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Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Assistant to the Editor:Dr.Razvan Mezei,Lander University,SC 29649, USA.

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fax 055-751-6117
yjcho@nongae.gsnu.ac.kr
Nonlinear operator Th.,Inequalities,
Geometry of Banach Spaces

5) Sever S.Dragomir
School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City M.C
Victoria 8001,Australia
tel 61 3 9688 4437,fax 61 3 9688 4050
sever.dragomir@vu.edu.au,
sever@sci.vu.edu.au
Math.Analysis,Inequalities,Approximation Th.,
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Università di Bologna
Dipartimento di Matematica
Piazza di Porta San Donato 5
40126 Bologna, ITALY
tel.++39 051 2094451
fax.++39 051 2094490
favini@dm.unibo.it
Partial Differential Equations, Control Theory,
Differential Equations in Banach Spaces

7) Claudio A. Fernandez
Facultad de Matematicas
Pontificia Unversidad Católica de Chile
Vicuna Mackenna 4860
Santiago, Chile
tel.++56 2 354 5922
fax.++56 2 552 5916
cfernand@mat.puc.cl
Partial Differential Equations,
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Scattering and Spectral Theory

8) A.M.Fink
Department of Mathematics
Iowa State University
Ames,IA 50011-0001,USA
tel.515-294-8150

25) Svetlozar T.Rachev
Dept.of Statistics and Applied Probability Program
University of California,Santa Barbara
CA 93106-3110,USA
tel.805-893-4869
rachev@pstat.ucsb.edu
AND
Chair of Econometrics and Statistics
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss,Bau II,20.12,R210
Postfach 6980,D-76128,Karlsruhe,Germany
tel.011-49-721-608-7535
rachev@lsoe.uni-karlsruhe.de
Mathematical and Empirical Finance,
Applied Probability, Statistics and Econometrics

26) John Michael Rassias
University of Athens
Pedagogical Department
Section of Mathematics and Infomatics
20, Hippocratous Str., Athens, 106 80, Greece

Address for Correspondence
4, Agamemnonos Str.
Aghia Paraskevi, Athens, Attikis 15342 Greece
jrassias@primedu.uoa.gr
jrassias@tellas.gr
Approximation Theory,Functional Equations,
Inequalities, PDE

27) Paolo Emilio Ricci
Universita' degli Studi di Roma "La Sapienza"
Dipartimento di Matematica-Istituto "G.Castelnuovo"
P.le A.Moro,2-00185 Roma,ITALY
tel.++39 0649913201,fax ++39 0644701007
riccip@uniroma1.it,Paoloemilio.Ricci@uniroma1.it
Orthogonal Polynomials and Special functions,
Numerical Analysis, Transforms,Operational Calculus,
Differential and Difference equations

28) Cecil C.Rousseau
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA
tel.901-678-2490,fax 901-678-2480
ccrousse@memphis.edu
Combinatorics,Graph Th.,
Asymptotic Approximations,
Applications to Physics

29) Tomasz Rychlik

fink@math.iastate.edu
Inequalities,Ordinary Differential Equations

9) Sorin Gal
Department of Mathematics
University of Oradea
Str.Armatei Romane 5
3700 Oradea,Romania
galso@uoradea.ro
Approximation Th.,Fuzzyness,Complex Analysis

10) Jerome A.Goldstein
Department of Mathematical Sciences
The University of Memphis,
Memphis,TN 38152,USA
tel.901-678-2484
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

11) Heiner H.Gonska
Department of Mathematics
University of Duisburg
Duisburg,D-47048
Germany
tel.0049-203-379-3542 office
gonska@informatik.uni-duisburg.de
Approximation Th.,Computer Aided Geometric Design

12) Dmitry Khavinson
Department of Mathematical Sciences
University of Arkansas
Fayetteville,AR 72701,USA
tel.(479)575-6331,fax(479)575-8630
dmitry@uark.edu
Potential Th.,Complex Analysis,Holomorphic PDE,
Approximation Th.,Function Th.

13) Virginia S.Kiryakova
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia 1090,Bulgaria
virginia@diogenes.bg
Special Functions,Integral Transforms,
Fractional Calculus

14) Hans-Bernd Knoop
Institute of Mathematics
Gerhard Mercator University
D-47048 Duisburg
Germany
tel.0049-203-379-2676

Institute of Mathematics
Polish Academy of Sciences
Chopina 12,87100 Torun, Poland
T.Rychlik@impan.gov.pl
Mathematical Statistics,Probabilistic Inequalities

30) Bl. Sendov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia 1090,Bulgaria
bsendov@bas.bg
Approximation Th.,Geometry of Polynomials,
Image Compression

31) Igor Shevchuk
Faculty of Mathematics and Mechanics
National Taras Shevchenko
University of Kyiv
252017 Kyiv
UKRAINE
shevchuk@univ.kiev.ua
Approximation Theory

32) H.M.Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria,British Columbia V8W 3P4
Canada
tel.250-721-7455 office,250-477-6960 home,
fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,Fractional Calculus and Appl.,
Integral Equations and Transforms,Higher Transcendental
Functions and Appl.,q-Series and q-Polynomials,
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33) Stevo Stevic
Mathematical Institute of the Serbian Acad. of Science
Knez Mihailova 35/I
11000 Beograd, Serbia
sstevic@ptt.yu; sstevo@matf.bg.ac.yu
Complex Variables, Difference Equations,
Approximation Th., Inequalities

34) Ferenc Szidarovszky
Dept.Systems and Industrial Engineering
The University of Arizona
Engineering Building,111
PO.Box 210020
Tucson,AZ 85721-0020,USA
szidar@sie.arizona.edu
Numerical Methods,Game Th.,Dynamic Systems,

knoop@math.uni-duisburg.de
Approximation Theory,Interpolation

15) Jerry Koliha
Dept. of Mathematics & Statistics
University of Melbourne
VIC 3010,Melbourne
Australia
koliha@unimelb.edu.au
Inequalities,Operator Theory,
Matrix Analysis,Generalized Inverses

16) Robert Kozma
Dept. of Mathematical Sciences
University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Mathematical Learning Theory,
Dynamic Systems and Chaos,
Complex Dynamics.

17) Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston,RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

18) Gerassimos Ladas
Department of Mathematics
University of Rhode Island
Kingston,RI 02881,USA
gladas@math.uri.edu
Differential and Difference Equations

19) V. Lakshmikantham
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901
e-mail: lakshmik@fit.edu
Ordinary and Partial Differential
Equations,
Hybrid Systems, Nonlinear Analysis

20) Rupert Lasser
Institut fur Biomathematik &
Biomertie,GSF
-National Research Center for
environment and health
Ingolstaedter landstr.1
D-85764 Neuherberg,Germany
lasser@gsf.de
Orthogonal Polynomials,Fourier Analysis,
Mathematical Biology

Multicriteria Decision making,
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35) Gancho Tachev
Dept.of Mathematics
Univ.of Architecture,Civil Eng. and Geodesy
1 Hr.Smirnenski blvd
BG-1421 Sofia,Bulgaria
gtt_fte@uacg.bg
Approximation Theory

36) Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock
Germany
manfred.tasche@mathematik.uni-rostock.de
Approximation Th.,Wavelet,Fourier Analysis,
Numerical Methods,Signal Processing,
Image Processing,Harmonic Analysis

37) Chris P.Tsokos
Department of Mathematics
University of South Florida
4202 E.Fowler Ave.,PHY 114
Tampa,FL 33620-5700,USA
profcpt@math.usf.edu,profcpt@chuma1.cas.usf.edu
Stochastic Systems,Biomathematics,
Environmental Systems,Reliability Th.

38) Lutz Volkmann
Lehrstuhl II fuer Mathematik
RWTH-Aachen
Templergraben 55
D-52062 Aachen
Germany
volkm@math2.rwth-aachen.de
Complex Analysis,Combinatorics,Graph Theory

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Department of Mathematical Sciences
University of Memphis
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THE METHOD OF LAPLACE AND WATSON'S LEMMA

RICHARD A. ZALIK

Abstract. in this paper we present a diferent proof of a well known asymptotic estimate for Laplace integrals. The novelty of our approach is that it emphasizes, and rigorously justifies, the appealing heuristic method of Laplace. As a bonus, we also obtain a simple and short proof of Watson's Lemma.

Let a be an element of the extended real number set $[-\infty, \infty]$. If

$$\lim_{x \longrightarrow a} f(x)/g(x) = 1$$

we write

$$f(x) \sim g(x), \quad x \longrightarrow a$$

and say that f is asymptotic to g, or that g is an asymptotic approximation to f. If there is no risk of ambiguity, we may also write $f \sim g$ for the sake of brevity.

Note that $f \sim g$ if and only if $\lim(f(x) - g(x))/g(x) = 0$. In other words, if and only if the relative error made in approximating f by g tends to zero. In some cases, in particular when the values involved are either very small or very large, it may be more appropriate to estimate the relative rather than the absolute error.

In this article we discuss the problem of obtaining asymptotic estimates for integrals of the form

$$I(x) := \int_J e^{-xp(t)} q(t)\, dt, \tag{1}$$

where J is a bounded or unbounded interval, $p(x)$ and $q(x)$ are functions satisfying certain properties, and $x \longrightarrow \infty$. Most authors, such as Bender and Orszag [1] use an appealing heuristic method attributed to Laplace to obtain an asymptotic estimate for $I(x)$. A rigorous proof may be found in, for example, Erdélyi [2, §2.4] (see also Olver [3, pp.80–82]). We give another rigorous proof of this estimate in Theorem 1. The novelty of our approach consists in breaking down the proof by means of two preliminary lemmas that highlight and rigorously justify the method of Laplace.

Under suitable conditions $I(x)$ has an infinite asymptotic expansion (see for example [3, pp.85–88]). In Theorem 2 we use Lemma 2 to give a simple proof of Watson's lemma, which gives an infinite asymptotic expansion for $I(x)$ when $p(t) = t$. We define asymptotic series in the paragraph preceding the statement of Theorem 2.

We begin with

Lemma 1. *Let J be an interval of the form $[a, \infty)$ or $[a, b]$, $a < b$, and assume that the following conditions are satisfied:*

Key words and phrases. Asymptotic methods; Laplace integrals; Watson's Lemma.
2010 Mathematics Classification: 34E05.

(a) *The function* $p(t)$ *is real–valued and measurable on* J *and for every point* $c > a$ *in* J,

$$\inf\{p(t); t \in J \cap [c, \infty)\} > p(a). \tag{2}$$

(b) *There is a number* $\sigma > 0$ *such that* $p(t)$ *is continuous and strictly increasing on* $[a, a + \sigma]$.

(c) *The function* $q(t)$ *is Lebesgue integrable on* J.

(d) *There are numbers* $\alpha > -1$ *and* $Q \neq 0$ *such that*

$$q(t) \sim Q(t - a)^{\alpha}, \quad t \to a^{+}.$$

Let $I(x)$ *be given by (1) and, for* $\delta > 0$ *such that* $a + \delta \in J$,

$$I(x, \delta) := \int_a^{a+\delta} e^{-xp(t)} q(t)\, dt. \tag{3}$$

Then $e^{-xp(t)} q(t)$ *is Lebesgue integrable on* J *for every positive* x *and there is a number* $\eta > 0$ *such that* $p(t)$ *is strictly increasing on* $[a, a + \eta]$, *and* $0 < \delta \leq \eta$ *implies that*

$$I(x) \sim I(x, \delta), \quad x \to \infty. \tag{4}$$

Remark 1: Equation (2) is needed when $J = [a, \infty)$ to ensure that $p(t)$ remains bounded away from $p(a)$ as t grows. If $J = [a, b]$, (2) is equivalent to $p(t)$ having a unique absolute minimum at a.

Proof. The integrability follows from

$$|e^{-xp(t)} q(t)| = e^{-xp(a)} e^{-x[p(t)-p(a)]} |q(t)| \leq e^{-xp(a)} |q(t)|.$$

From (b) and (d) there is a number $\eta > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a + \eta]$, and if $t \in (a, a + \eta]$ then

$$|Q^{-1}(t - a)^{-\alpha} q(t) - 1| < 1/2.$$

Thus $Q^{-1}(t - a)^{-\alpha} q(t) > 1/2$, and we conclude that $q(t)$ has constant sign on $(a, a + \eta]$ and that

$$|q(t)| > (1/2)|Q|(t - a)^{\alpha} \quad \text{on} \quad (a, a + \eta].$$

Let $\delta \in (0, \eta]$ be arbitrary but fixed, and let $J_\delta := J \setminus [a, a + \delta)$ and

$$M_\delta := \inf\{p(t); t \in J_\delta\}.$$

Assume $x > 0$. Since the hypotheses imply that $p(a) < M_\delta$ we have

$$\left| \int_{J_\delta} e^{-xp(t)} q(t)\, dt \right| \leq \int_{J_\delta} e^{-xp(t)} |q(t)|\, dt \leq K_1 e^{-M_\delta x}.$$

Let $C \in (p(a), M_\delta)$. By continuity, there is a $\delta_1 \in (0, \delta]$ such that $p(a + \delta_1) < C$. Since $p(t)$ is strictly increasing on $[a, a + \delta_1]$, we deduce that $p(t) < C$ for $t \in [a, a + \delta_1]$. Since $q(t)$ has constant sign on $(a, a + \delta]$, we have

$$|I(x, \delta)| = \left| \int_a^{a+\delta} e^{-xp(t)} q(t)\, dt \right| = \int_a^{a+\delta} e^{-xp(t)} |q(t)|\, dt \geq$$

$$\int_a^{a+\delta_1} e^{-xp(t)} |q(t)|\, dt \geq (1/2)|Q| \int_a^{a+\delta_1} e^{-xp(t)} (t - a)^{\alpha} |\, dt \geq K_2 e^{-Cx}.$$

Thus
$$\left|\frac{\int_{J_\delta} e^{-xp(t)}q(t)\,dt}{I(x,\delta)}\right| \le K_3 e^{(C-M_\delta)x} \longrightarrow 0, \quad x \longrightarrow \infty.$$
Since
$$I(x) = I(x,\delta) + \int_{J_\delta} e^{-xp(t)}q(t)\,dt,$$
the assertion follows. □

As a consequence of Lemma 1 we obtain the following basic proposition, which will be used to prove both theorems in this paper.

Lemma 2. *Let J be an interval of the form $[a,\infty)$ or $[a,b]$ with $a<b$, and assume that the function $q(t)$ satisfies conditions (c) and (d) of Lemma 1. Then*
$$\int_J e^{-xt}q(t)\,dt \sim Q\int_J e^{-xt}(t-a)^\alpha\,dt, \quad x\to\infty.$$

Proof. If $J=[a,\infty)$, Lemma 1 implies that there exists a number $b>a$ such that
$$\int_J e^{-xt}q(t)\,dt \sim \int_a^b e^{-xt}q(t)\,dt, \quad x\to\infty.$$
Thus we may assume without loss of generality that $J=[a,b]$.

Let
$$A(x) := \frac{\int_a^b e^{-xt}q(t)\,dt}{Q\int_a^b e^{-xt}(t-a)^\alpha\,dt}, \quad A_1(x,\delta) := \frac{\int_a^b e^{-xt}q(t)\,dt}{\int_a^{a+\delta} e^{-xt}q(t)\,dt},$$
$$A_2(x,\delta) := \frac{\int_a^{a+\delta} e^{-xt}q(t)\,dt}{Q\int_a^{a+\delta} e^{-xt}(t-a)^\alpha\,dt}, \quad A_3(x,\delta) := \frac{\int_a^{a+\delta} e^{-xt}(t-a)^\alpha\,dt}{\int_a^b e^{-xt}(t-a)^\alpha\,dt}.$$
Applying Lemma 1 we see that there is a $\delta_1>0$ such that if $0<\delta\le\delta_1$, then
$$\lim_{x\to\infty} A_1(x,\delta) = \lim_{x\to\infty} A_3(x,\delta) = 1. \tag{5}$$
Let $\varepsilon>0$ be given. Then there is a $\delta_2>0$ such that if $0<t-a<\delta_2$, then
$$|Q^{-1}(t-a)^{-\alpha}q(t)-1| < \varepsilon. \tag{6}$$
Let $\delta:=\min(\delta_1,\delta_2)$. Since the Generalized Mean Value Theorem implies there is a $\xi\in(a,a+\delta)$ such that $A_2(x,\delta)=Q^{-1}(\xi-a)^{-\alpha}q(\xi)$, we conclude from (6) that
$$-\varepsilon \le A_2(x,\delta)-1 \le \varepsilon.$$
Since
$$A(x) = A_1(x,\delta)A_2(x,\delta)A_3(x,\delta),$$
we deduce from (5) that
$$1-\varepsilon \le \liminf_{x\to\infty} A(x) \le \limsup_{x\to\infty} A(x) \le 1+\varepsilon.$$
Since ε is arbitrary, the assertion follows. □

Theorem 1. *Let J be an interval of the form $[a,\infty)$ or $[a,b]$, $a<b$, and assume that the following conditions are satisfied:*

(a) *The function $p(t)$ is real–valued and measurable on J and for every point $c>a$ in J, inequality (2) holds.*
(b) *There is a number $\sigma>0$ such that $p(t)$ is continuous and strictly increasing on $[a,a+\sigma]$, and $p\in C^1(a,a+\sigma]$.*

(c) *There are numbers* $P, \mu > 0$ *such that*

$$p(t) - p(a) \sim P(t-a)^{\mu}, \quad t \to a^+, \tag{7}$$

and

$$p'(t) \sim \mu P(t-a)^{\mu-1}, \quad t \to a^+. \tag{8}$$

(d) *The function* $q(t)$ *is Lebesgue integrable on* J.

(e) *There are numbers* $\lambda > 0$ *and* $Q \neq 0$ *such that*

$$q(t) \sim Q(t-a)^{\lambda-1}, \quad t \to a^+. \tag{9}$$

Then $e^{-xp(t)}q(t)$ *is Lebesgue integrable on* J *for every positive* x*, and if* $I(x)$ *is given by (1) then*

$$I(x) \sim \frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\frac{e^{-xp(a)}}{(Px)^{\lambda/\mu}}, \quad x \to \infty. \tag{10}$$

Remark 2: Conditions (b), (c) and (d) are satisfied if, for instance, there is a positive integer μ and a positive number σ such that $p \in C^{\mu}[a, a+\sigma]$,

$$p^{(\ell)}(a) = 0, 1 \leq \ell \leq \mu - 1, \quad \text{and} \quad p^{(\mu)}(a) < 0.$$

Proof. From Lemma 1 we deduce that there is a $\delta_0 > 0$ such that $p(t)$ is continuous and strictly increasing on $[a, a+\delta_0]$, $p \in C^1(a, a+\delta_0]$, and (4) is satisfied for any $\delta \in (0, \delta_0]$. Let δ be an arbitrary but fixed number in $(0, \delta_0]$, and let $I(x, \delta)$ be given by (3). Making the change of variable $s = p(t)$ we see that

$$I(x, \delta) = \int_{p(a)}^{p(a+\delta)} e^{-xs}\frac{q[p^{-1}(s)]}{p'[p^{-1}(s)]}\,ds.$$

Setting $c = p(a)$ and applying (7) we have

$$\lim_{s \to c^+}\frac{\left(p^{-1}(s) - a\right)^{\mu}}{s - c} = \lim_{t \to a^+}\frac{(t-a)^{\mu}}{p(t) - p(a)} = 1/P.$$

Therefore

$$p^{-1}(s) - a \sim \left(\frac{s-c}{P}\right)^{1/\mu}, \quad s \to c^+. \tag{11}$$

Moreover, (9) implies that

$$q[p^{-1}(s)] \sim Q[p^{-1}(s) - a]^{\lambda-1}, \quad s \to c^+.$$

Thus, from (11),

$$q[p^{-1}(s)] \sim Q\left(\frac{s-c}{P}\right)^{(\lambda-1)/\mu}, \quad s \to c^+. \tag{12}$$

On the other hand, (8) and (11) yield

$$p'[p^{-1}(s)] \sim \mu P(p^{-1}(s) - a)^{\mu-1} \sim \mu(P)^{1/\mu}(s-c)^{1-1/\mu}, \quad s \to c^+. \tag{13}$$

Combining (12) and (13) we conclude that

$$\frac{q[p^{-1}(s)]}{p'[p^{-1}(s)]} \sim (Q/\mu)(P)^{-(\lambda/\mu)}(s-c)^{\lambda/\mu-1}, \quad s \to c^+,$$

and Lemma 2 implies that

$$I(x,\delta) \sim \int_{p(a)}^{p(a+\delta)} e^{-xs}(Q/\mu)(P)^{-(\lambda/\mu)}(s-c)^{\lambda/\mu-1}\,ds =$$

$$(Q/\mu)(P)^{-(\lambda/\mu)} \int_{p(a)}^{p(a+\delta)} e^{-xs}(s-c)^{\lambda/\mu-1}\,ds =$$

$$(Q/\mu)(P)^{-\lambda/\mu}e^{p(a)x} \int_{0}^{p(a+\delta)-p(a)} e^{-xt}t^{\lambda/\mu-1}\,dt.$$

But Lemma 1 implies that there is a $\delta \in (0,\delta_0]$ such that

$$\int_0^{p(a+\delta)-p(a)} e^{-xt}t^{\lambda/\mu-1}\,dt \sim \int_0^{\infty} e^{-xt}t^{\lambda/\mu-1}\,dt = \Gamma\left(\frac{\lambda}{\mu}\right)x^{-\lambda/\mu}, \quad x \to \infty,$$

and the assertion follows. □

In the preceding theorem we have assumed that the minimum of $p(t)$ is unique and occurs at a. In other cases the interval of integration may be subdivided at the maxima and minima of $p(t)$. We may then apply Theorem 1 to intervals where the minimum is at a left end–point, and Corollary 1 below to intervals where the minimum is at a right end–point.

Making the change of variable $t \longrightarrow -t$ we obtain

Corollary 1. *Let J be an interval of the form $(-\infty, b]$ or $[a,b]$, $a < b$, and assume that the following conditions are satisfied:*

(a) *The function $p(t)$ is real–valued and measurable on J and for every point $c < b$ in J,*
$$\inf\{p(t); t \in J \cap (-\infty, c]\} > p(b).$$
(b) *There is a number $\sigma > 0$ such that $p(t)$ is continuous and strictly decreasing on $[b-\sigma, b]$, and $p \in C^1[b-\sigma, b)$.*
(c) *There are numbers $P, \mu > 0$ such that*
$$p(t) - p(b) \sim P(b-t)^{\mu}, \quad t \longrightarrow b^-.$$
and
$$p'(t) \sim -\mu P(b-t)^{\mu-1}, \quad t \to b^-.$$
(d) *The function $q(t)$ is Lebesgue integrable on J.*
(e) *There are numbers $\lambda > 0$ and $Q \neq 0$ such that*
$$q(t) \sim Q(b-t)^{\lambda-1}, \quad t \to b^-.$$

Then $e^{-xp(t)}q(t)$ is Lebesgue integrable on J for every positive x, and if $I(x)$ is given by (1) then

$$I(x) \sim \frac{Q}{\mu}\Gamma\left(\frac{\lambda}{\mu}\right)\frac{e^{-xp(b)}}{(Px)^{\lambda/\mu}}, \quad x \to \infty.$$

Before proceeding further, let us recall the definition of asymptotic expansion in the particular format that we require here. Let $(f_k)_{k=0}^{\infty}$ be a sequence of functions, let $(a(k))_{k=0}^{\infty}$ be a sequence of scalars such that the set of integers k for which $a(k) \neq 0$ is infinite, and let $m(k) := \inf\{r \geq k : a(r) \neq 0\}$. We say that

$$f(x) \sim \sum_{k=0}^{\infty} a(k) f_k, \quad x \longrightarrow a,$$

if for every $n > 0$

$$f(x) - \sum_{k=0}^{n-1} a(k) f_k \sim a(m(n)) f_{m(n))}, \quad x \longrightarrow a,$$

Fron Lemma 2 we obtain a simple proof of Watson's Lemma ([2, §2.4], [3, p.71]):

Theorem 2. *(Watson's Lemma) Let J be an interval of the form $[0, \infty)$ or $[0, b]$, $b > 0$, and assume that $q(t)$ is Lebesgue integrable on J and that there are $\alpha > -1$ and $\beta > 0$ such that*

$$q(t) \sim \sum_{k=0}^{\infty} a_k t^{\alpha+\beta k}, \quad t \to 0^+.$$

If

$$I(x) := \int_J e^{-xt} q(t)\, dt,$$

then

$$I(x) \sim \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha + \beta k + 1)}{x^{\alpha+\beta k+1}}, \quad x \to \infty.$$

Proof. Defining if necessary $q(t)$ to equal 0 on $[b, \infty)$ we may assume, without essential loss of generality, that $J = [0, \infty)$.

Let $n \geq 0$ be an arbitrary integer, and let N denote the smallest integer $k \geq n$ such that $a_{k+1} \neq 0$. By definition,

$$q(t) - \sum_{k=0}^{n} a_k t^{\alpha+\beta k} \sim a_{N+1} t^{\alpha+\beta(N+1)}, \quad t \to 0^+.$$

Applying Lemma 2 we conclude that

$$\int_0^{\infty} \left[q(t) - \sum_{k=0}^{n} a_k t^{\alpha+\beta k} \right] e^{-xt}\, dt \sim \int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt}\, dt, \quad x \to \infty,$$

i.e.,

$$I(x) - \sum_{k=0}^{n} a_k \frac{\Gamma(\alpha + \beta k + 1)}{x^{\alpha+\beta k+1}} \sim \frac{\Gamma(\alpha + \beta(N+1) + 1)}{x^{\alpha+\beta(N+1)+1}}, \quad x \to \infty,$$

and the assertion follows. □

The interested reader will have no difficulty in applying Lemma 1 and Lemma 2 to obtain a version of Watson's lemma for an arbitrary $p(t)$, as in [3, p.86].

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Department of Mathematics and Statistics, Auburn University, AL 36849–5310
E-mail address: `zalik@auburn.edu`

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On special strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh derivative

Alina Alb Lupaş
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro

Abstract

In the present paper we establish several strong differential subordinations regardind the new operator $RD_{\lambda,\alpha}^n$, given by $RD_{\lambda,\alpha}^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $RD_{\lambda,\alpha}^n f(z,\zeta) = (1-\alpha)R^n f(z,\zeta) + \alpha D_\lambda^n f(z,\zeta)$, where $R^n f(z,\zeta)$ denote the Ruscheweyh derivative, $D_\lambda^n f(z,\zeta)$ is the generalized Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$. A certain subclass, denoted by $\mathcal{RD}_n(\delta,\lambda,\alpha)$, of analytic functions is introduced by means of the new operator.

Keywords: strong differential subordination, univalent function, convex function, differential operator, best dominant, generalized Sălăgean operator, Ruscheweyh derivative.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$, with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Definition 1.1 *For $f \in \mathcal{A}_\zeta^*$, $\lambda \ge 0$ and $n \in \mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} D_\lambda^0 f(z,\zeta) &= f(z,\zeta), \\ D_\lambda^1 f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f'(z,\zeta) = D_\lambda f(z,\zeta), \dots \\ D_\lambda^{n+1} f(z,\zeta) &= (1-\lambda) D_\lambda^n f(z,\zeta) + \lambda z \left(D_\lambda^n f(z,\zeta)\right)' = D_\lambda\left(D_\lambda^n f(z,\zeta)\right), \quad z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark 1.1 *If $f \in \mathcal{A}_\zeta^*$ and $f(z) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $D_\lambda^n f(z,\zeta) = z + \sum_{j=2}^\infty [1 + (j-1)\lambda]^n a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.2 *For $f \in \mathcal{A}_\zeta^*$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'(z,\zeta), \dots \\ (n+1) R^{n+1} f(z,\zeta) &= z \left(R^n f(z,\zeta)\right)' + n R^n f(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $R^n f(z,\zeta) = z + \sum_{j=2}^\infty C_{n+j-1}^n a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [4] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [6], [5].

Definition 1.3 *[6] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\zeta)=H(w(z),\zeta)$ for all $\zeta\in\overline{U}$. In such a case we write $f(z,\zeta)\prec\prec H(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.3 *[6] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.3 is equivalent to $f(0,\zeta)=H(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f\left(U\times\overline{U}\right)\subset H\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma 1.1 *[3] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]$ and $p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, then $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, where $g(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$ is convex and it is the best dominant.*

Lemma 1.2 *[3] Let $g(z,\zeta)$ be a convex function in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and let $h(z,\zeta)=g(z,\zeta)+n\alpha zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $\alpha>0$ and n is a positive integer. If $p(z,\zeta)=g(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots$, $z\in U$, $\zeta\in\overline{U}$, is holomorphic in $U\times\overline{U}$ and $p(z,\zeta)+\alpha zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, then $p(z,\zeta)\prec\prec g(z,\zeta)$ and this result is sharp.*

2 Main results

Definition 2.1 *Let $\delta\in[0,1)$, $\alpha,\lambda\geq 0$ and $m\in\mathbb{N}$. A function $f\in\mathcal{A}^*_\zeta$ is said to be in the class $\mathcal{RD}_m(\delta,\lambda,\alpha,\zeta)$ if it satisfies the inequality*

$$\operatorname{Re}\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z>\delta,\qquad z\in U,\ \zeta\in\overline{U}. \tag{2.1}$$

Theorem 2.1 *The set $\mathcal{RD}_m(\delta,\lambda,\alpha,\zeta)$ is convex.*

Proof. Let the functions $f_j(z,\zeta)=z+\sum_{j=2}^{\infty}a_{jk}(\zeta)z^j$, $k=1,2$, $z\in U$, $\zeta\in\overline{U}$, be in the class $\mathcal{RD}_m(\delta,\lambda,\alpha,\zeta)$. It is sufficient to show that the function $h(z,\zeta)=\eta_1f_1(z,\zeta)+\eta_2f_2(z,\zeta)$ is in the class $\mathcal{RD}_m(\delta,\lambda,\alpha,\zeta)$, with η_1 and η_2 nonnegative such that $\eta_1+\eta_2=1$.

Since $h(z,\zeta)=z+\sum_{j=2}^{\infty}(\eta_1a_{j1}(\zeta)+\eta_2a_{j2}(\zeta))z^j$, $z\in U$, $\zeta\in\overline{U}$, then

$$RD^m_{\lambda,\alpha}h(z,\zeta)=z+\sum_{j=2}^{\infty}\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}(\eta_1a_{j1}(\zeta)+\eta_2a_{j2}(\zeta))z^j,\ \ z\in U,\zeta\in\overline{U}. \tag{2.2}$$

Differentiating (2.2) we obtain

$\left(RD^m_{\lambda,\alpha}h(z,\zeta)\right)'_z=1+\sum_{j=2}^{\infty}\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}(\eta_1a_{j1}(z)+\eta_2a_{j2}(\zeta))jz^{j-1}$, $z\in U,\zeta\in\overline{U}$.

Hence

$$\operatorname{Re}\left(RD^m_{\lambda,\alpha}h(z,\zeta)\right)'_z=1+\operatorname{Re}\left(\eta_1\sum_{j=2}^{\infty}j\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}a_{j1}(\zeta)z^{j-1}\right) \tag{2.3}$$

$$+\operatorname{Re}\left(\eta_2\sum_{j=2}^{\infty}j\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}a_{j2}(\zeta)z^{j-1}\right).$$

Taking into account that $f_1,f_2\in\mathcal{RD}_m(\delta,\lambda,\alpha,\zeta)$ we deduce

$$\operatorname{Re}\left(\eta_k\sum_{j=2}^{\infty}j\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}a_{jk}(\zeta)z^{j-1}\right)>\eta_k(\delta-1),\quad k=1,2. \tag{2.4}$$

Using (2.4) we get from (2.3)

$$\operatorname{Re}\left(RD^m_{\lambda,\alpha}h\left(z,\zeta\right)\right)'_z > 1+\eta_1\left(\delta-1\right)+\eta_2\left(\delta-1\right)=\delta, \quad z\in U, \zeta\in\overline{U},$$

which is equivalent that $\mathcal{RD}_m\left(\delta,\lambda,\alpha,\zeta\right)$ is convex. ■

Theorem 2.2 *Let $g\left(z,\zeta\right)$ be a convex function such that $g\left(0,\zeta\right)=1$ and let h be the function $h\left(z,\zeta\right)=g\left(z,\zeta\right)+\frac{1}{c+2}zg'_z\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, $c>0$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{RD}_m\left(\delta,\lambda,\alpha,\zeta\right)$ and $F\left(z,\zeta\right)=I_c\left(f\right)\left(z,\zeta\right)=\frac{c+2}{z^{c+1}}\int_0^z t^c f\left(t,\zeta\right)dt$, $z\in U$, $\zeta\in\overline{U}$, then*

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z \prec\prec h\left(z,\zeta\right), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.5}$$

implies

$$\left(RD^m_{\lambda,\alpha}F(z,\zeta)\right)'_z \prec\prec g\left(z,\zeta\right), \quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F\left(z,\zeta\right)=\left(c+2\right)\int_0^z t^c f\left(t,\zeta\right)dt. \tag{2.6}$$

Differentiating (2.6), with respect to z, we have $\left(c+1\right)F\left(z,\zeta\right)+zF'_z\left(z,\zeta\right)=\left(c+2\right)f\left(z,\zeta\right)$ and

$$\left(c+1\right)RD^m_{\lambda,\alpha}F\left(z,\zeta\right)+z\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z=\left(c+2\right)RD^m_{\lambda,\alpha}f\left(z,\zeta\right), \quad z\in U,\ \zeta\in\overline{U}. \tag{2.7}$$

Differentiating (2.7) with respect to z we have

$$\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z+\frac{1}{c+2}z\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)''_{z^2}=\left(RD^m_{\lambda,\alpha}f\left(z,\zeta\right)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{2.8}$$

Using (2.8), the strong differential subordination (2.5) becomes

$$\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z+\frac{1}{c+2}z\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)''_{z^2}\prec\prec g\left(z,\zeta\right)+\frac{1}{c+2}zg'_z\left(z,\zeta\right). \tag{2.9}$$

Denote

$$p\left(z,\zeta\right)=\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z, \quad z\in U,\ \zeta\in\overline{U}. \tag{2.10}$$

Replacing (2.10) in (2.9) we obtain

$$p\left(z,\zeta\right)+\frac{1}{c+2}zp'_z\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)+\frac{1}{c+2}zg'_z\left(z,\zeta\right), \quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2 we have

$$p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e.}\ \left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z\prec\prec g\left(z,\zeta\right), z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. ■

Theorem 2.3 *Let $h\left(z,\zeta\right)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$, $\delta\in[0,1)$ and $c>0$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$ and I_c is given by Theorem 2.2, then*

$$I_c\left[\mathcal{RD}_m\left(\delta,\lambda,\alpha,\zeta\right)\right]\subset\mathcal{RD}_m\left(\delta^*,\lambda,\alpha,\zeta\right), \tag{2.11}$$

where $\delta^*=2\delta-\zeta+2\left(c+2\right)\left(\zeta-\delta\right)\beta\left(c\right)$ and $\beta\left(x\right)=\int_0^1\frac{t^x}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that $p\left(z,\zeta\right)+\frac{1}{c+2}zp'_z\left(z,\zeta\right)\prec h\left(z,\zeta\right)$, where $p\left(z,\zeta\right)$ is defined in (2.10).

Using Lemma 1.1 we deduce that $p\left(z,\zeta\right)\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, that is $\left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z\prec\prec g\left(z,\zeta\right)\prec\prec h\left(z,\zeta\right)$, where $g\left(z,\zeta\right)=\frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\frac{\zeta+(2\delta-\zeta)t}{1+t}dt=\left(2\delta-\zeta\right)+\frac{2(c+2)(\zeta-\delta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{1+t}dt$. Since g is convex and $g\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re}\ \left(RD^m_{\lambda,\alpha}F\left(z,\zeta\right)\right)'_z\geq\min_{|z|=1}\operatorname{Re}\ g\left(z,\zeta\right)=\operatorname{Re}\ g\left(1,\zeta\right)=\delta^*=2\delta-\zeta+2\left(c+2\right)\left(\zeta-\delta\right)\beta\left(c\right). \tag{2.12}$$

From (2.12) we deduce inclusion (2.11). ■

Theorem 2.4 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.13}$$

holds, then

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\prec\prec g(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. By using the properties of operator $RD^m_{\lambda,\alpha}$, we have $RD^m_{\lambda,\alpha}f(z,\zeta)=z+\sum_{j=2}^\infty\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Consider $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}=\frac{z+\sum_{j=2}^\infty\left\{\alpha[1+(j-1)\lambda]^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j}{z}=1+p_1(\zeta)z+p_2(\zeta)z^2+\ldots$, $z\in U$, $\zeta\in\overline{U}$.

Let $RD^m_{\lambda,\alpha}f(z,\zeta)=zp(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Differentiating with respect to z we obtain $\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z=p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then (2.13) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e. } \frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

■

Theorem 2.5 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.14}$$

holds, then

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t)dt$ is convex and it is the best dominant.

Proof. With notation $p(z,\zeta)=\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}=1+\sum_{j=2}^\infty\left\{\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^{j-1}$ and $p(0,\zeta)=1$, we obtain for $f(z,\zeta)=z+\sum_{j=2}^\infty a_j(\zeta)z^j$, $p(z,\zeta)+zp'_z(z,\zeta)=\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z$.

We have $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Since $p(z,\zeta)\in\mathcal{H}^*[1,1,\zeta]$, using Lemma 1.1, for $n=1$ and $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{z}\int_0^z h(t)dt\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

Corollary 2.6 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, a convex function in $U\times\overline{U}$, $0\leq\beta<1$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and verifies the strong differential subordination*

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.15}$$

then

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

where g is given by $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.5 and considering $p(z,\zeta) = \frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z}$, the strong differential subordination (2.15) becomes

$$p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z}, \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.1 for $n = 1$ and $\gamma = 1$, we have $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.,

$$\frac{RD^m_{\lambda,\alpha}f(z,\zeta)}{z} \prec\prec g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)\,dt = \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)\,t}{1+t}dt = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z),$$

$z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.7 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha, \lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\left(\frac{zRD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)'_z \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.16}$$

holds, then

$$\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{RD^m_{\lambda,\alpha}f(z,\zeta)} \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta)\, z^j$ we have
$RD^m_{\lambda,\alpha}f(z,\zeta) = z + \sum_{j=2}^{\infty}\left\{\alpha\left[1+(j-1)\lambda\right]^m + (1-\alpha)\,C^m_{m+j-1}\right\} a_j(\zeta)\, z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider $p(z,\zeta) = \frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{RD^m_{\lambda,\alpha}f(z,\zeta)} = \frac{z+\sum_{j=2}^{\infty}\left(\alpha[1+(j-1)\lambda]^{m+1}+(1-\alpha)C^{m+1}_{m+j}\right)a_j(\zeta)z^j}{z+\sum_{j=2}^{\infty}\left(\alpha[1+(j-1)\lambda]^{m}+(1-\alpha)C^{m}_{m+j-1}\right)a_j(\zeta)z^j} = \frac{1+\sum_{j=2}^{\infty}\left(\alpha[1+(j-1)\lambda]^{m+1}+(1-\alpha)C^{m+1}_{m+j}\right)a_j(\zeta)z^{j-1}}{1+\sum_{j=2}^{\infty}\left(\alpha[1+(j-1)\lambda]^{m}+(1-\alpha)C^{m}_{m+j-1}\right)a_j(\zeta)z^{j-1}}$.

We have $p'_z(z,\zeta) = \frac{\left(RD^{m+1}_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)} - p(z,\zeta)\cdot\frac{\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z}{RD^m_{\lambda,\alpha}f(z,\zeta)}$.

Then $p(z,\zeta) + zp'_z(z,\zeta) = \left(\frac{zRD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{RD^m_{\lambda,\alpha}f(z,\zeta)}\right)'_z$.

Relation (2.16) becomes $p(z,\zeta) + zp'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and by using Lemma 1.2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\frac{RD^{m+1}_{\lambda,\alpha}f(z,\zeta)}{RD^m_{\lambda,\alpha}f(z,\zeta)} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.8 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. If $\alpha, \lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z,\zeta) -$$
$$\frac{\alpha\left[(m+1)(m+2) - \frac{1}{\lambda^2}\right]}{z}D^{m+2}_{\lambda}f(z,\zeta) + \frac{\alpha\left[(m+1)(2m+1) - \frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_{\lambda}f(z,\zeta) -$$
$$\frac{\alpha\left[m^2 - \frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_{\lambda}f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.17}$$

holds, then

$$\left[RD^m_{\lambda,\alpha}f(z,\zeta)\right]'_z \prec\prec g(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

This result is sharp.

Proof. Let

$$p(z,\zeta)=\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z=(1-\alpha)\left(R^mf(z,\zeta)\right)'_z+\alpha\left(D^m_\lambda f(z,\zeta)\right)'_z \tag{2.18}$$

$$=1+\sum_{j=2}^{\infty}\left(\alpha\left[1+(j-1)\lambda\right]^m+(1-\alpha)C^m_{m+j-1}\right)ja_j(\zeta)z^{j-1}=1+p_1(\zeta)z+p_2(\zeta)z^2+....$$

By using the properties of operators $RD^m_{\lambda,\alpha}$, R^m and D^m_λ, after a short calculation, we obtain
$p(z,\zeta)+zp'_z(z,\zeta)=\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z,\zeta)-$
$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z,\zeta)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z,\zeta)-\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z,\zeta)$.

Using the notation in (2.18), the strong differential subordination becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta).$$

By using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}\quad \left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. ∎

Theorem 2.9 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z,\zeta)-$$

$$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z,\zeta)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z,\zeta)-$$

$$\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{2.19}$$

holds, then

$$\left[RD^m_{\lambda,\alpha}f(z,\zeta)\right]'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{z}\int_0^z h(t)dt$ is convex and it is the best dominant.

Proof. Using the properties of operator $RD^m_{\lambda,\alpha}$ and considering $p(z)=\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\in\mathcal{H}^*[1,1,\zeta]$, we obtain
$p(z,\zeta)+zp'_z(z,\zeta)=\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z,\zeta)-$
$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z,\zeta)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z,\zeta)-\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then $\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z,\zeta)-$
$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z,\zeta)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z,\zeta)-\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, becomes $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.1, for $n=1$ and $\gamma=1$, we obtain

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}$$

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta)=\frac{1}{z}\int_0^z h(t)dt\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and $g(z,\zeta)$ is convex and it is the best dominant. ∎

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Corollary 2.10 *Let* $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ *a convex function in* $U\times\overline{U}$, $0\leq\beta<1$. *If* $\alpha,\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ *and verifies the strong differential subordination*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z,\zeta)+\frac{m^2}{z}RD^{m}_{\lambda,\alpha}f(z,\zeta)-$$
$$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z,\zeta)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z,\zeta)-$$
$$\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^{m}_\lambda f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{2.20}$$

then

$$\left[RD^m_{\lambda,\alpha}f(z,\zeta)\right]'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where g *is given by* $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z\in U$, $\zeta\in\overline{U}$. *The function* q *is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z,\zeta)=\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z$, the strong differential subordination (2.20) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z},\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1 for $n=1$ and $\gamma=1$, we have $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\left(RD^m_{\lambda,\alpha}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\,dt=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(1+z),$$

$z\in U$, $\zeta\in\overline{U}$. ■

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A note on special strong differential subordinations using a multiplier transformation and Ruscheweyh derivative

Alina Alb Lupaş
Department of Mathematics and Computer Science
University of Oradea
str. Universitatii nr. 1, 410087 Oradea, Romania
dalb@uoradea.ro

Abstract

In the present paper we establish several strong differential subordinations regardind the new operator $RI^{\alpha}_{m,\lambda,l}$, given by $RI^{\alpha}_{m,\lambda,l}:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$, $RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=(1-\alpha)R^m f(z,\zeta)+\alpha I(m,\lambda,l)f(z,\zeta)$, where $R^m f(z,\zeta)$ denote the Ruscheweyh derivative, $I(m,\lambda,l)$ is the multiplier transformation and $\mathcal{A}^*_{n\zeta}=\{f\in\mathcal{H}(U\times\overline{U}),\ f(z,\zeta)=z+a_{n+1}(\zeta)z^{n+1}+\dots,\ z\in U,\ \zeta\in\overline{U}\}$ is the class of normalized analytic functions. A certain subclass, denoted by $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$, of analytic functions is introduced by means of the new operator.

Keywords: strong differential subordination, univalent function, convex function, best dominant, differential operator, multiplier transformation, Ruscheweyh derivative.
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1 Introduction

Denote by U the unit disc of the complex plane $U=\{z\in\mathbb{C}:\ |z|<1\}$, $\overline{U}=\{z\in\mathbb{C}:\ |z|\le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U\times\overline{U})$ the class of analytic functions in $U\times\overline{U}$.

Let $\mathcal{A}^*_{n\zeta}=\{f\in\mathcal{H}(U\times\overline{U}),\ f(z,\zeta)=z+a_{n+1}(\zeta)z^{n+1}+\dots,\ z\in U,\zeta\in\overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k\ge 2$, and $\mathcal{H}^*[a,n,\zeta]=\{f\in\mathcal{H}(U\times\overline{U}),\ f(z,\zeta)=a+a_n(\zeta)z^n+a_{n+1}(\zeta)z^{n+1}+\dots,\ z\in U,\ \zeta\in\overline{U}\}$, for $a\in\mathbb{C}$, $n\in\mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k\ge n$.

We also extend the differential operators presented above to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [8].

Definition 1.1 *[1] For $f\in\mathcal{A}^*_{n\zeta}$, $n,m\in\mathbb{N}$, the operator S^m is defined by $S^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned}S^0f(z,\zeta)&=f(z,\zeta),\\ S^1f(z,\zeta)&=zf'_z(z,\zeta),...,\\ S^{m+1}f(z,\zeta)&=z\left(S^mf(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U}.\end{aligned}$$

Remark 1.1 *[1] If $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, then $S^mf(z,\zeta)=z+\sum_{j=n+1}^{\infty}j^m a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.*

Definition 1.2 *[3] For $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\ge 0$, $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, the operator $I(m,\lambda,l)f(z,\zeta)$ is defined by the following infinite series*

$$I(m,\lambda,l)f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j,\ \ z\in U,\zeta\in\overline{U}.$$

Remark 1.2 *[3] It follows from the above definition that*

$$(l+1)I(m+1,\lambda,l)f(z,\zeta)=[l+1-\lambda]I(m,\lambda,l)f(z,\zeta)+\lambda z\left(I(m,\lambda,l)f(z,\zeta)\right)'_z,\ z\in U,\zeta\in\overline{U}.$$

1

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [7] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [9], [8].

Definition 1.3 *[9] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\zeta)=H(w(z),\zeta)$ for all $\zeta\in\overline{U}$. In such a case we write $f(z,\zeta)\prec\prec H(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.3 *[9] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.3 is equivalent to $f(0,\zeta)=H(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f\left(U\times\overline{U}\right)\subset H\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma 1.1 *[1] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]$ and $p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, then $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, where $g(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)t^{\frac{\gamma}{n}-1}dt$ is convex and it is the best dominant.*

Lemma 1.2 *[1] Let $g(z,\zeta)$ be a convex function in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and let $h(z,\zeta)=g(z,\zeta)+n\alpha zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $\alpha>0$ and n is a positive integer. If $p(z,\zeta)=g(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots$, $z\in U$, $\zeta\in\overline{U}$, is holomorphic in $U\times\overline{U}$ and $p(z,\zeta)+\alpha zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, then $p(z,\zeta)\prec\prec g(z,\zeta)$ and this result is sharp.*

2 Main results

Definition 2.1 *Let $\alpha,\lambda,l\geq 0$, $n,m\in\mathbb{N}$. Denote by $RI^{\alpha}_{m,\lambda,l}$ the operator given by $RI^{\alpha}_{m,\lambda,l}:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,*

$$RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=(1-\alpha)R^mf(z,\zeta)+\alpha I(m,\lambda,l)f(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

Remark 2.1 *If $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, then*
$RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Remark 2.2 *For $\alpha=0$, $RI^0_{m,\lambda,l}f(z,\zeta)=R^mf(z,\zeta)$, where $z\in U$, $\zeta\in\overline{U}$, and for $\alpha=1$, $RI^1_{m,\lambda,l}f(z,\zeta)=I(m,\lambda,l)f(z,\zeta)$, where $z\in U$, $\zeta\in\overline{U}$, which was studied in [3], [4]. For $l=0$, we obtain $RI^{\alpha}_{m,\lambda,0}f(z,\zeta)=RD^m_{1,\alpha}f(z,\zeta)$ which was studied in [5], [6] and for $l=0$ and $\lambda=1$, we obtain $RI^{\alpha}_{m,1,0}f(z,\zeta)=L^m_{\alpha}f(z,\zeta)$ which was studied in [1], [2].*

For $m=0$, $RI^{\alpha}_{0,\lambda,l}f(z,\zeta)=(1-\alpha)R^0f(z,\zeta)+\alpha I(0,\lambda,l)f(z,\zeta)=f(z,\zeta)=R^0f(z,\zeta)=I(0,\lambda,l)f(z,\zeta)$, where $z\in U$, $\zeta\in\overline{U}$.

Definition 2.2 *Let $\delta\in[0,1)$, $\alpha,\lambda,l\geq 0$ and $n,m\in\mathbb{N}$. A function $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ is said to be in the class $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$ if it satisfies the inequality*

$$\operatorname{Re}\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z>\delta,\qquad z\in U,\ \zeta\in\overline{U}. \tag{2.1}$$

Theorem 2.1 *The set $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$ is convex.*

Proof. Let the functions $f_j(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_{jk}(\zeta)z^j$, $k=1,2$, $z\in U$, $\zeta\in\overline{U}$, be in the class $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$. It is sufficient to show that the function $h(z,\zeta)=\eta_1f_1(z,\zeta)+\eta_2f_2(z,\zeta)$ is in the class $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$, with η_1 and η_2 nonnegative such that $\eta_1+\eta_2=1$.

Since $h(z,\zeta)=z+\sum_{j=n+1}^{\infty}(\eta_1a_{j1}(\zeta)+\eta_2a_{j2}(\zeta))z^j$, $z\in U$, $\zeta\in\overline{U}$, then

$$RI^{\alpha}_{m,\lambda,l}h(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}(\eta_1a_{j1}(\zeta)+\eta_2a_{j2}(\zeta))z^j, \tag{2.2}$$

$z\in U$, $\zeta\in\overline{U}$.

Differentiating (2.2) with respect to z we obtain
$\left(RI_{m,\lambda,l}^{\alpha}h(z,\zeta)\right)_z' = 1+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C_{m+j-1}^m\right\}(\eta_1 a_{j1}(\zeta)+\eta_2 a_{j2}(\zeta))jz^{j-1}$, $z\in U$, $\zeta\in\overline{U}$.

Hence

$$\operatorname{Re}\left(RI_{m,\lambda,l}^{\alpha}h(z,\zeta)\right)_z' = 1+\operatorname{Re}\left(\eta_1\sum_{j=n+1}^{\infty}j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C_{m+j-1}^m\right\}a_{j1}(\zeta)z^{j-1}\right)$$
$$+\operatorname{Re}\left(\eta_2\sum_{j=n+1}^{\infty}j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C_{m+j-1}^m\right\}a_{j2}(\zeta)z^{j-1}\right). \tag{2.3}$$

Taking into account that $f_1, f_2\in\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$ we deduce

$$\operatorname{Re}\left(\eta_k\sum_{j=n+1}^{\infty}j\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C_{m+j-1}^m\right\}a_{jk}(\zeta)z^{j-1}\right)>\eta_k(\delta-1),\quad k=1,2.$$

Using (2.1) we get from (2.3)

$$\operatorname{Re}\left(RI_{m,\lambda,l}^{\alpha}h(z,\zeta)\right)_z' > 1+\eta_1(\delta-1)+\eta_2(\delta-1)=\delta,\quad z\in U,\ \zeta\in\overline{U},$$

which is equivalent that $\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$ is convex. ∎

Theorem 2.2 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let $h(z,\zeta)=g(z,\zeta)+\frac{1}{c+2}zg_z'(z,\zeta)$, where $z\in U$, $\zeta\in\overline{U}$, $c>0$.*

If $\alpha,\lambda,l\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)$ and $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, then

$$\left(RI_{m,\lambda,l}^{\alpha}f(z,\zeta)\right)_z' \prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{2.4}$$

implies

$$\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z' \prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. We obtain that

$$z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt. \tag{2.5}$$

Differentiating (2.5), with respect to z, we have $(c+1)F(z,\zeta)+zF_z'(z,\zeta)=(c+2)f(z,\zeta)$ and

$$(c+1)RI_{m,\lambda,l}^{\alpha}F(z,\zeta)+z\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z'=(c+2)RI_{m,\lambda,l}^{\alpha}f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}. \tag{2.6}$$

Differentiating (2.6) with respect to z we have

$$\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z'+\frac{1}{c+2}z\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_{z^2}''=\left(RI_{m,\lambda,l}^{\alpha}f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U}. \tag{2.7}$$

Using (2.7), the strong differential subordination (2.4) becomes

$$\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z'+\frac{1}{c+2}z\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_{z^2}''\prec g(z)+\frac{1}{c+2}zg_z'(z). \tag{2.8}$$

Denote

$$p(z,\zeta)=\left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U}. \tag{2.9}$$

Replacing (2.9) in (2.8) we obtain

$$p(z,\zeta)+\frac{1}{c+2}zp_z'(z,\zeta)\prec\prec g(z,\zeta)+\frac{1}{c+2}zg_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2 we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e.}\ \left(RI_{m,\lambda,l}^{\alpha}F(z,\zeta)\right)_z'\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. ∎

Theorem 2.3 *Let* $h(z,\zeta)=\frac{\zeta+(2\delta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$, $\delta\in[0,1)$ *and* $c>0$. *If* $\alpha,\lambda,l\geq 0$, $m\in\mathbb{N}$ *and* I_c *is given by Theorem 2.2, then*

$$I_c\left[\mathcal{RI}_m(\delta,\lambda,l,\alpha,\zeta)\right]\subset\mathcal{RI}_m(\delta^*,\lambda,l,\alpha,\zeta), \tag{2.10}$$

where $\delta^*=2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right)$ *and* $\beta(x)=\int_0^1\frac{t^{x-1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that, $p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, where $p(z,\zeta)$ is defined in (2.9).

Using Lemma 1.1 we deduce that $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, that is $\left(RI^{\alpha}_{m,\lambda,l}F(z,\zeta)\right)'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, where $g(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z t^{\frac{c+2}{n}-1}\frac{\zeta+(2\delta-\zeta)t}{1+t}dt=(2\delta-\zeta)+\frac{2(c+2)(\zeta-\delta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{1+t}dt$. Since g is convex and $g\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re}\left(RI^{\alpha}_{m,\lambda,l}F(z,\zeta)\right)'_z\geq\min_{|z|=1}\operatorname{Re}g(z,\zeta)=\operatorname{Re}g(1,\zeta)=\delta^*=2\delta-\zeta+\frac{2(c+2)(\zeta-\delta)}{n}\beta\left(\frac{c+2}{n}-2\right). \tag{2.11}$$

From (2.11) we deduce inclusion (2.10). ■

Theorem 2.4 *Let* $g(z,\zeta)$ *be a convex function such that* $g(0,\zeta)=1$ *and let* h *be the function* $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

If $\alpha,\lambda,l\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ *and satisfies the strong differential subordination*

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{2.12}$$

then

$$\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. By using the properties of operator $RI^{\alpha}_{m,\lambda,l}$, we have
$RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.
Consider $p(z,\zeta)=\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j}{z}=1+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots$, $z\in U$, $\zeta\in\overline{U}$.

Let $RI^{\alpha}_{m,\lambda,l}f(z,\zeta)=zp(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Differentiating with respect to z we obtain $\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z=p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then (2.12) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \text{i.e.}\ \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

■

Theorem 2.5 *Let* $h(z,\zeta)$ *be a convex function such that* $h(0,\zeta)=1$.

If $\alpha,\lambda,l\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ *and satisfies the strong differential subordination*

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}, \tag{2.13}$$

then

$$\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$ *is convex and it is the best dominant.*

Proof. Let $p(z,\zeta) = \frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z} = \frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^j}{z} =$ $1+\sum_{j=n+1}^{\infty}\left\{\alpha\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m+(1-\alpha)C^m_{m+j-1}\right\}a_j(\zeta)z^{j-1} = 1+\sum_{j=n+1}^{\infty}p_j(\zeta)z^{j-1},\quad z\in U,\ \zeta\in\overline{U}.$

Differentiating with respect to z, we obtain $\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z = p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and (2.13) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Since $p(z,\zeta)\in\mathcal{H}^*[1,n,\zeta]$, using Lemma 1.1 for $\gamma=1$, we have

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),z\in U,\ \zeta\in\overline{U},\quad \text{i.e.}$$

$$\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and $g(z,\zeta)$ is convex and it is the best dominant. ■

Corollary 2.6 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U\times\overline{U}$, $0\le\beta<1$. If $\alpha\ge 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.14}$$

then

$$\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where g is given by $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.5 and considering $p(z,\zeta)=\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}$, the strong differential subordination (2.14) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z},\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1 for $\gamma=1$, we have $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\frac{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt$$

$$=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt,\quad z\in U,\ \zeta\in\overline{U}.$$

■

Theorem 2.7 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

If $\alpha,\lambda,l\ge 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination

$$\left(\frac{zRI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\right)'_z\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.15}$$

holds, then

$$\frac{RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l}f(z,\zeta)}\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. For $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$ we have
$RI^{\alpha}_{m,\lambda,l} f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C^m_{m+j-1} \right\} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Consider

$$p(z,\zeta) = \frac{RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l} f(z,)} = \frac{z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^{m+1} + (1-\alpha) C^{m+1}_{m+j} \right\} a_j(\zeta) z^j}{z + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^{m} + (1-\alpha) C^{m}_{m+j-1} \right\} a_j(\zeta) z^j}.$$

We have $p'_z(z,\zeta) = \frac{\left(RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta)\right)'_z}{RI^{\alpha}_{m,\lambda,l} f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z}{RI^{\alpha}_{m,\lambda,l} f(z,\zeta)}$ and we obtain
$p(z,\zeta) + z \cdot p'_z(z,\zeta) = \left(\frac{z RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l} f(z,\zeta)} \right)'_z$.

Relation (2.15) becomes

$$p(z,\zeta) + z p'_z(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + z g'_z(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

By using Lemma 1.2, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \ z \in U, \ \zeta \in \overline{U}, \ \text{i.e.} \ \frac{RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta)}{RI^{\alpha}_{m,\lambda,l} f(z,\zeta)} \prec\prec g(z,\zeta), \ z \in U, \ \zeta \in \overline{U}.$$

■

Theorem 2.8 *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + z g'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

If $\alpha, \lambda, l \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}^*_{n\zeta}$ and the strong differential subordination

$$\frac{(m+1)(m+2)}{z} RI^{\alpha}_{m+2,\lambda,l} f(z,\zeta) - \frac{(m+1)(2m+1)}{z} RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta) +$$
$$\frac{m^2}{z} RI^{\alpha}_{m,\lambda,l} f(z,\zeta) + \frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2,\lambda,l) f(z,\zeta) -$$
$$\frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] I(m+1,\lambda,l) f(z,\zeta) +$$
$$\frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m,\lambda,l) f(z,\zeta) \prec\prec h(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \tag{2.16}$$

holds, then

$$\left[RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right]'_z \prec\prec g(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

This result is sharp.

Proof. Let

$$p(z,\zeta) = \left(RI^{\alpha}_{m,\lambda,l} f(z,\zeta)\right)'_z = (1-\alpha)\left(R^m f(z,\zeta)\right)'_z + \alpha \left(I(m,\lambda,l) f(z,\zeta)\right)'_z \tag{2.17}$$

$$= 1 + \sum_{j=n+1}^{\infty} \left\{ \alpha \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m + (1-\alpha) C^m_{m+j-1} \right\} j a_j(\zeta) z^{j-1} = 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots.$$

By using the properties of operators $RI^{\alpha}_{m,\lambda,l}$, R^m and $I(m,\lambda,l)$, after a short calculation, we obtain
$p(z,\zeta) + z p'_z(z,\zeta) = \frac{(m+1)(m+2)}{z} RI^{\alpha}_{m+2,\lambda,l} f(z,\zeta) - \frac{(m+1)(2m+1)}{z} RI^{\alpha}_{m+1,\lambda,l} f(z,\zeta) + \frac{m^2}{z} RI^{\alpha}_{m,\lambda,l} f(z,\zeta) +$
$\frac{\alpha}{z} \left[\frac{(l+1)^2}{\lambda^2} - (m+1)(m+2) \right] I(m+2,\lambda,l) f(z,\zeta) - \frac{\alpha}{z} \left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2} - (m+1)(2m+1) \right] \cdot$
$I(m+1,\lambda,l) f(z,\zeta) + \frac{\alpha}{z} \left[\frac{(l+1-\lambda)^2}{\lambda^2} - m^2 \right] I(m,\lambda,l) f(z,\zeta)$.

6

Using the notation in (2.17), the differential subordination becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta).$$

By using Lemma 1.2, we have

$$p(z,\zeta)\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\ \text{i.e.}\ \left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and this result is sharp. ■

Theorem 2.9 *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$.*
If $\alpha,\lambda,l\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and satisfies the strong differential subordination

$$\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)+$$
$$\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f(z,\zeta)+\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2}-(m+1)(m+2)\right]I(m+2,\lambda,l)f(z,\zeta)-$$
$$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2}-(m+1)(2m+1)\right]I(m+1,\lambda,l)f(z,\zeta)+$$
$$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2}-m^2\right]I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.18}$$

then

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt$ is convex and it is the best dominant.

Proof. Using the properties of operator $RI^{\alpha}_{m,\lambda,l}$ and considering $p(z,\zeta)=\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z$, we obtain $p(z,\zeta)+zp'_z(z,\zeta)=\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)+\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f(z,\zeta)+$ $\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2}-(m+1)(m+2)\right]I(m+2,\lambda,l)f(z,\zeta)-\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2}-(m+1)(2m+1)\right]\cdot$ $I(m+1,\lambda,l)f(z,\zeta)+\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2}-m^2\right]I(m,\lambda,l)f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then (2.18) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1, for $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and $g(z,\zeta)$ is convex and it is the best dominant. ■

Corollary 2.10 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U\times\overline{U}$, $0\leq\beta<1$. If $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$\frac{(m+1)(m+2)}{z}RI^{\alpha}_{m+2,\lambda,l}f(z,\zeta)-\frac{(m+1)(2m+1)}{z}RI^{\alpha}_{m+1,\lambda,l}f(z,\zeta)+$$
$$\frac{m^2}{z}RI^{\alpha}_{m,\lambda,l}f(z,\zeta)+\frac{\alpha}{z}\left[\frac{(l+1)^2}{\lambda^2}-(m+1)(m+2)\right]I(m+2,\lambda,l)f(z,\zeta)-$$
$$\frac{\alpha}{z}\left[\frac{2(l+1-\lambda)(\bar{l}+1)}{\lambda^2}-(m+1)(2m+1)\right]I(m+1,\lambda,l)f(z,\zeta)+$$
$$\frac{\alpha}{z}\left[\frac{(l+1-\lambda)^2}{\lambda^2}-m^2\right]I(m,\lambda,l)f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.19}$$

then

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z \prec\prec g(z,\zeta) \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U},$$

where g is given by $g(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function q is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z,\zeta)=\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z$, the strong differential subordination (2.19) becomes

$$p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z},\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.1 for $\gamma=1$, we have $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\left(RI^{\alpha}_{m,\lambda,l}f(z,\zeta)\right)'_z \prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt=$$

$$\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt,\ z\in U,\ \zeta\in\overline{U}.$$

■

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Global behavior of the max-type difference equation $x_{n+1} = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^{\alpha}}\}$

Taixiang Sun[*,1] **Hongjian Xi**[2] **Bin Qin** [2]

[1]College of Mathematics and Information Science, Guangxi University, Nanning 530004, P.R. China

[2]Department of Mathematics, Guangxi College of Finance and Economics, Nanning 530003, P.R. China

Abstract In this paper, we study global behavior of the following max-type difference equation

$$x_{n+1} = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^{\alpha}}\}, \quad n = 0, 1, \ldots,$$

where $A \in (0, +\infty)$, $\alpha \in (0, 1)$ and $m, k \in \{0, 1, 2, \cdots\}$, and initial values $x_{-l}, x_{-l+1}, \cdots, x_0 \in (0, +\infty)$ with $l = \max\{k, m\}$. The special case when $m = 0$ and $k = 2$ has been completely investigated by A.Gelisken and C.Cinar. Here we extend their results to the general case.

AMS Subject Classification: 39A10; 39A11.

Keywords: Max-type difference equation, Positive solution, Periodicity.

1 Introduction

In the recent years, there has been a lot of interest in studying the global behavior of, so called, max-type difference equations, see e.g.[1-24] (see also references therein). In [9], the second order max-type difference equation

$$x_{n+1} = \max\{\frac{A}{x_n}, \frac{1}{x_{n-2}^{\alpha}}\}, \quad n = 0, 1, \ldots,$$

has been studied. In this paper, we study the following max-type difference equation

$$x_{n+1} = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^{\alpha}}\}, \quad n = 0, 1, \ldots, \tag{1.1}$$

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$*$ E-mail address: stxhql@gxu.edu.cn

where $A \in R^+ \equiv (0, +\infty)$, $\alpha \in (0,1)$ and $m, k \in \{0, 1, 2, \cdots\}$, and initial values $x_{-l}, x_{-l+1}, \cdots, x_0 \in R^+$ with $l = \max\{k, m\}$.

2 The case $A \leq 1$

In this section, we investigate the equation (1.1) with $A \leq 1$. It is easy to see that if $\{x_n\}_{n=-l}^{\infty}$ is a positive solution of equation (1.1), then for all $n \geq 0$,

$$x_{n+1}x_{n-k}^{\alpha} \geq 1. \tag{2.1}$$

Lemma 2.1. *Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of equation (1.1) and $P_n = \max\{x_n, x_{n-1}, \cdots, x_{n-2l-1}, 1\}$ for all $n \geq l+1$. Then*

(1) $x_{n+1} \leq P_n$ for all $n \geq l+1$ and $\{P_n\}_{n=l+1}^{\infty}$ is non-increasing.

(2) x_n is bounded and moreover $A/P_{l+1} \leq x_n \leq P_{l+1}$ for any $n \geq m+l+2$.

Proof. By (2.1), we obtain that for any $n \geq l+1$,

$$\begin{aligned} x_{n+1} &= \max\{\frac{Ax_{n-m-k-1}^{\alpha}}{x_{n-m}x_{n-m-k-1}^{\alpha}}, \frac{x_{n-2k-1}^{\alpha^2}}{x_{n-k}^{\alpha}x_{n-2k-1}^{\alpha^2}}\} \\ &\leq \max\{Ax_{n-m-k-1}^{\alpha}, x_{n-2k-1}^{\alpha^2}\} \\ &\leq \max\{x_{n-m-k-1}, x_{n-2k-1}, 1\} \\ &\leq P_n. \end{aligned}$$

Hence

$$P_{n+1} = \max\{x_{n+1}, x_n, \cdots, x_{n-2l}, 1\} \leq P_n,$$

which implies that for all $n \geq l+1$,

$$x_n \leq P_{l+1}.$$

Furthermore, it follows that for all $n \geq m+l+1$,

$$x_{n+1} = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^{\alpha}}\} \geq \frac{A}{P_{l+1}}.$$

The proof is complete. □

Remark 2.2. Note that from definition P_n we have that $P_n \geq 1$ for all $n \geq l+1$.

Lemma 2.3. *Let $\{x_n\}_{n=-l}^{\infty}$ be a positive solution of equation (1.1) and P_n be as in Lemma 2.1. If $\lim_{n\longrightarrow\infty} P_n = S$, then $\limsup_{n\longrightarrow\infty} x_n = S \geq 1$.*

Proof. Since P_n is a subsequence of x_n, it follows

$$S \leq \limsup_{n\longrightarrow\infty} x_n.$$

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On the other hand, by $x_{n+1} \leq P_n$ for all $n \geq l+1$, we obtain

$$\limsup_{n \longrightarrow \infty} x_n \leq \limsup_{n \longrightarrow \infty} P_n = S.$$

The proof is complete. $\square$

Theorem 2.4. *Suppose that* $\{x_n\}_{n=-l}^{\infty}$ *is a positive solution of equation (1.1) with* $A \leq 1$, *then* $\lim_{n \longrightarrow \infty} x_n = 1$

Proof. By Lemma 2.1 we may assume that there exist $0 < m_1 < m_2 < \cdots < m_t < \cdots$ and $0 < n_1 < n_2 < \cdots < n_t < \cdots$ such that

$$\begin{aligned} x_{n_t+1} &\longrightarrow \liminf_{n \longrightarrow \infty} x_n = L > 0, \\ x_{n_t-m} &\longrightarrow L_1, \\ x_{n_t-k} &\longrightarrow L_2, \\ x_{m_t+1} &\longrightarrow \limsup_{n \longrightarrow \infty} x_n = S \geq 1, \\ x_{m_t-m} &\longrightarrow K_1, \\ x_{m_t-k} &\longrightarrow K_2. \end{aligned}$$

By taking the limit in the following relationship

$$x_{m_t+1} = \max\{\frac{A}{x_{m_t-m}}, \frac{1}{x_{m_t-k}^{\alpha}}\}$$

as $t \longrightarrow \infty$, it follows

$$\begin{aligned} S &= \max\{\frac{A}{K_1}, \frac{1}{K_2^{\alpha}}\} \\ &= \begin{cases} \frac{A}{K_1}, & \text{if } \frac{A}{K_1} \geq \frac{1}{K_2^{\alpha}} \\ \frac{1}{K_2^{\alpha}}, & \text{if } \frac{A}{K_1} \leq \frac{1}{K_2^{\alpha}} \end{cases} \\ &\leq \begin{cases} \frac{1}{K_1}, & \text{if } \frac{A}{K_1} \geq \frac{1}{K_2^{\alpha}} \\ \frac{1}{K_2}, & \text{if } \frac{A}{K_1} \leq \frac{1}{K_2^{\alpha}} \end{cases} \\ &\leq \frac{1}{L}, \end{aligned}$$

which implies $SL \leq 1$.

We claim $S = 1$. Indeed, if $S > 1$, then by taking the limit in the following relationship

$$x_{n_t+1} = \max\{\frac{A}{x_{n_t-m}}, \frac{1}{x_{n_t-k}^{\alpha}}\} \tag{2.2}$$

as $t \longrightarrow \infty$, we obtain

$$1 > \frac{1}{S} \geq L = \max\{\frac{A}{L_1}, \frac{1}{L_2^{\alpha}}\} \geq (\frac{1}{L_2})^{\alpha} > \frac{1}{L_2} \geq \frac{1}{S}.$$

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This is a contradiction. The claim is proven.

Furthermore by taking the limit in (2.2) as $t \longrightarrow \infty$, we obtain

$$1 = \frac{1}{S} \geq L = \max\{\frac{A}{L_1}, \frac{1}{L_2^\alpha}\} \geq (\frac{1}{L_2})^\alpha \geq \frac{1}{L_2} \geq \frac{1}{S} = 1,$$

which implies $\lim_{n \longrightarrow \infty} x_n = 1$. The proof is complete. □

3 The case $A > 1$ and $m = 0$

In this section, we investigate the equation (1.1) with $A > 1$ and $m = 0$. Let $x_n = \sqrt{A} y_n (n \geq -k)$, then the equation (1.1) implies the equation

$$y_{n+1} = \max\{\frac{1}{y_n}, \frac{B}{y_{n-k}^\alpha}\}, \tag{3.1}$$

where $B = A^{-\frac{1+\alpha}{2}} < 1$ and initial values $y_{-k}, y_{-k+1}, \cdots, y_0 \in R^+$. It is easy to see that if $\{y_n\}_{n=-k}^\infty$ is a positive solution of equation (3.1), then for all $n \geq 0$,

$$y_{n+1} y_n \geq 1. \tag{3.2}$$

Lemma 3.1. *Let $\{y_n\}_{n=-k}^\infty$ be a positive solution of equation (3.1) and $Q_n = \max\{y_n, y_{n-1}, \cdots, y_{n-k-1}, 1\}$ for all $n \geq k+1$. Then*

(1) $y_{n+1} \leq Q_n$ for all $n \geq k+1$ and $\{Q_n\}_{n=k+1}^\infty$ is non-increasing.

(2) y_n is bounded and moreover $1/Q_{k+1} \leq y_n \leq Q_{k+1}$ for any $n \geq k+2$.

Proof. By (3.2), we obtain that for any $n \geq k+1$,

$$\begin{aligned} y_{n+1} &= \max\{\frac{y_{n-1}}{y_n y_{n-1}}, \frac{B y_{n-k-1}^\alpha}{y_{n-k}^\alpha y_{n-k-1}^\alpha}\} \\ &\leq \max\{y_{n-1}, B y_{n-k-1}^\alpha\} \\ &\leq \max\{y_{n-1}, y_{n-k-1}, 1\} \\ &\leq Q_n. \end{aligned}$$

Hence

$$Q_{n+1} = \max\{y_{n+1}, y_n, \cdots, y_{n-k}, 1\} \leq Q_n,$$

which implies that for all $n \geq k+1$,

$$y_n \leq Q_{k+1}.$$

Furthermore, it follows that for all $n \geq k+1$,

$$y_{n+1} = \max\{\frac{1}{y_n}, \frac{B}{y_{n-k}^\alpha}\} \geq \frac{1}{Q_{k+1}}.$$

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The proof is complete. □

Remark 3.2. Note that from definition Q_n we have that $Q_n \geq 1$ for all $n \geq k+1$.

Lemma 3.3. *Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of equation (3.1) and Q_n be as in Lemma 3.1. If $\lim_{n\longrightarrow\infty} Q_n = S$, then $\limsup_{n\longrightarrow\infty} y_n = S \geq 1$.*

Proof. Since Q_n is a subsequence of y_n, it follows

$$S \leq \limsup_{n\longrightarrow\infty} y_n.$$

On the other hand, by $y_{n+1} \leq Q_n$ for all $n \geq k+1$, we obtain

$$\limsup_{n\longrightarrow\infty} y_n \leq \limsup_{n\longrightarrow\infty} Q_n = S.$$

The proof is complete. □

Theorem 3.4 *Let $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of equation (3.1) and Q_n be as in Lemma 3.1. If $\lim_{n\longrightarrow\infty} Q_n = S$, then there exists $N_1 > 0$ such that $y_{N_1+2p} \geq y_{N_1+2(p+1)} \geq S$ for all $p \geq 0$, and $\lim_{p\longrightarrow\infty} y_{N_1+2p} = S$ and $\lim_{p\longrightarrow\infty} y_{N_1+2p+1} = 1/S$.*

Proof For $\varepsilon = \frac{1-B}{1+B}S$, by (3.2) and Lemma 3.3 there exists N such that for any $n \geq N$,

$$y_{n-k-1}^{\alpha} \leq S + \varepsilon$$

and

$$\begin{aligned} y_{n+1} &= \max\{\frac{y_{n-1}}{y_n y_{n-1}}, \frac{By_{n-k-1}^{\alpha}}{y_{n-k}^{\alpha} y_{n-k-1}^{\alpha}}\} \\ &\leq \max\{y_{n-1}, By_{n-k-1}^{\alpha}\} \\ &\leq \max\{y_{n-1}, B(S+\varepsilon)\} \\ &= \max\{y_{n-1}, S-\varepsilon\}. \end{aligned}$$

It follows from Lemma 3.1 and Lemma 3.3 that there exist $N < n_1 < n_2 < \cdots < n_k < \cdots$ such that $y_{n_k} \geq S$. By taking a subsequence we may assume that $n_k = 2l_k + t$ $(0 \leq t < 2)$. Hence

$$\begin{aligned} S \leq y_{n_k} &= y_{2l_k+t} \leq \max\{y_{2(l_k-1)+t}, S-\varepsilon\} \\ &= y_{2(l_k-1)+t} \leq \max\{y_{2(l_k-2)+t}, S-\varepsilon\} \\ &= y_{2(l_k-2)+t} \leq \max\{y_{2(l_k-3)+t}, S-\varepsilon\} \\ &\qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \\ &= y_{n_1}. \end{aligned}$$

Choose $N_1 = n_1$, then $y_{N_1+2p} \geq y_{N_1+2(p+1)} \geq S$ for all $p \geq 0$.

On the other hand, for sufficiently large $p \geq 0$,

$$\max\{\frac{1}{y_{N_1+2p+1}}, S\} \leq y_{N_1+2p+2} = \max\{\frac{1}{y_{N_1+2p+1}}, \frac{By_{N_1+2p-k}^{\alpha}}{y_{N_1+2p+1-k}^{\alpha} y_{N_1+2p-k}^{\alpha}}\}$$

5

$$
\begin{aligned}
&\leq \max\{\frac{1}{y_{N_1+2p+1}}, By^{\alpha}_{N_1+2p-k}\} \\
&\leq \max\{\frac{1}{y_{N_1+2p+1}}, B(S+\varepsilon)\} \\
&= \max\{\frac{1}{y_{N_1+2p+1}}, S-\varepsilon\} \\
&= \frac{1}{y_{N_1+2p+1}}.
\end{aligned}
$$

Hence $y_{N_1+2p+2}y_{N_1+2p+1} = 1$ and $\lim_{n\longrightarrow\infty} y_{N_1+2p+1} = 1/S$. The proof is complete. □

From Theorem 3.4, we get

Theorem 3.5 *Suppose that* $\{x_n\}_{n=-l}^{\infty}$ *is a positive solution of equation (1.1) with* $A > 1$, *then* $\lim_{n\longrightarrow\infty} x_{2n}$ *and* $\lim_{n\longrightarrow\infty} x_{2n+1}$ *are convergent.*

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A General System Of Quadratic Functional Equations In Non-Archimedean Fuzzy Menger Normed Spaces

[1]M. B. Ghaemi and [2]H. Majani

[1,2]Department of Mathematics, Iran University of Science and Technology,

Narmak, Tehran, Iran.

E-mails:[1]mghaemi@iust.ac.ir, [2]majani.hamid@hotmail.com.

Abstract. In this paper, we prove the generalized Hyers–Ulam–Rassias stability for a general system of quadratic functional equations in non–Archimedean fuzzy Menger normed spaces. Our results notably generalize previous papers in this topic and by the method of our paper, one can prove many various general systems of n functional equations and n variables ($n \in \mathbb{N}$) in fuzzy normed spaces.

Keywords: Quadratic Functional Equations, Non-Archimedean Fuzzy Menger Normed spaces, Generalized Hyers–Ulam–Rassias stability.

1. Introduction and preliminaries

A. K. Katsaras[21] introduced an idea of a fuzzy norm on a linear space in 1984, in the same year Wu and Fang[34] introduced a notion of fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological linearspaces. In 1992, Felbin[10] introduced an alternative definition of a fuzzy norm on a linear space with an associated metric of Kaleva and Seikkala type(see [19]). Xiao and Zhu[35] studied the linear topological structures of fuzzy normed linear spaces. In 1994, Cheng and Mordeson introduced a definition of a fuzzy norm on a linear space in such a way that the corresponding induced fuzzy metric is of Kramosiland Michalek type[23]. In 2003, Bag and Samanta[5] modified the definition of Cheng and Mordeson[6] by removing a regular condition. Following D. Mihet[24] and modifying the definition of a fuzzy normed space in [4], A. K. Mirmostafaee and M.

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Sal Moslehian[26] introduced non–Archimedean fuzzy normed spaces. Recently D. Mihet[25] restated definition of them. Although many results in the classical normed space theory have a non–Archimedean counterpart, their proofs are different and require a rather new kind of intuition [3, 8, 27, 28, 33].

In 1897, Hensel[15] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non–Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p–adic strings and superstrings [22]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non–Archimedean normed space; cf. [27]. These facts show that the non–Archimedean framework is of special interest.

Definition 1.1. *Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have*

(i) $|a| \geq 0$ *and equality holds if and only if* $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping will be called a valued field. If the condition (iii) in the definition of a valuation mapping is replaced with

$$(iii)' \ |a + b| \leq \max\{|a|, |b|\}$$

then the valuation $|\cdot|$ is said to be non–Archimedean. The condition $(iii)'$ is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from $(iii)'$ that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$. The most important examples of non-Archimedean spaces are p–adic numbers.

Example 1.2. *Let p be a prime number. For any non–zero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p, define the p–adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a* *non–Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and is called the p–adic number field.*

Definition 1.3. *A triangular norm (briefly t–norm, [30]) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, non–decreasing in each variable and has 1 as the unit element. Basic examples are the Łukasiewicz t–norm T_L, $T_L(a,b) = \max(a+b-1, 0)$, the product t–norm T_P, $T_P(a,b) = ab$ and the strongest triangular norm T_M, $T_M(a,b) = \min(a,b)$.*

Now we recall definition of a non–Archimedean fuzzy Menger normed space which is given in [25] and [26].

Definition 1.4. *Let X be a linear space over a non–Archimedean field $\mathbb{K}$ and T be a continuous t–norm. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a non–Archimedean fuzzy Menger norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{K}$,*

(N1) $N(x,c) = 0$ *for* $c \leq 0$;

(N2) $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$;

(NA4) $N(x+y, \max\{s,t\}) \geq T(N(x,s), N(y,t))$;

(N5) $\lim_{t\to\infty} N(x,t) = 1$.

If N is a non–Archimedean fuzzy Menger norm on X, then the triple (X, N, T) is called a non–Archimedean fuzzy Menger normed space.

It follows from $(NA4)$ that $N(x,.)$ is non–decreasing for every $x \in X$. Also one can show that the condition $(NA4)$ is equivalent to the following condition:

$$N(x+y,t) \geq T(N(x,t), N(y,t)).$$

Definition 1.5. *Let (X, N, T) be a non–Archimedean fuzzy Menger normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - \lim x_n = x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

Let T be a given t–norm. Then (by associativity) a family of mappings $T^n : [0, 1] \to [0, 1]$, $n \in \mathbb{N}$, is defined as follows:

$$T^1(x) = T(x, x)\ ,\ T^n(x) = T(T^{n-1}(x), x)\ ,\ x \in [0, 1].$$

For three important t–norms T_M, T_P and T_L we have

$$T_M^n(x) = x\ ,\ T_P^n(x) = x^n\ ,\ T_L^n(x) = \max\{(n + 1)x - n, 0\}\ ,\ n \in \mathbb{N}.$$

Definition 1.6. *(Hadzić[13]) A t–norm T is said to be of H–type if a family of functions $\{T^n(t)\}$; $n \in \mathbb{N}$, is equicontinuous at $t = 1$, that is,*

$$\forall \varepsilon \in (0, 1)\ \exists \delta \in (0, 1)\ :\ t > 1 - \delta\ \Rightarrow\ T^n(t) > 1 - \varepsilon\ (n \geq 1).$$

The t-norm T_M is a trivial example of t–norm of H–type, but there are t-norms of H–type with $T \neq T_M$ (see, e.g., Hadzić[12]).

Lemma 1.7. *We consider the notations of the definition(1.5). Also assume that T is a t–norm of H–type. Then the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $N(x_{n+1} - x_n, t) > 1 - \varepsilon$.*

Proof. Due to

$$N(x_{n+p}-x_n,t) \geq T(N(x_{n+p}-x_{n+p-1},t), N(x_{n+p-1}-x_n,t)) \geq$$
$$T(N(x_{n+p}-x_{n+p-1},t), T(N(x_{n+p-1}-x_{n+p-2},t), N(x_{n+p-2}-x_n,t))) \geq$$
$$\vdots$$
$$\geq T(N(x_{n+p}-x_{n+p-1},t), T(N(x_{n+p-1}-x_{n+p-2},t), \cdots,$$
$$T(N(x_{n+2}-x_{n+1},t), N(x_{n+1}-x_n,t)))\cdots),$$

and by the assumption, T is an H–type t–norm, the sequence $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have $N(x_{n+1}-x_n,t) > 1-\varepsilon$. We will use this criterion in this paper. □

It is easy to see that every convergent sequence in a non–Archimedean fuzzy Menger normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy Menger norm is said to be complete and the non–Archimedean fuzzy Menger normed space is called a non–Archimedean fuzzy Menger Banach space.

The first stability problem concerning group homomorphisms was raised by Ulam[32] in 1940 and solved in the next year by Hyers[14]. Hyers' theorem was generalized by Aoki[2] for additive mappings and by Th. M. Rassias[29] for linear mappings by considering an unbounded Cauchy difference. In 1994, a generalization of the Rassias theorem was obtained by Găvruta[11] by replacing the unbounded Cauchy difference by a general control function.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric bi–additive function [1, 20]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation(1.1) is said to be a quadratic function. It is well known that a function $f : X \to Y$ where X and Y are real vector spaces, is quadratic if and only if there exists a unique symmetric bi–additive function B such that $f(x) = B(x,x)$ for all

x. The bi–additive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)).$$

The Hyers-Ulam stability problem for the quadratic functional equation was solved by F. Skof[31]. S. Czerwik[7] proved the Hyers–Ulam–Rassias stability of the equation(1.1). Later, S.-M. Jung[17] has generalized the results obtained by Skof and Czerwik. Eshaghi Gordji and Khodaei [9] obtained the general solution and the generalized Hyers–Ulam–Rassias stability of the following quadratic functional equation for $a,b\in\mathbb{Z}\backslash\{0\}$ with $a\neq\pm1,\pm b$,

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y) \tag{1.2}$$

Throughout this paper, unless otherwise explicitly stated, we assume that $u\in\mathbb{R}$, $i,m,n\in\mathbb{N}\cup\{0\}$, $\mathbb{K}$ is a non–Archimedean field, T is an H–type continuous t–norm, (Y,N,T) be a non–Archimedean fuzzy Menger Banach space over $\mathbb{K}$, (Z,M,T) be a non–Archimedean fuzzy Menger normed space over $\mathbb{K}$ and X be a vector space over $\mathbb{K}$. Also assume $f:X^n\to Y$ be a mapping. We consider following general system of quadratic functional equations:

$$\begin{cases} f(a_1x_1+b_1y_1,x_2,...,x_n)+f(x_1,...,x_{n-1},a_nx_n-b_ny_n)= \\ 2a_1^2f(x_1,x_2,...,x_n)+2b_1^2f(x_1,...,x_n); \\ \quad\vdots \\ f(x_1,...,x_{n-1},a_nx_n+b_ny_n)+f(x_1,...,x_{n-1},a_nx_n-b_ny_n)= \\ 2a_n^2f(x_1,...,x_n)+2b_n^2f(x_1,...,x_{n-1},y_n); \end{cases} \tag{1.3}$$

for all $x_i,y_i\in X$ and $a_i,b_i\in\mathbb{K}\setminus\{0\}$ with $a_i\neq\pm1,\pm b_i$, $i=1,...,n$.

In this paper, we establish the generalized Hyers–Ulam–Rassias stability of system(1.3) in non–Archimedean fuzzy Menger Banach space .

2. Main Results

In this section, we prove the fuzzy generalized Hyers–Ulam–Rassias stability of system(1.3).

Theorem 2.1. *Let $\varphi_i : X^{n+1} \to Z$ for $i \in \{1, ..., n\}$ be a mapping such that*

$$\begin{cases} \varphi_i^m = \varphi_i^m(x_1, ..., x_i, y_i, ..., x_n, u) = \\ T\Big\{M\Big(\frac{1}{2a_1^{2m+2}...a_i^{2m+2}a_{i+1}^{2m}...a_n^{2m}}\varphi_i(a_1^{m+1}x_1, ..., a_{i-1}^{m+1}x_{i-1}, a_i^m x_i, 0, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big), \\ \quad N\Big(\frac{b_i^2}{a_1^{2m+2}...a_i^{2m+2}a_{i+1}^{2m}...a_n^{2m}} f(a_1^{m+1}x_1, ..., a_{i-1}^{m+1}x_{i-1}, 0, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big)\Big\}; \\ \Phi_1^m = \Phi_1^m(x_1, y_1, x_2, ..., x_n, u) = \varphi_1^m(x_1, y_1, x_2, ..., x_n, u); \\ \Phi_i^m = \Phi_i^m(x_1, ..., x_i, y_i, ..., x_n, u) = \\ T\Big(\varphi_i^m(x_1, ..., x_i, y_i, ..., x_n, u), \Phi_{i-1}^m(x_1, ..., x_{i-1}, y_{i-1}, ..., x_n, u)\Big); \\ \lim_{m\to\infty} \Phi_n^m = 1; \end{cases} \tag{2.1}$$

and

$$\begin{cases} \Psi_1 = \Psi_1(x_1, ..., x_n, y_n, u) = \Phi_n^1(x_1, ..., x_n, y_n, u); \\ \Psi_m = \Psi_m(x_1, ..., x_n, y_n, u) = T\Big(\Phi_n^m(x_1, ..., x_n, y_n, u), \Psi_{m-1}(x_1, ..., x_n, y_n, u)\Big); \\ \Psi = \Psi(x_1, ..., x_n, y_n, u) = \lim_{m\to\infty} \Psi_m(x_1, ..., x_n, y_n, u) = 1. \end{cases} \tag{2.2}$$

and

$$\lim_{m\to\infty} M\Big(\frac{1}{a_1^{2m}...a_n^{2m}}\varphi_i(a_1^m x_1, ..., a_i^m x_i, a_i^m y_i, ..., a_n^m x_n), u\Big) = 1; \tag{2.3}$$

for all $u > 0$ and $x_i, y_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$ with $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$. Let $f : X^n \to Y$ be a mapping satisfying

$$\begin{cases} N\Big(f(a_1x_1 + b_1y_1, x_2, ..., x_n) + f(a_1x_1 - b_1y_1, x_2, ..., x_n) - \\ 2a_1^2 f(x_1, ..., x_n) - 2b_1^2 f(y_1, x_2, ..., x_n), u\Big) \geq M\Big(\varphi_1(x_1, y_1, x_2, ..., x_n), u\Big); \\ \qquad \vdots \\ N\Big(f(x_1, ..., x_{n-1}, a_nx_n + b_ny_n) + f(x_1, ..., x_{n-1}, a_nx_n - b_ny_n) - \\ 2a_n^2 f(x_1, ..., x_n) - 2b_n^2 f(x_1, ..., x_{n-1}, y_n), u\Big) \geq M\Big(\varphi_n(x_1, ..., x_n, y_n), u\Big); \end{cases}$$

for all $u > 0$ and $x_i, y_i \in X$ and $a_i, b_i \in \mathbb{K} \setminus \{0\}$ with $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$. Then there exists a unique mapping $F : X^n \to Y$ satisfying (1.3) and

$$N\Big(f(x_1, ..., x_n) - F(x_1, ..., x_n), u\Big) \geq \Psi \tag{2.4}$$

for all $u > 0$ and $x_i \in X$, $i = 1, ..., n$.

Proof. Fix $i \in \{1, 2, ..., n\}$ and consider the following inequality.

$$\begin{aligned} &N\Big(f(x_1, x_2, ..., a_i x_i + b_i y_i, ..., x_n) + f(x_1, x_2, ..., a_i x_i - b_i y_i, ..., x_n) - \\ &2a_i^2 f(x_1, ..., x_n) - 2b_i^2 f(x_1, x_2, ..., y_i, ..., x_n), u\Big) \geq M\Big(\varphi_i(x_1, ..., x_i, y_i, ..., x_n), u\Big). \end{aligned} \tag{2.5}$$

Let $y_i = 0$ in (2.5). Then we get

$$\begin{aligned} &N\Big(f(x_1, ..., x_n) - \frac{1}{a_i^2} f(x_1, x_2, ..., a_i x_i, ..., x_n), u\Big) \geq \\ &T\Big\{M\Big(\frac{1}{2a_i^2}\varphi_i(x_1, ..., x_i, 0, x_{i+1}, ..., x_n), u\Big), N\Big(\frac{b_i^2}{a_i^2} f(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n), u\Big)\Big\} \end{aligned}$$

Hence

$$\begin{aligned} &N\Big(\frac{1}{a_1^2 ... a_{i-1}^2} f(a_1 x_1, ..., a_{i-1} x_{i-1}, x_i, ..., x_n) - \frac{1}{a_1^2 ... a_i^2} f(a_1 x_1, ..., a_i x_i, x_{i+1}, ..., x_n), u\Big) \geq \\ &T\Big\{M\Big(\frac{1}{2a_1^2 ... a_i^2}\varphi_i(a_1 x_1, ..., a_{i-1} x_{i-1}, x_i, 0, x_{i+1}, ..., x_n), u\Big), \\ &N\Big(\frac{b_i^2}{a_1^2 ... a_i^2} f(a_1 x_1, ..., a_{i-1} x_{i-1}, 0, x_{i+1}, ..., x_n), u\Big)\Big\}. \end{aligned}$$

So we have

$$\begin{aligned} &N\Big(\frac{1}{a_1^{2m+2} ... a_{i-1}^{2m+2} a_i^{2m} ... a_n^{2m}} f(a_1^{m+1} x_1, ..., a_{i-1}^{m+1} x_{i-1}, a_i^m x_i, ..., a_n^m x_n) - \\ &\frac{1}{a_1^{2m+2} ... a_i^{2m+2} a_{i+1}^{2m} ... a_n^{2m}} f(a_1^{m+1} x_1, ..., a_i^{m+1} x_i, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big) \geq \\ &T\Big\{M\Big(\frac{1}{2a_1^{2m+2} ... a_i^{2m+2} a_{i+1}^{2m} ... a_n^{2m}}\varphi_i(a_1^{m+1} x_1, ..., a_{i-1}^{m+1} x_{i-1}, a_i^m x_i, 0, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big), \\ &N\Big(\frac{b_i^2}{a_1^{2m+2} ... a_i^{2m+2} a_{i+1}^{2m} ... a_n^{2m}} f(a_1^{m+1} x_1, ..., a_{i-1}^{m+1} x_{i-1}, 0, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big)\Big\} = \varphi_i^m. \end{aligned}$$

Therefore we obtain

$$N\Big(\frac{1}{a_1^{2m+2} ... a_n^{2m+2}} f(a_1^{m+1} x_1, ..., a_n^{m+1} x_n) - \frac{1}{a_1^{2m} ... a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n), u\Big) \geq \Phi_n^m. \tag{2.6}$$

for all $m \in \mathbb{N} \cup \{0\}$. Hence by (2.1) and (2.6) the sequence

$$\{\frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n)\}$$

is Cauchy. By completeness of Y, we conclude that it is convergent. Therefore we can define $F : X^n \to Y$ by

$$\lim_{m\to\infty} N\Big(F(x_1, ..., x_n) - \frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n), u\Big) = 1, \tag{2.7}$$

for all $u > 0$, $x_i \in X$ and $a_i \in \mathbb{K} \setminus \{0\}$ with $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$. Using induction with (2.6) and (2.2), we obtain

$$N\Big(f(x_1, ..., x_n) - \frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n), u\Big) \geq \Psi_m. \tag{2.8}$$

By taking m to approach infinity in (2.8) and using (2.2) one obtains (2.4).

