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A New System of variational Inclusions with (H, η)-accretive Operators in Banach Spaces

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Abstract In this paper, we introduce and study a new system of variational inclusions involving (H, η)-accretive operators, we prove the existence and uniqueness of solutions for this new system of variational inclusions. We also construct a new algorithm for approximating the solution of this system and discuss the convergence of the sequence of iterates generated by the algorithm.

Key words (H, η)-accretive operator, Resolvent operator technique, System of variational inclusion, Iterative algorithm

1.INTRODUCTION

Variational inclusions are important generalization of classical variational inequalities and thus, have wide applications to many field including, for example, mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences. For these reasons, variational inclusions have been intensively studied in recent years. For details, we refer the reader to [1-12] and the reference therein.

Recently, some interesting and important problems related to variational inequalities and complementarity problems have been considered by many authors. Ansari and Yao [1] studied a system of variational inequalities using a fixed point theorem. Huang and Fang [9] introduced a system of order complementurity problems and established some existence results for there using fixed-point theory. Kassay and Kolumbán [10] introduced a system of variational inequalities and proved an existence theorem using Fan's lemma. Kassay, Kolumbán and Páles [11] introduced and studied Minty and stampacchia variational inequality systems using the Kakutani-Fan-Glicksberg fixed-point theorem. In [12], Verma introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions

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of these systems of variational inequalities. Very recently, Fang and Huang [4] introduced a system of variational inclusions and developed a Mann iterative algorithm to approximate the unique solution of the system.

On the other hand, monotonicity technique were extended and applied in recent years because of there importance in theory of variational inequalities, complementarity problems, and variational inclusions. In [8], Huang and Fang introduced a class of generalized monotone operators, maximal η-monotone operators, and defined an associated resolvent operator. Using resolvent operator methods, they developed some iterative algorithms to approximate the solution of a class of general variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator due to Ding and Luo [2]. In [3], Fang and Huang introduced another class of generalized monotone operators, H-monotone operators, and defined an associated resolvent operator. They also established the Lipschitz continuity of the resolvent operator and studied a class of variational inclusions in Hilbert spaces using the resolvent operator. In recent paper [5], Fang and Huang further introduced a new class of generalized monotone operators, (H, η)-monotone operators, which provide a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. They also studied a class of variational inclusions using the resolvent operator associated with an (H, η)-monotone operator. In [6], Fang and Huang generalize the resolvent operator technique by introducing a new class of H-accretive operators in Banach spaces. They extend the concept of resolvent operators.

Motivated and inspired by above works, in this paper, we introduce and study a new system of variational inclusions involving (H, η)-accretive operators in Banach spaces. Using the resolvent operator method associated with (H, η)-accretive operators, we prove the existence and uniqueness of solutions for this new system of variational inclusions. We also construct a new algorithm for approximating the solution of the system of variational inclusions and discuss the convergence of iterative sequence generated by the algorithm. The present results improve and extend many known results in the literature.

2. PRELIMINARIES

Let X be a real Banach space endowed with a norm $\|\cdot\|$ and dual space $X^*, \langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and 2^X denote the family of all nonempty subsets of X. The generalized dual mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q \text{and } \|f^*\| = \|x\|^{q-1}\}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized dual mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$, for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that X is a real

Banach space such that J_q is single-valued. The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1; \|x\| < 1, \|y\| \le t\}.$$

A Banach space X is called uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_X(t)}{t} = 0$, X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \le ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. The following theorem gives characteristic inequalities in q-uniformly smooth Banach spaces.

Theorem X[**6**]

Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all $x, y \in X$,

$$\|x+y\|^q \le \|x\|^q + q < y, J_q(x) > + c_q\|y\|^q,$$

Definition 2.1 Let $T, A : X \to X$ be two single-valued operators. The operator T is said to be

(i) accretive if

$$\langle Tx - Ty, J_q(x-y)\rangle \ge 0, \forall x, y \in X;$$

(ii) strictly accretive if

$$\langle Tx - Ty, J_q(x-y)\rangle \ge 0, \forall x, y \in X,$$

and the equality holds if and only if $x = y$;

(iii) strongly accretive if

$$\langle Tx - Ty, J_q(x-y)\rangle \ge r\|x-y\|^q, \forall x, y \in X;$$

(iv) strongly accretive with respect to H if there exists some constant $\gamma > 0$, such that

$$\langle T_x - T_y, J_q(H(x) - H(y))\rangle \ge \gamma\|x-y\|^q, \forall x, y \in X;$$

(v) Lipschitz continuous if there exists some constant $s > 0$, such that

$$\|T_x - T_y\| \le s\|x-y\|, \forall x, y \in X$$

Remark 2.1 If T and H are Lipschitz continuous with constants τ and s, respectively, and T is strongly accretive with respect to H with a constant γ, then $\gamma \le \tau s^{q-1}$.

Example 2.1 Let $X = (-\infty, +\infty), Tx = -x$ and $Hx = -2x$, for all $x \in X$. Then, T is strongly accretive with respect to H, but T is not strongly accretive.

Example 2.1 Show that the strong accretivity of T with respect to H is a generalization of the strong accretivity of T.

Definition 2.2 A single-valued operator $\eta : X \times X \to X$ is said to be

(i) accretive if

$$\langle J_q(u-v), \eta(u,v)\rangle \geq 0, \forall u, v \in X;$$

(2) strictly accretive if

$$\langle J_q(u-v), \eta(u,v)\rangle \geq 0, \forall u, v \in X$$

and equality holds if and only if $u = v$;

(3)strongly accretive if there exists a constant $\delta > 0$, such that

$$\langle J_q(u-v), \eta(u,v)\rangle \geq \delta\|u-v\|^q, \forall u, v \in X;$$

Definition 2.3 Let $\eta : X \times X \to X$ and $H : X \to X$ be two single-valued operators and let $M : X \to 2^X$ be a multivalued operator. M is said to be:

(1)accretive if

$$\langle J_q(x-y), u-v\rangle \geq 0, \forall u, v \in X, x \in Mu, y \in Mv;$$

(2)η-accretive if

$$\langle J_q(x-y), \eta(u,v)\rangle \geq 0, \forall u, v \in X, x \in Mu, y \in Mv;$$

(3) strictly η-acretive if M is η-accretive and equality holds if and only if $u = v$;

(4) strongly η-accretive if there exists some constant $r > 0$, such that

$$\langle J_q(x-y), \eta(u,v)\rangle \geq r\|u-v\|^2, \forall u, v \in X, x \in M_u, y \in M_v;$$

(5) m-accretive if M is accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$, where I denotes the identity operator on X;

(6) (m, η)-accretive if M is η-accretive and $(I + \lambda M)(X) = X$, for all $\lambda > 0$;

(7) H-accretive if M is accretive and $(H + \lambda M)(X) = X$, for all $\lambda > 0$;

(8) (H, η)-accretive if M is η-accretive and $(H + \lambda M)(X) = X$, for all $\lambda > 0$.

Remark 2.1 If $X = H$, a Hilbert space, then we can obtain the corresponding definition of maximal η-monotone operators, H-monotone operators, and (H, η)-monotone operators, which were first introduced in Huang and Fang[8], Fang and Huang[3], respectively, obviously, the class of (H, η)-accretive operators provides a unifying framework for classes of M-accretive operators, $(m, \eta)-$accretive operators and $H-$ accretive operators.

For our results, we need the following lemmas.

Lemma 2.1 Let $\eta : X \times X \to X$ be a single-valued operator, $H : X \to X$ be strictly η-accretive operator and $M : X \to 2^X$ be an (H, η)-acretive operator. Then, the operator $(H + \lambda M)^{-1}$ is single-valued, where $\lambda > 0$ is a constant.

Proof. Let $u \in X, x, y \in (H+\lambda M)^{-1}(u)$. It follows that $-H(x)+u \in \lambda M(x)$ and $-H(y)+u \in \lambda M(y)$. Since M is (H,η)-accretive,

$$\langle \eta(x,y), J_q(H(x)-H(y))\rangle \geq 0$$

The strict accretiveness of H implies that $x=y$. Thus, $(H+\lambda M)^{-1}$ is simple-valued. The proof is complete.

Based on Lemma 2.1, we can define the resolvent operator $R^{H,\eta}_{M,\lambda}$ associated with H and M as follows.

Definition 2.4 Let $\eta : X \times X \to X$ be a single-valued operator, $H : X \to X$ be a strictly $\eta-$ accretive operator and $M : X \to 2^X$ be an (H,η)-accretive operator. The resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is defined by

$$R^{H,\eta}_{M,\lambda}(u) = (H+\lambda M)^{-1}(u), \forall u \in X$$

Lemma 2.2 Let $\eta : X \times X \to X$ be a single-valued Lipschitz continous operator with constant $\tau, H : X \to X$ be a strongly η-accretive operator with constant r and $M : X \to 2^X$ be an (H,η)-accretive operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : X \to X$ is Lipschitz continuous with constant τ^{q-1}/r, that is,

$$\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\tau^{q-1}}{r}\|u-v\|, \forall u, v \in X,$$

Proof. Let u, v be any given points in X. It follows that

$$R^{H,\eta}_{M,\lambda}(u) = (H+\lambda M)^{-1}(u)$$

and

$$R^{H,\eta}_{M,\lambda}(v) = (H+\lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}(u - H(R^{H,\eta}_{M,\lambda}(u))) \in M(R^{H,\eta}_{M,\lambda}(u))$$

and

$$\frac{1}{\lambda}(v - H(R^{H,\eta}_{M,\lambda}(v))) \in M(R^{H,\eta}_{M,\lambda}(v)).$$

Since M is η-accretive,

$$\begin{aligned}&\langle \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v)), J_q(\tfrac{1}{\lambda}(u - H(R^{H,\eta}_{M,\lambda}(u)))\\ &-\tfrac{1}{\lambda}(v - H(R^{H,\eta}_{M,\lambda}(v))))\rangle \geq 0.\end{aligned}$$

The inequality above implies that

$$\begin{aligned}&\tau^{q-1}\|u-v\| \cdot \|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^{q-1}\\ &\geq \|u-v\|\|J_q(\eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\|\\ &\geq \langle J_q(u-v), \eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\rangle\\ &\geq \langle J_q(H(R^{H,\eta}_{M,\lambda}(u)) - H(R^{H,\eta}_{M,\lambda}(v))),\\ &\eta(R^{H,\eta}_{M,\lambda}(u), R^{H,\eta}_{M,\lambda}(v))\rangle\\ &\geq r\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\|^q.\end{aligned}$$

Hence,

$$\|R_{M,\lambda}^{H,\eta}(u) - R_{M,\lambda}^{H,\eta}(v)\| \le \frac{\tau^{q-1}}{r}\|u-v\|, \forall u, v \in X.$$

3. A NEW SYSTEM OF VARIATIONAL INCLUSIONS

In this section, we shall introduce a new system of variational inclusions involuing (H,η)-accretive operators in Banach spaces. In what follows, unless otherwise specified, we always assume that X_1 and X_2 are two real Banach spaces, $A \subset X_1$ and $B \subset X_2$ are two nonempty, closed and convex sets. Let

$$F : X_1 \times X_2 \to X_1, G : X_1 \times X_2 \to X_2,$$

$H_1 : X_1 \to X_1, H_2 : X_2 \to X_2$, and $\eta : X \times X \to X$ be five operators. Moreover, let $M : X_1 \to 2^{X_1}$ be an (H_1, η)-accretive operator and $N : X_2 \to 2^X$ be an (H_2, η)-accretive operator. The system of variational inclusions is formulated as follows.

Find $(a, b) \in X_1 \times X_2$, such that

$$\begin{aligned} 0 &\in F(a,b) + M(a), \\ 0 &\in G(a,b) + N(b). \end{aligned} \tag{3.1}$$

Some examples of problem (3.1) include the following.

(I) If $X_1 = \mathcal{H}_1$, and $X_2 = \mathcal{H}_2$ two Hilbert spaces, then, problem (3.1) reduces to the following problem. Find $(a, b) \in A \times B$, such that

$$\begin{aligned} 0 &\in F(a,b) + M(a), \\ 0 &\in G(a,b) + N(b). \end{aligned} \tag{3.2}$$

(II) If $X_1 = \mathcal{H}_1$ and $X_2 = \mathcal{H}_2$, two Hilbert spaces, $M(x) = \partial\varphi(x)$ and $N(y) = \partial\Phi(y)$, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, where $\varphi : \mathcal{H}_1 \to R \cup \{+\infty\}$ and $\Phi : \mathcal{H}_2 \to R \cup \{+\infty\}$ are two proper, convex, lower semicontinuous functionals, and $\partial\varphi$ and $\partial\Phi$ denote the subdifferential operators of φ and Φ, respectively, then problem (3.1) reduces to the following problem: find $(a, b) \in A \times B$, such that

$$\begin{aligned} \langle F(a,b), x-a\rangle + \varphi(x) - \varphi(a) \ge 0, \forall x \in \mathcal{H}_1, \\ \langle G(a,b), y-b\rangle + \Phi(y) - \Phi(b) \ge 0, \forall y \in \mathcal{H}_2, \end{aligned} \tag{3.3}$$

which is called a system of nonlinear variational inequalities.

(III) If $X_1 = X_2 = \mathcal{H}_1$, a Hilbert space, $A = B, F(x,y) = \rho T(y) + x - y$, and $G(x,y) = \gamma T(x) + y - x$, for all $x, y \in \mathcal{H}$, where $T : A \to \mathcal{H}$ is a nonlinear mapping, and $\rho > 0$ and $\gamma > 0$ are two constants, then problem (3.3) reduces to the following system of variational inequalities: find $(a, b) \in A \times A$, such that

$$\begin{aligned} \langle \rho T(b) + a - b, x - a\rangle \ge 0, \forall x \in A, \\ \langle \gamma T(a) + b - a, x - b\rangle \ge 0, \forall x \in A, \end{aligned} \tag{3.5}$$

which is the system of nonlinear variational inequalities considered by verma [12].

4.EXISTENCE AND UNIQUENESS

In this section, we will prove existence and uniqueness for solutions of problem (3.1). For our main results, we have the following characterization of solutions of problem (3.1).

Lemma 4.1 Let $\eta : X \times X$ be a single-valued operator, $H_1 : X_1 \to X_1$ and $H_2 : X_2 \to X_2$ be two strictly η-accretive operators, $M : X_1 \to X_1$ be (H_1, η)-accretive, and $N : X_2 \to X_2$ be $(H_2, \eta)-$accretive. Then, for any given $(a, b) \in X_1 \times X_2, (a, b)$ is a solution of problem (3.1) if and only if (a, b) satisfies

$$\begin{aligned} a &= R^{H,\eta}_{M,\rho}[H_1(a) - \rho F(a, b)], \\ b &= R^{H,\eta}_{N,\lambda}[H_2(b) - \lambda G(a, b)], \end{aligned}$$

where $\lambda > 0$ and $\rho > 0$ are two constants.

Proof. The fact directly follows from Definition 2.4.

Theorem 4.1 let $\eta : X \times X \to X$ be a Lipschitz continuous operator with constant σ. Let $H_1 : X_1 \to X_1$ be a strongly η-accretive, Lipschitz continuous operator with constant γ_1, τ_1, and $H_2 : X_2 \to X_2$ be a strongly η-accretive, Lipschitz continous operator with constants γ_2 and τ_2. Let $M : X_1 \to 2^{X_1}$ be (H_1, η)-accretive and $N : X_2 \to 2^{X_2}$ be (H_2, η)-accretive. Let $F : X_1 \times X_2 \to X_1$ be an operator, such that, for any given $(a, b) \in X_1 \times X_2, F(\cdot, b)$ is strongly accretive with respect to H_1 and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a, \cdot)$ is Lipscaitz continuous with a constant θ, let $X_1 \times X_2 \to X_2$ be an operator, such that, for any given $(x, y) \in X_1 \times X_2, G(x, \cdot)$ is strongly accretive with a constant ξ. Let there exist constants $\rho > 0$ and $\lambda > 0$, such that

$$\begin{aligned} &\gamma_2\sigma^{q-1}\sqrt[q]{\tau_1^q - q\rho r_1 + c_q\rho^q s_1^q} + \gamma_1\sigma^{q-1}\lambda\xi < \gamma_1\gamma_2, \\ &\gamma_1\sigma^{q-1}\sqrt[q]{\tau_2^q - q\lambda r_2 + c_q\rho^q s_2^q} + \gamma_2\sigma^{q-1}\rho\theta < \gamma_1\gamma_2, \end{aligned}$$

Then, problem (3.1) admits a unique solutions.

Proof. For any given $\lambda > 0$ and $\rho > 0$, define $T_\rho : X_1 \times X_2 \to X_1$ and $S_\lambda : X_1 \times X_2 \to X_2$ by

$$T_\rho(u, v) = R^{H_1,\eta}_{M,\rho}[H_1(u) - \rho F(u, v)] \tag{4.2}$$

and

$$S_\lambda(u, v) = R^{H_2,\eta}_{N,\lambda}[H_2(v) - \lambda G(u, v)],$$

for all $(u, v) \in X_1 \times X_2$.

For any $(u_1, v_1), (u_2, v_2) \in X_1 \times X_2$, it follows from (4.2) and Lemma 2.2 that

$$\begin{aligned} &\|T_\rho(u_1, v_1) - T_\rho(u_2, v_2)\| \\ &\le \tfrac{\sigma^{q-1}}{\gamma_1}\|H_1(u_1) - H_1(u_2) - \rho(F(u_1, v_1) - F(u_2, v_2))\| \\ &\le \tfrac{\sigma^{q-1}}{\gamma_1}\|H_1(u_1) - H_1(u_2) - \rho(F(u_1, v_1) - F(u_2, v_1))\| \\ &\quad + \tfrac{\rho\sigma^{q-1}}{\gamma_1}\|F(u_2, v_1) - F(u_2, v_2)\| \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\|S_\lambda(u_1,v_1)-S_\lambda(u_2,v_2)\|\\ &\le \tfrac{\sigma^{q-1}}{\gamma_2}\|H_2(v_1)-H_2(v_2)-\lambda(G(u_1,v_1)-G(u_2,v_2))\|\\ &\le \tfrac{\sigma^{q-1}}{\gamma_2}\|H_2(v_1)-H_2(v_2)-\lambda(G(u_1,v_1)-G(u_1,v_2))\|\\ &\quad+\tfrac{\lambda\sigma^{q-1}}{\gamma_2}\|G(u_1,v_2)-G(u_2,v_2)\|\end{aligned}\tag{4.4}$$

By assumptions and Theorem X we have

$$\begin{aligned}&\|H_1(u_1)-H_1(u_2)-\rho(F(u_1,v_1)-F(u_2,v_1))\|^q\le\|(H_1(u_1)-H_1(u_2)\|^q\\ &-q\rho\langle F(u_1,v_1)-F(u_2,v_1),J_q(H_1(u_1)-H_1(u_2))\rangle\\ &\quad+\rho^q c_q\|F(u_1,v_1)-F(u_2,v_1)\|^q\\ &\le(\tau_1^q-q\rho r_1+c_q\rho^q s_1^q)\|u_1-u_2\|^q\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&\|H_2(v_1)-H_2(v_2)-\lambda(G(u_1,v_1)-G(u_1,v_2))\|^q\\ &\le(\tau_2^q-q\lambda r_2+c_q\lambda^q s_2^q)\|v_1-v_2\|^q.\end{aligned}\tag{4.6}$$

Furthermore,

$$\|F(u_2,v_1)-F(u_2,v_2)\|\le\theta\|v_1-v_2\|.\tag{4.7}$$

and

$$\|G(u_1,v_2)-G(u_2,v_2)\|\le\xi\|u_1-u_2\|.\tag{4.8}$$

It follows from (4.3)—(4.8) that

$$\begin{aligned}&\|T_\rho(u_1,v_1)-T_\rho(u_2,v_2)\|\\ &\le\tfrac{\sigma^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q-q\rho r_1+c_q\rho^q s_1^q}\|u_1-u_2\|\\ &\quad+\tfrac{\sigma^{q-1}\rho\theta}{\gamma_1}\|v_1-v_2\|,\\ &\|S_\lambda(u_1,v_1)-S_\lambda(u_2,v_2)\|\\ &\le\tfrac{\sigma^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q-q\lambda r_2+c_q\lambda^q s_2^q}\|v_1-v_2\|\\ &\quad+\tfrac{\sigma^{q-1}\lambda\xi}{\gamma_2}\|u_1-u_2\|.\end{aligned}\tag{4.9}$$

Now, (4.9) implies that

$$\begin{aligned}&\|T_\rho(u_1,v_1)-T_\rho(u_2,v_2)\|+\|S_\lambda(u_1,v_1)-S_\lambda(u_2,v_2)\|\\ &\le(\tfrac{\sigma^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q-q\rho r_1+c_q\rho^q s_1^q}+\tfrac{\sigma^{q-1}\lambda\xi}{\gamma_2})\|u_1-u_2\|\\ &+(\tfrac{\sigma^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q-q\lambda r_2+c_q\lambda^q s_2^q}+\tfrac{\sigma^{q-1}\rho\theta}{\gamma_1})\|v_1-v_2\|\\ &\le k(\|u_1-u_2\|+\|v_1-v_2\|),\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}&k=\max\{\tfrac{\sigma^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q-q\rho r_1+c_q\rho^q s_1^q}+\tfrac{\sigma^{q-1}\lambda\xi}{\gamma_2},\\ &\tfrac{\sigma^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q-q\lambda r_2+c_q\lambda^q s_2^q}+\tfrac{\sigma^{q-1}\rho\theta}{\gamma_1}\}\end{aligned}$$

Define $\|\cdot\|_1$ on $X_1\times X_2$ by

$$\|(u,v)\|_1=\|u\|+\|v\|,\forall(u,v)\in X_1\times X_2.$$

It is easy to see that $(X_1 \times X_2, \|\cdot\|_1)$ is a Banach space. For any given $\lambda > 0$ and $\rho > 0$, define $Q_{\lambda,\rho} : X_1 \times X_2 \to X_1 \times X_2$ by

$$Q_{\lambda,\rho}(u,v) = (T_\rho(u,v), S_\lambda(u,v)), \forall (u,v) \in X_1 \times X_2.$$

By (4.1), we know that $0 < k < 1$. It follows from (4.10) that

$$\|Q_{\lambda,\rho}(u_1,v_1) - Q_{\lambda,\rho}(u_2,v_2)\|_1 \leq k\|(u_1,v_1) - (u_2,v_2)\|_1.$$

This proves that $Q_{\lambda,\rho} : X_1 \times X_2 \to X_1 \times X_2$ is a contraction operator. Hence, there exists a unique $(a,b) \in X_1 \times X_2$, such that

$$Q_{\lambda,\rho}(a,b) = (a,b),$$

that is,

$$\begin{aligned} a &= R^{H_1,\eta}_{M,\rho}[H_1(a) - \rho F(a,b)], \\ b &= R^{H_2,\eta}_{N,\lambda}[H_2(b) - \lambda G(a,b)]. \end{aligned}$$

By Lemma 4.1, (a,b) is the unique solution of problem (3.1).

Remark 4.1 From Theorem 4.1 we can get the existence and uniqueness of solutions for problem (3.1)—(3.5).

Remark 4.2 From Definition 2.1 we know that

$$r_1 \leq s_1\tau_1^{q-1}, \quad r_2 \leq s_2\tau_2^{q-1}$$

Remark 4.3 If X is 2-uniformly smooth and there exists $\rho = \lambda > 0$, such that

$$\begin{aligned} &\left| \rho - \frac{r_1\gamma_2^2\sigma - \gamma_1^2\gamma_2\xi}{\sigma(c_2s_1^2\gamma_2^2 - \gamma_1^2\xi^2)} \right| \\ &< \frac{\sqrt{(r_1\gamma_2^2\sigma - \gamma_1^2\gamma_2\xi)^2 - (\gamma_2^2\tau_1^2\sigma^2 - \gamma_1^2\gamma_2^2)(c_2s_1^2\gamma_2^2 - \gamma_1^2\xi^2)}}{\sigma(c_2s_1^2\gamma_2^2 - \gamma_1^2\xi^2)} \\ &(\sigma^2\tau_1^2 - \gamma_1^2)(c_2s_1^2\gamma_2^2 - \gamma_1^2\xi^2) < (r_1\gamma_2\sigma - \gamma_1^2\xi)^2, \gamma_1^2\xi^2 < c_2s_1^2\gamma_2^2 \end{aligned}$$

and

$$\begin{aligned} &\left| \rho - \frac{r_2\gamma_1^2\sigma - \gamma_2^2\gamma_1\theta}{\sigma(c_2s_2^2\gamma_1^2 - \gamma_2^2\theta^2)} \right| \\ &< \frac{\sqrt{(r_2\gamma_1^2\sigma - \gamma_2^2\gamma_1\theta)^2 - (\gamma_1^2\tau_2^2\sigma^2 - \gamma_2^2\gamma_1^2)(c_2s_2^2\gamma_1^2 - \gamma_2^2\theta^2)}}{\sigma(c_2s_2^2\gamma_1^2 - \gamma_2^2\theta^2)}, \\ &(\sigma^2\tau_2^2 - \gamma_2^2)(c_2s_2^2\gamma_1^2 - \gamma_1^2\theta^2) < (r_1\gamma_1\sigma - \gamma_2^2\theta)^2, \gamma_2^2\theta^2 < c_2s_2^2\gamma_1^2, \end{aligned}$$

then condition (4.1) is satisfied. We note that all Hilbert spaces and L_p(or l_p) spaces $(2 \leq p < \infty)$ are 2-uniformly smooth.

5.ITERATIVE ALGORITHM AND CONVERGENCE

In this section, we will construct the Mann iterative algorithm for approximating the unique solution of problem (3.1) and discuss the convergence analysis of the algorithm.

Lemma 5.1 [7] Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions.

(i) $0 \leq k_n < 1, n = 0, 1, 2, \cdots$ and $\limsup_n k_n < 1$.

(ii) $c_{n+1} \leq k_n c_n, n = 0, 1, 2, \cdots$

Then, c_n converges to 0 as $n \to \infty$.

Algorithm 5.1 Let η, H_1, H_2, M, N, F and G be the same as in Theorem 4.1. For any given $(a_0, b_0) \in X_1 \times X_2$, define the Mann iterative sequence $\{(a_n, b_n)\}$ by

$$\begin{aligned} a_{n+1} &= \alpha_n a_n + (1-\alpha_n) R_{M,\rho}^{H_1,\eta}[H_1(a_n) - \rho F(a_n, b_n)], n = 0, 1, 2, \cdots \\ b_{n+1} &= \alpha_n b_n + (1-\alpha_n) R_{N,\lambda}^{H_2,\eta}[H_2(b_n) - \lambda G(a_n, b_n)], n = 0, 1, 2, \cdots \end{aligned} \tag{5.1}$$

where

$$0 \leq \alpha_n < 1 \quad \text{and} \quad \limsup_n \alpha_n < 1. \tag{5.2}$$

Theorem 5.1 Let η, H_1, H_2, M, N, F, and G be as in Theorem 4.1. Assume that all the conditions of Theorem 4.1 hold. Then, (a_n, b_n) generated by algorithm 5.1 converges strongly to the unique solution (a, b) of problem (3.1) and there exists $d \in [0, 1)$, such that

$$\|a_n - a\| + \|b_n - b\| \leq d^m(\|a_0 - a\| + \|b_0 - b\|), \text{ for all } n \geq 0.$$

Proof. By Theorem 4.1, problem (3.1) admits a unique solution, (a, b). It follows from lemma 4.1 that

$$\begin{aligned} a &= \alpha_n a + (1-\alpha_n) R_{M,\rho}^{H,\eta}[H_1(a) - \rho F(a, b)], \\ b &= \alpha_n b + (1-\alpha_n) R_{N,\lambda}^{H_2,\eta}[H_2(b) - \lambda G(a, b)] \end{aligned} \tag{5.3}$$

By (5.1) and (5.3),

$$\begin{aligned} &\|a_{n+1} - a\| \leq \alpha_n \|a_n - a\| + (1-\alpha_n) \\ &\|R_{M,\rho}^{H_1\eta}[H_1(a_n) - \rho F(a_n, b_n)] - R_{M,\rho}^{H_1,\eta}[H_1(a) - \rho F(a, b)]\| \\ &\leq \alpha_n \|a_n - a\| + (1-\alpha_n) \tfrac{\sigma^{q-1}}{\gamma_1} \\ &\|H_1(a_n) - H_1(a) - \rho(F(a_n, b_n) - F(a, b))\| \\ &\leq \alpha_n \|a_n - a\| + (1-\alpha_n) \tfrac{\sigma^{q-1}}{\gamma_1} \\ &\|H_1(a_n) - H_1(a) - \rho(F(a_n, b_n) - F(a, b_n))\| \\ &+(1-\alpha_n) \tfrac{\sigma^{q-1}\rho}{\gamma_1} \|F(a, b_n) - F(a, b)\| \\ &\leq \alpha_n \|a_n - a\| + (1-\alpha_n) \tfrac{\sigma^{q-1}}{\gamma_1} \sqrt[q]{\tau_1^q - q\rho r_1 + c_q \rho^q s_1^q} \\ &\|a_n - a\| + (1-\alpha_n) \tfrac{\sigma^{q-1}\rho\theta}{\gamma_1} \|b_n - b\| \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\|b_{n+1} - b\| \leq \alpha_n \|b_n - b\| + (1-\alpha_n) \\ &\|R_{N,\lambda}^{H_2,\eta}[H_2(b_n) - \lambda G(a_n, b_n)] - R_{N,\lambda}^{H_2,\eta}[H_2(b) - \lambda G(a, b)]\| \\ &\leq \alpha_n \|b_n - b\| + (1-\alpha_n) \tfrac{\sigma^{q-1}}{\gamma_2} \\ &\sqrt[q]{\tau_2^q - q\lambda r_2 + c_q \lambda^q s_2^q} \|b_n - b\| + (1-\alpha_n) \tfrac{\sigma\lambda\xi}{\gamma_2} \|a_n - a\|. \end{aligned}$$

It follows from (5.4) and (5.5) that

$$\begin{aligned}&\|a_{n+1}-a\|+\|b_{n+1}-b\| \le \alpha_n(\|a_n-a\|+\|b_n-b\|)\\&+(1-\alpha_n)k(\|a_n-a\|+\|b_n-b\|)\\&=(k+(1-k)\alpha_n)(\|a_n-a\|+\|b_n-b\|)\end{aligned}$$

where $0\le k<1$ is defined by

$$\begin{aligned}k\ &=\ \max\{\tfrac{\sigma^{q-1}}{\gamma_1}\sqrt[q]{\tau_1^q-q\rho r_1+c_q\rho^q s_1^q}\\&\quad+\tfrac{\sigma^{q-1}\lambda\xi}{\gamma_2},\tfrac{\sigma^{q-1}}{\gamma_2}\sqrt[q]{\tau_2^q-q\lambda r_1+c_q\lambda^q s_2^q}+\tfrac{\sigma^{q-1}\rho\theta}{\gamma_1}\}.\end{aligned}$$

Let $c_n=\|a_n-a\|+\|b_n-b\|$ and $k_n=k+(1-k)\alpha_n$. Then (5.6) can be rewritten as

$$c_{n+1}\le k_nc_n,\quad n=0,1,2,\cdots$$

By (5.2), we know that $\limsup_n k_n<1$, it follows from lemma 5.1 that $0\le k_n\le d<1$ and that

$$\|a_n-a\|+\|b_n-b\|\le d^n(\|a_n-a\|+\|b_n-b\|),\ \text{for all}\ \ n\ge 0.$$

Therefore, (a_n,b_n) converges geometrically to the unique solution (a,b) of problem (3.1).

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A Class of Generalized Set-valued Variational Inclusions in Smooth Banach Spaces *

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Abstract: In this paper, a class of generalized set-valued variational inclusions in Banach spaces are studied, which include many variational inclusions studied by others in recent years. By using an important inequality, several existence theorems for the generalized set-valued variational inclusions in smooth Banach spaces are established, and some perturbed iterative algorithms for solving this kind of set-valued variational inclusions are suggested and analyzed. Our results improve and generalize many known results.

Key Words and Phrases: Generalized set-valued variational inclusion; iterative algorithm with error; smooth Banach space.

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1 Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques, both for their own sake and for the applications. Useful and important generalizations of variational inequalities are set-valued variational inclusions, which have been studied by several authors.

Recently, Chidume, Zegeye and Kazmi [1] studied the following class of set-valued variational inclusion problems in a Banach space E. For a given m-accretive mapping $A : D(A) \subset E \to 2^E$, a nonlinear mapping $N(\cdot,\cdot) : E \times E \to E$, two set-valued mappings $T, F : E \to CB(E)$ (here $CB(E)$ denotes the family of all nonempty closed and bounded subsets of E) and a single-valued mapping $g : H \to H$, find $q \in E, w \in T(q), v \in F(q)$ such that

$$f \in N(w, v) + \lambda A(g(q)), \tag{1.1}$$

where $f \in E$ is a given point and $\lambda > 0$.

For a suitable choice of the mappings T, F, N, g, A and $f \in E$, a number of known and new variational inequalities, variational inclusions, and related optimization problems can be obtained from (1.1).

Inspired and motivated by the works of [1-3], in this paper, we introduce and study a class of more general set-valued variational inclusions in Banach spaces. By using the inequality of Liu [6], the existence theorem and approximate theorem of solutions of the set-valued variational inclusions in smooth Banach spaces are established and suggested. The results presented in this paper generalize, improve and unify the corresponding results of Chidume, Zegeye and Kazmi [1], Huang and Fang [2, 3], Huang [4], Fang and Huang [5], Liu and Kang [7].

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2 Preliminaries

Let E be a real Banach space, E^* be the topological dual space of E, $\langle \cdot, \cdot \rangle$ be the dual pair between E and E^*, $D(T)$ denotes the domain of T, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$.

Definition 2.1 Let $A : D(A) \subset E \to 2^E$ be a set-valued mapping, $\phi : [0, \infty) \to [0, \infty)$ a strictly increasing function with $\phi(0) = 0$. The mapping A is said to be

(1) accretive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq 0$$

for all $u \in Ax$ and $v \in Ay$;

(2) ϕ-strongly accretive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|$$

for all $u \in Ax$ and $v \in Ay$;

(3) ϕ-expansive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\|u - v\| \geq \phi(\|x-y\|)$$

for all $u \in Ax$ and $v \in Ay$;

(4) ϕ-strongly pseudocontractive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$$

for all $u \in Ax$ and $v \in Ay$;

(5) m-accretive, if A is accretive and $(I + \rho A)D(A) = E$ for all $\rho > 0$, where I is the identity mapping.

Definition 2.2 Let $T, F : E \to 2^E$ be two set-valued mappings, $N(\cdot, \cdot) : E \to E$ a nonlinear mapping, and $\phi : [0, \infty] \to [0, \infty]$ a strictly increasing function with $\phi(0) = 0$.

(1) The mapping $x \to N(x, y)$ is said to be ϕ-strongly accretive with respect to the mapping T if, for any $x_1, x_2 \in E$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle N(u_1, y) - N(u_2, y), j(x_1 - x_2) \rangle \geq \phi(\|x_1 - x_2\|)\|x_1 - x_2\|$$

for all $u_1 \in Tx_1, u_2 \in Tx_2$.

(2) The mapping $y \to N(x, y)$ is said to be accretive with respect to the mapping F if, for any $y_1, y_2 \in E$, there exists $j(y_1 - y_2) \in J(y_1 - y_2)$ such that

$$\langle N(x, v_1) - N(x, v_2), j(y_1 - y_2) \rangle \geq 0$$

for all $v_1 \in Fy_1, v_2 \in Fy_2$.

Definition 2.3 Let $T : E \to CB(E)$ be a set-valued mapping and $H(\cdot, \cdot)$ a Hausdorff metric in $CB(E)$, T is said to be ξ-Lipschitz continuous, if for any $x, y \in E$,

$$H(Tx, Ty) \leq \xi\|x-y\|$$

where $\xi > 0$ is a constant.

Definition 2.4 The set-valued mapping $T : E \to CB(E)$ is said to be uniformly continuous, if for any given $\varepsilon > 0$, there exists a $\delta > 0$, such that for any given $x, y \in E$, when $\|x - y\| < \delta$, we have

$$H(Tx, Ty) \leq \varepsilon,$$

where H is a Hausdorff metric in $CB(E)$.

We also need the following lemmas.

Lemma 2.1 [10] Let E is a real Banach space and $J : E \to 2^{E^*}$ is the normalized duality mapping, then for any given $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.2 [8] Let X and Y be two Banach spaces, $T : X \to 2^Y$ a lower semicontinuous nonempty closed convex valued function. Then T admits a continuous selection, i.e., there exists a continuous selection mapping $H : X \to Y$, such that $h(x) \in Tx$, for all $x \in X$.

Lemma 2.3 [9] Let E be a complete metric space, $T : E \to CB(E)$ a set-valued mapping. Then for any given $\varepsilon > 0$ and $x, y \in E, u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty).$$

3 Iterative Algorithms

Using Lemma 2.3, we suggest the following algorithms for generalized set-valued variational inclusion(1.1).

Algorithm 3.1 For any given $x_0 \in E, u_0 \in Tx_0, z_0 \in Fx_0$, compute the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{v_n\}$ by iterative schemes such as

$$\begin{cases} x_{n+1} \in a_n x_n + b_n(f + y_n - N(w_n, v_n) - \lambda W(g(y_n))) + c_n e_n, \\ y_n \in a'_n x_n + b'_n(f + x_n - N(u_n, z_n) - \lambda W(g(x_n))) + c'_n f_n, \\ u_n \in Tx_n, \|u_n - u_{n+1}\| \leq (1 + \frac{1}{n+1})H(Tx_n, Tx_{n+1}), \\ z_n \in Fx_n, \|z_n - z_{n+1}\| \leq (1 + \frac{1}{n+1})H(Fx_n, Fx_{n+1}), \\ w_n \in Ty_n, \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n+1})H(Ty_n, Ty_{n+1}), \\ v_n \in Fy_n, \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(Fy_n, Fy_{n+1}), \\ n = 0, 1, 2, \cdots, \end{cases} \tag{3.1}$$

where $\{e_n\}$ and $\{f_n\}$ are any bounded sequences in E, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are constants such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$.

The sequence $\{x_n\}$ generated by Algorithm 3.1 is called the Ishikawa iterative sequence with errors.

In Algorithm 3.1, if $b'_n = c'_n = 0$, for all $n \geq 0$, then $y_n = x_n$. Take $z_n = v_n$ and $w_n = u_n$ for all $n \geq 0$. Then we obtain the following algorithm.

Algorithm 3.2 For any given $x_0 \in E, u_0 \in Tx_0, z_0 \in Fx_0$, compute the sequences $\{x_n\}, \{w_n\}$ and $\{v_n\}$ by the iterative schemes such as

$$\begin{cases} x_{n+1} \in a_n x_n + b_n(f + x_n - N(w_n, v_n) - \lambda W(g(x_n))) + c_n e_n, \\ w_n \in Tx_n, \|w_n - w_{n+1}\| \leq (1 + \frac{1}{n+1})H(Tx_n, Tx_{n+1}), \\ v_n \in Fx_n, \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(Fx_n, Fx_{n+1}), \\ n = 0, 1, 2, \cdots. \end{cases} \tag{3.2}$$

The sequence $\{x_n\}$ generated by Algorithm 3.2 is called the Mann iterative sequence with errors.

4 An Existence Result

Lemma 4.1 Let E be a real Banach space and $T : E \to CC(E)$ a lower semicontinuous and ϕ-strongly psuedocontractive mapping, where $CC(E)$ denotes the family of all closed and convex subsets of E. Then T admits a continuous and ϕ-strongly psuedocontractive selection.

Proof : By Lemma 2.2, T admits a continuous selection such that $h(x) \in T(x)$ for all $x \in E$. Now we prove that $h : E \to E$ is ϕ-strongly pseudocontractive. In fact, since $T : E \to CC(E)$ is ϕ-strongly psuedocontractive, for any $x, y \in E$ and $u \in Tx, v \in Ty$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

Especially, letting $u = h(x) \in Tx, v = h(y) \in Ty$, we know that

$$\langle h(x) - h(y), j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|,$$

which implies that $h : E \to E$ is continuous and ϕ-strongly psuedocontractive. This completes the proof.

Theorem 4.1 Let E be a real smooth Banach space, $T, F : E \to CB(E)$ and $A : D(A) \subset E \to 2^E$ be three set-valued mappings, $g : E \to D(A)$ be a single-valued mapping, and $N(\cdot,\cdot) : E \times E \to E$ be a single-valued continuous mapping satisfying the following conditions:

(1) $A \circ g : E \to CC(E)$ is accretive and lower semicontinuous ;

(2) $T : E \to CB(E)$ is $\mu-$Lipschitz continuous;

(3) $F : E \to CB(E)$ is $\xi-$Lipschitz continuous;

(4) the mapping $x \to N(x, y)$ with respect to the mapping T is ϕ-strongly accretive , where $\phi : [0, \infty] \to [0, \infty]$ is a strictly increasing function with $\phi(0) = 0$;

(5) the mapping $y \to N(x, y)$ is accretive with respect to the mapping F;

(6) $N(Tx, Fx) \in CC(E), \forall x \in E$.

Then for any given $f \in E$ and $\lambda > 0$, there exist $q \in E$, $w \in Tq$ and $v \in Fq$, which is a solution of set-valued variational inclusion (1.1).

Proof. For any given $f \in E$, let $Sx = f - N(Tx, Fx) - \lambda A \circ g(x) + x$. Since E is a smooth Banach space, $J : E \to 2^{E^*}$ is single-valued. From the conditions (1), (4) and (5), we know that

$$\langle u - v, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|, \quad \forall u \in Sx, v \in Sy.$$

This implies that S is ϕ-strongly psuedocontractive. Since $N(\cdot,\cdot)$ is continuous, T and F are M-Lipschitz continuous, and $\lambda A \circ g$ is lower semicontinuous, it follows that S is lower semicontinuous and ϕ-strongly psuedocontractive. By condition (6), the operator $S : E \to CC(E)$ satisfies the conditions of Lemma 4.1 and so there exists a continuous ϕ-strongly psuedocontractive mapping $h : E \to E$ such that

$$h(x) \in Sx = f - N(Tx, Fx) - \lambda A(g(x)) + x.$$

From Lemma 2.2 in Liu and Kang [7], h have a unique fixed point q, i.e.,

$$q = h(q) \in S(q) = f - N(Tq, Fq) - \lambda A(g(q)) + q.$$

This implies that $f \in N(Tq, Fq) + \lambda A(g(q))$. Therefore, there exists $w \in Tq, v \in Fq$ such that $f \in N(w, v) + \lambda Ag(q)$. This completes the proof.

Remark 4.1 Theorem 4.1 generalizes Theorem 3.1 in Chidume, Zegeye and Kazmi [1] in the following sense: Theorem 4.1 requires the operator $A \circ g$ is accretive whereas Theorem 3.1 of Chidume, Zegeye and Kazmi [1] requires that $A \circ g$ is m-accretive.

Remark 4.2 The proof method in Theorem 4.1 is quite different from one in Theorem 3.1 in Chidume, Zegeye and Kazmi [1].

5 Convergence of Algorithms

Lemma 5.1 [8] Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three nonnegative real sequences satisfying the following inequality

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n + \gamma_n, \tag{5.1}$$

for all $n \in N$, where $\{\omega_n\} \subset [0,1]$, $\Sigma\omega_n = \infty$, $\beta_n = o(\omega_n)$ and $\Sigma\gamma_n < \infty$. Then $\lim_{n\to\infty}\alpha_n = 0$.

Theorem 5.1 Let E, T, F, A, g be the same as in Theorem 4.1, $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be real sequences in $[0,1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, b_n + c_n \in (0,1), n \geq 0$;

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} \frac{c_n}{b_n+c_n} = 0$;

(iii) $\Sigma b_n = +\infty$.

If $R(I - N(T(\cdot), F(\cdot)))$ and $R(A \circ g)$ are both bounded, $N(T(\cdot), F(\cdot))$ and $A \circ g$ are both uniformly continuous, then the sequences $\{x_n\}$, $\{w_n\}$ and $\{v_n\}$ generated by Algorithm 3.1 strongly converge to the solution q, w, v of set-valued variational inclusion (1.1).

Proof. By the proof of Theorem 4.1, for any $x, y \in E, \bar{x} \in Sx, \bar{y} \in Sy$, there exist $w \in Tx, v \in Fx$ and $u \in Ty, z \in Fy$ such that

$$\bar{x} = x - (N(w,v) + \lambda A(g(x))) + f,$$

$$\bar{y} = y - (N(u,z) + \lambda A(g(y))) + f$$

and

$$\begin{aligned}&\langle N(w,v) + \lambda A(g(x)) - N(u,z) - \lambda A(g(y)), j(x-y)\rangle \\ &= \langle x - \bar{x} - (y - \bar{y}), j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\| \geq A(x-y)\|x-y\|,\end{aligned}$$

where $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)}$ for all $x, y \in E$. This implies that

$$\langle x - \bar{x} - A(x,y)x - (y - \bar{y} - A(x,y)y), j(x-y)\rangle \geq 0$$

for all $x, y \in E, \bar{x} \in Sx, \bar{y} \in Sy$. From Lemma 2.1, we know that

$$\|x-y\| \leq \|x - y + r((x-\bar{x}) - A(x,y)x - (y - \bar{y} - A(x,y)y))\| \tag{5.2}$$

for all $x, y \in E$ and $\bar{x} \in Sx, \bar{y} \in Sy$, where $r > 0$ is a constant. Since $R(I - N(T(\cdot), F(\cdot)))$ and $R(A \circ g)$ are both bounded, letting $d_n = b_n + c_n, d'_n = b'_n + c'_n$ and

$$D = \max\{\sup\{\|w-q\| : w \in f + x - N(Tx, Fx) - \lambda A(g(x)), x \in E\},$$

$$\sup_{n\geq 0}\|e_n - q\|, \sup_{n\geq 0}\|f_n - q\|, \|x_0 - q\|\} < \infty,$$

by induction, we can prove that $\max\{\|x_n - q\|, \|y_n - q\|\} \leq D$. It follows from (3.1) and (3.2) that there exist $p_n \in Sy_n$ and $r_n \in Sx_n$ such that

$$\begin{aligned} x_{n+1} &= (1-d_n)x_n + b_n p_n + c_n e_n \\ &= (1-d_n)x_n + d_n p_n + c_n(e_n - p_n), \end{aligned} \tag{5.3}$$

and

$$y_n = (1-d'_n)x_n + b'_n r_n + c'_n f_n, \tag{5.4}$$

for all $n \geq 1$. From (5.3), we know that

$$(1-d_n)x_n = x_{n+1} - d_n p_n - c_n(e_n - p_n), \tag{5.5}$$

for all $n \geq 1$. By Lemma 2.3, there exists $p'_n \in Sx_{n+1}$ such that

$$\|p_n - p'_n\| \leq (1 + \frac{1}{n})H(Sx_{n+1}, Sy_n). \tag{5.6}$$

Therefore, from (5.5), we know that

$$\begin{aligned}(1 - d_n)x_n = [1 - (1 - A(x_{n+1}, q))d_n]x_{n+1} + d_n(1 - A(x_{n+1}, q))x_{n+1} \\ -d_n p'_n + d_n(p'_n - p_n) - c_n(e_n - p_n).\end{aligned} \tag{5.7}$$

Notice

$$(1 - d_n)q = [1 - (1 - A(x_{n+1}, q))d_n]q + d_n(1 - A(x_{n+1}, q))q - d_n q. \tag{5.8}$$

Combining (5.2), (5.7) and (5.8), we know that

$$\begin{aligned}(1 - d_n)\|x_n - q\| \quad &\geq \quad [1 - (1 - A(x_{n+1}, q))d_n]\|x_{n+1} - q \\ &\quad + \frac{d_n}{1 - (1 - A(x_{n+1}, q))d_n}[(1 - A(x_{n+1}, q))x_{n+1} \\ &\quad - p'_n - (1 - A(x_{n+1}, q))q + q]\| - d_n\|p'_n - p_n\| - c_n\|e_n - p_n\| \\ &\geq \quad [1 - (1 - A(x_{n+1}, q))d_n]\|x_{n+1} - q\| - d_n\|p'_n - p_n\| - 2Dc_n,\end{aligned}$$

which implies

$$\begin{aligned}\|x_{n+1} - q\| \quad &\leq \quad \frac{1 - d_n}{1 - (1 - A(x_{n+1}, q))d_n}\|x_n - q\| \\ &\quad + \frac{d_n}{1 - (1 - A(x_{n+1}, q))d_n}\|p'_n - p_n\| + \frac{2Dc_n}{1 - (1 - A(x_{n+1}, q))d_n} \\ &\leq \quad (1 - A(x_{n+1}, q)d_n)\|x_n - q\| + Md_n\|p'_n - p_n\| + 2DMc_n,\end{aligned} \tag{5.9}$$

where M is a constant. From (5.3),(5.4) and the boundedness of $\{x_n\}, \{y_n\}, \{e_n\}$ and $\{p_n\}$, we know that

$$\begin{aligned}\|x_{n+1} - y_n\| \quad &= \quad \|(1 - d_n)(x_n - y_n) + d_n(p_n - y_n) + c_n(e_n - p_n)\| \\ &\leq \quad (1 - d_n)\|(x_n - y_n)\| + d_n\|p_n - y_n\| + c_n\|e_n - p_n\| \to 0.\end{aligned}$$

From the uniformly continuity of S, we know that $H(Sy_n, Sx_{n+1}) \to 0$ as $n \to \infty$. Furthermore, from (5.6), we know that

$$\|p_n - p'_n\| \to 0. \tag{5.10}$$

Let $\inf\{A(x_{n+1}, q) : n \geq 0\} = r$. We assert that $r = 0$. Suppose that $r > 0$. Then (5.9) yields that

$$\|x_{n+1} - q\| \leq (1 - rd_n)\|x_n - q\| + Md_n\|p_n - p'_n\| + 2DMc_n, \quad \forall n \geq 0. \tag{5.11}$$

Taking $\alpha_n = \|x_n - q\|$, $\omega_n = rd_n$, $\beta_n = Md_n\|p_n - p'_n\|$ and $\gamma_n = 0$ in (5.1), it follows from the conditions (i)-(iii) and (5.10) that

$$\Sigma\omega_n = \infty, \quad \beta_n = o(\omega_n), \quad \Sigma\gamma_n < \infty.$$

Now (5.11) and Lemma 5.1 ensure that $\|x_n - q\| \to 0$, which means that $r = 0$. This is a contradiction. Therefore, $r = 0$ and there exists $\{\|x_{n_i+1} - q\|\}$ such that

$$\|x_{n_i+1} - q\| \to 0 \quad (i \to \infty).$$

Since $\{p_n\}$ and $\{e_n\}$ are both bounded, from $d_{n_i+1} \to 0, b_{n_i+1} \to 0, c_{n_i+1} \to 0$ and $x_{n_i+1} \to q$, we know that

$$x_{n_i+2} = (1 - d_{n_i+1})x_{n_i+1} + b_{n_i+1}p_{n_i+1} + c_{n_i+1}e_{n_i+1} \to q.$$

By induction, we can prove that $x_{n_i+j} \to q$ for all $j \geq 0$, which implies $x_n \to q$. Also, we know that $y_n \to q$. The rest proof is same as the proof of Huang [4]. This completes the proof.

Remark 5.1 Theorem 6.1 generalizes Theorem 3.2 in Chidume, Zegeye and Kazmi [1] in the following sense:

1. Theorem 5.1 requires the operator $A \circ g$ is accretive whereas Theorem 3.2 in Chidume, Zegeye and Kazmi [1] requires $A \circ g$ is m-accretive.

2. The Mann iterative scheme in Chidume, Zegeye and Kazmi [1] is replaced by a new Mann iterative scheme with errors.

Remark 5.2 The proof method in Theorem 5.1 is quite different from one in Theorem 3.2 in Chidume, Zegeye and Kazmi [1].

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Solving two-point boundary value problems by modified Adomian decomposition method *

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Abstract

In this paper, we present the modified Adomian decomposition method for solving two-point linear and nonlinear boundary value problems of the form

$$y'' = g(x) + f(x, y, y'),$$

$$y(0) = A, y(c) = B.$$

Theoretical considerations has been discussed and some examples were presented to show the ability of the method for linear and non-linear ordinary differential equations .

MSC: 65Lxx

Keywords: Modified Adomian decomposition method; Two-point boundary value problem.

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1 Introduction

In this paper, we study two-point boundary value problems of the form

$$y^{''} = g(x) + f(x, y, y^{'}), \tag{1}$$

subject to the boundary conditions

$$y(0) = A, y(c) = B.$$

Where f is continuous on the set $D = \{(x, y, y^{'})|x \in [0, c] \text{ or } x \in [c, 0], y, y^{'} \in \mathbb{R}\}$ and $g(x)$ is given function.

Two-point boundary value problems occur in applied mathematics, theoretical physics, engineering, control and optimization theory. Several numerical methods for solving two-point boundary value problems were studied in[3,5,6,8,9]. The Adomian decomposition method (ADM) has been studied by many scientists [1,2,11] for solving differential and integral problems in many scientific applications. It decomposes the solution into the series which converges rapidly.

In this work, anew modified of the ADM is proposed to overcome difficulties occurred in the standard ADM for solving two-point boundary value problems, namely, the modified ADM (MADM). Main idea of the MADM is to create a canonical form containing all boundary conditions so that the zeroth component is explicitly determined without additional calculations and all other components are also easily determined.

This paper is organized as follows: in sections 2, the proposed method is analyzed. Several numerical illustrations are demonstrated in section 3.

2 The method

We propose the new differential operator, as below

$$L = x^{-1}\frac{d}{dx}x^{2}\frac{d}{dx}x^{-1}, \tag{2}$$

so, the problem(1) can be written as,

$$Ly = g(x) + f(x, y, y^{'}). \tag{3}$$

The inverse operator L^{-1} is therefore considered a two-fold integrals operator, as below,

$$L^{-1}(.) = x \int_{c}^{x} x^{-2} \int_{0}^{x} x(.)dxdx. \tag{4}$$

By operating L^{-1} on problem(3), we have

$$y(x) = A + \frac{(B-A)}{c}x + L^{-1}g(x) + L^{-1}f(x, y, y^{'}), \tag{5}$$

where

$$y(c) = B, y(0) = A.$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $f(x, y, y^{'})$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{6}$$

and

$$f(x, y, y^{'}) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [12] to formulate Adomian polynomials. The following algorithm:

$$A_0 = F(u),$$

$$A_1 = F^{'}(u_0)u_1,$$

$$A_2 = F^{'}(u_0)u_2 + \frac{1}{2}F^{''}(u_0)u_1^2,$$

$$A_3 = F^{'}(u_0)u_3 + F^{''}(u_0)u_1u_2 + \frac{1}{3!}F^{'''}(u_0)u_1^3, \tag{8}$$

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can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting(6)and(7) into (5),

$$\sum_{n=0}^{\infty} y_n = A + \frac{(B-A)}{c}x + L^{-1}g(x) + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{9}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = A + \frac{(B-A)}{c}x + L^{-1}g(x), \tag{10}$$

$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

which gives

$$\begin{aligned} y_0 &= A + \frac{(B-A)}{c}x + L^{-1}g(x), \\ y_1 &= L^{-1}A_0, \\ y_2 &= L^{-1}A_1, \\ y_3 &= L^{-1}A_3, \end{aligned} \tag{11}$$

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From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (6) can be immediately obtained. For numerical purposes, the n-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} y_k, \tag{12}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial value problems.

3 Numerical examples

In this part we present four examples. The first and the second examples are considered to illustrate the method for linear two-point boundary value problems. While in third and fourth examples we solve a nonlinear two-point boundary value problem.

Example 1. Let us consider the following linear problem[4,7] :

$$y^{''} = y^{'} - e^{(x-1)} - 1, \quad 0 < x < 1, \tag{13}$$

$$y(0) = 0, y(1) = 0.$$

The exact solution is $y(x) = x(1 - e^{(x-1)})$.

In an operator form, Eq.(13) becomes

$$Ly = y^{'} - e^{(x-1)} - 1. \tag{14}$$

Applying L^{-1} on both sides of(14) we find

$$y = L^{-1}(-e^{(x-1)} - 1) + L^{-1}y^{'},$$

Using Adomian decomposition for $y(x)$ as given in(6)we obtain

$$\sum_{n=0}^{\infty} y_n(x) = \frac{1}{e} + \frac{3x}{2} - \frac{x}{e} - \frac{x^3}{2} - e^{x-1} + L^{-1} \sum_{n=0}^{\infty} y_n^{'}.$$

The components $y_n(x)$ can be recursively determined by using the relation

$$y_0 = \frac{1}{e} + \frac{3x}{2} - \frac{x}{e} - \frac{x^3}{2} - e^{x-1},$$

$$y_{n+1} = L^{-1}y_n^{'}, \qquad n \geq 0.$$

This in turn given

$$y_0 = \frac{1}{e} + \frac{3x}{2} - \frac{x}{e} - \frac{x^3}{2} - e^{x-1},$$

$$y_1 = \frac{1}{e} - e^{x-1} + \frac{5x}{12} - \frac{x}{2e} + \frac{3x^2}{4} - \frac{x^2}{2e} - \frac{x^3}{6},$$

$$y_2 = \frac{1}{e} - e^{x-1} + \frac{7x}{12e} + \frac{5x^2}{24} - \frac{x^2}{4e} + \frac{x^3}{4} - \frac{x^3}{6e} - \frac{x^4}{24},$$

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In Fig. 1 we have plotted $\sum_{i=0}^{3} y_i(x)$, which is almost equal to the exact solution $y(x) = x(1 - e^{(x-1)})$.

Example 2. Let us consider the following linear equation[9]:

$$y^{''} = y + \cos(x), \quad 0 < x < 1, \tag{15}$$

Subject to the boundary conditions

$$y(0) = 1, y(1) = 1.$$

The exact solution is $y(x) = c_1 e^x + c_2 e^{-x} - \cos(x)/2$,where
$c_1 = \frac{-3\cosh(1)+3\sinh(1)+\cos(1)+2}{4\sinh(1)}$,$c_2 = \frac{3\cosh(1)+3\sinh(1)-\cos(1)-2}{4\sinh(1)}$

In an operator form, Eq.(15) becomes

$$Ly = y + \cos(x). \tag{16}$$

Applying L^{-1} on both sides of(16) we find

$$y = 1 + L^{-1}(\cos(x)) + L^{-1}y,$$

Using Adomian decomposition for $y(x)$ as given in(6)we obtain

$$\sum_{n=0}^{\infty} y_n(x) = 2 - x + x\cos(1) - \cos(x) + L^{-1}\sum_{n=0}^{\infty} y_n.$$

The components $y_n(x)$ can be recursively determined by using the relation

$$y_0 = 2 - x + x\cos(1) - \cos(x),$$

$$y_{n+1} = L^{-1}y_n, \quad n \geq 0.$$

This in turn given

$$y_0 = 2 - x + x\cos(1) - \cos(x),$$

$$y_1 = -1 + \frac{x}{6} + x^2 - \frac{x^3}{6} - \frac{7}{6}x\cos(1) + \frac{1}{6}x^3\cos(1) + \cos(x),$$

$$y_2 = 1 - \frac{217x}{360} - \frac{x^2}{2} + \frac{x^3}{36} + \frac{x^4}{12} - \frac{x^5}{120} + \frac{427x}{360}\cos(1) - \frac{7x^3}{36}\cos(1),$$

$$y_3 = -1 + \frac{9649x}{15120} + \frac{x^2}{2} - \frac{217x^3}{2160} - \frac{x^4}{24} + \frac{x^5}{360} + \frac{x^6}{360} - \frac{x^7}{5040} - \frac{3593x}{3024}\cos(1)$$
$$+\frac{427x^3}{2160}\cos(1) - \frac{7x^5}{720}\cos(1) + \frac{x^7}{5040}\cos(1) + \cos(x),$$

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This means that the solution in a series form is given by

$$y(x) = y_0+y_1+y_2+y_3+... = 1-0.889108x+x^2-0.14755x^3+0.0416667x^4-0.00769486x^5$$
$$+0.00277778x^6 - 0.0000912099x^7 + ...$$

Note that the Taylor series of the exact solution with order 7 is as below

$$y(x) = 1 - 0.888758x + x^2 - 0.148126x^3 + 0.0416667x^4 - 0.00740632x^5$$
$$+0.00277778x^6 - 0.000176341x^7 + ...$$

Example 3. Let us consider the following nonlinear problem:

$$y'' = y^2 + 2 - x^4, \quad -2 < x < 0, \tag{17}$$

Subject to the boundary conditions

$$y(0) = 0, y(-2) = 4.$$

The solution is $y(x) = x^2$.

In an operator form, Eq.(17) becomes

$$Ly = y^2 + 2 - x^4. \tag{18}$$

Applying L^{-1} on both sides of(18) we find

$$y = -2x + L^{-1}(2 - x^4) + L^{-1}y^2,$$

proceeding as before we obtain

$$y_0 = -\frac{16x}{15} + x^2 - \frac{x^6}{30},$$
$$y_{n+1} = L^{-1}A_n, \quad n \geq 0.$$

By using A reliable modification in[10] we get

$$y_0 = x^2,$$
$$y_1 = -\frac{16x}{15} - \frac{x^6}{30} + L^{-1}A_0 = -\frac{16x}{15} - \frac{x^6}{30} + x\int_{-2}^{x} x^{-2}\int_{0}^{x} x(x^4)dxdx = 0, \tag{19}$$
$$y_{n+2} = 0, n \geq 0.$$

In view of (19), the solution is given by

$$y(x) = x^2.$$

Example 4. Consider the following nonlinear problem[8]:

$$y'' = y^2 + 2\pi^2\cos(2\pi x) - \sin^4(\pi x), \quad 0 < x < 1, \tag{20}$$

with the boundary conditions

$$y(0) = 0, y(1) = 0.$$

The exact solution is $y(x) = \sin^2(\pi x)$.

In an operator form, Eq.(20) becomes

$$Ly = y^2 + 2\pi^2\cos(2\pi x) - \sin^4(\pi x). \tag{21}$$

Applying L^{-1} on both sides of(21) we find

$$y = L^{-1}(2\pi^2 \cos(2\pi x) - \sin^4(\pi x)) + L^{-1}y^2,$$

proceeding as before we obtain

$$y_0 = \frac{1}{2} + \frac{15}{128\pi^2} - \frac{3x^2}{16} - \frac{\cos(2\pi x}{2} - \frac{\cos(2\pi x}{8\pi^2} + \frac{\cos(4\pi x}{128\pi^2} + \frac{3x}{16},$$

$$y_{n+1} = L^{-1}A_n, \quad n \geq 0.$$

In Fig. 2 we have plotted $\sum_{i=0}^{1} y_i(x)$, which is almost equal to the exact solution $y(x) = \sin^2(\pi x)$.

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Cauchy and Poison Integrals of Tempered Ultradistributions of Roumieu and Beurling Types

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Abstract

Equipped with a new topology, necessary conditions for convergence of sequences of ultradifferentiable functions of rapid descents are obtained .The Cauchy and Poison integrals of a certain space of tempered ultradistributions are shown to be well-defined. Boundedness theorem is , then, verified.

Keywords: Cauchy and Poison integrals, ultradifferentiable function, Roumieu type tempered ultradistribution, Beurling type tempered ultradistribution.

1. Introduction

Let C be an open convex cone, with vertex $\bar{0}$, in R^n, such that $\bar{C}$ does not contain any entire straight line. Let C' denote a compact subcone of C and, $T^c = R^n + iC$ is the corresponding tubular radial domain of C. Then, the Cauchy kernel corresponding to the tube T^c is defined by [cf. [2, 4]]

$$K(z-t) = \int_{C^*} exp(i(z-t,\eta))d\eta, \tag{1.1}$$

where, $z \in T^c, t \in R^n$, and $C^* = \{t \in R^n : \rangle t, y \langle \geq 0, \text{for all} \, y \in C\}$ is the dual of C.

The $a_i's$ and $b_j's$, wherever they appear, $i, j = 0,1,2,...,$ are to be considered as sequences of positive real numbers on which the constraints imposed are [cf.[9,p.66]]

(i) $a_i^2 \leq a_{i-1}a_{i+1}, j = 1,2,...;$

$b_j^2 \leq b_{j-1}b_{j+1}, j = 1,2,...;$

(ii) $S, S_1 > 0 \text{and} \; T, T_1 > 1$ are constants such that

$$a_i \leq ST^i \min_{0\leq k\leq i} a_k a_{i-k}, i,k = 0,1,2,...;$$

$$b_j \leq S_1 T_1^{\ j} \min_{0\leq k\leq j} a_k a_{j-k}, j,k = 0,1,2,...;$$

(iii) There are constants *S, T* such that

$$\sum_{k=i+1}^{\infty} \frac{a_{k-1}}{a_k} \leq Si \frac{a_i}{a_{i+1}}, i = 1,2,...;$$

$$\sum_{k=j+1}^{\infty} \frac{b_{k-1}}{b_k} \leq Tj \frac{b_j}{b_{j+1}}, j = 1,2,... \quad .$$

Let the sequence (b_j) satisfy (i), (ii) and (iii) and, that $b_j / (j b_j)$ is almost increasing. Then the associated function of (b_j) is defined by [1, p.31]

$$b^*(\rho) = \sup_i \log(\rho^j j! b_0 / b_j), 0 < j < \infty \quad . \tag{1.2}$$

For our main results we define on (b_j) and (a_j) the respective associated functions $b'(.)$ and $a'(.)$ where

$$\left.\begin{aligned} &b'(\rho) = \sup_{j\in N_0} \log(\rho^j / b_j) \\ &and \\ &a'(\rho) = \sup_{i\in N_0} \log(\rho^i / a_i) \end{aligned}\right\} . \tag{1.3}$$

At times, (ii) and (iii) are replaced by the weaker conditions

(ii)' $a_{i+1} \leq ST^i a_i, i = 0,1,2,...$;

$b_{j+1} \leq S_1 T_1^{\ j} b_j, j = 0,1,2,...$;

(iii)' $\sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i} < \infty$,

and

$$\sum_{j=1}^{\infty} \frac{b_{j-1}}{b_j} < \infty$$

respectively.

Functions possessing a property that, they and their partial derivatives decrease to zero, as $|x| \to \infty$, faster than every power of $1/|x|$, are said to be of rapid descent or rapidly decreasing functions. The ultradifferentiable functions are defined to be the set of all infinitely smooth functions (C^{∞}-functions) whose derivatives satisfy certain growth conditions as the order of the derivatives increase.

We denote by $S_{\{b_j\}}^{\{a_i\}}\left(R^n\right)$ the set of all infinitely smooth complex valued functions $\phi(x)$, on R^n ,such that for certain positive constants A , B and E, depending on ϕ , the relation

$$\left|x^{\alpha}\phi^{(\beta)}(x)\right| \leq E\, A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|} \qquad (1.4)$$

remains true for arbitrary $\alpha, \beta \in N_0^n\left(N \cup \{0\}\right)$.

Elements satisfying (1.4) are , indeed , ultradifferentiable functions of rapid descents in the sense of Roumieu and therefore are called ultradifferentiable functions of Roumieu type.

The dual space $S'^{\{a_i\}}_{\{b_j\}}\left(R^n\right)$ of the function space $S_{\{b_j\}}^{\{a_i\}}\left(R^n\right)$ is called the space of tempered ultradistributions of Roumieu type(see [5] , [6] and [5]).

However, it will be interesting to know that such ultradistribution space we define in this article and the ultradistributions we introduce in [9] as well as the spaces of ultradistributions constructed by Carmichael,R.D., Pathack ,R.S. and Pilipovic, S. in [4] generalize the Schwartz space S' of tempered distributions [10]. That is, $S' \subset S'^{\{a_i\}}_{\{b_j\}}$. This ,due to its construction, can not be applied for the ultradistributions appear in [3] , [7] and [8] which are duals of a Zemanian space Z of Fourier transforms of test functions of bounded support .

Norms on $S_{\{b_j\}}^{\{a_i\}}$ can be defined by

$$\xi_{A,B}(\phi) = \sup_{\substack{\alpha,\beta \in N_0^n \\ x \in R^n}} \frac{\left|x^{\alpha}\phi^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}}$$

taking into account that conditions on ϕ , A and B are already mentioned in (1.4) .

2. Convergence in the Space $S_{\{b_j\}}^{\{a_i\}}$

Let $\phi \in S_{\{b_j\}}^{\{a_i\}}(R^n)$. On $S_{\{b_j\}}^{\{a_i\}}$, we define a norm by virtue of the equation

$$\eta_{A,B}(\phi) = \sup_{\alpha,\beta} \frac{\left|\phi^{(\beta)}(x)\, exp\, a^{'}(|x|/A)\right|}{B^{|\beta|} b_{|\beta|}} \quad ,$$

where A , B are constants dependent on ϕ and $a^{'}(|x|/A)$ have significance of (1.3).

With this topology, we prove certain theorem justifying convergence of sequences as follows

Theorem 2.1. *Let* $a_0 = b_0 = 1$. *Then, a necessary condition for a sequence* $(\theta_\nu) \in S_{\{b_j\}}^{\{a_i\}}(R^n)$ *to converges to zero, as* $\nu \to \infty$, *is that*

$$\eta_{A,B}(\theta_\nu) \to 0 \quad as \ \nu \to \infty \ .$$

Proof. Let $\phi \in S_{\{b_j\}}^{\{a_i\}}(R^n)$. The relation (1.4) implies that

$$\frac{\left|x^\alpha \theta_\nu^{(\beta)}(x)\right|}{A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|}} < \infty \ , \ \alpha, \beta \in N_0^n \ .$$

Therefore,

$$\frac{\left|x^\alpha \theta_\nu^{(\beta)}(x)\right|}{A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|}} \to 0 \tag{1.5}$$

uniformly in x as $|\alpha| + |\beta| \to \infty$.

We have,

$$\frac{\left|x^\alpha \theta_\nu^{(\beta)}(x)\right|}{A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|}} \leq \frac{\left|x^{|\alpha|+1} \theta_\nu^{(\beta)}(x)\right|}{A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|}} . \frac{1}{|x|} .$$

Since $1/\left(A^{|\alpha|} a_{|\alpha|}\right)$ and $1/\left(B^{|\beta|} b_{|\beta|}\right)$ are both bounded, we have

$$\frac{\left|x^\alpha \theta_\nu^{(\beta)}(x)\right|}{A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|}} \leq \frac{e}{k} \ , \tag{1.6}$$

for all $|x| \geq k > 1$ and some constant e .

Let $|x| \to \infty$, right hand side of (1.6) converges to zero in $\alpha, \beta \in N_0^n$. This together with (1.5) implies that for some $x_0 \in R^n$, $\alpha_0, \beta_0 \in N_0^n$ we have

$$\sup_{\substack{\alpha,\beta\in N_o^n \\ x\in R^n}} \frac{\left|x^{\alpha}\phi_{\nu}^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}} = \frac{\left|x_0^{\alpha_0}\phi_{\nu}^{(\beta_0)}(x_0)\right|}{A^{|\alpha_0|}B^{|\beta_0|}a_{|\alpha_0|}b_{|\beta_0|}}. \tag{1.7}$$

By virtue of (1.3) and (1.7) we obtain

$$\frac{\left|x_0^{\alpha_0}\phi_{\nu}^{(\beta_0)}(x_0)\right|}{A^{|\alpha_0|}B^{|\beta_0|}a_{|\alpha_0|}b_{|\beta_0|}} \le \sup_{\substack{\alpha,\beta\in N_o^n \\ x\in R^n}} \frac{\left|\theta_{\nu}^{(\beta)}(x)\ exp\ a'\left(|x|/A\right)\right|}{B^{|\beta|}b_{|\beta|}}\ . \tag{1.8}$$

Combining (1.7) and (1.8) then yields

$$\xi_{A,B}(\theta_\nu) \le \eta_{A,B}(\theta_\nu). \tag{1.9}$$

Hence as $\nu \to \infty$, $\xi_{A,B}(\theta_\nu) \to 0$, whenever $\eta_{A,B}(\theta_\nu) \to 0$.

Conversely, due to analysis which is alike to that in the first part, we have

$$\frac{\left|x^{\alpha}\phi_{\nu}^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}} \to 0 \tag{1.10}$$

uniformly in $x \in R^n$, $\beta \in N_0^n$, as $|x| \to \infty$, similarly,

$$\frac{\left|x^{\alpha}\phi_{\nu}^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}} \to 0 \tag{1.11}$$

uniformly in $\alpha, \beta \in N_0^n$, as $|x| \to \infty$,

and,

$$\frac{\left|x^{\alpha}\phi_{\nu}^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}} \to 0 \tag{1.12}$$

uniformly in $\alpha \in N_0^n$, $x \in R^n$, as $|\beta| \to \infty$.

Owing to (1.10), (1.11), (1.12) and (1.3) we find $\alpha_1, \beta_1 \in N_0^n$ and $x_1 \in R^n$ such that

$$\sup_{\substack{\beta\in N_0^n \\ x\in R^n}} \frac{\left|\theta_{\nu}^{(\beta)}(x)\exp a'\left(|x|/A\right)\right|}{B^{|\beta|}b_{|\beta|}} = \frac{\left|x_1^{\alpha_1}\theta_{\nu}^{(\beta)}(x_1)\right|}{A^{|\alpha_1|}B^{|\beta_1|}a_{|\alpha_1|}b_{|\beta_1|}} \le \sup_{\substack{\alpha,\beta\in N_0^n \\ x\in R^n}} \frac{\left|x^{\alpha}\theta_{\nu}^{(\beta)}(x)\right|}{A^{|\alpha|}B^{|\beta|}a_{|\alpha|}b_{|\beta|}}.$$

Hence, $\eta_{A,B}(\theta_\nu) \le \xi_{A,B}(\theta_\nu)$. Allowing $\nu \to \infty$, completes the proof of the above theorem.

In view of Theorem 2.1, following theorem is true.

Theorem 2.2. *Let* $a_0 = b_0 = 1$. *For every* $f \in S_{\{b_j\}}^{\{a_i\}}\left(R^n\right)$, *there is a constant* $d > 0$ *such that*

$$\left|\left\langle f, \theta\right\rangle\right| \leq d \sup_{\substack{\beta \in N_0^n \\ x \in R^n}} \frac{\left|\theta^{(\beta)}(x) \exp a'\left(|x|/A\right)\right|}{B^{|\beta|} b_{|\beta|}} \quad ,$$

for all $\theta \in S_{\{b_j\}}^{\{a_i\}}\left(R^n\right)$.

3. Cauchy Integral of Tempered Ultradistributions of Roumieu Type

We mainly devote this section to theorems justifying the definition of the Cauchy and Poison integrals of tempered ultradistributions in a particular ultradistribution space and ,further, we verify a related restricted theorem for the boundedness of kernels as follows.

Theorem 3.1. .*Let the sequence* $\left(a_i\right)$ *satisfy* (i) *and* (iii)' *and* , $z \in T^c$ *but fixed* . *Then,*

$$K(z-t) \in S_{\{b_j\}}^{\{a_i\}}\left(R^n\right) \ ,$$

as a function of $t\left(t \in \Omega \subseteq R^n\right)$.

Proof. Let S_n be the surface area of the unit sphere in $R^n, \sigma = \sigma_y = \delta|y|$, $\delta = \delta\left(C'\right) > 0$ and $y(= \operatorname{Im} z) \in C'$ be a certain compact subcone of C . For a fixed $z \in T^c$ and *n-tuple* β of non-negative integers satisfying

$$\left|D_t^\beta K(z-t)\right| \leq S_n (2\pi\sigma)^{-n-|\beta|} \Gamma(n+|\beta|) \ , \tag{3.1}$$

Lemma 1 in [2] ,then, implies that

$$K(z-t) \in C^\infty \ ,$$

as a function of $t \in R^n$.

Set $\quad a_{|\alpha||\beta|} = \dfrac{a_{|\alpha|}}{a_{|\alpha|-1}} \dfrac{b_{|\beta|}}{b_{|\beta|-1}}, \ |\alpha|, |\beta| = 1,2,\ldots$.

Without lose of generality, we may assume $|\beta| > |\alpha|$. By virtue of [1, Lemma 4.1] and condition (iii)', we have

$$|\beta| / a_{|\alpha||\beta|} \to 0, \tag{3.2}$$

as $|\alpha|, |\beta| \to \infty$. Further

$$1/a_{|\alpha||\beta|} \to 0, \text{ as } |\alpha|, |\beta| \to \infty. \tag{3.3}$$

The fact that $a_{|\alpha|-1}/a_{|\alpha|}$ vanishes after $|\alpha|$-step and, properties of Gamma functions, $\Gamma(n+|\beta|) = (n+|\beta|-1)(n+|\beta|-2)\ldots n\Gamma(n)$, yields that

$$\frac{a_0 b_0 \Gamma(n+|\beta|)}{\gamma^{|\beta||\alpha|} L^{|\beta|} a_{|\alpha|} b_{|\beta|}} = \Gamma(n)\left(\frac{n a_0 b_0}{\gamma^{|\alpha|} L a_1 b_1}\right)\left(\frac{(n+1) a_1 b_1}{\gamma^{|\alpha|} L a_2 b_2}\right)\ldots\left(\frac{(n+|\beta|) b_{|\beta|-1}}{\gamma^{|\alpha|} L b_{|\beta|}}\right). \tag{3.4}$$

Employing (3.2) and (3.3) in (3.4) then yields

$$\frac{a_0 b_0 \Gamma(n+|\beta|)}{\gamma^{|\alpha||\beta|} L^{|\beta|} a_{|\alpha|} b_{|\beta|}} \to 0 \ ,$$

as $|\alpha|, |\beta| \to \infty$. Therefore,

$$\Gamma(n+|\beta|) \leq R \gamma^{|\alpha||\beta|} L^{|\beta|} a_{|\alpha|} b_{|\beta|} \ , \tag{3.5}$$

for some constant $R > 0$.

With the aid of (3.5) and (3.1) we have

$$\begin{aligned} |t|^{|\alpha|} \left| D_t K(z-t) \right| &\leq |t|^{|\alpha|} S_n \sigma^{-n-|\beta|} \Gamma(n+|\beta|) \\ &\leq R |t|^{|\alpha|} S_n (2\pi\sigma)^{-n-|\beta|} \gamma^{|\alpha||\beta|} L^{|\beta|} a_{|\alpha|} b_{|\beta|} \ . \end{aligned}$$

Setting $E = \left(|t|^{|\alpha|} S_n R\right) / (2\pi\sigma)^n$, $A = \gamma$ and $B = L\gamma / (2\pi\sigma)$ the above inequality is interpreted to mean

$$|t|^{|\alpha|} \left| D_t^{\beta} K(z-t) \right| \leq E \ A^{|\alpha|} B^{|\beta|} a_{|\alpha|} b_{|\beta|} \ .$$

This completes the proof of the theorem.

Now, let $U \in S'^{\{a_i\}}_{\{b_j\}}(\Omega), \Omega \subseteq R^n$, and z be an arbitrary but fixed in T^c. Then, with the aid of the above theorem, we may define the *Cauchy integral* of U to be the map

$$\mathsf{C}(U;z)=\langle U_t, K(z-t)\rangle, t\in\Omega\,. \tag{3.6}$$

Theorem 3.2. *Let* $U\in S'^{\{a_i\}}_{\{b_j\}}(R^n), \Omega\subseteq R^n$. *Let the sequence* (b_j) *satisfy conditions* (i) *and* (ii) *and,* (a_i) *satisfy* (i). *For an arbitrary compact subcone* C' *of* C *and some* $t=t(C')>0$, *we have the existence of constants* R *and* ρ *such that the inequality*

$$|\mathsf{C}(U;x)|\leq\rho\exp\left(b^*\left(R/|y|\right)+a'\left(|t|/A\right)\right) \tag{3.7}$$

holds true.

Proof. Upon employing Theorem 2.1, (3.1) then yields

$$\begin{aligned}|\mathsf{C}(U;x)| &= |\langle U_t, K(z-t)\rangle| \\ &\leq D \sup_{\substack{\beta\in N_0^n\\ t\in\Omega}} \frac{\left|D_t^\beta K(z-t)\, exp\, a'\left(|t|/A\right)\right|}{B^{|\beta|}b_{|\beta|}} \\ &\leq D\frac{d\,S_d\Gamma(d+|\beta|)\exp a'\left(|t|/A\right)}{B^{|\beta|}b_{|\beta|}\left(2\pi d|y|\right)^{|\beta|+d}} \quad ,\end{aligned}$$

where, d is the dimension and D is certain positive constant.

Let $p=|\beta|+d$, $q=d$ and the sequence (b_j) satisfy (ii) . We have

$$b_{|\beta|+d}\leq S_1^p b_d b_{|\beta|},$$

for some constant $S_1>0$.

$$i.e \qquad \left(1/b_{|\beta|}\right)\leq\left(S_1^p b_d\right)/b_p\,.$$

Hence, setting $\rho=\left(DS_dB^db_d\right)/\left(pb_0\right)$ and, employing the fact that $1/B^{|\beta|}=B^d\left(1/B^p\right)$ implies

$$\begin{aligned}|\mathsf{C}(U;Z)| &\leq D\left(\left(S_dB^db_d\Gamma(p)S_1^pb_d\exp a'\left(|t|/A\right)\right)/\left(2\pi\delta B|y|\right)^pb_p\right) \\ &= D\left(\left(S_dB^db_d\right)/\left(pb_0\right)\right)\left(\left(S_1^p\,p!b_0\right)/\left(2\pi B\delta|y|\right)^pb_p\right)\exp a'\left(|t|/A\right) \\ &\leq\rho\,\exp\sup_p\,\log\left(\left(S_1/\left(2\pi B\delta|y|\right)^p\,p!\,b_0\right)/b_p\right)\exp a'\left(|t|/A\right).\end{aligned} \tag{3.8}$$

The notations σ,δ have the usual meaning in Theorem 3.1. Setting $R=S_1/2\pi\delta B$ and considering (1.2), relation (3.8) produces (3.7) .

This completes the proof of the theorem.

4. Poison Integral of Tempered Ultradistributions of Roumieu Type

Let C be an open, convex cone and $z \in T^c$ be arbitrary but fixed, the Poison kernel related to a tubular radial domain T^c is defined by

$$Q(z;t) = \frac{K(z-t)\overline{K(z-t)}}{K(2iy)} \quad , \tag{4.1}$$

where $K(z;t)$ stands for the Cauchy kernel corresponding to the tube T^c .

Lemma 4.1. *Under the assumption that the sequence* (a_i) *satisfies* (i) *and* (iii)', *the Poison kernel*

$$Q(z;t) \in S_{\{b_j\}}^{\{a_i\}}(\Omega),$$

as a function of $t \in \Omega(\subseteq R^n)$, *where* $z \in T^c$, *but fixed.*

Proof. of the above Lemma is similar to the proof considered in Theorem 3.1 .The detailed analysis is thus avoided .

As a consequence of Lemma 4.1, the generalized *Poison integral* of tempered ultradistribution $U \in S_{\{b_j\}}^{'\{a_i\}}(\Omega)$ can be defined as

$$P(U;z) = \langle U_t, Q(z;t) \rangle \quad , \tag{4.2}$$

$z \in T^C$.

It is apparent from definitions that Theorems 3.2 and 3.1 can similarly be verified for the Poison integral .

Remark. Denoting by $S_{(b_j)}^{(a_i)}(R^n)$ the set of all complex valued infinitely differentiable functions $\phi(x)$ such that (1.4) holds for some constant E and arbitrary constants *A and B,* the elements of $S_{(b_j)}^{(a_i)}(R^n)$ are ultradifferentiable functions of rapid descent of *Beurling type* and ,therefore ,the corresponding dual, $S_{(b_j)}^{'(a_i)}(R^n)$, consists of tempered ultra-distributions of *Beurling type* .

It is apparent from definitions that $S_{(b_j)}^{(a_i)}(R^n) \subset S_{\{b_j\}}^{\{a_i\}}(R^n)$ and ,thus, continuous linear functionals on $S_{\{b_j\}}^{\{a_i\}}(R^n)$ are, indeed , continuous linear functionals on $S_{(b_j)}^{(a_i)}(R^n)$.

i.e

$$S'^{\{a_i\}}_{\{b_j\}}(R^n) \subset S'^{(a_i)}_{(b_j)}(R^n).$$

To consider the Beurling type ultradistributions , (3.5) and (4.2) can be confirmed similarly.

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INVERSE OF SOME CLASSES OF PERMUTATION BINOMIALS

AMELA MURATOVIĆ-RIBIĆ

Abstract. In this paper we will find inverse polynomials of some special classes of permutation binomials. Both considered polynomials and their inverses are easy to calculate and have only a few nonzero coefficients what makes them suitable for applications.

In this paper We consider three known classes of permutation binomials and find their inverses. Inverses of two considered classes are actually new classes of permutation polynomials (PP).

Let p be a prime, let n be a positive integer, let $q = p^n$ and let $\mathbb{F}_q$ denote the Galois field of order q. Let $f(X) = \sum_{i=0}^{q-2} a_i X^i \in \mathbb{F}_q$ be a PP then its inverse (see [3]) is $f^{-1}(X) = \sum_{i=0}^{q-2} b_i X^i$ where

$$b_j = \sum \frac{(q-1-j)!}{t_0!t_1!\dots t_{q-2}!} a_0^{t_0} a_1^{t_1} \dots a_{q-2}^{t_{q-2}}, \tag{1}$$

and sum is over all integers $t_j \geq 0$, $0 \leq j \leq q-2$ such that $t_0+t_1+\dots+t_{q-2} = q-1-j$ and $t_1 + 2t_2 + \dots + (q-2)t_{q-2} \equiv (q-2) \mod (q-1)$.

It is known that there are permutation polynomials of the form

$$f(X) = X^{\frac{q-1+d}{d}} + aX \tag{2}$$

where $d|(q-1)$ and $a \in \mathbb{F}_q$(see [1, Theorem 4.3]).

Proposition 1. *The inverse of the permutation polynomial (5) is of the form*

$$f^{-1}(X) = \sum_{s=0}^{d-1} b_{j_s} X^{j_s}$$

where

$$j_s = s\frac{q-1}{d} + 1, \qquad s = 0, 1, \dots, d-1,$$

and its coefficients are given by

$$b_{j_s} = \sum_{u=1}^{\lfloor \frac{q-1+d-s-j_s}{d} \rfloor} \binom{q-1-j_s}{q-1-j_s+d-ud-s} a^{(q-1-j_s+d-ud-s)}.$$

Proof. An application of the formula (1) to polynomial (5) gives

$$b_j = \sum \binom{q-1-j}{t} a^t \tag{3}$$

where $0 \leq t \leq q-1-j$ and

$$t + (q-1-j-t)\frac{q-1+d}{d} \equiv (q-2)(\mathrm{mod} q-1). \tag{4}$$

Last equation is equivalent to

$$t + (q-1-j-t)\frac{q-1+d}{d} = u(q-1) - 1$$

where u is a positive integer. Solve it for t to get

$$t = (q-1+d-ud-j) - d\frac{j-1}{q-1}.$$

As $q-1+d-du-j$ is an integer, the fraction $d\frac{j-1}{q-1}$ has to be an integer too. Therefore

$$d\frac{j-1}{q-1} = s \qquad s \in \mathbb{Z},$$

which gives

$$j = s\frac{q-1}{d} + 1.$$

But $0 \le j \le q-2$ implies that

$$j_s = s\frac{q-1}{d} + 1 \qquad s = 0, 1, \ldots, d-1.$$

Condition $0 \le t \le q-1-j_s$ implies that

$$0 \le q-1+d-ud-j_s-s \le q-1-j_s.$$

Therefore u satisfies

$$1 \le u \le \frac{q-1+d-s-j_s}{d}$$

which gives

$$b_{j_s} = \sum_{u=1}^{\lfloor \frac{q-1+d-s-j_s}{d} \rfloor} \binom{q-1-j_s}{q-1-j_s+d-ud-s} a^{(q-1-j_s+d-ud-s)}.\square$$

Specially (see [2, Theorem 7.11 and Remark 7.12]) consider the case $d = 2$. Then

$$f(X) = X^{\frac{q+1}{2}} + aX \tag{5}$$

is a PP if q is odd and $a = (c^2+1)(c^2-1)^{-1}$ for $c \in \mathbb{F}_q^*$ such that $c^2 \neq 1$.

Proposition 2. *The inverse of the PP (2) is of the form*

$$f^{-1}(X) = b_{\frac{q+1}{2}} X^{\frac{q+1}{2}} + b_1 X$$

where its coefficients are given by

$$b_{\frac{q+1}{2}} = \begin{cases} ((1+a)^{\frac{q-3}{2}} + (1-a)^{\frac{q-3}{2}})2^{-1}, & \text{if } \frac{q-1}{2} \text{ is even} \\ ((1+a)^{\frac{q-3}{2}} - (1-a)^{\frac{q-3}{2}})2^{-1}, & \text{if } \frac{q-1}{2} \text{ is odd} \end{cases}$$

and

$$b_1 = ((1+a)^{q-2} - (1-a)^{q-2})2^{-1}.$$

Proof. An application of the formula (1) to polynomial (2) gives

$$b_j = \sum \binom{q-1-j}{t} a^t \tag{6}$$

where $0 \le t \le q-1-j$ and

$$t + (q-1-j-t)\frac{q+1}{2} \equiv (q-2)(\mathrm{mod}\, q-1). \tag{7}$$

Last equation is equivalent to

$$t + (q-1-j-t)\frac{q+1}{2} = u(q-1) - 1$$

where u is a positive integer. Solve it for t to get

$$t = (q+1-2u-j) - 2\frac{j-1}{q-1}.$$

As $q+1-2u-j$ is an integer, the fraction $2\frac{j-1}{q-1}$ has to be an integer too. Also $0 \le j \le q-2$ implies that

$$j = \frac{q+1}{2} \qquad \text{or} \qquad j = 1.$$

For $j = 1$, $t = q - 2u$ and the condition $0 \le t \le q-1-j$ implies that $1 \le u \le \frac{q-1}{2}$ which gives

$$b_1 = \sum_{u=1}^{\frac{q-1}{2}} \binom{q-2}{q-2u} a^{q-2u} = ((1+a)^{q-2} - (1-a)^{q-2})2^{-1}.$$

For $j = \frac{q+1}{2}$, $t = \frac{q+1}{2} - 2u - 1$, and condition $0 \le t \le q-1-j$ implies $1 \le u \le \frac{q-1}{4}$ and therefore

$$b_{\frac{q+1}{2}} = \sum_{u=1}^{\lfloor \frac{q-1}{4} \rfloor} \binom{\frac{q-3}{2}}{\frac{q-1}{2} - 2u} a^{\frac{q-1}{2}-2u} = \begin{cases} ((1+a)^{\frac{q-3}{2}} + (1-a)^{\frac{q-3}{2}})2^{-1}, & \text{if } \frac{q-1}{2} \text{ is even} \\ ((1+a)^{\frac{q-3}{2}} - (1-a)^{\frac{q-3}{2}})2^{-1}, & \text{if } \frac{q-1}{2} \text{ is odd.} \end{cases}$$

□

Example 1. Let $q = 13$, $a = 2$, $\frac{13-1}{2} = 6$ is even. For the polynomial $f(X) = X^7 + 2X$ we have

$$\begin{aligned} b_1 &= ((1+2)^{-1} - (1-2)^{-1})2^{-1} = 5, \\ b_7 &= ((1+2)^{\frac{10}{2}} + (1-2)^{\frac{10}{2}})2^{-1} = 4. \end{aligned}$$

and therefore $f^{-1}(X) = 4X^7 + 5X$.

Example 2. Let $q = 7$, $a = 4$ then $\frac{6}{2} = 3$ is odd. For the polynomial $f(X) = X^4 + 4X$ we have

$$\begin{aligned} b_1 &= ((1+4)^{-1} - (1-4)^{-1})2^{-1} = 4, \\ b_4 &= ((1+4)^2 - (1-4)^2)2^{-1} = 1 \end{aligned}$$

and therefore $f^{-1}(X) = X^4 + 4X$.

Consider now a class of PP (see [4, Theorem 1.])

$$f(X) = X^u(X^v + 1) \tag{8}$$

where $3|(q-1)$, u and v are positive integers, $(v, q-1) = \frac{q-1}{3}$, $(u, \frac{q-1}{3}) = 1$, $u \neq v(\text{mod}3)$, and $2^{(q-1)/3} = 1$ in $\mathbb{F}_q$. Let $v = z\frac{q-1}{3}$, $z = 1, 2$.

Proposition 3. *The inverse of the polynomial (8) is a permutation trinomial*

$$f^{-1}(X) = b_{j_2}X_2^j + b_{j_1}X^{j_1} + b_{j_0}X^{j_0},$$

where for c_0 being the least positive solution of the Diophantine equation $\frac{q-1}{3}c - uY = 1$,

$$j_s = q - 1 - \frac{\frac{q-1}{3}c_0 - 1}{u} - s\frac{q-1}{3} \quad \textit{for} \quad s = 0, 1, 2,$$

$h = c_0 + us(\text{mod}2)$ and

$$b_{j_s} = M\Big(q - 1 - j_s, 3, \big(q - 1 - j_s + \frac{u(q-1-j_s)+1}{v} - \frac{3h}{z}\big)(\text{mod}3)\Big)(\text{mod}p).$$

Proof. An application of the formula (1) to PP (8) gives that

$$b_j = \sum \binom{q-1-j}{t} \tag{9}$$

where $0 \leq t \leq q - 1 - j$ and

$$ut + (u + v)(q - 1 - j - t) \equiv (q - 2)(\text{mod}q - 1)$$

which is equivalent to

$$ut + (u + v)(q - 1 - j - t) = -1 + m(q - 1), \qquad m \geq 1, m \in \mathbb{Z} \tag{10}$$

Solve this equation for t to get

$$t = (q - 1 - j) + \frac{u(q-1-j) - m(q-1) + 1}{v}. \tag{11}$$

Let $m = zk + \hat{h}$, $\hat{h} = 0, 1$, then

$$t = (q - 1 - j) - 3k + \frac{u(q-1-j)+1}{v} - \frac{3\hat{h}}{z}$$

which implies

$$\frac{u(q-1-j)+1}{v} - \frac{3\hat{h}}{z} \in \mathbb{Z}.$$

Condition $g.c.d.(u, \frac{q-1}{3}) = 1$ implies that Diophantine equation $\frac{q-1}{3}c - uY = 1$ has solutions of the form $c = c_0 + su$ and $Y = (q - 1 - j) = Y_0 + sv$, where c_0 is the least positive integer that solves this equation, y_0 correspond to c_0 and $s \in \mathbb{Z}$. Now

$$\frac{u(q-1-j)+1}{v} - \frac{3h}{z} = \frac{c}{z} - \frac{3\hat{h}}{z} \in \mathbb{Z}$$

which implies that $\hat{h} = c_0 + su(\text{mod}z) = h$ and

$$j = q - 1 - \frac{\frac{q-1}{3}c - 1}{u} \in \mathbb{Z}.$$

On the other hand

$$1 \leq \frac{\frac{q-1}{3}c - 1}{u} \leq (q - 1) \quad \text{implies} \quad 0 < c_0 + su \leq 3u + \frac{3}{q-1}.$$

If $q - 1 > 3$ then $0 \leq s \leq 2$ and thus the only non-zero coefficients of f^{-1} are for

$$j_s = q - 1 - \frac{\frac{q-1}{3}c_0 - 1}{u} - s\frac{q-1}{3} \quad \text{for} \quad s = 0, 1, 2.$$

If $q-1 \le 3$ the inverse has at most three nonzero coefficients for every PP. Formula (11) and $0 \le t \le q-1-j_s$ imply that

$$b_{j_s} = \sum_t \binom{q-1-j_s}{t}$$

where sum runs over all t, $0 \le t \le q-1-j_s$ of the form

$$t = (q-1-j_s) - 3k + \frac{u(q-1-j_s)+1}{v} - \frac{3h}{z} \quad k \ge 0, \quad k \in \mathbb{Z}.$$

Therefore, (see [4, Lemma 4.])

$$b_{j_s} = M\Big(q-1-j_s, 3, \big(q-1-j_s + \frac{u(q-1-j_s)+1}{v} - \frac{3h}{z})(\mathrm{mod} 3)\Big)(\mathrm{mod} p). \square$$

Example 3. Let $q = 25$, $f(X) = X(X^8+1)$. Then $u = 1$, $v = 8$ and $c_0 = 1$. Inverse has nonzero coefficients for indices $j_0 = 17$, $j_1 = 9$ and $j_2 = 1$. Coefficients are

$$\begin{aligned} b_{17} &= M(7,3,(7+1)(\mathrm{mod}3)) = \frac{2^7-2}{3}(\mathrm{mod}5) = 2 \\ b_9 &= M(15,3,(15+2)(\mathrm{mod}3)) = \frac{2^{15}+1}{3}(\mathrm{mod}5) = 3 \\ b_1 &= M(23,3,(23+3)(\mathrm{mod}3)) = \frac{2^{23}+1}{3}(\mathrm{mod}5) = 3 \end{aligned}$$

and thus $f^{-1}(X) = 2X^{17} + 3^9 + 3X$.

