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INVERSE OBSTACLE SCATTERING WITH CONDUCTIVE BOUNDARY CONDITION FOR A COATED DIELECTRIC CYLINDER

AHMET ALTUNDAG

ABSTRACT. The inverse problem under consideration is to reconstruct the conductive function of a coated dielectric infinite cylinder from the far field pattern for scattering of a time-harmonic E-polarized electromagnetic plane wave. We propose an inverse algorithm that extends the approach suggested by Akduman and Kress [1] for an impedance cylinder embedded in an homogeneous background medium. It is based on a system of nonlinear boundary integral equation associated with a single-layer potential approach to solve the forward scattering problem. We present the mathematical foundations of the method and exhibit its feasibility by numerical examples.

1. INTRODUCTION

The problem is to determine the conductive function defined on the coated boundary from scattering of time-harmonic E-polarized electromagnetic plane waves. In the current paper we deal with dielectric scatterers covered by a thin boundary layer described by a conductive boundary condition and confine ourselves to the case of infinitely long cylinders.

Let the simply connected bounded domain $D \subset \mathbb{R}^2$ with C^2 boundary ∂D represents the cross section of an infinitely long homogeneous dielectric cylinder having constant wave number k_d with $Im\{k_d\}, \mathbb{R}e\{k_d\} \geqslant 0$ embedded in a homogeneous background with positive wave number k_0. Denote by ν the outward unit normal vector to ∂D. Then, given an incident plane wave $u^i = e^{ik_0 x.d}$ with incident direction given by the unit vector d, the direct scattering problem for E-polarized electromagnetic waves by a coated dielectric is modeled by the following conductive boundary value problem for the Helmholtz equation: Find solutions $u \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{D})$ and $v \in H^1(D)$ to the Helmholtz equations

$$\Delta u + k_0^2 u = 0 \quad \text{in } R^2 \setminus \bar{D}, \qquad \Delta v + k_d^2 v = 0 \quad \text{in } D \tag{1.1}$$

with the conductive boundary conditions

$$u = v, \qquad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} + i\eta v \quad \text{on} \quad \partial D \tag{1.2}$$

for some complex valued function $\eta \in C^1(\partial D)$ with $Re\{\eta\} \leq 0$ and the total field is given by $u = u^i + u^s$ with the scattered wave u^s fulfilling the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial u^s}{\partial r} - ik_0 u^s \right) = 0, \quad r = |x|, \tag{1.3}$$

uniformly with respect to all directions.

The latter is equivalent to an asymptotic behavior of the form

$$u^s(x) = \frac{e^{ik_0|x|}}{\sqrt{|x|}} \left\{ u_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty, \tag{1.4}$$

Key words and phrases. inverse scattering; Helmholtz equation; transmission problem; singlelayer approach; nonlinear integral equations.

uniformly in all directions, with the far field pattern u_∞ defined on the unit circle S^1 in $\mathbb{R}^2$ (see[10]). In the above, u and v represent the electric field that is parallel to the cylinder axis, (1.1) corresponds to the time-harmonic Maxwell equations and the conductive boundary conditions (1.2) model the continuity of the tangential components of the electric and magnetic field across the interface ∂D with the term $i\eta v$ modelling the boundary layer.

The inverse obstacle problem we are interested in is, given the conductive layer and the far field pattern u_∞ for one incident plane wave with incident direction $d \in S^1$ to determine the conductive function η. More generally, we also consider the reconstruction of η from the far field patterns for a small finite number of incident plane waves with different incident directions. This inverse problem is nonlinear and ill-posed, since the solution of the scattering problem (1.1)–(1.3) is nonlinear with respect to the conductive function and since the mapping from the conductive function into the far field pattern is extremely smoothing.

For a stable solution of the inverse problem we propose an algorithm that extends the approach suggested by Akduman and Kress[1] for the case of an infinitely long impedance cylinder with arbitrarily shaped cross section embedded in a homogeneous background. Representing the solution v and u^s to the forward scattering problem in terms of single-layer potentials in D and in $\mathbb{R}^2 \setminus \bar{D}$ with densities φ_d and φ_0, respectively, the conductive boundary condition (1.2) provides a system of two boundary integral equations on ∂D for the corresponding densities, that in the sequel we will denote as field equations. For the inverse problem, the required coincidence of the far field of the single-layer potential representing u^s and the given far field u_∞ provides a further equation that we denote as data equation. The system of the field and data equations can be viewed as three equations for three unknowns, i.e., the two densities and the conductive function η. They are linear with respect to the densities and nonlinear with respect to the conductive function.

To some extend, the inverse problem consists in solving a certain Cauchy problem, i.e., extending a solution to the Helmholtz equation from knowing their Cauchy data on some boundary curve. With this respect we also mention the related work of Ben Hassen, Ivanyshyn and Sini [6], Cakoni and Colton [7], Cakoni, Colton and Monk [8], Eckel and Kress [11], Fang and Zeng [12], Ivanyshyn and Kress [14], Jakubik and Potthast [15]. For the simultaneous reconstruction of the shape and the impedance function in a homogeneous background we refer to Kress and Rundell [19], Liu, Nakamura and Sini[21], Nakamura and Sini [23], and Serranho [24].

The plan of the paper is as follows: In Section 2, as ingredient of our inverse algorithm we provide an existence proof for the solution of the forward scattering problem via a single-layer approach followed by a corresponding numerical solution method in Section 3. The details of the inverse algorithm are presented in Section 4 and in Section 5 we demonstrate the feasibility of the method by some numerical examples.

2. The direct problem

The forward scattering problem (1.1)–(1.3) has at most one solution (see Gerlach and Kress [13]). Existence can be proven via boundary integral equations by a combined single- and double-layer approach (see Gerlach and Kress [13]).

Here, as one of the ingredients of our inverse algorithm, we follow [4] and suggest a single-layer approach. For this we denote by

$$\Phi_k(x,y) := \frac{i}{4}\, H_0^{(1)}(k|x-y|), \quad x \neq y,$$

the fundamental solution to the the Helmholtz equation with wave number k in $\mathbb{R}^2$ in terms of the Hankel function $H_0^{(1)}$ of order zero and of the first kind. Adopting the notation of [10], in a

Sobolev space setting, for $k = k_d$ and $k = k_0$, we introduce the single-layer potential operators

$$S_k : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$$

by

$$(S_k\varphi)(x) := 2\int_{\partial D} \Phi_k(x,y)\varphi(y)\,ds(y), \quad x \in \partial D, \tag{2.5}$$

and the normal derivative operators

$$K'_k : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$$

by

$$(K'_k\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\,\varphi(y)\,ds(y), \quad x \in \partial D. \tag{2.6}$$

For the Sobolev spaces and the mapping properties of these operators we refer to [17, 22].

Then, from the jump relations it can be seen that the single-layer potentials

$$\begin{aligned} v(x) &= \int_{\partial D} \Phi_{k_d}(x,y)\varphi_d(y)\,ds(y), \quad x \in D, \\ u^s(x) &= \int_{\partial D} \Phi_{k_0}(x,y)\varphi_0(y)\,ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{D}, \end{aligned} \tag{2.7}$$

solve the scattering problem (1.1)–(1.3) provided the densities φ_d and φ_0 satisfy the system of integral equations

$$\begin{aligned} S_{k_d}\varphi_d - S_{k_0}\varphi_0 &= 2u^i|_{\partial D}, \\ \varphi_d + \varphi_0 + i\eta S_{k_d}\varphi_d + K'_{k_d}\varphi_d - K'_{k_0}\varphi_0 &= 2\,\frac{\partial u^i}{\partial \nu}\bigg|_{\partial D}. \end{aligned} \tag{2.8}$$

Theorem 2.1. *Provided k_0 is not a Dirichlet eigenvalue of the negative Laplacian for the domain D the system* (2.8) *has a unique solution in $H^{-1/2}(\partial D) \times H^{-1/2}(\partial D)$.*

Proof. We first establish that (2.8) has at most one solution. If φ_d and φ_0 satisfy the homogeneous form of (2.8) then the single-layer potentials (2.7) solve the scattering problem with zero incident field. Consequently, since the forward scattering problem has at most one solution, we have $u^s = 0$ in $\mathbb{R}^2 \setminus \bar{D}$ and $v = 0$ in D. Then u^s is also defined in D and has vanishing trace on ∂D. Therefore, the assumption on k_0 implies that $u^s = 0$ also in D and consequently $\varphi_0 = 0$ on ∂D by the jump relations. Analogously, v considered in $\mathbb{R}^2 \setminus \bar{D}$ also has vanishing trace on ∂D and consequently using the radiation condition if k_d is real valued or the exponential decay at infinity if $\operatorname{Im} k_d > 0$ we have $v = 0$ in $\mathbb{R}^2 \setminus \bar{D}$. Again the jump relations imply $\varphi_d = 0$ on ∂D and the uniqueness proof for the solution of the system (2.8) is completed.