For $i \in \{1, 2, ..., n\}$ and by (2.5) and (2.7), we get

$$\begin{aligned}
&N\Big(F(x_1, x_2, ..., a_i x_i + b_i y_i, ..., x_n) + F(x_1, x_2, ..., a_i x_i - b_i y_i, ..., x_n) - \\
&2a_i^2 F(x_1, ..., x_n) - 2b_i^2 F(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n), u\Big) = \\
&\lim_{m\to\infty} N\Big(\frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_i^m (a_i x_i + b_i y_i), ..., a_n^m x_n) + \\
&\frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_i^m (a_i x_i - b_i y_i), ..., a_n^m x_n) - \\
&\frac{2a_i^2}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_i^m x_i, ..., a_n^m x_n) - \frac{2b_i^2}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_i^m y_i, ..., a_n^m x_n), u\Big) \geq \\
&\lim_{m\to\infty} M\Big(\frac{1}{a_1^{2m}...a_n^{2m}} \varphi_i(a_1^m x_1, ..., a_i^m x_i, a_i^m y_i, ..., a_n^m x_n), u\Big).
\end{aligned} \tag{2.9}$$

By (2.3) and (2.9), we conclude that F satisfies (1.3).

Suppose that there exists another mapping $F' : X^n \to Y$ which satisfies (1.3) and (2.4). So we have

$$\begin{aligned}
&N\Big(F(x_1, ..., x_n) - F'(x_1, ..., x_n), u\Big) = \\
&\lim_{m\to\infty} N\Big(\frac{1}{a_1^{2m}...a_n^{2m}} F(a_1^m x_1, ..., a_n^m x_n) - \frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n) + \\
&\frac{1}{a_1^{2m}...a_n^{2m}} f(a_1^m x_1, ..., a_n^m x_n) - \frac{1}{a_1^{2m}...a_n^{2m}} F'(a_1^m x_1, ..., a_n^m x_n), u\Big) \geq \\
&T\Big\{\Psi\Big(a_1^m x_1, ..., a_n^m x_n, y_n, |a_1^{2m}...a_n^{2m}|u\Big), \Psi\Big(a_1^m x_1, ..., a_n^m x_n, y_n, |a_1^{2m}...a_n^{2m}|u\Big)\Big\},
\end{aligned}$$

which tends to 1 as $m \to \infty$ by (2.2). Therefore $F = F'$. This completes the proof. □

In the manner of proof of Theorem(2.1), one can prove the following corollary.

Corollary 2.2. *Let $\varphi_i : X^{n+1} \to Z$ for $i \in \{1, ..., n\}$ be a mapping such that*

$$\begin{cases}
\varphi_i^m = \varphi_i^m(x_1, ..., x_i, y_i, ..., x_n, u) = \\
M\Big(\frac{1}{2a_1^{2m+2}...a_i^{2m+2}a_{i+1}^{2m}...a_n^{2m}} \varphi_i(a_1^{m+1}x_1, ..., a_{i-1}^{m+1}x_{i-1}, a_i^m x_i, 0, a_{i+1}^m x_{i+1}, ..., a_n^m x_n), u\Big); \\
\Phi_1^m = \Phi_1^m(x_1, y_1, x_2, ..., x_n, u) = \varphi_1^m(x_1, y_1, x_2, ..., x_n, u); \\
\Phi_i^m = \Phi_i^m(x_1, ..., x_i, y_i, ..., x_n, u) = \\
T\Big(\varphi_i^m(x_1, ..., x_i, y_i, ..., x_n, u), \Phi_{i-1}^m(x_1, ..., x_{i-1}, y_{i-1}, ..., x_n, u)\Big); \\
\lim_{m\to\infty} \Phi_n^m = 1;
\end{cases}$$

and

$$\begin{cases}
\Psi_1 = \Psi_1(x_1, ..., x_n, y_n, u) = \Phi_n^1(x_1, ..., x_n, y_n, u); \\
\Psi_m = \Psi_m(x_1, ..., x_n, y_n, u) = T\Big(\Phi_n^m(x_1, ..., x_n, y_n, u), \Psi_{m-1}(x_1, ..., x_n, y_n, u)\Big); \\
\Psi = \Psi(x_1, ..., x_n, y_n, u) = \lim_{m\to\infty} \Psi_m(x_1, ..., x_n, y_n, u) = 1.
\end{cases}$$

and

$$\lim_{m\to\infty} M\Big(\frac{1}{a_1^{2m}...a_n^{2m}} \varphi_i(a_1^m x_1, ..., a_i^m x_i, a_i^m y_i, ..., a_n^m x_n), u\Big) = 1;$$

for all $u > 0$ *and* $x_i, y_i \in X$ *and* $a_i \in \mathbb{K} \setminus \{0\}$ *with* $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$. *Let* $f : X^n \to Y$ *be a mapping satisfying*

$$\begin{cases} N\Big(f(a_1x_1 + b_1y_1, x_2, ..., x_n) + f(a_1x_1 - b_1y_1, x_2, ..., x_n) - \\ 2a_1^2 f(x_1, ..., x_n) - 2b_1^2 f(y_1, x_2, ..., x_n), u\Big) \geq M\Big(\varphi_1(x_1, y_1, x_2, ..., x_n), u\Big); \\ \quad \vdots \\ N\Big(f(x_1, ..., x_{n-1}, a_nx_n + b_ny_n) + f(x_1, ..., x_{n-1}, a_nx_n - b_ny_n) - \\ 2a_n^2 f(x_1, ..., x_n) - 2b_n^2 f(x_1, ..., x_{n-1}, y_n), u\Big) \geq M\Big(\varphi_n(x_1, ..., x_n, y_n), u\Big); \end{cases}$$

for all $x_i, y_i \in X$ *and* $a_i, b_i \in \mathbb{K} \setminus \{0\}$ *with* $a_i \neq \pm 1, \pm b_i$, $i = 1, ..., n$. *Assume that* $f(x_1, x_2, ..., x_n) = 0$ *if* $x_i = 0$ *for some* $i = 1, ..., n$. *Then there exists a unique mapping* $F : X^n \to Y$ *satisfying (1.3) and*

$$N\Big(f(x_1, ..., x_n) - F(x_1, ..., x_n), u\Big) \geq \Psi$$

for all $u > 0$ *and* $x_i \in X$, $i = 1, ..., n$.

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A Note on Horner's Method

Tian-Xiao He[1] and Peter J.-S. Shiue [2]

[1]Department of Mathematics and Computer Science
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA

[2]Department of Mathematical Sciences, University of Nevada, Las Vegas
Las Vegas, NV 89154-4020, USA

Abstract

Here we present an application of Horner's method in evaluating the sequence of Stirling numbers of the second kind. Based on the method, we also give an efficient way to calculate the difference sequence and divided difference sequence of a polynomial, which can be applied in the Newton interpolation. Finally, we survey all of the results in Proposition 1.4.

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Key Words and Phrases: Horner's method, Stirling numbers of the second kind, divided difference, Newton interpolation.

1 Introduction

The number of ways of partition a set of n elements into k nonempty subsets is called the Stirling number of the second kind, denoted by $S(n,k)$. In other words, $S(n,k)$ is the number of equivalence relations with k classes on a finite set with n elements. From [3], $S(n,k)$ equals

$$\begin{aligned} S(n,k) &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n \\ &= \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n \\ &= \frac{1}{k!}\Delta^k t^n\Big|_{t=0}. \end{aligned} \tag{1}$$

As a division algorithm, Horner's method is a nesting technique requiring only n multiplications and n additions to evaluate an arbitrary nth-degree polynomial, which can be surveyed by Horner's theorem (see, for example, [1]).

Theorem 1.1 *Let*

$$P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0.$$

If $b_d = a_d$ *and*

$$b_k = a_k + b_{k+1}x_0, \quad k = n-1, n-2, \ldots, 1, 0,$$

then $b_0 = P(x_0)$, *and* $P(x)$ *can be written as*

$$P(x) = (x - x_0)Q(x) + b_0,$$

where

$$Q(x) = b_d x^{d-1} + b_{d-1}x^{d-2} + \cdots + b_2 x + b_1.$$

The theorem can be proved using a direct calculation. An additional advantage of Horner's method is the differentiation of $P(x)$:

$$P'(x) = Q(x) + (x - x_0)Q'(x).$$

Hence, $P'(x_0) = Q(x_0)$, which is very convenient when applying Newton's method to find roots of a polynomial.

Example 1 As an example, we use Horner's method to evaluate $P(x) = x^4 - 2x^2 + 3x - 4$ at $x_0 = -1$. First we construct the synthetic division as follows.

$x_0 = -1$	$a_4 = 1$	$a_3 = 0$	$a_2 = -2$	$a_1 = 3$	$a_0 = -4$
		$b_4 x_0 = -1$	$b_3 x_0 = 1$	$b_2 x_0 = 1$	$b_1 x_0 = -4$
	$b_4 = 1$	$b_3 = -1$	$b_2 = -1$	$b_1 = 4$	$b_0 = -8$

Hence,

$$x^4 - 2x^2 + 3x - 4 = (x+1)(x^3 - x^2 - x + 4) - 8.$$

In [6], Pathan and Collyer present an excellent survey on Horner's method and its application in solving polynomial equations by determining the location of roots. In this note, we shall give other applications of Horner's method in the calculation of Stirling numbers of the second kind, the difference sequences, and the divided difference sequences (or equivalently, the coefficients of Newton interpolation) of polynomials. There are numerous ways to evaluate a Stirling number sequence or Stirling matrix. For example, in [4], El-Mikkawy gives an algorithm based on Newton's divided difference interpolating polynomials. In [2], Cheon and Kim present a method based on the relationship between the Stirling matrix and other combinatorial sequences such as the Vandermonde matrix, the Bernoulli numbers, and Eulerian numbers. However, our algorithm of calculating Stirling number sequences based on Horner's method is different and efficient, which contains an idea suitable for constructing algorithms in calculation of many sequences. This general idea will be presented in Proposition 1.4.

From Proposition 1.4.2 of [7], if the polynomial $f(n)$ of degree $\leq d$ is expanded in terms of the basis $\binom{n}{k}$, $0 \leq k \leq d$, then the coefficients are $\Delta^k f(0)$, namely,

$$f(n) = \sum_{k=0}^{d} \Delta^k f(0) \binom{n}{k} = \sum_{k=0}^{d} \frac{\Delta^k f(0)}{k!} (n)_k, \tag{2}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ are the falling factorial polynomials. In particular, for $f(n) = n^d$, we have $\Delta^0 f(0) = f(0) = 0$ and

$$n^d = \sum_{k=1}^{d} \Delta^k 0^d \binom{n}{k} = \sum_{k=1}^{d} S(d,k)(n)_k, \tag{3}$$

where the rightmost equation comes from (1). Therefore, we may give the following algorithm to find out the kth order difference of f at 0 and Stirling numbers of the second order from (2) and (3) respectively.

Algorithm 1.2 *Write (2) as*

$$\begin{aligned} f(n) &= (n-0)\left(\Delta^0 f(0) + (n-1)\left(\frac{\Delta^1 f(0)}{1!} + (n-2)\left(\frac{\Delta^2 f(0)}{2!} + \cdots \right.\right.\right. \\ &\qquad \left.\left.\left. +(n-d+1)\frac{\Delta^d f(0)}{d!}\right)\right)\cdots\right). \end{aligned}$$

Use synthetic division to obtain $f(n)/(n-0)$, a polynomial of degree $d-1$, with the constant term $\Delta^0 f(0)$. Then, evaluate $(f(n)/(n-0)-\Delta^0 f(0))/(n-1)$ to find the quotient polynomial of degree $d-2$ including its constant term $\Delta^1 f(0)$. Continue this process until a single constant is left, which is $\Delta^d f(0)/d!$. Or equivalently, Use Horner's method to find

$$f(n) = (n-0)f_1(n), \quad deg\ f_1(n) \le d-1,$$

where the constant term of $f_1(n)$ is $\Delta^0 f(0)$. Then, use Horner's method again to evaluate

$$f_1(n) = (n-1)f_2(n), \quad deg\ f_2(n) \le d-2,$$

which contain the constant term $\Delta^1 f(0)/1!$. Continue the process and finally obtain

$$f_{d-1} = (n-d+1)f_d(n), \quad f_d(n) = \Delta^d f(0)/d!.$$

When $f(n) = n^d$, from (3) it can be seen that the above algorithm provides a way to evaluate the Stirling numbers of the second kind $S(d,1)$, $S(d,2), \ldots$, $S(d,d)$ defined by (1).

Example 2 Consider $f(n) = n^4 - 2n^2 + 3n - 4$. We use the following synthetic division to find out its difference sequence from order 0 to 4.

$$
\begin{array}{c|ccccc}
0 & 1 & 0 & -2 & 3 & -4 \\
 & & 0 & 0 & 0 & 0 \\ \hline
1 & 1 & 0 & -2 & 3 & \underline{-4} \\
 & & 1 & 1 & -1 & \\ \hline
2 & 1 & 1 & -1 & \underline{2} & \\
 & & 2 & 6 & & \\ \hline
3 & 1 & 3 & \underline{5} & & \\
 & & 3 & & & \\ \hline
 & \underline{1} & \underline{6} & & &
\end{array}
$$

Hence, $\Delta^0 f(0) = f(0) = -4$, $\Delta^1 f(0) = 2$, $\Delta^2 f(0) = 5(2!) = 10$, and $\Delta^3 f(0) = 6(3!) = 36$, and $\Delta^4 f(0) = 1(4!) = 24$, which can be read on the diagonal from the top right to the bottom line.

Example 3 From expansion (see, for examples, [3] and [7])

$$n^4 = \sum_{k=1}^{4} S(4,k)(n)_k,$$

or equivalently,

$$n^3 = S(4,1) + (n-1)(S(4,2) + (n-2)(S(4,3) + (n-3)S(4,4))),$$

we may use the following division to evaluate $S(4,k)$ $(k = 1,2,3,4)$.

$$
\begin{array}{c|cccc}
1 & 1 & 0 & 0 & 0 \\
 & & 1 & 1 & 1 \\ \hline
2 & 1 & 1 & 1 & \underline{1} \\
 & & 2 & 6 & \\ \hline
3 & 1 & 3 & \underline{7} & \\
 & & 3 & & \\ \hline
 & \underline{1} & \underline{6} & &
\end{array}
$$

Hence, we can read $S(4,1)=1$, $S(4,2)=7$, $S(4,3)=6$, and $S(4,4)=1$ diagonally from the top right to the bottom line. In addition, the first calculation gives $\{1,0,0,0,0\}$, the second calculation $\{1,1,1,1,\}$, the third calculation $\{1,3,7\}$, and the fourth calculation $\{1,6\}$, which are respectively the first, second, third, and fourth row of the table of the Stirling numbers of the second kind. In other words, the division of x^d by $x-j$ generates $\{S(d,j),S(d-1,j),\cdots,S(j,j)\}$.

From (3) we immediately know that $S(d,d)=1$ because it is the coefficient of n^d. Using our method, one may calculate the matrices related to Stirling numbers easily, for example, matrices T_n and W_n defined by (2) and (16) in [8].

Algorithm 1.2 can also be used to evaluate non-centeral Stirling numbers of the second kind (*cf.* [5]) defined by

$$(x-a)^d=\sum_{k=0}^{d} S_a(d,k)(x)_k$$

with a parameter a. In fact, a similar argument can be used to calculate $\{S_a(d,k)\}$ by using the transformation $x-a\mapsto x$ in Algorithm 1.2.

From Theorem 1.1 we also know that Horner's method provides simple algorithms to evaluate divided differences and derivatives of a polynomial, and the former can be used to find the coefficients of the Newton interpolation while the latter can be used to approximate the zeros of the polynomial with any required significant digits.

Let $X=\{x_0,x_1,\ldots,x_d\}$ be a set of $d+1$ distinct points, and let $f(x)$ be a polynomial of degree d. Then we can write $f(x)$ in terms of its Newton interpolation form on the set X as

$$\begin{aligned} f(x) &= f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)+\cdots \\ &\quad +f[x_0,x_1,\ldots,x_d](x-x_0)(x-x_1)\cdots(x-x_{d-1}), \end{aligned} \tag{4}$$

where $f[x_0]=f(x_0)$ and $f[x_0,x_1,\ldots x_k]$ is the kth order divided difference of f at $\{x_0,x_1,\ldots,x_k\}$ defined by

$$f[x_0,x_1,\ldots x_k]=\frac{1}{x_k-x_0}(f[x_1,x_2,\ldots,x_k]-f[x_0,x_1,\ldots,x_{k-1}])$$

for $k=1,2,\ldots,d$, and can be evaluated using the following algorithm.

Algorithm 1.3 *Write (4) as*

$$\begin{aligned} f(x) &= f[x_0] + (x - x_0)\left(f[x_0, x_1] + (x - x_1)\left(f[x_0, x_1, x_2] + \cdots\right.\right. \\ &\quad \left.\left. +(x - x_{d-1}) f[x_0, x_1, \ldots, x_d]\right)\right), \end{aligned}$$

where $f[x_0] = f(x_0)$. Use synthetic division to obtain $(f(x)-f(x_0))/(x-x_0)$, a polynomial of degree $d-1$, with the constant term $f[x_0, x_1]$. Then, evaluate $(f(x)-f(x_0))/(x-x_0)-f[x_0, x_1])/(x-x_1)$ to find the quotient polynomial of degree $d-2$ and its constant term $f[x_0, x_1, x_2]$. Continue this process until a single constant is left, which is $f[x_0, x_1, \ldots, x_d]$. Or equivalently, Use Horner's method to find

$$f(x) - f(x_0) = (x - x_0) f_1(x), \quad deg\ f_1(x) \le d - 1,$$

where the constant term of $f_1(x)$ is $f[x_0, x_1]$. Then, use Horner's method again to evaluate

$$f_1(x) = (x - x_1) f_2(x), \quad deg\ f_2(x) \le d - 2,$$

which contains the constant term $f[x_0, x_1, x_2]$. Continue the process and finally to obtain

$$f_{d-1} = (x - x_{d-1}) f_d(x), \quad f_d(n) = f[x_0, x_1, \ldots, x_d].$$

Example 4 To find the divided differences of $f(x) = x^4 - 2x^2 + 3x - 4$ on the set $\{-1, 0, 1, 3, 4\}$, we consider $f(x) - f(-1) = x^4 - 2x^2 + 3x + 4$ and use the following synthetic division to obtain its divided difference at the given knot points.

$$\begin{array}{c|ccccc}
-1 & 1 & 0 & -2 & 3 & 4 \\
 & & -1 & 1 & 1 & -4 \\
\hline
0 & 1 & -1 & -1 & 4 & 0 \\
 & & 0 & 0 & 0 & \\
\hline
1 & 1 & -1 & -1 & \underline{4} & \\
 & & 1 & 0 & & \\
\hline
3 & 1 & 0 & \underline{-1} & & \\
 & & 3 & & & \\
\hline
 & \underline{1} & \underline{3} & & &
\end{array}$$

Hence, $f[-1] = f(-1) = -8$, $f[-1, 0] = 4$, $f[-1, 0, 1] = -1$, $f[-1, 0, 1, 3] = 3$, and $f[-1, 0, 1, 3, 4] = 1$. It can be seen that the new method is much easier than the traditional method.

Let r be a real number, and let $f(x)$ be a polynomial of degree d. Then, using the Taylor expansion of $f(x)$ yields

$$f(x) = f(r) + f'(r)(x-r) + \frac{f''(r)}{2!}(x-r)^2 + \cdots + \frac{f^{(d)}(r)}{d!}(x-r)^d, \quad (5)$$

which can written recursively as

$$f(x) - f(r) = (x-r)f_1(x), \quad f_k(x) = (x-r)f_{k+1}(x), \quad k = 1, 2, \ldots, d-1,$$

and the constant term of $f_k(x)$ is $f^{(k)}(r)/k!$ $(k = 1, 2, \ldots, d)$. Thus we may apply Horner's method to find all derivatives of f at r. Obviously, for polynomial

$$g(x) = f(r) + f'(r)x + \frac{f''(r)}{2!}x^2 + \cdots + \frac{f^{(d)}(r)}{d!}x^d, \quad (6)$$

the roots of $g(x) = 0$ are the roots of equation $f(x) = 0$, each diminished by r. We can use the process to diminish a root of the proposed equation by its first digit. Then we apply it again to diminish the corresponding root of the resulting equation by its first digit, which is the second digit of the required root of the original equation. Using this process continuously, we finally approximate the root of the original equation $f(x) = 0$ to the required significant digits. More details can be found in [9]. Here is an example.

Example 5 Consider equation $f(x) = x^4 - 2x^2 + 3x - 4 = 0$, which has a root in the interval $(1, 2)$. We may use (6) to find $g(x)$, where $r = 1$. The process can be shown in the following synthetic division.

$$
\begin{array}{c|ccccc}
1 & 1 & 0 & -2 & 3 & -4 \\
 & & 1 & 1 & -1 & 2 \\
\hline
 & 1 & 1 & -1 & 2 & \underline{-2} \\
 & & 1 & 2 & 1 & \\
\hline
 & 1 & 2 & 1 & \underline{3} & \\
 & & 1 & 3 & & \\
\hline
 & 1 & 3 & \underline{4} & & \\
 & & 1 & & & \\
\hline
 & \underline{1} & \underline{4} & & &
\end{array}
$$

Hence, we obtain

$$f(1) = -2, f'(1) = 3, \frac{f''(1)}{2!} = 4, \frac{f'''(1)}{3!} = 4, \frac{f^{(4)}(1)}{4!} = 1,$$

and the corresponding

$$g(x) = -2 + 3x + 4x^2 + 4x^3 + x^4.$$

Therefore the new equation is $g(x) = 0$. Multiply the root by 10 and change the equation to be

$$x^4 + 40x^3 + 400x^2 + 3000x - 20000 = 0.$$

It is easy to see that $g(x)$ has a root between 3 and 4. Thus we may use (6) again to generate a new polynomial and solve the corresponding equation.

3	1	40	400	3000	−20000
		3	129	1587	13761
	1	43	529	4587	$\underline{-6239}$
		3	138	2001	
	1	46	667	$\underline{6588}$	
		3	147		
	1	49	$\underline{814}$		
		3			
	$\underline{1}$	$\underline{52}$			

The above table shows that an approximation of the original polynomial equation to its second significant digit is 1.3, and the third significant digit can be found using the polynomial equation

$$x^4 + 52x^3 + 814x^2 + 6588x - 6239 = 0.$$

Multiply the root by 10 to change the equation to be

$$x^4 + 520x^3 + 81400x^2 + 6588000x - 62390000 = 0,$$

which has a root in the interval $(8, 9)$. Thus, the original polynomial equation $f(x) = x^4 - 2x^2 + 3x - 4 = 0$ has a root of approximately 1.38, and its better approximation with more significant digits can be found from the equation

$$x^4 + 552x^3 + 94264x^2 + 7992288x - 4206064 = 0$$

generated by using the following table. Since $x^4 + 552x^3 + 94264x^2 + 7992288x - 4206064 = 0$ has a root between 5 and 6, we obtain the root of original equation with four significant digits as 1.385.

8	1	520	81400	6588000	−62390000
		8	4224	684992	58183936
	1	528	85624	7272992	$\underline{-4206064}$
		8	4288	719296	
	1	536	89912	$\underline{7992288}$	
		8	4352		
	1	544	$\underline{94264}$		
		8			
	$\underline{1}$	$\underline{552}$			

From Example 5, one may find Horner's method is not an efficient way to evaluate the roots of polynomial equations, but it is a faster way to find out the coefficients of the expansions of polynomials in terms of nested bases formed by products of linear polynomials.

Proposition 1.4 *Let $\phi_k(x) = a_k x - b_k$, $k = 1, 2, \ldots$, and let $f(x)$ be a polynomial of degree d. Then*

$$f(x) = c_0 + \sum_{k=1}^{d} c_k \Pi_{j=1}^{k} \phi_j(x), \tag{7}$$

where c_k $(k = 0, 1, \ldots, d)$ can be found using the synthetic division based on Horner's method.

One may see the examples of Proposition 1.4 from the algorithms applied to the expansions (2)-(5). Interested readers may also construct examples for any polynomial expansion defined by (7). For instance, we may calculate the binomial sequence $\binom{n}{k}$ $(k = 0, 1, \ldots, n)$ for $n \in \mathbb{N}$ by applying Horner's method to the expansion

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k.$$

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COMMON FIXED POINT RESULTS WITH APPLICATIONS IN CONVEX METRIC SPACES

SAFEER HUSSAIN KHAN and MUJAHID ABBAS

Abstract ___ The notion of metric convexity introduced by Takahashi [24] is employed to obtain sufficient conditions for the existence of a common fixed point for a Banach operator pair of mappings satisfying generalized contractive conditions. As an application, related results on best approximation are derived. Our results generalize various known results in contemporary literature.

Keywords and Phrases: Convex metric space, common fixed point, best approximation, Banach operator pair.

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1. INTRODUCTION and PRELIMINARIES

Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric conditions on mappings and/or underlying spaces play a crucial role in metric fixed point problems. Although it has a purely metric flavor, it is also a major branch of nonlinear functional analysis with close ties to Banach space geometry, see for example [10] and references mentioned therein. Several results concerning the existence and approximation of a fixed point of a mapping rely on convexity hypotheses and geometric properties of the Banach spaces. Takahashi [24] introduced the notion of a convexity on metric spaces. Afterwards, Ciric [6], Ding [7], Goebel and Kirk [9], Guay et al [11], Shimizu and Takahashi[20, 21] and other authors have studied fixed point theorems in convex metric spaces (see also [4]). On the other hand, Shahzad[19] introduced a class of noncommuting mappings called R-subweakly commuting mappings, and thus obtained common fixed points of S-nonexpansive mappings in normed spaces. Recently, Chen and Li [5] introduced the class of Banach operator pairs as a new class of noncommuting maps. In this paper, common fixed points for Banach operator pair of mappings which are more general than C_q-commuting mappings, are obtained in the setting of a convex metric space. As an application, invariant approximation results for these mappings are also derived.

For the sake of convenience, we gather some basic definitions and set out the terminology needed in the sequel.

Definition 1.1. Let (X, d) be a metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X, if, for each $(x, y, \lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space X together with a convex structure W is called a convex metric space. Obviously, $W(x, x, \lambda) = x$.

Let X be a convex metric space. A nonempty subset E of X is said to be *convex* if $W(x, y, \lambda) \in E$ whenever $(x, y, \lambda) \in E \times E \times [0,1]$. A subset E of a convex metric space is said to be *q-starshaped* or *starshaped with respect to q* if there exists q in E such that $W(x, q, \lambda) \in E$ whenever $(x, \lambda) \in E \times [0,1]$. Obviously, q- starshaped subsets of X contain all convex subsets of X as a proper subclass. Takahashi [24] has shown that open spheres $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex in a convex metric space X. A convex metric space X is said to have property (A) if: $d(W(y, x, \lambda), W(z, x, \lambda)) \leq \lambda d(y, z)$, for all $x, y, z \in X$ and $\lambda \in (0, 1)$. Property (A) is a convex metric space analogue of condition (I) for the starshaped metric spaces of Guay et al, see Definition 3.2 [11]. Throughout this paper, a convex metric space X is assumed to have property (A).

Note that every normed space is a convex metric space. Takahashi [24] has also shown that there are convex metric spaces which cannot be embedded in any normed space .

Example 1.2. Let $X = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 > 0\}$. For $x =$

$(x_1, x_2, x_3), y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in X, and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, define a mapping $W : X^3 \times [0,1]^3 \to X$ by,

$$W(x, y, z, \alpha, \beta, \gamma) = (\alpha x_1 + \beta x_2 + \gamma x_3, \alpha y_1 + \beta y_2 + \gamma y_3, \alpha z_1 + \beta z_2 + \gamma z_3),$$

and a metric $d : X \times X \to [0, \infty)$ by, $d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|$. Here X is a convex metric space but it is not a normed space.

Example 1.3. Let $X = \{(x_1, x_2) \in R^2 : x_1, x_2 > 0\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X, and $\alpha \in [0,1]$. Define a mapping $W : X \times X \times [0,1] \to X$ by

$$W(x, y, \alpha) = (\alpha x_1 + (1-\alpha) y_1, \frac{\alpha x_1 x_2 + (1-\alpha) y_1 y_2}{\alpha x_1 + (1-\alpha) y_1}),$$

and a metric $d : X \times X \to [0, \infty)$ by $d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|$. It can be verified that X is a convex metric space but not a normed space.

Definition 1.4. Let $T, S : X \to X$. A point $x \in X$ is called:

(1) *a fixed point of* T if $Tx = x$;

(2) *a coincidence point of the pair* $\{T, S\}$ if $Tx = Sx$;

(3) *a common fixed point of the pair* $\{T, S\}$ if $x = Tx = Sx$.

$F(T), C(T, S)$ and $F(T, S)$ denote the set of all fixed points of T, the set of all coincidence points of the pair $\{T, S\}$, and the set of all common fixed points of the pair $\{T, S\}$, respectively.

Definition 1.5. Let E be a q-starshaped subset of a convex metric space X. Let $T, S : X \to X$, $q \in F(S)$ and E be both T and S invariant. Put

$$Y_q^{Tx} = \{y_\lambda : y_\lambda = W(Tx, q, \lambda) \text{ and } \lambda \in [0,1]\},$$

and for each x in X, $d(Sx, Y_q^{Tx}) = \inf\{d(Sx, y_\lambda) : \lambda \in [0,1]\}$. The map T is said to be:

(1) an S-*contraction* if there exists $k \in (0,1)$ such that

$$d(Tx, Ty) \le k d(Sx, Sy);$$

(2) *asymptotically* S-*nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \ge 1$, with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \le k_n \, d(Sx, Sy)$ for each x, y in E

and $n \in \mathbb{N}$. If $k_n = 1$ for all $n \in \mathbb{N}$, then T is called an S- nonexpansive mapping. If $S = I$ (the identity map), then T is an asymptotically nonexpansive mapping;

(3) *R-weakly commuting* if there exists a real number $R > 0$ such that

$$d(STx, TSx) \leq Rd(Tx, Sx)$$

for all x in E;

(4) *R-subweakly commuting* if there exists a real number $R > 0$ such that

$$d(TSx, STx) \leq Rd(Sx, Y_q^{Tx})$$

for all $x \in E$;

(5) C_q-commuting if $STx = TSx$ for all $x \in C_q(S,T)$, where $C_q(S,T) = \cup\{C(S,T_k) : 0 \leq k \leq 1\}$ and $T_k x = W(Tx, q, k)$.

Clearly C_q-commuting maps are weakly compatible but the converse is not true in general (see for example [2]).

A self mapping T on a convex metric space X is said to be

(6) *affine* on E if

$$T(W(x,y,\lambda)) = W(Tx, Ty, \lambda)$$

for all $x, y \in E$ and $\lambda \in (0,1)$;

(7) *uniformly asymptotically regular* on E if for each $\varepsilon > 0$, there exists a positive integer N such that $d(T^n x, T^{n+1}x) < \varepsilon$ for all $n \geq N$ and for all x in E.

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a *Banach operator pair* if the set $F(I)$ is T-invariant, namely $T(F(I)) \subseteq F(I)$. Obviously, any commuting pair (T, I) is a Banach operator pair but not conversely in general, see [5]. If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (cf. Example 1 [5]). If the self-maps T and I of X satisfy

$$d(ITx, Tx) \leq kd(Ix, x)$$

for all $x \in X$ and $k \geq 0$, then (T, I) is a Banach operator pair.

Example 1.6. Let $X = \mathbb{R}$ with usual metric and $M = [1, \infty)$. Let $T(x) = x^2$ and $I(x) = 2x - 1$, for all $x \in M$. Let $q = 1$. Then M is convex with $q \in F(I)$, $F(I) = \{1\}$ and $C_q(I, T) = [1, \infty)$. Note that the pair (T, I) is Banach operator but T and I are not C_q-commuting maps and hence not R-subweakly commuting maps.

2. COMMON FIXED POINT RESULTS

In this section, the existence of common fixed points of Banach operator pair of mappings is established in a convex metric space. The following result is a consequence of ([14], Theorem 2.1).

Theorem 2.1. Let M be a subset of a metric space (X, d), and I and T be weakly compatible self-maps of M. Assume that $clT(M) \subset I(M)$, $clT(M)$ is complete, and T and I satisfy for all $x, y \in M$ and $0 \leq h < 1$,

$$d(Tx, Ty) \leq h \max \{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}.$$

Then $M \cap F(I) \cap F(T)$ is a singleton.

The following result extends and improves Lemma 3.1 of [5] and Theorem 1 in [16].

Lemma 2.2. Let M be a nonempty subset of a metric space (X, d), and (T, I) be a Banach operator pair on M. Assume that $clT(M)$ is complete, and T and I satisfy for all $x, y \in M$ and $0 \leq h < 1$,

$$d(Tx, Ty) \leq h \max \{d(Ix, Iy), d(Tx, Ix), d(Ty, Iy), d(Tx, Iy), d(Ty, Ix)\}. \tag{2.1}$$

If I is continuous and $F(I)$ is nonempty, then there exists a unique common fixed point of T and I.

Proof. By our assumptions, $T(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty and closed. Moreover, $cl(T(F(I)))$ being subset of $cl(T(M))$ is complete. Further, for all $x,\ y \in F(I)$, we have by inequality (2.1),

$$\begin{aligned} d(Tx, Ty) &\leq h \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Iy, Tx), d(Ix, Ty)\} \\ &= h \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}. \end{aligned}$$

Hence T is a generalized contraction on $F(I)$ and $cl(T(F(I))) \subseteq cl(F(I)) = F(I)$. By Theorem 2.1, T has a unique fixed point z in $F(I)$ and consequently $F(I) \cap F(T)$ is singleton.

The following result presents an analogue of Lemma 3.3 [3] for Banach operator pair without linearity of I.

Lemma 2.3. Let I and T be self-maps on a nonempty q-starshaped subset M of a convex metric space X. Assume that I is continuous and $F(I)$ is q-starshaped with $q \in F(I)$, (T, I) is a Banach operator pair on M and satisfy for each $n \geq 1$

$$d(T^n x, T^n y) \leq k_n max \left\{ \begin{array}{c} d(Ix, Iy), dist(Ix, Y_q^{T^n x}), dist(Iy, Y_q^{T^n y}), \\ dist(Ix, Y_q^{T^n y}, dist(Iy, Y_q^{T^n x}) \end{array} \right\} \quad (2.2)$$

for all $x, y \in M$, where $\{k_n\}$ is a sequence of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$. For each $n \geq 1$, define a mapping T_n on M by

$$T_n x = W(T^n x, q, \mu_n),$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. Then for each $n \geq 1$, T_n and I have exactly one common fixed point x_n in M such that

$$Ix_n = x_n = W(T^n x, q, \mu_n),$$

provided $cl(T_n(M))$ is complete for each n.

Proof. By definition,

$$T_n x = W(T^n x, q, \mu_n).$$

As (T, I) is a Banach operator pair, for each $n \geq 1$, $T^n(F(I)) \subseteq F(I)$ and $F(I)$ is nonempty and closed. Since $F(I)$ is q-starshaped and $T^n x \in F(I)$, for each $x \in F(I)$, $T_n x = W(T^n x, q, \mu_n) \in F(I)$. Thus (T_n, I) is Banach operator pair for each n. Also by (2.2),

$$\begin{aligned} d(T_n x, T_n y) &= d(W(T^n x, q, \mu_n), W(T^n y, q, \mu_n)) \\ &\leq \mu_n d(T^n x, T^n y) \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_n max\{d(Ix, Iy), dist(Ix, Y_q^{T^n x}), dist(Iy, Y_q^{T^n y}),\\
&\quad dist(Ix, Y_q^{T^n y}, dist(Iy, Y_q^{T^n x})\}\\
&\leq \lambda_n max\{d(Ix, Iy), d(Ix, T_n x), d(Iy, T_n y),\\
&\quad d(Ix, T_n y), d(Iy, T_n x)\}
\end{aligned}
$$

for each $x, y \in M$. By Lemma 2.2, for each $n \geq 1$, there exists a unique $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Thus for each $n \geq 1$, $M \cap F(T_n) \cap F(I) \neq \phi$.
The following result extends the recent results due to Chen and Li ([5], Theorems 3.2-3.3) to asymptotically I-nonexpansive maps.

Theorem 2.4. Let I and T be self-maps on a q-starshaped subset M of a convex metric space X. Assume that (T, I) is Banach operator pair on M, $F(I)$ is q-starshaped with $q \in F(I)$, I is continuous, T is uniformly asymptotically regular and asymptotically I-nonexpansive. Then $F(T) \cap F(I) \neq \phi$, provided $cl(T(M))$ is compact and T is continuous.

Proof. Notice that compactness of $cl(T(M))$ implies that $clT_n(M)$ is compact and thus complete. From Lemma 2.3, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Ix_n = W(T^n x, q, \mu_n)$. As $T(M)$ is bounded, so $d(x_n, T^n x_n) = d(W(T^n x, q, \mu_n), T^n x_n)) \leq (1 - \mu_n) d(T^n x_n, q) \to 0$ as $n \to \infty$. Since (T, I) is Banach operator pair and $Ix_n = x_n$, so $IT^n x_n = T^n Ix_n = T^n x_n$ and thus we get

$$
\begin{aligned}
d(x_n, Tx_n) &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n)\\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(IT^n x_n, Ix_n)\\
&= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n).
\end{aligned}
$$

Further, T is uniformly asymptotically regular, therefore we have

$$
d(x_n, Tx_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \to 0,
$$

as $n \to \infty$. Since $cl(T(M))$ is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. By the continuity of I and T and the fact $d(x_m, Tx_m) \to 0$, we have $y \in F(T) \cap F(I)$. Thus $F(T) \cap F(I) \neq \phi$.

Corollary 2.5([5], Theorems 3.2-3.3). Let I and T be self-maps on a q-starshaped subset M of a normed space X. Assume that (T, I) is Banach operator pair on M, $F(I)$ is q-starshaped with $q \in F(I)$, I is continuous and T is I-nonexpansive. Then $F(T) \cap F(I) \neq \phi$, provided $cl(T(M))$ is compact.

Corollary 2.6 ([1], Theorem 2.2 and [13], Theorem 6). Let I and T be self-maps on a q-starshaped subset M of a normed space X. Assume that (T, I) is commuting pair on M, $F(I)$ is q-starshaped with $q \in F(I)$, I is continuous and T is I-nonexpansive. Then $F(T) \cap F(I) \neq \phi$, provided $cl(T(M))$ is compact.

Meinardus [17] was the first to employ a fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results have appeared in the literature of approximation theory ([1], [19] and [22]).

Definition 2.7. Let X be a metric space and M be a closed subset of X. If there exists a $y_0 \in M$ such that $d(x, y_0) = d(x, M) = inf\{d(x, y) : y \in M\}$, then y_0 is called a *best approximation to x out of M*. We denote by $P_M(x)$, the set of all best approximations to x out of M.

Remark 2.8. Let M be a closed convex subset of a convex metric space. As $W(u, v, \lambda) \in M$ for $(u, v, \lambda) \in M \times M \times [0, 1]$, the definition of convex structure on X implies that $W(u, v, \lambda) \in P_M(x)$. Hence $P_M(x)$ is a convex subset of X. Also, $P_M(x)$ is a closed subset of X. Moreover, it can be shown that $P_M(x) \subset \partial M$, where ∂M stands for the boundary of M.

Now we obtain results on best approximation as a fixed point of Banach operator pair of mappings in a convex metric space.

Theorem 2.9. Let M be a subset of a convex metric space X and $I, T : X \to X$ be mappings such that $u \in F(I) \cap F(T)$ for some $u \in X$ and $T(\partial M \cap M) \subseteq M$. Suppose that $P_M(u)$ is nonempty and q-starshaped, I is continuous on $P_M(u)$, $d(Tx, Tu) \leq d(Ix, Iu)$ for each $x \in P_M(u)$ and $I(P_M(u)) \subseteq P_M(u)$. If (T, I) is a Banach operator pair on $P_M(u)$, $F(I)$ is nonempty and q-starshaped for $q \in F(I)$, T is uniformly asymptotically regular and asymptotically I-nonexpansive then $P_M(u) \cap F(I) \cap F(T) \neq \phi$, provided T is continuous and

$cl(T(P_M(u)))$ is compact.

Proof. Let $x \in P_M(u)$. Then for any $h \in (0,1)$, $d(W(u,x,h),u) \leq (1-h)d(x,u) < dist(u,C)$. It follows that $\{W(u,x,h) : 0 < h < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subseteq M$, Tx must be in M. Also $Ix \in P_M(u)$, $u \in F(I) \cap F(T)$ and I and T satisfy $d(Tx,Tu) \leq d(Ix,Iu)$, thus we have

$$\begin{aligned} d(Tx,u) &= d(Tx,Tu) \leq d(Ix,Iu) \\ &= d(Ix,u) = dist(u,M). \end{aligned}$$

It further implies that $Tx \in P_M(u)$. Therefore T is a self map of $P_M(u)$. The result now follows from Theorem 2.4.

The above result extends Theorem 3.2 of [1], Theorems 4.1-4.2 of [5], Theorem 7 of [13], Theorem 3 of [18], the corresponding results of [15], [16], [22], and[23].

Remarks 2.10.

(1) Theorem 2.4 extends and improves Theorems 1 and 2 of Dotson [8], Theorem 2.2 of Al-Thagafi [1], Theorem 4 of Habiniak [12] and Theorem 1 of Khan and Khan [16].

(2) As an application of Theorem 2.4, we can prove an analogue of recent invariant approximation results in [2], namely, Theorem 3.1–Theorem 4.4 for asymptotically I-nonexpansive map T for which (T,I) is a Banach operator pair.

(3) Theorem 2.7 extends and improves Theorem 3.4 of Beg et al [3] to convex metric spaces.

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Safeer Hussain Khan

Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar.

safeer@qu.edu.qa; safeerhussain5@yahoo.com

Mujahid Abbas

Department of Mathematics,

Lahore University of Management Sciences, 54792-Lahore, Pakistan.

mujahid@lums.edu.pk

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BASIC HYPERGEOMETRIC SERIES AND q-HARMONIC NUMBER IDENTITIES

WENCHANG CHU

Dipartimento di Matematica
Università del Salento
Lecce–Arnesano P. O. Box 193
Lecce 73100 Italy

NANCY S. S. GU

Center for Combinatorics, LPMC
Nankai University
Tianjin 300071, P.R. China

ABSTRACT. The derivative operator method is systematically employed in examining four typical basic hypergeometric series theorems, named as the q-Chu-Vandermonde identity, the q-Pfaff-Saalschütz theorem, the q-Dougall-Dixon formula and Watson's q-Whipple transformation, which establish numerous identities on q-harmonic numbers, including several surprising summation formulae.

1. INTRODUCTION AND NOTATIONS

Let $\mathbb{N}_0$ be the set of nonnegative integers. With the q-shifted factorial of order $n \in \mathbb{N}_0$ given by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n := \prod_{k=0}^{n-1}(1-xq^k) \quad \text{for} \quad n = 1,2,\cdots$$

we define the q-binomial coefficients

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{(q^{1+x-k};q)_k}{(q;q)_k} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

as well as the basic hypergeometric series

$${}_{1+r}\phi_r \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_r \end{matrix} \,\middle|\, q; z \right] = \sum_{n=0}^{\infty} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n \, (b_1;q)_n \cdots (b_r;q)_n} z^n$$

where for convenience, $0 < |q| < 1$ will be assumed throughout the paper. See Gasper and Rahman [6] for a comprehensive study of the theory of basic hypergeometric series.

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Email addresses: chu.wenchang@unisalento.it and gu@nankai.edu.cn.

For a differentiable function $f(x)$, denote the two derivative operators by

$$\mathcal{D}_x f(x) = \frac{d}{dx} f(x) \qquad \text{and} \qquad \mathcal{D}_0 f(x) = \frac{d}{dx} f(x)\Big|_{x=0}.$$

Then it is easy to check the following derivatives of q-binomial coefficients at $x = 0$

$$\mathcal{D}_0 \begin{bmatrix} x+m \\ n \end{bmatrix} = (\ln q) \begin{bmatrix} m \\ n \end{bmatrix} \{\mathcal{H}_{m-n}(q) - \mathcal{H}_m(q)\}, \tag{1.1}$$

$$\mathcal{D}_0 \begin{bmatrix} x+m \\ n \end{bmatrix}^{-1} = (\ln q) \begin{bmatrix} m \\ n \end{bmatrix}^{-1} \{\mathcal{H}_m(q) - \mathcal{H}_{m-n}(q)\}; \tag{1.2}$$

where the q-analogue of harmonic number [1] reads as

$$\mathcal{H}_0(q) = 0 \ \text{ and } \ \mathcal{H}_n(q) = \sum_{k=1}^{n} \frac{q^k}{1-q^k} \ \text{ for } \ n = 1, 2, \cdots \tag{1.3}$$

which will frequently be written as $\mathcal{H}_n$ instead of $\mathcal{H}_n(q)$ for brevity.

Richard Askey has pointed out that Isaac Newton was the first to notice that the partial sums of the harmonic series arise from the differentiation of a product [10]. In the sequel, many harmonic number identities were discovered by using this differentiation method (see [1, 3, 5, 11] for example).

In this paper, we will apply further the derivative operator method to four typical basic hypergeometric series formulae. Numerous identities on q-harmonic numbers will be established. Especially, we will examine, in detail, the q-Chu-Vandermonde identity, the q-Pfaff-Saalschütz summation theorem, the q-Dougall-Dixon formula and Watson's q-Whipple transformation. They will present a complete coverage on the q-harmonic numbers identities related to these four fundamental q-series summation formulae.

The method can briefly be described as the following three steps:

- For a given q-series theorem, reformulating it in terms of q-binomial sum equation.
- Applying the derivative operator $\mathcal{D}_0$ across the resulting q-binomial equation.
- Writing down the q-harmonic number identities by specifying parameters.

In most cases, this procedure can be carried out through quite routine computations. However there are several situations in which a limiting relation on finite q-harmonic number sums will be required, that will be proved in the sixth section. As a documentary source for further references on q-harmonic number identities, ninty examples will carefully be selected as consequences.

The rest of the paper will be organized as follows. From Section 2 to Section 5, we shall systematically investigate the q-harmonic number identities by utilizing the aforementioned four well-known basic hypergeometric series formulae. Then the sixth section will present a quite useful limiting relation concerning a class of finite q-sums. Finally, the paper will end up with Section 7 by comparing the classical harmonic number identities appeared in [5] and their q-analogues obtained in this paper.

2. The q-Chu-Vandermonde identity

Recall the q-Chu-Vandermonde convolution identity [6, Appendix II-6]

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}\,\middle|\, q; q\right] = \frac{(c/a;q)_n}{(c;q)_n} a^n. $$

Performing the parameter replacements

$$ \left.\begin{matrix} a \to q^{-n-\mu n} \\ c \to q^{1+\lambda n+x} \end{matrix}\right\} \quad \text{where} \quad \lambda,\mu \in \mathbb{N}_0 $$

we can equivalently restate it as the following q-binomial identity

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \begin{bmatrix} n+\mu n \\ k \end{bmatrix} \begin{bmatrix} x+n+\lambda n \\ n-k \end{bmatrix} = \begin{bmatrix} x+2n+\lambda n+\mu n \\ n \end{bmatrix}. \tag{2.1} $$

Applying the derivative operator $\mathcal{D}_0$ across this equation and then appealing to (1.1), we find the expression

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \begin{bmatrix} n+\mu n \\ k \end{bmatrix} \begin{bmatrix} n+\lambda n \\ n-k \end{bmatrix} \big\{\mathcal{H}_{\lambda n+n} - \mathcal{H}_{\lambda n+k}\big\} = \begin{bmatrix} 2n+\lambda n+\mu n \\ n \end{bmatrix} \big\{\mathcal{H}_{\lambda n+\mu n+2n} - \mathcal{H}_{\lambda n+\mu n+n}\big\}. $$

According to the factor inside the braces $\{\cdots\}$, splitting the left-hand side into two sums with respect to k and then evaluating the first one by (2.1), we get immediately the following q-harmonic number identity.

Theorem 1 ($\lambda,\mu \in \mathbb{N}_0$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \begin{bmatrix} n+\mu n \\ k \end{bmatrix} \begin{bmatrix} n+\lambda n \\ n-k \end{bmatrix} \mathcal{H}_{\lambda n+k} = \begin{bmatrix} 2n+\lambda n+\mu n \\ n \end{bmatrix} \big\{\mathcal{H}_{\lambda n+n} + \mathcal{H}_{\lambda n+\mu n+n} - \mathcal{H}_{\lambda n+\mu n+2n}\big\}. $$

One special case corresponding to $\mu = 0$ reads as

$$ \sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+\lambda n \\ n-k \end{bmatrix} \mathcal{H}_{\lambda n+k} = \begin{bmatrix} 2n+\lambda n \\ n \end{bmatrix} \big\{2\mathcal{H}_{\lambda n+n} - \mathcal{H}_{\lambda n+2n}\big\} \tag{2.2} $$

which can further be specialized by letting $\lambda = 0$ to the following identity

$$ \sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \mathcal{H}_k = \begin{bmatrix} 2n \\ n \end{bmatrix} \big\{2\mathcal{H}_n - \mathcal{H}_{2n}\big\}. \tag{2.3} $$

3. The q-Pfaff-Saalschütz theorem

Making the parameter replacements

$$ \left.\begin{matrix} a \to q^{-n-\mu n-\mu' x} \\ b \to q^{1+\lambda n+\lambda' x} \\ c \to q^{1+\nu n+\nu' x} \end{matrix}\right\} \quad \text{where} \quad \lambda,\mu,\nu \in \mathbb{N}_0 $$

in the q-Pfaff-Saalschütz theorem [6, Appendix II-12]

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix}\,\Big|\, q;q\right] = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/(ab);q)_n} $$

we can reformulate it as the q-binomial equation

$$ \sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k+\lambda n+\lambda' x \\ k \end{bmatrix}\begin{bmatrix} n+\mu n+\mu' x \\ k \end{bmatrix}}{\begin{bmatrix} k+\nu n+\nu' x \\ k \end{bmatrix}\begin{bmatrix} k+(\lambda-\mu-\nu-2)n+(\lambda'-\mu'-\nu')x \\ k \end{bmatrix}} q^{(n-k)(n-k+\mu n+\mu' x)} = \frac{\begin{bmatrix} (\lambda-\nu)n+(\lambda'-\nu')x \\ n \end{bmatrix}\begin{bmatrix} (\mu+\nu+2)n+(\mu'+\nu')x \\ n \end{bmatrix}}{\begin{bmatrix} n+\nu n+\nu' x \\ n \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-1)n+(\lambda'-\mu'-\nu')x \\ n \end{bmatrix}}. $$

In view of (1.1) and (1.2), we establish the following three theorems on q-harmonic sums by applying the derivative operator $\mathcal{D}_0$ respectively to the cases $\mu'=\nu'=0$, $\lambda'=\nu'=0$ and $\lambda'=\mu'=0$ of the last identity.

Theorem 2 ($\lambda,\mu,\nu \in \mathbb{N}_0$: $\lambda > 1+\mu+\nu$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} \lambda n+k \\ k \end{bmatrix}\begin{bmatrix} \mu n+n \\ k \end{bmatrix}}{\begin{bmatrix} \nu n+k \\ k \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-2)n+k \\ k \end{bmatrix}} \left\{\mathcal{H}_{\lambda n+k} - \mathcal{H}_{(\lambda-\mu-\nu-2)n+k}\right\} $$
$$ = \frac{\begin{bmatrix} (\lambda-\nu)n \\ n \end{bmatrix}\begin{bmatrix} (\mu+\nu+2)n \\ n \end{bmatrix}}{\begin{bmatrix} \nu n+n \\ n \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-1)n \\ n \end{bmatrix}} \left\{\mathcal{H}_{(\lambda-\nu)n} - \mathcal{H}_{(\lambda-\nu-1)n} + \mathcal{H}_{\lambda n} - \mathcal{H}_{(\lambda-\mu-\nu-1)n}\right\}. $$

Theorem 3 ($\lambda,\mu,\nu \in \mathbb{N}_0$: $\lambda > 1+\mu+\nu$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} \lambda n+k \\ k \end{bmatrix}\begin{bmatrix} \mu n+n \\ k \end{bmatrix}}{\begin{bmatrix} \nu n+k \\ k \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-2)n+k \\ k \end{bmatrix}} \left\{k - \mathcal{H}_{\mu n+n-k} + \mathcal{H}_{(\lambda-\mu-\nu-2)n+k}\right\} $$
$$ = \frac{\begin{bmatrix} (\lambda-\nu)n \\ n \end{bmatrix}\begin{bmatrix} (\mu+\nu+2)n \\ n \end{bmatrix}}{\begin{bmatrix} \nu n+n \\ n \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-1)n \\ n \end{bmatrix}} \left\{n - \mathcal{H}_{\mu n+n} - \mathcal{H}_{(1+\mu+\nu)n} + \mathcal{H}_{(2+\mu+\nu)n} + \mathcal{H}_{(\lambda-\mu-\nu-1)n}\right\}. $$

Theorem 4 ($\lambda,\mu,\nu \in \mathbb{N}_0$: $\lambda > 1+\mu+\nu$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)(n-k+\mu n)} \frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} \lambda n+k \\ k \end{bmatrix}\begin{bmatrix} \mu n+n \\ k \end{bmatrix}}{\begin{bmatrix} \nu n+k \\ k \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-2)n+k \\ k \end{bmatrix}} \left\{\mathcal{H}_{\nu n+k} - \mathcal{H}_{(\lambda-\mu-\nu-2)n+k}\right\} $$
$$ = \frac{\begin{bmatrix} (\lambda-\nu)n \\ n \end{bmatrix}\begin{bmatrix} (\mu+\nu+2)n \\ n \end{bmatrix}}{\begin{bmatrix} \nu n+n \\ n \end{bmatrix}\begin{bmatrix} (\lambda-\mu-\nu-1)n \\ n \end{bmatrix}} \left\{\begin{matrix} \mathcal{H}_{(1+\mu+\nu)n} - \mathcal{H}_{(\mu+\nu+2)n} + \mathcal{H}_{\nu n+n} \\ -\mathcal{H}_{(\lambda-\nu-1)n} + \mathcal{H}_{(\lambda-\nu)n} - \mathcal{H}_{(\lambda-\mu-\nu-1)n} \end{matrix}\right\}. $$

These identities contain the following interesting special cases.

Example 1 (Theorem 2: $\lambda=2$ and $\mu=\nu=0$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n+k \\ k \end{bmatrix} \left\{\mathcal{H}_{2n+k} - \mathcal{H}_k\right\} = 2\begin{bmatrix} 2n \\ n \end{bmatrix}^2 \left\{\mathcal{H}_{2n} - \mathcal{H}_n\right\}. $$

Example 2 (Theorem 2: $\lambda=3$, $\mu=1$ and $\nu=0$)**.**

$$ \sum_{k=0}^{n} q^{(n-k)(2n-k)} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} 2n \\ k \end{bmatrix}\begin{bmatrix} 3n+k \\ k \end{bmatrix} \left\{\mathcal{H}_{3n+k} - \mathcal{H}_k\right\} = \begin{bmatrix} 3n \\ n \end{bmatrix}^2 \left\{2\mathcal{H}_{3n} - \mathcal{H}_n - \mathcal{H}_{2n}\right\}. $$

Example 3 (Theorem 2: $\lambda = 3$, $\mu = 0$ and $\nu = 1$).

$$\sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 3n+k \\ 2n \end{bmatrix} \big\{\mathcal{H}_{3n+k} - \mathcal{H}_k\big\} = \begin{bmatrix} 3n \\ n \end{bmatrix}^2 \big\{\mathcal{H}_{2n} + \mathcal{H}_{3n} - 2\mathcal{H}_n\big\}.$$

Example 4 (Theorem 3: $\lambda = 2$ and $\mu = \nu = 0$).

$$\sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n+k \\ k \end{bmatrix} \big\{k + \mathcal{H}_k - \mathcal{H}_{n-k}\big\} = \begin{bmatrix} 2n \\ n \end{bmatrix}^2 \big\{n + \mathcal{H}_{2n} - \mathcal{H}_n\big\}.$$

Example 5 (Theorem 3: $\lambda = 3$, $\mu = 1$ and $\nu = 0$).

$$\sum_{k=0}^{n} q^{(n-k)(2n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 3n+k \\ k \end{bmatrix} \big\{k + \mathcal{H}_k - \mathcal{H}_{2n-k}\big\} = \begin{bmatrix} 3n \\ n \end{bmatrix}^2 \big\{n + \mathcal{H}_n + \mathcal{H}_{3n} - 2\mathcal{H}_{2n}\big\}.$$

Example 6 (Theorem 3: $\lambda = 3$, $\mu = 0$ and $\nu = 1$).

$$\sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 3n+k \\ 2n \end{bmatrix} \big\{k + \mathcal{H}_k - \mathcal{H}_{n-k}\big\} = \begin{bmatrix} 3n \\ n \end{bmatrix}^2 \big\{n + \mathcal{H}_{3n} - \mathcal{H}_{2n}\big\}.$$

Example 7 (Theorem 4: $\lambda = 3$, $\mu = 0$ and $\nu = 1$).

$$\sum_{k=0}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 3n+k \\ 2n \end{bmatrix} \big\{\mathcal{H}_{n+k} - \mathcal{H}_k\big\} = \begin{bmatrix} 3n \\ n \end{bmatrix}^2 \big\{3\mathcal{H}_{2n} - 2\mathcal{H}_n - \mathcal{H}_{3n}\big\}.$$

4. The terminating q-Dougall-Dixon theorem

In this section, we shall derive three summation theorems on q-harmonic numbers by examining the following terminating version of the q-Dougall-Dixon formula [6, Appendix II-21]

$${}_6\phi_5 \left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, & b, & d, & q^{-n} \\ a^{\frac12}, & -a^{\frac12}, & aq/b, aq/d, & q^{1+n}a \end{matrix} \,\middle|\, q; \frac{q^{1+n}a}{bd} \right] = \frac{(aq;q)_n (aq/(bd);q)_n}{(aq/b;q)_n (aq/d;q)_n}.$$

4.1. Firstly, making the parameter replacements

$$\left.\begin{matrix} a \to q^{-n-x} \\ b \to q^{1+bn} \\ d \to q^{1+dn} \end{matrix}\right\} \quad \text{where} \quad b, d \in \mathbb{N}_0$$

we can rewrite the terminating q-Dougall-Dixon formula as the q-binomial equation

$$\sum_{k=0}^{n} q^k \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} x+n \\ k \end{bmatrix} \begin{bmatrix} k+bn \\ k \end{bmatrix} \begin{bmatrix} k+dn \\ k \end{bmatrix}}{\begin{bmatrix} k-x \\ k \end{bmatrix} \begin{bmatrix} x+bn+n \\ k \end{bmatrix} \begin{bmatrix} x+dn+n \\ k \end{bmatrix}} (1 - q^{x+n-2k}) = (1 - q^x) \frac{\begin{bmatrix} x+n \\ n \end{bmatrix} \begin{bmatrix} 1+x+bn+dn+n \\ n \end{bmatrix}}{\begin{bmatrix} x+bn+n \\ n \end{bmatrix} \begin{bmatrix} x+dn+n \\ n \end{bmatrix}}$$

which yields, under the derivative operator $\mathcal{D}_0$, the following summation theorem.

Theorem 5 (q-harmonic number identity: $b, d \in \mathbb{N}_0$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} k+bn \\ k \end{bmatrix} \begin{bmatrix} k+dn \\ k \end{bmatrix}}{\begin{bmatrix} n+bn \\ k \end{bmatrix} \begin{bmatrix} n+dn \\ k \end{bmatrix}} \big\{1 + (1-q^{n-2k})(2\mathcal{H}_k - \mathcal{H}_{bn+k} - \mathcal{H}_{dn+k})\big\} = \frac{\begin{bmatrix} 1+bn+dn+n \\ n \end{bmatrix}}{\begin{bmatrix} n+bn \\ n \end{bmatrix} \begin{bmatrix} n+dn \\ n \end{bmatrix}}.$$

As applications, we display the following q-harmonic number identities.

Example 8 (Theorem 5: $b=0$ and $d\to\infty$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix} \big\{1+(1-q^{n-2k})\mathcal{H}_k\big\} = 1.$$

Example 9 (Theorem 5: $b,\ d\to\infty$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^2 \big\{1+2(1-q^{n-2k})\mathcal{H}_k\big\} = (q;q)_n.$$

Example 10 (Theorem 5: $b=1$ and $d\to\infty$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1+ {\scriptstyle (1-q^{n-2k})(2\mathcal{H}_k-\mathcal{H}_{n+k})}\big\} = 1.$$

Example 11 (Theorem 5: $b=0$ and $d=1$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1+ {\scriptstyle (1-q^{n-2k})(\mathcal{H}_k-\mathcal{H}_{n+k})}\big\} = \begin{bmatrix} 1+2n \\ n \end{bmatrix}.$$

Example 12 (Theorem 5: $b=d=1$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} 2n-k \\ n \end{bmatrix}^2 \big\{1+ {\scriptstyle 2(1-q^{n-2k})(\mathcal{H}_k-\mathcal{H}_{n+k})}\big\} = \begin{bmatrix} 1+3n \\ n \end{bmatrix}.$$

4.2. Secondly, under the parameter replacements

$$\left.\begin{aligned} a &\to q^{-n-x} \\ b &\to q^{1+bn} \\ d &\to q^{-n-dn} \end{aligned}\right\} \quad \text{where} \quad b,d\in\mathbb{N}_0$$

the terminating q-Dixon formula becomes the following q-binomial equation

$$\sum_{k=0}^{n}(1-q^{x+n-2k})\frac{\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} x+n \\ k \end{bmatrix}\begin{bmatrix} k+bn \\ k \end{bmatrix}\begin{bmatrix} n+dn \\ k \end{bmatrix}}{\begin{bmatrix} k-x \\ k \end{bmatrix}\begin{bmatrix} x+bn+n \\ k \end{bmatrix}\begin{bmatrix} k+dn-x \\ k \end{bmatrix}} q^{k+(n-k)(x-k)} = (-1)^n(1-q^x)\frac{\begin{bmatrix} x+n \\ n \end{bmatrix}\begin{bmatrix} x+bn-dn \\ n \end{bmatrix}}{\begin{bmatrix} n+bn+x \\ n \end{bmatrix}\begin{bmatrix} n+dn-x \\ n \end{bmatrix}} q^{dn^2+\binom{1+n}{2}}.$$

Applying the derivative operator $\mathcal{D}_0$ to it gives rise to the summation theorem.

Theorem 6 (q-harmonic number identity: $b,d\in\mathbb{N}_0$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} k+bn \\ k \end{bmatrix}\begin{bmatrix} n+dn \\ k \end{bmatrix}}{\begin{bmatrix} n+bn \\ k \end{bmatrix}\begin{bmatrix} k+dn \\ k \end{bmatrix}} \big\{1+ {\scriptstyle (1-q^{n-2k})(k+2\mathcal{H}_k-\mathcal{H}_{bn+k}+\mathcal{H}_{dn+k})}\big\} = (-1)^n \frac{\begin{bmatrix} bn-dn \\ n \end{bmatrix}}{\begin{bmatrix} n+bn \\ n \end{bmatrix}\begin{bmatrix} n+dn \\ n \end{bmatrix}} q^{dn^2+\binom{1+n}{2}}.$$

As applications, we display the following q-harmonic number identities.

Example 13 (Theorem 6: $b=0$ and $d\to\infty$).

$$\sum_{k=0}^{n} q^{(1+k)(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \big\{1+ {\scriptstyle (1-q^{n-2k})(k+\mathcal{H}_k)}\big\} = 1.$$

Example 14 (Theorem 6: $b = d = 0$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \big\{1 + (1-q^{n-2k})(k+2\mathcal{H}_k)\big\} = \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

Example 15 (Theorem 6: $b \to \infty$ and $d = 0$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^3 \big\{1 + (1-q^{n-2k})(k+3\mathcal{H}_k)\big\} = (-1)^n q^{\binom{1+n}{2}}.$$

Example 16 (Theorem 6: $b = 0$ and $d = 1$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1 + (1-q^{n-2k})(k+\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = \begin{bmatrix} 2n-1 \\ n \end{bmatrix} q^n.$$

Example 17 (Theorem 6: $b = 1$ and $d \to \infty$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1 + (1-q^{n-2k})(k+2\mathcal{H}_k-\mathcal{H}_{n+k})\big\} = q^{n^2+n}.$$

Example 18 (Theorem 6: $b \to \infty$ and $d = 1$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1 + (1-q^{n-2k})(k+2\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n q^{\frac{n}{2}(1+3n)}.$$

Example 19 (Theorem 6: $b = 1$ and $d = 0$).

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1 + (1-q^{n-2k})(k+3\mathcal{H}_k-\mathcal{H}_{n+k})\big\} = (-1)^n q^{\binom{1+n}{2}}.$$

4.3. Finally, carrying out the parameter replacements

$$\left.\begin{aligned} a &\to q^{-n-x} \\ b &\to q^{-n-bn} \\ d &\to q^{-n-dn} \end{aligned}\right\} \quad \text{where} \quad b, d \in \mathbb{N}_0$$

in the terminating q-Dougall-Dixon formula, we can express it as the q-binomial equation

$$\sum_{k=0}^{n} (1-q^{x+n-2k}) \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} x+n \\ k \end{bmatrix} \begin{bmatrix} n+bn \\ k \end{bmatrix} \begin{bmatrix} n+dn \\ k \end{bmatrix}}{\begin{bmatrix} k-x \\ k \end{bmatrix} \begin{bmatrix} k+bn-x \\ k \end{bmatrix} \begin{bmatrix} k+dn-x \\ k \end{bmatrix}} q^{\binom{n-2k}{2}+x(n-2k)} = (-1)^n (1-q^x) \frac{\begin{bmatrix} x+n \\ n \end{bmatrix} \begin{bmatrix} 2n+bn+dn-x \\ n \end{bmatrix}}{\begin{bmatrix} n+bn-x \\ n \end{bmatrix} \begin{bmatrix} n+dn-x \\ n \end{bmatrix}}$$

which results, under the derivative operator $\mathcal{D}_0$, in the following summation theorem.

Theorem 7 (q-harmonic number identity: $b, d \in \mathbb{N}_0$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} n+bn \\ k \end{bmatrix} \begin{bmatrix} n+dn \\ k \end{bmatrix}}{\begin{bmatrix} k+bn \\ k \end{bmatrix} \begin{bmatrix} k+dn \\ k \end{bmatrix}} \big\{1 + (1-q^{n-2k})(2k+2\mathcal{H}_k+\mathcal{H}_{bn+k}+\mathcal{H}_{dn+k})\big\} = (-1)^n \frac{\begin{bmatrix} 2n+bn+dn \\ n \end{bmatrix}}{\begin{bmatrix} n+bn \\ n \end{bmatrix} \begin{bmatrix} n+dn \\ n \end{bmatrix}}.$$

As applications, we display the following q-harmonic number identities.

Example 20 (Theorem 7: $b,\ d \to \infty$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^2 \big\{1 + 2(1-q^{n-2k})(k+\mathcal{H}_k)\big\} = (-1)^n (q;q)_n.$$

Example 21 (Theorem 7: $b = 0$ and $d \to \infty$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^3 \big\{1 + (1-q^{n-2k})(2k+3\mathcal{H}_k)\big\} = (-1)^n.$$

Example 22 (Theorem 7: $b = d = 0$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^4 \big\{1 + 2(1-q^{n-2k})(k+2\mathcal{H}_k)\big\} = (-1)^n \begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Example 23 (Theorem 7: $b = 1$ and $d \to \infty$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1 + (1-q^{n-2k})(2k+2\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n.$$

Example 24 (Theorem 7: $b = 0$ and $d = 1$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1 + (1-q^{n-2k})(2k+3\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n \begin{bmatrix} 3n \\ n \end{bmatrix}.$$

Example 25 (Theorem 7: $b = d = 1$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n \\ n+k \end{bmatrix}^2 \big\{1 + 2(1-q^{n-2k})(k+\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n \begin{bmatrix} 4n \\ n \end{bmatrix}.$$

5. Watson's q-Whipple transformation

Watson's q-Whipple transformation [6, Appendix III-18] is fundamental in the q-series theory

$$\begin{aligned} &{}_8\phi_7 \left[\begin{matrix} a,\ qa^{\frac{1}{2}},\ -qa^{\frac{1}{2}}, & b, & c, & d, & e, & q^{-n} \\ a^{\frac{1}{2}},\ -a^{\frac{1}{2}}, & aq/b, & aq/c, & aq/d, & aq/e, & q^{1+n}a \end{matrix} \,\middle|\, q;\ \frac{q^{2+n}a^2}{bcde} \right] \\ &= \frac{(aq;q)_n (aq/(bd);q)_n}{(aq/b;q)_n (aq/d;q)_n} \times {}_4\phi_3 \left[\begin{matrix} q^{-n}, b, d, aq/(ce) \\ aq/c, aq/e, q^{-n}bd/a \end{matrix} \,\middle|\, q;\ q \right]. \end{aligned}$$

It will systematically be explored in five different manners to prove five transformation theorems concerning q-binomial coefficients and q-harmonic numbers.

5.1. Making the parameter replacements

$$\left. \begin{aligned} a &\to q^{-x-n} \\ b &\to q^{1+bn} \\ c &\to q^{1+cn} \\ d &\to q^{1+dn} \\ e &\to q^{1+en} \end{aligned} \right\} \quad \text{where} \quad b, c, d, e \in \mathbb{N}_0$$

we can reformulate Watson's q-Whipple transformation as

$$\sum_{k=0}^{n}(1-q^{x+n-2k})\frac{\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}x+n\\k\end{bmatrix}\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}{\begin{bmatrix}k-x\\k\end{bmatrix}\begin{bmatrix}n+bn+x\\k\end{bmatrix}\begin{bmatrix}n+cn+x\\k\end{bmatrix}\begin{bmatrix}n+dn+x\\k\end{bmatrix}\begin{bmatrix}n+en+x\\k\end{bmatrix}}q^{k(1+x+n-k)}$$
$$=(1-q^{x})\frac{\begin{bmatrix}x+n\\n\end{bmatrix}\begin{bmatrix}1+x+bn+dn+n\\n\end{bmatrix}}{\begin{bmatrix}x+bn+n\\n\end{bmatrix}\begin{bmatrix}x+dn+n\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{x\ell}\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}\ell+dn\\\ell\end{bmatrix}\begin{bmatrix}1+x+cn+en+n\\\ell\end{bmatrix}}{\begin{bmatrix}x+cn+n\\\ell\end{bmatrix}\begin{bmatrix}x+en+n\\\ell\end{bmatrix}\begin{bmatrix}1+x+bn+dn+\ell\\\ell\end{bmatrix}}.$$

Under the derivative operator $\mathcal{D}_0$, this gives the following transformation theorem.

Theorem 8 (Transformation formula: $b,c,d,e\in\mathbb{N}_0$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}{\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}$$
$$\times\Big\{1-(1-q^{n-2k})(k+\mathcal{H}_{bn+k}+\mathcal{H}_{cn+k}+\mathcal{H}_{dn+k}+\mathcal{H}_{en+k}-2\mathcal{H}_k)\Big\}$$
$$=\frac{\begin{bmatrix}1+bn+dn+n\\n\end{bmatrix}}{\begin{bmatrix}n+bn\\n\end{bmatrix}\begin{bmatrix}n+dn\\n\end{bmatrix}}\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}\ell+dn\\\ell\end{bmatrix}\begin{bmatrix}1+cn+en+n\\\ell\end{bmatrix}}{\begin{bmatrix}n+cn\\\ell\end{bmatrix}\begin{bmatrix}n+en\\\ell\end{bmatrix}\begin{bmatrix}1+bn+dn+\ell\\\ell\end{bmatrix}}.$$

We collect the following q-harmonic number identities as consequences.

Example 26 (Theorem 8: $b=c=d=0$ and $e\to\infty$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\begin{bmatrix}n\\k\end{bmatrix}^{-1}\big\{1-(1-q^{n-2k})(k+\mathcal{H}_k)\big\}\;=\;(1-q^{1+n})\big\{1+n+\mathcal{H}_{n+1}\big\}.$$

Example 27 (Theorem 8: $b=c=d=e=0$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\begin{bmatrix}n\\k\end{bmatrix}^{-2}\big\{1-(1-q^{n-2k})(k+2\mathcal{H}_k)\big\}\;=\;\tfrac{(1-q^{1+n})^2}{1-q^{2+n}}\big\{1+n+2\mathcal{H}_{n+1}\big\}.$$

Example 28 (Theorem 8: $b=c=d=0$ and $e=1$)**.**

$$\sum_{k=0}^{n}\frac{q^{k(1+n-k)}}{\begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}2n\\n+k\end{bmatrix}}\big\{1-(1-q^{n-2k})(k+\mathcal{H}_k+\mathcal{H}_{n+k})\big\}=\frac{1-q^{1+2n}}{1+q^{1+n}}\begin{bmatrix}2n\\n\end{bmatrix}^{-1}\Big\{1+n+\tfrac{q^{1+n}}{1-q^{1+n}}+\mathcal{H}_{2n+1}\Big\}.$$

Example 29 (Theorem 8: $b=d=0$, $c=1$ and $e\to\infty$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\frac{\begin{bmatrix}n+k\\k\end{bmatrix}}{\begin{bmatrix}2n\\k\end{bmatrix}}\big\{1-(1-q^{n-2k})(k+\mathcal{H}_{n+k})\big\}=(1-q^{1+2n})\big\{1+n+\mathcal{H}_{2n+1}-\mathcal{H}_n\big\}.$$

Example 30 (Theorem 8: $b=d=0$ and $c=e=1$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\frac{\begin{bmatrix}n+k\\k\end{bmatrix}^2}{\begin{bmatrix}2n\\k\end{bmatrix}^2}\big\{1-(1-q^{n-2k})(k+2\mathcal{H}_{n+k})\big\}=\tfrac{(1-q^{1+2n})^2}{1-q^{2+3n}}\big\{1+n+2\mathcal{H}_{2n+1}-2\mathcal{H}_n\big\}.$$

Example 31 (Theorem 8: b, c, $d\to\infty$ and $e=0$)**.**

$$\sum_{k=0}^{n}q^{k(1+n-k)}\begin{bmatrix}n\\k\end{bmatrix}\big\{1-(1-q^{n-2k})(k-\mathcal{H}_k)\big\}=\sum_{\ell=0}^{n}\frac{(q;q)_n}{(q;q)_\ell}.$$

Example 32 (Theorem 8: $b,\ c,\ d,\ e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}^2\big\{1-(1-q^{n-2k})(k-2\mathcal{H}_k)\big\}=(q;q)_n\sum_{\ell=0}^{n}\begin{bmatrix} n\\ \ell\end{bmatrix}.$$

Example 33 (Theorem 8: $b=d=1$, $c=0$ and $e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}\frac{\begin{bmatrix} n+k\\ k\end{bmatrix}^2}{\begin{bmatrix} 2n\\ k\end{bmatrix}^2}\big\{1-(1-q^{n-2k})(k+2\mathcal{H}_{n+k}-\mathcal{H}_k)\big\}=\frac{\begin{bmatrix} 1+3n\\ n\end{bmatrix}}{\begin{bmatrix} 2n\\ n\end{bmatrix}^2}\sum_{\ell=0}^{n}\frac{\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}^2}{\begin{bmatrix} 1+2n+\ell\\ \ell\end{bmatrix}}.$$

Example 34 (Theorem 8: $b=c=d=1$ and $e=0$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}\frac{\begin{bmatrix} n+k\\ k\end{bmatrix}^3}{\begin{bmatrix} 2n\\ k\end{bmatrix}^3}\big\{1-(1-q^{n-2k})(k+3\mathcal{H}_{n+k}-\mathcal{H}_k)\big\}=\frac{\begin{bmatrix} 1+3n\\ n\end{bmatrix}}{\begin{bmatrix} 2n\\ n\end{bmatrix}^2}\sum_{\ell=0}^{n}\frac{(1-q^{1+2n})\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}^2}{(1-q^{1+2n-\ell})\begin{bmatrix} 1+2n+\ell\\ \ell\end{bmatrix}}.$$

Example 35 (Theorem 8: $b=c=d=1$ and $e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}^2\frac{\begin{bmatrix} n+k\\ k\end{bmatrix}^3}{\begin{bmatrix} 2n\\ k\end{bmatrix}^3}\big\{1-(1-q^{n-2k})(k+3\mathcal{H}_{n+k}-2\mathcal{H}_k)\big\}=\frac{\begin{bmatrix} 1+3n\\ n\end{bmatrix}}{\begin{bmatrix} 2n\\ n\end{bmatrix}^2}\sum_{\ell=0}^{n}\frac{\begin{bmatrix} n\\ \ell\end{bmatrix}\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}^2}{\begin{bmatrix} 2n\\ \ell\end{bmatrix}\begin{bmatrix} 1+2n+\ell\\ \ell\end{bmatrix}}.$$

Example 36 (Theorem 8: $b=c=d=e=1$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}^2\frac{\begin{bmatrix} n+k\\ k\end{bmatrix}^4}{\begin{bmatrix} 2n\\ k\end{bmatrix}^4}\big\{1-(1-q^{n-2k})(k+4\mathcal{H}_{n+k}-2\mathcal{H}_k)\big\}=\frac{\begin{bmatrix} 1+3n\\ n\end{bmatrix}}{\begin{bmatrix} 2n\\ n\end{bmatrix}^2}\sum_{\ell=0}^{n}\begin{bmatrix} n\\ \ell\end{bmatrix}\frac{\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}^2\begin{bmatrix} 1+3n\\ \ell\end{bmatrix}}{\begin{bmatrix} 2n\\ \ell\end{bmatrix}^2\begin{bmatrix} 1+2n+\ell\\ \ell\end{bmatrix}}.$$

Example 37 (Theorem 8: $b=0$, $d=1$ and $c,\ e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n+k\\ k\end{bmatrix}\begin{bmatrix} 2n-k\\ n\end{bmatrix}\big\{1-(1-q^{n-2k})(k+\mathcal{H}_{n+k}-\mathcal{H}_k)\big\}=\begin{bmatrix} 1+2n\\ n\end{bmatrix}\sum_{\ell=0}^{n}\begin{bmatrix} n\\ \ell\end{bmatrix}\frac{\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}}{\begin{bmatrix} 1+n+\ell\\ \ell\end{bmatrix}}(q;q)_\ell.$$

Example 38 (Theorem 8: $b,\ c,\ d\to\infty$ and $e=1$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n+k\\ k\end{bmatrix}\begin{bmatrix} 2n-k\\ n\end{bmatrix}\big\{1-(1-q^{n-2k})(k+\mathcal{H}_{n+k}-2\mathcal{H}_k)\big\}=\sum_{\ell=0}^{n}\begin{bmatrix} 2n-\ell\\ n\end{bmatrix}\frac{(q;q)_n}{(q;q)_\ell}.$$

Example 39 (Theorem 8: $b=d=1$ and $c,\ e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+n-k)}\begin{bmatrix} n+k\\ k\end{bmatrix}^2\begin{bmatrix} 2n-k\\ n\end{bmatrix}^2\big\{1-(1-q^{n-2k})(k+2\mathcal{H}_{n+k}-2\mathcal{H}_k)\big\}=\begin{bmatrix} 1+3n\\ n\end{bmatrix}\sum_{\ell=0}^{n}\begin{bmatrix} n\\ \ell\end{bmatrix}\frac{\begin{bmatrix} n+\ell\\ \ell\end{bmatrix}^2}{\begin{bmatrix} 1+2n+\ell\\ \ell\end{bmatrix}}(q;q)_\ell.$$

5.2. With the parameter replacements

$$\left.\begin{aligned} a&\to q^{-x-n}\\ b&\to q^{1+bn}\\ c&\to q^{1+cn}\\ d&\to q^{1+dn}\\ e&\to q^{-n-en}\end{aligned}\right\}\quad\text{where}\quad b,c,d,e\in\mathbb{N}_0$$

Watson's q-Whipple transformation can be rewritten as

$$\sum_{k=0}^{n}(1-q^{x+n-2k})\frac{\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}x+n\\k\end{bmatrix}\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}k-x\\k\end{bmatrix}\begin{bmatrix}n+bn+x\\k\end{bmatrix}\begin{bmatrix}n+cn+x\\k\end{bmatrix}\begin{bmatrix}n+dn+x\\k\end{bmatrix}\begin{bmatrix}k+en-x\\k\end{bmatrix}}q^k$$
$$=(1-q^x)\frac{\begin{bmatrix}x+n\\n\end{bmatrix}\begin{bmatrix}1+x+bn+dn+n\\n\end{bmatrix}}{\begin{bmatrix}x+bn+n\\n\end{bmatrix}\begin{bmatrix}x+dn+n\\n\end{bmatrix}}\sum_{\ell=0}^{n}\frac{(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}\ell+dn\\\ell\end{bmatrix}\begin{bmatrix}x+cn-en\\\ell\end{bmatrix}}{\begin{bmatrix}n+cn+x\\\ell\end{bmatrix}\begin{bmatrix}\ell+en-x\\\ell\end{bmatrix}\begin{bmatrix}1+x+bn+dn+\ell\\\ell\end{bmatrix}}q^{ne\ell+\binom{1+\ell}{2}}.$$

Applying the derivative operator $\mathcal{D}_0$ to it, we get the following transformation theorem.

Theorem 9 (Transformation formula: $b,c,d,e\in\mathbb{N}_0$).

$$\sum_{k=0}^{n}q^k\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}$$
$$\times\left\{1+(1-q^{n-2k})(2\mathcal{H}_k-\mathcal{H}_{bn+k}-\mathcal{H}_{cn+k}-\mathcal{H}_{dn+k}+\mathcal{H}_{en+k})\right\}$$
$$=\frac{\begin{bmatrix}1+bn+dn+n\\n\end{bmatrix}}{\begin{bmatrix}n+bn\\n\end{bmatrix}\begin{bmatrix}n+dn\\n\end{bmatrix}}\sum_{\ell=0}^{n}(-1)^\ell\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}\ell+dn\\\ell\end{bmatrix}\begin{bmatrix}cn-en\\\ell\end{bmatrix}}{\begin{bmatrix}n+cn\\\ell\end{bmatrix}\begin{bmatrix}\ell+en\\\ell\end{bmatrix}\begin{bmatrix}1+bn+dn+\ell\\\ell\end{bmatrix}}q^{ne\ell+\binom{1+\ell}{2}}.$$

We collect the following q-harmonic number identities as consequences.

Example 40 (Theorem 9: $b=c=d=0$ and $e\to\infty$).

$$\sum_{k=0}^{n}q^k\begin{bmatrix}n\\k\end{bmatrix}^{-1}\left\{1-(1-q^{n-2k})\mathcal{H}_k\right\}=(q^{-1}-q^n)\mathcal{H}_{n+1}.$$

Example 41 (Theorem 9: $b=1$, $c=d=0$ and $e\to\infty$).

$$\sum_{k=0}^{n}q^k\frac{\begin{bmatrix}n+k\\k\end{bmatrix}}{\begin{bmatrix}2n\\k\end{bmatrix}}\left\{1-(1-q^{n-2k})\mathcal{H}_{n+k}\right\}=(q^{-1-n}-q^n)\left\{\mathcal{H}_{2n+1}-\mathcal{H}_n\right\}.$$

Example 42 (Theorem 9: $b=d=0$, $c\to\infty$ and $e=1$ with $n>0$).

$$\sum_{k=0}^{n}q^k\frac{\begin{bmatrix}2n\\k\end{bmatrix}}{\begin{bmatrix}n+k\\k\end{bmatrix}}\left\{1+(1-q^{n-2k})\mathcal{H}_{n+k}\right\}=(1-q^{-n})\left\{\mathcal{H}_{n-1}-\mathcal{H}_{2n}\right\}.$$

Example 43 (Theorem 9: $b=c=d=0$ and $e=1$ with $n>1$).

$$\sum_{k=0}^{n}q^k\frac{\begin{bmatrix}2n\\k\end{bmatrix}}{\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n+k\\k\end{bmatrix}}\left\{1-(1-q^{n-2k})(\mathcal{H}_k-\mathcal{H}_{n+k})\right\}=\frac{(1-q^n)(1-q^{n+1})}{q-q^n}\left\{\mathcal{H}_{n+1}+\mathcal{H}_{n-1}-\mathcal{H}_{2n}\right\}.$$

Example 44 (Theorem 9: b, c, $d\to\infty$, $e=0$).

$$\sum_{k=0}^{n}q^k\begin{bmatrix}n\\k\end{bmatrix}^3\left\{1+3(1-q^{n-2k})\mathcal{H}_k\right\}=\sum_{\ell=0}^{n}(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\frac{(q;q)_n}{(q;q)_\ell}q^{\binom{1+\ell}{2}}.$$

Example 45 (Theorem 9: $b=d=1$, $c=0$ and $e\to\infty$).

$$\sum_{k=0}^{n}q^k\begin{bmatrix}n\\k\end{bmatrix}\frac{\begin{bmatrix}n+k\\k\end{bmatrix}^2}{\begin{bmatrix}2n\\k\end{bmatrix}^2}\left\{1+(1-q^{n-2k})(\mathcal{H}_k-2\mathcal{H}_{n+k})\right\}=\frac{\begin{bmatrix}1+3n\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}^2}\sum_{\ell=0}^{n}q^\ell\frac{\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^2}{\begin{bmatrix}1+2n+\ell\\\ell\end{bmatrix}}.$$

Example 46 (Theorem 9: $b = d = 1$, $c \to \infty$ and $e = 0$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^3 \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}^2}{\begin{bmatrix} 2n \\ k \end{bmatrix}^2} \{1 + (1 - q^{n-2k})(3\mathcal{H}_k - 2\mathcal{H}_{n+k})\} = \frac{\begin{bmatrix} 1+3n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} (-1)^\ell \frac{\begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}{\begin{bmatrix} 1+2n+\ell \\ \ell \end{bmatrix}} q^{\binom{1+\ell}{2}}.$$

Example 47 (Theorem 9: $b = c = d = 1$ and $e \to \infty$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}^3}{\begin{bmatrix} 2n \\ k \end{bmatrix}^3} \{1 + (1 - q^{n-2k})(2\mathcal{H}_k - 3\mathcal{H}_{n+k})\} = \frac{\begin{bmatrix} 1+3n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} q^{(1+n)\ell} \frac{\begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}{\begin{bmatrix} 2n \\ \ell \end{bmatrix} \begin{bmatrix} 1+2n+\ell \\ \ell \end{bmatrix}}.$$

Example 48 (Theorem 9: $b = c = d = 1$ and $e = 0$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^3 \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}^3}{\begin{bmatrix} 2n \\ k \end{bmatrix}^3} \{1 + 3(1 - q^{n-2k})(\mathcal{H}_k - \mathcal{H}_{n+k})\} = \frac{\begin{bmatrix} 1+3n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} (-1)^\ell \frac{\begin{bmatrix} n \\ \ell \end{bmatrix}^2 \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}{\begin{bmatrix} 2n \\ \ell \end{bmatrix} \begin{bmatrix} 1+2n+\ell \\ \ell \end{bmatrix}} q^{\binom{1+\ell}{2}}.$$

Example 49 (Theorem 9: $b = 0$, c, $d \to \infty$ and $e = 1$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \{1 + (1 - q^{n-2k})(\mathcal{H}_k + \mathcal{H}_{n+k})\} = \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} q^{n\ell + \binom{1+\ell}{2}}.$$

Example 50 (Theorem 9: b, c, $d \to \infty$ and $e = 1$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \{1 + (1 - q^{n-2k})(2\mathcal{H}_k + \mathcal{H}_{n+k})\} = \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} \frac{(q;q)_n}{(q;q)_\ell} q^{n\ell + \binom{1+\ell}{2}}.$$

Example 51 (Theorem 9: $b = 1$, c, $d \to \infty$ and $e = 0$).

$$\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \{1 + (1 - q^{n-2k})(3\mathcal{H}_k - \mathcal{H}_{n+k})\} = \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} q^{\binom{1+\ell}{2}}.$$

5.3. Performing the parameter replacements

$$\left.\begin{array}{l} a \to q^{-x-n} \\ b \to q^{1+bn} \\ c \to q^{-n-cn} \\ d \to q^{1+dn} \\ e \to q^{-n-en} \end{array}\right\} \quad \text{where} \quad b, c, d, e \in \mathbb{N}_0$$

we can express Watson's q-Whipple transformation as

$$\sum_{k=0}^{n} (1 - q^{x+n-2k}) \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} x+n \\ k \end{bmatrix} \begin{bmatrix} k+bn \\ k \end{bmatrix} \begin{bmatrix} n+cn \\ k \end{bmatrix} \begin{bmatrix} k+dn \\ k \end{bmatrix} \begin{bmatrix} n+en \\ k \end{bmatrix}}{\begin{bmatrix} k-x \\ k \end{bmatrix} \begin{bmatrix} n+bn+x \\ k \end{bmatrix} \begin{bmatrix} k+cn-x \\ k \end{bmatrix} \begin{bmatrix} n+dn+x \\ k \end{bmatrix} \begin{bmatrix} k+en-x \\ k \end{bmatrix}} q^{k(1+k-n-x)}$$

$$= (1 - q^x) \frac{\begin{bmatrix} x+n \\ n \end{bmatrix} \begin{bmatrix} 1+x+bn+dn+n \\ n \end{bmatrix}}{\begin{bmatrix} x+bn+n \\ n \end{bmatrix} \begin{bmatrix} x+dn+n \\ n \end{bmatrix}} \sum_{\ell=0}^{n} (-1)^\ell \frac{\begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} \ell+bn \\ \ell \end{bmatrix} \begin{bmatrix} \ell+dn \\ \ell \end{bmatrix} \begin{bmatrix} \ell+n+cn+en-x \\ \ell \end{bmatrix}}{\begin{bmatrix} \ell+cn-x \\ \ell \end{bmatrix} \begin{bmatrix} \ell+en-x \\ \ell \end{bmatrix} \begin{bmatrix} 1+x+bn+dn+\ell \\ \ell \end{bmatrix}} q^{\binom{1+\ell}{2} - n\ell}$$

which results, under the derivative operator $\mathcal{D}_0$, in the following transformation theorem.

Theorem 10 (Transformation formula: $b,c,d,e\in\mathbb{N}_0$).

$$\begin{aligned}&\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}\\&\times\Big\{1+(1-q^{n-2k})(k+2\mathcal{H}_k-\mathcal{H}_{bn+k}+\mathcal{H}_{cn+k}-\mathcal{H}_{dn+k}+\mathcal{H}_{en+k})\Big\}\\&=\frac{\begin{bmatrix}1+bn+dn+n\\n\end{bmatrix}}{\begin{bmatrix}n+bn\\n\end{bmatrix}\begin{bmatrix}n+dn\\n\end{bmatrix}}\sum_{\ell=0}^{n}(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}\ell+dn\\\ell\end{bmatrix}\begin{bmatrix}n+cn+en+\ell\\\ell\end{bmatrix}}{\begin{bmatrix}\ell+cn\\\ell\end{bmatrix}\begin{bmatrix}\ell+en\\\ell\end{bmatrix}\begin{bmatrix}1+bn+dn+\ell\\\ell\end{bmatrix}}q^{\binom{1+\ell}{2}-n\ell}.\end{aligned}$$

We collect the following q-harmonic number identities as consequences.