Example 4. Let $q = 25$, $u = 3$, $v = 16$ and $f(X) = X^3(X^{16}+1)$.
$j_0 = 19$, $j_1 = 11$ and $j_2 = 3$. Then

$$b_{19} = M(5,3,0) = 1, \quad b_{11} = M(13,3,2) = \frac{2^{13}-2}{3} = 0 \quad b_3 = M(21,3,1) = 1$$

and thus $f^{-1}(X) = f(X) = X^{19} + X^3$.

Remark 1. In ([4, Remark 1.]) it was shown that $f(X) = X^u(X^v+1)$ permutes the $\mathbb{F}_q$ if and only if $g(X) = X^u(X^v+a)$, $\quad a^3 = 1$ permutes $\mathbb{F}_q$. Inverse of the polynomial $g(X)$ has the same form and coefficients are given by

$$b_{j_s} = a^m M(q-1-j_s, 3, (q-1-j_s + \frac{u(q-1-j_s)+1}{v} - \frac{3h}{z})(\mathrm{mod}3))(\mathrm{mod}p)$$

where $m = (q-1-j_s + \frac{u(q-1-j_s)+1}{v} - \frac{3h}{z})(\mathrm{mod}3)$.

Consider now a new class of (PP). Let $5|(q-1)$, u and v are positive integers, $(v, q-1) = \frac{q-1}{5}$, $(u, \frac{q-1}{5}) = 1$, $v + 2u \ne v(\mod 5)$, and $2^{(q-1)/5} = 1$ in $\mathbb{F}_q$ and $(\frac{1+\sqrt{5}}{2})^{\frac{q-1}{5}} + (\frac{1-\sqrt{5}}{2})^{\frac{q-1}{5}} \equiv 2(\mathrm{mod}p)$. These conditions are sufficient and necessary for polynomial

$$f(X) = X^u(X^v+1) \tag{12}$$

to be a PP over field $\mathbb{F}_q$,(see [4, Theorem 2]). Let $v = \frac{q-1}{5}z$ where $1 \le z \le 4$.

Proposition 4. *The inverse of the polynomial (12) is a PP of the form*

$$f^{-1}(X) = b_{j_4}X_4^{j} + b_{j_3}X^{j_3} + b_{j_2}X^{j_2} + b_{j_1}X^{j_1} + b_{j_0}X^{j_0},$$

for c_0 being the least positive solution of the Diophantine equation $\frac{q-1}{5}c - uY = 1$,

$$j_s = q - 1 - \frac{\frac{q-1}{5}c_0 - 1}{u} - s\frac{q-1}{5} \quad \textit{for} \quad s = 0,1,2,3,4,$$

$h = c_0 + su(\mathrm{mod} z)$ *and*

$$b_{j_s} = M\Big(q - 1 - j_s, 5, \big(q - 1 - j_s + \frac{u(q-1-j_s)+1}{v} - \frac{5h}{z}\big)(\mathrm{mod} 5)\Big)(\mathrm{mod} p).$$

Proof. Proceeding in the same way as in the proof of Proposition 2. coefficients are given by

$$b_j = \sum \binom{q-1-j}{t} \tag{13}$$

where sum is over

$$t = (q-1-j) + \frac{u(q-1-j) - m(q-1) + 1}{z\frac{q-1}{5}}, m \in \mathbb{Z}, m \geq 1.$$

Diophantine equation $c\frac{q-1}{5} - Yu = 1$ has solutions in the form $c = c_0 + su$ and $Y = (q-1-j) = y_0 + sv$ where c_0 is the least positive such solution and $s \in \mathbb{Z}$. Let $m = zk + h$ where $0 \leq h < z$ and $k = 0,1,2,\ldots$. Now

$$t = q - 1 - j + \frac{u(q-1-j)+1}{v} - \frac{5h}{z} - 5k$$

and t can be integer if $\frac{u(q-1-j)+1}{\frac{q-1}{5}} = c_0 + su \in \mathbb{Z}$ and $h = c_0 + su(\bmod z)$. Therefore

$$j_s = q - 1 - \frac{c_0\frac{q-1}{5} - 1}{u} - s\frac{q-1}{5}.$$

As $0 \leq j_s < q-1$ then s can take values $s = 0,1,2,3,4$. Coefficients of the inverse (see [4, Lemma 5. and formula (1)]) are given by

$$b_{j_s} = M\Big(q - 1 - j_s, 5, \big(q - 1 - j_s + \frac{u(q-1-j_s)+1}{v} - \frac{5h}{z}\big)(\mathrm{mod} 5)\Big).\Box$$

Remark 2. In ([4, Remark 2.]) it was shown that $f(X) = X^u(X^v+1)$ permutes the $\mathbb{F}_q$ if and only if $g(X) = X^u(X^v + a), \quad a^5 = 1$ permutes $\mathbb{F}_q$. Inverse of $g(X)$ is having the same form as inverse of $f(X)$ just its coefficients are given by

$$b_{j_s} = a^m M(q - 1 - j_s, 5, (q - 1 - j_s + \frac{u(q-1-j_s)+1}{v} - \frac{5h}{z})(\mathrm{mod} 5))(\mod p)$$

where $m = (q - 1 - j_s + \frac{u(q-1-j_s)+1}{v} - \frac{5h}{z})(\mathrm{mod} 5)$.

Example 5. Let $q = 81$ and $f(X) = X(X^{16} + 1)$ so $u = 1$ and $v = 16$. Diophantine equation has the form $16X - Y = 1$ so $c_0 = 1$ and thus

$$j_0 = 65, \quad j_1 = 49, \quad j_2 = 33, \quad j_3 = 17, \quad j_4 = 1.$$

$$\begin{aligned}
b_{65} &= M(15,5,1) = M(14,5,0) + M(14,5,1) = 1 \\
b_{49} &= M(31,5,3) = M(30,5,2) + M(30,5,3) = 2 \\
b_{33} &= M(47,5,0) = M(46,5,0) + M(46,5,4) = 1 \\
b_{17} &= M(63,5,2) = M(62,5,1) + M(62,5,2) = 2 \\
b_{1} &= M(79,5,4) = M(78,5,3) + M(78,5,4) = 2.
\end{aligned}$$

Inverse polynomial is

$$f^{-1}(X) = X^{65} + 2X^{49} + X^{33} + 2X^{17} + 2X.$$

Example 6. Let $q = 81$ and $f(X) = X^3(X^{32} + 1)$, then $u = 3$, $v = 32$, $z = 2$ and $c_0 = 1$. We have

$$j_0 = 75, \quad j_1 = 59, \quad j_2 = 42, \quad j_3 = 27, \quad j_4 = 11.$$

Inverse polynomial is $f^{-1}(X) = X^{75} + 2X^{59} + X^{43} + 2X^{27} + 2X^{11}$.

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On the numerical solution of functional differential equations

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Abstract

In this paper, the aim is to solve the functional differential equations in the following form using multiquadric approximation scheme,

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t))), & t_1 \le t \le t_f, \\ u(t) = \phi(t), \quad t \le t_1, & \end{cases} \tag{1}$$

where $f : [t_1, t_f] \times R \times R \to R$, $\alpha(t)$ is a continuous function on $[t_1, t_f]$ and $\phi(t) \in C$ represents the initial point or the initial data.
We present the property of multiquadric approximation scheme and its advantage of using the data points in arbitrary locations with arbitrary ordering. In the sequel, presented numerical solutions of some experiments, illustrates the high accuracy and the efficiency of the proposed method.

Keywords: Multiquadric approximation scheme; Delay differential equations; Functional differential equations; Pantograph equations.
2000 Mathematics Subject Classification: 65N; 65L10; 65N55.

1. Introduction

The multiquadric (MQ) approximation scheme is an useful method for the numerical solution of ordinary and partial differential equations (ODEs and PDEs). It is a grid-free spatial approximation scheme which converges exponentially for the spatial terms of ODEs and PDEs.
The MQ approximation scheme was first introduced by Hardy [2] who successfully applied this method for approximating surface and bodies from field data. Hardy [3] has written a detailed review article summarizing its explosive growth in use since it was first introduced.

In 1972, Franke [4] published a detailed comparison of 29 different scattered data schemes against analytic problems. Of all the techniques tested, he concluded that MQ performed the best in accuracy, visual appeal, and ease of implementation, even against various finite element schemes.

Functional differential equations are considered as a branch of delay differential equations (DDEs). DDEs arise in many areas of mathematical modeling. For instance, population dynamics, infectious diseases, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircraft and control problems. There are many books to the application of DDEs which we can point out to the books of Driver [5], Gopalsamy [6], Halanay [7], Kolmanovskii and Myshkis [8], Kolmanovskii and Nosov [9] and Kuang [10]. Some modelers ignore the 'lag' effect and use an ODE model as a substitute for a DDE model. Kuang ([10], p.11) comments under the heading *"Small Delay Can Have Large Effects"*, on the dangers that researchers risk if they ignore lags which they think are small; see also El'sgol'ts and Norkin ([11], p.243 et seq.). Other modelers replace a scalar DDE by a system of ODE in an attempt to simulate phenomena more appropriately modeled by DDEs .There are inherit qualitative differences between DDEs and finite systems of ODEs that make such a strategy risky. The fact that many phenomena frequently modeled by ODEs can be bettered modeled by DDEs has not escaped the attention of the numerical analysis community. It is bettered to discuss about the DDEs independently and try not to enter issue of ODEs in the problem which it is a complete DDE problem. Many different methods have been presented for numerical solution of DDEs such that we can point out to the Radau IIA method ([12]), Runge-Kutta method and continuous Runge-Kutta method ([13] and [14]).

In the following of these methods, we are interested to solve functional differential equations (FDEs)(1) of DDEs by the MQ approximation scheme, because this method of solution works excellently, particularly when the data are scattered. The organization of this paper is as follows: Section 2 is devoted to introduce the MQ approximation scheme and its preliminary concepts. In Section 3 we have applied the MQ approximation scheme to equation (1). In Section 4 we have presented some experiments and their numerical results which illustrate the efficiency and accuracy of the proposed method.

2. MQ approximation scheme

The basic MQ approximate schem assumes that any function can be expanded as a finite series of upper hyperboloids,

$$u(t) = \sum_{j=1}^{N} a_j h(t - t_j), \qquad t \in R^d, \tag{2}$$

where N is the total number of data centers under consideration, and

$$h(t - t_j) = ((t - t_j)^2 + R^2)^{\frac{1}{2}}, \quad j = 1, 2, \dots, N.$$

$(t - t_j)^2$ is the square of Euclidean distance in R^d and $R^2 > 0$ is an input shape parameter. Note that, the basis function h is continuously differentiable, and is a type of spline approximation.

The expansion coefficients a_j are found by solving a set of full linear equations,

$$u(t_i) = \sum_{j=1}^{N} a_j h(t_i - t_j), i = 1, 2, \ldots, N. \tag{3}$$

Zerroukut et al [15] found that a constant shape parameter (R^2) has achieved a better accuracy. Mai-Duy and Tran-Cong [16] have developed new methods based on radial basis function networks (RBFN) for the approximation of both functions and their first and higher derivatives. The so called direct RBFN (DRBFN) and indirect RBFN (IRBFN) methods where studied and it was found that the IRBFN methods yields consistently better results for both functions and derivatives. Recently, Aminataei and Mazarei [17] stated that, in the numerical solution of elliptic PDEs using direct and indirect RBFN methods, the IRBFN method is very accurate than other methods and the error is very small. They have shown that, especially, on one dimensional equations, IRBFN method is more accurate than DRBFN method.

Micchelli [18] proved that MQ belongs to a class of conditionally positive definite RBFN. He showed that the equation (2) is always solvable for distinct points. Madych and Nelson [19] proved that the MQ interpolation always produces a minimal semi-norm error, and that the MQ interpolant and derivative estimates converge exponentially as the density of data centers increases.

In contrast, the MQ interpolant is continuously differentiable over the entire domain of data centers, and the spatial derivative approximations were found to be excellent, most especially in very steep gradient regions where traditional methods fail. This excellent ability to approximate spatial derivatives is due in large part by a slight modification of the original MQ scheme by permitting the shape parameter to vary with the basis function.

Instead of using the expansion in equation (2), we used from ($[20]-[22]$) the following:

$$u(t) = \sum_{j=1}^{N} a_j h(t - t_j), \qquad t \in R^d, \tag{4}$$

where

$$h(t - t_j) = ((t - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, N,$$

$$R_j^2 = R_{min}^2 (\frac{R_{max}^2}{R_{min}^2})^{(\frac{j-1}{N-1})}, \quad j = 1, 2, \ldots, N,$$

and

$$R_{min}^2 > 0.$$

R_{max}^2 and R_{min}^2 are two input parameters chosen so that the ratio

$$\frac{R_{max}^2}{R_{min}^2} \cong 10 \ \ to \ \ 10^6.$$

Madych [23] proved that under circumstances very large values of a shape parameter are desirable. The adhoc formula in equation (4) is a way to have at least one very large value of a shape parameter without incurring the onset of severe ill-conditioning problems.

Spatial partial derivatives of any function are formed by differentiating the spatial basis functions. Consider a one dimensional problem. The first derivative is given by simple differentiation:

$$u'(t_i) = \sum_{j=1}^{N} \frac{a_j(t_i - t_j)}{h_{ij}}, \quad h_{ij} = ((t_i - t_j)^2 + R_j^2)^{\frac{1}{2}}, \quad i = 1, 2, \ldots, N.$$

3. Numerical solution of FDEs

In this section, we are interested to solve equation (1), i.e.

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t))), & t_1 \leq t \leq t_f, \\ u(t) = \phi(t), \qquad t \leq t_1, \end{cases} \tag{5}$$

by the MQ approximation scheme mentioned in section 2. The aforesaid equation is one of the main delay differential equations including following important pantograph equations:

$$\begin{cases} u'(t) = au(t) + bu(rt), & 0 \leq t \leq t_f, \\ u(0) = u_1, \end{cases}$$

where $a, b \in C$ and $0 < r < 1$. This equation is a very special delay differential equation that arise in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics and electro-dynamics. In particular it was used by Okendon and Taylor [1] to study how to electric current collected by the pantograph of an electric locomotive.

For the solution of equation (5), it is sufficient to suppose that approximate solution is

$$u(t) = \sum_{j=1}^{N} a_j h(t - t_j), \qquad t \in [t_1, t_f],$$

with

$$u'(t_i) = \sum_{j=1}^{N} \frac{a_j(t_i - t_j)}{h_{ij}}, \quad i = 1, 2, \ldots, N.$$

Now we use the N collocation points to gain a system of N equations with N unknowns. Then we must solve this system to distinct the unknown coefficients.
By imposing the supplementary condition to the problem, we have following system for equation (5),

$$\begin{cases} \sum_{j=1}^{N} \frac{a_j(t_i - t_j)}{h_{ij}} = f(t_i, \sum_{j=1}^{N} a_j h(t_i - t_j), \sum_{j=1}^{N} a_j h(\alpha(t_i) - t_j)), \quad i = 2, 3, \ldots, N, \\ \sum_{j=1}^{N} a_j h(t_1 - t_j) = u_1. \end{cases} \tag{6}$$

Here, the differential equation yields $N - 1$ equations and initial condition produce one equation. Thus the system has N equations with N unknowns. This system must be solved to extract the unknown coefficients. Hence, we have used the Gauss elimination method with total pivoting to solve such a system.

Remark. It is noticeable that collocating points can be scattered. This is the main difference between this method of solution and other methods. In next section, the numerical results demonstrate this issue, easily and the efficiency of MQ approximation scheme in this sense, is observable.

4. Numerical experiments

In this part, we present some experiments in-which their numerical solutions illustrate the high accuracy and efficiency of MQ approximation scheme.
Problem 1. Consider following pantograph equation [14](p.167),

$$\begin{cases} u'(t) - u(\frac{1}{2}t) = 0, & t \in [0,1], \\ u(0) = 1. \end{cases}$$

The considered delay is $\alpha(t) = \frac{1}{2}t$ or $r = \frac{1}{2}$. The exact solution is $u(t) = \sum_{k=0}^{\infty} \frac{\frac{1}{2}^{\frac{1}{2}k(k-1)}}{k!} t^k$. The MQ approximate solution is obtained with $R_{max} = 150, R_{min} = .99$ and $N = 21$, and the results are given in Table I.

Table I

t_i	MQ approximate solution	Exact solution	Error
0	0.999999999978	1	2.2×10^{-11}
.05	1.050627608197	1.050627608238	4.12×10^{-11}
.1	1.102520898478	1.102520898518	4.09×10^{-11}
.15	1.155695642676	1.155695642708	3.23×10^{-11}
.2	1.210167710898	1.210167710940	4.22×10^{-11}
.25	1.265953071919	1.265953071922	3.48×10^{-12}
.3	1.323067793204	1.323067793243	3.98×10^{-11}
.35	1.381528041658	1.381528041681	2.3×10^{-11}
.4	1.441350083511	1.441350083507	3.9×10^{-11}
.45	1.502550284794	1.502550284799	5.02×10^{-12}
.5	1.565145111761	1.565145111746	1.4×10^{-11}
.55	1.629151130950	1.629151130963	1.34×10^{-11}
.6	1.694585009799	1.694585009792	6.31×10^{-11}
.65	1.761463516598	1.761463516621	2.34×10^{-11}
.7	1.829803521200	1.829803521189	1.09×10^{-11}
.75	1.899621994909	1.899621994899	9.81×10^{-11}
.8	1.970936011106	1.970936011130	2.49×10^{-11}
.85	2.043762745581	2.043762745551	2.95×10^{-11}
.9	2.118119476412	2.118119476428	1.61×10^{-11}
.95	2.194023584937	2.194023584941	4.59×10^{-12}
1	2.271492555499	2.271492555501	2.06×10^{-12}

Problem 2. Consider the following FDE,

$$\begin{cases} u'(t) + u(t) - u(rt) = t^2 + 2t - (rt)^2, & t \in [0, 2], \\ u(t) = 0, \quad t \le 0. \end{cases}$$

Here, the delay $\alpha(t) = rt$ is considered and the exact solution is $u(t) = t^2$. For $r = 0.5$ with $R_{max} = 1950, R_{min} = 50.26$ and $N = 6$, we have the following Table which illustrate the efficiency and accuracy of MQ approximation scheme.

Table II

t_i	MQ approximate solution	Exact solution	Error
0	0	0	0
.4	0.160000000	0.160000000	0
.8	0.640000000	0.640000000	0
1.2	1.440000000	1.440000000	0
1.6	2.560000000	2.560000000	0
2	4.000000000	4.000000000	0

For $r = .25$ with $R_{max} = 2950, R_{min} = 150.26$ and $N = 6$, we have MQ approximate solution in Table III.

Table III

t_i	MQ approximate solution	Exact solution	Error
0	0	0	0
.4	0.160000000	0.160000000	0
.8	0.640000000	0.640000000	0
1.2	1.440000000	1.440000000	0
1.6	2.560000000	2.560000000	0
2	3.999999999	4.000000000	1.0×10^{-9}

In the following, we have presented an almost complicated experiment which its numerical results shows that, in spite of complexity of problem, in MQ method, data can be scattered. Therefore this method isn't depend on the selection of points. Here, also we observe the high efficiency and accuracy of this method, too.

Problem 3. Consider the following FDE,

$$\begin{cases} \sqrt{e^t + sin(\sqrt{t})}\; u'(t) + \sqrt{cos(t)}\; u(\alpha(t)) = \sqrt{e^t + sin(\sqrt{t})}\; e^t + \sqrt{cos(t)}\; e^{\alpha(t)}, & t \in [0, 1], \\ u(t) = t^2 + t + 1, \quad t \le 0, \end{cases}$$

the exact solution is e^t.
By choosing $R_{max} = 50, R_{min} = .499, N = 15$ and $\alpha(t) = \sqrt{t}$, we have the following Table for MQ approximate solution.

Table IV

t_i	MQ approximate solution	Exact solution	Error
0	1.0000000000	1	0
.06	1.0618365460	1.0618365465	5.45×10^{-10}
.15	1.1618342422	1.1618342427	5.28×10^{-10}
.2	1.2214027577	1.2214027581	4.6×10^{-10}
.28	1.3231298118	1.3231298123	5.37×10^{-10}
.34	1.4049475901	1.4049475905	4.63×10^{-10}
.41	1.5068177847	1.5068177851	4.12×10^{-10}
.47	1.5999941928	1.5999941932	4.12×10^{-10}
.55	1.7332530174	1.7332530178	4.67×10^{-10}
.64	1.8964808790	1.8964808793	3.04×10^{-10}
.69	1.9937155328	1.9937155332	4.43×10^{-10}
.76	2.1382762201	2.1382762204	3.96×10^{-10}
.85	2.3396468515	2.3396468519	4.25×10^{-10}
.91	2.4843225330	2.4843225333	3.84×10^{-10}
1	2.7182818281	2.7182818284	3.59×10^{-10}

And by choosing $\alpha(t) = t^2$, when R_{max}, R_{min} and N are as before, we also have the following Table for MQ approximate solution.

Table V

t_i	MQ approximate solution	Exact solution	Error
0	1.0000000000	1	0
.06	1.0618365459	1.0618365465	6.45×10^{-10}
.15	1.1618342422	1.1618342427	5.28×10^{-10}
.2	1.2214027577	1.2214027581	4.6×10^{-10}
.28	1.3231298119	1.3231298123	4.37×10^{-10}
.34	1.4049475901	1.4049475905	4.63×10^{-10}
.41	1.5068177847	1.5068177851	4.12×10^{-10}
.47	1.5999941929	1.5999941932	3.12×10^{-10}
.55	1.7332530175	1.7332530178	3.67×10^{-10}
.64	1.8964808788	1.8964808793	5.04×10^{-10}
.69	1.9937155330	1.9937155332	2.43×10^{-10}
.76	2.1382762203	2.1382762204	1.96×10^{-10}
.85	2.3396468517	2.3396468519	2.25×10^{-10}
.91	2.4843225331	2.4843225333	2.84×10^{-10}
1	2.7182818283	2.7182818284	1.59×10^{-10}

We have observed that, in this problem there is a little difference between the results of Tables IV and V in spite of different delays. This is an excellent advantage for application of MQ method, because delay differential equations are very sensitive to the delays and their behaviors. In particular, when we apply a method which needs collocation points, if scattered data used, the round off error may occurs, soon. But this method (MQ method) isn't depend on collocating points in large scales.

Problem 4. Consider the following non-linear FDE,

$$\begin{cases} u'(t) + u^2(\frac{t}{2}) = 0, & t \in [0, 1], \\ u(t) = e^{-t}, & t \leq 0. \end{cases}$$

The exact solution is $u(t) = e^{-t}$. For $R_{max} = 40, R_{min} = 1.2$ and $N = 8$, we have the following Table which illustrate the efficiency and accuracy of MQ approximation scheme for non-linear FDEs, too.

Table VI

t_i	MQ approximate solution	Exact solution	Error
0	1.00000000	1	0
$\frac{1}{7}$	0.86688225	0.86687789	4.35×10^{-6}
$\frac{2}{7}$	0.75148083	0.75147729	3.53×10^{-6}
$\frac{3}{7}$	0.65144157	0.65143905	2.87×10^{-6}
$\frac{4}{7}$	0.56471978	0.56471812	1.66×10^{-6}
$\frac{5}{7}$	0.48954264	0.48954165	9.87×10^{-7}
$\frac{6}{7}$	0.42437329	0.42437284	4.53×10^{-7}
1	0.36787948	0.36787944	4.70×10^{-8}

5. Conclusion

In this study, the MQ approximation scheme is proposed for solving functional differential equations. This method of solution is easy to implement and yields desired accuracy only in few terms. As we have observed, the method works excellently for functional differential equations in spite of scattered data. The computations associated with the experiments discussed above, were performed by using Maple 10.

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Lacunary Strongly Almost Convergent Sequences of Fuzzy Numbers

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Abstract. In this paper we introduce and study strongly almost statistical convergence, lacunary strongly almost statistical convergence and lacunary strongly almost summable of sequences of fuzzy numbers.

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Key words: Fuzzy numbers; lacunary sequence; statistical convergence.

1. Introduction

Here we include only a brief idea of fuzzy numbers. For details one my refer [6], [7].

Let D denote the set of all closed bounded intervals A=$\left[\underline{A},\overline{A}\right]$ on the real line R. For A, B$\in$ D define A$\leq$B if and only if $\underline{A}\leq\underline{B}$ and $\overline{A}\leq\overline{B}$, d$(A,B)$= max$\left\{\left|\underline{A}-\underline{B}\right|,\left|\overline{A}-\overline{B}\right|\right\}$.

It is easy to see that defines a metric on D and (D,d) is complete metric space. Also $\leq$ is a partial order in D. A fuzzy number is a fuzzy subset of real line R which is bounded , convex and normal. Let $L(R)$denote the set of all fuzzy numbers which are upper semicontinuous and have compact support. In other words, if $X\in L(R)$ then for any $\alpha\in[0,1]$, X^{α} is compact , where

$$X^{\alpha}=\begin{cases} t, X(t)\geq\alpha & if\ \ \alpha\in(0,1] \\ t, X(t)>0 & if\ \ \alpha=0 \end{cases}.$$

For each $0<\alpha\leq 1$, the α-level set X^{α} is a non-empty compact subset of R. The linear structure of $L(R)$ induces addition $X+Y$ and scalar multiplication λX, $\lambda\in R$, in terms of α-level sets, by

$$[X+Y]^{\alpha}=[X]^{\alpha}+[Y]^{\alpha} \quad and \quad [\lambda X]^{\alpha}=\lambda[X]^{\alpha}$$

for each $0 \le \alpha \le 1$.

Define $\bar{d}: L(R) \times L(R) \to R$ by $\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d(X^\alpha, Y^\alpha)$. For $X, Y \in L(R)$ define $X \le Y$ if and only if $X^\alpha \le Y^\alpha$ for any $\alpha \in [0,1]$. It is know that $L(R)$ is a complete metric space with the metric $\bar{d}$ [5].

A metric on $L(R)$ is said to be translation invariant if $\bar{d}(X+Z, Y+Z) = \bar{d}(X,Y)$ for all $X, Y, Z \in L(R)$.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ [5].

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_o = 0$ and $h_r = k_r - k_{r-1} \to \infty \quad as\ r \to \infty$. Throughout this paper the intervals determined by $\theta = (k_r)$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated as q_r.

The idea of statistical convergence of a sequence of real numbers was introduced by Fast [1]. Schoenberg [8] studied statistical convergence as a summability method and listed some of elemantary properties of statistical convergence.Both these authors noted that if bounded sequence is statistically convergent to l, then it is Cesaro summable to l. Lacunary statistical convergent sequences were introduced by Fridy and Orhan [2].

A sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1} \left| \{k \le n : \ |x_k - l| \ge \varepsilon\} \right| = 0$$

where the vertical bars denote the cardinality of the set which they enclosed, in which case we write $S - \lim x = l$.

For sequences of fuzzy numbers Nuray and Savas [3] and Nuray [4] have investigated statistically convergent and lacunary statistical convergent sequences of fuzzy numbers.

In this paper we introduce and study strongly almost statistical convergence, lacunary strongly almost statistical convergence and lacunary strongly almost summable of sequences of fuzzy numbers.

2. Lacunary Strongly Almost Convergence

Definition 1. A sequence $X = (X_k)$ of fuzzy numbers is said to be strong almost statistically convergent to fuzzy number X_o if for every $\varepsilon > 0$

$$\lim_{n\to\infty} n^{-1} \left| \{k \le n : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\} \right| = 0 \text{ uniformly in } m,$$

where

$$t_{km}(X) = \frac{X_m + X_{m+1} + \ldots + X_{m+k}}{k+1}.$$

In this case we write $X_k \to X_o\ (\hat{S})$. The set of all strong almost statistically convergent sequences is denoted by $\hat{S}$.

Definition 2. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary strongly almost statistically convergent to fuzzy number X_o if for every $\varepsilon > 0$

$$\lim_{r\to\infty} h_r^{-1}\left|\left\{k \in I_r :\ \ \overline{d}(t_{km}(X), X_o) \geq \varepsilon\right\}\right| = 0 \text{ uniformly in } m.$$

In this case we write $X_k \to X_o\ (\hat{S}_\theta)$. The set of all lacunary strong almost statistically convergent sequences is denoted by $\hat{S}$.

Definition 3. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary strong almost summable if there is a fuzzy number X_o such that

$$\lim_{r\to\infty} h_r^{-1}\sum_{k\in I_r}\overline{d}(t_{km}(X), X_o) = 0 \text{ uniformly in } m.$$

We shall use $\hat{N}_\theta$ to denote the set of all lacunary strong convergent sequences of fuzzy numbers.

Theorem 4. $\hat{N}_\theta$ is complete metric space with the metric

$$\delta(X,Y) = \sup_r h_r^{-1}\sum_{k\in I_r}\overline{d}(t_{km}(X), t_{km}(Y)).$$

Proof. Let (X^n) be a Cauchy sequence in $\hat{N}_\theta$, where

$(X^n) = (X_i^n) = (X_1^n, X_2^n, X_3^n, \ldots) \in \hat{N}_\theta$ for each $n \in N$. Then

$$\delta(X^n, X^t) = \sup_r h_r^{-1}\sum_{k\in I_r}\overline{d}(t_{km}(X^n), t_{km}(X^t)) \to 0 \quad as\ n,t \to \infty.$$

Hence for each k and m, $as\ n,t \to \infty$, we have $\overline{d}(t_{km}(X^n), t_{km}(X^t)) \to 0$. In particular

$$\lim_{n,t\to\infty}\overline{d}(t_{0m}(X^n), t_{0m}(X^t)) = \lim_{n,t\to\infty}\overline{d}(X^n, X^t) = 0$$

for each fixed m. Hence $(X^n)_n$ is a Cauchy sequence in $L(R)$. Since $L(R)$ is complete, it is convergent

$$\lim_{n\to\infty} X_k^n = X_k$$

say, for each $k \in N$. Since $(X^n)_n$ is a Cauchy sequence, for each $\varepsilon > 0$, there exists $n_o = n_o(\varepsilon)$ such that

$$\delta(X^n, X^t) < \varepsilon \ , \quad \text{for all } n, t \geq n_o .$$

So, we have

$$\lim_{t\to\infty} h_r^{-1} \sum_{k\in I_r} \overline{d}(t_{km}(X^n), t_{km}(X^t)) = h_r^{-1} \sum_{k\in I_r} \overline{d}(t_{km}(X^n), t_{km}(X)) \leq \varepsilon$$

for all m and $n \geq n_o$. This implies that $\delta(X^n, X) < \varepsilon$, for all $n \geq n_o$, that is $X^n \to X \ as\ n \to \infty$, where $X = (X_k)$. Since

$$h_r^{-1} \sum_{k\in I_r} \overline{d}(t_{km}(X), X_o) \leq h_r^{-1} \sum_{k\in I_r} \overline{d}(t_{km}(X^{n_o}), X_o) + h_r^{-1} \sum_{k\in I_r} \overline{d}(t_{km}(X^{n_o}), t_{km}(X)) \to 0$$

$as\ r \to \infty$, uniformly in m. So, we obtain $X = (X_k) \in \hat{N}_\theta$. Therefore $\hat{N}_\theta$ is complete metric space. This completes the proof.

There is strong connection between $\hat{N}_\theta$ and the class of sequences of fuzzy numbers $\hat{w}$, which is defined by

$$\hat{w} = \left\{ X = (X_k) : \ \lim_n n^{-1} \sum_{k=1}^{n} \overline{d}(t_{km}(X), X_o) = 0, \quad uniformly\ in\ m \right\}.$$

In special case $\theta = (2^r)$, we have $\hat{N}_\theta = \hat{w}$.

Theorem 5. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. Then

(i) For $limsup_r q_r < \infty$, we have $\hat{N}_\theta \subset \hat{w}$.

(ii) For $liminf_r q_r > 1$, we have $\hat{w} \subset \hat{N}_\theta$.

(iii) $\hat{N}_\theta = \hat{w}$ if $1 < liminf_r q_r \leq limsup_r q_r < \infty$.

Proof. (i) If $limsup_r q_r < \infty$, then there is a $K > 0$ such $q_r < K$ for all r. Suppose that $X_k \to X_o(\hat{N}_\theta)$, and for each $m \geq 1$ let $N_{rm} = \left| \{ k \in I_r : \ \overline{d}(t_{km}(X), X_o) \geq \varepsilon \} \right|$. By the definition, given $\varepsilon > 0$, there is an $r_o \in N$ such that

$$\frac{N_{rm}}{h_r} < \varepsilon, \ for\ all\ r > r_o\ \ and\ m \geq 1. \qquad (1)$$

Now let $M = max\{N_{rm} : 1 \leq r \leq r_o\}$ and let n be any integer satisfying $k_{r-1} < n \leq k_r$,

then for each $m \geq 1$, we have

$$n^{-1}\left|\left\{k \le n : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\right\}\right|$$

$$\le \frac{1}{k_{r-1}}\left|\left\{k \le k_r : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}}\left\{N_{1m} + N_{2m} + \cdots + N_{r_o m} + N_{(r_o+1)m} + \cdots + N_{rm}\right\}$$
$$\le \frac{M}{k_{r-1}} r_o + \frac{1}{k_{r-1}}\left\{h_{r_o+1}\frac{N_{(r_o+1)m}}{h_{r_o+1}} + \cdots + h_r \frac{N_{rm}}{h_r}\right\}$$
$$\le \frac{M}{k_{r-1}} r_o + \frac{1}{k_{r-1}}\left(\sup_{r>r_o}\frac{N_{rm}}{h_r}\right)\left\{h_{r_o+1} + \cdots + h_r\right\}$$
$$\le \frac{M}{k_{r-1}} r_o + \varepsilon \frac{k_r - k_{r_o}}{k_{r-1}} \qquad \text{by (1)}$$
$$\le \frac{M}{k_{r-1}} r_o + \varepsilon q_r < \frac{M}{k_{r-1}} r_o + \varepsilon K$$

so, the result follows immediately.

(ii) Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \ge 1 + \delta$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{h_r}{k_r} \ge \frac{\delta}{1+\delta}$. If $X_k \to X_o(\hat{w})$, then for every $\varepsilon > 0$, for each $m \ge 1$ and sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k \le k_r : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\right\}\right|$$
$$\ge \frac{1}{k_{r-1}}\left|\left\{k \in I_r : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\right\}\right|$$
$$\ge \frac{\delta}{1+\delta} h_r^{-1}\left|\left\{k \in I_r : \ \bar{d}(t_{km}(X), X_o) \ge \varepsilon\right\}\right|$$

which yields that $X_k \to X_o(\hat{N}_\theta)$.

(iii) Follows from (i) and (ii).

The following theorem gives the relation between lacunary almost statistical convergence and lacunary strongly almost convergence of sequences of fuzzy numbers.

Theorem 5. Let $\theta = (k_r)$ be a lacunary sequence and let $X = (X_k)$ be a sequence of fuzzy numbers. Then

(i) $X_k \to X_o(\hat{N}_\theta)$ implies $X_k \to X_o(\hat{S}_\theta)$.
(ii) $X = (X_k) \in \hat{m}$ and $X_k \to X_o(\hat{S}_\theta)$ imply $X_k \to X_o(\hat{N}_\theta)$.
(iii) $\hat{N}_\theta = \hat{S}_\theta$ if $X = (X_k) \in \hat{m}$,

where $\hat{m} = \left\{X = (X_k) : \ \sup_{k,m} \bar{d}(t_{km}(X), X_o) < \infty\right\}$.

Proof.(i) If $\varepsilon > 0$ and $X_k \to X_o(\hat{N}_\theta)$, we can write

$$h_r^{-1}\sum_{k\in I_r}\overline{d}(t_{km}(X),X_o)\geq h_r^{-1}\sum_{\substack{k\in I_r\\ d(t_{km}(X),X_o)\geq\varepsilon}}\overline{d}(t_{km}(X),X_o)$$

$$\geq \varepsilon h_r^{-1}\left|\left\{k\in I_r:\ \overline{d}(t_{km}(X),X_o)\geq\varepsilon\right\}\right|.$$

It follows that $X_k \to X_o(\hat{S}_\theta)$.

(ii) Suppose that $X=(X_k)\in\hat{m}$ and $X_k \to X_o(\hat{S}_\theta)$. Since $X=(X_k)\in\hat{m}$, there is a constant $B>0$ such that $\overline{d}(t_{km}(X),X_o)<B$ for all $k,m\in N$. Therefore we have, for every $\varepsilon>0$

$$h_r^{-1}\sum_{k\in I_r}\overline{d}(t_{km}(X),X_o)=h_r^{-1}\sum_{\substack{k\in I_r\\ d(t_{km}(X),X_o)\geq\varepsilon}}\overline{d}(t_{km}(X),X_o)+h_r^{-1}\sum_{\substack{k\in I_r\\ d(t_{km}(X),X_o)<\varepsilon}}\overline{d}(t_{km}(X),X_o)$$

$$\leq B h_r^{-1}\left|\left\{k\in I_r:\ \overline{d}(t_{km}(X),X_o)\geq\varepsilon\right\}\right|+\varepsilon$$

Taking the limit as $\varepsilon\to 0$, the result follows.

(iii) Follows from (i) and (ii).

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NEWMARK METHOD APPLIED TO THE ELASTO-DYNAMIC PROBLEM WITH SLIP-RATE DEPENDENT FRICTION

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ABSTRACT. We consider the dynamic evolution of an elastic body in unilateral frictional contact with a rigid foundation. Friction is modeled by the Coulomb law with a coefficient that depends on the slip rate, which is a non-monotone function. This problem is ill posed; the solution is non-unique and shocks. We use the Newmark method for time discretization of the problem. We have obtained an elliptic variational inequality at each time step. We prove by the minimization problem the existence of the solution. An upper bound for time step size, which is not a (Courant, Friedrichs and Lewy) CFL condition, is deduced from the solution uniqueness criterion using the first eigenvalue of the eigenvalue problem.

1. INTRODUCTION

Duvaut and Lions [3] obtained the first existence and uniqueness results in the mathematical approach of the dynamic contact problems with friction in elasticity. They studied the special case of a prescribed normal pressure where the contact surface is known in advance. To obtain some existence results without this restrictive assumption, Martins and Oden [12], [11], relaxed the non-penetrability of mass by considering the normal compliance model of contact with friction.

Since the pure elastic problem with friction seems to be very irregular, they considered only the viscous case. As far as we know, there is no general existence result for dynamic elastic contact.

All the above results involve a fixed friction coefficient μ. In the study of many frictional processes (stick-slip motions, earthquakes modeling, etc...) the friction coefficient has to be considered variable during the slip. Usually choice of such a variation is the friction law in which the friction coefficient is dependent on the slip rate $\dot{u}_\tau(t)$ i.e. $\mu = \mu\left(\left|\dot{u}_\tau(t)\right|\right)$. The simplest function of μ versus the slip rate is the discontinuous jump from a "static" value (for $\dot{u}_\tau(t) = 0$) down to a "dynamic" value (for $\dot{u}_\tau(t) \neq 0$). The same frictional instabilities can occur when the coefficient of friction μ is a smooth and decreasing function of the slip rate (see [14] and [22]). Ionescu and Paumier [6], [7], studied this model of friction for the one-dimensional shearing problem of infinite elastic slab. They pointed out that the solution of the problem is not uniquely determined. However, since the problem is ill posed, a criterion to select the most appropriate solution with a physical interpretation is needed. Whatever is the selection rule to choose the solution, shocks will occur. A possible choice for this criterion is the so-called perfect delay convention: "the system only jumps when it has no other choice" (see [7] and [2]).

Key words and phrases. Eigenvalue, Elastodynamic, Friction, Inquality variational, Newmark method, Optimization, Slip-rate.

Three technique can be used to transform this ill posed problem into a well posed one and to justify the choice of the perfect delay criterion are currently under investigation. In all three cases, another problem (which depends on a small parameter) is considered. This new problem is well posed and its unique solution converges to the solution of the initial (ill posed) problem chosen by the perfect delay convention.

The first technique, due to Renard [19], [20], consists in adding a small mass concentrated on the friction surface. Using stability arguments, Renard has proved the convergence of the solution of a large class of loadings in the one-dimensional case.

The second one uses a rate and state dependent friction law of Dieterich and Ruina type (see [21], [18], [4] and [15]). Favreau and al. [5] have shown numerically that the limit solution, when the characteristic slip $L \to 0$ is the one corresponding to the rate dependent model assuming the perfect delay convention. For both methods, there are no existence or uniqueness results in two or three dimensional.

The third technique uses a viscoelastic constitutive law with small viscosity (see [9]). An existence and uniqueness result is obtained in the general three-dimensional case.

B. Nouiri and B. Benabderrahmane [13], they are presented a new technique, is based on the Newmark method for the time discretization of the dynamic evolution of an elastic body in unilateral frictional contact with a rigid foundation. An existence and uniqueness result is obtained in the one-dimensional case.

In this paper, we have generalized the above technique [13] in two or three-dimensional case. We have obtained an elliptic variational inequality at each time step. We prove by the minimization problem the result of solution existence. An upper bound for time step size, which is not a (Courant, Friedrichs and Lewy) CFL condition, is deduced from the solution uniqueness criterion using the first eigenvalue of the eigenvalue problem.

2. Problem statement

We take the geometry of an elastic body rubbing on a rigid surface, represented by a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$). The boundary $\Gamma$of Ω is assumed to be *Lipschitz* is divided as follows: $\Gamma = \bar{\Gamma}_d \cup \bar{\Gamma}_f \cup \bar{\Gamma}_c$ where Γ_d, Γ_f and Γ_c are three disjoint parts and $(meas(\Gamma_d) > 0)$. Γ_f is the surface on which the force $F(x)$ is exercised and Γ_c is the surface of contact with friction. To simplify the problem without losing in generality, we are going to suppose that the displacement field $U(x)$ on Γ_d is equal to zero. We denote by η and τ the unit outward normal vector on Γ_f and the tangent vector on Γ_c respectively.

It is important to specify the condition modeling slip-rate dependent friction with to imposed normal constraint:

$$\sigma_\eta\left(\bar{u}(t)\right) = -S \quad \text{on } \Gamma_c \tag{2.1}$$

$$\left|\sigma_\tau\left(\bar{u}(t)\right)\right| \leq S\mu(\left|\dot{\bar{u}}_\tau(t)\right|) \text{ with} \tag{2.2}$$

$$\left|\sigma_\tau\left(\bar{u}(t)\right)\right| \leq S\mu(0) \quad \text{if} \quad \dot{\bar{u}}_\tau(t) = 0 \text{ on } \Gamma_c \tag{2.3}$$

$$\sigma_\tau\left(\bar{u}(t)\right) = S\mu\left(\left|\dot{\bar{u}}_\tau(t)\right|\right)\frac{\dot{\bar{u}}_\tau(t)}{\left|\dot{\bar{u}}_\tau(t)\right|} \quad \text{if} \;\; \dot{\bar{u}}_\tau(t) \neq 0 \;\; \text{on} \;\; \Gamma_c \tag{2.4}$$

Where $\bar{u}$ is the displacement field, σ_η and σ_τ denote the normal constraint and the tangential constraint respectively. μ the friction coefficient on Γ_c, which is dependent of the slip rate $\dot{\bar{u}}_\tau(t)$ and S represents the normal load on Γ_c. The condition (2.3) describes the stick phenomenon, while the condition (2.4) describes the slip phenomenon.

The elasto-dynamic problem of contact with friction to imposed normal constraint considered here consists of finding the displacement field $\bar{u} :]0,T]\times\Omega \longrightarrow \mathbb{R}^n$ such that:

$$\operatorname{div}\sigma(\bar{u}(t)) + r(t) = \rho\, \ddot{\bar{u}}(t) \text{ in } \Omega \tag{2.5}$$

$$\sigma(\bar{u}(t)) = \mathcal{A}\varepsilon(\bar{u}(t)) \;\; \text{in } \Omega \tag{2.6}$$

$$\bar{u}(t) = 0 \text{ on } \Gamma_d \tag{2.7}$$

$$\sigma(\bar{u}(t))\eta = F(t) \text{ on } \Gamma_f \tag{2.8}$$

$$\bar{u}(t) \;\; \text{verifying} \;\; (2.1) - (2.4) \tag{2.9}$$

$$\bar{u}(0) = \bar{u}_0(x);\; \dot{\bar{u}}(0) = \bar{u}_1(x) \text{ in } \Omega \tag{2.10}$$

Where r represents the density of volume forces (for example the weight) and $\bar{u}_0$, $\bar{u}_1$ are the initial data. div denotes the divergence operator of the tensor valued functions and $\sigma = (\sigma_{ij})$ stands for the stress tensor field. This latter is obtained from the displacement field by the constitutive law of linear elasticity defined by (2.6). $\mathcal{A}$ is the fourth order symmetric and elliptic tensor of linear elasticity and $\varepsilon(v) = \frac{1}{2}\left(\nabla v + \nabla^T v\right)$ represents the linearized strain tensor field.

To simplify the problem (2.5) – (2.10), we homogenize the movement equation (2.5), the boundary conditions (2.1), (2.8) and the initial condition (2.10).

We denote by u^{ss} the "stick solution":

$$u^{ss} :]0,T] \times \Omega \longrightarrow \mathbb{R}^n$$

Satisfying the following auxiliary problem, which is corresponding to the stick condition (2.3):

$$\left\{\begin{array}{l} \operatorname{div}\sigma(u^{ss}(t)) + r(t) = \rho\ddot{u}^{ss}(t) \text{ in } \Omega \\ \sigma(u^{ss}(t)) = \mathcal{A}\varepsilon(u^{ss}(t)) \;\; \text{in } \Omega \\ u^{ss}(t) = 0 \text{ on } \Gamma_d \\ \sigma(u^{ss}(t))\eta = F(t) \;\; \text{on } \Gamma_f \\ \sigma_\eta(u^{ss}(t)) = -S \;\; \text{on } \Gamma_c \\ \dot{u}^{ss}_\tau = 0 \;\; \text{on } \Gamma_c \\ u^{ss}(0) = \bar{u}_0(x); \;\; \dot{u}^{ss}(0) = \bar{u}_1(x) \;\; \text{in } \Omega \end{array}\right. \tag{2.11}$$

We denote by $-q(t)$ the tangential constraint on Γ_c corresponding to the "stick solution" of the problem (2.11), i.e. $q(t) = -\sigma_\tau(u^{ss}(t))$ and we pose $u(t) = \bar{u}(t) - u^{ss}(t)$.

We deduce from the problems (2.5)−(2.10) and (2.11) the homogeneous dynamics problem with friction, which consists of finding the displacement field $u :]0,T] \times \Omega \longrightarrow \mathbb{R}^n$ such that:

$$\operatorname{div}\sigma(u(t)) = \rho\ddot{u}(t) \text{ in } \Omega \tag{2.12}$$

$$\sigma(u(t)) = \mathcal{A}\varepsilon(u(t)) \text{ in } \Omega \tag{2.13}$$

$$u(t) = 0 \text{ on } \Gamma_d \tag{2.14}$$

$$\sigma(u(t))\eta = 0 \text{ on } \Gamma_f \tag{2.15}$$

$$\sigma_\eta(u(t)) = 0 \text{ on } \Gamma_c \tag{2.16}$$

$$|\sigma_\tau(u(t)) - q(t)| \leq S\mu(0) \text{ if } \dot{u}_\tau(t) = 0 \text{ on } \Gamma_c \tag{2.17}$$

$$\sigma_\tau(u(t)) + S\mu(|\dot{u}_\tau(t)|)\frac{\dot{u}_\tau(t)}{|\dot{u}_\tau(t)|} = q(t) \text{ if } \dot{u}_\tau(t) \neq 0 \text{ on } \Gamma_c \tag{2.18}$$

$$u(0) = \dot{u}(0) = 0 \text{ in } \Omega \tag{2.19}$$

Where

$$\sigma_\eta(u) = (\sigma(u)\eta)\eta, \ \sigma_\tau(u) = \sigma(u)\eta - \sigma_\eta(u)\eta$$

Remark 1. *In the passage from the problem* (2.5) − (2.10) *to the problem* (2.12) − (2.19)*, we remark that the effects by the external force* r*, the parameters* F, S *and initial data* $\bar{u}_0$, $\bar{u}_1$*lead to the effects of* $q(t)$ *and* S.

It is easy to verify that the problem (2.11) is well posed, i.e. it admits a unique solution, and therefore the essential point in the following will be to study the problem (2.12) − (2.19).

3. Hypothesis and variational formulation

We suppose that the elastic tensor $\mathcal{A}$ is symmetrical and coercive, i.e. $\mathcal{A} = (\mathcal{A}_{ijkh})$, satisfying the following conditions:

$$\mathcal{A}_{ijkh} \in L^\infty(\Omega); \mathcal{A}_{ijkh} = \mathcal{A}_{jikh} = \mathcal{A}_{ijhk} \text{ a.e. in } \Omega \tag{3.1}$$

$$\exists\alpha_0 > 0 : \mathcal{A}_{ijkh}\tau_{ij} \cdot \tau_{kh} \geq \alpha_0\tau_{ij}\tau_{ij} \text{ a.e. in } \Omega, \forall\tau_{ij} = \tau_{ji} \in \mathbb{R} \tag{3.2}$$

It is supposed that the friction coefficient μ, satisfy the following hypothesis:

$$\mu : \Gamma_c \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

$$u \longrightarrow \mu(x,u) \text{ is continuously differentiable on } [0,+\infty[\text{ a.e. on } \Gamma_c \tag{3.3}$$

$$x \longrightarrow \mu(x,u); x \longrightarrow \partial_u\mu(x,u) \text{ is measurable a.e. on } \mathbb{R}_+ \tag{3.4}$$

$$\exists\ell_\mu \geq 0, \forall a,b \in \mathbb{R}_+ : |\mu(x,a) - \mu(x,b)| \leq \ell_\mu |a-b| \text{ a.e. on } \Gamma_c \tag{3.5}$$

$$\exists M_0 > 0 : |\partial_u\mu(x,u)| \leq M_0 \text{ a.e. on } \Gamma_c \tag{3.6}$$

We assume also that ρ, r , F, S and q have the following regularities:

$$\rho(x) \geq \rho_0 > 0; \ \rho(x) \in L^\infty(\Omega) \tag{3.7}$$

(3.8) $$r(t,x) \in L^2\left(0,T;\left[L^2(\Omega)\right]^n\right)$$

(3.9) $$F(t,x) \in L^2\left(0,T;\left[L^2(\Gamma_f)\right]^n\right)$$

(3.10) $$S(x) \geq 0; S \in L^\infty(\Gamma_c),\ q \in \mathcal{L}_p$$

We designate by $\mathcal{L}_p$ the subspace following of $[L^p(\Gamma_c)]^n$:

$$\mathcal{L}_p = \{z \in [L^p(\Gamma_c)]^n / z(x) \cdot \eta(x) = 0 \text{ a.e. } x \in \Gamma_c\}, 1 \leq p < \infty$$

Let us denote by $H = \left[L^2(\Omega)\right]^n$ the Hilbert space endowed with the inner product:

$$(u,v) = \int_\Omega \rho u \cdot v dx, \quad u,v \in H$$

which generates an equivalent norm given by:

$$|u|_H = \left(\int_\Omega \rho u^2 dx\right)^{\frac{1}{2}}$$

Let V_0 be the closed subspace of $V = \left[H^1(\Omega)\right]^n$ given by:

$$V_0 = \{v \in V, v = 0 \quad \text{on } \Gamma_d\}$$

equipped with the usual norm $\|\cdot\|$. We denote by $\langle \cdot,\cdot \rangle$ the duality pairing between V_0' and V_0.

For $p < 2(n-1)/(n-2)$ one designates by $\gamma_\tau : V_0 \to \mathcal{L}_p$ the compact operator that associated to all displacement$v$ the tangent component of her trace on Γ_c:

$$\gamma_\tau(v) = v - (v \cdot \tau)\tau$$

Let us introduce now the bilinear form $a(.,.)$ and functional f and j given by the following formulas:

$$\left\{\begin{array}{l} a : V_0 \times V_0 \longrightarrow \mathbb{R},\ f : V_0 \longrightarrow \mathbb{R} \text{ and } j : V_0 \times V_0 \longrightarrow \mathbb{R} \text{ such as:} \\ a(u,v) = \int_\Omega \mathcal{A}\varepsilon(u)\varepsilon(v)dx,\ \ u,v \in V_0, \\ \langle f(t), v\rangle = \int_{\Gamma_c} q(t) \cdot v_\tau dx,\ v \in V_0, \\ j(u(t),v) = \int_{\Gamma c} S\mu(|u_\tau(t)|)|v_\tau| dx,\ u,v \in V_0. \end{array}\right. ,$$

By using (3.1), (3.2) and the *Korn* inequality, one deduces that the bilinear form $a(\cdot,\cdot)$ is symmetric and coercive, i.e.

(3.11) $$a(u,v) = a(v,u), \quad \forall u,v \in V_0$$

(3.12) $$\exists \alpha_0 > 0 : a(v,v) \geq \alpha_0 \|v\|_V^2, \quad \forall v \in V_0$$

With the above notations, the problem(2.12) – (2.19) leads to the following second order hyperbolic variational inequality:

(3.13) $$\left\{\begin{array}{l} \text{Find } u :]0,T] \longrightarrow V_0, \text{ such that } \forall v \in V_0,\ \forall t \in]0,T],\ \text{we have:} \\ (\ddot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \langle f(t), v - \dot{u}(t)\rangle \\ u(0) = \dot{u}(0) = 0 \end{array}\right.$$

Remark 2. *The main difficulty in the study of the above evolution variational inequality is the non-monotone dependence of* $\mu(\cdot)$ *versus the slip-rate* $\left|\dot{u}_\tau(t)\right|$.

4. Elliptic problem at each time step

We consider here the *Newmark method*, with parameters $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$, for the time discretization of the variational problem (3.13). To this end, let $\Delta t > 0$ be the time step, M the maximum number of steps, and $T = M \cdot \Delta t$. We denote by $u^m, \dot{u}^m$ and $\ddot{u}^m$ the discretization of the solution at time $t = m \cdot \Delta t$, i.e. $u^m = u(m \cdot \Delta t), \dot{u}^m = \dot{u}(m \cdot \Delta t)$ and $\ddot{u}^m = \ddot{u}(m \cdot \Delta t)$ respectively, for all $0 \leq m \leq M$.

The initial condition in (3.13) become:

$$u^0 = \dot{u}^0 = \ddot{u}^0 = 0 \tag{4.1}$$

which is the starting of recursive problem. Suppose that we have established the solution up to $t = m \cdot \Delta t$, i.e. we have $u^k, \dot{u}^k, \ddot{u}^k$ for all $k \leq m$.

In the *Newmark method*, the numerical solution $u^{m+1}, \dot{u}^{m+1}$, $\ddot{u}^{m+1}$ of (3.13) at $t = (m+1) \cdot \Delta t$ is obtained from:

$$u^{m+1} = u^m + \Delta t \cdot \dot{u}^m + \frac{\Delta t^2}{4}\left(\ddot{u}^{m+1} + \ddot{u}^m\right) \tag{4.2}$$

$$\dot{u}^{m+1} = \dot{u}^m + \frac{\Delta t}{2}\left(\ddot{u}^{m+1} + \ddot{u}^m\right) \tag{4.3}$$

In terms of velocity, the above problem can be written as the following variational inequality:

$$\begin{cases} \text{Find } \dot{u}^{m+1} \in V_0 \text{ such that:} \\ \left(\dot{u}^{m+1}, v - \dot{u}^{m+1}\right) + \frac{\Delta t^2}{4} a\left(\dot{u}^{m+1}, v - \dot{u}^{m+1}\right) + \\ + \frac{\Delta t}{2}\left[j\left(\dot{u}^{m+1}, v\right) - j\left(\dot{u}^{m+1}, \dot{u}^{m+1}\right)\right] \geq G\left(v - \dot{u}^{m+1}\right), \forall v \in V_0 \end{cases} \tag{4.4}$$

where:

$$G(v) = \left\langle f^{m+1}, v\right\rangle + \left(\dot{u}^m + \frac{\Delta t}{2}\ddot{u}^m, v\right) - \frac{\Delta t}{2} a\left(\dot{u}^m + \frac{\Delta t}{2}\ddot{u}^m, v\right)$$

If $\dot{u}^{m+1}$ is obtained, then we can deduce u^{m+1} and $\ddot{u}^{m+1}$using:

$$u^{m+1} = u^m + \frac{\Delta t}{2}\left(\dot{u}^m + \dot{u}^{m+1}\right)$$

$$\ddot{u}^{m+1} = \frac{2}{\Delta t}\left(\dot{u}^{m+1} - \dot{u}^m\right) - \ddot{u}^m$$

An implicit scheme for the homogeneous problem $(2.12) - (2.19)$ is used to solve the nonlinear problem, given by a variational inequality (4.4) at each time step.

4.1. **Existence of the solution.** Let us introduce the energy functional $\Phi : V_0 \longrightarrow \mathbb{R}$ given by:

$$\Phi(v) = \frac{1}{2}(v, v) + \frac{\Delta t^2}{8} a(v, v) + \frac{\Delta t}{2}\int_{\Gamma_c} S \cdot \Psi\left(x, |v_\tau|\right) d\Gamma - G(v) \tag{4.5}$$

where $\Psi : \Gamma_c \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denotes the primitive of μ i.e.

$$\Psi(x, p) = \int_0^p \mu(x, \zeta)\, d\zeta, \forall \zeta \in \mathbb{R}_+, a.e. x \in \Gamma_c \tag{4.6}$$

The following result can be obtained using the same technique as in [8].

Theorem 1. *Under hypothesis* (3.3) – (3.7), (3.10). *We have:*

1) If $u = \dot{u}^{m+1} \in V_0$ *is a local minimum for* Φ *i.e. there exists* $\delta > 0$ *such that*

$$\Phi(u) \leq \Phi(v) \ \forall v \in V_0, \ \ \|u - v\|_V \leq \delta. \tag{4.7}$$

then u *is a solution of* (4.4).

2) There exists at least a global minimum for Φ, *i.e. there exists* $u_g \in V_0$ *such that:*

$$\Phi(u_g) \leq \Phi(v) \ \forall v \in V_0.$$

Proof. 1) Let us suppose that (4.7) holds. If $v \in V_0$ and $\theta_0 \in]0,1[$ such that $\theta_0 \|u - v\|_V < \delta$ then for all $\theta \in]0, \theta_0[$ we have:

$$\Phi(u) \leq \Phi(u + \theta\,(v - u))$$

Bearing in mind that $|\gamma_\tau\,(u + \theta\,(v - u))| \leq |\gamma_\tau\,(u)| - \theta\,(|\gamma_\tau\,(u)| - |\gamma_\tau\,(v)|)$ we deduce that:

$$\theta\,(u, v - u) + \frac{\theta^2}{2}\,(v - u, v - u) + \theta\frac{\Delta t^2}{4} a\,(u, v - u) + \frac{(\theta\Delta t)^2}{8} a\,(v - u, v - u) +$$
$$\frac{\Delta t}{2} \int_{\Gamma_c} S\,[\Psi(x, |u_\tau| + \theta(|v_\tau| - |u_\tau|)) - \Psi(x, |u_\tau|)]\,d\Gamma \geq \theta G(v - u)$$

Dividing this last inequality by θ and passing to the limit with $\theta \to 0$ we deduce (4.4).

2) We have:

$$\forall v \in V_0 : |\Phi(v)| \geq C\,\|v\|_V\,, \quad C > 0$$

Then

$$\underset{\|v\| \to +\infty}{Lim}\ \Phi(v) = \infty$$

By using the *Weierstrass* theorem, we deduce that Φ admits at least a global minimum. □

4.2. **Uniqueness of the solution.** Let us analyze here what are the conditions to be imposed on the parameters Δt , $\mathcal{A}$, ρ and μ, such that the functional Φ would be strongly coercive. We have to consider the following eigenvalue problem connected to (4.4) :

Find $\lambda \in \mathbb{R}$ and $v : \Omega \longrightarrow \mathbb{R}^n$, $v \neq 0$, such as:

$$\operatorname{div} \sigma(v) = \lambda \rho v \ ; \quad \sigma(v) = \mathcal{A}\varepsilon(v) \text{ in } \Omega \tag{4.8}$$

$$v = 0 \text{ on } \Gamma_d \ \ ; \sigma(v)\eta = 0 \text{ on } \Gamma_f \tag{4.9}$$

$$\sigma_\eta(v) = 0; \quad \sigma_\tau(v) = g v_\tau \ \text{ on } \ \Gamma_c \tag{4.10}$$

Where

$$g(x) = -\frac{2}{\Delta t} S(x) \underset{\zeta \in \mathbb{R}_+}{Inf}\, \partial_\zeta \mu(x, \zeta)$$

The variational formulation of (4.8) – (4.10) is:

$$v \in V_0, v \neq 0 \ \ a\,(v, w) + \lambda\,(v, w) = \int_{\Gamma_c} g v_\tau \cdot w_\tau d\Gamma\ , \quad \forall w \in V_0 \tag{4.11}$$

Theorem 2. *Let Ω be bounded. We have:*

1) The eigenvalues and the eigenfunctions of the problem (4.11) *consist of a sequence* $\left(\lambda_n^2, \varphi_n\right)_{n\in\mathbb{N}}$ *with*

$$\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots , \text{ and } \underset{n\longrightarrow+\infty}{Lim} \lambda_n = -\infty$$

2) Let $\beta > 0$ and let us denote by $\lambda_0(\beta)$ the first eigenvalue of (4.11) *in which g was replaced by βg. Then $\beta \longrightarrow \lambda_0(\beta)$ is convex and the following inequality holds:*

$$a\,(v,v) + \lambda_0(\beta)\,(v,v) \geq \beta \int_{\Gamma_c} g v_\tau^2 d\Gamma \quad , \quad \forall v \in V_0 \tag{4.12}$$

3) If $g \geq 0$ then $\beta \longrightarrow \lambda_0(\beta)$ is an increasing function.

Remark 3. *In general, $\lambda_0$is not negative, hence there exist at most a finite number of positive eigenvalues.*

Proof. **1)** Let γ be a positive constant, we rewrite (4.11) as follows :

$$b\,(v,w) = \delta\,(v,w) \quad , \quad \forall v,w \in V_0 \tag{4.13}$$

where:

$$\delta = \gamma - \lambda \text{ and } b\,(v,w) = a\,(v,w) + \gamma\,(v,w) - \int_{\Gamma_c} g v_\tau \cdot w_\tau d\Gamma$$

We are going to prove that the bilinear form $b(\cdot,\cdot)$ is coercive. We have:

$$\forall v \in V_0 : b\,(v,v) = a\,(v,v) + \gamma\,(v;v) - \int_{\Gamma_c} g v_\tau^2 d\Gamma$$

And by using the interpolation inequality (see [17]), we obtain:

$$\int_{\Gamma_c} v_\tau^2 dx \ \leq \bar{C}\,\|v\|_H\,\|v\|_V \ , \ \bar{C} > 0$$

Therefore we have:

$$b\,(v,v) \geq \alpha_0\,\|v\|_V^2 + \gamma\,\|v\|_H - \bar{\alpha}\bar{C}\,\|v\|_H\,\|v\|_V$$

Where:

$$\bar{\alpha} = \frac{2S}{\Delta t} M_0$$

Posing $\gamma = \frac{\bar{\alpha}^2\bar{C}^2}{4\alpha_0}$, we obtain:

$$b\,(v,v) \geq \alpha_0\,\|v\|_V^2 \tag{4.14}$$

The problem (4.13) admits a unique solution $v \in V_0$. Then, it exists an increasing sequence of eigenvalues (δ_n)**.** Moreover δ_n tends to $+\infty$. We denote by $V_n \subset V_0$ the finite dimension subspace of eigenfunctions associated to the eigenvalue δ_n.

Finally, for $\lambda_n = \gamma - \delta_n$ we deduce the first part of the theorem 2.

2) Let $\beta > 0$. We replace in (4.13) g by βg, we remark that δ_0, as a function of β, is the lower bound of a family of affine functions:

$$\delta_0(\beta) = \underset{v\in V_0, \|v\|_H = 1}{Inf} \left[a\,(v,v) + \gamma - \beta \int_{\Gamma_c} g v_\tau^2 d\Gamma\right] \tag{4.15}$$

Hence it is a concave function. Since $\lambda_0(\beta) = \gamma - \delta_0(\beta)$ we obtain that $\beta \longrightarrow \lambda_0(\beta)$ is convex. Moreover we have:

$$b(v;v) \geq \delta_0(\beta)(v,v), \quad \forall v \in V_0 \tag{4.16}$$

Which implies (4.12).

3) Let $\beta_1 > \beta_2$.We use (4.11) with $g = \beta g$, $v = w = \varphi$ and $\beta = \beta_2$ to get:

$$a(\varphi,\varphi) + \lambda_0(\beta_2)(\varphi,\varphi) = \beta_2 \int_{\Gamma_c} g\varphi_\tau^2 d\Gamma \tag{4.17}$$

From (4.12) with $v = \varphi$ and $\beta = \beta_1$ to obtain:

$$a(\varphi,\varphi) + \lambda_0(\beta_1)(\varphi,\varphi) \geq \beta_1 \int_{\Gamma_c} g\varphi_\tau^2 d\Gamma \tag{4.18}$$

Substitution of (4.18) from (4.17) yields to: $\lambda_0(\beta_1) > \lambda_0(\beta_2)$. Then the function $\beta \longrightarrow \lambda_0(\beta)$ is strictly increasing. □

We can now express the criteria of the uniqueness and the stability of the solution of the problem (4.4).

Theorem 3. *Let Ωbe bounded. We have:*

(i) $\nabla\Phi$ is a lipschitz functional, i.e. There exists a real constant $\ell_\Phi \geq 0$ such that:

$$\forall v_1, v_2 \in V_0 : \|\nabla\Phi(v_1) - \nabla\Phi(v_2)\|_{V'} \leq \ell_\Phi \|v_1 - v_2\|_V \tag{4.19}$$

(ii) If

$$\frac{(\Delta t)^2}{4}\lambda_0 < 1, \tag{4.20}$$

where $\lambda_0$is given by the above theorem, then Φ is a uniformly convex functional, i.e. there exists $\alpha > 0$ such that:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle \geq \alpha \|w\|_V^2 \ , \ \ \forall v, w \in V_0, \tag{4.21}$$

and (4.7) has a unique solution which is also the unique solution of (4.4), i.e. $u = \dot{u}^{m+1}$.

Remark 4. *The above condition (4.20) on the time step Δt is not a (Courant, Friedrichs and Lewy) CFL condition. If the process is stable, i.e. $\lambda_0 \leq 0$, then there is no condition (in terms of convergence and stability) on the time step. If the process is unstable, i.e. $\lambda_0 > 0$, then (4.20) which is equivalent to:*

$$\Delta t < \Delta t_{cr} = \frac{2}{\sqrt{\lambda_0}}$$

Proof. (i) Let $v_1, v_2, w \in V_0$, we have:

$$\begin{aligned} |\langle \nabla\Phi(v_1) - \nabla\Phi(v_2), w\rangle| \quad &\leq \quad |(v_1 - v_2, w)| + \frac{\Delta t^2}{4}|a(v_1 - v_2, w)| + \\ &+ \frac{\Delta t}{2}\int_{\Gamma_c} S \cdot |\mu(x, |v_{1\tau}|) - \mu(x, |v_{2\tau}|)| \, |w_\tau| \, d\Gamma \end{aligned}$$

By using the continuity of the trace operator and (3.5), we obtain:

$$\begin{aligned}|\langle \nabla\Phi(v_1) - \nabla\Phi(v_2), w\rangle| &\leq \|v_1 - v_2\|_H \|w\|_H + \frac{\Delta t^2}{4} \|\mathcal{A}\|_{L^\infty(\Omega)} \|v_1 - v_2\|_V \|w\|_V + \\ &+ \frac{S\Delta t}{2} \ell_\mu \int_{\Gamma_c} ||v_{1\tau}| - |v_{2\tau}|| \, |w_\tau| \, d\Gamma \\ &\leq \|v_1 - v_2\|_H \|w\|_H + \frac{\Delta t^2}{4} \|\mathcal{A}\|_{L^\infty(\Omega)} \|v_1 - v_2\|_V \|w\|_V + \\ &+ \frac{\ell_\mu S\Delta t}{2} C \|v_1 - v_2\|_V \|w\|_V \\ &\leq \ell_\Phi \|v_1 - v_2\|_V \|w\|_V\end{aligned}$$

Where:

$$\ell_\Phi = \frac{\ell_\mu S\Delta t}{2} C + Max\left\{1, \frac{\Delta t^2}{4} \|\mathcal{A}\|_{L^\infty(\Omega)}\right\}$$

Here ℓ_Φ is a constant such that (4.19) holds.
(ii) Let $v, w \in V_0$, we have:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle = \underset{\theta\to 0}{Lim} \frac{\langle \nabla\Phi(v + \theta w), w\rangle - \langle \nabla\Phi(v), w\rangle}{\theta}$$

By a simple calculation one gets:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle = (w, w) + \frac{\Delta t^2}{4} a(w, w) + \frac{\Delta t}{2} \int_{\Gamma_c} S\mu'(x, |v_\tau|) |w_\tau|^2 \, d\Gamma$$

We distinguish the two following cases:
1) If μ is an increasing function, then:

$$\forall v \in V_0 : \mu'(x, |v_\tau|) \geq 0$$

We have:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle \geq |w|_H^2 + \frac{\Delta t^2}{4} \alpha_0 \|w\|_V^2 \geq \alpha \|w\|_V^2 \ , \ \alpha = \frac{\Delta t^2}{4}\alpha_0$$

And by consequence , the functional Φ is uniformly convex, then Φ admits a unique global minimum.
2) If μ is a decreasing function, then:

$$\forall v \in V_0 : \mu'(x, |v_\tau|) \leq 0$$

We have:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle \geq (w, w) + \frac{\Delta t^2}{4} a(w, w) - \frac{\Delta t^2}{4} \int_{\Gamma_c} g w_\tau^2 d\Gamma \tag{4.22}$$

From (4.12), we obtain:

$$-\frac{\Delta t^2}{4} a(w, w) - \lambda_0(\beta) \frac{\Delta t^2}{4} (w, w) \leq -\beta \frac{\Delta t^2}{4} \int_{\Gamma_c} g w_\tau^2 d\Gamma \tag{4.23}$$

And by the use (4.22), (4.23) and (3.12), it comes:

$$\left\langle \nabla^2\Phi(v)w, w\right\rangle \geq \frac{\beta - \lambda_0(\beta)\frac{\Delta t^2}{4}}{\beta} |w|_H^2 + \frac{\Delta t^2}{4} \alpha_0 \left(\frac{\beta - 1}{\beta}\right) \|w\|_V^2 \tag{4.24}$$

By using (4.20) and The function $\beta \longrightarrow \lambda_0(\beta)$ is increasing, then, it exists $\bar{\beta} > 1$ such that $\lambda_0(\bar{\beta})\frac{\Delta t^2}{4} < 1$, therefore:

$$\left\langle \nabla^2 \Phi(v)w, w \right\rangle \geq \alpha \left\| w \right\|_{V_0}^2 \quad \text{when} \quad \alpha = \frac{\bar{\beta}-1}{\bar{\beta}} Min \left\{ 1, \frac{\Delta t^2}{4}\alpha_0 \right\}$$

Since the functional Φ is convex, problem (4.4) and (4.7) are equivalent. The uniqueness of u comes from the strict convexity of the functional Φ. □

Conclusion 1. *In this paper, we present a new technique able to transform this ill posed problem into a well posed one. This technique is based on the Newmark method and the minimization problem. The solution of well posed problem chosen by the criterion using the first eigenvalue of the eigenvalue problem. This criterion is not a (Courant, Friedrichs and Lewy) CFL condition.*

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RELATIVE FREDHOLM ALTERNATIVE TO BOUNDARY VALUE PROBLEM FOR THE ELLIPTIC EQUATIONS

BENABDERRAHMANE BENYATTOU AND NOUIRI BRAHIM

ABSTRACT. The aim of this study is to prove some regularity results by passing to the Fredholm alternative for the elliptic problem governed by the *Laplace* equations with contact without friction boundary conditions. This investigation permits to evaluete of the index of the *Laplace* operator.

1. INTRODUCTION

From the previous works [4] and [5], the Fredholm alternative for the Laplace equations and Maxwell equations, with the classical boundary conditions (*Dirichlet*, *Neumann* and mixed : *Dirichlet*-*Neumann*), has been studied using the fundamental results of functional analysis [8], [10], [11] and others concerning the *Sobolev* spaces and their properties [12], [13] and [6]. Using some theoretical results concerning the elliptic problems in [6], [8], [10], [11], [12], [13], this paper presents the analysis of the Fredholm alternative question for the elliptic problem governed by the Laplace operator with contact without friction boundary conditions. This problem is the limit of the elasticity problem when the sum of the elasticity coefficients, $(\lambda + \mu) > 0$, tends to zero.

Let Ω is homogeneous, elastic and isotropic medium occupying a bounded domain in IR^2, limited by straight polygonal boundary Γ which is supposed to be regular, $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$, $\Gamma_i \cap \Gamma_j = \varnothing, \forall i \neq j$, where $\Gamma_j =]S_j, S_{j+1}[$, and S_j are the different corners of Ω. In this study η^j and τ^j represent the unit outward normal vector and the tangent vector on Γ_j, respectively. ω_j represents the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω.

Let $f : \Omega \to \mathbb{R}^2$, the elliptic problem considered here consists of finding the displacement field $u : \Omega \longrightarrow \mathbb{R}^2$ satisfying:

$$(P) \quad \left\{ \begin{array}{c} \Delta u + f = 0 \text{ in } \Omega \\ \left(u.\eta^j, \left(\Sigma(u).\eta^j\right).\tau^j\right) = (0,0) \text{ on } \Gamma_j, \; j = 1, ..., J \end{array} \right. ,$$

The boundary conditions used in this problem are called contact without friction $(C.W.F)$.

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We denote by $\Sigma(u) = (\sigma_{ij}(u)), i, j = 1, 2$, where the elements $\sigma_{ij}(u)$ are given by Hook's law, in the case $(\lambda + \mu) \to 0$, using *Lamé* coefficient $\mu > 0$:

$$\sigma_{ij}(u) = \mu\left[2\varepsilon_{ij}(u) - \delta_{ij}\varepsilon_{kk}(u)\right],\ i, j = 1, 2$$

where δ_{ij} is *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_i x_j + \partial_j x_i)$ is the linearized tensor of linear elasticity. ∂_j is used as the partial derivative of u with respect to x_j.

Generally, the problem (P) hasn't sufficiently regular solution, hence we try to impose conditions on $f \in L^2(\Omega)^2$allowing variational solution included in the space V such as $V = \left\{v \in H^1(\Omega)^2;\ u.\eta^j = 0, \text{ on } \Gamma_j\right\}$ is in $H^{s+2}(\Omega)^2 \cap V\ (s \geq 0)$, where $H^{s+2}(\Omega)$ denotes $(s+2)$ order *Sobolev* space.

Studying of the $(C.W.F)$ boundary conditions, it is proved (see[2]) that the problem (P) amount to the two problems of oblique derivatives boundary conditions without coupling:

$$(P_k)\ \left\{\begin{array}{c} \Delta u_k + f_k = 0 \text{ in } \Omega \\ a_j D_x u_k + b_j D_y u_k = 0 \text{ on } \Gamma_j \quad j = 1, ..., J,\ k = 1, 2 \\ a_j^2 + b_j^2 \neq 0 \end{array}\right.$$

Then (see [2]) there is a constant C_s such that

$$\|u\|_{s+2} \leq C_s \|\Delta u\|_s \tag{1.1}$$

is satisfied for all $u \in W_s(\Omega)$, where

$$W_s(\Omega) = \left\{u \in H^2(\Omega)^2 \cap V;\ \left(\Sigma(u).\eta^j\right).\tau^j = 0 \text{ on } \Gamma_j,\ D_x u + D_y u \in H_0^s(\Omega)\right\}$$

2. *Fredholm* ALTERNATIVE

Let $\Re_s(\Omega)$ be the subspace of $H^s(\Omega)^2$ defined by

$$\Re_s(\Omega) = \left\{f \in H^s(\Omega)^2;\ f = \Delta u,\ u \in W_s(\Omega)\right\}$$

Using the inequality (1.1), it can be seen that $\Re_s(\Omega)$ is a closed subspace of $H^s(\Omega)^2$. Let $N_s(\Omega)$ be the orthogonal of $\Re_s(\Omega)$ in $H^{-s}(\Omega)^2$, i.e.

$$N_s(\Omega) = \left\{v \in H^{-s}(\Omega)^2,\ (v, f) = 0 \text{ for all } f \in \Re_s(\Omega)\right\}$$

According to a generalization of *Green* formula, it can be observed that the elements of $N_s(\Omega)$ are solutions to the homogeneous contact without friction problem:

$$\left\{\begin{array}{c} \Delta u = 0 \text{ in } \Omega \\ \left(u.\eta^j, \left(\Sigma(u).\eta^j\right).\tau^j\right) = (0, 0) \text{ on } \Gamma_j,\ j = 1, ..., J \end{array}\right.$$

2.1. Some results concerning the *Green* formula.

Let $D_s(\Omega) = \left\{v \in H^{-s}(\Omega)^2;\ \Delta v \in L^2(\Omega)^2\right\}$ be the subspace of $H^{-s}(\Omega)^2$ equipped with norm

$$\|v\|_{D_s(\Omega)} = \|v\|_{-s} + \|\Delta v\|_0$$

By using the techniques of Lions-Magenes [13] and P.Grisvard [5], we prove the following result:

Lemma 1. *$D(\overline{\Omega})$ is dense in $D_s(\Omega)$.*

Lemma 2. *We introduce the subspace $X_s(\Gamma)$, defined by*

$$X_s(\Gamma) = \{((\Sigma(u).\eta).\eta, -u.\tau) : \ u \in W_s(\Omega)\}$$

The application $v \longmapsto (v.\eta, (\Sigma(v).\eta).\tau)$ on Γ, which is well-defined for all $v \in D(\overline{\Omega})$, prolongs by continuity in a continuous linear application:

$$\gamma_s : D_s(\Omega) \longrightarrow X_s(\Gamma)^* \text{ such that}$$

$$(\Delta u, v) - (u, \Delta v) = \langle((\Sigma(u).\eta).\eta, -u.\tau), \gamma_s(v.\eta, (\Sigma(v).\eta).\tau)\rangle$$

for all $u \in W_s(\Omega)$ and $v \in D(\overline{\Omega})$, where the notation $\langle ., . \rangle$ represents the duality pairing between $X_s(\Gamma)$ and $X_s(\Gamma)^*$.

An analogous Lemma has been introduced by P. Grisvard [5] in the case of the *Dirichlet* problem.

Proof. For all $u \in W_s(\Omega)$ and $v \in D(\overline{\Omega})$ we have

$$(\Delta u, v) - (u, \Delta v) = \int_\Gamma (\Sigma(u).\eta).vd\sigma - \int_\Gamma (\Sigma(v).\eta).ud\sigma, \tag{1.2}$$

for all $u \in W_s(\Omega)$.

According to the boundary condition $(u.\eta, (\Sigma(u).\eta).\tau) = (0,0)$ in Γ, the Eq. (1.2) becomes

$$(\Delta u, v) - (u, \Delta v) = \int_\Gamma ((\Sigma(u).\eta).\eta)\, v.\eta d\sigma - \int_\Gamma ((\Sigma(v).\eta).\tau)\, u.\tau d\sigma,$$

Or equivalently

$$(\Delta u, v) - (u, \Delta v) = \int_\Gamma (\Sigma(u).\eta).\eta, -u.\tau)\,(v.\eta, (\Sigma(v).\eta).\tau)\, d\sigma \tag{1.3}$$

It is clear that the left term of Eq. (1.3) is bilinear continuous, therefore we finish the proof by using the density of $D(\overline{\Omega})$ in $D_s(\Omega)$. □

Remark 1. *From the Eq.* (1.3), *the following expression is obtained*

$$(u, \Delta v) = 0, \ \forall u \in D(\Omega), \ \forall v \in N_s(\Omega)$$

and consequently we have $\Delta v = 0 \in L^2(\Omega)^2$ what implies that v belongs to $D_s(\Omega)$ and gives a sense to $\gamma_s(v.\eta, (\Sigma(v).\eta).\tau) \in X_s(\Gamma)^*$.

Lemma 3. *We have*

$$N_s(\Omega) = \left\{v \in H^{-s}(\Omega)^2, \ \Delta v = 0; \ \gamma_s(v.\eta, (\Sigma(v).\eta).\tau) = (0,0)\right\}$$

where γ_s is a generalized of the trace map defined by duality (see Lemma 2).

Proof. Thanks to Lemma 2, it is easy to verify that $N_s(\Omega) \subseteq [\Re_s(\Omega)]^\perp$.

Reciprocally, let v be in $[\Re_s(\Omega)]^\perp$ then

$$(v, \Delta u) = 0, \ \forall u \in W_s(\Omega)$$

According to (1.3), it can be written

$$-(u, \Delta v) = \int_\Gamma ((\Sigma(u).\eta).\eta, -u.\tau)\,(v.\eta, (\Sigma(v).\eta).\tau)\, d\sigma$$

Or equivalently

$$(u, \Delta v) = 0, \ \forall u \in W_s(\Omega), \ \forall v \in ID(\Omega)$$

what implies $\Delta v = 0$.

Using the Lemma 2, we find

$$\langle ((\Sigma(u).\eta).\eta, -u.\tau), \gamma_s(v.\eta, (\Sigma(v).\eta).\tau)\rangle_{X_s(\Gamma)\times X_s(\Gamma)^*} = 0, \ \forall u \in W_s(\Omega)$$

consequently,

$$\gamma_s(v.\eta, (\Sigma(v).\eta).\tau) = 0$$

□

We have the necessary and sufficiencies conditions on $f \in H^s(\Omega)^2$, in order to allow for u, solution of (P) to be in $H^{s+2}(\Omega)^2 \cap V$. This condition is expressed as follows:

$$(f, v) = 0, \text{ for all } v \in N_s(\Omega)$$

3. Laplace index

3.1. **Dimension of $N_s(\Omega)$.** In this subsection, we are interested in the study of $N_s(\Omega)$. It is proved that $N_s(\Omega)$ is a finished dimension subspace, after having, we will determinate exactly his dimension.

We start with comparative study between the trace operator γ_s which is defined in Lemma 2 and usual trace operators.

Remark 2. *For $v \in N_s(\Omega)$ we have*
i) $\Delta v = 0$, i.e. v is analytic in $\overline{\Omega} \setminus S$;
ii) $\left(v.\eta^j, \left(\Sigma(v).\eta^j\right).\tau^j\right) = (0,0)$ on Γ_j, $j = 1, ..., J$.

Expression $ii)$ is signified that the function $\left(v.\eta^j, \left(\Sigma(v).\eta^j\right).\tau^j\right)$ which is continuous on $\overline{\Omega} \setminus S$ is null on $\overset{0}{\Gamma}_j$, $j = 1, ..., J$, where $\overset{0}{\Gamma}_j$ indicates the interior of Γ_j what doesn't imply that $\gamma_s(v.\eta, (\Sigma(v).\eta).\tau) = 0$.

This Remark permits us to deduce a regularity result in the neighborhood of the regular parts boundary. This result can be obtained by using the reflection processes, compared to each segments of Γ_j, $j = 1, ..., J$, and the analyticity of harmonic distributions.

These results permit us to introduce the space $M_s(\Omega)$, defined by

$$M_s(\Omega) = \left\{ v \in H^{-s}(\Omega)^2 \cap C^\infty\left(\overline{\Omega} \backslash S\right); \ \Delta v = 0; \ \ (v.\eta, (\Sigma(v).\eta).\tau) \diagup_{\Gamma_j} = (0,0) \right\}$$

An analogous space was constructed by P. Grisvard [5].

Consequently, it is clear that $N_s(\Omega) \subseteq M_s(\Omega)$ and besides we have the following Theorem:

Theorem 1. *If ω is satisfied the following condition*

$$\omega \neq \frac{\ell\pi}{k+2}; \ \ell, k \in \mathbb{N}, \ \ell \neq (k+2), \ k = 1, ..., s$$

then we obtain $N_s(\Omega) = M_s(\Omega)$.

We start giving some important results:

We denote by $H_{0,0}^{m+\frac{1}{2}}(\Gamma_j)$ the subspace $H^{m+\frac{1}{2}}(\Gamma_j)$ defined by the following conditions:

$$\begin{cases} f^{(k)}(S_j) = f^{(k)}(S_{j+1}) = 0 \\ \int\limits_{\Gamma_j} \frac{\left|f^{(m)}(t)\right|}{(t-S_j)(S_{j+1}-t)} dt < +\infty \end{cases}, k = 0, ..., m-1$$

Equipped with norm

$$\|f\|_{H_{0,0}^{m+\frac{1}{2}}(\Gamma_j)} = \|f\|_{m+\frac{1}{2}} + \left[\int\limits_{\Gamma_j} \frac{\left|f^{(m)}(t)\right|}{(t-S_j)(S_{j+1}-t)} dt\right]^{\frac{1}{2}}$$

Where $\Gamma_j =]S_j, S_{j+1}[$, $j = 1, ..., J$ and m is an integer.

We recall that $D(\Gamma_j)$ is dense in $H_{0,0}^{m+\frac{1}{2}}(\Gamma_j)$ for all integer m. Relatively to this space we have

Proposition 1. *The application*

$$u \longmapsto (\{f_k\}, \{g_j\})$$

defined by

$$\begin{cases} f_k(x) = D_y^k u(x,0), \ x > 0, \ k = 0, ..., s-1 \\ g_j(y) = D_x^j u(0,y), \ y > 0, \ k = 0, ..., s-1 \end{cases}$$

is linear, continuous and surjective (and admits a continuous linear relevement) of $H^s(\Omega)$ *in the subspace*

$$\prod_{k=0}^{s-1} H^{s-k-\frac{1}{2}}(\mathbb{R}_+) \times \prod_{j=0}^{s-1} H^{s-j-\frac{1}{2}}(\mathbb{R}_+)$$

and satisfying the following conditions

$$\begin{cases} f_k^{(j)}(0) = g_j^{(k)}(0), j+k \le s-2 \\ \int\limits_0^r \left|f_k^{(j)}(t) - g_j^{(k)}(t)\right|^2 \frac{dt}{t}, \textit{for } j+k = s-1 \end{cases}$$

where r *is a finished or infinite number.*

For the demonstration we invite the reading to consult P. Grisvard [5].

Lemma 4. *For* $\omega \neq \frac{\ell\pi}{k+2}$, $\ell, k \in \mathbb{N}$, $1 \le k \le s$, *we have*

$$X_s(\Gamma) = \prod_{j=1}^{J} H_{0,0}^{s+\frac{1}{2}}(\Gamma_j) \times \prod_{j=1}^{J} H_{0,0}^{s+\frac{3}{2}}(\Gamma_j)$$

We are of course identified $(\varphi, \psi) \in X_s(\Gamma)$ to $\left(\{\varphi_j\}_{j=1}^{J}, \{\psi_j\}_{j=1}^{J}\right)$, where φ_j and ψ_j indicate the restriction of φ and ψ to Γ_j, $j = 1, ..., J$, respectively .

Proof. Using the Proposition 1, we will determine the splice conditions between (φ_j, ψ_j) so that $(\varphi, \psi) = \left(\{\varphi_j\}_{j=1}^{J}, \{\psi_j\}_{j=1}^{J}\right)$ is element of $X_s(\Gamma)$. These conditions are obviously related to each corner S_j.

Essentially, we limit to search for those that are relative to the corner S_1, as amounting by the following affine change of variables :

$$\left\{ \begin{array}{c} X = x - y\cot\omega \\ Y = y \end{array} \right.$$

to the particular case: $S_1 = (0,0)$ and $\omega_1 = \frac{\pi}{2}$ or $\omega_1 = \frac{3\pi}{2}$. This change conserves the spaces

$$H^{s+2}(\Omega),\ V \text{ and } H_0^s(\Omega)$$

on the other hand it doesn't conserve the expression $D_x u + D_y u$ which becomes

$$D_x u - \cot\omega_1 D_x u + D_y u$$

In the same way the considered change doesn't conserve the boundary conditions $((\Sigma(u).\eta).\eta, -u.\tau)$ which become

$i)$ For $y = 0$,we have

$$-u.\tau = u_1(x,0) \tag{1.4}$$

$$(\Sigma(u).\eta).\eta = \begin{pmatrix} \sigma_{11}(u) & \sigma_{12}(u) \\ \sigma_{21}(u) & \sigma_{22}(u) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \text{ where}$$

$$\left\{ \begin{array}{c} \sigma_{11}(u) = \mu\left[D_x u_1(x,0) + \cot\omega_1 D_x u_2(x,0) - D_y u_2(x,0)\right] \\ \sigma_{12}(u) = \sigma_{21}(u) = \mu\left[D_y u_1(x,0) - \cot\omega_1 D_x u_1(x,0) + D_x u_2(x,0)\right] \\ \sigma_{22}(u) = -\sigma_{11}(u) \end{array} \right.$$

then

$$(\Sigma(u).\eta).\eta = \mu\left[D_x u_1(x,0) + \cot\omega_1 D_x u_2(x,0) - D_y u_2(x,0)\right] \tag{1.5}$$

$ii)$ For $x = 0$, we have

$$-u.\tau = -\left[\cos\omega_1 u_1(0,y) + \sin\omega_1 u_2(0,y)\right] \text{ and} \tag{1.6}$$

$$(\Sigma(u).\eta).\eta = \mu\left[\begin{array}{c} D_x u_1(0,y) - \cot\omega_1 D_x u_2(0,y) + \cos 2\omega_1 D_y u_2(0,y) - \\ \sin 2\omega_1 D_y u_1(0,y) \end{array} \right] \tag{1.7}$$

In the following, we are interested only to the case $0 < \omega < \frac{\pi}{2}$. The case $\frac{\pi}{2} < \omega < \frac{3\pi}{2}$ is treated similarly to the previous one.

We have $u \in W_s(\Omega)$, i.e., u must verify the following boundary conditions

$$(u.\eta, (\Sigma(u).\eta).\tau) = (0,0) \text{ on } \Gamma_j$$

which becomes, after the affine change of variables, as follows:

$i)$ For $y = 0$,

$$\left\{ \begin{array}{c} u.\eta = u_2(x,0) = 0 \\ (\Sigma(u).\eta).\tau = \mu\left[D_x u_2(x,0) + D_y u_1(x,0) - \cot\omega_1 D_x u_1(x,0)\right] = 0 \end{array} \right.$$

or

$$\left\{ \begin{array}{c} u_2(x,0) = 0 \\ D_y u_1(x,0) - \cot\omega_1 D_x u_1(x,0) = 0 \end{array} \right. \tag{1.8}$$

$ii)$ For $x = 0$,

$$\left\{ \begin{array}{c} u.\eta = \sin\omega_1 u_1(0,y) + \cos\omega_1 u_2(0,y) = 0 \\ (\Sigma(u).\eta).\tau = D_x u_2(0,y) - D_y u_1(0,y) + \cot\omega_1 D_x u_1(0,y) = 0 \end{array} \right. \tag{1.9}$$

Thanks to (1.8), (1.9) the expressions (1.4), (1.5) and (1.6), (1.7) become, respectively:

(1.10)
$$\left\{\begin{array}{c} u_1(x,0) \\ \mu D_y\left(u_2(x,0)-\tan\omega_1 u_1(x,0)\right)=\mu\left[D_y u_2(x,0)-D_x u_1(x,0)\right] \end{array}\right., \text{ for } y=0$$

$$\left\{\begin{array}{c} \frac{1}{\sin\omega_1}u_2(0,y)=\frac{1}{\cos\omega_1}u_1(0,y) \\ -2\mu\cot\omega_1 D_x u_2(0,y) \end{array}\right., \text{ for } x=0 \tag{1.11}$$

Therefore, we must specify the splice conditions between (1.10) and (1.10), when ω belongs to $W_s(\Omega)$ such that

$$\Omega=\mathbb{R}_+^*\times\mathbb{R}_+^*=\left\{(x,y)\in\mathbb{R}^2,\ x>0 \text{ and } y>0\right\}$$

In the following, we are going to search the conditions that we must impose so that the vectors $\left(\{\varphi_j\}_{j=1}^2,\{\psi_j\}_{j=1}^2\right)$ belong to $X_s(\Gamma)$.

We pose

$$\left\{\begin{array}{c} \psi_1(x)=u_1(x,0) \\ \psi_2(y)=u_1(0,y) \\ \varphi_1(x)=D_y u_2(x,0)-D_x u_1(x,0) \\ \varphi_2(y)=D_x u_2(0,y) \end{array}\right.$$

The functions φ_j and ψ_j, $j=1,2$, must satisfy the condition $D_x.+D_y.\in H_0^s(\Omega)$ using the affine change of variables.

We start by: ψ_j, $j=1,2$.

i) ψ_1 : The condition $D_x\psi_1+D_y\psi_1\in H_0^s(\Omega)$ becomes

$$D_x\psi_1(x)+D_y\psi_1(x)-\cot\omega_1 D_x\psi_1(x)\in H_0^s(\Omega)$$

Using the definition of $H_0^s(\Omega)$, it results

$$D_y^k\left[(1-\cot\omega_1)D_x\psi_1(x)+D_y\psi_1(x)\right],\ k=0,...,s-1$$

Or

$$(1-\cot\omega_1)D_x D_y^k u_1(x,0)+D_y^{k+1}u_1(x,0),\ k=0,...,s-1 \tag{1.12}$$

ii) ψ_2 : According to the condition $D_x\psi_2+D_y\psi_2\in H_0^s(\Omega)$, we obtain

$$(1-\cot\omega_1)D_x^{j+1}u_1(0,y)+D_y D_x^j u_1(0,y),\ j=0,...,s-1 \tag{1.13}$$

Now pass to the case of φ_j, $j=1,2$. As in the previous cases, it is easy to verify the following cases:

i) φ_1 :

$$\left\{\begin{array}{l} D_y^{k+2}u_2(x,0)+(1-\cot\omega_1)D_x D_y^{k+1}u_2(x,0)- \\ -(\cot\omega_1)^k D_x^2 D_y^k u_1(x,0)=0,\ k=0,...,s-1 \end{array}\right. \tag{1.14}$$

ii) φ_2 :

$$(1-\cot\omega_1)D_x^{j+2}u_2(0,y)+D_y D_x^{j+1}u_2(0,y)=0,\ j=0,...,s-1 \tag{1.15}$$

Posing

$$\left\{\begin{array}{l} g_j(y)=D_x^j u_1(0,y);\ \ell_j(y)=D_x^j u_2(0,y) \\ f_k(x)=D_y^k u_1(x,0);\ h_k(x)=D_y^k u_2(x,0) \end{array}\right.$$

the expressions $(1.12)-(1.15)$ become

$$\left\{\begin{array}{c} f_{k+1}(x)-(\cot\omega_1-1)\,f_k^{'}(x)=0 \\ -(\cot\omega_1-1)\,g_{j+1}(y)+g_j^{'}(y)=0 \end{array}\right. ,\ k,j=0,...,s-1 \tag{1.16}$$

$$\left\{\begin{array}{c} h_{k+2}(x)+(1-\cot\omega_1)\,h_{k+1}^{'}(x)-(\cot\omega_1)^k D_x^{k+2}u_1(x,0)=0 \\ (1-\cot\omega_1)\,\ell_{j+2}(y)+\ell_{j+1}^{'}(y)=0 \end{array}\right. \tag{1.17}$$

By recurrence, we can see that Eq. (1.16) is equivalent to the following relation

$$\left\{\begin{array}{c} f_k(x)=\alpha^k f_0^{(k)}(x)\,,\ k=0,...,s \\ g_j(y)=\left(\frac{1}{\alpha}\right)^j g_0^{(j)}(y)=0,\ j=0,...,s \end{array}\right. \tag{1.18}$$

Where $\alpha=(1-\cot\omega_1)$.