To establish existence of a solution, we note that due to the assumption on k_0 the inverse operator $S_{k_0}^{-1} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ exists and is bounded. With its aid, we can equivalently transform (2.8) into

$$\begin{aligned} \varphi_d + S_{k_0}^{-1}[S_{k_d} - S_{k_0}]\varphi_d - \varphi_0 &= 2S_{k_0}^{-1}u^i|_{\partial D}, \\ \varphi_d + \varphi_0 + i\eta S_{k_d}\varphi_d + K'_{k_d}\varphi_d - K'_{k_0}\varphi_0 &= 2\,\frac{\partial u^i}{\partial \nu}\bigg|_{\partial D}. \end{aligned} \tag{2.9}$$

Now (2.9) is of the form

$$\mathbb{A}\begin{pmatrix}\varphi_d\\ \\ \varphi_0\end{pmatrix}+\mathbb{B}\begin{pmatrix}\varphi_d\\ \\ \varphi_0\end{pmatrix}=2\begin{pmatrix}S_{k_0}^{-1}u^i|_{\partial D}\\ \\ \left.\dfrac{\partial u^i}{\partial\nu}\right|_{\partial D}\end{pmatrix}$$

with the matrix operators

$$\mathbb{A},\mathbb{B}: H^{-1/2}(\partial D)\times H^{-1/2}(\partial D)\to H^{-1/2}(\partial D)\times H^{-1/2}(\partial D)$$

given by

$$\mathbb{A}=\begin{pmatrix} I & -I\\ \\ I & I\end{pmatrix}\quad\text{and}\quad \mathbb{B}=\begin{pmatrix} S_{k_0}^{-1}[S_{k_d}-S_{k_0}] & 0\\ \\ i\eta S_{k_d}+K'_{k_d} & -K'_{k_0}\end{pmatrix}.$$

Clearly, $\mathbb{A}$ has a bounded inverse and $\mathbb{B}$ is compact since its components are compact. In particular, $S_{k_d}-S_{k_0}: H^{-1/2}(\partial D)\to H^{1/2}(\partial D)$ is compact because of cancellation of singularities in the two single-layer operators. Therefore existence of a solution follows from uniqueness by the Riesz–Fredholm theory for compact operators (see [17]). □

We note that instead of using Sobolev spaces the existence analysis can also be carried out in a classical Hölder space setting replacing $H^{-1/2}(\partial D)$ by $C^{0,\alpha}(\partial D)$.

3. Numerical solution

For the numerical solution of (2.8) and the presentation of our inverse algorithm we assume that the boundary curve ∂D is given by a regular 2π–periodic parameterization

$$\partial D=\{z(t): 0\le t\le 2\pi\}. \tag{3.1}$$

Then, via $\psi=\varphi\circ z$. emphasizing the dependence of the operators on the boundary curve, we introduce the parameterized single-layer operator

$$\widetilde{S}_k: H^{-1/2}[0,2\pi]\times C^2[0,2\pi]\to H^{1/2}[0,2\pi]$$

by

$$\widetilde{S}_k(\psi,z)(t):=\frac{i}{2}\int_0^{2\pi} H_0^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|\,\psi(\tau)\,d\tau$$

and the parameterized normal derivative operators

$$\widetilde{K}'_k: H^{-1/2}[0,2\pi]\times C^2[0,2\pi]\to H^{-1/2}[0,2\pi]$$

by

$$\widetilde{K}'_k(\psi,z)(t):=\frac{ik}{2}\int_0^{2\pi}\frac{[z'(t)]^\perp\cdot[z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\,H_1^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|\,\psi(\tau)\,d\tau$$

for $t\in[0,2\pi]$. Here we made use of $H_0^{(1)\prime}=-H_1^{(1)}$ with the Hankel function $H_1^{(1)}$ of order zero and of the first kind. Furthermore, we write $a^\perp=(a_2,-a_1)$ for any vector $a=(a_1,a_2)$, that is, $a^\perp$ is obtained by rotating a clockwise by 90 degrees. Then the parameterized form of (2.8) is given by

$$\begin{aligned}\widetilde{S}_{k_d}(\psi_d,z)-\widetilde{S}_{k_0}(\psi_0,z)\;&=\;2\,u^i\circ z,\\ \psi_d+\psi_0+\eta\circ z\widetilde{S}_{k_d}(\psi_d,z)+\widetilde{K}'_{k_d}(\psi_d,z)-\widetilde{K}'_{k_0}(\psi_0,z)\;&=\;\frac{2}{|z'|}\,[z']^\perp\cdot\operatorname{grad}u^i\circ z.\end{aligned}\tag{3.2}$$

The kernels

$$M(t,\tau) := \frac{i}{2}\, H_0^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|$$

and

$$L(t,\tau) := \frac{ik}{2}\, \frac{[z'(t)]^{\perp}\cdot[z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\, H_1^{(1)}(k|z(t)-z(\tau)|)\,|z'(\tau)|$$

of the operators $\widetilde{S}_k$ and $\widetilde{K}'_k$ can be written in the form

$$(3.3)\qquad \begin{aligned} M(t,\tau) &= M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),\\ L(t,\tau) &= L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau), \end{aligned}$$

where

$$\begin{aligned} M_1(t,\tau) &:= -\frac{1}{2\pi}\, J_0(k|z(t)-z(\tau)|)|z'(\tau)|,\\ M_2(t,\tau) &:= M(t,\tau) - M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right),\\ L_1(t,\tau) &:= -\frac{k}{2\pi}\, \frac{[z'(t)]^{\perp}\cdot[z(\tau)-z(t)]}{|z'(t)|\,|z(t)-z(\tau)|}\, J_1(k|z(t)-z(\tau)|)\,|z'(\tau)|,\\ L_2(t,\tau) &:= L(t,\tau) - L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right). \end{aligned}$$

The functions M_1, M_2, L_1, and L_2 turn out to be smooth with diagonal terms

$$M_2(t,t) = \left[\frac{i}{2} - \frac{C}{\pi} - \frac{1}{\pi}\ln\left(\frac{k}{2}|z'(t)|\right)\right]\,|z'(t)|$$

in terms of Euler's constant C and

$$L_2(t,t) = -\frac{1}{2\pi}\,\frac{[z'(t)]^{\perp}\cdot z''(t)}{|z_1'(t)|^2}\,.$$

For integral equations with kernels of the form (3.3) a combined collocation and quadrature methods based on trigonometric interpolation as described in Section 3.5 of [10] or in [20] is at our disposal. We refrain from repeating the details. For a related error analysis we refer to [17] and note that we have exponential convergence for smooth, i.e., analytic boundary curves ∂D.

For a numerical example, we consider the scattering of a plane wave by a dielectric cylinder with a non-convex kite-shaped cross section with boundary ∂D described by the parametric representation

$$(3.4)\qquad z(t) = (\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

The following conductive functions are chosen in our experiments.

- $$(3.5)\qquad \eta_1 = -\sin^4(0.5t) + i\cos^4(0.5t)$$
- $$(3.6)\qquad \eta_2 = -1.5 - \sin^3 t + i\sin t$$

•

$$\eta_3 = -0.5e^{-(t-pi)^2} + i(0.6 + 0.2\sin t) \tag{3.7}$$

From the asymptotics for the Hankel functions, it can be deduced that the far field pattern of the single-layer potential u^s with density φ_0 is given by

$$u_\infty(\hat{x}) = \gamma \int_{\partial D} e^{-ik_0\,\hat{x}\cdot y}\varphi_0(y)\,ds(y), \quad \hat{x} \in S^1, \tag{3.8}$$

where

$$\gamma = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k_0}}\,.$$

The latter expression can be evaluated by the composite trapezoidal rule after solving the system of integral equations (2.8) for φ_0, i.e., after solving (3.2) for ψ_0. Table 3.1 gives some approximate values for the far field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$ and the wave numbers are $k_0 = 2.8$ and $k_d = 1 + 1i$, and the conductive function η_1 is chosen. Note that the exponential convergence is clearly exhibited.