Example 52 (Theorem 10: $b=d=0$, $c=1$ and $e\to\infty$ with $n>0$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\frac{\begin{bmatrix}2n\\k\end{bmatrix}}{\begin{bmatrix}n+k\\k\end{bmatrix}}\Big\{1+(1-q^{n-2k})(k+\mathcal{H}_{n+k})\Big\}=(q^n-q^{2n})\Big\{1+n+\mathcal{H}_{2n}-\mathcal{H}_{n-1}\Big\}.$$

Example 53 (Theorem 10: $b=d=0$ and $c=e=1$ with $n>0$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\frac{\begin{bmatrix}2n\\k\end{bmatrix}^2}{\begin{bmatrix}n+k\\k\end{bmatrix}^2}\Big\{1+(1-q^{n-2k})(k+2\mathcal{H}_{n+k})\Big\}=q^n\frac{(1-q^n)^2}{1-q^{3n}}\Big\{1+n+2\mathcal{H}_{2n}-2\mathcal{H}_{n-1}\Big\}.$$

Example 54 (Theorem 10: $b=0$ and $c,\ d,\ e\to\infty$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}\Big\{1+(1-q^{n-2k})(k+\mathcal{H}_k)\Big\}=\sum_{\ell=0}^{n}(-1)^\ell(q;q)_\ell\begin{bmatrix}n\\\ell\end{bmatrix}q^{\binom{1+\ell}{2}-n\ell}.$$

Example 55 (Theorem 10: $b,\ d\to\infty$ and $c=e=0$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}^4\Big\{1+(1-q^{n-2k})(k+4\mathcal{H}_k)\Big\}=\sum_{\ell=0}^{n}(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}n+\ell\\\ell\end{bmatrix}\frac{(q;q)_n}{(q;q)_\ell}q^{\binom{1+\ell}{2}-n\ell}.$$

Example 56 (Theorem 10: $b=0$, $c=e=1$ and $d\to\infty$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}\frac{\begin{bmatrix}2n\\k\end{bmatrix}^2}{\begin{bmatrix}n+k\\k\end{bmatrix}^2}\Big\{1+(1-q^{n-2k})(k+\mathcal{H}_k+2\mathcal{H}_{n+k})\Big\}=\sum_{\ell=0}^{n}(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}3n+\ell\\\ell\end{bmatrix}}{\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^2}q^{\binom{1+\ell}{2}-n\ell}.$$

Example 57 (Theorem 10: $b=d=1$, $c=0$ and $e\to\infty$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}^3\frac{\begin{bmatrix}n+k\\k\end{bmatrix}^2}{\begin{bmatrix}2n\\k\end{bmatrix}^2}\Big\{1+(1-q^{n-2k})(k+3\mathcal{H}_k-2\mathcal{H}_{n+k})\Big\}=\frac{\begin{bmatrix}1+3n\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}^2}\sum_{\ell=0}^{n}(-1)^\ell\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^2}{\begin{bmatrix}1+2n+\ell\\\ell\end{bmatrix}}q^{\binom{1+\ell}{2}-n\ell}.$$

Example 58 (Theorem 10: $b=d=1$ and $c=e=0$).

$$\sum_{k=0}^{n}q^{k(1+k-n)}\begin{bmatrix}n\\k\end{bmatrix}^4\frac{\begin{bmatrix}n+k\\k\end{bmatrix}^2}{\begin{bmatrix}2n\\k\end{bmatrix}^2}\Big\{1+(1-q^{n-2k})(k+4\mathcal{H}_k-2\mathcal{H}_{n+k})\Big\}=\frac{\begin{bmatrix}1+3n\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}^2}\sum_{\ell=0}^{n}(-1)^\ell\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^3}{\begin{bmatrix}1+2n+\ell\\\ell\end{bmatrix}}q^{\binom{1+\ell}{2}-n\ell}.$$

Example 59 (Theorem 10: $b=0$, $d=1$ and $c,\ e\to\infty$)**.**

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1+(1-q^{n-2k})(k+\mathcal{H}_k-\mathcal{H}_{n+k})\big\}$$
$$= \begin{bmatrix} 1+2n \\ n \end{bmatrix} \sum_{\ell=0}^{n} (-1)^{\ell}(q;q)_{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} \frac{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} 1+n+\ell \\ \ell \end{bmatrix}} q^{\binom{1+\ell}{2}-n\ell}.$$

Example 60 (Theorem 10: $b,\ d\to\infty$ and $c=e=1$)**.**

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} 2n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n \\ n+k \end{bmatrix}^2 \big\{1+(1-q^{n-2k})(k+2\mathcal{H}_k+2\mathcal{H}_{n+k})\big\}$$
$$= \sum_{\ell=0}^{n} (-1)^{\ell} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix}^2 \frac{\begin{bmatrix} 3n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} n \\ \ell \end{bmatrix}} \frac{(q;q)_n}{(q;q)_{\ell}} q^{\binom{1+\ell}{2}-n\ell}.$$

Example 61 (Theorem 10: $b=d=1$ and $c,\ e\to\infty$)**.**

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} 2n-k \\ n \end{bmatrix}^2 \big\{1+(1-q^{n-2k})(k+2\mathcal{H}_k-2\mathcal{H}_{n+k})\big\}$$
$$= \begin{bmatrix} 1+3n \\ n \end{bmatrix} \sum_{\ell=0}^{n} (-1)^{\ell}(q;q)_{\ell} \frac{\begin{bmatrix} n \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}{\begin{bmatrix} 1+2n+\ell \\ \ell \end{bmatrix}} q^{\binom{1+\ell}{2}-n\ell}.$$

Example 62 (Theorem 10: $b,\ d\to\infty$, $c=0$ and $e=1$)**.**

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1+(1-q^{n-2k})(k+3\mathcal{H}_k+\mathcal{H}_{n+k})\big\}$$
$$= \sum_{\ell=0}^{n} (-1)^{\ell} \begin{bmatrix} 2n+\ell \\ \ell \end{bmatrix} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} \frac{(q;q)_n}{(q;q)_{\ell}} q^{\binom{1+\ell}{2}-n\ell}.$$

Example 63 (Theorem 10: $b=1$, $c=e=0$ and $d\to\infty$)**.**

$$\sum_{k=0}^{n} q^{k(1+k-n)} \begin{bmatrix} n \\ k \end{bmatrix}^3 \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \big\{1+(1-q^{n-2k})(k+4\mathcal{H}_k-\mathcal{H}_{n+k})\big\} = {}_{\ell=0}^{n}(-1)^{\ell} \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2 q^{\binom{1+\ell}{2}-n\ell}.$$

5.4. Under the parameter replacements

$$\left.\begin{array}{l} a\to q^{-x-n} \\ b\to q^{1+bn} \\ c\to q^{-n-cn} \\ d\to q^{-n-dn} \\ e\to q^{-n-en} \end{array}\right\} \quad \text{where} \quad b,c,d,e\in\mathbb{N}_0$$

Watson's q-Whipple transformation becomes the equation

$$\sum_{k=0}^{n}\left(1-q^{x+n-2k}\right)\frac{\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}x+n\\k\end{bmatrix}\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}k-x\\k\end{bmatrix}\begin{bmatrix}n+bn+x\\k\end{bmatrix}\begin{bmatrix}k+cn-x\\k\end{bmatrix}\begin{bmatrix}k+dn-x\\k\end{bmatrix}\begin{bmatrix}k+en-x\\k\end{bmatrix}}q^{\binom{n-2k}{2}+(n-2k)x}$$
$$=(-1)^n(1-q^x)\frac{\begin{bmatrix}x+n\\n\end{bmatrix}\begin{bmatrix}bn-dn+x\\n\end{bmatrix}}{\begin{bmatrix}x+bn+n\\n\end{bmatrix}\begin{bmatrix}n+dn-x\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{(n-\ell)(n-\ell+dn)}\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}n+dn\\\ell\end{bmatrix}\begin{bmatrix}\ell+n+cn+en-x\\\ell\end{bmatrix}}{\begin{bmatrix}\ell+cn-x\\\ell\end{bmatrix}\begin{bmatrix}\ell+en-x\\\ell\end{bmatrix}\begin{bmatrix}\ell-n+bn-dn+x\\\ell\end{bmatrix}}.$$

Applying the derivative operator $\mathcal{D}_0$ to it, we obtain the following transformation theorem.

Theorem 11 (Transformation formula: $b,c,d,e\in\mathbb{N}_0$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}$$
$$\times\left\{1+(1-q^{n-2k})(2k+2\mathcal{H}_k-\mathcal{H}_{bn+k}+\mathcal{H}_{cn+k}+\mathcal{H}_{dn+k}+\mathcal{H}_{en+k})\right\}$$
$$=\frac{(-1)^n\begin{bmatrix}bn-dn\\n\end{bmatrix}}{\begin{bmatrix}n+bn\\n\end{bmatrix}\begin{bmatrix}n+dn\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{(n-\ell)(n-\ell+dn)}\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}\ell+bn\\\ell\end{bmatrix}\begin{bmatrix}n+dn\\\ell\end{bmatrix}\begin{bmatrix}\ell+cn+en+n\\\ell\end{bmatrix}}{\begin{bmatrix}\ell+cn\\\ell\end{bmatrix}\begin{bmatrix}\ell+en\\\ell\end{bmatrix}\begin{bmatrix}\ell+bn-dn-n\\\ell\end{bmatrix}}.$$

We collect the following q-harmonic number identities as consequences.

Example 64 (Theorem 11: $b=0$ and $c,\ d,\ e\to\infty$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}\left\{1+(1-q^{n-2k})(2k+\mathcal{H}_k)\right\}=\sum_{\ell=0}^{n}(-1)^\ell(q;q)_\ell\begin{bmatrix}n\\\ell\end{bmatrix}q^{\binom{1+n-\ell}{2}}.$$

Example 65 (Theorem 11: $b\to\infty$ and $c=d=e=0$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}^5\left\{1+(1-q^{n-2k})(2k+5\mathcal{H}_k)\right\}=(-1)^n\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}^2\begin{bmatrix}n+\ell\\\ell\end{bmatrix}q^{(n-\ell)^2}.$$

Example 66 (Theorem 11: $b=0$, $c=d=1$ and $e\to\infty$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}\frac{\begin{bmatrix}2n\\k\end{bmatrix}^2}{\begin{bmatrix}n+k\\k\end{bmatrix}^2}\left\{1+(1-q^{n-2k})(2k+\mathcal{H}_k+2\mathcal{H}_{n+k})\right\}=\frac{\begin{bmatrix}2n-1\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}}\sum_{\ell=0}^{n}(-1)^\ell\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}2n\\\ell\end{bmatrix}}{\begin{bmatrix}n+\ell\\\ell\end{bmatrix}\begin{bmatrix}2n-1\\\ell\end{bmatrix}}q^{\binom{1+n-\ell}{2}}.$$

Example 67 (Theorem 11: $b=0$ and $c=d=e=1$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}\frac{\begin{bmatrix}2n\\k\end{bmatrix}^3}{\begin{bmatrix}n+k\\k\end{bmatrix}^3}\left\{1+(1-q^{n-2k})(2k+\mathcal{H}_k+3\mathcal{H}_{n+k})\right\}$$
$$=\frac{\begin{bmatrix}2n-1\\n\end{bmatrix}}{\begin{bmatrix}2n\\n\end{bmatrix}}\sum_{\ell=0}^{n}(-1)^\ell\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}2n\\\ell\end{bmatrix}\begin{bmatrix}3n+\ell\\\ell\end{bmatrix}}{\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^2\begin{bmatrix}2n-1\\\ell\end{bmatrix}}q^{\binom{1+n-\ell}{2}}.$$

Example 68 (Theorem 11: $b\to\infty$ and $c=d=e=1$)**.**

$$\sum_{k=0}^{n}q^{\binom{n-2k}{2}}\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}2n\\k\end{bmatrix}^3}{\begin{bmatrix}n+k\\k\end{bmatrix}^3}\left\{1+(1-q^{n-2k})(2k+2\mathcal{H}_k+3\mathcal{H}_{n+k})\right\}=\frac{(-1)^n}{\begin{bmatrix}2n\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{(n-\ell)(2n-\ell)}\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}2n\\\ell\end{bmatrix}\begin{bmatrix}3n+\ell\\\ell\end{bmatrix}}{\begin{bmatrix}n+\ell\\\ell\end{bmatrix}^2}.$$

Example 69 (Theorem 11: $b \to \infty$, $c = e = 1$ and $d = 0$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^3 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^2}{\begin{bmatrix} n+k \\ k \end{bmatrix}^2} \Big\{1 + {\scriptstyle (1-q^{n-2k})(2k+3\mathcal{H}_k+2\mathcal{H}_{n+k})}\Big\} = (-1)^n \sum_{\ell=0}^{n} q^{(n-\ell)^2} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \frac{\begin{bmatrix} 3n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}.$$

Example 70 (Theorem 11: $b \to \infty$, $d = 1$ and $c = e = 0$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^4 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}} \Big\{1 + {\scriptstyle (1-q^{n-2k})(2k+4\mathcal{H}_k+\mathcal{H}_{n+k})}\Big\} = (-1)^n \sum_{\ell=0}^{n} q^{(n-\ell)(2n-\ell)} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \frac{\begin{bmatrix} 2n \\ \ell \end{bmatrix}}{\begin{bmatrix} 2n \\ n+\ell \end{bmatrix}}.$$

Example 71 (Theorem 11: $b = 1$ and $c = d = e = 0$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^5 \frac{\begin{bmatrix} n+k \\ k \end{bmatrix}}{\begin{bmatrix} 2n \\ k \end{bmatrix}} \Big\{1 + (1 - q^{n-2k})\big({\scriptstyle 2k+5\mathcal{H}_k-\mathcal{H}_{n+k}}\big)\Big\} = \frac{(-1)^n}{\begin{bmatrix} 2n \\ n \end{bmatrix}} \sum_{\ell=0}^{n} q^{(n-\ell)^2} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2.$$

Example 72 (Theorem 11: $b = 0$, $c = 1$ and d, $e \to \infty$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix} \begin{bmatrix} 2n \\ n+k \end{bmatrix} \Big\{1 + (1 - q^{n-2k})(2k + \mathcal{H}_k + \mathcal{H}_{n+k})\Big\} = \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} q^{\binom{1+n-\ell}{2}}.$$

Example 73 (Theorem 11: $b = 1$ and c, d, $e \to \infty$).

$$\begin{aligned} &\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \Big\{1 + {\scriptstyle (1-q^{n-2k})(2k+2\mathcal{H}_k-\mathcal{H}_{n+k})}\Big\} \\ &= \sum_{\ell=0}^{n} (-1)^\ell (q;q)_\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} q^{n(n-\ell)+\binom{1+n-\ell}{2}}. \end{aligned}$$

Example 74 (Theorem 11: $b = 1$, $c = 0$ and d, $e \to \infty$).

$$\begin{aligned} &\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^2 \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \Big\{1 + {\scriptstyle (1-q^{n-2k})(2k+3\mathcal{H}_k-\mathcal{H}_{n+k})}\Big\} \\ &= \sum_{\ell=0}^{n} (-1)^\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} q^{n(n-\ell)+\binom{1+n-\ell}{2}}. \end{aligned}$$

Example 75 (Theorem 11: $b = 1$, $c = d = 0$ and $e \to \infty$).

$$\sum_{k=0}^{n} q^{\binom{n-2k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^3 \begin{bmatrix} n+k \\ k \end{bmatrix} \begin{bmatrix} 2n-k \\ n \end{bmatrix} \Big\{1 + {\scriptstyle (1-q^{n-2k})(2k+4\mathcal{H}_k-\mathcal{H}_{n+k})}\Big\} = (-1)^n \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \begin{bmatrix} n+\ell \\ \ell \end{bmatrix} q^{(n-\ell)^2}.$$

5.5. Carrying out the parameter replacements

$$\left.\begin{array}{l} a \to q^{-x-n} \\ b \to q^{-n-bn} \\ c \to q^{-n-cn} \\ d \to q^{-n-dn} \\ e \to q^{-n-en} \end{array}\right\} \quad \text{where} \quad b, c, d, e \in \mathbb{N}_0$$

we can display Watson's q-Whipple transformation as

$$\sum_{k=0}^{n}(1-q^{x+n-2k})\frac{\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}x+n\\k\end{bmatrix}\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}k-x\\k\end{bmatrix}\begin{bmatrix}k+bn-x\\k\end{bmatrix}\begin{bmatrix}k+cn-x\\k\end{bmatrix}\begin{bmatrix}k+dn-x\\k\end{bmatrix}\begin{bmatrix}k+en-x\\k\end{bmatrix}}q^{k(1+3k-3n)+x(n-3k)+\binom{n}{2}}$$
$$=(-1)^n(1-q^x)\frac{\begin{bmatrix}x+n\\n\end{bmatrix}\begin{bmatrix}2n+bn+dn-x\\n\end{bmatrix}}{\begin{bmatrix}n+bn-x\\n\end{bmatrix}\begin{bmatrix}n+dn-x\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{\ell(\ell-x-n)}\frac{\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}n+bn\\\ell\end{bmatrix}\begin{bmatrix}n+dn\\\ell\end{bmatrix}\begin{bmatrix}\ell+n+cn+en-x\\\ell\end{bmatrix}}{\begin{bmatrix}\ell+cn-x\\\ell\end{bmatrix}\begin{bmatrix}\ell+en-x\\\ell\end{bmatrix}\begin{bmatrix}2n+bn+dn-x\\\ell\end{bmatrix}}$$

which yields, under the derivative operator $\mathcal{D}_0$, the following transformation theorem.

Theorem 12 (Transformation formula: $b,c,d,e\in\mathbb{N}_0$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^2\frac{\begin{bmatrix}n+bn\\k\end{bmatrix}\begin{bmatrix}n+cn\\k\end{bmatrix}\begin{bmatrix}n+dn\\k\end{bmatrix}\begin{bmatrix}n+en\\k\end{bmatrix}}{\begin{bmatrix}k+bn\\k\end{bmatrix}\begin{bmatrix}k+cn\\k\end{bmatrix}\begin{bmatrix}k+dn\\k\end{bmatrix}\begin{bmatrix}k+en\\k\end{bmatrix}}$$
$$\times\Big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k+\mathcal{H}_{bn+k}+\mathcal{H}_{cn+k}+\mathcal{H}_{dn+k}+\mathcal{H}_{en+k})\Big\}$$
$$=\frac{(-1)^n\begin{bmatrix}2n+bn+dn\\n\end{bmatrix}}{\begin{bmatrix}n+bn\\n\end{bmatrix}\begin{bmatrix}n+dn\\n\end{bmatrix}}\sum_{\ell=0}^{n}q^{\ell(\ell-n)-\binom{n}{2}}\begin{bmatrix}n\\\ell\end{bmatrix}\frac{\begin{bmatrix}n+bn\\\ell\end{bmatrix}\begin{bmatrix}n+dn\\\ell\end{bmatrix}\begin{bmatrix}n+cn+en+\ell\\\ell\end{bmatrix}}{\begin{bmatrix}\ell+cn\\\ell\end{bmatrix}\begin{bmatrix}\ell+en\\\ell\end{bmatrix}\begin{bmatrix}2n+bn+dn\\\ell\end{bmatrix}}.$$

We collect the following q-harmonic number identities as consequences.

Example 76 (Theorem 12: $b,\ c,\ d,\ e\to\infty$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^2\big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k)\big\}=(-1)^n(q;q)_n\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 77 (Theorem 12: $b,\ c,\ d\to\infty$ and $e=0$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^3\big\{1+3(1-q^{n-2k})(k+\mathcal{H}_k)\big\}=(-1)^n\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}\frac{(q;q)_n}{(q;q)_\ell}q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 78 (Theorem 12: $b,\ e\to\infty$ and $c=d=0$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^4\big\{1+(1-q^{n-2k})(3k+4\mathcal{H}_k)\big\}=(-1)^n\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}^2q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 79 (Theorem 12: $b=c=d=0$ and $e\to\infty$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^5\big\{1+(1-q^{n-2k})(3k+5\mathcal{H}_k)\big\}=(-1)^n\sum_{\ell=0}^{n}q^{\ell(\ell-n)-\binom{n}{2}}\begin{bmatrix}n\\\ell\end{bmatrix}^2\begin{bmatrix}2n-\ell\\n\end{bmatrix}.$$

Example 80 (Theorem 12: $b=c=d=e=0$).

$$\sum_{k=0}^{n}q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^6\big\{1+3(1-q^{n-2k})(k+2\mathcal{H}_k)\big\}=(-1)^n\sum_{\ell=0}^{n}q^{\ell(\ell-n)-\binom{n}{2}}\begin{bmatrix}n\\\ell\end{bmatrix}^2\begin{bmatrix}n+\ell\\\ell\end{bmatrix}\begin{bmatrix}2n-\ell\\n\end{bmatrix}.$$

Example 81 (Theorem 12: $b=c=d=1$ and $e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^3}{\begin{bmatrix} n+k \\ k \end{bmatrix}^3} \big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k+3\mathcal{H}_{n+k})\big\}$$
$$=(-1)^n \frac{\begin{bmatrix} 4n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} \frac{\begin{bmatrix} n \\ \ell \end{bmatrix}\begin{bmatrix} 2n \\ \ell \end{bmatrix}^2}{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} 4n \\ \ell \end{bmatrix}} q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 82 (Theorem 12: $b=c=d=e=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^4}{\begin{bmatrix} n+k \\ k \end{bmatrix}^4} \big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k+4\mathcal{H}_{n+k})\big\}$$
$$=(-1)^n \frac{\begin{bmatrix} 4n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} q^{\ell(\ell-n)-\binom{n}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix} \frac{\begin{bmatrix} 2n \\ \ell \end{bmatrix}^2\begin{bmatrix} 3n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} 4n \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2}.$$

Example 83 (Theorem 12: $b=d=1$, $c=0$ and $e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^3 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^2}{\begin{bmatrix} n+k \\ k \end{bmatrix}^2} \big\{1+(1-q^{n-2k})(3k+3\mathcal{H}_k+2\mathcal{H}_{n+k})\big\} = (-1)^n \frac{\begin{bmatrix} 4n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}^2} \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix} \frac{\begin{bmatrix} 2n \\ \ell \end{bmatrix}^2}{\begin{bmatrix} 4n \\ \ell \end{bmatrix}} q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 84 (Theorem 12: $b=0$ and $c=d=e=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^3 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^3}{\begin{bmatrix} n+k \\ k \end{bmatrix}^3} \big\{1+3(1-q^{n-2k})(k+\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n \frac{\begin{bmatrix} 3n \\ n \end{bmatrix}}{\begin{bmatrix} 2n \\ n \end{bmatrix}} \sum_{\ell=0}^{n} q^{\ell(\ell-n)-\binom{n}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \frac{\begin{bmatrix} 2n \\ \ell \end{bmatrix}\begin{bmatrix} 3n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2\begin{bmatrix} 3n \\ \ell \end{bmatrix}}.$$

Example 85 (Theorem 12: $b=d=0$, $c=1$ and $e\to\infty$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^4 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}} \big\{1+(1-q^{n-2k})(3k+4\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix}^2 \frac{\begin{bmatrix} 2n \\ n+\ell \end{bmatrix}}{\begin{bmatrix} 2n \\ \ell \end{bmatrix}} q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 86 (Theorem 12: $b=d=0$ and $c=e=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^4 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}^2}{\begin{bmatrix} n+k \\ k \end{bmatrix}^2} \big\{1+(1-q^{n-2k})(3k+4\mathcal{H}_k+2\mathcal{H}_{n+k})\big\} = (-1)^n \begin{bmatrix} 2n \\ n \end{bmatrix} \sum_{\ell=0}^{n} q^{\ell(\ell-n)-\binom{n}{2}} \frac{\begin{bmatrix} n \\ \ell \end{bmatrix}^3\begin{bmatrix} 3n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2\begin{bmatrix} 2n \\ \ell \end{bmatrix}}.$$

Example 87 (Theorem 12: $b=c=d=0$ and $e=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}^5 \frac{\begin{bmatrix} 2n \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}} \big\{1+(1-q^{n-2k})(3k+5\mathcal{H}_k+\mathcal{H}_{n+k})\big\} = (-1)^n \begin{bmatrix} 2n \\ n \end{bmatrix} \sum_{\ell=0}^{n} q^{\ell(\ell-n)-\binom{n}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}^3 \frac{\begin{bmatrix} 2n+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}^2\begin{bmatrix} 2n \\ \ell \end{bmatrix}}.$$

Example 88 (Theorem 12: b, c, $d\to\infty$ and $e=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} 2n \\ k \end{bmatrix}\begin{bmatrix} 2n \\ n+k \end{bmatrix} \big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k+\mathcal{H}_{n+k})\big\}$$
$$=(-1)^n \sum_{\ell=0}^{n} \begin{bmatrix} 2n \\ n+\ell \end{bmatrix} \frac{(q;q)_n}{(q;q)_\ell} q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 89 (Theorem 12: b, $e\to\infty$ and $c=d=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)}\begin{bmatrix}2n\\k\end{bmatrix}^2\begin{bmatrix}2n\\n+k\end{bmatrix}^2\Big\{1+(1-q^{n-2k})(3k+2\mathcal{H}_k+2\mathcal{H}_{n+k})\Big\}=(-1)^n\sum_{\ell=0}^{n}\begin{bmatrix}2n\\\ell\end{bmatrix}\begin{bmatrix}2n\\n+\ell\end{bmatrix}q^{\ell(\ell-n)-\binom{n}{2}}.$$

Example 90 (Theorem 12: b, $e\to\infty$, $c=0$ and $d=1$).

$$\sum_{k=0}^{n} q^{k(1+3k-3n)}\begin{bmatrix}n\\k\end{bmatrix}^2\begin{bmatrix}2n\\k\end{bmatrix}\begin{bmatrix}2n\\n+k\end{bmatrix}\Big\{1+(1-q^{n-2k})(3k+3\mathcal{H}_k+\mathcal{H}_{n+k})\Big\}=(-1)^n\sum_{\ell=0}^{n}\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}2n\\\ell\end{bmatrix}q^{\ell(\ell-n)-\binom{n}{2}}.$$

6. Limiting Relation on finite sums of q-harmonic numbers

Recall the assumption $0<|q|<1$ in the introduction, that will be invoked in this section for deriving limiting relations. Let $\{P_k(w),Q_k(w)\}$ be two sequences of polynomials with $P_k(w)$ and $Q_k(w)$ being of degree k in w and $P_k(0)=Q_k(0)=1$ (i.e. the constant term being equal to one). Suppose that $\lambda,\nu,n,k\in\mathbb{N}_0$ with $k\le n$. Then there holds the limiting relation

$$\lim_{y\to\infty}\left\{\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}\mathcal{H}_{ny+k}(q)-\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\mathcal{H}_{ny+n-k}(q)\right\}=0. \tag{6.1}$$

To see this, rewrite the left-hand side of (6.1) in two terms

$$\begin{aligned}&\left\{\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}\mathcal{H}_{ny+k}(q)-\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\mathcal{H}_{ny+n-k}(q)\right\}\\&=\Big\{\mathcal{H}_{ny+k}(q)-\mathcal{H}_{ny+n-k}(q)\Big\}\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\\&+\mathcal{H}_{ny+k}(q)\left\{\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}-\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\right\}.\end{aligned}$$

Observe that both $\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}$ and $\mathcal{H}_{ny+k}(q)$ are bounded. More precisely, we have

$$\lim_{y\to\infty}\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}=\frac{P_{\lambda k+\nu}(0)}{Q_{\lambda k+\nu}(0)}=1\quad\text{and}\quad\Big|\lim_{y\to\infty}\mathcal{H}_{ny+k}(q)\Big|<\sum_{\ell=1}^{\infty}\frac{|q|^\ell}{1-|q|}=\frac{|q|}{(1-|q|)^2}.$$

They imply further the following two limiting relations

$$\lim_{y\to\infty}\Big\{\mathcal{H}_{ny+k}(q)-\mathcal{H}_{ny+n-k}(q)\Big\}=0\quad\text{and}\quad\lim_{y\to\infty}\left\{\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}-\frac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\right\}=0.$$

From these four relations, the limiting relation displayed in (6.1) follows consequently.

Now we are ready to prove the limiting relation on finite sums of the q-binomial coefficients and the q-harmonic numbers, similar to that in [5, Theorem 13] on classical harmonic numbers.

Theorem 13. *Let $\{P_k(w), Q_k(w)\}$ be two polynomial sequences with $P_k(w)$ and $Q_k(w)$ being of degree k in w and $P_k(0) = Q_k(0) = 1$. If $f_n(k)$ is a function independent of y which satisfies the reflection property $f_n(k) = -f_n(n-k)$, then there holds the limiting relation:*

$$\lim_{y\to\infty}\sum_{k=0}^{n} f_n(k)\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}\mathcal{H}_{ny+k}(q) = 0. \tag{6.2}$$

Proof. Reversing the summation order and applying the reflection property, we can reformulate the finite sum stated in the theorem as

$$2\sum_{k=0}^{n} f_n(k)\frac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}\mathcal{H}_{ny+k}(q) = \sum_{k=0}^{n} f_n(k)\Big\{\tfrac{P_{\lambda k+\nu}(q^y)}{Q_{\lambda k+\nu}(q^y)}\mathcal{H}_{ny+k}(q) - \tfrac{P_{\lambda(n-k)+\nu}(q^y)}{Q_{\lambda(n-k)+\nu}(q^y)}\mathcal{H}_{ny+n-k}(q)\Big\}.$$

According to (6.1), the last line tends to zero as $y \to \infty$, which confirms (6.2). □

There is a large class of functions satisfying the reflection property stated in Theorem 13. In particular for $\beta \in \mathbb{Z}$ and $\alpha, \gamma, \lambda, \mu, \nu \in \mathbb{N}_0$, the following functions

$$f_n(k) = \{1 - q^{n-2k}\}\begin{bmatrix} n \\ k \end{bmatrix}^{\lambda} \frac{\begin{bmatrix} n+\alpha n \\ k \end{bmatrix}^{\mu}}{\begin{bmatrix} k+\alpha n \\ k \end{bmatrix}^{\mu}} \frac{\begin{bmatrix} k+\gamma n \\ k \end{bmatrix}^{\nu}}{\begin{bmatrix} n+\gamma n \\ k \end{bmatrix}^{\nu}} q^{k+\beta k(n-k)}$$

have frequently appeared in the transformations proved in Sections 4 and 5. According to the theorems numbered from 2 to 12, we have found 90 summation and transformation formulae on q-binomial coefficients and q-harmonic numbers, that are presented as 90 examples. Even though the computations to deduce them are almost routine, Theorem 13 becomes often indispensable for simplifying finite q-harmonic number sums, when there exist one or more free parameters involved tending to infinity.

Now we take the summation formula displayed in Example 52 to illustrate how to derive q-harmonic number identities from the theorems established in this paper.

First, specifying with $b = d = 0$ and $c = 1$, we may state Theorem 10 as the following transformation

$$\sum_{k=0}^{n} q^{k(1+k-n)}\frac{\begin{bmatrix} 2n \\ k \end{bmatrix}\begin{bmatrix} n+ne \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}\begin{bmatrix} k+ne \\ k \end{bmatrix}}\{1+(1-q^{n-2k})(k+\mathcal{H}_{n+k}+\mathcal{H}_{k+ne})\} = \sum_{\ell=0}^{n}(-1)^{\ell}\frac{\begin{bmatrix} 1+n \\ n \end{bmatrix}\begin{bmatrix} n \\ \ell \end{bmatrix}\begin{bmatrix} 2n+ne+\ell \\ \ell \end{bmatrix}}{\begin{bmatrix} 1+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}\begin{bmatrix} ne+\ell \\ \ell \end{bmatrix}}q^{\binom{1+\ell}{2}-n\ell}.$$

Then letting $e \to \infty$ and applying Theorem 13, we derive

$$\sum_{k=0}^{n} q^{k(1+k-n)}\frac{\begin{bmatrix} 2n \\ k \end{bmatrix}}{\begin{bmatrix} n+k \\ k \end{bmatrix}}\big\{1+(1-q^{n-2k})(k+\mathcal{H}_{n+k})\big\} = \sum_{\ell=0}^{n}(-1)^{\ell}\frac{\begin{bmatrix} 1+n \\ n \end{bmatrix}\begin{bmatrix} n \\ \ell \end{bmatrix}}{\begin{bmatrix} 1+\ell \\ \ell \end{bmatrix}\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}}q^{\binom{1+\ell}{2}-n\ell}.$$

The right-hand side may be expressed in terms of basic hypergeometric series and then further simplified by means of the q-Chu-Vandermonde formula as follows

$$\begin{aligned}\frac{1-q^{n+1}}{1-q}{}_3\phi_2\left[\begin{matrix}q,\ q,\ q^{-n}\\ q^2,\, q^{n+1}\end{matrix}\,\middle|\, q;q\right] &= \lim_{\varepsilon\to1}\frac{q^{2n}-q^n}{1-\varepsilon}\left\{{}_2\phi_1\left[\begin{matrix}\varepsilon,\, q^{-n-1}\\ q^n\end{matrix}\,\middle|\, q;q\right]-1\right\}\\ &= \lim_{\varepsilon\to1}\frac{q^{2n}-q^n}{1-\varepsilon}\left\{\varepsilon^{n+1}\frac{(q^n/\varepsilon;q)_{n+1}}{(q^n;q)_{n+1}}-1\right\}\\ &= (q^n-q^{2n})\big\{1+n+\mathcal{H}_{2n}-\mathcal{H}_{n-1}\big\}.\end{aligned}$$

This confirms the q-harmonic number identity stated in Example 52.

7. Comparison with Classical Harmonic Number Identities

Generally, for a given classical binomial identity, there may exist more than one q-analogue. This phenomenon happens also for harmonic number identities. In order to facilitate reference for the reader, we tabulate the comparison between classical harmonic number identities and their q-analogues established respectively in [5] and the present paper. Here the entry numbers stand for that displayed in the two tables collected in [5] and the example numbers for the corresponding q-analogues found in this paper. We point out, by the way, that for the two formulae in Examples 42 and 52, their following common limiting case (the classical harmonic number identity) has been missed from reference [5]

$$\sum_{k=0}^{n}\frac{\binom{2n}{k}}{\binom{n+k}{k}}\big\{1+(n-2k)H_{n+k}\big\}=n\big\{H_{2n}-H_{n-1}\big\}.$$

Entry	Examples	Entry	Examples	Entry	Examples
I-1	1	I-17	22,55,78	II-7	56,66
I-2	4	I-18	24,62,90	II-8	63,75
I-3	2	I-19	25,60,89	II-9	57
I-4	5	I-20	26,40	II-10	58
I-5	3	I-21	27	II-11	70,85
I-6	6	I-22	28	II-12	71
I-7	7	I-23	29,41	II-13	69,83
I-8	8,13,31,54,64	I-24	30	II-14	67
I-9	9,14,20,32,76	I-25	43	II-15	68,81
I-10	11,37,59	I-26	53	II-16	65,79
I-11	12,39,61	II-1	34	II-17	80
I-12	16,49,72	II-2	36	II-18	87
I-13	18,23,50,88	II-3	33,45	II-19	86
I-14	10,17,38,73	II-4	46	II-20	84
I-15	19,51,74	II-5	35,47	II-21	82
I-16	15,21,44,77	II-6	48	Missed	42,52

Concluding Comments. In the previous paper [5], four classical hypergeometric series theorems are examined through the derivative operator method. Following the same scheme, their q-analogues are explored in the present work. Altogether both papers provide a comprehensive coverage to the identities on the classical and q-harmonic numbers. The authors hope that the summation and tranformation formulae shown in both papers may serve as a documentary source for further references.

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SOME RELATIONSHIP BETWEEN THE q-GENOCCHI NUMBERS AND BERNSTEIN POLYNOMIALS

N. S. Jung, H. Y. Lee, C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : Recently, we introduced some interesting relations between Genocchi number and Bernstein polynomials(see [9]). In this paper, we give some interesting identities on the q-Genocchi polynomials and Bernstein polynomials.

Key words : Genocchi numbers and polynomials, q-Genocchi numbers and polynomials, Bernstein polynomials

1. Introduction

Throughout this paper, let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r \frac{t}{s}$ with $(t,p) = (s,p) = (t,s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$.

We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ see [1, 2] .} \tag{1.1}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$ be translation. As well known equation, by (1.1), we have

$$I_{-1}(f_n) = \int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.2}$$

The Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{1.3}$$

In the special case, $x = 0$, $G_n(0) = G_n$ are called the n-th Genocchi numbers(see [1-9]).

We introduced the q-Genocchi polynomials as follows:

$$\frac{2t}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{1.4}$$

In the special case, $x = 0$, $G_{n,q}(0) = G_{n,q}$ are called the n-th q-Genocchi numbers.

From (1.4), we note that

$$G_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{l,q} x^{n-l}. \tag{1.5}$$

From (1.2) and (1.4), for $n = 1$, we have

$$t\int_{\mathbb{Z}_p} q^y e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}. \tag{1.6}$$

By (1.6), we obtain

$$G_{0,q}(x) = 0, \quad \int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1,q}(x)}{n+1}, \text{ for } n \in \mathbb{N}. \tag{1.7}$$

Bernstein polynomials of degree n are given by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in [0,1], n, k \in \mathbb{Z}_+, \text{see [3, 4, 5, 7, 8, 9]}. \tag{1.8}$$

In [1], Kim introduced p-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \text{ where } x \in \mathbb{Z}_p \text{ and } n, k \in \mathbb{Z}_+. \tag{1.9}$$

In this paper, we give some properties for the q-Genocchi numbers and polynomials. By using these properties, we investigate some interesting identities on the Bernstein polynomials and the q-Genocchi polynomials.

2. Some identities on the Bernstein and q-Genocchi polynomials

From (1.6), we can derive the following recurrence formula for the q-Genocchi numbers:

$$G_{0,q} = 0, \text{ and } q(G_q+1)^n + G_{n,q} = \begin{cases} 2, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{2.1}$$

with usual convention about replacing $(G_q)^n$ by $G_{n,q,w}$.

By (1.4), we easily get

$$\sum_{n=0}^{\infty} G_{n,q^{-1}}(1-x)(-1)^n \frac{t^n}{n!} = (-1)\frac{2tq}{qe^t+1}e^{xt} = (-1)q\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.2), we obtain the following theorem.

Theorem 1. Let $n \in \mathbb{Z}_+$. Then we have

$$G_{n,q}(x) = (-1)^{n-1} q^{-1} G_{n,q^{-1}}(1-x).$$

From (1.7), we note that

$$G_{0,q} = 0, \quad \int_{\mathbb{Z}_p} q^x x^n d\mu_{-1}(x) = \frac{G_{n+1,q}}{n+1}, \text{ for } n \in \mathbb{N}. \tag{2.3}$$

By (2.1), for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned}
G_{n,q}(2) &= (G_q+1+1)^n = \sum_{l=0}^{n}\binom{n}{l}G_{l,q}(1)\\
&= \frac{1}{q}\sum_{l=1}^{n}\binom{n}{l}qG_{l,q}(1)\\
&= \frac{1}{q}(nqG_{1,q}(1)) + \frac{1}{q}\sum_{l=2}^{n}\binom{n}{l}G_{l,q}(1)\\
&= \frac{2n}{q} - \frac{1}{q^2}qG_{n,q}(1)\\
&= \frac{2n}{q} + \frac{1}{q^2}G_{n,q}.
\end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2. For $n \in \mathbb{N}$ with $n > 1$, we have

$$qG_{n,q}(2) = 2n + \frac{1}{q}G_{n,q}.$$

By (2.3) and Theorem 2, we obtain the following corollary.

Corollary 3. For $n \in \mathbb{N}$ with $n > 1$, we have

$$\frac{1}{q}\int_{\mathbb{Z}_p} q^{-x}(x+2)^n d\mu_{-1}(x) = 2 + q\frac{G_{n+1,q^{-1}}}{n+1}.$$

By (1.7), (2.3) and Corollary 3, we know that

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^x(1-x)^n d\mu_{-1}(x) &= (-1)^n\int_{\mathbb{Z}_p} q^x(x-1)^n d\mu_{-1}(x)\\
&= (-1)^n\frac{G_{n+1,q}(-1)}{n+1}\\
&= \frac{1}{q}\frac{G_{n+1,q^{-1},w^{-1}}(2)}{n+1}\\
&= \frac{1}{q}\int_{\mathbb{Z}_p} q^{-x}(x+2)^n d\mu_{-1}(x)\\
&= 2 + q\frac{G_{n+1,q^{-1}}}{n+1}\\
&= 2 + q\int_{\mathbb{Z}_p} q^{-x}x^n d\mu_{-1}(x).
\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 4. For $n \in \mathbb{N}$ with $n > 1$, we have

$$\int_{\mathbb{Z}_p} q^x(1-x)^n d\mu_{-1}(x) = 2 + q\int_{\mathbb{Z}_p} q^{-x}x^n d\mu_{-1}(x).$$

In (1.9), we take the fermionic p-adic invariant integral on $\mathbb{Z}_p$ for one Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \int_{\mathbb{Z}_p} q^x x^{n-l} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G_{n-l+1,q}}{n-l+1} \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \frac{G_{k+l+1,q}}{k+l+1}, \text{ where } n,k \in \mathbb{Z}_+.\end{aligned} \tag{2.5}$$

From the reflection symmetric properties of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n,k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \tag{2.6}$$

For $n,k \in \mathbb{Z}_+$ with $n > k+1$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} q^x B_{n-k,n}(1-x) d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \int_{\mathbb{Z}_p} q^x (1-x)^{n-l} d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q \int_{\mathbb{Z}_p} q^{-x} x^{n-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 5. For $n,k \in \mathbb{Z}_+$ with $n > k+1$, we have

$$\int_{\mathbb{Z}_p} q^x B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q \frac{G_{n-l+1,q^{-1},w^{-1}}}{n-l+1}\right).$$

By (2.5) and Theorem 5, we obtain the following theorem.

Theorem 6. Let $n,k \in \mathbb{Z}_+$ with $n > k+1$. Then we have

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \frac{G_{k+l+1,q}}{k+l+1} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q \frac{G_{n-l+1,q^{-1},w^{-1}}}{n-l+1}\right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} q^x (1-x)^{n_1+n_2-l} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \frac{1}{q} \int_{\mathbb{Z}_p} q^{-x} (x+2)^{n_1+n_2-l} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + q \int_{\mathbb{Z}_p} q^{-x} x^{n_1+n_2-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 7. For $n_1, n_2\, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2 + q\frac{G_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right).\end{aligned}$$

By simple calculation, we easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p} q^x x^{l+2k} d\mu_{-1}(x) \qquad (2.7)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G_{l+2k+1,q}}{l+2k+1}, \text{ where } n_1, n_2, k \in \mathbb{Z}_+.\end{aligned}$$

Therefore, by (2.7) and Theorem 7, we obtain the following theorem.

Theorem 8. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$. Then we have

$$\begin{aligned}&\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2 + q\frac{G_{n_1+n_2-l+1,q^{-1},w^{-1}}}{n_1+n_2-l+1}\right)\\ &= \sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G_{l+2k+1,q}}{l+2k+1}.\end{aligned}$$

For $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k+1$, by the symmetry of Bernstein polynomials, we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\int_{\mathbb{Z}_p} q^x (1-x)^{n_1+n_2+n_3-l} d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\frac{1}{wq^h}\int_{\mathbb{Z}_p} q^{-hx} w^{-x} (x+2)^{n_1+n_2+n_3-l} d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2 + q\int_{\mathbb{Z}_p} q^{-hx} w^{-x} x^{n_1+n_2+n_3-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 9. For $n_1, n_2, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k+1$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2 + q\frac{G_{n_1+n_2+n_3-l,q^{-1},w^{-1}}}{n_1+n_2+n_3-l+1}\right).\end{aligned}$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^x B_{k,n_1}(x) B_{k,n_2}(x) B_{k,n_3}(x) d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1+n_2+n_3-3k}{l} \int_{\mathbb{Z}_p} q^x x^{l+3k} d\mu_{-1}(x) \\
&= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k} \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1+n_2+n_3-3k}{l} \frac{G_{l+3k+1,q}}{l+3k+1},
\end{aligned}$$

where $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$.

Therefore, by Theorem 9, we obtain the following theorem.

Theorem 10. Let $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$. Then we have

$$\begin{aligned}
&\sum_{l=0}^{3k} \binom{3k}{l} (-1)^{l+3k} \left(2 + q \frac{G_{n_1+n_2+n_3-l+1,q^{-1},w^{-1}}}{n_1+n_2+n_3-l+1}\right) \\
&= \sum_{l=0}^{n_1+n_2+n_3-3k} (-1)^l \binom{n_1+n_2+n_3-3k}{l} \frac{G_{l+3k+1,q}}{l+3k+1}.
\end{aligned}$$

Using the above theorem and mathematical induction, we have the following theorem.

Theorem 11. Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_m > mk + 1$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_m}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^x \left(\prod_{i=1}^{m} B_{k,n_i}(x)\right) d\mu_{-1}(x) \\
&= \left(\prod_{i=1}^{m} \binom{n_i}{k}\right) \sum_{l=0}^{mk} (-1)^{l+mk+1} \left(2 + q \frac{G_{n_1+\cdots+n_m-l+1,q^{-1},w^{-1}}}{n_1+\cdots+n_m-l+1}\right).
\end{aligned}$$

We also easily see that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^x \left(\prod_{i=1}^{m} B_{k,n_i}(x)\right) d\mu_{-1}(x) \\
&= \left(\prod_{i=1}^{m} \binom{n_i}{k}\right) \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1+\cdots+n_m-mk}{l} (-1)^l \frac{G_{l+mk+1,q}}{l+mk+1}.
\end{aligned} \tag{2.8}$$

By Theorem 11 and (2.8), we have the following corollary.

Corollary 12. Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_m > mk + 1$, we have

$$\begin{aligned}
&\sum_{l=0}^{mk} (-1)^{l+mk} \left(2 + q \frac{G_{n_1+\cdots+n_m-l+1,q^{-1},w^{-1}}}{n_1+\cdots+n_m-l+1}\right) \\
&= \sum_{l=0}^{n_1+\cdots+n_m-mk} \binom{n_1+\cdots+n_m-mk}{l} (-1)^l \frac{G_{l+mk+1,q}}{l+mk+1}.
\end{aligned}$$

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SOME THEOREMS IN CONE METRIC SPACES

Duran Turkoglu, Muhib Abuloha, Thabet Abdeljawad

Department of Mathematics, Faculty of Science and Arts, Gazi University, 06500, Ankara-Turkey. dturkoglu@gazi.edu.tr.

Department of Mathematics, Institute of Science and Technology, Gazi University, 06500, Ankara-Turkey. muhib2000@yahoo.com.

Department of Mathematics and Computer Science, ankaya University, 06530 Ankara-Turkey. thabet@cankaya.edu.tr.

Abstract

In this paper, some topological concepts and definitions are generalized to cone metric spaces. The distance between two sets in cone metric spaces is defined, where some examples are given. Moreover, it is proved that every cone metric space is a $T_2 - space$ and that Baire's category theorem is still valid in cone metric spaces. Regarding, the theory of linear operators in cone normed spaces, we defined bounded linear operators and scalar bounded linear operators. Examples are given to show that not necessarily all linear operators are bounded on finite dimensional cone normed spaces and to show that not every continuous linear operator between cone normed spaces is bounded.

Key words: Cone metric, cone normed, cone Banach, strongly minihedral, absolute value function, Meager (the first category), Nonmeager (the second category), Baire's category theorem, monotonic, semi-monotonic, linear operator, scalar bounded, continuous.

1 Introduction and Preliminaries

Cone metric spaces were first introduced in [4], where the authors described convergence in cone metric spaces and introduced completeness. Furthermore, in [1,5,7,13,14], they proved some common fixed point theorems in cone metric spaces. In [2], the authors introduced some generalized topological concepts and definitions in cone metric spaces and proved that every cone metric space is topological space as well as they proved some fixed point theorems in diametrically contractive mappings in cone metric spaces. Furthermore, cone metric spaces were studied by many authors (see [6,8–12]). However, in this paper, some topological concepts and definitions are generalized to cone metric spaces. We defined the distance between two sets in cone metric spaces supported by examples. Moreover, it is proved that every cone metric space is a $T_4 - space$. Also to prove Baire's category theorem in cone metric space, we defined the first category (Meager) and the second category (nonmeager). We also defined the cone normed space and proved that each cone normed space is a cone metric space. Furthermore, we defined cone Banach space and we gave an example about that and we defined the absolute value function and scalar bounded linear operator in cone metric spaces as well. Finally, we generalized bounded linear operators between two cone normed spaces and by an example showed that not necessarily all linear operators are bounded on finite dimensional cone normed spaces and to show that not every continuous linear operator between cone normed spaces is bounded.

Let E be a real Banach space and P a subset of E. Then, P is called a cone if and only if

P1) P is closed, non empty and $P \neq \{0\}$

P2) $a, b \in R \;\; a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$

P3) $x \in P$ and $-x \in P \Rightarrow x = 0$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in IntP$. ($IntP \cong interior\ of\ \ P$).

The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$, where K is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to \infty$.

Equivalently the cone P is called regular if every decreasing sequence which is bounded from below is convergent[4]. Throughout this article we assume that the cone P is normal with constant K.

P is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum and hence any subset of E which is bounded from below has an infimum . A norm $\| \cdot \|$ on E is called monotonic if $0 \leq x \leq y$ implies $\| x \| \leq \| y \|$, and semi-monotonic if $\| x \| \leq K \| y \|$ for some $K > 0$ and all x and y such that $0 \leq x \leq y$ [3]. It is know by [3] that P is normal if and only if $\| \, . \, \|$ is semi-monotonic.

Throughout this article we assume that P is a cone in E with $IntP \neq \varnothing$ and $\leq$ is partial ordering with respect to P. We also appeal to the following relations:

$$IntP + IntP \subset IntP \text{ and } \lambda IntP \subset IntP, \quad \lambda > 0$$

2 Cone Metric Spaces and Baire's Category Theorem

Definition 1 *[4] A cone metric space is an ordered pair (X, d), where X any set and $d : X \times X \to E$ is a mapping satisfying: d1) $0 < d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$. d2) $d(x, y) = d(y, x)$ for all $x, y \in X$. d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Example 2 *Let $E = R^n$ and $P = \{(x_1, x_2, ..., x_n) \in R^n : x_i \geq 0, i = 1, 2, 3, ..., n\}$ with $x = (x_1, x_2, ..., x_n) \leq y = (y_1, y_2, ..., y_n)$ if and only if $(y_i - x_i) \geq 0$ for all $i = 1, 2, 3, ..., n$. Then any subset of P has an infimum.*

Definition 3 *Let $A \neq \varnothing$ and $B \neq \varnothing$ be two subset of a cone metric space (X, d). Then the distance between A and B, denoted by $d(A, B)$, is defined by $d(A, B) = \inf \{d(x, y) : x \in A, \quad y \in B\}$. If $A = \{a\}$, we write $d(a, B)$ for $d(A, B)$.*

Example 4 *Let $E = R^2$ and $P = \{(x, y) : x \geq 0, \ y \geq 0\}$. Define $d : R^2 \times R^2 \to E$ by $d((x_1, x_2), (y_1, y_2)) = (| \, x_1 - y_1 \, |, | \, x_2 - y_2 \, |)$ and let $A = \{(x, y) \in R^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$, $B = \{(x, y) \in R^2 : 2 \leq x \leq 3, \ 0 \leq y \leq 1\}$, $C = \{(0, 0)\}$, $D = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$. Then $d(A, B) = (1, 0)$, $d(A, C) = (0, 0)$, $d(C, B) = (2, 0)$.*

Example 5 *Let $E = R^2$ and $P = \{(x, y) : x \geq 0, \ y \geq 0\}$. Define $d : R^2 \times R^2 \to E$ by $d((x_1, x_2), (y_1, y_2)) = (| \, x_1 - y_1 \, |, | \, x_2 - y_2 \, |)$ and let $A = \{(x, y) \in R^2 : 0 \leq x \leq 1, \ 2 \leq y \leq 3\}$, $B = \{(x, y) \in R^2 : 3 \leq x \leq 4, \ 0 \leq y \leq 3\}$. Then $d(A, B) = (2, 1)$.*

Lemma 6 *Let $c_1, c_2 \in P$ such that $c_1 < c_2 + s$ for all $s >> 0$. Then $c_1 \leq c_2$.*

Proposition 7 *(see prop.(1) in [2]) Every cone metric space (X, d) is a first countable topological space, whose topology τ_c is given by*

$$\tau_c = \{U \subset X : \forall x \in U, \ \exists c >> 0 \text{ such that } B(x, c) \subset U\} \cup \{\emptyset\} \tag{1}$$

where

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$$B(x,c) = \{y \in X : d(x,y) << c\} \tag{2}$$

Theorem 8 *Let (X,d) be a cone metric space and $A \neq \varnothing$ a subset of X. Then $x \in \overline{A}$ if and only if $d(x,A) = 0$.*

PROOF. Suppose $x \in \overline{A}$. Then, for fixed $c >> 0$ and each $n \in N$, $B\left(x, \frac{c}{n}\right) \cap A \neq \varnothing$. Therefore, for each $n \in N$ there exists $a_n \in A$ such that $0 \leq d(x,A) \leq d(x,a_n) < \frac{c}{n}$. Hence, $\frac{c}{n} - d(x,A) \in P$, for all $n \in N$. That P is closed implies that $-d(x,A) \in P$ and our conclusion then is that $d(x,A) = 0$. Conversely, $d(x,A) = 0 < \frac{c_0}{n}$ for all $n \in \mathbb{N}$ and some $c_0 >> 0$ implies the existence of a sequence $a_n \in A$ such that $d(x,a_n) < \frac{c_0}{n}$ for all $n \in \mathbb{N}$. Since P is a closed cone then we conclude that $-\lim_{n\to\infty} d(x,a_n) \in P$, $\lim_{n\to\infty} d(x,a_n) \in P$ and hence $\lim_{n\to\infty} d(x,a_n) = 0$. Now let $c >> 0$ be given and choose $\delta > 0$ such that $\|b\| < \delta$ implies that $b << c$. Since $\lim_{n\to\infty} d(x,a_n) = 0$ then for some $n_0 \in N$ and all $n > n_0$ we have $\|d(x,a_n)\| < \delta$ and so $d(x,a_n) << c$ for all $n > n_0$. Thus, $a_n \to x$ in (X,d). Since every cone metric space is first countable topological space then $x \in \overline{A}$.

As a consequence of the above Theorem, if a subset A of a cone metric space (X,d) is closed and $x \notin A$, then $d(x,A) > 0$.

Theorem 9 *Every cone metric space (X,d) is a T_2- space.*

PROOF. Let $x \neq y$ be two points in X. Suppose $B(x, \frac{c}{2}) \cap B(y, \frac{c}{2}) = \emptyset$ for all $c >> 0$, then by the triangle inequality $d(x,y) \leq c$ for all $c >> 0$. Hence $d(x,y) \leq \frac{c_0}{n}$ for all $n \in N$ and some $c_0 >> 0$. Since P is a closed cone then $d(x,y) = 0$ and so $x = y$ which is a contradiction.

Definition 10 *A subset M of a topological space (X, τ_c) is called: (I) Rare (nowhere dense) in X if $Int(\overline{M}) = \emptyset$. (II) Meager (the first category) in X if M is the union of countably many sets each of which is rare in X.(III) Nonmeager (second category) in X if M is not meager in X.*

Lemma 11 *[4]Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $y_n \to y$, $x_n \to x$ as $(n \to \infty)$, then $d(x_n,y_n) \to d(x,y)$ as $n \to \infty$.*

Lemma 12 *Let (X,d) be a cone metric space. If $a_n \in E$ such that $b \leq a_n$ for all n and $a_n \to a$ then $b \leq a$.*

Theorem 13 *(**Baire's Category Theorem in Cone Metric Space**) Every complete nonempty cone metric space (X, d) is nonmeager in itself. Hence $X = \cup_{k=1}^{\infty} M_k$, M_k closed, then at least one M_k contains a nonempty open subset.*

PROOF. Suppose the complete cone metric space (X, d) were first category. Then $X = \cup_{k=1}^{\infty} M_k$ with each M_k rare in X, we can shall construct a Cauchy sequence $\{x_k\}$ in X whose limit x (which exists by completeness) is no M_k and get a contradiction. By assumption, M_1 is rare in X, so that by definition, $\overline{M_1}$ does not contain a nonempty open set. But X does have. This implies that $\overline{M_1} \neq X$. Hence choose $x_1 \in \overline{M_1}^c$ and an open ball about it, say $B_1 = B(x_1, c_1) = \overline{M_1}^c$ with $c_1 < \frac{c}{2}$ where $c \in IntP$ is some fixed point. By assumption, M_2 is rare in X, so $\overline{M_2}$ does not contain a nonempty open set. Hence, it does not contain the open ball $B(x_1, \frac{c_1}{2})$. This implies that $\overline{M_2}^c \cap B(x_1, \frac{c_1}{2})$ is not empty and open, so that we may choose an open ball in this set say, $B_2 = B(x_2, c_2) \subset \overline{M_2}^c \cap B(x_1, \frac{c_1}{2})$, $c_2 < \frac{c_1}{2}$. By induction we thus construct a sequence of balls, $B_k = B(x_k, \frac{c_k}{2})$, $c_k < \frac{c}{2^k}$ such that $B_k \cap M_k = \varnothing$ and $B_{k+1} \subset B(x_k, \frac{c_k}{2}) \subset B_k$, $k = 1, 2, 3, \ldots$. Since $c_k < \frac{c}{2^k}$ the sequence x_k of the centers is Cauchy and hence converges, say $x_k \to x \in X$ because X is complete by assumption. Also, for every m and $n > m$ we have $B_n \subset B(x_m, \frac{c_m}{2})$, so that $d(x_m, x) \leq d(x_m, x_n) + d(x_n, x) < \frac{c_m}{2} + d(x_n, x)$. Since $x_n \to x$, then we have $\frac{c_0}{n} + \frac{c_m}{2} - d(x_m, x) \in P$ for all $n \in \mathbb{N}$ and some $c_0 >> 0$. Since P is a closed cone we conclude that $\frac{c_m}{2} - d(x_m, x) \in P$ and so $x \in B_m$ for every m. Since $B_m \subset \overline{M_m}^{\,c}$, we now see that $x \notin M_m$ for every m, so that $x \notin X = \cup_{m=1}^{\infty} M_m$. This contradicts that $x \in X$.

3 Cone Normed Spaces

Definition 14 *(see also [15], [16]) A cone normed space is an ordered pair $(X, \parallel . \parallel_c)$ where X is a vector space over R and $\parallel . \parallel_c$: $X \to E$ is a function satisfying:*

C1) $0 < \parallel x \parallel_c$, *for all* $x \in X$.
C2) $\parallel x \parallel_c = 0$ *if and only if* $x = 0$.
C3) $\parallel \alpha . x \parallel_c = \mid \alpha \mid \parallel x \parallel_c$, *for each* $x \in X$ *and* $\alpha \in R$.
C4) $\parallel x + y \parallel_c \leq \parallel x \parallel_c + \parallel y \parallel_c$, $x, y \in X$.

Proposition 15 *Each cone normed space is cone metric space. Namely,* $d : X \times X \to E$ *is defined by* $d(x,y) = \parallel x-y \parallel_c$ *.*

PROOF. for all $x, y, z \in X$

d1) $d(x,y) = 0$ if and only if $\parallel x-y \parallel_c = 0$ if and only if $x-y=0 \Leftrightarrow x=y$.

d2) $d(x,y) = \parallel x-y \parallel_c = \parallel -(y-x) \parallel_c$ by (C3) $\parallel y-x \parallel_c = d(y,x)$.

d3) $d(x,y) = \parallel x-y \parallel_c = \parallel x-z+z-y \parallel_c$ by (C4) $\parallel x-z+z-y \parallel_c \leq \parallel x-z \parallel_c + \parallel z-y \parallel_c = d\ (x,z) + d\ (z,y)$.

Remark 16 *Convergence in cone normed space is described by the cone metric induced by the norm. For example, a sequence* $x_n \in X$ *is said to converge to an element* $x \in X$*, if for all* $c >> 0$ *there exists* n_0 *such that* $d(x_n, x) = \parallel x_n - x \parallel_c << c$ *for all* $n > n_0$*. Hence, by [4], when the cone is normal, a sequence* $x_n \to x$ *if and only if* $\parallel d(x_n, x) \parallel = \parallel \parallel x_n - x \parallel_c \parallel \to 0$ *as* $n \to \infty$*.*

Definition 17 *A sequence* $x_n \in X$ *is called Cauchy sequence if for all* $c >> 0$ *there exists* n_0 *such that* $d(x_n, x_m) = \parallel x_n - x_m \parallel_c << c$ *for all* $m, n > n_0$*. Equivalently by [4] if* $\lim\limits_{m,n\to\infty} \parallel d(x_n, x_m) \parallel = \lim\limits_{m,n\to\infty} \parallel \parallel x_n - x_m \parallel_c \parallel = 0$*.*

Definition 18 *A cone normed space* $(X, \parallel . \parallel_c)$ *is called cone Banach space if every Cauchy sequence in* X *convergent in* X*.*

Example 19 *Let* $(E, \parallel . \parallel_c)$, $E = R^2$, $P = \{(x,y) : x \geq 0, y \geq 0\}$*. The function* $\parallel . \parallel_c$ *defined by* $\parallel (x,y) \parallel_c = (\alpha \mid x \mid, \beta \mid y \mid)$, $\alpha, \beta > 0$ *is a cone norm space and cone Banach space. Indeed,*

C1) $\parallel (x,y) \parallel_c > 0$, $\parallel (x,y) \parallel_c = 0 \Leftrightarrow (\alpha \mid x \mid, \beta \mid y \mid) = (0,0) \Leftrightarrow \alpha \mid x \mid = 0$ *and* $\beta \mid y \mid = 0 \Leftrightarrow x = 0,\ y = 0 \ \Leftrightarrow (x,y) = (0,0)$.

C2) $\parallel a(x,y) \parallel_c = \parallel (ax, ay) \parallel_c = (\alpha \mid ax \mid, \beta \mid ay \mid) = \mid a \mid (\alpha|x|, \beta|y|) = \mid a \mid . \parallel (x,y) \parallel_c$.

C3) $\parallel (x,y) + (z,w) \parallel_c = \parallel x+z, y+w \parallel_c = (\alpha \mid x+z \mid, \beta \mid y+w \mid) \leq (\alpha \mid x \mid + \alpha \mid z \mid, \beta \mid y \mid + \beta \mid w \mid) = (\alpha \mid x \mid, \beta \mid y \mid) + (\alpha \mid z \mid, \beta \mid w \mid) = \parallel (x,y) \parallel_c + \parallel (z,w) \parallel_c$.

This space is complete and hence cone Banach. Let $z_n = (x_n, y_n) \in R^2$ *be a Cauchy sequence.*

Hence by Lemma 4 in [4],

$$\lim_{m,n\to\infty} \| \| z_n - z_m \|_c \| = \lim_{m,n\to\infty} \| \| x_n - x_m, y_n - y_m \|_c \| =$$

$\lim_{m,n\to\infty} \| (\alpha \mid x_n - x_m \mid, \beta \mid y_n - y_m \mid) \| = \lim_{m,n\to\infty} \sqrt{\alpha^2 \mid x_n - x_m \mid^2 + \beta^2 \mid y_n - y_m \mid^2} = 0.$

Therefore, $\mid x_n - x_m \mid \to 0, \mid y_n - y_m \mid \to 0$ *as* $n, m \to \infty$ *and hence* $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequence in the field* R. *Find* $x, y \in R$ *such that* $\mid x_n - x \mid \to 0$, $\mid y_n - y \mid \to 0$ *as* $n \to \infty$.

We shall show that $z_n = (x_n, y_n) \to z = (x, y)$ *in cone norm space and hence* $(R^2, \| \ . \ \|_c)$ *is complete.* $\lim_{n\to\infty} \| \| z_n - z \|_c \| = \lim_{n\to\infty} \| \| (x_n - x, y_n - y) \|_c \| =$ $\lim_{n\to\infty} \| (\alpha \mid x_n - x \mid, \beta \mid y_n - y \mid \| \| = \lim_{n\to\infty} \sqrt{\alpha^2 \mid x_n - x \mid^2 + \beta^2 \mid y_n - y \mid^2} = 0.$

Proposition 20 *Every cone normed space is topological space.*

The result follows by [2] since each cone normed space is cone metric space. Actually, the topology is given by

$$\tau_c = \{U \subset X : \forall x \in U, \ \exists c >> 0 \ \ \text{such that} \ \ B(x, c) \subset U\} \cup \{\Phi\} \qquad (3)$$

where

$$B(x, c) = \{y \in X : \| x - y \|_c << c\} \qquad (4)$$

Proposition 21 *The cone metric d induced by a cone norm on a cone normed space satisfies :*

$$d\,(x + a, y + a) = d\,(x, y) \qquad (5)$$

$$d\,(\alpha x, \alpha y) = \mid \alpha \mid d\,(x, y) \qquad (6)$$

for all $x, y, a \in X$ *and every scalar* α.

PROOF. We have $d\,(x + a, y + a) = \| x + a - (y + a) \|_c = \| x - y \|_c = d\,(x, y)$ and $d\,(\alpha x, \alpha y) = \| \alpha x - \alpha y \|_c = \mid \alpha \mid \| x - y \|_c = \mid \alpha \mid d\,(x, y)$.

Definition 22 *The absolute value function* $abs : E \to E$ *is defined by:*

$$abs(a) = \left\{ \begin{array}{cc} a & a \geq 0 \\ -a & a < 0 \\ 0 & otherwise \end{array} \right\} \qquad (7)$$

Note that if $a < 0$ *then* $-a > 0$ *and* $abs(a) \in P$.

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Definition 23 *A cone norm* $\| \, . \, \|_n \colon E \to P$ *is said to satisfy the property (A) if :* $-c \leq a \leq c$ *if and only if* $\| \, a \, \|_n \leq c$, *for all* $a \in E$ *and* $c >> 0$.

Example 24 *Let* $E = R^2$ *and* $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$. *Then the norm* $\| \, . \, \|_n \colon E \to P$ *defined by* $\| \, (x, y) \, \|_n = (| \, x \, |, | \, y \, |)$, *satisfies the property (A). Indeed* $a = (x, y)$, *satisfies* $\| \, a \, \|_n \leq c = (c_1, c_2)$, $c_1, c_2 > 0$ *if and only if* $(| \, x \, |, | \, y \, |) \leq (c_1, c_2)$ *if and only if* $-c = (-c_1, -c_2) \leq a = (x, y) \leq (c_1, c_2) = c$. *Also it can be easily seen that* $\| \, . \, \|_n$ *is monotonic on* P *(i.e.* $0 \leq a \leq b$ *implies* $\| \, a \, \|_n \leq \| \, b \, \|_n$*).*

Remark 25 *Using (C4) and that E is normal cone with constant* K, *we can show that*

$$\| \, abs(\| \, y \, \|_c - \| \, x \, \|_c) \, \| \leq K \, \| \| \, x - y \, \|_c \| \, . \quad \forall x, y \in X \tag{8}$$

Indeed, $\| \, x \, \|_c = \| \, x - y + y \, \|_c \leq \| \, x - y \, \|_c + \| \, y \, \|_c$ *and* $\| \, y \, \|_c = \| \, y - x + x \, \|_c \leq \| \, y - x \, \|_c + \| \, x \, \|_c$, *hence,* $\| \, x \, \|_c - \| \, y \, \|_c \leq \| \, x - y \, \|_c$ *and* $\| \, y \, \|_c - \| \, x \, \|_c \leq \| \, x - y \, \|_c$ *. But then* $0 \leq abs(\| \, y \, \|_c - \| \, x \, \|_c) \leq \| \, x - y \, \|_c$ *implies that* $\| \, abs(\| \, y \, \|_c - \| \, x \, \|_c) \, \| \leq K \, \| \| \, x - y \, \|_c \|$ *. Furthermore, if* $\| \, . \, \|_n$ *is a cone norm on E with the property (A) then* $\| \| \| \| \, x \, \|_c - \| \, y \, \|_c \|_n \| \leq K \, \| \| \, x - y \, \|_c \|$ *.*

For more examples of cone normed spaces and the completion of cone normed spaces see [15] and [16], respectively.

4 Bounded Linear Operators Between Cone Normed Spaces

Definition 26 *A linear operator* $T : (X, \| \, . \, \|_{c_1}) \to (Y, \| \, . \, \|_{c_2})$ *between cone normed spaces over the same cone* P *in* E, *is called : (a) Bounded if there exists* $M > 0$ *such that for all* $x \in X$,

$$\| \, Tx \, \|_{c_2} \leq M \, \| \, x \, \|_{c_1} \tag{9}$$

(b) Scalar bounded if there exists $L > 0$ *such that for all* $x \in X$,

$$\| \| \, Tx \, \|_{c_2} \| \leq L \, \| \| \, x \, \|_{c_1} \| \tag{10}$$

When T *is scalar bounded, we define its scalar norm by*

$$\| \, T \, \|_s = \inf \{ L > 0 : \| \| \, Tx \, \|_{c_2} \| \leq L \, \| \| \, x \, \|_{c_1} \|, \forall x \in X \} \tag{11}$$

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From the above definition it easy to see that every bounded linear operator between cone normed spaces is scalar bounded with $L = MK$ and K is the normal constant of the cone P in E and

$$\| T \|_s = \sup_{0 \neq x \in X} \frac{\| \| Tx \|_{c_2} \|}{\| \| x \|_{c_1} \|} \tag{12}$$

Proposition 27 *If the norm $\| \, . \, \|$ on E is monotonic or the normal constant $K = 1$, then $(X, \| \| \, . \, \|_{c_1} \|)$ is a normed linear space.*

PROOF. If $\| x + y \|_{c_1} \leq \| x \|_{c_1} + \| y \|_{c_1}$, then monotonicity of $\| \, . \, \|$ implies that $\| \| x+y \|_{c_1} \| \leq \| \| x \|_{c_1} + \| y \|_{c_1} \| \leq \| \| x \|_{c_1} \| + \| \| y \|_{c_1} \|$ and the triangle inequality follows. The other axioms for the norm are trivially satisfied.

Remark 28 *If $\| \, . \, \|$ is monotonic on E then we call the above normed space $(X, \| \| \, . \, \|_{c_1} \|)$, the normed space associated to the cone normed space $(X, \| \, . \, \|_{c_1})$*

Definition 29 *A map $T : (X, \| \, . \, \|_{c_1}) \to (Y, \| \, . \, \|_{c_2})$ is called continuous at $x_0 \in X$, if for all $q \in E$ with $q >> 0$ there exists $p \in E$, $p >> 0$ such that for all $x \in X$, $\| x - x_0 \|_{c_1} < p$ implies $\| Tx - Tx_0 \|_{c_2} < q$. T is called continuous, if it is continuous at each $x \in X$.*

Theorem 30 *If $T : (X, \| \, . \, \|_{c_1}) \to (Y, \| \, . \, \|_{c_2})$ is a bounded linear operator then it is continuous.*

PROOF. Assume T is bounded and let $x_0 \in X$ then there exists $M > 0$ such that $\| Tx \|_{c_2} \leq M \| x \|_{c_1}$ for all $x \in X$. Now for $q >> 0$, choose $p = \frac{1}{M} q$. Then, $x \in X$, $\| x - x_0 \|_{c_1} < p$ implies $\| Tx - Tx_0 \|_{c_2} = \| T(x - x_0) \|_{c_2} \leq M \| x - x_0 \|_{c_1} < M \frac{q}{M} = q$.

Linear operators on finite dimensional normed linear spaces are necessary bounded. However, in general, it is not the case for cone normed spaces. Also, the converse of the above theorem may not be true.

Example 31 *Let $E = R^2$ and $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0\}$ and $X = E$. Define on E the cone norm $\| (x, y) \|_c = (| x |, | y |)$. Then ,*

(a) the linear operator $T : (E, \| \, . \, \|_c) \to (E, \| \, . \, \|_c)$ define by $T(x, y) = (x - y, y - x)$ is not bounded. In*deed, for any $m > 0$, $m \neq 1$, let $x_m = \left(\frac{1}{m}, m \right)$, then for $m > 1$, we have $\| Tx_m \|_c = \| \left(m - \frac{1}{m}, \frac{1}{m} - m \right) \|_c =$*

$\left(m - \frac{1}{m}, m - \frac{1}{m}\right)$ *is not comparable to* $m. \parallel (\frac{1}{m}, m) \parallel_c = (1, m^2)$, and for $0 < m \leq 1$, let x=$(1,-1)$ *then* $\parallel Tx \parallel_c = \parallel (2,-2) \parallel_c = (2,2) \geq m. \parallel (1,-1) \parallel_c = (m,m)$.

(b) The linear operator T in (a) is continuous. Since cone metric spaces are first countable [2], in order to prove continuity of T, it will be enough to show that it is sequentially continuous. To this end assume that $z_n = (x_n, y_n)$ is a sequence in E such that $z_n \to z = (x,y)$. Then, by Lemma (1) in [4], $\lim\limits_{n\to\infty} \parallel d(z_n, z) \parallel = \lim\limits_{n\to\infty} \parallel \parallel z_n - z \parallel_c \parallel = \lim\limits_{n\to\infty} \parallel (\mid x_n - x \mid, \mid y_n - y \mid) \parallel = 0$. *Equivalently, if* $\mid x_n - x \mid \to 0$ *and* $\mid y_n - y \mid \to 0$. *But then,* $0 \leq \parallel Tz_n - Tz \parallel_c = \parallel (x_n - y_n, y_n - x_n) - (x - y, y - x) \parallel_c = \parallel (x_n - y_n - x + y, y_n - x_n - y + x) \parallel_c = (\mid (x_n - y_n - x + y \mid, \mid y_n - x_n - y + x \mid) \leq$

$(\mid x_n - x \mid + \mid y - y_n \mid, \mid y_n - y \mid + \mid x - x_n \mid)$ *implies,*

$0 \leq \parallel \parallel Tz_n - Tz \parallel_c \parallel \leq \sqrt{(\mid x_n - x \mid + \mid y_n - y \mid)^2 + (\mid y_n - y \mid + \mid x - x_n \mid)^2} \to 0$, *hence* $\parallel \parallel Tz_n - Tz \parallel_c \parallel \to 0$, *that is* $Tz_n \to Tz$.

(c) It can be easily shown that any linear operator on E represented by a diagonal matrix is bounded. For example, if T is defined by $T(x,y) = (\alpha x, \beta y)$, $\alpha, \beta \in R$, then $\parallel T(x,y) \parallel_c \leq \max\{\mid \alpha \mid, \mid \beta \mid\} . \parallel (x,y) \parallel_c$.

Theorem 32 *If the norm of E is monotonic. Then any linear operator on a finite dimensional cone normed space is scalar bounded.*

PROOF. Let $(X, \parallel . \parallel_{c_1})$ be a finite dimensional cone normed space over a cone P in E with a monotonic norm $\parallel . \parallel$, and $(Y, \parallel . \parallel_{c_2})$ an arbitrary cone normed space over the same cone. Then any linear operator T from X into Y is bounded if treated as a linear operator between the normed linear spaces $(X, \parallel \parallel . \parallel_{c_1} \parallel)$, $(Y, \parallel \parallel . \parallel_{c_2} \parallel)$. But bounded linear operators between these two normed spaces are exactly scalar bounded operators from $(X, \parallel . \parallel_{c_1})$ into $(Y, \parallel . \parallel_{c_2})$.

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Stability of a mixed type additive and quadratic functional equation in random normed spaces

M. Eshaghi Gordji[1], M. Bavand Savadkouhi[2] and J. M. Rassias[3]

[1,2] **Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran;**
Center of Excellence in Nonlinear Analysis and Applications, Semnan University, Semnan, Iran
[3] **Section of Mathematics and Informatics, Pedagogical Department, National and Capodistrian University of Athens, 4, Agamemnonos St., Aghia Paraskevi, Athens 15342, Greece**
e-mail: madjid.eshaghi@gmail.com, bavand.m@gmail.com, jrassias@primedu.uoa.gr

Abstract. In this paper, we obtain the general solution and the stability result for the following functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+2f(3x)-2f(x)$$

in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms.

1. Introduction

The stability problem of functional equations originated from the question of Ulam [59] in 1940, concerning the stability of group homomorphisms. In 1941, D. H. Hyers [38] gave the first affirmative answer to the question of Ulam for Banach spaces. In 1978, Th. M. Rassias [53] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. (see [8]–[33] and [49]–[52]).

This new concept is known as generalized Hyers-Ulam stability of functional equations (see [1]-[4],[28, 29, 37],[39]-[46] and [48, 54, 57]). The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to symmetric bi-additive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all x (see [1, 43]). The bi-additive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)) \tag{1.2}$$

Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f:A\to B$, where A is normed space and B Banach space (see [58]). Cholewa [5] noticed that the Theorem of Skof is still true if relevant domain A is replaced by an abelian group. In the paper [7], Czerwik proved the Hyers-Ulam-Rassias stability of the equation (1.1). Grabiec [34] has generalized the above mentioned result.

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A. Najati and M. B. Moghimi in [47] introduced the following functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x)$$

with f(0)=0. It is easy to see that the mapping $f(x)=ax^2+bx+c$ is a solution of the functional equation. They established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation whenever f is a mapping between two quasi Banach spaces.

The aim of this paper is to investigate the stability of the additive–quadratic functional equation in random normed spaces (in the sense of Sherstnev), under arbitrary continuous t-norms.

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [6, 55, 56]. Throughout this paper, Δ^+ is the space of distribution maps; that is, the space of all mappings $F:\mathbb{R}\cup\{-\infty,\infty\}\to[0,1]$, such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0)=0$ and $F(+\infty)=1$. D^+ is a subset of Δ^+ consisting of all functions $F\in\Delta^+$ for which $l^-F(+\infty)=1$, where $l^-f(x)$ denotes the left limit of the function f at the point x; that is, $l^-f(x)=\lim_{t\to x^-}f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F\le G$ if and only if $F(t)\le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t)=\begin{cases}0, & \text{if } t\le 0,\\ 1, & \text{if } t>0.\end{cases}$$

Definition 1.1. *([55]). A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*
(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1)=a$ for all $a\in[0,1]$;
(d) $T(a,b)\le T(c,d)$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous t-norms are $T_P(a,b)=ab$, $T_M(a,b)=\min(a,b)$ and $T_L(a,b)=\max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [35], [36]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i=x_1$ and $T_{i=1}^n x_i=T(T_{i=1}^{n-1}x_i,x_n)$ for $n\ge 2$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. It is known ([36]) that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty}(T_L)_{i=1}^\infty x_{n+i}=1\iff\sum_{n=1}^\infty(1-x_n)<\infty. \tag{1.3}$$

Definition 1.2. *([56]). A* random normed space *(briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold:*
(RN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;
(RN2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for all $x\in X$, $\alpha\neq 0$;
(RN3) $\mu_{x+y}(t+s)\ge T(\mu_x(t),\mu_y(s))$ for all $x,y\in X$ and $t,s\ge 0$.

Every normed space $(X,\|.\|)$ defines a random normed space (X,μ,T_M) where

$$\mu_x(t)=\frac{t}{t+\|x\|},$$

for all $t>0$, and T_M is the minimum t-norm. This space is called induced random normed space.

Definition 1.3. *Let (X, μ, T) be an RN-space.*
(1) *A sequence $\{x_n\}$ in X is said to be* convergent *to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.*
(2) *A sequence $\{x_n\}$ in X is called* Cauchy sequence *if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.*
(3) *An RN-space (X, μ, T) is said to be* complete *if and only if every Cauchy sequence in X is convergent to a point in X.*

Theorem 1.4. *([55]). If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

In this paper we deal with the following functional equation:

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 2f(3x) - 2f(x) \tag{1.4}$$

on random normed spaces. It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (1.4). In section 2 we investigate the general solution of functional equation (1.4) when f is a mapping between vector spaces, and in section 3 we establish the stability of the functional equation (1.4) in RN-spaces.

2. General solution

We need the following two lemmas for the general solution of (1.4). Throughout this section X and Y are vector space.

Lemma 2.1. *If an even function $f : X \longrightarrow Y$ with $f(0) = 0$ satisfies (1.4) for all $x, y \in X$, then f is quadratic.*

Proof. Replacing y by $x + y$ in (1.4), by evenness of f, we obtain

$$f(4x + y) + f(2x - y) = f(2x + y) + f(y) + 2f(3x) - 2f(x) \tag{2.1}$$

for all $x, y \in X$. If we Replace y by $-y$ in (2.1), we get by evenness of f,

$$f(4x - y) + f(2x + y) = f(2x - y) + f(y) + 2f(3x) - 2f(x) \tag{2.2}$$

for all $x, y \in X$. If we add (2.1) to (2.2), we have

$$f(4x + y) + f(4x - y) - 2f(y) = 4f(3x) - 4f(x) \tag{2.3}$$

for all $x, y \in X$. Letting $y = 0$ in (2.3), we get

$$f(4x) = 2f(3x) - 2f(x) \tag{2.4}$$

for all $x \in X$. Once again letting $y = 2x$ in (2.3), we get

$$f(6x) = 4f(3x) + f(2x) - 4f(x) \tag{2.5}$$

for all $x \in X$. It follows from (2.4) and (2.5) that

$$f(6x) = 2f(4x) + f(2x) \tag{2.6}$$

for all $x \in X$. Once again letting $y = 4x$ in (2.3), we get

$$f(8x) = 2f(4x) + 4f(3x) - 4f(x) \tag{2.7}$$

for all $x \in x$. It follows from (2.4) and (2.7) that

$$f(8x) = 4f(4x). \tag{2.8}$$

If we replace x by $\frac{x}{4}$ in (2.8), we get that

$$f(2x) = 4f(x) \tag{2.9}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.6), we obtain

$$f(3x) = 2f(2x) + f(x) \tag{2.10}$$

for all $x \in X$. It follows from (2.9) and (2.10) that

$$f(3x) = 9f(x). \tag{2.11}$$

Replacing y by $3y$ in (1.4), we get

$$f(3x+3y) + f(3x-3y) = f(x+3y) + f(x-3y) + 2f(3x) - 2f(x) \tag{2.12}$$

for all $x, y \in X$. By (2.12) and using (1.4) and (2.11), we get

$$\begin{aligned} 9f(x+y) + 9f(x-y) = f(x+y) + f(x-y) + 2f(3y) \\ - 2f(y) + 2f(3x) - 2f(x) \end{aligned}$$

for all $x, y \in X$. Hence the function $f : X \to Y$ is quadratic. □

Corollary 2.2. *If an even function $f : X \to Y$ satisfies* (1.4) *for all $x, y \in X$, then mapping $g : X \to Y$ defined by $g(x) := f(x) - f(0)$ is quadratic.*

Lemma 2.3. *If an odd function $f : X \to Y$ satisfies* (1.4) *for all $x, y \in X$, then f is additive.*

Proof. By letting $y = x$ in (1.4), we get

$$f(4x) = 2f(3x) - 2f(x) \tag{2.13}$$

for all $x \in X$. If we let $y = 3x$ in (1.4), we get by the oddness of f,

$$f(6x) = 2f(3x) + f(4x) - 2f(x) - f(2x) \tag{2.14}$$

for all $x \in X$. It follows from (2.13) and (2.14) that

$$f(6x) = 2f(4x) - f(2x) \tag{2.15}$$

for all $x \in X$. Once again, by letting $y = 5x$ in (1.4), we get by the oddness of f,

$$f(8x) - f(2x) = f(6x) - f(4x) + 2f(3x) - 2f(x) \tag{2.16}$$

for all $x \in X$. By (2.13) and using (2.15) and (2.16), we get

$$f(8x) = 2f(4x) \tag{2.17}$$

for all $x \in X$. If we replace x by $\frac{x}{4}$ in (2.17), we obtain

$$f(2x) = 2f(x) \tag{2.18}$$

for all $x \in X$. Now, in (2.15), replacing x by $\frac{x}{2}$, we have

$$f(3x) = 2f(2x) - f(x) \tag{2.19}$$

for all $x \in X$. It follows from (2.18) and (2.19) that

$$f(3x) = 3f(x) \tag{2.20}$$

for all $x \in X$. By (1.4) we conclude that

$$f(3x+y) + f(3x-y) = f(x+y) + f(x-y) + 4f(x) \tag{2.21}$$

for all $x, y \in X$. Replacing x in (2.21) by $\frac{x}{3}$ and multiplying both sides of (2.21) by 2 we obtain

$$f(x+3y) + f(x-3y) = 3f(x+y) + 3f(x-y) - 4f(x) \tag{2.22}$$

for all $x, y \in X$. Replacing x and y by y and x in (2.21), respectively, we get

$$f(x+3y) - f(x-3y) = f(x+y) - f(x-y) + 4f(x) \tag{2.23}$$

for all $x, y \in X$. By adding (2.22) to (2.23), we obtain

$$f(x+3y) = 2f(x+y) + f(x-y) - 2f(x) + 2f(y) \tag{2.24}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.24)

$$f(x-3y) = 2f(x-y) + f(x+y) - 2f(x) + 2f(y) \tag{2.25}$$

for all $x, y \in X$. Once again, if we replace x in (2.24) by $x-2y$, we get that

$$f(x+y) = 2f(x-y) + f(x-3y) - 2f(x-2y) + 2f(y) \tag{2.26}$$

for all $x, y \in X$. It follows from (2.25) and (2.26),

$$f(x-2y) = 2f(x-y) - f(x) \tag{2.27}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.27), we lead to

$$f(x+2y) = 2f(x+y) - f(x) \tag{2.28}$$

for all $x, y \in X$. If we replace y by $x+y$ in (2.27), we get that

$$f(x+2y) = f(x) + 2f(y) \tag{2.29}$$

for all $x, y \in X$. It follows from (2.28) and (2.29),

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Hence mapping $f : X \to Y$ is additive. □

Theorem 2.4. *A function $f : X \to Y$ satisfies* (1.4) *for all $x, y \in X$ if and only if there exists a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$, such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.*

Proof. If there exists a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$, such that $f(x) = B(x, x) + A(x)$ for all $x \in X$. It is clear that function $f : X \to Y$ satisfies (1.4).
Conversely, let f satisfies (1.4). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)),$$

for all $x \in X$. By (1.4), we have

$$\begin{aligned}
f_e(3x+y) + f_e(3x-y) &= \frac{1}{2}[f(3x+y) + f(-3x-y) + f(3x-y) + f(-3x+y)] \\
&= \frac{1}{2}[f(3x+y) + f(3x-y)] + \frac{1}{2}[f(-3x+(-y)) + f(-3x-(-y))] \\
&= \frac{1}{2}[f(x+y) + f(x-y) + 2f(3x) - 2f(x)] \\
&+ \frac{1}{2}[f(-x-y) + f(-x-(-y)) + 2f(-3x) - 2f(-x)] \\
&= \frac{1}{2}(f(x+y) + f(-x-y)) + \frac{1}{2}(f(x-y) + f(x-(-y))) \\
&+ 2[\frac{1}{2}(f(3x) + f(-3x))] - 2[\frac{1}{2}(f(x) + f(-x))] \\
&= f_e(x+y) + f_e(x-y) + 2f_e(3x) - 2f_e(x),
\end{aligned}$$

for all $x, y \in X$. This means that f_e satisfies (1.4). Similarly we can show that f_o satisfies (1.4). By Corollary 2.2 and Lemma 2.3, we achive that the function $f_e - f(0)$ and f_o are quadratic and additive respectively. Therefore there exists a symmetric bi-additive function

$B : X\times X \to Y$ such that $f_e(x) = B(x,x)+f(0)$ for all $x \in X$. So $f(x) = B(x,x)+A(x)+f(0)$ for all $x \in X$, where $A(x) = f_o(x)$ for all $x \in X$. □

3. Stability

Throughout this section X will be a real linear space and (Y, μ, T) will be a complete RN-space.

Theorem 3.1. *Let $f : X \to Y$ be an even function with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{3.1}$$

for all $x,y \in X$ and all $t > 0$. If

$$\lim_{n\to\infty} T_{i=1}^{\infty}(2\rho_{\frac{2^{n+i-1}}{4}x,\frac{2^{n+i-1}}{4}x}(2^{2n+i}t) + \rho_{\frac{2^{n+i-1}}{4}x,\frac{5.2^{n+i-1}}{4}x}(2^{2n+i}t) + \rho_{\frac{2^{n+i-1}}{4}x,\frac{-3.2^{n+i-1}}{4}x}(2^{2n+i}t)) = 1 \tag{3.2}$$

and

$$\lim_{n\to\infty} \rho_{2^n x,2^n y}(2^{2n}t) = 1 \tag{3.3}$$

for all $x,y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{Q(x)-f(x)}(t) \geq T_{i=1}^{\infty}(2\rho_{\frac{2^{i-1}}{4}x,\frac{2^{i-1}}{4}x}(2^it)+\rho_{\frac{2^{i-1}}{4}x,\frac{5.2^{i-1}}{4}x}(2^it)+\rho_{\frac{2^{i-1}}{4}x,\frac{-3.2^{i-1}}{4}x}(2^{2i}t)), \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. By replacing y by $x + y$ in (3.1), we get

$$\mu_{f(4x+y)+f(2x-y)-f(2x+y)-f(y)-2f(3x)+2f(x)}(t) \geq \rho_{x,x+y}(t) \tag{3.5}$$

for all $x,y \in X$. If we Replace y by $-y$ in (3.5), we get

$$\mu_{f(4x-y)+f(2x+y)-f(2x-y)-f(y)-2f(3x)+2f(x)}(t) \geq \rho_{x,x-y}(t) \tag{3.6}$$

for all $x,y \in X$. If we add (3.5) to (3.6), we have

$$\mu_{f(4x+y)+f(4x-y)-2f(y)-4f(3x)+4f(x)}(t) \geq \rho_{x,x+y}(t) + \rho_{x,x-y}(t). \tag{3.7}$$

Letting $y = 0$ in (3.7), we get the inequality

$$\mu_{2f(4x)-4f(3x)+4f(x)}(t) \geq 2\rho_{x,x}(t) \tag{3.8}$$

for all $x \in X$. Once again by letting $y = 4x$ in (3.7), we get the inequality

$$\mu_{f(8x)-2f(4x)-4f(3x)+4f(x)}(t) \geq \rho_{x,5x}(t) + \rho_{x,-3x}(t) \tag{3.9}$$

for all $x \in X$. It follows from (3.8) and (3.9) that

$$\mu_{f(8x)-4f(4x)}(t) \geq 2\rho_{x,x}(t) + \rho_{x,5x}(t) + \rho_{x,-3x}(t) \tag{3.10}$$

for all $x \in X$. If we replace x by $\frac{x}{4}$ in (3.10), we obtain

$$\mu_{f(2x)-4f(x)}(t) \geq 2\rho_{\frac{x}{4},\frac{x}{4}}(t) + \rho_{\frac{x}{4},\frac{5x}{4}}(t) + \rho_{\frac{x}{4},\frac{-3x}{4}}(t) \tag{3.11}$$

for all $x \in X$. Letting

$$\psi_{x,x}(t) = 2\rho_{\frac{x}{4},\frac{x}{4}}(t) + \rho_{\frac{x}{4},\frac{5x}{4}}(t) + \rho_{\frac{x}{4},\frac{-3x}{4}}(t) \tag{3.12}$$

for all $x \in X$, then we get

$$\mu_{f(2x)-4f(x)}(t) \geq \psi_{x,x}(t) \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{f(2x)}{2^2} - f(x)}(t) \geq \psi_{x,x}(2^2 t) \tag{3.14}$$

for all $x \in X$ and all $t > 0$. Hence,

$$\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}} - \frac{f(2^k x)}{2^{2k}}}(t) \geq \psi_{2^k x, 2^k x}(2^{2(k+1)} t) \tag{3.15}$$

for all $x \in X$ and all $k \in \mathbb{N}$. This means that

$$\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}} - \frac{f(2^k x)}{2^{2k}}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^k x, 2^k x}(2^{k+1} t) \tag{3.16}$$

for all $x \in X$, $t > 0$ and all $k \in \mathbb{N}$. As $1 > \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}$, by the triangle inequality it follows

$$\begin{aligned}\mu_{\frac{f(2^n x)}{2^{2n}} - f(x)}(t) &\geq T_{k=1}^{n-1}\left(\mu_{\frac{f(2^{k+1}x)}{2^{2(k+1)}} - \frac{f(2^k x)}{2^{2k}}}\left(\frac{t}{2^{k+1}}\right)\right) \geq T_{k=1}^{n-1}(\psi_{2^k x, 2^k x}(2^{k+1} t)) \\ &= T_{i=1}^{n}(\psi_{2^{i-1}x, 2^{i-1}x}(2^i t))\end{aligned} \tag{3.17}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$, we replace x with $2^m x$ in (3.17) to find that

$$\mu_{\frac{f(2^{n+m}x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}}(t) \geq T_{i=1}^{n}(\psi_{2^{i+m-1}x, 2^{i+m-1}x}(2^{i+2m} t)). \tag{3.18}$$

Since the right hand side of the inequality (3.18) tends to 1 as m and n tend to infinity, sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is a Cauchy sequence. Therefore, we may define $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}$ for all $x \in X$. Now, we show that Q is a quadratic map. Replacing x, y with $2^n x$ and $2^n y$ respectively in (3.1), it follows that

$$\mu_{\frac{f(3.2^n x + 2^n y)}{2^{2n}} + \frac{f(3.2^n x - 2^n y)}{2^{2n}} - \frac{f(2^n x + 2^n y)}{2^{2n}} - \frac{f(2^n x - 2^n y)}{2^{2n}} - 2\frac{f(3.2^n x)}{2^{2n}} + 2\frac{f(2^n x)}{2^{2n}}}(t) \geq \rho_{2^n x, 2^n y}(2^{2n} t). \tag{3.19}$$

Taking limit as $n \to \infty$, we find that Q satisfies (1.4) for all $x, y \in X$. Therefore by Lemma 2.1 we get that mapping $Q : X \to Y$ is quadratic.
To prove (3.4), take limit as $n \to \infty$ in (3.17) and by (3.12). Finally, to prove the uniqueness of the quadratic function Q subject to (3.4), let us assume that there exists a quadratic function Q' which satisfies (3.4). Since $Q(2^n x) = 2^{2n} Q(x)$ and $Q'(2^n x) = 2^{2n} Q'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (3.4) it follows that

$$\begin{aligned}\mu_{Q(x)-Q'(x)}(2t) &= \mu_{Q(2^n x) - Q'(2^n x)}(2^{2n+1} t) \\ &\geq T(\mu_{Q(2^n x) - f(2^n x)}(2^{2n} t), \mu_{f(2^n x) - Q'(2^n x)}(2^{2n} t)) \\ &\geq T(T_{i=1}^{\infty}(2\rho_{\frac{2^{i+n-1}}{4}x, \frac{2^{i+n-1}}{4}x}(2^{2n+i} t) + \rho_{\frac{2^{i+n-1}}{4}x, \frac{5.2^{i+n-1}}{4}x}(2^{2n+i} t) \\ &+ \rho_{\frac{2^{i+n-1}}{4}x, \frac{-3.2^{i+n-1}}{4}x}(2^{2n+i} t)), T_{i=1}^{\infty}(2\rho_{\frac{2^{i+n-1}}{4}x, \frac{2^{i+n-1}}{4}x}(2^{2n+i} t) \\ &+ \rho_{\frac{2^{i+n-1}}{4}x, \frac{5.2^{i+n-1}}{4}x}(2^{2n+i} t) + \rho_{\frac{2^{i+n-1}}{4}x, \frac{-3.2^{i+n-1}}{4}x}(2^{2n+i} t)))\end{aligned} \tag{3.20}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (3.20), we find that $Q = Q'$. □

Theorem 3.2. *Let X be a real linear space, (Y, μ, T) be a complete RN-space and $f : X \to Y$ be an odd function which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{3.21}$$

for all $x, y \in X$ and all $t > 0$. If

$$\lim_{n\to\infty} T_{i=1}^{\infty}(2\rho_{\frac{2^{n+i-1}}{4}x,\frac{2^{n+i-1}}{4}x}(2^{n+i}t) + \rho_{\frac{2^{n+i-1}}{4}x,\frac{3.2^{n+i-1}}{4}x}(2^{n+i}t) + \rho_{\frac{2^{n+i-1}}{4}x,\frac{5.2^{n+i-1}}{4}x}(2^{n+i}t)) = 1 \tag{3.22}$$

and

$$\lim_{n\to\infty} \rho_{2^n x,2^n y}(2^n t) = 1 \tag{3.23}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{A(x)-f(x)}(t) \geq T_{i=1}^{\infty}(2\rho_{\frac{2^{i-1}}{4}x,\frac{2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x,\frac{3.2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x,\frac{5.2^{i-1}}{4}x}(2^i t)), \tag{3.24}$$

for all $x \in X$ and all $t > 0$.