So that ψ_1, ψ_2 belong to $H_{0,0}^{s+\frac{3}{2}}(\mathbb{R}_+)$, it is necessary that

$$\left\{\begin{array}{c} f_k^{(j)}(0)=g_j^{(k)}(0)\,,\ j+k\leq s \\ \int\limits_0^s\left|f_k^{(j)}(t)-g_j^{(k)}(t)\right|^2\frac{dt}{t}<+\infty,\ j+k=s+1 \end{array}\right. \tag{1.19}$$

φ_1,φ_2 belong to $H_{0,0}^{s+\frac{1}{2}}(\mathbb{R}_+)$ provided that the following condition $(\cot\omega_1)^k D_x^{k+2}u_1(x,0)=0$ is verified, which is possible, because in the case $y=0$, $u_1(x,0)$ is arbitrarily chosen. As a consequence, the condition (1.19) is always verified by replacing f by h and g by ℓ.

According to (1.18), from Eq. (1.19) it results

$$\alpha^m f_0^{(m)}(0)=g_0^{(m)}(0)\,,\text{for } m\leq s$$

such that $f_0(0)=g_0(0)=u_1(0,0)=0$.

This identity is verified when $\alpha^m=1$, for all m, $m=0,...,s-1$. Accordingly, thanks to (1.18), we see that it is the case when

$$\omega_1\neq\frac{\ell\pi}{k+2};\ \ell,k\in\mathbb{N},\ \ell\neq(k+2),\ k=1,...,s$$

Consequently, we deduce

$$\left\{\begin{array}{c} \psi_1^{(j)}(0)=\psi_2^{(j)}(0)=0,\ j=0,...,s-1 \\ \int\limits_0^s\left|\psi_1^{(s)}(t)\right|^2\frac{dt}{t}<+\infty,\ \int\limits_0^s\left|\psi_2^{(s)}(t)\right|^2\frac{dt}{t}<+\infty \end{array}\right.$$

what proves that ψ_1, ψ_2 belong to $H_{0,0}^{s+\frac{3}{2}}(\mathbb{R}_+)$. By the same techniques we demonstrate that $\varphi_1,\varphi_2\in H_{0,0}^{s+\frac{1}{2}}(\mathbb{R}_+)$. □

3.2. **Dimension of $M_s(\Omega)$.** We introduce the following subspace

$$M_s(\Omega)=\left\{v\in H^{-s}(\Omega)^2\cap C^\infty\left(\overline{\Omega}\backslash S\right);\ \Delta v=0;\ \left(v.\eta,(\Sigma(v).\eta).\tau\right)\diagup_{\Gamma_j}=(0,0)\right\}$$

and the functions:

$$N(\omega)=Card\left\{k\in\mathbb{N}^*;\ k<\frac{\omega}{\pi}(s+2)\right\}$$

$N(\omega)$ is the biggest number strictly inferior to $\frac{\omega}{\pi}(s+2)$. We have

Theorem 2. *Suppose Ω is a simply connected, the dimension of $M_s(\Omega)$ is exactly*

$$N = \sum_{j=1}^{J} N(\omega_j)$$

We immediately get the following Lemma.

Lemma 5. *Suppose Ω is a simply connected, $\Delta : W_s(\Omega) \to H^s(\Omega)^2$ is an operator with index. More precisely, Δ is injective (because $\dim(Ker\Delta) = 0 < \infty$), has a closed image of codimension lower or equal to $N < +\infty$ in $H^s(\Omega)^2$.*

In the particular case, where Ω doesn't have any angle ω of the following form

$$\frac{\ell\pi}{k+2};\ \ell, k \in \mathbb{N},\ \ell \neq (k+2),\ k = 1, ..., s$$

the codimension of the image of Δ in $H^s(\Omega)^2$ is exactly equal to N.

Before proving the Theorem 2, we need some Lemmas.

For more of convenience, we suppose that we are in the case where $S_1(0,0)$, and where the segment Γ_1 is carried on the axis O_x (with $\omega_1 > 0$), the plan is supposed to orient in the usual direction.
We set $z = x + iy = r\exp(i\theta)$.

Lemma 6. *Let $k \in \mathbb{N}^*$ satisfy $k < \frac{\omega_1}{\pi}(s+2)$, there exist $u_k \in M_s(\Omega)$ such that, if we set $w_k = u_k - v_k$, we have*

$$w_k \in H^1(\Omega)^2, \frac{w_k}{r} \in L^2(\Omega)^2 \ ,$$

where

$$v_k = 4(-1)^k \sin 2\omega \left[\operatorname{Re} z^{-\alpha_k}, \operatorname{Im} z^{-\alpha_k}\right],\ \textit{with } \alpha_k = \frac{k\pi}{\omega} - 1$$

Proof. As for $k < \frac{\omega_1}{\pi}(s+2)$, we have

$$r^s v_k = 4(-1)^k r^s \sin 2\omega \left[\operatorname{Re} z^{-\alpha_k}, \operatorname{Im} z^{-\alpha_k}\right] \in L^2(\Omega)^2$$

We know, thanks to P. Grisvard [7], if $u \in H_0^s(\Omega)$ then $\dfrac{u}{r^s} \in L^2(\Omega)$, where r designates the distance of $u \in \Omega$ to the set $S = \{S_1, S_2, ..., S_J\}$. As $r^s v_k \in L^2(\Omega)^2$, then $v_k \in H^{-s}(\Omega)^2$.

It is obvious that this function is harmonic and null on the segments Γ_1 and Γ_J, moreover it is regular the neighborhood of $\Gamma_1 \setminus S_1$.

Let (φ_k, ψ_k) be the trace of $(v_k.\eta, (\Sigma(v_k).\eta).\tau)$ on Γ_1, it is clear that $(\varphi_k, \psi_k) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$, therefore by applying the variational method we see that exists $w_k \in H^1(\Omega)^2$ harmonic and satisfying

$$w_k = (\varphi_k, \psi_k) \text{ on } \Gamma$$

Posing $u_k = w_k + v_k$, thanks to condition $w_k = 0$ on Γ_1 and Γ_J, we have $w_k \in H_0^1(\Omega)^2$, and consequently $\frac{w_k}{r} \in L^2(\Omega)^2$. □

Remark 3. *(Consequences of Lemma 7); Taking into account the fact that*

$$\frac{1}{r}\left[\operatorname{Re} z^{-\alpha_k}, \operatorname{Im} z^{-\alpha_k}\right] \notin L^2(\Omega)^2$$

and that functions defined by

$$\left[\operatorname{Re} z^{-\alpha_k}, \operatorname{Im} z^{-\alpha_k}\right],\ k \in \mathbb{N}^*,\ k < \frac{\omega_1}{\pi}(s+2)$$

are linearly independent in $M_s(\Omega)$*, we deduce that the functions* u_k*, such that* $k \in \mathbb{N}^*$ *and* $k < \frac{\omega_1}{\pi}(s+2)$*, are linearly independent in* $M_s(\Omega)$*. As we can repeat same construction for each corner, we deduce that*

$$\dim(M_s(\Omega)) \geq N$$

Lemma 7. *For* $\varphi \in \overset{0}{D}(]0, r_0[)$ *and* $\psi \in \overset{0}{D}(]0, \omega[)$*, if* $r_0 < 1$ *and* $r_0 < d(0, S)$*, then* f *belongs to* $H_0^s(\Omega)$*, where*

$$f(r,\theta) = \frac{r^{\frac{\ell\pi}{\omega}-2}}{|\ln r|}\varphi(r)\psi(\theta)$$

And if $\ell \geq \frac{\omega_1}{\pi}(s+2)$*, then there is a constant* C_ℓ *such that*

$$\|f\|_s \leq C_\ell \|\psi\|_{H^s(]0,\omega[)} \left\{\max_{j=0}^{s} \max_{0 \leq r \leq r_0} r^j \varphi^{(j)}(r)\right\}$$

The proof of this Lemma is obtained by using, as in P.Grisvard [5], the direct calculations.

Let's come back to the proof of the Theorem 2:

Proof. Let $w \in M_s(\Omega)$, we study w in the neighborhood of the corner S_1. Thanks to the regularity of w, we can develop the first and the second component of w, per report to θ, in series of cosine and sine respectively, as follows:

$$w = (u,v) = \left[\sum_{k\geq 1} u_k(r)\cos\alpha_k\theta, \sum_{k\geq 1} v_k(r)\sin\alpha_k\theta\right],\ \text{where } \alpha_k = \frac{k\pi}{\omega} - 1$$

As w is harmonic, u and v are also. Of $\Delta w = 0$, by passing in polar coordinates it can be written:

$$\begin{cases} u_k'' + \dfrac{u_k'}{r} - \alpha_k^2 \dfrac{u_k}{r^2} = 0 \\ v_k'' + \dfrac{v_k'}{r} - \alpha_k^2 \dfrac{v_k}{r^2} = 0 \end{cases}$$

By noticing that these differential equations are the same ones, we thus take only one of them, (for example: the one that depends on u).

It is easy to verify that the solution of this equation is given by

$$u_k(r) = a_k r^{\alpha_k} + b_k r^{-\alpha_k},\ \alpha_k = \frac{k\pi}{\omega} - 1$$

And like $w \in H^{-s}(\Omega)^2$ ($u \in H^{-s}(\Omega)$), we have

$$\langle u, f\rangle = \sum_{k\geq 1} \int_0^{r_0}\int_0^{\omega} u_k(r) f(r,\theta) J(r,\theta)\, dr\, d\theta$$

where $J(r,\theta) = r\sin(\alpha_k\theta)$ denotes the *Jacobian*.

By substituting of u_k, f and $J(r,\theta)$ by these values, it results

$$\langle u, f\rangle = \sum_{k\geq 1} \int_0^{r_0}\int_0^{\omega} \left(a_k r^{\frac{k\pi}{\omega}-1} + b_k r^{-\frac{k\pi}{\omega}+1}\right)\left(\frac{r^{\frac{\ell\pi}{\omega}-2}}{|\ln r|}\varphi(r)\psi(\theta)\right) r\sin\left(\theta\left(\frac{k\pi}{\omega}-1\right)\right) dr\, d\theta$$

or

$$\langle u,f\rangle = \left[\sum_{k\geq 1}\left(a_k\int_0^{r_0}\frac{r^{\frac{(\ell+k)\pi}{\omega}-2}}{|\ln r|}\varphi(r)\,dr + b_k\int_0^{r_0}\frac{r^{\frac{(\ell-k)\pi}{\omega}}}{|\ln r|}\varphi(r)\,dr\right)\right]. \left[\int_0^{\omega}\psi(\theta)\sin\left(\left(\frac{k\pi}{\omega}-1\right)\theta\right)d\theta\right]$$

Therefore

$$\langle u,f\rangle \leq C_\ell\,\|u\|_{-s}\,\|\psi\|_{H^s(]0,\omega[)}\left\{\max_{j=0}^{s}\max_{0\leq r\leq r_0} r^j\varphi^{(j)}(r)\right\}$$

by varying the functions φ and ψ, in the case when the function $\frac{r^{\frac{(\ell-k)\pi}{\omega}}}{|\ln r|}$ is not integrated in the neighborhood of zero, it can be seen that the last inequality is not possible that if $b_k = 0$.

In the particular case, when $\ell = \frac{\omega_1}{\pi}(s+1)$, it is noted that $b_k = 0$ for $k \geq \frac{\omega_1}{\pi}(s+2)$.

We have thus to prove that

$$u(r,\theta) = \sum_{k\geq 1}\left(a_k r^{\frac{k\pi}{\omega}-1}\cos\left(\frac{k\pi}{\omega}-1\right)\theta + b_k r^{-\frac{k\pi}{\omega}+1}\cos\left(\frac{k\pi}{\omega}-1\right)\theta\right)$$

and consequently, for $0 < r < r_0$ we have

$$v(r,\theta) = \sum_{k\geq 1} a_k r^{\frac{k\pi}{\omega}-1}\cos\left(\frac{k\pi}{\omega}-1\right)\theta + \sum_{1\leq k<\frac{\omega_1}{\pi}(s+2)} b_k r^{-\frac{k\pi}{\omega}+1}\cos\left(\frac{k\pi}{\omega}-1\right)\theta$$

This will permit to have the following expression

$$u + \sum_{1\leq k<\frac{\omega_1}{\pi}(s+2)} b_k u_k =$$

$$= \sum_{k\geq 1} a_k r^{\frac{k\pi}{\omega}-1}\cos\left(\frac{k\pi}{\omega}-1\right)\theta - \sum_{1\leq k<\frac{\omega_1}{\pi}(s+2)} b_k\left(\operatorname{Re}\left(z^{-\frac{k\pi}{\omega}+1}\right) - u_k\right)$$

Thanks to Lemma 7, the second sum in the right-hand member of this equality is an element of $H^1(\Omega)$, and consequently, the series defined an element of $H^1\left(\Omega'\right)$, where

$$\Omega' = \{z \in C : 0 < \operatorname{Re} z < r_0\} \cap \Omega$$

By the same techniques, we verify that

$$v + \sum_{1\leq k<\frac{\omega_1}{\pi}(s+2)} b_k' v_k =$$

$$= \sum_{k\geq 1} a_k' r^{\frac{k\pi}{\omega}-1}\sin\left(\frac{k\pi}{\omega}-1\right)\theta - \sum_{1\leq k<\frac{\omega_1}{\pi}(s+2)} b_k'\left(\operatorname{Im}\left(z^{-\frac{k\pi}{\omega}+1}\right) - v_k\right)$$

is an element of $H^1\left(\Omega'\right)$.

By repeating the same processes in the neighborhood of each corner of Ω, we see that is a vectorial subspace $M_s'(\Omega)$ of dimension N in $M_s(\Omega)$ such that

$$M_s(\Omega) \subseteq M_s'(\Omega) + H^1(\Omega)^2$$

where $M_s^{'}(\Omega)$ is a subspace generated by the elements of the Lemma 7 corresponding to each corner, i.e.:

$$M_s^{'}(\Omega) = \langle u_k \rangle\,, \text{ where } u_k = \left[\mathrm{Re}\left(\frac{1}{z^{\alpha_k}}\right), \mathrm{Im}\left(\frac{1}{z^{\alpha_k}}\right)\right]$$

According to the uniqueness of the solution of the variational problem, it results $M_s(\Omega) \cap H^1(\Omega)^2 = \{0\}$, what implies that $M_s(\Omega) \subseteq M_s^{'}(\Omega)$. This inclusion joined to the inequality $\dim M_s(\Omega) \geq \dim M_s^{'}(\Omega)$ involves $\dim M_s(\Omega) = \dim M_s^{'}(\Omega) = N$. □

Conclusion 1. *Thanks to the inequality a priori* (1.1) *which is checked for all* $u \in H^{s+2}(\Omega)^2 \cap V$, *it can be seen that the operator of Laplace,* $\Delta : H^{s+2}(\Omega)^2 \cap V \to H^s(\Omega)^2$ *is injective and has closed image of finished codimension. Consequently* Δ *is an operator with index which is worth* $-2N$.

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Integral Inequalities Regarding Double Integrals

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Abstract. New integral inequality regarding double integrals is presented.

1. Introduction

The following open question was proposed in [2]
Under what conditons does the inequality

(1.1) $$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx$$

hold for α *and* β ?

In [1], the authors gave an answer by establishing the following

Theorem . *If the function* f *satisfies*

(1.2) $$\int_0^1 f(t)\,dt \geq \frac{1-x^2}{2}, \qquad \forall x \in [0,1],$$

then

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx$$

for every real $\alpha \geq 1$ *and* $\beta > 0$.

The aim of this paper is that to give a new theorem concerning a similar result but for a double integrals and over the domain $[a,b]\times[a,b]$.

2. New inequality

We state and prove the following

Theorem 2.1. *Let f(x,y) , g(x) , h(y) be continuous functions defined on* $[a,b]\times[a,b]$, $[a,b]$ *respectively,* f *is nonnegative,* $g(a)=h(a)=0$, $g'(x), h'(y)\geq 1\ \forall\, x,y\in[a,b]$. *Let* $\alpha\geq 1$, $\beta,\gamma>0$. *If*

$$\int_y^b\int_x^b f(t,u)\,dt\,du \;\geq\; \int_y^b\int_x^b g(t)\,g'(t)\,h(u)\,h'(u)\,dt\,du \quad \forall\; x,y\in[a,b], \tag{2.1}$$

$$=\frac{1}{4}\left(g^2(b)-g^2(x)\right)\left(h^2(b)-h^2(y)\right),$$

then

$$\int_a^b\int_a^b f^{\alpha+\beta+\gamma}(x,y)\,dxy \;\geq\; \int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy\,, \tag{2.2}$$

provided one of the following holds :

(i) $g(x), h(y)\;\geq 1\quad \forall\, x,y\in[a,b]$.

(ii) $g^{\frac{\alpha\gamma}{\beta+\gamma}}(b)\,h^{\frac{\alpha\beta}{\beta+\gamma}}(b)\;\geq\;\dfrac{(\alpha+\beta+1)\,(\alpha+\gamma+1)}{\left[\beta\left(\dfrac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]\left[\gamma\left(\dfrac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]}$.

(iii) $\max\left\{\dfrac{g^{\beta}(b)}{h^{\alpha+\beta+1}(b)}\,,\;\dfrac{h^{\gamma}(b)}{g^{\alpha+\gamma+1}(b)}\right\}\;\leq\;\dfrac{\alpha+\beta+\gamma+1}{(\alpha+\beta+1)\,(\alpha+\gamma+1)\,(b-a)}$.

Proof . By changing the order of the integration, we have

$$\int_a^b\int_a^b\int_y^b\int_x^b f(t,u)\,g^{\beta-1}(x)\,g'(x)\,h^{\gamma-1}(y)\,h'(y)\,dt\,du\,dx\,dy$$

$$=\int_a^b\int_y^b\left(\int_a^b\int_x^b f(t,u)\,g^{\beta-1}(x)\,g'(x)\,dt\,dx\right)h^{\gamma-1}(y)\,h'(y)\,du\,dy$$

$$=\int_a^b\int_y^b\left(\int_a^b f(t,u)\,dt\int_a^t g^{\beta-1}(x)\,g'(x)\,dx\right)h^{\gamma-1}(y)\,h'(y)\,du\,dy$$

$$=\frac{1}{\beta}\int_a^b\int_y^b\left(\int_a^b f(t,u)\,g^{\beta}(t)\,dt\right)h^{\gamma-1}(y)\,h'(y)\,du\,dy$$

$$=\frac{1}{\beta}\int_a^b g^{\beta}(t)\,dt\int_a^b\int_y^b f(t,u)\,h^{\gamma-1}(y)\,h'(y)\,du\,dy$$

$$=\frac{1}{\beta}\int_a^b g^{\beta}(t)\,dt\int_a^b f(t,u)\,du\int_a^u h^{\gamma-1}(y)\,h'(y)\,dy$$

$$=\frac{1}{\beta\gamma}\int_a^b g^{\beta}(t)\,dt\int_a^b f(t,u)\,h^{\gamma}(u)\,du$$

$$= \frac{1}{\beta\gamma} \int_a^b \int_a^b f(t,u)\, g^{\beta}(t)\, h^{\gamma}(u)\, dt\, du .$$

Also,

$$\int_a^b \int_a^b \int_y^b \int_x^b f(t,u)\, g^{\beta-1}(x)\, g'(x)\, h^{\gamma-1}(y)\, h'(y)\, dt\, du\, dx\, dy$$

$$= \int_a^b \int_a^b \left(\int_y^b \int_x^b f(t,u)\, dt\, du \right) g^{\beta-1}(x)\, g'(x)\, h^{\gamma-1}(y)\, h'(y)\, dx\, dy$$

$$\geq \frac{1}{4} \int_a^b \int_a^b \left(g^2(b) - g^2(x)\right)\left(h^2(b) - h^2(y)\right) g^{\beta-1}(x)\, g'(x)\, h^{\gamma-1}(y)\, h'(y)\, dx\, dy$$

$$= \frac{1}{4} \int_a^b \left(g^2(b) - g^2(x)\right) g^{\beta-1}(x)\, g'(x)\, dx \int_a^b \left(h^2(b) - h^2(y)\right) h^{\gamma-1}(y)\, h'(y)\, dy$$

$$= \frac{1}{4} \left(\frac{g^{\beta+2}(b)}{\beta} - \frac{g^{\beta+2}(b)}{\beta+1} \right) \left(\frac{h^{\gamma+2}(b)}{\gamma} - \frac{h^{\gamma+2}(b)}{\gamma+2} \right)$$

$$= \frac{g^{\beta+2}(b)\, h^{\gamma+2}(b)}{\beta\gamma(\beta+2)(\gamma+2)}.$$

Therefore

$$\int_a^b \int_a^b f(t,u)\, g^{\beta}(t)\, h^{\gamma}(u)\, dt\, du \;\geq\; \frac{g^{\beta+2}(b)\, h^{\gamma+2}(b)}{(\beta+2)(\gamma+2)} .$$

Now

$$\frac{1}{\alpha} f^{\alpha}(x,y) + \frac{\alpha-1}{\alpha} g^{\alpha}(x)\, h^{\alpha}(y) \geq f(x,y)\, g^{\alpha-1}(x)\, h^{\alpha-1}(y).$$

Multiplying the above inequality by $\alpha\, g^{\beta}(x)\, h^{\gamma}(y)$, we obtain

$$f^{\alpha}(x,y)\, g^{\beta}(x)\, h^{\gamma}(y) \;\geq\; \alpha\, f(x,y)\, g^{\alpha+\beta-1}(x)\, h^{\alpha+\gamma-1}(y) - (\alpha-1)\, g^{\alpha+\beta}(x)\, h^{\alpha+\gamma}(y)$$

$$\geq \alpha\, f(x,y)\, g^{\alpha+\beta-1}(x)\, h^{\alpha+\gamma-1}(y) - (\alpha-1)\, g^{\alpha+\beta}(x)\, g'(x)\, h^{\alpha+\gamma}(y)\, h'(y)$$

Double integration get

$$\int_a^b \int_a^b f^{\alpha}(x,y)\, g^{\beta}(x)\, h^{\gamma}(y)\, dxdy \;\geq\; \alpha \int_a^b \int_a^b f(x,y)\, g^{\alpha+\beta-1}(x) h^{\alpha+\gamma-1}(y)\, dx\, dy$$

$$-(\alpha-1) \int_a^b \int_a^b g^{\alpha+\beta}(x)\, g'(x)\, h^{\alpha+\gamma}(y)\, h'(y)\, dxdy$$

$$\geq \alpha \frac{g^{\alpha+\beta+1}(b)\, h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} - (\alpha-1) \frac{g^{\alpha+\beta+1}(b)\, h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}$$

$$= \frac{g^{\alpha+\beta+1}(b)\, h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} .$$

We now deal with the three cases separately

Case 1. We claim that $\beta\left(\dfrac{\alpha+\beta+\gamma}{\beta+\lambda}\right)\le \alpha+\beta$ *and* $\gamma\left(\dfrac{\alpha+\beta+\gamma}{\beta+\gamma}\right)\le \alpha+\gamma$. In fact

$$\beta\le\beta+\gamma \Rightarrow 1+\frac{\alpha}{\beta+\gamma}\le 1+\frac{\alpha}{\beta} \Rightarrow \frac{\alpha+\beta+\gamma}{\beta+\gamma}\le\frac{\alpha+\beta}{\beta} \Rightarrow \beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)\le \alpha+\beta.$$

Therefore, we have

$$f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y) \le \frac{\alpha}{\alpha+\beta+\gamma}f^{\alpha+\beta+\gamma}(x,y)+\frac{\beta+\gamma}{\alpha+\beta+\gamma}\left(g^{\beta}(x)\,h^{\gamma}(y)\right)^{\frac{\alpha+\beta+\gamma}{\beta+\gamma}}$$

$$\text{(2.3)}\qquad \le \frac{\alpha}{\alpha+\beta+\gamma}f^{\alpha+\beta+\gamma}(x,y)+\frac{\beta+\gamma}{\alpha+\beta+\gamma}g^{\beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)}(x)\,g'(x)\;h^{\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)}(y)\,h'(y)$$

$$\le \frac{\alpha}{\alpha+\beta+\gamma}f^{\alpha+\beta+\gamma}(x,y)+\frac{\beta+\gamma}{\alpha+\beta+\gamma}g^{\alpha+\beta}(x)\,g'(x)\,h^{\alpha+\gamma}(y)\,h'(y).$$

Double integrating both sides the above inequality from a to b gives

$$\frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b f^{\alpha+\beta+\gamma}(x,y)\,dx\,dy \ge \int_a^b\!\!\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$-\frac{\beta+\gamma}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b g^{\alpha+\beta}(x)\,g'(x)\,h^{\alpha+\gamma}(y)\,h'(y)\,dx\,dy$$

$$\ge \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$+\frac{\beta+\gamma}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b\left(f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy-g^{\alpha+\beta}(x)\,g'(x)\,h^{\alpha+\gamma}(y)\,h'(y)\,dx\,dy\right)$$

$$\ge \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$+\frac{\alpha+\beta}{\alpha+\beta+\gamma}\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}-\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}\right)$$

$$=\frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\!\!\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy.$$

Case 2. Since g(x)>0, g is increasing. Hence $g(x)\ge g(a)=0$. It is not difficult to show that (ii) implies

$$\frac{g^{\beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1}(b)\,h^{\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1}(b)}{\left[\beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]\left[\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]} \le \frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}.$$

Therefore, from (2.3) we have

$$\frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha+\beta+\gamma}(x,y)\,dxdy \ge \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$
$$+\frac{\beta+\gamma}{\alpha+\beta+\gamma}\times$$
$$\left(\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy-\int_a^b\int_a^b g^{\beta\left(\frac{\alpha+\beta+\lambda}{\beta+\gamma}\right)}(x)\,g'(x)h^{\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)}(y)\,h'(y)\,dx\,dy\right)$$
$$\ge \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dxy \;+\; \frac{\beta+\gamma}{\alpha+\beta+\gamma}\times$$
$$\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}-\frac{g^{\beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1}(b)\,h^{\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1}(b)}{\left[\beta\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]\left[\gamma\left(\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right)+1\right]}\right)$$
$$\ge \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy \;+\; \frac{\beta+\gamma}{\alpha+\beta+\gamma}\times$$
$$\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}-\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}\right)$$
$$= \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy.$$

Case 3. We have, by (iii),

$$f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y) \le \frac{\alpha}{\alpha+\beta+\gamma}f^{\alpha+\beta+\gamma}(x,y)+\frac{\beta}{\alpha+\beta+\gamma}g^{\alpha+\beta+\gamma}(x)$$
$$+\frac{\gamma}{\alpha+\beta+\gamma}h^{\alpha+\beta+\gamma}(y)$$
$$\le\frac{\alpha}{\alpha+\beta+\gamma}f^{\alpha+\beta+\gamma}(x,y)+\frac{\beta}{\alpha+\beta+\gamma}g^{\alpha+\beta+\gamma}(x)\,g'(x)$$
$$+\frac{\gamma}{\alpha+\beta+\gamma}h^{\alpha+\beta+\gamma}(y)\,h'(y),$$

which implies

$$\frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha+\beta+\gamma}(x,y)\,dx\,dy \geq \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$+\frac{\beta}{\alpha+\beta+\gamma}\int_a^b\int_a^b\left(f^{\alpha}(x,y)\,g^{\beta}(x)h^{\gamma}(y)\,dx\,dy - g^{\alpha+\beta+\gamma}(x)\,g'(x)\,dx\,dy\right)$$

$$+\frac{\gamma}{\alpha+\beta+\gamma}\int_a^b\int_a^b\left(f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy - h^{\alpha+\beta+\gamma}(y)\,h'(y)\,dx\,dy\right)$$

$$\geq \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$+\frac{\beta}{\alpha+\beta+\gamma}\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} - \frac{g^{\alpha+\beta+\gamma+1}(b)(b-a)}{\alpha+\beta+\gamma+1}\right)$$

$$+\frac{\gamma}{\alpha+\beta+\gamma}\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\beta+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} - \frac{h^{\alpha+\beta+\gamma+1}(b)(b-a)}{\alpha+\beta+\gamma+1}\right)$$

$$\geq \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)\,h^{\gamma}(y)\,dx\,dy$$

$$+\frac{\beta}{\alpha+\beta+\gamma}\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} - \frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}\right)$$

$$+\frac{\gamma}{\alpha+\beta+\gamma}\left(\frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)} - \frac{g^{\alpha+\beta+1}(b)\,h^{\alpha+\gamma+1}(b)}{(\alpha+\beta+1)(\alpha+\gamma+1)}\right)$$

$$= \frac{\alpha}{\alpha+\beta+\gamma}\int_a^b\int_a^b f^{\alpha}(x,y)\,g^{\beta}(x)h^{\gamma}(y)\,dx\,dy.$$

3. Applications

Corollary 3.1. *Suppose that the statement of Theorem 2.1 is satisfied. If*

(3.1)
$$\int_y^b\int_x^b f(t,u)\,dt\,du \;\geq\; \int_y^b\int_x^b t\,u\,dt\,du \qquad \forall\, x,y\in[0,b],$$

then

(3.2)
$$\int_0^b\int_0^b f^{\alpha+\beta+\gamma}(x,y)\,dxdy \;\geq\; \int_0^b\int_0^b f^{\alpha}(x,y)\,x^{\beta}\,y^{\gamma}\,dx\,dy\;,$$

provided

$$b^{\alpha} \;\geq\; \frac{(\beta+\gamma)^2(\alpha+\beta+1)(\alpha+\gamma+1)}{\big((\beta+\gamma)(\beta+1)+\alpha\beta\big)\big((\beta+\gamma)(\gamma+1)+\alpha\gamma\big)}.$$

Proof . Follows from Theorem 2.1, using (ii) and putting $a=0,\; g(z)=h(z)=z.$

Corollary 3.2. *Supose that the statement of Theorem 2.1 is satisfied . If*

(3.3) $$\int_y^b\int_x^b f(t,u)\,dt\,du \ \geq\ \int_y^b\int_x^b (t-a)(u-a)\,dt\,du \qquad \forall\, x, y \in [a,b],$$

then

(3.4) $$\int_a^b\int_a^b f^{\alpha+\beta+\gamma}(x,y)\,dxdy \ \geq\ \int_a^b\int_a^b f^{\alpha}(x,y)(x-a)^{\beta}(y-a)^{\gamma}\,dx\,dy\,,$$

provided

$$(b-a)^{\alpha} \ \geq\ \frac{(\alpha+\beta+1)(\alpha+\gamma+1)}{(\alpha+\beta+\gamma+1)}\,.$$

Proof . Follows from Theorem 2.1, by putting g(z)=h(z)=z-a .

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Inclusion Theorems for Absolute Summability of Infinite Series

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Abstract. Two new general theorems concerning necessary and sufficient conditions for the series $\sum a_n \lambda_n \beta_n$ to be summable $\varphi - \left|R, q_n\right|_s$ whenever $\sum a_n$ is summable $\phi - \left|R, p_n\right|_k$, $s \geq k \geq 1$, are presented .

1. Introduction

Let (φ_n) be a sequence of positive real numbers, let $\sum a_n$ be an infinite series with the sequence of partial sums (s_n). Let (t_n) denote the n-th (C,1) means of the sequence (na_n). The series $\sum a_n$ is said to be summable $\left|C,1\right|_k$, $k \geq 1$, if (see[1])

(1.1) $$\sum_{n=1}^{\infty} \frac{1}{n}\left|t_n\right|^k < \infty .$$

and it is said to be summable $\varphi - \left|C,1\right|_k$, $k \geq 1$, if (see [5])

(1.2) $$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^k}\left|t_n\right|^k < \infty .$$

If we are taking $\varphi_n = n$, $\varphi - \left|C,1\right|_k$ reduces to $\left|C,1\right|_k$ summability .
Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \ \ n \to \infty \quad \left(P_{-1} = P_{-1} = 0\right).$$

The sequence-to-sequence transformation

(1.3) $$u_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (u_n) of the Riesz mean or simply the $(\overline{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) $(see[2])$. The series $\sum a_n$ is said to be summable $\left|R, p_n\right|_k$, $k \geq 1$ if

(1.4) $$\sum_{n=1}^{\infty} n^{k-1}\left|u_n - u_{n-1}\right|^k < \infty .$$

In the special case when $p_n = 1$ for all n, then $\left|R, p_n\right|_k$ summability is the same as $\left|C,1\right|_k$ summability . The series $\sum a_n$ is summable $\varphi - \left|R, p_n\right|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}\left|u_n-u_{n-1}\right|^k<\infty.$$

For $\varphi_n=n,\quad \varphi-\left|R,p_n\right|_k$ summability is the same as $\left|R,p_n\right|_k$ summability .

In what follows we are assuming (q_n) is a sequence of positive numbers such that

$$Q_n=\sum_{v=0}^{n}q_v\to\infty,\quad as\ n\to\infty\ ,\qquad (Q_{-1}=q_{-1}=0).$$

Concrning $\left|C,1\right|_k$ summability, Mazhar [3] has proved the following

Theorem 1.1. *If*

(1.5) $$\lambda_m=o(1),\quad as\ \ m\to\infty,$$

(1.6) $$\sum_{n=1}^{m}n\log n\left|\Delta^2\lambda_n\right|=\mathrm{O}(1),\quad as\ \ m\to\infty\ ,$$

(1.7) $$\sum_{v=1}^{m}\frac{\left|t_v\right|^k}{v}=\mathrm{O}(\log m)\ \ as\ \ m\to\infty\ ,$$

then the series $\sum a_n\lambda_n$ *is summable* $\left|C,1\right|_k,\ k\geq 1.$

Ozarslan [4], on the other hand , generalized the previous result by giving the following

Theorem 1.2. *Let* (φ_n) *be a sequence of positive real numbers and the conditions (1.5)-(1.6) of Theorem (1.1) are satisfied . If*

(1.8) $$\sum_{v=1}^{m}\frac{\varphi_v^{k-1}}{v^k}\left|t_v\right|^k=\mathrm{O}(\log m)\quad as\ \ m\to\infty,$$

(1.9) $$\sum_{n=v}^{\infty}\frac{\varphi_n^{k-1}}{n^{k+1}}=\mathrm{O}\left(\frac{\varphi_v^{k-1}}{v^k}\right),$$

then the series $\sum a_n\lambda_n$ *is summable* $\varphi-\left|C,1\right|_k,\ \ k\geq 1.$

It should be mentioned that on taking $\varphi_n=n$ in Theorem (1.2), we get Theorem 1.1 .

2. Lemmas

The following Lemmas are needed for our aim

Lemma 2.1. *Let* (β_n) *be nonnegative, nondecreasing sequence of real numbers satisfying*

(2.1) $$\sum_{n=1}^{m}n\,\beta_n\left|\Delta^2\lambda_n\right|<\infty\ \ .$$

Then conditions (1.5) and (3.1) implies

(2.2) $$\sum_{n=1}^{\infty} \beta_n |\Delta\lambda_n| = O(1),$$

(2.3) $$\sum_{n=1}^{\infty} |\lambda_n||\Delta\beta_n| = O(1),$$

(2.4) $$n\,\beta_n |\Delta\lambda_n| = O(1), \quad as\ n \to \infty,$$

(3.5) $$\beta_n |\lambda_n| = O(1), \quad as\ n \to \infty .$$

Proof. By virtue of (1.5),

$$\begin{aligned}
\sum_{n=1}^{\infty} \beta_n |\Delta\lambda_n| &= \sum_{n=1}^{\infty} \beta_n \sum_{v=n}^{\infty} \Delta|\Delta\lambda_v| \\
&\le \sum_{n=1}^{\infty} \beta_n \sum_{v=n}^{\infty} |\Delta^2\lambda_v| \\
&= \sum_{v=1}^{\infty} |\Delta^2\lambda_v| \sum_{n=1}^{v} \beta_v \\
&= O(1) \sum_{v=1}^{\infty} v\,\beta_v |\Delta^2\lambda_v| \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m} |\lambda_n||\Delta\beta_n| &= \sum_{n=1}^{m} |\Delta\beta_n| \left|\sum_{v=n}^{\infty} \Delta\lambda_v\right| \\
&\le \sum_{n=1}^{\infty} |\Delta\beta_n| \sum_{v=n}^{\infty} |\Delta\lambda_v| \\
&= \sum_{v=1}^{\infty} |\Delta\lambda_v| \sum_{n=1}^{v} |\Delta\beta_n| \\
&= \sum_{v=1}^{\infty} |\Delta\lambda_v| \left(\beta_{v+1} - \beta_1\right) \\
&\le O(1) \sum_{v=1}^{\infty} |\Delta\lambda_v|\, \beta_v \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}
n\,\beta_n |\Delta\lambda_n| &= n\,\beta_n \left|\sum_{v=n}^{\infty} \Delta|\Delta\lambda_v|\right| \\
&\le n\,\beta_n \sum_{v=n}^{\infty} \left|\Delta|\Delta\lambda_v|\right| \\
&\le n\,\beta_n \sum_{v=n}^{\infty} |\Delta^2\lambda_v| \\
&= O(1) \sum_{n=v}^{\infty} v\,\beta_v |\Delta^2\lambda_v| \\
&= O(1).
\end{aligned}$$

$$\begin{aligned}\beta_n|\lambda_n| &= \beta_n \sum_{v=n}^{\infty} \Delta|\lambda_v| \\ &\le \beta_n \sum_{v=n}^{\infty} |\Delta\lambda_v| \\ &= O(1) \sum_{v=n}^{\infty} \beta_v |\Delta\lambda_v| \\ &= O(1), \quad \text{by the first part .}\end{aligned}$$

Lemma 2.2. *Let* $Q_n \to \infty$, *as* $n \to \infty$, φ_n *is nonnegative such that and* $\left(\dfrac{\varphi_n q_n}{Q_n}\right)$ *is nondecreasing, let* $s \ge 1$. *Then*

$$\left(\frac{\varphi_v q_v}{Q_v}\right)^{s-1} \frac{1}{Q_{v-1}^s} = O(1) \sum_{n=v}^{\infty} \varphi_n^{s-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^s .$$

Proof. As $Q_n \to \infty$, then

$$\frac{1}{Q_{v-1}^s} = \sum_{n=v}^{\infty} \Delta\left(\frac{1}{Q_{n-1}^s}\right) = \sum_{n=v}^{\infty} \frac{Q_n^s - Q_{n-1}^s}{Q_n^s Q_{n-1}^s} .$$

Since by the mean value theorem,

$$\begin{aligned}Q_n^s - Q_{n-1}^s &= \left(Q_v^s\right)', \quad \textit{for some } n-1 \le v \le n \\ &= O(1)\, Q_v^{s-1} |\Delta Q_v| \\ &= O(1)\, Q_n^{s-1} q_n ,\end{aligned}$$

then we have

$$\frac{1}{Q_v^{s-1}} = O(1) \sum_{n=v}^{\infty} \frac{q_n}{Q_n Q_{n-1}^s} ,$$

and hence

$$\begin{aligned}\left(\frac{\varphi_v q_v}{Q_v}\right)^{s-1} \frac{1}{Q_{v-1}^s} &= O(1) \sum_{n=v}^{\infty} \left(\frac{\varphi_n q_n}{Q_n}\right)^{s-1} \frac{q_n}{Q_n Q_{n-1}^s} \\ &= O(1) \sum_{n=v}^{\infty} \varphi_n^{s-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^s .\end{aligned}$$

3. Main Results

Theorem 3.1. *Let* $(\varphi_n), (\phi_n), (\beta_n)$ *be sequences of positive real numbers such that* (β_n) *is nondecreasing satisfying (1.5) and (2.1) . Then sufficient conditions for the series* $\sum a_n \lambda_n \beta_n$ *to be summaqble* $\phi - |R, q_n|_s$, *whenever* $\sum a_n$ *is sumable* $\phi - |R, p_n|_k$, $s \ge k \ge 1$, *are*

(3.1) $$\sum_{n=v}^{m} \frac{\varphi_n^{s-1} q_n^s}{Q_n^s Q_{n-1}} = O\left(\frac{\varphi_v^{s-1} q_v^{s-1}}{Q_v^s} \right),$$

(3.2) $$\left(\varphi_n |X_n|\right)^{s-k} = O(1),$$

(3.3) $$\varphi_n = O(\phi_n),$$

(3.4) $$\frac{P_n q_n}{p_n Q_n} = O(1), \quad P_n = O(np_n),$$

(3.5) $$\Delta\beta_n = O(n^{-1}\beta_n).$$

Proof. Let (t_n), (T_n) denote the $(\overline{N}, p_n)$, $(\overline{N}, q_n)$ mean of the series $\sum a_n$, $\sum a_n \lambda_n \beta_n$ respectively. By definition, we have

(3.6) $$Y_n := T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \lambda_v \beta_v , \quad X_n := t_n - t_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v .$$

Then via Abel's transformation, we have

$$\begin{aligned} Y_n &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \frac{Q_{v-1}}{P_{v-1}} \lambda_v \beta_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} P_{r-1} a_r \right) \Delta \left(\frac{Q_{v-1}}{P_{v-1}} \lambda_v \beta_v \right) + \left(\sum_{v=1}^{n} P_{v-1} a_v \right) \frac{Q_{n-1}}{P_{n-1}} \lambda_n \beta_n \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \Big(Q_{v-1} X_v \lambda_v \beta_v + \frac{P_{v-1} q_v}{p_v} X_v \lambda_v \beta_v + \frac{P_{v-1} Q_v}{p_v} X_v \Delta\lambda_v \beta_v \\ &\qquad + \frac{P_{v-1} Q_v}{p_v} X_v \lambda_v \Delta\beta_v \Big) + \frac{P_n q_n}{p_n Q_n} X_n \lambda_n \beta_n \\ &= Y_{n1} + Y_{n2} + Y_{n3} + Y_{n4} + Y_{n5}. \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{s-1} |Y_{nr}|^s < \infty, \quad r = 1,2,3,4,5.$$

Applying Holder's inequality, and using lemma 2.1,

$$\begin{aligned} \sum_{n=1}^{m} \varphi_n^{s-1} |Y_{n1}|^s &= \sum_{n=1}^{m} \varphi_n^{s-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} Q_{v-1} X_v \lambda_v \beta_v \right|^s \\ &\le \sum_{n=1}^{m} \varphi_n^{s-1} \frac{q_n^s}{Q_n^s Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1}^s}{q_v^{s-1}} |X_v|^s |\lambda_v|^s \beta_v^s \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{s-1} \\ &= O(1) \sum_{v=1}^{m} \frac{Q_{v-1}^s}{q_v^{s-1}} |X_v|^s |\lambda_v|^s \beta_v^s \sum_{n=v}^{m} \frac{\varphi_n^{s-1} q_n^s}{Q_n^s Q_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{s-1} |X_v|^s |\lambda_v|^s \beta_v^s \end{aligned}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{s-1} |X_v|^s$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{k-1} |X_v|^k \left(\varphi_v |X_v|\right)^{s-k}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \phi_v^{k-1} |X_v|^k$$

$$= \mathrm{O}(1) .$$

$$\sum_{n=1}^{m} \varphi_n^{s-1} |Y_{n2}|^s = \sum_{n=1}^{m} \varphi_n^{s-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1} q_v}{p_v} X_v \lambda_v \beta_v \right|^s$$

$$\leq \sum_{n=1}^{m} \varphi_n^{s-1} \frac{q_n^s}{Q_n^s Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1}^s q_v}{p_v^s} |X_v|^s |\lambda_v|^s \beta_v^s \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{s-1}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{P_{v-1}^s q_v}{p_v^s} |X_v|^s |\lambda_v|^s \beta_v^s \sum_{n=v}^{m} \frac{\varphi_n^{s-1} q_n^s}{Q_n^s Q_{n-1}}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{s-1} \frac{P_v^s q_v^s}{p_v^s Q_v^s} |X_v|^s |\lambda_v|^s \beta_v^s$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{s-1} |X_v|^s$$

$$= \mathrm{O}(1), \quad \text{as in the previous case} .$$

$$\sum_{n=2}^{m} \varphi_n^{s-1} |Y_{n3}|^s = \sum_{n=2}^{m} \varphi_n^{s-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v-1} Q_v}{p_v} X_v \Delta\lambda_v \beta_v \right|^s$$

$$\leq \sum_{n=2}^{m} \varphi_n^{s-1} \frac{q_n^s}{Q_n^s Q_{n-1}^s} \sum_{v=1}^{n-1} \frac{P_{v-1}^s Q_v^s}{p_v^s} |X_v|^s |\Delta\lambda_v| \beta_v \left(\sum_{v=1}^{n-1} |\Delta\lambda_v| \beta_v \right)^{s-1}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{P_{v-1}^s Q_v^s}{p_v^s} |X_v|^s |\Delta\lambda_v| \beta_v \sum_{n=v+1}^{m} \frac{\varphi_n^{s-1} q_n^s}{Q_n^s Q_{n-1}^s}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \frac{P_{v-1}^s Q_v}{p_v^s} |X_v|^s |\Delta\lambda_v| \beta_v \sum_{n=v+1}^{m} \frac{\varphi_n^{s-1} q_n^s}{Q_n^s Q_{n-1}}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{s-1} \frac{P_v^s q_v^{s-1}}{p_v^s Q_v^{s-1}} |X_v|^s |\Delta\lambda_v| \beta_v$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \varphi_v^{s-1} \frac{P_v}{p_v} |X_v|^s |\Delta\lambda_v| \beta_v$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \phi_v^{k-1} |X_v|^k v |\Delta\lambda_v| \beta_v$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} \left(\sum_{r=1}^{v} \phi_r^{k-1} |X_r|^k \right) \Delta\left(v |\Delta\lambda_v| \beta_v\right) + \mathrm{O}(1) \left(\sum_{v=1}^{m} \phi_v^{k-1} |X_v|^k \right) \times m |\Delta\lambda_m| \beta_m$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| \beta_v + \mathrm{O}(1) \sum_{v=1}^{m-1} (v+1) |\Delta^2 \lambda_v| \beta_v$$

$$+ O(1)\sum_{v=1}^{m-1}(v+1)|\Delta\lambda_{v+1}||\Delta\beta_v| + O(1)$$

$$= O(1) + O(1)\sum_{v=1}^{m-1} v|\Delta^2\lambda_v|\beta_v + O(1)\sum_{v=1}^{m-1}(v+1)|\Delta\lambda_{v+1}|v^{-1}\beta_v$$

$$= O(1) + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|\beta_{v+1}$$

$$= O(1)\ .$$

$$\sum_{n=2}^{m}\varphi_n^{s-1}|Y_{n4}|^s = \sum_{n=2}^{m}\varphi_n^{s-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}Q_v}{p_v}X_v\lambda_{v+1}\Delta\beta_v\right|^s$$

$$= O(1)\sum_{n=2}^{m}\varphi_n^{s-1}\left(\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}Q_v}{p_v}|X_v||\lambda_{v+1}||\Delta\beta_v|\right)^s$$

$$= O(1)\sum_{n=2}^{m}\varphi_n^{s-1}\left(\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{P_{v-1}Q_v}{p_v}|X_v||\lambda_v||\Delta\beta_v|\right)^s$$

$$= O(1)\sum_{n=2}^{m}\varphi_n^{s-1}\frac{q_n^s}{Q_n^sQ_{n-1}^s}\sum_{v=1}^{n-1}\frac{P_{v-1}^sQ_v^s}{p_v^s}|X_v|^s|\lambda_v||\Delta\beta_v|\left(\sum_{v=1}^{n-1}|\lambda_v||\Delta\beta_v|\right)^{s-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{P_v^sQ_v^s}{p_v^s}|X_v|^s|\lambda_v|\ |\Delta\beta_v|\sum_{n=v+1}^{m}\frac{\varphi_n^{s-1}q_n^s}{Q_n^sQ_{n-1}^s}$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{s-1}\frac{P_v^sq_v^{s-1}}{p_v^sQ_v^{s-1}}|X_v||\lambda_v|\ |\Delta\beta_v|$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{s-1}|X_v|^s\ v|\lambda_v|v^{-1}\beta_v$$

$$= O(1)\sum_{v=1}^{m}\phi_v^{k-1}|X_v|^k$$

$$= O(1)\ .$$

Finally,

$$\sum_{n=1}^{m}\varphi_n^{s-1}|Y_{n5}|^s = \sum_{n=1}^{m}\varphi_n^{s-1}\left|\frac{P_nq_n}{p_nQ_n}X_n\lambda_n\beta_n\right|^s$$

$$= O(1)\sum_{n=1}^{m}\varphi_n^{s-1}\left(\frac{P_nq_n}{p_nQ_n}\right)^s|X_n|^s|\lambda_n|^s\beta_n^s$$

$$= O(1)\sum_{n=1}^{m}\varphi_n^{s-1}|X_n|^s$$

$$= O(1)\sum_{v=1}^{m}\phi_n^{k-1}|X_n|^k$$

$$= O(1)\ .$$

Theorem 2.2. *Let*

(3.7) $$\sum_{n=v+1}^{\infty}\phi_n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k = O\left(\frac{\phi_v^{k-1}}{P_v^k}\right),$$

then necessary conditions for the series $\sum a_n\lambda_n\beta_n$ *to be summable* $\varphi-|R,q_n|_s$ *, whenever* $\sum a_n$ *is summable* $\phi-|R,p_n|_k$, $1\le k\le s$, *are*

(3.8) $$\lambda_v\beta_v = O\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}}\left(\frac{p_v}{P_v}\right)^{k/s}\left(\frac{Q_v}{q_v}\right)\right),$$

(3.9) $$\lambda_v\beta_v = O\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}}\left(\frac{p_v}{P_v}\right)^{k/s}\left(\frac{Q_v}{q_v}\right)^{2-1/s}\right),$$

(3.10) $$\Delta\lambda_v\beta_v = O\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}}\left(\frac{p_v}{P_v}\right)^{k/s}\left(\frac{Q_v}{q_v}\right)^{1-1/s}\right),$$

(3.11) $$\lambda_{v+1}\Delta\beta_v = O\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}}\left(\frac{p_v}{P_v}\right)^{k/s}\left(\frac{Q_v}{q_v}\right)^{1-1/s}\right).$$

Proof. We are given that

(3.12) $$\sum_{n=1}^{\infty}\varphi_n^{s-1}|Y_n|^s < \infty ,$$

whenever

(3.13) $$\sum_{n=1}^{\infty}\phi_n^{k-1}|X_n|^k < \infty.$$

The space of sequences (a_n) satisfying (3.13) is a Banach space if normed by

(3.14) $$\|X\| = \left(|X_0|^k + \sum_{n=1}^{\infty}\phi_n^{k-1}|X_n|^k\right)^{1/k}.$$

We as well consider the space of the sequences satisfying (3.12) . This space is a BK-space with respect to the norm

(3.15) $$\|Y\| = \left(|Y_0|^s + \sum_{n=1}^{\infty}\varphi_n^{s-1}|Y_n|^s\right)^{1/s}.$$

Observe that Y_n as defined transform the space of sequences satisfying (3.13) into the space of sequences satisfying (3.9). By applying the Banach-Steinhaus theorem there exists a constant K>0 such that

(3.16) $$\|Y\| \le K\|X\|.$$

Applying (3.6) to $a_v = e_v - e_{v+1}$, where e_v is the vth coordinate vector, we have

$$X_n = \begin{cases} 0, & if \;\; n < v, \\ \dfrac{p_v}{P_v}, & if \;\; n = v, \\ -\dfrac{p_v p_n}{P_n P_{n-1}}, & if \;\; n > v \end{cases}$$

$$Y_n = \begin{cases} 0, & if \;\; n < v, \\ \dfrac{q_v}{Q_v}\lambda_v \beta_v , & if \;\; n = v,. \\ \dfrac{q_n}{Q_n Q_{n-1}}\Delta\left(Q_{v-1}\lambda_v \beta_v\right), & if \;\; n > v \end{cases}$$

By (3.14) and (3.15) it follows that

(3.17) $$\|X\| = \left(\phi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \sum_{n=v+1}^{\infty} \phi_n^{k-1}\left(\frac{p_v p_n}{P_n P_{n-1}}\right)^k \right)^{1/k},$$

(3.18) $$\|Y\| = \left(\varphi_v^{s-1}\left(\frac{q_v}{Q_v}\lambda_v \beta_v\right)^s + \sum_{n=v+1}^{\infty} \varphi_n^{s-1}\left(\frac{q_n}{Q_n Q_{n-1}}\Delta\left(Q_{v-1}\lambda_v \beta_v\right)\right)^s \right)^{1/s}$$

Applying (3.16) for (3.17) and (3.18) , it follows that

$$\varphi_v^{s-1}\left(\frac{q_v}{Q_v}\lambda_v \beta_v\right)^s + \sum_{n=v+1}^{\infty} \varphi_n^{s-1}\left(\frac{q_n}{Q_n Q_{n-1}}\Delta\left(Q_{v-1}\lambda_v \beta_v\right)\right)^s \le K^s \left(\phi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \sum_{n=v+1}^{\infty} \phi_n^{k-1}\left(\frac{p_v p_n}{P_n P_{n-1}}\right)^k \right)$$

The above inequality is true if and only if each term of the left-hand side is O (the right-hand side) . Therefore, we have

$$\begin{aligned} \varphi_v^{s-1}\left(\frac{q_v}{Q_v}\lambda_v \beta_v\right)^s &= \mathrm{O}(1) \left(\phi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k + \sum_{n=v+1}^{\infty} \phi_n^{k-1}\left(\frac{p_v p_n}{P_n P_{n-1}}\right)^k \right) \\ &= \mathrm{O}(1)\left(\phi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right) + \mathrm{O}(1)\, p_v^k \left(\frac{\phi_v^{k-1}}{P_v^k} \right) \\ &= \mathrm{O}(1)\left(\phi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k \right), \end{aligned}$$

which implies

$$\lambda_v \beta_v = \mathrm{O}\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right) \right),$$

and also

$$\sum_{n=v+1}^{\infty} \varphi_n^{s-1} \left(\frac{q_n}{Q_n Q_{n-1}} \Delta \left(Q_{v-1} \lambda_v \beta_v \right) \right)^s = \mathrm{O}(1) \left(\phi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \right)$$

or

$$\left(\Delta \left(Q_{v-1} \lambda_v \beta_v \right) \right)^s \sum_{n=v}^{\infty} \varphi_n^{s-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^s = \mathrm{O}(1) \left(\varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \right).$$

Making use of lemma 2.2, we obtain

$$\left(\Delta \left(Q_{v-1} \lambda_v \beta_v \right) \right)^s \left(\frac{\varphi_v q_v}{Q_v} \right)^{s-1} \frac{1}{Q_{v-1}^s} = \mathrm{O}(1) \left(\varphi_v^{k-1} \left(\frac{p_v}{P_v} \right)^k \right)$$

or

$$\Delta \left(Q_{v-1} \lambda_v \beta_v \right) = \mathrm{O}(1) \frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{1=1/s} Q_{v-1}\ .$$

That is

$$\left(-q_v \beta_v \lambda_v + Q_v \Delta \lambda_v \beta_v + Q_v \lambda_{v+1} \Delta \beta_v \right) = \mathrm{O}(1) \frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{1-1/s} Q_{v-1}.$$

Since λ_v , $\Delta\lambda_v$ are linearly independent and the same is true for β_v , $\Delta\beta_v$, the three terms in the L.H.S. of the above equality are linearly independent. Therefore each of these terms is $\mathrm{O}\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{1-1/s} Q_{v-1} \right)$. This implies

$$\lambda_v \beta_v = \mathrm{O}\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{2-1/s} \right),$$

$$\Delta \lambda_v \, \beta_v = \mathrm{O}\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{1-1/s} \right),$$

$$\lambda_{v+1} \Delta \beta_v = \mathrm{O}\left(\frac{\phi_v^{(k-1)/s}}{\varphi_v^{1-1/s}} \left(\frac{p_v}{P_v} \right)^{k/s} \left(\frac{Q_v}{q_v} \right)^{1-1/s} \right).$$

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Boundedness criteria for certain third order nonlinear delay differential equations

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Abstract: In this paper, we obtain boundedness criteria for third order nonlinear delay differential equation:

$$x'''(t)+f(x(t),x'(t),x''(t))x''(t)+g(x(t-r),x'(t-r))+h(x(t-r))$$
$$=p(t,x(t),x(t-r),x'(t),x'(t-r),x''(t)),$$

where $r>0$ is a constant delay. We introduce a Lyapunov functional as a basic tool and use it throughout the paper. Our result includes a new boundedness theorem in addition to the ones previously obtained for delay differential equations.

1. Introduction

It is well-known that delay differential equations or more generally functional differential equations are used as models to describe many physical and biological systems. In fact, many actual systems have the property aftereffect, i.e. the future states depend not only on the present, but also on the past history. Aftereffect is believed to occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory, etc. Therefore, it is very important to study the qualitative behaviors of solutions of delay differential equations or more generally functional differential equations.

In 1965, Ponzo [11] introduced a technique, involving integration by parts, to construct a Lyapunov function for nonlinear third order differential equation without delay:

$$x'''(t)+f(x(t),x'(t))x''(t)+g(x(t),x'(t))x'(t)+h(x(t))=0.$$

He constructed a Lyapunov function and gave some sufficient conditions to guarantee the asymptotic stability of trivial solution of this equation. Later, based on the result of Ponzo [11], Tunç [12] improved the result established by Ponzo [11] to nonlinear third order delay differential equation

$$x'''(t)+f(x(t),x'(t))x''(t)+g(x(t-r),x'(t-r))+h(x(t-r))=0$$

and proved the asymptotic stability of trivial solution of this equation.

Now, taking into consideration the result of Tunç [12], we deal with nonlinear third order delay differential equation of the form:

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$$x'''(t) + f(x(t), x'(t), x''(t))x''(t) + g(x(t-r), x'(t-r)) + h(x(t-r))$$
$$= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) \qquad \textbf{(1)}$$

or its equivalent system:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= -f(x(t), y(t), z(t))z(t) - g(x(t), y(t)) - h(x(t)) + \int_{t-r}^{t} \frac{\partial}{\partial x} g(x(s), y(s))y(s)ds \\ &\quad + \int_{t-r}^{t} \frac{\partial}{\partial y} g(x(s), y(s))z(s)ds + \int_{t-r}^{t} \frac{d}{dx} h(x(s))y(s)ds \\ &\quad + p(t, x(t), x(t-r), y(t), y(t-r), z(t)), \end{aligned} \qquad \textbf{(2)}$$

where r is a positive constant, that is, r is a constant delay; the functions f, g, h and p depend on only the arguments displayed explicitly and the primes in the equation (1) denote differentiation with respect to $t \in \Re^+$, $\Re^+ = [0, \infty)$. It is assumed that the functions f, g, h and p are continuous for all values of their respective arguments on $\Re^3$, $\Re^2$, $\Re$ and $\Re^+ \times \Re^5$, respectively. These acceptations guarantee the existence of the solution of equation (1) (See [2, pp.14]). Besides, it is supposed that the derivatives $\frac{d}{dx}h(x)$, $\frac{\partial}{\partial x}g(x, y)$, $\frac{\partial}{\partial y}g(x, y)$ and $\frac{\partial}{\partial z}f(x, y, z)$ exist and are continuous. Moreover, it is also assumed that the functions $f(x, y, z)$, $g(x(t-r), y(t-r))$, $h(x(t-r))$ and $p(t, x, x(t-r), y, y(t-r), z)$ satisfy a Lipschitz condition in x, y, z, $x(t-r)$, $y(t-r)$ and z. Then the solution is unique (See [2, pp.14]). All solutions considered are supposed to be real valued; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively

2. Preliminaries

In order to reach the main result of this paper, we give some important basic information for the general non-autonomous delay differential system (See Burton [1], Èl'sgol'ts [2], Èl'sgol'ts and Norkin [3], Gopalsamy [4], Hale [5], Hale and Verduyn Lunel [6], Kolmanovskii and Myshkis [7], Kolmanovskii and Nosov [8], Krasovskii [9] and Yoshizawa [13]). Now, we consider the general non-autonomous delay differential system:

$$\dot{x} = f(t, x_t), \; x_t = x(t+\theta), \; -r \le \theta \le 0, \; t \ge 0, \qquad \textbf{(3)}$$

where $f : [0, \infty) \times C_H \to \Re^n$ is a continuous mapping, $f(t, 0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \|.\|)$ is the Banach space of continuous function $\phi : [-r, 0] \to \Re^n$ with supremum norm, $r > 0$, C_H is the open H-ball in C; $C_H := \{\phi \in (C[-r, 0], \Re^n) : \|\phi\| < H\}$.

Definition 1. (See [13].) A function $x(t_0,\phi)$ is said to be a solution of (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \geq 0$, if there is a constant $A > 0$ such that $x(t_0,\phi)$ is a function from $[t_0 - r,\ t_0 + A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0,\phi) \in C_H$ for $t_0 \leq t < t_0 + A$,

(ii) $x_{t_0}(t_0,\phi) = \phi$,

(iii) $x(t_0,\phi)$ satisfies (3) for $t_0 \leq t < t_0 + A$.

Standard existence theory, see Burton [1], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t,t_0,\phi)$ such that on $[t_0, t_0 + \alpha)$ satisfying equation (3) for $t > t_0$, $x_t(t,\phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.|$ will denote a convenient norm in $\Re^n$ with $|x| = \max_{1 \leq i \leq n} |x_i|$. Now, let us assume that $C(t)$={ϕ: $[t-\alpha, t] \to \Re^n$ | ϕ is continuous} and ϕ_t denotes the ϕ in the particular $C(t)$, and that $\|\phi_t\| = \max_{t-\alpha \leq s \leq t} |\phi(t)|$. Clearly, equation (1) is also particular case of (3).

Definition 2. (See [1].) A continuous function $W : [0,\infty) \to [0,\infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 3. (See [1].) Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : [0,\infty) \times D \to [0,\infty)$ is called positive definite if $V(t,0) = 0$ and if there is a wedge W_1 with $V(t,x) \geq W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t,x) \leq W_2(|x|)$.

Definition 4. (See [1].) Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}$ and is defined by the following relation

$$\dot{V}(t,\phi) = \limsup_{h \to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

3. Main result

Our main result is the following theorem.

Theorem. In addition to the basic assumptions imposed on the functions f, g, h and p appearing in equation (1), we assume that there are positive constants a, b, c, c_1, λ, μ, L and M such that the following conditions hold for all x, y and z:

(i) $f(x,y,z) \geq a + 2\lambda$, $(y \neq 0)$, and $y \dfrac{\partial}{\partial z} f(x,y,z) \geq 0$.

(ii) $g(x,0) = 0$, $\dfrac{g(x,y)}{y} \geq b + 2\mu$, $(y \neq 0)$, $\left|\dfrac{\partial}{\partial x} g(x,y)\right| \leq L$ and $\left|\dfrac{\partial}{\partial y} g(x,y)\right| \leq M$.

(iii) $h(0)=0$, $0<c_1\le \frac{d}{dx}h(x)\le c$.

(iv) $ab-c>\frac{a}{y}\int_0^y \frac{\partial}{\partial x} f(x,\eta,0)\eta d\eta+\frac{1}{y}\int_0^y \frac{\partial}{\partial x} g(x,\eta)d\eta \ge 0$, $(y\neq 0)$.

(v) $|p(t,x,x(t-r),y,y(t-r),z)|\le q(t)$ for all t, x, $x(t-r)$, y, $y(t-r)$ and z, where $\max q(t)<\infty$ and $q\in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions. Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t)=\phi(t),\ x'(t)=\phi'(t),\ x''(t)=\phi''(t)$$

satisfies the inequalities

$$|x(t)|\le K,\ |x'(t)|\le K,\ |x''(t)|\le K$$

for all $t\ge t_0$, where $\phi\in C^2([t_0-r,t_0],\Re)$, provided that

$$r<\min\left\{\frac{4a\mu}{aL+aM+ac+(L+c)(1+a)},\frac{4\lambda}{L+M+c+M(1+a)}\right\}.$$

Proof. We introduce the Lyapunov functional $V=V(x_t,y_t,z_t)$ defined by

$$V(x_t,y_t,z_t)=\frac{1}{2}z^2+ayz+a\int_0^y f(x,\eta,0)\eta d\eta+\int_0^y g(x,\eta)d\eta+h(x)y$$

$$+a\int_0^x h(\xi)d\xi+\rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds+\gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \qquad \textbf{(4)}$$

where ρ and γ are some positive constants which will be determined later in the proof. Now, the assumptions $f(x,y,z)\ge a+2\lambda$, $\frac{g(x,y)}{y}\ge b+2\mu$, $(y\neq 0)$, and $0<c_1\le \frac{d}{dx}h(x)\le c$ imply

$$\int_0^y f(x,\eta,0)\eta d\eta \ge \left(\frac{a+2\lambda}{2}\right)y^2,$$

$$\int_0^y g(x,\eta)d\eta=\int_0^y \frac{g(x,\eta)}{\eta}\eta d\eta \ge \left(\frac{b+2\mu}{2}\right)y^2,$$

$$h^2(x)=2\int_0^x h(\xi)\frac{d}{dx}h(\xi)d\xi \le 2c\int_0^x h(\xi)d\xi,$$

respectively.

Making use of the inequalities obtained above, the Lyapunov functional $V = V(x_t, y_t, z_t)$ implies that

$$V(x_t, y_t, z_t) \geq \frac{1}{2}z^2 + ayz + \frac{a^2}{2}y^2 + \left(\frac{b+2\mu}{2}\right)y^2 + h(x)y + a\int_0^x h(\xi)d\xi$$

$$+ a\lambda y^2 + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$\geq \frac{1}{2}(z+ay)^2 + \left(\frac{ab-c}{2a}\right)y^2 + \frac{ac}{2}\left(\sqrt{2c^{-1}\int_0^x h(\xi)d\xi} - a^{-1}|y|\right)^2$$

$$+ (a\lambda + \mu)y^2 + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds \geq 0\,. \qquad \textbf{(5)}$$

Therefore, it can be seen from (5) that there exist some positive constants D_i, $(i = 1,\ 2, 3)$ such that

$$V \geq D_1x^2 + D_2y^2 + D_3z^2 + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$\geq D_1x^2 + D_2y^2 + D_3z^2 \geq D_4(x^2 + y^2 + z^2)\,, \qquad \textbf{(6)}$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since the integrals $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative.

Now, time derivative of functional $V(x_t, y_t, z_t)$ along system (2) gives that

$$\frac{d}{dt}V(x_t, y_t, z_t) = -\left[a\frac{g(x,y)}{y} - \frac{d}{dx}h(x) - \frac{a}{y}\int_0^y \frac{\partial}{\partial x}f(x,\eta,0)\eta d\eta - \frac{1}{y}\int_0^y \frac{\partial}{\partial x}g(x,\eta)d\eta\right]y^2$$

$$-\big(f(x,y,z) - a\big)z^2 + z\int_{t-r}^{t}\frac{\partial}{\partial x}g(x(s), y(s))y(s)ds$$

$$-a\big(f(x,y,z) - f(x,y,0)\big)yz$$

$$+z\int_{t-r}^{t}\frac{\partial}{\partial y}g(x(s), y(s))z(s)ds + ay\int_{t-r}^{t}\frac{\partial}{\partial x}g(x(s), y(s))y(s)ds$$

$$+ay\int_{t-r}^{t}\frac{\partial}{\partial y}g(x(s), y(s))z(s)ds + z\int_{t-r}^{t}\frac{d}{dx}h(x(s))y(s)ds$$

$$+ay\int_{t-r}^{t}\frac{d}{dx}h(x(s))y(s)ds + \rho y^2 r - \rho\int_{t-r}^{t}y^2(s)ds + \gamma z^2 r - \gamma\int_{t-r}^{t}z^2(s)ds$$

$$+(ay+z)\ p(t, x(t), x(t-r), y(t), y(t-r), z(t))\,. \qquad \textbf{(7)}$$

By noting the assumptions (i)-(v) of theorem and the inequality $2|uv| \le u^2 + v^2$, ones can easily get the followings:

$$-(f(x,y,z)-a)z^2 \le -2\lambda z^2,$$

$$-\left[a\frac{g(x,y)}{y} - \frac{d}{dx}h(x) - \frac{a}{y}\int_0^y \frac{\partial}{\partial x} f(x,\eta,0)\eta d\eta - \frac{1}{y}\int_0^y \frac{\partial}{\partial x} g(x,\eta) d\eta\right] y^2$$

$$\le -\left[ab - c - \frac{a}{y}\int_0^y \frac{\partial}{\partial x} f(x,\eta,0)\eta d\eta - \frac{1}{y}\int_0^y \frac{\partial}{\partial x} g(x,\eta) d\eta\right] y^2 - 2\mu a y^2$$

$$\le -2\mu a y^2,$$

$$z\int_{t-r}^{t} \frac{\partial}{\partial x} g(x(s),y(s))y(s)ds \le \frac{L}{2} rz^2(t) + \frac{L}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$z\int_{t-r}^{t} \frac{\partial}{\partial y} g(x(s),y(s))z(s)ds \le \frac{M}{2} rz^2(t) + \frac{M}{2}\int_{t-r}^{t} z^2(s)ds,$$

$$ay\int_{t-r}^{t} \frac{\partial}{\partial x} g(x(s),y(s))y(s)ds \le \frac{aL}{2} ry^2(t) + \frac{aL}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$ay\int_{t-r}^{t} \frac{\partial}{\partial y} g(x(s),y(s))z(s)ds \le \frac{aM}{2} ry^2(t) + \frac{aM}{2}\int_{t-r}^{t} z^2(s)ds,$$

$$z\int_{t-r}^{t} \frac{d}{dx} h(x(s))y(s)ds \le \frac{c}{2} rz^2(t) + \frac{c}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$ay\int_{t-r}^{t} \frac{d}{dx} h(x(s))y(s)ds \le \frac{ac}{2} ry^2(t) + \frac{ac}{2}\int_{t-r}^{t} y^2(s)ds,$$

$$(ay+z)\ p(t,x,x(t-r),y,y(t-r),z)$$

$$\le |ay+z|\ |p(t,x,x(t-r),y,y(t-r),z)|$$

$$\le (|z| + a|y|)q(t) \le D_5(|y|+|z|)q(t),$$

where $D_5 = \max\{1,\ a\}$.

Substituting the inequalities in (7), it is easily seen that

$$\frac{d}{dt}V(x_t,y_t,z_t) \le -\left[2a\mu - \left(\frac{aL + aM + ac + 2\rho}{2}\right)r\right]y^2$$

$$-\left[2\lambda - \left(\frac{L + M + c + 2\gamma}{2}\right)r\right]z^2$$

$$-a\big(f(x,y,z)-f(x,y,0)\big)yz$$
$$+\left[\frac{(L+aL+c+ac)}{2}-\rho\right]\int_{t-r}^{t} y^2(s)ds$$
$$+\left[\frac{(1+a)M}{2}-\gamma\right]\int_{t-r}^{t} z^2(s)ds$$
$$+D_5\big(|y|+|z|\big)q(t)\,. \qquad \textbf{(8)}$$

By choosing $\rho=\dfrac{(L+aL+c+ac)}{2}$ and $\gamma=\dfrac{(1+a)M}{2}$, we get

$$\frac{d}{dt}V(x_t,y_t,z_t)\le-\left[2a\mu-\left(\frac{aL+aM+ac+2\rho}{2}\right)r\right]y^2$$
$$-\left[2\lambda-\left(\frac{L+M+c+2\gamma}{2}\right)r\right]z^2$$
$$-a\big(f(x,y,z)-f(x,y,0)\big)yz$$
$$+D_5\big(|y|+|z|\big)q(t) \quad \text{by (8).}$$

The above inequality implies that

$$\frac{d}{dt}V(x_t,y_t,z_t)\le-\alpha\,y^2-\sigma\,z^2-a\big(f(x,y,z)-f(x,y,0)\big)yz+D_5\big(|y|+|z|\big)q(t) \qquad \textbf{(9)}$$

for some positive constants α and σ , provided that

$$r<\min\left\{\frac{4a\mu}{aL+aM+ac+(L+c)(1+a)},\frac{4\lambda}{L+M+c+M(1+a)}\right\}.$$

Now, we consider the term

$$a\big(f(x,y,z)-f(x,y,0)\big)yz\,,$$

which is contained in (9).

By using the mean value theorem (for derivative), we have

$$a\big(f(x,y,z)-f(x,y,0)\big)yz=a\left[\frac{f(x,y,z)-f(x,z,0)}{z}\right]yz^2$$
$$=ayz^2\frac{\partial}{\partial z}f(x,y,\theta z)\,,0\le\theta\le1\,.$$

By using the assumption (i), it also follows that

$$ayz^2 \frac{\partial}{\partial z} f(x, y, \theta z) \geq 0.$$

Now, obviously, ones can see that

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq D_5 (|y| + |z|) q(t). \quad \textbf{(10)}$$

Also by using the inequality $|u| < 1 + u^2$ and (10) , it is clear that

$$\frac{d}{dt} V(x_t, y_t, z_t) \leq D_5 (2 + y^2 + z^2) q(t). \quad \textbf{(11)}$$

Next, the inequality (6) implies that

$$(y^2 + z^2) \leq D_4^{-1} V(x_t, y_t, z_t).$$

Using this fact into (11), we obtain

$$\begin{aligned}\frac{d}{dt} V(x_t, y_t, z_t) &\leq D_5 (2 + D_4^{-1} V(x_t, y_t, z_t)) q(t)\\ &= 2D_5 q(t) + D_5 D_4^{-1} V(x_t, y_t, z_t) q(t). \quad \textbf{(12)}\end{aligned}$$

Now, integrating (12) from 0 to t and using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality, we get

$$\begin{aligned}V(x_t, y_t, z_t) &\leq V(x_0, y_0, z_0) + 2D_5 A + D_5 D_4^{-1} \int_0^t (V(x_s, y_s, z_s)) q(s) ds\\ &\leq (V(x_0, y_0, z_0) + 2D_5 A) \exp\left(D_5 D_4^{-1} \int_0^t q(s) ds \right)\\ &\leq (V(x_0, y_0, z_0) + 2D_5 A) \exp(D_5 D_4^{-1} A) = K_1 < \infty, \quad \textbf{(13)}\end{aligned}$$

where $K_1 > 0$ is a constant, $K_1 = (V(x_0, y_0, z_0) + 2D_5 A) \exp(D_5 D_4^{-1} A)$ and $A = \int_0^\infty q(s) ds$.

Thus, both inequalities (6) and (13) imply that

$$x^2(t) + y^2(t) + z^2(t) \leq D_4^{-1} V(x_t, y_t, z_t) \leq K,$$

where $K = K_1 D_4^{-1}$. Therefore, ones can conclude that

$$|x(t)| \leq K, \ |y(t)| \leq K, \ |z(t)| \leq K$$

for all $t \geq t_0$. That is,

$$|x(t)| \leq K, \ |x'(t)| \leq K, \ |x''(t)| \leq K$$

for all $t \geq t_0$. Now the proof is complete.

Example. We consider nonlinear third order delay differential equation:

$$x'''(t) + \left(8 + (x'(t))^2\right)x''(t) + 4x'(t-r) + \sin x'(t-r) + 2x(t-r) .$$

$$= \frac{1}{1 + t^2 + x^2(t) + x^2(t-r) + x'^2(t) + x'^2(t-r) + x''^2(t)} . \qquad \textbf{(14)}$$

Now, it can be seen that differential equation (14) has the form (1) and it may be expressed as:

$$x'(t) = y(t) ,$$

$$y'(t) = z(t) ,$$

$$z'(t) = -\left(8 + y^2(t)\right)z(t) - \left(4y(t) + \sin y(t)\right) - 2x(t)$$

$$+ 2\int_{t-r}^{t} y(s)ds + \int_{t-r}^{t} \left(4 + \cos y(s)\right)z(s)ds$$

$$+ \frac{1}{1 + t^2 + x^2(t) + x^2(t-r) + y^2(t) + y^2(t-r) + z^2(t)} . \qquad \textbf{(15)}$$

Clearly, by comparing (15) with (2) and viewing the assumptions of the theorem, it follows that

$$f(y) = 8 + y^2 ,$$

$$8 + y^2 \geq 8 = a + 2\lambda ,$$

$$g(y) = 4y + \sin y , \quad g(0) = 0 ,$$

$$\frac{g(y)}{y} = 4 + \frac{\sin y}{y} , \ (y \neq 0),$$

$$4 + \frac{\sin y}{y} \geq 3 = b + 2\mu ,$$

$$h(x) = 2x , \ h(0) = 0 , \ h'(x) = 2 ,$$

$c_1 \in (0,2]$, $c = 2$, $ab > 2$, $M = 5$ and $L = 0$ (or $L = \varepsilon$ for any $\varepsilon > 0$),

$$p(t, x, x(t-r), y, y(t-r), z)$$

$$= \frac{1}{1 + t^2 + x^2 + x^2(t-r) + y^2 + y^2(t-r) + z^2} \leq \frac{1}{1 + t^2}$$

and

$$\int_0^{\infty} q(s)ds = \int_0^{\infty} \frac{1}{1 + s^2} ds = \frac{\pi}{2} < \infty ,$$

that is, $q \in L^1(0,\infty)$.
Hence, the above facts show that all the conditions from (i) to (v) of theorem hold.
Now, we introduce the Lyapunov functional

$$V_1(x_t, y_t, z_t) = \frac{1}{2}z^2 + ayz + a\int_0^y (8+\eta^2)\eta d\eta + \int_0^y (4\eta + \sin\eta)d\eta$$

$$+2yx + 2a\int_0^x \xi d\xi + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$= \frac{1}{2}z^2 + ayz + 4ay^2 + \frac{a}{4}y^4 + 2y^2 + 1 - \cos y + 2yx$$

$$+ax^2 + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \qquad \textbf{(16)}$$

where ρ and γ are some positive constants, which will be determined later. Clearly, the Lyapunov functional $V_1(x_t, y_t, z_t)$ is a special case of $V(x_t, y_t, z_t)$, which is given by (4). Now, in particular, we can choose $a = 6$ and $\lambda = 1$ since $a + 2\lambda = 8$. Hence, clearly, it follows from (16) that

$$V_1(x_t, y_t, z_t) = \frac{1}{2}z^2 + 6yz + 26y^2 + \frac{3}{2}y^4 + 1 - \cos y + 2xy$$

$$+6x^2 + \rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$= \frac{2}{5}\left(z + \frac{15}{2}y\right)^2 + \frac{3}{2}y^4 + 3\left(|x| - 3^{-1}|y|\right)^2 + 3x^2 + \frac{19}{6}y^2 + \frac{1}{10}z^2$$

$$+\rho\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \gamma\int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds$$

$$\geq 3x^2 + \frac{19}{6}y^2 + \frac{1}{10}z^2 \geq \frac{1}{10}(x^2 + y^2 + z^2)$$

$$= D_6(x^2 + y^2 + z^2). \qquad \textbf{(17)}$$

Next, by differentiating the functional $V_1(x_t, y_t, z_t)$ and using (16) and (15), we find

$$\frac{d}{dt}V_1(x_t, y_t, z_t) = -\left(22 + 6\frac{\sin y}{y} - \rho r\right)y^2 - \left(2 + y^2 - \gamma r\right)z^2$$

$$+\frac{z + 6y}{1 + t^2 + x^2 + x^2(t-r) + y^2 + y^2(t-r) + z^2}$$

$$+z\int_{t-r}^{t}(4+\cos y(s))z(s)ds+6y\int_{t-r}^{t}(4+\cos y(s))z(s)ds$$

$$+2z\int_{t-r}^{t}y(s)ds+12y\int_{t-r}^{t}y(s)ds-\rho\int_{t-r}^{t}y^2(s)ds-\gamma\int_{t-r}^{t}z^2(s)ds\,. \qquad \mathbf{(18)}$$

By using the facts $|4+\cos y|\le 5$, $\left|\frac{\sin y}{y}\right|\le 1$ and the inequality $2|ab|\le a^2+b^2$, we obtain the following inequalities for some terms included in (18):

$$-\left(22+6\frac{\sin y}{y}-\rho r\right)y^2\le-(16-\rho r)y^2,$$

$$-(2+y^2-\gamma r)z^2\le-(2-\gamma r)z^2,$$

$$z\int_{t-r}^{t}(4+\cos y(s))z(s)ds\le\frac{5}{2}\,rz^2+\frac{5}{2}\int_{t-r}^{t}z^2(s)ds,$$

$$6y\int_{t-r}^{t}(4+\cos y(s))z(s)ds\le 15\,ry^2+15\int_{t-r}^{t}z^2(s)ds,$$

$$2z\int_{t-r}^{t}y(s)ds\le rz^2+\int_{t-r}^{t}y^2(s)ds$$

and

$$12y\int_{t-r}^{t}y(s)ds\le 6ry^2+6\int_{t-r}^{t}y^2(s)ds\,.$$

By gathering all discussions above into (18), we have

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le-(16-(\rho+21)r)y^2-\left(2-\left(\gamma+\frac{7}{2}\right)r\right)z^2$$

$$-(\rho-7)\int_{t-r}^{t}y^2(s)ds-\left(\gamma-\frac{35}{2}\right)\int_{t-r}^{t}z^2(s)ds$$

$$+\frac{z+6y}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}\,. \qquad \mathbf{(19)}$$

If we choose $\rho=7$ and $\gamma=\frac{35}{2}$, then the equality (18) implies that

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le-(16-28r)y^2-(2-21r)z^2$$

$$+\frac{|z|+6|y|}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}.$$

Now, ones can conclude that

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le-\alpha y^2-\sigma z^2+\frac{7+6y^2+z^2}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}$$

for some positive constants α and σ provided that $r<\frac{2}{21}$.

Hence, we can write

$$\frac{d}{dt}V_1(x_t,y_t,z_t)\le\frac{7+6y^2+z^2}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}$$

$$\le\frac{7+6y^2+z^2}{1+t^2}=\frac{7}{1+t^2}+\frac{6y^2+z^2}{1+t^2}$$

$$\le\frac{7}{1+t^2}+\frac{6(y^2+z^2)}{1+t^2}$$

$$\le\frac{7}{1+t^2}+\frac{6}{(1+t^2)D_6}V_1(x_t,y_t,z_t). \qquad \textbf{(20)}$$

Now, integrating (20) from 0 to t, using the fact $\frac{1}{1+t^2}\in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, it can be easily obtained the boundedness of all solutions of equation (14).

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SOME GENERALIZED CLASSES OF DIFFERENCE SEQUENCES OF FUZZY NUMBERS DEFINED BY A MODULUS FUNCTION

AYHAN ESI AND MEHMET AÇIKGÖZ

Abstract. The purpose of this paper is to introduce the concepts of lacunary strong convergence of generalized difference sequences of fuzzy numbers with respect to a modulus function. We give various properties and inclusion relations on these classes of sequences of fuzzy numbers.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [14] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy ordering, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [8] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [3], Nanda [11], Esi [4], Altin et. al [1], Mursaleen and Başarir [9] and many others.

Let D denote the set of all closed bounded intervals $A=[\underline{A},\overline{A}]$ on the real line R. For $A,B\in D$ define $A\leq B$ if and only if $\underline{A}\leq\underline{B}$ and $\overline{A}\leq\overline{B}$,

$$\overline{d}(A,B)=\max\left\{|\underline{A}-\underline{B}|,\left|\overline{A}-\overline{B}\right|\right\}.$$

It is easy to see that $\overline{d}$ defines a metric on D and $\left(D,\overline{d}\right)$ is complete metric space. Also $\leq$ is a partial order on D. A fuzzy number is a fuzzy subset of real line R which is bounded, convex and normal. Let $L\left(R\right)$ denote the set of all fuzzy numbers those are upper semicontinuous and have compact support. In other words, if X $\in$ then for any $\alpha\in[0,1]$, X^{α} is compact, where $X^{\alpha}\left(t\right)=\left\{t\in R:X\left(t\right)\geq\alpha\right\}$ for $\alpha>0$. The $\overline{0}-$cut is defined as the closure of the strong $\overline{0}-$cut, i.e. closure $\left\{t\in R:X\left(t\right)>\alpha\right\}$.

For each $0<\alpha\leq 1$, the α-level set X^{α} is a non-empty compact subset of R. The linear structure of $L\left(R\right)$ induces addition $X+Y$ and scalar multiplication λX, $X\in R$, in terms of α-level sets defined by

$$[X+Y]^{\alpha}=[X]^{\alpha}+[Y]^{\alpha}\quad\text{and}\quad[\lambda X]^{\alpha}=\lambda[X]^{\alpha}$$

for each $0\leq\alpha\leq 1$.

Define $d:L\left(R\right)\text{x}L\left(R\right)\rightarrow R$ by $d\left(X,Y\right)=\sup\limits_{0\leq\alpha\leq 1}\overline{d}\left(X^{\alpha},Y^{\alpha}\right)$ for $X,Y\in L\left(R\right)$. Define $X\leq Y$ if and only if $X^{\alpha}\leq Y^{\alpha}$ for any $\alpha\in[0,1]$. It is known that $L\left(R\right)$ is a complete metric space with the metric d (see for instance [8]).

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A metric on $L(R)$ is said to be a translation invariant if $d(X+Z, Y+Z) = d(X, Y)$ for $X, Y, Z \in L(R)$.

The metric d has the following properties:

$$d(cX, cY) = |c|\, d(X, Y) \tag{1.1}$$

for $c \in R$ and

$$d(X+Y, Z+W) \leq d(X, Z) + d(Y, W) \tag{1.2}$$

A sequence of fuzzy numbers is a function X from the set N of natural numbers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ [8]. We denote by $w(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded [8]. We denote by $l_\infty(F)$ the set of all bounded sequences $X = (X_k)$ of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there is a positive integer n_0 such that $d(X_k, X_0) < \varepsilon$ for $k > n_0$ [2]. We denote by $c(F)$ the set of all convergent sequences $X = (X_k)$ of fuzzy numbers.

It is straightforward to see that $c(F) \subset l_\infty(F) \subset w(F)$. For further studies, one may refer to [3] and [14].

In [11], it is shown that $c(F)$ and $l(\infty)$ are complete metric spaces.

By a lacunary sequence $\Theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ the intervals determined by Θ and $q_r = \frac{k_r}{k_{r-1}}$ for $r = 0, 1, 2, \ldots$. Lacunary sequences have been discussed in [2,5,6].

The notion of modulus function was introduced by Nakano [10]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$.

(ii) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at zero.

Since $|f(x) - f(y)| \leq f(x-y)$, it follows from (iv) f is continuous on $[0, \infty)$. Furthermore, we have $f(nx) \leq nf(x)$ for all $n \in N$ from condition (ii). A modulus function may be bounded or unbounded. This concept have been studied by Ruckle [13], Maddox [7], Pehlivan and Fisher [12] and many others.

In the present note we introduce and examine the concepts of lacunary strong convergence of generalized difference sequences of fuzzy numbers with respect to a modulus function.

Lemma 1. *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$ we have $f(x) \leq 2f(1)\,\delta^{-1}x$.*

Let $w(F)$ be the set of all sequences of fuzzy numbers. Let $r \in N$ be fixed, then the operation

$$\Delta^r : w(F) \to w(F)$$

is defined by

$$\Delta X_k = X_k - X_{k+1} \;\text{ and }\; \Delta^r X_k = \Delta\left(\Delta^{r-1} X_k\right) \;\; (r \geq 2)$$

for all $k \in N$. The generalized difference has the following binomial representation:

$$\Delta^r X_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} X_{k+v}, \text{ for all } k \in N.$$

Definition 1. *Let $\Theta = (k_r)$ be a lacunary sequence and f be a modulus function. We define the following classes of sequences of fuzzy numbers as follows:*

$$N_\Theta^0 (\Delta^m, F, f) = \left\{ X = (X_k) \in w(F) : \lim_{r\to\infty} h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, \overline{0})] = 0 \right\},$$

$$N_\Theta (\Delta^m, F, f) = \left\{ X = (X_k) \in w(F) : \sup_{r\to\infty} h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, X_0)] = 0 \right\}$$

and

$$N_\Theta^\infty (\Delta^m, F, f) = \left\{ X = (X_k) \in w(F) : \sup_r h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, \overline{0})] < \infty \right\}.$$

If $X = (X_k) \in N_\Theta (\Delta^m, F, f)$, then the sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary strongly Δ^m-convergent to the fuzzy number X_0 with respect to modulus function f.

If we take $f(x) = x$, we obtain the classes of sequences of fuzzy numbers $N_\Theta^0 (\Delta^m, F)$, $N_\Theta (\Delta^m, F)$ and $N_\Theta^\infty (\Delta^m, F)$ from the above classes of sequences of fuzzy numbers, respectively.

2. Main Results

In this section we state and prove the results of this paper.

Theorem 1. $N_\Theta^0 (\Delta^m, F, f)$, $N_\Theta (\Delta^m, F, f)$, *and* $N_\Theta^\infty (\Delta^m, F, f)$ *are closed under the operations of addition and scalar multiplication.*

Proof. We shall prove only $N_\Theta^0 (\Delta^m, F, f)$. The others can be treaated similarly. Let $X = (X_k)$, $Y = (Y_k) \in N_\Theta^0 (\Delta^m, F, f)$ and $\alpha, \beta \in R$.Then there exist positive K_α and K_β such that $|\alpha| \le K_\alpha$ and $|\beta| \le K_\beta$. From the definition of modulus f and by taking into account the properties (1.1) and (1.2) of the metric d, we have

$$\begin{aligned} h_r^{-1} \sum_{k \in I_r} f[d(\alpha\Delta^m X_k + \beta\Delta^m Y_k, \overline{0})] &\le K_\alpha h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, \overline{0})] \\ + K_\beta h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m Y_k, \overline{0})] &\to 0 \text{ as } r \to \infty. \end{aligned}$$

Hence $\alpha X + \beta Y \in N_\Theta^0 (\Delta^m, F, f)$. □

Theorem 2. *Let f be a modulus. Then* $N_\Theta (\Delta^m, F) \subset N_\Theta (\Delta^m, F, f)$.

Proof. Let $X = (X_k) \in N_\Theta (\Delta^m, F)$. Then we have

$$A_r = h_r^{-1} \sum_{k \in I_r} d(\Delta^m X_k, \overline{0}) \to 0 \text{ as } r \to \infty.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every t with $0 \leq t \leq \delta$. Then we can write

$$\begin{aligned} h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, X_0)] &= h_r^{-1} \sum_{k \in I_r, d(\Delta^m X_k, X_0) \leq \delta} f[d(\Delta^m X_k, X_0)] \\ &\quad + h_r^{-1} \sum_{k \in I_r, d(\Delta^m X_k, X_0) > \delta} f[d(\Delta^m Y_k, X_0)] \\ &\leq h_r^{-1} h_r \varepsilon + h_r^{-1} 2 f(1) \delta^{-1} h_r A_r \end{aligned}$$

from Lemma. Therefore $X = (X_k) \in N_\Theta(\Delta^m, F, f)$. □

Theorem 3. *Let f be a modulus. If $\lim_{t \to \infty} \frac{f(t)}{t} = \beta > 0$, then $N_\Theta(\Delta^m, F) = N_\Theta(\Delta^m, F, f)$.*

Proof. By Theorem 2, we need only to show that $N_\Theta(\Delta^m, F, f) \subset N_\Theta(\Delta^m, F)$. Let $\beta > 0$ and $X = (X_k) \in N_\Theta(\Delta^m, F, f)$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t \geq 0$. Hence we have

$$h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, X_0)] \geq \beta h_r^{-1} \sum_{k \in I_r} d(\Delta^m X_k, X_0).$$

Therefore we have $X = (X_k) \in N_\Theta(\Delta^m, F)$. □

Theorem 4. *Let $m \geq 1$ be a fixed integer and f be a modulus, then*

$$N_\Theta^0\left(\Delta^{m-1}, F, f\right) \subset N_\Theta^0(\Delta^m, F, f), \quad N_\Theta\left(\Delta^{m-1}, F, f\right) \subset N_\Theta(\Delta^m, F, f)$$

and

$$N_\Theta^\infty\left(\Delta^{m-1}, F, f\right) \subset N_\Theta^\infty(\Delta^m, F, f).$$

Proof. The proof of the inclusions follow from the following inequality

$$h_r^{-1} \sum_{k \in I_r} f[d(\Delta^m X_k, X_0)] \leq h_r^{-1} \sum_{k \in I_r} f[d\left(\Delta^{m-1} X_k, X_0\right)] + h_r^{-1} \sum_{k \in I_r} f[d\left(\Delta^{m-1} X_{k+1}, X_0\right)]$$

since $\Delta^m X_k = \Delta^{m-1} X_k - \Delta^{m-1} X_{k+1}$, properties of modulus f and by taking into account the property (1.2). □

Theorem 5. *Let $\Theta = (k_r)$ be a lacunary sequence and f be a modulus. If $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$, then $|\sigma_1|(\Delta^m, F, f)$, where*

$$|\sigma_1|(\Delta^m, F, f) = \left\{ X = (X_k) \in w(F) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f[d(\Delta^m X_k, X_0)] = 0 \right\}.$$

Proof. Suppose that $1 < \liminf_r q_r$. Then there exists a $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$ for sufficiently larger r. Since $h_r = k_r - k_{r-1}$, we have $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. Let $X = (X_k) \in |\sigma_1|(\Delta^m, F, f)$. We may write

$$\begin{aligned} h_r^{-1} \sum_{i \in I_r} f[d(\Delta^m X_i, X_0)] &= h_r^{-1} \sum_{i=1}^{k_r} f[d(\Delta^m X_i, X_0)] - h_r^{-1} \sum_{i=1}^{k_{r-1}} f[d(\Delta^m X_i, X_0)] \\ &= \frac{k_r}{h_r} \left(k_r^{-1} \sum_{i=1}^{k_r} f[d(\Delta^m X_i, X_0)] \right) - \frac{k_{r-1}}{h_r} \left(k_{r-1}^{-1} \sum_{i=1}^{k_{r-1}} f[d(\Delta^m X_i, X_0)] \right) \end{aligned}$$

Hence, we obtain $|\sigma_1|(\Delta^m, F, f) \subset N_\Theta(\Delta^m, F, f)$. Now suppose that $\limsup_r q_r < \infty$ and let $\varepsilon > 0$ be given. Then, there exists a constant $T > 0$ such that $q_r < T$ for all $r \in N$. Suppose that let $X = (X_k) \in N_\Theta(\Delta^m, F, f)$. Then there exists i_0 such that for every $i \geq i_0$

$$A_i = h_i^{-1} \sum_{j \in I_i} f[d(\Delta^m X_j, X_0)] < \varepsilon.$$

We can also choose a number $M > 0$ such that $A_i \leq M$ for all $i \in N$. Now let n be any integer with $k_{r-1} < n < k_r$. Then

$$\begin{aligned}
n^{-1} \sum_{j=1}^{n} f[d(\Delta^m X_j, X_0)] &\leq k_{r-1}^{-1} \sum_{j=1}^{k_r} f[d(\Delta^m X_j, X_0)] \\
&= k_{r-1}^{-1} \left\{ \sum_{j \in I_1}^{k_r} f[d(\Delta^m X_j, X_0)] + \sum_{j \in I_2}^{k_r} f[d(\Delta^m X_j, X_0)] + ... + \sum_{j \in I_r}^{k_r} f[d(\Delta^m X_j, X_0)] \right\} \\
&= k_{r-1}^{-1} \left\{ \sum_{i=1}^{i_0} \sum_{j \in I_i} f[d(\Delta^m X_j, X_0)] + \sum_{i=i_0+1}^{r} \sum_{j \in I_i} f[d(\Delta^m X_j, X_0)] \right\} \\
&\leq k_{r-1}^{-1} \left\{ \sum_{i=1}^{i_0} \sum_{j \in I_i} f[d(\Delta^m X_j, X_0)] + \varepsilon(k_r - k_{i_0}) \right\} \\
&= k_{r-1}^{-1} \{ h_1 A_1 + h_2 A_2 + ... + h_{i_0} A_{i_0} + \varepsilon(k_r - k_{i_0}) \} \\
&\leq k_{r-1}^{-1} \left\{ \sup_{1 \leq j \leq i_0} A_j \right\} + \varepsilon(k_r - k_{i_0}) k_{r-1}^{-1} < M k_{r-1}^{-1} k_{i_0} + \varepsilon T
\end{aligned}$$

since $k_{r-1} \to \infty$ as $n \to \infty$, it follows that $X = (X_k) \in |\sigma_1|(\Delta^m, F, f)$. This completes the proof. $\square$

3. Conclusion

Giving particular values to the sequence $\Theta = (k_r)$, modulus function f and $m \in N$, we obtain some classes of sequences of fuzzy numbers which are special cases of the classes of sequences that we have defined. The most of the results proved in the previous section will be true for these classes.

