \times X$ such that $(x, z) \perp_\beta (y, z)$ and (x,z) is not $(\alpha, \beta) - L^p$ 2 - norm orthogonolity to (y,z). By Corollary 1 we can choose $b \neq 0$ such that $((by + x, z) \perp_{L^p} (y, z))(\alpha, \beta)$. But then $(by + x, z) \perp_\beta (y, z)$. Thus we have $(x, z) \perp_\beta (y, z)$ and $(by + x, z) \perp_\beta (y, z)$ which cotradicts the left uniqueness of 2 - norm Birkhoff orthogonality in strict convex space [5], hence 2 - norm Birkhoff orthogonality implies $(\alpha, \beta) - L^p$ 2 - norm orthogonality, which is sufficient for X to be an inner product space and $p = 2$ by Theorem 5.

The other part is also obvious.

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JOURNAL OF APPLIED FUNCTIONAL ANALYSIS,VOL.3,NO.4,497-505,COPYRIGHT 2008 EUDOXUS PRESS, LLC

The "pseudo-remaining Cauchy equations"

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ABSTRACT. In this paper, given a pseudo-addition $\oplus$ and its corresponding $\oplus$-fitting pseudo-multiplication $\odot$, we consider and solve the *pseudo-remaining Cauchy equations.* They are the *remaining Cauchy equations,* considered by Aczél, where the ordinary sum and multiplication are replaced by the pseudo-operations. This is a further application of the pseudo-operations, already used in the solution of Cauchy equation and in the axiomatic theory of generalized integrals.

A.M.S. SUBJECT CLASSIFICATION (2000): 39B22, 39B62.

KEY WORDS: Cauchy equation, pseudo-operations.

1 Introduction

After the equation $f(x+y) = f(x) + f(y)$, which is considered as the *Cauchy equation,* Aczél in [1] studied the three following equations, which are known as the *remaining Cauchy equations:*

(I) $\quad f(x+y) = f(x) \cdot f(y)$,

(II) $\quad f(x \cdot y) = f(x+y)$,

(III) $\quad f(x \cdot y) = f(x) \cdot f(y)$.

We call them *classical* remaining equations because they are expressed using classical operations $+$ and $\cdot$.

In the previous paper [4], Benvenuti, Vivona and Divari have studied the Cauchy equation:

$$f(x \oplus y) = f(x) \oplus f(y) \quad ,$$

in which the classical operation $+$ is replaced by the pseudo-operation $\oplus$. We call it the *pseudo*-Cauchy equation.

In this paper, using the pseudo-arithmetical operations $\oplus$ and $\odot$, we shall study the following equations:

(I') $\quad f(x \oplus y) = f(x) \odot f(y)$,

(II') $\quad f(x \odot y) = f(x) \oplus f(y)$,

(III') $\quad f(x \odot y) = f(x) \odot f(y)$,

which we shall call the *pseudo*-remaining Cauchy equations.

We obtain the solutions of the pseudo-remaining equations by reduction to the classical remaining Cauchy equations. However, differently from the classical solutions of the remaining Cauchy equations, we consider only positive solutions.

2 Preliminaries.

Let $\mathcal{F}$ be the family of all continuous functions $f : [0, F] \to [0, F]$, with $0 < F \leq +\infty$.

A binary operation $\oplus : [0, F]^2 \to [0, F]$ is called *pseudo-addition* on $[0, F]$ ([2]) if the following properties are satisfied:

(A1) $\quad x \oplus y = y \oplus x \quad$ *(commutativity),*

(A2) $\quad x \leq x', y \leq y' \;\; \Rightarrow x \oplus y \leq \; x' \oplus y' \quad$ (*monotonicity*),

(A3) $\quad (x \oplus y) \oplus z = x \oplus (y \oplus z) \quad$ *(associativity),*

(A4) $\quad x_n \to x, \; y_n \to y \;\; \Rightarrow x_n \oplus y_n \to x \oplus y \quad$ *(continuity),*