Proof. By letting $y = x$ in (3.21), we get

$$\mu_{f(4x)-2f(3x)+2f(x)}(t) \geq \rho_{x,x}(t) \tag{3.25}$$

for all $x \in X$. If we let $y = 3x$ in (3.21), we get by the oddness of f,

$$\mu_{f(6x)-2f(3x)-f(4x)+2f(x)+f(2x)}(t) \geq \rho_{x,3x}(t) \tag{3.26}$$

for all $x \in X$. It follows from (3.25) and (3.26) that

$$\mu_{f(6x)-2f(4x)+f(2x)}(t) \geq \rho_{x,x}(t) + \rho_{x,3x}(t) \tag{3.27}$$

for all $x \in X$. Once again, by letting $y = 5x$ in (3.21), we get by the oddness of f,

$$\mu_{f(8x)-f(2x)-f(6x)+f(4x)-2f(3x)+2f(x)}(t) \geq \rho_{x,5x}(t) \tag{3.28}$$

for all $x \in X$. By (3.25) and using (3.27) and (3.28), we get

$$\mu_{f(8x)-2f(4x)}(t) \geq 2\rho_{x,x}(t) + \rho_{x,3x}(t) + \rho_{x,5x}(t) \tag{3.29}$$

for all $x \in X$. If we replace x by $\frac{x}{4}$ in (3.29), we get that

$$\mu_{f(2x)-2f(x)}(t) \geq 2\rho_{\frac{x}{4},\frac{x}{4}}(t) + \rho_{\frac{x}{4},\frac{3x}{4}}(t) + \rho_{\frac{x}{4},\frac{5x}{4}}(t) \tag{3.30}$$

for all $x \in X$. Let

$$\phi_{x,x}(t) = 2\rho_{\frac{x}{4},\frac{x}{4}}(t) + \rho_{\frac{x}{4},\frac{3x}{4}}(t) + \rho_{\frac{x}{4},\frac{5x}{4}}(t) \tag{3.31}$$

for all $x \in X$ and all $t > 0$, then we get

$$\mu_{f(2x)-2f(x)}(t) \geq \phi_{x,x}(t) \tag{3.32}$$

for all $x \in X$ and $t > 0$. Thus we have

$$\mu_{\frac{f(2x)}{2}-f(x)}(t) \geq \phi_{x,x}(2t) \tag{3.33}$$

for all $x \in X$. Therefore,

$$\mu_{\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}}(t) \geq \phi_{2^k x,2^k x}(2^{k+1}t) \tag{3.34}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Hence

$$\mu_{\frac{f(2^n x)}{2^n}-f(x)}(t) \geq T_{k=1}^{n-1}(\mu_{\frac{f(2^{k+1}x)}{2^{k+1}}-\frac{f(2^k x)}{2^k}}(t)) \geq T_{k=1}^{n-1}(\phi_{2^k x,2^k x}(2^{k+1}t)) = T_{i=1}^{n}(\phi_{2^{i-1}x,2^{i-1}x}(2^i t)) \tag{3.35}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(2^n x)}{2^n}\}$, we replace x with $2^m x$ in (3.35) to find that

$$\mu_{\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^m x)}{2^m}}(t) \geq T_{i=1}^{n}(\phi_{2^{i+m-1}x,2^{i+m-1}x}(2^{i+m}t)). \tag{3.36}$$

Since the right hand side of (3.36) tends to 1 as m and n tend to infinity, sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence. Therefore, we may define $A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in X$. Now, we show that A is a additive map. Replacing x, y with $2^n x$ and $2^n y$ respectively in (3.21), it follows that

$$\mu_{\frac{f(3.2^n x+2^n y)}{2^n}+\frac{f(3.2^n x-2^n y)}{2^n}-\frac{f(2^n x+2^n y)}{2^n}-\frac{f(2^n x-2^n y)}{2^n}-2\frac{f(3.2^n x)}{2^n}+2\frac{f(2^n x)}{2^n}}(t) \geq \rho_{2^n x, 2^n y}(2^n t). \tag{3.37}$$

Taking limit as $n \to \infty$, we find that A satisfies (1.4) for all $x, y \in X$. Therefore by Lemma 2.3 we get that mapping $A : X \to Y$ is additive.
To prove (3.24), take limit as $n \to \infty$ in (3.35) and by (3.31). Finally, to prove the uniqueness of the additive function A subject to (3.24), let us assume that there exists an additive function A' which satisfies (3.24). Since $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A'(x)$ for all $x \in X$ and $n \in \mathbb{N}$, from (3.24) it follows that

$$\begin{aligned}
\mu_{A(x)-A'(x)}(2t) &= \mu_{A(2^n x)-A'(2^n x)}(2^{n+1} t) \\
&\geq T(\mu_{A(2^n x)-f(2^n x)}(2^n t), \mu_{f(2^n x)-A'(2^n x)}(2^n t)) \\
&\geq T(T_{i=1}^{\infty}(2\rho_{\frac{2^{i+n-1}}{4}x, \frac{2^{i+n-1}}{4}x}(2^{i+n} t) + \rho_{\frac{2^{i+n-1}}{4}x, \frac{3.2^{i+n-1}}{4}x}(2^{i+n} t) \\
&+ \rho_{\frac{2^{i+n-1}}{4}x, \frac{5.2^{i+n-1}}{4}x}(2^{i+n} t)), T_{i=1}^{\infty}(2\rho_{\frac{2^{i+n-1}}{4}x, \frac{2^{i+n-1}}{4}x}(2^{i+n} t) \\
&+ \rho_{\frac{2^{i+n-1}}{4}x, \frac{3.2^{i+n-1}}{4}x}(2^{i+n} t) + \rho_{\frac{2^{i+n-1}}{4}x, \frac{5.2^{i+n-1}}{4}x}(2^{i+n} t)))
\end{aligned} \tag{3.38}$$

for all $x \in X$ and all $t > 0$. By letting $n \to \infty$ in (3.38), we find that $A = A'$. □

Theorem 3.3. *Let $f : X \to Y$ be a function with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-2f(3x)+2f(x)}(t) \geq \rho_{x,y}(t) \tag{3.39}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}(2\rho_{\frac{2^{n+i-1}}{4}x, \frac{2^{n+i-1}}{4}x}(2^{n+i} t) + \rho_{\frac{2^{n+i-1}}{4}x, \frac{3.2^{n+i-1}}{4}x}(2^{n+i} t) \\
&+ \rho_{\frac{2^{n+i-1}}{4}x, \frac{5.2^{n+i-1}}{4}x}(2^{n+i} t) + 2\rho_{\frac{-2^{n+i-1}}{4}x, \frac{-2^{n+i-1}}{4}x}(2^{n+i} t) + \rho_{\frac{-2^{n+i-1}}{4}x, \frac{-3.2^{n+i-1}}{4}x}(2^{n+i} t) \\
&+ \rho_{\frac{-2^{n+i-1}}{4}x, \frac{-5.2^{n+i-1}}{4}x}(2^{n+i} t)) = 1 = \lim_{n\to\infty} T_{i=1}^{\infty}(2\rho_{\frac{2^{n+i-1}}{4}x, \frac{2^{n+i-1}}{4}x}(2^{2n+i} t) \\
&+ \rho_{\frac{2^{n+i-1}}{4}x, \frac{5.2^{n+i-1}}{4}x}(2^{2n+i} t) + \rho_{\frac{2^{n+i-1}}{4}x, \frac{-3.2^{n+i-1}}{4}x}(2^{2n+i} t) + 2\rho_{\frac{-2^{n+i-1}}{4}x, \frac{-2^{n+i-1}}{4}x}(2^{2n+i} t) \\
&+ \rho_{\frac{-2^{n+i-1}}{4}x, \frac{-5.2^{n+i-1}}{4}x}(2^{2n+i} t) + \rho_{\frac{-2^{n+i-1}}{4}x, \frac{3.2^{n+i-1}}{4}x}(2^{2n+i} t))
\end{aligned} \tag{3.40}$$

and

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^{2n} t) = 1 = \lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^n t) \tag{3.41}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned}
\mu_{Q(x)-A(x)-f(x)}(t) &\geq T_{i=1}^{\infty}(2\rho_{\frac{2^{i-1}}{4}x, \frac{2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x, \frac{3.2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x, \frac{5.2^{i-1}}{4}x}(2^i t) \\
&+ 2\rho_{\frac{-2^{i-1}}{4}x, \frac{-2^{i-1}}{4}x}(2^i t) + \rho_{\frac{-2^{i-1}}{4}x, \frac{-3.2^{i-1}}{4}x}(2^i t) + \rho_{\frac{-2^{i-1}}{4}x, \frac{-5.2^{i-1}}{4}x}(2^i t)) \\
&+ T_{i=1}^{\infty}(2\rho_{\frac{2^{i-1}}{4}x, \frac{2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x, \frac{5.2^{i-1}}{4}x}(2^i t) + \rho_{\frac{2^{i-1}}{4}x, \frac{-3.2^{i-1}}{4}x}(2^i t) \\
&+ 2\rho_{\frac{-2^{i-1}}{4}x, \frac{-2^{i-1}}{4}x}(2^i t) + \rho_{\frac{-2^{i-1}}{4}x, \frac{-5.2^{i-1}}{4}x}(2^i t) + \rho_{\frac{-2^{i-1}}{4}x, \frac{3.2^{i-1}}{4}x}(2^i t)),
\end{aligned} \tag{3.42}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Let

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

for all $x \in X$. Then $f_e(0) = 0$, $f_e(-x) = f_e(x)$, and

$$\begin{aligned} \mu_{f_e(3x+y)+f_e(3x-y)-f_e(x+y)-f_e(x-y)-2f_e(3x)+2f_e(x)}(t) &\geq \rho_{x,y}(2t) + \rho_{-x,-y}(2t) \\ &\geq \rho_{x,y}(t) + \rho_{-x,-y}(t) \end{aligned} \tag{3.43}$$

for all $x, y \in X$. Hence, in view of Theorem 3.1, there exists a unique quadratic function $Q : X \to Y$ satisfying (3.4). Let

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and

$$\begin{aligned} \mu_{f_o(3x+y)+f_o(3x-y)-f_o(x+y)-f_o(x-y)-2f_o(3x)+2f_o(x)}(t) &\geq \rho_{x,y}(2t) + \rho_{-x,-y}(2t) \\ &\geq \rho_{x,y}(t) + \rho_{-x,-y}(t) \end{aligned} \tag{3.44}$$

for all $x, y \in X$. From Theorem 3.2, it follows that there exist a unique additive mapping $A : X \to Y$ satisfying (3.24). Now it is obvious that (3.42) holds true for all $x \in A$, and the proof of Theorem is complete. □

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Monotonic Random Walk on a One-Dimensional Lattice

Alexander P. Buslaev
Dept. of Higher Mathematics, State Technical University MADI
64, Leningradskiy prosp., Moscow, Russia, 125319
E-mail: busl@math.madi.ru.

Alexander G. Tatashev
Dept. of Higher Mathematics, State Technical University MADI
64, Leningradskiy prosp., Moscow, Russia, 125319
E-mail: a-tatashev@rambler.ru.

Abstract

Monotonic random walk of particles on a one-dimensional lattice is considered. This model is treated as the limit case of the model in which the particles move on a ring. In this limit case the ring is great. The average velocity of the particles has been found.

Key words: stochastic models, random walk, traffic, average velocity, Markov chains.

1. Introduction

Models of random walk on a lattice are used for the study of the road traffic.

The appearance of cell automata models in the problem of traffic flows is associated with works of Nagel and al. [11, 12], which were published in mid-nineties. In [11, 12] the dependence of the average velocity and intensity of traffic flow on the model parameters was studied. The following factors, which contributed to the activation of this approach, can be noted

1) desire to explain the discrepancy between the solutions of traffic equations and experimental observations, which exhibited chaotic behavior in the so-called regime of instability;

2) desire to create models discrete in time and space (or only on one coordinate), which would be independent on continuum models and could take into account the individual behavior of *"particles with motivated behavior"*.

It has been noted that the scheme considered in [11, 12] is similar to monotonic random walks on a lattice. This theme has its own history. In particular, the works of Soviet mathematician Yu.K. Belyaev and his students [1, 13] are devoted to traffic flows in the underground and contain exact results for one-dimensional random walk (not only monotonic walk).

As the Russian mathematician M. Blank noted, in subsequent development of the model of cellular automata, the Western science has been satisfied by numerical results only, which is apparently due to the relative availability of computational tools. M. Blank gave well-defined statements and found exact results in a number of important cases, [2, 3].

Here we are talking about exact estimates, while simulation results such as the statement of the decisive influence of the presence of heavy trucks on flow work certainly exist.

In [8] a model of movement of particles (vehicles) on a multi-lane road location was considered. In this model the velocity of movement is the sum of determinate and stochastic components. In this model the determinate component of movement corresponds to the background movement on lane and the stochastic component corresponds to individual manoeuvres of particles. Each lane is a sequence of cells. The dimension of the cell is determined by the *dynamic dimension i.e the length of the road segment occupied by a particle*. The dynamic dimension takes into account the safety requirement and depends on velocity of movement, [10]. Stochastic movement is described by monotonic random walk on cells of lane and the regular movement is described by uniform movement of all the cells of lane, [7, 8].

In [9] a model of random walk on one-lane ring has been considered. The formula has been found for the average velocity of particles. This formula is a generalization of the formula, found in [3], for the model of random walk, where randomness occurs only for the initial configuration of particles.

In [4–6] models of random walk on a discrete lattice, similar to models introduced in [7, 8], have been used to solve some traffic optimization problems.

The present work considers a stochastic model, which describes movement of particles (vehicles) on a one-dimensional lattice. The model is the limit case of the model considered in [9], in which the number of cells is great.

2. Formulation of problem

Let us describe the stochastic model of movement on the ring, which was considered in [9]. The ring contains n cells and $m < n$ particles. The $(i+1)$th cell follows the ith cell in the direction of movement, $i = 1, 2, \dots, n-1$. The 1st cell follows the nth cell. Transitions of particles occur at discrete times $1, 2, 3, \dots$. If at a discrete time the cell, which follows the particle in the direction of movement, is empty then the particle passes to the following cell with a probability p which does not depend on behavior of the other particles. The initial configuration of particles on the lattice is fixed.

States of the model are determined by configurations of particles on the lattice. The steady state probabilities and the average velocity of particles were found in [9]. The initial configuration of particles on the lattice is fixed.

In [9] a Markov chain was considered, each state of which corresponds to a state of the model. The number of states is equal to number C_n^m of m-combinations from a set of n elements. It is proved that the probabilities of states depend on the number of particle clusters only. "Cluster" is a group of occupied cells isolated from other clusters by empty cells.

By $u(n, m, p)$ denote the average individual velocity of movement of particles, i.e. the stable probability that some fixed particle passes at the current time to the next cell.

In accordance with results of work [9]

$$u(n, m, p) = \frac{n}{m} \sum_{k=1}^{\min(m,n-m)} \frac{Cp}{(1-p)^{k-1}} C_{m-1}^{k-1} C_{n-m-1}^{k-1},$$

where

$$C = \left(\sum_{k=1}^{\min(m,n-m)} \frac{n}{k} \cdot C_{m-1}^{k-1} C_{n-m-1}^{k-1} \frac{1}{(1-p)^{k-1}} \right)^{-1}.$$

Let r be the density of flow defined as $r = m/n$. In present work the limit of individual velocity $u(r, p)$ has been found for $n \to \infty$, $m \to \infty$ so that $m/n \to r$.

3. The average velocity of particles

Theorem 1. *The limit of individual velocity $u(r, p)$ for $n \to \infty$, $m \to \infty$, $m/n \to r$ exists and is calculated as*

$$\lim_{n\to\infty,\ m/n\to r} u(n, m, p) = u(r, p) = \frac{1 - \sqrt{1 - 4rp(1-r)}}{2r}. \tag{1}$$

Proof. Let A_i be the event that the ith cell is occupied and B_k be the event that there are only k clusters. Consider a pair of cells with numbers 1 and 2 as a Markov chain with four states $E_0 = (0, 0)$, $E_1 = (0, 1)$, $E_2 = (1, 0)$, $E_3 = (1, 1)$, where 0 or 1 in the ith position means that cell i is empty or occupied, $i = 1, 2$ (Fig. 1). Then $E_0 = \overline{A_1}\ \overline{A_2}$, $E_1 = \overline{A}_1 A_2$, $E_2 = A_1 \overline{A_2}$, $E_3 = A_1 A_2$.

Let p_0, p_1, p_2 and p_3 be the stable probabilities of states E_0, E_1, E_2 and E_3. The stable probabilities of both the original model and the considered pair of cells exist and do not depend on the choice of the initial state, [9].

If k is the number of clusters for the current state of model, then k particles have empty cells ahead in the direction of movement.

Then the ratio $a = a(n, m)$ of average number of clusters to the number of particles is equal to the probability $P(\overline{A_2}/A_1)$ that the cell is free in front of the fixed particle. Let $q(n, m)$ be *intensity of particles*, that is the number of particles passing the section per a time unit

$$q(n, m) = pra(n, m) = pma(n, m)/n. \tag{2}$$

We have

$$P(\overline{A_2}/A_1) = \frac{p_2}{p_2 + p_3} = a(n, m) = a. \tag{3}$$

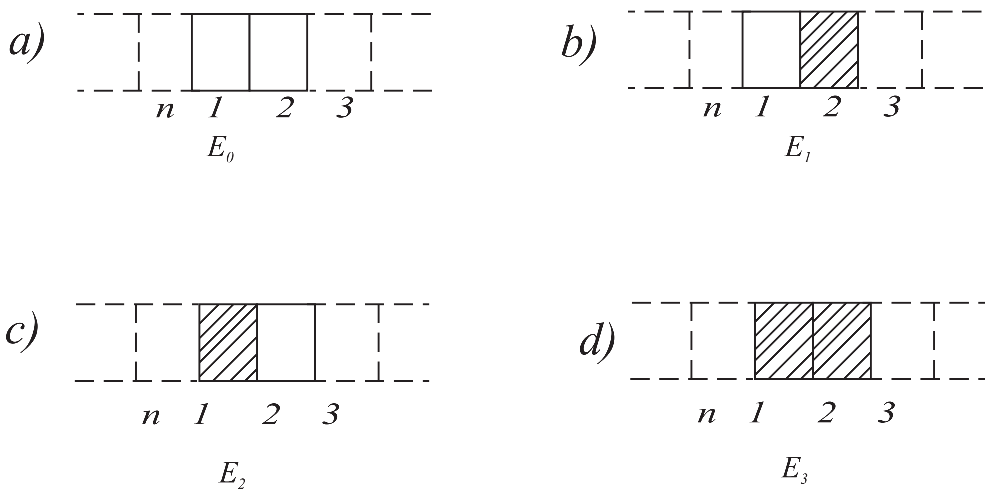

Figure 1: States of the pair of neighboring cells

It follows from (3) that

$$p_3 = \frac{1-a}{a} p_2. \tag{4}$$

We shall use the fact that for the considered model *all states with the same number of clusters have the same probability.* This fact is a consequence of formula found in [9]. In accordance with this formula

$$P(k) = \frac{C}{(1-p)^{k-1}}, \tag{5}$$

where $P(k)$ is the probability of a state with k clusters; C is the normalizing constant.

Let us use the notion of a *dual model.* A similar notion was introduced in [3]. In the dual model there are $n-m$ particles, which correspond to the *empty places* of the original model. Direction of movement in the dual model is opposite to direction of movement in the original model.

Lemma 1. *The following equality is true*

$$a(n, n-m) = \frac{r}{1-r} a(n,m), \;\; r = \frac{m}{n}. \tag{6}$$

Proof of Lemma 1. Flow intensities in the original and dual models are equal

$$q(n,m) = q(n, n-m).$$

From this equality and (2) it follows that

$$pma(n,m)/n = p(n-m)a(n,n-m)/n$$

and hence (6) is true.

Let us derive the formula for a.

Let p_{13} be probability of transition from state E_1 to state E_3 (Fig. 1b, 1d) and p_{32} probability of transition from state E_3 to state E_2 (Fig. 1c, 1d). From state E_3 the chain can come only to state E_2. The chain can come to state E_3 only from E_1. *For a stable state the number of entrances to a state per time unit is equal to the number of exits from this state*

$$p_3 p_{32} = p_1 p_{13}. \tag{7}$$

Next we derive formulas for transition probabilities

$$p_{32} = ap + o(1),\ n \to \infty,\ m/n \to r, \tag{8}$$

$$p_{13} = a\frac{r}{1-r}p(1-ap) + o(1),\ n \to \infty,\ m/n \to r. \tag{9}$$

Equations (7)–(9) allow to write the balance equation so that value of a could be found.

Lemma 2. *Equality (8) is true for the probabilities of transition from state E_3 to state E_2.*

Proof of Lemma 2. Transition of chain from state E_3 to state E_2 occurs if a particle passes from cell 2 to cell 3, Fig. 1c, 1d, and so

$$p_{32} = P(\overline{A_3}/A_1A_2)p. \tag{10}$$

The equality

$$P(\overline{A_3}/A_1A_2) = a(n-1, m-1)$$

is true. This equality follows from the one-to-one correspondence between two sets, one of which is the set of states of the original models, for which cells 1 and 2 are occupied, and the other set is the set of states of the model with $n-1$ cells and $m-1$ particles, for which cell 1 is occupied.

Indeed, a model state is described by vector $(\theta_1, \ldots, \theta_n)$ where the value of θ_i, $i = 1, \ldots, n$, is 1 for A_i and 0 for $\overline{A_i}$. State $(1, 1, \theta_3^{(n)}, \ldots, \theta_n^{(n)})$ of the original model corresponds to state $(1, \theta_2^{(n-1)}, \ldots, \theta_{n-1}^{(n-1)})$ of the model with $n-1$ cells and $\theta_i^{(n-1)} = \theta_{i+1}^{(n)}$, $i = 2, \ldots, n-1$. All the configurations of $m-2$ particles on the set of cells $3, \ldots, n$ have the same probability under condition $A_1A_2B_k$. The probabilities of all particle configurations on the set of cells $2, \ldots, n-1$ of the model with $n-1$ cells and $m-1$ particles under condition A_1B_k, $k = 1, \ldots, m-1$ are also equal. The states which are in one-to-one correspondence have the same conditional probabilities. Indeed, let $\alpha(k)$ be the number of model states for which cells 1, 2 are occupied and there are k clusters, $k = 1, \ldots,$
$\min(m-1, n-m)$. Then the probability of a configuration of $m-2$ particles

on the set of cells $3,\dots,n$, under the condition A_1A_2, if there are k clusters in the model, is calculated with formula (5) where

$$C=\left(\sum_{k=1}^{\min(m-1,n-m)}\frac{\alpha(k)}{(1-p)^{k-1}}\right)^{-1}.$$

The probability of a fixed configuration of $m-2$ particles on the set of particles $2,\dots,n-1$ in the model with $n-1$ cells and $m-1$ particles under the conditions A_1, if there are k clusters in the model, is the same. If states of two models are in one-to-one correspondence, then these states are characterized by the same number of clusters. So the considered conditional probabilities of these states are the same.

Value of $a(m,n)$ is continuous on density r so that

$$\lim_{n\to\infty,\ m/n\to r}(P(\overline{A_3}/A_1A_2)-a(n,m))=$$

$$=\lim_{n\to\infty,\ m/n\to r}(a(n-1,m-1)-a(n,m))=0.$$

Hence,

$$P(\overline{A_3}/A_1A_2)=a(n,m)+o(1),\ n\to\infty,\ m/n\to r. \tag{11}$$

Formula (8) follows from (10) and (11). Lemma 2 is proved.

Lemma 3. *For the transition from state E_1 to state E_3, equation (9) is true.*

Proof of Lemma 3. Transition of the chain from state E_1 to state E_3 (Fig. 1b, 1d) occurs if the particle occupying the n-th cell passes to cell 1 and the particle occupying cell 2 does not move. The transition occurs with probability p if cells n and 3 are occupied, and with probability $p(1-p)$ if cell n is occupied and cell 3 is empty. We have (Fig. 2)

$$p_{13}=P(A_3A_n/\overline{A_1}A_2)p+P(\overline{A_3}A_n/\overline{A_1}A_2)p(1-p)=$$

.

$$=P(A_3/\overline{A_1}A_2)P(A_n/\overline{A_1}A_2A_3)p+$$
$$+P(\overline{A_3}/\overline{A_1}A_2)P(A_n/\overline{A_1}A_2\overline{A_3})p(1-p). \tag{12}$$

We have

$$a(n,m)=\frac{1}{p_1+p_3}\cdot(p_1P(\overline{A_3}/\overline{A_1}A_2)+p_3P(\overline{A_3}/A_1A_2)). \tag{13}$$

From (11) and (13) we have

$$P(\overline{A_3}/\overline{A_1}A_2)=a(n,m)+o(1),\ n\to\infty,\ m/n\to r. \tag{14}$$

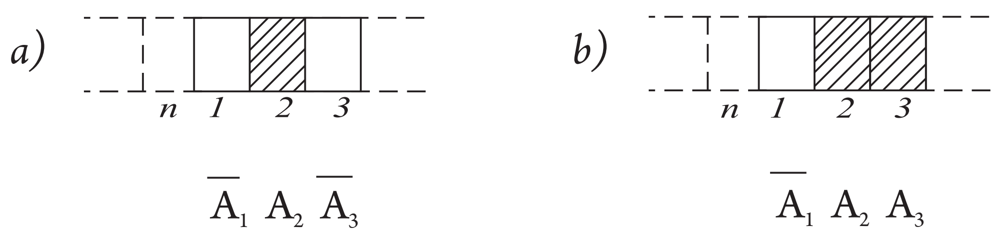

Figure 2: States of the four cells. Conditional probabilities of states of cell n are calculated

Comparing the behavior of the original and dual system and taking into account (6) and (14) we have

$$P(A_n/\overline{A_1}A_2) = P^*(\overline{A_3}/\overline{A_1}A_2) = a(n, n-m) + o(1) =$$
$$= \frac{r}{1-r}a(n,m) + o(1),\ n \to \infty,\ m/n \to r, \tag{15}$$

where the asterisk indicates that probability is calculated for the model for which the number of particles is $n-m$ and the number of cells is n still.

We have (Fig. 2a)

$$P(A_n/\overline{A_1}A_2\overline{A_3}) = a(n-2, n-m-1). \tag{16}$$

Equality (16) follows from the one-to-one correspondence between configurations of $n-m-2$ empty places on the set of cells $4,\dots,n$, for which cells 1 and 3 are empty, cell 2 is occupied and there are k clusters in the considered model, and the set of configurations of $n-m-2$ empty places on the set of cells $2,\dots,n-2$ of the model with $n-2$ cells and $n-m-1$ empty places, for which cell 1 is empty and there are $k-1$ clusters. State $(0,1,0,\theta_4^{(n)},\dots,\theta_n^{(n)})$ of the original model corresponds to state $(0,\theta_2^{(n-2)},\dots,\theta_{n-2}^{(n-2)})$ of the model with $n-2$ cells, where $\theta_i^{(n-2)} = \theta_{i+2}^{(n)}$, $i = 2,\dots,n-2$. In addition, in accordance with (5) the conditional probabilities of the states that are in correspondence are the same. Indeed, let $\beta(k)$ be the number of the original model states, for which cells 1,3 are free, cell 2 is occupied and there are k clusters, $k = 2,\dots,\min(n-m, m-2)$. Then the probability of a fixed configuration of $n-m-2$ empty places on cells $4,\dots,n$ of the original model under the condition $\overline{A_1}A_2\overline{A_3}$ is calculated with formula (5), where k is the number of clusters for this configuration,

$$C = \left(\sum_{l=2}^{\min(m,n-m)} \frac{\beta(l)}{(1-p)^{l-1}}\right)^{-1}.$$

This probability is equal to the probability of a fixed configuration of $n-m-2$ empty places on set of cells $2,\dots,n-2$ of the model with $n-2$ cells and $n-m-1$ empty places under the condition $\overline{A_1}$ if there are $k-1$ clusters in the model for this configuration. For a state of this model the number of clusters is less by one than for the corresponding state of the original model, therefore the considered conditional probabilities are the same for corresponding states of the models.

As

$$\lim_{n\to\infty,\ m/n\to r}(a(n,n-m)-a(n-2,n-m-1))=0$$

we have

$$a(n-2,n-m-1)=a(n,n-m)+o(1),\ n\to\infty,\ m/n\to r. \tag{17}$$

From (6), (16) and (17) we obtain (Fig. 2a)

$$P(A_n/\overline{A_1}A_2\overline{A_3})=\frac{r}{1-r}a(n,m)+o(1),\ n\to\infty,\ m/n\to r. \tag{18}$$

Taking into account (15) and (18) we have (Fig. 2b)

$$P(A_n/\overline{A_1}A_2A_3)=\frac{r}{1-r}a(n,m)+o(1),\ n\to\infty,\ m/n\to r. \tag{19}$$

From (12), (14), (18) and (19) we obtain (9). Lemma 3 has been proved.

Let us proceed to prove Theorem 1.

From the fact that probabilities of all the states with the same number of clusters are the same it follows that

$$p_1=p_2. \tag{20}$$

From (4), (7)–(9) and (20) we obtain

$$p_1(1-a)p=p_1a\frac{r}{1-r}p(1-ap)+o(1),\ n\to\infty,\ m/n\to r. \tag{21}$$

Passing to the limit in (21) we obtain

$$p_1(1-a)p=p_1a\frac{r}{1-r}p(1-ap). \tag{22}$$

From (22) we obtain the quadratic equation for a

$$a^2rp-a+1-r=0. \tag{23}$$

The solution of this equation, all values of which are not greater than 1, is

$$a=a(r,p)=\frac{1-\sqrt{1-4rp(1-r)}}{2rp}. \tag{24}$$

Some values of the second branch of the solution of equation (23) are greater than 1 and a pass from the first branch to the second one means that function $u(r,p)$ is discontinuous. In accordance with theory of Markov chain, steady state probabilities are unique and therefore solution (24) is unique.

Recall $u(r,p) = pa(r,p)$. Then (1) is true.
Theorem 1 is proved.

Table 1 shows values of average velocity $u(170, 170r)$ of particles for $n = 170$ and different values of p, r, and the corresponding limit values of $u(p,r)$.

Table 1. Average velocity $u(170, 170r)$ (on the left) of particles for $n = 170$ and different values of p, r, and the corresponding value of $u(r,p)$ (on the right)

	$r = 0.1$		$r = 0.3$		$r = 0.5$		$r = 0.7$		$r = 0.9$	
$p = 0.1$	0.091	0.091	0.072	0.072	0.052	0.051	0.031	0.031	0.010	0.010
$p = 0.3$	0.279	0.278	0.226	0.225	0.164	0.163	0.097	0.097	0.031	0.031
$p = 0.5$	0.474	0.472	0.399	0.397	0.294	0.293	0.171	0.170	0.053	0.052
$p = 0.7$	0.677	0.676	0.599	0.597	0.454	0.452	0.257	0.256	0.075	0.075
$p = 0.9$	0.890	0.889	0.845	0.843	0.686	0.684	0.362	0.361	0.099	0.099

4. Conclusion

In this paper the formula has been found for the average velocity of movement on a one-dimensional lattice as the limit case of random walk on a ring as the length of the ring tends to infinity.

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Editor in Chief: George Anastassiou
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Chinju 660-701
KOREA
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yjcho@nongae.gsnu.ac.kr
Nonlinear operator Th.,Inequalities,
Geometry of Banach Spaces

5) Sever S.Dragomir
School of Communications and Informatics
Victoria University of Technology
PO Box 14428
Melbourne City M.C
Victoria 8001,Australia
tel 61 3 9688 4437,fax 61 3 9688 4050
sever.dragomir@vu.edu.au,
sever@sci.vu.edu.au
Math.Analysis,Inequalities,Approximation Th.,
Numerical Analysis, Geometry of Banach Spaces,
Information Th. and Coding

6) Angelo Favini
Università di Bologna
Dipartimento di Matematica
Piazza di Porta San Donato 5
40126 Bologna, ITALY
tel.++39 051 2094451
fax.++39 051 2094490
favini@dm.unibo.it
Partial Differential Equations, Control Theory,
Differential Equations in Banach Spaces

7) Claudio A. Fernandez
Facultad de Matematicas
Pontificia Unversidad Católica de Chile
Vicuna Mackenna 4860
Santiago, Chile
tel.++56 2 354 5922
fax.++56 2 552 5916
cfernand@mat.puc.cl
Partial Differential Equations,
Mathematical Physics,
Scattering and Spectral Theory

8) A.M.Fink
Department of Mathematics
Iowa State University
Ames,IA 50011-0001,USA
tel.515-294-8150

25) Svetlozar T.Rachev
Dept.of Statistics and Applied Probability Program
University of California,Santa Barbara
CA 93106-3110,USA
tel.805-893-4869
rachev@pstat.ucsb.edu
AND
Chair of Econometrics and Statistics
School of Economics and Business Engineering
University of Karlsruhe
Kollegium am Schloss,Bau II,20.12,R210
Postfach 6980,D-76128,Karlsruhe,Germany
tel.011-49-721-608-7535
rachev@lsoe.uni-karlsruhe.de
Mathematical and Empirical Finance,
Applied Probability, Statistics and Econometrics

26) John Michael Rassias
University of Athens
Pedagogical Department
Section of Mathematics and Infomatics
20, Hippocratous Str., Athens, 106 80, Greece

<u>Address for Correspondence</u>
4, Agamemnonos Str.
Aghia Paraskevi, Athens, Attikis 15342 Greece
jrassias@primedu.uoa.gr
jrassias@tellas.gr
Approximation Theory,Functional Equations,
Inequalities, PDE

27) Paolo Emilio Ricci
Universita' degli Studi di Roma "La Sapienza"
Dipartimento di Matematica-Istituto "G.Castelnuovo"
P.le A.Moro,2-00185 Roma,ITALY
tel.++39 0649913201,fax ++39 0644701007
riccip@uniroma1.it,Paoloemilio.Ricci@uniroma1.it
Orthogonal Polynomials and Special functions,
Numerical Analysis, Transforms,Operational Calculus,
Differential and Difference equations

28) Cecil C.Rousseau
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152,USA
tel.901-678-2490,fax 901-678-2480
ccrousse@memphis.edu
Combinatorics,Graph Th.,
Asymptotic Approximations,
Applications to Physics

29) Tomasz Rychlik

fink@math.iastate.edu
Inequalities,Ordinary Differential Equations

9) Sorin Gal
Department of Mathematics
University of Oradea
Str.Armatei Romane 5
3700 Oradea,Romania
galso@uoradea.ro
Approximation Th.,Fuzzyness,Complex Analysis

10) Jerome A.Goldstein
Department of Mathematical Sciences
The University of Memphis,
Memphis,TN 38152,USA
tel.901-678-2484
jgoldste@memphis.edu
Partial Differential Equations,
Semigroups of Operators

11) Heiner H.Gonska
Department of Mathematics
University of Duisburg
Duisburg,D-47048
Germany
tel.0049-203-379-3542 office
gonska@informatik.uni-duisburg.de
Approximation Th.,Computer Aided Geometric Design

12) Dmitry Khavinson
Department of Mathematical Sciences
University of Arkansas
Fayetteville,AR 72701,USA
tel.(479)575-6331,fax(479)575-8630
dmitry@uark.edu
Potential Th.,Complex Analysis,Holomorphic PDE,
Approximation Th.,Function Th.

13) Virginia S.Kiryakova
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia 1090,Bulgaria
virginia@diogenes.bg
Special Functions,Integral Transforms,
Fractional Calculus

14) Hans-Bernd Knoop
Institute of Mathematics
Gerhard Mercator University
D-47048 Duisburg
Germany
tel.0049-203-379-2676

Institute of Mathematics
Polish Academy of Sciences
Chopina 12,87100 Torun, Poland
T.Rychlik@impan.gov.pl
Mathematical Statistics,Probabilistic Inequalities

30) Bl. Sendov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Sofia 1090,Bulgaria
bsendov@bas.bg
Approximation Th.,Geometry of Polynomials,
Image Compression

31) Igor Shevchuk
Faculty of Mathematics and Mechanics
National Taras Shevchenko
University of Kyiv
252017 Kyiv
UKRAINE
shevchuk@univ.kiev.ua
Approximation Theory

32) H.M.Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria,British Columbia V8W 3P4
Canada
tel.250-721-7455 office,250-477-6960 home,
fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis,Fractional Calculus and Appl.,
Integral Equations and Transforms,Higher Transcendental
Functions and Appl.,q-Series and q-Polynomials,
Analytic Number Th.

33) Stevo Stevic
Mathematical Institute of the Serbian Acad. of Science
Knez Mihailova 35/I
11000 Beograd, Serbia
sstevic@ptt.yu; sstevo@matf.bg.ac.yu
Complex Variables, Difference Equations,
Approximation Th., Inequalities

34) Ferenc Szidarovszky
Dept.Systems and Industrial Engineering
The University of Arizona
Engineering Building,111
PO.Box 210020
Tucson,AZ 85721-0020,USA
szidar@sie.arizona.edu
Numerical Methods,Game Th.,Dynamic Systems,

knoop@math.uni-duisburg.de
Approximation Theory,Interpolation

15) Jerry Koliha
Dept. of Mathematics & Statistics
University of Melbourne
VIC 3010,Melbourne
Australia
koliha@unimelb.edu.au
Inequalities,Operator Theory,
Matrix Analysis,Generalized Inverses

16) Robert Kozma
Dept. of Mathematical Sciences
University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Mathematical Learning Theory,
Dynamic Systems and Chaos,
Complex Dynamics.

17) Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston,RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

18) Gerassimos Ladas
Department of Mathematics
University of Rhode Island
Kingston,RI 02881,USA
gladas@math.uri.edu
Differential and Difference Equations

19) V. Lakshmikantham
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901
e-mail: lakshmik@fit.edu
Ordinary and Partial Differential
Equations,
Hybrid Systems, Nonlinear Analysis

20) Rupert Lasser
Institut fur Biomathematik &
Biomertie,GSF
-National Research Center for
environment and health
Ingolstaedter landstr.1
D-85764 Neuherberg,Germany
lasser@gsf.de
Orthogonal Polynomials,Fourier Analysis,
Mathematical Biology

Multicriteria Decision making,
Conflict Resolution,Applications
in Economics and Natural Resources
Management

35) Gancho Tachev
Dept.of Mathematics
Univ.of Architecture,Civil Eng. and Geodesy
1 Hr.Smirnenski blvd
BG-1421 Sofia,Bulgaria
gtt_fte@uacg.bg
Approximation Theory

36) Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock
Germany
manfred.tasche@mathematik.uni-rostock.de
Approximation Th.,Wavelet,Fourier Analysis,
Numerical Methods,Signal Processing,
Image Processing,Harmonic Analysis

37) Chris P.Tsokos
Department of Mathematics
University of South Florida
4202 E.Fowler Ave.,PHY 114
Tampa,FL 33620-5700,USA
profcpt@math.usf.edu,profcpt@chuma1.cas.usf.edu
Stochastic Systems,Biomathematics,
Environmental Systems,Reliability Th.

38) Lutz Volkmann
Lehrstuhl II fuer Mathematik
RWTH-Aachen
Templergraben 55
D-52062 Aachen
Germany
volkm@math2.rwth-aachen.de
Complex Analysis,Combinatorics,Graph Theory

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A New Approach of Statistical Hypothesis Verification

Iuliana F IATAN

Abstract

A new approach of statistical hypothesis verification is proposed. We shall prove a theorem which allow us to express our likelihood ratio test. Since its distribution is difficult to calculate we shall use the simulation with our algorithm in order to determine the critical value of the generalized likelihood ratio.

Keywords: hypothesis verification, parameter space, critical value, generalized likelihood ratio,

1 Introduction

The paper is organized as follows. The first section gives a brief review of the paper. The second section presents the proposed approach of statistical hypothesis verification. The third section focuses on computing critical value of the generalized likelihood ratio. The sections 4 introduces an algorithm for two classes comparing. The section 5 presents some conclusions of this paper. The paper one finishes with its references.

Assume that we have:

- a selection $X_1^{(1)}, \dots, X_{N_1}^{(1)}$ on a random vector $X^{(1)} \sim \mathcal{N}(\mu^{(1)}, \Sigma^{(1)})$ and
- a selection $X_1^{(2)}, \dots, X_{N_2}^{(2)}$ on a random vector $X^{(2)} \sim \mathcal{N}(\mu^{(2)}, \Sigma^{(2)})$,

where $\mu^{(1)}, \mu^{(2)}, \Sigma^{(1)}, \Sigma^{(2)}$ unknown parameters, $\mu^{(1)}$, $\mu^{(2)}$ being $d \times 1$ vectors and $\Sigma^{(1)}, \Sigma^{(2)}$ being a $d \times d$ definite positive matrix.

We propose to verify the hypothesis

$$\mathrm{H}_0 : \mu^{(1)} = \mu^{(2)},\ \Sigma^{(1)} = \Sigma^{(2)}.$$

The critical value V_α of our likelihood ratio test V (which is used for testing of the hypothesis H_0) has to be determined such that $P(V \leq V_\alpha | \mathrm{H}_0) = \alpha$, α being the risk of first order which has to be small. We shall prove a theorem which allow us to express V. Since the distribution of this ratio V is difficult to calculate we shall use the simulation with our algorithm in order to determine the critical value V_α of the generalized likelihood ratio.

2 Hypothesis Verification Concerning the Identity of Two Normal Distribution

We denote by:

- Ω the parameter space where $\Sigma^{(1)}, \Sigma^{(2)}$ are $d \times d$ defined positive matrices,
- Ω_1 is a subset of the parameter space for which $\Sigma^{(1)} = \Sigma^{(2)}$,
- Ω_2 is a subset of the parameter space for which $\Sigma^{(1)} = \Sigma^{(2)}$ and $\mu^{(1)} = \mu^{(2)}$.

Proposition[4] We consider the hypotheses

$$\mathrm{H}_1 : \Sigma^{(1)} = \Sigma^{(2)},\ \mu^{(1)} \neq \mu^{(2)},$$

$$\mathrm{H}_2 : \Sigma^{(1)} = \Sigma^{(2)},\ \mu^{(1)} = \mu^{(2)},$$

$$\mathrm{H}_{12} : \mu^{(1)} = \mu^{(2)},\ \Sigma^{(1)} \neq \Sigma^{(2)}.$$

If l_1, l_2 and l_{12} are the likelihood ratios for verification of the hypotheses $\mathrm{H}_1, \mathrm{H}_2$ and respective H_{12}, then

$$l_{12} = l_1 l_2.$$

Proof:
The likelihood function is

$$\mathrm{L} = \frac{1}{(2\pi)^{\frac{d}{2}(N_1+N_2)} \cdot \left|\Sigma^{(1)}\right|^{\frac{N_1}{2}} \cdot \left|\Sigma^{(2)}\right|^{\frac{N_2}{2}}} \cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \mu^{(1)}\right)^t (\Sigma^{(1)})^{-1}\left(X_\alpha^{(1)} - \mu^{(1)}\right)\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \mu^{(2)}\right)^t (\Sigma^{(2)})^{-1}\left(X_\alpha^{(2)} - \mu^{(2)}\right)\right]. \qquad (1)$$

The maximum likelihood estimates of parameter with respect to Ω are:

$$\hat{\mu}^{(j)} = \overline{X}^{(j)} = \frac{1}{N_j}\sum_{i=1}^{N_j} X_i{}^{(j)},\ j = \overline{1,2},$$

$$\hat{\Sigma}^{(j)} = \frac{1}{N_j}\sum_{i=1}^{N_j}\left(X_i{}^{(j)} - \overline{X}^{(j)}\right)\left(X_i{}^{(j)} - \overline{X}^{(j)}\right)^t,\ j = \overline{1,2},$$

or

$$\hat{\Sigma}^{(j)} = \frac{1}{N_j} S_j,\ j = \overline{1,2}. \qquad (2)$$

where

$$S_j = \sum_{i=1}^{N_j}\left(X_i{}^{(j)} - \overline{X}^{(j)}\right)\left(X_i{}^{(j)} - \overline{X}^{(j)}\right)^t,\ j = \overline{1,2}. \qquad (3)$$

The maximum value of the likelihood function from (1) with respect to Ω is

$$\mathrm{L}_\Omega = \frac{1}{(2\pi)^{\frac{d}{2}(N_1+N_2)} \cdot \left|\hat{\Sigma}^{(1)}\right|^{\frac{N_1}{2}} \cdot \left|\hat{\Sigma}^{(2)}\right|^{\frac{N_2}{2}}} \cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)} - \overline{X}^{(1)}\right)^t (\hat{\Sigma}^{(1)})^{-1}\left(X_\alpha^{(1)} - \overline{X}^{(1)}\right)\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)} - \overline{X}^{(2)}\right)^t (\hat{\Sigma}^{(2)})^{-1}\left(X_\alpha^{(2)} - \overline{X}^{(2)}\right)\right]. \qquad (4)$$

Taking into account [4] that

$$-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\overline{X}^{(1)}\right)^t(\hat{\Sigma}^{(1)})^{-1}\left(X_\alpha^{(1)}-\overline{X}^{(1)}\right)=-\frac{N_1}{2}d$$

and respectively

$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\overline{X}^{(2)}\right)^t(\hat{\Sigma}^{(2)})^{-1}\left(X_\alpha^{(2)}-\overline{X}^{(2)}\right)=-\frac{N_2}{2}d$$

the relation (4) may be written in the form

$$\mathrm{L}_\Omega=\frac{1}{(2\pi)^{\frac{d}{2}(N_1+N_2)}\cdot|\hat{\Sigma}^{(1)}|^{\frac{N_1}{2}}\cdot|\hat{\Sigma}^{(2)}|^{\frac{N_2}{2}}}\cdot\mathrm{e}^{\frac{-N_1}{2}d}\cdot\mathrm{e}^{\frac{-N_2}{2}d};$$

therefore

$$\mathrm{L}_\Omega=\frac{1}{(2\pi)^{\frac{d}{2}N}\cdot|\hat{\Sigma}^{(1)}|^{\frac{N_1}{2}}\cdot|\hat{\Sigma}^{(2)}|^{\frac{N_2}{2}}}\cdot\mathrm{e}^{\frac{-N}{2}d},\tag{5}$$

where $N=N_1+N_2$.

With respect to Ω_1, the maximum value of the likelihood function is

$$\mathrm{L}_{\Omega_1}=\frac{1}{(2\pi)^{\frac{d}{2}N}\cdot|\hat{\Sigma}|^{\frac{N}{2}}}\cdot\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\overline{X}^{(1)}\right)^t\hat{\Sigma}^{-1}\left(X_\alpha^{(1)}-\overline{X}^{(1)}\right)\right]\cdot$$

$$\cdot\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\overline{X}^{(2)}\right)^t\hat{\Sigma}^{-1}\left(X_\alpha^{(2)}-\overline{X}^{(2)}\right)\right],\tag{6}$$

where

$$\hat{\mu}^{(j)}=\overline{X}^{(j)},\ j=\overline{1,2},$$

$$\hat{\Sigma}^{(1)}=\hat{\Sigma}^{(2)}=\hat{\Sigma}=\frac{1}{N}S;$$

we have used the notation

$$S=S_1+S_2,\tag{7}$$

with S_1 and S_2 from (3).

With respect to Ω_2, the maximum value of the likelihood function is

$$\mathrm{L}_{\Omega_2}=\frac{1}{(2\pi)^{\frac{d}{2}N}\cdot|\hat{\Sigma}^{*}|^{\frac{N}{2}}}\cdot\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_1}\left(X_\alpha^{(1)}-\overline{X}\right)^t(\hat{\Sigma}^{*})^{-1}\left(X_\alpha^{(1)}-\overline{X}\right)\right]\cdot$$

$$\cdot\exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N_2}\left(X_\alpha^{(2)}-\overline{X}\right)^t(\hat{\Sigma}^{*})^{-1}\left(X_\alpha^{(2)}-\overline{X}\right)\right],\tag{8}$$

where

$$\hat{\mu}^{(1)} = \hat{\mu}^{(2)} = \overline{X},$$

$$\hat{\Sigma}^* = \frac{1}{N} S^*, \tag{9}$$

and

$$S^* = \sum_{\alpha=1}^{N_1} \left(X_\alpha^{(1)} - \overline{X} \right) \left(X_\alpha^{(1)} - \overline{X} \right)^t + \sum_{\alpha=1}^{N_2} \left(X_\alpha^{(2)} - \overline{X} \right) \left(X_\alpha^{(2)} - \overline{X} \right)^t.$$

Since, keeping with [4] we have

$$\sum_{\alpha=1}^{N_1} \left(X_\alpha^{(1)} - \overline{X} \right)^t (\hat{\Sigma}^*)^{-1} \left(X_\alpha^{(1)} - \overline{X} \right) + \sum_{\alpha=1}^{N_2} \left(X_\alpha^{(2)} - \overline{X} \right)^t (\hat{\Sigma}^*)^{-1} \left(X_\alpha^{(2)} - \overline{X} \right) = Nd,$$

we deduce

$$\mathrm{L}_{\Omega_2} = \frac{1}{(2\pi)^{\frac{d}{2}N} \cdot |\hat{\Sigma}^*|^{\frac{N}{2}}} \cdot \mathrm{e}^{\frac{-N}{2}d}. \tag{10}$$

We can write

$$S^* = \sum_{\alpha=1}^{N_1} \left(X_\alpha^{(1)} - \overline{X}^{(1)} + \overline{X}^{(1)} - \overline{X} \right) \left(X_\alpha^{(1)} - \overline{X}^{(1)} + \overline{X}^{(1)} - \overline{X} \right)^t +$$

$$+ \sum_{\alpha=1}^{N_2} \left(X_\alpha^{(2)} - \overline{X}^{(2)} + \overline{X}^{(2)} - \overline{X} \right) \left(X_\alpha^{(2)} - \overline{X}^{(2)} + \overline{X}^{(2)} - \overline{X} \right)^t =$$

$$= \sum_{\alpha=1}^{N_1} \left(X_\alpha^{(1)} - \overline{X}^{(1)} \right) \left(X_\alpha^{(1)} - \overline{X}^{(1)} \right)^t + N_1 \left(\overline{X}^{(1)} - \overline{X} \right) \left(\overline{X}^{(1)} - \overline{X} \right)^t +$$

$$+ \sum_{\alpha=1}^{N_2} \left(X_\alpha^{(2)} - \overline{X}^{(2)} \right) \left(X_\alpha^{(2)} - \overline{X}^{(2)} \right)^t + N_2 \left(\overline{X}^{(2)} - \overline{X} \right) \left(\overline{X}^{(2)} - \overline{X} \right)^t. \tag{11}$$

Based on the relation (7), from (11) we deduce

$$S^* = S + N_1 \left(\overline{X}^{(1)} - \overline{X} \right) \left(\overline{X}^{(1)} - \overline{X} \right)^t + N_2 \left(\overline{X}^{(2)} - \overline{X} \right) \left(\overline{X}^{(2)} - \overline{X} \right)^t. \tag{12}$$

Since l_1, l_2 and l_{12} are the likelihood ratios for verification of the hypotheses $\mathrm{H}_1, \mathrm{H}_2$ respective H_{12} it results

$$l_1 = \frac{\mathrm{L}_{\Omega_1}}{\mathrm{L}_\Omega},$$

$$l_2 = \frac{\mathrm{L}_{\Omega_2}}{\mathrm{L}_{\Omega_1}},$$

$$l_{12} = \frac{\mathrm{L}_{\Omega_2}}{\mathrm{L}_\Omega}.$$

We obtain

$$l_1 l_2 = \frac{\mathrm{L}_{\Omega_1}}{\mathrm{L}_{\Omega}} \cdot \frac{\mathrm{L}_{\Omega_2}}{\mathrm{L}_{\Omega_1}} = \frac{\mathrm{L}_{\Omega_2}}{\mathrm{L}_{\Omega}} = l_{12}. \tag{13}$$

Substituting (5) and (10) into (13) we shall obtain

$$l_{12} = \frac{\frac{1}{(2\pi)^{\frac{d}{2}N} \cdot |\hat{\Sigma}^*|^{\frac{N}{2}}} \cdot \mathrm{e}^{\frac{-N}{2}d}}{\frac{1}{(2\pi)^{\frac{d}{2}N} \cdot |\hat{\Sigma}^{(1)}|^{\frac{N_1}{2}} \cdot |\hat{\Sigma}^{(2)}|^{\frac{N_2}{2}}} \cdot \mathrm{e}^{\frac{-N}{2}d}} = \frac{\left|\hat{\Sigma}^{(1)}\right|^{\frac{N_1}{2}} \cdot \left|\hat{\Sigma}^{(2)}\right|^{\frac{N_2}{2}}}{\left|\hat{\Sigma}^*\right|^{\frac{N}{2}}}. \tag{14}$$

Using (9) and (2) in (14) we deduce

$$l_{12} = \frac{|S_1|^{\frac{N_1}{2}} \cdot |S_2|^{\frac{N_2}{2}}}{N_1^{\frac{N_1 d}{2}} \cdot N_2^{\frac{N_2 d}{2}}} \cdot \frac{|S^*|^{\frac{N}{2}}}{N^{\frac{Nd}{2}}}.$$

■

3 Computing Critical Value of the Generalized Likelihood Ratio

The critical value l_α of the likelihood ratio test l (which is used for testing of the hypothesis H_0) has to be determined such that $P(l \leq l_\alpha | \mathrm{H}_0) = \alpha$, α being the risk of first order which has to be small.

From [4] we know that

$$V_2 = \frac{|S|^{\frac{1}{2}n}}{|S^*|^{\frac{1}{2}n}} = l_2^{n/N},$$

where $n = n_1 + n_2$, and $n_j = N_j - 1,\ j = \overline{1,2}$ is the same with the likelihood ratio for verification of the hypothesis H_2.

It results that for verification of the hypothesis H_0 we can use the statistic

$$V = V_1 V_2 = \frac{|S_1|^{\frac{1}{2}n_1} \cdot |S_2|^{\frac{1}{2}n_2}}{|S^*|^{\frac{1}{2}n}}. \tag{15}$$

instead of the statistic l.

The distribution of this ratio V is difficult to calculate since:

- $|S_1|, |S_2|, |S^*|$ are generalized dispersions (see [7]); their distributions don't have a form which can permit us to calculate the critical values ;
- $|S_1|, |S_2|$ and $|S^*|$ are not independently.

Therefore, in order to determine the critical value V_α of the generalized likelihood ratio such that

$$P(V \leq V_\alpha | \mathrm{H}_0) = \alpha,$$

(α being the risk of first order which has to be small) one use the simulation with the following algorithm:

Algorithm 1
Step 0. Inputs $N_1, N_2, \mu^{(1)}, \mu^{(2)}, \Sigma^{(1)}, \Sigma^{(2)}, d$.
Step 1. Generate $X_1^{(1)}, \ldots, X_{N_1}^{(1)} \sim \mathcal{N}(\mu^{(1)}, \Sigma^{(1)})$.
Step 2. Generate $X_1^{(2)}, \ldots, X_{N_2}^{(2)} \sim \mathcal{N}(\mu^{(2)}, \Sigma^{(2)})$.
Step 3. Generate $S_1,\ S_2$ using (3).
Step 4. Calculate S^* using the relation (12).
Step 5. Calculate $|S_1|,\ |S_2|$.
Step 6. Calculate $|S^*|$.
Step 7. Calculate V with (15).
Step 8. We have to repeat for m times the steps 1-7 (for example $m \geq 2000$) and we shall obtain the selection $l_1, l_2, \ldots l_m$.
Step 9. We construct a histogram with $\{l_1, l_2, \ldots l_m\}$ (see [7]) such as:
Step 9.1. We choose a positive integer k which represents the number of the histogram intervals $I_1, I_2, \ldots I_k$.
Step 9.2. We determine the absolute frequencies $f_1, f_2, \ldots f_k$, namely the number of the selection values which are in the interval $I_j,\ 1 \leq j \leq k$.
Step 9.3. We determine the relative frequencies $r_1, r_2, \ldots r_k$, $r_i = \frac{f_i}{m},\ 1 \leq i \leq k$.

The graph from the Figure 1 represents the form of selection histogram $l_1, l_2, \ldots l_m$ which one obtains by simulation.

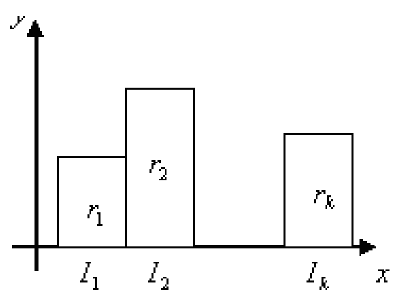

Figure 1. Histogram

The critical value for our test V_α one chooses such that the sum of rectangle area from the left side of l_α is α, where $\alpha = P(V \leq V_\alpha)$ is the risk of the first kind.

4 An Algorithm for Two Classes Comparing

In this section we shall describe an algorithm which can classify some new given d- dimensional objects, $Y_1, ..., Y_K$ drawn from $\mathcal{N}(\mu, \Sigma)$, testing if they apart into one of the two classes C_1 or C_2.

The class C_1 contains N_1 vectors: $X_1^{(1)}, \ldots, X_{N_1}^{(1)}$ while the class C_2 contains N_2 vectors: $X_1^{(2)}, \ldots, X_{N_2}^{(2)}$.

Algorithm 2
Step 0. Input $K, N_1, N_2, \mu, \mu^{(1)}, \mu^{(2)}, \Sigma, \Sigma^{(1)}, \Sigma^{(2)}, d$.
Step 1. We generate $Y_1, ..., Y_K \sim \mathcal{N}(\mu, \Sigma)$.
Step 2. We verify the hypothesis:

H_0: $\mu_1 = \mu_2,\ \Sigma^{(1)} = \Sigma^{(2)}$
using the Algorithm 1.

In the case when the hypothesis isn't true we have to go to Step 4.

Otherwise, go to Step 3.

Step 3. **For** $i := 1$ **to** K
do begin
We test the null hypothesis
$\mathrm{H} : Y_i, X_1^{(1)}, \dots, X_{N_1}^{(1)}$ *are drawn from* $\mathcal{N}(\mu^{(1)}, \Sigma^{(1)})$
against the alternative hypothesis
$\mathrm{NH} : X_1^{(1)}, \dots, X_{N_1}^{(1)}$ *are drawn from* $\mathcal{N}(\mu^{(1)}, \Sigma^{(1)}),\ \mu \neq \mu_1$
using [3] the Algorithm 3.1;
end

Step 4. We verify the hypothesis:
H_0: $\Sigma^{(1)} = \Sigma^{(2)},\ \mu_1 \neq \mu_2$, see [7], [4].

In the case when the hypothesis is true we have to go to Step 5.

Otherwise, go to Step 6.

Step 5. **For** $i := 1$ **to** K
do begin
We test the null hypothesis

$$\mathrm{H} : Y_i, X_1^{(1)}, \dots, X_{N_1}^{(1)} \textit{ are drawn from } \mathcal{N}(\mu^{(1)}, \Sigma) \textit{ and}$$

$$X_1^{(2)}, \dots, X_{N_2}^{(2)} \textit{ are drawn from } \mathcal{N}(\mu^{(2)}, \Sigma),$$

against the alternative hypothesis

$$\mathrm{NH} : X_1^{(1)}, \dots, X_{N_1}^{(1)} \textit{ are drawn from } \mathcal{N}(\mu^{(1)}, \Sigma) \textit{ and}$$

$$Y_i, X_1^{(2)}, \dots, X_{N_2}^{(2)} \textit{ are drawn from } \mathcal{N}(\mu^{(2)}, \Sigma),$$

using [3] the Algorithm 3.2;
end

Step 6. We verify the hypothesis:

- H_0: $\mu = \mu^{(1)},\ \Sigma = \Sigma^{(1)}$
- H_0': $\mu = \mu^{(2)},\ \Sigma = \Sigma^{(2)}$

using the Algorithm 1.

If one of the hypotheses is true, namely if the new objects belong to the same class C_1 or C_2 then the algorithm one finishes.

If either of the hypotheses is not true, namely if the new objects don't belong to the same class C_1 or C_2 then go to Step 7.

Step 7. **For** $i := 1$ **to** K
do begin
We test the null hypothesis

$$\mathrm{H} : Y_i, X_1^{(1)}, \dots, X_{N_1}^{(1)} \textit{ are drawn from } \mathcal{N}(\mu^{(1)}, \Sigma^{(1)}) \textit{ and}$$

$$X_1^{(2)}, \dots, X_{N_2}^{(2)} \textit{ are drawn from } \mathcal{N}(\mu^{(2)}, \Sigma^{(2)}),$$

against the alternative hypothesis

$$\text{NH}: X_1^{(1)}, \ldots, X_{N_1}^{(1)} \; are \; drawn \; from \; \mathcal{N}(\mu^{(1)}, \Sigma^{(1)}) \; and$$

$$Y_i, X_1^{(2)}, \ldots, X_{N_2}^{(2)} \; are \; drawn \; from \; \mathcal{N}(\mu^{(2)}, \Sigma^{(2)}),$$

using [3] the Algorithm 3.3.

end

5 Conclusions

The paper discusses a new approach of statistical hypothesis verification. We have proved a theorem which allow us to express our likelihood ratio test. Since its distribution is difficult to calculate we have used the simulation with our algorithm in order to determine the critical value of the generalized likelihood ratio. This algorithm is used in order to construct a complex algorithm for two classes comparing. In conclusion, the classification problem of the new given objects $Y_1, ..., Y_K$, drawn from $\mathcal{N}(\mu, \Sigma)$ is a complex problem which one reduces to application of the Algorithm 1 and Algorithms 3.1, 3.2 or 3.3 (see [3]).

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Elements of right delta fractional Calculus on Time Scales

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we develop the right delta fractional calculus on time scales.

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1 Background

For the basics of times scales please read [1], [2], [3], [5], [7], [8], [9], [10], [11], [12], [13].

Let $\mathbb{T}$ be a time scale, and ([9], p. 38) g_k, $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $s, t \in \mathbb{T} : g_0(t,s) = h_0(t,s) = 1$,

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\,\Delta\tau, \tag{1}$$

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \quad \forall\ s,t \in \mathbb{T}.$$

We have

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s), \tag{2}$$

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t), s), \quad k \in \mathbb{N},\ t \in \mathbb{T}^k.$$

Also

$$g_1(t,s) = h_1(t,s) = t - s, \quad \forall\ s,t \in \mathbb{T}. \tag{3}$$

Here g_k, h_k are continuous in t.

Example 1 *(see [9], p. 39-40)*

i) When $\mathbb{T} = \mathbb{R}$,

$$g_k(t,s) = h_k(t,s) = \frac{(t-s)^k}{k!}, \quad \forall\ s,t \in \mathbb{R}. \tag{4}$$

ii) When $\mathbb{T} = \mathbb{Z}$, *we get*

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!},$$

and

$$g_k(t,s) = \frac{(t-s+k-1)^{(k)}}{k!}, \tag{5}$$

furthermore it holds

$$h_k(t,s) = (-1)^k g_k(s,t), \quad k \in \mathbb{N}_0,\ \forall\ s,t \in \mathbb{Z}. \tag{6}$$

We need

Theorem 2 *([9], p. 45) We have that*

$$h_n(t,s) = (-1)^n g_n(s,t), \tag{7}$$

for every $t \in \mathbb{T}$ *and every* $s \in \mathbb{T}^{k^n}$.

If $\mathbb{T} = \mathbb{T}^k$, *then (7) is true for every* $s,t \in \mathbb{T}$.

We need delta Taylor formula on time scales.

Theorem 3 *([6], [11]) Assume* $\mathbb{T} = \mathbb{T}^k$, $f \in C_{rd}^m(\mathbb{T})$, $m \in \mathbb{N}$, $s,t \in \mathbb{T}$. *Then*

$$f(t) = \sum_{k=0}^{m-1} f^{\Delta^k}(s)\, h_k(t,s) + R_m^s(f)(t), \tag{8}$$

where

$$R_m^s(f)(t) = \int_s^t h_{m-1}(t,\sigma(\tau))\, f^{\Delta^m}(\tau)\, \Delta\tau. \tag{9}$$

We notice that

$$\begin{aligned} R_m^s(f)(t) &= -\int_t^s h_{m-1}(t,\sigma(\tau))\, f^{\Delta^m}(\tau)\, \Delta\tau \\ &\overset{(7)}{=} (-1)^m \int_t^s g_{m-1}(\sigma(\tau),t)\, f^{\Delta^m}(\tau)\, \Delta\tau. \end{aligned} \tag{10}$$

In this article we assume $\mathbb{T} = \mathbb{T}^k$.

We make

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Definition 4 *Let $\alpha \geq 0$. We consider the continuous functions*

$$g_\alpha : \mathbb{T}^2 \to \mathbb{R} :$$

$$g_0 (t,s) = 1,$$

$$g_{\alpha+1} (t,s) = \int_s^t g_\alpha (\sigma (\tau), s) \Delta\tau, \quad \forall\ s,t \in \mathbb{T}. \tag{11}$$

We are motivated a lot by the formula

$$\int_t^x \frac{(x-s)^{\mu-1}}{\Gamma(\mu)} \frac{(s-t)^{\nu-1}}{\Gamma(\nu)} ds = \frac{(x-t)^{\mu+\nu-1}}{\Gamma(\mu+\nu)}, \tag{12}$$

where $\mu, \nu > 0$ and Γ is the gamma function.

Assumption 5 *Let $\alpha, \beta > 1$ and $x \leq t \leq \tau$, $x, t, \tau \in \mathbb{T}$. We assume that*

$$\int_x^{\sigma(\tau)} g_{\alpha-1} (\sigma (t), x) g_{\beta-1} (\sigma (\tau), t) \Delta t = g_{\alpha+\beta-1} (\sigma (\tau), x). \tag{13}$$

We call for $\alpha, \beta > 1$ and $x \leq t \leq \tau$,

$$\theta (x, \tau) := \int_\tau^{\sigma(\tau)} g_{\alpha-1} (\sigma (t), x) g_{\beta-1} (\sigma (\tau), t) \Delta t. \tag{14}$$

By Theorem 1.75, p. 28, [9] for $f \in C_{rd} (\mathbb{T})$ and $t \in \mathbb{T}^k$, we have

$$\int_t^{\sigma(t)} f (\tau) \Delta\tau = \mu (t) f (t), \tag{15}$$

where $\mu (t) := \sigma (t) - t$.

So by (15) we get that

$$\theta (x, \tau) = g_{\alpha-1} (\sigma (\tau), x) g_{\beta-1} (\sigma (\tau), \tau) \mu (\tau). \tag{16}$$

Definition 6 *Let $f \in L_1 ([a,b) \cap \mathbb{T})$. We define the right forward graininess deviation functional of f as follows:*

$$E (f, \alpha, \beta, b, \mathbb{T}, x) = \int_x^b f (\tau) \theta (x, \tau) \Delta\tau = \tag{17}$$

$$\int_x^b f (\tau) g_{\alpha-1} (\sigma (\tau), x) g_{\beta-1} (\sigma (\tau), \tau) \mu (\tau) \Delta\tau.$$

If $\mathbb{T} = \mathbb{R}$, then $\mu (\tau) = 0$ and hence $E (f, \alpha, \beta, b, \mathbb{R}, x) = 0$.

2 Results

We give

Definition 7 *Let $a,b \in \mathbb{T}$, $\alpha \geq 1$ and $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$. Here $f \in L_1([a,b) \cap \mathbb{T})$ (Lebesgue Δ-integrable function on $[a,b) \cap \mathbb{T}$). We define the right Δ-Riemann-Liouville type fractional integral*

$$I_{b-}^{\alpha} f(t) := \int_t^b g_{\alpha-1}(\sigma(\tau), t) f(\tau) \Delta\tau, \tag{18}$$

for $t \in [a,b] \cap \mathbb{T}$. Here $\int_t^b \cdot \Delta\tau = \int_{[t,b)} \cdot \Delta\tau$.

By [8] we get that $I_{b-}^{1} f(t) = \int_t^b f(\tau) \Delta\tau$ is absolutely continuous in $t \in [a,b] \cap \mathbb{T}$.

Lemma 8 *Let $\alpha > 1$, $f \in L_1([a,b) \cap \mathbb{T})$, $f : [a,b] \cap \mathbb{T} \to \mathbb{R}$. Assume that $g_{\alpha-1}(\sigma(\tau), t)$ is Lebesgue Δ-measurable on $([a,b] \cap \mathbb{T})^2$; $a,b \in \mathbb{T}$. Then $I_{b-}^{\alpha} f \in L_1([a,b] \cap \mathbb{T})$, that is $I_{b-}^{\alpha} f$ is finite a.e.*

Proof. By Tietze's extension theorem of General Topology we easily derive that the continuous function $g_{\alpha-1}$ on $([a,b] \cap \mathbb{T})^2$ is bounded, since its continuous extension $F_{\alpha-1}$ on $[a,b]^2$ is bounded. Notice here $([a,b] \cap \mathbb{T})^2$ is a closed subset of $[a,b]^2$.

So there exists $M > 0$ such that $|g_{a-1}(s,t)| \leq M$, $\forall\ (s,t) \in ([a,b] \cap \mathbb{T})^2$.

Let here *id* denote the identity map. We see that
$(\sigma, id)(([a,b) \cap \mathbb{T}) \times ([a,b] \cap \mathbb{T})) \subseteq ([a,b] \cap \mathbb{T})^2$.

Therefore $|g_{\alpha-1}(\sigma(\tau), t)| \leq M$, $\forall\ (\tau, t) \in ([a,b) \cap \mathbb{T}) \times ([a,b] \cap \mathbb{T})$, since $(\sigma(\tau), t) \in ([a,b] \cap \mathbb{T})^2$.

Define $K : \Omega := ([a,b] \cap \mathbb{T})^2 \to \mathbb{R}$, by

$$K(\tau, t) := \begin{cases} g_{\alpha-1}(\sigma(\tau), t), & \text{if } a \leq t \leq \tau < b, \\ 0, & \text{if } a \leq \tau < t \leq b, \end{cases}$$

where $t, \tau \in \mathbb{T}$.

Clearly here K is Lebesgue Δ-measurable on Ω, since the restriction of a measurable function to a measurable subset of its domain is a measurable function, and the union of two measurable functions over disjoint domains is measurable. Notice here that $|K(\tau,t)| \leq M$, $\forall\ (\tau,t) \in ([a,b] \cap \mathbb{T})^2$.

Next we consider the repeated double Lebesgue Δ-integral

$$\int_a^b \left(\int_a^b |K(\tau,t)| \, |f(\tau)| \, \Delta t \right) \Delta\tau = \int_a^b |f(\tau)| \left(\int_a^b |K(\tau,t)| \, \Delta t \right) \Delta\tau$$

$$\leq M(b-a)\int_a^b |f(\tau)|\,\Delta\tau = M(b-a)\,\|f\|_{L_1([a,b)\cap\mathbb{T})} < \infty.$$

By Tonelli's theorem we derive that $(\tau,t) \to K(\tau,t)\,f(\tau)$ is Lebesgue Δ-integrable over Ω.

Let now the characteristic function

$$\chi_{[t,b)}(\tau) = \begin{cases} 1, & \text{if } \tau \in [t,b) \\ 0, & \text{else,} \end{cases}$$

where $\tau \in [a,b] \cap \mathbb{T}$.

Then the function $(\tau,t) \to \chi_{[t,b)}(\tau)\,K(\tau,t)\,f(\tau)$ is Lebesgue Δ-integrable over Ω.

Hence by Fubini's theorem we get that

$$\int_a^b \chi_{[t,b)}(\tau)\,K(\tau,t)\,f(\tau)\,\Delta\tau = \int_t^b g_{\alpha-1}(\sigma(\tau),t)\,f(\tau)\,\Delta\tau = I_{b-}^{\alpha} f(t)$$

is Lebesgue Δ-integrable on $[a,b]\cap\mathbb{T}$, proving the claim. ■

From now on we make

Assumption 9 *We suppose that $g_{\alpha-1}(\sigma(\cdot),\cdot)$ is continuous on $([a,b]\cap\mathbb{T})^2$, for any $\alpha > 1$.*

We give the following semigroup property of right Δ-Riemann-Liouville type fractional integrals.

Theorem 10 *Let $a,b \in \mathbb{T}$, $f \in L_1([a,b)\cap\mathbb{T})$; $\alpha,\beta > 1$. Then*

$$I_{b-}^{\alpha} I_{b-}^{\beta} f(x) = I_{b-}^{\alpha+\beta} f(x) - E(f,\alpha,\beta,b,\mathbb{T},x), \quad \forall\, x \in [a,b]\cap\mathbb{T}. \tag{19}$$

Proof. Here we have

$$I_{b-}^{\beta} f(t) = \int_t^b g_{\beta-1}(\sigma(\tau),t)\,f(\tau)\,\Delta\tau.$$

We observe that

$$I_{b-}^{\alpha} I_{b-}^{\beta} f(x) = \int_x^b g_{\alpha-1}(\sigma(t),x)\,I_{b-}^{\beta} f(t)\,\Delta t =$$

$$\int_x^b g_{\alpha-1}(\sigma(t),x)\left(\int_t^b g_{\beta-1}(\sigma(\tau),t)\,f(\tau)\,\Delta\tau\right)\Delta t =$$

$$\int_x^b \left(\int_t^b g_{\alpha-1}(\sigma(t),x)\,g_{\beta-1}(\sigma(\tau),t)\,f(\tau)\,\Delta\tau\right)\Delta t =: (*).$$

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Clearly here it holds

$$|g_{\alpha-1}(\sigma(t), x)| \leq M_1, \quad \forall\ t, x \in [a, b] \cap \mathbb{T},$$

and

$$|g_{\beta-1}(\sigma(\tau), t)| \leq M_2, \quad \forall\ \tau, t \in [a, b] \cap \mathbb{T},$$

where $M_1, M_2 > 0$.

Hence

$$\left|I_{b-}^{\alpha} I_{b-}^{\beta} f(x)\right| \leq \int_x^b \left(\int_t^b |g_{\alpha-1}(\sigma(t), x)| \, |g_{\beta-1}(\sigma(\tau), t)| \, |f(\tau)| \, \Delta\tau\right) \Delta t \leq$$

$$M_1 M_2 \left(\int_x^b \left(\int_t^b |f(\tau)| \, \Delta\tau\right) \Delta t\right) \leq M_1 M_2 \left(\int_x^b \left(\int_a^b |f(\tau)| \, \Delta\tau\right) \Delta t\right) \leq$$

$$M_1 M_2 (b-a) \|f\|_{L_1([a,b) \cap \mathbb{T})} < \infty.$$

So that $I_{b-}^{\alpha} I_{b-}^{\beta} f(x)$ exists, $\forall\ x \in [a, b] \cap \mathbb{T}$. Consequently by Fubini's theorem we have

$$(*) = \int_x^b \left(\int_x^\tau g_{\alpha-1}(\sigma(t), x) \, g_{\beta-1}(\sigma(\tau), t) \, f(\tau) \, \Delta t\right) \Delta\tau =$$

$$\int_x^b f(\tau) \left(\int_x^\tau g_{\alpha-1}(\sigma(t), x) \, g_{\beta-1}(\sigma(\tau), t) \, \Delta t\right) \Delta\tau$$

(here $x \leq t < \tau$)

$$= \int_x^b f(\tau) \left(\int_x^{\sigma(\tau)} g_{\alpha-1}(\sigma(t), x) \, g_{\beta-1}(\sigma(\tau), t) \, \Delta t - \right.$$

$$\left. \int_\tau^{\sigma(\tau)} g_{\alpha-1}(\sigma(t), x) \, g_{\beta-1}(\sigma(\tau), t) \, \Delta t\right) \Delta\tau$$

$$\overset{\text{by } ((13), (14))}{=} \int_x^b g_{\alpha+\beta-1}(\sigma(\tau), x) \, f(\tau) \, \Delta\tau - \int_x^b f(\tau) \, \theta(x, \tau) \, \Delta\tau$$

$$= I_{b-}^{\alpha+\beta} f(x) - \int_x^b f(\tau) \, \theta(x, \tau) \, \Delta\tau$$

$$\overset{(17)}{=} I_{b-}^{\alpha+\beta} f(x) - E(f, \alpha, \beta, b, \mathbb{T}, x),$$

proving the claim (19). ∎

We make

Remark 11 *Let $\mu > 2 : m-1 < \mu \leq m$, $m \in \mathbb{N}$, i.e. $m = \lceil \mu \rceil$ (ceiling of the number), $\widetilde{\nu} = m - \mu$ ($0 \leq \widetilde{\nu} < 1$). Let $f \in C_{rd}^{m}([a,b] \cap \mathbb{T})$. Clearly here ([10]) $f^{\Delta^{m}}$ is Lebesgue Δ-integrable function. We define the right delta fractional derivative on $\mathbb{T}$ of order $\mu - 1$ as follows*

$$\Delta_{b-}^{\mu-1} f(t) = (-1)^{m} \left(I_{b-}^{\widetilde{\nu}+1} f^{\Delta^{m}} \right)(t) = (-1)^{m} \int_{t}^{b} g_{\widetilde{\nu}}(\sigma(\tau), t) f^{\Delta^{m}}(\tau) \Delta\tau, \quad (20)$$

$\forall\ t \in [a,b] \cap \mathbb{T}$.

Notice $\Delta_{b-}^{\mu-1} f \in C([a,b] \cap \mathbb{T})$, by a simple argument using the dominated convergence theorem in Lebesgue Δ-sense.

If $\mu = m$, then $\widetilde{\nu} = 0$, then

$$\Delta_{b-}^{m-1} f(t) = (-1)^{m} \int_{t}^{b} f^{\Delta^{m}}(\tau) \Delta\tau = (-1)^{m} \left(f^{\Delta^{m-1}}(b) - f^{\Delta^{m-1}}(t) \right). \quad (21)$$

More generally, by [8], given that $f^{\Delta^{m-1}}$ is everywhere finite and absolutely continuous on $[a,b] \cap \mathbb{T}$, then $f^{\Delta^{m}}$ exists Δ-a.e. and is Lebesgue Δ-integrable on $[t,b) \cap \mathbb{T}$, $\forall\ t \in [a,b] \cap \mathbb{T}$, and one can plug it into (20).

Remark 12 *We observe that*

$$\begin{aligned} \left(I_{b-}^{\mu-1} \Delta_{b-}^{\mu-1} f \right)(t) &= (-1)^{m} \left(I_{b-}^{\mu-1} I_{b-}^{\widetilde{\nu}+1} f^{\Delta^{m}} \right)(t) \\ &\overset{(19)}{=} (-1)^{m} \left(\left(I_{b-}^{\mu-1+\widetilde{\nu}+1} f^{\Delta^{m}} \right)(t) - E\left(f^{\Delta^{m}}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t \right) \right) \\ &= (-1)^{m} \left(\left(I_{b-}^{m} f^{\Delta^{m}} \right)(t) - E\left(f^{\Delta^{m}}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t \right) \right). \end{aligned} \quad (22)$$

Therefore

$$\begin{aligned} &\left(I_{b-}^{\mu-1} \Delta_{b-}^{\mu-1} f \right)(t) + (-1)^{m} E\left(f^{\Delta^{m}}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t \right) = \\ &(-1)^{m} \left(I_{b-}^{m} f^{\Delta^{m}}(t) \right) = (-1)^{m} \int_{t}^{b} g_{m-1}(\sigma(\tau), t) f^{\Delta^{m}}(\tau) \Delta\tau \quad (23) \\ &\overset{(10)}{=} R_{m}^{b}(f)(t). \end{aligned}$$

Now we can use (8) with $s = b$.

We have established the following delta time scales right fractional Taylor formula.

Theorem 13 *Assume $\mathbb{T} = \mathbb{T}^{k}$, $f \in C_{rd}^{m}(\mathbb{T})$, $m \in \mathbb{N}$, $a, b \in \mathbb{T}$, and $\mu > 2 : m-1 < \mu \leq m$, $\widetilde{\nu} = m - \mu$; also suppose Assumption 5, Assumption 9. Then*

$$f(t) = \sum_{k=0}^{m-1} f^{\Delta^{k}}(b) h_{k}(t,b) + \left(I_{b-}^{\mu-1} \Delta_{b-}^{\mu-1} f \right)(t) +$$

$$(-1)^m E\left(f^{\Delta^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t\right), \tag{24}$$

$\forall\ t \in [a,b] \cap \mathbb{T}$.

Remark 14 *One can rewrite (24) as follows*

$$f(t) = \sum_{k=0}^{m-1} f^{\Delta^k}(b)\, h_k(t,b) + \int_t^b g_{\mu-2}(\sigma(\tau), t)\left(\Delta_{b-}^{\mu-1} f\right)(\tau)\, \Delta\tau$$

$$+(-1)^m \int_t^b f^{\Delta^m}(\tau)\, g_{\mu-2}(\sigma(\tau), t)\, g_{\widetilde{\nu}}(\sigma(\tau), \tau)\, \mu(\tau)\, \Delta\tau, \tag{25}$$

$\forall\ t \in [a,b] \cap \mathbb{T}$.

Corollary 15 *In the assumptions of Theorem 13, additionally assume that* $f^{\Delta^k}(b) = 0$, $k = 0, 1, ..., m-1$. *Then*

$$B(t) := f(t) + (-1)^{m+1} E\left(f^{\Delta^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t\right)$$

$$= \int_t^b g_{\mu-2}(\sigma(\tau), t)\left(\Delta_{b-}^{\mu-1} f\right)(\tau)\, \Delta\tau, \tag{26}$$

$\forall\ t \in [a,b] \cap \mathbb{T}$.

Remark 16 *Notice (by [8]) that* $\left(I_{b-}^{\mu-1} \Delta_{b-}^{\mu-1} f\right)(t)$ *and* $E(f^{\Delta^m}, \mu-1, \widetilde{\nu}+1, b, \mathbb{T}, t)$ *are absolutely continuous functions on* $[a,b] \cap \mathbb{T}$.

One can use (25) and (26) to establish right fractional delta inequalities on time scales of Poincaré type, Sobolev type, Opial type, Ostrowski type, Hilbert-Pachpatte type, etc, analogous to [4].

To keep article short we avoid this similar task.

Our theory is not void because it is fulfilled when $\mathbb{T} = \mathbb{R}$, etc, see also [4].

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Center manifolds for some partial functional differential equations with infinite delay in fading memory spaces[1]

Mostafa ADIMY*, **Khalil EZZINBI**† **and Catherine MARQUET**‡

*INRIA, Dracula team, Institut Camille Jordan
CNRS UMR 5208, Université Claude Bernard Lyon 1
43 boulevard du 11 novembre 1918
69622 Villeurbanne cedex, France

†Université Cadi Ayyad
Faculté des Sciences Semlalia
Département de Mathématiques, B.P. 2390
Marrakesh, Morocco

‡Université de Pau et des Pays de l'Adour
Laboratoire de Mathématiques Appliquées CNRS UMR 5142
Avenue de l'université 64000
Pau, France

Abstract. We study the existence of a center manifold for some semilinear partial functional differential equations with infinite delay in fading memory spaces. We assume that the unbounded linear part of the equation satisfies the Hille-Yosida condition. The existence of this centre manifold is obtained, under sufficiently small nonlinearity, as the graph of a fixed point for an integral operator given by a variation-of-constants formula. We use a new reduction principle to prove that the flow on the center manifold is completely determined by an ordinary differential equation in a finite dimensional space. When the nonlinear perturbation is only locally Lipschitzian, we obtain the existence of a local center manifold.

Keys words: Partial differential equations, infinite delay, Hille-Yosida operator, integral solution, semigroup, variation-of-constants formula, fading memory space, center manifold, reduction principle.

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*ezzinbi@ucam.ac.ma, †mostafa.adimy@inria.fr, ‡catherine.marquet@univ-pau.fr

1. Introduction

In this paper we prove the existence of a center manifold for the following class of partial functional differential equations with infinite delay

$$(1.1) \qquad \begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + g(u_t), \ t \geq \sigma, \ \sigma \in \mathbb{R}, \\ u_\sigma = \varphi \in \mathcal{B}, \end{cases}$$

where $A : D(A) \to E$ is a linear operator defined on a Banach space $(E, |.|)$. We suppose that A satisfies the following Hille-Yosida condition.

($\mathbf{H_1}$) There exist $\omega \in \mathbb{R}$ and $M_0 \geq 1$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$(1.2) \qquad |R(\lambda, A)^n| \leq \frac{M_0}{(\lambda - \omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda > \omega$.

Without loss of generality, we assume that $M_0 = 1$. Otherwise, one can renorm the space $(E, |.|)$ with an equivalent norm for which we get the estimation (1.2) with $M_0 = 1$. Remark that with the condition ($\mathbf{H_1}$) the domain of the operator A is not necessarily dense in E.

$\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into E satisfying the fundamental axioms introduced by Hale and Kato in [20] (see also [22] and section 2). As usual, for every $t \in \mathbb{R}$, the history function $u_t \in \mathcal{B}$ is defined for $\theta \in (-\infty, 0]$ by $u_t(\theta) = u(t + \theta)$. L is a bounded linear operator from $\mathcal{B}$ to E and g is Lipschitzian from $\mathcal{B}$ into E with $g(0) = 0$.

The center manifold theory is a powerful tool in the analysis of the qualitative behavior of solutions for infinite dimensional dynamical systems, because the interesting behavior takes place on the center manifold. The existence and other proprieties (bifurcation, stability, etc.) of centre manifolds for partial functional differential equations have been considered in many works. For further references on the development and applications of the center manifold theory, we refer the reader to [10, 12–14, 17–19, 21, 23–30] and the references therein.

Partial functional differential equations with infinite delay have been studied extensively in the literature, for the reader, we refer to [1–3, 8, 9, 11, 20, 22, 32] , and the references therein. In [3], the authors proved when A satisfies ($\mathbf{H_1}$) the existence, regularity of solutions and stability of equilibriums of equation (1.1). They obtained that the phase space of (1.1) is given by $Y := \left\{\varphi \in \mathcal{B} : \varphi(0) \in \overline{D(A)}\right\}$. If g is differentiable at 0 with $g'(0) = 0$, then the linearized equation of (1.1) around zero is given by

$$(1.3) \qquad \begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t), \ t \geq \sigma, \\ v_\sigma = \phi \in \mathcal{B}. \end{cases}$$

If all characteristic values of equation (1.3) have negative real part and $\mathcal{B}$ is uniform fading memory space, then the zero equilibrium of (1.1) is uniformly asymptotically stable (more details can be found in [3]). However, if there exists at least one characteristic value with a positive real part, then the zero solution of (1.1) is unstable. In the critical case, when exponential stability is not possible and there exists a characteristic value with zero real part, the situation is more complicated since either stability or instability may hold. The existence of a center manifold brings up the question about the dynamics of the system in this critical case.

In [17] and [29], the authors studied the existence of a center manifold for equation (1.1) when $\overline{D(A)} = E$ and the delay is finite. In this paper, we consider equation (1.1) when the domain $D(A)$ is not necessarily dense in E and the delay is infinite. The non-density of the operator A in equation (1.1) occurs, in many situations, from restrictions made on the space where the equations are considered (for example, periodic continuous solutions, Hölder continuous functions) or from boundary conditions (the space C^1 with null value on the boundary is not dense in the space of continuous functions). For more details, we refer to [4–7] .

As a simple example of equation (1.1), let us consider the following Lotka–Volterra model with diffusion, which has been studied in [5] and [11] (see also [32, p.10])

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = D\dfrac{\partial^2}{\partial x^2}v(t,x) + \displaystyle\int_{-\infty}^{0} K(\theta)v\left(t+\theta,x\right)d\theta + \int_{-\infty}^{0} G\left(\theta, v(t+\theta,x\right)d\theta, \\ \hfill \text{for } t \geq 0 \text{ and } 0 \leq x \leq \pi, \\ v(t,0) = v(t,\pi) = 0, \ \text{for } t \geq 0, \\ v(0,\theta) = v_0(\theta), \ \text{for } -\infty < \theta \leq 0 \text{ and } 0 \leq x \leq \pi. \end{cases}$$

The organization of this work is as follows. In section 2, we recall some results about integral solutions and the semigroup generated by the linearized equation (1.3). We describe the variation-of-constants formula for non-homogeneous problems. We also give some results on the spectral analysis of (1.3). In section 3, we prove the existence of a global center manifold for equation (1.1) with sufficiently small nonlinearities. In section 4, we study the existence of a local center manifold when the function g is only C^1-function in a neighborhood of zero. In section 5, we prove an important result namely that the flow on the center manifold is governed by an ordinary differential equation in a finite dimensional space.

2. Fading memory spaces, variation-of-constants formula and spectral decomposition

We use the axiomatic approach introduced by Hale and Kato [20] for the phase space $\mathcal{B}$. That is the best way to treat the infinite delay differential equations, since properties of solutions depend especially on the choice of the space $\mathcal{B}$. We assume that $(\mathcal{B}, |\cdot|)$ is a linear normed space of functions mapping $(-\infty, 0]$ into the Banach space E and satisfying the following fundamental axioms.

(**A**) There exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to E$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,
(*i*) $x_t \in \mathcal{B}$,
(*ii*) $t \to x_t$ is continuous with respect to $|\cdot|$ on $[\sigma, a]$,
(*iii*) $N\,|x(t)| \leq |x_t| \leq K(t-\sigma) \sup\limits_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma)\,|x_\sigma|$.

(**B**) $\mathcal{B}$ is a Banach space.

As a consequence of axiom (**A**), we deduce the following result.