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Optimal Design of Artificial Blending Phosphorus Ore[1]

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Abstract. Through an orthogonal experiment, the effect of adding MgO, Al_2O_3, $Fe_2(SO_4)_3$, SiO_2 and CaO to Jinhe phosphorus ore decomposition is studied. A class of response curved surface mathematical models, which concern the rate of phosphorus ore decomposition and the receiving rate of phosphoric anhydride with each orthogonal factor are also established by using the software Statistics Package for Social Science (in short, SPSS). Furthermore, the optimal components of the phosphorus ore are calculated by the mathematical software Matlab. The results presented in this paper can provide a theoretical foundation for artificial blending rock and the exploitation of phosphorus ore with middle or low grade in producing wet-process phosphoric acid.

Key words and phrases: Orthogonal experiment design, response curved surface mathematical model, mathematical softs SPSS and Matlab, optimal components of artificial blending phosphorus ore, wet-process phosphoric acid.

2000 Mathematics Subject Classification: 62K20, 94C30, 65D17

1 Introduction

Recently, the exploitation and applications of phosphorus ore with middle or low grade have become very fashionable and important for studying the utilization ratio of resource, the quality of ultimate production and the economy benefit of blending rock. See, for example, [1-8] and the references therein.

In [9], Lu studied the problem of rational ore matching for enhancing the utilization ratio of resources, stabilifying the quality of ultimate production and improving economy benefit. But the author did not let us know how to select the proportion to blend rock. Very recently, Cai [2] introduced some laboratory tests for 10 kinds of imported iron ore and obtained the research results of classifying them into three categories. Through the single factor experiments, Cui [3, 4] studied the effects of adding potassium sulfate, sodium sulfate, copper sulfate and zinc sulfate in phosphorus ore on decomposition rate of phosphorus ore, recovery ratio of phosphoric acid and the crystallization of calcium sulfate. Furthermore, we observed the crystallizations of calcium sulfate by electron micrograph and analyzed the mechanism of effect.

On the other hand, based on the chemistry expert systems, Pan et al. [12] constructed the basic theory systems of intelligent mathematics. Further, the authors pointed that "the

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intelligent mathematics can provide some necessary mathematical methods and models for the expression, consequence and getting of knowledge, and can offer some golden rule for the design of expert systems". By using the response curved surface methodology and technique, Sun et al. [13] and Xu et al. [15] investigated the main effects and interactions of individual factors to each evaluating indicator, and established the response curved surface regression equations and the significant results of the regression coefficient. Further, Xu et al. [15] pointed out "the response curved regression is chosen for making up for the deficiency of the orthogonal experiments in the research". For more details of mathematical modelling and experiment design methods, the readers can refer to [7, 8, 11, 14] and the references therein.

Motivated and inspired by the above works, in this paper, based on an orthogonal experiment, the effect of adding salts and oxides to Jinhe phosphorus ore decomposition is studied. A class of response curved surface mathematical models, which concern the rate of phosphorus ore decomposition and the receiving rate of phosphoric anhydride with each orthogonal factor are also established by using SPSS. Moreover, the optimal components of the phosphorus ore are calculated by Matlab. Theory basis is provided for artificial blending rock, and the approach can be applied to exploit the phosphorus ore with middle or low grade in producing wet-process phosphoric acid.

2 Single Factor Experiments

In this section, based on the optimal reaction conditions for the single factor experiments in [5], the effect of adding salts and oxides of different configuration and different consistence on phosphorus ore shall be studied in the production of wet-process phosphoric acid. The mathematical models of influence of relevant factors, which concern the ratio of phosphorus ore decomposition (DR) and the receiving ratio of phosphoric anhydride (RR) will also be obtained.

2.1 Material and Reagents

Jinhe phosphorus ore (purchased from Sichuan Shifang Chemical Industry Co., Ltd.) was selected as material, which is high grade. Its chemical components are shown in Table 1. Sulfuric acid is industrial vitriol (98%). Magnesia, aluminium oxide, ferric oxide and cupric

Table 1: Chemical components of the phosphorus ore

Components	P_2O_5	CaO	MgO	Fe_2O_3	Al_2O_3	F	acid-insoluble substance
Content (%)	32.73	45.59	0.69	0.72	0.22	2.61	7.20

sulfate are AR, and ferric sulfate is CP.

2.2 Results for Single Factors Problems

Throughout this paper, we always suppose that x_1, x_2, x_3 and x_4 denote magnesia, alumina oxide, ferric oxide and ferric sulfate, respectively. For $i = 1, 2, 3, 4$, let y_{i1} and y_{i2} be the DR and RR with respect to x_i, respectively and let R_{i1}^2 and R_{i2}^2 be the square of goodness of fit in the models of the DR and RR with respect to x_i, respectively.

To jinhe phosphorus ore, in producing wet-process phosphoric acid, by the figures in [3, 4] and the linear (nonlinear) regressions of each evaluating indicator against individual factors treated from SPSS (13.0 for windows), the single-factor experiment shows as follows:

(1) The effect of artificial adding magnesia to phosphorus ore decomposition is great. See, Fig. 1.

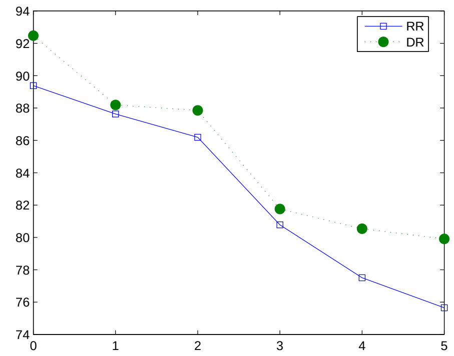


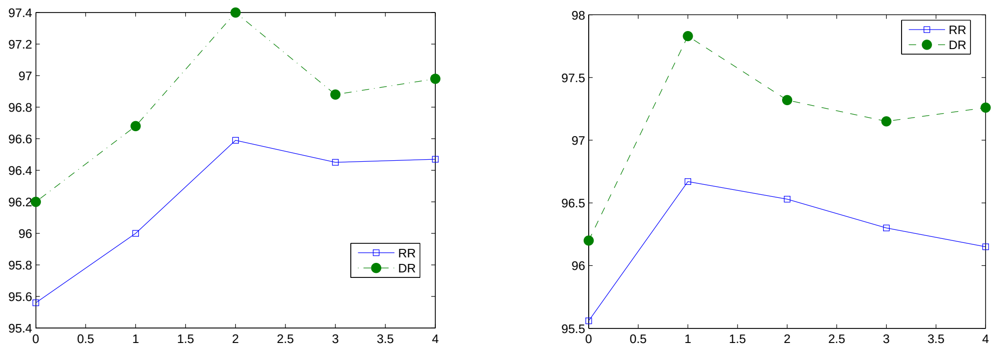
Figure 1: The effect of magnesia content.

The DR and RR are of negative linear with respective to the content of magnesia, it is because

$$y_{11} = 90.32 - 2.98x_1, \quad R_{11}^2 = 0.966, \quad y_{12} = 91.68 - 2.62x_1, \quad R_{12}^2 = 0.925.$$

Hence, magnesia is harmful.

(2) The effect of adding alumina and ferric oxide is dissimilar to that of magnesia (see, Fig. 2).



(a) alumina (b) ferric oxide

Figure 2: The effect of alumina and ferric oxide content.

The proper alumina and ferric oxide are favorable to phosphorus ore decomposition. Indeed,

$$y_{21} = 95.54 + 0.68x_2 - 0.11x_2^2, \quad R_{21}^2 = 0.939,$$
$$y_{22} = 96.23 + 0.52x_2 - 0.09x_2^2, \quad R_{22}^2 = 0.938$$

and

$$y_{31} = 95.57 + 1.78x_3 - 0.85x_3^2 + 0.11x_3^3, \quad R_{31}^2 = 0.981,$$
$$y_{32} = 96.24 + 2.65x_3 - 1.40x_3^2 + 0.20x_3^3, \quad R_{32}^2 = 0.934.$$

(3) The effect of adding ferric sulfate to phosphorus ore decomposition is prominent. See, Fig. 3.

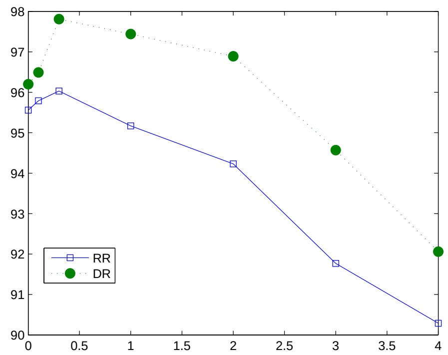


Figure 3: The effect of ferric sulfate content.

In fact,

$$y_{41} = 95.68 + 0.74x_4 - 1.08x_4^2 + 0.14x_4^3, \quad R_{41}^2 = 0.991,$$
$$y_{42} = 96.41 + 2.92x_4 - 1.72x_4^2 + 0.18x_4^3, \quad R_{42}^2 = 0.972.$$

On the other hand, the scatter graphs show when the content of ferric sulfate is less than 0.3%, the DR and RR are of positive linear with respect to the content of ferric sulfate. Otherwise, they will become negative linear.

The single factor experiment results can provide theoretical foundation for studying phosphorus ore components by orthogonal experiment design and can guide the exploitation of phosphorus ore with middle or low grade.

3 Optimal Experiment Design

Based on an orthogonal experiments with the same material and reagents as in above single factors experiment, we shall find the optimal components of Jinhe phosphorus ore by adding oxides in producing wet-process phosphoric acid, and obtain the function of concerning the DR and RR with each orthogonal factor by using SPSS.

3.1 Orthogonal Experiment Design

The DR and RR were chosen as two main indicators to evaluate the capability of the artificial blending phosphorus ore. According to earlier single factors experiments results, MgO (A), Al_2O_3 (B), $Fe_2(SO_4)_3$ (C), SiO_2 (D) and CaO (E), which have the significant influence on the two indicators, were served as five factors and each factor was contained three levels. Therefore, $L_{27}(3^{13})$ orthogonal table was selected for the multi-factors experiment. The arrangements of each experiment and the design of the orthogonal table head are displayed in Tables 2 and 3, respectively.

Table 2: Factors and levels of orthogonal experiment

	Factors ($wt\%$)				
Levels	A	B	C	D	E
1	0	1	0	0	0
2	1	3	5	10	4
3	3	5	10	20	8

Table 3: Design of the table head

Factors	A	B	empty	empty	C	empty	empty
Column No.	1	2	3	4	5	6	7
Factors	empty	empty	D	empty	E	empty	
Column No.	8	9	10	11	12	13	

3.2 Experimental Approach

With Jinhe phosphorus ore as the material, the mixture with water according to the proportion 1:3.5 and the adding oxides according to orthogonal table was put in a reactor. The loaded reactor was put into constant temperature water trough and controlled the temperature at $70°C$, which fluctuant range is $0.5°C$. Then, the measured sulfuric acid was appended continuously and whisking intension was controlled at 200rpm by electromotive beater. Next, the liquid and solid were separated by using vacuum filtration when the reaction time is 3 hours. Finally, the content of the phosphoric anhydride was analyzed, and the Evaluating indicators (in short, EI), i.e., the ratio of phosphorus ore decomposition and the receiving ratio of phosphoric anhydride were computed.

In this process, the concentration of the sulfuric acid is 30% and its actual dosage is 100%-105% of the theoretic dosage in producing wet-process phosphoric acid. Further, the content of the phosphoric anhydride was determined by using gravimetric quinolinium molybdophosphate method.

3.3 Experimental Results

The experiment arrangements and results are listed in Table 4. These values were evaluated using SPSS. Two different statistical methods were carried out in this study. A direct observation was carried out to compare the different effect of each factor to artificial adding oxides for the phosphorus ore. Compared with the above traditional data analysis method, the multiple regressions was conducted to optimize the regression model for each evaluating indicator in addition to picking out effective factors. Eventually we used both direct observation and multiple regressions method to predict process parameters separately and some experiments were carried out to compare the two predicted results.

3.3.1 Direct Observation

The statistical parameter I_i, II_i, III_i and R_i were calculated following the procedures described by Yang [16], where I_i, II_i and III_i represent the sum of DR or RR at level I, II and III, respectively, i represents the factors A, B, C, D and E, and R_i represents the range of DR or RR between level I, II and III. The results (see Table 5) reveal that the effect of all the factors to DR was in the order of E>A>B>C>D. As for RR, effect of all the factors was in the order of E>B>A>D>C.

Table 4: Results of orthogonal experiment

	Factors ($wt\%$)					EI (%)			Factors ($wt\%$)					EI (%)	
No.	A	B	C	D	E	DR	RR	No.	A	B	C	D	E	DR	RR
1	1	1	1	1	1	97.46	97.08	15	2	2	3	3	3	94.14	94.01
2	1	1	2	2	2	95.74	94.96	16	2	3	1	2	3	97.73	96.21
3	1	1	3	3	3	92.79	90.52	17	2	3	2	3	1	96.45	97.89
4	1	2	1	2	3	96.67	96.65	18	2	3	3	1	2	93.48	94.69
5	1	2	2	3	1	96.61	96.86	19	3	1	1	2	3	82.82	82.23
6	1	2	3	1	2	91.45	94.55	20	3	1	2	3	1	93.83	95.27
7	1	3	1	3	2	96.89	94.89	21	3	1	3	1	2	91.99	93.98
8	1	3	2	1	3	94.45	93.82	22	3	2	1	3	2	91.03	90.22
9	1	3	3	2	1	94.04	94.92	23	3	2	2	1	3	89.65	92.41
10	2	1	1	3	2	97.81	97.16	24	3	2	3	2	1	96.31	96.45
11	2	1	2	1	3	86.89	83.61	25	3	3	1	1	1	96.61	95.42
12	2	1	3	2	1	95.34	94.70	26	3	3	2	2	2	95.76	96.22
13	2	2	1	1	1	97.33	96.37	27	3	3	3	3	3	91.55	92.71
14	2	2	2	2	2	95.98	97.08								

3.3.2 Response Surface Methodology

Mathematical model for response surface of each evaluating indicator against m individual factors were treated as follows:

$$Y = b_0 + \sum_{j=1}^{m} b_j x_j + \sum_{1=i<j}^{m} b_{ij} x_i x_j + \sum_{j=1}^{m} b_{jj} x_j^2.$$

By using enter method and the soft SPSS, multiple regressions of each evaluating indicator against individual factors and some possible interactions were treated. The significant ($P < 0.1$) regressions are illustrated in Table 6 and the significance analysis of each evaluating indicator against individual factors is listed in Table 7, where R^2 is the square of goodness of fit and sig. denotes the level of significance.

3.3.3 Optimization Confirmation Experiments

It follows from Table 7 that the response curved surface equations are as follows:

$$\begin{aligned}
DR \quad &= -73.74A + 5.16A*C + 2.38A*D + 5.99A*E + 62.11B - 4.04B*C \\
&\quad -2.07B*D - 4.86B*E + 0.36C*C - 0.21C*D + 0.97C*E \\
&\quad +9.62D - 0.22D*D - 0.26D*E + 0.52E*E, \\
RR \quad &= -73.15A + 5.12A*C + 2.35A*D + 6.05A*E + 61.55B - 3.98B*C \\
&\quad -2.07B*D - 4.77B*E + 0.36C*C - 0.21C*D + 0.98C*E \\
&\quad +9.65D - 0.21D*D - 0.27D*E + 0.48E*E.
\end{aligned}$$

Hence, the optimal adding contents, i.e. the stationary points based on the above regression models were calculated by the Matlab Solver, which is incorporated into Matlab (version 7.0.1). The optimal parameters and levels for DR and RR through direct observation and multiple regressions are displayed in Table 8.

Further, the confirmation experiments were conducted based on the predicted optimum levels of the enter regressions and the results for DR and RR are 94.87 and 95.77, respectively.

Table 5: Direct observation results on DR and RR

	EI	Factors A	B	C	D	E
	I_i	856.1	834.67	854.35	839.31	863.98
	II_i	855.15	849.17	845.36	850.39	850.13
DR	III_i	829.55	856.96	841.09	851.1	826.69
	R_i	26.55	22.29	13.26	11.79	37.29
	rather excellent level	A_1	B_3	C_1	D_3	E_1
	effect of all the factors	E>A>B>C>D				
	I_i	853.25	829.51	845.23	841.93	864.96
	II_i	851.72	853.6	848.12	848.42	853.75
RR	III_i	834.91	856.77	846.53	849.53	821.17
	R_i	18.34	27.26	2.89	7.6	43.79
	rather excellent level	A_1	B_3	C_2	D_3	E_1
	effect of all the factors	E>B>A>D>C				

Table 6: SPSS output for regression models

EI	R^2	F value for model	$P < 0.1$
DR	0.989	69.919	0.000
RR	0.988	68.409	0.000

On the other hand, through direct observation, two indexes were synthetically considered and the more excellent technical adding oxides scheme $A_1B_3C_2D_3E_1$ was selected. Furthermore, the validatory experiment based on the obtained adding oxides scheme were done and the results for the direct observation are as follows: DR and RR are 94.57 and 95.23, respectively.

Therefore, the experimental results show that the phosphorus ore with high grade becomes middle or low grade by adding salts and oxides in producing wet-process phosphoric acid, the results of the regression experiments is better than traditional experiments on the optimum adding content.

3.4 Discussion

It is well known that orthogonal experiments are especially useful in statistical analysis of multiple factors problems (see, for example, [16]). However, this method is able to keep error acceptable to a certain extent. If the error caused by linearization becomes unacceptable, it is worth-while to add the quadratic of the Taylor expansion into the linear equation. Therefore, the approximate equation of the nonlinear relationship has a form of quadratic polynomial of the independent variable. For example, for a five-factor issue, a regression model applied to both linear and nonlinear model is:

$$Y = b_0 + \sum_{i=1}^{5} b_i x_i + \sum_{1=j<k}^{5} b_{jk} x_j x_k + \sum_{n=1}^{5} b_{nn} x_n^2.$$

But, this leads to much larger scale of data to be collected and parameters to be regressed. From the function above, the regression coefficients to be calculated are 21 in total. In fact, some items in function above are not significant in F test as some values are not

Table 7: Test for significance of regression coefficient on DR and RR

	Term	Unstandardized Coefficients β	Unstandardized Coefficients Std. Error	Standardized Coefficients γ	T value	sig.
DR	A	−73.744	14.737	−1.430	−5.004	0.000
	B	62.113	8.209	2.253	7.567	0.000
	D	9.619	1.940	1.319	4.959	0.000
	A∗C	5.156	1.084	0.645	4.755	0.000
	A∗D	2.382	0.542	0.596	4.393	0.001
	A∗E	5.991	1.356	0.600	4.420	0.001
	B∗C	−4.044	0.710	−0.947	−5.698	0.000
	B∗D	−2.069	0.355	−0.969	−5.829	0.000
	B∗E	−4.861	0.887	−0.911	−5.479	0.000
	C∗C	0.362	0.166	0.229	2.182	0.050
	C∗D	−0.208	0.104	−0.184	−2.010	0.068
	C∗E	0.970	0.340	0.343	2.850	0.015
	D∗D	−0.215	0.089	−0.544	−2.428	0.032
	D∗E	−0.257	0.130	−0.182	−1.987	0.0702
	E∗E	0.522	0.259	0.211	2.018	0.066
RR	A	−73.148	14.892	−1.419	−4.912	0.000
	B	61.553	8.296	2.233	7.420	0.000
	D	9.649	1.960	1.323	4.922	0.000
	A∗C	5.119	1.096	0.641	4.671	0.001
	A∗D	2.352	0.548	0.589	4.292	0.001
	A∗E	6.049	1.370	0.606	4.416	0.001
	B∗C	−3.980	0.717	−0.932	−5.548	0.000
	B∗D	−2.067	0.359	−0.968	−5.763	0.000
	B∗E	−4.771	0.897	−0.894	−5.322	0.000
	C∗C	0.362	0.167	0.229	2.162	0.051
	C∗D	−0.209	0.105	−0.185	−1.991	0.070
	C∗E	0.975	0.344	0.345	2.833	0.015
	D∗D	−0.213	0.090	−0.539	−2.381	0.035
	D∗E	−0.271	0.131	−0.192	−2.068	0.061
	E∗E	0.482	0.262	0.195	1.842	0.090

Table 8: Optimal adding contents

Factors	Adding contents			
	DR		RR	
	Direct observation	Enter regression	Direct observation	Enter regression
A	0	0.4214	0	1.0345
B	5	0.3928	5	1.2729
C	0	3.6181	5	5.0237
D	20	20.4207	20	19.5741
E	0	0.548	0	0.2303

significant in F test in analysis of variance (in short, ANOVA) and in t test as some values are not significant in t test in regression coefficients table. If only F test significant items are not in the regression function, it could be concise. But it involves independent variables filtering issue. While in resolving multiple regressions problems, not all the independent variables have exactly the same effect on dependent variable. Some of them are significant while others can be ignored. Significant independent variables should be filtered out through certain approach. Methods such as forwards regressions, backwards regressions, stepwise regressions, enter regressions and optimization subclass regressions are commonly used. In this paper, by using F test to model and t test to regression coefficients, the enter multiple regressions are carried out with considering individual factors, their square and interaction items.

4 Conclusions

In this paper, based on the optimal reaction conditions of [5] and the single factor experiments in producing wet-process phosphoric acid, the effect of adding salts and oxides of different configuration and different consistence on phosphorus ore reaction is first studied and the corresponding mathematical models for single factors problems on the ratio of phosphorus ore decomposition (DR) and the receiving ratio of phosphoric anhydride (RR) are obtained by using the mathematical soft SPSS (13.0 for windows).

Secondly, based on the orthogonal experiments design and response curved surface technique, the optimal components of Jinhe phosphorus ore in producing wet-process phosphoric acid are obtained by adding oxides. The response curved surface mathematical models, which concern the DR and RR with each orthogonal factor are also established by SPSS as follows:

$$
\begin{aligned}
DR \quad &= -73.74A + 5.16A * C + 2.38A * D + 5.99A * E + 62.11B - 4.04B * C \\
&\quad -2.07B * D - 4.86B * E + 0.36C * C - 0.21C * D + 0.97C * E \\
&\quad +9.62D - 0.22D * D - 0.26D * E + 0.52E * E, \\
RR \quad &= -73.15A + 5.12A * C + 2.35A * D + 6.05A * E + 61.55B - 3.98B * C \\
&\quad -2.07B * D - 4.77B * E + 0.36C * C - 0.21C * D + 0.98C * E \\
&\quad +9.65D - 0.21D * D - 0.27D * E + 0.48E * E.
\end{aligned}
$$

Further, the optimal adding contents, i.e. the stationary points based on the above regression models are calculated by using the mathematical soft Matlab (version 7.0.1). The optimal parameters and levels for DR and RR through direct observation and multiple regressions are compared.

Finally, the validation experiments indicate that the optimized conditions by enter multiple regressions are better than by direct observation analysis, and the components of Jinhe phosphorus ore will be improved under the optimized artificial blending rock conditions. Experiment results suggest that multiple regressions can avoid the weakness of direct observation. The results presented in this paper provide a theoretical foundation for artificial blending rock and the exploitation of phosphorus ore with middle or low grade in producing wet-process phosphoric acid.

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STRONG CONVERGENCE OF MODIFIED MANN ITERATION FOR FINITE NONEXPANSIVE MAPPINGS

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Abstract. Strong convergence theorems on modified Mann iteration for finite nonexpansive mappings are established in Banach spaces. The main theorems generalize the corresponding result of Kim and Xu [12] to the viscosity approximation method for finite nonexpansive mappings in a reflexive Banach space having a weakly sequentially continuous duality mapping. Our results also improve the corresponding results of [8,9,10,21,22].

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Key words and phrases: Mann iteration; Viscosity approximation method; Nonexpansive mapping; Common fixed points; Contraction; Variational inequality; Weakly sequentially continuous duality mapping

1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x,\ y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

Now let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x,\ y \in C$), and $Fix(T)$ denote the set of fixed points of T, that is, $Fix(T) = \{x \in C : x = Tx\}$.

We consider the Mann iterative scheme for nonexpansive mapping: for T nonexpansive mapping and $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily.

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The Mann iterative scheme for nonlinear mappings has extensively been studied over the last forty years for constructions of fixed points of nonlinear mappings and of solutions of nonlinear operator equations involving nonexpansive mappings, monotone, accretive and pseudo-contractive operators and others. (see, e,g., [3,4,11,13]). In particular, the construction of fixed points of nonexpansive mappings by Mann iterative scheme is important and useful in the theory of nonexpansive mappings and its applications in a number of applied areas, for instance, in image recovery and signal processing (see e.q., [2,16,17]).

In 2003, Nakajo and Takahashi [15] proposed the following modification of the Mann iterative scheme (1.1) in a Hilbert space H:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where P_K denotes the nearest point (metric) projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then $\{x_n\}$ generated by (1.2) converges strongly to $P_{Fix(T)}(x_0)$. Their argument does not work outside the Hilbert space setting.

Recently, without any additional projection in the scheme, Kim and Xu [12] provided a simpler modification of Mann iterative scheme (1.1) in a uniformly smooth Banach space as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1). They proved that $\{x_n\}$ generated by (1.3) converges to a fixed point of T under the control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0;$

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (or, equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$), $\sum_{n=0}^{\infty} \beta_n = \infty;$

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty.$

On the other hand, as the viscosity approximation method, Moudafi [14] and Xu [20] considered the iterative scheme: for T a nonexpansive mapping, $f \in \Sigma_C$ and $\alpha_n \in (0,1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \ge 0. \tag{1.4}$$

Under the conditions (C1), (C2) and (C3) on $\{\alpha_n\}$, Xu [20] showed in a uniformly smooth Banach space that $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T, which solves certain variational inequality. The results of Xu [20] extended the results of Moudafi [14] to a Banach space setting. In 2006, Jung [9] considered the iterative scheme: for $N > 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $f \in \Sigma_C$ and $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_{n+1} x_n, \quad n \geq 0, \tag{1.5}$$

where $T_n := T_{n \bmod N}$, and extended results of Xu [20] (and Moudafi [14]) to the case of a family of finite nonexpansive mappings. In particular, under the conditions (C1), (C2) and the perturbed control condition on $\{\alpha_n\}$

$$\text{(C4)} \qquad |\alpha_{n+N} - \alpha_n| \leq \circ(\alpha_{n+N}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty,$$

he obtained the strong convergence of the sequence $\{x_n\}$ generated by (1.5) to a solution in $\bigcap_{i=1}^{N} Fix(T_i)$ of certain variational inequality in either a reflexive Banach space having a uniformly Gâteaux differentiable norm with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings or a reflexive Banach space having a weakly sequentially continuous duality mapping, and gave an example which satisfies the conditions (C1), (C2) and (C4) , but fails to satisfy the condition (C3) for $N > 1$; $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$.

Very recently, Yao et al. [21] proposed the following modified Mann iterative scheme: for T nonexpansive mapping, $f \in \Sigma_C$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$,

$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n. \end{cases}$$

By using Lemma 2 of Suzuki [19], they studied strong convergence of this iterative scheme in a uniformly smooth Banach space under the following conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$

$$\text{(C5)} \qquad \alpha_n \to 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

In this paper, motivated by [9,20,21], as a viscosity approximation method, we introduce modified Mann iterative scheme for finite nonexpansive mappings : for $N > 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $f \in \Sigma_C$ and $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$,

$$\text{(IS)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases}$$

and establish the strong convergence of the sequence $\{x_n\}$ generated by (IS) in a reflexive Banach space having a weakly sequentially continuous duality mapping under certain appropriate conditions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$ and the sequence $\{x_n\}$. Moreover, we show that this strong limit is a solution of certain variational inequality. The main results improve the recent result of Kim and Xu [12] to the viscosity approximation method for finite nonexpansive mappings. Our results also improve the corresponding results of [8,9,10,21,22].

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0,\infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}, \quad \text{for all } x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J, is referred to as the *normalized duality mapping*. The following property of duality mapping is well-known:

$$J_\varphi(\lambda x) = \operatorname{sign} \lambda\left(\frac{\varphi(|\lambda|\cdot\|x\|)}{\|x\|}\right)J(x) \quad \text{for all } x \in E\setminus\{0\}, \quad \lambda \in \mathbb{R}, \tag{2.2}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $J(-x) = -J(x)$ for all $x \in E$ ([5]).

Following Browder [1], we say that a Banach space E has a weakly sequential continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J_\varphi(x_n) \overset{*}{\rightharpoonup} J_\varphi(x)$.

For example, every l^p space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \text{for all } t \in \mathbb{R}^+.$$

Then it is known [1] that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. Thus it is easy to see that the normalized duality mapping $J(x)$ can also be defined as the subdifferential of the convex functional $\Phi(\|x\|) = \|x\|^2/2$, that is, for all $x \in E$

$$J(x) = \partial\Phi(\|x\|) = \{f \in E^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, f\rangle \ \text{ for all } y \in E\}.$$

It is well-known that if E is smooth, then the normalized duality mapping J is single-valued and norm to weak* continuous. Also, if E has a uniformly Gâteaux differentiable norm, the normalized duality mapping J is uniformly norm to weak* continuous on each bounded subsets of E ([5,6]).

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D. Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be a *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $t \geq 0$ and $x + t(x - Qx) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows [7. p.48]: If E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.3}$$

(Note that this fact still holds by (2.2) if the normalized duality mapping J is replaced by a general duality mapping J_φ with gauge function φ.)

We need the following lemmas for the proof of our main results, For these lemmas, we refer to [5,11,13].

Lemma 2.1. *Let E be a real Banach space and φ a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \text{for all } t \in \mathbb{R}^+.$$

Then the following inequality holds

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \quad \text{for all } x, y \in E,$$

where $j_\varphi(x+y) \in J_\varphi(x+y)$*. In particular, if* E *is smooth, then one has*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \text{for all } x, y \in E.$$

Lemma 2.2. *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \ge 0,$$

where $\{\lambda_n\}$*,* $\{\delta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions*:
(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$*;*
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\delta_n < \infty$*;*
(iii) $\gamma_n \ge 0$ $(n \ge 0)$*,* $\sum_{n=0}^{\infty} \gamma_n < \infty$*.*
Then $\lim_{n\to\infty} s_n = 0$*.*

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \cdots) \in l^\infty$. We write $u_n(a_n)$ instead of $\mu((a_0, a_1, \cdots))$. μ is said to be *Banach limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $u_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \cdots) \in l^\infty$. If μ is a Banach limit, the following are well-known:
(i) for all $n \ge 1, a_n \le c_n$ implies $\mu(a_n) \le \mu(c_n)$,
(ii) $\mu(a_{n+N}) = \mu(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \le \mu_n(a_n) \le \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \cdots) \in l^\infty$.

The following lemma was given in [22] as the revision of [18, Proposition 2].

Lemma 2.3. *Let* $a \in \mathbb{R}$ *be a real number and a sequence* $\{a_n\} \in l^\infty$ *satisfy the condition* $\mu_n(a_n) \le a$ *for all Banach limit* μ*. If* $\limsup_{n\to\infty}(a_{n+N} - a_n) \le 0$ *for* $N \ge 1$*, then* $\limsup_{n\to\infty} a_n \le a$*.*

Finally, the sequence $\{x_n\}$ in E is said to be *weakly asymptotically regular* if for $N \ge 1$,

$$w - \lim_{n\to\infty}(x_{n+N} - x_n) = 0, \quad \text{that is,} \quad x_{n+N} - x_n \rightharpoonup 0$$

and *asymptotically regular* if for $N \ge 1$,

$$\lim_{n\to\infty} \|x_{n+N} - x_n\| = 0,$$

respectively.

3. Main results

Now, we study the strong convergence results for modified Mann iterative scheme (IS) in Banach spaces.

We consider N mappings $T_1,\ T_2,\ \cdots,T_N$. For $n > N$, set $T_n := T_{n \mod N}$, where $n \mod N$ is defined as follows: if $n = kN + l,\ 0 \le l < N$, then

$$n \mod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases}$$

For any $n \ge 1$, $T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ is nonexpansive and so, for any $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1} : C \to C$ defines a strict contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point $x_t^k(f)$ satisfying

$$\text{(A)} \qquad x_t^n(f) = tf(x_t^n(f)) + (1-t)T_{n+N}T_{n+N-1}\cdots T_{n+1}x_t^n(f).$$

For simplicity we will write x_t^n for $x_t^n(f)$ provided no confusion occurs.

The following result for the existence of $Q(f)$ which solves a variational inequality

$$\langle (I-f)(Q(f)), J_\varphi(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$$

was obtained by Jung [10].

Theorem J [10]**.** *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T_1,\cdots,T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ and*

$$\begin{aligned} F = Fix(T_NT_{N-1}\cdots T_1) = {} & Fix(T_1T_N\cdots T_3T_2) \\ = {} & \cdots = Fix(T_{N-1}T_{N-2}\cdots T_1T_N). \end{aligned}$$

Then $\{x_t^n\}$ defined by (A) *converges strongly to a point in F. If we define $Q : \Sigma_C \to F$ by*

$$Q(f) := \lim_{t\to 0} x_t^n, \quad f \in \Sigma_C, \tag{3.1}$$

then $Q(f)$ is independent of n and $Q(f)$ solves a variational inequality

$$\langle (I-f)(Q(f)), J_\varphi(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F.$$

Remark 3.1. (1) In Theorems J, if $f(x) = u$, $x \in C$, is a constant, then it follows from (2.3) that (3.1) is reduced to the sunny nonexpansive retraction from C onto F,

$$\langle Q(u) - u, J_\varphi(Q(u)-p)\rangle \le 0, \quad u \in C, \quad p \in F.$$

(2) We point out that in Theorem 3.1, Lemma 3.1 and Theorem 3.2 of [10], for abbreviation, the normalized duality mapping J instead of the duality mapping J_φ was indeed assumed. The normalized duality mapping J in [10] should be replaced by the duality mapping J_φ. But even if we assume the duality mapping J_φ, we can obtain the conclusion in Theorem J by using the same methods as Proposition 3.1 and Theorem 3.2 below.

Using Theorem J, we have the following result.

Proposition 3.1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ satisfying the following conditions:*

(i) $$T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N;$$

(ii) $$\begin{aligned} F = Fix(T_N T_{N-1} \cdots T_1) = & \ Fix(T_1 T_N \cdots T_3 T_2) \\ = & \ \cdots = Fix(T_{N-1} \cdots T_1 T_N). \end{aligned}$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the condition:
(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0.$
Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by

(IS) $$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0. \end{cases}$$

and μ a Banach limit. Then

$$\mu_n(\langle (I-f)(Q(f)), J_\varphi(Q(f) - x_n)\rangle) \leq 0,$$

where $Q(f) = \lim_{t\to 0^+} x_t$ and x_t is defined by $x_t = tf(x_t) + (1-t)Sx_t$ for $S = T_N T_{N-1} \cdots T_1$ and $t \in (0,1)$.

Proof. Note that the definition of the weak sequential continuity of duality mapping J_φ implies that E is smooth. Let $x_t = tf(x_t) + (1-t)Sx_t$ for $S = T_N T_{N-1} \cdots T_1$ and $t \in (0,1)$. Then by Theorem J, $\{x_t\}$ strongly converges to a point in F, which is also denoted by $Q(f) := \lim_{t\to 0^+} x_t$.

Now, since

$$x_t - x_{n+N} = (1-t)(Sx_t - x_{n+N}) + t(f(x_t) - x_{n+N}),$$

by Lemma 2.1, we have

$$\begin{aligned} \Phi(\|x_t - x_{n+N}\|) \leq & \Phi((1-t)\|Sx_t - x_{n+N}\|) \\ & + t\langle f(x_t) - x_{n+N}, J_\varphi(x_t - x_{n+N})\rangle. \end{aligned} \tag{3.2}$$

Let $p \in F$. Then we have

$$\begin{aligned}\|x_t - p\| &\le t\|f(x_t) - p\| + (1-t)\|Sx_t - Sp\| \\ &\le t\|f(x_t) - p\| + (1-t)\|x_t - p\|\end{aligned}$$

This gives that

$$\begin{aligned}\|x_t - p\| &\le \|f(x_t) - p\| \le \|f(x_t) - f(p)\| + \|f(p) - P\| \\ &\le k\|x_t - p\| + \|f(p) - p\|\end{aligned}$$

and so

$$\|x_t - p\| \le \frac{1}{1-k}\|f(p) - p\|, \qquad t \in (0,1),$$

and hence $\{x_t\}$ is bounded. We also have

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$$

for all $n \ge 0$ and all $p \in F$ and so $\{x_n\}$ is bounded. Indeed, let $p \in F$ and $d = \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$. Noting that

$$\|y_n - p\| \le \beta_n\|x_n - p\| + (1-\beta_n)\|T_{n+1}x_n - p\| \le \|x_n - p\|,$$

we have

$$\begin{aligned}\|x_1 - p\| &\le (1-\alpha_0)\|y_0 - p\| + \alpha_0\|f(x_0) - p\| \\ &\le (1-\alpha_0)\|x_0 - p\| + \alpha_0(\|f(x_0) - f(p)\| + \|f(p) - p\|) \\ &\le (1-(1-k)\alpha_0)\|x_0 - p\| + \alpha_0\|f(p) - p\| \\ &\le (1-(1-k)\alpha_0)d + \alpha_0(1-k)d = d.\end{aligned}$$

Using an induction, we obtain $\|x_{n+1} - p\| \le d$. Hence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$. As a consequence with the control condition (C1), we get

$$\begin{aligned}&\|x_{n+1} - T_{n+1}x_n\| \\ &\le \|x_{n+1} - y_n\| + \|y_n - T_{n+1}x_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|y_n\|) + \beta_n(\|y_n\| + \|T_{n+1}x_n\|) \to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

By using the same method, we have

$$\|x_{n+N} - T_{n+N} \cdots T_{n+1}x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Indeed, noting that each T_i is nonexpansive and using just above fact, we obtain the finite table

$$\begin{aligned} x_{n+N} - T_{n+N}x_{n+N-1} &\to 0, \\ T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} &\to 0, \\ &\vdots \\ T_{n+N}\cdots T_{n+2}x_{n+1} - T_{n+N}\cdots T_{n+1}x_n &\to 0. \end{aligned}$$

Adding up this table yields

$$x_{n+N} - T_{n+N}\cdots T_{n+1}x_n \to 0 \quad (\text{as } n \to \infty).$$

Moreover, we prove that

$$x_{n+N} - T_N T_{N-1}\cdots T_1 x_n = x_{n+N} - Sx_n \to 0 \;\; (\text{as } n \to \infty). \tag{3.3}$$

Indeed, we can see the following:
If $n \mod N = 1$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_1 T_N \cdots T_2$;
If $n \mod N = 2$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_2 T_1 T_N \cdots T_3$;
$\vdots$
If $n \mod N = N$, then $T_{n+N}T_{n+N-1}\cdots T_{n+1} = T_N T_{N-1}\cdots T_1$.
In view of the condition (i)

$$T_N T_{N-1}\cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots T_{N-1}T_{N-2}\cdots T_1 T_N,$$

so we have

$$T_N T_{N_1}\cdots T_1 = T_{n+N}T_{n+N-1}\cdots T_{n+1}, \quad \text{for all } n \geq 1.$$

This implies that

$$\begin{aligned} x_{n+N} - Sx_n &= x_{n+N} - T_N T_{N-1}\cdots T_1 x_n \\ &= x_{n+N} - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n \to 0 \;\; (\text{as } n \to \infty). \end{aligned}$$

Observe also with (3.3) that

$$\|Sx_t - x_{n+N}\| \leq \|x_t - x_n\| + e_n,$$

where $e_n = \|x_{n+N} - Sx_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned} &\langle f(x_t) - x_{n+N}, J_\varphi(x_t - x_{n+N})\rangle \\ &\quad = \langle f(x_t) - x_t, J_\varphi(x_t - x_{n+N})\rangle + \|x_t - x_{n+N}\|\varphi(\|x_t - x_{n+N}\|). \end{aligned}$$

Thus it follows from (3.2) that

$$\begin{aligned}&\Phi(\|x_t - x_{n+N}\|)\\ \le&\Phi((1-t)(\|x_t - x_n\| + e_n))\\ &+ t(\langle f(x_t) - x_t, J_\varphi(x_t - x_{n+N})\rangle + \|x_t - x_{n+N}\|\varphi(\|x_t - x_{n+N}\|))\end{aligned} \tag{3.4}$$

Applying the Banach limit μ to (3.4), we have

$$\begin{aligned}\mu_n(\Phi(\|x_t - x_{n+N}\|)) \le \mu_n(\Phi((1-t)(\|x_t - x_n\| + e_n)))&\\ + t\mu_n(\langle f(x_t) - x_t, J_\varphi(x_t - x_{n+N})\rangle)&\\ + t\mu_n(\|x_t - x_{n+N}\|\varphi(\|x_t - x_{n+N}\|))&\end{aligned} \tag{3.5}$$

and it follows from (3.5) that

$$\begin{aligned}\mu_n(\langle x_t - f(x_t), &J_\varphi(x_t - x_n)\rangle)\\ &\le \frac{1}{t}\mu_n(\Phi((1-t)\|x_t - x_n\|) - \Phi(\|x_t - x_n\|))\\ &\quad + \mu_n(\|x_t - x_{n+N}\|\varphi(\|x_t - x_{n+N}\|))\\ &= -\frac{1}{t}\mu_n\left\{\int_{(1-t)\|x_t - x_n\|}^{\|x_t - x_n\|}\varphi(\tau)d\tau\right\}\\ &\quad + \mu_n(\|x_t - x_{n+N}\|\varphi(\|x_t - x_{n+N}\|))\\ &= \mu_n(\|x_t - x_n\|(\varphi(\|x_t - x_n\|) - \varphi(\tau_n))),\end{aligned} \tag{3.6}$$

for some τ_n satisfying $(1-t)\|x_t - x_n\| \le \tau_n \le \|x_t - x_n\|$. Since φ is uniformly continuous on compact intervals of $\mathbb{R}^+$,

$$\begin{aligned}\|x_t - x_n\| - \tau_n &\le t\|x_t - x_n\|\\ &\le t\left(\frac{2}{1-k}\|f(p) - p\| + \|x_0 - p\|\right) \to 0 \quad (\text{as } t \to 0),\end{aligned}$$

and $Q(f) = \lim_{t\to 0} x_t$, we conclude from (3.6) that

$$\begin{aligned}\mu_n(\langle (I-f)(Q(f)), &J_\varphi(Q(f) - x_n)\rangle)\\ &\le \limsup_{t\to 0}\mu_n(\langle x_t - f(x_t), J_\varphi(x_t - x_n)\rangle) \le 0.\end{aligned}$$

□

Theorem 3.1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ satisfying the following conditions:*

(i) $T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N;$

(ii)
$$F = Fix(T_N T_{N-1} \cdots T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} \cdots T_1 T_N).$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by

(IS)
$$\begin{cases} x_0 = x \in C, \\ y_n = \beta_n x_n + (1-\beta_n) T_{n+1} x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0. \end{cases}$$

If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves a variational inequality

$$\langle (I-f)(Q(f)), J_\varphi(Q(f)-p) \rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F.$$

Proof. First, we note that by Theorem J, there exists a solution $Q(f)$ of a variational inequality

$$\langle (I-f)(Q(f)), J_\varphi(Q(f)-p) \rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F,$$

where $Q(f) = \lim_{t\to 0} x_t$, and x_t is defined by $x_t = tf(x_t) + (1-t)Sx_t$ for $S = T_N T_{N-1} \cdots T_1$ and $t \in (0,1)$.

We proceed with the following steps:

Step 1. $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$ for all $n \geq 0$ and all $p \in Fix(T)$ as in the proof of Proposition 3.1. Hence $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$.

Step 2. $\limsup_{n\to\infty} \langle (I-f)(Q(f)), J_\varphi(Q(f)-x_n) \rangle \leq 0$. To this end, put

$$a_n := \langle (I-f)(Q(f)), J_\varphi(Q(f)-x_n) \rangle, \quad n \geq 1.$$

Then Proposition 3.1 implies that $\mu_n(a_n) \leq 0$ for any Banach limit μ. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} (a_{n+N} - a_n) = \lim_{j\to\infty} (a_{n_j+N} - a_{n_j})$$

and $x_{n_j} \rightharpoonup q \in E$. This implies that $x_{n_j+N} \rightharpoonup q$ since $\{x_n\}$ is weakly asymptotically regular. From the weak sequential continuity of duality mapping J_φ, we have

$$w - \lim_{j\to\infty} J_\varphi(Q(f) - x_{n_j+N}) = w - \lim_{j\to\infty} J_\varphi(Q(f) - x_{n_j}) = J_\varphi(Q(f) - q),$$

and so

$$\begin{aligned}&\limsup_{n\to\infty}(a_{n+N}-a_n)\\&\quad=\lim_{j\to\infty}\langle (I-f)(Q(f)), J_\varphi(Q(f)-x_{n_j+N})-J_\varphi(Q(f)-x_{n_j})\rangle=0.\end{aligned}$$

Then Lemma 2.3 implies that $\limsup_{n\to\infty} a_n \leq 0$, that is,

$$\limsup_{n\to\infty}\langle (I-f)(Q(f)), J_\varphi(Q(f)-x_n)\rangle \leq 0.$$

Step 3. $\lim_{n\to\infty}\|x_n-Q(f)\|=0$. By using (IS), we have

$$x_{n+1}-Q(f)=\alpha_n(f(x_n)-Q(f))+(1-\alpha_n)(y_n-Q(f)).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}&\|x_{n+1}-Q(f)\|^2\\&\leq (1-\alpha_n)^2\|y_n-Q(f)\|^2+2\alpha_n\langle f(x_n)-Q(f), J(x_{n+1}-Q(f))\rangle\\&\leq (1-\alpha_n)^2\|x_n-Q(f)\|^2+2\alpha_n\langle f(x_n)-f(Q(f)), J(x_{n+1}-Q(f))\rangle\\&\quad+2\alpha_n\langle f(Q(f))-Q(f), J(x_{n+1}-Q(f))\rangle\\&\leq (1-\alpha_n)^2\|x_n-Q(f)\|^2+2k\alpha_n\|x_n-Q(f)\|\|x_{n+1}-Q(f)\|\\&\quad+2\alpha_n\langle f(Q(f))-Q(f), J(x_{n+1}-Q(f))\rangle\\&\leq (1-\alpha_n)^2\|x_n-Q(f)\|^2+k\alpha_n(\|x_n-Q(f)\|^2+\|x_{n+1}-Q(f)\|^2)\\&\quad+2\alpha_n\langle f(Q(f))-Q(f), J(x_{n+1}-Q(f))\rangle.\end{aligned}$$

Now, by using gauge function φ, we define for every $n\geq 0$

$$\theta_n:=\begin{cases}\frac{\|Q(f)-x_n\|}{\varphi(\|Q(f)-x_n\|)}, & \text{if } Q(f)\neq x_n\\ 0, & \text{if } Q(f)=x_n.\end{cases}$$

From $\sup\{\frac{\|Q(f)-x_n\|}{\varphi(\|Q(f)-x_n\|)}: Q(f)\neq x_n\}<\infty$, we obtain $\limsup_{n\to\infty}\theta_n<\infty$. Also from (2.2), we have

$$J(Q(f)-x_{n+1})=\theta_{n+1}J_\varphi(Q(f)-x_{n+1}),\quad \text{for all } n\geq 0.$$

It then follows that

$$\begin{aligned}\|x_{n+1}-Q(f)\|^2&\leq \frac{1-(2-k)\alpha_n+\alpha_n^2}{1-k\alpha_n}\|x_n-Q(f)\|^2\\&\quad+\frac{2\alpha_n}{1-k\alpha_n}\langle (I-f)(Q(f)), J(Q(f)-x_{n+1})\rangle\\&\leq \frac{1-(2-k)\alpha_n}{1-k\alpha_n}\|x_n-Q(f)\|^2+\frac{\alpha_n^2}{1-k\alpha_n}M\\&\quad+\frac{2\alpha_n}{1-k\alpha_n}\theta_{n+1}\langle (I-f)(Q(f)), J_\varphi(Q(f)-x_{n+1})\rangle,\end{aligned}\tag{3.7}$$

where $M = \sup_{n\geq 0} \|x_n - Q(f)\|^2$. Put

$$\begin{aligned}\lambda_n =& \frac{2(1-k)\alpha_n}{1-k\alpha_n} \text{ and}\\ \delta_n =& \frac{M\alpha_n}{2(1-k)} + \frac{1}{1-k}\theta_{n+1}\langle (I-f)(Q(f)), J_\varphi(Q(f)-x_{n+1})\rangle.\end{aligned}$$

From (C1), (C2) and Step 2, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Since (3.7) reduces to

$$\|x_{n+1} - Q(f)\|^2 \leq (1-\lambda_n)\|x_n - Q(f)\|^2 + \lambda_n\delta_n,$$

from Lemma 2.2, we conclude that $\lim_{n\to\infty} \|x_n - Q(f)\| = 0$. This completes the proof. $\square$

Corollary 3.1. *Let E be a reflexive Banach space with a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $T_1, \cdots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ satisfying the following conditions:*

(i) $\qquad T_N T_{N-1} \cdots T_1 = T_1 T_N \cdots T_3 T_2 = \cdots = T_{N-1} T_{N-2} \cdots T_1 T_N;$

(ii) $$\begin{aligned} F = Fix(T_N T_{N-1} \cdots T_1) =& \ Fix(T_1 T_N \cdots T_3 T_2) \\ =& \ \cdots = Fix(T_{N-1} \cdots T_1 T_N). \end{aligned}$$

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,1)$ which satisfy the conditions;
(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0$*;*
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by (IS). *If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ is the unique solution of a variational inequality*

$$\langle (I-f)(Q(f)), J_\varphi(Q(f)-p)\rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F.$$

Remark 3.2. If $\{\alpha_n\}$ and $\{\beta_n\}$ in Corollary 3.1 satisfy conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0, \quad \lim_{n\to\infty} \beta_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(A1) $\sum_{n=0}^{\infty} |\beta_{n+N} - \beta_n| < \infty$; and
(A2) $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$, or the perturbed control condition:
(A3) $|\alpha_{n+N} - \alpha_n| \leq \circ(\alpha_{n+N}) + \sigma_n, \quad \sum_{n=0}^{\infty} \sigma_n < \infty$,
then the sequence $\{x_n\}$ generated by (IS) is asymptotically regular. Now we give only the proof in case when $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (C1), (C2),

(A1) and (A3). Indeed, by Step 1 in the proof of Proposition 3.1, there exists a constant $L > 0$ such that

$$L = \max\{\sup_{n\geq 0}\{\|f(x_n)\| + \|T_{n+1}x_n\|\}, \sup_{n\geq 0}\{\|x_n\| + \|T_{n+1}x_n\|\}\}.$$

Since for all $n \geq 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned}
x_{n+N} - x_n = {} & (\alpha_{n+N-1} - \alpha_{n-1})(f(x_{n-1}) - T_n x_{n-1}) \\
& + (1-\alpha_{n+N-1})\beta_{n+N-1}(x_{n+N-1} - x_{n-1}) \\
& + [(\beta_{n+N-1} - \beta_{n-1})(1-\alpha_{n+N-1}) \\
& \qquad - (\alpha_{n+N-1} - \alpha_{n-1})\beta_{n-1}](x_{n-1} - T_n x_{n-1}) \\
& + (1-\beta_{n+N-1})(1-\alpha_{n+N-1})(T_{n+N}x_{n+N-1} - T_{n+N}x_{n-1}) \\
& + \alpha_{n+N-1}(f(x_{n+N-1}) - f(x_{n-1})),
\end{aligned}$$

and so

$$\begin{aligned}
& \|x_{n+N} - x_n\| \\
\leq {} & (1-\beta_{n+N-1})(1-\alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| \\
& + k\alpha_{n+N-1}\|x_{n+N-1} - x_{n-1}\| + (1-\alpha_{n+N-1})\beta_{n+N-1}\|x_{n+N-1} - x_{n-1}\| \\
& + |\beta_{n+N-1} - \beta_{n-1}|L + 2|\alpha_{n+N-1} - \alpha_{n-1}|L \\
\leq {} & (1-(1-k)\alpha_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + |\beta_{n+N-1} - \beta_{n-1}|L \\
& + 2(\circ(\alpha_{n+N-1}) + \sigma_{n-1})L.
\end{aligned}$$

By taking $s_{n+1} = \|x_{n+N} - x_n\|$, $\lambda_n = (1-k)\alpha_{n+N-1}$, $\lambda_n\delta_n = 2\circ(\alpha_{n+N-1})L$ and $\gamma_n = |\beta_{n+N-1} - \beta_{n-1}|L + 2\sigma_{n-1}L$, we have

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + \gamma_n.$$

Hence, by (C1), (C2), (A1), (A3) and Lemma 2.2,

$$\lim_{n\to\infty}\|x_{n+N} - x_n\| = 0.$$

In view of this observation, we have the following result.

Corollary 3.2. *Let E, C and $T_i, \cdots, T_N$ be as in Corollary 3.1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in* $(0,1)$ *which satisfy the conditions* (C1), (C2), (A1) *and* (A3) (*or the conditions* (C1), (C2), (A1) *and* (A2)) *in Remark 3.2. Let $f \in \Sigma_C$ and let $\{x_n\}$ be the sequence generated by* (IS). *Then $\{x_n\}$ converges strongly to $Q(f) \in F$, where $Q(f)$ is the unique solution of a variational inequality*

$$\langle (I-f)(Q(f)), J_\varphi(Q(f) - p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F$$

Remark 3.3 (1) Proposition 3.1 and Theorem 3.1 improve the corresponding results of Jung [8,9,10], Kim and Xu [12] and Zhou et al. [22] (that is, Theorem 10 of [8], Theorem 2 in [9], Proposition 3.1 and Theorem 3.3 in [10], Theorem 1 in [12], and Theorem 5 and Theorem 6 in [22]) in several aspects.

(2) Theorem 3.1 also extends Theorem 1 of Yao et al. [21] to the case of a family of finite mappings under the different conditions on the parameter $\{\beta_n\}$ and the sequence $\{x_n\}$ in the space having a weakly sequentially continuous duality mapping.

(3) Corollary 3.1 (and Corollary 3.2) also improves Theorem 1 of Kim and Xu [12] to the viscosity approximation method for finite nonexpansive mappings together with different condition from the condition (A2) on $\{\alpha_n\}$.

(4) Even the case of $\beta_n = 0$, Corollary 3.2 generalizes Theorem 10 of Jung [8] and Theorem 2 of Jung [9] since the assumption of uniformly Gâteaux differentiable norm and the fixed point property (that is, the uniform smoothness assumption) was removed.

(5) Even the case of $f(x) = u$, $x \in C$, a constant, Theorem 3.1 also works in a Banach space setting as opposed to iterative scheme of Nakajo and Takahashi [15], which works in only in the framework of Hilbert spaces.

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LINEAR COMBINATIONS OF COMPOSITION OPERATORS ON $H^\infty(B_N)$

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Abstract. We investigate the compactness of linear combinations of composition operators acting on bounded holomorphic function space $H^\infty(B_N)$ in the unit ball of $\mathbb{C}^N$, and completely characterize the association of compactness and coefficients of linear combinations of composition operators.

1. Introduction

The algebra of all holomorphic functions on the unit ball B_N in $\mathbb{C}^N$ will be denoted by $H(B_N)$. Let $S(B_N)$ be the set of holomorphic self- maps of B_N, and $H^\infty(B_N)$ denote the space of all bounded holomorphic functions on B_N endowed with the norm of $\|f\| = \sup\limits_{z\in B_N} |f(z)|$, the closed unit ball of $H^\infty(B_N)$ is written by H^∞.

For $z, w \in B_N$, it is well known that the pseudo-hyperbolic distance between z and w is

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z - P_z(w) - S_z Q_z(w)}{1 - \langle w, z\rangle}\right|$$

where $S_z = \sqrt{1-|z^2|}$, P_z is the orthogonal projection from $\mathbb{C}^N$ onto the one dimensional subspace $[z]$ generated by z, and Q_z is the orthogonal projection from $\mathbb{C}^N$ onto $\mathbb{C}^N - [z]$.

Let $\varphi \in S(B_N)$, the composition operator C_φ induced by φ is defined by

$$(C_\phi f)(z) = f(\phi(z)),$$

for z in B_N and $f \in H(B_N)$. It is easy to see that the composition operator C_φ is always bounded on $H^\infty(B_N)$ with norm 1.

During the past few decades much effort has been devoted to the research of such operators on a variety of Banach spaces of holomorphic functions with the goal of explaining the operator-theoretic behavior of C_φ, such as compactness and spectra, in terms of the function-theoretic properties of the symbol φ. We recommend the interested readers refer to the books by J. H. Shapiro [8] and Cowen and MacCluer [1], which are good sources for information on much of the developments in the theory of composition operators up to the middle of last decade.

In the past few years, many authors are interested in studying the mapping properties of the difference of two composition operators, i.e., an operator of the form

$$T = C_\varphi - C_\psi.$$

The primary motivation for this has been the desire to understand the topological structure of the whole set of composition operators acting on a given functions.

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Most papers in this area have focused on the classic reflexive spaces, however, some classical non-reflexive spaces have also been discussed lately in the unit disc in the complex plane. In [6], MacCluer,Ohno and Zhao characterized the compactness of the difference of composition operator on H^∞ spaces by Poincaré distance. Their work was extended to the setting of weighted composition operators by Hosokawa, Izuchi and Ohno [3]. In [7], Moorhouse characterized the compact difference of composition operators acting on the standard weighted bergman spaces and necessary conditions on a large scale of weighted Dirichlet spaces. Hosokawa and Ohno [3] and [4] gave a characterization of compact difference on Bloch space in the unit disc. In [9] and [2], Carl and Gorkin et al., independently extended the results to $H^\infty(B_n)$ spaces, they described compact difference by Carathéodory pseudo-distance on the ball, which is the generalization of Poinaré distance on the disc.

Lately, Izuchi and Ohno [5] characterized the compactness of linear combinations of composition operators on the Banach algebra of bounded analytic functions on the open unit disk.

Motivated by [5] on the disk, we generalize the results to the unit ball, investigate the compactness of linear combinations of composition operators acting on bounded holomorphic function space $H^\infty(B_N)$ in the unit ball of $\mathbb{C}^N$, and completely characterize the association of compactness and coefficients of linear combinations of composition operators. For the proof, we need some complex calculation skills.

2. Main Results

For our discussion of compactness, we will need the following, minor modification of that of Theorem 3.4 in [1].

Proposition 1. *(Compactness Criterion) Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be distinct functions in $S(B_N)$ and $\lambda_i \in C$ with $\lambda_i \neq 0$. Then the linear combination of composition operators $\sum\limits_{i=1}^{n} \lambda_i C_{\varphi_i}$ is compact on $H^\infty(B_N)$ if and only if whenever $\{f_m\}_m$is a bounded sequence in $H^\infty(B_N)$ such that $\{f_m\}_m$ converges to 0 uniformly on any compact subset of B_N, then $\left\| \sum\limits_{i=1}^{n} \lambda_i C_{\varphi_i} \right\|_\infty$ tends to 0 as $n \to \infty$.*

Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be distinct functions in $S(B_N)$ and $n \geq 2$. Let $Z = Z(\varphi_1, \varphi_2, \cdots, \varphi_n)$ be the family of sequence$\{z_k\}_k$ in B_N satisfying the following three conditions:

(a) $|\varphi_i(z_k)| \to 1$ as $k \to \infty$ for some i;

(b) $\{\varphi_i(z_k)\}_k$ is a convergent sequence for every i;

(c) $\frac{\varphi_j(z_k) - P_{\varphi_j(z_k)}(\varphi_i(z_k)) - S_{\varphi_j(z_k)} Q_{\varphi_j(z_k)}(\varphi_i(z_k))}{1 - \langle \varphi_i(z_k), \varphi_j(z_k) \rangle}$ is a convergent sequence for every i, j.

(c') $\rho(\varphi_i(z_k), \varphi_j(z_k))_k$ is a convergent sequence for every i, j.

Note that if $|\varphi_i(z_k)| \to 1$ as $k \to \infty$ for some i, then it is easy to see that there exists a subsequence $\{z_{k_j}\}_j \in Z$.

For $\{z_k\}_k \in Z$, we write

$$I(z_k) = \{i : 1 \leq i \leq n, |\varphi_i(z_k)| \to 1 \text{ as } k \to \infty\}.$$

By condition (a), $I(z_k) \neq \emptyset$. By condition (b), there exists δ with $0 < \delta < 1$ such that $|\varphi_j(z_k)| < \delta < 1$ for every $j not \in I(z_k)$ and k. For each $t \in I(z_k)$, let

$$I_0(z_k, t) = \{j \in I(z_k) : \rho(\varphi_j(z_k), \varphi_t(z_k)) \to 0 \text{ as } k \to \infty\}.$$

For $s,t\in I(z_k)$, it is clear that either $I_0(z_k,s)=I_0(z_k,t)$ or $I_0(z_k,s)\bigcap I_0(z_k,t)=\emptyset$. Hence there is a subset $t_1,t_2,t_l\subset I(z_k)$ such that

$$I(z_k)=\bigcup_{p=1}^{l} I_0(\{z_k\},t_p)\ \ and\ \ \bigcup_{p=1}^{l} I_0(\{z_k\},t_p)\bigcap\bigcup_{p=1}^{l} I_0(\{z_k\},t_q)=\emptyset$$

for $p\neq q$.

When we consider the compactness of linear combinations $\sum\limits_{i=1}^{n}\lambda_i C_{\varphi_i}$, some C_{φ_i} could be compact, that is $\|\varphi_i\|_\infty<1$ (see Exercise 4.1.8(b) in [1]). We may exclude such trivial ones from our linear combinations.

Theorem 1. *Let $\varphi_1,\varphi_2,\cdots,\varphi_n$ be distinct functions in $S(B_N)$ with $\|\varphi_i\|_\infty=1$, and $\lambda_i\in\mathbb{C}$ with $\lambda_i\neq 0$ for every i. Then the following conditions are equivalent.*

(1) $\sum\limits_{i=1}^{n}\lambda_i C_{\varphi_i}$ is compact on H^∞.

(2) $\sum\{\lambda_i : i\in I_0(z_k,t)\}=0$ for every $\{z_k\}_k\in Z=Z(\varphi_1,\varphi_2,\cdots,\varphi_n)$ and $t\in I(z_k)$.

Proof. (1) $\Rightarrow$ (2).

Suppose that $\sum\limits_{i=1}^{n}\lambda_i C_{\varphi_i}$ is compact on H^∞. Let $\{z_k\}_k\in Z$ and $t\in I(z_k)$, that is $|\varphi_t(z_k)|\to 1$ as $k\to\infty$. For each positive integer k, denoting

$$f_k(z)=\frac{1-|\varphi_t(z_k)|^2}{1-\langle z,\varphi_t(z_k)\rangle}\prod_{j\notin I_0(\{z_k\},t)}\left\langle \varphi_{\varphi_j(z_k)}(z),\varphi_{\varphi_j(z_k)}(\varphi_t(z_k))\right\rangle \varphi_{\varphi_{j_i}(z_k)}(z),$$

where j_i can be any integer $\in I_0(z_k,t)$. It is easy to check that $f_k(z)\in H^\infty$, $\|f_k\|_\infty\le 2$ and $\{f_k\}_k$ converges to 0 uniformly on every compact subset of B_N. It follows that

$$\left\|\sum_{i=1}^{n}\lambda_i C_{\varphi_i} f_k\right\|_\infty \ge \left|\sum_{i=1}^{n}\lambda_i f_k(\varphi_i(z_k))\right|$$

$$=\left|\sum_{i\in I_0(\{z_k\},t)}\frac{1-|\varphi_t(z_k)|^2}{1-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle}\prod_{j\notin I_0(\{z_k\},t)}\left\langle\varphi_{\varphi_j(z_k)}(\varphi_i(z_k)),\varphi_{\varphi_j(z_k)}(\varphi_t(z_k))\right\rangle\varphi_{\varphi_{j_i}(z_k)}(\varphi_i(z_k))\right|.$$

Here by the definition of $I_0(z_k,t)$, it follows that $\rho(\varphi_i(z_k),\varphi_t(z_k))\to 0$ as $k\to\infty$. Hence

$$\left\langle\varphi_{\varphi_t(z_k)}(\varphi_i(z_k)),\varphi_t(z_k)\right\rangle=\frac{|\varphi_t(z_k)|^2-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle}{1-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle}\to 0,$$

furthermore

$$\frac{1-|\varphi_t(z_k)|^2}{1-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle}\to 1$$

as $k\to\infty$.

With the same proof, when $k\to\infty$, we have

$$\frac{|\varphi_i(z_k)|^2-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle}{1-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle}\to 0$$

and

$$\frac{1-|\varphi_i(z_k)|^2}{1-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle}\to 1.$$

Note that $i \in I_0(z_k,t)$, $\frac{1-|\varphi_i(z_k)|^2}{1-|\varphi_t(z_k)|^2} \to 1$, it follows that $1-|\varphi_i(z_k)|^2$, $1-|\varphi_t(z_k)|^2$,$1-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle$ and $1-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle$ are equivalent as $k\to\infty$.

On the other hand

$$\begin{aligned}
&\left|\varphi_{\varphi_j(z_k)}(\varphi_i(z_k))-\varphi_{\varphi_j(z_k)}(\varphi_t(z_k))\right|^2\\
&=\left\langle\varphi_{\varphi_j(z_k)}(\varphi_i(z_k))-\varphi_{\varphi_j(z_k)}(\varphi_t(z_k)),\varphi_{\varphi_j(z_k)}(\varphi_i(z_k))-\varphi_{\varphi_j(z_k)}(\varphi_t(z_k))\right\rangle\\
&=1-\frac{\left(1-|\varphi_j(z_k)|^2\right)\left(1-|\varphi_i(z_k)|^2\right)}{|1-\langle\varphi_i(z_k),\varphi_j(z_k)\rangle|^2}-\left[1-\frac{\left(1-|\varphi_j(z_k)|^2\right)(1-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle)}{(1-\langle\varphi_i(z_k),\varphi_j(z_k)\rangle)(1-\langle\varphi_j(z_k),\varphi_t(z_k)\rangle)}\right]\\
&-\left[1-\frac{\left(1-|\varphi_j(z_k)|^2\right)(1-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle)}{(1-\langle\varphi_j(z_k),\varphi_i(z_k)\rangle)(1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle)}\right]+1-\frac{\left(1-|\varphi_j(z_k)|^2\right)\left(1-|\varphi_t(z_k)|^2\right)}{|1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle|^2},
\end{aligned}$$

from the equality above, it follows that

$$\begin{aligned}
&\left(1-|\varphi_j(z_k)|^2\right)\left(1-|\varphi_i(z_k)|^2\right)\times\left|\frac{1}{1-\langle\varphi_i(z_k),\varphi_j(z_k)\rangle}-\frac{1}{1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle}\right|^2\\
&=\left(1-|\varphi_j(z_k)|^2\right)\left(1-|\varphi_i(z_k)|^2\right)\times\left|\frac{\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_j(z_k)\rangle}{(1-\langle\varphi_i(z_k),\varphi_j(z_k)\rangle)(1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle)}\right|^2\\
&\le\left(1-|\varphi_j(z_k)|^2\right)\left(1-|\varphi_i(z_k)|^2\right)\times\frac{1}{(1-|\varphi_j(z_k)|)(1-|\varphi_i(z_k)|)}\times\left|\frac{\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_j(z_k)\rangle}{(1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle)}\right|^2\\
&\le 4\left|\frac{\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_t(z_k)\rangle+\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_j(z_k)-\varphi_t(z_k)\rangle}{(1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle)}\right|^2\\
&\le 8\frac{|\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_t(z_k)\rangle|^2+|\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_j(z_k)-\varphi_t(z_k)\rangle|^2}{|1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle|^2}.
\end{aligned}$$

Where

$$\begin{aligned}
&\left|\frac{\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_t(z_k)\rangle}{1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle}\right|^2\\
&\le\left|\frac{|\varphi_t(z_k)|^2-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle}{1-|\varphi_t(z_k)|^2}\right|^2\to 0
\end{aligned}$$

as $k\to\infty$ and

$$\begin{aligned}
&\left|\tfrac{\langle\varphi_i(z_k)-\varphi_t(z_k),\varphi_j(z_k)-\varphi_t(z_k)\rangle}{1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle}\right|^2\\
&\le|\varphi_i(z_k)-\varphi_t(z_k)|^2\times|\varphi_j(z_k)-\varphi_t(z_k)|^2\times\tfrac{1}{|1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle|^2}\\
&\le\left[\left||\varphi_i(z_k)|^2-\langle\varphi_t(z_k),\varphi_i(z_k)\rangle\right|+\left||\varphi_t(z_k)|^2-\langle\varphi_i(z_k),\varphi_t(z_k)\rangle\right|\right]\\
&\times\tfrac{\left||\varphi_j(z_k)|^2-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle\right|+\left||\varphi_t(z_k)|^2-\langle\varphi_j(z_k),\varphi_t(z_k)\rangle\right|}{|1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle|^2}.
\end{aligned}\tag{d}$$

If follows from condition (c) that

$$\left\langle\varphi_{\varphi_j(z_k)}(\varphi_t(z_k)),\varphi_j(z_k)\right\rangle=\frac{|\varphi_j(z_k)|^2-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle}{1-\langle\varphi_t(z_k),\varphi_j(z_k)\rangle}\to\alpha,\quad|\alpha|<1$$

and

$$\left\langle\varphi_{\varphi_t(z_k)}(\varphi_j(z_k)),\varphi_t(z_k)\right\rangle=\frac{|\varphi_t(z_k)|^2-\langle\varphi_j(z_k),\varphi_t(z_k)\rangle}{1-\langle\varphi_j(z_k),\varphi_t(z_k)\rangle}\to\gamma,\quad|\gamma|<1.$$

While

$$\frac{|\varphi_i(z_k)|^2 - \langle \varphi_t(z_k), \varphi_i(z_k)\rangle}{1-|\varphi_t(z_k)|} \to 0 \text{ and } \frac{|\varphi_t(z_k)|^2 - \langle \varphi_i(z_k), \varphi_t(z_k)\rangle}{1-|\varphi_t(z_k)|} \to 0,$$

therefore $(d) \to 0$ as $k \to \infty$.

That is

$$\varphi_{\varphi_j(z_k)}(\varphi_i(z_k)) - \varphi_{\varphi_j(z_k)}(\varphi_t(z_k)) \to 0$$

as $k \to \infty$. Therefore

$$\lim_{k\to\infty} \varphi_{\varphi_j(z_k)}(\varphi_i(z_k)) = \lim_{k\to\infty} \varphi_{\varphi_j(z_k)}(\varphi_t(z_k))$$

since$j \notin I_0(\{z_k\}, t)$. By the definition of $I_0(z_k, t)$ and (c)

$$\lim_{k\to\infty} \varphi_{\varphi_j(z_k)}(\varphi_t(z_k)) = \beta_{j,t} \neq 0$$

for some $\beta_{j,t} \in C^N$.

By condition (1) and Proposition 1,

$$\left\| \sum_{i=1}^{n} \lambda_i C_{\varphi_i} f_k \right\|_{\infty} \to 0$$

as $k \to \infty$. Therefore we get

$$\left(\sum_{i\notin I_0(\{z_k\},t)} \lambda_i \right) \prod_{j\notin I_0(\{z_k\},t)} \beta_{j_i,t} |\beta_{j,t}|^2 = 0.$$

Consequently, we have

$$\sum_{i\notin I_0(\{z_k\},t)} \lambda_i = 0.$$

(2) $\Rightarrow$ (1).

Suppose that $\sum\limits_{i=1}^{n} \lambda_i C_{\varphi_i}$ is not compact on H^{∞}, then there exists a sequence $\{f_m\}_m$ in the ball of $H^{\infty}(B_N)$ such that $f_m \to 0$ uniformly on every compact subset of B_N and $\left\| \sum\limits_{i=1}^{n} \lambda_i f_m(\varphi_i) \right\|_{\infty} \to 0$ as $m \to \infty$.

For some $\epsilon > 0$, considering a sequence of $\{f_m\}_m$, we may assume that

$$\left\| \sum_{i=1}^{n} \lambda_i f_m(\varphi_i) \right\|_{\infty} > \epsilon > 0$$

for every m.

Take $z_k \in B_N$ with $|z_k| \to 1$ and $\left| \sum\limits_{i=1}^{n} \lambda_i f_m(\varphi_i)(z_k) \right| > \epsilon$.

Considering a subsequence of $\{z_k\}_k$, we may assume that $\varphi_i(z_k) \to \alpha_i$ as $k \to \infty$ for every i. Since $f_m \to 0$ and uniformly on every compact subset of B_N, $|\alpha_i| = 1$ for some i. Moreover we may assume that$\{z_k\}_k \in Z$. Also we have

$$\lim_{m\to\infty} \inf \left| \sum_{i\in I(\{z_k\})} \lambda_i f_m(\varphi_i(z_k)) \right| \geq \epsilon.$$

Recall that there exists a subset $\{t_1, t_2, \cdots, t_l\} \subset I\{z_k\}$ such that

$$I\{z_k\} = \bigcup_{p=1}^{l} I_0(\{z_k\}, t_p),$$

and

$$I_0\left(\{z_k\}, t_p\right) \cap I_0\left(\{z_k\}, t_q\right) = \emptyset$$

for $p \neq q$.

Let $i \in I_0\left(\{z_k\}, t_p\right)$, then

$$\rho\left(\varphi_i\left(z_k\right), \varphi_{t_p}\left(z_k\right)\right) \to 0$$

as $k \to \infty$.

By Schwarz's lemma, it follows that

$$\rho\left(f_m\left(\varphi_i\left(z_k\right)\right), f_m\left(\varphi_{t_p}\left(z_k\right)\right)\right) \leq \rho\left(\varphi_i\left(z_k\right), \varphi_{t_p}\left(z_k\right)\right) \to 0 \qquad (e)$$

as $k \to \infty$.

Since $\{f_m\left(\varphi_i\left(z_k\right)\right)\}_m$ is bounded, considering a sequence of $\{z_k\}_k$, we may assume that $f_m\left(\varphi_i\left(z_k\right)\right) \to \beta_i$ as $k \to \infty$ for every i. By (e), it follows that $\beta_i = \beta_{t_p}$ for every $i \in I_0\left(\{z_k\}, t_p\right)$. Therefore by condition (2),

$$\lim_{k\to\infty} \sum_{i\in I(\{z_k\})} \lambda_i f_m\left(\varphi_i(z_k)\right) = \lim_{k\to\infty} \sum_{p=1}^{l} \sum_{i\in I_0(\{z_k\},t_p)} \lambda_i f_m\left(\varphi_i(z_k)\right)$$

$$= \sum_{p=1}^{l} \sum_{i\in I_0(\{z_k\},t_p)} \lambda_i \beta_{t_p} = \sum_{p=1}^{l} \beta_{t_p} \sum_{i\in I_0(\{z_k\},t_p)} \lambda_i = 0.$$

This contradicts condition. This completes the proof of Theorem 1. □

The following corollaries follow from Theorem 1.

Corollary 1. *Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be distinct functions in $S\left(B_N\right)$ with $\|\varphi_i\|_\infty = 1$ and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i if $\sum_{i\in J} \lambda_i \neq 0$ for every $J \in \{1, 2, \cdots, n\}$, then $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$ is not compact on H^∞. This says that the sum $\sum_{i=1}^{n} C_{\varphi_i}$ is never compact on H^∞ for every $\varphi_i \in S\left(B_N\right)$ with $\|\varphi_i\|_\infty = 1$, $i = 1, 2, \cdots, n$.*

Corollary 2. *Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be distinct functions in $S\left(B_N\right)$ with $\|\varphi_i\|_\infty = 1$ and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Suppose that $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i\in J} \lambda_i \neq 0$ for every non-empty subset J of $\{1, 2, \cdots, n\}$. Then $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$ is compact on H^∞ if and only if $C_{\varphi_i} - C_{\varphi_j}$ is compact on H^∞ for every i, j with $i \neq j$.*

Proof. Suppose that $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$ is compact on H^∞. Then by Theorem 1, for every $\{z_k\}_k \in Z$, $I\left(\{z_k\}\right) = \{1, 2, \cdots\cdots, n\}$ and $I_0\left(\{z_k\}, t\right) = \{1, 2, \cdots\cdots, n\}$.

For every $t \in \left(\{z_k\}\right)$. Hence $\lim_{|\varphi_i(z)|\to 1} \rho\left(\varphi_i\left(z\right), \varphi_j\left(z\right)\right) \to 0$, by [9], $C_{\varphi_i} - C_{\varphi_j}$ is compact for every i, j. Suppose that $C_{\varphi_i} - C_{\varphi_j}$ is compact for every i, j. Since

$$\sum_{i=1}^{n} \lambda_i C_{\varphi_i} = \sum_{i=1}^{n} \lambda_i C_{\varphi_1} + \sum_{i=1}^{n} \lambda_i \left(C_{\varphi_i} - C_{\varphi_1}\right) = \sum_{i=2}^{n} \lambda_i \left(C_{\varphi_i} - C_{\varphi_1}\right),$$

it follows that $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$ is compact. □

Under the assumption of Corollary 2, we obtain the following corollary by Theorem 3 in [9].

Corollary 3. *Let $\varphi_1, \varphi_2, \cdots, \varphi_n$ be distinct functions in $S(B_N)$ with $\|\varphi_i\|_\infty = 1$ and $\lambda_i \in \mathbb{C}$ with $\lambda_i \neq 0$ for every i. Suppose that $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i \in J} \lambda_i \neq 0$ for every non-empty proper subset J of $\{1, 2, \cdots, n\}$. Then the following conditions are equivalent:*

(1) $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$*:* $H^\infty \to H^\infty$ *is compact;*

(2) $\sum_{i=1}^{n} \lambda_i C_{\varphi_i}$*:* $B \to H^\infty$ *is compact.*

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An Extension of Some Common Fixed Point Theorems for Selfmappings in Uniform Space

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Abstract

In this paper, we establish some common fixed point theorems for selfmappings in uniform space by employing both the concepts of an $A-$distance and an $E-$distance introduced by Aamri and El Moutawakil [1], the notion of the comparison functions as well as a contractive condition of the integral type.

Our results are generalizations and extensions of some results of Aamri and El Moutawakil [1], Branciari [5], Jungck [7] and Olatinwo [10, 11, 12].

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1. Introduction

Let (X, Φ) be a uniform space, where X is a nonempty set equipped with a nonempty family Φ of subsets of $X \times X$ satisfying certain properties. Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.* Interested readers can consult Bourbaki [4] and Zeidler [19] for the definition of uniform space. The definition is also available on internet (by Wikipedia, the free encyclopedia).

The concept of a $W-$distance on metric space was introduced by Kada et al [8] to generalize some important results in nonconvex minimizations and in fixed point theory for both $W-$contractive and $W-$expansive maps. The theory of fixed point or common fixed point for contractive or expansive selfmappings in complete metric space has been well-developed. Interested readers can consult Berinde [2, 3], Jachymski [6], Kada et al [8], Kang [9], Rhoades [13, 14], Rus [16], Rus et al [17], Wang et al [18] and Zeidler [19] for further study of fixed point or common fixed point theory.

Using the ideas of Kang [9], Montes and Charris [15] established some results on fixed and coincidence points of maps by means of appropriate $W-$contractive or $W-$expansive assumptions in uniform space. Furthermore, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an $A-$distance and an $E-$distance.

In [1], the following contractive definition was employed:
Let $f,\ g : X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))),\ \forall\ x,\ y \in X, \tag{1}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function satisfying
(i) for each $t \in (0, +\infty),\ 0 < \psi(t)$,
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0,\ \forall\ t \in (0, +\infty)$.
ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

In this paper, we shall establish some common fixed point theorems by employing the concepts of an $A-$distance and an $E-$distance, the notion of the comparison functions, as well as a contractive condition of the integral type. Literature abounds with several contractive conditions that have been employed by various researchers over the years to obtain different fixed point theorems. For the various contractive definitions that have been employed over the years and the notion of the comparison functions, we refer our interested readers to Berinde [2, 3], Branciari [5], Rhoades [13, 14], Rus [16] and Rus et al [17]. Both Branciari [5] and Rhoades [13] used contractive conditions of the integral type to extend the Banach's fixed point theorem.

2. Preliminaries

We shall require the following definitions and lemma in the sequel. The Remark 2.1, Definitions 2.2 - 2.6 and Lemma 2.7 are contained in [1, 9, 15]. Let (X, Φ) be a uniform space.

Remark 2.1: When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

Definition 2.2: If $V \in \Phi$ and $(x, y) \in V,\ (y, x) \in V,\ x$ and y are said to be $V-$*close.* A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $V \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are $V-$close for $n,\ m \geq N$.

Definition 2.3: A function $p : X \times X \to \mathbb{R}^+$ is said to be an $A-$*distance* if for any $V \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.4: A function $p : X \times X \to \mathbb{R}^+$ is said to be an $E-$*distance* if

(p_1) p is an $A-$distance,

(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall\ x,\ y \in X.$

Definition 2.5: A uniform space (X, Φ) is said to be *Hausdorff* if and only if the intersection of all $V \in \Phi$ reduces to the diagonal $\{\ (x, x) \mid x \in X\}$, i.e. if $(x, y) \in V$ for all $V \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \Phi$ is said to be *symmetrical* if $V = V^{-1} = \{\ (y, x) \mid (x, y) \in V\}$.

Definition 2.6: Let (X, Φ) be a uniform space and p be an $A-$distance on X.

(i) X is said to be $S-$*complete* if for every $p-$Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim\limits_{n\to\infty} p(x_n, x) = 0$.

(ii) X is said to be $p-$*Cauchy complete* if for every $p-$Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim\limits_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.

(iii) $f : X \to X$ is $p-$*continuous* if $\lim\limits_{n\to\infty} p(x_n, x) = 0$ implies $\lim\limits_{n\to\infty} p(f(x_n), f(x)) = 0$.

(iv) $f : X \to X$ is $\tau(\Phi)-$*continuous* if $\lim\limits_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim\limits_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.

(v) X is said to be $p-$*bounded* if $\delta_p(X) = \sup\ \{\ \mathrm{p(x, y)} \mid \mathrm{x,\ y \in X}\} < \infty$.

We shall require the following lemma in the sequel.

Lemma 2.7: *Let (X, Φ) be a Hausdorff uniform space and p be an $A-$distance on X. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x,\ y,\ z \in X$, the following hold:*

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n,\ \forall\ n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n,\ \forall\ n \in \mathbb{N}$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n\ \forall\ m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).

Remark 2.8: A sequence in X is $p-$Cauchy if it satisfies the usual metric condition. See [1] for this remark.

Definition 2.9 [Berinde [2, 3]]: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *comparison function* if:

(i) ψ is monotone increasing ; (ii) $\lim_{n\to\infty} \psi^n(t) = 0, \forall\, t \geq 0$.

See Rus [16] and Rus et al [17] for more on the comparison functions.

Remark 2.10: Every comparison function satisfies the condition $\psi(0) = 0$. Also, both conditions (i) and (ii) imply that $\psi(t) < t,\ \forall\, t > 0$.

In this paper, we shall employ the following contractive condition of the integral type: Let $f,\ g : X \to X$ be selfmappings of X. There exist a comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Phi(0) = 0$ such that $\forall\ x,\ y \in X$, we have

$$(2) \qquad \int_0^{p(f(x),f(y))} \varphi(t)dt \leq \Phi\left(\int_0^{p(x,g(x))} \varphi(t)dt\right) + \psi\left(\int_0^{p(g(x),g(y))} \varphi(t)dt\right),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0,\ \int_0^\epsilon \varphi(t)dt > 0$.

Apart from condition (2), we shall also use the following contractive condition of the integral type: Let $f,\ g : X \to X$ be selfmappings of X. There exist a constant $L \geq 0$ and a comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\forall\ x,\ y \in X$, we have

$$(3) \qquad \int_0^{p(f(x),f(y))} \varphi(t)dt \leq L \int_0^{p(x,g(x))} \varphi(t)dt + \psi\left(\int_0^{p(g(x),g(y))} \varphi(t)dt\right),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping as defined in (2).

Although, condition (2) is more general than (3), but we shall state without proof a commmon fixed point result involving condition (3) and which also extends some results of [1, 5, 7, 10].

Remark 2.11: The contractive condition (2) is more general than (1) in the sense that if in (2), $\varphi(t) = 1,\ \forall\ t \in \mathbb{R}^+$, and $\Phi(u) = 0,\ \forall\ u \in \mathbb{R}^+$, then we obtain (1).

Our results are generalizations and extensions of some results of Aamri and El Moutawakil [1], Branciari [5], Jungck [7] and Olatinwo [10, 11].

3. The Main Results

The main results of this paper are the following:

Theorem 3.1: Let (X, Φ) be a Hausdorff uniform space and p an $E-$distance on X. Suppose that X is $p-$bounded and $S-$complete. Let f and g be commuting $p-$continuous or $\tau(\Phi)-$continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$;

(ii) $p(f(x_i), f(x_i)) = 0, \ \forall \ x_i \in X, \ i = 0, 1, 2, \cdots;$
(iii) both $f, \ g : X \to X$ satisfy the contractive condition (2).
Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a comparison function and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ a monotone increasing function such that $\Phi(0) = 0$. Suppose that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0, \ \int_0^\epsilon \varphi(t)dt > 0$. Then, f and g have a unique common fixed point.

Proof: We shall first establish the existence of the common fixed point of f and g by using property (p_1) of $E-$distance:
Consider $x_n = f(x_{n-1})$ with $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$,
choose $x_1 \in X$ such that $f(x_1) = g(x_2)$, and in general,
choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$.
Therefore, we obtain by the repeated application of (2) that

$$\begin{aligned}
\int_0^{p(f(x_n),f(x_{n+m}))} \varphi(t)dt &\leq \Phi\left(\int_0^{p(x_n,g(x_n))} \varphi(t)dt\right) + \psi\left(\int_0^{p(g(x_n),g(x_{n+m}))} \varphi(t)dt\right) \\
&= \Phi\left(\int_0^{p(x_n,f(x_{n-1}))} \varphi(t)dt\right) + \psi\left(\int_0^{p(f(x_{n-1}),f(x_{n+m-1}))} \varphi(t)dt\right) \\
&= \psi\left(\int_0^{p(f(x_{n-1}),f(x_{n+m-1}))} \varphi(t)dt\right) \\
&\leq \psi\left(\Phi\left(\int_0^{p(x_{n-1},g(x_{n-1}))} \varphi(t)dt\right) + \psi\left(\int_0^{p(g(x_{n-1}),g(x_{n+m-1}))} \varphi(t)dt\right)\right) \\
&= \psi\left(\Phi\left(\int_0^{p(x_{n-1},f(x_{n-2}))} \varphi(t)dt\right) + \psi\left(\int_0^{p(f(x_{n-2}),f(x_{n+m-2}))} \varphi(t)dt\right)\right) \\
&= \psi^2\left(\int_0^{p(f(x_{n-2}),f(x_{n+m-2}))} \varphi(t)dt\right) \\
&\leq \cdots \leq \psi^n\left(\int_0^{p(f(x_0),f(x_m))} \varphi(t)dt\right) \leq \psi^n\left(\int_0^{\delta_p(X)} \varphi(t)dt\right),
\end{aligned}$$

from which we have that

$$\int_0^{p(f(x_n),f(x_{n+m}))} \varphi(t)dt \leq \psi^n\left(\int_0^{\delta_p(X)} \varphi(t)dt\right), \tag{4}$$

where $p(f(x_0), f(x_m)) \leq \delta_p(X), \ \delta_p(X) = \sup \ \{ \ \mathrm{p(x, y)} \mid \mathrm{x}, \ \mathrm{y} \in \mathrm{X}\} < \infty$
and $\int_0^{\delta_\mathrm{p}(\mathrm{X})} \varphi(\mathrm{t})\mathrm{dt} > 0$ (by the condition on φ).
Therefore, using the definition of comparison function in (5) yields

$$\psi^n\left(\int_0^{\delta_p(X)} \varphi(t)dt\right) \to 0 \text{ as n} \to \infty,$$

from which it follows that

$$\int_0^{p(f(x_n),f(x_{n+m}))} \varphi(t)dt \to 0 \text{ as n} \to \infty,$$

so that $\lim\limits_{n\to\infty} p(f(x_n), f(x_{n+m})) = 0$ since $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$.
Hence, by applying Lemma 2.7 (c), we have that $\{f(x_n)\}_{n=0}^\infty$ is a p-Cauchy sequence.
Since X is $S-$complete, $\lim\limits_{n\to\infty} p(f(x_n), u) = 0$, for some $u \in X$. Therefore $\lim\limits_{n\to\infty} p(g(x_n), u) = 0$. Since f and g are $p-$continuous, then
$\lim\limits_{n\to\infty} p(f(g(x_n)), f(u)) = \lim\limits_{n\to\infty} p(g(f(x_n)), g(u)) = 0$.
Also, since f and g are commuting, then $fg = gf$, so that we have
$\lim\limits_{n\to\infty} p(f(g(x_n)), f(u)) = \lim\limits_{n\to\infty} p(f(g(x_n)), g(u)) = 0,$

and by Lemma 2.7 (a), we obtain $f(u) = g(u)$.
Since $f(u) = g(u),\ fg = gf$, we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$.
Suppose that $p(f(u), f(f(u))) \neq 0$. Using condition (3), then we have that

$$\begin{aligned}\int_0^{p(f(u),f(f(u)))} \varphi(t)dt &\leq \Phi\left(\int_0^{p(u,g(u))} \varphi(t)dt\right) + \psi\left(\int_0^{p(g(u),g(f(u)))} \varphi(t)dt\right)\\ &= \Phi\left(\int_0^{p(u,f(u))} \varphi(t)dt\right) + \psi\left(\int_0^{p(f(u),f(f(u)))} \varphi(t)dt\right)\\ &= \psi\left(\int_0^{p(f(u),f(f(u)))} \varphi(t)dt\right) < \int_0^{p(f(u),f(f(u)))} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Thus, $\int_0^{p(f(u),f(f(u))} \varphi(t)dt = 0$ (by the fact that $\int_0^{\epsilon} \varphi(t)dt > 0$ for each $\epsilon > 0$). Therefore, it implies that $p(f(u), f(f(u)) = 0$.
Also, by using condition (ii) of the theorem, then we have that $p(f(u), f(u)) = 0$.
Since $p(f(u), f(u)) = 0$ and p(f(u), f(f(u))) = 0, then using Lemma 2.7 (a) yields $f(f(u)) = f(u)$.
Thus, we have $g(f(u)) = f(f(u)) = f(u)$. Hence, $f(u)$ is a common fixed point of f and g.
The proof is similar when f and g are $\tau(\Phi)-$continuous as $S-$completeness implies $p-$Cauchy completeness.
We now prove the uniqueness of the common fixed point of f and g :
Suppose not. Then, there exist $u,\ v \in X$ such that $f(u) = g(u) = u$ and
$f(v) = g(v) = v$. Let $p(u, v) \neq 0$. Then, we have

$$\begin{aligned}\int_0^{p(u,v)} \varphi(t)dt = \int_0^{p(f(u),f(v))} \varphi(t)dt &\leq \Phi\left(\int_0^{p(u,g(u))} \varphi(t)dt\right) + \psi\left(\int_0^{p(g(u),g(v))} \varphi(t)dt\right)\\ &= \Phi\left(\int_0^{p(u,f(u))} \varphi(t)dt\right) + \psi\left(\int_0^{p(f(u),f(v))} \varphi(t)dt\right)\\ &= \psi\left(\int_0^{p(u,v)} \varphi(t)dt\right) < \int_0^{p(u,v)} \varphi(t)dt,\end{aligned}$$

from which we have that
$\int_0^{p(u,v)} \varphi(t)dt = 0$, by the condition on φ. Therefore, it implies that $p(u, v) = 0$.
In a similar manner, we also have that $p(v, u) = 0$.
Using condition (p_2) of $E-$distance, we have $p(u, u) \leq p(u, v) + p(v, u)$,
from which it follows that $p(u, u) = 0$.
Since $p(u, u) = 0$ and $p(u, v) = 0$, then by Lemma 2.7 (a), we have that $u = v$.

Corollary 3.2: Let (X, Φ) be a Hausdorff uniform space and p an $E-$distance on X. Suppose that X is $p-$bounded and $S-$complete. Let f and g be commuting $p-$continuous or $\tau(\Phi)-$continuous selfmappings of X such that
(i) $f(X) \subseteq g(X)$;
(ii) $p(f(x_i), f(x_i)) = 0,\ \forall\ x_i \in X,\ i = 0, 1, 2, \cdots$;
(iii) both $f,\ g : X \to X$ satisfy the contractive condition (3).
Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a comparison function and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a Lebesgue-integrable mapping which is summable, nonnegative and such that for each $\epsilon > 0,\ \int_0^{\epsilon} \varphi(t)dt > 0$.

Then, f and g have a unique common fixed point.
Proof: The proof of Corollary 3.2 follows a similar argument as in Theorem 3.1.
Remark 3.3: The results in this paper are generalizations and extensions of both Theorem 3.1 and Theorem 3.2 of Aamri and El Moutawakil [1], Theorem 2.1 of Branciari [5] as well as some results of Olatinwo [10, 11, 12].

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STATISTICAL APPROXIMATION TO PERIODIC FUNCTIONS BY A GENERAL CLASS OF LINEAR OPERATORS

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Abstract. In this paper, we are considering A-statistical convergence and by using various matrix summability methods we present an approximation theorem, which is a non-trivial generalization of Baskakov's result [5] regarding the approximation to periodic functions by a general class of linear operators.

1. Introduction

Recent studies demonstrate that the notion of statistical convergence, which was first introduced by Fast [1], plays an important role in the approximation theory (see, e.g., [2, 3, 4]). This type of convergence method is quite effective, especially when the classical limit fails. The aim of this study is to obtain a general approximation theorem via statistical convergence, which generalizes Baskakov's results (see [5]) on the approximation to periodic functions by means of a general class of linear operators.

Consider the sequence of linear operators

$$L_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)U_n(t)dt, \qquad f \in C_{2\pi} \text{ and } n = 1, 2, ..., \tag{1.1}$$

where

$$U_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos kt.$$

As usual, $C_{2\pi}$ denotes the space of all 2π-periodic and continuous functions on the whole real line, endowed with the norm

$$\|f\|_{C_{2\pi}} := \sup_{x\in\mathbb{R}} |f(x)|, \quad f \in C_{2\pi}.$$

If $U_n(t) \geq 0$, $t \in [0, \pi]$, then the operators (1.1) are positive. In this case, Korovkin [6] proved the following approximation theorem:

Theorem A [6]. *If* $\lim_{n\to\infty} \lambda_1^{(n)} = 1$ *and* $U_n(t) \geq 0$ *for all* $t \in [0, \pi]$ *and* $n \in N$, *then, for all* $f \in C_{2\pi}$,

$$\lim_{n\to\infty} L_n(f;x) = f(x) \text{ uniformly with respect to all } x \in \mathbb{R}.$$

Observe that Theorem A is valid for the positive linear operators (1.1) we consider. However, Baskakov [5] shows that an analogous result is also valid for a more general class of linear operators that are not necessarily positive. In this paper,

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using the concept of statistical convergence we give a generalization of both of the results of Korovkin and Baskakov.