Table 3.1. Numerical results for direct scattering problem

n	$\mathrm{Re}\,u_\infty(d)$	$\mathrm{Im}\,u_\infty(d)$	$\mathrm{Re}\,u_\infty(-d)$	$\mathrm{Im}\,u_\infty(-d)$
8	-2.5727739209	0.4381005402	-2.4613551693	0.6118414535
16	-2.6086999117	0.5099897666	-2.4645038927	0.5815194742
32	-2.6087198359	0.5099895695	-2.4645127478	0.5815282913
64	-2.6087198065	0.5099895747	-2.4645127414	0.5815282789

4. The inverse problem

The inverse scattering problem that we are concerned with is, given the shape of the scatterer, to determine the conductive function η from a knowledge of the far field pattern for one or several incident plane waves. The inverse problem is ill-posed since the mapping taking conductive function η into the farfield pattern associated with the scattering problem (1.1) and (1.2) is highly smoothing since the far field is an analytic function. We will handle this issue of ill-posedness by using Tikhonov regularization. We note that the far field pattern for one incident plane wave uniquely determine the conductive function η. As a consequence of Rellich's lemma (see [10]), the far field pattern uniquely determine u^s in $\mathbb{R}^2 \setminus \bar{D}$ provided $\mathrm{Im}k_d > 0$. Then from the first condition in (1.2) using $\mathrm{Im}k_d > 0$ we observe that v is also uniquely determined in D. From (1.2) we can read off the uniqueness of conductive function η if we assume that ∂D is analytic since in this case v can not vanish on open intervals of ∂D.

We now proceed describing an algorithm for approximately solving the inverse scattering problem by extending the method proposed by Akduman and Kress [1]. After introducing the far field operator $S_\infty : H^{-1/2}(\partial D) \to L^2(S^1)$ by

$$(S_\infty\varphi)(\hat{x}) := \gamma \int_{\partial D} e^{-ik_0\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in S^1, \tag{4.1}$$

from (2.7) and (3.8) we observe that the far field pattern for the solution to the scattering problem (1.1)–(1.3) is given by

$$u_\infty = S_\infty\varphi_0 \tag{4.2}$$

in terms of the solution to (2.8). Here S^1 denotes the unit circle in $\mathbb{R}^2$. Therefore we can state the following theorem as theoretical basis of our inverse algorithm.

Theorem 4.1. *For a given incident field u^i and a given far field pattern u_∞, assume that ∂D and the densities φ_d and φ_0 satisfy the system of three integral equations*

$$\begin{aligned} S_{k_d}\varphi_d - S_{k_0}\varphi_0 &= 2u^i, \\ \varphi_d + \varphi_0 + i\eta S_{k_d}\varphi_d + K'_{k_d}\varphi_d - K'_{k_0}\varphi_0 &= 2\,\frac{\partial u^i}{\partial \nu}\,, \\ S_\infty\varphi_0 &= u_\infty. \end{aligned} \tag{4.3}$$

Then η solves the inverse problem.

Given the far field pattern u_∞, the density φ_0 is found by solving the third equation in (4.3), i.e., the data equation

$$S_\infty\varphi_0 = u_\infty. \tag{4.4}$$

Since the operator $S_\infty : L^2(\partial D) \to L^2(S^1)$ is compact, it can not have bounded inverse. Therefore the equation (4.4) is ill-posed.

We also require the parameterized version

$$\widetilde{S}_\infty : H^{-1/2}[0, 2\pi] \times C^2[0, 2\pi] \to L^2(S^1)$$

of the far field operator as given by

$$\widetilde{S}_\infty(\psi, z)(\hat{x}) := \gamma \int_0^{2\pi} e^{-ik_0\,\hat{x}\cdot z(\tau)}\psi(\tau)\,d\tau, \quad \hat{x} \in S^1. \tag{4.5}$$

Then the parameterized form of (4.3) is given by

$$\begin{aligned} \widetilde{S}_{k_d}(\psi_d, z) - \widetilde{S}_{k_0}(\psi_0, z) &= 2\,u^i \circ z, \\ \psi_d + \psi_0 + i\eta \circ z\widetilde{S}_{k_d}(\psi_d, z) + \widetilde{K}'_{k_d}(\psi_d, z) - \widetilde{K}'_{k_0}(\psi_0, z) &= \frac{2}{|z'|}\,[z']^\perp \cdot \operatorname{grad} u^i \circ z, \\ \widetilde{S}_\infty(\psi_0, z) &= u_\infty. \end{aligned} \tag{4.6}$$

The third equation of (4.6) requires stabilization and for this we use Tikhonov regularization, i.e., the ill-posed data equation is replaced by

$$\alpha\psi_0 + \widetilde{S}^*_\infty\widetilde{S}_\infty\psi_0 = \widetilde{S}^*_\infty u_\infty, \tag{4.7}$$

with some positive regularization parameter α and the adjoint operator $\widetilde{S}^* : L^2(S^1) \to L^2[0, 2\pi]$ of $\widetilde{S}_\infty$. After finding the density ψ_0 from the (4.7) we can now find density ψ_d from the first equation of (4.6).

$$\psi_d = \widetilde{S}^{-1}_{k_d}(2u^i \circ z - \widetilde{S}_{k_d}(\psi_0, z)) \tag{4.8}$$

Now it remains to find the conductive function η from the second equation of (4.6).

$$\eta \circ z = -i\frac{2\frac{\partial u^i \circ z}{\partial \nu} - \psi_0 - \psi_d - \widetilde{K}'_{k_d}(\psi_d, z) + \widetilde{K}'_{k_0}(\psi_0, z)}{\widetilde{S}_{k_d}(\psi_d, z)} \tag{4.9}$$

The reconstruction of the conductive function from equation (4.9) will be sensitive to errors due to the fact that it blows up in the vicinity of zeros of the $\widetilde{S}_{k_d}(\psi_d, z)$. To obtain a more stable solution (see Akduman and Kress [1]), we express the unknown conductive function in terms of some basis functions μ_j, $j = 0, \pm 1, \pm 2, \dots, \pm N$ as a linear combination

$$\eta = \sum_{j=-N}^{N} a_j \mu_j \quad \text{on} \quad \partial D. \tag{4.10}$$

A possible choice of basis functions consists of splines or trigonometric polynomials. We satisfy the second equation of (4.6) in a least square sense, penalized via Tikhonov regularization, for the unknown coefficient $a_{-N}, \dots, a_N$ i.e., we determine the coefficients $a_{-N}, \dots, a_N$ in (4.10) such that for a set of grid points $z_1, \dots, z_M$ on ∂D the least square sum

$$\sum_{m=1}^{M} \left| \psi_d(z_m) + \psi_0(z_m) + i \sum_{j=-N}^{N} a_j \mu_j(z_m) \widetilde{S}_{k_d} \psi_d(z_m) + \widetilde{K}'_{k_d} \psi_d(z_m) - \widetilde{K}'_{k_0} \psi_0(z_m) - \frac{\partial u^i}{\partial \nu}(z_m) \right|^2 \tag{4.11}$$

is minimized.

The above algorithm has a straightforward extension for the case of more than one incident wave. Assume that $u_1^i, \dots, u_P^i$ are P incident waves with different incident directions and $u_{\infty,1}, \dots, u_{\infty,P}$ the corresponding far field patterns for scattering from ∂D. Then the inverse problem to determine the unknown conductive function η from these given far field patterns and incident fields is equivalent to solving

$$\begin{aligned} \widetilde{S}_{k_d}(\psi_{d,p}, z) - \widetilde{S}_{k_0}(\psi_{0,p}, z) \; &= \; 2\, u_p^i \circ z, \\ \psi_{d,p} + i\eta \circ z \widetilde{S}_{k_d}(\psi_{d,p}, z) + \widetilde{K}'_{k_d}(\psi_{d,p}, z) + \psi_{0,p} - \widetilde{K}'_{k_0}(\psi_{0,p}, z) \; &= \; \frac{2}{|z'|}\, [z']^{\perp} \cdot \operatorname{grad} u_p^i \circ z, \\ \widetilde{S}_{\infty}(\psi_{0,p}, z) \; &= \; u_{\infty,p}. \end{aligned} \tag{4.12}$$

for $p = 1, \dots, P$. We first solve the first field and data equations in (4.12) for $p = 1, \dots, P$ to obtain $2P$ densities $\psi_{d,1}, \dots, \psi_{d,P}$ and $\psi_{0,1}, \dots, \psi_{0,P}$. We satisfy second equation of (4.12) in a least square sense, penalized via Tikhonov regularization, for the unknown coefficient $a_{-N}, \dots, a_N$ such that for a set of grid points $z_1, \dots, z_M$ on ∂D the least square sum

$$\sum_{p=1}^{P} \sum_{m=1}^{M} \left| \psi_{d,p}(z_m) + \psi_{0,p}(z_m) + i \sum_{j=-N}^{N} a_j \mu_j(z_m) \widetilde{S}_{k_d} \psi_{d,p}(z_m) + \widetilde{K}'_{k_d} \psi_{d,p}(z_m) - \widetilde{K}'_{k_0} \psi_{0,p}(z_m) - \frac{\partial u_p^i}{\partial \nu}(z_m) \right|^2 \tag{4.13}$$

is minimized.