Lemma 2.1. *[20] Let $C_{00} := C_{00}((-\infty, 0]; E)$ be the space of continuous functions mapping $(-\infty, 0]$ into E with compact support. Then, $C_{00} \subset \mathcal{B}$. More precisely, for $a < 0$, we have*

$$|\varphi| \leq K(-a) \sup_{a \leq \theta \leq 0} |\varphi(\theta)|,$$

for any $\varphi \in C_{00}$ with $supp(\varphi) \subset [a, 0]$.

Our next objective is to write equation (1.1) as an integral equation (variation-of-constants formula). We need the following lemma.

Lemma 2.2. *[7] Assume that $(\mathbf{H}_1)$ holds. Let A_0 be the part of the operator A in $D(A)$, which is defined by*

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax. \end{cases}$$

Then A_0 generates a C_0-semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.

Definition 2.3. [3] Let $\phi \in \mathcal{B}$. A function $u : \mathbb{R} \to E$ is called an integral solution of equation (1.1) on $\mathbb{R}$ if the following conditions hold.
(i) u is continuous on $[\sigma, \infty)$,
(ii) $u_\sigma = \phi$,
(iii) $\displaystyle\int_\sigma^t u(s)\, ds \in D(A)$ for $t \geq \sigma$,
(iv) $\displaystyle u(t) = \phi(0) + A\int_\sigma^t u(s)\, ds + \int_\sigma^t L(u_s) ds + \int_\sigma^t g(x_s)\, ds$ for $t \geq \sigma$.

For the next, we assume that $\mathcal{B}$ satisfies axioms $(\mathbf{A})$ and $(\mathbf{B})$.

Theorem 2.4. *[3] Assume that $(\mathbf{H}_1)$ holds and g is Lipschitzian. Then, for all $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A)}$, equation (1.1) has a unique integral solution which is given by*

$$u(t) = \begin{cases} T_0(t-\sigma)\phi(0) + \lim\limits_{\lambda \to +\infty} \displaystyle\int_\sigma^t T_0(t-s)\lambda R(\lambda, A)\left[L(u_s) + g(x_s)\right] ds \text{ for } t \geq \sigma, \\ \phi(t) \text{ for } t \leq \sigma, \end{cases}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.

Let $Y = \left\{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\right\}$ be the phase space corresponding to equation (1.1). For $t \geq 0$, we define the operator $U(t)$ by

$$U(t)\phi = u_t(\cdot, \phi, L, 0), \ \phi \in Y,$$

where $u(\cdot, \phi, L, 0)$ is the integral solution of equation (1.1) with $g = 0$ and $\sigma = 0$.

Theorem 2.5. *[3] Assume that $(\mathbf{H}_1)$ holds. Then $(U(t))_{t \geq 0}$ is a C_0-semigroup on Y. Moreover, $(U(t))_{t \geq 0}$ satisfies, for $t \geq 0$, $\phi \in Y$, the following translation property.*

$$(U(t)\phi)(\theta) = \begin{cases} (U(t+\theta)\phi)(0) \text{ for } t+\theta \geq 0, \\ \phi(t+\theta) \text{ for } t+\theta \leq 0. \end{cases}$$

In order to give a variation-of-constants formula for equation (1.1), we need to introduce the following sequence of linear operators $\widetilde{B}_n : E \to \mathcal{B}$ defined, for $n > \omega$ and $x \in E$, by

$$(\widetilde{B}_n x)(\theta) = \begin{cases} n(n\theta + 1)R(n, A)x, \ -\dfrac{1}{n} \leq \theta \leq 0, \\ 0, \ \theta < -\dfrac{1}{n}. \end{cases}$$

For $x \in E$, the functions $\widetilde{B}_n x$ belong to C_{00} with the support included in $[-1, 0]$. By Lemma 2.1, we deduce that

$$\left|\widetilde{B}_n x\right| \leq K(1)|x| \text{ for } x \in E \text{ and } n > \omega.$$

The following integral formula of equation (1.1) was obtained in [9].

Theorem 2.6. *[9] Assume that* $(\mathbf{H}_1)$ *holds. Then, for all* $\phi \in Y$*, the integral solution* u *of equation* (1.1) *satisfies the following variation-of-constants formula*

$$u_t = U(t-\sigma)\phi + \lim_{n\to\infty} \int_\sigma^t U(t-s)\widetilde{B}_n g\left(x_s\right) ds \text{ for } t \geq \sigma. \tag{2.1}$$

Moreover, for any $T > \sigma$*, the limit in (2.1) exists uniformly for* $t \in [\sigma, T]$*.*

To get some interesting properties of partial functional differential equations with infinite delay, we need to suppose the following axiom.

$(\mathbf{C})$ If a uniformly bounded sequence $(\varphi_n)_n$ in C_{00} converges to a function φ compactly in $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $|\varphi_n - \varphi| \to 0$ as $n \to \infty$.

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$$

by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta), \ t+\theta \leq 0, \\ 0, \ t+\theta \geq 0. \end{cases}$$

Definition 2.7. Assume that $\mathcal{B}$ satisfies $(\mathbf{C})$.
(i) $\mathcal{B}$ is said to be a fading memory space if for all $\phi \in \mathcal{B}_0$,

$$S_0(t)\phi \underset{t\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{B}.$$

(ii) Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$|S_0(t)| \underset{t\to\infty}{\longrightarrow} 0.$$

The following results give some properties of fading memory spaces.

Lemma 2.8. *[22]* (i) *If* $\mathcal{B}$ *is a fading memory space, then the functions* $K(\cdot)$ *and* $M(\cdot)$ *in* $(\mathbf{A})$ *can be chosen to be constants.*
(ii) *If* $\mathcal{B}$ *is a uniform fading memory space, then the functions* $K(\cdot)$ *and* $M(\cdot)$ *can be chosen such that* $K(\cdot)$ *is constant and* $M(t) \to 0$ *as* $t \to \infty$*.*

Proposition 2.9. *[22] If* $\mathcal{B}$ *is a fading memory space, then the space* $\mathcal{BC}((-\infty, 0]; E)$ *of all bounded and continuous* E*-valued functions on* $(-\infty, 0]$*, endowed with the uniform norm topology, is continuously embedding in* $\mathcal{B}$*.*

In order to study the qualitative behavior of the semigroup $(U(t))_{t\geq 0}$, we suppose the following assumption.

$(\mathbf{H}_2)$ $T_0(t)$ is compact on $\overline{D(A)}$ for each $t>0$.

Let V be a bounded subset of a Banach space Z. The Kuratowski measure of non-compactness $\alpha(V)$ of V is defined by

$$\alpha(V)=\inf\left\{\begin{array}{c} d>0 \text{ such that there exists a finite number of sets } V_1,...,V_n \text{ with} \\ diam(V_i)\leq d \text{ such that } V\subseteq \bigcup\limits_{i=1}^{n} V_i \end{array}\right\}.$$

Moreover, for a bounded linear operator P on Z, we define $|P|_\alpha$ by

$$|P|_\alpha=\inf\{k>0:\alpha(P(V))\leq k\alpha(V) \text{ for any bounded set } V \text{ in } Z\}.$$

For the semigroup $(U(t))_{t\geq 0}$, we define the essential growth bound $\omega_{ess}(U)$ by

$$\omega_{ess}(U):=\lim_{t\to\infty}\frac{1}{t}\log|U(t)|_\alpha.$$

We have the following fundamental result.

Theorem 2.10. *[11] Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, and* $\mathcal{B}$ *is a uniform fading memory space. Then*

$$\omega_{ess}(U)<0.$$

Definition 2.11. Let $\mathcal{C}$ be a densely defined operator on Z. The essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda\in\sigma(\mathcal{C})$ such that one of the following conditions holds.

(i) $Im(\lambda I-\mathcal{C})$ is not closed,

(ii) the generalized eigenspace $M_\lambda(\mathcal{C}):=\bigcup\limits_{k\geq 1}Ker(\lambda I-\mathcal{C})^k$ is of infinite dimension,

(iii) λ is a limit point of $\sigma(\mathcal{C})\setminus\{\lambda\}$.

The essential radius of any bounded operator P on Z is defined by

$$r_{ess}(P):=\sup\{|\lambda|:\lambda\in\sigma_{ess}(P)\}.$$

Let A_U denote the infinitesimal generator of $(U(t))_{t\geq 0}$. Then

$$\begin{cases} D(A_U)=\{\varphi\in Y:\varphi'\in Y,\ \varphi(0)\in D(A) \text{ and } \varphi'(0)=A\varphi(0)+L(\varphi)\}, \\ A_U\varphi=\varphi'. \end{cases}$$

Let

$$\sigma^+(A_U)=\{\lambda\in\sigma(A_U):\mathcal{R}e(\lambda)\geq 0\}.$$

As an immediate consequence of Theorem 2.10, we have the following spectral property of A_U.

Lemma 2.12. *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, and* $\mathcal{B}$ *is a uniform fading memory space. Then,* $\sigma^+(A_U)$ *is a finite set of eigenvalues of* A_U *which are not in the essential spectrum. More precisely,* $\lambda\in\sigma^+(A_U)$ *if and only if there exists* $x\in D(A)\backslash\{0\}$ *solving the following characteristic equation*

$$\Delta(\lambda)x:=\lambda x-Ax-L(e^{\lambda\cdot}x)=0,$$

where $e^{\lambda\cdot}x$ *is the element of* $\mathcal{B}$ *defined for all* $\theta\leq 0$ *by* $(e^{\lambda\cdot}x)(\theta):=e^{\lambda\theta}x$.

Proof. Theorem 2.10 implies that $\omega_{ess}(U) < 0$. By Corollary 2.11 of ([16], page 258), we know that $\sigma^+(A_U)$ is a finite subset of the point spectrum $\sigma_p(A_U)$. On the other hand, we have for $t \geq 0$

$$r_{ess}(U(t)) = e^{t\omega_{ess}(U)} < 1 \text{ and } e^{t\sigma_{ess}(A_U)} \subset \sigma_{ess}(U(t)).$$

It follows that

$$\sigma_{ess}(A_U) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}.$$

Consequently, $\sigma^+(A_U)$ is a finite subset of the point spectrum $\sigma_p(A_U)$. Let $\lambda \in \sigma^+(A_U)$. Then, there exists $\varphi \in D(A_U)$, $\varphi \neq 0$ such that $A_U\varphi = \lambda\varphi$, which implies that

$$\lim_{t \to 0^+} \frac{1}{t}(U(t)\varphi - \varphi) = \lambda\varphi.$$

By axiom $(\mathbf{A}) - (ii)$, we deduce that $\lim_{t\to 0^+} \left(\frac{1}{t}(U(t)\varphi - \varphi)\right)(0) = \lambda\varphi(0)$. On the other hand,

$$\left(\frac{1}{t}(U(t)\varphi - \varphi)\right)(0) = A\left(\frac{1}{t}\int_0^t (U(s)\varphi)(0)ds\right) + \frac{1}{t}\int_0^t L(U(s)\varphi)ds.$$

Let t go to zero. From the closedness of the operator A, we obtain that

$$\varphi(0) \in D(A) \text{ and } A\varphi(0) + L(\varphi) = \lambda\varphi(0).$$

By the spectral mapping theorem ([16], page 277), we have

$$e^{\lambda t} \in \sigma_p(U(t)) \text{ and } U(t)\varphi = e^{\lambda t}\varphi.$$

Using the translation property of the semigroup solution, we obtain that

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) \text{ for } \theta \leq 0.$$

Since $\varphi \neq 0$, then $\varphi(0) \neq 0$ and

$$\ker \Delta(\lambda) \neq \{0\}.$$

Conversely, let $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) \geq 0$. Then, for all $x \in E$ we have $e^{\lambda \cdot}x \in \mathcal{B}$. If there exists $a \in D(A) \setminus \{0\}$ such that $\Delta(\lambda)a = 0$, then $\varphi = e^{\lambda \cdot}a \in Y$. Hence $\varphi \in C^1((-\infty, 0]; E)$ and $\varphi'(0) = \lambda a = A\varphi(0) + L(\varphi) \in D(A)$. By Theorem 3 in [3], we conclude that

$$\varphi \in D(A_U) \text{ and } A_U\varphi = \lambda\varphi. \ \square$$

As $\omega_{ess}(U) < 0$, we have the following spectral decomposition of the phase space Y.

Theorem 2.13. *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, and* $\mathcal{B}$ *is a uniform fading memory space. Then, there exist linear subspaces of* Y *denoted by* Y_-, Y_0 *and* Y_+ *respectively with*

$$Y = Y_- \oplus Y_0 \oplus Y_+$$

such that
(i) $A_U(Y_-) \subset Y_-$, $A_U(Y_0) \subset Y_0$ *and* $A_U(Y_+) \subset Y_+$;
(ii) Y_0 *and* Y_+ *are finite dimensional;*
(iii) $\sigma(A_U|Y_0) = \{\lambda \in \sigma(A_U) : \mathrm{Re}\,\lambda = 0\}$, $\sigma(A_U|Y_+) = \{\lambda \in \sigma(A_U) : \mathrm{Re}\,\lambda > 0\}$;
(iv) $U(t)Y_- \subset Y_-$ *for* $t \geq 0$ *and* $U(t)|Y_0 \cup Y_+$ *can be extended to* $t < 0$ *such that* $U(t)Y_0 \subset Y_0$, $U(t)Y_+ \subset Y_+$ *for* $t \in \mathbb{R}$;

(v) for any $0 < \gamma < \inf\{|\mathrm{Re}\,\lambda| : \lambda \in \sigma(A_U) \text{ and } \mathrm{Re}\,\lambda \neq 0\}$, *there exists* $K \geq 1$ *such that, for* $\varphi \in Y$,

$$\begin{array}{l} |U(t)P_-\varphi| \leq Ke^{-\gamma t}\,|P_-\varphi| \text{ for } t \geq 0, \\ |U(t)P_0\varphi| \leq Ke^{\frac{\gamma}{3}|t|}\,|P_0\varphi| \text{ for } t \in \mathbb{R}, \\ |U(t)P_+\varphi| \leq Ke^{\gamma t}\,|P_+\varphi| \text{ for } t \leq 0, \end{array}$$

where P_-, P_0 *and* P_+ *are the projections of* Y *into* Y_-, Y_0 *and* Y_+.
Y_-, Y_0 *and* Y_+ *are called respectively the stable, center and unstable subspaces of the semigroup* $(U(t))_{t\geq 0}$.

3. Global existence of the center manifold

Theorem 3.1. *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold, and there exists* $\varepsilon > 0$ *such that*

$$Lip(g) = \sup_{\varphi_1 \neq \varphi_2} \frac{|g(\varphi_1) - g(\varphi_2)|}{|\varphi_1 - \varphi_2|} < \varepsilon.$$

Then, there exists a bounded Lipschitz map $h_g : Y_0 \to Y_- \oplus Y_+$ *such that* $h_g(0) = 0$ *and the Lipschitz manifold*

$$M_g := \{\varphi + h_g(\varphi) : \varphi \in Y_0\}$$

is globally invariant under the flow of equation (1.1) on Y.
M_g *is called the center manifold of equation (1.1).*

Proof. Let $\mathcal{M} = BC(Y_0, Y_- \oplus Y_+)$ denote the Banach space of bounded continuous maps $h : Y_0 \to Y_- \oplus Y_+$ endowed with the uniform norm topology. We define

$$\mathcal{F} := \{h \in \mathcal{M} : h \text{ is Lipschitz, } h(0) = 0 \text{ and } Lip(h) \leq 1\}.$$

Let $h \in \mathcal{F}$ and $\varphi \in Y_0$. Using the strict contraction principle, one can prove the existence of $v_t^\varphi \in Y_0$ solution of the following equation

$$v_t^\varphi = U(t)\varphi + \lim_{n\to\infty} \int_0^t U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^0 d\tau, \; t \in \mathbb{R}. \tag{3.1}$$

We now introduce the mapping $T_g : \mathcal{F} \to \mathcal{M}$ defined, for $h \in \mathcal{F}$ and $\varphi \in Y_0$, by

$$\begin{aligned} T_g(h)\varphi = {} & \lim_{n\to\infty} \int_{-\infty}^0 U(-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^- d\tau \\ & - \lim_{n\to\infty} \int_0^{+\infty} U(-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^+ d\tau. \end{aligned}$$

The first step is to prove that T_g maps $\mathcal{F}$ into itself. Let $\varphi_1, \varphi_2 \in Y_0$, $0 < \gamma < \inf\{|\mathrm{Re}\,\lambda| : \lambda \in \sigma(A_U) \text{ and } \mathrm{Re}\,\lambda \neq 0\}$ and $t \in \mathbb{R}$. Suppose that $Lip(g) < \varepsilon$. Then,

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq Ke^{\frac{\gamma}{3}|t|}\,|\varphi_1 - \varphi_2| + 2K\,|P_0|\,\varepsilon \left| \int_0^t e^{\frac{\gamma}{3}|t-\tau|}\,|v_\tau^{\varphi_1} - v_\tau^{\varphi_2}|\,d\tau \right|,$$

where K is the positive constant given by Theorem (2.13). By Gronwall's lemma, we get that

$$e^{-\frac{\gamma}{3}|t|}\,|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K\,|\varphi_1 - \varphi_2|\,e^{2K|P_0|\varepsilon|t|}.$$

Then

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K\,|\varphi_1 - \varphi_2|\,e^{\left[\frac{\gamma}{3} + 2K|P_0|\varepsilon\right]|t|}, \; t \in \mathbb{R}.$$

If we choose ε such that

$$2K\,|P_0|\,\varepsilon < \frac{\gamma}{6}, \tag{3.2}$$

we obtain

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K\,|\varphi_1 - \varphi_2|\,e^{\frac{\gamma}{2}|t|}. \tag{3.3}$$

Moreover,

$$\begin{aligned}|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq & \int_{-\infty}^{0} |U(-\tau)P_-|\,|g(v_\tau^{\varphi_1} + h(v_\tau^{\varphi_1})) - g(v_\tau^{\varphi_2} + h(v_\tau^{\varphi_2}))|\,d\tau \\ & + \int_0^{+\infty} |U(-\tau)P_+|\,|g(v_\tau^{\varphi_1} + h(v_\tau^{\varphi_1})) - g(v_\tau^{\varphi_2} + h(v_\tau^{\varphi_2}))|\,d\tau.\end{aligned}$$

Consequently,

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq 2\int_{-\infty}^{0} Ke^{\gamma\tau}\,|P_-|\,\varepsilon\,|v_\tau^{\varphi_1} - v_\tau^{\varphi_2}|\,d\tau + 2\int_0^{+\infty} Ke^{-\gamma\tau}\,|P_+|\,\varepsilon\,|v_\tau^{\varphi_1} - v_\tau^{\varphi_2}|\,d\tau.$$

Using the inequality (3.3), we obtain

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq 2K^2\,|P_-|\,\varepsilon\,|\varphi_1 - \varphi_2| \int_{-\infty}^{0} e^{\frac{\gamma}{2}\tau}d\tau + 2K^2\,|P_+|\,\varepsilon\,|\varphi_1 - \varphi_2| \int_0^{+\infty} e^{-\frac{\gamma}{2}\tau}d\tau.$$

It follows that

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq \frac{4\varepsilon}{\gamma}K^2\left(|P_-| + |P_+|\right)|\varphi_1 - \varphi_2|\,.$$

We choose ε such that

$$\frac{4\varepsilon}{\gamma}K^2\left(|P_-| + |P_+|\right) < 1.$$

Then T_g maps $\mathcal{F}$ into itself.

The next step is to show that T_g is a strict contraction on $\mathcal{F}$. Let $h_1, h_2 \in \mathcal{F}$ and $\varphi \in Y_0$. Let $v_t^i,\ i = 1, 2$, denote the solution of the following equation

$$v_t^i = U(t)\varphi + \lim_{n\to\infty} \int_0^t U(t-\tau)\left(\widetilde{B}_n g(v_\tau^i + h_i(v_\tau^i))\right)^0 d\tau \text{ for } t \in \mathbb{R}.$$

Then,

$$\left|v_t^1 - v_t^2\right| \leq \varepsilon K\,|P_0| \left|\int_0^t e^{\frac{\gamma}{3}|t-\tau|}\left(\left|v_\tau^1 - v_\tau^2\right| + \left|h_1(v_\tau^1) - h_1(v_\tau^2)\right| + \left|h_1(v_\tau^2) - h_2(v_\tau^2)\right|\right) d\tau\right|.$$

Consequently,

$$\left|v_t^1 - v_t^2\right| \leq 2\varepsilon K\,|P_0| \left|\int_0^t e^{\frac{\gamma}{3}|t-\tau|}\left|v_\tau^1 - v_\tau^2\right| d\tau\right| + \varepsilon K\,|P_0|\,|h_1 - h_2| \left|\int_0^t e^{\frac{\gamma}{3}|t-\tau|} d\tau\right|.$$

By Gronwall's lemma, we obtain

$$\left|v_t^1 - v_t^2\right| \leq \frac{3\varepsilon K}{\gamma}\,|P_0|\,|h_1 - h_2|\,e^{\left[\frac{\gamma}{3} + 2K|P_0|\varepsilon\right]|t|}.$$

Thanks to (3.2), we state

$$\left|v_t^1 - v_t^2\right| \leq \frac{3\varepsilon K}{\gamma}\,|P_0|\,|h_1 - h_2|\,e^{\frac{\gamma}{2}|t|} \text{ for all } t \in \mathbb{R}.$$

For $i = 1, 2$, we have

$$T_g(h_i)\varphi = \lim_{n\to\infty} \int_{-\infty}^{0} U(-\tau) \left(\widetilde{B}_n g(v_\tau^i + h_i(v_\tau^i)) \right)^- d\tau$$
$$- \lim_{n\to\infty} \int_{0}^{+\infty} U(-\tau) \left(\widetilde{B}_n g(v_\tau^i + h_i(v_\tau^i)) \right)^+ d\tau.$$

It follows that

$$|T_g(h_1)\varphi - T_g(h_2)\varphi| \leq 2K\,|P_-| \frac{6\varepsilon^2 K}{\gamma^2} |P_0| \, |h_1 - h_2| + \frac{K\,|P_-|\,\varepsilon}{\gamma} |h_1 - h_2|$$
$$+ 2K\,|P_+| \frac{6\varepsilon^2 K}{\gamma^2} |P_0| \, |h_1 - h_2| + \frac{K\,|P_+|\,\varepsilon}{\gamma} |h_1 - h_2| \, .$$

Consequently,

$$|T_g(h_1) - T_g(h_2)| \leq (|P_-| + |P_+|) \frac{K\varepsilon}{\gamma} \left[|P_0| \frac{12\varepsilon K}{\gamma} + 1 \right] |h_1 - h_2| \, .$$

We choose ε such that

$$(|P_-| + |P_+|) \frac{K\varepsilon}{\gamma} \left[|P_0| \frac{12\varepsilon K}{\gamma} + 1 \right] < 1.$$

Then T_g is a strict contraction on $\mathcal{F}$. Consequently, it has a unique fixed point h_g in $\mathcal{F}$ ($T_g(h_g) = h_g$).
Finally, we show that

$$M_g := \{ \varphi + h_g(\varphi) : \varphi \in Y_0 \}$$

is globally invariant under the flow on Y. Let $\varphi \in Y_0$ and v be the solution of equation (3.1). We claim that $t \to v_t^\varphi + h_g(v_t^\varphi)$ is an integral solution of equation (1.1) with initial condition $\varphi + h_g(\varphi)$. In fact, we have $T_g(h_g)(v_t^\varphi) = h_g(v_t^\varphi)$ for $t \in \mathbb{R}$. Moreover,

$$T_g(h)(v_t^\varphi) = \lim_{n\to\infty} \int_{-\infty}^{0} U(-\tau) \left(\widetilde{B}_n g(v_{t+\tau}^\varphi + h_g(v_{t+\tau}^\varphi)) \right)^- d\tau$$
$$- \lim_{n\to\infty} \int_{0}^{+\infty} U(-\tau) \left(\widetilde{B}_n g(v_{t+\tau}^\varphi + h_g(v_{t+\tau}^\varphi)) \right)^+ d\tau.$$

Therefore

$$h_g(v_t^\varphi) = \lim_{n\to\infty} \int_{-\infty}^{t} U(t-\tau) \left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^- d\tau$$
$$- \lim_{n\to\infty} \int_{t}^{+\infty} U(t-\tau) \left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^+ d\tau.$$

Then, for $t \in \mathbb{R}$, we obtain

$$v_t^\varphi + h_g(v_t^\varphi) = U(t)\varphi + \lim_{n\to\infty} \int_{0}^{t} U(t-\tau) \left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^0 d\tau$$
$$+ \lim_{n\to\infty} \int_{-\infty}^{t} U(t-\tau) \left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^- d\tau$$
$$- \lim_{n\to\infty} \int_{t}^{+\infty} U(t-\tau) \left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^+ d\tau.$$

For any $t \geq a$, we have

$$\lim_{n\to\infty}\int_{-\infty}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau$$
$$= \lim_{n\to\infty}\int_{-\infty}^{a} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau$$
$$+\lim_{\lambda\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau,$$

and

$$\lim_{\lambda\to\infty}\int_{-\infty}^{a} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau$$
$$= U(t-a)\left(\lim_{n\to\infty}\int_{-\infty}^{a} U(a-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau\right).$$

By the same argument, we have

$$\lim_{n\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau$$
$$= \lim_{n\to\infty}\int_{a}^{+\infty} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau$$
$$-\lim_{n\to\infty}\int_{t}^{+\infty} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau,$$

and

$$\lim_{n\to\infty}\int_{a}^{+\infty} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau$$
$$= U(t-a)\left(\lim_{n\to\infty}\int_{a}^{+\infty} U(a-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau\right).$$

Moreover, note that

$$h_g(v_a^\varphi) = \lim_{n\to\infty}\int_{-\infty}^{a} U(a-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau$$
$$- \lim_{n\to\infty}\int_{a}^{+\infty} U(a-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau.$$

In particular

$$v_t^\varphi = U(t-a)v_a^\varphi + \lim_{n\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau.$$

Consequently, for any $t \geq a$, we obtain

$$v_t^\varphi + h_g(v_t^\varphi) = U(t-a)\left(v_a^\varphi + h_g(v_a^\varphi)\right) + \lim_{n\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau$$
$$+ \lim_{n\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau$$
$$+ \lim_{n\to\infty}\int_{a}^{t} U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau.$$

Therefore, for any $t \geq a$,

$$v_t^\varphi + h_g(v_t^\varphi) = U(t-a)\left(v_a^\varphi + h_g(v_a^\varphi)\right) + \lim_{n\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B}_n g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)d\tau.$$

Finally, we conclude that $v_t^\varphi + h_g(v_t^\varphi)$ is an integral solution of equation (1.1) on $\mathbb{R}$ with initial value $\varphi + h_g(\varphi)$. $\square$

Theorem 3.2. *Let $h \in \mathcal{F}$ and $\varphi \in Y_0$. If v^φ is the solution of equation (3.1) on $\mathbb{R}$, then*

$$|v_t^\varphi| \leq K|\varphi| e^{\frac{\gamma}{2}|t|} \text{ and } |v_t^\varphi + h_g(v_t^\varphi)| \leq 2K|\varphi| e^{\frac{\gamma}{2}|t|}.$$

Conversely, if we choose ε such that

$$\frac{2K\varepsilon}{\gamma}\left(|P_-| + |P_+| + 3|P_0|\right) < 1,$$

then for any integral solution u of equation (1.1) on $\mathbb{R}$ with $u_t = O(e^{\frac{\gamma}{2}|t|})$, we have

$$u_t \in M_g \text{ for all } t \in \mathbb{R}.$$

Proof. Let v^φ be the solution of equation (3.1). Then, using the estimate (3.3) we obtain

$$|v_t^\varphi| \leq K|\varphi| e^{\frac{\gamma}{2}|t|} \text{ for } t \in \mathbb{R}.$$

Moreover, from the fact that $Lip(h) \leq 1$ and $h_g(0) = 0$, we obtain

$$|v_t^\varphi + h_g(v_t^\varphi)| \leq 2K|\varphi| e^{\frac{\gamma}{2}|t|} \text{ for } t \in \mathbb{R}.$$

Let u be an integral solution of equation (1.1) such that $u_t = O(e^{\frac{\gamma}{2}|t|})$. Then there exists a positive constant K_0 such that $|u_t| \leq K_0 e^{\frac{\gamma}{2}|t|}$ for all $t \in \mathbb{R}$. Note that

$$u_t = U(t-s)u_s + \lim_{n\to\infty}\int_s^t U(t-\tau)\widetilde{B}_n g(u_\tau)d\tau,\ t \geq s.$$

On the other hand,

$$u_t^+ = U(t-s)u_s^+ + \lim_{n\to\infty}\int_s^t U(t-\tau)\left(\widetilde{B}_n g(u_\tau)\right)^+ d\tau,\ s \geq t.$$

Moreover, for $s \geq t$ and $s \geq 0$ we have

$$\left|U(t-s)u_s^+\right| \leq Ke^{\gamma(t-s)}|P_+ u_s| \leq K_0 K|P_+| e^{\gamma(t-s)} e^{\frac{\gamma}{2}|s|} = K_0 K|P_+| e^{\gamma t} e^{-\frac{\gamma}{2}s}.$$

Therefore,

$$\lim_{s\to\infty}\left|U(t-s)u_s^+\right| = 0.$$

It follows that

$$u_t^+ = -\lim_{n\to\infty}\int_t^{+\infty} U(t-\tau)\left(\widetilde{B}_n g(u_\tau)\right)^+ d\tau.$$

Similarly, we can prove that

$$u_t^- = \lim_{n\to\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B}_n g(u_\tau)\right)^- d\tau.$$

We conclude that

$$u_t = u_t^+ + u_t^- + U(t)u_0^0 + \lim_{n\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B}_n g(u_\tau)\right)^0 d\tau \text{ for } t \in \mathbb{R}.$$

Let $\phi \in Y_0$ such that $\phi = u_0$. By Theorem 3.1, there exists an integral solution w of equation (1.1) on $\mathbb{R}$ with initial value $\phi + h_g(\phi)$ such that, for all $t \in \mathbb{R}$, $w_t \in M_g$ and

$$\begin{aligned} w_t = U(t)\phi &+ \lim_{n\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B}_n g(w_\tau)\right)^0 d\tau \\ &+ \lim_{n\to\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B}_n g(w_\tau)\right)^+ d\tau \\ &- \lim_{n\to\infty}\int_t^{+\infty} U(t-\tau)\left(\widetilde{B}_n g(w_\tau)\right)^- d\tau. \end{aligned}$$

Then

$$\begin{aligned} |u_t - w_t| \le &\left|\lim_{n\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B}_n\left(g(u_\tau)-g(w_\tau)\right)\right)^0 d\tau\right| \\ &+\left|\lim_{n\to\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B}_n\left(g(u_\tau)-g(w_\tau)\right)\right)^+ d\tau\right| \\ &+\left|\lim_{n\to\infty}\int_t^{+\infty} U(t-\tau)\left(\widetilde{B}_n\left(g(u_\tau)-g(w_\tau)\right)\right)^- d\tau\right|. \end{aligned}$$

This implies that

$$\begin{aligned} |u_t - w_t| \le K\varepsilon &\left(|P_0|\left|\int_0^t e^{\frac{\gamma}{3}|t-\tau|}\left|u_\tau - w_\tau\right| d\tau\right|\right. \\ &\left.+\left|P_-\right|\int_{-\infty}^t e^{-\gamma(t-\tau)}\left|u_\tau - w_\tau\right| d\tau + |P_+|\int_t^{+\infty} e^{\gamma(t-\tau)}\left|u_\tau - w_\tau\right| d\tau\right). \end{aligned}$$

Let $N(t) = e^{-\frac{\gamma}{2}|t|}\left|u_t - w_t\right|$ for all $t \in \mathbb{R}$. Then, $\widetilde{N} = \sup_{t\in\mathbb{R}}\left(N(t)\right) < \infty$. On the other hand, we have

$$N(t) \le K\varepsilon\widetilde{N}\left(|P_0|\left|\int_0^t e^{-\frac{\gamma}{6}|t-\tau|}d\tau\right| + |P_-|\int_{-\infty}^t e^{-\frac{\gamma}{2}(t-\tau)}d\tau + |P_+|\int_t^{+\infty} e^{\frac{\gamma}{2}(t-\tau)}d\tau\right).$$

Finally, we obtain

$$\widetilde{N} \le \frac{2K\varepsilon}{\gamma}\left(3\left|P_0\right| + |P_-| + |P_+|\right)\widetilde{N}.$$

Consequently, $\widetilde{N} = 0$ and $u_t = w_t$ for $t \in \mathbb{R}$. In particular, M_g contains all bounded solutions of equation (1.1). □

4. **Local existence of the center manifold**

In this section, we prove the existence of a local center manifold when g is defined in a neighborhood of zero. We suppose that

($\mathbf{H}_3$) There exists $\rho_1 > 0$ such that $g : B(0,\rho_1) \to E$ is C^1-function, $g(0) = 0$ and $g'(0) = 0$, where $B(0,\rho_1) = \{\varphi \in \mathcal{B} : |\varphi| < \rho_1\}$.

For $\rho < \rho_1$, we define the cut-off function $g_\rho : \mathcal{B} \to E$ by

$$g_\rho(\varphi) = \begin{cases} g(\varphi) \text{ if } |\varphi| \le \rho, \\ g(\frac{\rho}{|\varphi|}\varphi) \text{ if } |\varphi| > \rho. \end{cases}$$

We consider the following partial functional differential equation

$$
\left\{
\begin{array}{l}
\dfrac{d}{dt}u(t) = Au(t) + L(u_t) + g_\rho(u_t), \ t \geq \sigma, \\
u_\sigma = \varphi \in \mathcal{B}.
\end{array}
\right.
\tag{4.1}
$$

Theorem 4.1. *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold. Then there exist* $0 < \rho < \rho_1$ *and a Lipschitz continuous a mapping* $h_{g_\rho} : Y_0 \to Y_- \oplus Y_+$ *such that* $h_{g_\rho}(0) = 0$ *and the local Lipschitz manifold*

$$M_{g_\rho} = \left\{ \varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0 \right\}$$

is globally invariant under the flow associated to equation (4.1).

Proof. Using the same arguments as in [31], Proposition 3.10, p.95, one can show that g_ρ is Lipschitz continuous with

$$Lip(g_\rho) \leq 2 \sup_{|\varphi| < \rho} |g'(\varphi)| .$$

It follows that $Lip(g_\rho)$ goes to zero when ρ goes to zero. According to Theorem 3.1, we deduce that there exist $\rho \geq 0$ and mapping $h_{g_\rho} : Y_0 \to Y_- \oplus Y_+$ with $h_{g_\rho}(0) = 0$ such that the Lipschitz manifold

$$M_{g_\rho} = \left\{ \varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0 \right\}$$

is globally invariant under the flow of equation (4.1). □

Remark 4.2. (i) The set

$$\widetilde{M}_{g_\rho} := \left\{ \varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0 \right\} \cap B(0, \rho)$$

is called the local center manifold associated to equation (1.1).
(ii) $\widetilde{M}_{g_\rho}$ contains all bounded integral solutions by ρ of equation (1.1).

5. Reduction principle: flow on the center manifold

We establish that the flow on the center manifold is governed by an ordinary differential equation in a finite dimensional space. We assume that the conditions of Theorem 3.1 are satisfied. We also assume that the unstable space Y_+ is reduced to zero. This means that $\sigma(A_U) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\}$ and

$$Y = Y_- \oplus Y_0.$$

Let d be the dimension of the center space Y_0 and $\Phi = (\phi_1, ..., \phi_d)$ be a basis of Y_0. Then, there exist d elements $\phi_1^*, ..., \phi_d^*$ of Y^*, the dual space of Y, such that

$$
\left\{
\begin{array}{l}
\langle \phi_i^*, \phi_j \rangle := \phi_i^*(\phi_j) = \delta_{ij}, \ 1 \leq i, j \leq d, \\
\phi_i^* = 0 \text{ on } Y_-.
\end{array}
\right.
$$

Denote by Ψ the transpose of $(\phi_1^*, ..., \phi_d^*)$. Let $\phi \in Y$. Define

$$\langle \Psi, \phi \rangle := \left(\phi_1^*(\phi), ..., \phi_d^*(\phi) \right)^T .$$

Then, the projection operator $P_0 : Y \to Y_0$ is given by

$$P_0 \phi = \Phi \langle \Psi, \phi \rangle .$$

Put $U^0(t) := U(t)|Y_0$. As Y_0 is a finite dimensional space, $(U^0(t))_{t\in\mathbb{R}}$ is a strongly continuous group. Then, there exists a $d\times d$ matrix G ([15], Theorem 2.15, p.102) such that

$$U^0(t)\Phi = \Phi e^{tG},\ t\in\mathbb{R}.$$

Let $n\in\mathbb{N}$, $n\geq n_0>\omega$ and $i\in\{1,....,d\}$. We define the function $x^*_{n,i}$ on E by

$$x^*_{n,i}(y) = \left\langle \phi^*_i, \widetilde{B}_n y\right\rangle,\ y\in E.$$

Then, $x^*_{n,i}$ is a bounded linear operator on E. Let x^*_n be the transpose of $(x^*_{n,1},...,x^*_{n,d})$. We obtain

$$\langle x^*_n, y\rangle = \left\langle \Psi, \widetilde{B}_n\left(y\right)\right\rangle.$$

Consequently,

$$\sup_{n\geq n_0}|x^*_n| < \infty.$$

This implies that $(x^*_n)_{n\geq n_0}$ is a bounded sequence in $\mathcal{L}(E,\mathbb{R}^d)$. Then, we get the following important result.

Theorem 5.1. *There exists $x^*\in\mathcal{L}(E,\mathbb{R}^d)$ such that $(x^*_n)_{n\geq n_0}$ converges weakly to x^* in the sense that*

$$\langle x^*_n, y\rangle \to \langle x^*, y\rangle \text{ as } n\to\infty \text{ for } y\in E.$$

To prove this result, we need the following fundamental theorem [33, p. 776]

Theorem 5.2. *Let X be a separable Banach space and $(z^*_n)_{n\geq p}$ be a bounded sequence in X^*. Then there exists a subsequence $\left(z^*_{n_k}\right)_{k\geq 0}$ of $(z^*_n)_{n\geq p}$ which converges weakly in X^* in the sense that there exists $z^*\in X^*$ such that*

$$\left\langle z^*_{n_k}, y\right\rangle \to \langle z^*, y\rangle \text{ as } k\to\infty \text{ for } y\in X.$$

Proof of Theorem 5.1. Let Z_0 be a closed separable subspace of E. Since $(x^*_n)_{n\geq n_0}$ is a bounded sequence, thanks to Theorem 5.2 there is a subsequence $\left(x^*_{n_k}\right)_{k\in\mathbb{N}}$ which converges weakly to some $x^*_{Z_0}$ in Z_0. We claim that all the sequence $(x^*_n)_{n\geq n_0}$ converges weakly to $x^*_{Z_0}$ in Z_0. This can be done by contradiction. Suppose that there exists a subsequence $\left(x^*_{n_q}\right)_{q\in\mathbb{N}}$ of $(x^*_n)_{n\geq n_0}$ which converges weakly to some $\widetilde{x}^*_{Z_0}$ with $\widetilde{x}^*_{Z_0}\neq x^*_{Z_0}$. Let $\widetilde{u}_t(.,\sigma,\varphi,f)$ be the integral solution of the following equation

$$\begin{cases} \dfrac{d}{dt}\widetilde{u}(t) = A\widetilde{u}(t) + L(\widetilde{u}_t) + f(t),\ t\geq\sigma,\\ \widetilde{u}_\sigma = \varphi\in\mathcal{B}, \end{cases}$$

where f is a continuous function from $\mathbb{R}$ to E. By using the variation-of-constants formula and the spectral decomposition, we obtain

$$P_0\widetilde{u}_t(.,\sigma,0,f) = \lim_{n\to+\infty}\int_\sigma^t U\left(t-\xi\right)\left(\widetilde{B}_n f(\xi)\right)^0 d\xi$$

and

$$P_0\left(\widetilde{B}_n f(\xi)\right) = \Phi\left\langle \Psi, \widetilde{B}_n f(\xi)\right\rangle = \Phi\left\langle x^*_n, f(\xi)\right\rangle.$$

It follows that

$$P_0\widetilde{u}_t(.,\sigma,0,f) = \lim_{n\to+\infty}\Phi\int_\sigma^t e^{(t-\xi)G}\left\langle \Psi, \widetilde{B}_n f(\xi)\right\rangle d\xi,$$
$$= \lim_{n\to+\infty}\Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x_n^*, f(\xi)\right\rangle d\xi.$$

Let $f = c$ (for a fixed $c \in Z_0$) be a constant function. Then

$$\lim_{k\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_k}^*, c\right\rangle d\xi = \lim_{p\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_p}^*, c\right\rangle d\xi.$$

Consequently,

$$\int_\sigma^t e^{(t-\xi)G}\left\langle x_{Z_0}^*, c\right\rangle d\xi = \int_\sigma^t e^{(t-\xi)G}\left\langle \widetilde{x}_{Z_0}^*, c\right\rangle d\xi.$$

Hence $x_{Z_0}^* = \widetilde{x}_{Z_0}^*$. This yields to a contradiction.

We now conclude that the whole sequence $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_0}^*$ in Z_0. Let Z_1 be another closed separable subspace of E. By using the same argument as above, we obtain that $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_1}^*$ in Z_1. Since $Z_0 \cap Z_1$ is a closed separable subspace of E, we get that $x_{Z_1}^* = x_{Z_0}^*$ in $Z_0 \cap Z_1$. For any $y \in E$, we define x^* by

$$\langle x^*, y\rangle = \langle x_Z^*, y\rangle,$$

where Z is an arbitrary given closed separable subspace of E such that $y \in Z$. Then x^* is a bounded linear operator from E to $\mathbb{R}^d$ such that

$$|x^*| \leq \sup_{n\geq n_0}|x_n^*| < \infty$$

and $(x_n^*)_{n\geq n_0}$ converges weakly to x^* in E. □

As an immediate consequence, we obtain the following result.

Corollary 5.3. *For any continuous function $f : \mathbb{R} \to E$, we have*

$$\lim_{n\to+\infty}\int_\sigma^t U(t-\xi)\left(\widetilde{B}_n f(\xi)\right)^0 d\xi = \Phi\int_\sigma^t e^{(t-\xi)G}\langle x^*, f(\xi)\rangle d\xi,\ t,\sigma \in \mathbb{R}.$$

Let $\varphi \in Y_0$. Then from the properties of the center manifold, we know that the integral solution starting from $\varphi + h(\varphi)$ is given by $v_t^\varphi + h_g(v_t^\varphi)$, where v_t^φ is the solution of

$$v_t^\varphi = U(t)\varphi + \lim_{\lambda\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B}_\lambda g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau,\ t \in \mathbb{R}.$$

Let $z(t)$ be the component of $v_t^\varphi \in Y_0$. Then

$$\Phi z(t) = v_t^\varphi \text{ for } t \in \mathbb{R}.$$

By Theorem 5.1 and Corollary 5.3, we have for $t \in \mathbb{R}$

$$\Phi z(t) = \Phi e^{Gt}z(0) + \Phi\int_0^t e^{(t-\tau)G}\langle x^*, g(v_\tau^\varphi + h_g(v_\tau^\varphi))\rangle d\tau.$$

We conclude that z satisfies

$$z(t) = e^{Gt}z(0) + \lim_{n\to\infty}\int_0^t e^{(t-\tau)G}\langle x_n^*, g(v_\tau^\varphi + h_g(v_\tau^\varphi))\rangle d\tau.$$

Finally, we obtain at the following ordinary differential equation (the reduced system)

$$z'(t) = Gz(t) + \langle x^*, g(\Phi z(t) + h_g(\Phi z(t)))\rangle \text{ for } t \in \mathbb{R}. \tag{5.1}$$

This determines the flow on the center manifold.

Remark: In this paper, the center manifold reduction technique has been proposed for some partial functional differential equations with infinite delay in fading memory spaces. We expect many applications of this important result, especially to study the stability of equilibriums in critical cases when the classical methods (like the linearization principle) do not work. We also expect to use it to develop new results on bifurcation theory for partial functional differential equations with infinite delay.

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Fuzziffied Random Generalized Nonlinear Variational Like Inequalities

G. Anastassiou
Department of Mathematical Sciences
University of Memphis, Memphis, TN 38152, USA

Salahuddin and M.K. Ahmad
Department of Mathematics
Aligarh Muslim University, Aligarh-202002, India
salahuddin12@mailcity.com; ahmad_kalimuddin@yahoo.co.in

Abstract

In this paper, we introduced and studied a new class of generalized nonlinear variational like inequality for fuzzy random mappings and have proved existence theorem for auxiliary problem of the fuzziffied random generalized nonlinear variational like inequalities. By exploiting the theorem, we construct and analyze a new random iterative algorithm for finding the selections of the fuzziffied random generalized nonlinear variational like inequality. Further more, we prove the existence of unique solutions of the fuzzified random iterative algorithm for finding the selection of the fuzziffied generalized nonlinear variational like inequality problems. The convergence analysis of fuzziffied random iterative sequences generated by the random iterative algorithm is also discussed.

Keywords: Fuzziffied random generalized nonlinear variational like inequalities, fuzziffied random iterative sequences, random iterative algorithm, Hausdorff space, Hilbert spaces, measurable spaces, Borel set.
Mathematics Subject Classification: 49J40

1 Introduction

Variational inequality theory has been a very effective and powerful tool for studying a wide range of problems arising in many diverse fields of pure and applied sciences. It is well known that one of the most important problems in variational inequality theory is the development of efficient and implementable iterative algorithms for solving various class of variational inequalities and variational inclusions. In [8] and [20, 27, 28, 30, 31, 32] there are lot of iterative algorithms for finding the approximate solutions of various variational inequalities. Glowinski, Lions and Tremolieres [16] had developed the auxiliary principle techniques. By using the auxiliary principle technique, Ding [13, 14, 15] suggested several iterative algorithms to compute approximate solutions for some class of general nonlinear mixed variational inequalities and variational like inequalities in reflexive Banach spaces. Variational inequalities have been widely used as a mathematical programming tool in modeling, optimization and decision making problems. However, facing uncertainty is a constant challenge for optimization and decision making problems. Treating uncertainty for fuzzy mathematical results in the study of fuzzy optimization and decision making, the variational inequalities have been generalized and extended in multi direction using novel techniques of fuzzy environment.

It is well known that fuzzy set theory which was introduced by Zadeh [35] in 1965 has gained importance in analysis, from both theoretical and practical point of view. The fuzzy sets are distinguished from ordinary or crisp sets in that the degree of membership of an element in a fuzzy set can be any number in the unit interval [0,1] as opposed to a number from the binary pair {0,1} for crisp sets. This property of fuzzy sets enable us to represent realistically imprecise concepts in which the transition from membership to membership is gradual.

In 1989, Chang and Zhu [4] introduced the concepts of variational inequality for fuzzy mappings which were later developed by Chang and Huang [5], Noor [29], Lee et al. [24], Ding [10], Ding and Park [11] etc. in Hilbert spaces. Recently Huang and Lan [17] considered nonlinear equations with fuzzy mappings in fuzzy normed spaces and Lan and Verma [22] considered fuzzy variational inclusion problems in Banach spaces. The concepts of random fuzzy mapping was first introduced by Huang [18]. The random variational inclusion problem for fuzzy random mappings is studied by Lee et al. [25], Dai [9] and Ahmad and Faraj [1]. The random variational inequality (inclusion) problems and random quasi-variational inclusion problems have been studied by Chang [3], Khan and Salahuddin [21], Cho et al. [7], Lan [23] and Huang and Cho [20] etc.

Inspired and motivated by recent works [1, 6, 9, 16, 25, 26, 31, 33, 36], in this paper we introduced and studied a new class of fuzzified random generalized nonlinear variational like inequalities. An existence theorem for auxiliary problem of the fuzzified random generalized nonlinear variational like inequality is established. By exploiting the theorem, we construct and analyze a new random iterative algorithm for finding the solution of the fuzzified random generalized nonlinear variational like inequality. Further, we prove the existence of unique solutions of the fuzzified random generalized nonlinear variational like inequality problems and discuss the convergence analysis of fuzzified random iterative sequences generated by the random iterative algorithm. Our results improve and generalize many known corresponding results presented in [12, 14, 18, 20].

2 Preliminaries

Throughout this paper, let H be a real Hilbert space endowed with dual space H^*, pairing between norm denoted by $\|.\|$, inner product $\langle u, v\rangle$ for $u \in H$, $v \in H^*, D$ be a nonempty closed convex subset of H. We denote by 2^H and $CB(H)$ the family of all nonempty subsets and the families of all the nonempty bounded closed subsets of H respectively, $\hat{H}(.,.)$ represents the Hausdorff metric on $CB(H)$. Let (Ω, Σ) be a measurable space, where Ω is a set and Σ is σ-algebra of subsets of Ω. We denote by $\mathcal{B}(H)$ the class of Borel σ-fields in H.

Definition 2.1. A mapping $f : \Omega \to H$ is said to be measurable if for any $C \in \mathcal{B}(H)$, $f^{-1}(C) = \{t \in \Omega : \ f(t) \in C\} \in \Sigma$.

Definition 2.2. A multivalued mapping $A : \Omega \to CB(H)$ is said to be measurable if for any $C \in \mathcal{B}(H)$,

$$A^{-1}(C) = \{t \in \Omega : \ A(t) \cap C \neq \emptyset\} \in \Sigma.$$

Definition 2.3. A mapping $f : \Omega \times H \to H$ is called a random operator if for any $w \in H$, $f(t, w) = w(t)$ is measurable. A random operator $f : \Omega \times H \to H$ is said to be continuous if for any $t \in \Omega$, the mapping $f(t, \cdot) : H \to H$ is continuous.

Definition 2.4. A mapping $u : \Omega \to H$ is called a measurable selection of the multivalued measurable mapping $A : \Omega \to CB(H)$ if u is a measurable mapping and $t \in \Omega$, $u(t) \in A(t)$.

Definition 2.5. A mapping $T : \Omega \times H \to CB(H)$ is called a random multivalued mapping if for any $w \in H$, $T(\cdot, w)$ is measurable. A random multivalued mapping $T : \Omega \times H \to CB(H)$ is said to be $\hat{H}$-continuous if for any $t \in \Omega$, $T(t, \cdot)$ is continuous in the Hausdorff metric.

Let $\mathcal{F}(H)$ be a collection of fuzzy sets over H. A mapping F from H into $\mathcal{F}(H)$ is called a fuzzy mapping. If F is a fuzzy mapping on H, for any $u \in H$, $F(u)$ (denoted by F_u in what follows) is a fuzzy set on H and $F_u(z)$ is the membership function of z in F_u.

Let $M \in \mathcal{F}(H)$, $q \in [0, 1]$, then the set

$$(M)_q \ = \ \{u \in H : \ M(u) \geq q\}$$

is called q-cut of M.

Definition 2.6. A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is called measurable if for any $\alpha \in (0, 1]$, $(F(\cdot))_\alpha : \Omega \to 2^H$ is measurable multivalued mapping.

Definition 2.7. A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is called a random fuzzy mapping if for any $w \in H$, $F(\cdot, w) : \Omega \to \mathcal{F}(H)$ is a measurable mapping.

Clearly, the random fuzzy mapping includes multivalued mapping, random multivalued mappings and fuzzy mappings as the special cases:

Let $A, T : \Omega \times H \to \mathcal{F}(H)$ be two fuzzy random mappings satisfying the following condition (S):

(S): There exist two mappings $a, c : H \to (0,1]$ such that for all $(t,w) \in \Omega \times H$, $(A_{t,w})_{a(w)} \in CB(H)$, $(T_{t,w})_{c(w)} \in CB(H)$. By using the fuzzy random mappings A, T, we can define two random multivalued mappings $\tilde{A}$ and $\tilde{T}$ as follows:

$$\begin{aligned} &\forall\ (t,w) \in \Omega \times H, \qquad \tilde{A} : \Omega \times H \to CB(H), \quad (t,w) \to (\tilde{A}_{t,w})_{a(w)}, \\ &\forall\ (t,w) \in \Omega \times H, \qquad \tilde{T} : \Omega \times H \to CB(H), \quad (t,w) \to (\tilde{T}_{t,w})_{c(w)}, \end{aligned}$$

where $A_{(t,w)} = A(t, w(t))$.

So, A and T are called the random multivalued mappings induced by the fuzzy random mappings $\tilde{A}$ and $\tilde{T}$ respectively.

Given mappings $a, c : H \to (0,1]$ the fuzzy random mappings $A, T : \Omega \times H \to \mathcal{F}(H)$ satisfying the condition (S). Let $N : \Omega \times H \times H \to H$ and $\eta : \Omega \times H \times H \to H^*$ be the mappings. Let $b : H \times H \to (-\infty, \infty)$ be a real valued function. For given $f \in H$ and any measurable mapping $v : \Omega \to H$, find measurable mappings $u, x, y : \Omega \to H$ such that $A_{t,u(t)}(x(t)) \geq a(u(t))$, $T_{t,u(t)}(y(t)) \geq c(u(t))$ and

$$\langle N_t(x(t), y(t)) - f, \eta_t(v(t), u(t)) \rangle + a(u(t), v(t) - u(t)) \geq b(u(t), u(t)) - b(u(t), v(t)) \quad (2.1)$$

for all $v(t) \in H$, $t \in \Omega$ and $\eta_t(u(t), v(t)) = \eta(t, u(t), v(t))$, where $a : H \times H \to (-\infty, +\infty)$ is a coercive continuous bilinear form, that is there exists the measurable mappings $e, d : \Omega \to (0,1)$ such that

$$\begin{aligned} &(\mathrm{C}_1) \quad a(v(t), v(t)) \geq d_t \|v(t)\|^2 \quad \text{for all } v(t) \in H \\ &(\mathrm{C}_2) \quad |a(u(t), v(t))| \leq e_t \|u(t)\|.\|v(t)\| \quad \text{for all } u(t), v(t) \in H. \end{aligned}$$

It follows from (C_1) and (C_2) that $d(t) \leq e(t)$. Again function $b(.,.)$ is non differentiable and satisfies the following conditions:

(C_3) for any measurable mapping $v : \Omega \to H$, $b(., v(t))$ is linear;

(C_4) for each measurable mapping $w : \Omega \to H$, $b(w(t), .)$ is a convex function;

(C_5) for any measurable mappings $w, v : \Omega \to H$, $b(w(t), v(t))$ is bounded, that is there exists a measurable function $\gamma : \Omega \to (0, +\infty)$ such that

$$b(w(t), v(t)) \leq \gamma_t \|w(t)\|.\|v(t)\|;$$

(C_6) for any measurable mappings $w, v, z : \Omega \to H$

$$b(w(t), v(t)) - b(w(t), z(t) \leq b(w(t), v(t) - z(t)).$$

The inequality (2.1) is called fuzzified random generalized nonlinear variational like inequalities.

Remark 2.1.

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1. For any measurable mapping $w, v : \Omega \to H$

$$b(-w(t), v(t)) = -b(w(t), v(t)) \text{ and } b(-w(t), v(t)) \le \gamma_t \|w(t)\|.\|v(t)\|,$$

hold from condition (C_3) and (C_5), respectively. So

$$|b(w(t), v(t))| \le \gamma_t \|w(t)\| \, \|v(t)\|.$$

2. For any measurable mappings $w, v, z : \Omega \to H$

$$|b(w(t), v(t)) - b(w(t), z(t))| \le \gamma_t \|w(t)\| \, \|v(t) - z(t)\|,$$

follows from conditions (C_4) and (C_6). So, $b(w(t), v(t))$ is continuous with respect to second variable.

Special Cases:

1. If a is zero operator $f \equiv 0$. Then problem (2.1) is equivalent to the problem (2.1) of Dai [9]. For any measurable mapping $v : \Omega \to H$, finding measurable mappings $u, x, y : \Omega \to H$ such that

$$A_{t,u(t)}(x(t)) \ge a(u(t)), \quad T_{t,u(t)}(y(t)) \ge c(u(t))$$

and

$$\langle N(x(t), y(t)), \eta(v(t), u(t))\rangle + b(u(t), v(t)) - b(u(t), u(t)) \ge 0 \tag{2.2}$$

where $N, \eta : H \times H \to H$.

2. If $\tilde{A}, \tilde{T} : H \to CB(H)$ are classical set valued mappings we can define the fuzzy mappings $A, T : H \to \mathcal{F}(H)$ by

$$u \to \chi A(u), \quad u \to \chi T(u)$$

where $\chi A(u)$ and $\chi T(u)$ are characteristic functions of $A(u)$ and $T(u)$ respectively, taking $a(u) = c(u) = 1$ for all $u \in H$. For given $f \in H$ and any measurable mapping $v : \Omega \to H$, find measurable mappings $u, x, y : \Omega \to H$ such that

$$\langle N_t(x(t), y(t)) - f, \eta_t(v(t), u(t))\rangle + a(u(t), v(t) - u(t)) - b(u(t), v(t)) \ge b(u(t), u(t)) \tag{2.3}$$

for all $v(t) \in H$. It is called random generalized nonlinear variational like inequalities, which is the variant form of a problem studied by Liu et al. [26].

3. If a is zero operator, $A, T : D \to H$ and $\eta : D \times D \to H^*$ be the single valued mappings, then (2.3) reduces to the following nonlinear mixed variational like inequality:

For a given $f \in H$, find $u \in D$ such that

$$\langle N(A(u), T(u)) - f, \eta(v, u)\rangle + b(u, v) - b(u, u) \le 0, \ \forall\, v \in H, \tag{2.4}$$

considered by Ding [14].

4. If $N(Au, Tu) = Au - Tu$, $f \equiv 0$ and $b(u, u) = \varphi(u)$, where $\varphi : H \to (-\infty, \infty)$ is a functional. Then (2.4) is equivalent to a problem for finding $u \in D$ such that

$$\langle A(u) - T(u),\ \eta(v, u)\rangle \geq \varphi(u) - \varphi(v), \quad \forall\ v \in D. \tag{2.5}$$

   It is called mixed nonlinear variational like inequalities, considered and studied by Ding [13].

5. Again, $\eta(v, u) = gv - gu$, for all $u, v \in D$ where $g : D \to D$ is a mapping, then (2.5) is equivalent to finding $u \in K$ such that

$$\langle Au - Tu,\ gv - gu\rangle \geq \varphi(u) - \varphi(v), \quad \forall\ v \in D, \tag{2.6}$$

   which was studied by Yao [34].

Definition 2.8. A random multivalued mapping $\tilde{A} : \Omega \times H \to CB(H)$ is said to be $\hat{H}$-Lipschitz continuous, if there exists a measurable function $\lambda : \Omega \to (0, +\infty)$ such that

$$\hat{H}(\tilde{A}(t, u_1(t)), \tilde{A}(t, u_2(t))) \leq \lambda(t)\|u_1(t) - u_2(t)\|^2\ \forall\ u_1(t), u_2(t) \in H.$$

We give the following Lemmas.

Lemma 2.1[1]. Let $\tilde{A} : \Omega \times H \to CB(H)$ be a $\hat{H}$-continuous random multivalued mapping, then for measurable mapping $u : \Omega \to H$, the multivalued mapping $\tilde{A}(., u(t)) : \Omega \to CB(H)$ is measurable.

Lemma 2.2[2]. Let $\tilde{A}_1,\ \tilde{A}_2 : \Omega \to CB(H)$ be two measurable multivalued mappings, $\epsilon > 0$ be a constant and $x_1 : \Omega \to H$ be a measurable selection of $\tilde{A}_1$, then there exists a measurable selection $x_2 : \Omega \to H$ of $\tilde{A}_2$ such that for all $t \in \Omega$,

$$\|x_1(t) - x_2(t)\| \leq (1 + \epsilon)\hat{H}(\tilde{A}_1(t), \tilde{A}_2(t)).$$

Lemma 2.3[3]. Let D be a nonempty closed convex subset of a Hausdorff linear topological space E and $\phi, \psi : D \times D \to R$ be the mappings satisfying the following conditions:

(a) $\psi(u, v) \leq \phi(u, v)$ for all $u, v \in D$ and $\psi(u, u) \geq 0$ for all $u \in D$;

(b) for each $u \in D$ $\phi(u, .)$ is upper semicontinuous on D;

(c) for each $v \in D$, the set $\{u \in D : \psi(u, v) < 0\}$ is a convex set;

(d) there exists a nonempty compact set $K \subset D$ and $u_0 \in K$ such that $\psi(u_0, y) < 0$ for all $v \in D \backslash K$.

Then there exists $\hat{v} \in K$ such that $\phi(u, \hat{v}) \geq 0$ for all $u \in D$.

Definition 2.9. Let $N : \Omega \times H \times H \to H$ and $\eta : \Omega \times D \times D \to H^*$ be the mappings.

(i) N is said to be random η-strongly monotone with respect to the first variable if there exists a measurable function $s_{t,N} : \Omega \to (0,1)$ such that

$$\langle N_t(u(t),.) - N_t(v(t),.),\ \eta_t(u(t),v(t))\rangle \geq s_{t,N}\|u(t)-v(t)\|^2$$

for all $u, v \in D$, $t \in \Omega$;

(ii) N is said to be randomly η-monotone with respect to the second variable if

$$\langle N_t(.,u(t)) - N_t(.,v(t)),\ \eta_t(u(t),v(t))\rangle \geq 0$$

for all $u, v \in D$ and $t \in \Omega$;

(iii) N is said to be randomly Lipschitz continuous if there exists a measurable mapping $\delta : \Omega \to (0,1)$ such that

$$\|N_t(u(t),.) - N_t(v(t),.)\| \leq \delta_{t,N}\|u(t)-v(t)\| \ \ \forall\, u, v \in D,\ t \in \Omega;$$

(iv) N is said to be η-hemicontinuous with respect to A and T of the first and second variables if for any $u, v \in D$, the mapping $g : [0,1] \to (-\infty,\infty)$ defined by

$$g(t) = \langle N_t(\alpha(t)u(t)+(1-\alpha(t))v(t), \alpha(t)u(t)+(1-\alpha(t))v(t)), \eta_t(u(t),v(t))\rangle$$

is continuous at 0^+;

(v) η is said to be randomly Lipschitz continuous with measurable mapping $\sigma : \Omega \to (0,1)$ such that

$$\|\eta_t(u(t),v(t))\| \leq \sigma_{t,\eta}\|u(t)-v(t)\| \ \ \forall\, u, v \in D,\ t \in \Omega;$$

(v) η is said to be randomly strongly monotone with a measurable mapping $\mu : \Omega \to (0,1)$ such that

$$\langle u(t)-v(t), \eta_t(u(t),v(t))\rangle \geq \mu_{t,\eta}\|u(t)-v(t)\|^2 \ \ \forall\, u(t), v(t) \in D,\ t \in \Omega.$$

Definition 2.10. Let $N : \Omega \times H \times H \to H$, $\eta : \Omega \times D \times D \to H^*$ be the mappings. A random multivalued mapping $\tilde{A} : \Omega \times H \to CB(H)$ is said to be

(i) randomly monotone if

$$\langle N_t(x(t),.) - N_t(y(t),.),\ \eta_t(u(t),v(t))\rangle \geq 0$$

for all $x(t) \in \tilde{A}(t,u(t))$, $y(t) \in \tilde{A}(t,v(t))$, $u(t), v(t) \in H$, $t \in \Omega$;

(ii) $\tilde{A}$ is said to be random τ-strong monotone with respect to the first variable of N if there exists a measurable function $\tau : \Omega \to (0, \infty)$ such that

$$\langle N_t(x(t), .) - N_t(y(t), .),\ \eta_t(u(t), v(t))\rangle \geq \tau_{t,N}\|u(t) - v(t)\|^2$$

for all $x(t) \in \tilde{A}(t, u(t))$, $y(t) \in \tilde{A}(t, v(t))$, $u(t), v(t) \in H$, $t \in \Omega$.

Lemma 2.4. Let (Ω, Σ) be a measurable space and D be a nonempty convex subset of a topological vector space. Let $\varphi : \Omega \times D \times D \to (-\infty, \infty)$ be a real valued function such that

(i) for each $(v, u) \in D \times D$, $t \to \varphi(t, v, u)$ is measurable mapping;

(ii) for each $(t, v) \in \Omega \times D$, $u \to \varphi(t, v, u)$ is continuous on each nonempty compact subset of D;

(iii) for each $(t, u) \in \Omega \times D$, $v \to \varphi(t, v, u)$ is lower semicontinuous on each nonempty compact subset of D;

(iv) for each $t \in \Omega$, each nonempty finite set $\{v_1, v_2, \cdots, v_n\} \subset D$ and for each

$$u = \sum_{i=1}^{m} \alpha_i v_i \ \ (\alpha_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1), \quad \min_{1\leq i\leq m} \varphi(t, v_i, u) \leq 0;$$

(v) for each $t \in \Omega$, there exists a nonempty compact subset D_0 of D and a nonempty compact subset K of D such that for each $u \in D\backslash K$ there is a $v \in \mathrm{co}(D_0 \cup \{u\})$ with $\varphi(t, v, u) > 0$.

Then there exists a measurable mapping $u : \Omega \to D$ such that $\varphi(t, v, u(t)) \leq 0$ for all $v \in D$ and $t \in \Omega$.

Definition 2.11. A random multivalued mapping $\tilde{T} : \Omega \times H \to CB(H)$ is said to be randomly $\hat{H}$-Lipschitz continuous, if there exists a measurable function $\kappa : \Omega \to (0, \infty)$ such that

$$\hat{H}(\tilde{T}(t, u(t)), \tilde{T}(t, v(t))) \leq \kappa_{\tilde{T}}\|u(t) - v(t)\| \ \forall\ u(t), v(t) \in H,\ t \in \Omega.$$

3 Auxiliary Problem and Algorithm

Now we consider the following auxiliary problem with respect to the fuzziffied random generalized nonlinear variational like inequality problem (2.1).

For any given measurable mapping $u : \Omega \to D$, for measurable mapping $v : \Omega \to D$, find measurable mapping $\hat{w} : \Omega \to D$ such that for all $t \in \Omega$ $\hat{x}(t) \in \tilde{A}(t, \hat{w}(t))$, $\hat{y}(t) \in \tilde{T}(t, \hat{w}(t))$ and

$$\begin{aligned}\langle \hat{w}(t), \eta_t(v(t), \hat{w}(t))\rangle \geq \langle u(t), \eta_t(v(t), \hat{w}(t))\rangle - \rho_t\langle N_t(\hat{x}(t), \hat{y}(t)) - f, \eta_t(v(t), \hat{w}(t))\rangle \\ -\rho_t a(\hat{w}(t), v(t) - \hat{w}(t)) - \rho_t b(u(t), v(t)) + \rho_t b(u(t), \hat{w}(t))\end{aligned} \quad (3.1)$$

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where $\rho : \Omega \to (0, \infty)$ is a measurable mapping.

Theorem 3.1. Let (Ω, Σ) be a measurable space, D be a nonempty convex closed subset of H and $f \in H$. Let random fuzzy mappings $A, T : \Omega \times H \to \mathcal{F}(H)$ satisfy the condition (S) and $\tilde{A}$ and $\tilde{T}$, the random multivalued mappings induced by the fuzzy random mappings A and T respectively. Let $N : \Omega \times H \times H \to H$ be the mapping such that N is random η-hemicontinuous with respect to $\tilde{A}$ and $\tilde{T}$ in first and second variables. Let $\eta : \Omega \times D \times D \to H^*$ be the random Lipschitz continuous with a measurable function $\sigma : \Omega \to (0, 1)$ and random η strongly monotone with respect to the measurable mappings $\mu : \Omega \to (0, 1)$. For each $v \in D$, $\eta_t(., v(t))$ be continuous and semi continuous in second variables and $\eta_t(v(t), u(t)) = -\eta_t(u(t), v(t))$ for all $u, v \in D$, $t \in \Omega$. Suppose $a : D \times D \to (-\infty, +\infty)$ satisfies (C_1) and (C_2). Let $b : D \times D \to (-\infty, +\infty]$ be a real valued function satisfying (C_3).

(1) For each $t \in \Omega$, the mappings $A(t, .)$, $T(t, .)$ are randomly $\hat{H}$-Lipschitz continuous with the measurable functions $\lambda, \kappa : \Omega \to (0, 1)$ respectively;

(2) the measurable mappings $N_t(., .)$ is randomly Lipschitz continuous and randomly η-strongly monotone with respect to the random multivalued mappings $\tilde{A}$ in the first variable with the measurable functions $\delta, s : \Omega \to (0, +\infty)$ respectively and $N(., .)$ is randomly Lipschitz continuous and randomly η-relaxed monotone with respect to the random multivalued mapping $\tilde{T}$ in the second variable with the measurable functions $\xi, r : \Omega \to (0, +\infty)$ respectively.

Then the auxiliary problem (3.1) has a unique random solutions.

Proof. For any fixed measurable mapping $u : \Omega \to D$, for any measurable mappings $v, w : \Omega \to D$, we define the functional $\phi, \psi : \Omega \times D \times D \to (-\infty, +\infty]$ by

$$\begin{aligned}\phi_t(v(t), w(t)) = &\langle v(t), \eta_t(v(t), w(t))\rangle - \langle u(t), \eta_t(v(t), w(t))\rangle \\ &+ \rho_t\langle N_t(x_1(t), y_1(t)) - f, \eta_t(v(t), w(t))\rangle \\ &+ \rho_t a(v(t), v(t) - w(t)) - \rho_t b(u(t), w(t)) + \rho_t b(u(t), v(t))\end{aligned}$$

for all $x_1(t) \in A(t, v(t))$, $y_1(t) \in T(t, v(t))$ and

$$\begin{aligned}\psi_t(v(t), w(t)) = &\langle w(t), \eta_t(v(t), w(t))\rangle - \langle u(t), \eta_t(v(t), w(t))\rangle \\ &+ \rho_t\langle N_t(x_2(t), y_2(t)) - f, \eta_t(v(t), w(t))\rangle \\ &+ \rho_t a(w(t), v(t) - w(t)) - \rho_t b(u(t), w(t)) + \rho_t b(u(t), v(t)),\end{aligned}$$

for all $x_2(t) \in A(t, w(t))$, $y_2(t) \in T(t, w(t))$.

Since $\tilde{A}$ and $\tilde{T}$ are random multivalued mappings induced by the fuzzy random mappings A and T respectively i.e., for each $t \in \Omega$, $u(t) \in D$, $\tilde{A}(t, u(t))$ and $\tilde{T}(t, u(t))$ are measurable mappings. From Lemma 2.4, for any fixed $(v(t), u(t)) \in D \times D$, $t \to$

$\varphi_t(v(t), u(t))$ is measurable. Now we verify the functional ϕ and ψ satisfy all conditions of Lemma 2.3 in the weak topology. It is easy to see that for all $v(t), w(t) \in D$, $t \in \Omega$,

$$\begin{aligned}\phi_t(v(t), w(t)) - \psi_t(v(t), w(t)) &= \langle v(t) - w(t), \eta_t(v(t), w(t))\rangle \\ &+ \rho_t\langle N_t(x_1(t), y_1(t)) - N_t(x_2(t), y_1(t)), \eta_t(v(t), w(t))\rangle \\ &+ \rho_t\langle N_t(x_2(t), y_1(t)) - N_t(x_2(t), y_2(t)), \eta_t(v(t), w(t))\rangle \\ &+ \rho_t a(v(t) - w(t), v(t) - w(t)) \\ &\geq [\mu_{t,\eta} + \rho_t(s_{t,N_{\tilde{A}},\eta} - r_{t,N_{\tilde{T}},\eta} + d_t)]\|v(t) - w(t)\|^2 \geq 0,\end{aligned}$$

which implies that ϕ_t and ψ_t satisfy the condition (1) of Lemma 2.3. Given a is a coercive continuous bilinear from, therefore $a(v(t), v(t) - w(t))$ is weakly upper semi continuous with respect to $w(t)$. Since b is convex, lower semi continuous in the second variable. For any measurable mapping $v : \Omega \to D$, the mapping $u(t) \to \eta_t(u(t), v(t))$ for $t \in \Omega$ is convex and upper semi continuous. Therefore $\phi_t(v(t), .)$ is weakly upper semi continuous in the second variable and the set $\{t \in \Omega,\ v(t) \in D : \psi_t(v(t), w(t)) < 0\}$ is convex for each measurable mapping $w : \Omega \to D$. Therefore the Assumption (b) and (c) of Lemma 2.3, and (iv) of Lemma 2.4 hold. Let $\hat{v} : \Omega \to D$ be a measurable mapping. Assume

$$\begin{aligned}L_t = [\mu_{t,\eta} + \rho_t(s_{t,N_{\tilde{A}},\eta} - r_{t,N_{\tilde{T}},\eta} + d_t)]^{-1}[\sigma_{t,\eta}\|u(t) - \bar{v}(t)\| + \rho_t e_t\|\bar{v}(t)\| \\ + \rho_t\sigma_{t,\eta}\|N_t(x_1(t), y_1(t)) - f\| + \rho_t\gamma_t\|u(t)\|]\end{aligned}$$

and

$$M_t = \{t \in \Omega,\ w(t) \in D :\ \|w(t) - \bar{v}(t)\| \leq L_t\}.$$

Since M_t is weakly compact subset of D and for any $w(t) \subseteq D\backslash M$ and $x_2(t) \in \tilde{A}(t, w(t))$, $y_2(t) \in \tilde{T}(t, w(t))$, $x_1(t) \in \tilde{A}(t, \bar{v}(t))$, $y_1(t) \in \tilde{T}(t, \bar{v}(t))$

$$\begin{aligned}\psi_t(\bar{v}(t), w(t)) &= \langle w(t), \eta_t(\bar{v}(t), w(t))\rangle - \langle u(t), \eta_t(\bar{v}(t), w(t))\rangle \\ &+ \rho_t\langle N_t(x_2(t), y_2(t)) - f, \eta_t(\bar{v}(t), w(t))\rangle \\ &+ \rho_t a(w(t), \bar{v}(t) - w(t)) - \rho_t b(u(t), w(t)) + \rho_t b(u(t), \bar{v}(t)) \\ &\leq -\langle w(t) - \bar{v}(t), \eta_t(w(t), \bar{v}(t))\rangle + \langle u(t) - \bar{v}(t), \eta_t(w(t), \bar{v}(t))\rangle \\ &- \rho_t\langle N_t(x_2(t), y_2(t)) - N_t(x_1(t), y_2(t)), \eta_t(w(t), \bar{v}(t))\rangle \\ &- \rho_t\langle N_t(x_1(t), y_2(t)) - N_t(x_1(t), y_1(t)), \eta_t(w(t), \bar{v}(t))\rangle \\ &- \rho_t\langle N_t(x_1(t), y_1(t)) - f, \eta_t(w(t), \bar{v}(t))\rangle - \rho_t a(w(t) - \bar{v}(t), w(t) - \bar{v}(t)) \\ &- \rho_t a(\bar{v}(t), w(t) - \bar{v}(t)) + \rho_t b(u(t), \bar{v}(t) - w(t)) \\ &\leq -\|w(t) - \bar{v}(t)\|\{[\mu_{t,\eta} + \rho_t(s_{t,N_{\tilde{A}},\eta} - r_{t,N_{\tilde{T}},\eta} + d_t)]\|w(t) - \bar{v}(t)\| \\ &- \sigma_{t,\eta}\|u(t) - \bar{v}(t)\| - \rho_t e_t\|\bar{v}(t)\| - \rho_t\sigma_{t,\eta}\|N_t(x_1(t), y_1(t)) - f\| - \rho_t\gamma_t\|u(t)\|\} \\ &< 0.\end{aligned}$$

Hence condition (d) of Lemma 2.3 holds. By Lemma 2.3, for $t \in \Omega$, there exists a measurable mapping $\hat{w} : \Omega \to D$ such that $\phi_t(u(t), \hat{w}(t)) \geq 0$ for each measurable mapping $u : \Omega \to D$.

We know that mapping $N_t(.,.)$ is random Lipschitz continuous in first and second variable and the mapping $\tilde{A}(t,.)$, $\tilde{T}(t,.)$ are random $\hat{H}$-Lipschitz continuous. Based on Lemma 2.1, for any measurable mapping $v:\Omega\to D$, we obtain $\hat{w}:\Omega\to D$, there exist $x_1(t)\in\tilde{A}(t,v(t))$, $y_1(t)\in\tilde{T}(t,v(t))$ such that

$$\begin{aligned}\langle v(t),\eta_t(v(t),\hat{w}(t))\rangle \ &\geq \langle u(t),\eta_t(v(t),\hat{w}(t))\rangle-\rho_t\langle N_t(x_1(t),y_1(t))-f,\eta_t(v(t),\hat{w}(t))\rangle\\ &\quad -\rho_t a(v(t),v(t)-\hat{w}(t))-\rho_t b(u(t),v(t))+\rho_t b(u(t),\hat{w}(t)) \quad (3.2)\end{aligned}$$

for each given measurable mappings $u:\Omega\to D$.

Let β be in $(0,1]$ and a measurable point $v:\Omega\to D$. Replacing $v(t)$ by $v_\beta(t)=\beta v(t)+(1-\beta)\hat{w}(t)$ in (3.2), we have

$$\begin{aligned}\langle v_\beta(t),\eta_t(v_\beta(t),\hat{w}(t))\rangle \ &\geq \langle u(t),\eta_t(v_\beta(t),\hat{w}(t))\rangle-\rho_t\langle N_t(x_\beta(t),y_\beta(t))-f,\\ &\quad \eta_t(v_\beta(t),\hat{w}(t))\rangle-\rho_t a(v_\beta(t),v_\beta(t)-\hat{w}(t))\\ &\quad -\rho_t b(u(t),v_\beta(t))+\rho_t b(u(t),\hat{w}(t)) \quad (3.3)\end{aligned}$$

for all $v(t)\in D$, $t\in\Omega$, for $x_\beta(t)\in\tilde{A}(t,v_\beta(t))$ and $y_\beta(t)\in\tilde{T}(t,v_\beta(t))$.

Since b is convex in the second variable, $\langle N_t(x(t),y(t)),\eta_t(v(t),.)\rangle$ is concave and upper semicontinuous. From (C_6) and (3.3), we have

$$\begin{aligned}[\langle v_\beta(t),\eta_t(v(t),\hat{w}(t))\rangle] &\geq \beta[\langle u(t),\eta_t(v(t),\hat{w}(t))\rangle-\rho_t\langle N_t(x_\beta(t),y_\beta(t))-f,\\ &\quad \eta_t(v(t),\hat{w}(t))\rangle-\rho_t a(v_\beta(t),v(t)-\hat{w}(t))-\rho_t b(u(t),v(t))\\ &\quad +\rho_t b(u(t),\hat{w}(t))]\ \forall\ v(t)\in D,\ t\in\Omega,\end{aligned}$$

implies that

$$\begin{aligned}[\langle v_\beta(t),\eta_t(v(t),\hat{w}(t))\rangle] &\geq \langle u(t),\eta_t(v(t),\hat{w}(t))\rangle-\rho_t\langle N_t(x_\beta(t),y_\beta(t))-f,\\ &\quad \eta_t(v(t),\hat{w}(t))\rangle-\rho_t a(v_\beta(t),v(t)-\hat{w}(t))-\rho_t b(u(t),v(t))\\ &\quad +\rho_t b(u(t),\hat{w}(t))\ \forall\ v(t)\in D,\ t\in\Omega,\end{aligned}$$

where $\rho:\Omega\to(0,1)$ is a measurable mapping.

Letting $\beta\to 0^+$ in the above inequality, thus

$$\begin{aligned}\langle \hat{w}(t),\eta_t(v(t),\hat{w}(t))\rangle &\geq \langle u(t),\eta_t(v(t),\hat{w}(t))\rangle-\rho_t\langle N_t(x(t),y(t))-f,\eta_t(v(t),\hat{w}(t))\rangle\\ &\quad -\rho_t a(\hat{w}(t),v(t)-\hat{w}(t))-\rho_t b(u(t),v(t))+\rho_t b(u(t),\hat{w}(t))\end{aligned}$$

for all $v(t)\in D,\ t\in\Omega$, $x(t)\in\tilde{A}(t,\hat{w}(t))$ and $y(t)\in\tilde{T}(t,\hat{w}(t))$.

This shows that for any $t\in\Omega$ and given each measurable mapping $u:\Omega\to D$, for each $v:\Omega\to D$ measurable mapping, the measurable mapping $\hat{w}:\Omega\to D$, $\hat{x}(t)\in\tilde{A}(t,\hat{w}(t))$, $\hat{y}(t)\in\tilde{T}(t,\hat{w}(t))$ is the random solution of the auxiliary problem (3.1).

We prove that for any $t \in \Omega$, the measurable mapping $t \to w(t)$, $x(t) \in \tilde{A}(t, w(t))$, $y(t) \in \tilde{T}(t, w(t))$ is unique random solutions of the auxiliary problem (3.1). Supposing the measurable mapping $w_1(t), w_2(t) \in D$, $x_1(t) \in \tilde{A}(t, w_1(t))$, $y_1(t) \in \tilde{T}(t, w_1(t))$, $x_2(t) \in \tilde{A}(t, w_2(t))$, $y_2(t) \in \tilde{T}(t, w_2(t))$ are two random solutions of the auxiliary problem (3.1), we have the conclusion that for all $v : \Omega \to D$, $t \in \Omega$

$$\begin{aligned}\langle w_1(t), \eta_t(v(t), w_1(t))\rangle \geq \langle u(t), \eta_t(v(t), w_1(t))\rangle &- \rho_t\langle N_t(x_1(t), y_1(t)) - f,\\ &\eta_t(v(t), w_1(t))\rangle - \rho_t a(w_1(t), v(t) - w_1(t))\\ &- \rho_t b(u(t), v(t)) + \rho_t b(u(t), w_1(t)) \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}\langle w_2(t), \eta_t(v(t), w_2(t))\rangle \geq \langle u(t), \eta_t(v(t), w_2(t))\rangle &- \rho_t\langle N_t(x_2(t), y_2(t)) - f,\\ &\eta_t(v(t), w_2(t))\rangle - \rho_t a(w_2(t), v(t) - w_2(t))\\ &- \rho_t b(u(t), v(t)) + \rho_t b(u(t), w_2(t)). \qquad (3.5)\end{aligned}$$

Putting $v(t) = w_2(t)$ in (3.4) and $v(t) = w_1(t)$ in (3.5), we have

$$\begin{aligned}\langle w_1(t), \eta_t(w_2(t), w_1(t))\rangle \geq \langle u(t), \eta_t(w_2(t), w_1(t))\rangle &- \rho_t\langle N_t(x_1(t), y_1(t)) - f,\\ &\eta_t(w_2(t), w_1(t))\rangle - \rho_t a(w_1(t), w_2(t) - w_1(t))\\ &- \rho_t b(u(t), w_2(t)) + \rho_t b(u(t), w_1(t))\end{aligned}$$

and

$$\begin{aligned}\langle w_2(t), \eta_t(w_1(t), w_2(t))\rangle \geq \langle u(t), \eta_t(w_1(t), w_2(t))\rangle &- \rho_t\langle N_t(x_2(t), y_2(t)) - f,\\ &\eta_t(w_1(t), w_2(t))\rangle - \rho_t a(w_2(t), w_1(t) - w_2(t))\\ &- \rho_t b(u(t), w_1(t)) + \rho_t b(u(t), w_2(t)).\end{aligned}$$

$$\begin{aligned}\langle w_1(t) - w_2(t), \eta_t(w_1(t), w_2(t))\rangle \leq &-\rho_t\langle N_t(x_1(t), y_1(t)) - N_t(x_2(t), y_1(t)),\\ &\eta_t(w_1(t), w_2(t))\rangle - \rho_t\langle N_t(x_2(t), y_1(t))\\ &- N_t(x_2(t), y_2(t)), \eta_t(w_1(t), w_2(t))\rangle\\ &- \rho_t a(w_1(t) - w_2(t), w_1(t) - w_2(t)).\end{aligned}$$

Noting that $N(.,.)$ is random η-strongly monotone with respect to random multivalued mapping $\tilde{A}$ in first variable with measurable function $s_t : \Omega \to (0, +\infty)$ and random η-relaxed monotone with respect to the random multivalued mapping $\tilde{T}$ in the second argument with $r : \Omega \to (0, +\infty)$, $a(.,.)$ a coercive we have

$$\mu_t\|w_1(t) - w_2(t)\|^2 \leq -\rho_t(r_{t,N_{\tilde{T}},\eta} + s_{t,N_{\tilde{A}},\eta} + d_t)\|w_1(t) - w_2(t)\|^2 \leq 0$$

which yield that $w_1(t) = w_2(t)$.

Further let $x_1(t) \in \tilde{A}(t, w_1(t))$, $y_1(t) \in \tilde{T}(t, w_1(t))$, $x_2(t) \in \tilde{A}(t, w_2(t))$, $y_2(t) \in \tilde{T}(t, w_2(t))$ by Lemma 2.2, we get

$$\|x_1(t) - x_2(t)\| \leq (1 + \epsilon)\hat{H}(\tilde{A}(t, w_1(t)), \tilde{A}(t, w_2(t))) \leq (1 + \epsilon)\lambda_{t,\hat{H}_{\tilde{A}}}\|w_1(t) - w_2(t)\|$$

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$$\|y_1(t) - y_2(t)\| \leq (1+\epsilon)\hat{H}(\tilde{T}(t, w_1(t)), \tilde{T}(t, w_2(t))) \leq (1+\epsilon)\kappa_{t,\hat{H}_{\tilde{T}}}\|w_1(t) - w_2(t)\|.$$

So we get $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$, which imply that for any $t \in \Omega$ and the measurable mapping $u : \Omega \to D$, the measurable mapping $\hat{w}, \hat{x}, \hat{y} : \Omega \to D$ such that $t \in \Omega$, $\hat{x}(t) \in \tilde{A}(t, \hat{w}(t))$, $\hat{y}(t) \in \tilde{T}(t, \hat{w}(t))$ is a unique random solutions of the auxiliary problem (3.1).

4 Existence and Convergence Analysis

In this section, we first construct an algorithm for solving the fuzzified random generalized nonlinear variational like inequality (2.1). Then we prove the existence of solutions for the fuzzified random generalized nonlinear variational like inequalities (2.1) and also discuss the convergence of the sequence generated by the Algorithm 4.1. From Theorem 3.1, we suggest the following algorithm for the fuzzified random generalized nonlinear variational like inequalities (2.1).

Algorithm 4.1. Suppose that $a : D \times D \to (-\infty, \infty)$ satisfies $(C_1) - (C_2)$, $b : D \times D \to (-\infty, \infty)$ satisfies $(C_3) - (C_6)$, $N : \Omega \times H \times H \to H$, $\eta : \Omega \times D \times D \to H^*$ are mappings and $f \in H$. For any given measurable mapping $u_0 : \Omega \to D$ by Lemma 2.1, the multivalued mappings $\tilde{A}(., u_0(.)), \tilde{T}(., u_0(.)) : \Omega \times H \to CB(H)$ are measurable, hence there exist measurable selection $x_0 : \Omega \to D$ of $\tilde{A}(., u_0(.))$ and $y_0 : \Omega \to D$ of $\tilde{T}(., u_0(.))$. From Theorem 3.1 for given each measurable mapping $v : \Omega \to D$, there exists measurable mapping $u_1 : \Omega \to D$, the measurable selection $x_1 : \Omega \to D$ of $\tilde{A}(., u_1(.))$ and $y_1 : \Omega \to D$ of $\tilde{T}(., u_1(.))$ such that for all $t \in \Omega$,

$$\begin{aligned}\langle u_1(t), \eta_t(v(t), u_1(t))\rangle \geq \langle u_0(t), \eta_t(v(t), u_1(t))\rangle &- \rho_t\langle N_t(x_1(t), y_1(t)) - f,\\ &\eta_t(v(t), u_1(t))\rangle - \rho_t a(u_1(t), v(t) - u_1(t)) - \rho_t b(u_0(t), v(t))\\ &+ \rho_t b(u_0(t), u_1(t)) + \langle e_0(t), \eta_t(v(t), u_1(t))\rangle\end{aligned}$$

and

$$\begin{aligned}\|x_1(t) - x_0(t)\| &\leq (1+1)\hat{H}(\tilde{A}(t, u_1(t)), \tilde{A}(t, u_0(t)))\\ \|y_1(t) - y_0(t)\| &\leq (1+1)\hat{H}(\tilde{T}(t, u_1(t)), \tilde{T}(t, u_0(t))).\end{aligned}$$

Continuing the above process inductively, we can define the following random iterative sequences $\{u_n(t)\}$ and $\{x_n(t)\}$ and $\{y_n(t)\}$ for solving problem (2.1) as follows:

$$\begin{aligned}\langle u_{n+1}(t), \eta_t(v(t), u_{n+1}(t))\rangle \geq \langle u_n(t), \eta_t(v(t), u_{n+1}(t))\rangle &- \rho_t\langle N_t(x_{n+1}(t), y_{n+1}(t)) - f,\\ &\eta_t(v(t), u_{n+1}(t))\rangle - \rho_t a(u_{n+1}(t), v(t) - u_{n+1}(t)) - \rho_t b(u_n(t), v(t))\\ &+ \rho_t b(u_n(t), u_{n+1}(t)) + \langle e_n(t), \eta_t(v(t), u_{n+1}(t))\rangle,\end{aligned}$$

$$x_{n+1}(t) \in \tilde{A}(t, u_{n+1}(t)), \qquad y_{n+1}(t) \in \tilde{T}(t, u_{n+1}(t)),$$

$$\begin{aligned}\|x_{n+1}(t) - x_n(t)\| &\leq (1+(1+n)^{-1})\hat{H}(\tilde{A}(t, u_{n+1}(t)), \tilde{A}(t, u_n(t))),\\ \|y_{n+1}(t) - y_n(t)\| &\leq (1+(1+n)^{-1})\hat{H}(\tilde{T}(t, u_{n+1}(t)), \tilde{T}(t, u_n(t))),\end{aligned} \tag{4.1}$$

for $n \geq 0$, where $\{e_n(t)\}_{n\geq 0} \subset H$ and $\rho : \Omega \to (0, \infty)$ is measurable mapping.

Theorem 4.1. Let (Ω, Σ) be a measurable space and D be a nonempty closed convex subset of H and $f \in H$. Let the measurable mapping $\eta : \Omega \times D \times D \to D$ be random strongly η monotone and random Lipschitz continuous with the measurable functions $\mu, \sigma : \Omega \to (0, +\infty)$ respectively. Let the measurable function $b : D \times D \to (-\infty, +\infty]$ be a real valued function satisfy $(C_3) - (C_6)$. Suppose $a : D \times D \to (-\infty, +\infty)$ satisfy (C_1) and (C_2). Suppose $A, T : \Omega \times H \to \mathcal{F}(H)$ be two random fuzzy mappings satisfying the condition (S). Let $\tilde{A}, \tilde{T} : \Omega \times H \to CB(H)$ be random $\hat{H}$-Lipschitz continuous multivalued mappings induced by $\tilde{A}$ and $\tilde{T}$ respectively. The measurable function N is randomly Lipschitz continuous and random strongly monotone in the first variable with respect to the random multivalued mapping $\tilde{A}$ with the measurable functions $\delta, s : \Omega \to (0, +\infty)$ respectively. $N_t(.,.)$ is randomly Lipschitz continuous and random relaxed monotone with respect to the random multivalued mappings $\tilde{T}$ in the second variable with measurable functions $\xi, r : \Omega \to (0, +\infty)$ respectively too. Suppose that $\tilde{A}$ and $\tilde{T}$ are Lipschitz $\hat{H}$-continuous with $\lambda_{\hat{H}_{\tilde{A}}}, \kappa_{\hat{H}_{\tilde{T}}} : \Omega \to (0, 1)$ respectively and

$$(\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}}) > s_{t,N_{\tilde{A}}}, \;\; q_t = \frac{\gamma_t - d_t}{\sigma_t}, \;\; p_t = \frac{\mu_t}{\sigma_t}, \;\; \lim_{n\to\infty} \|e_n(t)\| = 0. \tag{4.2}$$

If there exists any measurable function $\rho : \Omega \to (0, +\infty)$ satisfying

$$0 < \rho_t < \frac{\mu_t}{\gamma_t - d_t} \tag{4.3}$$

and one of the following conditions:

$$\begin{aligned} &\left| \rho_t - \frac{s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}} - q_t p_t}{(\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2 - q_t^2} \right| \\ &\quad < \frac{\sqrt{(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}} - q_t p_t)^2 - ((\lambda_{t,\hat{H}_{\tilde{A}}} \delta_t + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2 - q_t^2)(1 - p_t^2)}}{(\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2 - q_t^2} \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}}) &> q_t, \\ |s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}} - q_t p_t| &> \sqrt{((\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2 - q_t^2)(1 - p_t^2)}, \\ p_t &< 1, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\left| \rho_t - \frac{r_{t,N_{\tilde{T}}} - q_t p_t - s_{t,N_{\tilde{A}}}}{q_t^2 - (\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2} \right| \\ &\quad < \frac{\sqrt{(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}} - q_t p_t)^2 + (q_t^2 - (\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2)(1 - p_t^2)}}{q_t^2 - (\delta_t \lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t \kappa_{t,\hat{H}_{\tilde{T}}})^2} \end{aligned}$$

$$(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}}) < q_t.$$

Then there exist measurable mappings $u, x, y : \Omega \to D$ such that for all $t \in \Omega$, $x(t) \in \tilde{A}(t, u(t))$, $y(t) \in \tilde{T}(t, u(t))$ has a unique solutions of (2.1). Moreover the random iterative sequences $\{u_n(t)\}_{n\geq 0}$, $\{x_n(t)\}_{n\geq 0}$ and $\{y_n(t)\}_{n\geq 0}$ obtained by random Algorithm 4.1 converges to $u(t)$, $x(t)$ and $y(t)$, respectively.