Before proceeding further we recall some basic definitions and notation used in the paper.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. Then, the asymptotic density of K, denoted by $\delta(K)$, is given by

$$\delta\{K\} := \lim_j \frac{1}{j} \left|\{n \leq j : n \in K\}\right|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set B. A number sequence (x_n) is statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta\left\{n : |x_n - L| \geq \varepsilon\right\} = 0,$$

or, equivalently,

$$\lim_j \frac{1}{j} \left|\{n \leq j : |x_n - L| \geq \varepsilon\}\right| = 0$$

for every $\varepsilon > 0$. In this case, we write $st - \lim_n x_n = L$. Note that convergent sequences are statistically convergent, but the converse is not always true. Although every convergent sequence is bounded, a statistically convergent sequence does not need to be bounded. If

$$\delta\left\{n : |x_n| > M\right\} = 0 \;\text{ for some } M > 0,$$

then we say that (x_n) is statistically bounded. Of course, every bounded sequence is statistically bounded but not conversely. However, we can easily see that every statistically convergent sequence must be statistically bounded. Connor [7] proved the following useful characterization for statistical convergence.

Theorem B [7]**.** *$st - \lim_n x_n = L$ if and only if there exists an index set K with $\delta\{K\} = 1$ such that $\lim\limits_{n \in K} x_n = L$, i.e., for every $\varepsilon > 0$, there is a number $n_0 \in K$ such that $|x_n - L| < \varepsilon$ holds for all $n \geq n_0$ with $n \in K$.*

Other important properties of statistical convergence may be found in the papers [1, 8, 9]. Now let $A := (a_{jn})$, $j, n = 1, 2, ...$, be an infinite summability matrix. For a given sequence (x_n), the A- transform of x, denoted by $((Ax)_j)$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$$

provided the series converges for each $j \in \mathbb{N}$. We say that A is regular [10] if

$$\lim_j (Ax)_j = L \quad \text{ whenever } \quad \lim_n x_n = L\,.$$

Assume that A is a non-negative regular summability matrix. Using this type matrices Freedman and Sember [11] extent the statistical convergence to the concept of A-statistical convergence as follows:

The A-density of a subset K of $\mathbb{N}$ is defined by

$$\delta_A\{K\} = \lim_j \sum_{n \in K} a_{jn}$$

provided that the limit exists. Of course, if we take $A = C_1 = (c_{jn})$, the Cesáro matrix given by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise,} \end{cases}$$

then $\delta_{C_1}\{K\} = \delta\{K\}$. As in the definition of statistical convergence, we say that (x_n) is A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\delta_A \{n : |x_n - L| \geq \varepsilon\} = 0,$$

or, equivalently,

$$\lim_j \sum_{n \,:\, |x_n - L| \geq \varepsilon} a_{jn} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$ (see, e.g., [9, 11, 12]). Observe that if $A = C_1$, then C_1-statistical convergence coincides with statistical convergence. Also, if

$$\delta_A \{n : |x_n| > M\} = 0 \ \text{ for some } M > 0,$$

then (x_n) is called A-statistically bounded. It is not hard to see that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A. This follows from the well-known regularity conditions of A introduced by Silverman and Toeplitz (see, for instance, Hardy [13, pp. 43-45]); but its converse is not always true. Actually, if $A = (a_{jn})$ is any nonnegative regular summability matrix satisfying the condition

$$\lim_j \max_n \{a_{jn}\} = 0,$$

then A-statistical convergence is stronger than convergence (see [9]). We should note that Theorem B is also valid for A-statistical convergence (see [12]).

2. A Statistical Approximation Theorem

We denote by E the class of operators L_n as in (1.1) such that the integrals

$$\int_t^{\pi/2} \int_{t_1}^{\pi} U_n(t_2)dt_2dt_1, \quad 0 \leq t < \frac{\pi}{2},$$

$$\int_{\pi/2}^{t} \int_{t_1}^{\pi} U_n(t_2)dt_2dt_1, \quad \frac{\pi}{2} \leq t \leq \pi,$$

are non-negative. Obviously, the class E contains the class of positive linear operators L_n with $U_n(t) \geq 0$, $t \in [0, \pi]$.

Now we are ready to give our main result.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. If the sequence of operators (1.1) belongs to the class E, and if the following conditions*

(a) $st_A - \lim_n \lambda_1^{(n)} = 1$,

(b) $\delta_A \left\{ n : \|L_n\| = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} |U_n(t)|\, dt > M \right\} = 0$

hold for some $M > 0$, *then, for all* $f \in C_{2\pi}$, *we have*

$$st_A - \lim_n \|L_n(f) - f\|_{C_{2\pi}} = 0.$$

Proof. Since the functions $\cos t$ and $U_n(t)$ are even, we may write from (1.1) that

$$1 - \lambda_1^{(n)} = \frac{2}{\pi}\int_0^{\pi}(1-\cos t)U_n(t)dt.$$

Now integrating twice by parts of the above integral we have

$$\begin{aligned} 1 - \lambda_1^{(n)} &= \frac{2}{\pi}\int_0^{\pi} \sin t \left(\int_t^{\pi} U_n(t_1)dt_1\right) dt \\ &= \frac{2}{\pi}\int_0^{\pi} \cos t \left(\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right) dt. \end{aligned}$$

By the hypothesis (a), we see that

$$st_A - \lim_n \left\{\int_0^{\pi} \cos t \left(\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right)\right\} dt = 0. \tag{2.1}$$

Since the operators belong to E, the sign of the term inside the brackets is the same as the function $\cos t$ for all $t \in [0, \pi]$. So, it follows from (2.1) that

$$st_A - \lim_n \left\{\int_0^{\pi} \left|\cos t \left(\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right)\right| dt\right\} = 0. \tag{2.2}$$

Now we claim that

$$st_A - \lim_n \left\{\int_0^{\pi} \left|\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right| dt\right\} = 0. \tag{2.3}$$

To establish this, for any $\varepsilon > 0$, we first choose $\delta = \delta(\varepsilon)$ such that $0 < \delta < \sqrt{\dfrac{\varepsilon}{M\pi}}$. Since

$$\begin{aligned} \int_0^{\pi} \left|\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right| dt &\le \int_{|t-\pi/2|\le\delta} \left|\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right| dt \\ &\quad + \int_{|t-\pi/2|>\delta} \left|\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right| dt, \end{aligned}$$

we write

$$\int_0^{\pi} \left|\int_t^{\pi/2}\int_{t_1}^{\pi} U_n(t_2)dt_2 dt_1\right| dt \le J_{n,1} + J_{n,2}, \tag{2.4}$$

where

$$J_{n,1} := \int\limits_{|t-\pi/2|\le\delta} \left| \int\limits_t^{\pi/2}\int\limits_{t_1}^{\pi} U_n(t_2)dt_2dt_1 \right| dt$$

and

$$J_{n,2} := \int\limits_{|t-\pi/2|>\delta} \left| \int\limits_t^{\pi/2}\int\limits_{t_1}^{\pi} U_n(t_2)dt_2dt_1 \right| dt$$

Setting, for some $M > 0$,

$$K := \{n : \|L_n\| > M\},$$

we get from (b) that $\delta_A\{\mathbb{N}\backslash K\} = 1$. Also, observe that

$$\left| \int\limits_{t_1}^{\pi} U_n(t_2)dt_2 \right| \le \frac{M\pi}{2}$$

holds for all $n \in \mathbb{N}\backslash K$ and for all $t_1 \in [0,\pi]$. Since $0 < \delta < \sqrt{\frac{\varepsilon}{M\pi}}$, we have

$$J_{n,1} < \varepsilon$$

for every $\varepsilon > 0$ and for all $n \in \mathbb{N}\backslash K$. This means that

$$\lim_{\substack{n\to\infty \\ (n\in\mathbb{N}\backslash K)}} J_{n,1} = 0$$

Since $\delta_A\{\mathbb{N}\backslash K\} = 1$, it follows from Theorem B that

$$st_A - \lim_n J_{n,1} = 0. \tag{2.5}$$

On the other hand, we get

$$J_{n,2} \le \left| \int\limits_{|t-\pi/2|>\delta} \frac{\cos t}{\cos(\pi/2-\delta)} \left(\int\limits_t^{\pi/2}\int\limits_{t_1}^{\pi} U_n(t_2)dt_2dt_1 \right) dt \right|,$$

which implies that

$$J_{n,2} \le \frac{1}{\cos(\pi/2-\delta)} \int\limits_0^{\pi} \cos t \left(\int\limits_t^{\pi/2}\int\limits_{t_1}^{\pi} U_n(t_2)dt_2dt_1 \right) dt$$

for all $n \in \mathbb{N}$. By (2.2), it is clear that

$$st_A - \lim_n J_{n,2} = 0. \tag{2.6}$$

Now, for a given $r > 0$, define the sets

$$\begin{aligned} D \quad &: \quad = \left\{ n : \int\limits_0^{\pi} \left| \int\limits_t^{\pi/2}\int\limits_{t_1}^{\pi} U_n(t_2)dt_2dt_1 \right| dt \ge r \right\}, \\ D_1 \quad &: \quad = \left\{ n : J_{n,1} \ge \frac{r}{2} \right\}, \\ D_2 \quad &: \quad = \left\{ n : J_{n,2} \ge \frac{r}{2} \right\}. \end{aligned}$$

Then, by (2.4), we immediately get that

$$D \subseteq D_1 \cup D_2,$$

and hence

$$\sum_{n \in D} a_{jn} \leq \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn} \tag{2.7}$$

holds for all $j \in \mathbb{N}$. Letting $j \to \infty$ in both sides of (2.7) and also using (2.5), (2.6), we conclude that

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which proves our claim (2.3). Now let m be an arbitrary non-negative integer. Since

$$\begin{aligned} \left|1 - \lambda_m^{(n)}\right| &= \left|\frac{2}{\pi}\int_0^{\pi}(1-\cos mt)U_n(t)dt\right| \\ &= \left|\frac{2m^2}{\pi}\int_0^{\pi}\cos mt\left(\int_t^{\pi/2}\int_{t_1}^{\pi}U_n(t_2)dt_2dt_1\right)dt\right| \\ &\leq \frac{2m^2}{\pi}\int_0^{\pi}\left|\int_t^{\pi/2}\int_{t_1}^{\pi}U_n(t_2)dt_2dt_1\right|dt, \end{aligned}$$

(2.3) implies, for every $m \geq 0$, that

$$st_A - \lim_n \lambda_m^{(n)} = 1.$$

The operators (1.1) can be written as follows:

$$L_n(f;x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\frac{1}{2} + \sum_{k=1}^{n}\cos k(t-x)\right\}dt,$$

see, e.g., [6, p. 68]. Then we observe that

$$L_n(1;x) = 1$$

and

$$\begin{aligned} L_n(\cos kt;x) &= \lambda_k^{(n)}\cos kx, \\ L_n(\sin kt;x) &= \lambda_k^{(n)}\sin kx \end{aligned}$$

for $k = 1, 2, ...$, and for all $n \in \mathbb{N}$, see, e.g., [6, p. 69]. Thus, we have

$$st_A - \lim_n \|L_n(f_m) - f_m\|_{C_{2\pi}} = 0,$$

where the set $\{f_m : m = 0, 1, 2, ...\}$ denotes the class $\{1, \cos x, \sin x, \cos 2x, \sin 2x, ...\}$. Since $\{f_0, f_1, f_2, ...\}$ is a fundamental system of $C_{2\pi}$ (see, for instance, [6]), for a given $f \in C_{2\pi}$, we can find a trigonometric polynomial P given by

$$P(x) = a_0 f_0(x) + a_1 f_1(x) + ... + a_m f_m(x)$$

such that for any $\varepsilon > 0$ the inequality

$$\|f - P\|_{C_{2\pi}} < \varepsilon \tag{2.8}$$

holds. By linearity of the operators L_n, we have

$$\|L_n(f) - L_n(P)\|_{C_{2\pi}} = \|L_n(f-P)\|_{C_{2\pi}} \leq \|L_n\| \, \|f-P\|_{C_{2\pi}} . \tag{2.9}$$

It follows from (2.8), (2.9) and (b) that, for all $n \in \mathbb{N} \backslash K$,

$$\|L_n(f) - L_n(P)\|_{C_{2\pi}} \leq M\varepsilon. \tag{2.10}$$

On the other hand, since

$$L_n(P;x) = a_0 L_n(f_0;x) + a_1 L_n(f_1;x) + ... + a_m L_n(f_m;x),$$

we obtain, for every $n \in \mathbb{N}$, that

$$\|L_n(P) - P\|_{C_{2\pi}} \leq C \sum_{i=0}^{m} \|L_n(f_i) - f_i\|_{C_{2\pi}} , \tag{2.11}$$

where $C = \max\{|a_0|, |a_1|, ..., |a_m|\}$. Thus, for every $n \in \mathbb{N} \backslash K$, we get from (2.8), (2.10) and (2.11) that
(2.12)

$$\begin{aligned} \|L_n(f) - f\|_{C_{2\pi}} &\leq \|L_n(f) - L_n(P)\|_{C_{2\pi}} + \|L_n(P) - P\|_{C_{2\pi}} + \|f-P\|_{C_{2\pi}} \\ &\leq (M+1)\varepsilon + C \sum_{i=0}^{m} \|L_n(f_i) - f_i\|_{C_{2\pi}} \end{aligned}$$

Now, for a given $r > 0$, choose $\varepsilon > 0$ such that $0 < (M+1)\varepsilon < r$. Then define the following sets:

$$\begin{aligned} E &: = \left\{ n \in \mathbb{N} \backslash K : \|L_n(f) - f\|_{C_{2\pi}} \geq r \right\}, \\ E_i &: = \left\{ n \in \mathbb{N} \backslash K : \|L_n(f_i) - f_i\|_{C_{2\pi}} \geq \frac{r - (M+1)\varepsilon}{(m+1)C} \right\}, \ i = 0, 1, ..., m. \end{aligned}$$

From (2.12), we easily check that

$$E \subseteq \bigcup_{i=0}^{m} E_i,$$

which yields, for every $j \in \mathbb{N}$,

$$\sum_{n \in E} a_{jn} \leq \sum_{i=0}^{m} \sum_{n \in E_i} a_{jn}. \tag{2.13}$$

Taking limit as $j \to \infty$ in both sides of (2.13) and using the hypothesis (a) we obtain that

$$\lim_j \sum_{n \in E} a_{jn} = 0.$$

So we have

$$st_A - \lim_n \|L_n(f) - f\|_{C_{2\pi}} = 0.$$

Theorem is proved. □

Concluding Remarks. If we replace the matrix A with the identity matrix, then our Theorem 2.1 reduces to Baskakov's result (see [5, Theorem 1]). Observe that if the matrix $A = (a_{jn})$ satisfies the condition $\lim_j \max_n \{a_{jn}\} = 0$, then Baskakov's result does not necessarily hold while Theorem 2.1 still holds. Furthermore, taking the Cesáro matrix C_1 instead of A, one can obtain the statistical version of Theorem 2.1.

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New Ideas Concerning Some Integral Inequality

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Abstract. In this paper general new theorems concerning integral inequalities are given. These theorems cover known results, generalizations and new results.

1. Introduction

In [4] the following result was proved

If $f \geq 0$ *is a continuous function on [0,1] such that*

$$\int_x^1 f(t)\,dt \geq \int_x^1 t\,dt, \qquad \forall x \in [0,1], \tag{1.1}$$

then

$$\int_0^1 f^{\alpha+1}(x)\,dx \geq \int_0^1 x^{\alpha} f(x)\,dx, \quad \forall \alpha > 0. \tag{1.2}$$

The following question was raised in [4]

If f *satisfies the above assumptions, under what additional assumptions can one claim that*

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\alpha} f^{\beta}(x)\,dx, \quad \forall \alpha, \beta > 0 \ ? \tag{1.3}$$

It was proved in [3] that *if* $f \geq 0$ *is a continuous function on [0,b] satisfying*

$$\int_x^b f^{\alpha}(t)\,dt \geq \int_x^b t^{\alpha}\,dt, \quad \alpha, b > 0, \quad \forall x \in [0,b], \tag{1.4}$$

then

$$\int_0^b f^{\alpha+\beta}(x)\,dx \geq \int_0^b x^{\alpha} f^{\beta}(x)\,dx, \quad \forall \beta > 0. \tag{1.5}$$

In [1] the authors gave an answer to the posed question of [4] by establishing the following

Theorem 1.1. *If* f *is nonnegative continuous satisfies (1.1), then*

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx \tag{1.6}$$

for every $\alpha \geq 1$ *and* $\beta > 0$.

Finally, the author in [2] generalized the result of [1] by giving the following

Theorem 1.2. *Suppose* $f, g \in L^1[a,b]$, $f, g \geq 0$, *g is nondecreasing. If*

$$\int_x^b f(t)\,dt \geq \int_x^b g(t)\,dt, \quad \forall x \in [a,b], \tag{1.7}$$

then

$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx, \quad \forall \alpha, \beta \geq 0,\ \alpha+\beta \geq 1. \tag{1.8}$$

2. Main Result

We state and prove the following

Theoem 2.1. *Suppose* $f, g, \varphi \geq 0$, $f, g : [a,b] \to \Re$, $\Phi, \phi, \varphi \in C([a,b], \Re)$, φ, g *are nondecreasing . If*

$$\int_x^b \Phi\big(f(t)\big)\,dt \geq \int_x^b \phi\big(g(t)\big)\,dt \quad \forall x \in [a,b], \tag{2.1}$$

then

$$\int_a^b \varphi\big(g(x)\big)\,\Phi\big(f(x)\big)\,dx \geq \int_a^b \varphi\big(g(x)\big)\phi\big(g(x)\big)dx\,. \tag{2.2}$$

If (2.1) reverses, then (2.2) reverses .

Proof . Integration by parts gives

$$\int_a^b \varphi'\big(g(x)\big)g'(x)\int_x^b \big(\Phi\big(f(t)\big)-\phi\big(g(t)\big)\big)dt\,dx$$

$$= \left[\varphi\big(g(x)\big)\int_x^b \big(\Phi\big(f(t)\big)-\phi\big(g(t)\big)\big)dt\right]_a^b - \int_a^b \varphi\big(g(x)\big)\big[-\big(\Phi\big(f(x)\big)-\phi\big(g(x)\big)\big)\big]dx$$

$$= -\varphi\big(g(a)\big)\int_a^b \big(\Phi\big(f(t)\big)-\phi\big(g(t)\big)\big)dt + \int_a^b \varphi\big(g(x)\big)\big(\Phi\big(f(x)\big)-\phi\big(g(x)\big)\big)dx\,.$$

Therefore

$$\int_a^b \varphi\big(g(x)\big)\big(\Phi\big(f(x)\big)-\phi\big(g(x)\big)\big)\,dx$$

$$= \int_a^b \varphi'\big(g(x)\big)g'(x)\int_x^b \big(\Phi\big(f(t)\big)-\phi\big(g(t)\big)\big)dt\,dx + \varphi\big(g(a)\big)\int_a^b \big(\Phi\big(f(t)\big)-\phi\big(g(t)\big)\big)dt$$

$$\geq 0\,.$$

Theoem 2.2. *Suppose* $f, g, \varphi \geq 0$, $f, g : [a,b] \to \Re$, $\Phi, \phi, \varphi \in C([a,b], \Re)$, φ, g *are nonincreasing . If*

(2.3) $$\int_a^x \Phi(f(t))\,dt \ \geq \int_a^x \phi(g(t))\,dt \qquad \forall x \in [a,b],$$

then

(2.4) $$\int_a^b \varphi(g(x))\,\Phi(f(x))\,dx \geq \int_a^b \varphi(g(x))\phi(g(x))dx.$$

If (2.3) reverses, then (2.4) reverses .

Proof . Integration by parts gives

$$\int_a^b \varphi'(g(x))g'(x)\int_a^x (\Phi(f(t)-\phi(g(t)))dt\,dx$$

$$= \left[\varphi(g(x))\int_a^x (\Phi(f(t))-\phi(g(t)))dt\right]_a^b - \int_a^b \varphi(g(x))(\Phi(f(x))-\phi(g(x)))dx$$

$$= \ \varphi(g(b))\int_a^b (\Phi(f(t))-\phi(g(t)))dt - \int_a^b \varphi(g(x))(\Phi(f(x))-\phi(g(x)))dx\,.$$

Therefore

$$\int_a^b \varphi(g(x))(\Phi(f(x))-\phi(g(x)))\ dx$$

$$= -\int_a^b \varphi'(g(x))g'(x)\int_a^x (\Phi(f(t)-\phi(g(t)))dt\,dx + \ \varphi(g(b))\int_a^b (\Phi(f(t))-\phi(g(t)))dt$$

$$\geq 0\ .$$

Theorem 2.3. *Suppose* $f, g, \varphi \geq 0, \quad f, g : [a,b] \to \Re,\ \Phi, \phi, \varphi \in C([a,b],\Re),$ $f, g, \varphi,$ *are nondecreasing*. *If*

(2.5) $$\int_{f(x)}^{f(b)} \Phi(t)\ dt \ \geq \int_{f(x)}^{f(b)} \phi(t)\ dt \qquad \forall x \in [a,b],$$

then

(2.6) $$\int_a^b \varphi(g(x))\,\Phi(f(x))\,dx \geq \int_a^b \varphi(g(x))\phi(f(x))dx\ .$$

If (2.5) reverses, then (2.6) reverses .

Proof . We have

$$\int_a^b \varphi'(g(x))g'(x)\ \int_{f(x)}^{f(b)} (\Phi(t)-\phi(t))\,dt\,dx$$

$$= \left[\varphi(g(x))\int_{f(x)}^{f(b)} (\Phi(t)-\phi(t))dt\right]_a^b - \int_a^b \varphi(g(x))[-(\Phi(f(x))-\phi(f(x)))]\,dx$$

$$= -\ \varphi(g(a))\int_{f(a)}^{f(b)} (\Phi(t)-\phi(t))dt + \ \int_a^b \varphi(g(x))(\Phi(f(x))-\phi(f(x)))\,dx\,,$$

which implies

$$\int_a^b \varphi(g(x))\big(\Phi(f(x)) - \phi(f(x))\big)\,dx$$

$$= \int_a^b \varphi'(g(x))\,g'(x) \int_{f(x)}^{f(b)} \big(\Phi(t) - \phi(t)\big)\,dt\,dx + \varphi(g(a)) \int_{f(a)}^{f(b)} \big(\Phi(t) - \phi(t)\big)dt$$

$$\geq 0\,.$$

Theorem 2.4. *Suppose* $f, g, \varphi \geq 0$, $f, g : [a,b] \to \Re$, $\Phi, \phi, \varphi \in C([a,b], \Re)$, f, g *are nondecreasing ,* φ *is nonincreasing . If*

(2.7) $$\int_{f(a)}^{f(x)} \Phi(t)\,dt \;\geq\; \int_{f(a)}^{f(x)} \phi(t)\,dt \qquad \forall x \in [a,b],$$

then

(2.8) $$\int_a^b \varphi(g(x))\,\Phi(f(x))\,dx \geq \int_a^b \varphi(g(x))\phi(g(x))dx\ .$$

If (2.7) reverses, then (2.8) reverses .

Proof. It is similar to the proof of theorem 2.3 and therefore it is omitted .

3. Applications

We start with the following new result which generalized a similar result to theorem 1.1 , but in this case we have interchanged the domains of α and β .

Theorem 3.1. *Let f be nonnegative continuous satisfying (1.7), then*

(3.1) $$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b f^{\beta}(x)\,g^{\alpha}(x)\,dx\,, \quad \forall\ \alpha > 0,\ \ \beta \geq 1.$$

Proof. Making use of theorem 2.1 by putting

$$\Phi = \phi = I, \quad \varphi(x) = x^{\gamma}, \quad \gamma > 0\,,$$

we have

(3.2) $$\int_a^b f(x)\,g^{\gamma}(x)\,dx \geq \int_a^b g^{\gamma+1}(x)\,dx, \quad \gamma > 0.$$

Applying the AG inequality, we have, for $\beta \geq 1$,

$$f^{\beta}(x) \geq (1-\beta)\,g^{\beta}(x) + \beta\, f(x)\, g^{\beta-1}(x)\,,$$

which implies

(3.3) $$f^{\beta}(x)\,g^{\alpha}(x) \geq (1-\beta)\,g^{\alpha+\beta}(x) + \beta\, f(x)\, g^{\alpha+\beta-1}(x)\,.$$

Since for $\alpha, \beta > 0$, we have

(3.4) $$\left(f^{\alpha}(x)-g^{\alpha}(x)\right)\left(f^{\beta}(x)-g^{\beta}(x)\right)\geq 0,$$

then via (3.3), the above implies

$$f^{\alpha+\beta}(x)-f^{\alpha}(x)\,g^{\beta}(x)\geq f^{\beta}(x)\,g^{\alpha}(x)-g^{\alpha+\beta}(x)$$
$$\geq \beta\left(f(x)\,g^{\alpha+\beta-1}(x)-g^{\alpha+\beta}(x)\right).$$

The result follows by integrating the above and making use of (3.2).

Theorem 1.2 can also be obtained from theorem 2.1 as follows

Theorem 3.2 [2] . *Let f,g be nonnegative continuous , g is nondecreasing. If (1.7) satisfied, then (1.8) valid .*

Proof. Making use of the AG inequality, we have

(3.5) $$f^{\alpha+\beta}(x)-g^{\alpha+\beta}(x)\geq(\alpha+\beta)\left(f(x)\,g^{\alpha+\beta-1}(x)-g^{\alpha+\beta}(x)\right),\quad \alpha+\beta\geq 1.$$

By (3.2), on integrating the above, we obtain

(3.6) $$\int_a^b\left(f^{\alpha+\beta}(x)-g^{\alpha+\beta}(x)\right)dx\geq(\alpha+\beta)\int_a^b\left(f(x)\,g^{\alpha+\beta-1}(x)-g^{\alpha+\beta}(x)\right)dx\geq 0\ .$$

Now, by the AG inequality again, we have

$$f^{\alpha+\beta}(x)-f^{\alpha}(x)\,g^{\beta}(x)\geq\frac{\beta}{\alpha}\left(f^{\alpha}(x)\,g^{\beta}(x)-g^{\alpha+\beta}(x)\right).$$

Integrating the above with the using of (3.6) we get

$$\int_a^b f^{\alpha+\beta}(x)\,dx-\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx\geq\frac{\beta}{\alpha}\left(\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx-\int_a^b g^{\alpha+\beta}(x)\,dx\right)$$
$$\geq\frac{\beta}{\alpha}\left(\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx-\int_a^b f^{\alpha+\beta}(x)\,dx\right).$$

The above implies

$$\left(1+\beta/\alpha\right)\left(\int_a^b f^{\alpha+\beta}(x)\,dx-\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx\right)\geq 0\ .$$

The following is a generalization of the result of [1] .

Theorem 3.3. *If* $f,g\geq 0$ *are continuous functions on [a,b] satisfying*

(3.7) $$\int_x^b f^{\alpha}(t)\,dt\geq\int_x^b g^{\alpha}(t)\,dt,\qquad\forall x\in[a,b],$$

then

(3.8) $$\int_a^b f^{\alpha+\beta}(x)\,dx\geq\int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx,\quad \alpha,\beta>0.$$

Proof. The inequality

$$\left(f^{\beta}(x)-g^{\beta}(x)\right)\left(f^{\alpha}(x)-g^{\alpha}(x)\right)\geq 0$$

implies

(3.9) $$f^{\beta}(x)\left(f^{\alpha}(x)-g^{\alpha}(x)\right)\geq g^{\beta}(x)\left(f^{\alpha}(x)-g^{\alpha}(x)\right).$$

On putting $\Phi(x)=\phi(x)=x^{\alpha}$, $\varphi(x)=x^{\beta}$, in theorem 2.1 and using (3.9), we obtain

$$\int_a^b f^{\beta}(x)\left(f^{\alpha}(x)-g^{\alpha}(x)\right)dx \ge \int_a^b g^{\beta}(x)\left(f^{\alpha}(x)-g^{\alpha}(x)\right)dx \ge 0 .$$

The result of [1] follows by putting in theorem 3.3, $g(x)=x$.

Another kind of such integral inequalities, the following reverse inequality .

Theorem 3.4. *Let f, g be nonnegative continuous functions defined on [a,b] and let* $0<\beta<\alpha<1$. *If*

(3.10) $$\int_x^b f^{-\beta}(t)\,dt \le \int_x^b g^{-\beta}(t)\,dt, \quad \forall x\in[a,b],$$

then

(3.11) $$\int_a^b f^{\alpha-\beta}(x)\,dx \le \int_a^b f^{-\beta}(x)\,g^{\alpha}(x)\,dx$$

Proof. The inequality

$$\left(f^{\alpha}(x)-g^{\alpha}(x)\right)\left(f^{-\beta}(x)-g^{-\beta}(x)\right)\le 0,$$

implies

$$\int_a^b f^{\alpha}(x)\left(f^{-\beta}(x)-g^{-\beta}(x)\right)dx \le \int_a^b g^{\alpha}(x)\left(f^{-\beta}(x)-g^{-\beta}(x)\right)dx .$$

But the RHS of the above inequality is ≤ 0, which follows from theorem 2.1, by putting $\Phi(x)=\phi(x)=x^{-\beta}$, $\varphi(x)=x^{\alpha}$, the result follows .

Theorem 3.5. *Let f, g be nonnegative continuous and let* $0<\beta<\alpha<1$.*If*

(3.12) $$\int_a^x f(t)\,dt \le \int_a^x g(t)\,dt, \quad \forall x\in[a,b],$$

then

(3.13) $$\int_a^b f^{\alpha-\beta}(x)\,dx \le \int_a^b g^{\alpha}(x)\,f^{-\beta}(x)\,dx .$$

Proof. Making use of the AG inequality, we have for $0<\alpha<1$,

$$f^{\alpha}(x)\le(1-\alpha)\,g^{\alpha}(x)+\alpha\,f(x)\,g^{\alpha-1}(x) ,$$

which implies

(3.14) $$f^{\alpha}(x)\,g^{-\beta}(x)\le(1-\alpha)\,g^{\alpha-\beta}(x)+\ f(x)\,g^{\alpha-\beta-1}(x) .$$

The inequality $\left(f^{\alpha}(x)-g^{\alpha}(x)\right)\left(f^{-\beta}(x)-g^{-\beta}(x)\right)\le 0$, with (3.14) implies

$$\int_a^b \left(f^{\alpha-\beta}(x)-g^{\alpha}(x)\,f^{-\beta}(x)\right)dx \le \int_a^b \left(f^{\alpha}(x)\,g^{-\beta}(x)-g^{\alpha-\beta}(x)\right)dx$$

$$\le\alpha\int_a^b \left(f(x)\,g^{\alpha-\beta-1}(x)-g^{\alpha-\beta}(x)\right)dx .$$

The last integral is nonpositive, which follows from theorem 2.2, by putting

$$\Phi=\phi=I, \quad \varphi(x)=x^{\alpha-\beta-1} .$$

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Exact Orders in Simultaneous Approximation by Complex Bernstein Polynomials

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Abstract

In this paper we obtain the exact orders in approximation by complex Bernstein polynomials and their derivatives on compact disks.

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Keywords and Phrases : Complex Bernstein polynomials, exact orders in simultaneous approximation.

1 Introduction

The following Bernstein's result is classical.

Theorem 1.1. (see e.g. [4, p. 88]) *Denoting* $\mathbb{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$, *if* $G \subset \mathbb{C}$ *is open, so that* $\overline{\mathbb{D}}_1 = \{z \in \mathbb{C}; |z| \leq 1\} \subset G$ *and if* $f : G \to \mathbb{C}$ *is analytic in* G, *then the complex Bernstein polynomials* $B_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^k (1-z)^{n-k} f(k/n)$, *uniformly converge to* f *in* $\overline{\mathbb{D}}_1$.

An upper estimate of this uniform convergence was found in several recent works, see e.g. [1-3], [5] and can be expressed by the following.

Theorem 1.2. *Let* $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ *denote the open disk of radius* $R > 1$ *and center* 0, *and let us suppose that* $f : \mathbb{D}_R \to \mathbb{C}$ *is analytic in* $\mathbb{D}_R$, *that is we can write*

$f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. Then for all $n\in\mathbb{N}$ and $1\leq r<R$ we have

$$\|B_n(f)-f\|_r\leq O[1/n],$$

with explicit constants depending on f and r in $O\left[\frac{1}{n}\right]$ (for example, by e.g. [3] we can write $\|B_n(f)-f\|_r\leq\frac{M_r(f)}{n}$, where $0<M_r(f)=2\sum_{j=2}^{\infty}j(j-1)|c_j|r^j<\infty$). Here $\|\cdot\|_r$ denotes the uniform norm in $\overline{\mathbb{D}}_r$.

Also, recently we proved the following Voronovskaja's theorem in complex setting.

Theorem 1.3. ([2])*Let $R>1$ and suppose that $f:\mathbb{D}_R\to\mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. For any $r\in[1,R)$ and $n\in\mathbb{N}$ we have*

$$\|B_n(f)-f-\frac{e_1(1-e_1)}{2n}f''\|_r\leq\frac{5K_r(f)(1+r)^2}{2n^2},$$

where $e_1(z)=z$ and $K_r(f)=\sum_{k=3}^{\infty}|c_k|k(k-1)(k-2)^2r^{k-2}<\infty$.

In Section 2 we prove that if the analytic function f is not a polynomial of degree ≤ 1, then we have $\|B_n(f)-f\|_r\geq\frac{C_r(f)}{n}$, $n\in\mathbb{N}$, that is in Theorem 1.2 in fact the equivalence $\|B_n(f)-f\|_r\sim\frac{1}{n}$ holds. In Section 3 we prove that for any $p\in\mathbb{N}$, $r\geq 1$, if f is not a polynomial of degree $\leq\max\{1,p-1\}$, then we have $\|B_n^{(p)}(f)-f^{(p)}\|_r\sim\frac{1}{n}$, where the constants in the equivalence depend on f, r and p.

2 Approximation by Complex Bernstein Polynomials

The main result of this section is the following.

Theorem 2.1. *Let $R>1$, $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$ and let us suppose that $f:\mathbb{D}_R\to\mathbb{C}$ is analytic in $\mathbb{D}_R$, that is we can write $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. If f is not a polynomial of degree ≤ 1, then for any $r\in[1,R)$ we have*

$$\|B_n(f)-f\|_r\geq\frac{C_r(f)}{n},n\in\mathbb{N},$$

where the constant $C_r(f)$ depends only on f and r.

Proof. For all $z\in\mathbb{D}_R$ and $n\in\mathbb{N}$ we have

$$B_n(f)(z)-f(z)=$$

$$\frac{1}{n}\left\{\frac{z(1-z)}{2}f''(z)+\frac{1}{n}\left[n^2\left(B_n(f)(z)-f(z)-\frac{z(1-z)}{2n}f''(z)\right)\right]\right\}.$$

In what follows we will apply to this identity the following obvious property :

$$\|F+G\|_r\geq|\,\|F\|_r-\|G\|_r\,|\geq\|F\|_r-\|G\|_r.$$

It follows

$$\|B_n(f)-f\|_r \geq \frac{1}{n}\left\{\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|B_n(f)-f-\frac{e_1(1-e_1)}{2n}f''\right\|_r\right]\right\}.$$

Taking into account that by hypothesis f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, we get $\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z(1-z)}{2}f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r$, which implies $f''(z) = 0$ for all $z \in \overline{\mathbb{D}}_r \setminus \{0, 1\}$. Since f is supposed to be analytic, from the identity theorem of analytic (holomorphic) functions this necessarily implies that $f''(z) = 0$, for all $z \in \mathbb{D}_R$, i.e. that f is a polynomial of degree ≤ 1, which is a contradiction.

But by Theorem 1.3 we have

$$n^2\left\|B_n(f)-f-\frac{e_1(1-e_1)}{2n}f''\right\|_r \leq \frac{5K_r(f)(1+r)^2}{2}.$$

Therefore, there exists an index n_0 depending only on f and r, such that for all $n \geq n_0$ we have

$$\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r - \frac{1}{n}\left[n^2\left\|B_n(f)-f-\frac{e_1(1-e_1)}{2n}f''\right\|_r\right] \geq \frac{1}{2}\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r,$$

which immediately implies

$$\|B_n(f)-f\|_r \geq \frac{1}{n}\cdot\frac{1}{2}\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r, \forall n \geq n_0.$$

For $n \in \{1, ..., n_0 - 1\}$ we obviously have $\|B_n(f)-f\|_r \geq \frac{M_{r,n}(f)}{n}$ with $M_{r,n}(f) = n\cdot\|B_n(f)-f\|_r > 0$, which finally implies $\|B_n(f)-f\|_r \geq \frac{C_r(f)}{n}$ for all n, where $C_r(f) = \min\{M_{r,1}(f), ..., M_{r,n_0-1}(f), \frac{1}{2}\left\|\frac{e_1(1-e_1)}{2}f''\right\|_r\}$. This completes the proof.

Combining now Theorem 2.1 with Theorem 1.2 we immediately get the following.

Corollary 2.2. *Let $R > 1$, $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$. If f is not a polynomial of degree ≤ 1, then for any $r \in [1, R)$ we have*

$$\|B_n(f)-f\|_r \sim \frac{1}{n}, n \in \mathbb{N},$$

where the constants in the equivalence depend on f and r.

3 Approximation by Derivatives of Complex Bernstein Polynomials

The main result of this section is the following.

Theorem 3.1. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{D}_R \to \mathbb{C}$ is analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \le r < r_1 < R$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\le \max\{1, p-1\}$, then we have*

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend on f, r, r_1 and p.

Proof. Denoting by Γ the circle of radius $r_1 >$ and center 0 (where $r_1 > r \ge 1$), by the Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$ we have

$$B_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_\Gamma \frac{B_n(f)(v) - f(v)}{(v-z)^{p+1}} dv,$$

which by Theorem 1.2 and by the inequality $|v - z| \ge r_1 - r$ valid for all $|z| \le r$ and $v \in \Gamma$, immediately implies

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \le \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}} \|B_n(f) - f\|_{r_1} \le M_{r_1}(f) \frac{p! r_1}{n(r_1 - r)^{p+1}}.$$

It remains to prove the lower estimate for $\|B_n^{(p)}(f) - f^{(p)}\|_r$. For this purpose, as in the proof of Theorem 2.1, for all $v \in \Gamma$ and $n \in \mathbb{N}$ we have

$$B_n(f)(v) - f(v) =$$

$$\frac{1}{n}\left\{\frac{v(1-v)}{2} f''(v) + \frac{1}{n}\left[n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n} f''(v)\right)\right]\right\},$$

which replaced in the above Cauchy's formula implies

$$B_n^{(p)}(f)(z) - f^{(p)}(z) = \frac{1}{n}\left\{\frac{p!}{2\pi i} \int_\Gamma \frac{v(1-v)f''(v)}{2(v-z)^{p+1}} dv + \right.$$

$$\left.\frac{1}{n} \cdot \frac{p!}{2\pi i} \int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n} f''(v)\right)}{(v-z)^{p+1}} dv\right\} =$$

$$\frac{1}{n}\left\{\left[\frac{z(1-z)}{2} f''(z)\right]^{(p)} + \right.$$

$$\left.\frac{1}{n} \cdot \frac{p!}{2\pi i} \int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n} f''(v)\right)}{(v-z)^{p+1}} dv\right\}.$$

Passing now to $\|\cdot\|_r$ it follows

$$\|B_n^{(p)}(f) - f^{(p)}\|_r \ge \frac{1}{n}\left\{\left\|\left[\frac{e_1(1-e_1)}{2} f''\right]^{(p)}\right\|_r - \right.$$

$$\left.\frac{1}{n}\left\|\frac{p!}{2\pi} \int_\Gamma \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n} f''(v)\right)}{(v-z)^{p+1}} dv\right\|_r\right\},$$

where by using Theorem 1.3 we get

$$\left\| \frac{p!}{2\pi} \int_{\Gamma} \frac{n^2\left(B_n(f)(v) - f(v) - \frac{v(1-v)}{2n}f''(v)\right)}{(v-z)^{p+1}} dv \right\|_r \leq$$

$$\frac{p!}{2\pi} \cdot \frac{2\pi r_1 n^2}{(r_1-r)^{p+1}} \left\| B_n(f) - f - \frac{e_1(1-e_1)}{2n} f'' \right\|_{r_1} \leq$$

$$\frac{5K_{r_1}(f)(1+r_1)^2}{2} \cdot \frac{p! r_1}{(r_1-r)^{p+1}}.$$

But by hypothesis on f we have $\left\| \left[\frac{e_1(1-e_1)}{2} f'' \right]^{(p)} \right\|_r > 0$. Indeed, supposing the contrary it follows that $\frac{z(1-z)}{2} f''(z)$ is a polynomial of degree $\leq p-1$. Now, if $p=1$ and $p=2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq 1 = \max\{1, p-1\}$, which contradicts the hypothesis. If $p > 2$ then the analyticity of f obviously implies that f necessarily is a polynomial of degree $\leq p-1 = \max\{1, p-1\}$, which again contradicts the hypothesis.

In continuation reasoning exactly as in the proof of Theorem 2.1, we immediately get the desired conclusion.

Remark. Let us suppose that $f^{(p)} \in C[0,1]$, $p \in \mathbb{N}$. By taking $r=1$ and $\lambda = 1$ in [6, Theorem 2], we immediately obtain the following upper estimate for the derivatives of the real Bernstein polynomials attached to f, valid for all $n \geq n_p$

$$\|B_n^{(p)}(f) - f^{(p)}\| \leq A_p[\omega_1(f^{(p)}; 1/n) + \omega_2^{\varphi}(f^{(p)}; 1/\sqrt{n}) + \|f^{(p)}\|/n],$$

where $\|\cdot\|$ denotes the uniform norm on $C[0,1]$, $n_p \in \mathbb{N}$ depends only on p, ω_1 denotes the uniform modulus of continuity, $\varphi(x) = \sqrt{x(1-x)}$ and ω_2^{φ} denotes the Ditzian-Totik second order modulus of smoothness.

Then, the above Theorem 3.1 suggests the following open question : for any $p \in \mathbb{N}$, there exist the positive constants C_p and n_p depending only on p, such that for all $n \geq n_p$

$$C_p[\omega_1(f^{(p)}; 1/n) + \omega_2^{\varphi}(f^{(p)}; 1/\sqrt{n}) + \|f^{(p)}\|/n] \leq \|B_n^{(p)}(f) - f^{(p)}\|.$$

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Modified Adomian decomposition method for Solving singular two-point boundary value problems *

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Abstract

In this paper, modified Adomian decomposition method for solving singular two-point boundary value problems is formulated. The proposed method can be applied to linear and nonlinear problems. The scheme is tested for some examples and the obtained results demonstrate efficiency of the proposed method.

AMS:65L10

Keywords: Adomian decomposition method; Singular two-point boundary value problems

1 Introduction

The singular two-point boundary value problems arise from many engineering and physics applications. Several numerical methods for solving singular non-linear differential equations were studied in [2,4,5,10]. Russell and Shampine [9] have shown

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that for(linear) $f(x,y) = kx + g(x)$ has a unique solution. Manoj Kumar [6,7,8] have suggested a new finite difference method, the fourth-order finite difference method and a higher order method for solving singular two-point boundary value problems(1), where $\alpha \in (0,1)$. Al-Gahatani[1] proposed integral formulation for the solution of a class of second-order boundary value problems which are described by the equation $y'' + P(x,y,y',y'') = 0$,$x \in (0,a)$. He solved the resulting integral equation by expressing the dependent variable y as a power series which made the computation of various integrals possible. The goal of this paper is to introduce a new reliable modification of Adomian decomposition method. For this reason, a new differential operator is proposed which can be used for singular two-point boundary value problem

$$y'' + \frac{\alpha}{x}y' = g(x) + f(x,y), \tag{1}$$

under the boundary condition

$$y(0) = A, y(c) = B, c \neq 0$$

where $\alpha \leq 1$ and A, B are finite constants. We assume that, $f(x,y)$ is a continuous real values function for every $x \in (0,1)$.

Main idea of the method is to create a canonical form containing all boundary conditions so that the zeroth component is explicitly determined without additional calculations and all other components are also easily determined. Recently, A modified form of the decomposition method was developed by Wazwaz [11,12]. Convergence of Adomian's method proved by Cherruault et al.[3].

2 Analysis of the method

We propose the new differential operator, as below

$$L = x^{-1}\frac{d}{dx}x^{2-\alpha}\frac{d}{dx}x^{-1+\alpha}, \tag{2}$$

so, the problem(1) can be written as,

$$Ly = g(x) + f(x,y), \tag{3}$$

The inverse operator L^{-1} is therefore considered a two-fold integrals operator, as below,

$$L^{-1}(.) = x^{1-\alpha} \int_c^x x^{-2+\alpha} \int_0^x x(.)dxdx. \tag{4}$$

By operating L^{-1} on problem(3), we have

$$y(x) = A + (B - A)c^{-1+\alpha}x^{1-\alpha} + L^{-1}g(x) + L^{-1}f(x, y), \tag{5}$$

where

$$y(c) = B, y(0) = A.$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $f(x, y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{6}$$

and

$$f(x, y) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [11] to formulate Adomian polynomials. The following algorithm:

$$\begin{aligned} A_0 &= F(u), \\ A_1 &= F^{'}(u_0)u_1, \\ A_2 &= F^{'}(u_0)u_2 + \frac{1}{2}F^{''}(u_0)u_1^2, \\ A_3 &= F^{'}(u_0)u_3 + F^{''}(u_0)u_1u_2 + \frac{1}{3!}F^{'''}(u_0)u_1^3, \\ &. \\ &. \\ &. \end{aligned} \tag{8}$$

can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting(6)and(7) into (5),

$$\sum_{n=0}^{\infty} y_n = A + (B - A)c^{-1+\alpha}x^{1-\alpha} + L^{-1}g(x) + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{9}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = A + (B - A)c^{-1+\alpha}x^{1-\alpha} + L^{-1}g(x), \tag{10}$$

$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

which gives

$$\begin{aligned} y_0 &= A + (B - A)c^{-1+\alpha}x^{1-\alpha} + L^{-1}g(x), \\ y_1 &= L^{-1}A_0, \\ y_2 &= L^{-1}A_1, \\ y_3 &= L^{-1}A_3, \\ &. \\ &. \\ &. \end{aligned} \tag{11}$$

From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (6) can be immediately obtained. For numerical purposes, the n-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} y_k, \tag{12}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial value problems.

3 Numerical illustrations

Example 1. We consider the linear boundary value problem :

$$y'' + \frac{\rho}{x} y' = -x^{1-\rho} \cos x - (2-\rho) x^{-\rho} \sin x, \tag{13}$$

$$y(0) = 0, y(1) = \cos 1.$$

We put

$$L(.) = x^{-1} \frac{d}{dx} x^{2-\rho} \frac{d}{dx} x^{-1+\rho}(.),$$

so

$$L^{-1}(.) = x^{1-\rho} \int_1^x x^{-2+\rho} \int_0^x x(.)dxdx.$$

In an operator form, Eq.(13) becomes

$$Ly = -x^{1-\rho} \cos x - (2-\rho) x^{-\rho} \sin x. \tag{14}$$

By applying L^{-1} to both sides of(14) we have

$$y = y(0) + (y(1) - y(0))x^{1-\rho} + L^{-1}(-x^{1-\rho} \cos x - (2-\rho) x^{-\rho} \sin x)$$
$$= x^{1-\rho} \cos 1 + x^{1-\rho} \int_1^x x^{-2+\rho} \int_0^x x(-x^{1-\rho} \cos x - (2-\rho) x^{-\rho} \sin x)dxdx,$$

and it implies,

$$y(x) = x^{1-\rho} \cos 1 - x^{1-\rho} \cos 1 + x^{1-\rho} \cos x = x^{1-\rho} \cos x.$$

so, the exact solution is easily obtained by this method.

Example 2. Consider the linear boundary value problem:

$$(x^{\alpha} y')' = \beta x^{\alpha+\beta-2}((\alpha+\beta-1) + \beta x^{\beta})y, \tag{15}$$

$$y(0) = 1, y(-1) = \frac{1}{e} = 0.367879,$$

with exact solution $y(x) = e^{x^{\beta}}$.

We solve the problem(15)for $\alpha = -2$,$\beta = 3$, rewrite (15)as

$$y'' - \frac{2}{x}y' = 9x^4 y, \tag{16}$$

we put

$$L() = x^{-1}\frac{d}{dx}x^4\frac{d}{dx}x^{-3}(),$$

so

$$L^{-1}(.) = x^3\int_{-1}^{x} x^{-4}\int_{0}^{x} x(.)dxdx.$$

In an operator form, Eq. (16) becomes

$$Ly = 9x^4 y. \tag{17}$$

Applying L^{-1} on both sides of (17)we find

$$y = y(1) + (y(0) - y(-1))x^3 + 9L^{-1}x^4 y.$$

Proceeding as before we obtained the recursive relationship

$$y_0 = 1 + 0.632121x^3,$$

$$y_{k+1} = 9L^{-1}x^4 y_k, k \geq 0.$$

This in turn gives

$$y_0 = 1 + 0.632121x^3,$$

$$y_1 = 0.394647x^3 + 0.5x^6 + 0.105353x^9,$$

$$y_2 = -0.0293754x^3 + 0.0657744x^9 + 0.0416666x^{12} + 0.0052676x^{15},$$

$$y_3 = 0.0028707x^3 - 0.0048959x^9 + 0.0032887x^{15} + 0.0013889x^{18} + 0.0001254x^{21},$$

$$y_4 = -0.0002889x^3 + 0.0004784x^9 - 0.0002448x^{15} + 0.0000783x^{21} + 0.0000248x^{24} + 1.741956 \times 10^{-6}x^{27}.$$

Consequently, the series solution is

$$y(x) = 1 + 0.9999738x^3 + 0.5x^6 + 0.1667104x^9 + 0.0416667x^{12} + 0.0083116x^{15} + 0.0013889x^{18}$$

$$+0.0002037x^{21} + 0.0000248x^{24} + 1.741956 \times 10^{-6}x^{27}$$

Not that the Taylor series of the exact solution $y(x) = e^{x^3}$ with order 27. is as below

$$e^{x^3} = 1+x^3+0.5x^6+0.166667x^9+0.0416667x^{12}+0.00833333x^{15}+0.00138889x^{18}+0.000198413x^{21}$$
$$+0.0000248016x^{24} + 2.75573 \times 10^{-6}x^{27}$$

Example 3. We consider the non-linear boundary value problem:

$$(x^{\alpha}y^{'})^{'} = \beta x^{\alpha+\beta-2}(\beta x^{\beta}e^{y} - (\alpha+\beta-1))/(4+x^{\beta}), \tag{18}$$
$$y(0) = \ln(\frac{1}{4}), y(1) = \ln(\frac{1}{5}),$$

with exact solution $y(x) = \ln(\frac{1}{4+x^{\beta}})$.

We solve (18) for $\alpha = -1, \beta = 2$, so we can rewrite (18)as

$$y^{''} - \frac{1}{x}y^{'} = \frac{4x^2}{4+x^2}e^{y}, \tag{19}$$
$$y(0) = \ln(\frac{1}{4}), y(1) = \ln(\frac{1}{5}),$$

We put

$$L() = x^{-1}\frac{d}{dx}x^3\frac{d}{dx}x^{-2}(),$$

so

$$L^{-1}(.) = x^2\int_1^x x^{-3}\int_0^x x(.)dxdx.$$

In an operator form, Eq. (19) becomes

$$Ly = \frac{4x^2}{4+x^2}e^{y}. \tag{20}$$

Applying L^{-1} on both sides of (20)we find

$$y = y(0) + (y(1) - y(0))x^2 + 4L^{-1}\frac{x^2}{4+x^2}e^{y}.$$

By modified Adomian decomposition method [12]we obtain

$$y_0 = \ln(\frac{1}{4}),$$

$$y_1 = \ln(\frac{4}{5})x^2 + 4L^{-1}\frac{x^2}{4+x^2}A_0,$$

$$y_{k+1} = 4L^{-1}\frac{x^2}{4+x^2}A_k, k \geq 1. \tag{21}$$

The Adomian polynomials for the nonlinear term $F(y) = e^y$ are computed as follows:

$$A_0 = e^{y_0},$$

$$A_1 = y_1 e^{y_0},$$
$$A_2 = (y_2 + \frac{1}{2}y_1^2)e^{y_0}, \tag{22}$$
$$A_3 = (y_3 + y_1 y_2 + \frac{1}{3!}y_1^3)e^{y_0},$$

Substituting (22) into (21) and it must noted that, to compute y_1 we use the Taylor series of $\frac{1}{4+x^2}$ with order 10. In this case we obtain

$$y_0 = \ln(\frac{1}{4}),$$

$$y_1 = -0.2520728x^2 + 0.03125x^4 - 0.0026042x^6 + 0.0003255x^8 - 0.0000488x^{10}$$
$$+8.1380208 \times 10^{-6}x^{12} - 1.4532180 \times 10^{-6}x^{14},$$

$$y_2 = 0.0098820x^2 - 0.0105030x^6 + 0.0006510x^8 - 0.0000326x^{10} + 2.7126736 \times 10^{-6}x^{12}$$
$$-2.9064360 \times 10^{-7}x^{14} + 3.6330450 \times 10^{-8}x^{16} - 5.0458959 \times 10^{-9}x^{18},$$

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In Fig.1 we have plotted $\sum_{i=0}^{4} y_i(x)$, which is almost equal to the exact solution $y(x) = \ln(\frac{1}{4+x^2})$.

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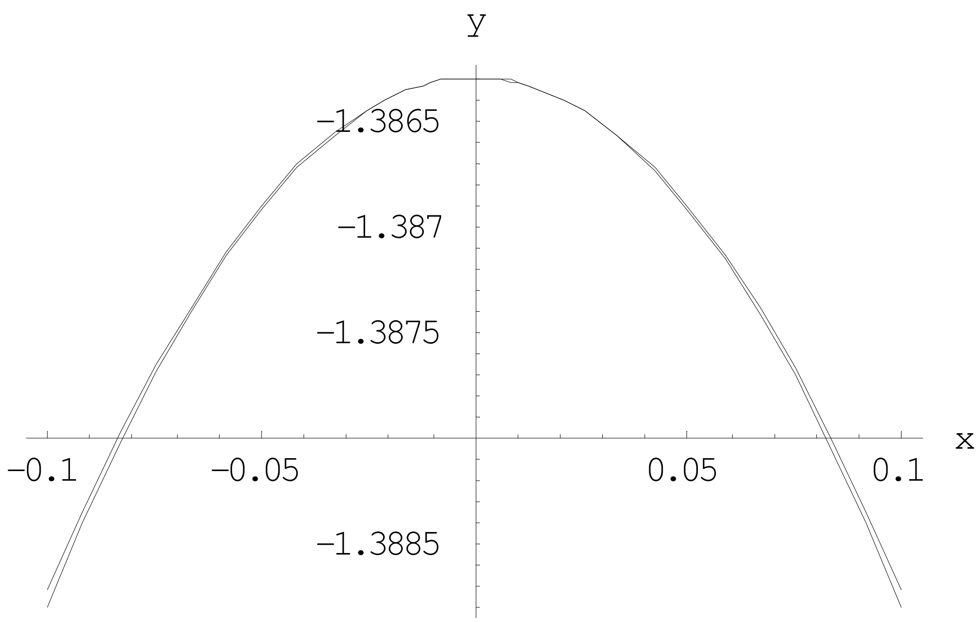

Fig. 1. The exact solution $y = ln(\frac{1}{4+x^2})$ and the Adomian decomposition solution $y = \sum_{i=0}^{4} y_{i(x)}$.

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Modified Adomian decomposition method for solving nonlinear oscillatory systems *

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Abstract

In this paper, an efficient modification of Adomian decomposition method is introduced for solving nonlinear oscillatory equations of the form

$$y''(x) + cy'(x) + \epsilon y(x) = f(x, y).$$

The proposed method can be applied to linear and nonlinear problems. The scheme is tested for some examples and the obtained results demonstrate efficiency of the proposed method.

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1 Introduction

The decomposition method has been shown[1-3,8-14] to solve eectively, easily and accurately a large class of linear and nonlinear, ordinary, partial, deterministic or stochastic dierential equations with approximate solutions which converge rapidly to accurate solutions. In recent years, many papers were devoted to the problem of approximate solution of nonlinear oscillatory equation [4-7]. The basic motivation of this work is to apply the modied Adomian decomposition method to the non-linear oscillatory equations. For this reason, a new differential operator is proposed which can be used for nonlinear oscillatory equations. In addition, the proposed method is tested for some examples and the obtained results show the advantage of using this method.

2 Modified Adomian decomposition method

Consider the non-linear oscillator equation written in the form

$$y^{''}(x) + cy^{'}(x) + \epsilon y(x) = f(x, y), \tag{1}$$

$$y(0) = a, y^{'}(0) = b,$$

where c is real number and ϵ is a parameter (not necessarily small). We propose the new differential operator, as below

$$L(.) = e^{-mx}\frac{d}{dx}e^{-hx}\frac{d}{dx}e^{(m+h)x}(.), \tag{2}$$

where $2m + h = c$, $m(m + h) = \epsilon$,

so, the problem (1) can be written as,

$$Ly = f(x, y). \tag{3}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(.) = e^{-(m+h)x}\int_0^x e^{hx}\int_0^x e^{mx}(.)dxdx. \tag{4}$$

By applying L^{-1} on (3), we have

$$y(x) = \frac{1}{h}y^{'}(0)e^{-mx} + \frac{(m+h)}{h}y(0)e^{-mx} - \frac{1}{h}y^{'}(0)e^{-(m+h)x} - \frac{m}{h}y(0)e^{-(m+h)x} + L^{-1}f(x,y). \tag{5}$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $f(x,y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{6}$$

and

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [8,12] to formulate Adomian polynomials. The following algorithm:

$$\begin{aligned} A_0 &= F(u), \\ A_1 &= F^{'}(u_0)u_1, \\ A_2 &= F^{'}(u_0)u_2 + \frac{1}{2}F^{''}(u_0)u_1^2, \\ A_3 &= F^{'}(u_0)u_3 + F^{''}(u_0)u_1u_2 + \frac{1}{3!}F^{'''}(u_0)u_1^3, \\ &. \\ &. \\ &. \end{aligned} \tag{8}$$

can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting(6)and(7) into (5),

$$\sum_{n=0}^{\infty} y_n = \frac{1}{h}y^{'}(0)e^{-mx} + \frac{(m+h)}{h}y(0)e^{-mx} - \frac{1}{h}y^{'}(0)e^{-(m+h)x} - \frac{m}{h}y(0)e^{-(m+h)x} + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{9}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = \frac{1}{h}y^{'}(0)e^{-mx} + \frac{(m+h)}{h}y(0)e^{-mx} - \frac{1}{h}y^{'}(0)e^{-(m+h)x} - \frac{m}{h}y(0)e^{-(m+h)x},$$

$$y_{n+1} = L^{-1}A_n, n \geq 0, \tag{10}$$

which gives

$$y_0 = \frac{1}{h}y^{'}(0)e^{-mx} + \frac{(m+h)}{h}y(0)e^{-mx} - \frac{1}{h}y^{'}(0)e^{-(m+h)x} - \frac{m}{h}y(0)e^{-(m+h)x},$$

$$\begin{aligned} y_1 &= L^{-1}A_0, \\ y_2 &= L^{-1}A_1, \\ y_3 &= L^{-1}A_3, \\ &. \\ &. \\ &. \end{aligned} \tag{11}$$

From (8) and (11), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (6) can be immediately obtained. For numerical purposes, the n-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} y_k, \tag{12}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial value problems.

3 Numerical examples

In this section, three oscillatory equations are considered and then are solved by standard and modified Adomian decomposition methods.

Example 1. The Linear Damping Oscillator Equation

Consider the linear damping oscillator equation:

$$y^{''} + 2\epsilon y^{'} + y = 0, \tag{13}$$

$$y(0) = a, y^{'}(0) = 0.$$

Standard Adomian decomposition method: we put

$$L(.) = \frac{d^2}{dx^2}(.),$$

so

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

In an operator form, Eq.(13) becomes

$$Ly = -2\epsilon y^{'} - y. \tag{14}$$

By applying L^{-1} to both sides of (14) we have

$$y = y(0) + xy^{'}(0) - L^{-1}(2\epsilon y^{'} + y).$$

Proceeding as before we obtained the recursive relationship

$$y_0 = y(0) + xy^{'}(0),$$

$$y_{n+1} = -L^{-1}(y + 2\epsilon y^{'}), n \geq 0,$$

and the first few components are as follows

$$y_0 = a,$$

$$y_1 = -a\frac{x^2}{2},$$

$$y_2 = 2a\epsilon\frac{x^3}{3!} + a\frac{x^4}{4!},$$

$$y_3 = -4\epsilon^2\frac{x^4}{4!} - 4a\epsilon\frac{x^5}{5!} - a\frac{x^6}{6!},$$

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$$y = y_0 + y_1 + y_2 + y_3 + ... = a - a\frac{x^2}{2} + 2a\epsilon\frac{x^3}{3!} + a(-4\epsilon^2+1)\frac{x^4}{4!} - 4a\epsilon\frac{x^5}{5!} - a\frac{x^6}{6!} + ...$$

Modified Adomian decomposition method:we put $2m+h=2\epsilon$, $m(m+h)=1$, it follows that $h=\pm 2i\sqrt{1-\epsilon^2}$,$m=\epsilon \mp i\sqrt{1-\epsilon^2}$, $i=\sqrt{-1}$.

Substitution of $h=-2i\sqrt{1-\epsilon^2}$, $m=\epsilon+i\sqrt{1-\epsilon^2}$, in Eq.(2) yields the operator

$$L() = e^{-(\epsilon+i\sqrt{1-\epsilon^2})x}\frac{d}{dx}e^{2i\sqrt{1-\epsilon^2}x}\frac{d}{dx}e^{(\epsilon-i\sqrt{1-\epsilon^2})x}(),$$

so

$$L^{-1}(.) = e^{(-\epsilon+i\sqrt{1-\epsilon^2})x}\int_0^x e^{-2i\sqrt{1-\epsilon^2}x}\int_0^x e^{(\epsilon+i\sqrt{1-\epsilon^2})x}(.)dxdx.$$

In an operator form, Eq.(13) becomes

$$Ly = 0. \tag{15}$$

Now, by applying L^{-1} to both sides of (15), we have

$$L^{-1}Ly = 0,$$

and it, implies that

$$y = \frac{\epsilon - i\sqrt{1-\epsilon^2}}{-2i\sqrt{1-\epsilon^2}}ae^{(-\epsilon-i\sqrt{1-\epsilon^2})x} + \frac{\epsilon + i\sqrt{1-\epsilon^2}}{2i\sqrt{1-\epsilon^2}}ae^{(-\epsilon+i\sqrt{1-\epsilon^2})x}$$

$$\Rightarrow y = ae^{-\epsilon x}(\cos\sqrt{1-\epsilon^2}x + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}x.$$

So the exact solution is easily obtained by proposed Adomian decomposition method .

Example 2. Consider the following Duffing equation:

$$y''(x) + 3y(x) - 2y^3(x) = \cos x \sin 2x, \tag{16}$$

with initial conditions

$$y(0) = 0, y^{'}(0) = 1.$$

The analytic solution of this equation is $y(x) = \sin x$.

Standard Adomian decomposition method: we put

$$L(.) = \frac{d^2}{dx^2}(.),$$

so

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

In an operator form, Eq.(16) becomes

$$Ly = \cos x \sin 2x + 2y^3 - 3y. \tag{17}$$

By applying L^{-1} to both sides of (17) we have

$$y = y(0) + xy^{'}(0) + L^{-1}(\cos x \sin 2x) + L^{-1}(2y^3 - 3y).$$

Proceeding as before we obtained the recursive relationship

$$y_0 = y(0) + xy^{'}(0) + L^{-1}(\cos x \sin 2x) = \frac{5x}{3} - \frac{\sin x}{2} - \frac{1}{18}\sin 3x,$$

$$y_{n+1} = L^{-1}(2A_n - 3y_n), n \geq 0, \tag{18}$$

when A_n's are Adomian polynomials of nonlinear term y^3, as below

$$A_0 = y_0^3,$$

$$A_1 = 3y_0^2 y_1,$$

$$A_2 = 3y_0^2 y_2 + 3y_0 y_1^2, \tag{19}$$

$$.$$

$$.$$

$$.$$

Substituting (19) into (18) gives the components y_0, y_1,...

We take

$$y = y_0 + y_1.$$

By using Taylor series of $y = y_0 + y_1$ with order 7 we get

$$y = x - \frac{x^3}{6} - \frac{x^5}{15} + \frac{101x^7}{1260} + ...$$

Modified Adomian decomposition method: we put $2m + h = 0$, $m(m + h) = 3$, it follows that $h = \pm 2i\sqrt{3}$, $m = \mp i\sqrt{3}$, $i = \sqrt{-1}$.

Substitution of $h = -2i\sqrt{3}$, $m = i\sqrt{3}$, in Eq. (2)yields the operator

$$L(.) = e^{-i\sqrt{3}x}\frac{d}{dx}e^{2i\sqrt{3}x}\frac{d}{dx}e^{-i\sqrt{3}x}(.),$$

so

$$L^{-1}(.) = e^{i\sqrt{3}x}\int_0^x e^{-2i\sqrt{3}x}\int_0^x e^{i\sqrt{3}x}(.)dxdx.$$

In an operator form, Eq.(16) becomes

$$Ly = 2y^3 + \cos x \sin 2x. \tag{20}$$

Applying L^{-1} to both sides of (20) we find

$$y = \frac{\sin\sqrt{3}x}{\sqrt{3}} + L^{-1}(\cos x \sin 2x) + 2L^{-1}y^3.$$

Proceeding as before we obtain

$$y_0 = \frac{\sin\sqrt{3}x}{\sqrt{3}} + \frac{\sin^3 x}{3},$$

$$y_{n+1} = 2L^{-1}A_n, n \geq 0. \tag{21}$$

when A_n's are Adomian polynomials of nonlinear term y^3,mentioned in (19), we obtain y_0, y_1,...

By using Taylor series of$y = y_0 + y_1$ we get

$$y = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + ...$$

Not that the Taylor series of the exact solution $y(x) = \sin x$ with order 7 is as below

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + ...$$

So the rate of convergence of modified Adomian is faster than standard Adomian method for this problem.

Example 3. Consider the non-linear equation:

$$y^{''}(x) - 2y^{'}(x) + y(x) + y^2(x) = \cos^2 x + 2\sin x, \tag{22}$$

subject to the initial conditions

$$y^{'}(0) = 0, y(0) = 1.$$

The analytic solution of this equation is $y(x) = \cos x$.

Standard Adomian decomposition method: we put

$$L(.) = \frac{d^2}{dx^2}(.),$$

so

$$L^{-1}(.) = \int_0^x \int_0^x (.)dxdx.$$

In an operator form, Eq.(22) becomes

$$Ly = \cos^2 x + 2\sin x - y^2 - y + 2y^{'}. \tag{23}$$

By applying L^{-1} to both sides of (23) we have

$$y = y(0) + xy^{'}(0) + L^{-1}(\cos^2 x + 2\sin x) + L^{-1}(-y^2 - y + 2y^{'}).$$

Proceeding as before we obtained the recursive relationship

$$y_0 = y(0) + xy^{'}(0) + L^{-1}(\cos^2 x + 2\sin x) = 1 + 2x + \frac{x^2}{4} - 2\sin x + \frac{\sin^2 x}{4},$$

$$y_{n+1} = L^{-1}(-A_n - y_n + 2y_n^{'}), n \geq 0, \tag{24}$$

when A_n's are Adomian polynomials of nonlinear term y^2, as below

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

$$A_2 = 2y_0y_2 + y_1^2, \tag{25}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

Substituting (25) into (24) gives the components y_0, y_1, y_2,...

We take

$$y = y_0 + y_1 + y_2$$

Modified Adomian decomposition method: we put $2m + h = 0$, $m(m + h) = 1$, it follows that $h = 0$, $m = -1$.

Substitution of $h = 0$, $m = -1$, in Eq. (2)yields the operator

$$L(.) = e^x \frac{d^2}{dx^2} e^{-x}(.),$$

so

$$L^{-1}(.) = e^x \int_0^x \int_0^x e^{-x}(.)dxdx.$$

In an operator form, Eq.(22) becomes

$$Ly = \cos^2 x + 2\sin x - y^2. \tag{26}$$

Applying L^{-1} to both sides of (26) we find

$$y = e^x - xe^x + L^{-1}(\cos^2 x + 2\sin x) - L^{-1}y^2.$$

Proceeding as before we obtain

$$y_0 = \frac{1}{2} - \frac{11}{25}e^x + \frac{3}{5}xe^x + \cos x - \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x,$$

$$y_{n+1} = -L^{-1}A_n, n \geq 0. \tag{27}$$

When A_n's are Adomian polynomials of nonlinear term y^2 mentioned in (25), we obtain, y_0, y_1, y_2,....

The graph of $y = \sum_{i=0}^{2} y_i$ is sketched in Fig 1 and compared with the solution of the standard Adomian decomposition method.

The comparison between the results mentioned in Examples 1-3 show the power of the proposed method of this paper for these nonlinear oscillator equations.

4 Conclusion

Adomian decomposition method has been known to be powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equation and so on. In this paper, we proposed an efficient modification of the standard Adomian decomposition method for solving nonlinear oscillator equations. In Example 1, the system was a linear systems and we derived the exact solution. For non-linear system we usually derive a very good approximations to the the solution, as in Example 3, and some times the exact solutions can be found, as in Example 2(Duffing equation). The study showed that the modified Adomian decomposition method is simple and easy to use and produces reliable results with few iterations used.

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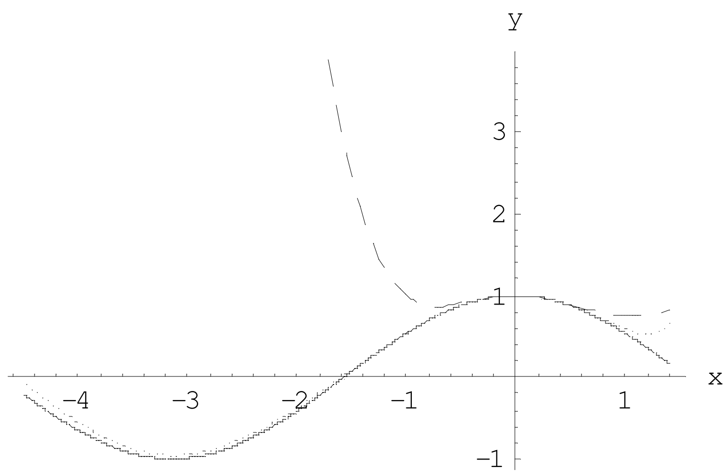

Fig.1. The exact solution $y = Cos(x)$ (____), modified Adomian decomposition solution (……) and Adomian decomposition solution(_ _ _ _ _)

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SMOOTH DEPENDENCE BY PARAMETER FOR DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Using the Perov's fixed point theorem and the theorem of fiber generalized contractions is obtained the smooth dependence by parameter of the solution of initial value problems associated to neutral delay integro-differential equations.

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Keywords and phrases: Perov's fixed point theorem, Picard operators, fiber contraction principle, smooth dependence by parameter.

1. INTRODUCTION

An efficient technique to approach systems of operatorial equations (see [14]) and operatorial (differential and integro-differential) equations of neutral type (see [4], [1], [2], [3] and [5]) can be obtained using the Perov's fixed point theorem (see [7], [9] and [11]). In the study of the smooth dependence by parameters of the solution of operatorial equations is very useful the notions of Picard and weakly Picard operators (see [14] and [13]) and the theorem of fiber generalized contractions (see [10], [12] and [11]).

In what follows we apply the Perov's fixed point theorem to the initial value problem associated to the following delay Volterra integro-differential equation of neutral type:

$$(1)\qquad \begin{cases} x'(t) = f(t, x(t), x'(t-\tau)) + \int\limits_{t-\tau}^{t} g(t, s, x(s), x'(s))\,ds, \ t \in [0, b] \\ x(t) = \varphi(t), \ t \in [-\tau, 0] \end{cases}$$

This equation generalize the following delay integral equation from [6] and [8]:

$$\begin{cases} x(t) = f(t, x(t)) + \int\limits_{t-\tau}^{t} g(t, s, x(s))\,ds, \ t \in [0, b] \\ x(t) = \varphi(t), \ t \in [-\tau, 0] \end{cases}$$

In this paper we will study the dependence of the solution by a parameter λ. We recall the following notions and results:

Definition 1. *([11], [14] and [13]) Let (X, d) be a metric space.*
An operator $A : X \to X$ is Picard operator if there exists $x^* \in X$ such that:
(i) x^* is the unique fixed point of A;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$, where $A^0 = Id(X)$ and $A^{n+1} = A \circ A^n$, $\forall\ n \in \mathbb{N}$.

Definition 2. *([11], [14] and [13]) Let (X, d) be a metric space. An operator $A : X \to X$ is weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.*

Theorem 1. *(The Perov's fixed point theorem - A.I.Perov - [9]) Let (X, d) be a complete generalized metric space such that $d(x, y) \in \mathbb{R}^n$. Suppose that $T : X \to X$ is a map for which exists a matrix $Q \in M_n(\mathbb{R})$ with the property:*

$$d(T(x), T(y)) \leqslant Q d(x, y), \ \forall\ x, y \in X.$$

If all the eigenvalues of Q lies in the open unit disc of $\mathbb{R}^2$, then T is a Q-contraction (that is $\lim\limits_{m\to\infty} Q^m = 0$) and has an unique fixed point x^* so that the sequence of successive approximations, $x_m = A^m(x_0)$, converges to x^* for any $x_0 \in X$.

Moreover, for any $m \in \mathbb{N}^*$ the following estimation holds:

$$d(x_m, x^*) \leqslant Q^m (I_n - Q)^{-1} d(x_0, x_1).$$

The Perov's fixed point theorem was used to obtain a technique for the existence, uniqueness and approximation of the solution of initial value problems associated to neutral differential equations and neutral integro-differential equations, in [1], [2], [4] and [5]. The same technique was applied in [3] to obtain the existence and uniqueness of the solution of the initial value problem (1).

Theorem 2. *(A fiber generalized contraction principle - I.A.Rus - [10], [11] and [12]) Let (X, d) be a metric space (generalized or not) and (Y, ρ) be a complete generalized metric space ($\rho(x, y) \in \mathbb{R}^n_+$). Let $A : X \times Y \to X \times Y$ be a continuous operator and $B : X \to X$, $C : X \times Y \to Y$ operators. Suppose that:*
(i) the operator B has an unique fixed point x^* and for any $x_0 \in X$ the sequence given by $x_{n+1} = B(x_n)$ converges in X to x^*;
(ii) $A(x, y) = (B(x), C(x, y))$, for all $x \in X$, $y \in Y$;
(iii) there exists a matrix $Q \in M_n(\mathbb{R}_+)$, with $Q^m \to 0$ as $m \to \infty$, such that $\rho(C(x, y_1), C(x, y_2)) \leqslant Q\rho(y_1, y_2)$, for all $x \in X$ and $y_1, y_2 \in Y$. Then, the operator A has an unique fixed point (x^*, y^*) and for any $(x_0, y_0) \in X \times Y$ the sequence given by $(x_{n+1}, y_{n+1}) = A((x_n, y_n))$ converges to (x^*, y^*) in $X \times Y$.

2. Main result

Consider the integro-differential equation:

$$(2)\qquad \begin{cases} x'_t(t, \lambda) = f(t, x(t, \lambda), x'(t-\tau, \lambda), \lambda) + \displaystyle\int_{t-\tau}^{t} g(t, s, x(s, \lambda), x'(s, \lambda), \lambda)\, ds, \ t \in [0, b] \\ x(t, \lambda) = \varphi(t, \lambda), \ t \in [-\tau, 0], \ \lambda \in [a, c] \end{cases}$$

where $\varphi \in C([-\tau, 0] \times [a, c])$ in the following conditions:

(i) (continuity): $g \in C([-\tau, b] \times [-\tau, b] \times \mathbb{R} \times \mathbb{R} \times [a, c])$, $f \in C([0, b] \times \mathbb{R} \times \mathbb{R} \times [a, c])$ and $\varphi \in C^1([-\tau, 0] \times [a, c])$;

(ii) (boundedness): $\varphi(t, \lambda) \geqslant 0$, for all $(t, \lambda) \in [-\tau, 0] \times [a, c]$ and there exists $m_1, M_1 \geqslant 0, m_2, M_2 \geqslant 0$ such that:

$$m_1 \leqslant g(t, s, u, v, \lambda) \leqslant M_1, \text{ for all } (t, s, u, v, \lambda) \in [-\tau, b] \times [-\tau, b] \times \mathbb{R}_+ \times \mathbb{R} \times [a, c]$$

and

$$m_2 \leqslant f(t, u, v, \lambda) \leqslant M_2, \text{ for all } (t, u, v, \lambda) \in [0, b] \times \mathbb{R}_+ \times \mathbb{R} \times [a, c]$$

(iii) (first compatibility condition): $\varphi(0, \lambda) = 0$, $\forall\, \lambda \in [a, c]$ and

$$\varphi'_t(0, \lambda) = f(0, 0, \varphi'_t(-\tau, \lambda), \lambda) + \int_{-\tau}^{t} g(0, s, \varphi(s, \lambda), \varphi'_t(s, \lambda), \lambda), \ \lambda \in [a, c]$$

(iv) (Lipschitz property): there exists $\alpha_1, \beta_1 > 0$, $\alpha_2, \beta_2 > 0$ such that:

$$|f(t, x_1, y_1, \lambda) - f(t, x_2, y_2, \lambda)| \leqslant \alpha_1 |x_1 - x_2| + \beta_1 |y_1 - y_2|$$

and

$$|g(t, s, x_1, y_1, \lambda) - g(t, s, x_2, y_2, \lambda)| \leqslant \alpha_2 |x_1 - x_2| + \beta_2 |y_1 - y_2|$$

for all $(t, s, \lambda) \in [-\tau, b] \times \mathbb{R} \times [a, c]$ and $(x_i, y_i) \in \mathbb{R}_+ \times \mathbb{R}$, $i = \overline{1, 2}$;

(v) (smoothness): $f(s, \cdot, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R} \times [a, c])$, and $g(t, s, \cdot, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R} \times [a, c])$, for all t,$s \in [-\tau, b]$, $\varphi \in C^2([-\tau, 0] \times [a, c])$ and there exists $M_3, M_4, M_5, M_6 \geqslant 0$ such that:

$$\left|\frac{\partial f}{\partial x}(s, u, v, \lambda)\right| \leqslant M_3 \ ; \ \left|\frac{\partial f}{\partial y}(s, u, v, \lambda)\right| \leqslant M_4;$$
$$\left|\frac{\partial g}{\partial x}(t, s, u, v, \lambda)\right| \leqslant M_5 \ ; \ \left|\frac{\partial g}{\partial y}(t, s, u, v, \lambda)\right| \leqslant M_6;$$

(vi) (second compatibility condition): $\varphi'_\lambda(0, \lambda) = 0$,

$$\varphi''_{t\lambda}(0, \lambda) = \frac{\partial f}{\partial \lambda}(0, \varphi(0, \lambda), \varphi'_t(-\tau, \lambda), \lambda) + \frac{\partial f}{\partial x}(0, \varphi(0, \lambda), \varphi'_t(-\tau, \lambda), \lambda) \cdot \varphi'_\lambda(0, \lambda) +$$

$$+\frac{\partial f}{\partial y}\left(0,\varphi\left(0,\lambda\right),\varphi_s'\left(-\tau,\lambda\right),\lambda\right)\cdot\varphi_{t\lambda}''\left(0,\lambda\right)+\int\limits_{-\tau}^{0}\left[\frac{\partial g}{\partial\lambda}\left(0,s,\varphi\left(s,\lambda\right),\varphi_t'\left(s,\lambda\right),\lambda\right)+\right.$$

$$\left.+\frac{\partial g}{\partial x}\left(0,s,\varphi\left(s,\lambda\right),\varphi_t'\left(s,\lambda\right),\lambda\right)\cdot\varphi_\lambda'\left(0,\lambda\right)+\frac{\partial g}{\partial y}\left(0,s,\varphi\left(0,\lambda\right),\varphi_t'\left(0,\lambda\right),\lambda\right)\cdot\varphi_{ts}''\left(0,\lambda\right)\right]ds$$

for all $\lambda\in[a,c]$.

Consider the following generalized metric spaces (X,d) and (Y,ρ), where:

$$X=C\left([-\tau,b]\times[a,c]\right)\times C\left([-\tau,b]\times[a,c]\right)$$

and $Y=C\left([-\tau,b]\times[a,c]\right)\times C\left([-\tau,b]\times[a,c]\right)$ and the metrics are:

$\rho:Y\times Y\to\mathbb{R}^2$

$$\rho\left(\left(x_1,y_1\right),\left(x_2,y_2\right)\right)=\Bigg(\max_{t\in[-\tau,b],\lambda\in[a,c]}\left|x_1\left(t\right)-x_2\left(t\right)\right|\cdot e^{-\theta[(t+\tau)+(\lambda-a)]},$$

$$\max_{t\in[-\tau,b],\lambda\in[a,c]}\left|x_1\left(t\right)-x_2\left(t\right)\right|\cdot e^{-\theta[(t+\tau)+(\lambda-a)]}\Bigg)$$

$d:X\times X\to\mathbb{R}^2,\ d=\rho|_{X\times X}$.

If we differentiate the equation from (2) in respect with t and denoting $x_t'\left(t,\lambda\right)=y\left(t,\lambda\right)$, we obtain:

$$(3)\qquad\begin{cases}(2.1.)\begin{cases}x\left(t,\lambda\right)=\displaystyle\int\limits_0^t f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)ds+\\ \quad+\displaystyle\int\limits_0^t\left(\int\limits_{\eta-\tau}^{\eta}g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)ds\right)d\eta,\ t\in[0,b]\\ y\left(t,\lambda\right)=f\left(t,x\left(t,\lambda\right),y\left(t-\tau,\lambda\right),\lambda\right)+\displaystyle\int\limits_{t-\tau}^{t}g\left(t,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)ds\end{cases}\\ (2.2.)\ \left(x\left(t,\lambda\right),y\left(t,\lambda\right)\right)=\left(\varphi\left(t,\lambda\right),\varphi_t'\left(t,\lambda\right)\right),\ t\in[-\tau,0]\end{cases}$$

Denoting $u\left(t,\lambda\right)=\frac{\partial x}{\partial\lambda}\left(t,\lambda\right)$ and $v\left(t,\lambda\right)=\frac{\partial y}{\partial\lambda}\left(t,\lambda\right)$, we have:

$$(4)\qquad\begin{cases}(3.1.)\begin{cases}u\left(t,\lambda\right)=\displaystyle\int\limits_0^t\left[\frac{\partial}{\partial\lambda}f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)+\frac{\partial}{\partial x}f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot\right.\\ \quad\left.\cdot u\left(s,\lambda\right)+\dfrac{\partial}{\partial y}f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot v\left(s,\lambda\right)\right]ds+\\ \quad+\displaystyle\int\limits_0^t\left\{\int\limits_{\eta-\tau}^{\eta}\left[\frac{\partial}{\partial\lambda}g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot u\left(s,\lambda\right)+\right.\right.\\ \quad\left.\left.+\ \dfrac{\partial}{\partial y}g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot v\left(s,\lambda\right)\right]ds\right\}d\eta\\ v\left(t,\lambda\right)=\dfrac{\partial}{\partial\lambda}f\left(t,x\left(t,\lambda\right),y\left(t-\tau,\lambda\right),\lambda\right)+\dfrac{\partial}{\partial x}f\left(t,x\left(t,\lambda\right),y\left(t-\tau,\lambda\right),\lambda\right)\cdot\\ \quad\cdot u\left(t,\lambda\right)+\dfrac{\partial}{\partial y}f\left(t,x\left(t,\lambda\right),y\left(t-\tau,\lambda\right),\lambda\right)\cdot v\left(t-\tau,\lambda\right)+\\ \quad+\displaystyle\int\limits_{t-\tau}^{t}\left[\frac{\partial}{\partial\lambda}g\left(t,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)+\frac{\partial}{\partial x}g\left(t,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot\right.\\ \quad\left.\cdot u\left(t,\lambda\right)+\dfrac{\partial}{\partial y}g\left(t,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot v\left(s,\lambda\right)\right]ds,\ t\in[0,b]\end{cases}\\ (3.2.)\ \left(u\left(t,\lambda\right),v\left(t,\lambda\right)\right)=\left(\varphi_\lambda'\left(t,\lambda\right),\varphi_{t\lambda}''\left(t,\lambda\right)\right),\ t\in[-\tau,0]\end{cases}$$

Define the operators, $B : X \to X,\ C : X \times Y \to Y,\ A : X \times Y \to X \times Y$ by $A(x,y) := (B(x), C(x,y))$, where

(5)
$$B(x,y)(t,\lambda) := (B_1(x,y)(t,\lambda), B_2(x,y)(t,\lambda)) =$$
$$= \begin{cases} (\text{second part of } x(t,\lambda) \text{ from } (2.1.), \text{second part of } y(t,\lambda) \text{ from } (2.1.)), \\ t \in [0,b] \\ (\varphi(t,\lambda), \varphi'_t(t,\lambda)),\ t \in [-\tau, 0] \end{cases}$$
and

(6)
$$C((x,y),(u,v))(t,\lambda) := (C_1((x,y),(u,v))(t,\lambda), C_2((x,y),(u,v))(t,\lambda)) =$$
$$= \begin{cases} (\text{second part of } u(t,\lambda) \text{ from } (3.1.),\ \text{second part of v}(t,\lambda) \text{ from } (3.1.)), \\ t \in [0,b] \\ (\varphi'_\lambda(t,\lambda), \varphi''_{t\lambda}(t,\lambda)),\ t \in [-\tau, 0] \end{cases}$$

Theorem 3. *a) In the conditions (i)-(iv) the equation (3) has in X an unique solution (x^*, y^*). Moreover, for any $(x_0, y_0) \in X$, the sequence $((x_n, y_n))_{n\in\mathbb{N}}$ defined by:*
$$x_{n+1}(t,\lambda) = \int_0^t f(s, x_n(s,\lambda), y_n(s-\tau,\lambda), \lambda)\, ds +$$
$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x_n(s,\lambda), y_n(s,\lambda), \lambda)\, ds \right) d\eta,\ t \in [0,b]$$
$$x_n(t,\lambda) = \varphi'(t,\lambda),\ \textit{for all } n \in \mathbb{N},\ \textit{and } t \in [-\tau, 0]$$
$$y_{n+1}(t,\lambda) = f(t, x_n(t,\lambda), y_n(t,\lambda), \lambda) + \int_{t-\tau}^{t} g(t, x_n(s,\lambda), y_n(s,\lambda), \lambda)\, ds, t \in [0,b]$$
$$y_n(t,\lambda) = \varphi''_{t\lambda}(t,\lambda),\ t \in [-\tau, 0]$$
uniformly converges to (x^*, y^*), for any $t \in [-\tau, b]$ and $\lambda \in [a,c]$.
b) In the conditions (i)-(vi) the solution (x^*, y^*) has the property $x^*(t,\cdot) \in C^1[a,c]$, $y^*(t,\cdot) \in C^1[a,c]$ and the pair $\left(\frac{\partial x^*}{\partial \lambda}, \frac{\partial y^*}{\partial \lambda}\right)$ is the unique solution of the equation (4) on Y.

Proof. Conditions (i) and (ii) imply that $B(X) \subset X$. and condition (i) imply that $B(x,y) \in C^1([-\tau,0] \times [a,c]) \times C([-\tau,b] \times [a,c])$.
From condition (ii) we have
$$B_1(x,y)(t,\lambda) \geqslant 0,\ \text{for all } (t,\lambda) \in [-\tau,b] \times [a,c]$$
$$B_1(x,y)(t,\lambda) \leqslant bM_2 + b\tau M_1,\ \text{for all } (t,\lambda) \in [-\tau,b] \times [a,c]$$
$$B_2(x,y)(t,\lambda) \geqslant 0,\ \text{for all } (t,\lambda) \in [-\tau,b] \times [a,c]$$
$$B_2(x,y)(t,\lambda) \leqslant M_2 + \tau M_1,\ \text{for all } (t,\lambda) \in [-\tau,b] \times [a,c].$$
From condition (iii), we have:
$$B_1(x,y) \in C([-\tau,b] \times [a,c], [0, bM_2 + b\tau M_1])$$
$$B_2(x,y) \in C([-\tau,b] \times [a,c], [0, M_2 + \tau M_1]),\ \forall\ (x,y) \in X.$$
So, $B(X) \subset X$. Then,
$$d_B(B(x_1,y_1), B(x_2,y_2)) =$$
$$= (\|B_1(x_1,y_1) - B_1(x_2,y_2)\|_B, \|B_2(x_1,y_1) - B_2(x_2,y_2)\|_B) =$$
$$= \Bigg(\max_{t\in[-\tau,T],\lambda\in[a,c]} \Bigg| \int_0^t f(s, x_1(s,\lambda), y_1(s-\tau,\lambda), \lambda)\, ds +$$
$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x_1(s,\lambda), y_1(s,\lambda), \lambda)\, ds \right) d\eta - \int_0^t f(s, x_2(s,\lambda), y_2(s-\tau,\lambda), \lambda)\, ds -$$

$$-\int_0^t \left(\int_{\eta-\tau}^{\eta} g\left(\eta, s, x_2\left(s,\lambda\right), y_2\left(s,\lambda\right), \lambda\right) ds\right) d\eta \Bigg| \cdot e^{-\theta[(t+\tau)+(\lambda-a)]},$$

$$\max_{t\in[-\tau,T],\lambda\in[a,c]} \left| f\left(t, x_1\left(t,\lambda\right), y_1\left(t-\tau,\lambda\right), \lambda\right) + \right.$$

$$+\int_{t-\tau}^{t} g\left(t, s, x_1\left(s,\lambda\right), y_1\left(s,\lambda\right), \lambda\right) ds - f\left(t, x_2\left(t,\lambda\right), y_2\left(t-\tau,\lambda\right), \lambda\right) -$$

$$\left. -\int_{t-\tau}^{t} g\left(t, s, x_2\left(s,\lambda\right), y_2\left(s,\lambda\right), \lambda\right) ds \right| \cdot e^{-\theta[(t+\tau)+(\lambda-a)]}\Bigg).$$

We have that:

$$\Bigg|\int_0^t f\left(s, x_1\left(s,\lambda\right), y_1\left(s-\tau,\lambda\right), \lambda\right) ds + \int_0^t \left(\int_{\eta-\tau}^{\eta} g\left(\eta, s, x_1\left(s,\lambda\right), y_1\left(s,\lambda\right), \lambda\right) ds\right) d\eta -$$

$$-\int_0^t f\left(s, x_2\left(s,\lambda\right), y_2\left(s-\tau,\lambda\right), \lambda\right) ds - \int_0^t \left(\int_{\eta-\tau}^{\eta} g\left(\eta, s, x_2\left(s,\lambda\right), y_2\left(s,\lambda\right), \lambda\right) ds\right) d\eta\Bigg| \leqslant$$

$$\leqslant \int_0^t \left|f\left(s, x_1\left(s,\lambda\right), y_1\left(s-\tau,\lambda\right), \lambda\right) - f\left(s, x_2\left(s,\lambda\right), y_2\left(s-\tau,\lambda\right), \lambda\right)\right| ds +$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} \left|g\left(\eta, s, x_1\left(s,\lambda\right), y_1\left(s,\lambda\right), \lambda\right) - g\left(\eta, s, x_2\left(s,\lambda\right), y_2\left(s,\lambda\right), \lambda\right)\right| ds\right) d\eta \leqslant$$

$$\leqslant \int_0^t \left[\alpha_1 \left\|x_1 - x_2\right\|_B \cdot e^{\theta[(s+\tau)+(\lambda-a)]} + \beta_1 \left\|y_1 - y_2\right\|_B \cdot e^{\theta[(s+\tau)+(\lambda-a)]} \cdot e^{-\theta\tau}\right] ds +$$

$$+\int_0^t \left[\int_{\eta-\tau}^{\eta} \left(\alpha_2 \left\|x_1 - x_2\right\|_B \cdot e^{\theta[(s+\tau)+(\lambda-a)]} + \beta_2 \left\|y_1 - y_2\right\|_B \cdot e^{\theta[(s+\tau)+(\lambda-a)]}\right) ds\right] d\eta \leqslant$$

$$\leqslant \int_0^t \left(\alpha_1 \left\|x_1 - x_2\right\|_B + \beta_1 \left\|y_1 - y_2\right\|_B \cdot e^{-\theta\tau}\right) \cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds +$$

$$+\int_0^t \left[\int_{\eta-\tau}^{\eta} \left(\alpha_2 \left\|x_1 - x_2\right\|_B + \beta_2 \left\|y_1 - y_2\right\|_B\right) \cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds\right] d\eta \leqslant$$

$$\leqslant \left(\frac{\alpha_1}{\theta} \left\|x_1 - x_2\right\|_B + \frac{\beta_1}{\theta} \cdot e^{-\theta\tau} \left\|y_1 - y_2\right\|_B\right) \cdot e^{\theta[(s+\tau)+(\lambda-a)]} +$$

$$+\left(\frac{\alpha_2}{\theta^2} \left\|x_1 - x_2\right\|_B + \frac{\beta_2}{\theta^2} \left\|y_1 - y_2\right\|_B\right) \int_0^t e^{\theta[(s+\tau)+(\lambda-a)]}\Big|_{\eta-\tau}^{\eta} d\eta \leqslant$$

$$\leqslant \left(\frac{\alpha_1}{\theta} \left\|x_1 - x_2\right\|_B + \frac{\beta_1}{\theta} \cdot e^{-\theta\tau} \left\|y_1 - y_2\right\|_B\right) \cdot e^{\theta[(s+\tau)+(\lambda-a)]} +$$

$$+\left(\frac{\alpha_2}{\theta^2} \left\|x_1 - x_2\right\|_B + \frac{\beta_2}{\theta^2} \left\|y_1 - y_2\right\|_B\right) \left(e^{\theta[(t+\tau)+(\lambda-a)]} - e^{\theta[t+(\lambda-a)]} - e^{\theta[\tau+(\lambda-a)]} + e^{\lambda-a}\right) \leqslant$$

$$\leqslant \left[\left(\frac{\alpha_1}{\theta} + \frac{\alpha_2}{\theta^2}\right) \left\|x_1 - x_2\right\|_B + \left(\frac{\beta_1}{\theta} \cdot e^{-\theta\tau} + \frac{\beta_2}{\theta^2}\right) \left\|y_1 - y_2\right\|_B\right] \cdot e^{\theta[(t+\tau)+(\lambda-a)]}.$$

So,

$$\left\|B_1\left(x_1, y_1\right) - B_2\left(x_2, y_2\right)\right\|_B \leqslant$$

$$\leqslant \left(\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2}\right)\|x_1-x_2\|_B+\left(\frac{\beta_1}{\theta}\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta^2}\right)\|y_1-y_2\|_B.$$

On the other hand, for $t\in[0,b]$, we have:

$$\left|f\left(t,x_1\left(t,\lambda\right),y_1\left(t-\tau,\lambda\right),\lambda\right)+\int\limits_{t-\tau}^{t} g\left(t,s,x_1\left(t,\lambda\right),y_1\left(t-\tau,\lambda\right),\lambda\right)ds-\right.$$

$$\left.-f\left(t,x_2\left(t,\lambda\right),y_2\left(t-\tau,\lambda\right),\lambda\right)-\int\limits_{t-\tau}^{t} g\left(t,s,x_2\left(t,\lambda\right),y_2\left(t-\tau,\lambda\right),\lambda\right)ds\right|\leqslant$$

$$\leqslant\left|f\left(t,x_1\left(t,\lambda\right),y_1\left(t-\tau,\lambda\right),\lambda\right)-f\left(t,x_2\left(t,\lambda\right),y_2\left(t-\tau,\lambda\right),\lambda\right)\right|+$$

$$+\int\limits_{t-\tau}^{t}\left|g\left(t,s,x_1\left(t,\lambda\right),y_1\left(t-\tau,\lambda\right),\lambda\right)-g\left(t,s,x_2\left(t,\lambda\right),y_2\left(t-\tau,\lambda\right),\lambda\right)\right|ds\leqslant$$

$$\leqslant\alpha_1\left|x_1\left(t,\lambda\right)-x_2\left(t,\lambda\right)\right|\cdot e^{-\theta[(t+\tau)+(\lambda-a)]}\cdot e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\beta_1\left|y_1\left(t-\tau,\lambda\right)-y_2\left(t-\tau,\lambda\right)\right|\cdot e^{-\theta[t+(\lambda-a)]}\cdot e^{\theta[t+(\lambda-a)]}\cdot e^{-\theta\tau}\cdot e^{\theta\tau}+$$

$$+\int\limits_{t-\tau}^{t}\left(\alpha_2\left|x_1\left(s,\lambda\right)-x_2\left(s,\lambda\right)\right|\cdot e^{-\theta[(s+\tau)+(\lambda-a)]}\cdot e^{\theta[(s+\tau)+(\lambda-a)]}+\right.$$

$$\left.+\beta_2\left|y_1\left(s,\lambda\right)-y_2\left(s,\lambda\right)\right|\cdot e^{-\theta[(s+\tau)+(\lambda-a)]}\cdot e^{\theta[(s+\tau)+(\lambda-a)]}\right)ds\leqslant$$

$$\leqslant\alpha_1\|x_1-x_2\|_B\cdot e^{-\theta[(t+\tau)+(\lambda-a)]}+\beta_1\|y_1-y_2\|_B\cdot e^{\theta[(t+\tau)+(\lambda-a)]}\cdot e^{-\theta\tau}+$$

$$+\int\limits_{t-\tau}^{t}\left(\alpha_2\|x_1-x_2\|_B\, e^{\theta[(s+\tau)+(\lambda-a)]}+\beta_2\|y_1-y_2\|_B\cdot e^{\theta[(s+\tau)+(\lambda-a)]}\right)ds\leqslant$$

$$\leqslant\left(\alpha_1\|x_1-x_2\|_B+\beta_1\cdot e^{-\theta\tau}\|y_1-y_2\|_B\right)\cdot e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\left(\frac{\alpha_2}{\theta}\|x_1-x_2\|_B+\frac{\beta_2}{\theta}\|y_1-y_2\|_B\right)\int\limits_{t-\tau}^{t}\theta\cdot e^{\theta[(s+\tau)+(\lambda-a)]}ds\leqslant$$

$$\leqslant\left[\left(\alpha_1+\frac{\alpha_2}{\theta}\right)\|x_1-x_2\|_B+\left(\beta_1\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta}\right)\|y_1-y_2\|_B\right]\cdot e^{\theta[(t+\tau)+(\lambda-a)]}.$$

So,

$$\|B_2\left(x_1,y_1\right)-B_2\left(x_2,y_2\right)\|_B\leqslant\left(\alpha_1+\frac{\alpha_2}{\theta}\right)\|x_1-x_2\|_B+\left(\beta_1\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta}\right)\|y_1-y_2\|_B.$$

We infer that:

$$d_B\left(B\left(x_1,y_1\right),B\left(x_2,y_2\right)\right)=\begin{pmatrix}\|B_1\left(x_1,y_1\right)-B_1\left(x_2,y_2\right)\|_B\\ \|B_2\left(x_1,y_1\right)-B_2\left(x_2,y_2\right)\|_B\end{pmatrix}\leqslant$$

$$\leqslant\begin{pmatrix}\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2} & \frac{\beta_1}{\theta}\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta^2}\\ \alpha_1+\frac{\alpha_2}{\theta} & \beta_1\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta}\end{pmatrix}\begin{pmatrix}\|x_1-x_2\|_B\\ \|y_1-y_2\|_B\end{pmatrix},\ \forall\ (x_1,x_2)\in X,\ \ (y_1,y_2)\in X.$$

The eigenvalues of the matrix:

$$Q=\begin{pmatrix}\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2} & \frac{\beta_1}{\theta}\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta^2}\\ \alpha_1+\frac{\alpha_2}{\theta} & \beta_1\cdot e^{-\theta\tau}+\frac{\beta_2}{\theta}\end{pmatrix}$$

are $\lambda_1=0$ and $\lambda_2=\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2}+\beta_1 e^{-\theta\tau}+\frac{\beta_2}{\theta}>0$. We infer that:

$0<\lambda_2<1\Leftrightarrow h\left(\theta\right)=\theta^2-\left(\alpha_1+\beta_2\right)\theta-\alpha_2>\theta^2\cdot\beta_1 e^{-\theta\tau}$

The equation $\theta^2-\left(\alpha_1+\beta_2\right)\theta-\alpha_2=0$ have the solutions $\theta_1<0$ and $\theta_2>0$, and the top of the graph for the associate function by second order is $V\left(\frac{\alpha_1+\beta_2}{2},-\frac{\Delta}{4}\right)$, where $\Delta=\left(\alpha_1+\beta_2\right)^2+4\alpha_2$.