5. Numerical examples

To avoid an inverse crime, in our numerical examples the synthetic far field data were obtained by a numerical solution of the boundary integral equations based on a combined single- and double-layer approach (see [9, 18]) using the numerical schemes as described in [10, 16, 17]. In each iteration step of the inverse algorithm for the solution of the field equations we used the numerical method described in Section 3 using 64 quadrature points. The equation (4.11) was solved by Tikhonov regularization with an H^2 Sobolev penalty term and with regularization parameter λ. The regularized equation is solved by Nyström's method with the composite trapezoidal rule again using 64 quadrature points. In addition, the regularized data was solved by Tikhonov regularization with an L^2 penalty term and with regularization parameter α.

In all our example we used $N = 10$ as degree for the approximating trigonometric polynomials in (4.10), and the wave numbers $k_0 = 4.5$ and $k_d = 2 + 2i$, and the regularization parameters $\alpha = 10^{-7}$ and $\lambda = 1$. In order to obtain noisy data, random errors are added point-wise to u_∞,

$$\widetilde{u}_\infty = u_\infty + \delta\xi \frac{||u_\infty||}{|\xi|} \tag{5.1}$$

with the random variable $\xi \in C$ and $\{Re\xi, Im\xi\} \in (0, 1)$.

TABLE 5.2. Parametric representation of boundary curves.

Counter type		Parametric representation
Apple-shaped	:	$z(t) = \{\frac{0.5+0.4\cos t+0.1\sin 2t}{1+0.7\cos t}(\cos t, \sin t) : t \in [0, 2\pi]\}$
Kite-shaped	:	$z(t) = \{(\cos t + 1.3\cos^2 t - 1.3, 1.5\sin t) : t \in [0, 2\pi]\}$
Peanut-shaped	:	$z(t) = \{\sqrt{\cos^2 t + 0.25\sin t}\,(\cos t, \sin t) : t \in [0, 2\pi]\}$
Rounded triangle	:	$z(t) = \{(2 + 0.3\cos 3t)(\cos t, \sin t) : t \in [0, 2\pi]\}$

In all the following examples, the green curve represents exact graph of conductive function η, the blue curve represents the reconstruction obtained from noisy data and the red curve represents the reconstruction obtained from noiseless data. In all examples we represent reconstructions from exact data and perturbed data with 1% relative error in the L^2 norm. In the figures 1, 2,3, and 4, the scatterers are apple-shaped, kite-shaped, peanut-shaped, and rounded-triangle-shaped respectively.

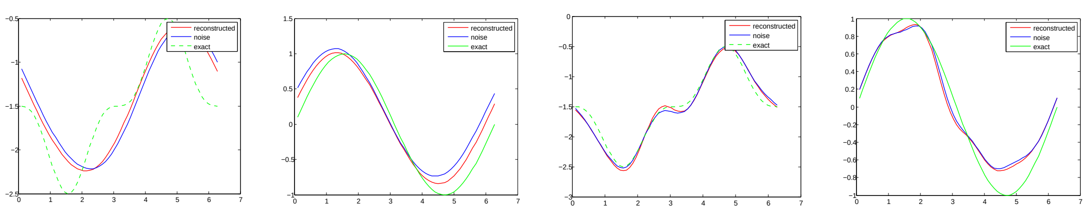

FIGURE 1. Reconstruction of η_2 (3.6). Real part of η_2 obtained for P=1 (the left figure), imaginary part of η_2 obtained for P=1 (the middle left fig.), real part of η_1 obtained for P=8 (the middle right fig.), imaginary part of η_2 obtained for P=8 (the right fig.).

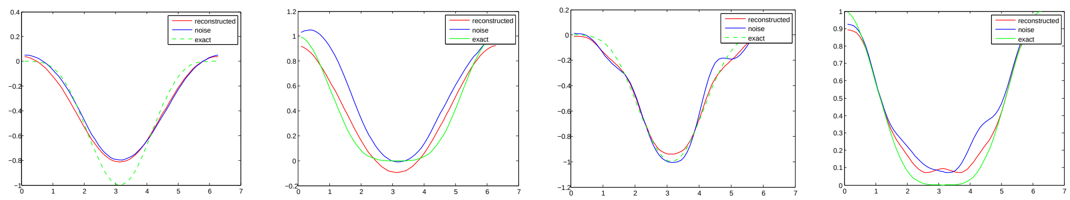

FIGURE 2. Reconstruction of η_1 (3.5). Real part of η_1 obtained for P=1 (the left figure), imaginary part of η_1 obtained for P=1 (the middle left fig.), real part of η_1 obtained for P=8 (the middle right fig.), imaginary part of η_1 obtained for P=8 (the right fig.).

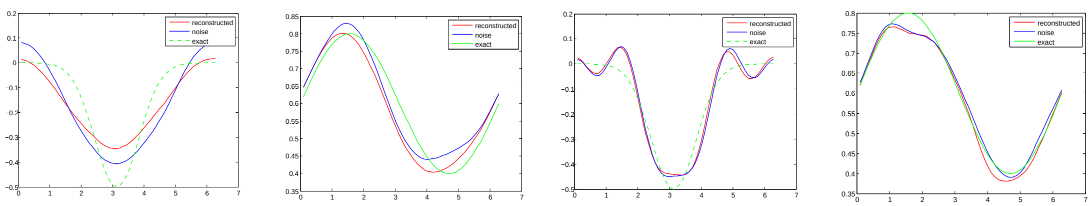

FIGURE 3. Reconstruction of η_3 (3.7). Real part of η_3 obtained for P=1 (the left figure), imaginary part of η_3 obtained for P=1 (the middle left fig.), real part of η_3 obtained for P=8 (the middle right fig.), imaginary part of η_3 obtained for P=8 (the right fig.).

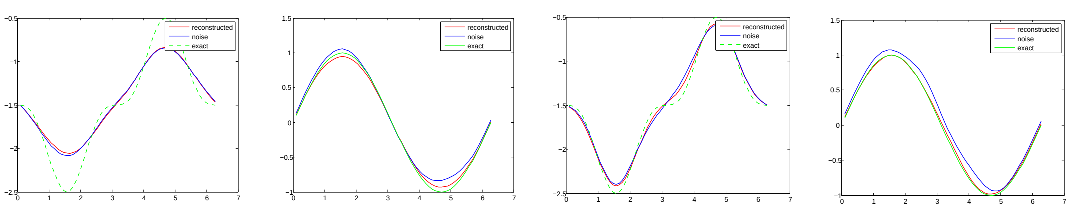

FIGURE 4. Reconstruction of η_2 (3.6). Real part of η_2 obtained for P=1 (the left figure), imaginary part of η_2 obtained for P=1 (the middle left fig.), real part of η_2 obtained for P=8 (the middle right fig.), imaginary part of η_2 obtained for P=8 (the right fig.).

Our examples clearly indicate the feasibility of the proposed algorithm with a reasonable stability against noise. From our further numerical experiments it became obvious that using more than one incident wave improved on the accuracy of the reconstruction and the stability.

Further research will be directed towards applying the algorithm to real data, to extend the numerics to the three dimensional case. One can also extend this inverse scattering problem to a simultaneous reconstruction of the conductive function and the shape of the scatterer. Similar problems have recently been considered by Kress and Rundell [19] and Serranho [24] for impenetrable scatterers. The shape reconstruction of transmision problem via single-layer potential approach was investigated by Altundag and Kress [4] and [5] and by Altundag [2], [3]. Uniqueness in inverse obstacle scattering with the conductive boundary condition was established by Gerlach and Kress [13]. However, there is no uniqueness result for one or finitely many incident fields for conductive transmission problem for shape reconstruction.

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Dynamic Analysis of Some Impulsive Fractional Neural Network with Mixed Delay*

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Abstract: In this paper, the authors study some impulsive fractional-order neural network with mixed delay. By the fractional integral and the definition of stability, the existence of solutions of the network is proved, the sufficient conditions for stability of the system are presented. Some examples are given to illustrate the main results.