(A5) $\quad x \oplus 0 = x \quad$ *(neutral element).*

These axioms ensure that the pair $([0, F], \oplus)$ is an I-semigroup. The structure of I-semigroup is known by virtue of the representation theorem of Mostert and Shield in [6] (see also [5, 3]). This theorem asserts that there always exist a finite or countable system of open disjoint intervals $(\alpha_h, \beta_h), h \in K, K$ and corresponding continuous, bijective and increasing maps $g_h : [\alpha_h, \beta_h] \to [0, +\infty]$, with $g_h(\alpha_h) = 0$, such that

$$x \oplus y = \begin{cases} g_h^{-1}\Big[\Big(g_h(x) + g_h(y)\Big) \wedge g_h(\beta_h)\Big] , & (x, y) \in (\alpha_h, \beta_h)^2 \\ x \vee y , & \text{otherwise} \end{cases} \tag{1}$$

and the function g_h is determined uniquely up to a multiplicative positive constant.

The interval $I_h = (\alpha_h, \beta_h)$ is of two different types with respect to the operation $\oplus$: in fact, we can have $g_h(\beta_h) = +\infty$ or $g_h(\beta_h) < +\infty$ [4, 2].

We recall, now, the definition of a $\oplus$-fitting pseudo-multiplication $\odot$ [2].

Let $\oplus$ be a given pseudo-addition on $[0, F]$. A binary operation $\odot : [0, F] \times [0, F] \to [0, F]$ is called *a $\oplus$-fitting pseudo-multiplication* if the following properties are satisfied:

(M1) $x \odot 0 = 0 \odot x = 0$ *(zero element),*

(M2) $x \le x', y \le y' \Rightarrow x \odot y \le x' \odot y'$ *(monotonicity),*

(M3) $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ *(left distributivity),*

(M4) $\sup_{n,m} (x_n \odot y_m) = \sup_n(x_n) \odot \sup_m(y_m)$ *(left continuity).*

More, we shall assume

(M5) there exists $u \in (0, F]$ such that $u \odot y = y$ *(left unit element).*

As we have seen in [2], if $\oplus \neq \vee$ and the operation $\odot$ is a $\oplus$-fitting pseudo-multiplication with left unit u, there exists k such that $u \in I_k$, $g_k(\beta_k) = +\infty$ and the pseudo-multiplication is uniquely determined on the strip $I_k \times [0, F]$ by :

$$x \odot y = g_h^{-1}\Big[g_k(x) \cdot g_h(y)/g_k(u)\Big] , \qquad (x, y) \in I_k \times I_h, \tag{2}$$

$h \in K$.

We look for solutions of the equations (I'), (II'), (III') satisfying the following condition:

$$(*) \qquad \text{if} \quad x \in I_k \quad \text{then} \quad f(x) \in I_k \quad .$$

Moreover, we shall find only the restrictions of $f|_{I_k}$.

For the particular index k, the formula (1) becomes

$$x \oplus y = g_k^{-1}\Big[g_k(x) + g_k(y)\Big] \ , \qquad (x, y) \in (\alpha_k, \beta_k)^2. \tag{3}$$

3 The equation $f(x \oplus y) = f(x) \odot f(y)$

From now on we shall assume $x, y \in I_k$.

The equation (I'), using (2) and (3), becomes

$$f\Big[g_k^{-1}\Big(g_k(x) + g_k(y)\Big)\Big] = g_k^{-1}\Big[g_k(f(x)) \cdot g_k(f(y))/g_k(u)\Big].$$

Putting

$$g_k(x) = \xi \quad \text{and} \quad g_k(y) = \eta \ \ , \tag{4}$$

we obtain

$$f\Big(g_k^{-1}(\xi + \eta)\Big) = g_k^{-1}\Big[g_k f(g_k^{-1}(\xi)) \cdot g_k f(g_k^{-1}(\eta))/g_k(u)\Big],$$

and, immediately,

$$g_k f g_k^{-1}(\xi + \eta) = g_k f g_k^{-1}(\xi) \cdot g_k f g_k^{-1}(\eta)/g_k(u). \tag{5}$$

Set, now,

$$g_k f g_k^{-1} = \Psi_k \ \text{ and } \ g_k(u) = \lambda_k > 0 \ \ . \tag{6}$$

So, the equation (5) becomes

$$\Psi_k(\xi + \eta) = \frac{1}{\lambda_k}\ \Big(\Psi_k(\xi) \cdot \Psi_k(\eta)\Big) \ \ , \tag{7}$$

which reduces to the remaining equation (I), when $\lambda_k = 1$.