Plotting geometrical the graphs for the functions $h\left(\theta\right)$ and $u\left(\theta\right)=\theta^2\beta_1 e^{-\theta\tau}$, we see that exists an unique point $\theta^*>\theta_2$ such that:

$h\left(\theta^*\right)=u\left(\theta^*\right)$ and $h\left(\theta\right)>\theta^2\beta e^{-\theta\tau}$, $\forall\ \theta>\theta^*$.

On the other hand, this can be obtained from the properties:
$h(\theta) < 0,\ \forall\ \theta \in [0, \theta_2),\ u(\theta) > 0, \forall\ \theta > 0$
$\lim\limits_{\theta\to\infty} h(\theta) = \infty,\ \lim\limits_{\theta\to\infty} \theta^2\beta_1 e^{-\theta\tau} = 0$
and because the function $u(\theta) = \theta^2\beta_1 e^{-\theta\tau}$ have in $\theta = 0$ overall minimum (minim global), and in $\theta = \frac{2}{\tau}$ local maximum.

If we choose a value $\theta > \theta^*$ then the operator $B = (B_1, B_2)$ given by (5) is Q-contraction, and from the Perov's fixed point theorem, it has an unique fixed point $(x^*, y^*) \in X$.

The pair (x^*, y^*) is the unique solution of initial value problem (3), because for any $t \in [0, b]$ and $\eta \in [0, b]$

$$x^*(t, \lambda) = \int_0^t f(s, x^*(s, \lambda), y^*(s - \tau, \lambda), \lambda) ds + \tag{7}$$

$$+ \int_0^t \left(\int_{\eta-\tau}^{\eta} g(\eta, s, x^*(s, \lambda), y^*(s, \lambda), \lambda) ds \right) d\eta$$

and

$$y^*(t, \lambda) = f(t, x^*(t, \lambda), y^*(t - \tau, \lambda), \lambda) + \int_0^t g(t, s, x^*(s, \lambda), y^*(s, \lambda), \lambda) ds \tag{8}$$

Using the continuity and compatibility, from $x^*, y^* \in C[-\tau, b]$ and $x^*(t) = \varphi(t),\ \forall\ t \in [-\tau, 0]$ we infer that $x^* \in C^1[-\tau, b]$.

Differentiating in respect with t the equation (7), we obtain:

$$(x^*)'(t, \lambda) = f(t, x^*(t, \lambda), y^*(t - \tau, \lambda), \lambda) +$$

$$+ \int_{t-\tau}^{t} g(t, s, x^*(s, \lambda), y^*(s, \lambda), \lambda) ds\ ,\ \forall\ t \in [0, b]$$

which together with equation (8) give us $(x^*)' = y^*$.

We propose to obtain the point θ^*. It observed that:
$h(\theta) = \theta^2\beta_1 e^{-\theta\tau} \Leftrightarrow \theta = H(\theta) = \alpha_1 + \beta_2 + \theta\beta_1 e^{-\theta\tau} + \frac{\alpha_2}{\theta}$,
so θ^* is fixed point of H.

Moreover, $H'(\theta) < 0 \Leftrightarrow -\frac{\alpha_2}{\theta^2} + \beta_1 e^{-\theta\tau}(1 - \theta\tau) < 0$.

- If $\theta \geqslant \frac{1}{\tau}$ then $H'(\theta) < 0$ and $H'\left(\frac{1}{\tau}\right) = -\tau^2\beta_1 < 0$. Then, $H'(\theta) < 0,\ \forall\ \theta \geqslant \frac{1}{\tau}$.
- If $\frac{1}{\tau} < \theta_2$ then we'll consider $\overline{\theta} = H(\theta_2) > \theta^*$ and then, for any $\theta > \overline{\theta}$ we have $0 < \lambda_2 < 1$.
- If $\frac{1}{\tau} > \theta_2$ exists two possibilities:

1) If $h\left(\frac{1}{\tau}\right) < \frac{1}{\tau^2}\beta e^{-1}$ then we take $\overline{\theta} = H\left(\frac{1}{\tau}\right) > \theta^*$ and then, for any $\theta > \overline{\theta}$ we have $0 < \lambda_2 < 1$.
2) If $h\left(\frac{1}{\tau}\right) > \frac{1}{\tau^2}\beta e^{-1}$ then is clear that $\frac{1}{\tau} > \theta^*$ and for any $\theta > \frac{1}{\tau}$ we have $0 < \lambda_2 < 1$.

So, in given conditions, we can find $\overline{\theta}$ (which can be $H(\theta_2)$, or $H\left(\frac{1}{\tau}\right)$, or $\frac{1}{\tau}$) such that $0 < \lambda_2 < 1$, $\forall\ \theta > \overline{\theta}$.

b) The smoothness condition $f(s, \cdot, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R} \times [a, c]), \forall\ s \in [-\tau, b]$, $g(s, \cdot, \cdot, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [a, c])$, $\forall\ s \in [-\tau, b]$ and condition (iv) allow us to take the constants $\alpha_1, \alpha_2, \beta_1$ and β_2 from Lipschitz property such as:
$\alpha_1 = \left\|\frac{\partial f}{\partial x}\right\|\ ;\ \beta_1 = \left\|\frac{\partial f}{\partial y}\right\|\ ;\ \alpha_2 = \left\|\frac{\partial g}{\partial x}\right\|\ ;\ \beta_2 = \left\|\frac{\partial g}{\partial y}\right\|$.

We show that $C((x, y), \cdot) : Y \to Y$ is contraction, for any $(x, y) \in X$.

For any $(u_1, v_1),\ (u_2, v_2) \in Y$ we have:

$$\rho_B(C((x, y), (u_1, v_1)), C((x, y), (u_2, v_2))) =$$

$$= (\|C_1((x, y), (u_1, v_1)) - C_1((x, y), (u_2, v_2))\|_B\ ,$$

$$\|C_2((x, y), (u_1, v_1)) - C_2((x, y), (u_2, v_2))\|_B)\ .$$

Then,

$$\left|\int_0^t \left[\frac{\partial}{\partial\lambda} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right) + \frac{\partial}{\partial x} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot u_1\left(s,\lambda\right)+\right.\right.$$

$$\left.+\frac{\partial}{\partial y} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot v_1\left(s-\tau,\lambda\right)\right] ds+$$

$$+\int_0^t \left\{\int_{\eta-\tau}^{\eta} \left[\frac{\partial}{\partial\lambda} g\ \left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right) + \frac{\partial}{\partial x} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot u_1\left(s,\lambda\right)+\right.\right.$$

$$\left.\left.+\frac{\partial}{\partial y} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot v_1\left(s,\lambda\right)\right] ds\right\} d\eta-$$

$$-\int_0^t \left[\frac{\partial}{\partial\lambda} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right) + \frac{\partial}{\partial x} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot u_2\left(s,\lambda\right)+\right.$$

$$\left.+\frac{\partial}{\partial y} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\cdot v_2\left(s-\tau,\lambda\right)\right] ds+$$

$$-\int_0^t \left\{\int_{\eta-\tau}^{\eta} \left[\frac{\partial}{\partial\lambda} g\ \left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right) + \frac{\partial}{\partial x} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot u_2\left(s,\lambda\right)+\right.\right.$$

$$\left.\left.\left.+\frac{\partial}{\partial y} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\cdot v_2\left(s,\lambda\right)\right] ds\right\} d\eta\right| \leqslant$$

$$\leqslant \int_0^t \left|\frac{\partial}{\partial x} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\right|\cdot\left|u_1\left(s,\lambda\right)-u_2\left(s,\lambda\right)\right| ds+$$

$$+\int_0^t \left|\frac{\partial}{\partial y} f\left(s,x\left(s,\lambda\right),y\left(s-\tau,\lambda\right),\lambda\right)\right|\cdot\left|v_1\left(s-\tau,\lambda\right)-v_2\left(s-\tau,\lambda\right)\right| ds+$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} \left|\frac{\partial}{\partial x} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\right|\cdot\left|u_1\left(s,\lambda\right)-u_2\left(s,\lambda\right)\right| ds\right) d\eta+$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} \left|\frac{\partial}{\partial y} g\left(\eta,s,x\left(s,\lambda\right),y\left(s,\lambda\right),\lambda\right)\right|\cdot\left|v_1\left(s,\lambda\right)-v_2\left(s,\lambda\right)\right| ds\right) d\eta \leqslant$$

$$\leqslant \int_0^t \alpha_1\left|u_1\left(s,\lambda\right)-u_2\left(s,\lambda\right)\right|\cdot e^{-\theta[(s+\tau)+(\lambda-a)]}\cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds+$$

$$+\int_0^t \beta_1\left|v_1\left(s-\tau,\lambda\right)-v_2\left(s-\tau,\lambda\right)\right|\cdot e^{-\theta[s+(\lambda-a)]}\cdot e^{\theta[s+(\lambda-a)]}\cdot e^{-\theta\tau}\cdot e^{\theta\tau} ds+$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} \alpha_2\left|u_1\left(s,\lambda\right)-u_2\left(s,\lambda\right)\right|\cdot e^{-\theta[(s+\tau)+(\lambda-a)]}\cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds\right) d\eta+$$

$$+\int_0^t \left(\int_{\eta-\tau}^{\eta} \beta_2\left|v_1\left(s,\lambda\right)-v_2\left(s,\lambda\right)\right|\cdot e^{-\theta[(s+\tau)+(\lambda-a)]}\cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds\right) d\eta \leqslant$$

$$\leqslant \frac{1}{\theta}\left(\alpha_1\left\|u_1-u_2\right\|_B+\beta_1 e^{-\theta\tau}\left\|v_1-v_2\right\|_B\right) e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\int_0^t \left[\left(\alpha_2 \left\|u_1-u_2\right\|_B+\beta_2 \left\|v_1-v_2\right\|_B\right) \frac{1}{\theta} \cdot e^{\theta[(s+\tau)+(\lambda-a)]}\Big|_{\eta-\tau}^{\eta} \right] d\eta =$$

$$\leqslant \frac{1}{\theta}\left(\alpha_1 \left\|u_1-u_2\right\|_B+\beta_1 e^{-\theta\tau} \left\|v_1-v_2\right\|_B\right) e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\frac{1}{\theta}\left(\alpha_2 \left\|u_1-u_2\right\|_B+\beta_2 \left\|v_1-v_2\right\|_B\right) \int_0^t e^{\theta[(\eta+\tau)+(\lambda-a)]} d\eta \leqslant$$

$$\leqslant \frac{1}{\theta}\left(\alpha_1 \left\|u_1-u_2\right\|_B+\beta_1 e^{-\theta\tau} \left\|v_1-v_2\right\|_B\right) e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\frac{1}{\theta^2}\left(\alpha_2 \left\|u_1-u_2\right\|_B+\beta_2 \left\|v_1-v_2\right\|_B\right) \cdot e^{\theta[(t+\tau)+(\lambda-a)]} \leqslant$$

$$\leqslant \left[\left(\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2}\right)\left\|u_1-u_2\right\|_B+\left(\frac{\beta_1}{\theta}e^{-\theta\tau}+\frac{\beta_2}{\theta^2}\right)\left\|v_1-v_2\right\|_B\right] \cdot e^{\theta[(t+\tau)+(\lambda-a)]}.$$

So,

$$\left\|C_1\left((x,y),(u_1,v_1)\right)-C_1\left((x,y),(u_2,v_2)\right)\right\|_B \leqslant$$
$$\leqslant \left(\frac{\alpha_1}{\theta}+\frac{\alpha_2}{\theta^2}\right)\left\|u_1-u_2\right\|_B+\left(\frac{\beta_1}{\theta}e^{-\theta\tau}+\frac{\beta_2}{\theta^2}\right)\left\|v_1-v_2\right\|_B .$$

On the other hand, we have:

$$\left|\frac{\partial}{\partial\lambda} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right)+\frac{\partial}{\partial x} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right) \cdot u_1(t,\lambda)+\right.$$

$$+\frac{\partial}{\partial y} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right) \cdot v_1(t-\tau,\lambda)+$$

$$+\int_{t-\tau}^t \left[\frac{\partial}{\partial\lambda} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right)+\frac{\partial}{\partial x} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right) \cdot u_1(s,\lambda)+\right.$$

$$\left.+\frac{\partial}{\partial y} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right) \cdot v_1(s,\lambda)\right] ds-$$

$$-\frac{\partial}{\partial\lambda} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right)-\frac{\partial}{\partial x} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right) \cdot u_2(t,\lambda)-$$

$$-\frac{\partial}{\partial y} f\left(t, x(t,\lambda), y(t-\tau,\lambda), \lambda\right) \cdot v_2(t-\tau,\lambda)-$$

$$-\int_{t-\tau}^t \left[\frac{\partial}{\partial\lambda} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right)+\frac{\partial}{\partial x} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right) \cdot u_2(s,\lambda)+\right.$$

$$\left.\left.+\frac{\partial}{\partial y} g\left(t,s,x(s,\lambda), y(s,\lambda), \lambda\right) \cdot v_1(s,\lambda)\right] ds\right| \leq$$

$$\leqslant \alpha_1\left|u_1(t,\lambda)-u_2(t,\lambda)\right| \cdot e^{-\theta[(t+\tau)+(\lambda-a)]} \cdot e^{\theta[(t+\tau)+(\lambda-a)]}+$$

$$+\beta_1\left|v_1(t-\tau,\lambda)-v_2(t-\tau,\lambda)\right| \cdot e^{-\theta[t+(\lambda-a)]} \cdot e^{\theta[t+(\lambda-a)]} \cdot e^{-\theta\tau} \cdot e^{\theta\tau}+$$

$$+\int_{t-\tau}^t \alpha_2\left|u_1(t,\lambda)-u_2(t,\lambda)\right| \cdot e^{-\theta[(s+\tau)+(\lambda-a)]} \cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds+$$

$$+\int_{t-\tau}^t \beta_2\left|v_1(t,\lambda)-v_2(t,\lambda)\right| \cdot e^{-\theta[(s+\tau)+(\lambda-a)]} \cdot e^{\theta[(s+\tau)+(\lambda-a)]} ds \leqslant$$

$$\leqslant \left(\alpha_1 \left\|u_1-u_2\right\|_B+\beta_1 e^{-\theta\tau} \left\|v_1-v_2\right\|_B\right) \cdot e^{[(t+\tau)+(\lambda-a)]}+$$

$$+\frac{1}{\theta}\left(\alpha_2 \left\|u_1-u_2\right\|_B+\beta_2 \left\|v_1-v_2\right\|_B\right) \cdot e^{[(t+\tau)+(\lambda-a)]}.$$

So,

$$\left\|C_2\left((x,y),(u_1,v_1)\right)-C_2\left((x,y),(u_2,v_2)\right)\right\|_B =$$

$$= \left(\alpha_1 + \frac{\alpha_2}{\theta}\right) \|u_1 - u_2\|_B + \left(\beta_1 e^{-\theta\tau} + \frac{\beta_2}{\theta}\right) \|v_1 - v_2\|_B .$$

For any $(u_1, v_1), (u_2, v_2) \in Y$ and $\lambda \in [a, c]$, we obtain:

$$\rho_B \left(C\left((x, y), (u_1, v_1)\right), C\left((x, y), (u_2, v_2)\right)\right) \leqslant$$
$$\leqslant \begin{pmatrix} \frac{\alpha_1}{\theta} + \frac{\alpha_2}{\theta^2} & \frac{\beta_1}{\theta} e^{-\theta\tau} + \frac{\beta_2}{\theta^2} \\ \alpha_1 + \frac{\alpha_2}{\theta} & \beta_1 e^{-\theta\tau} + \frac{\beta_2}{\theta} \end{pmatrix} \rho\left((u_1, v_1), (u_2, v_2)\right).$$

We can choose $\theta > \theta^*$ and then $C((x, y), \cdot)$ is Q-contraction on Y, $\forall\ (x, y) \in X$, and $Q^n \to 0$, when $n \to \infty$.

From a fiber generalized contraction, we infer the existence and uniqueness of fixed point $((x^*, y^*), (u^*, v^*)) \in X \times Y$ of the operator A, that is

$B(x^*, y^*) = (x^*, y^*)$ and $C((x^*, y^*), (u^*, v^*)) = (u^*, v^*)$

and moreover, for $x_0 \in C^2([-\tau, b] \times [a, c], \mathbb{R}_+)$, $y_0 = \frac{\partial x_0}{\partial t}$, $u_0 = \frac{\partial x_0}{\partial \lambda}$, $v_0 = \frac{\partial y_0}{\partial \lambda}$, the sequence

$(A^n((x_0, y_0), (u_0, v_0)))_n = ((x_n, y_n), (u_n, v_n))$

in which $(u_{n+1}, v_{n+1}) = C((x_n, y_n), (u_n, v_n))$, converges uniformly to $((x^*, y^*), (u^*, v^*))$.

It is clear that in the condition (i)-(v) we have $x_n \in C^1([-\tau, b] \times [a, c])$, $\forall\ n \in \mathbb{N}^*$ and $y_n(t, \cdot) \in C^1([a, c])$, $\forall\ t \in [-\tau, b]$, $\forall\ n \in \mathbb{N}^*$.

So, $y^*(t, \cdot) \in C^1([a, c])$, $\forall\ t \in [-\tau, b]$ and

$x_n \xrightarrow{unif} x^*$, $y_n = \frac{\partial x_n}{\partial t} \xrightarrow{unif} y^*$

$u_n = \frac{\partial x_n}{\partial \lambda} \xrightarrow{unif} u^*$, $v_n = \frac{\partial y_n}{\partial \lambda} \xrightarrow{unif} v^*$

Then, $y^* = \frac{\partial x^*}{\partial t}$, $u^* = \frac{\partial x^*}{\partial \lambda}$, $v^* = \frac{\partial y^*}{\partial \lambda}$ and the pair $\left(\frac{\partial x^*}{\partial \lambda}, \frac{\partial y^*}{\partial \lambda}\right)$ is the unique solution of the equation (3) on Y. □

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Approximations by a Hybrid Method for Equilibrium Problems and Fixed Point Problems for a Nonexpansive Mapping in Hilbert Spaces

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Abstract

In this paper, we introduce an iterative scheme by a new hybrid method for finding a common element the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the above two sets by the new hybrid method in the mathematical programming.

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1 Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|,$$

for all $x, y \in C$. We denote by $\mathrm{F}(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $\mathrm{F}(S)$ is nonempty (see [6]). We write $x_n \longrightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. Let F be a bifunction of $C \times C$

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into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see [1, 3, 8, 12]). In 2005, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and they also proved a strong convergence theorem.

In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n, \tag{1.2}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [10]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [4]. Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. For finding an element of $EP(F) \cap F(S)$, Tada and Takahashi [11] introduced the following iterative scheme by the hybrid method in a Hilbert space: $x_0 = x \in H$ and let

$$\begin{cases} u_n \in C \text{ such that } \quad f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1 - \alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.3}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \longrightarrow \infty} r_n > 0$. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} x_0$.

On the other hand, Takahashi, Takeuchi and Kubota [14] proved the following strong convergence theorem by using the hybrid method in mathematical programming. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence as follows:

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $0 \leq \alpha_n < \alpha < 1$ for all $n \in \mathbb{N}$. They proved a strongly convergence theorem in a Hilbert space.

In this paper, motivated and inspired by the above results, we introduce a new iterative scheme, as follows for $C_1 = C$, $x_1 = P_{C_1} x_0$, and let

$$\begin{cases} u_n \in C \text{ such that } \quad f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

for finding a common element of the set of solutions of an equilibrium problem and the set of solutions of fixed points of a nonexpansive mappings in a Hilbert space. Moreover, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(S)\cap EP(F)}x_1$ by the hybrid method in the mathematical programming.

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the *Opial's condition* [9], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty}\|x_n - x\| < \liminf_{n\longrightarrow\infty}\|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. Hilbert space H satisfies the *Kadec-Klee property* [5, 13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \longrightarrow \|x\|$ together imply $\|x_n - x\| \longrightarrow 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [1]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$, $\limsup_{t\longrightarrow 0^+} F(tz+(1-t)x, y) \leq F(x, y)$;

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [1]

Lemma 2.1. [1] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [2].

Lemma 2.2. [2] *Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single- valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3 Strong convergence theorems

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and let S be a nonexpansive mappings from C into H such that* $\mathrm{F}(S) \cap EP(F) \neq \emptyset$*. For $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows:*

$$\begin{cases} u_n \in C \text{ such that } \; f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Su_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\longrightarrow\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(S)\cap EP(F)}x$.

Proof. We show first that the sequence $\{x_n\}$ is well defined. By induction we can show that $\mathrm{F}(S) \cap EP(F) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathrm{F}(S) \cap EP(F) \subset C = C_1$. Suppose that $\mathrm{F}(S) \cap EP(F) \subset C_k$ for each $k \in \mathbb{N}$. Hence, for $u \in F(S) \cap EP(F) \subset C_k$ and from $u_n = T_{r_n} x_n$, we note that

$$\begin{aligned} \|u_n - u\| &= \|T_{r_n} x_n - T_{r_n} u\|, \\ &\leq \|x_n - u\|, \end{aligned} \tag{3.2}$$

for every $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|y_n - u\| &= \|\alpha_n x_n + (1 - \alpha_n) S u_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|u_n - u\| \\ &\leq \alpha_n \|x_n - u\| + (1 - \alpha_n) \|x_n - u\| \\ &= \|x_n - u\|. \end{aligned} \tag{3.3}$$

Hence $u \in C_{k+1}$. This implies that

$$\mathrm{F}(S) \cap EP(F) \subset C_n \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Next, we prove that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, we know that $\|y_k - z\| \leq \|x_k - z\|$ is equivalent to

$$\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z\rangle \geq 0. \quad (\text{ by } (2.1))$$

So, C_{k+1} is closed and convex. Then, for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined. From Lemma 2.1, the sequence $\{u_n\}$ is also well defined.

From $x_n = P_{C_n} x_0$, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0$$

for each $y \in C_n$. Using $\mathrm{F}(S) \cap EP(F) \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for each } u \in \mathrm{F}(S) \cap EP(F) \quad \text{and} \quad n \in \mathbb{N}.$$

Hence, for $u \in \mathrm{F}(S) \cap EP(F)$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\ &= -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } \ u \in F(S) \cap EP(F) \quad \text{and} \quad n \in \mathbb{N}.$$

From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we obtain

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \tag{3.5}$$

It follow that, for $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= -\|x_o - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_o - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\| \end{aligned}$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|.$$

Since $\{\|x_n - x_0\|\}$ is bounded, $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists. Next we can show that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$. In deed, from (3.5) we get

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n + x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \|x_0 - x_n\|$ exists, this implies that

$$\lim_{n\longrightarrow\infty}\|x_n - x_{n+1}\| = 0. \tag{3.6}$$

Since $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_n - x_{n+1}\|.$$

By (3.6), we obtain

$$\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0. \tag{3.7}$$

For $v \in F(S) \cap EP(F)$, from Lemma 2.2, we get

$$\begin{aligned} \|u_n - u\|^2 &\leq \langle T_{r_n} x_n - T_{r_n} u, x_n - u\rangle \\ &= \langle u_n - u, x_n - u\rangle \\ &= \frac{1}{2}(\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2) \end{aligned}$$

and hence $\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2$.
From (3.2), we obtain

$$\begin{aligned} \|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|Su_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|u_n - u\|^2 \\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 - \|x_n - u_n\|^2\} \\ &= \|x_n - u\|^2 - (1-\alpha_n)\|x_n - u_n\|^2. \end{aligned}$$

Since $\{\alpha_n\} \subset [a, 1]$, we obtain

$$\begin{aligned} (1-a)\|x_n - u_n\|^2 &\leq (1-\alpha_n)\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|y_n - v\|^2 \\ &\leq \|x_n - y_n\|\{\|x_n - u\| - \|y_n - u\|\}. \end{aligned}$$

From this and (3.7), implies

$$\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0. \tag{3.8}$$

Since $\liminf_{n\longrightarrow\infty} r_n > 0$, we get

$$\lim_{n\longrightarrow\infty}\|\frac{x_n - u_n}{r_n}\| = \lim_{n\longrightarrow\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.9}$$

Since $(1-\alpha_n)Su_n = y_n - \alpha_n x_n$, we have

$$\begin{aligned}(1-a)\|Su_n - u_n\| &\leq (1-\alpha_n)\|Su_n - u_n\| = \|y_n - \alpha_n x_n - (1-\alpha_n)u_n\|\\ &\leq \alpha_n\|u_n - x_n\| + \|y_n - u_n\|\\ &\leq \|u_n - x_n\| + \|y_n - x_n\| + \|x_n - u_n\|\\ &\leq 2\|x_n - u_n\| + \|x_n - y_n\|.\end{aligned}$$

From (3.8) and (3.7), we obtain

$$\lim_{n\longrightarrow\infty}\|Su_n - u_n\| = 0. \tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. From $\|x_n - u_n\| \longrightarrow 0$, we obtain also that $u_{n_i} \rightharpoonup z$. Since $u_{n_i} \subset C$ and C is closed and convex, we obtain $z \in C$. From (3.10) we obtain $Su_{n_i} \rightharpoonup z$. Let us show $z \in EP(F)$. Since $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

From $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \longrightarrow 0$, it follows by (A4) that $0 \geq F(y, z)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)z$. Since $y \in C$ and $z \in C$, we have $y_t \in C$ and hence $F(y_t, z) \leq 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, z) \leq tF(y_t, y)$$

and hence $0 \leq F(y_t, y)$. From (A3), we have $0 \leq F(z, y)$ for all $y \in C$ and hence $z \in EP(F)$. Let us show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty}\|u_{n_i} - z\| &< \liminf_{n\longrightarrow\infty}\|u_{n_i} - Sz\|\\ &= \liminf_{n\longrightarrow\infty}\|u_{n_i} - Su_{n_i} + Su_{n_i} - Sz\|\\ &= \liminf_{n\longrightarrow\infty}\|Su_{n_i} - Sz\|\\ &\leq \liminf_{n\longrightarrow\infty}\|u_{n_i} - z\|\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(S)$. Hence $z \in F(S) \cap EP(F)$.

Finally, we show that $x_n \longrightarrow z$, where $z = P_{F(S)\cap VI(C,A)EP(F)}x_0$. Since $x_n = P_{C_n}x_0$ and $z \in F(S) \cap EP(F) \subset C_n$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|.$$

It follows from $z' = P_{F(S)\cap EP(F)}x_0$ and the lower semicontinuity of the norm that

$$\|z'-x_0\| \le \|z-x_0\| \le \liminf_{i\longrightarrow\infty}\|x_{n_i}-x_0\| \le \limsup_{i\longrightarrow\infty}\|x_{n_i}-x_0\| \le \|z'-x_0\|.$$

Thus, we obtain that $\lim_{k\longrightarrow\infty}\|x_{n_i}-x_0\| = \|z-x_0\| = \|z'-x_0\|$.
Using the Kadec-Klee property of a Hilbert space H, we obtain that

$$\lim_{k\longrightarrow\infty}x_{n_i} = z = z'.$$

Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to z, where $z = P_{F(T)\cap EP(F)}x$. □

As direct consequences of Theorem 3.1, we can obtain two corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *such that $EP(F)\neq\emptyset$. For $C_1 = C$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}$ and $\{u_n\}$ of C as follows:*

$$\begin{cases} u_n \in C \text{ such that } & f(u_n,y) + \dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y\in C,\\ C_{n+1} = \{z\in C_n : \|y_n-z\| \le \|x_n-z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n\in\mathbb{N}, \end{cases}$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$, and $\{\lambda_n\}\subset[a,b]\subset(0,2\alpha)$ and $\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\longrightarrow\infty}r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{EP(F)}x$.

Proof. Putting $S = I$ and $\alpha_n = 0$ in Theorem 3.1, the conclusion follows. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying* (A1)-(A4) *and let S be a nonexpansive mappings from C into H such that* $\mathrm{F}(S)\neq\emptyset$. *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x\in H$ and let*

$$\begin{cases} u_n\in C \text{ such that } \langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y\in C,\\ y_n = \alpha_n x_n + (1-\alpha_n)Su_n,\\ C_{n+1} = \{z\in C_n : \|y_n-z\| \le \|x_n-z\|\},\\ x_{n+1} = P_{C_{n+1}}x_0, \quad n\in\mathbb{N}, \end{cases}$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$, $\{\lambda_n\}\subset[a,b]\subset(0,2\alpha)$ and $\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\longrightarrow\infty}r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(S)}x$.

Proof. Putting $F(x,y) = 0$ for all $x,y\in C$ and $r_n = 1$ in Theorem 3.1, the conclusion follows. □

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A New Method for Solving Unconstrained Optimization Problems

Liliu Mo [1,2] and Ling Hong [1]

In this paper, a new conjugate gradient formula β_k^{New} is given to compute the search directions for unconstrained optimization problems. General convergence results for the proposed formula with some line searches such as the exact line search, the Grippo-Lucidi line search and the Wolfe-Powell line search are discussed. Under the above line searches and some assumptions, the global convergence properties of the given methods are discussed. The given formula $\beta_k^{New} \geq 0$, and the search directions d_k which are generated by the given β_k^{New} under the strong Wolfe-Powell line search satisfy the sufficient descent condition. Preliminary numerical results show that the proposed methods are efficient.

KEY WORDS: Unconstrained optimization; Conjugate gradient; Exact line search; Inexact line search; Global convergence.

1. INTRODUCTION

Due to the simplicity of its iteration and its very low memory requirements, the conjugate gradient method has played a special role for solving large-scale nonlinear optimization problems. It is designed to solve the following unconstrained nonlinear optimization problem:

$$\min\{f(x) \mid x \in \Re^n\}, \tag{1.1}$$

where $f : \Re^n \to \Re$ is a smooth, nonlinear function whose gradient will be denoted by $g(x)$. The iterative formula of the conjugate gradient method is given by

$$x_{k+1} = x_k + t_k d_k, \tag{1.2}$$

where t_k is a steplength which is computed by carrying out a line search, and d_k is the search direction defined by

$$d_k = \begin{cases} -g_k, & if \quad k = 1, \\ -g_k + \beta_k d_{k-1}, & if \quad k \geq 2, \end{cases} \tag{1.3}$$

where β_k is a scalar and g_k denotes $g(x_k)$. There are at least six formulas for β_k which are given below:

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}} \qquad (Hestenses - Stiefel[7], 1952);$$

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$$\beta_k^{FR} = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}} \qquad (Fletcher - Reeves[4], 1964);$$

$$\beta_k^{PRP} = \frac{g_k^T (g_k - g_{k-1})}{g_{k-1}^T g_{k-1}} \qquad (Polak - Ribière - Polyak[9,10], 1969);$$

$$\beta_k^{CD} = -\frac{g_k^T g_k}{d_{k-1}^T g_{k-1}} \qquad (Conjugate \quad Descent[3], 1987);$$

$$\beta_k^{LS} = -\frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T g_{k-1}} \qquad (Liu - Storey[8], 1991);$$

$$\beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}} \qquad (Dai - Yuan[1], 1999).$$

Generally, the PRP method performs much better than the FR method from the computation point of view. When the objective function is convex, Polak and Ribière[12] proved that the PRP method with the exact line search is globally convergent. But Powell [11] showed that there exist nonconvex functions on which the PRP method does not converge globally. He suggested that β_k should not be less than zero. Under the sufficient descent condition, Gilbert and Nocedal [5] proved that the modified PRP method $\beta_k = max\{0, \beta_k^{PRP}\}$ is globally convergent with the Wolfe-Powell line search.

In the already-existing convergence analysis of the conjugate gradient methods, the following line searches are often used.

The exact line search is to find t_k such that the cost function is minimized along the direction d_k, that is t_k satisfying

$$f(x_k + t_k d_k) = \lim_{t \geq 0} f(x_k + t d_k). \tag{1.4}$$

The weak Wolfe-Powell (WWP) line search is to find t_k satisfying

$$f(x_k + t_k d_k) - f(x_k) \leq \delta \ t_k g_k^T d_k, \tag{1.5}$$

$$g(x_k + t_k d_k)^T d_k \geq \sigma g_k^T d_k, \tag{1.6}$$

where $\delta \in (0, \frac{1}{2})$, $\sigma \in (\delta, 1)$.

The strong Wolfe-Powell (SWP) line search is to find t_k satisfying (1.5) and

$$\mid g(x_k + t_k d_k)^T d_k \mid \leq \sigma \mid g_k^T d_k \mid, \tag{1.7}$$

where $\delta \in (0, \frac{1}{2})$, $\sigma \in (\delta, 1)$.

This paper is organized as follow: In section 2, a new conjugate gradient formula and the corresponding algorithm are given. The global convergence results of the new formula with the exact line search, the Grippo-Lucidi line search and the Wolfe-Powell line search are given in section 3. The preliminary numerical results are contained in section 4.

2. THE NEW FORMULA AND THE CORRESPONDING ALGORITHM

It is well known that the global convergence of the most efficient PRP method can not be established with the weak Wolfe-Powell conditions (1.5) and (1.6). However, some methods which posses the global convergence property with the weak Wolfe-Powell(WWP), such as the FR method, do not perform better than the PRP method in numerical results. Powell[12] has given some arguments that explain the poor performance of the FR method for some problems. and also showed that the PRP method behaves quite differently. However, the PRP method with the exact line search is not globally convergent for some nonconvex functions, see Powell's counter example in [11].

Therefore, many of the variants of the PRP method had been widely studied.

In recent years, Z. X. Wei *etc.* proposed a new conjugate gradient method in [13] as follows:

$$\beta_k^{WYL} = \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1})}{g_{k-1}^T g_{k-1}}. \tag{2.1}$$

The new conjugate gradient method with the given β_k^{WYL} under some line search is global convergent, good numerical results are obtained. Motivated by [13], we modify the famous PRP conjugate gradient method as follows:

$$\beta_k^{New} = \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|}g_{k-1})}{\| g_{k-1} \|^2 + \mu(g_k^T d_{k-1})^2}, \mu > 0. \tag{2.2}$$

Hence, we obtion a new conjugate gradient formula.

Now we give the following algorithm firstly.

Algorithm 2.1

step 1: Given $x_1 \in R^n$, $\varepsilon > 0$, set $d_1 = -g_1$, $k = 1$, if $\| g_1 \| \leq \varepsilon$, then stop;

step 2: Compute t_k by some line searches;

step 3: Let $x_{k+1} = x_k + t_k d_k$, $g_{k+1} = g(x_{k+1})$, if $\| g_{k+1} \| \leq \varepsilon$, then stop;

step 4: Compute β_{k+1} by (2.2) and generate d_{k+1} by (1.3);

step 5: Set $k := k + 1$, go to step 2.

The following assumptions are often used in the studies of the conjugate gradient methods.

Assumption A: The level set $\Omega = \{x | f(x) \leq f(x_1)\}$ at x_1 is bounded, namely, there exists a constant ($a > 0$) such that

$$\| x \| \leq a \quad for \ \ all \ x \in \Omega. \tag{2.3}$$

Assumption B: In some neighborhood N of Ω, f is continuously differentiable, and its gradient g is Lipschitz continuous, namely, for all $x, y \in N$, there exists a constant $L \geq 0$ such that

$$\| g(x) - g(y) \| \leq L \| x - y \| . \tag{2.4}$$

3. THE GLOBAL CONVERGENCE PROPERTIES

In this section, the convergence properties of the new formula with the exact line search will be studied firstly; secondly, in order to ensure the sufficient descent condition, the Grippo-Lucidi line search is introduced, and the behaviors of the new formula with this line search are discussed; in the end, some arguments about the convergence properties of the new formula with the Wolfe-Powell line search are also given.

3.1. The convergence properties with the exact line search

The following lemmas are very useful in the process of the studies on the conjugate gradient methods.

Lemma 3.1.1 Suppose that assumptions A and B hold. Consider the methods in the form of (1.2) and (1.3), where d_k satisfies

$$g_k^T d_k < 0$$

for all k, and t_k is obtained by WWP (1.5) and (1.6), then,

$$\sum_{k\geq 1} \frac{(g_k^T d_k)^2}{\| d_k \|^2} < +\infty. \tag{3.1}$$

By the way, the inequation (3.1) also holds for the exact line seach, the Armijo-Goldstein line search and the strong Wolfe-Powell (SWP) line search. The proofs had been given in [15].

Theorem 3.1.2. Suppose that assumptions A and B hold, the sequence $\{x_k\}$ is generated by Algorithm 2.1, t_k is computed by exact line search. if $\| s_k \| = \| t_k d_k \| \to 0$ while $k \to \infty$, then

$$\lim_{k\to\infty} inf \| g_k \| = 0. \tag{3.2}$$

Proof. let θ_k be the angle between $-g_k$ and d_k, then, by the exact line search, we have $g_k^T d_{k-1} = 0$ and $d_k = -g_k + \beta_k d_{k-1}$. The above two equations indicate $\| d_k \| = \sec\theta_k \| g_k \|$ and $\beta_{k+1} \| d_k \| = \tan\theta_{k+1} \| g_{k+1} \|$. So we have

$$\begin{aligned}
\tan\theta_{k+1} &= \beta_{k+1}^{New} \sec\theta_k \frac{\| g_k \|}{\| g_{k+1} \|} \\
&= \sec\theta_k \frac{\| g_k \| g_{k+1}^T (g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k)}{\| g_{k+1} \| \cdot (\| g_k \|^2 + \mu(g_{k+1}^T d_k)^2)} \\
&\leq \sec\theta_k \frac{\| g_k \| \cdot \| g_{k+1} \| \cdot \| g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k \|}{\| g_{k+1} \| \cdot \| g_k \|^2} \\
&= \sec\theta_k \frac{\| g_{k+1} - g_k + g_k - \frac{\|g_{k+1}\|}{\|g_k\|} g_k \|}{\| g_k \|} \\
&\leq \sec\theta_k \frac{\| g_{k+1} - g_k \| + |\| g_k \| - \| g_{k+1} \||}{\| g_k \|} \\
&\leq \sec\theta_k \frac{2 \| g_{k+1} - g_k \|}{\| g_k \|}.
\end{aligned} \tag{3.3}$$

If (3.2) does not hold, that is to say, for all k, there exists $\gamma > 0$ such that

$$\| g_k \| \geq \gamma. \tag{3.4}$$

By $\| s_k \| \to 0$ and Lipschitz condition (2.4), there must exist an integer $M \geq 0$ for all $k \geq M$, such that

$$\| g_{k+1} - g_k \| \leq \frac{1}{4}\gamma. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$\tan\theta_{k+1} \leq \frac{1}{2}\sec\theta_k. \tag{3.6}$$

Note that, for all $\theta \in [0, \frac{1}{2})$, the following inequation holds

$$\sec\theta \leq 1 + \tan\theta. \tag{3.7}$$

(3.6) and (3.7) induce

$$\tan\theta_{k+1} \leq \frac{1}{2} + \frac{1}{4} + \cdots + (\frac{1}{2})^{k+1-m}(1+\tan\theta_m) \leq 1 + \tan\theta_m.$$

This result indicates that the angle θ_k must be always less than some angle $\overline{\theta}$ which is less than $\frac{\pi}{2}$. But by the lemma 3.1.1, we have $\sum\limits_{k\geq 1} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = \sum\limits_{k\geq 1} \| g_k \|^2 \cdot(\cos\theta_k)^2 < +\infty$.
This implies $\liminf\limits_{k\to\infty} \| g_k \| = 0$, which contradicts (3.4). The proof is completed.

When t_k is determined by the exact line search, Yuan [14] has proved that if the cost function is uniformly convex, for all k, the following inequation

$$f(x_k) - f(x_k + t_k d_k) \geq C \| s_k \|^2, \tag{3.8}$$

holds, where $C > 0$ is a constant.

By this property, we have the following theorem which shows that the new formula (2.2) with the exact line search is convergent to the uniformly convex functions.

Theorem 3.1.3 Suppose that assumptions A and B hold, $f(x)$ is uniformly convex and $\{x_k\}$ is generated by Algorithm 2.1. If the steplength t_k is determined by the exact line search, then (3.2) holds.

Proof. Since $f(x)$ is uniformly convex on the level set $\Omega = \{x \in R^n : f(x) \leq f(x_1)\}$, when $k \to \infty$, by (3.8) we have $\| s_k \| \to 0$, combining this result with theorem 3.1.2, (3.2) holds immediately.

3.2. The convergence properties with the Grippo-Lucidi line search

On some studies of the conjugate gradient methods, the sufficient descent condition

$$g_k^T d_k \leq -C \| g_k \|^2, \quad C > 0, \tag{3.9}$$

plays an important role. Unfortunately, this condition is hard to hold. It has been showed that the PRP method with the strong Wolfe-Powell line search does not ensure this condition at each iteration. So, Grippo and Lucidi [6] managed to find some line searches which ensure the sufficient descent condition, and they presented a new line search which ensures this condition. The convergence of the PRP method with this line search had been established.

In this subsection, we will show that β_k^{New} with the Grippo-Lucidi line seach is convergent.
Grippo-Lucidi line search . Compute

$$t_k = max\{\sigma^j \frac{\tau \mid g_k^T d_k \mid}{\| d_k \|^2}; \quad j = 0, \quad 1, \quad \cdots\} \tag{3.10}$$

satisfying the conditions:

$$f(x_k + t_k d_k) \le f(x_k) - \delta t_k^2 \parallel d_k \parallel^2, \tag{3.11}$$

$$-C_2 \parallel g_{k+1} \parallel^2 \le g_{k+1}^T d_{k+1} \le -C_1 \parallel g_{k+1} \parallel^2, \tag{3.12}$$

where $\delta > 0$, $\tau > 0$, $\sigma \in (0, \ \ 1)$ and $0 < C_1 < 1 < C_2$.

The following theorem shows that the Grippo-Lucidi line search is suitable for the new formula β_k^{New}.

Theorem 3.2.1 Suppose that assumptions A and B hold. Consider the method of form (1.2) and (1.3), t_k is computed by (3.10). Then for all k, there exists $t_k > 0$ such that (3.11) and (3.12) hold. Furthermore, there exists a constant $c > 0$ such that

$$t_k \ge c \frac{| g_k^T d_k |}{\parallel d_k \parallel^2}. \tag{3.13}$$

Proof. We prove this theorem by induction. Since $d_1 = -g_1$, (3.12) holds for $k = 1$. Suppose that (3.12) holds for some $k \ge 1$. Denote

$$C_3 = \frac{min(1 - C_1, \ \ C_2 - 1)}{2LC_2} > 0. \tag{3.14}$$

By Lipschitz condition (2.4) and (3.12), for any $t_k \in (0, \ \ C_3 \frac{|g_k^T d_k|}{\|d_k\|^2})$, we have

$$\begin{array}{rcl}
| g_{k+1}^T d_{k+1} + \parallel g_{k+1} \parallel^2 | & \le & | \beta_{k+1}^{New} | \cdot | g_{k+1}^T d_k | \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \frac{\|g_{k+1} - \frac{\|g_{k+1}\|}{\|g_k\|} g_k\| \cdot \|d_k\|}{\|g_k\|^2 + \mu (g_{k+1}^T d_k)^2} \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \frac{\|g_{k+1} - g_k + g_k - \frac{\|g_{k+1}\|}{\|g_k\|} g_k\| \cdot \|d_k\|}{\|g_k\|^2} \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \frac{(\|g_{k+1} - g_k\| + |\|g_k\| - \|g_{k+1}\||) \cdot \|d_k\|}{\|g_k\|^2} \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \frac{2\|g_{k+1} - g_k\| \cdot \|d_k\|}{\|g_k\|^2} \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \frac{2Lt_k \cdot \parallel d_k \parallel^2}{\parallel g_k \parallel^2} \\
& \le & \parallel g_{k+1} \parallel^2 \cdot \min(1 - C_1, C_2 - 1).
\end{array}$$

So (3.12) holds for $k = k + 1$.

On the other hand, by the mean value theorem and Lipschitz condition, we can obtain

$$\begin{array}{rcl}
f(x_{k+1}) - f(x_k) & = & \int_0^1 g(x_k + t \cdot t_k d_k)^T (t_k d_k) d(t) \\
& = & t_k g_k^T d_k + \int_0^1 [g(x_k + t \cdot t_k d_k) - g(x_k)]^T (t_k d_k) d(t) \\
& \le & t_k g_k^T d_k + \frac{1}{2} L t_k^2 \parallel d_k \parallel^2 \\
& = & -\frac{t_k^2 \mid g_k^T d_k \mid \cdot \parallel d_k \parallel^2}{\parallel d_k \parallel^2 \, t_k} + \frac{1}{2} t_k^2 L \parallel d_k \parallel^2 \\
& = & t_k^2 \parallel d_k \parallel^2 (-\frac{\mid g_k^T d_k \mid}{\parallel d_k \parallel^2 \, t_k} + \frac{L}{2}) \\
& \le & t_k^2 \parallel d_k \parallel^2 (-\frac{L + 2\delta}{2} + \frac{L}{2}) \\
& = & -\delta t_k^2 \parallel d_k \parallel^2 .
\end{array}$$

So, (3.11) holds for $t_k \in (0,\ \frac{2}{L+2\delta}\frac{|g_k^T d_k|}{\|d_k\|^2})$.

The existence of t_k satisfying (3.11) and (3.12)has been proved. Furthermore, (3.13) holds for $C = \min(\tau,\ C_3,\ \frac{2}{L+2\delta})$. The proof is completed.

With the above results, the global convergence of the formula β_k^{New} with the Grippo-Lucidi line search can be established by the following theorem.

Theorem 3.2.2 Suppose that assumptions A and B hold, consider the method of form (1.2) and (1.3), t_k is computed by Grippo-Lucidi line search, β_k^{New} is determined by (2.2). Then

$$\lim_{k\to\infty} \| g_k \|= 0. \tag{3.15}$$

Proof. By the Lipschitz condition (2.4), (3.10) and (3.12), we have

$$\begin{aligned} \| d_k \| &\leq \| g_k \| + | \beta_k^{New} | \cdot \| d_{k-1} \| \\ &\leq \| g_k \| (1 + \frac{\| g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1} \|}{\| g_{k-1} \|^2 + \mu(g_k^T d_{k-1})^2} \| d_{k-1} \|) \\ &\leq \| g_k \| (1 + \frac{\| g_k - g_{k-1} \| + |\| g_{k-1} \| - \| g_k \||}{\| g_{k-1} \|^2} \| d_{k-1} \|) \\ &\leq \| g_k \| (1 + \frac{2Lt_{k-1}}{\| g_{k-1} \|^2} \| d_{k-1} \|^2) \\ &\leq \| g_k \| (1 + \frac{2\tau L \mid g_{k-1}^T d_{k-1} \mid}{\| g_{k-1} \|^2}) \\ &\leq (1 + 2C_2\tau L) \| g_k \| . \end{aligned} \tag{3.16}$$

Because of the assumption A, it is obviously that the zoutendijk condition (3.1) holds. Combining (3.1), (3.12) and (3.16), we have

$$\infty > \sum_{k\geq 1} \frac{(g_k^T d_k)^2}{\| d_k \|^2} \geq C_1^2 (1 + 2C_2\tau L)^{-2} \sum_{k\geq 1} \| g_k \|^2 .$$

This result implies $\lim_{k\to\infty} \| g_k \|= 0$.

From the theorem 3.2.2, we know that any accumulation point of $\{x_k\}$ which is generated by (2.2) with the Grippo-Lucidi line search is a stationary point. This result is contributed to the property: the directions given by β_k^{New} approach to $-g_k$ while $\| s_{k-1} \|$ tend to zero.

3.3. The convergence properties with the Wolfe-Powell line search

In above section, we introduced the Grippo-Lucidi line search which were designed to match the requirements of the convergence of the PRP method. In fact, the main contribution of the Grippo-Lucdi line search is that it can ensure the sufficient descent conditions. But the global convergence study might not yield a better conjugate gradient method from the numerical computational point of view. In fact, the method given by Grippo and Lucidi did not perform better than the PRP method which employed (3.17).

Powell [11] suggested that β_k should not be less than zero. This suggestion is useful to the PRP method, see the detail in [11]. Unedr the sufficient descent conditon, Gilbert and Nocedal [5] proved that the modified PRP method

$$\beta_k = max(0, \beta_k^{PRP}) \tag{3.17}$$

is globally convergent with the Wolfe-Powell line search.

In this section, we will prove that, under the strong Wolfe-Powell line search, by restricting the parameter $\sigma < \frac{1}{4}$, the given β_k^{New} possess the sufficient condition which deduces the global convergent result of the new method under the strong Wolfe-Powell line search.

Now, we firstly prove the following lemma which shows that the sequence $\{d_k\}$ generated by algorithm 2.1 satisfies sufficient descent condition.

Lemma 3.3.1 Suppose that the sequences $\{g_k\}$ and $\{d_k\}$ are generated by the method of the form (1.2),(1.3) in which β_k was computed by (2.2), and the steplength t_k is determined by the Wolfe-Powell line search (1.5) and (1.7), if $\sigma \in (0, \frac{1}{4})$, then there exists a positive constant C such that the sufficient descent condition (3.9) and $\beta_k^{New} \geq 0$ hold.

Proof. We prove this lemma by induction. We firstly prove the descent property , namely $g_k^T d_k < 0$. Suppose that for all k, $g_k^T d_k \neq 0$. Since $g_1^T d_1 = - \| g_1 \|^2 < 0$, and set

$$g_{k-1}^T d_{k-1} < 0, \tag{3.18}$$

hold for $i = k-1$. By Cauchy-Schwarz inequality, we get

$$0 \leq 1 - \frac{g_k^T g_{k-1}}{\| g_k \| \| g_{k-1} \|} \leq 2. \tag{3.19}$$

Form (1.3) and $\beta_k = \beta_k^{New}$, we have

$$\frac{g_k^T d_k}{\| g_k \|^2} = -1 + \frac{g_k^T d_{k-1}}{\| g_{k-1} \|^2 + \mu (g_k^T d_{k-1})^2}(1 - \frac{g_k^T g_{k-1}}{\| g_k \| \| g_{k-1} \|}) \leq -1 + \frac{g_k^T d_{k-1}}{\| g_{k-1} \|^2}(1 - \frac{g_k^T g_{k-1}}{\| g_k \| \| g_{k-1} \|}). \tag{3.20}$$

Combining(3.19),(3.20) and (1.7), we can deduce that

$$\frac{g_k^T d_k}{\| g_k \|^2} \leq -1 - 2\sigma \frac{g_{k-1}^T d_{k-1}}{\| g_{k-1} \|^2}. \tag{3.21}$$

By repeating this process and the fact $g_1^T d_1 = - \| g_1 \|^2$, we have

$$\frac{g_k^T d_k}{\| g_k \|^2} \leq -2 + \sum_{j=0}^{k-1} (2\sigma)^j. \tag{3.22}$$

Since

$$\sum_{j=0}^{k-1} (2\sigma)^j < \sum_{j=0}^{\infty} (2\sigma)^j = \frac{1}{1-2\sigma},$$

(3.22) can be written as

$$\frac{g_k^T d_k}{\| g_k \|^2} \leq -2 + \frac{1}{1-2\sigma}. \tag{3.23}$$

Since $\sigma \in (0, \frac{1}{4})$, so $-2 + \frac{1}{1-2\sigma} < 0$ which implies $g_k^T d_k < 0$, namely (3.18) holds for $i = k$.

Now we prove the sufficient edscent property as follows. In fact, if $\sigma \in (0, \frac{1}{4})$, set $c = 2 - \frac{1}{1-2\sigma}$, then $0 < c < 1$, and (3.23) implies that

$$g_k^T d_k \leq -c \| g_k \|^2 . \tag{3.24}$$

So $g_1^T d_1 = -\| g_1 \|^2 \leq -c \| g_1 \|^2$ and (3.24) immediately deduce that the sufficient descent condition (3.9) hold for all $k \geq 1$.

For β_k^{New} , we can prove that β_k^{New} are always not less than zero. namely

$$\beta_k^{New} = \frac{g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1})}{\| g_{k-1} \|^2 + \mu (g_k^T d_{k-1})^2} \geq \frac{(\| g_k \|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} \| g_k \| \| g_{k-1} \|)}{\| g_{k-1} \|^2 + \mu (g_k^T d_{k-1})^2} = 0.$$

The proof is finished.

Gilbert and Nocedal [5] introduced the following property(*) which pertains to the PRP method under the sufficient descent condition. Now we will show that this property(*) pertains to the new method.

Property(*): Consider a method of form (1.2) and (1.3). Suppose that

$$0 < \gamma \leq \| g_k \| \leq \overline{\gamma}. \tag{3.25}$$

We say that the method has property(*), if for all k, there exist constants $b > 1,\ \lambda > 0$ such that $| \beta_k | \leq b$ and if $\| s_{k-1} \| \leq \lambda$ we have $|\beta_k| \leq \frac{1}{2b}$.

The following lemma shows that the new method has the property(*).

Lemma 3.3.2 Consider the method of form (1.2) and (1.3) in which $\beta_k = \beta_k^{New}$. Suppose that assumptions A and B hold, then, the method has property(*).

Proof. set $b = \frac{\overline{\gamma}^2(\gamma + \overline{\gamma})}{\gamma^3} > 1, \lambda = \frac{\gamma^2}{4L\overline{\gamma}b}$. By (2.2)and (3.25) we have

$$\begin{aligned} | \beta_k^{New} | &\leq \frac{| g_k^T(g_k - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1}) |}{\| g_{k-1} \|^2 + \mu (g_k^T d_{k-1})^2} \\ &\leq \frac{\| g_k \| (\| g_k \| + \frac{\overline{\gamma}}{\gamma} \| g_{k-1} \|)}{\| g_{k-1} \|^2} \\ &\leq \frac{\overline{\gamma}(\overline{\gamma} + \frac{\overline{\gamma}^2}{\gamma})}{\gamma^2} = \frac{\overline{\gamma}^2(\gamma + \overline{\gamma})}{\gamma^3} = b. \end{aligned} \tag{3.26}$$

From the assumption B and (2.4)holds. If $\| s_{k-1} \| \leq \lambda$ then,

$$\begin{aligned} | \beta_k^{New} | &\leq \frac{(\| g_k - g_{k-1} \| + \| g_{k-1} - \frac{\|g_k\|}{\|g_{k-1}\|} g_{k-1} \|) \| g_k \|}{\| g_{k-1} \|^2} \\ &\leq \frac{(L\lambda + | \| g_{k-1} \| - \| g_k \| |) \| g_k \|}{\| g_{k-1} \|^2} \\ &\leq \frac{(L\lambda + \| g_k - g_{k-1} \|) \| g_k \|}{\| g_{k-1} \|^2} \\ &\leq \frac{2L\lambda \| g_k \|}{\| g_{k-1} \|^2} \\ &\leq \frac{2L\overline{\gamma}\lambda}{\gamma^2} = \frac{1}{2b}. \end{aligned} \tag{3.27}$$

The proof is finished.

If (3.25) holds and the methods have property(*), then, the small steplength should not be too many. The following lemma shows this property.

Lemma 3.3.3 Suppose that assumptions A, B and (3.9) hold. Let $\{x_k\}$ and $\{d_k\}$ be generated by (1.2) and (1.3) in which t_k satisfies the Wolfe-Powell line search (1.5) and (1.6),

β_k has property(*). If (3.25) holds, then, for any $\lambda > 0$, there exist $\Delta \in N^+$ and $k_0 \in N^+$, for all $k \geq k_0$, such that

$$| \kappa_{k,\Delta}^{\lambda} | \geq \frac{\Delta}{2},$$

where $\kappa_{k,\Delta}^{\lambda} = \{i \in Z^+ : k \leq i \leq k + \Delta - 1, \| s_{i-1} \| \geq \lambda\}$, $| \kappa_{k,\Delta}^{\lambda} |$ denotes the numbers of the $\kappa_{k,\Delta}^{\lambda}$.

Lemma 3.3.4 Suppose that assumptions A, B and (3.9) hold. Let $\{x_k\}$ be generated by (1.2) and (1.3), t_k satisfies Wolfe-Powell line search (1.5) and (1.6), and $\beta_k \geq 0$ has property(*). Then,

$$\liminf_{k \to \infty} \| g_k \| = 0.$$

The proofs of lemma 3.3.3 and lemma 3.3.4 had been given in [2]. By the above three lemmas, we have the following convergence result.

Theorem 3.3.5 Suppose that assumptions A, B and (3.9) hold. Let $\{x_k\}$ be generated by (1.2) and (1.3), t_k satisfies the Wolfe-Powell line search (1.5) and (1.6), β_k is computed by (2.2), then

$$\liminf_{k \to \infty} \| g_k \| = 0.$$

This theorem is the immediate result of the above three lemmas, so the proof is omitted.

Theorem 3.3.5 shows that under some assumptions, the new formula with the Wolfe-Powell line search is globally convergent.

4. NUMERICAL RESULTS

In this section, we report the detailed numerical results of a number of problems by algorithm 2.1. Furthermore, the original PRP methods are given. we test the following four conjugate gradient methods :

$PRPSWP$: PRP method under the strong Wolfe-Powell conditions;

PRP^+SWP: conjugate gradient method with $\beta_k = max\{0, \beta_k^{PRP}\}$under the strong Wolfe-Powell conditions;

$WYLSWP$: conjugate gradient method β_k defined by (2.1)under the strong Wolfe-Powell conditions;

$NEWSWP$: New conjugate gradient method β_k defined by (2.2)under the strong Wolfe-Powell conditions, $\mu = 3.13$;

In our implementation, we choose parameters for line search conditions as follows: $\delta = 0.01, \sigma = 0.1$. The termination condition is

$$\| g_k \| \leq 10^{-5}.$$

The experiments were carried out on some famous test problems which can be obtained on net. In the following tables, the numerical results are written in the form NI/NF/NG, where NI, NF, NG denote the number of iterations, function evaluations and gradient evaluations respectively. Dim denotes the dimension of the test problems.

The numerical result of each method under strong Wolfe-Powell condition is showed in Table 1.

Table 1 Tests Results for the PRPSWP/PRP$^+$SWP/WYLSWP/NEWSWP Methods

		$PRPSWP$	PRP^+SWP	$WYLSWP$	$NEWSWP$
$Problem$	Dim	$NI/NF/NG$	$NI/NF/NG$	$NI/NF/NG$	$NI/NF/NG$
ROSE	2	29/502/65	22/394/60	42/464/102	32/353/27
FROTH	2	12/30/20	10/28/20	17/87/28	21/50/38
BADSCP	2	-	34/396/84	44/423/125	54/341/114
BADSCB	2	13/80/22	11/123/22	-	26/50/38
BEALE	3	9/126/21	9/173/20	17/136/29	17/89/30
JENSAM	4	-	-	11/31/21	9/121/16
HELIX	2	49/255/83	32/265/55	74/396/123	32/269/53
BARD	2	23/98/37	27/152/43	43/136/66	26/245/41
GAUSS	2	4/57/6	4/57/6	4/9/5	4/9/5
GULF	2	1/2/2	1/2/2	1/2/2	1/2/2
SING	3	181/706/300	49/155/79	44/191/74	70/201/115
WOOD	3	195/1278/353	84/461/170	145/649/252	93/509/171
KOWOSB	4	108/528/188	51/249/79	52/343/84	48/241/78
BIGGS	6	174/648/284	-	122/584/194	13/173/18
OSB2	11	255/1306/420	192/854/322	221/910/363	334/1596/549
WATSON	20	1887/5053/2961	753/2155/1216	1613/4132/2516	2055/5420/3171
ROSEX	8	23/402/59	25/371/62	39/488/88	38/626/80
	50	31/533/76	25/398/59	37/333/82	45/365/93
	100	29/479/77	33/566/101	43/399/94	47/415/96
SINGX	4	181/706/300	49/155/79	44/191/74	70/201/115
PEN1	2	5/18/12	6/20/14	5/18/12	5/18/12
PEN2	4	12/134/28	12/136/27	10/128/25	8/125/24
	50	467/1824/775	547/1916/944	121/1023/230	104/884/207
VARDIM	2	3/9/7	3/9/7	3/9/7	3/9/7
	50	10/52/36	10/52/36	10/52/36	10/52/36
TRIG	3	12/81/24	14/131/25	14/271/26	14/271/26
	50	41/279/72	41/230/72	38/219/66	38/219/66
	100	46/342/87	46/341/85	47/434/86	47/434/86
BV	3	12/25/16	12/25/16	12/25/16	12/25/16
	10	75/241/117	75/241/117	52/151/85	52/151/85
IE	3	5/12/7	6/14/8	5/12/7	5/12/7
	50	6/13/7	5/11/6	6/13/7	6/13/7
	100	6/13/8	6/13/8	6/13/8	6/13/8
	200	6/13/8	6/13/8	5/59/7	5/59/7
	500	6/13/8	6/13/8	6/13/8	6/13/8
RRID	3	10/75/13	113/33/19	13/31/17	14/31/18
	50	26/55/31	26/65/31	27/57/31	25/52/29
	100	30/67/36	30/67/36	30/67/36	27/59/33
	200	30/66/36	30/66/36	30/66/37	30/66/37
BAND	3	9/68/13	10/23/17	7/64/12	7/64/12
	50	18/183/24	16/331/25	19/670/26	18/572/25
	100	18/183/24	16/373/26	18/712/27	18/713/28
	200	19/283/27	117/340/27	18/677/26	18/629/26
LIN	2	1/3/3	1/3/3	1/3/3	1/3/3
	50	1/3/3	1/3/3	1/3/3	1/3/3
	500	1/3/3	1/3/3	1/3/3	1/3/3
	1000	1/3/3	1/3/3	1/3/3	1/3/3
LIN1	2	1/51/2	1/51/2	1/51/2	1/51/2
	10	1/3/3	1/3/3	1/3/3	1/3/3
LIN0	4	1/3/3	1/3/3	1/3/3	1/3/3

In order to rank these methods, we compute the total number of function and gradient evaluations by the following formula

$$N_{total} = NF + 5 * NG. \tag{4.1}$$

In the part, we compare the PRP$^+$SWP, WYLSWP and the NEWSWP with the PRPSWP as follow: for each testing example i, compute the total numbers of function evaluation and gradient evaluations required by the evaluated method $j(EM(j))$ and the PRPSWP method by the formula (4.1), and denote them by $N_{total,i}(EM(j))$ and $N_{total,i}(PRP)$; then calculate the radio

$$r_i(EM(j)) = \frac{N_{total,i}(EM(j))}{N_{total,i}(PRP)}. \tag{4.2}$$

If $EM(j_0)$ does not work for example i_0, but $EM(PRP)$ works for example i_0, we replace the $r_{i_0}(EM(j_0))$ by a constant τ_1 which is defined as follows:

$$\tau_1 = max\{r_i(EM(j_0)) :\ (i, j_0) \notin S_1\},$$

where $S_1 = \{(i, j_0)$: method j_0 does not work for example $i\}$.

If $EM(PRP)$ does not work for example i_0, but $EM(j_0)$ works for example i_0, we replace the $r_{i_0}(EM(j_0))$ by a constant τ_2 which is defined as follows:

$$\tau_2 = min\{r_i(EM(j_0)) :\ (i, j_0) \notin S_1\}.$$

If $EM(PRP)$ and $EM(j_0)$ do not work for example i_0, then we define $r_{i_0}(EM(j_0)) = 1$. The geometric mean of these ratios for method j over all the test problems is defined by:

$$r(EM(j)) = (\prod_{i\in S} r_i(EM(j)))^{\frac{1}{|S|}}, \tag{4.3}$$

where S denotes the set of the test problems and $\mid S \mid$ the number of elements in S. One advantage of the above rule is that, the comparison is relative and hence does not be dominated by a few problems for which the method requires a great deal of function evaluations and gradient functions.

According to the above rule, it is clear that r(PRPSWP)= 1. The values of r(PRP^+SWP), r(WYLSWP), r(NEWSWP)are listed in table 2.

Table 2 Relative Efficiency of the PRPSWP, PRP$^+$SWP, WYLSWP, NEWSWP Methods

PRPSWP	PRP^+SWP	WYLSWP	NEWSWP
1	0.9149	0.9529	0.8774

From Table 2, we observe that the average performances of the new conjugate gradient formula are the best among the four methods. Therefore, the new formula most efficient for unconstrained minimization problems.

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On Best Simultaneous Approximation in ε-Chainable Metric Spaces

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Abstract. *In this note, the existence of invariant best simultaneous approximation in ε-chainable metric space is proved. In doing so, we have used a recent result of Xu regarding the fixed points of set- valued mappings.*
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1. Introduction :

In the realm of Best Approximation Theory, it is viable, meaningful and potentially productive to know whether some useful properties of the function being approximated are inherited by the approximating function. In this perspective, Meinardus [8] observed the general principle that could be applied, while doing so.The author has employed a fixed point theorem as a tool to establish it . The result of Meinardus was further generalized by Habiniak [5], Smo1uk [15] and Subrahmanyam [16].

On the other hand, Beg and Shahzad [2], Fan [4], Hicks and Humphries [6], Reich [10], Singh [13, 14] and many others have used fixed point theorems in approximation theory, to prove existence of best approximation. Various types of applications of fixed point theorems may be seen in Klee [7], Meinardus [8] and Vlasov [18]. Some applications of the fixed point theorem to best simultaneous approximation is given by Sahney and Singh [11]. For the detail survey of the subject, we refer the reader to Cheney [3].

In this note, we use a recent result of Xu [19] regarding the fixed points of set-valued mappings and prove the existence of invariant best simultaneous approximation in ε-chainable metric space.

2. Prelimineries:

Let ε be a positive number. Recall that a metric space (X, d) is said to be ε-chainab1e if for every $x, y \in X$, there is an ε-chain from x to y ,that is, a finite set of points $x = x_0, x_1, x_2, \dots, x_n = y$ such that $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, 2, \dots, n$.

We denote by $CB(X)$ the family of all nonempty closed bounded subsets of X, by $C(X)$ the family of all nonempty compact subsets of X, and by H the Hausdorff metric on $CB(X)$ induced by d. The metric H is defined as follows:

$$H(A,B)= \max \{\sup_{a\in A} d(a, B), \sup_{b\in B} D(b, A)\}$$

for $A,B\in C(X)$, where $d(x, K) = \inf \{d(x, y)$ for $x\in X$, $y\in K$ and $K\subseteq X$ }.

Suppose that I denotes the unit interval [0,1].Then a continuous mapping $W: X x X x I \to X$ is said to be a convex structure on X if the inequality

$$d(u, W(x, y,\lambda) \leq \lambda\, d(u, x)+(1-\lambda)\, d(u, y)$$

holds for all $x, y, u\in X$ and $\lambda\in I$. The metric space X together with a convex structure W is called a convex metric space. Banach space and each of its convex subsets are simple examples of convex metric spaces. There are many convex metric spaces which can not be imbedded in any Banach space. For examples and other details we refer to Takahashi [7].

A subset K of a convex metric space X is said to be convex if $W(x, y,\lambda)\in K$ for all $x, y\in K$ and $\lambda\in I$. The set K is said to be starshaped if there exists $p\in K$ such that $W(x,p,\lambda)\in K$ for all $x\in K$ and $\lambda\in I$. Here p is called the star centre of K. Evidently, starshaped subsets of X contain all convex subsets of X as a proper subclass.

Let X be a convex metric space, and K be a starshaped subset of X with p as its star centre. Then X is said to satisfy condition (1) at $p\in K$ if for any $x, y \in K$, $\lambda\in I$,

$$d(W(x, p, \lambda), W(y, p,\lambda))\leq\lambda\, d(x, y) .$$

It may be observed that any normed space X always satisfies condition (l) (Beg and Shahzad [2]). It is well-known that the hyperspace (C(X),H) is an ε-chainable, whenever (X, d) is an ε-chainable metric space ([19],Theorem 1).

We shall make use of the following theorem due to Xu ([19],Theorem 2) in our main result.

Theorem 1. Let (X, d) be a complete ε-chainable metric space, and let $T: X\to C(X)$ be a set-valued mapping such that

$$H(Tx, Ty) \leq k\, d(x, y))\, d(x, y)$$

for all $x, y\in X$ with $0< d(x, y)< \varepsilon$, where $k : (0,\infty)\to[0,1)$ is a real-valued function with the property

(P) For each $0< t< \varepsilon$, there exist $\varepsilon (t)>0$ and $s(t)<1$ such that $k(r)\leq s(t)$ provided $t\leq r< t + \varepsilon(t)$.

Then T has a fixed point.

Remarks.

(l) The property (P) is equivalent to saying that $\lim\sup k(s)<1$ for all $0<t<\varepsilon$.

$s \to t^+$

(2) If we replace C(X) by CB(X), taking k(s)=k(a constant) $\varepsilon \in [0,1)$ and omitting the condition (P) in Theorem 1, the conclusion still holds (Beg and Azam [1], Beg and Shahzad [2, Theorem 1]).

3.**Results :**

Let (X, d) be a metric space, and G a nonempty subset of X. Suppose $A \in B(X)$, the set of nonempty bounded subset of X. Then we define

$$r_G(A) = \inf \{\sup d(a,g) : g \in G \;\; a \in A\}$$
$$\mathrm{cent}_G (A) = \{g_0 \in G : \sup d(a,g_0) = r_G (A),\ a \in A \}.$$

The number $r_G(A)$ is called the Chebyshev radius of A with respect to G. Also, an element $g \in \mathrm{cent}_G (A)$ is called a best simultaneous approximation of A with respect to G. If $A=\{x\}$, $x \in X$, then $r_G(A) = d(x,G)$ and $\mathrm{cent}_G (A)$ is the set of all best approximations of x out of G. We refer the readers to Milman [9] for further details of these concepts.

Now, we present our main result of this note.

Theorem 2. Let X be an ε-chainable convex metric space satisfying condition (I), and $T:X \to C(X)$ be a set-valued mapping. Suppose that $\mathrm{cent}_G(A)$ is nonempty compact, starshaped and T-invariant, where $G \in C(X)$ and $A \in C(X)$.Further, suppose that T satisfies the following conditions:

(i) T is continuous on $\mathrm{cent}_G(A)$, and

(ii) $d(x, y) \le H(A,G)$ implies $H(Tx, Ty) \le \Phi (d(x, y))\, d(x, y)$ for all $x, y \in A \cup \mathrm{cent}_G(A)$, with $0<d(x, y)<\varepsilon$,where $\Phi: (0,\infty) \to [0,1)$ is a real-valued function with $\limsup_{s \to t+} \Phi(s)<1$ for all t with $0<t<\varepsilon$.

Then $\mathrm{cent}_G(A)$ contains a T-invariant point.

Proof. Let p be the star-centre of $\mathrm{cent}_G(A)$. Then $W(x,p,\lambda) \in \mathrm{cent}_G(A)$ for each $x \in \mathrm{cent}_G(A)$. Let $\{ k_n \}$ be a real sequence with $0 \le k_n < l$ such that $k_n \to 1$ as $n \to \infty$.

Now define $T_n:\mathrm{cent}_G(A) \to C(\mathrm{cent}_G(A))$ by $T_n (x) = W(Tx,p,k_n)$ for all $x \in \mathrm{cent}_G(A)$. Now applying condition (I), we obtain

$$\begin{aligned} H(T_n x, T_n y) &= H(W(Tx,p, k_n),W(Ty,p, k_n)) \\ &\le k_n H(Tx, Ty) \\ &\le k_n \Phi(d(x,y))\, d(x,y) \qquad \text{(using (ii))} \end{aligned}$$

Thus we get

$$H(T_n x, T_n y) \le \Phi_n(d(x,y))\, d(x,y)$$

for all $d(x,y) \le H(A,G)$, where $\Phi_n = k_n\Phi:(0,\infty) \to [0,1)$ is a real-valued function with

$\limsup_{s\to t^+} \Phi(s)<l$ for all $0< t <\varepsilon$. Then it follows from Theorem 1 that each T_n has a fixed point, say z_n. Since $\mathrm{cent}_G(A)$ is compact, $\{z_n\}$ has a convergent subsequence $\{z_{n_i}\}$ such that $z_{n_i} \to z$ as $i \to \infty$ for some $z\in X$. Since

$$z_{n_i} \in T_{n_i} z_{n_i} = W(Tz_{n_i}, p, k_{n_i}),$$

and $k_{n_i} \to 1$ as $i \to\infty$, it follows that $z\in Tz$.
This completes the proof of our theorem.
Open Question. It is not known whether the condition of Theorem 2 is valid if C(X) is replaced by CB(X).

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Models based on Normal and Laplace Random Variables

by

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Abstract: The normal and Laplace distributions are the oldest known continuous distributions in statistics. The distributions of their linear combinations, products and ratios arise in many areas of the sciences and engineering. In this note, the exact distributions of $\alpha X + \beta Y$, $\mid XY \mid$ and $\mid X/Y \mid$ are derived when X and Y are independent normal and Laplace random variables. Several motivating applications are discussed.

Keywords and Phrases: Laplace distribution; Linear combination; Normal distribution; Product; Ratio.

1 Introduction

In this note, we derive the exact distributions of $\alpha X + \beta Y$, $\mid XY \mid$, and $\mid X/Y \mid$ when X and Y are independent random variables having the normal and Laplace distributions with the pdfs

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \tag{1}$$

and

$$f_Y(y) = \frac{1}{2\phi} \exp\left(-\frac{\mid y-\lambda \mid}{\phi} \right), \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu < \infty$, $-\infty < \lambda < \infty$, $\sigma > 0$ and $\phi > 0$. We assume without loss of generality that $\alpha > 0$.

Why study the linear combinations, products and ratios of normal and Laplace random variables? Four motivating applications from speech enhancement, option pricing, Bayesian statistics and discrimination theory are discussed in Section 2. The exact expressions for the pdfs and the cdfs of $\alpha X + \beta Y$, $\mid XY \mid$ and $\mid X/Y \mid$ are given in Sections 3, 4 and 5, respectively. The calculations involve several special functions, including the complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt$$

and the hypergeometric function defined by

$${}_1F_3\left(a; b, c, d; x\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\,(c)_k\,(d)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* [1] and Gradshteyn and Ryzhik [2].

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2 Motivating Applications

2.1 Speech Enhancement

Over the past four decades, the problem of speech enhancement (SE) has been discussed by many researchers, see e.g. [3]–[11]. The main objective of SE is to improve the performance of speech communication systems in a noisy environment.

The noisy component of noisy speech is the sum of the independent components of clean speech and noise. Usually, statistical models are used to estimate the clean speech and noise components. In the case of spectral representation of the signal, these components will be complex variables and one would model the statistical behavior of the real and imaginary parts. It is widely acknowledged that the two most popular statistical models for speech data are the normal and Laplace distributions.

Suppose X and Y are independent random variables representing the clean speech and noise components (the real and imaginary parts of the components, in the case of the spectral representation), respectively. The problem of SE requires estimation of the noisy component given by the sum $Z = X + Y$.

2.2 Option Pricing

The famous option pricing formula introduced by Kou [12] is derived by taking the sum of normal and Laplace random variables with zero means.

2.3 Posterior Density of the Normal Standard Deviation

The most popular posterior distribution for the standard deviation of the normal distribution can be obtained as follows. Suppose u is an observation from a normal distribution with mean λ and standard deviation ω. Suppose too that the observer has some prior knowledge about λ given by $p(\lambda, \omega) = g(\lambda)$, where $g(\cdot)$ denotes a symmetric probability density function (pdf). Then the joint posterior of λ and ω will be

$$p\left(\lambda, \omega \mid u\right) \quad \propto \quad \exp\left\{-\frac{(u-\lambda)^2}{2\omega^2}\right\} g(\lambda).$$

Thus, the marginal posterior of ω will be

$$p\left(\omega \mid u\right) \quad \propto \quad \int_{-\infty}^{\infty} \exp\left\{-\frac{(u-\lambda)^2}{2\omega^2}\right\} g(\lambda) d\lambda. \tag{3}$$

If we take $g(\cdot)$ to be a Laplace pdf then the posterior distribution given by (3) is the same as that of the product XY when $X \sim$ Normal and $Y \sim$ Laplace are independent of each other.

2.4 Discrimination Problem

It is well–known that the normal distribution is used to analyze symmetric data with short tails, while the Laplace distribution is used for data with long tails. Although these two distributions may provide similar fit for moderate sample sizes, it is still desirable to choose the better fitting model since inference procedures often involve tail probabilities, where the distributional assumption becomes crucial. Hence, it is important to make the best possible decision based on the available data.

Suppose an experimenter has n observations and the elementary data analysis, say a histogram, stem and left plot, or the box plot, suggests that they have come from a symmetric distribution. The experimenter wants to determine which of normal or Laplace distributions fits the data better. The usual approach for this is based on discriminant analysis. In fact, Kundu [13] used the ratio of maximized likelihoods to discriminate between normal and Laplace distributions.

A different approach is as follows. Let $\widehat{F}_X$ denote the estimate of the cdf of X by assuming that the n observations come from (1). Similarly, let $\widehat{F}_Y$ denote the estimate of the cdf of Y by assuming that the n observations come from (2). For every real number c plot a point $T(c)$ in a Cartesian coordinate system with the coordinates $(\widehat{F}_X(c), \widehat{F}_Y(c))$. Note that the coordinates of $T(c)$ lie between 0 and 1, so that the graph is always located within the unit square $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Moreover, by letting $c = \pm\infty$ we see that the graph always starts at $(0, 0)$ and ends at $(1, 1)$. See Figure 1.

[Figure 1 about here.]

The above graph can be used to discriminate between the two distributions. Note that the area above the graph is equal to

$$
\begin{aligned}
A(X, Y) &= \int_0^1 \Pr(X \leq c) d \Pr(Y \leq c) \\
&= \int_0^1 \Pr(X \leq c) f_Y(c) dc \\
&= \Pr(X < Y).
\end{aligned}
$$

In view of this relation, the area $A(X, Y)$ can be utilized as a measure of the size difference between the two distributions with $A(X, Y) = 1$ if and only if the distribution of X lies entirely below the distribution of Y. On the other hand, if X and Y are identically distributed, $A(X, Y) = 1/2$. Clearly, the computation of $A(X, Y)$ requires the study of the distribution of the ratio X/Y.

3 Distribution of the Linear Combination

Here, we consider the distribution of $Z = \alpha X + \beta Y$ when X and Y are independent random variables, distributed according to (1) and (2), respectively. Theorem 1 derives explicit expressions for the pdf and the cdf of Z in terms of the complementary error function.

Theorem 1 *Suppose X and Y are independent random variables, distributed according to (1) and (2), respectively. Then, the cdf of $Z = \alpha X + \beta Y$ can be expressed as*

$$
\begin{aligned}
F_Z(z) = \frac{1}{4}\Bigg[& 2\mathrm{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma}\right) - \exp\left\{\frac{1}{\phi^2} + \frac{2\beta(\beta\lambda + \alpha\mu - z)}{\alpha^2\sigma^2\phi}\right\} \mathrm{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma} + \frac{\alpha\sigma}{\sqrt{2}\beta\phi}\right) \\
& - \exp\left\{\frac{1}{\phi^2} - \frac{2\beta(\beta\lambda + \alpha\mu - z)}{\alpha^2\sigma^2\phi}\right\} \mathrm{erfc}\left(\frac{\beta\lambda + \alpha\mu - z}{\sqrt{2}\alpha\sigma} - \frac{\alpha\sigma}{\sqrt{2}\beta\phi}\right)\Bigg] \qquad (4)
\end{aligned}
$$

for $-\infty < z < \infty$*. The corresponding pdf can be expressed as*

$$f_Z(z) = \frac{A(z)+B(z)}{4\beta\phi} \exp\left\{-\frac{(z-\beta\lambda-\alpha\mu)^2}{2\alpha^2\sigma^2}\right\} \tag{5}$$

for $-\infty < z < \infty$*, where*

$$A(z) = \exp\left\{\frac{\left(\alpha^2\sigma^2-\phi\beta z+\phi\beta^2\lambda+\phi\alpha\beta\mu\right)^2}{2\beta^2\phi^2\alpha^2\sigma^2}\right\} \operatorname{erfc}\left\{\frac{\alpha^2\sigma^2-\phi\beta z+\phi\beta^2\lambda+\phi\alpha\beta\mu}{\sqrt{2}\beta\phi\alpha\sigma}\right\}$$

and

$$B(z) = \exp\left\{\frac{\left(\alpha^2\sigma^2+\phi\beta z-\phi\beta^2\lambda-\phi\alpha\beta\mu\right)^2}{2\beta^2\phi^2\alpha^2\sigma^2}\right\} \operatorname{erfc}\left\{\frac{\alpha^2\sigma^2+\phi\beta z-\phi\beta^2\lambda-\phi\alpha\beta\mu}{\sqrt{2}\beta\phi\alpha\sigma}\right\}.$$

Proof: The cdf $F_Z(z) = \Pr(\alpha X + \beta Y \le z)$ can be expressed as

$$\begin{aligned} F_Z(z) &= \frac{1}{2\phi}\int_{-\infty}^{\infty} \exp\left(-\frac{\mid y-\lambda\mid}{\phi}\right) \Phi\left(\frac{z-\beta y-\alpha\mu}{\alpha\sigma}\right) dy \\ &= \frac{1}{2\phi}\int_{-\infty}^{\infty} \exp\left(-\frac{\mid w\mid}{\phi}\right) \Phi\left(\frac{z-\beta w-\beta\lambda-\alpha\mu}{\alpha\sigma}\right) dw, \end{aligned} \tag{6}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the relationship

$$\Phi(-x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right),$$

(6) can be further rewritten as

$$\begin{aligned} F_Z(z) &= \frac{1}{4\phi}\int_{-\infty}^{\infty} \exp\left(-\frac{\mid w\mid}{\phi}\right) \operatorname{erfc}\left(\frac{\beta w+\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right) dw \\ &= \frac{1}{4\phi}\left\{\int_0^{\infty} \exp\left(-\frac{w}{\phi}\right) \operatorname{erfc}\left(\frac{\beta w+\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right) dw\right. \\ &\qquad \left. + \int_0^{\infty} \exp\left(-\frac{w}{\phi}\right) \operatorname{erfc}\left(\frac{-\beta w+\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right) dw\right\}. \end{aligned} \tag{7}$$

The two integrals in (7) can be calculated by direct application of equation (2.8.9.1) in volume 2 of Prudnikov *et al.* [1]. The result follows. ■

The following corollaries provide the cdfs for the sum and the difference of the normal and Laplace random variables.

Corollary 1 *Suppose* X *and* Y *are independent random variables, distributed according to (1) and (2), respectively. Then, the cdf of* $Z = X + Y$ *can be expressed as*

$$\begin{aligned} F_Z(z) &= \frac{1}{4}\left[2\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}\right) - \exp\left\{\frac{1}{\phi^2}+\frac{2(\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\phi}\right)\right. \\ &\qquad \left. - \exp\left\{\frac{1}{\phi^2}-\frac{2(\lambda+\mu-z)}{\sigma^2\phi}\right\}\operatorname{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}-\frac{\sigma}{\sqrt{2}\phi}\right)\right] \end{aligned} \tag{8}$$

for $-\infty < z < \infty$.

Proof: set $\alpha = 1$ and $\beta = 1$ into (4). ■

Corollary 2 *Suppose X and Y are independent random variables, distributed according to (1) and (2), respectively. Then, the cdf of* $Z = X - Y$ *can be expressed as*

$$\begin{aligned} F_Z(z) &= \frac{1}{4}\Bigg[2\mathrm{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}\right) - \exp\left\{\frac{1}{\phi^2}+\frac{2(-\lambda+\mu-z)}{\sigma^2\phi}\right\}\mathrm{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\phi}\right) \\ &\qquad - \exp\left\{\frac{1}{\phi^2}-\frac{2(-\lambda+\mu-z)}{\sigma^2\phi}\right\}\mathrm{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}-\frac{\sigma}{\sqrt{2}\phi}\right)\Bigg] \end{aligned} \quad (9)$$

for $-\infty < z < \infty$.

Proof: set $\alpha = 1$ and $\beta = -1$ into (4). ■

Note that the parameters in (4), (8) and (9) are functions of μ/σ (coefficient of variation for the normal model), λ/ϕ (coefficient of variation for the Laplace model), ϕ/σ (ratio of scale parameters), and ϕ.

4 Distribution of the Product

Theorem 2 derives an explicit expression for the cdf of $\mid XY \mid$ in terms of the hypergeometric function.

Theorem 2 *Suppose X and Y are independent random variables, distributed according to (1) and (2), respectively, with* $\lambda = \mu = 0$. *Then, the cdf of* $Z = \mid XY \mid$ *can be expressed as*

$$F_Z(z) = \frac{z}{\sqrt{2}\phi\sigma}\left\{\frac{3C}{\sqrt{\pi}}\,{}_1F_3\left(\frac{1}{2};\frac{3}{2},1,\frac{1}{2};-\frac{z^2}{8\phi^2\sigma^2}\right)+\frac{z}{\sqrt{2}\phi\sigma}\,{}_1F_3\left(1;2,\frac{3}{2},\frac{3}{2};-\frac{z^2}{8\phi^2\sigma^2}\right)\right\} \quad (10)$$

for $-\infty < z < \infty$, *where C denotes Euler's constant.*

Proof: The cdf $F_Z(z) = \Pr(\mid XY \mid \le z)$ can be expressed as

$$F_Z(z) = \frac{1}{2\phi}\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{z}{\sigma\mid y\mid}\right)-\Phi\left(-\frac{z}{\sigma\mid y\mid}\right)\right\}\exp\left(-\frac{\mid y\mid}{\phi}\right)dy, \quad (11)$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the relationship

$$\Phi(-x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right),$$

(11) can be rewritten as

$$\begin{aligned} F_Z(z) &= \frac{1}{2\phi}\int_{-\infty}^{\infty}\exp\left(-\frac{y}{\phi}\right)\mathrm{erfc}\left(-\frac{z}{\sqrt{2}\sigma\mid y\mid}\right)dy - 1 \\ &= \frac{1}{\phi}\int_0^{\infty}\exp\left(-\frac{y}{\phi}\right)\mathrm{erfc}\left(-\frac{z}{\sqrt{2}\sigma y}\right)dy - 1 \\ &= \frac{1}{\phi}\int_0^{\infty}y^{-2}\exp\left(-\frac{1}{\phi y}\right)\mathrm{erfc}\left(-\frac{yz}{\sqrt{2}\sigma}\right)dy - 1. \end{aligned} \quad (12)$$

The integral in (12) can be calculated by direct application of equation (2.8.5.14) in volume 2 of Prudnikov *et al.* [1]. The result follows. ■

Note that the parameters in (10) are functions of $\phi\sigma$ (product of scale parameters).

5 Distribution of the Ratio

Theorem 3 derives explicit expressions for the pdf and the cdf of $\mid X/Y \mid$ in terms of the complementary error function.

Theorem 3 *Suppose X and Y are independent random variables, distributed according to (1) and (2), respectively, with $\lambda = 0$. Then, the cdf of $Z = \mid X/Y \mid$ can be expressed as $F_Z(z) = G(z) - G(-z)$, where*

$$G(z) = \frac{1}{2} \exp\left(\frac{\sigma^2 + 2\phi\mu z}{2\phi^2 z^2}\right) \operatorname{erfc}\left(\frac{\mu}{\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2}\phi z}\right) \tag{13}$$

for $-\infty < z < \infty$. The corresponding pdf is $f_Z(z) = g(z) + g(-z)$, where g is the derivative of G given by

$$\begin{aligned} g(z) = & \frac{1}{2\sqrt{\pi}\phi z^3} \exp\left(\frac{\sigma^2 + 2\phi\mu z}{2\phi^2 z^2}\right) \left[\sqrt{2}\sigma z \exp\left\{-\left(\frac{\mu}{\sqrt{2}z} + \frac{\sigma}{\sqrt{2}\phi z}\right)^2\right\}\right. \\ & \left. -\frac{\sqrt{\pi}\sigma^2}{\phi} \operatorname{erfc}\left(\frac{\mu}{\sqrt{2}z} + \frac{\sigma}{\sqrt{2}\phi z}\right) - \sqrt{\pi}\mu z \operatorname{erfc}\left(\frac{\mu}{\sqrt{2}z} + \frac{\sigma}{\sqrt{2}\phi z}\right)\right] \end{aligned} \tag{14}$$

for $-\infty < z < \infty$.

Proof: The cdf $F_Z(z) = \Pr(\mid X/Y \mid \leq z)$ can be expressed as

$$F_Z(z) = \frac{1}{2\phi} \int_{-\infty}^{\infty} \left\{\Phi\left(\frac{\mu + z \mid y \mid}{\sigma}\right) - \Phi\left(\frac{\mu - z \mid y \mid}{\sigma}\right)\right\} \exp\left(-\frac{\mid y \mid}{\phi}\right), \tag{15}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the relationship

$$\Phi(-x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right),$$

(15) can be rewritten as

$$\begin{aligned} F_Z(z) &= \frac{1}{4\phi} \int_0^{\infty} \left\{\operatorname{erfc}\left(\frac{\mu - z \mid y \mid}{\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{\mu + z \mid y \mid}{\sqrt{2}\sigma}\right)\right\} \exp\left(-\frac{\mid y \mid}{\phi}\right) dy \\ &= \frac{1}{2\phi} \left\{\int_0^{\infty} \operatorname{erfc}\left(\frac{\mu - zy}{\sqrt{2}\sigma}\right) \exp\left(-\frac{y}{\phi}\right) dy - \int_0^{\infty} \operatorname{erfc}\left(\frac{\mu + zy}{\sqrt{2}\sigma}\right) \exp\left(-\frac{y}{\phi}\right) dy\right\}. \end{aligned} \tag{16}$$

The two integrals in (16) can be calculated by direct application of equation (2.8.9.1) in volume 2 of Prudnikov *et al.* [2]. The result follows. ■

Note that the parameters in both (13) and (14) are functions of μ/σ (coefficient of variation) and σ/ϕ (ratio of scale parameters).

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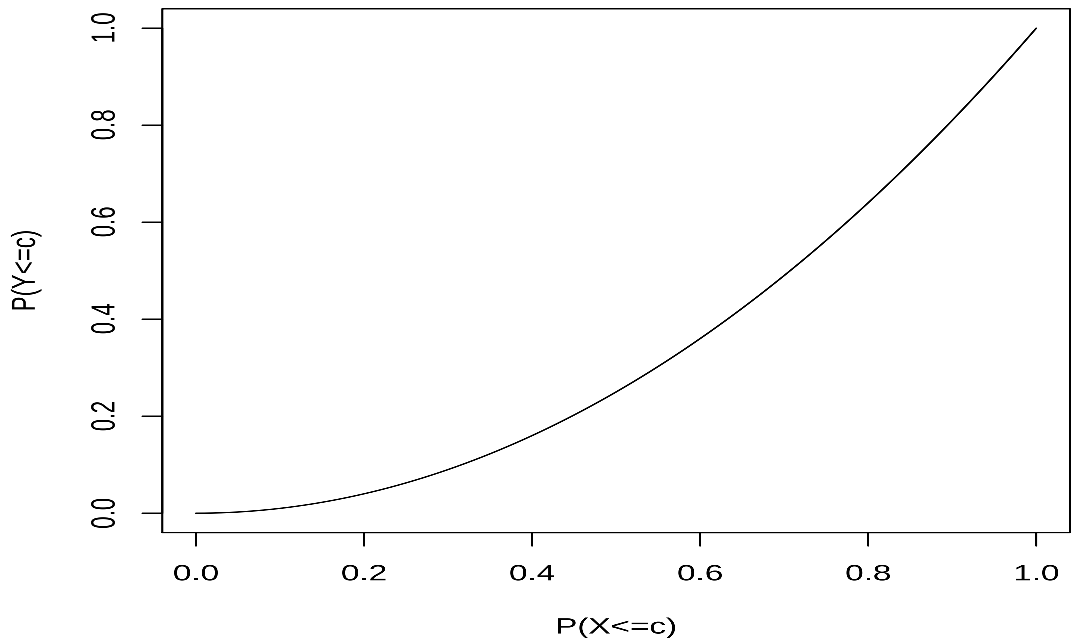

Figure 1. Plot of $\widehat{F}_Y(c)$ versus $\widehat{F}_X(c)$.