Keywords: fractional-order neural network,mixed delay,fixed point theorem

Clc number: 0175.13

Introduction

In this paper, we study some impulsive fractional-order neural network with mixed delay

$$\begin{cases} {}^cD_t^{\alpha}x(t) = -Cx(t) + AF(x(t)) + BG(x(t-\tau)) \\ \qquad\qquad + D\int_{-\infty}^{t} K(t-\eta)H(x(\eta))d\eta + J, & 0 < t \leq b, t \neq t_k, \\ \triangle x(t_k) = I(x(t_k^-)), \qquad k = 1,2,3,\cdots,m \\ x(\eta) = \varphi(\eta), & \eta \in (-\infty, 0] \end{cases} \tag{1}$$

where ${}^cD_t^{\alpha}$ is the standard Caputo fractional derivative of order α, $\alpha \in (0,1)$,$x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the neuron state vector of the neural network; $C = diag(c_1, c_2, \cdots, c_n)$ is a diagonal matrix and $c_i > 0$,$i \in N = 1, 2, \cdots, n$, $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, D = (d_{ij})_{n\times n}$ are the connection weight matrix, the delayed weight matrix and the distributively delayed connection weight matrix, respectively; $J = (J_1, J_2, \cdots, J_n)^T$ is an external input; $F(x(\cdot)) = (f_1(x_1(\cdot)), f_2(x_2(\cdot)), \cdots, f_n(x_n(\cdot)))^T$, $G(x(\cdot)) = (g_1(x_1(\cdot)), g_2(x_2(\cdot)), \cdots, g_n(x_n(\cdot)))^T$, $H(x(\cdot)) = (h_1(x_1(\cdot)), h_2(x_2(\cdot)), \cdots, h_n(x_n(\cdot)))^T$ represents the neuron activation function; $\tau = (\tau_{ij})_{n\times n}$

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is the transmission delay of the neural network satisfies $0 \leq \tau_{ij} \leq \delta$, where δ is a positive constant; $K(\cdot) = diag(k_1(\cdot), k_2(\cdot), \cdots, k_n(\cdot))$ is the delay kernel function and satisfies $\int_0^\infty k_i(t)dt = 1$, the function $I_k : X \to X$ is continous, and $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m = b, \Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k$, $\varphi(\eta) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T$ is the initial function and $\varphi_i(\eta) \in C([-\infty, 0], R), i \in N, \varphi_i(0) = 0$, the norm of $C([-\infty, 0], R)$ is denoted by $||\varphi(t)|| = \sum_{i=1}^n sup\{|\varphi_i(t)|\}$.

It is well known that the delayed and impulsive neural networks exhibiting the rich and colorful dynamical behaviors are important part of the delayed neural systems. The delayed and impulsive neural networks can exhibit some complicated dynamics and even chaotic behaviors. Due to their important and potential applications in signal processing, image processing, artificial intelligence as well as optimizing problems and so on, the dynamical issues of delayed and impulsive neural networks have attracted worldwide attention , many interesting stability criteria for the equilibriums and periodic solutions of delayed or impulsive neural networks have been derived via Lyapunov-type function or functional approaches. For example, Wang and Zheng[1]investigated the global asymptotic stability of the equilibrium point of a class of mixed recurrent neural networks with time delay in the leakage by using the Lyapunov functional method, linear matrix inequality approach and general convex combination technique term under impulsive perturbations. Sebdani et al.[2]considered bifurcations and chaos in a discrete-time-delayed Hopfield neural network with ring structures and different internal decays. M.U. Akhmet, E. ylmaz[3]got a criteria for the global asymptotic stability of the impulsive Hopfield-type neural networks with piecewise constant arguments of generalized type by using linearization.

For the last decades, fractional differential equations [4−11] have received intensive attention because they provide an excellent tool for the description of memory and hereditary properties of various materials and processes, such as physics, mechanics, chemistry, engineering, etc. Therefore, it may be more meaningful to model by fractional-order derivatives than integer- order ones. Recently, fractional calculus is introduced into artificial neural network . For example, Boroomand and Menhajas [12] investigated stability of fractional-order Hopfield-type neural networks through energy-like function analysis, Chen et al.[13] studied uniform stability and the the existence,uniqueness and stability of its equilibrium point of a class of fractional-order neural networks with constant delay . The authors [14−17] analyzed the stability of some other neural networks with delay. We all know that the delay is not always a constant, it maybe change in the network. Time-verying delays and distributed delays may occur in neural processing and signal transmission, which can cause instability, oscillations, there are few papers that consider the problems for fractional-order neural network with mixed delay and impulse. Thus, it is worth investigating some impulsive fractional-order neural network with mixed delay

To the best of our knowledge, the system(1) is still untreated in the literature and it is the motivation of the present work. The rest of this paper is organized as follows: In section 2, some notations and preparations are given. In section 3, some main results of (1) are obtained. At last, some examples are given to demonstrate the main results.

1 Preliminaries

In this section,we will give some definitions and preliminaries which will be used in the paper.

In order to define the solution of (1), the authors consider the following spaces $PC(J,R) = \{x : J \to R : x(t) \in C(t_k, t_{k+1}], k = 0, \cdots, m$, there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k^-) = x(t_k), k = 1, \cdots, m\}$. The norm $||x||_{PC} = sup\{||x(t)||_c : t \in J\}$.$PC^1(J,X)$=$\{x : J \to X, x \in C^1((t_k, t_{k+1}], X),\ k = 0, 1, 2, \cdots, n$, there exist $x'(t_k^+),\ k = 1, 2, \cdots, n,\ \}$. The norm $||x||_{PC^1} = sup\{||x(t)||_{pc}, ||x'(t)||_{pc} : t \in J\}$. Obviously $PC(J,X)$ and $PC^1(J,X)$ are Banach spaces.

Let's recall some known definitions of fractional calculus. For more details, one can see[4,5,6].

Definition 1 The integral

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \qquad \alpha > 0,$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

For a function $f(t)$ given in the interval $[0, \infty)$, the expression

$${}^L D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)dt,$$

where $n = [\alpha] + 1,\ [\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order $\alpha > 0$.

Definition 2 Caputo's derivative for a function $f : [0, \infty) \to R$ can be written as

$${}^c D_t^\alpha f(t) = {}^L D_t^\alpha [f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)], \qquad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of real number α.

Theorem 1 According to the Lemma2.6[18], one can get that if $u(t) \in PC^1(J, X)$, then

$$I^q\ {}^cD^q u(t) = \begin{cases} u(t) - u(0), & t \in [0, t_1], \\ u(t) - \sum_{i=1}^{k} I_i(u(t_i)) - u(0), & t \in (t_k, t_{k+1}], k \geq 1. \end{cases}$$

Proof: If $t \in [0, t_1]$, then

$$\begin{aligned} I^q\ {}^cD^q u(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{1}{\Gamma(1-q)} \int_0^s (s-\tau)^{-q} u'(\tau) d\tau ds \\ &= \frac{1}{\Gamma(q)\Gamma(1-q)} \int_0^t u'(\tau) \int_\tau^t (t-s)^{q-1}(s-\tau)^{-q} ds d\tau \\ &= \int_0^t u'(\tau) d\tau \\ &= u(t) - u(0). \end{aligned}$$

If $t \in (t_k, t_{k+1}], k \geq 1$, then

$$\begin{aligned} I^q\ {}^cD^q u(t) &= \frac{1}{\Gamma(q)\Gamma(2-q)} \int_0^t (t-s)^{q-1} \int_0^s (s-\tau)^{-q} u^{'}(\tau) d\tau ds \\ &= \frac{1}{\Gamma(q)\Gamma(1-q)} \int_0^{t_1} (t-s)^{q-1} \int_0^s (s-\tau)^{-q} u^{'}(\tau) d\tau ds \\ &\quad + \sum_{i=1}^{k-1} \frac{1}{\Gamma(q)\Gamma(1-q)} \int_{t_i}^{t_{i+1}} (t-s)^{q-1} \int_0^s (s-\tau)^{-q} u^{'}(\tau) d\tau ds \\ &\quad + \frac{1}{\Gamma(q)\Gamma(1-q)} \int_{t_k}^t (t-s)^{q-1} \int_0^s (s-\tau)^{-q} u^{'}(\tau) d\tau ds \\ &= \frac{1}{\Gamma(q)\Gamma(1-q)} \int_0^{t_1} (t-s)^{q-1} \int_0^s (s-\tau)^{-q} u^{'}(\tau) d\tau ds \\ &\quad + \sum_{i=1}^{k-1} \frac{1}{\Gamma(q)\Gamma(1-q)} \int_{t_i}^{t_{i+1}} (t-s)^{q-1} [\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{-q} u^{'}(\tau) d\tau \\ &\quad + \int_{t_i}^s (s-\tau)^{-q} u^{'}(\tau) d\tau] ds + \frac{1}{\Gamma(q)\Gamma(1-q)} \int_{t_k}^t (t-s)^{q-1} [\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (s-\tau)^{-q} u^{'}(\tau) d\tau \\ &\quad + \int_{t_k}^s (s-\tau)^{-q} u^{'}(\tau) d\tau] ds \\ &= \frac{1}{\Gamma(q)\Gamma(1-q)} \int_0^{t_1} u'(\tau) d\tau \int_\tau^{t_1} (t-s)^{q-1}(s-\tau)^{-q} ds \\ &\quad + \sum_{i=1}^{k-1} \sum_{j=1}^{i-1} \frac{1}{\Gamma(q)\Gamma(1-q)} \int_{t_j}^{t_{j+1}} u'(\tau) d\tau \int_{t_i}^{t_{i+1}} (t-s)^{q-1}(s-\tau)^{-q} ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}\frac{1}{\Gamma(q)\Gamma(1-q)}\int_{t_i}^{t_{i+1}}u^{'}(\tau)d\tau\int_{\tau}^{t_{i+1}}(t-s)^{q-1}(s-\tau)^{-q}ds\\
&+\sum_{j=0}^{k-1}\frac{1}{\Gamma(q)\Gamma(1-q)}\int_{t_j}^{t_{j+1}}u^{'}(\tau)d\tau\int_{t_k}^{t}(t-s)^{q-1}(s-\tau)^{-q}ds\\
&+\frac{1}{\Gamma(q)\Gamma(1-q)}\int_{t_k}^{t}u'(\tau)d\tau\int_{\tau}^{t}(t-s)^{q-1}(s-\tau)^{-q}ds\\
=&\,u(t)-\sum_{i=1}^{k}I_i(u(t_i))-u(0),
\end{aligned}
$$