In order to solve the equation (7), it is easy to prove the following

Lemma 3.1 *The function $h(\xi)$ is a solution of the equation* (I)

$$h(\xi + \eta) = h(\xi) \cdot h(\eta) \tag{8}$$

if and only if the function

$$\Psi_k(\xi) = \lambda_k \cdot h(\xi) \quad , \tag{9}$$

is a solution of the equation (7)*, with $\lambda_k = g_k(u)$.*

We know [1] that the solutions of (8) are given by

$$\forall\ \xi > 0 : \quad h(\xi) = e^{c\xi}\ ,\ c \in I\!R \quad . \tag{10}$$

Now, we are ready to get the first main result:

Theorem 3.2 *The class of the solutions of the equation*

$$(\mathrm{I}') \quad f(x \oplus y) = f(x) \odot f(y)$$

is given by the functions

$$\forall\ x \in I_k \ : \quad f(x) = g_k^{-1}\left(\lambda_k\ e^{c\ g_k(x)}\right),\ c \in I\!R, \quad \text{or} \quad f(x) \equiv a_k\ . \tag{11}$$

Proof . From (6) and (9), we get

$$g_k f g_k^{-1}(\xi) = \Psi_k(\xi) = \lambda_k \cdot h(\xi)\ ,$$

and from (4)

$$g_k f g_k^{-1}(g_k(\xi)) = g_k f(x) = \lambda_k \cdot h(g_k(x))\ .$$

So, the class of the solutions of the equation (I') is given by

$$f(x) = g_k^{-1}\left(\lambda_k \cdot h(g_k(x))\right) \quad .$$

Replacing the function $h(x)$ with the functions (11), we obtain the assertion. $\square$

It is easy to see that the functions (11) satisfy the condition (*).

4 The equation $f(x \odot y) = f(x) \oplus f(y)$

The equation (II') , using the pseudo-operations (2) and (3), becomes

$$f\left[g_k^{-1}\left(g_k(x) \cdot g_k(y)/g_k(u)\right)\right] = g_k^{-1}\Big[g_k(f(x)) + g_k(f(y))\Big],$$

and so

$$g_k f\left[g_k^{-1}\left(g_k(x) \cdot g_k(y)/g_k(u)\right)\right] = g_k(f(x)) + g_k(f(y)).$$

With the same notations as in (4) and (6), we obtain

$$\Psi_k\left(\frac{\xi \cdot \eta}{\lambda_k}\right) = \Psi_k(\xi) + \Psi_k(\eta) \ , \tag{12}$$

which is the remaining equation (II), when $\lambda_k = 1$.

In order to solve the equation (12), it is easy to prove the following

Lemma 4.1 *The function $h(\xi)$ is a solution of the equation* (II)

$$h(\xi \cdot \eta) = h(\xi) + h(\eta) \tag{13}$$

if and only if the function

$$\Psi_k(\xi) = h\left(\frac{\xi}{\lambda_k}\right) \quad , \tag{14}$$

is a solution of the equation (12)*, with $\lambda_k = g_k(u)$.*

We know [1] that the solutions of (13) are given by

$$\forall \ \xi > 0 : \quad h(\xi) = c \ \log \ \xi \ , \ c \in I\!R \ . \tag{15}$$

Now, we are ready to get the second main result:

Theorem 4.2 *The class of the solutions of the equation*

$$(\mathrm{II}') \quad f(x \odot y) = f(x) \oplus f(y)$$

is given by the functions

$$\forall x \in I_k : \quad f(x) = g_k^{-1}\left(c \ \log \frac{g_k(x)}{\lambda_k}\right) \ , \ c \in I\!R \ . \tag{16}$$

Proof. From (6) and (14), we get

$$g_k f g_k^{-1}(\xi) = \Psi_k(\xi) = h\left(\frac{\xi}{\lambda_k}\right)$$

and from (4)

$$g_k f g_k^{-1}(g_k(x)) = g_k f(x) = h\left(\frac{g(x)}{\lambda_k}\right) .$$

So, the class of the solutions of the equation (II') is given by

$$f(x) = g_k^{-1} h\left(\frac{g(x)}{\lambda_k}\right) .$$

Replacing the function $h(x)$ with the functions (15), we obtain the assertion. □

It is easy to see that the functions (16) satisfy the condition (*).

5 The equation $f(x \odot y) = f(x) \odot f(y)$

The equation (III') , using (2) and (3), becomes

$$f\left[g_k^{-1}\Big(g_k(x)\cdot g_k(y)/g_k(u)\Big)\right] = g_k^{-1}\Big(g_k(f(x))\cdot g_k(f(y))/g_k(u)\Big),$$

and so

$$g_k f\left[g_k^{-1}\Big(g_k(x)\cdot g_k(y)/g_k(u)\Big)\right] = g_k(f(x))\cdot g_k(f(y))/g_k(u).$$

With the same notations as in (4), we obtain

$$\Psi_k\left(\frac{\xi\cdot\eta}{\lambda_k}\right) = \frac{\Psi_k(\xi)\cdot\Psi_k(\eta)}{\lambda_k} , \tag{17}$$

which is the remaining equation (III), when $\lambda_k = 1$.

In order to solve the equation (17), it is easy to prove the following

Lemma 5.1 *The function $h(\xi)$ is a solution of the equation* (III)

$$h(\xi\cdot\eta) = h(\xi)\cdot h(\eta) \tag{18}$$

if and only if the function

$$\Psi_k(\xi) = \lambda_k\ h\left(\frac{\xi}{\lambda_k}\right) , \tag{19}$$

is a solution of the equation (17)*, with $\lambda_k = g_k(u)$.*

We know [1] that class of the solutions of (18) are given by the functions

$$\forall\ \xi > 0 : \quad h(\xi) = \xi^c\ ,\ c \in I\!R\ \ . \tag{20}$$

Now, we are ready to get the third main result:

Theorem 5.2 *The class of the solutions of the equation*

$$\text{(III}'\text{)} \quad f(x \odot y) = f(x) \odot f(y)$$

is given by the functions

$$\forall x \in I_k : \quad f(x) = g_k^{-1}(\lambda_k^{1-c}\ g_k^c(x)),\ c \in I\!R_0^+\ . \tag{21}$$

with $\lambda_k = g_k(u)$.

Proof. From (6) and (19), we get

$$g_k f g_k^{-1}(\xi) = \Psi_k(\xi) = \lambda_k\ h\left(\frac{\xi}{\lambda_k}\right)$$

and from (4)

$$g_k f g_k^{-1}(g_k(x)) = g_k f(x) = \lambda_k\ h\left(\frac{g_k(x)}{\lambda_k}\right)\ .$$

So, the class of the solution of the equation (III') is given by

$$f(x) = g_k^{-1}\left[\lambda_k\ h\left(\frac{g_k(x)}{\lambda_k}\right)\right]\ .$$

Replacing the function $h(x)$ with the functions (20), we obtain the assertion. $\square$

It is easy to see that the functions (21) satisfy the condition (*).

6 Conclusion

In this paper we have solved the so called "Pseudo-Remaining Cauchy Equation", using the pseudo-operations. We have given three main theorems, based on three lemmas, which express the solutions of these equations using the solutions of the classical remaining Cauchy equations.

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Journal of Applied Functional Analysis

A quartely international publication of Eudoxus Press, LLC of TN.

Editor in Chief: George Anastassiou

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Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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