Proof. From the proof of Theorem 3.1, for each $t \in \Omega$ and the measurable mapping $u : \Omega \to D$, there exists a unique solution w (i.e., $u(t) \in D$, $x(t) \in \tilde{A}(t, u(t))$, $y(t) \in \tilde{T}(t, u(t)))$ satisfying the auxiliary problem (3.1). Defining a random mapping $G : \Omega \times D \to D$ satisfying $G_t(u(t)) = w(t)$ (i.e. $u(t) \to w(u(t))$ where $w(t)$ (i.e. $u(t) \in D$, $x(t) \in \tilde{A}(t, u(t))$, $y(t) \in \tilde{T}(t, u(t))))$ is the unique solution of (3.1). Now we will prove that the random mapping G is a contraction mapping. Let any $u_1(t)$ and $u_2(t)$ in D, there exists unique $w_1(t) = G_t(u_1(t))$, $w_2(t) = G_t(u_2(t))$ for all $v(t) \in D$ and $t \in \Omega$, such that

$$\begin{aligned}\langle G_t(u_1(t)), \eta_t(v(t), G_t(u_1(t)))\rangle \geq{}& \langle u_1(t), \eta_t(v(t), G_t(u_1(t)))\rangle - \rho_t\langle N_t(x_1(t), y_1(t)) \\ &- f, \eta_t(v(t), G_t(u_1(t)))\rangle - \rho_t a(G_t(u_1(t)), v(t) \\ &- G_t(u_1(t))) - \rho_t b(u_1(t), v(t)) \\ &+ \rho_t b(u_1(t), G_t(u_1(t))), \end{aligned} \tag{4.6}$$

for all $x_1(t) \in \tilde{A}(t, u_1(t))$ and $y_1(t) \in \tilde{T}(t, u_1(t))$ and

$$\begin{aligned}\langle G_t(u_2(t)), \eta_t(v(t), G_t(u_2(t)))\rangle \geq{}& \langle u_2(t), \eta_t(v(t), G_t(u_2(t)))\rangle - \rho_t\langle N_t(x_2(t), y_2(t)) \\ &- f, \eta_t(v(t), G_t(u_2(t)))\rangle - \rho_t a(G_t(u_2(t)), v(t) \\ &- G_t(u_2(t))) - \rho_t b(u_2(t), v(t)) \\ &+ \rho_t b(u_2(t), G_t(u_2(t))), \end{aligned} \tag{4.7}$$

for all $x_2(t) \in \tilde{A}(t, u_2(t))$ and $y_2(t) \in \tilde{T}(t, u_2(t))$ for all $t \in \Omega$. Taking $v(t) = G_t(u_2(t))$ in (4.6) and $v(t) = G_t(u_1(t))$ in (4.7) and adding these inequalities with $\eta_t(u(t), v(t)) = -\eta_t(v(t), v(t))$ and the assumption of $b(.,.)$, we arrive at

$$\begin{aligned}&\mu_{t,\eta}\|G_t(u_1(t)) - G_t(u_2(t))\|^2 \\ &\qquad\leq \langle G_t(u_1(t)) - G_t(u_2(t)), \eta_t(G_t(u_1(t)), G_t(u_2(t)))\rangle \\ &\qquad\leq \langle u_1(t) - u_2(t) - \rho_t(N_t(x_1(t), y_1(t)) \\ &\qquad- N_t(x_2(t), y_2(t))), \eta_t(G_t(u_1(t)), G_t(u_2(t))\rangle \\ &\qquad- \rho_t a(G_t(u_1(t)) - G_t(u_2(t)), G_t(u_1(t)) - G_t(u_2(t))) \\ &\qquad+ \rho_t b(u_1(t) - u_2(t), G_t(u_2(t)) - G_t(u_1(t))) \\ &\qquad\leq \|u_1(t) - u_2(t) - \rho_t(N_t(x_1(t), y_1(t)) \\ &\qquad- N_t(x_2(t), y_2(t)))\|\|\eta_t(G_t(u_1(t)), G_t(u_2(t)))\| \\ &\qquad- \rho_t d_t\|G_t(u_1(t)) - G_t(u_2(t))\|^2 \\ &\qquad+ \rho_t\gamma_t\|u_1(t) - u_2(t)\|\|G_t(u_1(t)) - G_t(u_2(t))\|. \end{aligned} \tag{4.8}$$

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Now from the random Lipschitz $\hat{H}$-continuity of N both variables and strong random monotonicity of N with first variable and random relaxed monotonicity of N with second variables, we have

$$
\begin{aligned}
&\|u_1(t)-u_2(t)-\rho_t(N_t(x_1(t),y_1(t))-N_t(x_2(t),y_2(t)))\|^2\\
&\quad = \|u_1(t)-u_2(t)\|^2 - 2\rho_t\langle N_t(x_1(t),y_1(t)) - N_t(x_2(t),y_2(t)), u_1(t)-u_2(t)\rangle\\
&\quad + \rho_t^2\|N_t(x_1(t),y_1(t)) - N_t(x_2(t),y_2(t))\|^2\\
&\quad \le \|u_1(t)-u_2(t)\|^2 - 2\rho_t\langle N_t(x_1(t),y_1(t)) - N_t(x_2(t),y_1(t)), u_1(t)-u_2(t)\rangle\\
&\quad - 2\rho_t\langle N_t(x_2(t),y_1(t)) - N_t(x_2(t),y_2(t)), u_1(t)-u_2(t)\rangle\\
&\quad + \rho_t^2\Big[\|N_t(x_1(t),y_1(t)) - N_t(x_2(t),y_1(t))\| + \|N_t(x_2(t),y_1(t)) - N_t(x_2(t),y_2(t))\|\Big]^2\\
&\quad \le \|u_1(t)-u_2(t)\|^2 - 2\rho_t s_{t,N_{\tilde{A}}}\|u_1(t)-u_2(t)\|^2 + 2\rho_t r_{t,N_{\tilde{T}}}\|u_1(t)-u_2(t)\|^2\\
&\quad + \rho_t^2\Big[\delta_t\|x_1(t)-x_2(t)\| + \xi_t\|y_1(t)-y_2(t)\|\Big]^2\\
&\quad \le \Big[1-2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}})\Big]\|u_1(t)-u_2(t)\|^2\\
&\quad + \rho_t^2\Big[\delta_t\hat{H}(\tilde{A}(t,u_1(t)),\tilde{A}(t,u_2(t))) + \xi_t\hat{H}(\tilde{T}(t,u_1(t)),\tilde{T}(t,u_2(t)))\Big]^2\\
&\quad \le \Big[1-2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}})\Big]\|u_1(t)-u_2(t)\|^2\\
&\quad + \rho_t^2\Big[\delta_t\lambda_{t,\hat{H}_{\tilde{A}}}\|u_1(t)-u_2(t)\| + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}}\|u_1(t)-u_2(t)\|\Big]^2\\
&\qquad\quad \le \Big[1-2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2\Big]\|u_1(t)-u_2(t)\|^2. \qquad (4.9)
\end{aligned}
$$

From (4.8) and (4.9), we get

$$
\begin{aligned}
&\mu_t\|G_t(u_1(t)) - G_t(u_2(t))\|^2\\
&\quad \le \Big[\sigma_t\sqrt{1-2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2} + \rho_t\gamma_t\Big]\|u_1(t)-u_2(t)\|\\
&\|G_t(u_1(t)) - G_t(u_2(t))\| - \rho_t d_t\|G_t(u_1(t)) - G_t(u_2(t))\|^2,
\end{aligned}
$$

that is

$$\|G_t(u_1(t)) - G_t(u_2(t))\| \le \theta(t)\|u_1(t)-u_2(t)\|,$$

where

$$\theta(t) = \frac{\sigma_t\sqrt{1-2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2} + \rho_t\gamma_t}{\mu_t + \rho_t d_t} < 1. \qquad (4.10)$$

By (4.3), and one of (4.4) and (4.5), therefore $G : \Omega \times D \to D$ is a contraction mapping. Hence there exists a unique point $\hat{u}(t) \in D$ such that $\hat{u}(t) = G_t(\hat{u}(t))$ and for each measurable mapping $v : \Omega \to D$,

$$
\begin{aligned}
\langle \hat{u}(t), \eta_t(v(t),\hat{u}(t))\rangle \ge{}& \langle \hat{u}(t), \eta_t(v(t),\hat{u}(t))\rangle - \rho_t\langle N_t(\hat{x}(t),\hat{y}(t))\\
&- f, \eta_t(v(t),\hat{u}(t))\rangle - \rho_t a(\hat{u}(t), v(t)-\hat{u}(t))\\
&- \rho_t b(\hat{u}(t), v(t)) + \rho_t b(\hat{u}(t),\hat{u}(t)), \qquad (4.11)
\end{aligned}
$$

where $\rho : \Omega \to (0,1)$ is a measurable mapping, which implies that

$$\langle N_t(\hat{x}(t), \hat{y}(t)) - f, \eta_t(v(t), \hat{u}(t))\rangle + a(\hat{u}(t), v(t) - \hat{u}(t)) \geq b(\hat{u}(t), \hat{u}(t)) - b(\hat{u}(t), v(t)),$$

for all $v(t) \in D$, that is $\hat{u}(t) \in D$, $\hat{x}(t) \in \tilde{A}(t, \hat{u}(t))$, $\hat{y}(t) \in \tilde{T}(t, \hat{u}(t))$ is the unique random solutions of the problem (2.1).

Next we consider the convergence of random iterative sequence generated by Algorithm 4.1. Taking $v(t) = u_{n+1}(t)$ in (4.11) and $v(t) = u_n(t)$ in (4.1) and adding these inequalities we have

$$\begin{aligned}\mu_t\|u_{n+1}(t) - u_n(t)\|^2 &\leq \langle u_{n+1}(t) - u_n(t), \eta_t(u_{n+1}(t), u_n(t))\rangle \\ &\leq \langle u_{n+1}(t) - u_n(t) - \rho_t(N_t(x_{n+1}(t), y_{n+1}(t)) \\ &\quad - N_t(x_n(t), y_n(t))), \eta_t(u_{n+1}(t), u_n(t))\rangle \\ &\quad - \rho_t a(u_{n+1}(t) - u_n(t), u_{n+1}(t) - u_n(t)) \\ &\quad + \rho_t b(u_{n+1}(t) - u_n(t), u_n(t) - u_{n+1}(t)) \\ &\quad + \langle e_n(t), \eta_t(u_{n+1}(t), u_n(t))\rangle\end{aligned}$$

$$\begin{aligned}&\leq \Big[\sigma_t\sqrt{1 - 2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2} + \rho_t\gamma_t\Big]\|u_{n+1}(t) - u_n(t)\| \\ &\|u_{n+1}(t) - u_n(t)\| - \rho_t d_t\|u_{n+1}(t) - u_n(t)\|^2 + \|e_n(t)\|\|u_{n+1}(t) - u_n(t)\|, \quad \forall\, n \geq 1.\end{aligned}$$

That is

$$\|u_{n+1}(t) - u_n(t)\| \leq \theta_n(t)\|u_{n-1}(t) - u_n(t)\| + \|e_n(t)\| \tag{4.12}$$

where

$$\theta_n(t) = \frac{\sigma_t\sqrt{1 - 2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2(1 + n^{-1})^2} + \rho_t\gamma_t}{\mu_t + \rho_t d_t}.$$

Let

$$\theta(t) = \frac{\sigma_t\sqrt{1 - 2\rho_t(s_{t,N_{\tilde{A}}} - r_{t,N_{\tilde{T}}}) + \rho_t^2(\delta_t\lambda_{t,\hat{H}_{\tilde{A}}} + \xi_t\kappa_{t,\hat{H}_{\tilde{T}}})^2} + \rho_t\gamma_t}{\mu_t + \rho_t d_t}, \quad \forall\, t \in \Omega.$$

We know that for each $t \in \Omega$, $\theta_n(t) \to \theta(t)$ and $\|e_n(t)\| \to 0$ as $n \to \infty$. By Condition (4.2), it follows that $\theta(t) \in (0,1)$ and hence (4.12) implies that $\{u_n(t)\}$ is a Cauchy sequence in D. Since D is complete, there exists a measurable mapping $u : \Omega \to D$ such that $u_n(t) \to u(t)$ as $n \to \infty$.

Further, from Algorithm 4.1, we have

$$\|x_{n+1}(t) - x_n(t)\| \leq \lambda_{t,\hat{H}_{\tilde{A}}}\Big(1 + \frac{1}{n}\Big)\|u_{n+1}(t) - u_n(t)\|$$

$$\|y_{n+1}(t) - y_n(t)\| \leq \kappa_{t,\hat{H}_{\tilde{T}}}\Big(1 + \frac{1}{n}\Big)\|u_{n+1}(t) - u_n(t)\|,$$

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which implies that $\{x_n(t)\}$, $\{y_n(t)\}$ are also Cauchy sequences in H. Let $x_n(t) \to x(t)$, $y_n(t) \to y(t)$, $n \to \infty$. Since $\{u_n(t)\}$, $\{x_n(t)\}$ and $\{y_n(t)\}$ are sequences of measurable mappings, we know that $u, x, y : \Omega \to H$ are measurable. Now we prove that $x(t) \in \tilde{A}(t, u(t))$ and $y(t) \in \tilde{T}(t, u(t))$ for any $t \in \Omega$, we have

$$\begin{aligned} d(x(t),\, \tilde{A}(t, u(t)) = &\ \inf\{\|x(t) - z\| : z \in \tilde{A}(t, u(t))\} \\ \leq &\ \|x(t) - x_n(t)\| + d(x_n(t),\, \tilde{A}(t, u(t))) \\ \leq &\ \|x(t) - x_n(t)\| + \hat{H}(\tilde{A}(t, u_n(t)),\, \tilde{A}(t, u(t))) \\ \leq &\ \|x(t) - x_n(t)\| + \lambda_{t, \hat{H}_{\tilde{A}}} \|u_n(t) - u(t)\| \to 0. \end{aligned}$$

Hence, $x(t) \in \tilde{A}(t, u(t))$, for all $t \in \Omega$. Similarly, we can prove that $y(t) \in \tilde{T}(t, u(t))$. So we have

$$\langle N_t(x(t), y(t)) - f, \eta(v(t), u(t))\rangle + a(u(t), v(t) - u(t)) \leq b(u(t), u(t)) - b(u(t), v(t)),$$

for all $v(t) \in D$, $t \in \Omega$. This completes the proof.

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Spaces of D_{L^p} type and a convolution product associated with a singular second order differential operator

M.Dziri
Department of Mathematics, Faculty of Sciences of Tunis
2092 Elmanar 2, Tunis, Tunisia
moncef.dziri@iscae.rnu.tn

M.Jelassi
Department of Mathematics, College of Sciences, Al Jouf University
Al Jouf 2014, Saudi Arabia
mourad.jelassi@fst.rnu.tn

L.T.Rachdi
Department of Mathematics, Faculty of Sciences of Tunis
2092 Elmanar 2, Tunis, Tunisia
LakhdarTannech.Rachdi@fst.rnu.tn

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Abstract We define and study the spaces $\mathcal{M}_{\rho,p}$, $1 \leq p \leq \infty$, that are of D_{L^p}-type. Using the harmonic analysis associated with a singular second order differential operator, we give new characterization of the dual space $\mathcal{M}'_{\rho,p}$ and we describe its bounded subsets. Next, we define the convolution product in $\mathcal{M}'_{\rho,p} \times M_{\rho,r}$, $1 \leq r \leq p < \infty$ and we prove some new results.

Keywords: D_{L^p} type, convolution product, differential operator.
2000 MSC codes: 43-xx, 46-Exx.

1 Introduction

The spaces D_{L^p}; $1 \leq p \leq \infty$, have been studied by many authors ([1], [2], [3], [4], [9]). In this paper, we consider the second order differential operator defined on $]0, +\infty[$, by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

1

where A is a non negative function satisfying some conditions and ρ is a non negative real number.

This operator plays an important role in Mathematical Analysis. For example, many special functions (orthogonal polynomials) are eigenfunctions of operator of Δ type. The radial part of the Laplacian-Beltrami on symmetric space is also of Δ type.

The operator Δ has been studied by many authors and from many points of view (see [5],[8] [13], [14], [16], [17]). In particular, in [5] and [14] many properties of harmonic analysis related to the operator Δ have been studied (translation operators, convolution product, Fourier transform...).

In this work, we introduce the function spaces $\mathcal{M}_{\rho,p}$, $1 \leq p \leq \infty$, similar to D_{L^p}, but replacing the usual derivatives by the operator Δ.

The main result of this paper consists to give a new characterization of the dual space $\mathcal{M}'_{\rho,p}$ of the space $\mathcal{M}_{\rho,p}$ and a description of its bounded subsets. More precisely, in the first section, we recall some harmonic analysis results related to the convolution product and the Fourier transform connected with the differential operator Δ, that we use in the next sections.

In the second section, we define the space $\mathcal{M}_{\rho,p}$, $1 \leq p \leq \infty$, to be the set of measurable functions f on $[0, +\infty[$ such that for all $k \in \mathbb{N}; \Delta^k f$ belongs to the space $L^p(d\nu)$ (the space of measurable functions on $[0, +\infty[$ of pth power integrable on $[0, +\infty[$ with respect to the measure $d\nu(x) = A(x)dx$). We give some properties of this space, in particular we prove that it is a Frechet space.

The third section is consecrated to the study of the dual space $\mathcal{M}'_{\rho,p}$. We give a nice description of the elements of this space and we characterize its bounded subsets.

In the last section, we define and study the convolution product in $\mathcal{M}'_{\rho,p} \times \mathrm{M}_{\rho,r}$, $1 \leq r \leq p < \infty$, where $\mathrm{M}_{\rho,r}$ is the closure of the space $\mathcal{D}_*(\mathbb{R})$ (the space of even infinitely differentiable functions on $\mathbb{R}$, with compact support) in $\mathcal{M}_{\rho,r}$.

2 The operator Δ

In this section, we define and recall some properties of the harmonic analysis related to the operator Δ, defined on $]0, +\infty[$, by

$$\Delta u = u'' + \frac{A'}{A}u' + \rho^2 u,$$

where

$$A(x) = x^{2\alpha+1}B(x); \qquad \alpha > \frac{-1}{2}, \tag{2.1}$$

and B is a positive even infinitely differentiable function on $\mathbb{R}$, with $B(0) = 1$.
We assume that the functions A and B satisfy the following conditions

i) A is increasing, and $\lim_{x\to+\infty} A(x) = +\infty$.

ii) $\dfrac{A'}{A}$ is decreasing, and $\lim\limits_{x\to+\infty}\dfrac{A'(x)}{A(x)} = 2\rho$.

iii) there exists a constant $\delta > 0$, satisfying

$$\begin{cases} \dfrac{B^{'}(x)}{B(x)} & = 2\rho - \frac{2\alpha+1}{x} + e^{-\delta x}F(x),\ \text{for}\rho > 0 \\ \dfrac{B^{'}(x)}{B(x)} & = e^{-\delta x}F(x)\ ,\text{for}\rho = 0 \end{cases}$$

where F is C^∞ on $]0,\infty[$, bounded together with its derivatives on all interval $[x_0,\infty[$, $x_0 > 0$.

This operator has been studied by many authors ([6],[8],[14]). In particular, the following results have been established :
For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} \Delta u = -\lambda^2 u \\ u(0) = 1,\ u'(0) = 0 \end{cases} \tag{2.2}$$

admits a unique solution denoted by φ_λ.
The function φ_λ satisfies the following properties :

- product formula

$$\forall x, y \geqslant 0;\ \varphi_\lambda(x)\varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z)w(x,y,z)A(z)dz \tag{2.3}$$

  where $w(x,y,.)$ is a measurable positive function on $[0,+\infty[$, with support in $[|x-y|, x+y]$, satisfying

- $\displaystyle\int_0^\infty w(x,y,z)A(z)dz = 1$
- $\forall z \geqslant 0;\ w(x,y,z) = w(y,x,z)$
- $\forall z{>}0;\ w(x,y,z) = w(x,z,y)$
- $\forall x \geqslant 0$, the function

$$\lambda \longmapsto \varphi_\lambda(x)$$

  is analytic on $\mathbb{C}$.

- $\forall \lambda \geq 0$ and $x \in \mathbb{R}$ $\ |\varphi_\lambda(x)| \leq 1$.
- $\forall \lambda \in \mathbb{C}$, the function

$$x \longmapsto \varphi_\lambda(x)$$

is even infinitely differentiable on $\mathbb{R}$, and for all $k \in \mathbb{N}$ there exist $m_k \in \mathbb{N}$ and $c_{A,k} > 0$ such that

$$\forall x \in \mathbb{R}; \ \left|(\frac{d}{dx})^k(\varphi_\lambda(x))\right| \leqslant c_{A,k}(1+x^2)\lambda^{2m_k}\varphi_0(x). \tag{2.4}$$

- For $\rho > 0$, and from ([13], p.99) we have

$$\forall x \geq 0, \forall \lambda \in \mathbb{R}, \ \ |\varphi_\lambda(x)| \leq \varphi_0(x) \leq m(1+x)exp(-\rho x), \tag{2.5}$$

where m is a positive constant.

- We have the integral representation of Mehler type,

$$\forall x > 0, \forall \lambda \in \mathbb{C}, \ \ \varphi_\lambda(x) = \int_0^x k(x,t)\cos(\lambda t)dt, \tag{2.6}$$

where $k(x,.)$ is an even positive C^∞ function on $]-x,x[$, with support in $[-x,x]$. Also, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the equation

$$\Delta u = -\lambda^2 u$$

has a solution Φ_λ, satisfying

$$\Phi_\lambda(x) = A^{-\frac{1}{2}}(x)\exp(i\lambda x)\,V(x,\lambda),$$

with

$$\lim_{x\to+\infty} V(x,\lambda) = 1.$$

Consequently there exists a function (spectral function)

$$\lambda \longmapsto c(\lambda),$$

such that,

$$\forall \lambda \in \mathbb{C}, \lambda \neq 0; \ \varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}.$$

The function

$$\lambda \longrightarrow c(\lambda)$$

satisfies the following property

- For $\lambda \in \mathbb{R}$,we have $c(-\lambda) = \overline{c(\lambda)}$.
- The function $|c(\lambda)|^{-2}$ is continuous on $[0,+\infty[$.
- there exist positive constants k_1, k_2, and k_3, such that $\forall \lambda \in \mathbb{C}, \ Im\lambda \leqslant 0, |\lambda| \geqslant k_3$;

$$k_1\,|\lambda|^{\alpha+1/2} \leqslant |c(\lambda)|^{-1} \leqslant k_2\,|\lambda|^{\alpha+1/2}. \tag{2.7}$$

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We denote by

- $d\nu(x)$ the measure defined on $[0,+\infty[$, by

$$d\nu(x) = A(x)dx$$

- $L^p(d\nu), 1 \leqslant p \leqslant +\infty$; the space of measurable functions on $[0,+\infty[$, satisfying

$$\begin{array}{rcll} \|f\|_{p,\nu} & = & \left(\displaystyle\int_0^{+\infty} |f(x)|^p \, d\nu(x)\right)^{1/p} < +\infty; & 1 \leqslant p < +\infty \\ \|f\|_{\infty,\nu} & = & \displaystyle\operatorname*{ess\ sup}_{x\in[0,+\infty[} |f(x)| < +\infty; & p = +\infty \end{array}$$

- $d\gamma(\lambda)$ the measure defined on $[0,+\infty[$, by

$$d\gamma(\lambda) = \frac{d\lambda}{2\pi\,|c(\lambda)|^2}$$

- $L^p(d\gamma), 1 \leqslant p \leqslant +\infty$; the space of measurable functions on $[0,+\infty[$, satisfying

$$\begin{array}{rcll} \|f\|_{p,\gamma} & = & \left(\displaystyle\int_0^{+\infty} |f(\lambda)|^p \, d\gamma(\lambda)\right)^{1/p} < +\infty; & 1 \leqslant p < +\infty \\ \|f\|_{\infty,\gamma} & = & \displaystyle\operatorname*{ess\ sup}_{\lambda\in[0,+\infty[} |f(\lambda)| < +\infty; & p = +\infty \end{array}$$

Definition 2.1 *i) The translation operator associated with the operator Δ is defined on $L^1(d\nu)$, by*

$$\forall x, y \geqslant 0; \ \mathcal{T}_x f(y) = \int_0^{+\infty} f(z) w(x,y,z) d\nu(z)$$

where w is the function defined by the relation (2.3).

ii) The convolution product, associated with the operator Δ of $f, g \in L^1(d\nu)$ is defined by

$$f * g(x) = \int_0^{+\infty} \mathcal{T}_x f(y) g(y) d\nu(y).$$

We have the following properties

- $\mathcal{T}_x \varphi_\lambda(y) = \varphi_\lambda(x)\varphi_\lambda(y)$
- If $f \in L^p(d\nu), 1 \leqslant p \leqslant +\infty$; then for all $x \in [0,+\infty[$, the function $\mathcal{T}_x f$ belongs to $L^p(d\nu)$, and we have

$$\|\mathcal{T}_x f\|_{p,\nu} \leqslant \|f\|_{p,\nu}\,.$$

- Let $p, q, r \in [1, \infty]$ such that $1/r = 1/p + (1/q) - 1$. If $f \in L^p(d\nu)$ and $g \in L^q(d\nu)$ then the function $f * g \in L^r(d\nu)$ and we have

$$\|f * g\|_{r,\nu} \leqslant \|f\|_{p,\nu} \|g\|_{q,\nu} . \tag{2.8}$$

Definition 2.2 *The Fourier transform associated with the operator Δ is defined on $L^1(d\nu)$, by*

$$\forall \lambda \in \mathbb{R};\ \mathcal{F}f(\lambda) = \int_0^{+\infty} f(x)\varphi_\lambda(x)d\nu(x).$$

We have the following properties
• For $f \in L^1(d\nu)$, such that $\mathcal{F}f \in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost every $x \in [0, +\infty[$,

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}f)(x) \tag{2.9}$$

where $\mathcal{F}^{-1}$ is the application defined on $L^1(d\gamma)$ by

$$\mathcal{F}^{-1}(g)(x) = \int_0^{+\infty} g(\lambda)\varphi_\lambda(x)d\gamma(\lambda). \tag{2.10}$$

• Let f be in $L^1(d\nu)$. For all $x \in [0, +\infty[$, we have

$$\forall \lambda \in \mathbb{R};\ \mathcal{F}(\mathcal{T}_x f)(\lambda) = \varphi_\lambda(x)\mathcal{F}f(\lambda).$$

• For $f, g \in L^1(d\nu)$, we have

$$\forall \lambda \in \mathbb{R},\ \mathcal{F}(f * g)(\lambda) = \mathcal{F}f(\lambda)\mathcal{F}g(\lambda).$$

• The Fourier transform $\mathcal{F}$, can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$. This means that

$$\forall f \in L^2(d\nu);\ \|\mathcal{F}f\|_{2,\gamma} = \|f\|_{2,\nu} . \tag{2.11}$$

$$\forall f \in L^2(d\gamma);\ \left\|\mathcal{F}^{-1}f\right\|_{2,\nu} = \|f\|_{2,\gamma} . \tag{2.12}$$

• For all $f \in D_*(\mathbb{R})$ we have

$$\mathcal{F}(f) = \mathcal{F}_0 \circ^t \chi(f) \tag{2.13}$$

where $\mathcal{F}_0$ is the usual Fourier transform on $\mathcal{D}_*(\mathbb{R})$ defined by
$\mathcal{F}_0(f)(\lambda) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(x)\cos(\lambda x)dx$ and ${}^t\chi$ is the generalized Weyl transform associated with Δ, given by

$${}^t\chi(f)(x) = \int_x^\infty k(y, x)f(y)A(y)dy$$

(see [14]).

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Proposition 2.1 *Let $p \in [1,2]$, $f \in L^p(d\nu)$ and $g \in L^p(d\gamma)$. Then $\mathcal{F}f \in L^{p'}(d\gamma)$, $\mathcal{F}^{-1}(g) \in L^{p'}(d\nu)$ and we have*

$$\|\mathcal{F}f\|_{p',\gamma} \leqslant \|f\|_{p,\nu}\,;\ \left\|\mathcal{F}^{-1}g\right\|_{p',\nu} \leqslant \|g\|_{p,\gamma}$$

with $p' = \frac{p}{p-1}$.

This result is an immediate consequence of the Riesz-Thorin theorem ([10], [11]).

We denote by
• $\mathcal{E}_*(\mathbb{R})$ the space of even infinitely differentiable functions on $\mathbb{R}$.
• $S_*(\mathbb{R})$ the subspace of $\mathcal{E}_*(\mathbb{R})$, consisting of functions f rapidly decreasing together with their derivatives.
• $S^2_*(\mathbb{R}) = \varphi_0 S_*(\mathbb{R})$, where φ_0 is the eigenfunction of the operator Δ associated with the value $\lambda = 0$. For $\rho = 0$ the space $S^2_*(\mathbb{R})$ is $S_*(\mathbb{R})$.
• $\mathcal{D}'_*(\mathbb{R})$, $S'_*(\mathbb{R})$ and $(S^2_*)'(\mathbb{R})$ are respectively the dual spaces of $\mathcal{D}_*(\mathbb{R})$, $S_*(\mathbb{R})$ and $S^2_*(\mathbb{R})$.

From [13], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $S^2_*(\mathbb{R})$ onto $S_*(\mathbb{R})$. The inverse mapping is given by the relation (2.10).

Definition 2.3 *The Fourier transform $\mathcal{F}$ is defined on $(S^2_*)'(\mathbb{R})$ by*

$$< \mathcal{F}(T), \varphi > = < T, \mathcal{F}^{-1}(\varphi) >, \quad \varphi \in S_*(\mathbb{R}).$$

Then $\mathcal{F}$ is an isomorphism from $(S^2_*)'(\mathbb{R})$ onto $S'_*(\mathbb{R})$.

3 The space $\mathcal{M}_{\rho,p}$

We denote by

- For $f \in L^p(d\nu)$, $p \in [1, \infty]$, T_f is the element of $\mathcal{D}'_*(\mathbb{R})$ defined by

$$< T_f, \varphi > = \int_0^\infty f(x)\varphi(x)d\nu(x), \;\; \varphi \in \mathcal{D}_*(\mathbb{R}).$$

- For $g \in L^p(d\gamma)$, $p \in [1, \infty]$, T_g is the element of $S'_*(\mathbb{R})$ defined by

$$< T_g, \psi > = \int_0^\infty g(\lambda)\psi(\lambda)d\gamma(\lambda), \;\; \psi \in S_*(\mathbb{R}).$$

Remark 3.1

i) Using the relation (2.5) and the fact that $A(x) \sim \exp(2\rho x)$ $(x \to +\infty)$ for $\rho > 0$, we deduce that for all $p \in [1, 2]$,
$L^p(d\nu) \subset (S^2_*)'(\mathbb{R})$.

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ii) From proposition 2.1, we deduce that for all $f \in L^p(d\nu)$, $1 \leq p \leq 2$; $\mathcal{F}f$ belongs to the space $L^{p'}(d\gamma)$ and we have

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)} \tag{3.1}$$

Definition 3.1 *Let $p \in [1, \infty]$. We define $\mathcal{M}_{\rho,p}$ to be the set of even measurable functions f on $\mathbb{R}$ such that for all $k \in \mathbb{N}$ there exists $g_k \in L^p(d\nu)$ satisfying*

$$\Delta^k T_f = T_{g_k} \tag{3.2}$$

in $\mathcal{D}'_*(\mathbb{R})$,

where, for all $T \in \mathcal{D}'_*(\mathbb{R})$ $(resp.(S^2_*)'(\mathbb{R}))$,

$$< \Delta T, \varphi >=< T, \Delta\varphi >; \;\; \varphi \in \mathcal{D}_*(\mathbb{R}) \;(\text{resp.}\mathrm{S}^2_*(\mathbb{R})).$$

The space $\mathcal{M}_{\rho,p}$ is equipped with the topology generated by the family of norms

$$\gamma_{m,p}(f) = \max_{0 \leq k \leq m} \|g_k\|_{p,\nu}, \;\; m \in \mathbb{N},$$

where g_k, $k \in \mathbb{N}$, is the function given by the relation (3.2).
Let

$$d_p : \; \mathcal{M}_{\rho,p} \times \mathcal{M}_{\rho,p} \to [0, \infty[$$

$$(f, g) \mapsto d_p(f, g) = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\gamma_{m,p}(f - g)}{1 + \gamma_{m,p}(f - g)}.$$

Then d_p is a distance on $\mathcal{M}_{\rho,p}$. Moreover the sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0 in $(\mathcal{M}_{\rho,p}, d_p)$ if and only if

$$\forall \; m \in \mathbb{N}; \qquad \gamma_{m,p}(f_k) \underset{k\to\infty}{\longrightarrow} 0$$

In the following, we will give some properties of the space $\mathcal{M}_{\rho,p}$.

Proposition 3.1 *$(\mathcal{M}_{\rho,p}, d_p)$ is a Frechet space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{M}_{\rho,p}, d_p)$ and $(g_{n,k})_{k\in\mathbb{N}} \subset L^p(d\nu)$ such that

$$\Delta^k T_{f_n} = T_{g_{n,k}}, \;\; k \in \mathbb{N}. \tag{3.3}$$

Then for all $k \in \mathbb{N}$, $(g_{n,k})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(d\nu)$.
We put

$$f = g_0 = \lim_{n\to\infty} f_n$$

and

$$g_k = \lim_{n\to\infty} g_{n,k}, \;\; k \in \mathbb{N}^*,$$

in $L^p(d\nu)$. Thus

$$\forall\ k \in \mathbb{N}\,; \qquad T_{g_{n,k}} \underset{n\to\infty}{\longrightarrow} T_{g_k} \tag{3.4}$$

in $\mathcal{D}'_*(\mathbb{R})$. Since Δ is a continuous operator from $\mathcal{D}'_*(\mathbb{R})$ into itself, we deduce that

$$\Delta^k T_{f_n} \underset{n\to\infty}{\longrightarrow} \Delta^k T_f \ ,$$

in $D'_*(\mathbb{R})$.

From the relations (3.3) and (3.4), we deduce that

$$\forall\ k \in \mathbb{N}\,; \quad \Delta^k T_f = T_{g_k}.$$

This proves that $f \in \mathcal{M}_{\rho,p}$ and

$$f_n \underset{n\to\infty}{\longrightarrow} f \ ,$$

in $(\mathcal{M}_{\rho,p}, d_p)$.

Proposition 3.2 *Let $p \in [1,2]$ and $f \in \mathcal{M}_{\rho,p}$. Then*

i) For all $k \in \mathbb{N}$, the function

$$\lambda \to (1+\lambda^2)^k \mathcal{F}(f)(\lambda)$$

belongs to the space $L^{p'}(d\gamma)$, with $p' = \dfrac{p}{p-1}$.

ii) $\mathcal{M}_{\rho,p} \cap \mathcal{C}_*(\mathbb{R}) \subset \mathcal{E}_*(\mathbb{R})$, where $\mathcal{C}_*(\mathbb{R})$ is the space of even continuous functions on $\mathbb{R}$.

Proof. i) Let $f \in \mathcal{M}_{\rho,p},\ 1 \le p \le 2$, and $g_k \in L^p(d\nu)$ such that

$$\Delta^k T_f = T_{g_k}\ k \in \mathbb{N}$$

From the relation (3.1), we have

$$\mathcal{F}(T_{g_k}) = T_{\mathcal{F}(g_k)},$$

which gives

$$\mathcal{F}(\Delta^k T_f) = T_{\mathcal{F}(g_k)}$$

On the other hand

$$\mathcal{F}(\Delta^k T_f) = \lambda^{2k} \mathcal{F}(T_f) \tag{3.5}$$

$$= T_{\lambda^{2k}\mathcal{F}(f)}$$

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hence

$$\lambda^{2k}\mathcal{F}(f) = \mathcal{F}(g_k)$$

this equality, together with the fact that the function $\mathcal{F}(g_k)$ belongs to the space $L^{p'}(d\gamma)$ implies i).

ii) Let $f \in \mathcal{M}_{\rho,p} \cap \mathcal{C}_*(\mathbb{R})$. From the assertion i), we deduce that $\mathcal{F}(f) \in L^1(d\gamma) \cap L^2(d\gamma)$.

On the other hand, the transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$, then from the inversion formula for $\mathcal{F}$ and using the continuity of the function f, we have

$$\forall x \in \mathbb{R};\ f(x) = \int_0^{+\infty} \mathcal{F}f(\lambda)\varphi_\lambda(x)d\gamma(\lambda). \tag{3.6}$$

Consequently, ii) follows from the relation (2.4) and the fact that for all $k \in \mathbb{N}$,, the function

$$\lambda \to \lambda^k \mathcal{F}f(\lambda)$$

belongs to the space $L^1(d\gamma)$.

Proposition 3.3 *Let $p \in [1,2]$, then, for all $r \in [2,\infty]$,*

$$\mathcal{M}_{\rho,p} \cap \mathcal{C}_*(\mathbb{R}) \subset \mathcal{M}_{\rho,r}.$$

Proof. Let $f \in \mathcal{M}_{\rho,p} \cap \mathcal{C}_*(\mathbb{R}),\ p \in [1,2],\ r \geq 2$ and $r' = \dfrac{r}{r-1}$.
From the proposition 3.2, we deduce that $f \in \mathcal{E}_*(\mathbb{R})$ and that for all $k \in \mathbb{N}$, the function

$$\lambda \to \lambda^{2k}\mathcal{F}(f)(\lambda)$$

belongs to the space $L^{p'}(d\gamma)$. By applying Holder's inequality it follows that this last function belongs to the space $L^{r'}(d\gamma)$. On the other hand, from the relation (3.6), we deduce that

$$\begin{aligned} \forall\ x \in \mathbb{R},\ \ \Delta^k f(x) &= \int_0^{\infty} (-\lambda^2)^k \mathcal{F}(f)(\lambda)\varphi_\lambda(x)d\gamma(\lambda) \\ &= \mathcal{F}^{-1}((-\lambda^2)^k\mathcal{F}(f))(x), \end{aligned}$$

Consequently, from proposition 2.1 it follows that for all $k \in \mathbb{N}$, the function $\Delta^k f$ belongs to the space $L^r(d\nu)$.

4 The Dual space$\mathcal{M}'_{\rho,p}$

In this section we will give a new characterization of the dual space $\mathcal{M}'_{\rho,p}$ of $\mathcal{M}_{\rho,p}$. It is clear that for every $f \in \mathcal{M}_{\rho,p}$, the family $\{V_{m,p,\varepsilon}(f), m \in \mathbb{N}, \varepsilon > 0\}$ is a basic of neighborhoods of f in $(\mathcal{M}_{\rho,p}, d_p)$, where

$$V_{m,p,\varepsilon}(f) = \{g \in \mathcal{M}_{\rho,p}, \gamma_{m,p}(f-g) < \varepsilon\}.$$

In addition, $T \in \mathcal{M}'_{\rho,p}$ if and only if there exist $m \in \mathbb{N}$ and $c > 0$ such that

$$\forall \ f \in \mathcal{M}_{\rho,p}; \quad | < T, f > | \leq c\gamma_{m,p}(f). \tag{4.1}$$

For $f \in L^{p'}(d\nu)$ and $\psi \in \mathcal{M}_{\rho,p}$, we put

$$< \Delta^k(T_f), \psi >= \int_0^\infty f(x)\psi_k(x)d\nu(x) \tag{4.2}$$

with $T_{\psi_k} = \Delta^k T_\psi$. Then

$$\begin{aligned} | < \Delta^k(T_f), \varphi > | &\leq \|f\|_{p',\nu}\|\psi_k\|_{p,\nu} \\ &\leq \|f\|_{p',\nu}\gamma_{k,p}(\psi) \end{aligned}$$

this proves that for all $f \in L^{p'}(d\nu)$ and $k \in \mathbb{N}$, the functional $\Delta^k T_f$ defined by the relation (4.2) belongs to the space $\mathcal{M}'_{\rho,p}$.

In the following, we shall prove that every element of $\mathcal{M}'_{\rho,p}$ is also of this type.

Theorem 4.1 *Let $T \in \mathcal{D}'_*(\mathbb{R})$. Then T belongs to $\mathcal{M}'_{\rho,p}$, $1 \leq p < \infty$, if and only if there exist $m \in \mathbb{N}$ and $\{f_0, ..., f_m\} \subset L^{p'}(d\nu)$ such that*

$$T = \sum_{k=0}^{m} \Delta^k T_{f_k} \tag{4.3}$$

where $\Delta^k T_{f_k}$ is given by the relation (4.2).

Proof. It is clear that if

$$T = \sum_{k=0}^{m} \Delta^k T_{f_k}; \quad \{f_0, ..., f_m\} \subset L^{p'}(d\nu)$$

then T belongs to the space $\mathcal{M}'_{\rho,p}$.

Conversely, suppose that $T \in \mathcal{M}'_{\rho,p}$. From the relation (4.1) there exist $m \in \mathbb{N}$ and $c > 0$ such that

$$\forall \ \varphi \in \mathcal{M}_{\rho,p}, \ \ | < T, \varphi > | \leq c\gamma_{m,p}(\varphi)$$

Let

$$(L^p(d\nu))^{m+1} = \{(f_0, ..., f_m), \ f_k \in L^p(d\nu), \ 0 \leq k \leq m\}$$

equipped with the norm

$$\|(f_0, ..., f_m)\|_{(L^p(d\nu))^{m+1}} = \max_{0 \leq k \leq m} \|f_k\|_{p,\nu}$$

Now, we consider the mappings

$$\mathcal{A} : \mathcal{M}_{\rho,p} \to (L^p(d\nu))^{m+1}$$

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$$\varphi \mapsto (\varphi, g_1, ..., g_m)$$

where

$$\Delta^k T_\varphi = T_{g_k}, \quad k \geq 1$$

and

$$\mathcal{B} : \mathrm{Im}(\mathcal{A}) \to \mathbb{C}$$

$$\mathcal{B}(A\varphi) =< T, \varphi > .$$

From the relation (4.1) we deduce that

$$\begin{aligned} |\mathcal{B}\mathcal{A}(\varphi)| &= | < T, \varphi > | \\ &\leq c \, \|\mathcal{A}(\varphi)\|_{(L^p(d\nu))^{m+1}} \end{aligned}$$

this means that $\mathcal{B}$ is a continuous functional on the subspace $\mathrm{Im}(\mathcal{A})$ of the space $(L^p(d\nu))^{m+1}$. From Hahn Banach theorem's there exists a continuous extension of $\mathcal{B}$ to $(L^p(d\nu))^{m+1}$, denoted again by $\mathcal{B}$.
By Riez's theorem there exist $(f_0, ..., f_m) \in (L^{p'}(d\nu))^{m+1}$ such that
$\forall \ (\varphi_0, ..., \varphi_m) \in (L^p(d\nu))^{m+1}$,

$$\mathcal{B}(\varphi_0, ..., \varphi_m) = \sum_{k=0}^{m} \int_0^\infty f_k(x)\varphi_k(x)d\nu(x).$$

By means of the relation (4.2), we deduce that for $\varphi \in \mathcal{M}_{\rho,p}$, we have

$$< T, \varphi >= \sum_{k=0}^{m} \int_0^\infty f_k(x)\varphi_k(x)d\nu(x)$$

$$= \sum_{k=0}^{m} < \Delta^k T_{f_k}, \varphi > .$$

This completes the proof of the theorem 4.1.

Proposition 4.1 *Let $p \geq 2$. Then for all $T \in \mathcal{M}'_{\rho,p}$, $T \in (S^2_*)'(\mathbb{R})$ and there exist $m \in \mathbb{N}$ and $F \in L^p(d\gamma)$ such that*

$$\mathcal{F}(T) = T_{(1+\lambda^2)^m F}$$

Proof. Let $T \in \mathcal{M}'_{\rho,p}$. From the theorem 4.1 there exist $m \in \mathbb{N}$ and $(f_0, ..., f_m) \in (L^{p'}(d\nu))^{m+1}$, $p' = \dfrac{p}{p-1}$, such that

$$T = \sum_{k=0}^{m} \Delta^k T_{f_k}.$$

From the remark 3.1 i) and the fact that the operator Δ is continuous from $(S^2_*)'(\mathbb{R})$ into itself, we deduce that T belongs to $(S^2_*)'(\mathbb{R})$. Consequently, by virtue of the relation (3.5), we have

$$\mathcal{F}(T) = T_{(1+\lambda^2)^m F},$$

where

$$F = \sum_{k=0}^{m} \frac{\lambda^{2k}}{(1+\lambda^2)^m} \mathcal{F}(f_k)$$

which proves the result.

Proposition 4.2 *Let $T \in \mathcal{D}'_*(\mathbb{R})$, then $T \in \mathcal{M}'_{\rho,2}$ if and and only if there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma)$ such that*

$$\mathcal{F}(T) = T_{(1+\lambda^2)^m F}$$

Proof. From the proposition 4.1, we deduce that if $T \in \mathcal{M}'_{\rho,2}$ then there exist $m \in \mathbb{N}$ and $F \in L^2(d\gamma)$ verifying

$$\mathcal{F}(T) = T_{(1+\lambda^2)^m F}.$$

Conversely, suppose that

$$\mathcal{F}(T) = T_{(1+\lambda^2)^m F},$$

with $F \in L^2(d\gamma)$. Since $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$, then there exists $G \in L^2(d\nu)$ such that $\mathcal{F}(G) = F$ and from the relation (3.1) we have

$$\mathcal{F}(T_G) = T_F.$$

Consequently

$$\mathcal{F}(T) = \mathcal{F}((I-\Delta)^m T_G)$$

thus

$$T = \sum_{k=0}^{m} (-1)^k C_m^k \Delta^k T_G$$

and the theorem 4.1 implies that T belongs to $\mathcal{M}'_{\rho,2}$.

For $a > 0$, we denote by
- $\mathcal{D}_{*,a}(\mathbb{R})$ the subspace of $\mathcal{D}_*(\mathbb{R})$ consisting of function f such that $supp f \subset [-a, a]$.
- $\mathcal{D}'_{*,a}(\mathbb{R})$ the dual space of $\mathcal{D}_{*,a}(\mathbb{R})$.
- $\mathcal{W}_a^m(\mathbb{R})$, $m \in \mathbb{N}$ the space of function $f : \mathbb{R} \to \mathbb{C}$, of class C^{2m} on $\mathbb{R}$, even and with support in $[-a, a]$, normed by

$$Q_{m,a}(f) = \max_{0 \le k \le m} \|\Delta^k(f)\|_{\infty,\nu}.$$

Lemma 4.1 *For all $m \in \mathbb{N}$, there exists $s_0 \in \mathbb{N}$ such that for all $s \ge s_0$, the function g_s defined by*

$$\forall\, x \in \mathbb{R};\ g_s(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+\lambda^2)^s}\right)(x).$$

is even, of class C^{2m} on $\mathbb{R}$ and infinitely differentiable on $\mathbb{R}\backslash\{0\}$.

Proof. Let $m \in \mathbb{N}$. From the relations (2.4) and (2.10), we deduce that there exists $s_0 \in \mathbb{N}$, sufficiently large such that for $s \geq s_0$, the functions g_s and $f_s = \mathcal{F}_0(\dfrac{1}{(1+\lambda^2)^s})$ are of class C^{2m} on $\mathbb{R}$.

In the following we shall prove that for $s \geq s_0$, the function g_s is infinitely differentiable on $\mathbb{R}\backslash\{0\}$.
Let $\psi \in S_*(\mathbb{R})$; $\psi \geq 0$ and $\displaystyle\int_{\mathbb{R}} \psi(x)dx = 1$.Then,

$$\lim_{n\to+\infty} f_s *_\circ \psi_n = f_s$$

in $\mathrm{L}^2(\mathbb{R}, \mathrm{dx})$, where $\psi_n(x) = n\psi(nx)$ and $*_\circ$ is the usual convolution product on $\mathbb{R}$.

Now, we take $k = E(\alpha) + 2$. Since s is sufficiently large, then for every $j \in \mathbb{N}$, $0 \leq j \leq 2k$, the function $f_s^{(j)}$ belongs to $L^2(\mathbb{R}, dx)$ and

$$\lim_{n\to+\infty} f_s^{(j)} *_\circ \psi_n = (f_s *_\circ \psi_n)^{(j)} = f_s^{(j)} \tag{4.4}$$

in $\mathrm{L}^2(\mathbb{R}, \mathrm{dx})$.
Consequently

$$\forall j \in \mathbb{N}, 0 \leq j \leq k; \quad \lim_{n\to+\infty} \lambda^{2j}\mathcal{F}_0(f_s *_\circ \psi_n) = \frac{2}{\pi}\frac{\lambda^{2j}}{(1+\lambda^2)^s} \tag{4.5}$$

in $\mathrm{L}^2(\mathbb{R}, \mathrm{dx})$.
Using the relations (2.7) and (4.5), we deduce that

$$\lim_{n\to+\infty} \mathcal{F}_0(f_s *_\circ \psi_n) = \frac{2}{\pi}\frac{1}{(1+\lambda^2)^s}$$

in $\mathrm{L}^2(\mathrm{d}\gamma)$, and the relations (2.12); (2.13) lead to

$$\frac{2}{\pi}g_s = \lim_{n\to+\infty} ({}^t\chi)^{-1}(f_s *_\circ \psi_n) \text{ in } \mathrm{L}^2(\mathrm{d}\nu). \tag{4.6}$$

Let $[a, b] \subset]0, \infty[$ and $\theta \in \mathcal{D}_*(\mathbb{R})$ such that

$$\begin{cases} \theta(x) = 1 \text{ if } x \in [\dfrac{-a}{4}, \dfrac{a}{4}] \\ supp(\theta) \subset]\dfrac{-a}{2}, \dfrac{a}{2}[. \end{cases}$$

and φ an infinitely differentiable function on $\mathbb{R}$ such that

$$\begin{cases} \varphi(x) = 1 \text{ if } \mathrm{x} \in [\mathrm{a}, \mathrm{b}] \\ \mathrm{supp}\varphi \subset]\dfrac{\mathrm{a}}{2}, \mathrm{b} + 1[\end{cases}$$

using the relation (4.4), we deduce that

$$\lim_{n\to+\infty} ((1-\theta)(f_s *_\circ \psi_n))^{(2k)} = ((1-\theta)f_s)^{(2k)}$$

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in $\mathrm{L}^2(\mathbb{R}, \mathrm{dx})$, and by the same way as before, we have

$$\lim_{n\to+\infty} ({}^t\chi)^{-1}((1-\theta)(f_s *_\circ \psi_n)) = \mathcal{F}^{-1}(\mathcal{F}_0((1-\theta)f_s) \tag{4.7}$$

in $\mathrm{L}^2(\mathrm{d}\nu)$. However, using the expression of $({}^t\chi)^{-1}$ (see[14]) and the relation (4.6) we get

$$\frac{2}{\pi}\varphi g_s = \lim_{n\to+\infty} \varphi({}^t\chi)^{-1}((1-\theta)(f_s *_\circ \psi_n)) \text{ in } \mathrm{L}^2(\mathrm{d}\nu). \tag{4.8}$$

On the other hand, by a standart calculus, for all $s \geq s_0$, we have

$$f_s(x) = e^{-|x|}P_s(|x|),$$

where, P_s is a real polynomial; which implies that the function $(1-\theta)f_s$ belongs to $\mathcal{S}_*(\mathbb{R})$.
Since, $\mathcal{F}_0$ is an isomorphism from $S_*(\mathbb{R})$ onto itself and $\mathcal{F}$ is an isomorphism from $S^2_*(\mathbb{R})$ onto $S_*(\mathbb{R})$, we deduce that the function $\mathcal{F}^{-1}(\mathcal{F}_0(f_s(1-\theta)))$ belongs to $S^2_*(\mathbb{R})$.
Hence, the relations (4.7) and (4.8) lead to

$$\frac{2}{\pi}\varphi g_s = \varphi\mathcal{F}^{-1}(\mathcal{F}_0(f_s(1-\theta)))$$

this shows that the function g_s is infinitely differentiable on all interval $[a, b] \subset]0, \infty[$ and by parity it is infinitely differentiable on $\mathbb{R} \setminus \{0\}$.

Proposition 4.3 *Let $a > 0$ and $m \in \mathbb{N}$. Then there exists $s_o \in \mathbb{N}$ such that for every $s \in \mathbb{N},\ s \geq s_o$, we can find $\psi_s \in \mathcal{W}^m_a(\mathbb{R})$ and $F_s \in \mathcal{D}_{*,a}(\mathbb{R})$ satisfying*

$$\delta = (I - \Delta)^s T_{\psi_s} + T_{F_s}$$

in $\mathcal{D}'_*(\mathbb{R})$.

Proof. From lemma 4.1 there exists $s_o \in \mathbb{N}$ such that for every $s \in \mathbb{N},\ s \geq s_o$, the function g_s is of class C^{2m} on $\mathbb{R}$, and we have

$$(I - \Delta)^s T_{g_s} = \delta.$$

Let $h \in \mathcal{D}_{*,a}(\mathbb{R})$, satisfying

$$\forall\, x \in [-\frac{a}{2}, \frac{a}{2}];\ h(x) = 1.$$

Then,

$$h(I - \Delta)^s T_{g_s} = \delta.$$

On the other hand, the lemma 4.1 involves that for every $s \geq s_o$, the function

$$F_s(x) = (h-1)(I-\Delta)^s g_s + (I-\Delta)^s((1-h)g_s) \tag{4.9}$$

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belongs to the space $\mathcal{D}_{*,a}(\mathbb{R})$.

Moreover, we have

$$T_{(h-1)(I-\Delta)^s g_s} = (h-1)(I-\Delta)^s T_{g_s} = 0$$

and this implies, by using the relation (4.9) that

$$\begin{aligned} T_{F_s} &= T_{(I-\Delta)^s((1-h)g_s)} \\ &= (I-\Delta)^s T_{((1-h)g_s)}. \end{aligned}$$

Hence,

$$T_{F_s} + (I-\Delta)^s T_{hg_s} = (I-\Delta)^s T_{g_s} = \delta.$$

We complete the proof of the proposition by taking $\psi_s = hg_s$.

For $f \in \mathcal{D}_{*,a}(\mathbb{R}),\ a > 0$, we denote by

- $P_m(f) = \underset{0\le k\le m}{max} \|(\frac{d}{dx})^k f\|_{\infty,\nu}$
- $N_{p,m}(f) = \underset{0\le k\le m}{max} \|\Delta^k(f)\|_{p,\nu}$, $p \in [1,\infty]$.

Lemma 4.2 *Let $p \in [1,\infty]$. For all $m \in \mathbb{N}$, there exist $c > 0$ and $m' \in \mathbb{N}$ such that*

$$\forall\, \varphi \in \mathcal{D}_{*,a}(\mathbb{R}),\ P_m(\varphi) \le c N_{p,m'}(\varphi).$$

Proof. The case $p = \infty$ is proved in [6], then for all $m \in \mathbb{N}$, there exist $c > 0$ and $m' \in \mathbb{N}$ such that for all $0 \le k \le m$, we have

$$\|(\frac{d}{dx})^k \varphi\|_{\infty,\nu} \le c \underset{0\le k\le m'}{max} \|\Delta^k \varphi\|_{\infty,\nu} \tag{4.10}$$

On the other hand, from the inversion formula for $\mathcal{F}$, we have

$$\begin{aligned} \Delta^k \varphi(x) &= \int_0^\infty \mathcal{F}(\Delta^k \varphi)(\lambda)\varphi_\lambda(x)d\gamma(\lambda) \\ &= \int_0^\infty (-\lambda^2)^k \mathcal{F}(\varphi)(\lambda)\varphi_\lambda(x)d\gamma(\lambda) \end{aligned}$$

Now, from the relation (2.7), it follows that

$$\int_0^\infty \frac{d\gamma(\lambda)}{(1+\lambda^2)^{\alpha+2}} < \infty.$$

This involves that

$$\begin{aligned} \|\Delta^k \varphi\|_{\infty,\nu} &\le c\|(1+\lambda^2)^{k+E(\alpha)+3} \mathcal{F}(\varphi)\|_{\infty,\gamma} \\ &\le c\|\mathcal{F}((I-\Delta)^{k+E(\alpha)+3}\varphi)\|_{\infty,\gamma} \end{aligned}$$

$$\leq \quad c\|(I-\Delta)^{k+E(\alpha)+3}\varphi\|_{1,\nu} \tag{4.11}$$

According to the relation (4.10) and (4.11), we deduce that for every k; $0 \leq k \leq m$;

$$\|(\frac{d}{dx})^k\varphi\|_{\infty,\nu} \leq c_m N_{1,m'+E(\alpha)+3}(\varphi),$$

where $c_m = c\,2^{m'+E(\alpha)+3}$. Hence,

$$P_m(\varphi) \leq c_m N_{1,m'+E(\alpha)+3}(\varphi). \tag{4.12}$$

For $p \in]1,\infty[$, the result follows by using Holder's inequality and the relation (4.12).

Theorem 4.2 *Let $a > 0$ and B' a weakly* bounded set of $\mathcal{D}'_{*,a}(\mathbb{R})$. Then, there exists $m \in \mathbb{N}$ such that the elements of B' can be continuously extended to $\mathcal{W}^m_a(\mathbb{R})$. Moreover, the family of these extensions is equicontinuous.*

Proof. Let $p \in [1,\infty]$. Since B' is weakly* bounded in $\mathcal{D}'_{*,a}(\mathbb{R})$, then from [12] and lemma 4.2 there exist a positive constant c and $m \in \mathbb{N}$ such that
$\forall\, T \in B'$, $\forall\, \varphi \in \mathcal{D}_{*,a}(\mathbb{R})$,

$$| < T, \varphi > | \leq c N_{p,m}(\varphi) \tag{4.13}$$

Let's consider the mappings

$$\mathcal{A} : \mathcal{W}^m_a(\mathbb{R}) \to (L^p(d\nu))^{m+1}$$

$$\varphi \mapsto (\Delta^k\varphi)_{0\leq k\leq m}$$

and for all $T \in B'$,

$$L_T : \mathcal{A}(\mathcal{D}_{*,a}(\mathbb{R})) \to \mathbb{C}$$

$$< L_T, \mathcal{A}\varphi > = < T, \varphi > .$$

From the relation (4.13), we deduce that
$\forall\, \varphi \in \mathcal{D}_{*,a}(\mathbb{R})$,

$$| < L_T, \mathcal{A}\varphi > | \leq c\|\varphi\|_{(L^p(d\nu))^{m+1}}$$

this means that L_T is a continuous functional on the subspace $\mathcal{A}(\mathcal{D}_{*,a}(\mathbb{R}))$ of the space $(L^p(d\nu))^{m+1}$ and that for all $T \in B'$

$$\|L_T\|_{(D_{*,a}(\mathbb{R}))} = \sup_{\|\mathcal{A}\varphi\|_{(L^p(d\nu))^{m+1}}\leq 1} | < L_T, \mathcal{A}\varphi > | \leq c$$

From the Hahn Banach theorem's, L_T can be continuously extended on $(L^p(d\nu))^{m+1}$ denoted again by L_T. Furthermore, for all $T \in B'$

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \sup_{\|\psi\|_{(L^p(d\nu))^{m+1}}\leq 1} | < L_T, \psi > |$$

$$= \quad \|L_T\|_{\mathcal{A}(\mathcal{D}_{*,a}(\mathbb{R}))} \leq c \tag{4.14}$$

Now, from the Riez's theorem, there exists $(f_{T,k})_{0\leq k\leq m} \subset L^{p'}(d\nu)$ such that $\forall\, \psi = (\psi_0, ..., \psi_m) \in (L^p(d\nu))^{m+1}$,

$$< L_T, \psi >= \sum_{k=0}^{m} \int_0^{\infty} f_{T,k}(x)\psi_k(x)d\nu$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \max_{0\leq k\leq m} \|f_{T,k}\|_{p',\nu}.$$

Thus, from (4.14) it follows that
$\forall T \in B',\ \forall\, 0 \leq k \leq m$

$$\|f_{T,k}\|_{p',\nu} \leq c. \tag{4.15}$$

In particular, for $\varphi \in \mathcal{W}_a^m(\mathbb{R})$ we have

$$< L_T, \mathcal{A}\varphi >= \sum_{k=0}^{m} \int_0^{\infty} f_{T,k}(x)\Delta^k(\varphi)(x)d\nu(x)$$

using Holder inequality and the relation (4.15), we get
$\forall T \in B',\ \forall\, \varphi \in \mathcal{W}_a^m(\mathbb{R})$;

$$| < L_T, \mathcal{A}\varphi > | \leq (m+1)c[\nu(B(0,a))]^{1/p}\|\varphi\|_{\mathcal{W}_a^m(\mathbb{R})}$$

this shows that the mapping $L_T o\mathcal{A}$ is a continuous extension of T on $\mathcal{W}_a^m(\mathbb{R})$ and that the family $\{L_T o\mathcal{A}\}_{T\in B'}$ is equicontinuous, when applied to $\mathcal{W}_a^m(\mathbb{R})$. This completes the proof of the theorem 4.2.

In the following, we will give a new characterization of the space $\mathcal{M}'_{\rho,p}$.

Theorem 4.3 *Let $T \in \mathcal{D}'_*(\mathbb{R})$. Then for $p \in [1, \infty[$ and $p' = \frac{p}{p-1}$, T belongs to $\mathcal{M}'_{\rho,p}$ if and only if for every $\varphi \in \mathcal{D}_*(\mathbb{R})$, the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$, where*

$$T * \varphi(x) =< T, \mathcal{T}_x\varphi > .$$

Proof. Let $T \in \mathcal{M}'_{\rho,p}$. From the theorem 4.1 there exist $m \in \mathbb{N}$ and $f_0, ..., f_m \in L^{p'}(d\nu)$ such that

$$T = \sum_{k=0}^{m} \Delta^k(T_{f_k}),$$

in $\mathcal{M}'_{\rho,p}$. Thus, for every $\varphi \in \mathcal{D}_*(\mathbb{R})$

$$\begin{aligned} T * \varphi &= \sum_{k=0}^{m} T_{f_k} * \Delta^k\varphi \\ &= \sum_{k=0}^{m} f_k * \Delta^k\varphi \end{aligned}$$

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Since, for all $k \in \mathbb{N}$, $0 \leq k \leq m$, $f_k \in L^{p'}(d\nu)$ and $\Delta^k\varphi \in L^1(d\nu)$, then we deduce that $f_k * \Delta^k\varphi \in L^{p'}(d\nu)$. This implies that the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$.

Conversely, let $T \in \mathcal{D}'_*(\mathbb{R})$ such that for every $\varphi \in \mathcal{D}_*(\mathbb{R})$ the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$. For φ, ψ in $\mathcal{D}_*(\mathbb{R})$, we have

$$\begin{aligned} < T_{T*\varphi}, \psi > &= < T, \varphi * \psi > \\ &= < T, \psi * \varphi > \\ &= < T_{T*\psi}, \varphi > \end{aligned}$$

From Holder inequality and using the hypothesis we obtain

$$| < T_{T*\varphi}, \psi > | \leq \|T * \psi\|_{p',\nu} \|\varphi\|_{p,\nu},$$

from which, we deduce that the set

$$B' = \{T_{T*\varphi},\ \varphi \in \mathcal{D}_*(\mathbb{R});\ \|\varphi\|_{p,\nu} \leq 1\}$$

is bounded in $\mathcal{D}'_*(\mathbb{R})$.

Now, using the theorem 4.2, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R})$; $\|\varphi\|_{p,\nu} \leq 1$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}^m_a(\mathbb{R})$ and the family of these extensions is equicontinuous, which means that there exists $c > 0$ such that
$\forall\, \varphi \in \mathcal{D}_*(\mathbb{R})$; $\|\varphi\|_{p,\nu} \leq 1$, $\forall\, \psi \in \mathcal{W}^m_a(\mathbb{R})$

$$| < T_{T*\varphi}, \psi > | \leq c\|\psi\|_{\mathcal{W}^m_a(\mathbb{R})}.$$

This involves that
$\forall\, \varphi \in \mathcal{D}_*(\mathbb{R})$; $\forall\, \psi \in \mathcal{W}^m_a(\mathbb{R})$

$$| < T_{T*\varphi}, \psi > | \leq c\|\psi\|_{\mathcal{W}^m_a(\mathbb{R})} \|\varphi\|_{p,\nu}. \tag{4.16}$$

On the other hand, we have
$\forall\, \varphi \in \mathcal{D}_*(\mathbb{R})$, $\forall\, \psi \in \mathcal{W}^m_a(\mathbb{R})$

$$< T_{T*\varphi}, \psi > = < T * T_\psi, \varphi > \tag{4.17}$$

where, for all $\varphi \in \mathcal{D}_*(\mathbb{R})$

$$\begin{aligned} < T * T_\psi, \varphi > &= < T, T_\psi * \varphi > \\ &= < T, \psi * \varphi > . \end{aligned}$$

The relations (4.16) and (4.17) lead to
$\forall\, \varphi \in \mathcal{D}_*(\mathbb{R})$,

$$| < T * T_\psi, \varphi > | \leq c\|\psi\|_{\mathcal{W}^m_a(\mathbb{R})} \|\varphi\|_{p,\nu}$$

this last inequality shows that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem there exists
$g \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_g \tag{4.18}$$

Furthermore, from the proposition 4.3 , there exist $s \in \mathbb{N}$, $\psi_s \in \mathcal{W}_a^m(\mathbb{R})$ and $\varphi_s \in \mathcal{D}_{\star,a}(\mathbb{R})$ satisfying

$$\delta = (I - \Delta)^s T_{\psi_s} + T_{\varphi_s}$$

then

$$\begin{aligned} T &= (I - \Delta)^s (T * T_{\psi_s}) + T * T_{\varphi_s} \\ &= (I - \Delta)^s (T * T_{\psi_s}) + T_{T*\varphi_s}. \end{aligned} \tag{4.19}$$

We complete the proof by using the hypothesis, the relations (4.18), (4.19) and the theorem 4.1.

In the following, we will give a characterization of the bounded sets in $\mathcal{M}'_{\rho,p}$.

Theorem 4.4 *Let $p \in [1, \infty[$ and B' a subset of $\mathcal{M}'_{\rho,p}$. The following assertions are equivalent.*

i) B' is weakly bounded in $\mathcal{M}'_{\rho,p}$.

ii) There exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$ we can find $f_{0,T}, ..., f_{m,T} \subset L^{p'}(d\nu)$ satisfying

$$T = \sum_{k=0}^{m} \Delta^k T_{f_k} \ \textit{with} \ \max_{0 \le k \le m} \|f_k\|_{p',\nu} \le c$$

iii) For every $\varphi \in \mathcal{D}_*(\mathbb{R})$, the set $\{T * \varphi\}_{T \in B'}$ is bounded in $L^{p'}(d\nu)$.

Proof. 1) Suppose that B' is weakly* bounded in $\mathcal{M}'_{\rho,p}$, then from [12] B' is equicontinuous. There exist $c > 0$ and $m \in \mathbb{N}$ such that

$$\forall\, T \in B', \forall\, f \in \mathcal{M}_{\rho,p}, \ | < T, f > | \le c\gamma_{m,p}(f). \tag{4.20}$$

As in the proof of the theorem 4.2, we consider the mappings

$$\begin{aligned} \mathcal{A} : \mathcal{M}_{\rho,p} &\to (L^p(d\nu))^{m+1} \\ f &\mapsto (f, g_1, ..., g_m), \end{aligned}$$

with

$$\Delta^k T_f = T_{g_k};\ 0 \le k \le m$$

and for all $T \in B'$,

$$\begin{aligned} L_T : \mathcal{A}(\mathcal{M}_{\rho,p}) &\to \mathbb{C} \\ < L_T, \mathcal{A}(f) > &= < T, f > . \end{aligned}$$

Then, the relation (4.20) implies that
$\forall\, \varphi \in \mathcal{M}_{\rho,p}$,

$$|L_T(\mathcal{A}\varphi)| \le c\|\mathcal{A}\varphi\|_{(L^p(d\nu))^{m+1}}.$$

Using Hahn Banach's theorem and Riez's theorem, we deduce that L_T can be continuously extended on $(L^p(d\nu))^{m+1}$, denoted again by L_T, and that there exists $(f_{T,k})_{0\leq k\leq m} \subset L^{p'}(d\nu)$ verifying
$\forall\, \psi = (\psi_0, ..., \psi_m) \in (L^p(d\nu))^{m+1}$,

$$< L_T, \psi >= \sum_{k=0}^{m} \int_0^\infty f_{T,k}(x)\psi_k(x)d\nu(x)$$

with

$$\|L_T\|_{(L^p(d\nu))^{m+1}} = \max_{0\leq k\leq m} \|f_{T,k}\|_{p',\nu} \leq c.$$

In particular, if $\psi = \mathcal{A}(f)$, $f \in \mathcal{M}_{\rho,p}$,

$$< L_T, \mathcal{A}(f) >=< T, f >= \sum_{k=0}^{m} < \Delta^k T_{f_{T,k}}, f >$$

this proves that i) implies ii).

2) Suppose that there exist $c > 0$ and $m \in \mathbb{N}$ such that for every $T \in B'$ we can find $\{f_{0,T}, ..., f_{m,T}\} \subset L^{p'}(d\nu)$ satisfying

$$\forall\, T \in B',\ T = \sum_{k=0}^{m} \Delta^k T_{f_{T,k}} \text{ and } \max_{0\leq k\leq m} \|f_{T,k}\|_{p',\nu} \leq c$$

then
$\forall\, f \in \mathcal{M}_{\rho,p}$, $\forall\, T \in B'$

$$< T, f >= \sum_{k=0}^{m} \int_0^\infty f_{T,k}(x)g_k(x)d\nu(x)$$

consequently,
$\forall\, T \in B'$, $\forall\, f \in \mathcal{M}_{\rho,p}$,

$$| < T, f > | \leq (m + 1)c\gamma_{m,p}(f)$$

which means that the set B' is weakly* bounded in $\mathcal{M}'_{\rho,p}$ and proves that ii) implies i).

3) Suppose that ii) holds. Let $\varphi \in \mathcal{D}_*(\mathbb{R})$, then from theorem 4.3 we know that for all $T \in B'$ the function $T * \varphi$ belongs to the space $L^{p'}(d\nu)$. But

$$T * \varphi = \sum_{k=0}^{m} T_{f_k} * \Delta^k \varphi$$

consequently,
$\forall\, T \in B'$,

$$\|T * \varphi\|_{p',\nu} \leq (m + 1)c\gamma_{m,p}(\varphi).$$

Which shows that the set $\{T * \varphi\}_{T\in B'}$ is bounded in $L^{p'}(d\nu)$ and therefore ii) involves iii).

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4) Suppose that iii) holds. Let $T \in B'$, for all φ, $\psi \in \mathcal{D}_*(\mathbb{R})$, we have

$$\begin{aligned} |< T_{T*\varphi}, \psi >| &= |< T_{T*\psi}, \varphi >| \\ &\leq \|T * \psi\|_{p',\nu} \|\varphi\|_{p,\nu} \end{aligned}$$

from which, we deduce that the set

$$B' = \{T_{T*\varphi},\, T \in B',\, \varphi \in \mathcal{D}_*(\mathbb{R});\, \|\varphi\|_{p,\nu} \leq 1\}$$

is bounded in $\mathcal{D}'_*(\mathbb{R})$.

Now, using the theorem 4.2, it follows that for all $a > 0$ there exists $m \in \mathbb{N}$ such that for all $\varphi \in \mathcal{D}_*(\mathbb{R})$; $\|\varphi\|_{p,\nu} \leq 1$, and $T \in B'$, the mapping $T_{T*\varphi}$ can be continuously extended on the space $\mathcal{W}_a^m(\mathbb{R})$ and the family of these extensions is equicontinuous, which means that there exists $c > 0$ satisfying
$\forall T \in B'$, $\forall \varphi \in \mathcal{D}_*(\mathbb{R})$; $\forall \psi \in \mathcal{W}_a^m(\mathbb{R})$

$$|< T_{T*\varphi}, \psi >| \leq c\|\psi\|_{\mathcal{W}_a^m(\mathbb{R})} \|\varphi\|_{p,\nu} \tag{4.21}$$

On the other hand, for every $T \in B'$, we have
$\forall \varphi \in \mathcal{D}_*(\mathbb{R})$, $\forall \psi \in \mathcal{W}_a^m(\mathbb{R})$

$$< T_{T*\varphi}, \psi > = < T * T_\psi, \varphi > \tag{4.22}$$

from the relations (4.21) and (4.22) we deduce that the functional $T * T_\psi$ can be continuously extended on the space $L^p(d\nu)$ and from Riez's theorem there exist $g_{T,\psi} \in L^{p'}(d\nu)$ such that

$$T * T_\psi = T_{g_{T,\psi}} \tag{4.23}$$

However, the relations (4.21) and (4.23) involve that
$\forall T \in B'$,

$$\|g_{T,\psi}\|_{p',\nu} \leq c\|\psi\|_{W_a^m(\mathbb{R})} \tag{4.24}$$

By using the proposition 4.3, it follows that there exist $s \in \mathbb{N}$,
$\psi_s \in \mathcal{W}_a^m(\mathbb{R})$ and $\varphi_s \in \mathcal{D}_{*,a}(\mathbb{R})$ verifying, for all $T \in B'$,

$$T = T * \delta = (I - \Delta)^s (T * T_{\psi_s}) + T_{T*\varphi_s}$$

and by the relation (4.23) we get

$$T = (I - \Delta)^s T_{g_{T,s}} + T_{T*\varphi_s}$$

thus, from the hypothesis, we obtain,

$$\forall T \in B',\ \|T * \varphi_s\|_{p',\nu} \leq c_s,$$

and using the relation (4.24), we have

$$\forall T \in B',\ \|g_{T,s}\|_{p',\nu} \leq c\|\varphi_s\|_{\mathcal{W}_a^m(\mathbb{R})}$$

this completes the proof.

5 Convolution product on the space $\mathcal{M}'_{\rho,p} \times \mathrm{M}_{\rho,r}$

In this section, we define and study the convolution product on the space $\mathcal{M}'_{\rho,p} \times \mathrm{M}_{\rho,r}$, $1 \leq r \leq p < \infty$, where $\mathrm{M}_{\rho,r}$ is the closure of the space $\mathcal{D}_*(\mathbb{R})$ in $\mathcal{M}_{\rho,r}$.

Proposition 5.1 *Let $p \in [1,\infty[$. For every $x \in [0,\infty[$, the operator $\mathcal{T}_x$ given by the definition 2.1 i), is a continuous mapping from $\mathcal{M}_{\rho,p}$ into itself.*

Proof. Let $f \in \mathcal{M}_{\rho,p}$ and $g_k \in L^p(d\nu)$ such that

$$T_{g_k} = \Delta^k T_f, \ k \in \mathbb{N}$$

then,
$\forall\, \varphi \in \mathcal{D}_*(\mathbb{R})$;

$$< \Delta^k T_{\mathcal{T}_x f}, \varphi > = < T_{\mathcal{T}_x g_k}, \varphi > .$$

Since the operator $\mathcal{T}_x$ is continuous from $L^p(d\nu)$ into itself, we deduce that for all $f \in \mathcal{M}_{\rho,p}$ and $x \in [0,\infty[$, the function $\mathcal{T}_x f$ belongs to the space $\mathcal{M}_{\rho,p}$. Moreover,

$$\begin{aligned}\gamma_{m,p}(\mathcal{T}_x f) &= \max_{0\leq k\leq m} \|\mathcal{T}_x g_k\|_{p,\nu} \\ &\leq \max_{0\leq k\leq m} \|g_k\|_{p,\nu} = \gamma_{m,p}(f)\end{aligned}$$

which shows that the operator $\mathcal{T}_x$ is continuous from $\mathcal{M}_{\rho,p}$ into itself.

Definition 5.1 *The convolution product of $T \in \mathcal{M}'_{\rho,p}$ and $f \in \mathcal{M}_{\rho,p}$ is defined by*
$\forall\, x \in [0,\infty[$

$$T * f(x) = < T, \mathcal{T}_x f > .$$

Let $T \in \mathcal{M}'_{\rho,p}$; $T = \sum_{k=0}^{m} \Delta^k T_{f_k}$ with $\{f_k\}_{0\leq k\leq m} \subset L^{p'}(d\nu)$ and $\phi \in \mathrm{M}_{\rho,r}, 1 \leq r \leq p$, then for all $k \in \mathbb{N}$ there exists $\phi_k \in L^r(d\nu)$ such that

$$T_{\phi_k} = \Delta^k T_\phi.$$

It follows that for $0 \leq k \leq m$ the function $f_k * \phi_k$ belongs to the space $L^q(d\nu)$ with, $\frac{1}{q} = \frac{1}{r} + \frac{1}{p'} - 1 = \frac{1}{r} - \frac{1}{p}$ and by using the density of $\mathcal{D}_*(\mathbb{R})$ in $\mathrm{M}_{\rho,r}$, we deduce that the expression $\sum_{k=0}^{m} f_k * \phi_k$ is independent of the sequence $\{f_k\}_{0\leq k\leq m}$. Then, we put

$$T * \phi = \sum_{k=0}^{m} f_k * \phi_k. \tag{5.1}$$

This allows us to say that

$$\mathcal{M}'_{\rho,p} * \mathrm{M}_{\rho,r} \subset L^q(d\nu). \tag{5.2}$$

Lemma 5.1 *Let* $1 \leq r \leq p < \infty$, $T \in \mathcal{M}'_{\rho,p}$ *and* $\phi \in \mathrm{M}_{\rho,r}$. *Then, for all* $k \in \mathbb{N}$

$$\Delta^k T_{T*\phi} = T_{T*\phi_k}$$

with

$$T_{\phi_k} = \Delta^k T_\phi.$$

Proof. If $\phi \in \mathcal{D}_*(\mathbb{R})$, then the function $T * \phi$ is infinitely differentiable and we have

$$\Delta^k(T_{T*\phi}) = T_{\Delta^k(T*\phi)} = T_{T*\Delta^k\phi}$$

therefore, the result follows from the density of $\mathcal{D}_*(\mathbb{R})$ in $\mathrm{M}_{\rho,r}$.

Proposition 5.2 *Let* $1 \leq r \leq p < \infty$ *and* $q \in [1, \infty]$ *such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}$$

then for every $T \in \mathcal{M}'_{\rho,p}$, *the mapping*

$$\phi \to T * \phi$$

is continuous from $\mathrm{M}_{\rho,r}$ *into* $\mathcal{M}_{\rho,q}$.

Proof. Let $T \in \mathcal{M}'_{\rho,p}$; $T = \sum_{k=0}^{m} \Delta^k T_{f_k}$ with $\{f_k\}_{0\leq k\leq m} \subset L^{p'}(d\nu)$ then for $\phi \in \mathrm{M}_{\rho,r}, 1 \leq r \leq p$,

$$T * \phi = \sum_{k=0}^{m} f_k * \phi_k$$

where $\phi_k \in L^r(d\nu)$ and

$$T_{\phi_k} = \Delta^k T_\phi.$$

From the lemma 5.1, we have
$\forall\, s \in \mathbb{N}, \forall\, \phi \in \mathrm{M}_{\rho,r}$

$$\Delta^s T_{T*\phi} = T_{T*\phi_s}$$

using the relation (5.2), we deduce that the function $T * \phi$ belongs to the space $\mathcal{M}_{\rho,q}$.

On the other hand, from the relation (5.1), we obtain

$$\gamma_{l,q}(T * \phi) = \max_{0\leq s\leq l} \|T * \phi_s\|_{q,\nu}$$

But

$$T * \phi_s = \sum_{k=0}^{m} f_k * \phi_{k+s}$$

consequently,

$$\begin{aligned}\|T * \phi_s\|_{q,\nu} &\leq \sum_{k=0}^{m} \|f_k\|_{p',\nu} \|\phi_{k+s}\|_{r,\nu} \\ &\leq \Big(\sum_{k=0}^{m} \|f_k\|_{p',\nu}\Big) \gamma_{m+l,r}(\phi).\end{aligned}$$

Hence,

$$\gamma_{l,q}(T * \phi) \leq \Big(\sum_{k=0}^{m} \|f_k\|_{p',\nu}\Big) \gamma_{m+l,r}(\phi)$$

which proves the result.

Definition 5.2 *Let* $1 \leq p, q, r < \infty$ *such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}.$$

The convolution product of $T \in \mathcal{M'}_{\rho,p}$ *and* $S \in \mathcal{M'}_{\rho,q}$ *is defined by*
$\forall\, \phi \in \mathrm{A}_{\rho,r}$

$$< S * T, \phi >=< S, T * \phi > .$$

From this definition and the proposition 5.2 we deduce the following result

Proposition 5.3 *Let* $1 \leq p, q, r < \infty$ *such that*

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{p}.$$

Then, for all $T \in \mathcal{M'}_{\rho,p}$ *and* $S \in \mathcal{M'}_{\rho,q}$*, the functional* $S * T$ *is continuous on* $\mathrm{M}_{\rho,r}$.

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Multivariate complex general singular integral operators simultaneous approximation

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present complex multivariate simultaneous approximation for general smooth singular integral operators converging with rates to the unit operator. The associated and presented inequalities are in $\|\cdot\|_p$, $1 \leq p \leq \infty$ norm and they involve multivariate related moduli of smoothness. At the end we list as our theory's applicators the special cases of multivariate complex Picard, Gauss-Weierstrass, Poisson-Cauchy and Trigonometric singular integral operators.

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Key Words and Phrases: Complex multivariate Approximation, multivariate singular integral, simultaneous approximation, multivariate modulus of smoothness, rate of convergence.

1 Introduction

Here we are motivated by [1]-[3] and expand these works to complex valued functions. We present simultaneous approximation in $\|\cdot\|_p$, $1 \leq p \leq \infty$, of multivariate general smooth singular integral operators to the unit operator with rates. At the end we list specific operators where our theory can be applied. From our approximation results one can derive interesting convergence properties of these general operators. Our expansion to complex case is based on basic properties of complex numbers and complex valued functions.

2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, ..., r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 0, \end{cases} \tag{1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the real multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, ..., x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, d\mu_{\xi_n} (s), \tag{5}$$

where $s := (s_1, ..., s_N)$, $x := (x_1, ..., x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Above operators $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [1].

Definition 1 *Let $f \in C(\mathbb{R}^N)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by*

$$\omega_m (f; h)_p := \sup_{\|t\|_2 \leq h} \|\Delta_t^m f (x)\|_{p,x}, \tag{6}$$

$h > 0$, where

$$\Delta_t^m f (x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f (x + jt). \tag{7}$$

Denote

$$\omega_m (f; h)_\infty = \omega_m (f, h). \tag{8}$$

Above, $x, t \in \mathbb{R}^N$.

We make

Remark 2 *We consider here complex valued Borel measurable functions* $f : \mathbb{R}^N \to \mathbb{C}$ *such that* $f = f_1 + if_2$, $i = \sqrt{-1}$, *where* $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ *are implied to be real valued Borel measurable functions.*

We define the complex singular operators

$$\theta_{r,n}^{[m]}(f;x) := \theta_{r,n}^{[m]}(f_1;x) + i\theta_{r,n}^{[m]}(f_2;x), \quad x \in \mathbb{R}^N. \tag{9}$$

We assume that $\theta_{r,n}^{[m]}(f_j;x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$, $j = 1,2$.

One notices easily that

$$\left|\theta_{r,n}^{[m]}(f;x) - f(x)\right| \leq \tag{10}$$

$$\left|\theta_{r,n}^{[m]}(f_1;x) - f_1(x)\right| + \left|\theta_{r,n}^{[m]}(f_2;x) - f_2(x)\right|$$

also

$$\left\|\theta_{r,n}^{[m]}(f;x) - f(x)\right\|_{\infty,x} \leq \tag{11}$$

$$\left\|\theta_{r,n}^{[m]}(f_1;x) - f_1(x)\right\|_{\infty,x} + \left\|\theta_{r,n}^{[m]}(f_2;x) - f_2(x)\right\|_{\infty,x}$$

and

$$\left\|\theta_{r,n}^{[m]}(f) - f\right\|_p \leq \tag{12}$$

$$\left\|\theta_{r,n}^{[m]}(f_1) - f_1\right\|_p + \left\|\theta_{r,n}^{[m]}(f_2) - f_2\right\|_p, \quad p \geq 1.$$

Furthermore it holds

$$f_\alpha(x) = f_{1,\alpha}(x) + if_{2,\alpha}(x), \tag{13}$$

where α *denotes a partial derivative of any order and arrangement.*

Here based on Theorem 9 of [1] we obtain

Theorem 3 *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $N \geq 1$, *such that* $f = f_1 + if_2$, $j = 1,2$. *Here* $m \in \mathbb{N}$, $f_j \in C^m(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. *Assume* $\|f_{j,\alpha}\|_\infty < \infty$, *for all* $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N : |\alpha| = \sum_{k=1}^N \alpha_k = m$; $j = 1,2$. *Let* μ_{ξ_n} *be a Borel probability measure on* $\mathbb{R}^N$, *for* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence. Assume that for all* $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^N \alpha_k = m$ *we have that*

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^N |s_k|^{\alpha_k}\right)\left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty. \tag{14}$$

For $\widetilde{j} = 1, ..., m$, *and* $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = \widetilde{j}$, *call*

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s_1, ..., s_N). \tag{15}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,...,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{\infty,x} \leq$$

$$\sum_{\left(\substack{\alpha_1,...,\alpha_N \geq 0 \\ |\alpha| = m}\right)} \frac{(\omega_r(f_{1,\alpha}, \xi_n) + \omega_r(f_{2,\alpha}, \xi_n))}{\left(\prod_{k=1}^{N} \alpha_k!\right)}. \tag{16}$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right), \quad x \in \mathbb{R}^N.$$

The $m = 0$ case follows

Corollary 4 *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $N \geq 1$. *Here* $j = 1, 2$. *Let* $f_j \in C_B(\mathbb{R}^N)$ *(continuous and bounded functions). Then*

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \cdot \tag{17}$$

$$(\omega_r(f_1, \xi_n) + \omega_r(f_2, \xi_n)),$$

by assuming

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{18}$$

Proof. By Theorem 11 of [1]. ∎

Theorem 5 *([3]) Let* $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. *Here* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ *a bounded sequence. Let* $\beta := (\beta_1, ..., \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, ..., N$; $|\beta| := \sum_{i=1}^{N} \beta_i = l$. *Here* $f(x + sj)$, $x, s \in \mathbb{R}^N$, *is* μ_{ξ_n}*-integrable wrt* s, *for* $j = 1, ..., r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, ..., h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j} \geq 0$ $(j = 1, ..., r)$ *on* $\mathbb{R}^N$ *such that*

$$\left| \frac{\partial^{i_1} f(x + sj)}{\partial x_1^{i_1}} \right| \leq h_{i_1,j}(s), \quad i_1 = 1, ..., \beta_1, \tag{19}$$

$$\left| \frac{\partial^{\beta_1+i_2} f(x+sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1, ..., \beta_2,$$

$$\vdots$$

$$\left| \frac{\partial^{\beta_1+\beta_2+...+\beta_{N-1}+i_N} f(x+sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} ... \partial x_2^{\beta_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,\beta_2,...,\beta_{N-1},i_N,j}(s), \quad i_N = 1, ..., \beta_N,$$

$\forall\ x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[\widetilde{m}]}(f;x) \right)_\beta = \theta_{r,n}^{[\widetilde{m}]}(f_\beta;x). \tag{20}$$

We give

Theorem 6 *Let $f : \mathbb{R}^N \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$. For f_j the assumptions of Theorem 5 are valid. Call $\gamma = 0, \beta$. Assume $\|f_{(\gamma+\alpha)}\|_\infty < \infty$ and*

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty, \tag{21}$$

where μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is bounded sequence; for all $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N : |\alpha| = \sum_{k=1}^{N} \alpha_k = m$.

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s).$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f;\cdot) \right)_\gamma - f_\gamma(\cdot) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,...,\alpha_N \geq 0: \\ |\alpha| = \widetilde{j}}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}(\cdot)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_\infty \leq$$

$$\sum_{\left(\substack{\alpha_1,...,\alpha_N \geq 0 \\ |\alpha| = m} \right)} \frac{(\omega_r(f_{1,\gamma+\alpha}, \xi_n) + \omega_r(f_{2,\gamma+\alpha}, \xi_n))}{\left(\prod_{k=1}^{N} \alpha_k! \right)} \cdot \tag{22}$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right).$$

Proof. By Theorem 10 of [3]. ■

Also we have

Theorem 7 *Let $f : \mathbb{R}^N \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C_B^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 5 are valid for f_j. Call $\gamma = 0, \beta$. Assume*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f\right)_\gamma - f_\gamma \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) \right) \cdot \tag{23}$$

$$\left(\omega_r(f_{1,\gamma}, \xi_n) + \omega_r(f_{2,\gamma}, \xi_n)\right).$$

Proof. By Theorem 11 of [3]. ■

By Theorem 4 of [2] we get

Theorem 8 *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_{j,\alpha} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Assume for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^N \alpha_k = m$, we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^N |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{24}$$

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^N \alpha_k = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^N s_k^{\alpha_k} d\mu_{\xi_n}(s). \tag{25}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\widetilde{j}=1}^m \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod_{k=1}^N \alpha_k!} \right) \right\|_{p,x} \leq$$

$$\left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{k=1}^N \alpha_k! \right)} \right) \cdot$$

6

$$\left[\int_{\mathbb{R}^N}\left[\left(\prod_{k=1}^{N}|s_k|^{\alpha_k}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right]^p d\mu_{\xi_n}(s)\right]^{\frac{1}{p}}\cdot$$

$$\left(\omega_r(f_{1,\alpha},\xi_n)_p+\omega_r(f_{2,\alpha},\xi_n)_p\right). \tag{26}$$

We further get

Theorem 9 *Let $f:\mathbb{R}^N\to\mathbb{C}:f=f_1+if_2$, $j=1,2$. Let $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_p\left(\mathbb{R}^N\right)\right)$, $N\geq 1$, $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}>0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)<\infty. \tag{27}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_p\leq\left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}}\cdot \tag{28}$$

$$\left(\omega_r(f_1,\xi_n)_p+\omega_r(f_2,\xi_n)_p\right).$$

Proof. By Theorem 6 of [2]. ■

Based on Theorem 8 of [2] we get

Theorem 10 *Let $f:\mathbb{R}^N\to\mathbb{C}$, $f=f_1+if_2$, $j=1,2$. Let $f_j\in\left(C\left(\mathbb{R}^N\right)\cap L_1\left(\mathbb{R}^N\right)\right)$, $N\geq 1$. Assume μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}}>0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{r} d\mu_{\xi_n}(s)<\infty. \tag{29}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_1\leq\left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{r} d\mu_{\xi_n}(s)\right)\cdot \tag{30}$$

$$\left(\omega_r(f_1,\xi_n)_1+\omega_r(f_2,\xi_n)_1\right).$$

Based on Theorem 10 of [2] we get

Theorem 11 *Let $f:\mathbb{R}^N\to\mathbb{C}$, $f=f_1+if_2$, $j=1,2$. Let $f_j\in C^m\left(\mathbb{R}^N\right)$, $m,N\in\mathbb{N}$, with $f_{j,\alpha}\in L_1\left(\mathbb{R}^N\right)$, $|\alpha|=m$, $x\in\mathbb{R}^N$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n>0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Assume for all $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_k\in\mathbb{Z}^+$, $k=1,...,N$, $|\alpha|:=\sum_{k=1}^{N}\alpha_k=m$ that we have*

$$\int_{\mathbb{R}^N}\left(\left(\prod_{k=1}^{N}|s_k|^{\alpha_k}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right) d\mu_{\xi_n}(s)<\infty. \tag{31}$$

For $\widetilde{j} = 1, ..., m,$ *and* $\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_k \in \mathbb{Z}^+,\ k = 1, ..., N,\ |\alpha| := \sum_{k=1}^{N} \alpha_k = \widetilde{j},$ *call*

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s). \tag{32}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_\alpha(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{1,x} \leq$$

$$\sum_{|\alpha|=m} \left(\frac{1}{\prod_{k=1}^{N} \alpha_k!} \right) \left(\omega_r(f_{1,\alpha}, \xi_n)_1 + \omega_r(f_{2,\alpha}, \xi_n)_1 \right) \cdot \tag{33}$$

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s).$$

Based on Theorem 12 of [3] we get

Theorem 12 *Let* $f : \mathbb{R}^N \to \mathbb{C},\ f = f_1 + if_2;\ j = 1, 2,$ *with* $f_j \in C^{m+l}(\mathbb{R}^N),$ $m, l, N \in \mathbb{N}$. *The assumptions of Theorem 5 are valid for* f_j. *Call* $\gamma = 0, \beta$. *Let* $f_{j,(\gamma+\alpha)} \in L_p(\mathbb{R}^N),\ |\alpha| = m,\ x \in \mathbb{R}^N,\ p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *is a bounded sequence. Assume for all* $\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_k \in \mathbb{Z}^+,\ k = 1, ..., N,\ |\alpha| := \sum_{k=1}^{N} \alpha_k = m$ *we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{34}$$

For $\widetilde{j} = 1, ..., m,$ *and* $\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_k \in \mathbb{Z}^+,\ k = 1, ..., N,\ |\alpha| := \sum_{k=1}^{N} \alpha_k = \widetilde{j},$ *call*

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s).$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f;x)\right)_{\gamma} - f_{\gamma}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_{\gamma+\alpha}(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{p,x} \leq \tag{35}$$

$$\left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}}\right) \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{k=1}^{N} \alpha_k!\right)} \right) \cdot$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{k=1}^{N} |s_k|^{\alpha_k}\right) \left(1+\frac{\|s\|_2}{\xi_n}\right)^r \right]^p d\mu_{\xi_n}(s)\right]^{\frac{1}{p}} \cdot$$

$$\left(\omega_r\left(f_{1,\gamma+\alpha},\xi_n\right)_p + \omega_r\left(f_{2,\gamma+\alpha},\xi_n\right)_p\right).$$

Based on Theorem 13 of [3] we get

Theorem 13 *Let $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. Let $f_j \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 5 are valid. Call $\gamma = 0, \beta$. Let $f_{j,\gamma} \in L_p\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f\right)_{\gamma} - f_{\gamma} \right\|_p \leq \left(\int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \cdot \tag{36}$$

$$\left(\omega_r\left(f_{1,\gamma},\xi_n\right)_p + \omega_r\left(f_{2,\gamma},\xi_n\right)_p\right).$$

By Theorem 14 of [3] we get

Theorem 14 *Let $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. Let $f_j \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 5 are valid for f_j. Call $\gamma = 0, \beta$. Let $f_{j,\gamma} \in L_1\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$. Assume μ_{ξ_n} probability Borel measures on $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^{r} d\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f \right)_{\gamma} - f_{\gamma} \right\|_{1} \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \cdot \tag{37}$$
$$\left(\omega_r (f_{1,\gamma}, \xi_n)_1 + \omega_r (f_{2,\gamma}, \xi_n)_1 \right).$$

Finally we have

Theorem 15 *Let $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + i f_2$, $j = 1, 2$, with $f_j \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. The assumptions of Theorem 5 are valid for f_j. Call $\gamma = 0, \beta$. Let $f_{j,(\gamma+\alpha)} \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is bounded. Assume for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = m$, we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \tag{38}$$

For $\widetilde{j} = 1, ..., m$, and $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, ..., N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = \widetilde{j}$, call

$$c_{\alpha,n,\widetilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s). \tag{39}$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f; x) \right)_{\gamma} - f_{\gamma}(x) - \sum_{\widetilde{j}=1}^{m} \delta_{\widetilde{j},r}^{[m]} \left(\sum_{|\alpha|=\widetilde{j}} \frac{c_{\alpha,n,\widetilde{j}} f_{(\gamma+\alpha)}(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{1,x} \leq \tag{40}$$

$$\sum_{|\alpha|=m} \left(\frac{1}{\prod_{k=1}^{N} \alpha_k!} \right) \left(\omega_r (f_{1,\gamma+\alpha}, \xi_n)_1 + \omega_r (f_{2,\gamma+\alpha}, \xi_n)_1 \right) \cdot$$
$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s).$$

Proof. By Theorem 15 of [3]. ■

3 Applications

Let all entities as in section 2. We define the following specific operators for $f : \mathbb{R}^N \to \mathbb{C}$.

i) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f; x_1, ..., x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{41}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, e^{-\frac{\left(\sum_{i=1}^{N} |s_i|\right)}{\xi_n}} ds_1 ... ds_N.$$

ii) The general multivariate Gauss-Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f; x_1, ..., x_N) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{42}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, ..., x_N + s_N j) \, e^{-\frac{\left(\sum_{i=1}^{N} s_i^2\right)}{\xi_n}} ds_1 ... ds_N.$$

iii) The general multivariate Poisson-Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f; x_1, ..., x_N) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{43}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, ..., x_N + s_N j) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 ... ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta) \, \alpha \xi_n^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}. \tag{44}$$

iv) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f; x_1, ..., x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{45}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, ..., x_N + s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 ... ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta}\pi\left(-1\right)^{\beta}\beta\sum_{k=1}^{\beta}\left(-1\right)^{k}\frac{k^{2\beta-1}}{\left(\beta-k\right)!\left(\beta+k\right)!}. \tag{46}$$

One can apply the results of this article to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results. We intend to do that in a future article.