with the help of the substitution $s=z(t-\tau)+\tau$,

$$
\begin{aligned}
\int_{\tau}^{t}(t-s)^{q-1}(s-\tau)^{-q}ds &= \int_0^1(1-z)^{q-1}z^{-q}dz\\
&= B(1-q,q)=\frac{\Gamma(1-q)\Gamma(q)}{\Gamma(1-q+q)}\\
&= \frac{\Gamma(1-q)\Gamma(q)}{\Gamma(1)}=\Gamma(1-q)\Gamma(q).
\end{aligned}
$$

The proof is completed.

Let recollect the definition of stability which can be found in [13] and will be used in our main results.

Definition 3 The solution of system (1.1) is said to be stable if for any $\varepsilon>0$, there exist $\delta(t_0,\varepsilon)>0$, such that $0\le t_0\le t$, $||\varphi(t)-\phi(t)||<\delta$ imply $||y(t,t_0,\varphi(t))-x(t,t_0,\phi(t))||<\varepsilon$ for any two solutions $y(t,t_0,\varphi(t)),x(t,t_0,\phi(t))$ associated with $\varphi(t),\phi(t)$ that are the initial function. It is uniformly stable if the above δ is independent of t_0.

2 Existence and uniqueness of solution

In this section, we will investigate the existence and uniqueness of solution for impulsive fractional-order neural network with mixed delay. Without loss of generality, let $t\in(t_k,t_{k+1}],1\le k\le m-1$.

For sake of convenience ,the authors adopt the following notations and assumptions:
H(1): For $j=1,2,\cdots,n$, the functions $f_j,g_j,h_j,I_k:X\to X$ satisfy:

There exist Lipschitz constants $L_{fj}>0$,$L_{gj}>0$,$L_{hj}>0$, $L_{jk}>0$, such that $|f_j(x)-f_j(y)|\le L_{fj}|x-y|$, $|g_j(x)-g_j(y)|\le L_{gj}|x-y|$, $|h_j(x)-h_j(y)|\le L_{hj}|x-y|$, $|I_k(x)-I_k(y)|\le$

$L_{jk}|x-y|$, for all $x,y\in X$.

H(2): The delay kernel function $K(\cdot)=diag(k_1(\cdot),k_2(\cdot),\cdots,k_n(\cdot))$ satisfies

$$\int_0^\infty k_i(t)dt=1,\qquad k^*=\int_0^\infty k_i(t)e^{-t}dt<\infty,\qquad i=1,2,\cdots,n.$$

H(3):c_j,a_{ij},b_{ij},d_{ij} and $L_{fj},L_{gj},L_{hj},L_{jk}$ satisfy the following conditions:

(i)$||A^*||=\sum_{i=1}^n|a_i^*|=\sum_{i=1}^n\sup_{\forall j}\{|a_{ij}|L_{fj}\}$, $||B^*||=\sum_{i=1}^n|b_i^*|=\sum_{i=1}^n\sup_{\forall j}\{|b_{ij}|L_{gj}\}$, $||D^*||=\sum_{i=1}^n|d_i^*|=\sum_{i=1}^n\sup_{\forall j}\{|d_{ij}|L_{hj}\}$, $||I^*||=\sum_{k=1}^m|L_k^*|=\sum_{k=1}^m\sup_{\forall j}\{|L_{jk}|\},m\le n$;

(ii)$C_{\max}=\max\{c_j\}$,$C_{\min}=\min\{c_j\}$;

(iii)$M=||I^*||+\frac{b^\alpha}{\Gamma(\alpha+1)}(C_{\max}+||A^*||+||B^*||+||D^*||)<1$.

Theorem 2 If the assumption H(1),H(2) and H(3) hold , for each $x^0=(x_1^0,x_2^0,\cdots,x_n^0)$, the system(1) has a unique solution.

Proof: Consider the system (1), we will study the solvability and stability of it.

(1)Existence.

By Theorem 1, it is showed that the (1) is equivalent to the following integral equation

$$\begin{aligned}x_i(t)&=x_i(0)+\sum_{l=1}^k I(x_i(t_l^-))+I^\alpha[-c_ix_i(t)+\sum_{j=1}^n a_{ij}f_j(x_j(t))+\sum_{j=1}^n b_{ij}g(x_j(t-\tau_{ij}))\\&\quad+\sum_{j=1}^n d_{ij}\int_{-\infty}^t k_j(t-\eta)h_j(x_j(\eta))d\eta+J_i]\\&=x_i(0)+\sum_{l=1}^k I(x_i(t_l^-))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[-c_ix_i(s)+\sum_{j=1}^n a_{ij}f_j(x_j(s))+\sum_{j=1}^n b_{ij}g(x_j(t-\tau_{ij}))\\&\quad+\sum_{j=1}^n d_{ij}\int_{-\infty}^s k_j(s-\eta)h_j(x_j(\eta))d\eta+J_i]ds.\qquad(2)\end{aligned}$$

Constract the following sequences z_i^n, $z_i^0=x_i^0$,

$$\begin{aligned}z_i^{n+1}(t)&=x_i^0+\sum_{l=1}^k I(z_i^n(t_l^-))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[-c_iz_i^n(s)+\sum_{j=1}^n a_{ij}f_j(z_j^n(s))\\&\quad+\sum_{j=1}^n b_{ij}g(z_j^n(t-\tau_{ij}))+\sum_{j=1}^n d_{ij}\int_{-\infty}^s k_j(s-\eta)h_j(z_j^n(\eta))d\eta+J_i]ds,\end{aligned}$$

we can calculate that

$$||z_i^{n+1}(t)-z_i^n(t)||=sup\{|z_i^{n+1}(t)-z_i^n(t)|\}$$

$$
\begin{aligned}
&\leq sup|\sum_{l=1}^{k}\left(I(z_i^n(t_l^-))-I(z_i^{n-1}(t_l^-))\right)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[c_i|z_i^n(s)-z_i^{n-1}(s)|+\sum_{j=1}^{n}|a_{ij}||(f_j(z_i^n(s))-f_j(z_i^{n-1}(s)))|\\
&\quad+\sum_{j=1}^{n}|b_{ij}||(g_j(z_i^n(s-\tau_{ij}))-g(z_i^{n-1}(s-\tau_{ij})))|\\
&\quad+\sum_{j=1}^{n}|d_{ij}|\int_{-\infty}^{s}k_j(s-\eta)|h_j(z_i^n(\eta))-h_j(z_i^{n-1}(\eta))|d\eta]ds|\\
&\leq(||I^*||+\frac{b^\alpha}{\Gamma(\alpha+1)}(C_{\max}+||A^*||+||B^*||+||D^*||))||z_i^n(t)-z_i^{n-1}(t)||\\
&\leq M^n||z_i^1(t)-z_i^0(t)||\\
&\leq M^n(M-||I^*||)||z_i^0||,
\end{aligned}
$$

so the sequences z_i^n are convergent and the limits satisfy (2), the existence is proved .

(2) Stability.