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2. G.G.Lorentz, (title of book in italics) ***Bernstein Polynomials*** **(2nd ed.), Chelsea,New York,1986.**

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) ***Progress in Approximation Theory*** **(P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.**

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A Generalization Of An Integral Inequality

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Abstract. A generalization of integral inequality presented by [1] is given. Other new results are also proved.

1. Introduction

The following open question was proposed in [2]
Under what conditons does the inequality

(1.1) $$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx$$

hold for α *and* β ?

In [1], the authors gave an answer by establishing the following

Theorem . *If the function* f *satisfies*

(1.2) $$\int_0^1 f(t)\,dt \geq \frac{1-x^2}{2}, \qquad \forall x \in [0,1],$$

then

$$\int_0^1 f^{\alpha+\beta}(x)\,dx \geq \int_0^1 x^{\beta} f^{\alpha}(x)\,dx$$

for every real $\alpha \geq 1$ *and* $\beta > 0$.

The aim of this paper is that to give some generalization of the previous result as well as to establish another new result .

2. New Results

We state and prove the folowing

Theorem 2.1. *Let* f, g, h *be continuous functions defined on* $h([a,b]), [a,b]$ *respectively,* f *is nonnegative,* $g(h(a)) = 0$, $g'(h(x))h'(x) \geq 1$, $\forall\, x \in [a,b]$. *h is nondecreasing and let* $\alpha \geq 1$, $\beta > 0$. *If*

$$\int_x^b f(h(t))dt \geq \int_x^b g(h(t))g'(h(t))h'(t)\,dt \qquad \forall\, x \in [a,b], \tag{2.1}$$

then

$$\int_a^b f^{\alpha+\beta}(h(t))dt \geq \int_a^b f^{\alpha}(h(t))g^{\beta}(h(t))dt\,. \tag{2.2}$$

Proof. Since $(g(h(t)))' = g'(h(t))h'(t) \geq 0$, and h is nondecreasing, then $g(h(t))$ is increasing. That is $g(h(t)) > g(h(a)) = 0$. We have for $\gamma > 0$, via changing the order of integration

$$\int_a^b\int_x^b f(h(t))g^{\gamma-1}((h(x))g'(h(x))h'(x)\,dt\,dx$$

$$= \int_a^b f(h(t)) \int_a^t g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx\,dt$$

$$= \frac{1}{\gamma}\int_a^b f(h(t))g^{\gamma}(h(t))dt\,.$$

Also

$$\int_a^b\int_x^b f(h(t))g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dt\,dx$$

$$= \int_a^b\left(\int_x^b f(h(t))dt\right) g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx$$

$$\geq \int_a^b\left(\int_x^b g(h(t))g'(h(t))h'(t)\,dt\right) g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx$$

$$= \frac{1}{2}\int^b \left(g^2(h(b)) - g^2(h(x))\right)g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx$$

$$= \frac{1}{2}\left(\frac{g^{\gamma+2}(h(b))}{\gamma} - \frac{g^{\gamma+2}(h(b))}{\gamma+2}\right) = \frac{g^{\gamma+2}(h(b))}{\gamma(\gamma+2)}\,.$$

Therefore,

$$\int_a^b f(h(t))g^{\gamma}(h(t))dt \geq \frac{g^{\gamma+2}(h(b))}{\gamma(\gamma+2)}\,.$$

By using the AG inequality, for $\alpha \geq 1$,

$$\frac{1}{\alpha}f^{\alpha}(h(t)) + \frac{\alpha-1}{\alpha}g^{\alpha}(h(t)) \geq f(h(t))g^{\alpha-1}(h(t))$$

Multiplying the above inequality by $g^{\beta}(h(t))$ gives

$$\frac{1}{\alpha}f^{\alpha}(h(t))g^{\beta}(h(t)) + \frac{\alpha-1}{\alpha}g^{\alpha+\beta}(h(t)) \geq f(h(t))g^{\alpha+\beta-1}(h(t)),$$

which implies

$$f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big) \geq \alpha\ f\big(h(t)\big)g^{\alpha+\beta-1}\big(h(t)\big) - (\alpha-1)\, g^{\alpha+\beta}\big(h(t)\big)$$

$$\geq \alpha\ f\big(h(t)\big)g^{\alpha+\beta-1}\big(h(t)\big) - (\alpha-1)\, g^{\alpha+\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t)\,dt\,.$$

Integrating both sides of the above inequality implies

$$\int_a^b f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big)dt \geq \alpha\int_a^b f\big(h(t)\big)g^{\alpha+\beta-1}\big(h(t)\big)\,dt$$

$$-(\alpha-1)\int_a^b g^{\alpha+\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t)\,dt$$

$$\geq \alpha\frac{g^{\alpha+\beta+1}\big(h(b)\big)}{\alpha+\beta+1`} - (\alpha-1)\frac{g^{\alpha+\beta+1}\big(h(b)\big)}{\alpha+\beta+1}$$

$$= \frac{g^{\alpha+\beta+1}\big(h(b)\big)}{\alpha+\beta+1}\,.$$

Now making use of the AG inequality again, to have

$$\frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}\big(h(t)\big) + \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}\big(h(t)\big) \geq f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big),$$

and this implies

$$\frac{\alpha}{\alpha+\beta} f^{\alpha+\beta}\big(h(t)\big) \geq f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big) - \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}\big(h(t)\big)$$

$$\geq f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big) - \frac{\beta}{\alpha+\beta} g^{\alpha+\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t)\,.$$

On integrating the above from a to b, we obtain

$$\frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha+\beta}\big(h(t)\big)dt \geq \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big)dt$$

$$+\frac{\beta}{\alpha+\beta}\left(\int_a^b f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big)dt - \int_a^b g^{\alpha+\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t)\,dt\right)$$

$$\geq \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big)dt$$

$$+\frac{\beta}{\alpha+\beta}\left(\frac{g^{\alpha+\beta+1}\big(h(b)\big)}{\alpha+\beta+1} - \frac{g^{\alpha+\beta+1}\big(h(b)\big)}{\alpha+\beta+1}\right)$$

$$= \frac{\alpha}{\alpha+\beta}\int_a^b f^{\alpha}\big(h(t)\big)g^{\beta}\big(h(t)\big)dt\,.$$

The result follows.

Theorem 2.2. *Let* f, g, h *be continuous functions defined on* $h([a,b]),[a,b]$ *respectively,* f *is nonnegative,* $g(h(a))=0,\ \ g'(h(x))h'(x)\ge 1,\ \forall\, x\in[a,b]$. h *is nondecreasing and let* $\alpha\ge 1,\ \beta>0.$ *If*

(23) $$\int_a^x f(h(t))dt \le \int_a^x g(h(t))g'(h(t))h'(t)\,dt \qquad \forall\, x\in[a,b),$$

and

(2.4) $$\int_a^b f(h(t))dt \ge \frac{1}{2}g^2(h(b)),$$

then (2.2) is satisfied .

Proof . Let $\gamma>0.$ Then we have

$$\int_a^b\int_a^x f(h(t))g^{\gamma-1}(h(x))g'(h(x))h(x)\,dt\,dx$$

$$=\int_a^b f(h(t))\int_t^b g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx\,dt$$

$$=\frac{1}{\gamma}\int_a^b f(h(t))\left(g^{\gamma}(h(b))-g^{\gamma}(h(t))\right)dt$$

$$\ge \frac{1}{2\gamma}g^{\gamma+2}(h(t))-\frac{1}{\gamma}\int_a^b f(h(t))g^{\gamma}(h(t))dt\ .$$

Also,

$$\int_a^b\int_a^x f(h(t))g^{\gamma-1}(h(x))g(h(x))h(x)\,dt\,dx$$

$$=\int_a^b\left(\int_a^x f(h(t))dt\right)g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx$$

$$\le\int_a^b\left(\int_a^x g(h(t))g'(h(t))h'(t)\,dt\right)g^{\gamma-1}(h(x))g'(h(x))h'(x)\,dx$$

$$=\frac{1}{2}\int_a^b g^{\gamma+1}(h(x))g'(h(x))h'(x)\,dx$$

$$=\frac{g^{\gamma+2}(h(b))}{2(\gamma+2)}\ .$$

Therefore, we have

$$\frac{g^{\gamma+2}(h(b))}{2(\gamma+2)}\ge\frac{1}{2\gamma}g^{\gamma+2}(h(b))-\frac{1}{\gamma}\int_a^b f(g(t))g^{\gamma}(h(t))dt\ ,$$

which implies

$$\int_a^b f(g(t))g^{\gamma}(h(t))dt\ge\frac{g^{\gamma+2}(h(b))}{\gamma+2}.$$

The rest if the proof is exactly the same as what has been done in theorem 2.1.

Theorem 2.3. *Let* f, g, h *be continuous functions defined on* $h([a,b]),[a,b]$ *respectively,* f *is nonnegative,* $g(h(a))=0$, $g'(h(x))h'(x)\geq 1$, $\forall\, x\in[a,b]$. *h is nondecreasing and let* $0<\beta<\alpha<1$. *If*

$$\int_x^b f(h(t))dt \leq \int_x^b g(h(t))g'(h(t))h'(t)\,dt \qquad \forall\, x\in[a,b], \tag{2.5}$$

then

$$\alpha\int_a^b f^{\alpha-\beta}(h(t))dt + \beta\int_a^b f^{\alpha}(h(t))g^{-\beta}(h(t))dt \leq \frac{\alpha+\beta}{\alpha-\beta+1}g^{\alpha-\beta+1}(h(b)). \tag{2.6}$$

Proof . As before, it is not difficult to show that for $\gamma>0$,

$$\int_a^b f(h(t))g^{\gamma}(h(t))dt \leq \frac{g^{\gamma+2}(h(b))}{\gamma+2}.$$

By the AG inequality, we have for $0<\alpha<1$,

$$\frac{1}{\alpha}f^{\alpha}(h(t)) + \frac{\alpha-1}{\alpha}g^{\alpha}(h(t)) \leq f(h(t))g^{\alpha-1}(h(t))$$

Multiplying the above inequality by $g^{-\beta}(h(t))$ gives

$$\frac{1}{\alpha}f^{\alpha}(h(t))g^{-\beta}(h(t)) + \frac{\alpha-1}{\alpha}g^{\alpha-\beta}(h(t)) \leq f(h(t))g^{\alpha-\beta-1}(h(t)).$$

and this implies

$$\begin{aligned} f^{\alpha}(h(t))g^{-\beta}(h(t)) &\leq \alpha\, f(h(t))g^{\alpha-\beta-1}(h(t)) - (\alpha-1)\,g^{\alpha-\beta}(h(t)) \\ &\leq \alpha\, f(h(t))g^{\alpha-\beta-1}(h(t)) - (\alpha-1)\,g^{\alpha-\beta}(h(t))g'(h(t))h(t), \end{aligned}$$

and hence

$$\begin{aligned} \int_a^b f^{\alpha}(h(t))g^{-\beta}(h(t))dt &\leq \alpha\int_a^b f(h(t))g^{\alpha-\beta-1}(h(t))\,dt \\ &\quad -(\alpha-1)\int_a^b g^{\alpha-\beta}(h(t))g'(h(t))h'(t)\,dt \\ &\leq \alpha\frac{g^{\alpha-\beta+1}(h(b))}{\alpha-\beta+1`} - (\alpha-1)\frac{g^{\alpha-\beta+1}(h(b))}{\alpha-\beta+1} \\ &= \frac{g^{\alpha-\beta+1}(h(b))}{\alpha-\beta+1}. \end{aligned}$$

Now making use of the AG inequality again, to have

$$\frac{\alpha}{\alpha-\beta}f^{\alpha-\beta}(h(t)) - \frac{\beta}{\alpha-\beta}g^{\alpha-\beta}(h(t)) \leq f^{\alpha}(h(t))g^{-\beta}(h(t)),$$

and this implies

$$\frac{\alpha}{\alpha-\beta}f^{\alpha-\beta}(h(t)) \leq f^{\alpha}(h(t))g^{-\beta}(h(t)) + \frac{\beta}{\alpha-\beta}g^{\alpha-\beta}(h(t))$$

$$\leq f^{\alpha}\big(h(t)\big)g^{-\beta}\big(h(t)\big)+\frac{\beta}{\alpha-\beta}g^{\alpha-\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t) \ .$$

On integrating the above from a to b, we obtain

$$\alpha\int_a^b f^{\alpha-\beta}\big(h(t)\big)dt+\beta\int_a^b f^{\alpha}\big(h(t)\big)g^{-\beta}\big(h(t)\big)dt$$

$$\leq \alpha\int_a^b f^{\alpha}\big(h(t)\big)g^{-\beta}\big(h(t)\big)dt+\beta\int_a^b g^{\alpha-\beta}\big(h(t)\big)g'\big(h(t)\big)h'(t)\,dt$$

$$\leq \alpha\frac{g^{\alpha-\beta+1}\big(h(b)\big)}{\alpha-\beta+1}+\beta\frac{g^{\alpha-\beta+1}\big(h(b)\big)}{\alpha-\beta+1}$$

$$=\frac{\alpha+\beta}{\alpha-\beta+1}g^{\alpha-\beta+1}\big(h(b)\big) \ .$$

3. Applications

Remark 3.1. The result of [1] follows from Theorem 2.1 by puttimg

$$g(x)=h(x)=x, \quad a=0,\ b=1.$$

Corollary 3.2. *Let* f, g *be continuous functions defined on [a,b],* f *is nonnegative,*
$g(a)=0,\ g'(x)\geq 1\ \ \forall x\in[a,b],$ *and let* $\alpha\geq 1,\ \ \beta>0.$ *If*

(3.1) $$\int_x^b f(t)\,dt \ \geq \int_x^b g(t)\,g'(t)\,dt \qquad \forall x\in[a,b],$$

then

(3.2) $$\int_a^b f^{\alpha+\beta}(x)\,dx \ \geq \int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx.$$

Proof. Follows from Theorem 2.1 by putting $h(t)=t.$

Corollary 3.3. *Let* f *be nonnegative continuous function defined on [a,b], and let*
$\alpha\geq 1,\ \ \beta>0.$ *If*

(3.3) $$\int_x^b f(t)\,dt \ \geq \frac{(x-a)^2}{2} \qquad \forall x\in[a,b],$$

then

(3.4) $$\int_a^b f^{\alpha+\beta}(x)\,dx \ \geq \ \int_a^b (x-a)^{\beta}\ f^{\alpha}(x)\,dx\ .$$

Proof. Follows from Corollary 3.2 by putting $g(t)=t-a$.

Corollary 3.4. *If f is continuous in the domain stated, and if $\alpha \geq 1,\ \beta > 0,$ then the following inequalities holds (follows from Corollary 3.2).*

(1) $$\int_0^{\pi/2} f^{\alpha+\beta}(x)\,dx \geq \int_0^{\pi/2} f^{\alpha}(x)\sin^{\beta} x\,dx\ ,$$

provided

$$\int_x^{\pi/2} f(t)\,dt \ \geq \ \tfrac{1}{2}\cos 2x \qquad \forall x \in [0,\pi/2].$$

(2) $$\int_0^1 f^{\alpha+\beta}(x)\,dx \ \geq \ \int_0^1 f^{\alpha}(x)\left(\sin^{-1} x\right)^{\beta} dx,$$

provided

$$\int_x^1 f(t)\,dt \ \geq \tfrac{1}{2}\left(\left(\tfrac{\pi}{2}\right)^2 - \left(\sin^{-1} x\right)^2\right).$$

(3) $$\int_1^b f^{\alpha+\beta}(x)\,dx \geq \int_1^b f(x)\ln^{\beta} x\,dx\ ,$$

provided

$$\int_x^b f(t)\,dt \geq \tfrac{1}{2}\left(\ln^2 b - \ln^2 x\right) \qquad \forall x \in [1,b]\ .$$

(4) $$\int_{-\infty}^b f^{\alpha+\beta}(x)\,dx \geq \int_{-\infty}^b f^{\alpha}(x)e^{\beta x}\,dx,$$

provided

$$\int_x^b f(t)\,dt \ \geq \ \tfrac{1}{2}\left(e^{2b} - e^{2x}\right) \qquad \forall x \in (-\infty,b]\ .$$

Corollary 3.5. *Let f, g be continuous functions defined on* [a,b], *f is nonnegative,* $g(a)=0,\ g'(x)\geq 1\ \ \forall x \in [a,b],$ *and let* $\alpha \geq 1,\ \beta > 0.$ *If*

(3.5) $$\int_a^x f(t)\,dt \ \leq \ \int_a^x g(t)\,g'(t)\,dt \quad \forall x \in [a,b],$$

and

(3.6) $$\int_a^b f(t)\,dt \ \geq \ \tfrac{1}{2}g^2(b),$$

then

(3.7)
$$\int_a^b f^{\alpha+\beta}(x)\,dx \geq \int_a^b f^{\alpha}(x)\,g^{\beta}(x)\,dx\ .$$

Proof. Follows from Theorem 2.2 by putting $h(t)=t$.

Corollary 3.6. *Let* f, k *be nonnegative continuous functions defined on* [0,b], k is nonincreasing, $k'(t)\geq 1\ \forall\ t\in[0,b]$, *and let* $\alpha\geq 1$, $\beta>0$. *If*

(3.8)
$$\int_0^x f(t)\,dt \ \leq\ \int_0^x\int_0^t k(u)\,k(t)\,du\,dt \quad \forall x\in[0,b],$$

and

(3.9)
$$\int_0^b f(t)dt \ =\ \frac{1}{2}\left(\int_0^b k(u)du\right)^2,$$

then

(3.10)
$$\int_0^b f^{\alpha+\beta}(x)\,dt \ \geq\ \int_0^b f^{\alpha}(x)\,k^{\beta}(x)\,x^{\beta}\,dx.$$

Proof. Follows rom Corollary 3.5 by putting $g(t)=\int_0^t k(u)\,du$ as follows

$$\int_0^b f^{\alpha+\beta}(x)\,dx \geq \int_0^b f^{\alpha}(x)\left(\int_0^x k(t)\,dt\right)^{\beta} dx \geq \int_0^b f^{\alpha}(x)\,k^{\beta}(x)\left(\int_0^x dt\right)^{\beta} dx$$
$$= \int_0^b f^{\alpha}(x)\,k^{\beta}(x)\,x^{\beta}\,dx\ .$$

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Certain Class of Kernels for Roumieu –Type Convolution Transform of Ultra- Distributions of Compact Support

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Abstract

This paper deals with certain class of kernels and investigates the convolution transform of ultra-distributions of Roumieu-type supported inside compact subsets. Inversion sequence of operators of the transform is discussed.

Keywords: convolution transform, class of kernels, ultra-differentiable function, ultra-distribution, Roumieu-type ultra-distribution.

1. Introduction

The generalized convolution transform [cf. [2]]

$$F(x) = \langle f(t), G(x-t) \rangle , \qquad (1.1)$$

for a given x, is defined to be the number that f assigns to a test function space containing the kernel $G(x-t)$ as a function of t. The transform (1.1) is investigated by Hirschman and Widder [3,4] by restricting the class of kernels $G(t)$ to a fairly wide class of functions . Related formulae ,therein, are obtained. For real numbers c and d, among others, [2] define a space $l_{c,d}$ of test functions by means of a sequence $(\gamma_k)_{k=o}^{\infty}$ of seminorms , where

$$\gamma_k(\phi) \equiv \gamma_{k,c,d}(\phi) = sup\left|K(t)\phi^{(k)}(t)\right| \quad , \qquad (1.2)$$

$K(t)$ is infinitely smooth, $K(t) \neq 0$, and

$$K(t) = \begin{cases} e^{ct} & \text{for } t \in (1,\infty) \\ e^{dt} & \text{for } t \in (-\infty,-1) \end{cases} .$$

However, the above test function space $l_{c,d}$ meets the needs for quite variety of kernels given by [3], [4]. In this research work, we consider the class of kernels defined by [1], [5],[3]

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left[F(s)\right]^{-1} e^{st} ds$$
$$\equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left[\prod_{k=1}^{\infty}\left\{\left(1-s/a_k\right)/\left(1-s/c_k\right)\right\} X\right.$$
$$\left. exp\left(s\left(1/a_k - 1/c_k\right)\right)\right]^{-1} e^{st}\, ds, \tag{1.3}$$

where, $Re\, a_k = a_k$,$Re\, c_k = c_k$,$0 \leq a_k / c_k < 1$,$\sum a_k^{-2} < \infty$, and $N_+ + N_- = \infty$, where

$$N_\pm = \lim_{x\to\pm\infty}\ inf\left\{N\left(\{a_k\},a\right) - N\left(\{c_k\},a\right)\right\},$$

and, $N\left(\{\cdot\},x\right)$ is the number of $a_k's$, $b_k's$ between 0 and x.

2. The Convolution Transform of $\wp_{\{e_p\}}(R)$ and its Dual of Compact Support

The $e_i's$, $i = 0,1,2,\ldots$, wherever they appear, are to be considered as a sequence of positive real numbers on which the following constraint is emposed [cf.[7,p.66]]

(i) $e_i \leq ST^i \min_{0\leq k\leq i} e_k e_{i-k}$, (Stability under ultradifferentiable operators)

Where, $S > 0$,$T > 1$ being constants.

An infinitely differentiable function ϕ over R is said to be an ultradifferentiable function of Roumieu type if and only if for every compact subset K of R there are positive constants A and C depending on ϕ and K such that

$$\sup_{x\in K}\left|\frac{d^k}{dx_k}\phi(x)\right| \leq CA^k e_k .$$

The set of all such a ϕ is denoted by $\wp_{\{e_p\}}(R)$.

We denote by $\wp'_{\{e_p\}}(R)$ the strong dual of $\wp_{\{e_p\}}(R)$. Its elements so-called ultradistributions of Roumieu type of compact support.

Lemma 2.1. *Let α_1 and α_2 be defined by* [cf. [1,(2.1)]]

$$\alpha_1 = max\left(a_k, -\infty \middle| a_k < 0\right), \alpha_2 = min\left(a_k, \infty \middle| a_k > 0\right), \tag{2.1}$$

such that $c < \alpha_2$, $d > \alpha_1$, then for real x and $N_+ + N_- = \infty$ we have

$$G(x-t) \in \wp_{\{e_p\}}(R) \quad ,$$

where $G(t)$ is that defined by (1.3).

Proof. Conditions $c < \alpha_2$, and $d < \alpha_1$, the fact that $G(x-t) \in l_{c,d}$ [1, p.182] for any real x, imply that

$$\gamma_{k,c,d}\left(G(x-t)\right) = \sup_{t\in(1,\infty)} \left| e^{ct} G^{(k)}(x-t) \right| \tag{2.2}$$

and

$$\gamma_{k,c,d}\left(G(x-t)\right) = \sup_{t\in(-\infty,-1)} \left| e^{dt} G^{(k)}(x-t) \right| \tag{2.3}$$

are both finite.

Considering $t \in (1,\infty)$, we, for any $x \in R$, choose sufficiently small constant $E > 0$ such that $\left|G^{(k)}(x-t)\right| \le E\, e^{-ct}$.

Allowing K vary over all compact subsets of $(1,\infty)$ and considering supremum over all $t \in K$ we have, by the structure of the sequence $\left(e_p\right)$, the existence of constants $A_1, h_1 > 0$ dependent on $G(t)$ such that

$$\sup_{t\in K\subset(1,\infty)} \left|G^{(k)}(x-t)\right| \le E_1 A_1^k e_k . \tag{2.4}$$

Analogous argument for $t \in (-\infty,-1)$, yields that

$$\sup_{t\in K\subset(-\infty,-1)} \left|G^{(k)}(x-t)\right| \le E_2 A_2^k e_k . \tag{2.5}$$

Combing (2.4) and (2.5) yields that,

$$\sup_{t\in K\subset R}\left|G^{(k)}(x-t)\right|\le E\,A^k e_k,$$

for some positive constants E and A depending on $G(t)$.

In view of above lemma and conditions employed on α_1,α_2 and $G(t)$, we define the convolution transform of ultradistribution $f\in \wp'_{\{e_p\}}(R)$ to be the map

$$F(x)=\left\langle f(t),G(x-t)\right\rangle, \tag{2.6}$$

for any real x.

Theorem 2.2. *Let* $c<\alpha_2, d>\alpha_1,\alpha_1$ *and* α_2 *are as in* (2.1), *and* $N_+ + N_- = \infty$. *Let the sequence* (e_p) *satisfy* (i), *then for* $f\in \wp'_{\{e_p\}}(R)$ *we have*

$$D_x^k F(x)=\left\langle f(t),D_x^k G(x-t)\right\rangle, \tag{2.7}$$

Proof. Conditions $c<\alpha_2, d>\alpha_1$ and $N_+ + N_- = \infty$ implies that $G(x-t)\in \wp_{\{e_p\}}(R)$. Following [2], we attempt to prove the theorem by induction on the order of the derivatives k.

For $k=0$, (2.7) is that (2.6). Assume (2.7) is true for $(k-1)$ derivatives. Let x be fixed and $\Delta x\neq 0$. consider,

$$\left(\frac{1}{\Delta x}\right)\left[\frac{d^{k-1}}{dx^{k-1}}F(x+\Delta x)-\frac{d^{k-1}}{dx^{k-1}}F(x)\right]-\left\langle f(t),\frac{d^k}{dx^k}G(x-t)\right\rangle=\left\langle f(t),\eta_{\Delta x}(t)\right\rangle,$$

where

$$\eta_{\Delta x}(t)=\left(\frac{1}{\Delta x}\right)\left[\frac{d^{k-1}}{dx^{k-1}}G(x+\Delta x-t)-\frac{d^{k-1}}{dx^{k-1}}G(x-t)\right]-\frac{d^k}{dx^k}G(x-t). \tag{2.8}$$

To prove the theorem enough to prove that $\eta_{\Delta x}(t)\to 0$ uniformly in the topology of $\wp_{\{e_p\}}(R)$. For any non negative integer *m*, (2.8) can be written as

$$\eta_{\Delta x}^{(m)}(t)=\frac{(-1)^m}{\Delta x}\int_{x-t}^{x-t+\Delta x}dy\int_{x-t}^{y}\frac{d^{m+k-1}}{dx^{m+k-1}}G(\xi)\,d\xi. \tag{2.9}$$

Let $I=\{\xi: x-t-|\Delta x|<\xi<x-t+|\Delta x|\}$. Then, Lemma 2.1, condition (i) and (2.9) imply

$$\left|\eta_{\Delta x}^{(m)}(t)\right| \le \frac{|\Delta x|}{2} \sup_{\xi\in I}\left|\frac{d^{m+k-1}}{dx^{m+k-1}}G(\xi)\right| \le C\frac{|\Delta x|}{2}(AT)^{k-1}\, Se_{k-1}(AT)^m e_m. \tag{2.10}$$

Hence,

$$\sup_{t\in K}\left|\eta_{\Delta x}^{(m)}(t)\right| \le C' A'^m e^m , \tag{2.11}$$

where, $C' = C\frac{|\Delta x|}{2}(AT)^{k-1} Se_{k-1}$ and $A' = AT$.

Allowing $\Delta x \to 0$ in (2.10) together with (2.11) prove the theorem.

3. Inversion Formula.

Definition 3.1. Let $R_m(D)$ be the inversion sequence of operators [1], [2]

$$R_m(D)= e^{-b_m D} \prod_{k=1}^{m}\left(1-\frac{D}{a_n}\right)\left(1-\frac{D}{c_k}\right)^{-1} \exp\left(\left(a_k^{-1}+c_k^{-1}\right)D\right),$$

where, $e^{kD} f(x)= f(x+k)$, $D=\frac{d}{dx}$. $\left(1-\frac{D}{c}\right)^{-1}$ is given by

$$\left(1-\frac{D}{c}\right)^{-1} F(x)=\begin{cases} c\, e^{cx}\int_x^{\infty} e^{-cy} f(y)dy & for\, c>0 \\ -c\, e^{cx}\int_{-\infty}^{x} e^{-cy} f(y)dy & for\, c<0 \end{cases} \tag{3.1}$$

and $\lim_{m\to\infty} b_m = 0$.

Theorem 3.2. *Under the hypothesis of* (2.6) *and for a sequence* (e_i) *satisfying* (i) *we have,*

$$F(x)\in \wp_{\{e_p\}}(R) ,$$

for any real x.

Proof. Let K be a compact subset of R. and (e_p) satisfy (i). Then, there is $r\in N$ such that

$$\left|D_x^n F(x)\right| \le C' \max_{o\le k\le r} \sup_{t\in K}\left|D^{n+k} G(x-t)\right|$$

$$< C' \max_{o\le k\le r} \left|C\, A^{n+k} e_{n+k}\right|$$

$$< D\; A'^{n} e_n ,$$

where, $D = C' C\, A^r S T^r e_r$ and $A' = AT$. Constants S,T are that in (i). This completes the proof of the theorem.

Theorem 3.3. *Let* $F(x) = \langle f(t), G(x-t)\rangle$. *We have,*

$$R_m(D)F(x) \in \wp_{\{e_p\}}(R)$$

where $f \in \wp'_{\{e_p\}}(R)$ *and* (e_p) *satisfies* (i).

Proof. Presence of condition (i) is to ensure that $F(x) \in \wp'_{\{e_p\}}(R)$, see the above Theorem. The definition of $e^{kD} F(x)$ and properties of $\wp_{\{e_p\}}(R)$ implies that $e^{kD} F(x)$ and $\left(1-\frac{D}{a}\right)F(x)$ are both in $\wp_{\{e_p\}}(R)$. We only need to prove that

$$\left(1-\frac{D}{a}\right)F(x) \in \wp_{\{e_p\}}(R) \; .$$

For, let $a > 0$. By Theorem 3.2, we find positive constants C and A such that

$$\sup_{t\in K}\left|D_x^n F(x)\right| < CA^n e_n , \tag{3.2}$$

for any compact subset K of R. Employing (3.1) and integrating by parts, n-times, we have

$$\frac{d^n}{dx^n}\left(1-\frac{D}{a}\right)^{-1} F(x) = \frac{d^n}{dx^n}\left[a\, e^{ax} \int_x^{\infty} e^{-ay} F(y)dy\right]$$

$$= a\, e^{ax} \int_x^{\infty} e^{-ay} D_x^n F(y)dy \; .$$

Thus, invoking (3.2) we have

$$\left|\frac{d^n}{dx^n}\left(1-\frac{D}{a}\right)^{-1} F(x)\right| \le \left|a\, e^{ax}\right| \int_x^{\infty} \left|e^{-ay} D_x^n F(y)\right| dy < CA^n e_n \; .$$

Analogous proof can be followed for $a < 0$. Hence, our theorem is completely proved.

Theorem 3.4. *Let* α_1, α_2 *have similar conditions as in Theorem* 2.2. *Then, for* $f(t) \in \wp'_{\{e_p\}}(R)$ *and condition* (i) *holds, we have*

$$\lim_{m\to\infty} \left\langle R_m(D) F^{(n)}(x), \theta(x) \right\rangle = \left\langle f^{(n)}(t), \theta(t) \right\rangle,$$

Where $\theta \in \mathrm{D}$ (*Schwartz test function space*), $n \in 2N$.

Proof is similar to that in [8, Theo. 3.2].

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Convergence and Gibbs Phenomenon for Generalized Fourier Series*

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Abstract

In contrast with the convergence and Gibbs phenomenon for the classical Fourier series, the same problems are treated for generalized Fourier series in this paper. More precisely, the pointwise and uniform convergence are discussed; It turns out that the Gibbs phenomenon can be removed at rational discontinuity under some conditions. Finally, some numerical experiments are presented to illustrate our theory.

Key Words: generalized Fourier series, pointwise convergence, uniform convergence, Gibbs phenomenon

2000 MS Classification 42C40

1 Introduction

Fourier series is important in both mathematics and engineering. One of fundamental problems in that area is the convergence in some senses. Let $f \in L^2([0,1])$ be 1-periodic. Then its Fourier series is defined by $S(f,\cdot) := \sum_{n\in\mathbb{Z}} c_n e^{\mathrm{i}2\pi n\cdot}$ on $\mathbb{R}$, where the Fourier coefficients c_n are given by the formula $c_n = \int_0^1 f(x)e^{-\mathrm{i}2\pi nx}\mathrm{d}x$ for $n \in \mathbb{Z}$. A Fourier series is called convergent (uniformly convergent), if the partial sums

$$S_N(f,\cdot) = \sum_{n=-N}^{N} c_n e^{\mathrm{i}2\pi n\cdot}$$

is convergent (uniformly convergent). For the convergence of Fourier series, the following well known theorems are of importance.

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Theorem A. *(Jordan's test) Let f be 1-periodic and of bounded variation in some neighborhood $N(x)$ of $x \in \mathbb{R}$. Then*

$$\lim_{N\to\infty} S_N(f,\, x) = \frac{1}{2}\left[f(x+0) + f(x-0)\right].$$

If, in addition, f is continuous at x, then $\lim_{N\to\infty} S_N(f,\, x) = f(x)$ holds automatically; Moreover, if f is continuous in $N(x)$, then $\lim_{N\to\infty} S_N(f,\, \cdot) = f(\cdot)$ uniformly on any closed subinterval of $N(x)$.

Theorem B. *Let f be a 1-periodic function and of bounded variation on $[0,\, 1]$. If f has a jump discontinuity at $x \in (0,\, 1)$ and is continuous on $(x-\delta,\, x) \cup (x,\, x+\delta)$ for some $\delta > 0$, then there exists the Gibbs phenomenon at x, i.e., the Fourier series $S(f,\, \cdot)$ does not converge uniformly to f in $N(x)$.*

From Theorem B, we find that when a Fourier series is used to approximate a function f with a jump discontinuity, the N-th partial sum of f overshoots the function value at the jump point. This Gibbs phenomenon was studied by Wilbraham ([16]), Michelson ([11]) and Gibbs ([2]) long time ago. More investigations can be found in many references, e. g. in [1], [12], [15], [17] and etc. Two dimensional cases were treated very recently by G. Helmberg in [3], [4], [5].

Although wavelet analysis shows many advantages over the classical Fourier analysis, all standard wavelet expansions do exhibit Gibbs phenomenon ([7], [13] and [14]). This is not good, because in many applications, Gibbs phenomenon represents an undesirable effect. Many efforts have been made to eliminate or remove this behavior. For example, to avoid it, either Cesàro or the Abel means are used instead of the partial sums of the Fourier series.

This paper is devoted to the convergence and Gibbs phenomenon for generalized Fourier series. Before proceeding, let us first briefly introduce generalized Fourier series. It is well known that the classical Fourier basis does not give a satisfactory representation of a nonlinear and non-stationary signal ([6]). To better deal with such signals and to overcome shortcoming of Fourier basis, the authors of [10] introduce a class of orthonormal exponential bases, which includes the classical Fourier basis, the Walsh system and others: Let

$$g_n(x) := \begin{cases} a_n x + b_n, & x \in \left[0,\, \frac{1}{2}\right), \\ c_n x + d_n, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

be real-valued functions for $n \in \mathbb{Z}$. Then a characterization is given for $\{e^{\mathrm{i}2\pi g_n} :\ n \in \mathbb{Z}\}$ to be an orthonormal basis for $L^2([0,1])$. Li and Yan ([8]) extend that result to multi-knot piecewise linear functions:

Theorem C. *Let $q \in \mathbb{N}$ be a positive integer and $\mathbb{N}_q := \{0,\, 1,\, \ldots,\, q-1\}$. For $n \in \mathbb{Z}$ and $l \in \mathbb{N}_q$, define 1-periodic function $g_{n,l}$ by*

$$g_{n,l}(x) := a_n^j x + b_{n,l}^j,\ x \in I_j, \tag{1}$$

where $I_j = \left[\frac{j}{q}, \frac{j+1}{q}\right)$ $(0 \le j \le q-1)$, a_n^j depends on n, j and $\left\{a_n^j :\ n \in \mathbb{Z}\right\} = q\mathbb{Z}$ for $j \in \mathbb{N}_q$. Then the set

$$\mathcal{G} := \left\{e^{i2\pi g_{n,l}} :\ n \in \mathbb{Z},\, l \in \mathbb{N}_q\right\} \tag{2}$$

forms an orthonormal basis for the space $L^2([0,1])$ *if and only if*

$$\mathcal{B}_n := \frac{1}{\sqrt{q}}\left(e^{i2\pi b^j_{n,\,l}}\right)_{l,\,j\in\mathbb{N}_q} \tag{3}$$

is unitary for each $n \in \mathbb{Z}$.

Many examples are given by this theorem ([8]). In particular, Theorem C leads to the results in [10], when $q = 2$. As the Fourier basis, $\mathcal{G}$ constitutes bases, but not unconditional bases, for $L^p([0,1])$ with $1 < p < \infty$, $p \neq 2$ ([9]). We call the function system $\mathcal{G}$ defined by (2) the generalized Fourier system. Similar to the classical Fourier series, the *generalized Fourier series* of f is defined by

$$G(f,\,\cdot) := \sum_{n\in\mathbb{Z}}\sum_{l=0}^{q-1} \tilde{c}_{n,\,l}e^{\mathrm{i}2\pi g_{n,\,l}(\cdot)}, \tag{4}$$

where the generalized Fourier coefficients $\tilde{c}_{n,\,l}$ are given by

$$\tilde{c}_{n,\,l} = \int_0^1 f(x)e^{-\mathrm{i}2\pi g_{n,\,l}(x)}\mathrm{d}x \tag{5}$$

for $n \in \mathbb{Z}$ and $l \in \mathbb{N}_q$. The partial sums of generalized Fourier series of function f are well-defined on the whole real line by

$$G_N(f,\,\cdot) := \sum_{n=-N}^{N}\sum_{l=0}^{q-1} \tilde{c}_{n,\,l}e^{\mathrm{i}2\pi g_{n,\,l}(\cdot)} \tag{6}$$

for $N = 0, 1, \cdots$. Throughout the paper, we always assume that $q \geq 2$, because $q = 1$ essentially reduces to the classical Fourier series.

In the next section, some sufficient conditions are provided for pointwise and uniform convergence of $G_N(f,\,\cdot)$; Further, it is shown that Gibbs phenomenon for generalized Fourier series can be removed at rational discontinuity under some conditions. Examples and numerical experiments are presented in Section 3 to illustrate our theory.

2 Convergence and Gibbs Phenomenon

We always use generalized Fourier bases given in Theorem C, when discuss the convergence of a generalized Fourier series. The following simple lemma will be frequently used in this section.

Lemma 1 *For* $f \in L^2([0,1])$ *and* $k \in \mathbb{N}_q$, *define 1-periodic function* h_k *by*

$$h_k(\cdot) := f\left(\frac{\cdot + k}{q}\right) \tag{7}$$

on $[0,\,1)$. *Then, on* $I_k =: \left[\frac{k}{q},\,\frac{k+1}{q}\right)$,

$$\lim_{N\to\infty} G_N(f,\,\cdot) = \lim_{N\to\infty} S_N\left(h_k,\,q\cdot - k\right). \tag{8}$$

Proof Recall that for $x \in I_k$, $g_{n,l}(x) = a_n^k x + b_{n,l}^k$ and

$$G_N(f,\,x) = \sum_{n=-N}^{N} \sum_{l=0}^{q-1} \tilde{c}_{n,l} e^{\mathrm{i}2\pi b_{n,l}^k} e^{\mathrm{i}2\pi a_n^k x}$$

with $\tilde{c}_{n,l} = \int_0^1 f(y) e^{-\mathrm{i}2\pi g_{n,l}(y)} \mathrm{d}y$. Then $\tilde{c}_{n,l} = \sum_{j=0}^{q-1} e^{-\mathrm{i}2\pi b_{n,l}^j} \int_{I_j} f(y) e^{-\mathrm{i}2\pi a_n^j y} \mathrm{d}y$. Substituting this into $G_N(f,\,x)$ and using the unitary property of $\mathcal{B}_n$(Theorem C), one has

$$G_N(f,\,x) = q \sum_{n=-N}^{N} \left(\int_{I_k} f(y) e^{-\mathrm{i}2\pi a_n^k y} \mathrm{d}y \right) e^{\mathrm{i}2\pi a_n^k x} = \sum_{n=-N}^{N} \left(\int_0^1 h_k(t) e^{-\mathrm{i}2\pi a_n^k \cdot \frac{t+k}{q}} \mathrm{d}t \right) e^{\mathrm{i}2\pi a_n^k x}.$$

Note that $\left\{ a_n^j : n \in \mathbb{Z} \right\} = q\mathbb{Z}$ for each $j \in \mathbb{N}_q$. Then

$$G_N(f,\,x) = \sum_{n=-N}^{N} \left(\int_0^1 h_k(t) e^{-\mathrm{i}2\pi \frac{a_n^k}{q} \cdot t} \mathrm{d}t \right) e^{\mathrm{i}2\pi \frac{a_n^k}{q}(qx-k)},$$

which is a classical Fourier partial sum of function h_k at $qx - k$. Finally (8) follows easily . $\square$

By Theorem A and Lemma 1, we obtain the following pointwise convergence of the generalized Fourier series of a 1-periodic function f of bounded variation.

Proposition 1 *If f is 1-periodic and of bounded variation in some neighborhood $N(x)$ of x, then*

$$\lim_{N\to\infty} G_N(f,x) = \begin{cases} \frac{1}{2}\left[f(x+0) + f(x-0)\right], & x \in (0,\,1)\backslash \frac{\mathbb{N}_q}{q}, \\ \frac{1}{2}\left[f(x+0) + f\left(x + \frac{1}{q} - 0\right)\right], & x \in \frac{\mathbb{N}_q}{q}. \end{cases}$$

Proof For $x \in [0,\,1)$, there exists uniquely $k \in \mathbb{N}_q$ such that $x \in I_k$. Define 1-periodic function h_k by

$$h_k(\cdot) =: f\left(\frac{\cdot + k}{q}\right)$$

on $[0,\,1)$, as in (7). Then h_k is of bounded variation in some $N(qx-k)$. Hence, one can conclude $\lim_{N\to\infty} S_N(h_k,\,qx-k) = \frac{1}{2}\left[h_k(qx-k+0) + h_k(qx-k-0)\right]$, according to Theorem A. When $x \in (\frac{k}{q},\,\frac{k+1}{q})$, $qx-k \in (0,1)$ and $h_k(qx-k+0) = f\,(x+0)\,, h_k(qx-k-0) = f\,(x-0)\,.$ Therefore, the above limit value reduces to $\frac{1}{2}\left[f\,(x+0) + f\,(x-0)\right]$. Similarly, when $x = \frac{k}{q}$, $h_k(qx-k-0) = h_k(0-0) = h_k(1-0) = f\left(x + \frac{1}{q} - 0\right)$ and $h_k(qx-k+0) = h_k(0+0) = f\,(x+0)\,.$ Hence, $\lim_{N\to\infty} S_N(h_k,\,qx-k) = \frac{1}{2}\left[f\,(x+0) + f\left(x + \frac{1}{q} - 0\right)\right]$. These together with Lemma 1 leads to the final result. $\square$

Proposition 1 gives point-wise convergence of a generalized Fourier series. Comparing with Theorem A, the difference behaves on the set $\frac{\mathbb{N}_q}{q}$. Now, we turn to the uniform convergence of a generalized Fourier series.

Theorem 1 *Let f be 1-periodic, of bounded variation on $[0,\,1]$ and continuous in some neighborhood $N(x)$ of x. Then $\lim_{N\to\infty} G_N(f,\,\cdot) = f(\cdot)$ uniformly in each closed subinterval of $N(x)$ if one of the following two conditions holds:*

(i) $N(x) \subseteq (0,\,1)\backslash\frac{\mathbb{N}}{q}$;

(ii) $x = \frac{k}{q}$ with $k \in \mathbb{N}_q$, f is continuous on $(x - \frac{1}{q}, \delta_1) \cup (\delta_2, x + \frac{1}{q})$ with $\delta_1, \delta_2 > 0$ and $f\left(x - \frac{1}{q}\right) = f\left(x - \frac{1}{q} + 0\right) = f(x) = f\left(x + \frac{1}{q} - 0\right)$.

Proof One proves case 1 firstly: Since $N(x) \subseteq (0,\,1)\backslash\frac{\mathbb{N}}{q}$, $N(x) \subseteq (\frac{k}{q}, \frac{k+1}{q}) \subseteq I_k$ for some $k \in \mathbb{N}_q$. Assume $N(x) = (x-\delta, x+\delta) =: N(x,\delta)$ for some $\delta > 0$. Define 1−periodic function h_k by

$$h_k(\cdot) := f\left(\frac{\cdot + k}{q}\right)$$

on $[0,\,1)$. Then, h_k is of bounded variation on and continuous in $N(qx-k, q\delta)$ due to that of f in $N(x,\delta)$. By Theorem A, $\lim_{N\to\infty} S_N(h_k,\cdot) = h_k(\cdot)$ uniformly on each closed subinterval of $N(qx-k, q\delta)$. Equivalently, $\lim_{N\to\infty} S_N(h_k, q\cdot -k) = h_k(q\cdot -k)$ uniformly of $N(x,\delta)$. Using Lemma 1 and the fact $h_k(q\cdot -k) = f(\cdot)$ on I_k, one has that

$$\lim_{N\to\infty} G_N(f,\cdot) = \lim_{N\to\infty} S_N(h_k, q\cdot -k) = f(\cdot)$$

uniformly in each closed subinterval of $N(x,\delta)$. This completes the first part.

For $x = \frac{k}{q}$, one observes $h_k(0+0) = h_k(0)$ due to the continuity of f in $N(x)$. On the other hand, $h_k(0-0) = h_k(1-0) = h_k(0)$ because of the assumption $f(x) = f\left(x + \frac{1}{q} - 0\right)$. Hence h_k is continuous at 0. Moreover, the continuity of f on $(x, x+\delta) \cup (\delta_2, x+\frac{1}{q})$ implies that h_k is continuous in $(-\epsilon, q\delta)$ for some $\epsilon > 0$. Again using Theorem A, one knows that

$$\lim_{N\to\infty} S_N(h_k,\cdot) = h_k(\cdot)$$

uniformly on each closed subinterval of $(-\epsilon, q\delta)$. Then it follows from Lemma 1 and the definition of h_k that $\lim_{N\to\infty} G_N(f,\cdot) = \lim_{N\to\infty} S_N(h_k, q\cdot -k) = f(\cdot)$ uniformly on each closed subinterval of $(x - \frac{\epsilon}{q}, x+\delta)$.

To finish case 2, it is sufficient to show $\lim_{N\to\infty} G_N(f,\cdot) = f(\cdot)$ uniformly on each closed subinterval of $(x-\delta, x+\frac{\epsilon}{q})$, for which one considers similarly 1-periodic function

$$h_{k-1}(\cdot) := f\left(\frac{\cdot + k - 1}{q}\right) \tag{9}$$

on $[0,\,1)$. Note that the continuity of h_{k-1} on $(1-q\delta, 1+\epsilon)$ (for some $\epsilon > 0$) comes from that of f on $(x-\frac{1}{q}, \delta_1) \cup (x-\delta, x)$ and $f\left(x - \frac{1}{q}\right) = f\left(x - \frac{1}{q} + 0\right) = f(x)$. Then $\lim_{N\to\infty} S_N(h_{k-1},\cdot) = h_{k-1}(\cdot)$ uniformly on each closed subinterval of $(1-q\delta, 1+\epsilon)$. Moreover, by Lemma 1 and the definition for h_{k-1}, one concludes that $\lim_{N\to\infty} G_N(f,t) = \lim_{N\to\infty} S_N(h_{k-1}, qt-k+1) = f(t)$ uniformly on each closed subinterval of $(x-\delta, x+\frac{\epsilon}{q})$. Finally, the desired result follows. □

Remark 1 *The condition $f(x) = f\left(x + \frac{1}{q} - 0\right)$ in Theorem 1 is necessary: In fact, the continuity of f at x implies $f(x) = f(x+0)$. Combining this with Proposition 1, one receives the desired.*

Theorem 1 shows that the uniform convergence for generalized Fourier series keeps the same as the classical Fourier series, when a neighborhood $N(x)$ doesn't intersect with $\frac{\mathbb{N}}{q}$; If $N(x)\cap\frac{\mathbb{N}}{q}\neq\emptyset$, the situation changes. However, it provides a chance to avoid Gibbs phenomenon, as seen in Theorem 2. This is an advantage over the classical Fourier series. Now, we give a simple lemma to introduce the next theorem.

Lemma 2 *Let $q\in 2\mathbb{N}$ and φ be a 1-periodic function defined by*

$$\varphi(\cdot) =: \begin{cases} 1, & x\in I_{2j}, \\ -1, & x\in I_{2j+1} \end{cases} \tag{10}$$

for $j\in\mathbb{N}_{q/2}$. Then $\lim_{N\to\infty} G_N(\varphi,\,\cdot) = \varphi(\cdot)$ uniformly on $[0,\,1]$.

Proof Note that $\varphi(\cdot)$ is continuous on $(0,1)\backslash\frac{\mathbb{N}_q}{q}$ and $\varphi(x)=\varphi(x+0)=\varphi(x+\frac{1}{q}-0)$ for $x\in\frac{\mathbb{N}_q}{q}$. Then $\lim_{N\to\infty} G_N(\varphi,\,\cdot)=\varphi(\cdot)$ pointwise by Proposition 1. Now, it is sufficient to show the uniform convergence:

For $j\in\mathbb{N}_q$ and $p\in q\mathbb{Z}$, let $m := \sharp\left\{n\in\mathbb{Z}:\ a_n^j = p\right\}$ stand for the cardinality of the set. Because the generalized Fourier system $\left\{e^{\mathrm{i}2\pi g_{n,\,l}}\,:\,n\in\mathbb{Z},\,l\in\mathbb{N}_q\right\}$ forms an orthonormal basis for $L^2([0,\,1])$, for $f(\cdot) =: e^{2\pi i p\cdot}\chi_{I_j}(\cdot)\in L^2([0,\,1])$,

$$\frac{1}{q} = \int_0^1 |f(x)|^2\,\mathrm{d}x = \sum_{n\in\mathbb{Z}}\sum_{l=0}^{q-1}\left|\int_0^1 f(x)e^{-\mathrm{i}2\pi g_{n,\,l}(x)}\mathrm{d}x\right|^2 = q\sum_{n\in\mathbb{Z}}\left|\int_{I_j} e^{\mathrm{i}2\pi\left(p-a_n^j\right)x}\mathrm{d}x\right|^2.$$

This with $p-a_n^j\in q\mathbb{Z}$ leads to $\frac{1}{q}=qm(\frac{1}{q})^2$ and $m=1$. In particular, $m =: \sharp\left\{n\in\mathbb{Z}:\ a_n^j=0\right\}=1$. It follows that there exists uniquely $n_j\in\mathbb{Z}$ such that $a_{n_j}^j=0$ for each $j\in\mathbb{N}_q$. Since φ is a constant on I_k and $a_n^j\in q\mathbb{Z}$,

$$\tilde{c}_{n,\,l} = \int_0^1 \varphi(x)e^{-\mathrm{i}2\pi g_{n,\,l}(x)}\mathrm{d}x = 0$$

for $n\neq n_j$, $j\in\mathbb{N}_q$. Define $N_0=\max\{\,|n_j|\,:\,j\in\mathbb{N}_q\}$. Then, for $N>N_0$,

$$G_N(\varphi,\,\cdot) = G_{N_0}(\varphi,\,\cdot) = \sum_{j=0}^{q-1}\sum_{l=0}^{q-1}\tilde{c}_{n_j,\,l}e^{-\mathrm{i}2\pi g_{n_j,\,l}(\cdot)}.$$

Therefore $G_N(\varphi,\,\cdot)$ converges uniformly on $[0,\,1]$ as $N\to\infty$. This completes the proof. $\Box$

Using Theorem 1 and Lemma 2, we prove the following result: It tells us that the Gibbs phenomenon doesn't happen, when a family of functions are represented by generalized Fourier series; While it does by the classical Fourier series.

Theorem 2 *Suppose that $x=\frac{k}{q}\in[0,\,1)$ be a rational number. Let f be 1-periodic, of bounded variation on $[0,\,1]$ and have a jump discontinuity at x. If f is continuous on $(x-\frac{1}{q},\delta_1)\cup(x-\delta,\,x)\cup(x,\,x+\delta)\cup(\delta_2,x+\frac{1}{q})$ for some $\delta,\delta_1,\delta_2>0$ and $f\left(x-\frac{1}{q}\right)=f\left(x-\frac{1}{q}+0\right)=f(x-0)$, $f\,(x)=f(x+0)=f\left(x+\frac{1}{q}-0\right)$, then $\lim_{N\to\infty} G_N(f,\,\cdot)=f(\cdot)$ uniformly in each closed subinterval of $(x-\delta,x+\delta)$.*

Proof Let $l := \frac{1}{2}[f(x+0) - f(x-0)]$ and φ be the function in (10). Since $x = \frac{k}{q} \in [0, 1)$ is a rational number, one can assume $q \in 2\mathbb{N}$ and $k \in \mathbb{N}_q$. Moreover, define

$$h(\cdot) := f(\cdot) \pm l\varphi(\cdot)$$

on $\mathbb{R}$, where one takes "$-$" if $k \in 2\mathbb{N}_{q/2}$ and "+" otherwise. Then h is 1-periodic and of bounded variation on $[0, 1]$. Since $\varphi(x+0) = 1$, when $k \in 2\mathbb{N}_{q/2}$ and $\varphi(x+0) = -1$ otherwise,

$$h(x+0) = f(x+0) \pm l\varphi(x+0) = f(x+0) - l = \frac{1}{2}[f(x+0) + f(x-0)].$$

Similarly, $h(x-0) = f(x-0) \pm l\varphi(x-0) = f(x-0) + l = \frac{1}{2}[f(x+0) + f(x-0)]$. This shows that $h(x+0) = h(x-0)$. Moreover, one can conclude $h(x+0) = h(x)$, because φ is right continuous at x and the assumption $f(x) = f(x+0)$. Consequently, h is continuous at x. Moreover, the continuity of f and φ on $(x - \frac{1}{q}, \delta_1) \cup (x - \delta_0, x) \cup (x, x + \delta_0) \cup (\delta_2, x + \frac{1}{q})$ implies that h is continuous on $(x - \frac{1}{q}, \delta_1) \cup (x - \delta_0, x + \delta_0) \cup (\delta_2, x + \frac{1}{q})$.

Note that $\varphi(x+0) = \varphi\left(x + \frac{1}{q} - 0\right)$ and the given condition $f(x+0) = f\left(x + \frac{1}{q} - 0\right)$. Then $h(x+0) = h\left(x + \frac{1}{q} - 0\right)$. Similar arguments shows $h(x-0) = h\left(x - \frac{1}{q} + 0\right) = h\left(x - \frac{1}{q}\right)$. Combining this with the continuity of h at x, one has $h(x) = h\left(x + \frac{1}{q} - 0\right) = h\left(x - \frac{1}{q} + 0\right) = h\left(x - \frac{1}{q}\right)$. By Theorem 1, $\lim_{N\to\infty} G_N(h, \cdot) = h(\cdot)$ uniformly in each closed subinterval of $(x - \delta, x + \delta)$. Using $G_N(h, \cdot) = G_N(f, \cdot) \pm lG_N(\varphi, \cdot)$ and Lemma 2, one receives the desired. $\square$

Remark 2 *An example for f is the function ϕ given in* (10)*. There are many others, as seen in the next section. For arbitrary rational number $x \in [0, 1)$, there always exist $q \in 2\mathbb{N}$ and $k \in \mathbb{N}_q$ such that $x = \frac{k}{q}$. By Theorem 2, we can choose adaptively the generalized Fourier system according to q such that the Gibbs phenomenon is removed at the discontinuity x.*

Remark 3 *It should be pointed out that the condition $f(x) = f(x+0) = f\left(x + \frac{1}{q} - 0\right)$ in Theorem 2 is necessary: In fact, the uniform convergence of $\lim_{N\to\infty} G_N(f, \cdot) = f(\cdot)$ implies that of $\lim_{N\to\infty} S_N(h_k, q\cdot -k) = h_k(q\cdot -k)$ on $[x, x+\epsilon]$, thanks to Lemma 1 and the definition of h_k in (7). Using the continuity of $S_N(h_k, q\cdot -k)$, one knows that $h_k(q\cdot -k)$ is continuous on $[x, x+\epsilon]$ and furthermore $f(x) = f(x+0)$. This together with Proposition 1 leads to $f(x+0) = f\left(x + \frac{1}{q} - 0\right)$.*

A disadvantage of Theorem 2 is the assumption of x being rational. However, the following proposition shows that the Gibbs phenomenon always happens at the irrational jump discontinuity using a generalized Fourier system given in [8] and [10]. It is a good idea to study generalized Fourier bases with irrational knots. However, it seems to us not so easy.

Proposition 2 *Let f be* 1*-periodic and of bounded variation on* $[0, 1]$*. If f has a irrational jump discontinuity at $x \in (0, 1)$ and is continuous on $(x - \delta, x) \cup (x, x + \delta)$ for some $\delta > 0$. Then $G(f, \cdot)$ shows Gibbs phenomenon at x.*

Proof Since $x \in (0, 1)$ is irrational, there exist $2 \le q \in \mathbb{N}$ and $k \in \mathbb{N}_q$ such that $x \in (\frac{k}{q}, \frac{k+1}{q})$. Let h_k be defined as in (7). Then $qx - k$ is a discontinuity of h_k and h_k is continuous on $(qx - k - q\delta, qx - k) \cup (qx - k, qx - k + q\delta)$. By Theorem B, $S_N(h_k, q \cdot -k)$ does not converge uniformly to $h_k(q \cdot -k)$ in $N(x)$ as $N \to \infty$. This together with Lemma 1 yields that $G_N(f, \cdot)$ does not converge uniformly to $f(\cdot)$ in $N(x)$. Finally, the desired follows. $\square$

3 Numerical Results

In this section, some numerical results are given to illustrate our theory established in Section 2. In all examples, we take $N = 50$, $q = 2$ and

$$g_{n,0}(x) = \begin{cases} 2nx, & x \in [0, \frac{1}{2}), \\ 2nx + \frac{1}{2}, & x \in [\frac{1}{2}, 1), \end{cases}$$

$$g_{n,1}(x) = \begin{cases} 2nx, & x \in [0, \frac{1}{2}), \\ 2nx + 1, & x \in [\frac{1}{2}, 1). \end{cases}$$

Figure 1 and Figure 2 shows that the Gibbs phenomenon can be completely removed at 0, $\frac{1}{2}$, 1 by using generalized Fourier system and can't be by the classical Fourier basis; Gibbs phenomenon still exists at 0, $\frac{1}{2}$, 1 in Figure 3, because the conditions in Theorem 2 are not satisfied; From Figure 4 and Figure 5, we see that a generalized Fourier system is not necessarily better than the Fourier system. Figure 4 shows that the Gibbs phenomenon exists even at the continuity $\frac{1}{2}$, if $f(\frac{1}{2} - 0) \neq f(0 + 0)$. When that condition holds, the Gibbs phenomenon does't happen at the continuity $\frac{1}{2}$, see Figure 5.

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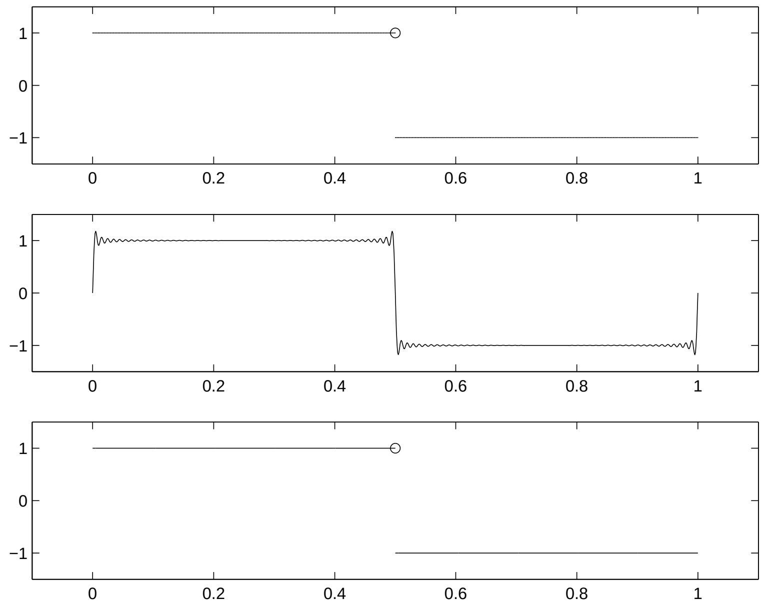

Figure 1: (1): original signal f; (2): $S_N(f,\,x)$; (3): $G_N(f,\,x)$

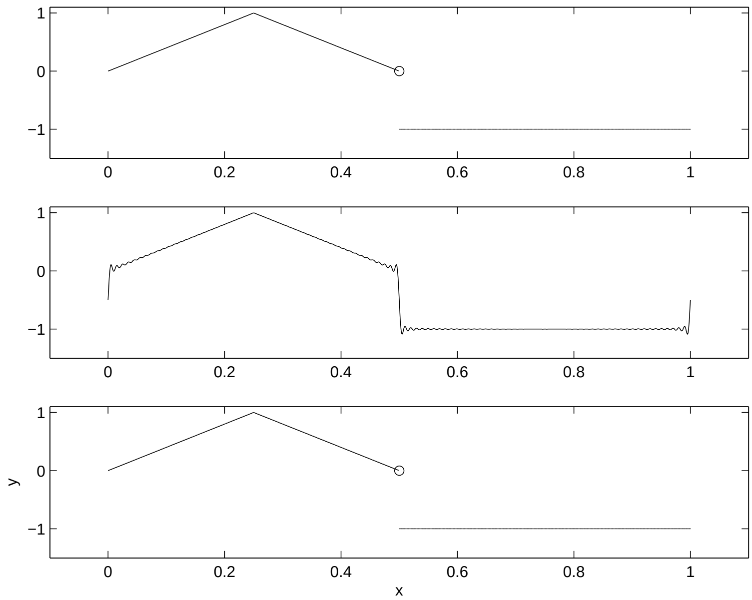

Figure 2: (1): original signal f; (2): $S_N(f, x)$; (3): $G_N(f, x)$

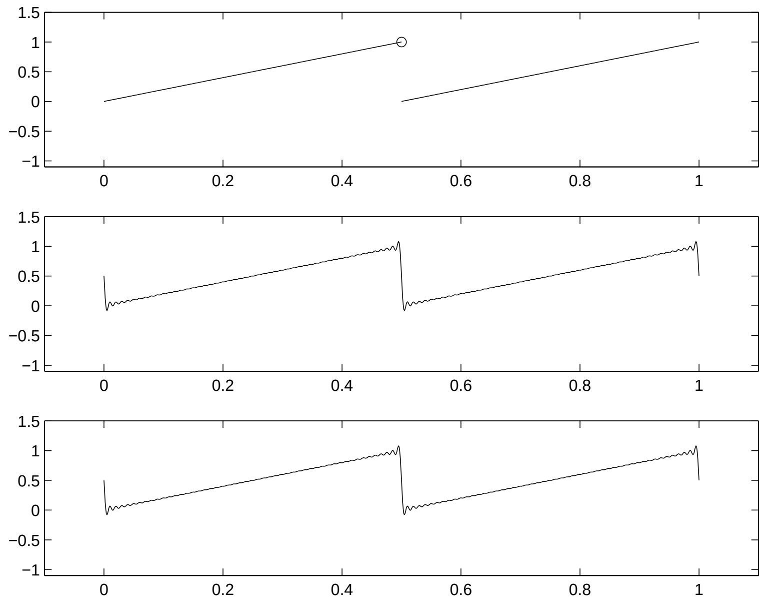

Figure 3: (1): original signal f; (2): $S_N(f, x)$; (3): $G_N(f, x)$

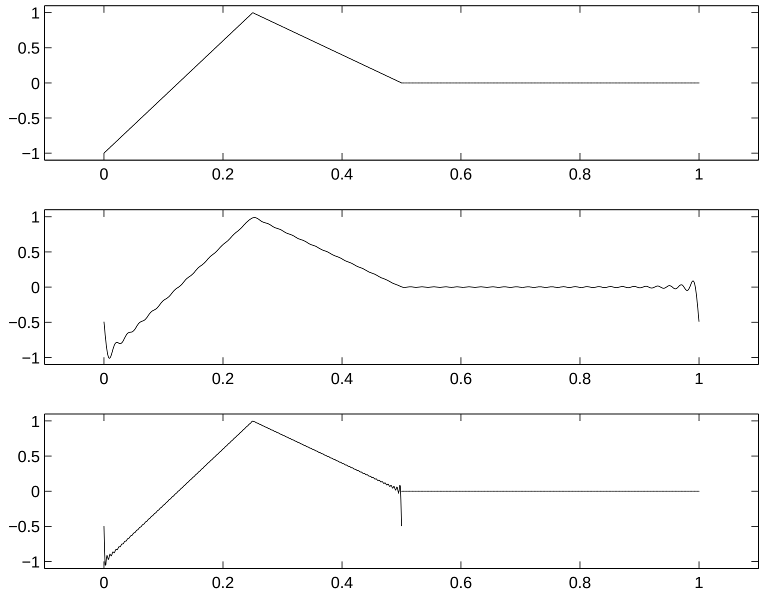

Figure 4: (1): original signal f; (2): $S_N(f,\, x)$; (3): $G_N(f,\, x)$

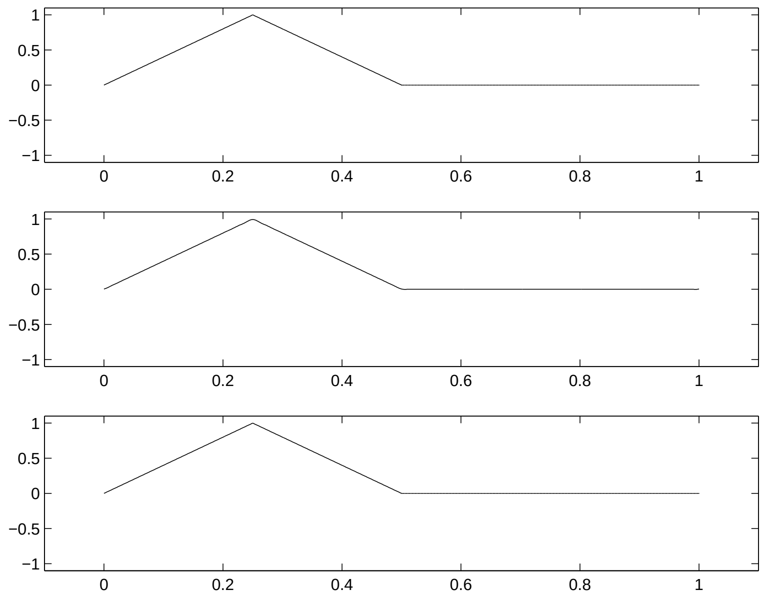

Figure 5: (1): original signal f; (2): $S_N(f,\, x)$; (3): $G_N(f,\, x)$

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Positive Semigroups on General Ordered Banach spaces

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Abstract

A one parameter semi-group of operators is a function $F : [0, \infty] \rightarrow L(X, X)$, such that: (i) $F(s + t) = F(s)F(t)$and (ii) $F(0) = I$, where X is a Banach space and $L(X, X)$ is the space of all bounded linear operators on X.

In this paper we define a novel order on general Banach spaces that induces a continuous half-norm different from the well known canonical half-norm considered by Arendt. Such half-norm defines a positive cone X^+ which enables one to define positive semi-groups and get results similar to those in the case of C[a,b], the Banach space of continuos functions on the compact interval [a,b]. We try to characterize semigroups which leave such X^+ invariant.

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0. Introduction.

Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X. A semigroup on X is a map $T : [0, \infty) \to L(X)$ that satisfies (i) $T(0) = I$, the identity operator on X, (ii) $T(s+t) = T(s)T(t)$. We write $(T(t))$ to denote such a semigroup. The semigroup $(T(t))$ is called a c_0 semigroup if T is a continuous map when $L(X)$ has the strong operator topology. The infinitesimal generator of $(T(t))$ is the linear operator A defined by

$D(A) = \left\{ x \in X : \lim\limits_{t \to 0^+} \frac{T(t)x - x}{t} \ exists \right\}$ and $Ax = \lim\limits_{t \to 0^+} \frac{T(t)x - x}{t}$ for all $x \in D(A)$.

If X is an ordered Banach space and X^+ is the cone of positive elements in X, [5], then the semigroup $(T(t))$ is called positive if $T(t)X^+ \subseteq X^+$ for all $t \geq 0$, [12]. Positive semigroups on general ordered Banach spaces have been investigated by many authors,[1], [2], [3] and [4]. The Banach space $C(K)$, the continuous functions on a compact metric space K, is an ordered Banach space with the natural order, $f \geq g$ if $f(t) \geq g(t)$ for all $t \in K$. Such order gives $C(K)$ a rich structure that enabled researchers to prove deep and more results on positive semigroups on $C(K)$ than on general ordered Banach spaces, [1], [8].

It is the object of this paper to define a new order on general Banach spaces, X, using the extreme points of the unit ball of X^* that produces rich structure on X, similar in some way to that on $C(K)$. **This enabled us to introduce new concepts on Banach spaces with our new order, that is known to hold only for Banach lattices, such as the positive minimum principle,and we prove similar results on semigroups on X as some of those on $C(K)$ that does not hold for general ordered Banach spaces.**

I. The New Order on Banach Spaces.

Let X^* be the dual of the Banach space X, and $B_1(X^*)$ be the unit ball of X^*. It is Known [6], that $B_1(X^*)$ is the w^*-closed convex hull of its extreme points. Let K be the collection of all sets of **linearly independent extreme points of** $B_1(X^*)$. K is ordered by inclusion : $A \leq B$ if $A \subseteq B$ for $A, B \in K$.

Now let C be a chain in K, and $\hat{C} = \bigcup\limits_{A \in C} A$. Since C is a chain, the elements of $\hat{C}$ are linearly independent extreme points, and $A \subseteq \hat{C}$ for all $A \in C$. Thus C has an upper bound. By Zorn's Lemma, K has a maximal element say $\mathfrak{A}$. Thus $\mathfrak{A}$ is a **maximal** subset of linearly independent extreme points of $B_1(X^*)$.

Two subsets A and B of the set $\mathfrak{A}$ is said to form a **cut** of $\mathfrak{A}$ if and only if $\mathfrak{A} = A \cup B$ and $A \cap B = \emptyset$ and $X^* = \overline{[A]} \oplus \overline{[B]}$.

Example 1.1. The natural basis (δ_n) of $l^1 = c_0^*$ is a maximal subset of linearly independent extreme points of $B_1(l^1)$. The same holds for $l^p = (l^{p*})^*$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$.

Example 1.2. Let (e_n), (f_n) be two orthonormal basis of l^2, and $X = K(l^2)$, the space of compact operators on l^2. It is known, [10], that $X^* = C_1(l^2)$, trace class operators. The set $\mathfrak{A} = \{e_i \otimes f_j : i, j \in N\}$ is a maximal subset of linearly independent extreme points of $B_1(X^*)$, [16].

Example 1.3. Let $X = C(K)$, space of continuous functions on the compact set K. For $t \in K$, let δ_t be the point mass measure on K. The set $\mathfrak{A} = \{\delta_t : t \in K\}$ is a maximal subset of linearly independent extreme points of $B_1(X^*)$.

It should be remarked that the above sets $\mathfrak{A}$ give the natural ordering on the corresponding space.

Now we use the set $\mathfrak{A}$ to define a new order on any Banach space X.

Definition 1.4. Let $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. An element $x \in X$ will be called positive (strictly positive) if $\langle x\, ,\, x^* \rangle \geq 0$ $(\langle x, x^* \rangle > 0)$ for all $x^* \in \mathfrak{A}$. We write $x \geq 0$ $(x > 0)$ for positive (strictly positive) $x \in X$.

Definition 1.5. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. For $x, y \in X$, we say $x \leq y$ if $y - x \geq 0$.

We should remark that it could happen that for certain maximal subsets A , X may has no strictly positive elements. But, since if $x^* \in extB_1(X^*)$, then $-x^* \in extB_1(X^*)$, we can choose A such that X has strictly positive elements.

Proposition 1.6. For any Banach space X, the relation " $\leq$ " is a partial order on X.

Proof. The reflexivity and transitivity of "$\leq$ " is clear. We only prove the antisymmetric..

For $x, y \in X$, let $x \leq y$ and $y \leq x$. Then $\langle x - y, x^* \rangle \geq 0$ and $\langle y - x, x^* \rangle \geq 0$ for all $x^* \in \mathfrak{A}$. This implies that $\langle y - x, x^* \rangle = 0$ for all $x^* \in \mathfrak{A}$. But $\mathfrak{A}$ is a maximal linearly independent subset of extreme points of $(B_1(X^*))$. Hence $\langle y - x, x^* \rangle = 0$ for all $x^* \in Ext(B_1(X^*))$ and so $\langle y - x, x^* \rangle = 0$ for $x^* \in Conv(Ext(B_1(X^*)))$. Consequently $\langle y - x, x^* \rangle = 0$ for $x^* \in w^*-$closure of $Conv(Ext(B_1(X^*)))$. By Krein-Milman Theorem, [6], $\langle y - x, x^* \rangle = 0$ for all $x^* \in B_1(X^*)$. Thus $y - x = 0$ and $y = x$.

Definition 1.7. For a Banach space X, X^* is called $\mathfrak{A}$ decomposable if there exists $\lambda > 0$ such that for any cut A and B of $\mathfrak{A}$ the subspaces $\overline{[A]}$ and $\overline{[B]}$ are complemented in X^*, and the projection $P : X^* = \overline{[A]} + \overline{[B]} \longrightarrow \overline{[A]}$ has norm $\|p\| \leq \lambda$, where $\overline{[A]}$ and $\overline{[B]}$ are the w^*-closure of the subspaces of X^* generated by A and B respectively.

Proposition 1.8. The spaces l^p, $1 \leq p < \infty$ and $M(K)$ are $\mathfrak{A}$ decomposable with $\lambda = 1$ and $\mathfrak{A} = \{\delta_n\}$, $\{\delta_t : t \in K\}$ respectively.

Proof. We will prove that $M(K)$ is decomposable with $\lambda = 1$.

Let A and B be two disjoint subsets of $\mathfrak{A}$. Then there exist two disjoint subsets E_1, E_2 in K such that $K = E_1 \dot{\cup} E_2$ and $A = \{\delta_t : \ t \in E_1\}$, $B = \{\delta_t : t \in E_2\}$. Consequently $[A]$ and $[B]$ are subspaces of $M(K)$ such that $M(K) = M(E_1) \oplus M(E_2)$, with $M(E_1) = w^* - cl(A)$, and $M(E_2) = w^* - cl(B)$.Further for any $x \in M(K) : x = y + z$, where, $y = w^*-\lim \sum\limits_{t \in D \subset E_1} x_t \delta_t$ and $z = w^*-\lim \sum\limits_{t \in Q \subset E_2} x_t \delta_t$, where D and Q are finite sets in E_1 and E_2,respectively. Further $\|y\| \leq \|x\|$. Thus the projection P onto $w^* - cl(A)$ has norm ≤ 1.

Now : let X be a Banach space with $\mathfrak{A}$ a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$ such that X^* is $\mathfrak{A}$ decomposable. For x, y in X let : $\mathfrak{A}_1 = \{x^* \in \mathfrak{A} : \langle y - x\, ,\, x^* \rangle \geq 0\}$ and $\mathfrak{A}_2 = \{x^* \in \mathfrak{A} : \langle y - x\, ,\, x^* \rangle < 0\}$. Then $\mathfrak{A}_1$, $\mathfrak{A}_2$ form a cut of $\mathfrak{A}$. Further : $x \leq y$ on $\mathfrak{A}_1$ and $y \leq x$ on $\mathfrak{A}_2$. Define :

$$x \vee y = \left\{ \begin{array}{ccc} y & on & \overline{[\mathfrak{A}_1]} \\ x & on & \overline{[\mathfrak{A}_2]} \end{array} \right\} \quad \text{and} \quad x \wedge y = \left\{ \begin{array}{ccc} x & on & \overline{[\mathfrak{A}_1]} \\ y & on & \overline{[\mathfrak{A}_2]} \end{array} \right\}.$$

Let $P : X^* \to \overline{[\mathfrak{A}_1]}$ be the projection on $\overline{[\mathfrak{A}_1]}$. Since X^* is decomposable, then $P(x^*) = x^*$ for $x^* \in \overline{[\mathfrak{A}_1]}$ and $P(x^*) = 0$ for $x^* \in \overline{[\mathfrak{A}_2]}$. Thus $x \vee y$ can be extended to a continuous linear functional on X^* by setting $x \vee y(x^*) = x \vee y(P(x^*))$. Similarly for $x \wedge y$. Let $x^+ = x \vee 0$, $x^- = x \wedge 0$, $\hat{X} = span\ X \cup \{x^+, x^- : x \in X\}$ in X^{**}, and $|x| = x^+ + x^-$.

Definition 1.9. A Banach space X will be called absolute if for every $x \in X$ both x^+ and x^- are in X. In other words $X = \hat{X}$.

Proposition 1.10. The spaces, c_0 , l^p, $1 < p < \infty$, and $C(K)$ are absolute Banach spaces.

Proof. We prove the lemma for l^p, $1 < p < \infty$.

Let $x = (x_n) \in l^p$. Let $\mathfrak{A} = \{\delta_n : n \in N\}$ be the fixed maximal subset of linearly independent extreme points of $B_1(l^{p*})$. So

$$\mathfrak{A}_1 = \{\delta_n \in \mathfrak{A} : \ x_n \geq 0\} \qquad \text{and} \qquad \mathfrak{A}_2 = \{\delta_n \in \mathfrak{A} : \ x_n < 0\}.$$

Now : let $I = \{n \in N : \ \delta_n \in \mathfrak{A}_1\}$ and $J = \{n \in N : \delta_n \in \mathfrak{A}_2\}$. Clearly $I \cap J = \emptyset$, so $\mathfrak{A}_1$ and $\mathfrak{A}_2$ form a cut of $\mathfrak{A}$. For $k \in N$

$$\langle x^+, \delta_k \rangle = \left\{ \begin{array}{cc} x_k & k \in I \\ 0 & k \in J \end{array} \right\} \quad \text{and} \quad \langle x^-, \delta_k \rangle = \left\{ \begin{array}{cc} -x_k & k \in J \\ 0 & k \in I \end{array} \right\}.$$

Thus :

$$\|x^+\|_p = \left(\sum_{k \in I} |x_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} < \infty,$$

$$\|x^-\|_p = \left(\sum_{k \in J} |x_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} < \infty.$$

So both x^+ and x^- are in l^p. Hence $|x| = x^+ + x^- \in l^p$ and l^p is an absolute Banach space.

Proposition 1.11. Let X be an absolute Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$ such that X^* is $\mathfrak{A}$ decomposable. Then :

(i) $|x| \geq 0$ for all $x \in X$.

(ii) $-|x| \leq x \leq |x|$ for all $x \in X$.

Proof. Since X is an absolute Banach space, then $|x| \in X$ for all $x \in X$. Now for $x^* \in \mathfrak{A}$ either $x^* \in \mathfrak{A}_1 = \{x^* \in \mathfrak{A} : \langle x , x^* \rangle \geq 0\}$ or $x^* \in \mathfrak{A}_2 = \{x^* \in \mathfrak{A} : \langle x , x^* \rangle < 0\}$. If $x^* \in \mathfrak{A}_1$, then $\langle |x| , x^* \rangle = \langle x, x^* \rangle \geq 0$. If $x^* \in \mathfrak{A}_2$, then $\langle |x| , x^* \rangle = \langle -x, x^* \rangle = - \langle x, x^* \rangle \geq 0$. So $|x| \geq 0$ and $x \leq |x|$ for all $x \in X$. Similarly $-|x| \leq x$ for all $x \in X$. Hence $-|x| \leq x \leq |x|$ for all $x \in X$.

Remark 1.12. For $x, y \in X$, let $\sup\{x, y\}$, $\inf\{x, y\}$ denote the least upper bound and the greatest lower bound of the set $\{x, y\}$ respectively.

Definition 1.13. The ordered Banach space $(X, \leq)$ is called a Banach lattice if for each pair $(x, y) \in X \times X$, the elements $\sup\{x, y\} = x \vee y$ and $\inf\{x, y\} = x \wedge y$ exist in X.

Proposition 1.14. An absolute Banach space is a Banach lattice and reflexivity guarantees the absoluteness.

Proof. Let X be a reflexive Banach space and $\mathfrak{A}$ be a fixed maximal subset of independent extreme points of $B_1(X^*)$ with X^* be $\mathfrak{A}$ decomposable. Then $x \vee y$ and $x \wedge y$ are in X. For $x^* \in \mathfrak{A}_1$, one has : $\langle x \vee y, x^* \rangle = \langle y, x^* \rangle \geq \langle x, x^* \rangle$. For

$x^* \in \mathfrak{A}_2 : \langle x \vee y, x^* \rangle = \langle x, x^* \rangle$. Hence $\langle x, x^* \rangle \leq \langle x \vee y, x^* \rangle$ for all $x^* \in \mathfrak{A}$ which implies that $x \leq x \vee y$. Similarly $y \leq x \vee y$. So $x \vee y$ is an upper bound for $\{x, y\}$.

To prove that $x \vee y$ is the least upper bound for $\{x, y\}$, suppose there exists $w \in X$ such that $x \leq w$, $y \leq w$ and $w \leq x \vee y$. Let $x^* \in \mathfrak{A}$. Then :

$$\langle x, x^* \rangle \leq \langle w, x^* \rangle \leq \langle x \vee y, x^* \rangle \quad \text{and} \quad \langle y, x^* \rangle \leq \langle w, x^* \rangle \leq \langle x \vee y, x^* \rangle .$$

But either $x^* \in \mathfrak{A}_1$ or $x^* \in \mathfrak{A}_2$. If $x^* \in \mathfrak{A}_1$, then $\langle y, x^* \rangle = \langle x \vee y, x^* \rangle \leq \langle w, x^* \rangle$, and so $\langle x \vee y, x^* \rangle = \langle w, x^* \rangle$. If $x^* \in \mathfrak{A}_2$, then $\langle x, x^* \rangle = \langle x \vee y, x^* \rangle \leq \langle w, x^* \rangle$, and so $\langle x \vee y, x^* \rangle = \langle w, x^* \rangle$. Consequently $x \vee y = \sup \{x, y\}$.

Similarly $x \wedge y = \inf \{x, y\}$.

Corollary 1.15. l^p, $1 < p < \infty$ is a lattice.

One also can prove :

Proposition 1.16. The Banach space c_0 with $\mathfrak{A} = \{\delta_n : n \in N\}$ is a lattice, and the Banach space $C(K)$ with $\mathfrak{A} = \{\delta_t : t \in K\}$ is a lattice.

Definition 1.17. Let X be a Banach space ordered by $\mathfrak{A}$ a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$ and B be a subset of X. We say that :

(i) B is bounded above if there exists $a \in X$ such that $x \leq a$ for all $x \in B$.

(ii) B is bounded below if there exists $b \in X$ such that $b \leq x$ for all $x \in B$.

Proposition 1.18. (a) every set B in c_0 and l^p , $1 < p < \infty$, that is bounded above has a supremum.

(b) every set B in c_0 and l^p , $1 < p < \infty$, that is bounded below has an infimum.

(c) If a Banach space X contains a strictly positive element x_0 , then it contains infinitely many strictly positive elements.

(d) $int(c_0^+) = \emptyset$, $int({l^p}^+) = \emptyset$ and $int(C(K)^+) \neq \emptyset$.

Proof. (**a**) Let $B \subseteq l^p$ and B be bounded above. Then there exists $y = (y_n) \in l^p$ such that $x = (x_n) \leq y$ for all $x \in B$. So $x_n \leq y_n$ for all $n \in N$ and for all $x \in B$.

Now : let $H_n = \{\langle x, \delta_n \rangle : x \in B\}$. Then for each $n \in N$, H_n is a bounded set of real numbers. So it has a supremum. Let $z = (z_n)$ be defined as $z_n = \sup H_n$. So $x_n \leq z_n \leq y_n$ for all $n \in N$. That $z \in l^p$ follows from

$$\sum_{n=1}^{\infty} |z_n|^p \leq \sum_{n=1}^{\infty} |x_n|^p + |y_n|^p < \infty.$$

That $z = \sup B$ follows from $z_n = \sup \{\langle x, \delta_n \rangle : x \in B\}$. The proof for c_0 is similar and it will be omitted.

(b) The proof is similar to(a).

(c) In fact for all positive integers n, $nx_0 \in X$ and $\langle nx_0, x^* \rangle = n \langle x_0, x^* \rangle > 0$.

(d) We will prove $int(c_0^+) = \emptyset$. Let $x_0 = (x_n) \in int(c_0^+)$. Then there exists $\epsilon > 0$ such that the ball $B(x_0, \epsilon) \subseteq c_0^+$. Since $x_0 = (x_n) \in c_0$, then $\lim\limits_{n \to \infty} x_n = 0$. So there exists $n_0 \in N$ such that $|x_n| < \frac{\epsilon}{3}$ for all $n > n_0$. Let $y = (y_n)$ be such that :

$$y_n = \left\{ \begin{array}{ll} x_n & n \leq n_0 \\ -\frac{\epsilon}{3} & n_0 < n < 2n_0 \\ 0 & n \geq 2n_0 \end{array} \right\} .$$

Then :

$$\|y - x_0\|_\infty = \sup_n |y_n - x_n| \leq \sup_{n > n_0} \left| x_n - -\tfrac{\epsilon}{3} \right| \leq \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} < \epsilon.$$

This implies that $y \in B(x_0 , \epsilon)$ and $\langle y, \delta_n \rangle = y_n = -\frac{\epsilon}{3}$ for all $n_0 < n < 2n_0$. So $y \notin c_0^+$ and so $x_0 \notin int(c_0^+)$. Thus $int(c_0^+) = \emptyset$.

Proposition 1.19. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. Then set $X^+ = \{x \in X : x \geq 0\}$ is a normed closed cone in X.

Proposition 1.20. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. Then the function $\alpha : X \longrightarrow R$ such that $\alpha(x) = \sup\limits_{x^* \in \mathfrak{A}} \langle x, x^* \rangle$ satisfies the following :

(1) $\alpha(x+y) \leq \alpha(x) + \alpha(y)$ for all $x, y \in X$.
(2) $\alpha(\lambda x) = \lambda \alpha(x)$ for all $x \in X$, for all $\lambda \in R$, $\lambda \geq 0$.
(3) $\alpha(-x)$ or $\alpha(x) > 0$ for all $x \in X$, $x \neq 0$

Proof. We only prove (3) : Let $x \in X$, $x \neq 0$. Then $\alpha(x)$, $\alpha(-x)$ can't be both negative. For if so, then $\alpha(x) = \sup\limits_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \leq 0$ and $\alpha(-x) = \sup\limits_{x^* \in \mathfrak{A}} \langle -x, x^* \rangle \leq 0$. Hence $\langle x, x^* \rangle \leq 0$ and $\langle -x, x^* \rangle \leq 0$ for all $x^* \in \mathfrak{A}$. This implies that $\langle x, x^* \rangle = 0$ for all $x^* \in \mathfrak{A}$, and so $x = 0$. It follows that either $\alpha(x)$ or $\alpha(-x)$ is strictly positive otherwise x would be zero.

Proposition 1.21. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. For $x \in X$, let $\alpha(x) = \sup\limits_{x^* \in \mathfrak{A}} \langle x, x^* \rangle$. Then $\|.\|_\alpha$ is a norm on X, where $\|x\|_\alpha = \max(\alpha(x), \alpha(-x))$.

Proof. Let $x, y \in X$. Then

(i) $\|0\|_\alpha = \max(\alpha(0), \alpha(-0))$.

If $\|x\|_\alpha = 0$, then $\max(\alpha(x), \alpha(-x)) = 0$, so both $\alpha(x)$ and $\alpha(-x) = 0$, and hence $x = 0$.

(ii) $\|\lambda x\|_\alpha = \max(\alpha(\lambda x), \alpha(-\lambda x))$. Now : if $\lambda \geq 0$, then $\alpha(\lambda x) = \lambda \alpha(x)$ and $\alpha(-\lambda x) = \lambda \alpha(-x)$. Consequently

$$\|\lambda x\|_\alpha = \lambda \|x\|_\alpha = |\lambda| \, \|x\|_\alpha .$$

If $\lambda < 0$, then $\lambda = -\mu$, $\mu > 0$. So

$$\alpha(\lambda x) = \alpha(-\mu x) = \mu \alpha(-x) = |\lambda| \, \alpha(-x),$$

and

$$\alpha(-\lambda x) = \alpha(\mu x) = \mu \alpha(x) = |\lambda| \, \alpha(x).$$

Thus

$$\|\lambda x\|_\alpha = \max(\alpha(\mu x), \mu\alpha(-x)) = \mu \max(\alpha(x), \alpha(-x)) = |\lambda| \, \|x\|_\alpha .$$

(iii) Since $\|x+y\|_\alpha = \max(\alpha(x+y), \alpha(-(x+y)))$, then either

$$\|x+y\|_\alpha = \alpha(x+y) \leq \alpha(x) + \alpha(y) \leq \|x\|_\alpha + \|y\|_\alpha ,$$

or

$$\|x+y\|_\alpha = \alpha(-(x+y)) \leq \alpha(-x) + \alpha(-y) \leq \|x\|_\alpha + \|y\|_\alpha .$$

Thus in either case $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$.

Remark 1.22. Let X be an absolute Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. Then $|\alpha(x)| \leq \|x\|_\alpha$ for all $x \in X$ and $\|.\|_\alpha$ is monotone ($\|x\|_\alpha \leq \|y\|_\alpha$ if $0 \leq x \leq y$).

Proof. Let $x \in X$. Since $\|x\|_\alpha = \max(\alpha(x), \alpha(-x))$, then $\alpha(x) \leq \|x\|_\alpha$ and $\alpha(-x) \leq \|x\|_\alpha$. By Lemma 1.20, either $\alpha(x) > 0$ or $\alpha(-x) > 0$. If $\alpha(x) > 0$, then $|\alpha(x)| = \alpha(x) \leq \|x\|_\alpha$. If $\alpha(x) < 0$, then $\alpha(-x) > 0$. This implies $\langle x, x^* \rangle < 0$ for all $x^* \in \mathfrak{A}$. But :

$$\alpha(-x) = \sup_{x^* \in \mathfrak{A}} \langle -x, x^* \rangle = \sup_{x^* \in \mathfrak{A}} - \langle x, x^* \rangle = - \inf_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \geq - \sup_{x^* \in \mathfrak{A}} \langle x, x^* \rangle = -\alpha(x)$$

So $-\alpha(x) \leq \alpha(-x) \leq \|x\|_\alpha$. Hence $|\alpha(x)| \leq \|x\|_\alpha$.

To prove that $\|.\|_\alpha$ is monotone, let $x, y \in X$ such that $0 \leq x \leq y$. Then $0 \leq \langle x, x^* \rangle \leq \langle y, x^* \rangle$ for all $x^* \in \mathfrak{A}$. Hence $\sup_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \leq \sup_{x^* \in \mathfrak{A}} \langle y, x^* \rangle$. Thus $\alpha(x) \leq \alpha(y)$. This implies that $\|x\|_\alpha \leq \|y\|_\alpha$.

Proposition 1.23. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. Then $\|x\|_\alpha \leq \|x\|$ for all $x \in X$. The two norms $\|.\|_\alpha$ and $\|.\|$ are not equivalent norms in general.

Proof. Let $x \in X$. Then

$$\alpha(x) = \sup_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \leq \sup_{x^* \in \mathfrak{A}} |\langle x, x^* \rangle| \leq \sup_{x^* \in B_1(X^*)} |\langle x, x^* \rangle| = \|x\|.$$

$$\alpha(-x) = \sup_{x^* \in \mathfrak{A}} \langle -x, x^* \rangle \leq \sup_{x^* \in \mathfrak{A}} |\langle x, x^* \rangle| \leq \sup_{x^* \in B_1(X^*)} |\langle x, x^* \rangle| = \|x\|.$$

But : $\|x\|_\alpha = \max(\alpha(x), \alpha(-x)) \leq \|x\|$.

Now : let $X = l^2$ and suppose there exists $c > 0$ such that $c\|x\| \leq \|x\|_\alpha$. For $n \in N$, let $x = \sum_{k=1}^{n} c\,\delta_k \in l^2$. Then

$$\|x\|_\alpha = \max(\alpha(x), \alpha(-x)) = c.$$

But :

$$c\|x\| = c\left(\sum_{k=1}^{n} c^2\right)^{\frac{1}{2}} = \sqrt{n}c^2.$$

Choose n large enough such that $c < \sqrt{n}c^2$. But this contradicts $c\|x\| \leq \|x\|_\alpha$. So $\|.\|$ and $\|.\|_\alpha$ are not equivalent in general.

Proposition 1.24. Let X be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$. Define $X^\alpha = \{x \in X : \alpha(-x) \leq 0\}$. Then $X^\alpha = X^+$.

Proof. Let $x \in X^\alpha$. Then :

$$\alpha(-x) = \sup_{x^* \in \mathfrak{A}} \langle -x, x^* \rangle = \sup_{x^* \in \mathfrak{A}} -\langle x, x^* \rangle = -\inf_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \leq 0.$$

So $\inf_{x^* \in \mathfrak{A}} \langle x, x^* \rangle \geq 0$, which implies $\langle x, x^* \rangle \geq 0$ for all $x^* \in \mathfrak{A}$. Hence $X^\alpha \subseteq X^+$.

Now let $x \in X^+$. Then $\langle x, x^* \rangle \geq 0$ for all $x^* \in \mathfrak{A}$, which implies $\langle -x, x^* \rangle \leq 0$ for all $x^* \in \mathfrak{A}$. Hence $\alpha(-x) = \sup_{x^* \in \mathfrak{A}} \langle -x, x^* \rangle \leq 0$. This implies $X^+ = X^\alpha$.

II Positive Semigroups.

In this section we will study positive semigroups on Banach spaces ordered by maximal linearly independent subset of extreme points of $B_1(X^*)$. Through out of this section X will be a Banach space and $\mathfrak{A}$ be a fixed maximal subset of linearly independent extreme points of $B_1(X^*)$ and X is ordered by $\mathfrak{A}$, and by what we remarked, following definition 1.5, we can choose $\mathfrak{A}$ such that $extB_1(X^*) \neq \varphi$.

An operator $T \in L(X)$ is called positive if $Tx \geq 0$ if $x \geq 0$. The following theorem holds true for general ordered Banach spaces, and the proof will be omitted. The same holds true for Theorem2.2.

Theorem 2.1. Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on X with infinitesimal generator A. Then the following are equivalent :

(i) $T(t)$ is positive for all $t \geq 0$.

(ii) $R(\lambda, A)$ is positive for sufficiently large λ.

We remark that if A is a densely defined linear operator on X and $int(X^+) \neq \emptyset$, then $int(X^+) \cap D(A) \neq \emptyset$. Further : $int(X^+) \cap Range(R(0, A)) \neq \emptyset$ if $0 \in \rho(A)$.