Conclusion: Our approximation results here imply important convergence properties of operators $\theta_{r,n}^{[m]}$ to the unit operator.

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Stability and superstability of $*-$bihomomorphisms on C^*-ternary algebras

M. Eshaghi Gordji

Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Iran
e-mail: madjid.eshaghi@gmail.com

A. Fazeli

Islamic AZAD University (IAU), Science and Research Branch, Tehran, Iran.
e-mail: asad.fazeli@gmail.com

Abstract. In this paper, we establish the stability and superstability of $*-$bihomomorphisms on C^*-ternary algebras associated with the following functional equation

$$f(x-y,t)+f(x,t-s)=2f(x,t)-f(y,t)-f(x,s).$$

1. Introduction

Let f be a mapping such that in general case its domain is a semigroup and its range is a topological vector space. Suppose that we are given a functional equation $E(f)=0$ so that the boundedness of f gives the boundedness of $E(f)$. More precisely, in the category of normed spaces, the functional equation $E(f)=0$ is stable if for every $\varepsilon>0$ there exists a $\delta>0$ that whenever $\|E(g)\|<\delta$ then there exists a function f such that $E(f)=0$ *and* $\|f-g\|<\varepsilon$. The equation $E(f)=0$ is superstable whenever every approximate solution of $E(f)\geq 0$ *or* $E(f)\leq\ 0$
is a true solution of $E(f)=0$. In other words, the boundedness of $E(f)$ shows that either f is bounded or $E(f)=0$. In question, stability was originated from Ulam conjecture by the following explanation.
Given a metric group (G,d), a number $\varepsilon>0$ and a mapping $f:G\to G$ which satisfies the inequality
$d(f(x.y),f(x).f(y))<\varepsilon$, for all x,y in G, does there exist an automorphism g of G and a constant $k>0$, depending only on G, such that $d(g(x),f(x))\leq k\varepsilon$ for all x in G?
Hyers proved the Ulam conjecture in the category of Banach spaces under the following theorem. Suppose that E_1,E_2 are two Banach spaces and $f:E_1\to E_2$ is a function that for all $x,y\in E_1$ satisfy the inequality
$\|f(x+y)-f(x)-f(y)\|<\varepsilon$ then there exists an additive function $g:E_1\to E_2$ that for all $x\in E_1$, $\|g(x)-f(x)\|\leq\varepsilon$.
So far we have the stability of Hyers–Ulam. Th. M. Rassias extended the Ulam's theorem by setting $\varepsilon(\|x\|^p+\|y\|^p)$ instead of ε where $0\leq p<1$, that, at present, is called Hyers–Ulam–Rassias stability. Thence the stability theory was extended day by day and several applications were obtained in a variety of branches of physics and mathematical sciences among others in Supersymmetric theories and Yang-Baxter equation and Cubic analog of Laplace and d'Alembert equations. The primitive work of Ulam has come in [69] and partial solution of Hyers is shown in [49]. The generalization of Hyers theorem by Rssias appears in [66]. The expression Hyers-Ulam-Rassias stability was propounded in [66]. For the history and various aspects of stability theory we refer the reader to [2, 34], [5]–[47],

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[52, 53] and [62]–[65].

Ternary algebraic operations have propounded originally in 19th century in Cayley [4] and J.J.Silvester's paper [68]. The application of ternary algebra in supersymmetry is presented in [51] and in Yang-Baxter equation in [55]. Cubic analogue of Laplace and d'alembert equations have been considered for first order by Himbert in [48],[50]. The previous definition of C^*-ternary algebras has been propounded by H.Zettle in [70]. In relation to homomorphisms and isomorphisms between various spaces we refer readers to [56]–[61], [67].

2. Preliminaries

Assume A is a linear space over a complex field equipped with a mapping $[\ ]: A^3 = A \times A \times A \to A$ with $(x, y, z) \to [x, y, z]$ that is linear in variables x, y, z and satisfies the associative identity, i.e. $[x, y, [z, u, v]] = [x, [y, z, u], v] = [[x, y, z], u, v]$ for all $x, y, z, u, v \in A$. The pair $(A, [\ ])$ is called a ternary algebra. The ternary algebra $(A, [\ ])$ is called unital if it has an identity element, i.e. an element $e \in A$ such that $[x, e, e] = [e, e, x] = x$ for every $x \in A$. A $* - ternary$ algebra is a ternary algebra together with a mapping $* : A \to A$ which satisfies $(x^*)^* = x$, $(\lambda x)^* = \bar{\lambda} x^*$, $(x + y)^* = x^* + y^*$, $[x, y, z]^* = [z^*, y^*, x^*]$ for all $x, y, z \in A$ and all $\lambda \in \mathbb{C}$. In the case that A is unital and e is its unit, we assume that $e^* = e$.

A is normed ternary algebra if A is a ternary algebra and there exists a norm $\|.\|$ on A which satisfies $\|[x, y, z]\| \leq \|x\|\ \|y\|\ \|z\|$ for all $x, y, z \in A$. Whenever the ternary algebra A is unital with unit element e, we repute $\|e\| = 1$. A normed ternary algebra A is called a Banach ternary algebra, if $(A, \|\ \|)$ is a Banach space. A C^*-ternary algebra is a Banach $* - ternary\ algebra$ if $\|[x, x^*, x]\| = \|x\|^3$ for all $x \in A$.
We suggest [54] for definition of C^*-ternary algebra.

Theorem 2.1. *Assume that A and B are two C^*-ternary algebras, then the cartesian product $A \times B$ is a C^*-ternary algebra.*

Proof. We define in $A \times B$ sum and scaler product and ternary product, pointwise ie
$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ and $\lambda(a, b) = (\lambda a, \lambda b)$ and $[(a_1, b_1), (a_2, b_2), (a_3, b_3)] = ([a_1, a_2, a_3], [b_1, b_2, b_3])$ for all $a, a_1, a_2, a_3 \in A$ and $b, b_1, b_2, b_3 \in B$ and $\lambda \in \mathbb{C}$. Also we define $(a, b)^* = (a^*, b^*)$ and $||(a, b)|| = max(||a||, ||b||)$ for all $a \in A$ and $b \in B$. The only subject that is not clear is C^*-ternary algebra identity,
$||[(a, b), (a, b)^*, (a, b)]|| = ||([a, a^*, a], [b, b^*, b])|| = \max(||a||^3, ||b||^3) = \max(||a||, ||b||)^3 = ||(a, b)||^3$ □

Now we proceed to the resumption of definitions.
Let X, Y, Z be linear spaces. A mapping $f : X \times Y \to Z$ is said to biadditive if for each fixed x in X, the map $y \mapsto f(x, y)$ is additive and for each fixed y in Y the map $x \mapsto f(x, y)$ is additive. We say that $f : X \times Y \to Z$ is bilinear if f is biadditive and $f(\lambda x, \mu y) = \lambda\mu f(x, y)$ for all $\lambda, \mu \in \mathbb{C}$ and $x \in X, y \in Y$.

Definition 2.2. [1] Suppose A and B are ternary algebras. A bilinear mapping $f : A \times A \to B$ is called ternary algebra left[right] bihomomorphism if it satisfies

$$f([x, y, z], t) = [f(x, t), f(y, t), f(z, t)] \qquad [f(x, [y, z, t]) = [f(x, y), f(x, z), f(x, t)]]$$

for all $x, y, z, t \in A$. f is called bihomomorphism if it is right and left bihomomorphism. In the case the function f is bijective and bihomomorphism, it will be called ternary algebra biisomorphism.

Definition 2.3. Let A and B be C^*-ternary algebras and $f : A \times A \to B$ be a bihomomorphism. f is called $*-$bihomomorphism if $f(x^*) = f(x)^*$ for every $x \in A \times A$.

3. **Biadditivity**

We need the following theorem to prove the main results of the present paper.

Theorem 3.1. *Let X and Y and Z be linear spaces and $f : X \times Y \to Z$ be a mapping. Then f is biadditive if and only if*

$$f(x - y, t) + f(x, t - s) = 2f(x, t) - f(y, t) - f(x, s) \tag{3.1}$$

for all $x, y \in X$ and $t, s \in Y$.

Proof. If f is a biadditive mapping, it is obvious that it satisfies (3.1). Conversely suppose that f satisfies (3.1). In (3.1) we set $x = y = s = t = 0$, we have $f(0, 0) = 0$. Letting $x = y = s = 0$ in (3.1), we obtain $f(0, t) = 0$. Putting $y = s = t = 0$ we get $f(x, 0) = 0$. Setting $s = 0$ in (3.1) we arrive at

$$f(x - y, t) = f(x, t) - f(y, t). \tag{3.2}$$

Letting $x = 0$ in (3.2), shows that $f(-y, t) = -f(y, t)$. Thus (3.2) gets out in the following form

$$f(x - y, t) = f(x, t) + f(-y, t). \tag{3.3}$$

By changing y to $-y$, (3.3) shows that f is additive with respect to first variable. Putting $y = 0$ in (3.1), we obtain

$$f(x, t - s) = f(x, t) - f(x, s). \tag{3.4}$$

Setting $t = 0$ in (3.4), we arrive at $f(x, -s) = -f(x, s)$. Interchanging s by $-s$ in (3.4) and the last equality we see that f is additive with respect to the second variable. So f is biadditive . $\square$

Theorem 3.2. *Let X and Y and Z be linear spaces and $f : X \times Y \to Z$ be a biadditive mapping. Then f is bilinear if and only if*

$$f(\lambda x, \mu y) = \lambda\mu f(x, y) \tag{3.5}$$

for each x in X and y in Y and $\lambda, \mu \in T^1_{\frac{1}{4}} = \{e^{i\theta}; 0 \le \theta \le \frac{\pi}{2}\}$.

Proof. At the beginning suppose that (3.5) is satisfied. Since f is biadditive, $f(rx, sy) = rsf(x, y)$ for all $r, s \in \mathbb{Q}$ and $x \in X$ and $y \in Y$. Now we consider some equalities:

If $\frac{\pi}{2} \le \theta \le \pi$ then $0 \le \theta - \frac{\pi}{2} \le \frac{\pi}{2}$ and $e^{i\theta} = e^{i(\theta - \frac{\pi}{2})} . e^{i\frac{\pi}{2}} = ie^{i(\theta - \frac{\pi}{2})}$.

If $\pi \le \theta \le \frac{3\pi}{2}$ then $0 \le \theta - \pi \le \frac{\pi}{2}$ and $e^{i\theta} = e^{i(\theta - \pi)} . e^{i\pi} = (-1)e^{i(\theta - \pi)}$.

If $\frac{-\pi}{2} \le \theta \le 0$ then $0 \le \theta + \frac{\pi}{2} \le \frac{\pi}{2}$ and $e^{i\theta} = e^{i(\theta + \frac{\pi}{2})} . e^{\frac{-i\pi}{2}} = (-i)e^{i(\theta + \frac{\pi}{2})}$.

So by using the above equalities and (3.5), we arrive at $f(\lambda x, \mu y) = \lambda\mu f(x, y)$

for all $\lambda, \mu \in T^1 = \{z \in \mathbb{C}; |z| = 1\}$ and $x \in X$ and $y \in Y$.

If $0 \leq r, s \leq 1$ letting $s_1 = s + \sqrt{1-s^2}\ i \in T^1$ and $r_1 = r + \sqrt{1-r^2}\ i \in T^1$.

So $r = \frac{r_1 + \bar{r_1}}{2}, s = \frac{s_1 + \bar{s_1}}{2}$. Therefore, by paying attention to what come in the above

$f(rx, sy) = rsf(x, y)$. Now if λ, μ belong to $\mathbb{C}$ then

$\lambda = |\lambda|e^{i\lambda_1}, \mu = |\mu|e^{i\mu_1}, |\lambda| = [|\lambda|] + \lambda_2, |\mu| = [|\mu|] + \mu_2$ that in those $0 \leq \lambda_2, \mu_2 < 1$. Thus

$f(\lambda x, \mu y) = f(([|\lambda|] + \lambda_2)e^{i\lambda_1}x, ([|\mu|] + \mu_2)e^{i\mu_1}y) = e^{i\lambda_1}e^{i\mu_1}f(([|\lambda|] + \lambda_2)x, ([|\mu|] + \mu_2)y) = \lambda\mu f(x, y)$

for all $x \in X$ and $y \in Y$. Hence f is bilinear. The converse is clear. □

Theorem 3.3. *Let X, Y, Z be linear spaces and $f : X \times Y \to Z$ be a mapping. Then f is bilinear if and only if*

$$f(\lambda x - \lambda y, \mu t) + f(\lambda x, \mu t - \mu s) = \lambda\mu(2f(x, t) - f(y, t) - f(x, s)) \tag{3.6}$$

for all $x, y \in X$ and $t, s \in Y$ and $\lambda, \mu \in T^1_{\frac{1}{4}}$.

Proof. If f is bilinear, then it is obvious that f satisfies (3.6).
On the other hand, suppose that f satisfies (3.6). Setting $\lambda = \mu = 1$ in (3.6). By Theorem 3.1, f is biadditive. Putting $y = s = 0$ in (3.6), we achieve $f(\lambda x, \mu t) = \lambda\mu f(x, t)$ for all $\lambda, \mu \in T^1_{\frac{1}{4}}$ and $t \in Y$. Therefore, according to theorem 3.2 f is bilinear. □

Notation: Let X, Y, Z be linear spaces. For a given mapping $f : X \times Y \to Z$, we set

$$E_{\lambda,\mu}f(x, y, t, s) = f(\lambda x - \lambda y, \mu t) + f(\lambda x, \mu t - \mu s) - \lambda\mu(2f(x, t) - f(y, t) - f(x, s))$$

for all $x, y \in X$ and $s, t \in Y$ and $\lambda, \mu \in \mathbb{C}$.

4. Stability of $*-$bihomomorphisms of C^*-ternary algebras

In this section we investigate the Stability of $*-$bihomomorphisms between C^*-ternary algebras.

Theorem 4.1. *Let A and B be two C^*-ternary algebras and $\varphi : A^4 \to [0, \infty)$ be a function such that*

$$\varphi(0, 0, 0, 0) = 0 \qquad \lim_{n\to\infty} \frac{1}{4^{nl}}\varphi(2^{nl}x, 2^{nl}y, 2^{nl}t, 2^{nl}s) = 0, \tag{4.1}$$

$$\tilde{M}_l(x, y) = \sum_{n=\frac{1-l}{2}}^{\infty} \frac{1}{4^{nl}} M(2^{nl}x, 2^{nl}y) < \infty \tag{4.2}$$

where $x, y, t, s \in A$ and $l \in \{+1, -1\}$ and

$M(x, y) = \varphi(0, x, 2y, 0) + \varphi(x, -x, 2y, y) + \varphi(0, 0, 2y, 0) + 3(\varphi(x, 0, y, -y) + \varphi(x, 0, 0, y) + \varphi(x, 0, 0, 0) + \varphi(0, 0, y, 0))$

and $f : A \times A \to B$ be a mapping which satisfies

$$||f(x, y) - f(x^*, y^*)|| \leq \varphi(x, y, 0, 0), \tag{4.3}$$

$$\|E_{\lambda,\mu}f(x, y, t, s)\| \leq \varphi(x, y, t, s), \tag{4.4}$$

$$max(||f([x,y,t],s)-[f(x,s),f(y,s),f(t,s)]||,||f(x,[y,t,s])-[f(x,y),f(x,t),f(x,s)]||\leq\varphi(x^3,y^3,t^3,s^3), \quad (4.5)$$

$$\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny)=\lim_{n\to\infty}\frac{1}{4^{3n}}f(2^nx,2^{3n}y)=\lim_{n\to\infty}\frac{1}{4^{3n}}f(2^{3n}x,2^ny), \quad (4.6)$$

$$[\lim_{n\to\infty}4^nf(\frac{x}{2^n},\frac{y}{2^n})=\lim_{n\to\infty}4^{3n}f(\frac{x}{2^n},\frac{y}{2^{3n}})=\lim_{n\to\infty}4^{3n}f(\frac{x}{2^{3n}},\frac{y}{2^n})\ ] \quad (4.7)$$

for all $x,y,t,s\in A$ and $\lambda,\mu\in T^1_{\frac{1}{4}}$. Then there exists a unique $*-$bihomomorphism of ternary algebras $H:A\times A\to B$ such that

$$||H(x,y)-f(x,y)||\leq\frac{1}{4}\tilde{M}_l(x,y) \quad (4.8)$$

$$\textit{and}\quad H(x,y)=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny)\qquad[H(x,y)=\lim_{n\to\infty}4^nf(\frac{x}{2^n},\frac{y}{2^n})]$$

for all $x,y\in A$.

Proof. Setting $\lambda=\mu=1$ in (4.4) we have

$$||f(x-y,t)+f(x,t-s)-2f(x,t)+f(y,t)+f(x,s)||\leq\varphi(x,y,t,s). \quad (4.9)$$

Putting $x=y=t=s=0$ in (4.9), we obtain $f(0,0)=0$ by (4.1).
Letting $x=y=s=0$ in (4.9), shows that

$$||f(0,t)||\leq\varphi(0,0,t,0).$$

Assuming $y=t=s=0$ in (4.9) we get

$$\|f(x,0)\|\leq\varphi(x,0,0,0).$$

Setting $y=-x,t=2s$ in (4.9) we arrive at

$$||f(2x,2s)+2f(x,s)-2f(x,2s)+f(-x,2s)||\leq\varphi(x,-x,2s,s). \quad (4.10)$$

Putting $x=s=0$ in (4.9) we conclude that

$$||f(-y,t)+f(y,t)||\leq\varphi(0,y,t,0)+\varphi(0,0,t,0). \quad (4.11)$$

Letting $y=x,t=2s$ in (4.11) we get

$$||f(-x,2s)+f(x,2s)||\leq\varphi(0,x,2s,0)+\varphi(0,0,2s,0). \quad (4.12)$$

By using (4.10) and (4.12) we come to

$$||f(2x,2s)+2f(x,s)-3f(x,2s)||\leq\varphi(0,x,2s,0)+\varphi(x,-x,2s,s)+\varphi(0,0,2s,0). \quad (4.13)$$

In (4.9) setting $y=0$ and $s=-t$, we achieve

$$||f(x,2t)-f(x,t)+f(0,t)+f(x,-t)||\leq\varphi(x,0,t,-t). \quad (4.14)$$

Assuming $y=t=0$ in (4.9) we have

$$\|f(x,s)+f(x,-s)\|\leq\varphi(x,0,0,s)+\varphi(x,0,0,0). \quad (4.15)$$

Substituting s by t in (4.15) shows that

$$||f(x,-t)+f(x,t)|| \leq \varphi(x,0,0,t)+\varphi(x,0,0,0). \tag{4.16}$$

(4.14) and (4.16) show that

$$||f(x,2t)-2f(x,t)|| \leq \varphi(x,0,t,-t)+\varphi(x,0,0,t)+\varphi(x,0,0,0)+\varphi(0,0,t,0). \tag{4.17}$$

Letting $t=s$ in (4.17) and multiply both sides by 3, gives

$$||3f(x,2s)-6f(x,s)|| \leq 3(\varphi(x,0,s,-s)+\varphi(x,0,0,s)+\varphi(x,0,0,0)+\varphi(0,0,s,0)). \tag{4.18}$$

(4.18) and (4.13) give

$$||f(2x,2s)-4f(x,s)|| \leq M(x,s). \tag{4.19}$$

Now let $l=1$. By replacing x by 2^jx and s by 2^js, and multiplying both sides of (4.19) by $\frac{1}{4^{j+1}}$, we get

$$||\frac{1}{4^{j+1}}f(2^{j+1}x,2^{j+1}s)-\frac{1}{4^j}f(2^jx,2^js)|| \leq \frac{1}{4^{j+1}}M(2^jx,2^js).$$

Now for given $m,p \in \mathbb{N}$, we have

$$||\frac{1}{4^{m+p}}f(2^{m+p}x,2^{m+p}s)-\frac{1}{4^m}f(2^mx,2^ms)|| \leq$$

$$\sum_{j=m}^{m+p-1}||\frac{1}{4^{j+1}}f(2^{j+1}x,2^{j+1}s)-\frac{1}{4^j}f(2^jx,2^js)|| \leq$$

$$\sum_{j=m}^{m+p-1}\frac{1}{4^{j+1}}M(2^jx,2^js). \tag{4.20}$$

So by (4.20) and (4.2), the sequence $\frac{1}{4^n}f(2^nx,2^ny)$ is a Cauchy sequence in Banach space B for all $x,y \in A$, and hence it is convergent. Define $H: A\times A \to B$ by $H(x,y)=\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny)$ for all $x,y \in A$. Letting $m=0$ and $p\to\infty$ in (4.20), so we obtain (4.8) with $l=1$. From (4.4) and (4.1), we get

$$||E_{\lambda,\mu}H(x,y,t,s)|| \leq \lim_{n\to\infty}\frac{1}{4^n}||E_{\lambda,\mu}f(2^nx,2^ny,2^nt,2^ns)|| \leq$$

$$\lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny,2^nt,2^ns)=0.$$

Thus $E_{\lambda,\mu}H(x,y,t,s)=0$ for all $x,y,t,s \in A$ and $\lambda,\mu \in T^1_{\frac{1}{4}}$. So, according to Theorem 3.3 H is a bilinear mapping. By (4.5), (4.6) and (4.1), we conclude that

$$\max(||H([x,y,t],s)-[H(x,s),H(y,s),H(t,s)]\ ||, ||H(x,[y,t,s])-[H(x,y),H(x,t),H(x,s)]\ ||)=$$

$$\max(||\lim_{n\to\infty}\frac{1}{4^n}f(2^n[x,y,t],2^ns)-\lim_{n\to\infty}\frac{1}{4^{3n}}[f(2^nx,2^ns),f(2^ny,2^ns),f(2^nt,2^ns)]||,$$

$$||\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^n[y,t,s])-\lim_{n\to\infty}\frac{1}{4^{3n}}[f(2^nx,2^ny),f(2^nx,2^nt),f(2^nx,2^ns)]||)=$$

$$\lim_{n\to\infty}\frac{1}{4^{3n}}max(||f([2^nx,2^ny,2^nt],2^ns)-[f(2^nx,2^ns),f(2^ny,2^ns),f(2^nt,2^ns)]||,$$

$$||f(2^nx,[2^ny,2^nt,2^ns])-[f(2^nx,2^ny),f(2^nx,2^nt),f(2^nx,2^ns)]||) \leq$$

$$\lim_{n\to\infty}\frac{1}{4^{3n}}\varphi(2^{3n}x^3,2^{3n}y^3,2^{3n}t^3,2^{3n}s^3)=0.$$

So $H([x,y,t],s) = [H(x,s),H(y,s),H(t,s)]$ and $H(x,[y,t,s]) = [H(x,y),H(x,t),H(x,s)]$. Therefore, H is a bihomomorphism. On the other hand, by (4.3), we have

$$||H(x,y) - H(x^*,y^*)|| = \lim_{n\to\infty} \frac{1}{4^n}||f(2^n x, 2^n y) - f((2^n x)^*, (2^n y)^*)|| \leq \frac{1}{4^n}\varphi(2^n x, 2^n y, 0, 0) = 0.$$

Hence H is $*-$bihomomorphism. Now let $T : A \times A \to B$ be another $*-$bihomomorphism satisfying (4.8), then by (4.2) we have

$$||H(x,y) - T(x,y)|| = \lim_{n\to\infty} ||\frac{1}{4^n} f(2^n x, 2^n y) - T(x,y)|| = \lim_{n\to\infty} \frac{1}{4^n}||f(2^n x, 2^n y) - T(2^n x, 2^n y)|| \leq$$

$$\lim_{n\to\infty} \frac{1}{4^{n+1}}\tilde{M_1}(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4}\sum_{j=n}^{\infty} \frac{1}{4^n} M(2^n x, 2^n y) = 0.$$

So, we conclude that $H(x,y) = T(x,y)$ for all $x, y \in A$.

Now assume that $l = -1$.

Interchanging x by $\frac{x}{2^{j+1}}$ and s by $\frac{s}{2^{j+1}}$ in(4.19) and multiplying its both sides by 4^j, we achieve

$$||4^j f(\frac{x}{2^j}, \frac{s}{2^j}) - 4^{j+1} f(\frac{x}{2^{j+1}}, \frac{s}{2^{j+1}})|| \leq 4^j M(\frac{x}{2^{j+1}}, \frac{s}{2^{j+1}}).$$

Now for given $m, p \in \mathbb{N}$, we see that

$$||4^{m+p} f(\frac{x}{2^{m+p}}, \frac{s}{2^{m+p}}) - 4^m f(\frac{x}{2^m}, \frac{s}{2^m})|| \leq \sum_{j=m}^{m+p-1} ||4^j f(\frac{x}{2^j}, \frac{s}{2^j}) - 4^{j+1} f(\frac{x}{2^{j+1}}, \frac{s}{2^{j+1}})|| \leq$$

$$\sum_{j=m}^{m+p-1} 4^j M(\frac{x}{2^{j+1}}, \frac{s}{2^{j+1}}). \tag{4.21}$$

By (4.21) and (4.2) the sequence $4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ is Cauchy in Banach space B. Then it is convergent for all $x, y \in A$. Define $H_1 : A \times A \to B$ by $H_1(x,y) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ for all x,y in A. The rest of the proof is similar to the case that $l = 1$. □

Theorem 4.2. *Let $\theta, p_1, p_2, p_3, p_4$ be real numbers such that $\theta \geq 0$ and all of p_1, p_2, p_3, p_4 be located in $(0, 2)$ [or all in $(2, \infty)$] and A,B be C^*ternary algebras and $f : A \times A \to B$ be a mapping that satisfies*

$$||f(x,y) - f(x^*,y^*)|| \leq \theta(||x||^{p_1} + ||y||^{p_2}),$$

$$\|E_{\lambda,\mu} f(x,y,t,s)\| \leq \theta(||x||^{p_1} + ||y||^{p_2} + ||t||^{p_3} + ||s||^{p_4}),$$

$$max(||f([x,y,t],s) - [f(x,s), f(y,s), f(t,s)]||, ||f(x,[y,t,s]) - [f(x,y), f(x,t), f(x,s)]||,$$

$$\leq \theta(||x^3||^{p_1} + ||y^3||^{p_2} + ||t^3||^{p_3} + ||s^3||^{p_4}),$$

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^n x, 2^{3n} y) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^{3n} x, 2^n y),$$

$$[\lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) = \lim_{n\to\infty} 4^{3n} f(\frac{x}{2^n}, \frac{y}{2^{3n}}) = \lim_{n\to\infty} 4^{3n} f(\frac{x}{2^{3n}}, \frac{y}{2^n})]$$

for all $x, y, t, s \in A$ and $\lambda, \mu \in T^1_{\frac{1}{4}}$. Then there exists a unique $*-$bihomomorphism $H : A \times A \to B$ which satisfies the following inequality

$$||H(x,y) - f(x,y)|| \leq$$

$$\theta(\frac{10}{|4 - 2^{p_1}|}||x||^{p_1} + \frac{2}{|4 - 2^{p_2}|}||x||^{p_2} + \frac{6 + 3 \times 2^{p_3}}{|4 - 2^{p_3}|}||y||^{p_3} + \frac{7 + 2^{p_4}}{|4 - 2^{p_4}|}||y||^{p_4}),$$

and

$$H(x,y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) \qquad [H(x,y) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})]$$

for all $x, y \in A$.

Proof. We set in Theorem 4.1

$$\varphi(x,y,t,s) = \theta(||x||^{p_1} + ||y||^{p_2} + ||t||^{p_3} + ||s||^{p_4}).$$

□

Theorem 4.3. *Let* A *and* B *be two* C^*-*ternary algebras and* $\varphi : A^4 \to [0,\infty)$ *be a function such that* $\varphi(0,0,0,0) = 0$ *and*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n t, s) = 0 \qquad [\ \lim_{n\to\infty} \frac{1}{2^n}\varphi(x, 2^n y, 2^n t, 2^n s) = 0\ ]$$

$$\tilde{M_1}(x,t) = \sum_{j=0}^{\infty} \frac{1}{2^j} M(2^j x, t) < \infty \qquad [\ \tilde{M_2}(x,t) = \sum_{j=0}^{\infty} \frac{1}{2^j} M(x, 2^j t) < \infty\ ] \tag{4.22}$$

where $x, y, t, s \in A$ *and*

$$M(x,t) = \varphi(x,-x,t,t) + \varphi(x,0,0,0) + \varphi(0,x,t,0) + \varphi(0,0,t,0),$$

$$[\ M(x,t) = \varphi(x,x,t,-t) + \varphi(x,0,0,0) + \varphi(x,0,0,t) + \varphi(0,0,t,0)\ ]$$

and $f : A \times A \to B$ *be a mapping satisfying*

$$||f(x,y) - f(x^*,y^*)|| \le \varphi(x,x,x,y) \qquad [\ ||f(x,y) - f(x^*,y^*)|| \le \varphi(x,y,y,y)\ ]$$

$$||E_{\lambda,\mu} f(x,y,t,s)|| \le \varphi(x,y,t,s)$$

$$||f([x,y,t],s) - [f(x,s), f(y,s), f(t,s)]|| \le \varphi(x^3,y^3,t^3,s^3)$$

$$[\ ||f(x,[y,t,s]) - [f(x,y), f(x,t), f(x,s)]|| \le \varphi(x^3,y^3,t^3,s^3)\ ]$$

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x, y) = \lim_{n\to\infty} \frac{1}{2^{3n}} f(2^{3n} x, y) \qquad [\lim_{n\to\infty} \frac{1}{2^n} f(x, 2^n y) = \lim_{n\to\infty} \frac{1}{2^{3n}} f(x, 2^{3n} y)]$$

for all $x, y, t, s \in A$ *and* $\lambda, \mu \in T^1_{\frac{1}{4}}$. *Then there exist a unique* $*-$*bihomomorphism* $H : A \times A \to B$ *such that*

$$||H(x,y) - f(x,y)|| \le \frac{1}{2}\tilde{M_1}(x,y) \qquad [\ ||H(x,y) - f(x,y)|| \le \frac{1}{2}\tilde{M_2}(x,y)\ ] \tag{4.23}$$

and

$$H(x,y) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x, y) \qquad [\ H(x,y) = \lim_{n\to\infty} \frac{1}{2^n} f(x, 2^n y)\ ]$$

for all $x, y \in A$.

Proof. Putting $y = t = s = 0$ in (4.9) shows that

$$||f(x,0)|| \le \varphi(x,0,0,0). \tag{4.24}$$

Setting $y = -x$ and $s = t$ in (4.9) we get

$$||f(2x,t) + f(x,0) - f(x,t) + f(-x,t)|| \le \varphi(x,-x,t,t). \tag{4.25}$$

Substituting y with x in (4.11) we obtain

$$||f(-x,t) + f(x,t)|| \le \varphi(0,x,t,0) + \varphi(0,0,t,0). \tag{4.26}$$

By using (4.24) and (4.25) and (4.26) we come to

$$||f(2x,t) - 2f(x,t)|| \le M(x,t). \tag{4.27}$$

Replacing x with 2^jx and multiply its both sides by $\frac{1}{2^{j+1}}$ we get

$$||\frac{1}{2^{j+1}}f(2^{j+1}x,t)-\frac{1}{2^j}f(2^jx,t)||\leq\frac{1}{2^{j+1}}M(2^jx,t).$$

Now let m,p be in $\mathbb{N}$. We have

$$||\frac{1}{2^{m+p}}f(2^{m+p}x,t)-\frac{1}{2^m}f(2^mx,t)||\leq\sum_{j=m}^{m+p-1}||\frac{1}{2^{j+1}}f(2^{j+1}x,t)-\frac{1}{2^j}f(2^jx,t)||\leq\sum_{j=m}^{m+p-1}\frac{1}{2^{j+1}}M(2^jx,t). \tag{4.28}$$

From (4.22) and (4.28) the sequence $\{\frac{1}{2^n}f(2^nx,y)\}$ is Cauchy and hence is convergent. Define $H:A\times A\to B$ by $H(x,y)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx,y)$ for all x,y in A.
Letting $m=0$ and $p\to\infty$ in (4.28), it follows that (4.23) is satisfied. The rest of the proof is similar to the proof of Theorem 4.1 . □

Theorem 4.4. *Suppose that θ,p_1,p_2,p_3,p_4 be real numbers such that $\theta\geq0$, p_1,p_2,p_3 lie in $(0,1)$ $[p_2,\ p_3\ ,p_4$ lie in $(0,1)]$. Let A and B be two C^*-ternary algebras and $f:A\times A\to B$ be a mapping satisfying*

$$||f(x,y)-f(x^*,y^*)||\leq\theta(||x||^{p_1}+||x||^{p_2}+||x||^{p_3}+||y||^{p_4}) \qquad [\ ||f(x,y)-f(x^*,y^*)||\leq\theta(||x||^{p_1}+||y||^{p_2}+||y||^{p_3}+||y||^{p_4})\ ]$$

$$||E_{\lambda,\mu}f(x,y,t,s)||\leq\theta(||x||^{p_1}+||y||^{p_2}+||t||^{p_3}+||s||^{p_4})$$

$$||f([x,y,t],s)-[f(x,s),f(y,s),f(t,s)]||\leq\theta(||x^3||^{p_1}+||y^3||^{p_2}+||t^3||^{p_3}+||s^3||^{p_4})$$

$$[\ ||f(x,[y,t,s])-[f(x,y),f(x,t),f(x,s)]||\leq\theta(||x^3||^{p_1}+||y^3||^{p_2}+||t^3||^{p_3}+||s^3||^{p_4})\ ]$$

$$\lim_{n\to\infty}\frac{1}{2^n}f(2^nx,y)=\lim_{n\to\infty}\frac{1}{2^{3n}}f(2^{3n}x,y) \qquad [\ \lim_{n\to\infty}\frac{1}{2^n}f(x,2^ny)=\lim_{n\to\infty}\frac{1}{2^{3n}}f(x,2^{3n}y)\ ]$$

for all $x,y,t,s\in A$ and $\lambda,\mu\in T^1_{\frac{1}{4}}$. Then there exist a unique left $[right]$ $*-$bihomomorphism $H:A\times A\to B$ such that

$$||H(x,y)-f(x,y)||\leq\theta(\frac{2}{2-2^{p_1}}||x||^{p_1}+\frac{2}{2-2^{p_2}}||x||^{p_2}+3||y||^{p_3}+||y||^{p_4})$$

$$[\ ||H(x,y)-f(x,y)||\leq\theta(3||x||^{p_1}+||x||^{p_2}+\frac{2}{2-2^{p_3}}||y||^{p_3}+\frac{2}{2-2^{p_4}}||y||^{p_4})\ ]$$

and

$$H(x,y)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx,y) \qquad [H(x,y)=\lim_{n\to\infty}\frac{1}{2^n}f(x,2^ny)]$$

for all $x,y\in A$.

Proof. It follows by Theorem 4.3 by putting $\varphi(x,y,t,s)=\theta(||x||^{p_1}+||y||^{p_2}+||t||^{p_3}+||s||^{p_4})$. □

5. **Superstability**

In this section, we investigate the Superstability of $*-$bihomomorphisms of C^*-ternary algebras.

Theorem 5.1. *Let A and B be two C^*-ternary algebras and $\varphi:A^4\to[0,\infty)$ be a function such that*

$$\varphi(0,y,t,s)=\varphi(x,0,t,s)=\varphi(x,y,t,0)=0 \tag{5.1}$$

$$\lim_{n\to\infty}\frac{1}{4^{nl}}\varphi(2^{nl}x,2^{nl}y,2^{nl}t,2^{nl}s)=0 \tag{5.2}$$

for all $x,y,t,s\in A$ and $l\in\{1,-1\}$ and let $f:A\times A\to B$ be a mapping satisfying

$$||f(x,y)-f(x^*,y^*)||\leq\varphi(x,y,x,y), \tag{5.3}$$

$$||E_{\lambda,\mu}f(x,y,t,s)|| \leq \varphi(x,y,t,s), \tag{5.4}$$

$$max(||f([x,y,t],s)-[f(x,s),f(y,s),f(t,s)]||,||f(x,[y,t,s])-[f(x,y),f(x,t),f(x,s)]||) \leq \varphi(x^3,y^3,t^3,s^3), \tag{5.5}$$

$$\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^ny)=\lim_{n\to\infty}\frac{1}{4^{3n}}f(2^{3n}x,2^ny)=\lim_{n\to\infty}\frac{1}{4^{3n}}f(2^nx,2^{3n}y), \tag{5.6}$$
$$[\lim_{n\to\infty}4^nf(\frac{x}{2^n},\frac{y}{2^n})=\lim_{n\to\infty}4^{3n}f(\frac{x}{2^{3n}},\frac{y}{2^n})=\lim_{n\to\infty}4^{3n}f(\frac{x}{2^n},\frac{y}{2^{3n}})]$$

for all $x,y,t,s\in A$ and $\lambda,\mu\in T^1_{\frac{1}{4}}$. Then f is a $*-$bihomomorphism.

Proof. Letting $\lambda=\mu=1$ in (5.4) to get

$$||f(x-y,t)+f(x,t-s)-2f(x,t)+f(y,t)+f(x,s)|| \leq \varphi(x,y,t,s). \tag{5.7}$$

Reputing $x=y=t=s=0$ in (5.7) to obtain $f(0,0)=0$. Putting $x=y=s=0$ in (5.7) to obtain $f(0,t)=0$ for all $t\in A$. putting $y=s=t=0$ in (5.7) to get $f(x,0)=0$ for all $x\in A$. In (5.7), put $x=0$ and $s=t$, we arrive at $f(-y,t)=-f(y,t)$, for all $y,t\in A$. Letting $t=y=0$ in (5.7), we conclude that $f(x,-s)=-f(x,s)$ for all $x,s\in A$. Put $y=0$ and $s=-t$ in (5.7) to obtain $f(x,2t)=2f(x,t)$ for all $x,t\in A$. Suppose that $s=0$ and $y=-x$ in (5.7) we arrive at $f(2x,t)=2f(x,t)$ for all $x,t\in A$. Thus we receive $f(2x,2y)=4f(x,y)$ for all $x,y\in A$. By induction, we get $f(2^nx,2^ny)=4^nf(x,y)$ for all $x,y\in A$ and $n\in\mathbb{Z}$ .Now let $l=1$. From (5.4), we get

$$||E_{\lambda,\mu}f(x,y,t,s)||=\lim_{n\to\infty}\frac{1}{4^n}||E_{\lambda,\mu}f(2^nx,2^ny,2^nt,2^ns)|| \leq \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny,2^nt,2^ns)=0.$$

Thus $E_{\lambda,\mu}f(x,y,t,s)=0$ for all $x,y,t,s\in A$ and $\lambda,\mu\in T^1_{\frac{1}{4}}$. So, according to Theorem 3.3, f is a bilinear map. By (5.2), (5.5) and (5.6) we conclude that

$$\max(||f([x,y,t],s)-[f(x,s),f(y,s),f(t,s)]\ ||,||f(x,[y,t,s])-[f(x,y),f(x,t),f(x,s)]\ ||)=$$
$$\max(||\lim_{n\to\infty}\frac{1}{4^n}f(2^n[x,y,t],2^ns)-\lim_{n\to\infty}\frac{1}{4^{3n}}[f(2^nx,2^ns),f(2^ny,2^ns),f(2^nt,2^ns)]||,$$
$$||\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,2^n[y,t,s])-\lim_{n\to\infty}\frac{1}{4^{3n}}[f(2^nx,2^ny),f(2^nx,2^nt),f(2^nx,2^ns)]||)=$$
$$\lim_{n\to\infty}\frac{1}{4^{3n}}max(||f([2^nx,2^ny,2^nt],2^ns)-[f(2^nx,2^ns),f(2^ny,2^ns),f(2^nt,2^ns)]||,$$
$$||f(2^nx,[2^ny,2^nt,2^ns])-[f(2^nx,2^ny),f(2^nx,2^nt),f(2^nx,2^ns)]||) \leq$$
$$\lim_{n\to\infty}\frac{1}{4^{3n}}\varphi(2^{3n}x^3,2^{3n}y^3,2^{3n}t^3,2^{3n}s^3)=0.$$

So $f([x,y,t],s)=[f(x,s),f(y,s),f(t,s)]$ and $f(x,[y,t,s])=[f(x,y),f(x,t),f(x,s)]$.
Therefore f is a bihomomorphism. On the other hand, by (5.3), we have

$$||f(x,y)-f(x^*,y^*)||=\lim_{n\to\infty}\frac{1}{4^n}||f(2^nx,2^ny)-f((2^nx)^*,(2^ny)^*)|| \leq \frac{1}{4^n}\varphi(2^nx,2^ny,2^nx,2^ny)=0.$$

Hence f is $*-$bihomomorphism. The proof of other case is similar. □

Theorem 5.2. *Let $\theta, p_1, p_2, p_3, p_4$ be real numbers such that $\theta \geq 0$, $p_1+p_2+p_3+p_4$ lie in $(0,2)$ $[p_1+p_2+p_3+p_4$ lie in $(2,\infty)]$. Let A and B be two ternary C^*-algebras and $f : A \times A \to B$ be a mapping satisfying*

$$||f(x,y) - f(x^*,y^*)|| \leq \theta(||x||^{p_1+p_3}||y||^{p_2+p_4}),$$

$$||E_{\lambda,\mu}f(x,y,t,s)|| \leq \theta(||x||^{p_1}||y||^{p_2}||t||^{p_3}||s||^{p_4}),$$

$$max(||f([x,y,t],s) - [f(x,s),f(y,s),f(t,s)]||, ||f(x,[y,t,s]) - [f(x,y),f(x,t),f(x,s)]||$$

$$\leq \theta(||x^3||^{p_1}||y^3||^{p_2}||t^3||^{p_3}||s^3||^{p_4}),$$

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^{3n}x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^{3n}} f(2^n x, 2^{3n} y)$$

$$[\lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) = \lim_{n\to\infty} 4^{3n} f(\frac{x}{2^{3n}}, \frac{y}{2^n}) = \lim_{n\to\infty} 4^{3n} f(\frac{x}{2^n}, \frac{y}{2^{3n}})]$$

for all $x,y,t,s \in A$ and $\lambda, \mu \in T^1_{\frac{1}{4}}$. Then f is a $*-$bihomomorphism.

Proof. It follows from Theorem 5.1 by putting $\varphi(x,y,t,s) = \theta(||x||^{p_1}||y||^{p_2}||t||^{p_3}||s||^{p_4})$. □

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Convergence of Complex General Singular Integral Operators

George A. Anastassiou & Razvan A. Mezei

Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu
rmezei@memphis.edu

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Abstract. In this article we study the general complex-valued singular integral operators over the real line regarding their convergence to the unit operator with rates in the L_p norm, $1 \leq p \leq \infty$. The related established inequalities involve the higher order L_p modulus of smoothness of the engaged function or its higher order derivative. Also we study the complex-valued fractional general singular integral operators on the real line, regarding their convergence to the unit operator with rates in the uniform norm. The related established inequalities involve the higher order moduli of smoothness of the associated right and left Caputo fractional derivatives of the engaged function.We finish with applications to trigonometric singular integral operators. The related simultaneous approximations are also studied extensively.

1 Convergence of Complex General Singular Integral Operators Background

We consider here complex valued Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, $i := \sqrt{-1}$. Here $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are implied to be real valued

Borel measurable functions.

Let $\xi > 0$ and μ_ξ be a Borel probability measure on $\mathbb{R}$. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}^+$ we put

$$\alpha_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 1, \dots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-n}, & j = 0. \end{cases} \tag{1}$$

Here we study the convergence of smooth general singular integral operators

$$\Theta_{r,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) d\mu_\xi(t), \forall \xi > 0. \tag{2}$$

Clearly by the definition of R.H.S.(2) we have

$$\Theta_{r,\xi}(f;x) = \Theta_{r,\xi}(f_1;x) + i\Theta_{r,\xi}(f_2;x). \tag{3}$$

We assume that $\Theta_{r,\xi}(f_j;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $j = 1, 2$.

I) Let $f_1, f_2 \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ with the rth modulus of smoothness finite, i.e.

$$\omega_r(f_{\widetilde{j}}^{(n)}, h) := \sup_{|t| \le h} \|\Delta_t^r f_{\widetilde{j}}^{(n)}(x)\|_{\infty,x} < \infty, \tag{4}$$

$h > 0$, where

$$\Delta_t^r f_{\widetilde{j}}^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f_{\widetilde{j}}^{(n)}(x+jt), \tag{5}$$

$\widetilde{j} = 1, 2$.

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \;\; k = 1, \dots, n \in \mathbb{N}.$$

The integrals

$$c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$$

are assumed to be finite, $k = 1, \dots, n$.

One notices easily that

$$|\Theta_{r,\xi}(f;x) - f(x)| \le |\Theta_{r,\xi}(f_1;x) - f_1(x)| + |\Theta_{r,\xi}(f_2;x) - f_2(x)| \tag{6}$$

also

$$\|\Theta_{r,\xi}(f;x) - f(x)\|_{\infty,x} \le \|\Theta_{r,\xi}(f_1;x) - f_1(x)\|_{\infty,x} + \|\Theta_{r,\xi}(f_2;x) - f_2(x)\|_{\infty,x}, \tag{7}$$

and

$$\|\Theta_{r,\xi}(f;x)-f(x)\|_{p,x} \leq \|\Theta_{r,\xi}(f_1;x)-f_1(x)\|_{p,x} + \|\Theta_{r,\xi}(f_2;x)-f_2(x)\|_{p,x}, p \geq 1. \tag{8}$$

Furthermore it holds

$$f^{(k)}(x) = f_1^{(k)}(x) + if_2^{(k)}(x), \tag{9}$$

for all $k = 1, \ldots, n$.

2 Main Results

By using Theorem 9 of [1] we obtain

Theorem 1. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$. Suppose that

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Assume also that $\omega_r(f_j^{(n)}, h) < \infty, \forall h > 0$. Then

$$\begin{aligned} &\left\|\Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{\infty,x} \\ &\leq \frac{1}{n!}\left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t)\right]\left(\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi)\right). \end{aligned} \tag{10}$$

When $n = 0$ the sum in L.H.S.(10) collapses.
Proof. By Theorem 9 of [1] and (6), (7), (9) we get

$$\begin{aligned} \left\|\Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{\infty,x} &= \left\|\left[\Theta_{r,\xi}(f_1;x) - f_1(x) - \sum_{k=1}^{n} \frac{f_1^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right]\right. \\ &\left.+ i\left[\Theta_{r,\xi}(f_2;x) - f_2(x) - \sum_{k=1}^{n} \frac{f_2^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right]\right\|_{\infty,x} \\ &\leq \left\|\Theta_{r,\xi}(f_1;x) - f_1(x) - \sum_{k=1}^{n} \frac{f_1^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{\infty,x} \\ &+ \left\|\Theta_{r,\xi}(f_2;x) - f_2(x) - \sum_{k=1}^{n} \frac{f_2^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{\infty,x} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\omega_r(f_1^{(n)}, \xi)}{n!} \left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \\
&\quad + \frac{\omega_r(f_2^{(n)}, \xi)}{n!} \left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \\
&= \frac{1}{n!} \left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \left(\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi) \right),
\end{aligned}
$$

proving the claim. ■

The $n = 0$ case follows.

Corollary 2 (to Theorem 1). *Let* $f : \mathbb{R} \to \mathbb{C} : f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C(\mathbb{R})$. Suppose

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Assume also $\omega_r(f_j, h) < \infty, \forall h > 0$. Then

$$\|\Theta_{r,\xi}(f) - f\|_\infty \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] (\omega_r(f_1, \xi) + \omega_r(f_2, \xi)). \qquad (11)$$

In [4] we proved

Theorem 3([4]). *Let $g \in C^{n-1}(\mathbb{R})$, such that $g^{(n)}$ exists, $n, r \in \mathbb{N}$. Furthermore suppose that for each $x \in \mathbb{R}$ the function $g^{(\widetilde{j})}(x + jt) \in L_1(\mathbb{R}, \mu_\xi)$ as a function of t, for all $\widetilde{j} = 0, 1, \ldots, n-1$; $j = 1, \ldots, r$. Suppose that there exist $\lambda_{\widetilde{j},j} \geq 0$, $\widetilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, with $\lambda_{\widetilde{j},j} \in L_1(\mathbb{R}, \mu_\xi)$ such that for each $x \in \mathbb{R}$ we have*

$$|g^{(\widetilde{j})}(x + jt)| \leq \lambda_{\widetilde{j},j}(t), \qquad (12)$$

for $\mu_\xi-$almost all $t \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, 2, \ldots, r$. Then $g^{(\widetilde{j})}(x+jt)$ defines a $\mu_\xi-$integrable function with respect to t for each $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, and

$$\left(\Theta_{r,\xi}(g; x)\right)^{(\widetilde{j})} = \Theta_{r,\xi}\left(g^{(\widetilde{j})}; x\right), \qquad (13)$$

for all $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$.

We present the following simultaneous approximation result.

Theorem 4. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^{n+\rho}(\mathbb{R})$, $n, \rho \in \mathbb{Z}^+$, and $\omega_r(f_j^{(n+\widetilde{j})}, h) < \infty, \forall h > 0$, for $\widetilde{j} = 0, 1, \ldots, \rho$. Suppose*

$$\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

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We consider the assumptions of Theorem 3 valid regarding f_1, f_2 for $n = \rho$. Then

$$\left\| (\Theta_{r,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(k+\widetilde{j})}(x)}{k!} \delta_k c_{k,\xi} \right\|_{\infty,x}$$
$$\leq \frac{1}{n!} \left[\int_{-\infty}^{\infty} |t|^n \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) \right] \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi) + \omega_r(f_2^{(n+\widetilde{j})}, \xi) \right), \quad (14)$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$. When $n = 0$ the sum in L.H.S.(14) collapses.

Proof. Similar to Theorem 1 here, and based on Theorem 8 of [4]. ■

II) Here let $f_1,\ f_2 \in C^n(\mathbb{R})$ with $f_1^{(n)}, f_2^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$. We need the rth L_p-modulus of smoothness

$$\omega_r(f_{\widetilde{j}}^{(n)}, h)_p := \sup_{|t| \leq h} \|\Delta_t^r f_{\widetilde{j}}^{(n)}(x)\|_{p,x}, \quad h > 0, \text{ with } \widetilde{j} = 1, 2. \quad (15)$$

Here we assume that $\omega_r(f_{\widetilde{j}}^{(n)}, h)_p < \infty$, $h > 0$, $\widetilde{j} = 1, 2$.

We present

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{C}$* such that $f = f_1 + i f_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that

$$\int_{-\infty}^{\infty} \left[\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right] |t|^{np-1} d\mu_\xi(t) < \infty,$$

and $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. Then

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{p,x}$$
$$\leq \frac{1}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1 \right) |t|^{np-1} d\mu_\xi(t) \right]^{\frac{1}{p}}$$
$$\cdot \xi^{\frac{1}{p}} \left(\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p \right). \quad (16)$$

Proof. By Theorem 1 of [2] and (6), (8), (9) we get

$$\left\| \Theta_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{p,x}$$
$$= \left\| \left[\Theta_{r,\xi}(f_1;x) - f_1(x) - \sum_{k=1}^{n} \frac{f_1^{(k)}(x)}{k!} \delta_k c_{k,\xi} \right] \right.$$

$$
\begin{aligned}
&+i\left[\Theta_{r,\xi}(f_2;x)-f_2(x)-\sum_{k=1}^{n}\frac{f_2^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right]\Bigg\|_{p,x} \\
\leq &\left\|\Theta_{r,\xi}(f_1;x)-f_1(x)-\sum_{k=1}^{n}\frac{f_1^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{p,x} \\
&+\left\|\Theta_{r,\xi}(f_2;x)-f_2(x)-\sum_{k=1}^{n}\frac{f_2^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{p,x} \\
\leq &\ R.H.S(16),
\end{aligned}
$$

proving the claim. ∎

Based on Proposition 3 of [2], similarly we give

Proposition 6. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$, where $f_1,f_2\in(C(\mathbb{R})\cap L_p(\mathbb{R}))$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Assume that

$$\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp}d\mu_\xi(t)<\infty.$$

Then

$$\|\Theta_{r,\xi}(f)-f\|_p\leq\left[\int_{-\infty}^{\infty}\left(1+\frac{|t|}{\xi}\right)^{rp}d\mu_\xi(t)\right]^{\frac{1}{p}}\left(\omega_r(f_1,\xi)_p+\omega_r(f_2,\xi)_p\right). \tag{17}$$

Based on Theorem 2 of [2] we get similarly

Theorem 7. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^n(\mathbb{R})$ with $f_j^{(n)}\in L_1(\mathbb{R})$, $n\in\mathbb{N}$. Assume that

$$\int_{-\infty}^{\infty}\left[\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right]|t|^{n-1}d\mu_\xi(t)<\infty,$$

and $c_{k,\xi}\in\mathbb{R}$, $k=1,\ldots,n$. Then

$$
\begin{aligned}
&\left\|\Theta_{r,\xi}(f;x)-f(x)-\sum_{k=1}^{n}\frac{f^{(k)}(x)}{k!}\delta_k c_{k,\xi}\right\|_{1,x} \\
\leq &\frac{1}{(n-1)!\,(r+1)}\left[\int_{-\infty}^{\infty}\left(\left(1+\frac{|t|}{\xi}\right)^{r+1}-1\right)|t|^{n-1}d\mu_\xi(t)\right] \\
&\cdot\xi\left(\omega_r(f_1^{(n)},\xi)_1+\omega_r(f_2^{(n)},\xi)_1\right).
\end{aligned}
\tag{18}
$$

Based on Proposition 4 of [2], we give similarly

Proposition 8. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$. Assume

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t) < \infty.$$

Then

$$\|\Theta_{r,\xi}(f) - f\|_1 \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^r d\mu_\xi(t)\right] (\omega_r(f_1, \xi)_1 + \omega_r(f_2, \xi)_1). \tag{19}$$

Next we give simultaneous approximation results.

Theorem 9. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}, \rho \in \mathbb{Z}^+$, *with* $f_j^{(n+\widetilde{j})} \in L_p(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots, \rho$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right) |t|^{np-1} d\mu_\xi(t) < \infty$, *and* $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. *We consider the assumptions of Theorem 3 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\begin{aligned} &\left\|(\Theta_{r,\xi}(f; x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{(k+\widetilde{j})}(x)}{k!} \delta_k c_{k,\xi}\right\|_{p,x} \leq \frac{1}{((n-1)!)} \\ &\frac{1}{(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_p + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_p\right) \\ &\cdot \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi}\right)^{rp+1} - 1\right) |t|^{np-1} d\mu_\xi(t)\right]^{\frac{1}{p}} \xi^{\frac{1}{p}}, \end{aligned} \tag{20}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.

Proof. By Theorem 11 of [4], and as in the proof of Theorem 5 here. ■

We give the related

Proposition 10. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Let* $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume that*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t) < \infty.$$

We consider the assumptions of Theorem 3 valid regarding f_1, f_2 *for* $n = \rho$. *Then*

$$\begin{aligned} &\left\|(\Theta_{r,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})}\right\|_p \\ &\leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi}\right)^{rp} d\mu_\xi(t)\right]^{\frac{1}{p}} \left(\omega_r(f_1^{(\widetilde{j})}, \xi)_p + \omega_r(f_2^{(\widetilde{j})}, \xi)_p\right), \end{aligned} \tag{21}$$

$\widetilde{j} = 0, 1, \ldots, \rho.$
Proof. By Proposition 12 of [4]. ■

Next comes the related L_1 result.

Theorem 11. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$, and $j = 1, 2$. *Let* $f_1, f_2 \in C^{n+\rho}(\mathbb{R})$, *with* $f_j^{(n+\widetilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *Assume that*

$$\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) < \infty,$$

and $c_{k,\xi} \in \mathbb{R}$, $k = 1, \ldots, n$. *We consider the assumptions of Theorem 3 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\begin{aligned} &\left\| (\Theta_{r,\xi}(f; x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{n} \frac{f^{\left(k+\widetilde{j}\right)}(x)}{k!} \delta_k c_{k,\xi} \right\|_{1,x} \\ &\leq \frac{1}{(n-1)!\,(r+1)} \left[\int_{-\infty}^{\infty} \left(\left(1 + \frac{|t|}{\xi} \right)^{r+1} - 1 \right) |t|^{n-1} d\mu_\xi(t) \right] \xi \qquad (22) \\ &\cdot \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_1 \right), \end{aligned}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. Based on Theorem 15 of [4]. ■

The last simultaneous approximation result follows

Proposition 12. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. *Assume*

$$\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^r d\mu_\xi(t) < \infty.$$

We consider the assumptions of Theorem 3 valid regarding f_1, f_2 *for* $n = \rho$. *Then*

$$\left\| (\Theta_{r,\xi}(f))^{(j)} - f^{(j)} \right\|_1 \leq \left[\int_{-\infty}^{\infty} \left(1 + \frac{|t|}{\xi} \right)^r d\mu_\xi(t) \right] \left(\omega_r(f_1^{(j)}, \xi)_1 + \omega_r(f_2^{(j)}, \xi)_1 \right), \tag{23}$$

for all $j = 0, 1, \ldots, \rho$.
Proof. Based on Proposition 16 of [4]. ■

III) We need

Definition 13. Let $r \in \mathbb{N}$, $\gamma > 0$. We define

$$\overline{\alpha}_j = \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-\gamma}, & j = 1, \ldots, r, \\ 1 - \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-\gamma}, & j = 0. \end{cases} \tag{24}$$

Denote

$$\overline{\delta}_k = \sum_{j=1}^{r} \alpha_j j^k, k = 1, \ldots, m-1, \tag{25}$$

where $m = \lceil \gamma \rceil$, $\lceil \cdot \rceil$ is the ceiling of a number.

In the next let $\xi > 0$, $x, x_0 \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$ Borel measurable, such that $f = f_1 + i f_2$. Suppose $f_1,\ f_2 \in C^m(\mathbb{R})$, with $\left\| f_1^{(m)} \right\|_\infty < \infty$, $\left\| f_2^{(m)} \right\|_\infty < \infty$.

Let μ_ξ probability Borel measure on $\mathbb{R}$, $\forall \xi > 0$. Consider the fractional integral

$$\begin{aligned} \overline{\Theta}_{r,\xi}(f;x) \ : &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f(x+jt) \right) d\mu_\xi(t) \qquad (26) \\ &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_1(x+jt) \right) d\mu_\xi(t) + i \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_2(x+jt) \right) d\mu_\xi(t) \\ &= \overline{\Theta}_{r,\xi}(f_1;x) + i\overline{\Theta}_{r,\xi}(f_2;x). \end{aligned}$$

We assume here that $\overline{\Theta}_{r,\xi}(f_1,x), \overline{\Theta}_{r,\xi}(f_2,x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Assume existence of $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, m-1$. Also suppose the existence of $\int_{-\infty}^{\infty} |t|^{\gamma+k} d\mu_\xi(t)$, $k = 0, 1, \ldots, r$.

Using Theorem 18 of [3], we obtain similarly, as in Theorem 1 here, the next result.

Theorem 14. It holds

$$\begin{aligned} & \left\| \overline{\Theta}_{r,\xi}(f,\cdot) - f(\cdot) - \sum_{k=1}^{m-1} \frac{f^{(k)}(\cdot)}{k!} \overline{\delta}_k c_{k,\xi} \right\|_\infty \\ & \leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma+k+1)\xi^k} \int_{-\infty}^{\infty} |t|^{\gamma+k} d\mu_\xi(t) \right] \\ & \cdot \left(\sup_{x\in\mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^{\gamma} f_1, \xi \right), \omega_r \left(D_{*x}^{\gamma} f_1, \xi \right) \right] \right\} \right. \qquad (27) \\ & \left. + \sup_{x\in\mathbb{R}} \left\{ \max \left[\omega_r \left(D_{x-}^{\gamma} f_2, \xi \right), \omega_r \left(D_{*x}^{\gamma} f_2, \xi \right) \right] \right\} \right). \end{aligned}$$

If $m = 1$ the sum disappears in L.H.S.(27).

Definition 15 (for Theorem 14)**.** Let $j = 1, 2$. Above $D_{x_0-}^{\gamma} f_j$ is the right Caputo fractional derivative of order $\gamma > 0$ is given by

$$D_{x_0-}^{\gamma} f_j(x) := \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^{x_0} (\zeta - x)^{m-\gamma-1} f_j^{(m)}(\zeta) d\zeta, \tag{28}$$

$\forall x \leq x_0 \in \mathbb{R}$ fixed.

We assume $D^{\gamma}_{x_0-} f(x) = 0, \forall x > x_0$.

Also $D^{\gamma}_{*x_0} f_j$ is the left Caputo fractional derivative of order $\gamma > 0$ is given by

$$D^{\gamma}_{*x_0} f_j(x) := \frac{1}{\Gamma(m-\gamma)} \int_{x_0}^{x} (x-t)^{m-\gamma-1} f_j^{(m)}(t) dt, \tag{29}$$

$\forall x \geq x_0 \in \mathbb{R}$ fixed, where $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \nu > 0$.

We assume $D^{\gamma}_{*x_0} f_j(x) = 0$, for $x < x_0$.

3 Convergence of Complex Trigonometric Singular Integral Operators

In this section we apply the general theory of this article to the complex-valued trigonometric smooth general singular integral operator $T_{r,\xi}(f,x)$ defined below.

We make

Remark 16. We need the following preliminary result.

Let $j, m \in \mathbb{Z}$, $m \geq 1$ such that $0 \leq j < 2m-1$. The integral

$$\int_{-\infty}^{\infty} x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \begin{cases} 2\int_0^\infty x^j \left(\frac{\sin x}{x}\right)^{2m} dx, & \text{if } j \text{ is even} \\ 0, & \text{if } j \text{ is odd} \end{cases}, \tag{30}$$

is an (absolutely) convergent integral (See also Note 17).

According to [5], page 210, item 1033, we obtain

case 1: j is even, $j < 2m-1$

$$\int_0^\infty x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \frac{\pi(-1)^{\frac{2m-j}{2}}(2m)!}{2^{j+1}(2m-j-1)!} \sum_{k=1}^{m} (-1)^k \frac{k^{2m-j-1}}{(m-k)!(m+k)!}, \tag{31}$$

and

case 2: j is odd, $j < 2m-1$

$$\int_0^\infty x^j \left(\frac{\sin x}{x}\right)^{2m} dx = \frac{(-1)^{\frac{j-1}{2}}(2m)!}{2^{j}(2m-j-1)!} \sum_{k=1}^{m} (-1)^{m-k} \frac{k^{2m-j-1}[\ln(2k)]}{(m-k)!(m+k)!}. \tag{32}$$

In particular, for $j = 0$ the formula (31) becomes

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^{2m} dx = \pi(-1)^m m \sum_{k=1}^{m} (-1)^k \frac{k^{2m-1}}{(m-k)!(m+k)!}. \tag{33}$$

We consider here complex valued Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, $i := \sqrt{-1}$.

Let $\beta \in \mathbb{N}$ and $\xi > 0$. Here we study the convergence of smooth trigonometric singular integral operators

$$T_{r,\xi}(f;x) := \frac{1}{W}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)\left(\frac{\sin(t/\xi)}{t}\right)^{2\beta}dt, \tag{34}$$

where

$$\begin{aligned} W &= \int_{-\infty}^{\infty}\left(\frac{\sin(t/\xi)}{t}\right)^{2\beta}dt \\ &= 2\xi^{1-2\beta}\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^{2\beta}dt \qquad (35)\\ &\overset{(33)}{=} 2\xi^{1-2\beta}\pi(-1)^{\beta}\beta\sum_{k=1}^{\beta}(-1)^{k}\frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!}. \end{aligned}$$

Clearly by the definition of R.H.S.(34) we have

$$T_{r,\xi}(f;x) = T_{r,\xi}(f_1;x) + iT_{r,\xi}(f_2;x). \tag{36}$$

We assume that $T_{r,\xi}(f_j;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $j = 1, 2$.

Let $\lfloor\cdot\rfloor$ denote the integer part of a real number and let

$$c_{k,\xi} := \frac{1}{W}\int_{-\infty}^{\infty}t^k\left(\frac{\sin(t/\xi)}{t}\right)^{2\beta}dt, k = 1,\ldots,n. \tag{37}$$

Note 17. As in the proof of Theorem 6 from [2], inequality (54) there, for $\beta > \frac{k+1}{2}$ we have

$$\int_{0}^{\infty}t^k\left(\frac{\sin t}{t}\right)^{2\beta}dt < \infty. \tag{38}$$

Hence

$$\begin{aligned} c_{k,\xi} &= \frac{1}{W}\xi^{k+1-2\beta}\int_{-\infty}^{\infty}t^k\left(\frac{\sin t}{t}\right)^{2\beta}dt \\ &\overset{(35)}{=} \begin{cases} \xi^k\frac{\int_0^\infty t^k\left(\frac{\sin t}{t}\right)^{2\beta}dt}{\int_0^\infty\left(\frac{\sin t}{t}\right)^{2\beta}dt}, & k = even \\ 0, & k = odd \end{cases} \\ &< \infty, \qquad (39) \end{aligned}$$

for any k such that $\beta > \frac{k+1}{2}$.

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I) Let $f_1, f_2 \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ with the rth modulus of smoothness finite, i.e. $\omega_r(f_{\tilde{j}}^{(n)}, h) < \infty, h > 0$ and let $\beta \in \mathbb{N}$, $\beta > \frac{n+1}{2}$.

One notices easily that

$$|T_{r,\xi}(f;x) - f(x)| \leq |T_{r,\xi}(f_1;x) - f_1(x)| + |T_{r,\xi}(f_2;x) - f_2(x)| \tag{40}$$

also

$$\|T_{r,\xi}(f;x) - f(x)\|_{\infty,x} \leq \|T_{r,\xi}(f_1;x) - f_1(x)\|_{\infty,x} + \|T_{r,\xi}(f_2;x) - f_2(x)\|_{\infty,x}, \tag{41}$$

and

$$\|T_{r,\xi}(f;x) - f(x)\|_{p,x} \leq \|T_{r,\xi}(f_1;x) - f_1(x)\|_{p,x} + \|T_{r,\xi}(f_2;x) - f_2(x)\|_{p,x}, p \geq 1. \tag{42}$$

By using Theorem 14 of [1] we obtain

Theorem 18. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ and $\beta \geq 1 + \left\lfloor \frac{n+r+1}{2} \right\rfloor$. Assume also $\omega_r(f_j^{(n)}, h) < \infty, \forall h > 0$. Then

$$\left\| T_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k)}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{\infty,x} \leq \frac{1}{\pi(-1)^\beta \beta \left[\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!} \right]}$$
$$\cdot \left[\int_0^\infty t^n (1+t)^r \left(\frac{\sin t}{t} \right)^{2\beta} dt \right] \left(\omega_r(f_1^{(n)}, \xi) + \omega_r(f_2^{(n)}, \xi) \right) \frac{\xi^n}{n!}. \tag{43}$$

When $n = 0, 1$ the sum in L.H.S.(43) collapses.
Proof. By Theorem 14 of [1] and (40), (41), (9), similar to the proof of Theorem 1. ∎

The $n = 0$ case follows.

Corollary 19 (to Theorem 18). *Let* $f : \mathbb{R} \to \mathbb{C} : f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C(\mathbb{R})$ and $\beta \geq 1 + \left\lfloor \frac{r+1}{2} \right\rfloor$. Assume also $\omega_r(f_j, h) < \infty, \forall h > 0$. Then

$$\|T_{r,\xi}(f) - f\|_\infty \leq \frac{\left[\int_0^\infty (1+t)^r \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]}{\pi(-1)^\beta \beta \left[\sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!(\beta+k)!} \right]} (\omega_r(f_1, \xi) + \omega_r(f_2, \xi)). \tag{44}$$

In [4] we mentioned

Theorem 20 ([4, Theorem 22 there])**.** *Let* $g \in C^{n-1}(\mathbb{R})$*, such that* $g^{(n)}$ *exists,* $n, r \in \mathbb{N}$*. Furthermore suppose that for each* $x \in \mathbb{R}$ *the function* $g^{(\tilde{j})}(x +$

$jt) \in L_1\left(\mathbb{R}, \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt\right)$ as a function of t, for all $\widetilde{j} = 0, 1, \ldots, n-1$; $j = 1, \ldots, r$. Suppose that there exist $\lambda_{\widetilde{j},j} \geq 0$, $\widetilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, with $\lambda_{\widetilde{j},j} \in L_1\left(\mathbb{R}, \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt\right)$ such that for each $x \in \mathbb{R}$ we have

$$|g^{(\widetilde{j})}(x + jt)| \leq \lambda_{\widetilde{j},j}(t), \tag{45}$$

for almost every $t \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, 2, \ldots, r$. Then $g^{(\widetilde{j})}(x + jt)$ defines a Lebesgue integrable function with respect to t for each $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$; $j = 1, \ldots, r$, and

$$(T_{r,\xi}(f; x))^{(\widetilde{j})} = T_{r,\xi}\left(f^{(\widetilde{j})}; x\right), \tag{46}$$

for all $x \in \mathbb{R}$, all $\widetilde{j} = 1, \ldots, n$.

We present the following simultaneous approximation result.

Theorem 21. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^{n+\rho}(\mathbb{R})$, $n, \rho \in \mathbb{Z}^+$, $\beta \in \mathbb{N}$, $\beta \geq 1 + \left\lfloor \frac{r+n+1}{2} \right\rfloor$ and $\omega_r(f_j^{(n+\widetilde{j})}, h) < \infty, \forall h > 0$, for $\widetilde{j} = 0, 1, \ldots, \rho$. We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\left\| (T_{r,\xi}(f; x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{\left(2k+\widetilde{j}\right)}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{\infty,x}$$

$$\leq \frac{\xi^n}{n!} \frac{\left[\int_0^\infty t^n (1+t)^r \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]} \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi) + \omega_r(f_2^{(n+\widetilde{j})}, \xi)\right), \tag{47}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$. When $n = 0, 1$ the sum in L.H.S.(47) collapses.

Proof. Similar to Theorem 4 here, and based on Theorem 27 of [4]. ■

II) Here let $f_1, f_2 \in C^n(\mathbb{R})$ with $f_1^{(n)}, f_2^{(n)} \in L_p(\mathbb{R})$, $1 \leq p < \infty$. We assume that $\omega_r(f_{\widetilde{j}}^{(n)}, h)_p < \infty$, $h > 0$, $\widetilde{j} = 1, 2$.

We present

Theorem 22. *Let $f : \mathbb{R} \to \mathbb{C}$* such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_p(\mathbb{R})$, $n \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \beta \in \mathbb{N}, \beta > \frac{\lceil rp \rceil + np + 1}{2}$. Then

$$\left\| T_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k)}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{p,x}$$

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$$\leq \left[\frac{1}{\left(\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right)} \sum_{h=1}^{\lceil rp \rceil + 1} \int_0^\infty t^{np-1+h} \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]^{\frac{1}{p}} \tag{48}$$

$$\cdot \frac{\xi^n}{((n-1)!)(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(\omega_r(f_1^{(n)}, \xi)_p + \omega_r(f_2^{(n)}, \xi)_p \right).$$

When $n = 0, 1$ the sum in L.H.S.(48) collapses.
Proof. By Theorem 6 of [2] and (40), (42), (9), similar to Theorem 5 here. ■

Based on Proposition 8 of [2], similarly we give

Proposition 23. *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \beta \in \mathbb{N}, \beta > \frac{\lceil rp \rceil + 1}{2}$.Then*

$$\|T_{r,\xi}(f) - f\|_p \leq \left[\frac{1}{\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt} \sum_{h=0}^{\lceil rp \rceil} \int_0^\infty t^h \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]^{\frac{1}{p}} (\omega_r(f_1, \xi)_p + \omega_r(f_2, \xi)_p). \tag{49}$$

Based on Theorem 7 of [2] we get similarly

Theorem 24. *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^n(\mathbb{R})$ with $f_j^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}, \beta \in \mathbb{N}, \beta > \frac{r+1+n}{2}$. Then*

$$\left\| T_{r,\xi}(f; x) - f(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k)}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{1,x} \tag{50}$$

$$\leq \frac{\xi^n \left(\omega_r(f_1^{(n)}, \xi)_1 + \omega_r(f_2^{(n)}, \xi)_1 \right)}{(n-1)!\,(r+1) \left[\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \sum_{h=1}^{r+1} \left[\int_0^\infty t^{n-1+h} \left(\frac{\sin t}{t} \right)^{2\beta} dt \right].$$

When $n = 0, 1$ the sum in L.H.S.(50) collapses.

Based on Proposition 9 of [2], we give similarly

Proposition 25. *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_1(\mathbb{R})), \beta \in \mathbb{N}, \beta > \frac{r+1}{2}$. Then*

$$\|T_{r,\xi}(f) - f\|_1 \leq \frac{(\omega_r(f_1, \xi)_1 + \omega_r(f_2, \xi)_1)}{\left[\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \sum_{h=0}^{r} \left[\int_0^\infty t^h \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]. \tag{51}$$

Next we give simultaneous approximation results.

Theorem 26. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $j = 1,2$. Let $f_j \in C^{n+\rho}(\mathbb{R})$, $n \in \mathbb{N}, \rho \in \mathbb{Z}^+$, with $f_j^{(n+\widetilde{j})} \in L_p(\mathbb{R})$, $\widetilde{j} = 0,1,\ldots,\rho$. Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\beta \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + np + 1}{2}$. We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\left\| (T_{r,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k+\widetilde{j})}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{p,x} \leq \frac{1}{((n-1)!)}$$
$$\cdot \frac{1}{(q(n-1)+1)^{\frac{1}{q}} (rp+1)^{\frac{1}{p}}} \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_p + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_p \right)$$
$$\cdot \left[\frac{1}{\left[\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \sum_{h=1}^{\lceil rp \rceil + 1} \int_0^\infty t^{np-1+h} \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]^{\frac{1}{p}} \xi^n, \tag{52}$$

for all $\widetilde{j} = 0,1,\ldots,\rho$. When $n = 0,1$ the sum in L.H.S.(52) collapses.

Proof. By Theorem 30 of [4], and as in the proof of Theorem 5 here. ∎

We give the related

Proposition 27. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Let $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_p(\mathbb{R}))$, $\widetilde{j} = 0,1,\ldots,\rho \in \mathbb{Z}^+$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\beta \in \mathbb{N}$, $\beta > \frac{\lceil rp \rceil + 1}{2}$. We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\left\| (T_{r,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})} \right\|_p \leq \left[\frac{\sum_{h=0}^{\lceil rp \rceil} \int_0^\infty t^h \left(\frac{\sin t}{t} \right)^{2\beta} dt}{\left[\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \right]^{\frac{1}{p}} \left(\omega_r(f_1^{(\widetilde{j})}, \xi)_p + \omega_r(f_2^{(\widetilde{j})}, \xi)_p \right), \tag{53}$$

$\widetilde{j} = 0,1,\ldots,\rho$.

Proof. By Proposition 31 of [4]. ∎

Next comes the related L_1 result.

Theorem 28. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$,* and *$j = 1,2$. Let $f_1, f_2 \in C^{n+\rho}(\mathbb{R})$, with $f_j^{(n+\widetilde{j})} \in L_1(\mathbb{R})$, $n \in \mathbb{N}$, $\widetilde{j} = 0,1,\ldots,\rho \in \mathbb{Z}^+$, $\beta \in \mathbb{N}$, $\beta > \frac{r+n+1}{2}$. We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\left\| (T_{r,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{f^{(2k+\widetilde{j})}(x)}{(2k)!} \delta_{2k} c_{2k,\xi} \right\|_{1,x}$$
$$\leq \frac{1}{(n-1)!\,(r+1) \left[\int_0^\infty \left(\frac{\sin t}{t} \right)^{2\beta} dt \right]} \left[\sum_{h=1}^{r+1} \int_0^\infty t^{n-1+h} \left(\frac{\sin t}{t} \right)^{2\beta} dt \right] \xi^n \tag{54}$$
$$\cdot \left(\omega_r(f_1^{(n+\widetilde{j})}, \xi)_1 + \omega_r(f_2^{(n+\widetilde{j})}, \xi)_1 \right),$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$. *When* $n = 0, 1$ *the sum in L.H.S.(54) collapses.*
Proof. Based on Theorem 34 of [4]. ■

The last simultaneous approximation result follows

Proposition 29. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$, $\beta \in \mathbb{N}$, $\beta > \frac{r+1}{2}$. *We consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\| (T_{r,\xi}(f))^{(j)} - f^{(j)} \right\|_1 \leq \frac{1}{\left[\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt\right]} \left[\sum_{h=0}^{r} \int_0^\infty t^h \left(\frac{\sin t}{t}\right)^{2\beta} dt\right] \tag{55}$$
$$\cdot \left(\omega_r(f_1^{(j)}, \xi)_1 + \omega_r(f_2^{(j)}, \xi)_1\right),$$

for all $j = 0, 1, \ldots, \rho$.
Proof. Based on Proposition 35 of [4]. ■

III) In the next let $\xi > 0$, $x, x_0 \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{C}$ Borel measurable, such that $f = f_1 + if_2$. Suppose $f_1, f_2 \in C^m(\mathbb{R})$, with $\left\| f_1^{(m)} \right\|_\infty < \infty$, $\left\| f_2^{(m)} \right\|_\infty < \infty$.

Consider the fractional integral

$$\begin{aligned} \overline{T}_{r,\xi}(f; x) \; : \; &= \frac{1}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f(x + jt)\right) \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt \qquad (56) \\ &= \frac{1}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_1(x + jt)\right) \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt \\ &\quad + \frac{i}{W} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \overline{\alpha}_j f_2(x + jt)\right) \left(\frac{\sin(t/\xi)}{t}\right)^{2\beta} dt \\ &= \overline{T}_{r,\xi}(f_1; x) + i\overline{T}_{r,\xi}(f_2; x). \end{aligned}$$

We assume here that $\overline{T}_{r,\xi}(f_1, x), \overline{T}_{r,\xi}(f_2, x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$.

Let $\beta \in \mathbb{N}$, $\beta > \frac{\gamma + r + 1}{2}$, $\gamma > 0$, $r \in \mathbb{N}$.

Using Theorem 23 of [3], we obtain similarly, as in Theorem 1 here, the next result.

Theorem 30. It holds

$$\left\| \overline{T}_{r,\xi}(f, \cdot) - f(\cdot) - \sum_{k=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \frac{f^{(2k)}(\cdot)}{(2k)!} \overline{\delta}_{2k} c_{2k,\xi} \right\|_\infty$$
$$\leq \left[\sum_{k=0}^{r} \frac{r!}{(r-k)!\Gamma(\gamma + k + 1)} \frac{\int_0^\infty t^{\gamma+k} \left(\frac{\sin t}{t}\right)^{2\beta} dt}{\int_0^\infty \left(\frac{\sin t}{t}\right)^{2\beta} dt}\right]$$

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$$\cdot\xi^{\gamma}\left(\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_1,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_1,\xi\right)\right]\right\}\right. \tag{57}$$
$$\left.+\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_r\left(D_{x-}^{\gamma}f_2,\xi\right),\omega_r\left(D_{*x}^{\gamma}f_2,\xi\right)\right]\right\}\right).$$

If $m=1,2$ the sum disappears in L.H.S.(57).

4 Applications to Particular Complex Trigonometric Singular Operators

In this section we work on the approximation results given in the previous section, for some particular values of r, n, p and β.

Case $\beta=2$. We have the following results

Theorem 31. *Let $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^1(\mathbb{R})$. Assume $\omega_r(f_j',h)<\infty,\forall h>0$. Then*

$$\|T_{1,\xi}(f)-f\|_{\infty}\leq\frac{3\xi}{\pi}\left[\ln 2+\frac{\pi}{4}\right]\left(\omega_1(f_1',\xi)+\omega_1(f_2',\xi)\right). \tag{58}$$

Proof. By Theorem 18, with $n=r=1$, $\beta=2$. ■

Corollary 32. *Let $f:\mathbb{R}\to\mathbb{C}$* such that $f=f_1+if_2$, where $f_1,f_2\in(C(\mathbb{R})\cap L_1(\mathbb{R}))$. *Then*

$$\|T_{1,\xi}(f)-f\|_1\leq(\omega_1(f_1,\xi)_1+\omega_1(f_2,\xi)_1)\left(\frac{3\ln 2}{\pi}+1\right). \tag{59}$$

Proof. By Proposition 25, with $r=1,\beta=2$. ■

Corollary 33. *Let $f:\mathbb{R}\to\mathbb{C}$* such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^1(\mathbb{R})$ with $f_j'\in L_1(\mathbb{R})$. *Then*

$$\|T_{1,\xi}(f;x)-f(x)\|_1\leq\frac{3}{2\pi}\left(\ln 2+\frac{\pi}{4}\right)\xi\left(\omega_1(f_1',\xi)_1+\omega_1(f_2',\xi)_1\right). \tag{60}$$

Proof. By Theorem 24, with $r=n=1,\beta=2$. ■

Corollary 34. *Let $f:\mathbb{R}\to\mathbb{C}$, such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^{1+\rho}(\mathbb{R}),\rho\in\mathbb{Z}^+$ and $\omega_1(f_j^{(1+\widetilde{j})},h)<\infty,\forall h>0$, for $\widetilde{j}=0,1,\ldots,\rho$. We consider the assumptions of Theorem 20 valid regarding f_1,f_2 for $n=\rho$. Then*

$$\left\|(T_{1,\xi}(f;x))^{(\widetilde{j})}-f^{(\widetilde{j})}(x)\right\|_{\infty,x}\leq\frac{3\xi}{\pi}\left(\ln 2+\frac{\pi}{4}\right)\left(\omega_1(f_1^{(1+\widetilde{j})},\xi)+\omega_1(f_2^{(1+\widetilde{j})},\xi)\right), \tag{61}$$

for all $\widetilde{j}=0,1,\ldots,\rho$.

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Proof. We are applying Theorem 21 here for $n = r = 1$, $\beta = 2$. ■

Corollary 35. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$, and $j = 1, 2$. Let $f_1, f_2 \in C^{1+\rho}(\mathbb{R})$ and $f^{(\widetilde{j}+1)} \in L_1(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$. We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{1,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x)\|_1 \le \frac{3\xi}{2\pi}\left(\ln 2 + \frac{\pi}{4}\right)\left(\omega_1(f_1^{(1+\widetilde{j})},\xi)_1 + \omega_1(f_2^{(1+\widetilde{j})},\xi)_1\right), \tag{62}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. We are applying Theorem 28 here for $n = r = 1$, $\beta = 2$. ■

Corollary 36. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Let $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_2(\mathbb{R}))$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{1,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})}\|_2 \le \left(\omega_1(f_1^{(\widetilde{j})},\xi)_2 + \omega_1(f_2^{(\widetilde{j})},\xi)_2\right)\sqrt{\frac{7}{4} + \frac{3}{\pi}\ln 2}. \tag{63}$$

for all $\widetilde{j} = 0, 1, \ldots \rho$.
Proof. By Proposition 27, with $p = 2, r = 1, \beta = 2$. ■

Corollary 37. *Let $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_2(\mathbb{R}))$. Then*

$$\|T_{1,\xi}(f) - f\|_2 \le (\omega_1(f_1,\xi)_2 + \omega_1(f_2,\xi)_2)\sqrt{\frac{7}{4} + \frac{3}{\pi}\ln 2}. \tag{64}$$

Proof. By Proposition 23, with $p = 2, r = 1, \beta = 2$. ■

Corollary 38. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{1,\xi}(f))^{(j)} - f^{(j)}\|_1 \le \left(\omega_1(f_1^{(j)},\xi)_1 + \omega_1(f_2^{(j)},\xi)_1\right)\left(\frac{3\ln 2}{\pi} + 1\right), \tag{65}$$

for all $j = 0, 1, \ldots \rho$.
Proof. By Proposition 29, with $r = 1, \beta = 2$. ■

Corollary 39. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{2,\xi}(f))^{(j)} - f^{(j)}\|_1 \le \left(\omega_2(f_1^{(j)},\xi)_1 + \omega_2(f_2^{(j)},\xi)_1\right)\left(\frac{7}{4} + \frac{3}{\pi}\ln 2\right), \tag{66}$$

for all $j = 0, 1, \ldots \rho$.

Proof. By Proposition 29, with $r = \beta = 2$. ■

Corollary 40. *Consider the assumptions of Theorem 30 are fulfilled. Then*

$$\left\|\overline{T}_{1,\xi}(f,\cdot) - f(\cdot)\right\|_{\infty} \leq \left(\frac{56\sqrt{2}-16}{15\pi}\right)\xi^{\frac{1}{2}} \cdot \left(\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_1\left(D_{x-}^{\frac{1}{2}}f_1,\xi\right),\omega_1\left(D_{*x}^{\frac{1}{2}}f_1,\xi\right)\right]\right\} + \sup_{x\in\mathbb{R}}\left\{\max\left[\omega_1\left(D_{x-}^{\frac{1}{2}}f_2,\xi\right),\omega_1\left(D_{*x}^{\frac{1}{2}}f_2,\xi\right)\right]\right\}\right). \tag{67}$$

Proof. By Theorem 30, with $\beta = 2, \gamma = \frac{1}{2}, r = 1$. ■

Case $\beta = 3$. We have the following results

Corollary 41. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$. *Then*

$$\|T_{2,\xi}(f) - f\|_1 \leq (\omega_2(f_1,\xi)_1 + \omega_2(f_2,\xi)_1)\left(\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{16}{11}\right). \tag{68}$$

Proof. By Proposition 25, with $r = 2, \beta = 3$. ■

Corollary 42. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$, where $f_1, f_2 \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$. *Then*

$$\|T_{3,\xi}(f) - f\|_1 \leq (\omega_3(f_1,\xi)_1 + \omega_3(f_2,\xi)_1)\left(\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{16}{11} + \frac{15}{22\pi}\ln\frac{256}{27}\right). \tag{69}$$

Proof. By Proposition 25, with $r = 3, \beta = 3$. ■

Corollary 43. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^1(\mathbb{R})$ with $f_j' \in L_2(\mathbb{R})$. *Then*

$$\|T_{1,\xi}(f;x) - f(x)\|_2 \leq \left(\sqrt{\frac{5}{22\pi}\ln\frac{256}{27} + \frac{25}{66}}\right)\xi\left(\omega_1(f_1',\xi)_2 + \omega_1(f_2',\xi)_2\right). \tag{70}$$

Proof. By Theorem 22, with $p = 2, r = n = 1, \beta = 3$. ■

Corollary 44. *Let* $f : \mathbb{R} \to \mathbb{C}$ *such that* $f = f_1 + if_2$, *where* $f_1, f_2 \in (C(\mathbb{R}) \cap L_2(\mathbb{R}))$. *Then*

$$\|T_{2,\xi}(f) - f\|_2 \leq (\omega_2(f_1,\xi)_2 + \omega_2(f_2,\xi)_2)\sqrt{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{15}{22\pi}\ln\frac{256}{27} + \frac{47}{22}}. \tag{71}$$

Proof. By Proposition 23, with $p = r = 2, \beta = 3$. ■

Corollary 45. *Let* $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^1(\mathbb{R})$ with $f_j' \in L_1(\mathbb{R})$. *Then*

$$\|T_{1,\xi}(f;x) - f(x)\|_1 \le \left(\frac{20}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right) + \frac{5}{22}\right)\xi\left(\omega_1(f_1', \xi)_1 + \omega_1(f_2', \xi)_1\right). \quad (72)$$

Proof. By Theorem 24, with $r = n = 1, \beta = 3$. ■

Corollary 46. $f : \mathbb{R} \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^2(\mathbb{R})$ with $f_j'' \in L_1(\mathbb{R})$. *Then*

$$\|T_{2,\xi}(f;x) - f(x) - \frac{f''(x)}{2}\delta_2 c_{2,\xi}\|_1 \le \left(\frac{25}{66} + \frac{5}{22\pi}\ln\frac{256}{27}\right)\xi^2\left(\omega_2(f_1'', \xi)_1 + \omega_2(f_2'', \xi)_1\right). \quad (73)$$

Proof. By Theorem 24, with $r = n = 2, \beta = 3$. ■

Corollary 47. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{1+\rho}(\mathbb{R})$, $\rho \in \mathbb{Z}^+$. *Assume also* $\omega_1(f_j^{(1+\widetilde{j})}, h) < \infty, \forall h > 0$, *for* $\widetilde{j} = 0, 1, \ldots, \rho$. *We consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\|(T_{1,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x)\right\|_{\infty,x} \le \frac{5}{22\pi}\left[2\pi - 32\ln 2 + 27\ln 3\right] \cdot \left(\omega_1(f_1^{(1+\widetilde{j})}, \xi) + \omega_1(f_2^{(1+\widetilde{j})}, \xi)\right)\xi, \quad (74)$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. We are applying Theorem 21 here for $n = r = 1$, $\beta = 3$. ■

Corollary 48. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{\rho}(\mathbb{R})$, $\rho \in \mathbb{Z}^+$. *Assume also* $\omega_3(f_j^{(\widetilde{j})}, h) < \infty, \forall h > 0$, *for* $\widetilde{j} = 0, 1, \ldots, \rho$. W*e consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\|(T_{1,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x)\right\|_{\infty,x} \le \frac{2}{11\pi}\left[13\pi - 90\ln 2 + 90\ln 3\right] \cdot \left(\omega_3(f_1^{(\widetilde{j})}, \xi) + \omega_3(f_2^{(\widetilde{j})}, \xi)\right), \quad (75)$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. We are applying Theorem 21 here for $n = 0, r = 3$, $\beta = 3$. ■

Corollary 49. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{1+\rho}(\mathbb{R})$, *with* $f_j^{(1+\widetilde{j})} \in L_2(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots \rho \in \mathbb{Z}^+$. *We consider the* *assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n=\rho$. *Then*

$$\|\left(T_{1,\xi}(f;x)\right)^{(\widetilde{j})}-f^{(\widetilde{j})}(x)\|_2 \leq \left(\sqrt{\frac{5}{22\pi}\ln\frac{256}{27}+\frac{25}{66}}\right) \cdot\xi\left(\omega_1(f_1^{(1+\widetilde{j})},\xi)_2+\omega_1(f_2^{(1+\widetilde{j})},\xi)_2\right), \tag{76}$$

for all $\widetilde{j}=0,1,\ldots,\rho$.