Assume that $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T$ and $y(t)=(y_1(t),y_2(t),\cdots,y_n(t))^T$ are the two solutions of (1) with the different initial condition $x_i(\eta)=\phi_i(\eta)\in C((-\infty,0],R)$, $\phi_i(0)=0,\ \ y_i(\eta)=\varphi_i(\eta)\in C((-\infty,0],R),\varphi_i(0)=0,i\in N$. We have

$$
{}^cD_t^\alpha(y_i(t)-x_i(t))=-c_i(y_i(t)-x_i(t))+\sum_{j=1}^{n}a_{ij}(f_j(y_j(t))-f_j(x_j(t)))+
$$

$$
\sum_{j=1}^{n}b_{ij}(g(y_j(t-\tau_{ij}))-g(x_j(t-\tau_{ij})))+\sum_{j=1}^{n}d_{ij}\int_{-\infty}^{t}k_j(t-\eta)(h_j(y_j(\eta))-h_j(x_j(\eta)))d\eta.
$$

According to the Definition 2 and the initial function $\varphi_i(0)=0$, if $n=1,0<q<1$,

$$
I^qD^qu(t)=u(t)-u(0).
$$

So the solution of the (1) can be denoted by the following form

$$
\begin{aligned}
y_i(t)-x_i(t)&=\sum_{l=1}^{k}I(y_i(t_l^-)-x_i(t_l^-))+I^\alpha[-c_i(y_i(t)-x_i(t))+\sum_{j=1}^{n}a_{ij}(f_j(y_j(t))-f_j(x_j(t)))+\\
&\sum_{j=1}^{n}b_{ij}(g_j(y_j(t-\tau_{ij}))-g(x_j(t-\tau_{ij})))+\sum_{j=1}^{n}d_{ij}\int_{-\infty}^{t}k_j(t-\eta)(h_j(y_j(\eta))-h_j(x_j(\eta)))d\eta]
\end{aligned}
$$

$$
=\sum_{l=1}^{k}I(y_i(t_l^-)-x_i(t_l^-))+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[-c_i(y_i(s)-x_i(s))+\sum_{j=1}^{n}a_{ij}(f_j(y_j(s))-f_j(x_j(s)))+
$$

$$\sum_{j=1}^{n} b_{ij}(g_j(y_j(s-\tau_{ij}))-g(x_j(s-\tau_{ij})))+\sum_{j=1}^{n} d_{ij}\int_{-\infty}^{s} k_j(s-\eta)(h_j(y_j(\eta))-h_j(x_j(\eta)))d\eta]ds.$$

Then

$$\begin{aligned}
|y_i(t)-x_i(t)| \leq & \sum_{l=1}^{k} I(y_i(t_l^-)-x_i(t_l^-)) \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[c_i|y_i(s)-x_i(s)|+\sum_{j=1}^{n}|a_{ij}||(f_j(y_j(s))-f_j(x_j(s)))| \\
& +\sum_{j=1}^{n}|b_{ij}||(g_j(y_j(s-\tau_{ij}))-g(x_j(s-\tau_{ij})))| \\
& +\sum_{j=1}^{n}|d_{ij}|\int_{-\infty}^{s} k_j(s-\eta)|h_j(y_j(\eta))-h_j(x_j(\eta))|d\eta]ds \\
\leq & \sum_{l=1}^{m} L_{jk}|y_i(t_l^-)-x_i(t_l^-)| \\
& +\frac{1}{\Gamma(\alpha)}c_i\int_0^t (t-s)^{\alpha-1}|y_i(s)-x_i(s)|ds \\
& +\sum_{j=1}^{n}|a_{ij}|L_{fj}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y_j(s)-x_j(s)|ds \\
& +\sum_{j=1}^{n}|b_{ij}|L_{gj}\frac{1}{\Gamma(\alpha)}\int_0^{\tau_{ij}} (t-s)^{\alpha-1}|\varphi_j(s-\tau_{ij})-\phi_j(s-\tau_{ij})|ds \\
& +\sum_{j=1}^{n}|b_{ij}|L_{gj}\frac{1}{\Gamma(\alpha)}\int_{\tau_{ij}}^t (t-s)^{\alpha-1}|y_j(s-\tau_{ij})-x_j(s-\tau_{ij})|ds \\
& +\sum_{j=1}^{n}|d_{ij}|L_{hj}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y_j(\eta_1)-x_j(\eta_1)|ds \quad 0\leq\eta_1\leq t \\
& +\sum_{j=1}^{n}|d_{ij}|L_{hj}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\varphi_j(\eta)-\phi_j(\eta)|ds \quad -\infty\leq\eta\leq 0 \\
\leq & \sum_{l=1}^{m} L_{jk}|y_i(t_l^-)-x_i(t_l^-)| \\
& +c_i\sup\{|y_i(t)-x_i(t)|\}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
& +\sum_{j=1}^{n}|a_{ij}|L_{fj}\sup\{|y_j(t)-x_j(t)|\}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
& +\sum_{j=1}^{n}|b_{ij}|L_{gj}\sup\{|\varphi_j(t)-\phi_j(t)|\}\frac{1}{\Gamma(\alpha)}\int_0^{\tau_{ij}} (t-s)^{\alpha-1}ds \qquad (3)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}|b_{ij}|L_{gj}\sup\{|y_j(t)-x_j(t)|\}\frac{1}{\Gamma(\alpha)}\int_{\tau_{ij}}^{t}(t-s)^{\alpha-1}ds\\
&+\sum_{j=1}^{n}|d_{ij}|L_{hj}\sup\{|y_j(\eta_1)-x_j(\eta_1)|\}\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}ds \qquad 0\le\eta_1\le s\\
&+\sum_{j=1}^{n}|d_{ij}|L_{hj}\sup\{|\varphi_j(\eta)-\phi_j(\eta)|\}\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}ds, \qquad -\infty\le\eta\le 0.
\end{aligned}
$$

From(3), one can get

$$
\begin{aligned}
||y(t)-x(t)|| = \sum_{i=1}^{n} sup\{|y_i-x_i|\} \le \big(||I^*|| + \frac{b^\alpha}{\Gamma(\alpha+1)}(C_{\max}+||A^*||+||B^*||+||D^*||)\big)||y(t)-x(t)||\\
+\frac{b^\alpha}{\Gamma(\alpha+1)}(||B^*||+||D^*||)||\varphi(t)-\phi(t)||,
\end{aligned}
$$

which implies that

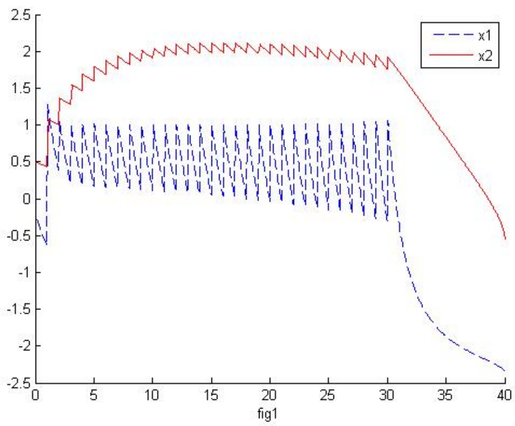
$$
||y(t)-x(t)|| \le \frac{b^\alpha(||B^*||+||D^*||)}{\Gamma(\alpha+1)-\Gamma(\alpha+1)||I^*||-b^\alpha(C_{\max}+||A^*||+||B^*||+||D^*||)}||\varphi(t)-\phi(t)||.
$$

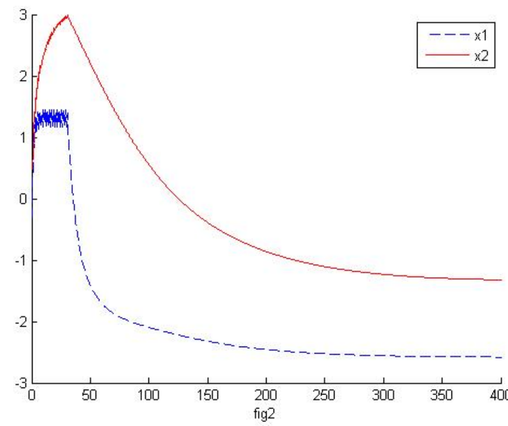
For $\forall\epsilon>0$, there exists $\delta=\frac{\Gamma(\alpha+1)-\Gamma(\alpha+1)||I^*||-b^\alpha(C_{\max}+||A^*||+||B^*||+||D^*||)}{b^\alpha(||B^*||+||D^*||)}\epsilon$, such that $||y(t)-x(t)||<\epsilon$, when $||\varphi(t)-\phi(t)||<\delta$, so the solution $x(t)$ is uniformly stable, which means that the network(1) is uniformly stable.

3 Some examples

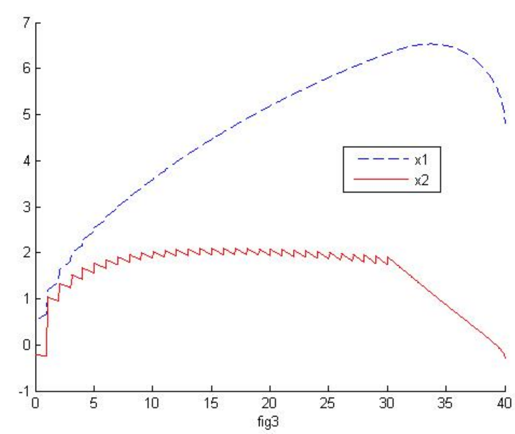
In this section, according to the impulsive fractional-order neural network (1), some examples are given to illustrate the main results.