Theorem 2.2. Suppose X^+ is normal and $int(X^+) \neq \emptyset$. Let A be a densely defined linear operator on X and $[\omega, \infty) \subseteq \rho(A)$ for some $\omega \in R$. If $R(\lambda, A)$ satisfies

$R(\lambda, A)x \geq 0$ if and only if $x \geq 0$ for all $\lambda > \omega$, then A is the infinitesimal generator of a strongly continuous positive semigroup.

Definition 2.3. An unbounded operator A on X is said to satisfy the positive minimum principle if for every $x \in D(A) \cap X^+$ and $x^* \in \mathfrak{A} : \langle x , x^* \rangle = 0$ implies that $\langle Ax , x^* \rangle \geq 0$.

Definition 2.4. Let B be a positive bounded linear operator on X. An element $x_0 \in X$ is called semi-maximal for B if for every $x \in X$ there exists $\lambda > 0$ such that $|\langle x, x^* \rangle| \leq \lambda \langle Bx_0 , x^* \rangle$ for every $x^* \in \mathfrak{A}$.

Example 2.5. Let $X = C([0,1])$. Let $\mathfrak{A} = \{\delta_t : t \in [0,1]\}$ be the fixed maximal subset of extreme points of $B_1(X^*)$. Define :

$B : C([0,1]) \longrightarrow C([0,1])$

$Bx(t) = (t+1)x(t)$.

Clearly B is a positive bounded linear operator on X. The constant function $x_0(t) = 1$ is a semi-maximal element for B. To see that : Let $x \in C([0,1])$. Then $|\langle x, \delta_t \rangle| = |x(t)|$ and $\langle Bx_0 , \delta_t \rangle = t+1$.

Now : since $[0,1]$ is compact, then there exists $M > 0$ such that $|x(t)| \leq M$ for all $t \in [0,1]$. Hence

$$|x(t)| \leq M(t+1) = M \langle Bx_0, \delta_t \rangle .$$

Theorem 2.6. Let X be an absolute real Banach space such that X^+ is normal and $int(X^+) \neq \emptyset$. Let A be a densely defined linear operator on X that satisfies :

(i) The positive minimum principle.

(ii) There exists $x_0 \in X$ and $\lambda_0 \in (\omega, \infty) \subseteq \rho(A)$ such that x_0 is strictly positive and semi-maximal for $R(\lambda_0, A)$.

(iii) $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

Then A is the infinitesimal generator of a strongly continuous positive semigroup.

Proof. The main idea of the proof is how one can define a norm on X under which $\lambda R(\lambda, A)$ is a contraction.

First suppose that $\omega < 0$ and $\lambda_0 = 0$. Let $y = R(0, A)x_0$. Then $y > 0$. Indeed if y is not strictly positive, then there exists $x^* \in \mathfrak{A}$ such that $\langle R(0, A)x_0 , x^* \rangle = 0$. Since A satisfies the positive minimum principle, then $\langle AR(0, A)x_0 , x^* \rangle \geq 0$. But $x_0 = -AR(0, A)x_0$. This implies that

$$0 < \langle x_0, x^* \rangle = -\langle AR(0, A)x_0, x^* \rangle \leq 0.$$

This cannot be true. Hence $y > 0$. Since x_0 is semi-maximal for $R(0, A)$, then for all $x \in X$ there exists $\lambda > 0$ such that $|\langle x, x^* \rangle| \leq \lambda \langle y, x^* \rangle$ for all $x^* \in \mathfrak{A}$. That means $|x| \leq \lambda y$.

Now we for $x \in X$, we define $\|x\|_0 = \inf \{\lambda > 0 : |x| \leq \lambda y\}$ .We claim that $\|.\|_0$ is a norm on X. Indeed :

(1) If $\|x\|_0 = 0$, then

$$\inf \{\lambda > 0 : -\lambda \langle y, x^* \rangle \leq \langle x, x^* \rangle \leq \lambda \langle y, x^* \rangle , \ x^* \in \mathfrak{A}\} = 0.$$

Hence $\langle x, x^* \rangle = 0$ for all $x^* \in \mathfrak{A}$. Thus $x = 0$.

(2) For $x \in X$ and $\beta \in R$ we have :

$$\begin{aligned}
\|\beta x\|_0 &= \inf \{\lambda > 0 : -\lambda y \leq \beta x \leq \lambda y\} . \\
&= \inf \{\lambda > 0 : -\lambda \langle y, x^* \rangle \leq \langle \beta x, x^* \rangle \leq \lambda \langle y, x^* \rangle , \ x^* \in \mathfrak{A}\} \\
&= \inf \left\{\lambda > 0 : -\tfrac{\lambda}{|\beta|} \langle y, x^* \rangle \leq \tfrac{\beta}{|\beta|} \langle x, x^* \rangle \leq \tfrac{\lambda}{|\beta|} \langle y, x^* \rangle , \ x^* \in \mathfrak{A}\right\} \\
&= |\beta| \inf \left\{ \tfrac{\lambda}{|\beta|} > 0 : -\tfrac{\lambda}{|\beta|} \langle y, x^* \rangle \leq \tfrac{\beta}{|\beta|} \langle x, x^* \rangle \leq \tfrac{\lambda}{|\beta|} \langle y, x^* \rangle , \ \ x^* \in \mathfrak{A}\right\} \\
&= |\beta| \, \|x\|_0 .
\end{aligned}$$

(3) For x_1 , x_2 $\in X$ we have :

$$\begin{aligned}\|x_1+x_2\|_0 &= \inf\{\lambda>0 : -\lambda y \le x_1+x_2 \le \lambda y\}\\ &= \inf\{\lambda>0 : -\lambda\langle y,x^*\rangle \le \langle x_1+x_2,x^*\rangle \le \lambda\langle y,x^*\rangle\,,\ x^*\in\mathfrak{A}\}\\ &= \sup\left\{\begin{array}{c}\lambda>0 : there\ exists\ no\ x^*\in\mathfrak{A}\ with\\ |\langle x_1,x^*\rangle+\langle x_2,x^*\rangle| \ge \lambda\langle y,x^*\rangle\end{array}\right\}\end{aligned}$$

Since $|\langle x_1,x^*\rangle|+|\langle x_2,x^*\rangle| \ge |\langle x_1,x^*\rangle+\langle x_2,x^*\rangle| \ge \lambda\langle y,x^*\rangle$ we get :

$$\begin{aligned}\|x_1+x_2\|_0 &= \sup\left\{\begin{array}{c}\lambda>0 : there\ exists\ no\ x^*\in\mathfrak{A}\ with\\ |\langle x_1,x^*\rangle|+|\langle x_2,x^*\rangle| \ge \lambda\langle y,x^*\rangle\end{array}\right\}\\ &\le \sup\left\{\begin{array}{c}\lambda>0 : there\ exists\ no\ x^*\in\mathfrak{A}\ with\\ |\langle x_1,x^*\rangle| \ge \lambda\langle y,x^*\rangle\end{array}\right\}\\ &\quad+\sup\left\{\begin{array}{c}\lambda>0 : there\ exists\ no\ x^*\in\mathfrak{A}\ with\\ |\langle x_2,x^*\rangle| \ge \lambda\langle y,x^*\rangle\end{array}\right\}\\ &= \|x_1\|_0+\|x_2\|_0\,.\end{aligned}$$

Further $\|.\|_0$ is equivalent to $\|.\|$ on X. Indeed : Let $x\in X$, $y\in int(X^+)$ and $\epsilon>0$ such that $B(y,\epsilon)\subseteq X^+$. Then $\left\|y \mp \frac{\epsilon x}{2\|x\|}-y\right\| = \left\|\frac{\epsilon x}{2\|x\|}\right\| = \frac{\epsilon}{2} < \epsilon$. This implies that $y \mp \frac{\epsilon x}{2\|x\|}\in X^+$, and so $y \mp \frac{\epsilon x}{2\|x\|}\ge 0$. Hence $-\frac{2\|x\|y}{\epsilon}\le x\le\frac{2\|x\|y}{\epsilon}$. It follows that from $(*)$ that $\|x\|_0\le\frac{2\|x\|}{\epsilon}$.

Now : since X^+ is normal, then by Proposition A.2.2 in [5], $[-y,y]$ is norm bounded, . So there exists $M>0$ such that $\|x\|\le M$ for all $x\in[-y,y]$. For $x\in X$ and $v=\frac{x}{\|x\|_0}$, we have : $\|v\|_0=1$. But $\|v\|_0=\inf\{\lambda>0 : -\lambda y\le v\le\lambda y\}$. So $\|v\|_0\,y\le v\le\|v\|_0\,y$. Consequently $-y\le v\le y$ and $v\in[-y,y]$. By normality of X^+ we get : $\|v\|=\left\|\frac{x}{\|x\|_0}\right\|\le M$. This implies that $\|x\|\le M\,\|x\|_0$. Thus $\|.\|_0$ and $\|.\|$ are equivalent norms on X.

Now : for $x\in[-y,y]$, one has $y-x$ and $y+x$ are positive. Further, if $\lambda>0$, then $R(\lambda,A)$ is a positive operator. Hence $R(\lambda,A)(y-x)\ge 0$, $R(\lambda,A)(y+x)\ge 0$, and this implies $-\lambda R(\lambda,A)y\le\lambda R(\lambda,A)x\le\lambda R(\lambda,A)y$. The resolvent identity gives :

$$\begin{aligned}\lambda R(\lambda,A)y &= \lambda R(\lambda,A)R(0,A)x_0\\ &= R(0,A)x_0-R(\lambda,A)x_0\\ &\le R(0,A)x_0 = y.\end{aligned}$$

It follows that :

$$-y\le-\lambda R(\lambda,A)y\le\lambda R(\lambda,A)x\le\lambda R(\lambda,A)y\le y$$

and $-y\le\lambda R(\lambda,A)x\le y$. Hence $\|\lambda R(\lambda,A)x\|_0\le 1$ and $\|\lambda R(\lambda,A)\|_0\le 1$. Thus $\lambda R(\lambda,A)$ is a contraction with respect to $\|.\|_0$ norm. By Hille-Yosida Theorem A is the infinitesimal generator of a C_0 semigroup of contractions on X with respect to $\|.\|_0$. Since $\|.\|_0$ and $\|.\|$ are equivalent, then there exists a , $b>0$ such that : $a\,\|x\|_0\le\|x\|\le b\,\|x\|_0$ for every $x\in X$. Hence :

$$\|T(t)x\|\le b\,\|T(t)x\|_0\le b\,\|x\|_0\le\tfrac{b}{a}\,\|x\|\,.$$

This implies that $T(t)$ is uniformly bounded with respect to $\|.\|$ norm. But $R(\lambda,A)\ge 0$ for all $\lambda>\omega$. Theorem 2.1 implies $(T(t))_{t\ge 0}$ is a positive semigroup.

Finally : for $\alpha>\omega>0$ consider the linear operator $A-\alpha$. We claim $[\omega-\alpha,\infty)\subseteq\rho(A-\alpha)$. Indeed : If $\mu\in[\omega-\alpha,\infty)$, then $\mu+\alpha\in\rho(A)$. Since

$$0\le R(\mu+\alpha,A)=R(\mu,A-\alpha),$$

then $\mu \in \rho(A-\alpha)$ and $R(\mu : A-\alpha) \geq 0$ for all $\mu \in [\omega-\alpha, \infty)$. By the first part of the proof, $A-\alpha$ is the generator of a strongly continuous positive bounded semigroup $(T(t))_{t\geq 0}$. Thus $A = A-\alpha+\alpha$ is the infinitesimal generator of the positive semigroup $S(t) = T(t)e^{\alpha t}$.

Now : for the case $\lambda_0 > \omega > 0$. Since $R(\lambda_0, A) = R(0, A-\lambda_0)$, then using (ii) we get : for all $x \in X$ there exists $\lambda > 0$ such that

$$\begin{aligned} |\langle x\,,\,x^*\rangle| &\leq \lambda \,\langle R(\lambda_0, A)x_0\,,\,x^*\rangle \\ &= \lambda\,\langle R(0, A-\lambda_0)x_0\,,\,x^*\rangle \end{aligned}$$

for all $x^* \in \mathfrak{A}$. So x_0 is a semi-maximal for $R(0, A-\lambda_0)$. Now :

$$y = R(\lambda_0, A)x_0 = R(0, A-\lambda_0)x_0 > 0.$$

For if not true, there exists $x^* \in \mathfrak{A}$ such that $\langle R(\lambda_0, A)x_0, x^*\rangle = 0$. By the positive minimum principle $\langle AR(\lambda_0, A)x_0, x^*\rangle \geq 0$. But $x_0 = (\lambda_0 - A)R(\lambda_0, A)x_0$. Thus

$$0 < \langle x\,,\,x^*\rangle = \lambda_0\ \langle R(\lambda_0, A)x_0\,,\,x^*\rangle - \langle AR(\lambda_0, A)x_0\,,\,x^*\rangle \leq 0,$$

which cannot be true. Hence $y > 0$ and x_0 is a semi-maximal element for $R(0, A-\lambda_0)$.

Theorem 2.7. Let $(T(t))_{t\geq 0}$ be a C_0 semigroup on X with infinitesimal generator A. Suppose that the set $G = \{x \in X : x > 0\} \cap int(X^+) \neq \emptyset$. Then the following are equivalent :

(i) $T(t)$ is positive for all $t > 0$.

(ii) The infinitesimal generator A satisfies the positive minimum principle.

Proof. $(i) \longrightarrow (ii)$ Let $x \in D(A)$, $x \geq 0$ and $x^* \in \mathfrak{A}$ such that $\langle x\,,\,x^*\rangle = 0$. Then :

$$\begin{aligned} \langle Ax\,,\,x^*\rangle &= \left\langle \lim_{t\to 0^+} \tfrac{T(t)x-x}{t}\,,\,x^* \right\rangle \\ &= \lim_{t\to 0^+} \tfrac{1}{t}\,\langle T(t)x\,,\,x^*\rangle - \tfrac{1}{t}\,\langle x\,,\,x^*\rangle \\ &= \lim_{t\to 0^+} \tfrac{1}{t}\,\langle T(t)x\,,\,x^*\rangle \geq 0. \end{aligned}$$

Conversely $(ii) \longrightarrow (i)$ Let $s = \inf\{\lambda \in R : [\lambda, \infty) \subseteq \rho(A)\}$. Let $x \in int(X^+)$ and $x > 0$. Then

$$\lambda_0 = \inf\{\lambda > s : R(\mu, A)x > 0,\ \text{for all } \mu \in (\lambda, \infty)\}$$

is finite since $\lim\limits_{\mu\to\infty} \mu R(\mu, A)x = x$. We claim that $\lambda_0 = s$. In fact if this is not true, then $[\lambda_0, \infty) \subseteq \rho(A)$, $R(\lambda_0, A)x$ is not strictly positive. But since

$$\lambda_0 = \inf\{\lambda > s : R(\mu, A)x > 0,\ \text{for all } \mu \in (\lambda, \infty)\},$$

then for all positive integers n, $R(\lambda_0 + \frac{1}{n}, A)x > 0$. This implies that $R(\lambda_0, A)x \geq 0$. Consequently there exists $x^* \in \mathfrak{A}$ such that $\langle R(\lambda_0, A)x\,,\,x^*\rangle = 0$. But A satisfies the positive minimum principle. So $\langle AR(\lambda_0, A)x\,,\,x^*\rangle \geq 0$. However

$$x = (\lambda_0 - A)R(\lambda_0, A)x = \lambda_0 R(\lambda_0, A)x - AR(\lambda_0, A)x.$$

Hence

$$\begin{aligned} \langle x\,,\,x^*\rangle &= \langle \lambda_0 R(\lambda_0, A)x\,,\,x^*\rangle - \langle AR(\lambda_0, A)x\,,\,x^*\rangle \\ &= \lambda_0.0 - \langle AR(\lambda_0, A)x\,,\,x^*\rangle \leq 0. \end{aligned}$$

Hence $\langle x\,,\,x^*\rangle \leq 0$. But $\langle x\,,\,x^*\rangle > 0$ by assumption. This is a contradiction. Thus $s = \lambda_0$. Thus $\langle R(\lambda, A)x\,,\,x^*\rangle > 0$ if $x \in int(X^+)$ and $\langle x\,,\,x^*\rangle > 0$ for every $x^* \in \mathfrak{A}$ for every $\lambda > s$.

Now : if $x > 0$ and $x \notin int(X^+)$, then x is in the boundary of X^+ and so there exists a sequence (x_n) in $int(X^+)$ such that x_n converges to x. Thus $\langle R(\lambda, A)x\,,\,x^*\rangle \geq 0$ for every $x^* \in \mathfrak{A}$ for every $\lambda > s$. if $x \in X$ such that $x \geq 0$ and for some $x^* \in \mathfrak{A}$, $\langle x\,,\,x^*\rangle = 0$. Choose a strictly positive element $x_0 \in int(X^+)$. Put $x_n = x + \frac{x_0}{n}$. Then :

$$\langle x_n\,,\,x^*\rangle = \left\langle \tfrac{x_0}{n}\,,\,x^*\right\rangle + \langle x\,,\,x^*\rangle = \tfrac{1}{n}\,\langle x_0\,,\,x^*\rangle + \langle x\,,\,x^*\rangle > 0$$

for all $n \in N$ for all $x^* \in \mathfrak{A}$. Hence the set of strictly positive elements in X is dense in X^+. It follows that $R(\lambda, A) \geq 0$ for all $\lambda > s$. Theorem 2.1 implies that $T(t) \geq 0$ for all $t \geq 0$.

Definition 2.8. An operator $T \in L(X)$ is called inner norm map if

$$|\langle Tx\,,\,x^*\rangle| \leq \|T\|\,\langle x\,,\,x^*\rangle$$

for all $x \in X^+$ and all $x^* \in \mathfrak{A}$.

Examples 2.9.

(1) Let $T : l^2 \longrightarrow l^2$ be such that $T\left(\sum\limits_{i=1}^{\infty} x_i\delta_i\right) = x_1\delta_1$. Then

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|\leq 1} \|x_1\delta_1\| \leq 1.$$

Since $T\delta_1 = \delta_1$, then $\|T\| = 1$. For $x = \sum\limits_{i=1}^{\infty} x_i\delta_i \in l^2$, we have :

$$\langle Tx\,,\,\delta_k\rangle = \left\{\begin{array}{ll} x_k & k = 1 \\ 0 & k \neq 1 \end{array}\right\}.$$

So $|\langle Tx\,,\,\delta_k\rangle| \leq |x_k| = |\langle\, x\,,\,\delta_k\rangle|$ for all $k \in N$. Hence $|\langle Tx\,,\,x^*\rangle| \leq \|T\|\,|\langle\, x\,,\,x^*\rangle|$ for all $x^* \in \mathfrak{A}$.

(2) Let $\mathfrak{A} = \{\delta_n\, :\, n \in N\} \subset \ell^2$, and T be any projection operator on any subspaces generated by any subset of $\mathfrak{A}$. Then T is an inner norm operator.

(3) Let g be any positive function in $C[0,1]$, and $T : C[0,1] \longrightarrow C[0,1]$, with $Tf = gf$. Then T is an inner product map.

Example (3) holds true for all positive multipliers on all classical sequences spaces.

We should remark that one can give many other examples.

Theorem 2.10. Let A be a bounded Schwarz map. Then the following are equivalent :

(i) $e^{tA} \geq 0$ for all $t \geq 0$.

(ii) For $0 \leq x \in X$; if $\langle x, x^*\rangle = 0$, then $\langle Ax\,,\,x^*\rangle \geq 0$ for all $x^* \in \mathfrak{A}$.

(iii) $A + \|A\| \geq 0$.

Proof. $(i) \longrightarrow (ii)$ Let $x^* \in \mathfrak{A}$ and $\langle x, x^*\rangle = 0$ with $x \in X^+$. Then :

$$\begin{aligned}\langle Ax\,,\,x^*\rangle &= \left\langle \lim_{t\to 0^+} \tfrac{e^{tA}x - x}{t}\,,\,x^*\right\rangle \\ &= \lim_{t\to 0^+} \left\langle \tfrac{e^{tA}x}{t}\,,\,x^*\right\rangle - \tfrac{\langle x\,,\,x^*\rangle}{t} \\ &= \lim_{t\to 0^+} \tfrac{1}{t}\left\langle e^{tA}x\,,\,x^*\right\rangle \geq 0.\end{aligned}$$

$(ii) \longrightarrow (iii)$ Let $x \in X^+$ and $x^* \in \mathfrak{A}$. Then :

$$\langle (A + \|A\|)x\,,\,x^*\rangle = \langle Ax\,,\,x^*\rangle + \|A\|\,\langle x\,,\,x^*\rangle\,.$$

If $\langle x\,,\,x^*\rangle = 0$, then by (ii) $\langle Ax\,,\,x^*\rangle \geq 0$. If $\langle x\,,\,x^*\rangle > 0$ then, since

$$|\langle Ax\,,\,x^*\rangle| \leq \|A\|\,\langle x\,,\,x^*\rangle\,,$$

for all $x^* \in \mathfrak{A}$ and $x \in X^+$, $\langle Ax\,,\,x^*\rangle + \|A\|\,\langle x\,,\,x^*\rangle \geq 0$. Thus $A + \|A\| \geq 0$.

$(iii) \longrightarrow (i)$ For $x^* \in \mathfrak{A}$ and $x \in X^+$ we have :

$$\begin{aligned}\left\langle e^{tA}x\,,\,x^*\right\rangle &= \left\langle e^{-t\|A\|}e^{t(A+\|A\|)}x\,,\,x^*\right\rangle \\ &= e^{-t\|A\|}\left\langle e^{t(A+\|A\|)}x\,,\,x^*\right\rangle \\ &= e^{-t\|A\|}\left\langle \sum_{n=0}^{\infty} \tfrac{t^n}{n!}\,(A+\|A\|)^n\,x\,,\,x^*\right\rangle \\ &= e^{-t\|A\|}\sum_{n=0}^{\infty} \tfrac{t^n}{n!}\,\langle (A+\|A\|)^n\,x\,,\,x^*\rangle\,.\end{aligned}$$

Since $e^{-t\|A\|} \geq 0$ and $(A + \|A\|)^n \geq 0$ for all $n \in N$, then $e^{tA} \geq 0$.

Definition 2.11. Let $(T(t))_{t\geq 0}$ be a positive semigroup on an absolute Banach space X ordered by the set $\mathfrak{A}$ with infinitesimal generator A and $(S(t))_{t\geq 0}$ be a semigroup in $L(X)$ with infinitesimal generator B. We say that $(T(t))_{t\geq 0}$ dominates $(S(t))_{t\geq 0}$ if $|S(t)x| \leq T(t)\,|x|$, for all $x \in X$ and for all $t \geq 0$.

Example 2.12. Let $X = C([0,1])$. Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be defined on $C([0,1])$ as follows :

$$(T(t)f)(s) = e^{st}f(s) \text{ and } (S(t)f)(s) = e^{ist}f(s).$$

Clearly $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are semigroups on $C([0,1])$ and $(T(t))_{t\geq 0}$ is a positive semigroup. Now :

$$|S(t)f(s)| = \left|e^{ist}f(s)\right| = \left|e^{ist}\right| |f(s)| = |f(s)| \leq e^{st}\,|f(s)| = T(t)\,|f(s)|\,.$$

Hence $T(t)$ dominates $S(t)$.

The following theorem is known for Banach lattices. However, **we don't assume in this theorem that our Banach space is absolute**.

Theorem 2.13. Let $(T(t))_{t\geq 0}$, $(S(t))_{t\geq 0}$ be two positive semigroups in $L(X)$ with infinitesimal generators A and B respectively. Suppose that $D(B) \subseteq D(A)$. Then the following are equivalent :

(i) $S(t) \leq T(t)$ for all $t \geq 0$,

(ii) $Bx \leq Ax$ for $0 \leq x \in D(B)$.

Proof. $(i) \longrightarrow (ii)$ Let $x \in D(B)$, $x \in X^+$ and $x^* \in \mathfrak{A}$. Then :

$$\begin{aligned}
\langle Bx\,,\,x^*\rangle &= \left\langle \lim_{t\to 0^+} \tfrac{S(t)x-x}{t}\,,\,x^*\right\rangle \\
&= \lim_{t\to 0^+} \left\langle \tfrac{S(t)x}{t}\,,\,x^*\right\rangle - \tfrac{\langle x\,,\,x^*\rangle}{t} \\
&= \lim_{t\to 0^+} \tfrac{1}{t}\left(\langle S(t)x\,,\,x^*\rangle - \langle x\,,\,x^*\rangle\right) \\
&\leq \lim_{t\to 0^+} \tfrac{1}{t}\left(\langle T(t)x\,,\,x^*\rangle - \langle x\,,\,x^*\rangle\right) \\
&= \lim_{t\to 0^+} \tfrac{1}{t}\,\langle T(t)x - x\,,\,x^*\rangle \\
&= \left\langle \lim_{t\to 0^+} \tfrac{T(t)x-x}{t}\,,\,x^*\right\rangle = \langle Ax\,,\,x^*\rangle\,.
\end{aligned}$$

Hence $Bx \leq Ax$ for all $x \in D(B) = D(A) \cap D(B)$.

Conversely $(ii) \longrightarrow (i)$ Since $(T(t))_{t\geq 0}$, $(S(t))_{t\geq 0}$ are positive semigroups, then by Theorem 2.1 both $R(\lambda, A)$ and $R(\lambda, B)$ are positive for large λ.

Now : Choose $\lambda > \max(\omega(A), \omega(B))$ and $0 \leq x \in D(B)$. Using Theorem 5.3 in [13], we have : $R(\lambda : B)x \in D(B)$. Using (ii), $D(B) \subseteq D(A)$ and $R(\lambda, B)$ is positive we get : $(A - B)R(\lambda, B)x \geq 0$. But $R(\lambda, A)$ is positive. So

$$R(\lambda, A)(A - B)R(\lambda, B)x \geq 0.$$

The identity :

$$R(\lambda, A)x - R(\lambda, B)x = R(\lambda, A)(A - B)R(\lambda, B)x$$

gives

$$R(\lambda, A)x - R(\lambda, B)x \geq 0,$$

and therefore $R(\lambda, A)x \geq R(\lambda, B)x$ for all $0 \leq x \in D(B)$. Since $\overline{D(B)} = X$, then $R(\lambda, A) \geq R(\lambda, B)$ for large λ. Consequently for $x \in X^+$ we have :

$$R(\lambda, A)(R(\lambda, A)x - R(\lambda, B)x) \geq 0.$$

Hence

$$R^2(\lambda, A)x \geq R(\lambda, A)R(\lambda, B)x \geq R(\lambda, B)R(\lambda, B)x = R^2(\lambda, B)x\,.$$

By mathematical induction we get : $R^n(\lambda, A)x \geq R^n(\lambda, B)x$. Theorem 8.3 in [13], now gives :

$$\begin{aligned} S(t)x = & \lim_{n\to\infty} (\tfrac{n}{t}R(\tfrac{n}{t}, B))^n x \\ \leq & \lim_{n\to\infty} (\tfrac{n}{t}R(\tfrac{n}{t}, A))^n x \\ = & \, T(t)x. \end{aligned}$$

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A note on Euler numbers and polynomials

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Abstract : In this paper, we give a formula on relationship between the Bernoulli numbers and Euler numbers. Finally, we investigate the zeros of the Euler polynomials $E_n(x)$.

Key words : Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, alternating sums of powers

2000 Mathematics Subject Classification : 11S80, 11B68

1 Introduction

In the 21st century, the computing environment would make more and more rapid progress and there has been increasing interest in solving mathematical problems with the aid of computers. By using software, mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Many mathematicians have studied Bernoulli polynomials, Bernoulli numbers, Euler polynomials, and Euler numbers. Bernoulli polynomials, Bernoulli numbers, Euler polynomials, and Euler numbers posses many interesting properties and arising in many areas of mathematics and physics. In this paper, we give a formula on relationship between the Bernoulli numbers and Euler numbers. We observe the structure of the real roots of our Euler polynomials, $E_n(x)$, using numerical investigation. By computer experiments, we demonstrate a remarkably regular structure of the complex roots of $E_n(x)$. Finally, we give a table for the solutions of our Euler polynomials $E_n(x)$. This numerical investigation is especially exciting because these steps are essential for most students to truly understand even basic concept of Euler numbers E_n and Euler polynomials $E_n(x)$.

2 Relationship between the Bernoulli and Euler numbers

Throughout this paper $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will be denoted by the ring of rational integers, the field of rational numbers, the field of real numbers and the complex number field, respectively. First, we introduce the ordinary Bernoulli numbers and Bernoulli polynomials. The usual Bernoulli numbers B_n are defined by

$$e^{Bt} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1},$$

where the symbol B_k is interpreted to mean that B^k must be replaced by B_k when we expand the one on the left. This relation can be written as

$$e^{(B+1)t} - e^{Bt} = t.$$

Hence we obtain

$$B_0 = 1, \quad (B+1)^k - B^k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention about replacing B^k by $B_k, (i \geq 0)$. The Bernoulli polynomials $B_n(x)$ are defined by the generating function:

$$e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt},$$

It is easily see that

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}, B_k(0) = B_k.$$

Bernoulli polynomials and Bernoulli numbers posses many interesting properties and arise in many areas of mathematics. Bernoulli numbers play an important role in mathematics. They first appeared in *Ars Conjectandi,* a famous and posthumous treaties published in 1713, by Jakob Bernoulli when he studied the sums of powers of consecutive integers $s_p(n) = \sum_{k=1}^{n-1} k^p$, where p and n are two positive integers(cf. [1], [2], [3]). The sums $s_p(n)$ can be written in the form

$$s_p(n) = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}.$$

Thanks to Bernoulli's polynomials, it's possible to rewrite the expression of the sums $s_p(n)$ as

$$s_p(n) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1}(B_{p+1}(n) - B_{p+1}).$$

Bernoulli numbers also appear in the computation of the numbers

$$\zeta(2p) = \sum_{k=1}^{\infty} \frac{1}{k^{2p}}.$$

We also have

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}, \quad k > 0.$$

Next, we introduce the ordinary Euler numbers and Euler polynomials. The usual Euler numbers E_n and the usual Euler polynomials $E_n(x)$ are defined by means of the following generating functions:

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < 2\pi), \tag{1}$$

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (|t| < 2\pi),$$

respectively. Let u be algebraic in complex number field. Then Frobenius-Euler numbers are defined by

$$e^{H(u)t} = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, (|t| < 2\pi), \tag{2}$$

This relation can be written as

$$H_0(u) = 1, \quad (H(u)+1)^k - uH_k(u) = 0 \quad (1 \leq k).$$

Therefore we have

$$uH_k(u) = \sum_{i=0}^{k} \binom{k}{i} H_i(u), H_k(u) = \frac{1}{u-1} \sum_{i=0}^{k-1} \binom{k}{i} H_i(u), \text{ for } u \neq 1.$$

By (1) and (2), note that $H_n(-1) = E_n$. Let n, k be positive integers ($k > 1$), and let

$$S_n(k) = 1^n + 2^n + 3^n + 4^n \cdots + k^n.$$

It was well known that

$$S_n(k) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i (k+1)^{n+1-i}, \text{ cf. [1], [2], [3],} \tag{3}$$

where $\binom{n}{k}$ is binomial coefficients. Since

$$S_n(2k) = 1^n + 2^n + \cdots + (2k)^n = (1^n + 3^n + + \cdots + (2k-1)^n) + (2^n + 4^n + + \cdots + (2k)^n),$$

we obtain

$$\sum_{m=1}^{k} (2m-1)^n = S_n(2k) - 2^n S_n(k). \tag{4}$$

Now, we consider the alternating sums of powers of consecutive integers

$$T_n(k) = \sum_{k=1}^{2k} (-1)^k k^n, \tag{5}$$

where k and n are two given positive integers. The sums $T_n(k)$ can be written in the form

$$\begin{aligned} T_n(k) &= -1^n + 2^n - 3^n + 4^n - 5^n + \cdots + (-1)^{2k-1}(2k-1)^n + (-1)^{2k}(2k)^n \\ &= (2^n + 4^n + + \cdots + (2k)^n) - (1^n + 3^n + 5^n + \cdots + (2k-1)^n) \\ &= 2^n(1^n + 2^n + \cdots + k^n) - \sum_{m=1}^{k} (2m-1)^n. \end{aligned}$$

By (4), we have the following theorem.

Theorem 1. Let $k, n (n \geq 1)$ be positive integers. Then we have

$$T_n(k) = 2^{n+1} S_n(k) - S_n(2k).$$

By simple calculations and (3), we have the following corollary.

Corollary 2. Let $k, n(n \geq 1)$ be positive integers. Then we have

$$T_n(k) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i \left\{ 2^{n+1}(k+1)^{n+1-i} - (2k+1)^{n+1-i} \right\}.$$

Since

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt} = e^{Et} e^{xt} = e^{(E+x)t} = \sum_{n=0}^{\infty} (E+x)^n \frac{t^n}{n!},$$

we have

$$E_n(x) = \sum_{n=0}^{n} \binom{n}{k} E_k x^{n-k}.$$

In order to obtain the relationship between the Bernoulli and Euler numbers, we introduce some interesting properties. We derive each of the following results:

$$\begin{aligned} 2\sum_{l=0}^{n-1}(-1)^l e^{lt} &= 2(1 - e^t + e^{2t} - e^{3t} + \cdots + (-1)^{n-1} e^{(n-1)t}) \\ &= \frac{2}{e^t+1} + 2\frac{(-1)^{n-1}e^{nt}}{e^t+1} \\ &= \frac{2}{e^t+1} - (-1)^n \frac{2e^{nt}}{e^t+1} \\ &= \sum_{k=0}^{\infty} E_k \frac{t^k}{k!} - (-1)^n \sum_{k=0}^{\infty} E_k(n) \frac{t^k}{k!}, \end{aligned} \tag{6}$$

and

$$2\sum_{l=0}^{n-1}(-1)^l e^{lt} = \sum_{k=0}^{\infty} 2 \sum_{l=0}^{n-1} (-1)^l l^k \frac{t^k}{k!}. \tag{7}$$

Next, by combining (6) and (7), we have the following theorem.

Theorem 3. Let k, n be given positive integers. Then we obtain

$$E_k + (-1)^{n+1} E_k(n) = 2\sum_{l=0}^{n-1}(-1)^l l^k = 2(0 - 1^k + 2^k - 3^k + \cdots + (-1)^{n-1}(n-1)^k). \tag{8}$$

Setting $n = 2n+1$ in Theorem 3, we obtain

$$E_k + (-1)^{2n+2} E_k(2n+1) = 2(-1^k + 2^k - 3^k + \cdots + (-1)^{2n}(2n)^k).$$

We now derive an interesting formula:

$$E_k + E_k(2n+1) = 2(-1^k + 2^k - 3^k + \cdots + (2n)^k).$$

By (5), (9) and Theorem 1, we obtain

$$E_k + E_k(2n+1) = 2T_n(k) = 2^{n+2} S_n(k) - 2S_n(2k). \tag{9}$$

Finally, by combining Corollary 2 and (9), we have the relationship between the Bernoulli and Euler numbers.

Theorem 4. Let k, n be given positive integers. Then we obtain

$$E_k + E_k(2n+1) = \frac{2}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i \left\{ 2^{n+1}(k+1)^{n+1-i} - (2k+1)^{n+1-i} \right\}.$$

3 Distribution and Structure of the zeros

Because

$$\frac{\partial F}{\partial x}(x,t) = tF(x,t) = \sum_{n=0}^{\infty} \frac{dE_n}{dx}(x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{dE_k}{dx}(x) = kE_{k-1}(x).$$

Then, it is easy to deduce that $E_k(x)$ are polynomials of degree k. Here is the list of the first Euler's polynomials.

$$\begin{aligned}
&E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2},\\
&E_2(x) = x^2 - x,\\
&E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4},\\
&E_4(x) = x^4 - 2x^3 + x\\
&E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}\\
&E_6(x) = x^6 - 3x^5 + 5x^3 - 3x,\\
&E_7(x) = x^7 - \frac{7}{2}x^6 + \frac{35}{4}x^4 - \frac{21}{2}x^2 + \frac{17}{8},\\
&E_8(x) = x^8 - 4x^7 + 14x^5 - 28x^3 + 17x,\\
&E_9(x) = x^9 - \frac{9}{2}x^8 + 21x^6 - 63x^4 + \frac{153}{2}x^2 - \frac{31}{2},\\
&\qquad \cdots
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} E_n(1-x)\frac{(-1)^n t^n}{n!} = \frac{2}{e^{-t}+1}e^{(1-x)(-t)} = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

we obtain

$$E_n(x) = (-1)^n E_n(1-x). \tag{10}$$

The question is: what happens with the reflexive symmetry (10), when one considers Euler polynomials? Prove that $E_n(x), x \in \mathbb{C}$, has $Re(x) = \frac{1}{2}$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions (Figure 1). Prove that $E_n(x) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_n(x)}$ of $E_n(x), Im(x) \neq 0$. Since n is the degree of the polynomial $E_n(x)$, the number of real zeros $R_{E_n(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{E_n(x)} = n - C_{E_n(x)}$, where $C_{E_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_n(x)}$ and $C_{E_n(x)}$. Next, we investigate the zeros of the Euler polynomials by using computer. Figure 1 displays the zeros of Euler polynomials $E_n(x), n = 40, 60, 80, 100, x \in \mathbb{C}$.

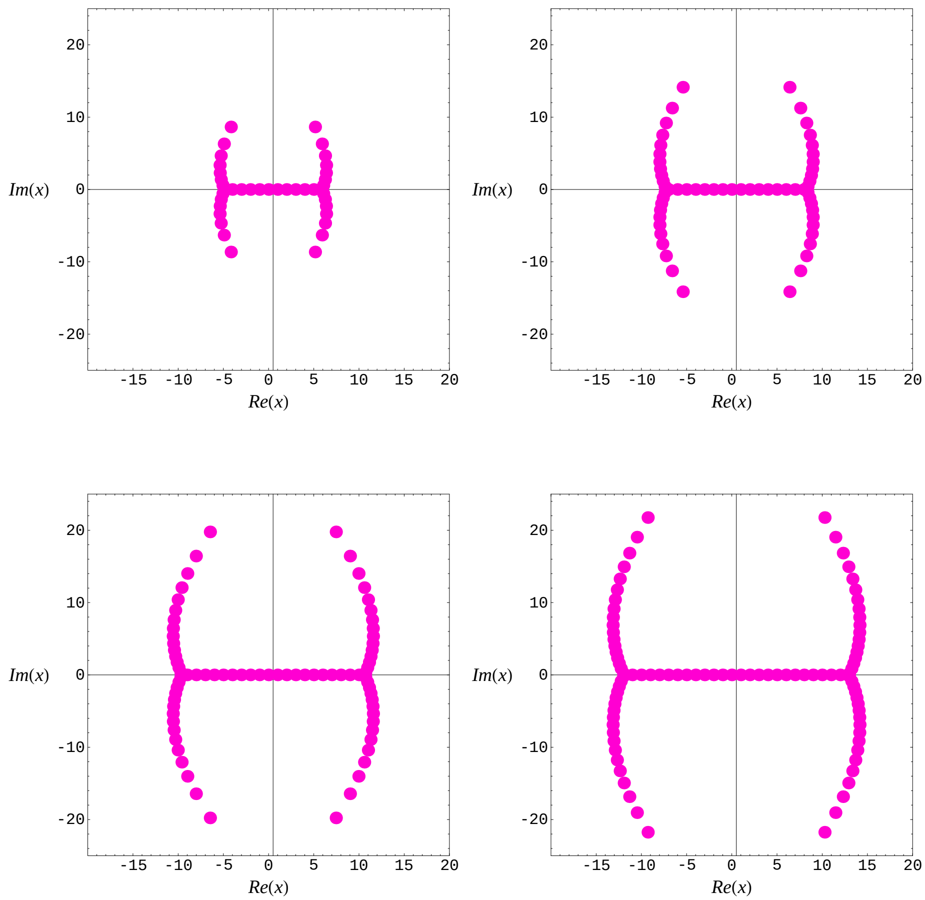

Figure 1 : Zeros of $E_n(x)$

Figure 2 displays the stacks of zeros of $E_n(x), 1 \leq n \leq 100$ from a 3-D structure.

Our numerical results for approximate solutions of real zeros of $E_n(x)$ are displayed. The results are obtained by Mathematica software. We observe a remarkably regular structure of the complex roots of Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of Euler polynomials(See Table 1).

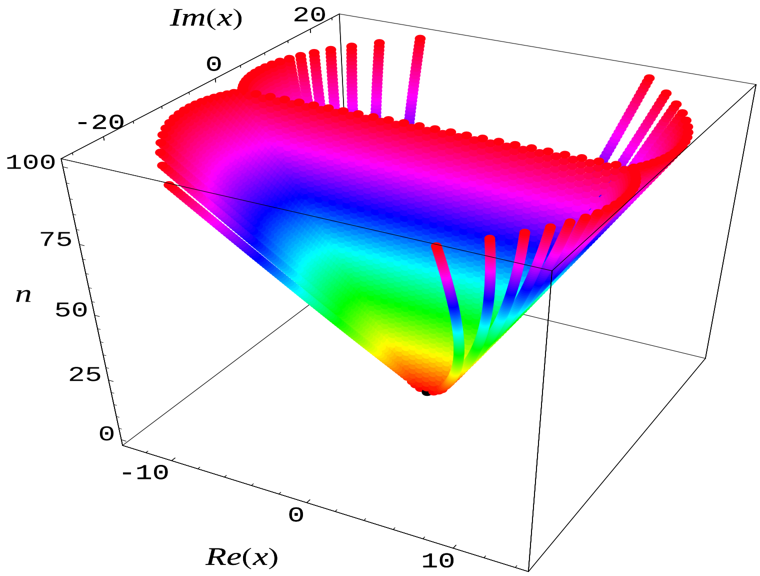

Figure 2 : Stacks of zeros of $E_n(x), 1 \leq n \leq 100$

Table 1. Numbers of real and complex zeros of $E_n(x)$

degree n	real zeros	complex zeros
26	10	16
27	7	20
28	8	20
29	9	20
30	10	20
31	11	20
32	8	24
33	9	24
34	10	24
35	11	24
36	12	24
37	9	28
38	10	28
39	11	28
40	12	28

Next, we calculated an approximate solution satisfying $E_n(x), x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E_n(x) = 0, x \in \mathbb{R}$

degree n	x
1	0.5000
2	0.0000, 1.000000
3	-0.366025404, 0.500000000, 1.366025404
4	-0.6180339, 0.0000, 1.0000, 1.61803
5	$-0.61803, -0.61803, 0.5000, 1.6180, 1.6180$
6	0.00000, 1.00000
7	-0.497731435, 0.500000000, 1.49773143
8	-0.932327751, 0.00000, 1.0000, 1.93232775
9	$-1.21973, -0.50008, 0.5000, 1.50008, 2.2197$
10	$-1.3652, -1.01497, 0.000, 1.0000, 2.0149, 2.3652$

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Exploring the q-Euler numbers and polynomials

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Abstract

In this paper we observe the structure of the roots of q-Euler polynomials $E_n(x,h|q)$, using numerical investigation. We study that the q-Euler polynomial are analytic continued to $E_q(s)$. A new formula for the q-Euler Zeta function $\zeta_{E,q}(s|h)$, in terms of nested series of $\zeta_{E,q}(n|h)$ is derived.

2000 Mathematics Subject Classification - 11B68, 11S40

Key words- Euler number, Euler polynomials, q-Euler numbers, q-Euler polynomial, q-Euler Zeta function

1 Introduction

Throughout this paper, $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ will denote the ring of integers, the field of real numbers and the complex numbers, respectively.

When one talks of q-extension, q is variously considered as an indeterminate, a complex numbers or p-adic numbers. In complex number field, we will assume that $|q| < 1$ or $|q| > 1$. The q-symbol $[x]_q$ denotes $[x]_q = \frac{1-q^x}{1-q}$.

In this paper we observe an interesting phenomenon of 'scattering' of the zeros of $E_n(x,h|q)$. Also, we study that the q-Euler polynomials due to T.Kim (see [1,8]) are analytic continued to $E_q(s)$. By those results, we give a new formula for the q-Euler zeta function due to T.Kim, cf. [1,5,7].

2 Generating q-Euler polynomials and numbers

For $h \in \mathbb{Z}$, the q-Euler polynomials were defined as

$$\sum_{n=0}^{\infty} \frac{E_n(x,h|q)}{n!} t^n = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t}, \tag{2.1}$$

for $x, q \in \mathbb{C}$, cf. [1,7]. In the special case $x = 0$, $E_n(0, h|q) = E_n(h|q)$ are called q-Euler numbers, cf. [1,2,3,4]. By (2.1), we easily see that

$$E_n(x,h|q) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{l+h}} q^{lx}, \quad cf.[7,8], \tag{2.2}$$

where $\binom{n}{j}$ is binomial coefficient. From (2.1), we derive

$$E_{n,q}(x,h|q) = (q^x E(h|q) + [x]_q)^n$$

with the usual convention of replacing $E^n(h|q)$ by $E_n(h|q)$. In the case $h = 0$, $E_n(x, 0|q)$ will be symbolically written as $E_{n,q}(x)$. Let $G_q(x,t)$ be generating function of q-Euler polynomials as follows:

$$G_q(x,t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

Then we easily see that

$$G_q(x,t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t}. \tag{2.4}$$

For $x = 0, E_{n,q} = E_{n,q}(0)$ will be called q-Euler numbers.

From (2.3), (2.4), we easily derive the following: For $k(=\text{even})$ and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) - E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n. \tag{2.5}$$

For $k(=\text{odd})$ and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) + E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n. \tag{2.6}$$

By (2.4), we easily see that

$$E_{m,q}(x) = \sum_{l=0}^{m} \binom{m}{l} q^{xl} E_{l,q} [x]_q^{m-l}. \tag{2.7}$$

From (2.5), (2.6), and (2.7), we derive

$$[2]_q \sum_{l=0}^{k-1} (-1)^{l-1} [l]_q^n = (q^{kn} - 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q} [k]_q^{n-l}, \tag{2.8}$$

where $k(=\text{even}) \in \mathbb{N}$. For $k(=\text{odd})$ and $n \in \mathbb{Z}_+$, we have

$$[2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n = (q^{kn} + 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q} [k]_q^{n-l}. \tag{2.9}$$

3 Zeros of q-Euler polynomials

In this section, we plot the zeros of q-Euler polynomials are solutions of $E_m(x,h|q), m = 40, q = 1/2, x \in \mathbb{C}$ (Figure 2).

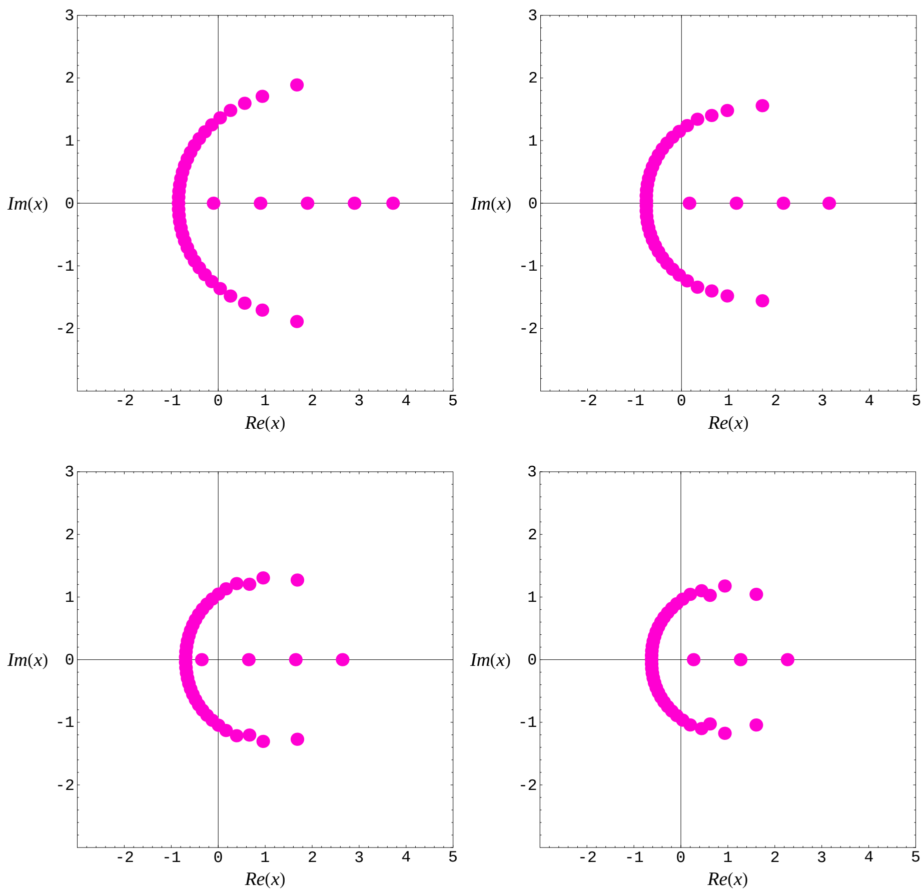

Figure 1: Zeros of q-Euler polynomials $E_{40}(x,h|\frac{1}{2}), h = 1,3,5,7$

The behavior of the zero of q-Euler polynomials $E_m(x,h|q), m = 40, q = 1/2, x \in \mathbb{C}$ for h is presented (Figure 3).

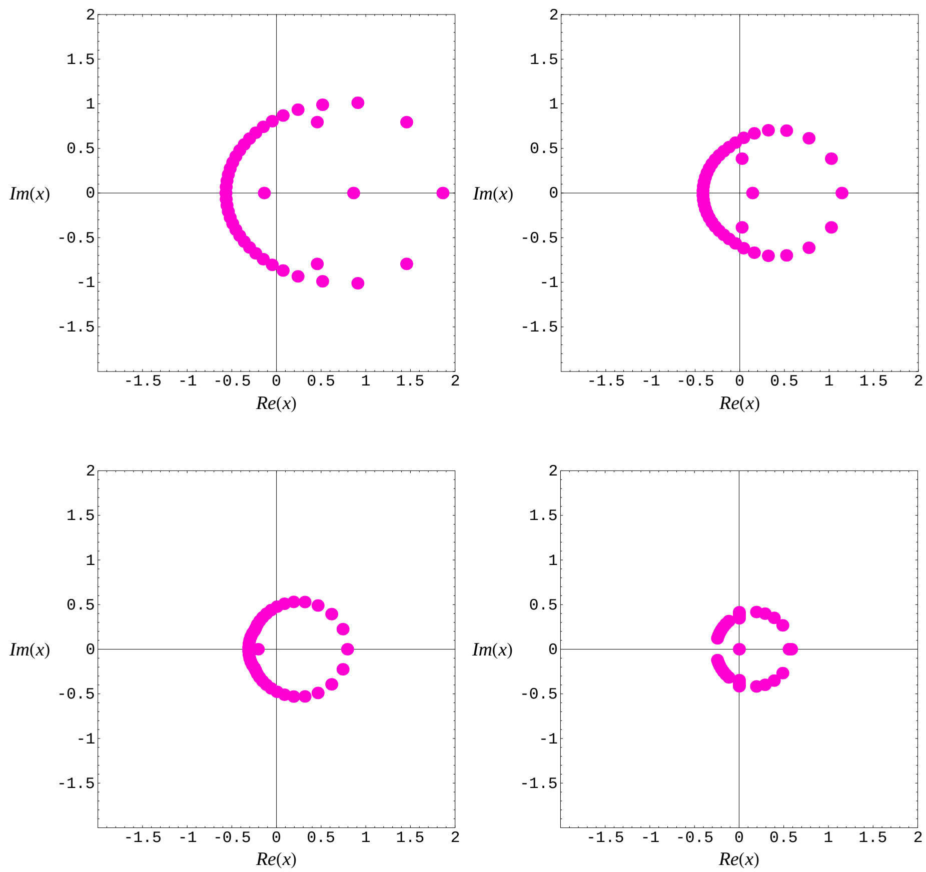

Figure 2: Zeros of q-Euler polynomials $E_{40}(x,h|\frac{1}{2}), h = 10, 20, 30, 40$

We plot the zeros of q-Euler polynomials are solutions of $E_m(x,h|q), m = 40, q = -1/2, x \in \mathbb{C}$(Figure 4).

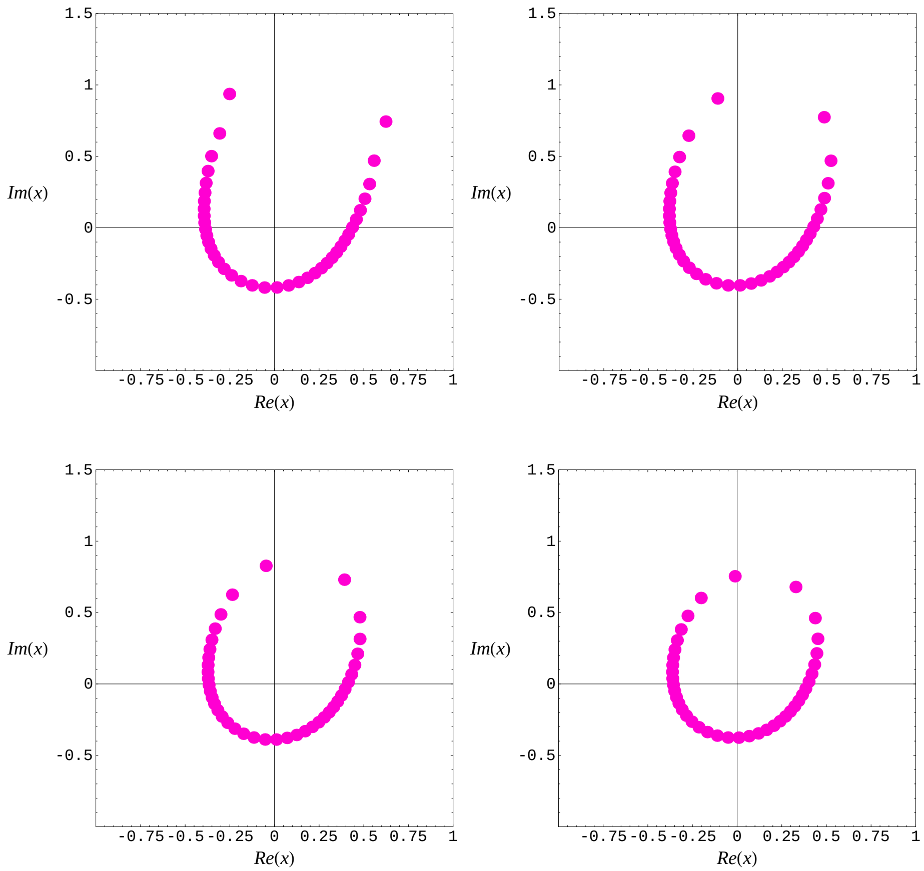

Figure 3: Zeros of q-Euler polynomials $E_{40}(x, h| - \frac{1}{2}), h = 1, 3, 5, 7$

The behavior of the zero of q-Euler polynomials $E_m(x, h|q), m = 40, q = -1/2, x \in \mathbb{C}$ for h is presented (Figure 5).

4 q-Euler zeta function

It was known that the Euler polynomials are defined as

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n, \quad |t| < \pi. \tag{4.1}$$

For $s \in \mathbb{C}, x \in \mathbb{R}$ with $0 \leq x < 1$, define

$$\zeta_E(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s}, \text{ and } \zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \tag{4.2}$$

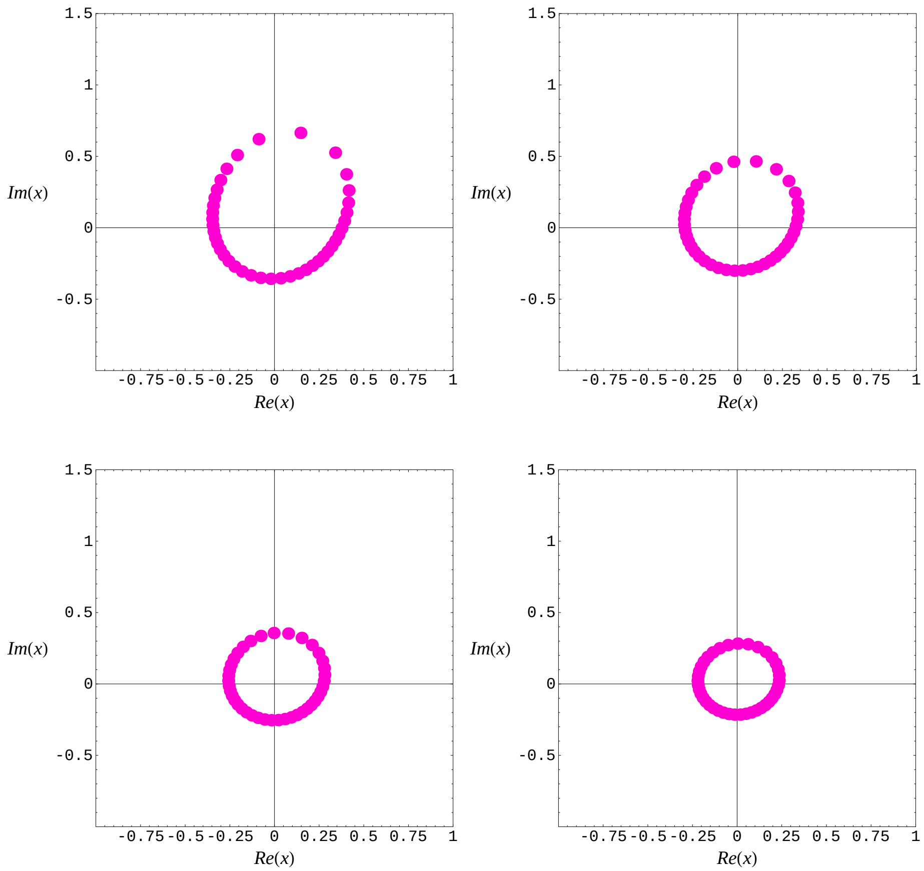

Figure 4: Zeros of q-Euler polynomials $E_{40}(x, h| -\frac{1}{2}), h = 10, 20, 30, 40$

Euler numbers are related to the Euler zeta function as

$$\zeta_E(-n) = E_n, \quad \zeta_E(-n, x) = E_n(x).$$

For $s, q, h \in \mathbb{C}$ with $|q| < 1$, we define q-Euler zeta function as follows:

$$\zeta_{E,q}(s, x|h) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{[n+x]_q^s}, \text{ and } \zeta_{E,q}(s|h) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^{nh}}{[n]_q^s}. \tag{4.3}$$

For $k \in \mathbb{N}, h \in \mathbb{Z}$, we have

$$\zeta_{E,q}(-n|h) = E_n(h|q).$$

In the special case $h = 0$, $\zeta_{E,q}(s|0)$ will be written as $\zeta_{E,q}(s)$. For $s \in \mathbb{C}$, we note that

$$\zeta_{E,q}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_q^s}.$$

We now consider the function $E_q(s)$ as the analytic continuation of Euler numbers. All the q-Euler numbers $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of Euler numbers evaluated at n,

$$E_q(n) = E_{n,q} \text{ for } n \geq 0.$$

Ordinary Euler numbers are defined by

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi. \tag{4.4}$$

From the definition of Euler numbers, it is easy to show that

$$E_0 = 1, \text{ and } E_n = -\frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l, \ n = 0, 1, 2, \cdots .$$

From (4.1), we can consider the q-extension of Euler numbers E_n as follows:

$$E_{0,q} = \frac{[2]_q}{2}, \text{ and } E_{n,q} = -\frac{1}{[2]_{q^n}} \sum_{l=0}^{n-1} \binom{n}{l} q^l E_{l,q}, n = 1, 2, 3, \cdots , \tag{4.5}$$

In fact, we can express $E_q'(s)$ in terms of $\zeta_{E,q}'(s)$, the derivative of $\zeta_{E,q}(s)$.

$$E_q(s) = \zeta_{E,q}(-s), E_q'(s) = \zeta_{E,q}'(-s), E_q'(2n+1) = \zeta_{E,q}'(-2n-1), \tag{4.6}$$

for $n \in \mathbb{N} \cup \{0\}$. This is just the differential of the functional equation and so verifies the consistency of $E_q(s)$ and $E_q'(s)$ with $E_{n,q}$ and $\zeta(s)$.

From the above analytic continuation of q-Euler numbers, we derive

$$\begin{aligned} &E_q(s) = \zeta_{E,q}(-s), E_q(-s) = \zeta_{E,q}(s) \\ &\Rightarrow E_{-n,q} = E_q(-n) = \zeta_{E,q}(n), n \in \mathbb{Z}_+. \end{aligned} \tag{4.7}$$

The curve $E_q(s)$ runs through the points $E_{-n,q}$ and grows $\sim n$ asymptotically as $(-n) \to -\infty$. The curve $E_q(s)$ runs through the point $E_q(-n)$ and $\lim_{n\to\infty} E_q(-n) = \lim_{n\to\infty} \zeta_{E,q}(n) = -2$ (Figure 7). From these results, we note that

$$\zeta_{E,q}(-n) = E_q(n) \mapsto \zeta_{E,q}(-s) = E_q(s).$$

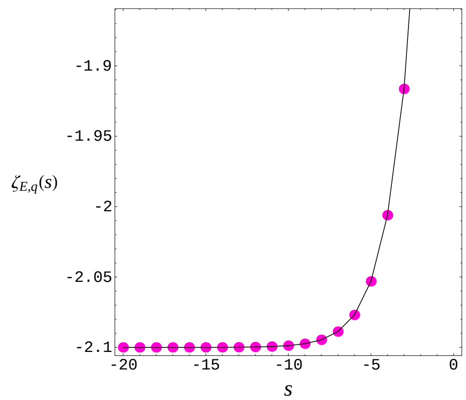

Figure 5: The curve of $E_q(s)$ runs through the point$\zeta_{n,q}, q = 11/10$

5 Analytic continuation of q-Euler polynomials

For consistency with the redefinition of $E_{n,q} = E_q(n)$ in (4.5) and (4.6), we have

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} E_{k,q} q^{kx} [x]_q^{n-k}.$$

The analytic continuation can be then obtained as

$$\begin{aligned}
&n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C}, \\
&E_{k,q} \mapsto E_q(k + s - [s]) = \zeta_{E,q}(-(k + (s - [s]))), \\
&\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)} \\
&\Rightarrow E_{n,q}(s) \mapsto E_q(s, w) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s) E_q(k + s - [s]) q^{(k+s-[s])w} [w]_q^{[s]-k}}{\Gamma(1 + k + (s - [s]))\Gamma(1 + [s] - k)} \\
&= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s) E_q((k-1) + s - [s]) q^{((k-1)+s-[s])w} [w]_q^{[s]+1-k}}{\Gamma(k + (s - [s]))\Gamma(2 + [s] - k)},
\end{aligned}$$

where $[s]$ gives the integer part of s, and so $s - [s]$ gives the fractional part.

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An existence and uniqueness result for a boundary value problem associated to a semilinear PDE

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Abstract - In this paper we investigate a boundary value problem for a semilinear equation of the form $-\Delta u + \lambda u + f(u) = g$, when the nonlinearity f satisfies a Lipschitz condition

Key words and phrases: *maximal monotone operator, strongly positive operator, strongly monotone operator, Lipschitz operator, Minty-Browder theorem, Banach fixed point theorem*

Mathematics Subject Classification (2000): 35J65, 47H05, 47J05

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ be a bounded domain and $g \in L^2(\Omega)$. We consider the boundary value problem

$$-\Delta u(x) + \lambda u(x) + f(u(x)) = g(x); \qquad x \in \Omega; \tag{1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{2}$$

where $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfies the Lipschitz condition $|f(u) - f(v)| \leqslant \alpha |u - v|$ for all $u, v \in \mathbf{R}$ $(\alpha > 0)$, $f(0) = 0$ and Δ is the Laplacian operator. We assume that the positive parameter λ satisfies the condition $\lambda > \alpha$.

The problems of the type $(1), (2)$ are motivated by the stationary diffusion phenomenon and have been investigated by many authors.

In a great number of papers the problem $(1), (2)$ is studied when the nonlinearity f satisfies an inequality of the type $|f(u)| \leq a |u|^p + b$, or in the case $f(u) = |u|^p$ for some $p \in \mathbf{R} - \{1\}$.

In this paper we investigate the existence and the uniqueness of the solution of the problem (1), (2), when the nonlinearity f satisfies only a Lipschitz condition.

We will prove using the Minty-Browder theorem, that the problem (1), (2), in the said conditions for f, g and the positive parameters λ and α, has a unique weak solution. The proof of the principal result of this paper uses essentially the monotonicity properties of the linear differential operator generated by the term $-\Delta u$ of the equation (1).

Theorem 1.1. *Let* $f : \mathbf{R} \longrightarrow \mathbf{R}$, $g : \Omega \longrightarrow \mathbf{R}$ *and* λ, α *positive parameters so that:*

(i) $|f(x) - f(y)| \leqslant \alpha |x - y|$ *for all* $x, y \in \mathbf{R}$*;*
(ii) $f(0) = 0$*;*
(iii) $g \in L^2(\Omega)$*;*
(iv) $\lambda > \alpha$.

Then the problem (1), (2) *has a unique weak solution.*

2. Proof of Theorem 1.1.

We denote by E the real Hilbert space $L^2(\Omega)$. The inner product and the corespondent norm in E will be denoted by $\langle \cdot, \cdot \rangle_2$ and $\|\cdot\|_2$.

Let $A : D(A) \subset E \longrightarrow E$ defined by $Au(x) = -\Delta u(x)$ for $x \in \Omega$, where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is known that the linear operator A is a maximal monotone operator.

Let $L : D(L) = D(A) \longrightarrow E$ defined by $Lu(x) = -\Delta u(x) + \lambda u(x)$ for $x \in \Omega$, i.e. $L = A + \lambda I$ where I is the identity of E.

$Rg(I + \theta A) = \{u + \theta Au / u \in D(A)\} = E$ for all $\theta > 0$, because A is a maximal monotone operator. It results that $Rg(A + \lambda I) = Rg(L) = E$, because A is linear.

Also we have

$$\langle Lu, u \rangle_2 = \langle Au, u \rangle_2 + \lambda \langle u, u \rangle_2 \geqslant \lambda \langle u, u \rangle_2 = \lambda \|u\|_2^2 \tag{3}$$

for all $u \in D(L)$, i.e. L is a strongly positive linear operator.

Let $F : E \longrightarrow E$ defined by $Fu(x) = f(u(x))$; $x \in \Omega$ (the definition is correct, because from the properties of f, it results that $Fu \in L^2(\Omega)$ for all $u \in L^2(\Omega)$). We have

$$\|Fu - Fv\|_2^2 = \int_\Omega |f(u(x)) - f(v(x))|^2 \, d\mu \leqslant \alpha^2 \int_\Omega |u(x) - v(x)|^2 \, d\mu = \alpha^2 \|u - v\|_2^2$$

for all $u, v \in E$(μ is the Lebesgue measure in $\mathbf{R}^N$). It results that the nonlinear operator F is a Lipschitz operator with the constant α.

Now we can writte the problem $(1),(2)$ in the equivalently operatorial form

$$Lu + Fu = g. \tag{4}$$

From (3) we obtain

$$\|Lu\|_2 \geq \lambda \|u\|_2 \quad \text{for all } u \in D(L).$$

Consequently there exists $L^{-1} : E \longrightarrow D(L) \subset E$ which is linear and bounded, $L^{-1} \in \mathcal{L}(E)$, the Banach space of all linear and bounded operators from E to E. Moreover we have

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda}.$$

The equation (4) can be written now as

$$u + L^{-1}Fu = L^{-1}g. \tag{5}$$

We consider the operator $T : E \longrightarrow E$ defined by

$$Tu = u + L^{-1}Fu = \left(I + L^{-1}F\right)u.$$

Therefore the equation (5) becomes

$$Tu = L^{-1}g, \tag{6}$$

and our problem is reduced to the study of the properties of the operator T. We will prove now that the operator T satisfies an inequality of the type

$$\langle Tu - Tv, u - v\rangle_2 \geq \beta \|u - v\|_2^2 \text{ for all } u, v \in E \ \ (\beta > 0)$$

i.e. T is a strongly monotone operator. With the Cauchy-Schwartz inequality we obtain

$$-\left\langle L^{-1}Fu - L^{-1}Fv, u - v\right\rangle_2 = -\left\langle L^{-1}(Fu - Fv), u - v\right\rangle_2 \leq$$
$$\leq \left|\left\langle L^{-1}(Fu - Fv), u - v\right\rangle_2\right| \leq$$
$$\leq \left\|L^{-1}(Fu - Fv)\right\|_2 \cdot \|u - v\|_2 \leq \frac{\alpha}{\lambda}\|u - v\|_2^2$$

and then

$$\langle Tu - Tv, u - v\rangle_2 = \left\langle u + L^{-1}Fu - v - L^{-1}Fv, u - v\right\rangle_2 =$$
$$= \|u - v\|_2^2 + \left\langle L^{-1}Fu - L^{-1}Fv, u - v\right\rangle_2$$
$$\geq \|u - v\|_2^2 - \frac{\alpha}{\lambda}\|u - v\|_2^2 = \left(1 - \frac{\alpha}{\lambda}\right)\|u - v\|_2^2,$$

for all $u, v \in E$. From the assumption $\lambda > \alpha$ it results that $1 - \frac{\alpha}{\lambda} > 0$ and our assertion is proved.

It is known that every strongly monotone operator is coercive (i.e. $\lim_{\|u\|_2 \to \infty} \frac{\langle Tu,u \rangle_2}{\|u\|_2} = \infty$). It's easy to observe that T is continuous and strictly monotone in the sense that $\langle Tu - Tv, u - v\rangle_2 > 0$ for all $u, v \in E$ with $u \neq v$. According to the Minty-Browder theorem, the equation (6) has a unique solution. Therefore the proof of the Theorem 1.1. is complete.

3. Remarks

1. The conclusion of the Theorem 1.1. can be established also using the Banach fixed point theorem. The operatorial form (5) of the studied problem can be written as $Vu = u$, where the operator $V : E \to E$ is defined by $Vu = -L^{-1}Fu + L^{-1}g$. We have

$$\|Vu - Vv\|_2 = \left\|L^{-1}Fu - L^{-1}Fv\right\|_2 = \left\|L^{-1}(Fu - Fv)\right\|_2 \leq$$

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \|Fu - Fv\|_2 \leq \frac{\alpha}{\lambda} \|u - v\|_2 \quad \text{for all } u, v \in E.$$

It results that V is a strict contraction from E to E because $\lambda > \alpha$. According to the Banach fixed point theorem, V has a unique fixed point, and this fact justifies the existence of the unique solution of the problem $(1), (2)$.

2. The Theorem 1.1. implies the fact that the problem

$$-\Delta u(x) + \lambda u(x) + f(u(x)) = 0; \qquad x \in \Omega;$$

$$u = 0 \text{ on } \partial\Omega$$

has only the trivial solution $u \equiv 0$, for all $\lambda > \alpha$.

It results that there are no eigenvalues of the nonlinear eigenvalues problem

$$\Delta u(x) - f(u(x)) = \lambda u(x); \qquad x \in \Omega;$$

$$u = 0 \text{ on } \partial\Omega$$

in the interval $(\alpha, +\infty)$.

3. For fixed $g \in L^2(\Omega)$, let $u(\lambda)$ be the unique weak solution of the problem $(1), (2)$ for all $\lambda > \alpha$. From (5) we obtain

$$\|u(\lambda)\|_2 = \left\|-L^{-1}Fu(\lambda) + L^{-1}g\right\|_2 \leq$$

$$\left\|L^{-1}\right\|_{\mathcal{L}(E)} \|Fu(\lambda)\|_2 + \left\|L^{-1}\right\|_{\mathcal{L}(E)} \|g\|_2 \leq \frac{\alpha}{\lambda} \|u(\lambda)\|_2 + \frac{1}{\lambda} \|g\|_2 .$$

So we have obtained the following estimation:

$$\|u(\lambda)\|_2 \leq \frac{1}{\lambda - \alpha} \|g\|_2 .$$

It results that $\|u(\lambda)\|_2 \longrightarrow 0$ when $\lambda \longrightarrow \infty$ and this fact signifies that for large values of λ, the solution $u(\lambda)$ has only very small values.

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COMMON FIXED POINT RESULTS IN FUZZY METRIC SPACE WITH NON- COMPATIBLE CONDITION

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Abstract

In this paper we obtained common fixed point theorems in fuzzy metric space under the Lipschitz type analogue of a strict contractive condition by using the notion of weak compatible of type A. In the setting of our results, we provide pair of mappings, which ensures the existence of a common fixed point.

Key words and Phrases: Weak compatible of type A, Contractive condition, Common fixed point.
AMS Subject Classification: 54H25.

Introduction

The study of fixed points for non-compatible mappings can be extended to the class of non-expansive or Lipschitz mapping pairs even without continuity of the mappings involved or completeness of the space.

The notion of weak commutativity generalized by Junjck [3] and another generalization is given by Pant [6] as R-weak commutative of type (A_g). Two self maps S and T of a metric space X are called *compatible* if $\lim_n d(STx_n,TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Tx_n = \lim_n Sx_n = p$ for some p in X. It follows that the maps S and T are called *non-compatible* if they are not compatible. Thus S and T are non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n Tx_n = \lim_n Sx_n = p$ for some p in X but $\lim_n d(STx_n,TSx_n) \neq 0$ or $\lim_n d(STx_n,TSx_n)$ does not exists.

In 1965, Zaded introduce the notation of fuzzy set. George and Veermani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [4]. They also showed that every metric space induces a fuzzy metric.

Definition 1.1[7]. A binary operation: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if ([0,1],) is an abelian topological monoid with unit 1 such that $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2 [4]. The 3-tuple (X,M,*) is called a fuzzy metric space (FM space) if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions; for all x, y, z in X and s, t >0,

(FM-1) $M(x,y,0) = 0$,

(FM-2) $M(x,y,t) = 1 \ \forall \ t > 0$ iff $x = y$,

(FM-3) $M(x,y,t) = M(y,x,t)$,

(FM-4) $M(x,y,t)*M(x,z,s) \leq M(x,z,t+s)$,

(FM-5) $M(x,y,.):[0,1) \rightarrow [0,1]$ is left continuous.

Note that M(x,y,t) can be thought of as the degree of nearness between x and y with respect to t. We identify x = y with M(x,y,t) = 1 for all t > 0 and M(x,y,t) = 0 with ∞ and we can fined some topological properties and example of fuzzy metric space .

Lemma 1.1 ([2]) For all x, y in X M(x,y,t) be non-decreasing.

Definition 1.3 ([2]) Let (X,M,*) be a fuzzy metric space:

(1) A sequence $\{x_n\}$ in X is said to be a convergent to a point x in X if $\lim_n M(x_n, x, t) = 1$ for all $t > 0$

(2) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_n M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

If in a fuzzy metric space every cauchy sequence is convergent then the FM space is complete.

Remark: Since * is continuous, it follows from FM-4 that the limit of the sequence in FM space is uniquely determined.

FM 6: Let (X,M,*) be a fuzzy metric space then $\lim_t M(x_n, x, t) = 1 \ \forall$ x, y in X.

Lemma 1.2([5]) If all $x, y \in X$, $t > 0$ and for a number $k \in (0,1)$,

$M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Definition 1.4 Let $\{x_n\}$ be a sequence in FM space (X,M,*). The self maps S and T are called *non-compatible* if there exists at least one sequence $\{x_n\}$ in X such that $\lim_n Tx_n = \lim_n Sx_n = p$ for some p in X but $\lim_n M(STx_n,TSx_n,t) \neq 1$ or $\lim_n d(STx_n,TSx_n,t)$ does not exists.

Main Result

Theorem 1. Let S and T be non-compatible self mappings of a fuzzy metric space (M,X,*) such that

(I) $\overline{T(X)} \subset S(X)$ where $\overline{T(X)}$ is closure of the range of T

(II) M(Tx,Ty,kt)

$$\geq \min\left\{M(Sx,Sy,t), M(Sx,Ty,t), \frac{M(Tx,Sy,t)*M(Ty,Sy,t)}{M(Tx,Sx,t)*M(Ty,Sx,t)}, \frac{M(Tx,Sx,t)*M(Ty,Sx,t)}{M(Tx,Sy,t)*M(Ty,Sy,t)} -\right\}$$

If T and be weak compatible of type A, then S and T have a unique common fixed point.

Proof : Since S and T are non-compatible mappings the there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p \qquad \text{for some } p \in X \ldots\ldots\ldots \quad (1)$$

but either $\lim_{n\to\infty} M(TSx_n, STx_n,t) \neq 1$ or limit does not exist. Again $\overline{T(X)} \subset S(X)$ and $p \in \overline{T(X)}$ therefore there exists u in X such that Su = p , (where $\lim_{n\to\infty} Sx_n = p$). If $Tu \neq Su$ then by (II)

M(Tx_n,Tu,kt)

$$\geq \min\left\{M(Sx_n,Su,t), M(Sx_n,Tu,t), \frac{M(Tx_n,Su,t)*M(Tu,Su,t)}{M(Tx_n,Sx_n,t)*M(Tu,Sx_n,t)}, \frac{M(Tx_n,Sx_n,t)*M(Tu,Sx_n,t)}{M(Tx_n,Su,t)*M(Tu,Su,t)} -\right\}$$

On taking limit $n\to\infty$

M(Su,Tu,kt)

$$> \min\left\{M(Su,Su,t), M(Su,Tu,t), \frac{M(Su,Su,t)*M(Tu,Su,t)}{M(Su,Su,t)*M(Tu,Su,t)}, \frac{M(Su,Su,t)*M(Tu,Su,t)}{M(Su,Su,t)*M(Tu,Su,t)} -\right\}$$

$\Rightarrow$ M(Su,Tu,kt)> M(Su,Tu,t) $\Rightarrow$ Su = Tu.

Now, S and T are weak compatible of type (A), therefore TTu = STu.

If Tu $\neq$ TTu the by (II)

M(Tu,TTu,kt)

$$> \min\left\{M(Su,STu,t), M(Su,TTu,t), \frac{M(Tu,STu,t)*M(TTu,STu,t)}{M(Tu,Su,t)*M(TTu,Su,t)}, \frac{M(Tu,Su,t)*M(Su,TTu,t)}{M(Tu,STu,t)*M(TTu,STu,t)} - \right\}$$

$$> \min\left\{M(Tu,TTu,t), M(Tu,TTu,t), \frac{M(Tu,TTu,t)*M(TTu,TTu,t)}{M(Tu,Tu,t)*M(TTu,Tu,t)}, \frac{M(Tu,Tu,t)*M(Tu,TTu,t)}{M(Tu,TTu,t)*M(TTu,TTu,t)} - \right\}$$

M(Tu,TTu,kt) > M(Tu,TTu,t).

$\Rightarrow$ Tu = TTu = STu.

Hence, Tu is common fixed point of S and T.

Uniqueness: Let for u $\neq$ v, Tu and Tv are two common fixed points of S and T.

Assume that Tu $\neq$ Tv, then by (II)

M(Tu,Tv,kt)

$$> \min\left\{M(Su,Sv,t), M(Su,Tv,t), \frac{M(Tu,Sv,t)*M(Tv,Sv,t)}{M(Tu,Su,t)*M(Tv,Su,t)}, \frac{M(Tu,Su,t)*M(Su,Tv,t)}{M(Tu,Sv,t)*M(Tv,Sv,t)} - \right\}$$

$$= \min\left\{M(Tu,Tv,t), M(Tu,Tv,t), \frac{M(Tu,Tv,t)*M(Tv,Tv,t)}{M(Tu,Tu,t)*M(Tv,Tu,t)}, \frac{M(Tu,Tu,t)*M(Tu,Tv,t)}{M(Tu,Tv,t)*M(Tv,Tv,t)} - \right\}$$

M(Tu,Tv,kt) > M(Tu,Tv,t) $\Rightarrow$ Tu = Tv.

In the theorem, we make modification in the condition of theorem 1 with R-weak commutative of type (A_g),we get discontinuity at common fixed point. Let S and T satisfying following condition:

$$\lim_{n\to\infty} TTx_n = Tp \quad \text{and} \quad \lim_{n\to\infty} STx_n = Sp \ldots\ldots\ldots\ldots(2),$$

whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = p$ for some p in X

Theorem 2. Let S and T be non-compatible self mappings of a fuzzy metric space (M,X,*) satisfying (II) of theorem 1 and the above condition (2). If T and S were R-weak commutative of type (A_g), then S and T have a unique common fixed point and fixed point is a fixed point of discontinuity.

Proof: Since S and T are non-compatible mappings the there exists a sequence $\{x_n\}$ in X such that by (1) $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = p$ for some $p \in X$

but either $\lim_{n\to\infty} Sx_n$ $M(TSx_n, STx_n, t) \neq 1$ or limit does not exist. Again by (2) we have $\lim_{n\to\infty} TTx_n = Tp$ and $\lim_{n\to\infty} STx_n = Sp$.

Further, R-weak commutative of type (A_g) yields

$M(TTx_n, STx_n, kt) \geq R\, M(Tx_n, Sx_n, t)$

On taking limit, we get Tp = Sp and TTp = STp.

Now, we claim that TTp=Tp. Let TTp≠Tp, then by (II)

$$M(Tp,TTp,kt)$$

$$> \min\left\{M(Sp,STp,t), M(Sp,TTp,t), \frac{M(Tp,STp,t)*M(TTp,STp,t)}{M(Tp,Sp,t)*M(TTp,Sp,t)}, \frac{M(Tp,Sp,t)*M(Sp,TTp,t)}{M(Tp,STp,t)*M(TTp,STp,t)} -\right\}$$

$= M(Tp,TTp,t)$

Which is contradiction, therefore TTp=Tp.

$\Rightarrow$ Tp = TTp = STp. Hence, Tp is common fixed point of T and S.

Now, we show that S and T are discontinuous at the common fixed point Tp=Sp=p. If possible, suppose T is continuous. Then considering sequence $\{x_n\}$ of (1) we get $\lim_{n\to\infty} TTx_n = Tp = p$. R-weak commutativity of type (A_g) implies that

$M(TTx_n, STx_n, kt) \geq R\, M(Tx_n, Sx_n, t)$.

On taking limit $n\to\infty$ this yields $\lim_{n\to\infty} STx_n = Tp = p$.

Therefore, $\lim_{n\to\infty} M(TSx_n, STx_n, t) = 1$. This contradiction the fact that either $\lim_{n\to\infty} M(TSx_n, STx_n, t) \neq 1$ or nonexistent for the sequence $\{x_n\}$ of (1). Hence T is discontinuous at the fixed point. Next, suppose that S is continuous. Then for the sequence $\{x_n\}$ of (1), we get $\lim_{n\to\infty} STx_n = Sp = p$ and $\lim_{n\to\infty} SSx_n = Sp = p$. In view of these limits, the inequality $M(Tp,TSx_n,kt)$

$$\geq \min\left\{M(Sp,Sx_n,t), M(Sp,Tx_n,t), \frac{M(Tp,SSx_n,t)*M(Tx_n,SSx_n,t)}{M(Tp,Sp,t)*M(TSx_n,Sp,t)}, \frac{M(Tp,Sp,t)*M(TSx_n,Sp,t)}{M(Tp,SSx_n,t)*M(TSx_n,SSx_n,t)} -\right\}$$

yields a contradiction unless $\lim_{n\to\infty} TSx_n = Tp = Sp$. But $\lim_{n\to\infty} TSx_n = Sp$ and $\lim_{n\to\infty} STx_n = Sp$ contradicts the fact that either $\lim_{n\to\infty} M(TSx_n, STx_n,t) \neq 1$ or nonexistent for the sequence $\{x_n\}$ of (1). Hence both T and S are discontinuous at their common fixed point.

Example1: Let X = [1,10] and M(x, y,t) = $\frac{t}{t+d(x,y)}$ and d is usual metric on X.

Define S,T:X→X by Tx = $\begin{cases} 1 & \text{if } 1 \le x < 3 \\ \frac{2+x}{4} & \text{if } x \ge 3 \end{cases}$ and Sx = $\begin{cases} \frac{x^2+1}{2} & \text{if } 1 \le x < 2 \\ \frac{2x+1}{5} & \text{if } x \ge 2 \end{cases}$

Also consider the sequence $x_n = 2+(1/n)$. $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$.

$\lim_{n\to\infty} TSx_n = 3/4$ and $\lim_{n\to\infty} STx_n = 3/5$.

Clearly, T & S are noncompatible but T & S are weak compatible of type (A).

$\Rightarrow$ TT2=ST2 and T2 = S2 = 1 (also TT1=ST1 & T1=S1=1). We observe that S and T satisfying the conditions of **theorem 1** and hence **1** is the fixed point.

Example2: Let X = [1,10] and M(x, y,t) = $\frac{t}{t+d(x,y)}$ [by 1] and d is usual metric on X. Define S,T:X→X by Tx = $\begin{cases} 1 & \text{if } x = 1 \text{ or } x > 3 \\ 4 & \text{if } 1 < x \le 3 \end{cases}$ and

Sx = $\begin{cases} 1 & \text{if } x = 1 \\ 5 & \text{if } 1 < x \le 3 \\ \frac{x+1}{4} & \text{if } x > 3 \end{cases}$ Also consider the sequence $\{x_n = 3+(1/n): n\geq 1\}$.

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$. $\lim_{n} TSx_n = 4$ and $\lim_{n} STx_n = 1$. Clearly, T & S are noncompatible but it can be verify T & S are R-weak commutative of type (A_g). We observe that S and T satisfying the conditions of **theorem 2** and hence **1** is the fixed point.

Conclusion: Firstly, we ensure the unique fixed point without compatible mappings but weak compatible of type A with Lipschiz type analogue of a plane contractive. Secondly, we provide a pair of mappings (R-weak commutative of type (A_g)) which ensures the existence of a common fixed point, however, both the mappings are discontinuous at the common fixed point.

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Solving second order ordinary differential equations with constant coefficients by Adomian decomposition method *

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Abstract

In this paper, an efficient modification of Adomian decomposition method is introduced for solving second-order ordinary differential equations with constant coefficients. The proposed method can be applied to linear and nonlinear problems. Some examples were presented to show the ability of the method for linear and nonlinear ordinary differential equations.

Keywords: Adomian decomposition method; second-order ordinary differential equations .

1 Introduction

In this paper, we consider the second order ordinary differential equation with constant coefficients of the form

$$y'' + ay' + by = g(x) + f(x, y), \tag{1}$$

$$y(0) = A, y'(0) = B.$$

*

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Where $f(x,y)$ is nonlinear function , $g(x)$ is given function and A, B, a, b are constants. The purpose of this paper to introduce a new differential operator to study the problem(1).

In recent years a large amount of literature developed concerning Adomian decomposition method [1-5,7], and the related modification [6,8,9,11]to investigate various scientific models. The method is tested for some examples.

2 Analysis of the method

Under the transformation $a = 2n+k$ and $b = n(n+k)$ the equation (1) is transformed to

$$y'' + (2n+k)y' + n(n+k)y = g(x) + f(x,y), \tag{2}$$

where n,k are constants.
We propose the new differential operator, as below

$$L() = e^{-nx}\frac{d}{dx}e^{-k}\frac{d}{dx}e^{n+k}(), \tag{3}$$

so, the problem (2) can be written as,

$$Ly = g(x) + f(x,y). \tag{4}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(.) = e^{-(n+k)x}\int_0^x e^{kx}\int_0^x e^{nx}(.)dxdx. \tag{5}$$

Applying L^{-1} of (5) to the third terms $y'' + (2n+k)y' + n(n+k)y$ of Eq.(2) we find

$$L^{-1}(y'' + (2n+k)y' + n(n+k)y$$
$$= e^{-(n+k)x}\int_0^x e^{kx}\int_0^x e^{nx}(y'' + (2n+k)y' + n(n+k)y)dxdx$$
$$= e^{-(n+k)x}\int_0^x e^{kx}(e^{nx}y' + (n+k)e^{nx}y - y'(0) - (n+k)y(o))dx$$

$$= y - \frac{1}{k}y^{'}(0)e^{-nx} - \frac{(n+k)}{k}y(0)e^{-nx} + \frac{1}{k}y^{'}(0)e^{-(n+k)x} + \frac{n}{k}y(0)e^{-(n+k)x}.$$

Operating with L^{-1} on (4), it follows

$$y(x) = \frac{1}{k}y^{'}(0)e^{-nx} + \frac{(n+k)}{k}y(0)e^{-nx} - \frac{1}{k}y^{'}(0)e^{-(n+k)x} - \frac{n}{k}y(0)e^{-(n+k)x}$$

$$+L^{-1}g(x) + L^{-1}f(x,y). \tag{6}$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $f(x,y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{7}$$

and

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \tag{8}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [7,10] to formulate Adomian polynomials. The following algorithm:

$$A_0 = F(u),$$

$$A_1 = F^{'}(u_0)u_1,$$

$$A_2 = F^{'}(u_0)u_2 + \frac{1}{2}F^{''}(u_0)u_1^2,$$

$$A_3 = F^{'}(u_0)u_3 + F^{''}(u_0)u_1u_2 + \frac{1}{3!}F^{'''}(u_0)u_1^3, \tag{9}$$

$$.$$

$$.$$

$$.$$

can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting(7)and(8) into (6),

$$\sum_{n=0}^{\infty} y_n = \frac{1}{k}y^{'}(0)e^{-nx} + \frac{(n+k)}{k}y(0)e^{-nx} - \frac{1}{k}y^{'}(0)e^{-(n+k)x} - \frac{n}{k}y(0)e^{-(n+k)x}$$

$$+L^{-1}g(x) + L^{-1}\sum_{n=0}^{\infty} A_n. \tag{10}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$\begin{aligned} y_0 &= \frac{1}{k}y^{'}(0)e^{-nx} + \frac{(n+k)}{k}y(0)e^{-nx} - \frac{1}{k}y^{'}(0)e^{-(n+k)x} - \frac{n}{k}y(0)e^{-(n+k)x} \\ &\quad +L^{-1}g(x), \qquad (11) \\ y_{n+1} &= L^{-1}A_n, n \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} y_0 &= \frac{1}{k}y^{'}(0)e^{-nx} + \frac{(n+k)}{k}y(0)e^{-nx} - \frac{1}{k}y^{'}(0)e^{-(n+k)x} - \frac{n}{k}y(0)e^{-(n+k)x} \\ &\quad +L^{-1}g(x), \\ y_1 &= L^{-1}A_0, \\ y_2 &= L^{-1}A_1, \\ y_3 &= L^{-1}A_3, \qquad (12) \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

From (9) and (12), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (7) can be immediately obtained. For numerical purposes, the n-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} y_k, \tag{13}$$

can be used to approximate the exact solution. The approach presented above can be validated by testing it on a variety of several linear and nonlinear initial value problems.

3 Numerical illustrations

Example 1. We consider the linear homogenous initial value problem :

$$y^{''} - 2y^{'} + 2y = 0, \tag{14}$$

$$y(0) = 0, y^{'}(0) = 5.$$

We put $2n + k = -2$ and $n(n + k) = 2$,
it follows that $k = \mp 2i$, $n = -1 \pm i$, where $i = \sqrt{-1}$,
substitution of $k = 2i$ and $n = -1 - i$ in Eq.(3) yields the operator

$$L() = e^{(1+i)x} \frac{d}{dx} e^{-2ix} \frac{d}{dx} e^{(-1+i)x}(),$$

so

$$L^{-1}(.) = e^{(1-i)x} \int_0^x e^{2ix} \int_0^x e^{(-1-i)x}(.)dxdx.$$

In an operator form, Eq.(14) becomes

$$Ly = 0. \tag{15}$$

Applying L^{-1} on both sides of(15) we find

$$L^{-1}Ly = 0,$$

and it implies,

$$y = \frac{1}{2i} y^{'}(0) e^{(1+i)x} + \frac{-1+i}{2i} y(0) e^{(1+i)x} + y(0) e^{(1-i)x} - \frac{1}{2i} y^{'}(0) e^{(1-i)x} - \frac{-1+i}{2i} y(0) e^{(1-i)x}$$
$$= \frac{5i}{2} e^x (\cos x - i \sin x) - \frac{5i}{2} e^x (\cos x + i \sin x) = 5e^x \sin x$$

So, the exact solution is easily obtained by this method.

Example 2. We consider the linear non-homogenous initial value problem:

$$y^{''} - 3y^{'} + 2y = x, \tag{16}$$

$$y(0) = 1, y^{'}(0) = 0.$$

We put $2n+k=-3$ and $n(n+k)=2$

it follows that $k=-1, k=1$ and $n=-1, n=-2$,

substitution of $k=-1$ and $n=-1$ in Eq.(3) yields the operator

$$L()=e^{x}\frac{d}{dx}e^{x}\frac{d}{dx}e^{-2x}(),$$

so

$$L^{-1}(.)=e^{2x}\int_0^x e^{-x}\int_0^x e^{-x}(.)dxdx.$$

In an operator form, Eq.(16) becomes

$$Ly=x. \tag{17}$$

Applying L^{-1} to both sides of (17) we find

$$L^{-1}Ly=e^{2x}\int_0^x e^{-x}\int_0^x e^{-x}(x)dxdx,$$

and it implies,

$$y(x)=-y^{'}(0)e^{x}+2y(0)e^{x}+y(0)e^{2x}+y^{'}(0)e^{2x}-2y(0)e^{2x}+\frac{3}{4}-e^{x}+\frac{1}{4}e^{2x}+\frac{x}{2}$$

$$\Longrightarrow y(x)=\frac{3}{4}+e^{x}-\frac{3}{4}e^{2x}+\frac{x}{2}.$$

So, the exact solution is easily obtained by this method.

Example 3. We consider the nonlinear initial value problem:

$$y^{''}-4y=8y\ln y, \tag{18}$$

$$y(0)=1, y^{'}(0)=0.$$

put $2n+k=0$ and $n(n+k)=-4$

it follows that $k=\mp 4, n=\pm 2$,

substitution of $k=-4$,$n=2$ in Eq.(3) yields the operator

$$L()=e^{-2x}\frac{d}{dx}e^{4x}\frac{d}{dx}e^{-2x}(),$$

so

$$L^{-1}(.)=e^{2x}\int_0^x e^{-4x}\int_0^x e^{2x}(.)dxdx.$$

According to (18) we have,

$$Ly = 8y \ln y,$$

proceeding as before we obtain

$$y_0 = -\frac{1}{4}y^{'}(0)e^{-2x} + \frac{1}{2}y(0)e^{-2x} + y(0)e^{2x} + \frac{1}{4}y^{'}(0)e^{2x} - \frac{1}{2}y(0)e^{2x}$$

$$= \frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x},$$

$$y_{n+1} = L^{-1}A_n, n \geq 0$$

when A_n's are Adomian polynomials of nonlinear term $y \ln y$ as below[6]

$$A_0 = y_0 \ln y_0,$$

$$A_1 = y_1(1 + \ln y_0,$$

$$A_2 = y_2((1 + \ln y_0) + \frac{y_1^2}{2}(1 + \frac{1}{y_0}),$$

.

.

.

It must be noted that, to compute y we use the Taylor series of $e^{\mp 2x}$ with order 8.In this case we obtain

$$y_0 = 1 + 2x^2 + \frac{2}{3}x^4 + \frac{4}{45}x^6 + \frac{2}{315}x^8 + ...$$

$$y_1 = \frac{4}{3}x^4 + \frac{8}{9}x^6 + \frac{8}{105}x^8 + \frac{16}{405}x^8 + ...$$

$$y_2 = \frac{16}{45}x^6 + \frac{8}{15}x^8 + \frac{16}{525}x^{10} + ...$$

$$y_3 = \frac{16}{315}x^8 + \frac{256}{945}x^{10} + \frac{608}{7425}x^{12}...$$

.

.

.

This means that the solution in a series form is given by

$$y(x) = y_0 + y_1 + y_2 + y_3 + ... = 1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \frac{2}{3}x^8 + ...,$$

and in the closed form

$$y(x) = e^{2x^2}.$$

4 Conclusion

Adomian decomposition method has been known to be a powerful device for solving many functional equation as algebraic equations, ordinary and partial differential equations, integral equations and so on . Here we used this method for solving second-order ordinary differential equation with constant coefficients. It is demonstrated that this method has the ability of solving equations of both linear and non-linear differential equations. For linear equations we derived the exact solutions and for nonlinear equations, we usually derive a very good approximations to the solutions, and some times the exact solution can be found.

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Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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