Proof. By Theorem 26, with $p=2, r=n=1, \beta=3$. ■

Corollary 50. *Let* $f:\mathbb{R}\to\mathbb{C}$, *such that* $f=f_1+if_2$, and $j=1,2$. *Let* $f_1,f_2\in C^{2+\rho}(\mathbb{R})$, *with* $f_j^{(2+\widetilde{j})}\in L_1(\mathbb{R})$, $\widetilde{j}=0,1,\ldots\rho\in\mathbb{Z}^+$. *We consider the assumptions of Theorem 20 valid regarding* f_1,f_2 *for* $n=\rho$. *Then*

$$\|\left(T_{2,\xi}(f;x)\right)^{(\widetilde{j})}-f^{(\widetilde{j})}(x)+\frac{5}{22}\xi^2 f^{(\widetilde{j}+2)}(x)\|_1 \leq \left(\frac{25}{66}+\frac{5}{22\pi}\ln\frac{256}{27}\right) \cdot\xi^2\left(\omega_2(f_1^{(2+\widetilde{j})},\xi)_1+\omega_2(f_2^{(2+\widetilde{j})},\xi)_1\right), \tag{77}$$

for all $\widetilde{j}=0,1,\ldots,\rho$.

Proof. By Theorem 28, with $r=n=2, \beta=3$. ■

Corollary 51. *Let* $f:\mathbb{R}\to\mathbb{C}$, *such that* $f=f_1+if_2$. *Let* $f_1^{(\widetilde{j})},f_2^{(\widetilde{j})}\in\left(C(\mathbb{R})\cap L_2(\mathbb{R})\right)$, $\widetilde{j}=0,1,\ldots,\rho\in\mathbb{Z}^+$; *We consider the assumptions of Theorem 20 valid regarding* f_1,f_2 *for* $n=\rho$. *Then*

$$\|\left(T_{2,\xi}(f)\right)^{(\widetilde{j})}-f^{(\widetilde{j})}\|_2 \leq \left(\omega_2(f_1^{(\widetilde{j})},\xi)_2+\omega_2(f_2^{(\widetilde{j})},\xi)_2\right) \cdot\sqrt{\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right)+\frac{15}{22\pi}\ln\frac{256}{27}+\frac{47}{22}}. \tag{78}$$

for all $\widetilde{j}=0,1,\ldots,\rho$.

Proof. By Proposition 27, with $p=r=2, \beta=3$. ■

Corollary 52. *Let* $f:\mathbb{R}\to\mathbb{C}$, *such that* $f=f_1+if_2$. *Here* $f_1^{(j)},f_2^{(j)}\in\left(C(\mathbb{R})\cap L_1(\mathbb{R})\right)$, $j=0,1,\ldots,\rho\in\mathbb{Z}^+$; *We consider the assumptions of Theorem 20 valid regarding* f_1,f_2 *for* $n=\rho$. *Then*

$$\|\left(T_{1,\xi}(f)\right)^{(j)}-f^{(j)}\|_1 \leq \left(\omega_1(f_1^{(j)},\xi)_1+\omega_1(f_2^{(j)},\xi)_1\right)\left(1+\frac{40}{11\pi}\ln\left(\frac{3^{\frac{27}{16}}}{4}\right)\right), \tag{79}$$

for all $j=0,1,\ldots,\rho$.

Proof. By Proposition 29, with $r=1, \beta=3$. ■

Corollary 53. *Consider the assumptions of Theorem 30 are fulfilled. Then*

$$\left\|\overline{T}_{2,\xi}(f,\cdot)-f(\cdot)\right\|_{\infty}\leq\frac{64}{1155\pi}\left(-95+252\sqrt{2}-127\sqrt{3}\right)\cdot\xi^{\frac{3}{2}}\left(\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_2\left(D_{x-}^{\frac{3}{2}}f_1,\xi\right),\omega_2\left(D_{*x}^{\frac{3}{2}}f_1,\xi\right)\right]\right\}+\sup_{x\in\mathbb{R}}\left\{\max\left[\omega_2\left(D_{x-}^{\frac{3}{2}}f_2,\xi\right),\omega_2\left(D_{*x}^{\frac{3}{2}}f_2,\xi\right)\right]\right\}\right).\tag{80}$$

Proof. By Theorem 30, with $\beta=3,\gamma=\frac{3}{2},r=2.$ ■

Case $\beta=4$. We have the following results

Corollary 54. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$, where $f_1,f_2\in(C(\mathbb{R})\cap L_1(\mathbb{R}))$. *Then*

$$\|T_{1,\xi}(f)-f\|_1\leq(\omega_1(f_1,\xi)_1+\omega_1(f_2,\xi)_1)\left(1+\frac{630}{151\pi}\ln\frac{2^{104/15}}{3^{81/20}}\right).\tag{81}$$

Proof. By Proposition 25, with $r=1,\beta=4.$ ■

Corollary 55. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^1(\mathbb{R})$ with $f_j'\in L_2(\mathbb{R})$. *Then*

$$\|T_{1,\xi}(f;x)-f(x)\|_2\leq\left(\sqrt{\frac{35}{151}+\frac{210}{151\pi}\ln\left(\frac{3^{27/8}}{32}\right)}\right)\xi\left(\omega_1(f_1',\xi)_2+\omega_1(f_2',\xi)_2\right).\tag{82}$$

Proof. By Theorem 22, with $p=2,r=n=1,\beta=4.$ ■

Corollary 56. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$, where $f_1,f_2\in(C(\mathbb{R})\cap L_2(\mathbb{R}))$. *Then*

$$\|T_{2,\xi}(f)-f\|_2\leq(\omega_2(f_1,\xi)_2+\omega_2(f_2,\xi)_2)\sqrt{\frac{630}{151\pi}\ln\frac{2^{29/15}}{3^{27/40}}+\frac{256}{151}}.\tag{83}$$

Proof. By Proposition 23, with $p=r=2,\beta=4.$ ■

Corollary 57. *Let* $f:\mathbb{R}\to\mathbb{C}$ such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^1(\mathbb{R})$ with $f_j'\in L_1(\mathbb{R})$. *Then*

$$\|T_{5,\xi}(f;x)-f(x)\|_1\leq\left(\frac{105}{151\pi}\ln\frac{2^{29/15}}{3^{27/40}}+\frac{945}{1208\pi}\ln\frac{4}{3}+\frac{1085}{4832}\right)\cdot\xi\left(\omega_5(f_1',\xi)_1+\omega_5(f_2',\xi)_1\right).\tag{84}$$

Proof. By Theorem 24, with $r=5,n=1,\beta=4.$ ■

Corollary 58. *Let* $f:\mathbb{R}\to\mathbb{C}$, *such that* $f=f_1+if_2$. *Here* $j=1,2$. *Let* $f_j\in C^{2+\rho}(\mathbb{R})$, $\rho\in\mathbb{Z}^+$. *Assume also* $\omega_1(f_j^{(2+\widetilde{j})},h)<\infty,\forall h>0$, *for* $\widetilde{j}=0,1,\ldots\rho$.

W*e consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\| (T_{1,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \frac{105}{604}\xi^2 f^{(2+\widetilde{j})}(x) \right\|_{\infty,x} \leq [2\pi - 120\ln 2 + 81\ln 3] \cdot \frac{105}{1208\pi}\left(\omega_1(f_1^{(2+\widetilde{j})},\xi) + \omega_1(f_2^{(2+\widetilde{j})},\xi)\right)\xi^2, \tag{85}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. We are applying Theorem 21 here for $n = 2$, $r = 1$, $\beta = 4$. ■

Corollary 59. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Here* $j = 1, 2$. *Let* $f_j \in C^{1+\rho}(\mathbb{R})$, *with* $f_j^{(1+\widetilde{j})} \in L_2(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\| (T_{1,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x)\|_2 \leq \left(\sqrt{\frac{35}{151} + \frac{210}{151\pi}\ln\left(\frac{3^{27/8}}{32}\right)}\right) \cdot \xi\left(\omega_1(f_1^{(1+\widetilde{j})},\xi)_2 + \omega_1(f_2^{(1+\widetilde{j})},\xi)_2\right), \tag{86}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. By Theorem 26, with $p = 2, r = n = 1, \beta = 4$. ■

Corollary 60. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$, and $j = 1, 2$. *Let* $f_1, f_2 \in C^{3+\rho}(\mathbb{R})$, *with* $f_j^{(3+\widetilde{j})} \in L_1(\mathbb{R})$, $\widetilde{j} = 0, 1, \ldots \rho \in \mathbb{Z}^+$. *We consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\left\| (T_{3,\xi}(f;x))^{(\widetilde{j})} - f^{(\widetilde{j})}(x) - \frac{385}{1208}\xi^2 f^{(\widetilde{j}+2)}(x) \right\|_1 \leq \left(\frac{315}{604\pi}\ln\frac{3^{9/4}}{2^{11/4}} + \frac{2415}{19328}\right) \cdot \xi^3\left(\omega_3(f_1^{(3+\widetilde{j})},\xi)_1 + \omega_3(f_2^{(3+\widetilde{j})},\xi)_1\right), \tag{87}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. By Theorem 28, with $r = n = 3, \beta = 4$. ■

Corollary 61. *Let* $f : \mathbb{R} \to \mathbb{C}$, *such that* $f = f_1 + if_2$. *Let* $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_2(\mathbb{R}))$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; *We consider the assumptions of Theorem 20 valid regarding* f_1, f_2 *for* $n = \rho$. *Then*

$$\| (T_{2,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})}\|_2 \leq \left(\omega_2(f_1^{(\widetilde{j})},\xi)_2 + \omega_2(f_2^{(\widetilde{j})},\xi)_2\right)\sqrt{\frac{630}{151\pi}\ln\frac{2^{29/15}}{3^{27/40}} + \frac{256}{151}}. \tag{88}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.
Proof. By Proposition 27, with $p = r = 2, \beta = 4$. ■

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Corollary 62. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Let $f_1^{(\widetilde{j})}, f_2^{(\widetilde{j})} \in (C(\mathbb{R}) \cap L_2(\mathbb{R}))$, $\widetilde{j} = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{1,\xi}(f))^{(\widetilde{j})} - f^{(\widetilde{j})} \|_2 \leq \left(\omega_1(f_1^{(\widetilde{j})}, \xi)_2 + \omega_1(f_2^{(\widetilde{j})}, \xi)_2 \right) \sqrt{\frac{630}{151\pi} \ln \frac{2^{104/15}}{3^{81/20}} + \frac{407}{302}}. \tag{89}$$

for all $\widetilde{j} = 0, 1, \ldots, \rho$.

Proof. By Proposition 27, with $p = 2, r = 1, \beta = 4$. ■

Corollary 63. *Let $f : \mathbb{R} \to \mathbb{C}$, such that $f = f_1 + if_2$. Here $f_1^{(j)}, f_2^{(j)} \in (C(\mathbb{R}) \cap L_1(\mathbb{R}))$, $j = 0, 1, \ldots, \rho \in \mathbb{Z}^+$; We consider the assumptions of Theorem 20 valid regarding f_1, f_2 for $n = \rho$. Then*

$$\| (T_{1,\xi}(f))^{(j)} - f^{(j)} \|_1 \leq \left(\omega_1(f_1^{(j)}, \xi)_1 + \omega_1(f_2^{(j)}, \xi)_1 \right) \left(1 + \frac{630}{151\pi} \ln \frac{2^{104/15}}{3^{81/20}} \right), \tag{90}$$

for all $j = 0, 1, \ldots, \rho$.

Proof. By Proposition 29, with $r = 1, \beta = 4$. ■

Corollary 64. *Consider the assumptions of Theorem 30 are fulfilled. Then*

$$\begin{aligned} \left\| \overline{T}_{2,\xi}(f, \cdot) - f(\cdot) \right\|_\infty \leq & \frac{1024}{971685\pi} \left(-10011 - 15848\sqrt{2} + 21303\sqrt{3} \right) \xi^{\frac{1}{2}} \\ & \cdot \left(\sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_2 \left(D_{x-}^{\frac{1}{2}} f_1, \xi \right), \omega_2 \left(D_{*x}^{\frac{1}{2}} f_1, \xi \right) \right] \right\} \right. \\ & \left. + \sup_{x \in \mathbb{R}} \left\{ \max \left[\omega_2 \left(D_{x-}^{\frac{1}{2}} f_2, \xi \right), \omega_2 \left(D_{*x}^{\frac{1}{2}} f_2, \xi \right) \right] \right\} \right). \end{aligned} \tag{91}$$

Proof. By Theorem 30, with $\beta = 4, \gamma = \frac{1}{2}, r = 2$. ■

Note 65. *All the approximation results of this article lead to interesting and useful convergence conclusions regarding our general and trigonometric operators as approximators to the unit operator.*

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On the MAC solution for a circular elastic plate

Igor Neygebauer
University of Dodoma, Tanzania
Department of Mathematics and Statistics
Email:newigor52r@yahoo.com

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Abstract

The method of additional conditions or MAC is applied to the boundary value problems of mathematical physics, where the classical solution does not exist or a nonphysical generalized solution is obtained. The circular elastic plate under a transversal force at the center of the plate is considered. The MAC solution is obtained using MAC transformation for Laplace equation.
KEY WORDS: Laplace and biharmonic equations, membrane, plate, MAC solution.

1 Introduction

Let us consider an isotropic elastic body. If one point of a body is given a small displacement which could present a displacement of one atom then the reaction at that point should be a force in general case and it is not a stress. That follows immediately from the balance of forces acting on the body. It seems that we have the small displacements and also small deformations in the whole body and the classical linear elasticity theory should be applied. But it is well known that an elastic body under applied force has a infinite displacement of the point of application of the force (1). So we obtain an unsolvable problem in the theory of linear elasticity. It could be possible to create a MAC solution of the given problem but it is not the goal of this paper. The method how to introduce the MAC solution could be understandable from what follows.
Some classical boundary value problems from elasticity could be considered. Sometimes it is easy to obtain the general solution of the differential equations of the problems. But we could meet difficulties to satisfy the prescribed boundary conditions. We will see that the solutions of some problems do not exist. In this case it is possible to create a generalized solution, using a limit of existing solutions. These solutions can be easily verified in experiments. If the experiment shows, that the physical solution exists and differs from the obtained generalized solution then the MAC or the method of additional conditions can be applied.

This method allows to transform the obtained generalized solution to the physically acceptable form. Examples of MAC solutions are solution in scheme of Dugdale-Barrenblat (5) in fracture mechanics. This scheme was applied to the linear elastic crack problem. The linear elastic solution has singularity near the tip of a crack. To avoid this singularity Dugdale and Barrenblat introduced additional yield stresses near the tip. The applied nonsingular condition gave the size of the zone, there the stresses are applied. This scheme was developed in (4), where the second additional condition of zero J-integral was introduced. This new condition gave the value of the applied additional stresses, which are 6 times more then the given stresses at infinity. This stress concentration factor corresponds to the experiments of Griffith's and Inglis (4). The usual strength criteria, which are used in the elastic stress fields without singularities, can be used here.
Sen-Venan method in elasticity could be considered as another example of MAC. Let us apply the method of additional conditions (MAC) to consider the displacements of an elastic plate under a transversal force. The equations and boundary conditions of the plates with some solved problems can be found in (2), (6), (7). Our goal is to present the MAC in this problem.

2 Circular plate

2.1 Nonexistent solution

Consider the bending of a circular elastic plate. The differential equation of the problem is

$$\Delta\Delta u = 0, \tag{1}$$

where u is the transversal displacements of the points of a circular plate, Δ is the Laplace operator. We consider an axis symmetric case, then the equation (1) will take the form

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial}{\partial r}\right)\right)\right) = 0, \tag{2}$$

where the plate occupies the domain $\Omega:\ 0 \le r \le R$. Suppose the following boundary conditions

$$u(0) = u_0, \tag{3}$$

$$u(R) = 0. \tag{4}$$

$$\frac{\partial u}{\partial r}(0) = 0 \tag{5}$$

$$M(R) = -D\cdot\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial u}{\partial r}\right)(R) = 0 \tag{6}$$

The condition (5) is natural and it is one of two values which are possible in classical case. The second possible value is infinity. The MAC theory could allow to consider any finite value of the condition (5). It will not be considered

in this paper.
The condition (6) is taken just for simplicity to use the result of the work (5) and to show the MAC approach in case of elastic plate.
The general solution of the equation (2) is

$$u = C_1 \cdot \ln r + C_2 \cdot r^2 \cdot \ln r + C_3 + C_4 \cdot r^2, \tag{7}$$

where the arbitrary constants C_1, C_2, C_3, C_4 could be found using the boundary conditions (3)-(6). The derivative of the function (7) is

$$\frac{\partial u}{\partial r} = \frac{C_1}{r} + C_2 \cdot r \cdot (2 \cdot \ln r + 1) + 2 \cdot C_4 \cdot r. \tag{8}$$

We obtain at $r = 0$ from (8):

$$\frac{\partial u}{\partial r}(0) = 0. \tag{9}$$

The constant $C_1 = 0$ because of the finiteness of that derivative in the experiments with the plate. If the condition (5) is not zero then the solution of the given problem does not exist.
If we put the function (7) into the condition (3) then

$$C_3 = u(0) = u_0. \tag{10}$$

Then the condition (4) gives the following equation

$$C_2 \cdot R^2 \cdot \ln R + u_0 + C_4 \cdot R^2 = 0. \tag{11}$$

If the force P is applied ar $r = 0$ then the equilibrium equation allows to find

$$Q(R) = \frac{P}{2 \cdot \pi \cdot R}. \tag{12}$$

The expression for Q in the theory of elastic plate is

$$Q(r) = D \cdot \frac{\partial}{\partial r} \left(\frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial u}{\partial r} \right) \right), \tag{13}$$

where D is the bending stiffness of the plate. The equations (12) and (13) create

$$C_2 = \frac{r}{4} \cdot Q(r) = \frac{R}{4} \cdot Q(R) = \frac{P}{8 \cdot \pi \cdot D}. \tag{14}$$

Then the equation(11) gives the constant C_4

$$C_4 = -C_2 \cdot \ln R - \frac{u_0}{R^2} = -\frac{P}{8 \cdot \pi \cdot D} \cdot \ln R - \frac{u_0}{R^2}. \tag{15}$$

The solution (7) and its derivative (8) will take the form

$$u(r) = \frac{P}{8 \cdot \pi \cdot D} \cdot r^2 \cdot \ln r + u_0 + \left(\frac{P \cdot \ln R}{8 \cdot \pi \cdot D} - \frac{u_0}{R^2} \right) \cdot r^2. \tag{16}$$

$$\frac{\partial u}{\partial r}(r) = \frac{P}{8 \cdot \pi \cdot D} \cdot r \cdot (2 \cdot \ln r + 1) + 2 \cdot \left(\frac{P \cdot \ln R}{8 \cdot \pi \cdot D} - \frac{u_0}{R^2} \right) \cdot r. \tag{17}$$

The bending moments are from (8):

$$M_r = -D \cdot \left(\frac{\partial^2 u}{\partial r^2} + \frac{\nu}{r} \cdot \frac{\partial u}{\partial r} \right), \tag{18}$$

$$M_t = -D \cdot \left(\frac{1}{r} \cdot \frac{\partial u}{\partial r} + \nu \cdot \frac{\partial^2 u}{\partial r^2} \right). \tag{19}$$

Using (16), (18), and (19) we obtain

$$M_r = -2 \cdot D \cdot (C_2 \cdot (1+\nu) \cdot (2 \cdot \ln r + 1) + C_2 + C_4 \cdot (1+\nu)), \tag{20}$$

$$M_t = -2 \cdot D \cdot (C_2 \cdot (1+\nu) \cdot (2 \cdot \ln r + 1) + \nu \cdot C_2 + C_4 \cdot (1+\nu)). \tag{21}$$

We can see in the experiment with the circular plate under the force P in the middle of a plate, that there are only the finite bending moments and corresponding stresses. It means that the logarithmic terms in the expressions (20) and (21) must be avoided. Then it should be

$$C_2 = \frac{P}{8 \cdot \pi \cdot D} = 0. \tag{22}$$

Therefore we obtain from (22) that

$$P = 0. \tag{23}$$

The force $P \neq 0$ according to the stated problem. This contradicts to the value (23). We can conclude from the obtained contradiction that the solution of the stated problem does not exist.
As we can see that the situation with the plate is similar to the situation with the membrane (3): the classical solution of the problem does not exist but the physical solution of the problem exists evidently.

2.2 Generalized solution

We can obtain the generalized solution of the problem using the similar way as in the membrane problem (3). It means that can suppose the distribution of the force P near the origin in the small circle. Then we can find the solution of the stated problem. And after that the radius of the small circle must tend to zero. This approach is presented in (6).
Let us consider for instance consider the case of boundary condition

$$\frac{du}{dr}(R) = 0 \tag{24}$$

instead of the condition (6). Then the generalized solution of the problem according to (8) is:

$$u(r) = \frac{P \cdot r^2}{8 \cdot \pi \cdot D} \cdot \ln \frac{r}{R} + \frac{P}{16 \cdot \pi \cdot D} \cdot (R^2 - r^2). \tag{25}$$

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The bending moments are (8):

$$M_r = \frac{P}{4 \cdot \pi} \cdot ((1+\nu) \cdot \ln \frac{R}{r} - 1), \tag{26}$$

$$M_t = \frac{P}{4 \cdot \pi} \cdot ((1+\nu) \cdot \ln \frac{R}{r} - \nu). \tag{27}$$

We can see that the expression for the displacement u according to (25) is well enough. But the expressions (26), (27) for the bending moments have singularities at the point of application of the force. They can not be used to determine the stresses near the origin. Another models have to be considered to determine the real stresses. We will use for that the MAC-solution of our problem.

2.3 MAC-solution

The method of Marcus (6) will help to obtain the MAC-solution for our plate problem.
Consider the differential equation of the plate (1) in the form

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 0. \tag{28}$$

The bending moments are

$$M_x = -D \cdot (\frac{\partial^2 u}{\partial x^2} + \nu \cdot \frac{\partial^2 u}{\partial y^2}), \tag{29}$$

$$M_y = -D \cdot (\frac{\partial^2 u}{\partial y^2} + \nu \cdot \frac{\partial^2 u}{\partial x^2}), \tag{30}$$

The sum of the expressions (29), (30) gives

$$M_x + M_y = -D \cdot (1+\nu) \cdot (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}). \tag{31}$$

If we introduce a new notation

$$M = \frac{M_x + M_y}{1+\nu} = -D \cdot (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \tag{32}$$

the equations (28) and (32) can be written in the form:

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = 0, \tag{33}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{M}{D}. \tag{34}$$

We consider the symmetric case and therefore the equations (33), (34) in polar coordinates will take the form

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial M}{\partial r}\right) = 0, \tag{35}$$

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) = -\frac{M}{D}. \tag{36}$$

The equation (35) is the classical membrane equation and its general MAC-solution according to (3) can be written in the form:

$$M(r) = C_1 + C_2 \cdot r, \tag{37}$$

where C_1 and C_2 are arbitrary constants.
Then the equation (36) yields

$$\frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) = -\frac{1}{D} \cdot (C_1 + C_2 \cdot r). \tag{38}$$

This equation could be considered in two ways: to find the classical solution and to find the MAC solution. Let us consider the classical solution of (38). Consider the general solution of the equation (38)

$$u(r) = C_3 \cdot \ln(r) + C_4 - \frac{1}{D} \cdot \left(C_1 \cdot \frac{r^2}{4} + C_2 \cdot \frac{r^3}{9}\right), \tag{39}$$

where C_3 and C_4 are arbitrary constants.
Using the boundary conditions (3)-(6) the solution (38) will be

$$u(r) = u_0 \cdot \left(1 - \frac{9}{5} \cdot \left(\frac{r}{R}\right)^2 + \frac{4}{5} \cdot \left(\frac{r}{R}\right)^3\right) \tag{40}$$

The bending moments M_r, M_t could be obtained from Eqs. (18), (19) and the force Q - from the equilibrium equation (12). Using classical equation

$$Q(r) = -\frac{\partial M}{\partial r} \tag{41}$$

at $r = R$ we find the connection between applied force P and displacement u_0:

$$u_0 = \frac{5 \cdot P \cdot R^2}{72 \cdot \pi \cdot D}. \tag{42}$$

3 Conclusion

The nonexistent, generalized and MAC solutions of the circular elastic plate were considered. Some problems of mathematical physics with nonexistent solutions can have the MAC-solutions. These MAC-solutions are corresponding and explaining the real physical situation of applied force.

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Approximate bi-homomorphisms and bi-derivations in normed Lie triple systems

[1]Javad Shokri, [2]Ali Ebadian, [3]Rasoul Aghalari and Madjid Eshaghi Gordji

[1,2,3]Department of Mathematics, Urmia University,
P.O.Box 165, Urmia, Iran. [4] Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran
Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Iran

Abstract. We prove the generalized Hyers–Ulam stability of the following 2-dimensional quadratic functional equation:

$$f(x+y,z-w)+f(x-y,z+w)=2f(x,z)+2f(y,w).$$

Moreover, we investigate the stability of bi-homomorphisms and bi-derivations on normed Lie triple systems.

1. Introduction

In 1940 S.Ulam [36] proposed the general Ulam stability problem:

Let G_1 be a group and let G_2 be a matric group with the matric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies inequality $d\Big(h(xy) - h(x)h(y)\Big) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

D. H. Hyers [24] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, a generalized version of Hyers' theorem for approximate additive mappings was given by T. Aoki [2]. In 1978, Th. M. Rassias [30] extended the theorem of Hyers by considering the unbounded cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p) \quad (\varepsilon \geq 0,\, p\in[0,1))$$

In 1990, Th. M. Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [22] following the same approach as in Th. M. Rassias [30] gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [22], as well as by Th. M. Rassias' and P. Semrl [34] that one can prove Th. M. Rassias' type theorem when $p = 1$. Th. M. Rassias Theorem for the stability for stability of the linear mappings between Banach spaces provided some influence for the development of the concept of generalized Hyers–Ulam stability, a fact which rekindled interest in the subject of stability of functional equations. This concept is known today as generalized Hyers–Ulam stability or Hyers-Ulam-Rassias stability of functional equations; cf. [4, 33]. Several mathematicians following the spirit of the approach in the paper of Th. M. Rassias [30] for the unbounded Cauchy difference obtained various results. For example in 1982, J. M. Rassias [29] obtained an analogous stability theorem in which he replace the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p.\|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$. In 1994, P. Găvruta [23] provided a further generalization of Th. M. Rassias' theorem in which he replaced the bound $\varepsilon(\|x\|^p+\|y\|^p)$ in (1.1) by

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[0]**E-mail:** [1]j.shokri@urmia.ac.ir, [2]a.ebadian@urmia.ac.ir, [3]r.aghalary@urmia.ac.ir,madjid.eshaghi@gmail.com
[0]The corresponding author: madjid.eshaghi@gmail.com (Madjid Eshaghi Gordji)

a general control function $\varphi(x, y)$. In 1996, G. Isac and Th. M. Rassias [25] applied the generalized Hyers-Ulam stability theory to prove fixed point theorems and obtained some new applications in nonlinear Analysis. During the last decades several stability problems of functional equations have been investigated by a number of mathematicians; cf. [4]– [32] and references therein.

Ternary algebraic operations were considered in 19th century by several mathematicians such as Cayley [3] how introduced the notion of cubic matrix which in turn was generalized by Kapranov [?]*et. al.* in 1990. there are some applications, although still hypothetical, in the fractional quantum Hall effect, the nonstandard statistics, supersymmetric theory, and Yang-Baxter equation. the comments on physical applications of ternary structures can be found in Refs. [1, 26, 35, 37].

A normed (Banach) Lie triple system is a normed (Banach) space $(\mathsf{A}; \|.\|)$ with a trilinear mapping $(x, y, z) \longmapsto [xyz]$ from $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ satisfying the following axioms

$$\begin{aligned}
&[xyz] = -[yxz],\\
&[xyz] + [yzx] + [zxy] = 0\\
&[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]],\\
&\|[xyz]\| \leqslant \|x\|\|y\|\|z\|,
\end{aligned}$$

for all $u, v, x, y, z \in \mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be normed Lie triple systems. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is said to be a homomorphism if

$$H([xyz]) = [H(x)H(y)H(z)]$$

for all $x, y, z \in \mathcal{A}$. A $\mathbb{C}$-linear mapping $\delta : \mathcal{A} \to \mathcal{B}$ is called a derivation if

$$\delta([xyz]) = [\delta(x)yz] + [x\delta(y)z] + [xy\delta(z)]$$

for all $x, y, z \in \mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be normed Lie triple systems. A $\mathbb{C}$-bilinear $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is said to be a bihomomorphism if it satisfies

$$\begin{aligned}
H([xyz], w) &= [H(x, w)H(y, w)H(z, w)],\\
H(x, [yzw]) &= [H(x, y)H(x, z)H(x, w)]
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. A $\mathbb{C}$-bilinear $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ is said to be a biderivation if it satisfies

$$\begin{aligned}
\delta([xyz], w) &= [\delta(x, w)yz] + [x\delta(y, w)z] + [xy\delta(z, w)],\\
\delta(x, [yzw]) &= [\delta(x, y)zw] + [y\delta(x, z)w] + [yz\delta(x, w)]
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$.

In this paper, we have analyzed the Hyers-Ulam-Rassias stability of bi-homomorphism and bi-derivation associated with the following quadratic functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) + 2f(y, w), \tag{1.1}$$

in Lie triple systems, which can be regarded as ternary structures. The reader is referred to [27, 28] for some other related results.

Throughout this paper, suppose that $\mathcal{A}$ is normed Lie triple system with norm $\|.\|_{\mathcal{A}}$ and that $\mathcal{B}$ is a Banach Lie triple system with norm $\|.\|_{\mathcal{B}}$. For given mapping $f : \mathcal{A} \times \mathcal{A} :\to \mathcal{B}$ and given subset $\mathbb{E}$ of $\mathbb{C}$, we define $D_{\lambda,\mu} : \mathcal{A}^4 \to \mathcal{B}$ by

$$D_{\lambda,\mu} f(x, y, z, w) := f(\lambda x + \lambda y, \mu z - \mu w) + f(\lambda x - \lambda y, \mu z + \mu w) - 2\lambda\mu f(x, z) - 2\lambda\mu f(y, w)$$

for all $\lambda, \mu \in \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and all $x, y, z, w \in \mathcal{A}$.

2. **Stability of bi–linear mappings**

Throughout this section $\mathcal{X}$ denotes a normed space and $\mathcal{Y}$ is a Banach space. We aim to prove the generalized Hyers–Ulam stability of 2-dimensional linear mappings.

Lemma 2.1. *Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{C}$-linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be a bi-additive mapping, such that $f(\lambda x, \mu y) = \lambda\mu f(x, y)$ for all $\lambda, \mu \in \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and all $x, y, z, w \in \mathcal{V}$, then f is $\mathbb{C}$-bilinear.*

Proof. Since f is a bi-additive, we get $f(\frac{1}{2}x, \frac{1}{2}y) = \frac{1}{4}f(x, y)$ for all $x, y \in \mathcal{X}$. Obviously, $f(0x, 0y) = 0f(x, y)$, now let $\mu_1, \mu_2 \in \mathbb{C}(\mu_1 \neq 0, \mu_2 \neq 0)$, and let M_1 and M_2 are two natural numbers, respectively, greater than $|\mu_1|$ and $|\mu_2|$. By an easily geometric argument, one can conclude that there exist numbers $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22} \in \mathbb{T}^1$ such that $2\frac{\mu_1}{M_1} = \lambda_{11} + \lambda_{12}$ and $2\frac{\mu_2}{M_2} = \lambda_{21} + \lambda_{22}$. Therefore

$$\begin{aligned} f(\mu_1 x, \mu_2 y) &= f(\frac{M_1}{2}.2.\frac{\mu_1}{M_1}x, \frac{M_2}{2}.2.\frac{\mu_2}{M_2}y) = \frac{M_1}{2}.\frac{M_2}{2}f(2.\frac{\mu_1}{M_1}x, 2.\frac{\mu_2}{M_2}y) \\ &= \frac{M_1}{2}.\frac{M_2}{2}f((\lambda_{11} + \lambda_{12})x, (\lambda_{21} + \lambda_{22})y) \\ &= \frac{M_1}{2}(\lambda_{11} + \lambda_{12}).\frac{M_2}{2}(\lambda_{21} + \lambda_{22})f(x, y) = \mu_1\mu_2 f(x, y) \end{aligned}$$

for all $x, y \in \mathcal{V}$. Thus the mapping $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ is a $\mathbb{C}$-bilinear. □

Lemma 2.2. *Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{C}$-linear spaces and $f : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ be a mapping satisfies $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $\lambda, \mu \in \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and all $x, y, z, w \in \mathcal{V}$, then f is $\mathbb{C}$-bilinear.*

Proof. Letting $\lambda = \mu = 1$, Theorem 2.1 in [**?**], f is bi-additive. Putting $y = w = 0$ in $D_{\lambda,\mu}(x, y, z, w) = 0$, we get $f(\lambda x, \mu z) = \lambda\mu f(x, y)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{X}$. So by Lemma 2.1, the mapping f is $\mathbb{C}$-bilinear. □

Theorem 2.3. *Let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ with $f(0, 0) = 0$ be a mapping for which there exist a function $\varphi : \mathcal{X}^4 \to [0, \infty)$, such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_{\mathcal{Y}} \leqslant \varphi(x, y, z, w), \tag{2.1}$$

$$\tilde{\varphi}(x, y, z, w) := \frac{1}{4}\sum_{n=0}^{\infty} 4^{-n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) < \infty \tag{2.2}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and $x, y, z, w \in \mathcal{X}$. then there exist a unique bi-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x, y) - T(x, y)\|_{\mathcal{Y}} \leqslant \frac{3}{2}\tilde{\varphi}(x.x, y, -y) + \frac{1}{2}\tilde{\varphi}(x, -x, y, y) + \tilde{\varphi}(0, x, 0, y) \tag{2.3}$$

for all $x, y \in \mathcal{X}$.

Proof. Assume that $\lambda = \mu = 1$. Put $w = z$ and $y = -x$ in (2.1) to obtain

$$\|f(2x, 2z) - 2f(x, z) - 2f(-x, z)\|_{\mathcal{Y}} \leqslant \varphi(x, -x, z, z) \tag{2.4}$$

for all $x, z \in \mathcal{X}$. Letting $x = z = 0$ in (2.1), we get

$$\|f(y, -w) + f(-y, w) - 2f(y, w)\|_{\mathcal{Y}} \leqslant \varphi(0, y, 0, w) \tag{2.5}$$

for all $y, w \in \mathcal{X}$. Replacing y by x and w by z in (2.5)

$$\|f(x, -z) + f(-x, z) - 2f(x, z)\|_{\mathcal{Y}} \leqslant \varphi(0, x, 0, z) \tag{2.6}$$

for all $x, z \in \mathcal{X}$. Putting $x = y$ and $w = -z$ in (2.1), we get

$$\|f(2x, 2z) - 2f(x, z) - 2f(x, -z)\|_{\mathcal{Y}} \leqslant \varphi(x, x, z, -z) \tag{2.7}$$

for all $x, z \in \mathcal{X}$. By inequalities (2.4) and (2.7), we get

$$\|2f(x, -z) - 2f(-x, z)\|_{\mathcal{Y}} \leqslant \varphi(x, x, z, -z) + \varphi(x, -x, z, z) \tag{2.8}$$

for all $x, z \in \mathcal{X}$. By (2.6) and (2.7), we have

$$\|f(2x, 2z) - 4f(x, z) - f(x, -z) + f(-x, z)\|_{\mathcal{Y}} \leqslant \varphi(x, x, z, -z) + \varphi(0, x, 0, z) \tag{2.9}$$

for all $x, z \in \mathcal{X}$. By two inequalities (2.8) and (2.9), we get

$$\|f(2x, 2y) - 4f(x, z) \leqslant \frac{3}{2}\varphi(x, x, z, -z) + \frac{1}{2}\varphi(x, -x, z, z) + \varphi(0, x, 0, z)$$

for all $x, z \in \mathcal{X}$. Setting $z = y$ in the above inequality and put $\psi(x, y) := \frac{3}{2}\varphi(x, x, y, -y) + \frac{1}{2}\varphi(x, -x, y, y) + \varphi(0, x, 0, y)$ to get

$$\|f(2x, 2y) - 4f(x, y)\|_{\mathcal{Y}} \leqslant \psi(x, y), \tag{2.10}$$

or

$$\left\|\frac{1}{4}f(2x, 2y) - f(x, y)\right\|_{\mathcal{Y}} \leqslant \frac{1}{4}\psi(x, y),$$

for all $x, y \in \mathcal{X}$. Replacing x by$2^j x$ and y by $2^j y$ and dividing 4^j in the above inequality, we obtain that

$$\left\|\frac{1}{4^{j+1}}f(2^{j+1}x, 2^{j+1}y) - \frac{1}{4^j}f(2^j x, 2^j y)\right\|_{\mathcal{Y}} \leqslant \frac{1}{4^{j+1}}\psi(2^j x, 2^j y)$$

for all $x, y \in \mathcal{X}$ and all $j = 0, 1, \ldots$. One can use induction to show that

$$\left\|\frac{1}{4^n}f(2^n x, 2^n y) - \frac{1}{4^m}f(2^m x, 2^m y)\right\|_{\mathcal{Y}} \leqslant \frac{1}{4}\sum_{j=m}^{n-1} 4^{-j}\psi(2^j x, 2^j y) \tag{2.11}$$

for all nonnegative integers $n > m$ and all $x, y \in \mathcal{X}$. It follows from the convergence of the series (2.2) and (2.11) that the sequence $\left\{\frac{1}{4^n}f(2^n x, 2^n y)\right\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Due to the completeness of $\mathcal{Y}$, this sequence is convergent. Define $T : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$T(x, y) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x, 2^n y)$$

for all $x, y \in \mathcal{X}$. By (2.1) and (2.2), we have

$$\begin{aligned} &\|T(x+y,z-w)+T(x-y,z+w)-2T(x,z)-2T(y,w)\|_{\mathcal{Y}} \\ &= \lim_{n\to\infty}\Big\|\frac{1}{4^n}f(2^n(x+y),2^n(z-w))+\frac{1}{4^n}f(2^n(x-y),2^n(z+w)) \\ &-\frac{2}{4^n}f(2^nx,2^nz)-\frac{2}{4^n}f(2^ny,2^nw)\Big\|_{\mathcal{Y}} \\ &\leqslant \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^nx,2^ny,2^nz,2^nw)=0 \end{aligned}$$

for all $x, y, z, w \in \mathcal{X}$. Then T satisfies (1.1), so by Lemma 2.2, the mapping $T : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ is $\mathbb{C}$-bilinear. Setting $m = 0$ and taking $n \to \infty$ in (2.11), one can obtain the inequality (2.3).

Now, let $T' : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ be another bi-linear mapping satisfy (2.3). Then we have

$$\begin{aligned} \|T(x,y)-T'(x,y)\| &= \frac{1}{4^n}\|T(2^nx,2^ny)-T'(2^nx,2^ny)\|_{\mathcal{Y}} \\ &\leqslant \frac{1}{4^n}\|T(2^nx,2^ny)-f(2^nx,2^ny)\|_{\mathcal{Y}}+\frac{1}{4^n}\|f(2^nx,2^ny)-T'(2^nx,2^ny)\|_{\mathcal{Y}} \\ &\leqslant \frac{2}{4^n}\Big(\frac{3}{2}\tilde{\varphi}(2^nx,2^nx,2^ny,-2^ny)+\frac{1}{2}\tilde{\varphi}(2^nx,-2^nx,2^ny,2^ny)+\tilde{\varphi}(0,2^nx,0,2^ny)\Big) \\ &\leqslant \frac{1}{4^n}\Big(\frac{1}{2}\sum_{k=0}^{\infty}4^{-k}\psi(2^k2^nx,2^k2^ny)\Big)=\frac{1}{2}\sum_{k=n}^{\infty}4^{-k}\psi(2^kx,2^ky)\to 0 \end{aligned}$$

as $n \to \infty$. Hence $T = T'$.

The next result is a dual to the previous theorem in some sense. □

Theorem 2.4. *Let $f : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ with $f(0,0) = 0$ be a mapping for which there exist a function $\varphi : \mathcal{X}^4 \to [0,\infty)$, such that*

$$\|D_{\lambda,\mu}f(x,y,z,w)\| \leqslant \varphi(x,y,z,w),$$

$$\tilde{\varphi}(x,y,z,w) := \frac{1}{4}\sum_{n=0}^{\infty}4^n\varphi(2^{-n}x,2^{-n}y,2^{-n}z,2^{-n}w) < \infty \tag{2.12}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and $x, y, z, w \in \mathcal{X}$. Then there exists a unique bi-linear mapping $T : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x,y)-T(x,y)\| \leqslant \frac{3}{2}\tilde{\varphi}(x.x,y,-y)+\frac{1}{2}\tilde{\varphi}(x,-x,y,y)+\tilde{\varphi}(0,x,0,y)$$

for all $x, y \in \mathcal{X}$.

Proof. It follows from(2.10) that

$$\Big\|f(x,y)-4f(\frac{1}{2}x,\frac{1}{2}y)\Big\|_{\mathcal{Y}} \leqslant \psi(\frac{1}{2}x,\frac{1}{2}y)$$

for all $x, y \in \mathcal{X}$. So

$$\begin{aligned} &\Big\|4^mf(\frac{x}{2^m},\frac{y}{2^m})-4^nf(\frac{x}{2^n},\frac{y}{2^n})\Big\|_{\mathcal{Y}} \\ &\leqslant \sum_{j=m}^{n-1}\Big\|4^jf(\frac{x}{2^j},\frac{y}{2^j})-4^{j+1}f(\frac{x}{2^{j+1}},\frac{y}{2^{j+1}})\Big\|_{\mathcal{Y}} \leqslant \frac{1}{4}\sum_{j=m}^{n-1}4^j\psi(\frac{x}{2^j},\frac{y}{2^j}) \end{aligned} \tag{2.13}$$

for all nonnegative integers m and n with $n > m$ and all $x, y \in \mathcal{X}$. It follows from convergency of two series (2.12) and (2.13) that the sequence $\{4^nf(\frac{x}{2^n},\frac{y}{2^n})\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$

is complete, the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ is convergence for all $x, y \in \mathcal{X}$. So we can define the mapping $T : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$T(x, y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$$

for all $x, y \in \mathcal{X}$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.13), we get desired inequality. The rest of the proof is similar to the proof of Theorem 2.3. $\square$

3. Stability of bi-homomorphisms in normed Lie triple systems

In this section, we prove the stability of bi-homomorphisms in normed Lie triple systems associated with the bi–linear functional equation.

Theorem 3.1. *Let θ and $p < 2$ be positive real numbers, and let $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ with $f(0, 0) = 0$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_{\mathcal{B}} \leqslant \theta(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}} + \|w\|^p_{\mathcal{A}}), \tag{3.1}$$

$$\begin{aligned}&\|f([x, y, z], w) - [f(x, w)f(y, w)f(z, w)]\|_{\mathcal{B}}\\ &\quad+\|f(x, [y, z, w]) - [f(x, y)f(x, z)f(x, w)]\|_{\mathcal{B}}\\ &\quad\leqslant \theta(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}} + \|w\|^p_{\mathcal{A}})\end{aligned} \tag{3.2}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and $x, y, z, w \in \mathcal{A}$. Then there exist a unique bi-homomorphism $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x, y) - H(x, y)\|_{\mathcal{B}} \leqslant \frac{5\theta}{4 - 2^p}(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}}) \tag{3.3}$$

for all $x, y \in \mathcal{A}$.

Proof. First let us assume $\|0\|^p = 0$ for $p < 0$. Put $\varphi(x, y, z, w) = \theta(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}} + \|w\|^p_{\mathcal{A}})$ in Theorem 2.3, to get a unique $\mathbb{C}$-bilinear mapping H given by $H(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$ satisfying (3.3). It follows from (3.1) that

$$\begin{aligned}&\|H([x, y, z], w) - [H(x, w)H(y, w)H(z, w)]\|_{\mathcal{B}}\\ &\quad+ \|H(x, [y, z, w]) - [H(x, y)H(x, z)H(x, w)]\|_{\mathcal{B}}\\ &= \lim_{n\to\infty} \frac{1}{4^n}(\|f([2^n x, 2^n y, 2^n z], 2^n w) - [f(2^n x, 2^n w), f(2^n y, 2^n w), f(2^n z, 2^n w)]\|_{\mathcal{B}}\\ &\quad+ \|f(2^n x, [2^n y, 2^n z, 2^n w]) - [f(2^n x, 2^n y)f(2^n x, 2^n z)f(2^n x, 2^n w)]\|_{\mathcal{B}})\\ &\leqslant \lim_{n\to\infty} \frac{2^{np}}{4^n}\theta(\|x\|^p_{\mathcal{A}} + \|y\|^p_{\mathcal{A}} + \|z\|^p_{\mathcal{A}} + \|w\|^p_{\mathcal{A}}) = 0\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. So

$$H([x, y, z], w) = [H(x, w), H(y, w), H(z, w)]$$

and

$$H(x, [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in \mathcal{A}$. $\square$

Theorem 3.2. *Let θ and $p > 2$ a positive real number, and let $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ be a mapping satisfying (3.1) and (3.2). Then there exists a unique bi-homomorphism $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x,y) - H(x,y)\|_{\mathcal{B}} \leqslant \frac{5\theta}{2^p - 4}(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p).$$

Proof. Put $\varphi(x,y,z,w) = \theta(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p + \|z\|_{\mathcal{A}}^p + \|w\|_{\mathcal{A}}^p)$ in Theorem 2.4 and note that inequality (3.2) implies that $f(0,0) = 0$. One can define $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$

$$H(x,y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$$

for all $x, y \in \mathcal{A}$. Then desired inequality for H and f is obtained. The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 3.3. *Let θ and $p \neq \frac{1}{2}$ be positive real numbers, and $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$, with $f(0,0) = 0$, be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\|_{\mathbb{B}} \leqslant \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p),$$

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w), f(y,w), f(z,w)]\|_{\mathcal{B}} \\ &+\|f(x,[y,z,w]) - [f(x,y), f(x,z), f(x,w)]\|_{\mathcal{B}} \\ &\leqslant \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p) \end{aligned} \tag{3.4}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a bi-homomorphism $H : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x,y) - H(x,y)\|_{\mathcal{B}} = \begin{cases} \frac{2\theta}{4-2^{4p}}\|x\|_{\mathcal{B}}^{2p}.\|y\|_{\mathcal{B}}^{2p} & ; \quad p < \frac{1}{2} \\ \frac{2\theta}{2^{4p}-4}\|x\|_{\mathcal{B}}^{2p}.\|y\|_{\mathcal{B}}^{2p} & ; \quad p > \frac{1}{2} \end{cases}, \tag{3.5}$$

Proof. In case $p < \frac{1}{2}$, we first assume $\|0\|^p = 0$ for $p < 0$. Put $\varphi(x,y,z,w) = \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p)$ in Theorem 2.3, to get a unique $\mathbb{C}$-bilinear mapping H given by $H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$ satisfying (3.5). Eq. (3.4) implies that

$$\begin{aligned} &\|H([x,y,z],w) - [H(x,w)H(y,w)H(z,w)]\|_{\mathcal{B}} \\ &\quad + \|H(x,[y,z,w]) - [H(x,y)H(x,z)H(x,w)]\|_{\mathcal{B}} \\ &= \lim_{n\to\infty} \frac{1}{4^n}(\|f([2^n x, 2^n y, 2^n z], 2^n w) - [f(2^n x, 2^n w), f(2^n y, 2^n w), f(2^n z, 2^n w)]\|_{\mathcal{B}} \\ &\quad + \|f(2^n x, [2^n y, 2^n z, 2^n w]) - [f(2^n x, 2^n y) f(2^n x, 2^n z) f(2^n x, 2^n w)]\|_{\mathcal{B}}) \\ &\leqslant \lim_{n\to\infty} \frac{2^{4np}}{4^n}\theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p) = 0 \end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. So

$$H([x,y,z],w) = [H(x,w), H(y,w), H(z,w)]$$

and

$$H(x,[y,z,w]) = [H(x,y), H(x,z), H(x,w)]$$

for all $x, y, z, w \in \mathcal{A}$.

The rest of proof is similar to the proof of Theorems 3.1 and 2.3.

In the case $p > \frac{1}{2}$, put $\varphi(x,y,z,w) = \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p)$ in Theorem 2.4, to get unique $\mathbb{C}$-bilinear mapping H given by $H(x,y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ satisfies inequality in (3.5). The rest of proof is similar to the proof of Theorems 3.2 and 2.4 □

4. Stability of bi-derivations in normed Lie triple systems

In this section we prove the Hyers-Ulam-Rassias stability of bi-derivation of (1.1).

Theorem 4.1. *Let θ and $p \neq \frac{1}{2}$ be positive real numbers, and let $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ with $f(0,0) = 0$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x,y,z,w)\|_{\mathcal{B}} \leqslant \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p),$$

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)]\|_{\mathcal{B}} \\ &+\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w][y,z,f(x,w)]\|_{\mathcal{B}} \\ &\leqslant \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p) \end{aligned} \tag{4.1}$$

for all $x, y, z, w \in \mathcal{A}$. If f satisfies

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^{3n} y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^{3n} x, 2^n y)$$

for all $x, y, \in \mathcal{A}$. Then there exists a unique bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x,y) - \delta(x,y)\|_{\mathcal{B}} \leqslant \begin{cases} \frac{2\theta}{4-2^{4p}}(\|x\|_{\mathcal{A}}^{2p}.\|y\|_{\mathcal{A}}^{2p}) & ; \quad p < \frac{1}{2} \\ \frac{2\theta}{2^{4p}-4}(\|x\|_{\mathcal{A}}^{2p}.\|y\|_{\mathcal{A}}^{2p}) & ; \quad p > \frac{1}{2} \end{cases}, \tag{4.2}$$

for all $x, y \in \mathcal{A}$.

Proof. For the case $p < \frac{1}{2}$ by putting $\varphi(x,y,z,w) = \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p)$ in Theorem 2.3, it follows that there exists $\mathbb{C}$-bilinear mapping δ given by $\delta(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$ which satisfies inequality (4.2).

It follows from (4.1) that

$$\begin{aligned} &\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)]\|_{\mathcal{B}} \\ &+ \|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w][y,z,\delta(x,w)]\|_{\mathcal{B}} \\ &= \lim_{n\to\infty} \Big(\Big\| \frac{1}{4^{3n}} f(2^{3n}[x,y,z], 2^{3n}w) - \Big[\frac{1}{4^n} f(2^n x, 2^n w), y, z\Big] \\ &\quad - \Big[x, \frac{1}{4^n} f(2^n x, 2^n w), z\Big] - \Big[x, y, \frac{1}{4^n} f(2^n z, 2^n w)\Big] \Big\|_{\mathcal{B}} \\ &\quad + \Big\| \frac{1}{4^{3n}} f(2^{3n}x, 2^{3n}[y,z,w]) - \Big[\frac{1}{4^n} f(2^n x, 2^n y), z, w\Big] \\ &\quad - \Big[y, \frac{1}{4^n} f(2^n x, 2^n z), w\Big] - \Big[y, z, \frac{1}{4^n} f(2^n x, 2^n w)\Big] \Big\|_{\mathcal{B}} \Big) \\ &= \lim_{n\to\infty} \Big(\Big\| \frac{1}{4^{3n}} f([2^n x, 2^n y, 2^n z], 2^{3n} w) - \frac{1}{4^{3n}} [f(2^n x, 2^{3n} w), 2^n y, 2^n z] \\ &\quad - \frac{1}{4^{3n}} [2^n x, f(2^n y, 2^{3n} w), 2^n z] - \frac{1}{4^{3n}} [2^n x, 2^n y, f(2^n z, 2^{3n} w)] \Big\|_{\mathcal{B}} \\ &\quad + \Big\| \frac{1}{4^{3n}} f(2^{3n} x, [2^n y, 2^n z, 2^n w]) - \frac{1}{4^{3n}} [f(2^{3n} x, 2^n y), 2^n z, 2^n w] \\ &\quad - \frac{1}{4^{3n}} [2^n y, f(2^{3n} x, 2^n z), 2^n w] - \frac{1}{4^{3n}} [2^n y, 2^n z, f(2^{3n} x, 2^n w)] \Big\|_{\mathcal{B}} \Big) \\ &\leqslant \lim_{n\to\infty} \Big(\frac{\theta}{4^{3n}} (2^{np}\|x\|_{\mathcal{A}}^p.2^{np}\|y\|_{\mathcal{A}}^p.2^{np}\|z\|_{\mathcal{A}}^p.2^{3np}\|w\|_{\mathcal{A}}^p) \\ &\quad + \frac{\theta}{4^{3n}} (2^{3np}\|x\|_{\mathcal{A}}^p.2^{np}\|y\|_{\mathcal{A}}^p.2^{np}\|z\|_{\mathcal{A}}^p.2^{np}\|w\|_{\mathcal{A}}^p) \Big) \\ &= \lim_{n\to\infty} \frac{2\theta 2^{6np}}{4^{3n}} (\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p) = 0 \end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. Therefore

$$\delta([xyz], w) = [\delta(x, w)yz] + [x\delta(y, w)z] + [xy\delta(z, w)],$$
$$\delta(x, [yzw]) = [\delta(x, y)zw] + [y\delta(x, z)w] + [yz\delta(x, w)]$$

for all $x, y, z, w \in \mathcal{A}$.

For the case $p > \frac{1}{2}$, similar to the first case by putting $\varphi(x, y, z, w) = \theta(\|x\|_{\mathcal{A}}^p.\|y\|_{\mathcal{A}}^p.\|z\|_{\mathcal{A}}^p.\|w\|_{\mathcal{A}}^p)$ in Theorem 2.4 and by the same argument in Theorem 3.2 one can get a unique $\mathbb{C}$-bilinear mapping δ given by $\delta(x, y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ which satisfies inequality (4.2). $\square$

Theorem 4.2. *Let θ and $p \neq 2$ be positive real numbers, and let $f : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ with $f(0, 0) = 0$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_{\mathcal{B}} \leqslant \theta(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p + \|z\|_{\mathcal{A}}^p + \|w\|_{\mathcal{A}}^p),$$

$$\begin{aligned}&\|f([x, y, z], w) - [f(x, w), y, z] - [x, f(y, w), z] - [x, y, f(z, w)]\|_{\mathcal{B}}\\ &+\|f(x, [y, z, w]) - [f(x, y), z, w] - [y, f(x, z), w][y, z, f(x, w)]\|_{\mathcal{B}}\\ &\leqslant \theta(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p + \|z\|_{\mathcal{A}}^p + \|w\|_{\mathcal{A}}^p)\end{aligned} \tag{4.3}$$

for all $x, y, z, w \in \mathcal{A}$. If f satisfies

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^{3n} y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^{3n} x, 2^n y)$$

for all $x, y, \in \mathcal{A}$. Then there exists a unique bi-derivation $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x, y) - \delta(x, y)\|_{\mathcal{B}} \leqslant \begin{cases} \frac{5\theta}{4-2^p}(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p) & ; \quad p < 2 \\ \frac{5\theta}{2^p-4}(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p) & ; \quad p > 2 \end{cases},$$

for all $x, y \in \mathcal{A}$.

Proof. If $p < 2$, we first assume that $\|0\|^p = 0$ for $p < 0$. Put $\varphi(x, y, z, w) = \theta(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p + \|z\|_{\mathcal{A}}^p + \|w\|_{\mathcal{A}}^p)$ in Theorem 2.3, to get a unique $\mathbb{C}$-bilinear mapping δ given by $\delta(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$ satisfies desired inequality. The rest of the proof is similar to the proof of Theorem 4.1.

If $p > 2$, putting $\varphi(x, y, z, w) = \theta(\|x\|_{\mathcal{A}}^p + \|y\|_{\mathcal{A}}^p + \|z\|_{\mathcal{A}}^p + \|w\|_{\mathcal{A}}^p)$ in Theorem 2.4, to get a unique $\mathbb{C}$-bilinear mapping δ given by $\delta(x, y) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n}, \frac{y}{2^n})$ satisfies desired inequality. The rest of the proof is similar to the proof of Theorem 4.1. $\square$

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SOME INEQUALITIES FOR THE q-DIGAMMA FUNCTIONS

W. T. Sulaiman
Department of Computer Engineering
College of Engineering
University of Mosul, Iraq.

Abstract. In [7] the authors proved many inequalities concerning the q-digamma function of the form $f^2(xy) \le (\ge) f(x) f(y)$ and $f(x+y) \le (\ge) f(x) + f(y)$. In the present paper we reproof some of the above results by simpler method using different technique as well as we present many other new results.

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1. Introduction

Let c be a complex number, the q-shifted factorial are defined by

(1.1) $$(c;q)_0 = 1, \quad (c,q)_n = \prod_{k=0}^{n-1}\left(1-cq^k\right), \quad n = 1,2,.....$$

(1.2) $$(c;q)_\infty = \lim_{n\to\infty}(c;q)_n = \prod_{k=o}^{\infty}\left(1-cq^k\right).$$

For x complex we denote

(1.3) $$[x]_q = \frac{1-q^x}{1-q}.$$

The q-Jakson integrals from 0 to c are defined by [5,6]

(1.4) $$\int_0^c f(x)\, d_q x = (1-q)c\sum_{n=0}^{\infty} f\left(cq^n\right)q^n,$$

and

(1.5) $$\int_0^\infty f(x)\, d_q x = (1-q)\sum_{n=-\infty}^{\infty} f\left(q^n\right)q^n,$$

provided the sum converges absolutely .

The q-analogue of the Gamma function is defined by Jakson [6] as follows

(1.6) $$\Gamma_q(x) = \frac{(q;q)_\infty}{\left(q^x;q\right)_\infty}(1-q)^{1-x}, \qquad x \ne 0,-1,-2,...... ,$$

and it is satisfying the following

(1.7) $$\Gamma(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1,$$

and tends to $\Gamma(x)$ as $q \to 1.$

The q-integral representation of the Gamma function is (see [2,4]) as follows

$$\Gamma_q(x) = K_q(x)\int_0^\infty t^{x-1}e_q^{-t}d_qt, \tag{1.8}$$

where

$$e_q^t = \frac{1}{\left((1-q)t;q\right)_\infty},$$

and

$$K_q(t) = \frac{(1-q)^{-x}}{1+(1-q)^{-1}}\times\frac{\left(-(1-q);q\right)_\infty\left(-(1-q)^{-1};q\right)_\infty}{\left(-(1-q)q^t;q\right)_\infty\left(-(1-q)^{-1}q^{1-t};q\right)_\infty}.$$

The q-analogue of the psi function $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ is defined as the logarithmic derivative of the q-gamma function, that is $\psi_q(x) = \dfrac{\Gamma_q'(x)}{\Gamma_q(x)}$.

From (1.6), we obtain for $x > 0$

$$\psi_q(x) = -\log(1-q) + \log q\sum_{n=0}^\infty \frac{q^{n+x}}{1-q^{n+x}}$$

$$= -\log(1-q) + \log q\sum_{n=1}^\infty \frac{q^{nx}}{1-q^n}, \tag{1.9}$$

$$= -\log(1-q) + \frac{\log q}{1-q}\int_0^q \frac{t^{x-1}}{1-t}d_qt. \tag{1.10}$$

For $x > 0$, we put

$$\alpha(x) = \frac{\log(x)}{\log(q)} - E\left(\frac{\log(x)}{\log(q)}\right)$$

and

$$\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}},$$

where $E\left(\dfrac{\log(x)}{\log(q)}\right)$ is the integer part of $\dfrac{\log(x)}{\log(q)}$.

From (1.9) one can deduce that

$$\psi_q^{(k)}(x) = \log^{k+1} q\sum_{n=1}^\infty \frac{n^k q^{nx}}{1-q^n}. \tag{1.11}$$

2. results

We start with the following Lemma

Lemma 2.1. *Let* $x, y > 0,\ \ 0 \le a < 1,$ *then*

$$a^x + a^y \ge a^{x+y}, \tag{2.1}$$

and the function $f(x) = a^x + a^y - a^{x+y}$ is non-increasing.
Proof. On keeping y fixed, we have

$$f'(x) = \log a(a^x - a^{x+y}) \le 0.$$

Therefore f is non-increasing. Since $\lim_{x\to\infty} f(x) = a^y > 0$, then $f(x) \ge 0$.

A short proof is giving for the following Lemma

Lemma 2.2[7]. *For* $0 < q < \frac{1}{2}$ *and* $0 < x < 1$, *we have* $\psi_q(x) < 0$.
Proof. From (1.9),

$$\begin{aligned}\psi(x) &= -\log(1-q) + \log q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n} \\ &\le -\log(1-q) + \log q \sum_{n=1}^{\infty} q^{nx} \\ &= -\log(1-q) + \log q \frac{q^x}{1-q^x} \\ &\le -\log(1-q) + \log q \frac{q}{1-q} = \log \frac{q^q}{(1-q)^{1-q}} < 0,\end{aligned}$$

as $q^q < (1-q)^{1-q}$ for $0 < q < \frac{1}{2}$.

The following results extend the scope of Lemma 2.2 to $0 < q < 1$.
Lemma 2.3. *For* $0 < q < 1$. $0 < x < 1$, *we have* $\psi_q(x) < 0$.
Proof. We have

$$\begin{aligned}\psi(x) &= -\log(1-q) + \log q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n} \\ &< -\log(1-q) + \log q \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \\ &= -\log(1-q) + \log q \int_1^{\infty} \frac{q^x}{1-q^x} dx \\ &= -\log(1-q) + \int_0^q \frac{-du}{1-u} \\ &= -\log(1-q) + \log(1-q) = 0.\end{aligned}$$

The following result is proved in three different methods

Theorem 2.4[7]. *For all* $0 < q < 1$ *and* $0 < x, y < 1$,
(2.2) $$\psi(x+y) < \psi(x) + \psi(y).$$

Proof.
Method 1. By Lemma 2.1, $q^{n(x+y)} - q^x - q^y$ is non-decreasing in both x and y.

$$\psi_q(x+y)-\psi_q(x)-\psi_q(y) = \log(1-q)+\log q\sum_{n=1}^{\infty}\frac{q^{n(x+y)}-q^{nx}-q^{ny}}{1-q^n}$$
$$\geq \log(1-q)+\log q\sum_{n=1}^{\infty}\frac{q^{2n}-2q^n}{1-q^n}$$
$$=\log(1-q)-\log q\sum_{n=1}^{\infty}q^n-\log q\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}$$
$$=\log(1-q)-\frac{q}{1-q}\log q-\int_1^{\infty}\frac{q^z\log q}{1-q^z}\,dz$$
$$=\log(1-q)-\frac{q}{1-q}\log q+\int_0^q\frac{du}{1-u}$$
$$=-\frac{q}{1-q}\log q>0.$$

Method 2. This method is restricted to $x>0,\ 0<y<\infty.$
Write
$$f(x)=\psi(x+y)-\psi(x)-\psi(y).$$
On keeping y fixed, we have
$$f'(x)=\psi'(x+y)-\psi'(x)$$
$$=\log^2 q\sum_{n=1}^{\infty}\frac{n}{1-q^n}\left(q^{n(x+y)}-q^x\right)\leq 0.$$
Therefore f is non-increasing .
As $\lim\limits_{x\to\infty} f(x)=-\psi(y)>0,$ then $f(x)\geq 0.$

Method 3. We have
$$\psi_q^{(1)}(x+y)\leq\psi_q^{(1)}(x)+\psi_q^{(1)}(y),$$
Let a,c be chosen such that $0<a<x,\ \ y<c<y+a$ and
$$(a-x)\psi_q^{(1)}(y)\geq a\psi_q'(c)-\psi_q(a),$$
then we have
$$\int_x^a\psi_q^{(1)}(x+y)\,dx\leq\int_x^a\psi_q^{(1)}(x)\,dx+\int_x^a\psi_q^{(1)}(y)\,dx$$
that is
$$\psi_q(a+y)-\psi_q(x+y)\leq\psi_q(a)-\psi_q(x)+(a-x)\psi_q^{(1)}(y)$$
or
$$\psi_q(x+y)\geq\psi_q(x)+\psi(y)+\psi_q(a+y)-\psi_q(y)-\psi_q(a)+(x-a)\psi_q^{(1)}(y).$$
Since, by the mean value theorem
$$\psi_q(a+y)-\psi_q(y)=a\psi_q'(c),\quad\text{for some}\quad y<c<a+y,$$
then
$$\psi_q(x+y)\geq\psi_q(x)+\psi(y)+a\psi_q'(c)-\psi_q(a)+\left(x-a\right)\psi_q'(y)$$
$$\geq\psi_q(x)+\psi(y).$$

$$\geq \psi_q(y)+\psi_q(x)-\psi_q(a)+(x-a)\psi_q^{(1)}(y)$$
$$\geq \psi_q(y)+\psi_q(x).$$

An easy proof is given for the following theorem

Theorem 2.5[7]. *Let* $q \in (0,1)$. *Let* $k \geq 1$ *be an integer.*

(1) *If k is even, then*

(2.3) $$\psi^{(k)}(x+y) \geq \psi^{(k)}(x)+\psi^{(k)}(y).$$

(2) *If k is odd, then*

(2.4) $$\psi^{(k)}(x+y) \leq \psi^{(k)}(x)+\psi^{(k)}(y).$$

Proof. Making use of (1.11), we have via Lemma 2.1, when k is even

$$\psi^{(k)}(x+y)-\psi^{(k)}(x)-\psi^{(k)}(y)$$
$$=\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k}{1-q^n}\left(q^{n(x+y)}-q^{nx}-q^{ny}\right) \geq 0.$$

If k is odd, the above inequality reverses .

The following results are all new :

Theorem 2.6. *Let* $k \geq 1$ *be an integer,* $s>1$, $\frac{1}{s}+\frac{1}{t}=1$, *then*

(i) *If* k *odd,*

(2.5) $$\psi_q^{(k)}\left(\tfrac{x}{s}+\tfrac{y}{t}\right) \leq \left(\psi_q^{(k)}(x)\right)^{1/s}\left(\psi_q^{(k)}(y)\right)^{1/t}.$$

(ii) *If k even* (2.5) *reverses.*

In particular, for odd k

$$\left(\psi_q^{(k)}\left(\tfrac{x+y}{2}\right)\right)^2 \leq \psi_q^{(k)}(x)\psi_q^{(k)}(y),$$

the above inequality reverses if k is even. If k is odd, then

$$\left(\psi_q^{(k)}(x+y)\right)^2 \leq \psi_q^{(k)}(x)\psi_q^{(k)}(y).$$

Proof.

$$\psi_q^{(k)}\left(\tfrac{x}{s}+\tfrac{y}{t}\right)= \log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{n\left(\frac{x}{s}+\frac{y}{t}\right)}}{1-q^n}$$
$$=\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^{\frac{k}{s}} q^{\frac{nx}{s}}}{\left(1-q^n\right)^{\frac{1}{s}}} \frac{n^{\frac{k}{t}} q^{\frac{ny}{t}}}{\left(1-q^n\right)^{\frac{1}{t}}}$$
$$\leq \log^{k+1} q \left(\sum_{n=1}^{\infty} \frac{n^k q^{nx}}{1-q^n}\right)^{1/s}\left(\sum_{n=1}^{\infty} \frac{n^k q^{ny}}{1-q^n}\right)^{1/t}$$

$$= \left(\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{nx}}{1-q^n} \right)^{1/s} \left(\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{ny}}{1-q^n} \right)^{1/t}$$

$$= \left(\psi_q^{(k)}(x)\right)^{1/s} \left(\psi_q^{(k)}(y)\right)^{1/t}.$$

On putting $s = t = \frac{1}{2}$, we obtain

$$\left(\psi_q^{(k)}\left(\tfrac{x+y}{2}\right)\right)^2 \le \psi_q^{(k)}(x)\, \psi_q^{(k)}(y).$$

If k is odd, $\psi_q^{(k)}(x)$ is decreasing and hence

$$\left(\psi(x+y)\right)^2 \le \left(\psi_q^{(k)}\left(\tfrac{x+y}{2}\right)\right)^2 \le \psi_q^{(k)}(x) \psi_q^{(k)}(y).$$

Theorem 2.7. *Let* $k \ge 1$ *be an odd integer. Then*

(2.6) $$\psi_q^{(k)}(xy) \le \left(\psi_q^{(k)}(x^s)\right)^{1/s} \left(\psi_q^{(k)}(y^t)\right)^{1/t}.$$

In particular for $x,\ y \ge 1$,

(2.7) $$\left(\psi_q^{(k)}(xy)\right)^2 \le \psi_q^{(k)}(x) \psi_q^{(k)}(y).$$

Proof.

$$\psi_q^{(k)}(xy) = \log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{nxy}}{1-q^n}$$

$$\le \log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{n\left(\frac{x^s}{s}+\frac{y^t}{t}\right)}}{1-q^n}$$

$$\le \left(\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{nx^s}}{1-q^n} \right)^{1/s} \left(\log^{k+1} q \sum_{n=1}^{\infty} \frac{n^k q^{ny^t}}{1-q^n} \right)^{1/t}$$

$$= \left(\psi_q^{(k)}(x^s)\right)^{1/s} \left(\psi_q^{(k)}(y^t)\right)^{1/t}.$$

In particular, If $x, y \ge 1$, then $\psi_q^{(k)}(x^s) < \psi_q^{(k)}(x)$, and the same for y and hence we have

$$\psi_q^{(k)}(xy) \le \left(\psi_q^{(k)}(x)\right)^{1/s} \left(\psi_q^{(k)}(y)\right)^{1/t}.$$

On putting $s = t = 2$, we obtain

$$\left(\psi_q^{(k)}(xy)\right)^2 \le \psi_q^{(k)}(x) \psi_q^{(k)}(y).$$

Theorem 2.8. *Let* $a, b > 0$. *Define*

$$B_q\left(1+ax, 1+by\right) = \frac{\Gamma_q(1+ax)\Gamma_q(1+by)}{\Gamma_q(2+ax+by)}.$$

Then the function B_q *is non-increasing*

Proof. Define

$$f(x) = B_q\left(1+ax, 1+by\right) = \frac{\Gamma_q(1+ax)\Gamma_q(1+by)}{\Gamma_q(2+ax+by)}.$$

$$\log f(x) = \log \Gamma_q(1+ax) + \log \Gamma_q(1+by) - \log \Gamma_q(2+ax+by)$$

On keeping y fixed and differentiating with respect to x, we have

$$\frac{f'(x)}{f(x)} = \frac{a\Gamma'(1+ax)}{\Gamma(1+ax)} - \frac{a\Gamma'(2+ax+by)}{\Gamma(2+ax++by)}$$
$$= a\psi_q(1+ax) - a\Psi_q(2+ax+by)$$
$$= a\log q\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\left(q^{nax} - q^{n(1+ax+by)}\right) \le 0.$$

Theorem2.9. *Let* $s>1,\ \frac{1}{s}+\frac{1}{t}=1$. *Then*

(2.8) $$\Psi_q(xy)+\log(1-q) \ge \left(\Psi(x^s)+\log(1-q)\right)^{1/s}\left(\Psi(y^t)+\log(1-q)\right)^{1/t}.$$

Proof. We have

$$\Psi_q(xy) = -\log(1-q)+\log q\sum_{n=1}^{\infty}\frac{q^{n\left(\frac{x^s}{s}+\frac{y^t}{t}\right)}}{1-q^n}$$
$$= -\log(1-q)+\log q\sum_{n=1}^{\infty}\frac{q^{\frac{nx^s}{s}}}{\left(1-q^n\right)^{1/s}}\frac{q^{\frac{ny^t}{t}}}{\left(1-q^n\right)^{1/t}}$$
$$\ge -\log(1-q)+\log q\left(\sum_{n=1}^{\infty}\frac{q^{nx^s}}{1-q^n}\right)^{1/s}\left(\sum_{n=1}^{\infty}\frac{q^{ny^t}}{1-q^n}\right)^{1/t}$$
$$= -\log(1-q)+\left(\log q\sum_{n=1}^{\infty}\frac{q^{nx^s}}{1-q^n}\right)^{1/s}\left(\log q\sum_{n=1}^{\infty}\frac{q^{ny^t}}{1-q^n}\right)^{1/t},$$

which implies

$$\Psi_q(xy)+\log(1-q) \ge \left(\Psi(x^s)+\log(1-q)\right)^{1/s}\left(\Psi(y^t)+\log(1-q)\right)^{1/t}$$

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Some Product Summability Of An Infinite Series

W. T. Sulaiman

Abstract. New results concerning product summability of an infinite series are given. Some special cases are also deduced.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n. Let u_n^α denote the nth Cesaro mean of order $\alpha > -1$ of the sequence (s_n). The series $\sum a_n$ is summable $\left|C,\alpha\right|_k,\ k \geq 1,$ if

(1) $$\sum_{n=1}^{\infty} n^{k-1}\left|u_n^\alpha - u_{n-1}^\alpha\right|^k < \infty \qquad (Flett\ [1]).$$

For $\alpha = 1,$ $\left|C,\alpha\right|_k$ reduces to $\left|C,1\right|_k$ summability.

Let (p_n) be a sequence of positive real constants such that $P_n = p_0 + ... + p_n \to \infty$ As $n \to \infty$ $(P_{-1} = p_{-1} = 0)$. The (N, p) transform ϕ_n of (s_n) generated by (p_n) is defined by

(2) $$\phi_n = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}s_v .$$

The sequence - to – sequence transformation

$$\Phi_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (Φ_n) of $(\overline{N}, p_n)$ transform of (s_n) generated by (p_n). The series $\sum a_n$ is summable $\left|R, p_n\right|_k,\ k \geq 1,$ if

(3) $$\sum_{n=1}^{\infty} n^{k-1}\left|\Phi_n - \Phi_{n-1}\right|^k < \infty .$$

In the special case when $p_n = 1$ for all n (resp. k=1) , $\left|R, p_n\right|_k$ summability reduces to $\left|C,1\right|_k$ (resp. $\left|R, p_n\right|$) summability .

The series $\sum a_n$ is said to be summable $\left|(N, p)(N, q)\right|$, when the (N, p) transform of the (N, q) transform of (s_n) is a sequence of bounded variation (see Das [2]) .

We give the following new definition :
Let (T_n) defines the sequence of the $(\overline{N}, q_n)$ transform of the $(\overline{N}, p_n)$ transform of (s_n) generated by the sequences (q_n) and (p_n) respectively. The series $\sum a_n$ is said to be summable $\left|(R, q_n)(R, p_n)\right|_k$, $k \ge 1$, *if*

(4) $$\sum_{n=1}^{\infty} n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty .$$

We may assume through the paper that $Q_n = q_0 + ... + q_n \to \infty,\ as\, n \to \infty,$

2. New Results

We state and prove the following

Theorem 1. Let $k \ge 1$, (λ_n) be a sequence of constants. Define

$$f_v = \sum_{r=v}^{n} \frac{q_r}{P_r}, \quad F_v = \sum_{r=v}^{n} p_r f_r .$$

Let

(5) $$p_v Q_v = \mathrm{O}(P_v),$$

(6) $$\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = \mathrm{O}\left(\frac{(vq_v)^{k-1}}{Q_v^k}\right).$$

Then sufficient conditions for the implication

(7) $$\sum a_n \textit{ is summable } \left|R, p_n\right|_k \quad \Rightarrow \quad \sum a_n \lambda_n \textit{ is summable } \left|(R, q_n)(R, p_n)\right|_k$$

are

(8) $|\lambda_n| F_v = \mathrm{O}(Q_v)$,

(9) $P_v |\lambda_v| f_v = \mathrm{O}(1)$,

(10) $p_v |\lambda_v| = \mathrm{O}(Q_v)$,

(11) $P_n |\lambda_n| = \mathrm{O}(p_n)$,

(12) $P_v |\Delta\lambda_v| F_v = \mathrm{O}(p_v Q_v)$,

and

(13) $P_v |\Delta\lambda_v| = \mathrm{O}(p_v Q_v)$.

Proof. Let (S_n) be the sequence of partial sums of $\sum a_n \lambda_n$. Let v_n, V_n be the $(\overline{N}, p_n)$, $(\overline{N}, q_n)(\overline{N}, p_n)$ transforms of the sequences (s_n), (S_n) respectively. We write $t_n = v_n - v_{n-1}$, $T_n = V_n - V_{n-1}$. Therefore

(14) $$t_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v ,$$

and

$$\begin{aligned} V_n &= \frac{1}{Q_n} \sum_{r=0}^{n} q_r \frac{1}{P_r} \sum_{v=0}^{r} p_v S_v \\ &= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v \sum_{r=v}^{n} \frac{q_r}{P_r} \\ &= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v f_v . \end{aligned}$$

$$\begin{aligned} T_n &= V_n - V_{n-1} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{n} p_r S_r f_r + \frac{p_n S_n f_n}{Q_{n-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{v} p_r f_r \sum_{v=0}^{r} a_v \lambda_v + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \qquad (15) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \frac{\lambda_v}{P_{v-1}} \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \frac{\lambda_v}{P_{v-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} Q_{r-1} a_r \right) \Delta_v \left(\frac{\lambda_v}{Q_{v-1}} \sum_{r=v}^{n} p_r f_r \right) + \left(\sum_{v=1}^{n} Q_{v-1} a_v \right) \frac{\lambda_n p_n f_n}{Q_{n-1}} \right) \\ &\qquad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} P_{r-1} a_r \right) \Delta \left(\frac{\lambda_v}{P_{v-1}} \right) + \left(\sum_{v=1}^{n} P_{v-1} a_v \right) \frac{\lambda_n}{P_{n-1}} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v F_v + P_{v-1} t_v \lambda_v f_v + \frac{P_{v-1}}{p_v} t_v \Delta\lambda_v F_{v+1} \right) \right) + P_n t_n \lambda_n f_n \\ &\qquad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(p_v t_v \lambda_v + \frac{P_{v-1}}{p_v} t_v \Delta\lambda_v \right) \right) + \frac{P_n}{p_n} t_n \lambda_n , \\ &= \sum_{j=1}^{7} T_{nj} . \end{aligned}$$

In order to complete the proof, it sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{nj} \right|^k < \infty, \quad j = 1,2,3,4,5,6,7.$$

Applying Holder's inequality,

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n1}|^k = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} t_v \lambda_v F_v\right|^k$$
$$\leq \sum_{n=2}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}\sum_{v=1}^{n-1}\frac{1}{q_v^{k-1}}|t_v|^k|\lambda_v|^k F_v^{\,k}\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1}$$
$$= \mathrm{O}(1)\sum_{v=1}^{m}\frac{1}{q_v^{k-1}}|t_v|^k|\lambda_v|^k F_v^k \sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}$$
$$= \mathrm{O}(1)\sum_{v=1}^{m} v^{k-1}|t_v|^k \frac{|\lambda_v|^k F_v^k}{Q_v^k} = \mathrm{O}(1).$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1} P_{v-1}\, t_v \lambda_v f_v\right|^k$$
$$\leq \sum_{n=2}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_v^k}{q_v^{k-1}}|t_v|^k|\lambda_v|^k f_v^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1}$$
$$= \mathrm{O}(1)\sum_{v=1}^{m}\frac{P_v^k}{q_v^{k-1}}|t_v|^k|\lambda_v|^k f_v^k \sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}$$
$$= \mathrm{O}(1)\sum_{v=1}^{m} v^{k-1}|t_v|^k \frac{P_v^k}{Q_v^k}|\lambda_v|^k f_v^k$$
$$= \mathrm{O}(1)\sum_{v=1}^{m} v^{k-1}|t_v|^k = \mathrm{O}(1).$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n3}|^k = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v} t_v\, \Delta\lambda_v\, F_{v+1}\right|^k$$
$$\leq \sum_{n=1}^{m+1} n^{k-1}\frac{q_n^k}{Q_n^k Q_{n-1}}\sum_{v=1}^{n-1}\frac{P_v^k}{p_v^k q_v^{k-1}}|t_v|^k|\Delta\lambda_v|^k F_v^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^k$$
$$= \mathrm{O}(1)\sum_{v=1}^{m}\frac{P_v^k}{p_v^k q_v^{k-1}}|t_v|^k|\Delta\lambda_v|^k F_v^k\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}}$$
$$= \mathrm{O}(1)\sum_{v=1}^{m} v^{k-1}|t_v|^k\,|\Delta\lambda_v|^k F_v^k \frac{P_v^k}{p_v^k Q_v^k} = \mathrm{O}(1).$$

$$\sum_{n=1}^{m} n^{k-1}|T_{n4}|^k = \sum_{n=1}^{m} n^{k-1}|P_n\, t_n \lambda_n f_n|^k$$
$$= \mathrm{O}(1)\sum_{n=1}^{m} n^{k-1}|t_n|^k\,|\lambda_n|^k P_n^k f_n^k = \mathrm{O}(1). \quad ,$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n5}|^k = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=1}^{n-1} p_v t_v \lambda_v\right|^k$$

$$= O(1)\sum_{v=1}^{m} p_v^k |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}}$$
$$= O(1)\sum_{v=1}^{m} v^{k-1} |t_v|^k |\lambda_v|^k \frac{p_v^k}{Q_v^k} = O(1).$$

$$\sum_{n=2}^{m+1} n^{k-1} |T_{n6}|^k = \sum_{n=2}^{m+1} n^{k-1} \left| \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} t_v \, \Delta\lambda_v \right|^k$$
$$\le \sum_{n=2}^{m+1} n^{k-1} \frac{p_n^k q_n^k}{P_n^k Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_v^k}{p_v^k q^{k-1}} |t_v|^k |\Delta\lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1}$$
$$= O(1)\sum_{v=1}^{m} \frac{P_1^k}{p_v^k q_v^{k-1}} |t_v|^k |\Delta\lambda_v|^k \sum_{n=v+1}^{m+1} n^{k-1} \frac{p_n^k q_n^k}{P_n^k Q_{n-1}}$$
$$= O(1)\sum_{v=1}^{m} v^{k-1} |t_v|^k |\Delta\lambda_v|^k \frac{P_v^k}{p_v^k Q_v^k} = O(1).$$

Finally

$$\sum_{n=1}^{m} n^{k-1} |T_{n7}|^k = \sum_{n=1}^{m} n^{k-1} \left| \frac{P_n}{p_n} t_n \lambda_n \right|^k$$
$$= O(1)\sum_{n=1}^{m} n^{k-1} |t_n|^k |\lambda_n|^k \frac{P_n^k}{p_n^k} = O(1).$$

This completes the proof of the theorem .

Theorem 2 . Let

(16) $$\sum_{n=v+1}^{\infty} \frac{n^{k-1} p_n^k}{P_v^k P_{n-1}} = O\left(\frac{(vp_v)^{k-1}}{P_v^k} \right),$$

(17) $$P_v = O(p_v Q_v),$$

(18) $$Q_n = O(nq_n).$$

Then necessary conditions for the implication (8) to be satisfied are

$$\lambda_v = O\left(\frac{Q_v Q_{v-1}}{(1+F_v) q_v} \frac{p_v}{P_v} \right), \quad \lambda_v = O\left(\frac{v^{1-1/k} Q_v}{P_v f_v} \right), \qquad \Delta\lambda_v = O\left(\frac{v^{1-1/k} Q_v}{(1+F_{v+1})} \frac{p_v}{P_v} \right).$$

Proof . For $k \ge 1$ define

$$A^* = \left\{ (a_j) : \sum a_j \ is \ summable \ |R, p_n|_k \right\},$$
$$B^* = \left\{ (b_j) : \sum b_j \lambda_j \ is \ summable \ |(R, q_n)(R, p_n)|_k \right\}.$$

From (15), we have

(19) $$T_n = \sum_{v=1}^{n} \left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) a_v \lambda_v$$

With t_n *and* T_n as defined by (14) and (19), the spaces A^* *and* B^* are BK-spaces with norms defined by

(20) $$\|c\|_1 = \left\{ |t_0|^k + \sum_{n=1}^{\infty} n^{k-1} |t_n|^k \right\}^{1/k},$$

(21) $$\|c\|_2 = \left\{ |T_0|^k + \sum_{n=1}^{\infty} n^{k-1} |T_n|^k \right\}^{1/k}.$$

respectively. By the hypothesis of the theorem,

(22) $$\|c\|_1 < \infty \;\Rightarrow\; \|c\|_2 < \infty .$$

The inclusion map $i: A^* \to B^*$ defined by i(a) = a is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exist a constant K>0 such that

(23) $$\|c\|_2 \le K \|c\|_1 .$$

Let e_n denote the nth coordinate vector. From (14) and (19), with (a_n) defined by $a_n = e_n - e_{n+1}$, $n = v$, $a_n = 0$ otherwise, we have

$$t_n = \begin{cases} 0, & n < v \\ \dfrac{p_v}{P_v}, & n = v \\ -\dfrac{p_n p_v}{P_n P_{n-1}}, & n > v . \end{cases}$$

and

$$T_n = \begin{cases} 0, & n < v \\ \left(\dfrac{q_v F_v}{Q_v Q_{v-1}} + \dfrac{p_v q_v}{P_v Q_{v-1}} \right) \lambda_v & n = v \\ \Delta_v \left(\left(\dfrac{q_n F_v}{Q_n Q_{n-1}} + \dfrac{p_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right), & n > v . \end{cases}$$

From (20) and (21), we have

(24) $$\|c\|_1 = \left\{ v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{p_n p_v}{P_n P_{n-1}}\right)^k \right\}^{1/k},$$

(25) $$\|c\|_2 = \left\{ v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k \right\}^{1/k}$$

Applying (23), we obtain

(26) $$v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k$$

$$= O(1)\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{p_n p_v}{P_n P_{n-1}}\right)^k \right).$$

As the R.H.S of (26), by (16), is

$$= O(1)\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \frac{p_v^k}{P_v^{k-1}} \sum_{n=v+1}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} \right)$$

$$= O(1)\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \left(\frac{p_v}{P_v}\right)^{k-1} v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right)$$

$$= O\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right),$$

and the fact that each term of the L.H.S of (26) is $O\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right)$, we obtain

$$v^{k-1}\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)^k |\lambda_v|^k = O\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right),$$

which implies by (17)

$$\left(\frac{q_v}{Q_v Q_{v-1}}\right)^k (1+F_v)^k |\lambda_v|^k = O\left(\frac{p_v}{P_v}\right)^k,$$

that is

$$|\lambda_v| = O\left(\frac{Q_{v-1} Q_v}{(1+F_v) q_v} \frac{p_v}{P_v}\right).$$

Also, we have, by (26),

(27) $$\sum_{n=v+1}^{\infty} n^{k-1}\left|\left(\frac{q_n p_v f_v}{Q_n Q_{n-1}}\right)\lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\Delta\lambda_v\right|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right).$$

The above , via the linear independence of λ_v *and* $\Delta\lambda_v$, implies

$$\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}}+\frac{p_n q_n}{P_n Q_{n-1}}\right)^k |\Delta\lambda_v|^k = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right)$$

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right) \qquad \text{(by (17))}$$

As, by (18), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty} \Delta\left(\frac{1}{Q_{n-1}^k}\right) = \mathrm{O}(1)\sum_{n=v+1}^{\infty}\frac{|\Delta Q_{n-1}^k|}{Q_n^k Q_{n-1}^k} = \mathrm{O}(1)\sum_{n=v+1}^{\infty}\frac{Q_{n-1}^{k-1} q_n}{Q_n^k Q_{n-1}^k} = \mathrm{O}(1)\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k,$$

we get

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \frac{1}{Q_v} = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right),$$

which implies

$$\Delta\lambda_v = \mathrm{O}\left(\frac{v^{1-1/k} Q_v}{1+F_{v+1}}\frac{p_v}{P_v}\right).$$

Also, by (26),

$$\sum_{n=v+1} n^{k-1}\left|\frac{q_n p_v f_v}{Q_n Q_{n-1}}\lambda_v\right|^k = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right),$$

$$p_v^k f_v^k |\lambda_v|^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right),$$

$$p_v^k f_v^k |\lambda_v|^k \frac{1}{Q_v^k} = \mathrm{O}\left(v^{k-1}\left(\frac{p_v}{P_v}\right)^k\right),$$

which implies

$$\lambda_v = \mathrm{O}\left(\frac{v^{1-1/k} Q_v}{P_v f_v}\right).$$

3. Applications

Corollary 3. Let $k \geq 1$. Define

(28) $$f_v = \sum_{r=v}^{n}\frac{q_r}{r}, \qquad F_v = \sum_{r=v}^{n} f_r .$$

Let

(29) $Q_v = \mathrm{O}(v)$,

(30) $f_v = \mathrm{O}(q_v)$.

Then sufficient conditions for the implication

(31) $\sum a_n$ is summable $|R, q_n|_k \Rightarrow \sum a_n \lambda_n$ is summable $|(R, q_n)(C,1)|_k$

are

(32) $|\lambda_n| < Q_n$,

(33) $|\lambda_v| f_v = \mathrm{O}(1)$,

(34) $|\lambda_v| F_v = \mathrm{O}(Q_v)$,

(35) $|\Delta\lambda_v| F_v = \mathrm{O}(q_v)$,

(36) $|\Delta\lambda_v| = \mathrm{O}(q_v)$.

Proof. Follows from theorem 1 by putting $p_n = 1$ for all n.

Corollary 4. Let $k \geq 1$. Define

(37) $$f_v = \sum_{r=v}^{n} \frac{1}{P_r}, \qquad F_v = \sum_{r=v}^{n} p_r f_r .$$

Let

(38) $vp_v = \mathrm{O}(P_v)$,

(39) $P_v f_v = \mathrm{O}(v)$.

Then sufficient conditions for the implication

(40) $\sum a_n$ is summable $|C,1|_k \Rightarrow \sum a_n \lambda_n$ is summable $|(C,1)(R, p_n)|_k$

are

(41) $|\lambda_n| < n$,

(42) $vp_v |\lambda_v| = \mathrm{O}(P_v)$,

(43) $|\lambda_v| F_v = \mathrm{O}(v)$,

(44) $|\Delta\lambda_v| F_v = \mathrm{O}(1)$,

(45) $|\Delta\lambda_v| = \mathrm{O}(1)$.

Proof. This follows from theorem 1, by putting $q_n = 1$ for all n, noticing that (3) is obviously satisfied as

(46) $$\sum_{n=v+1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=v+1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{v}.$$

Corollary 5. Let f_v, F_v be as defined in (28) . Let (18) be satisfied and

(47) $n = \mathrm{O}(Q_n)$.

Then necessary conditions for the implication (31) are

$$\lambda_v = \mathrm{O}\left(\frac{Q_v Q_{v-1}}{(1+F_v) v q_v} \right), \quad \lambda_v = \mathrm{O}\left(\frac{v^{1-1/k} Q_v}{v f_v} \right), \qquad \Delta\lambda_v = \mathrm{O}\left(\frac{Q_v}{(1+F_{v+1}) v^{1/k}} \right).$$

Proof. Follows from theorem 4 by putting $p_n = 1$ for all n, noticing that (16) is satisfied, the same as (46) .

Corollary 6. Let f_v , F_v be as defined in (36) . Let (16) be satisfied and

(48) $$P_v = \mathrm{O}\left(vp_v\right).$$

Then necessary conditions for the implication (40) are

$$\lambda_v = \mathrm{O}\left(\frac{v^2}{1+F_v}\frac{p_v}{P_v}\right), \quad \lambda_v = \mathrm{O}\left(\frac{v^{2-1/k}}{P_v f_v}\right), \quad \Delta\lambda_v = \mathrm{O}\left(\frac{v^{2-1/k}}{1+F_{v+1}}\frac{p_v}{P_v}\right).$$

Proof. Follows from theorem 4 by putting $q_n = 1$ for all n .

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