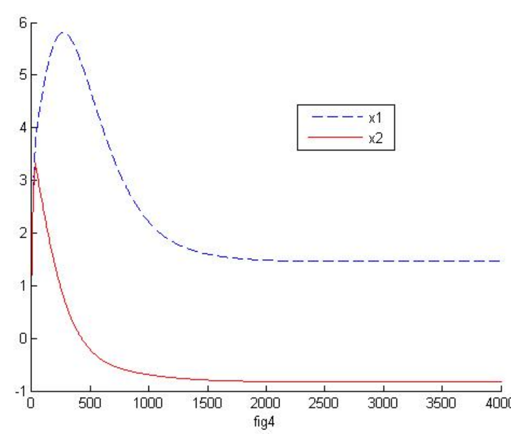
Example 3.1 Choose $C=diag(-0.01,0.01)$, $A=\begin{pmatrix}2&-1\\5&6\end{pmatrix}$, $B=\begin{pmatrix}-10&1\\-5&1\end{pmatrix}$, $D=\begin{pmatrix}-10&1\\-5&1\end{pmatrix}$, $I=\begin{pmatrix}0.8\\0.8\end{pmatrix}$, $\tau=\begin{pmatrix}0.2\\0.1\end{pmatrix}$, $f_j(x_j(t))=g_j(x_j(t))=h_j(x_j(t))=\frac{1}{1+e^{-x_j(t)}}$, let $L_{fj}=L_{gj}=L_{hj}=\frac{1}{1000}, m=30$, for the convenience, the authors take the interval 1 , it is easily to prove that the assumption (H1)(H3) hold, so example 3.1 is uniformly stable . For the convenience of studying the local and the whole conditions, the authors take the time t=40(fig1) and t=400(fig2) and get the unique equilibrium point x^*=(-2.6047,-1.3012).





Example 3.2 Choose $C = diag(-0.02, 0.01)$, $A = \begin{pmatrix} 1 & -1 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $I = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$, $\tau = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$, $f_j(x_j(t)) = g_j(x_j(t)) = \frac{1+e^{-x_j(t)}}{1+e^{-x_j(t)}}, h_j(x_j(t)) = \frac{1}{1+e^{-x_j(t)}}$, let $L_{fj} = L_{gj} = L_{hj} = \frac{1}{1000}, m = 30$, the interval is 1, it is proved that the assumption (H1)(H3) hold, so example 3.2 is uniformly stable . The authors also choose the time t=40(fig3) and t=4000(fig4) and get the unique equilibrium point x^*=(1.4574,-0.8297).





4 Conclusions

In this paper,by the fractional integral, the authors changed the derivative equation to integral one, for the convergence of sequences and the definition of stabilty, the existence of solutions of the network has been proved, the sufficient conditions for stability of the system have been presented. The authors also gave two examples and designed the relevant experimental procedures, after some experiments, the results have been illustrated .I think that the design of impulsive item is difficult.The finite item is proved to be feasible, but how the infinite one or the variable one , this can be our future work.

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Basins of Attraction of Certain Homogeneous Second Order Quadratic Fractional Difference Equation

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Abstract

We investigate the basins of attraction of equilibrium points and period-two solution of the difference equation of the form

$$x_{n+1} = \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_n x_{n-1}}, \quad n = 0, 1, \ldots,$$

where the parameters a, b, C, B are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. We show that this equation exhibits global period-two bifurcation, as certain parameters are pasing through the critical value.

Keywords: attractivity, basin, difference equation, invariant sets, periodic solutions, stable set
AMS 2000 Mathematics Subject Classification: 39A10, 39A11

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1 Introduction and preliminaries

We investigate global behavior of the equation

$$x_{n+1} = \frac{Bx_n x_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_n x_{n-1}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where the parameters a, b, C, B are positive numbers and the initial conditions $x_{-1}, x_0, x_{-1} + x_0 > 0$ are arbitrary nonnegative numbers. It is a special case of the second order homogeneous quadratic fractional equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2}, \quad n = 0, 1, 2, ... \tag{2}$$

and of the general second order quadratic fractional equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, ... \tag{3}$$

which global dynamics is under investigation. Some special cases of Eq.(3) have been considered in the series of papers [3, 4, 9, 10, 17]. Some special second order quadratic fractional difference equations have appeared in analysis of the systems of linear fractional difference equations in the plane, see [5, 7, 8, 14, 15, 16]. Describing the global dynamics of Eq.(3) is formidable task as this equation contains as a special cases many equations with well known but complicated dynamics, such as Lynes' equation. Equation (3) contains as a special case linear fractional equation which dynamics was investigated in great details in [11] and in many papers which solved some conjectures and open problems posed in [11]. Equation (2) can be brought to the form

$$x_{n+1} = \frac{A\left(\frac{x_n}{x_{n-1}}\right)^2 + B\left(\frac{x_n}{x_{n-1}}\right) + C}{a\left(\frac{x_n}{x_{n-1}}\right)^2 + b\left(\frac{x_n}{x_{n-1}}\right) + c}, \quad n = 0, 1, 2, ... \tag{4}$$

and one can take the advantage of this auxiliary equation to describe the dynamics of Eq.(2). In this paper we take a different approach based on the monotonic properties of the right hand side of Eq.(2) and theory of monotone maps, and use it to describe precisely the basins of attraction of all attractors of that equation. The special cases of Eq.(1) when $C = 0$ or $a = 0$ are linear fractional difference equations which global dynamics is described in [11]. We show that Eq.(1) exhibits three types of global behavior characterized by the existence of a unique positive equilibrium solution and a unique minimal period-two solution, which stable manifold serves as the boundary of the basins of attraction of locally stable equilibrium and points at infinity $(0, \infty)$ and $(\infty, 0)$. In fact, Eq.(1) exhibits period-two bifurcation studied in great details in [12].

Equation (1) is a special case of the general second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, 2, ..., \tag{5}$$

where

$$f(u, v) = \frac{Buv + Cv^2}{au^2 + buv}.$$

The next result is important for the general second order difference equation of the form (5), see [2].

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for ever solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad x_{-1}, x_0 \in I, \ n = 0, 1, 2, ... \tag{6}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *of even and odd terms of the solution do exactly one of the following:*

(i) Eventually they are both monotonically increasing.

(ii) Eventually they are both monotonically decreasing.

(iii) One of them is monotonically increasing and the other is monotonically decreasing.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{7}$$

provided that $\mathcal{R}$ is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next five results, from [13, 12], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [18, 19].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 4 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve $\mathcal{C}$ of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [20] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *(A) Assume the hypotheses of Theorem 2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{8}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \mathrm{int}\, Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n\to-\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 2 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Eq.(6) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Eq.(6) is a strictly competitive map on $I \times I$, see [13].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Eq.(6) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots .$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [13].

Remark 2 *The characteristic equation of Eq.(6) at an equilibrium point $(\bar{x}, \bar{x})$:*

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{9}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decraesing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the nonexistence of minimal period-two solution.

There are several attractivity results for Eq. (6). Some of these results give the sufficient conditions for all solutions to approach a unique equilibrium and they were used efficiently in [11]. See also [1, 6]. Next result is from [6].

Theorem 6 *Consider Eq. (6) where $f : I \times I \to I$ is a continuous function and f is decreasing in the first argument and increasing in the second argument. Assume that $\overline{x}$ is a unique equilibrium point which is locally asymptotically stable and assume that (ϕ, ψ) and (ψ, ϕ) are minimal period-two solutions which are saddle points such that*

$$(\phi, \psi) \preceq_{se} (\overline{x}, \overline{x}) \preceq_{se} (\psi, \phi) .$$

Then, the basin of attraction $\mathcal{B}((\overline{x}, \overline{x}))$ of $(\overline{x}, \overline{x})$ is the region between the global stable sets $\mathcal{W}^s((\phi, \psi))$ and $\mathcal{W}^s((\psi, \phi))$. More precisely

$$\mathcal{B}((\overline{x}, \overline{x})) = \{(x, y) : \exists y_u, y_l : y_u < y < y_l, (x, y_l) \in \mathcal{W}^s((\phi, \psi)), (x, y_u) \in \mathcal{W}^s((\psi, \phi))\} .$$

The basins of attraction $\mathcal{B}((\phi, \psi)) = \mathcal{W}^s((\phi, \psi))$ and $\mathcal{B}((\psi, \phi)) = \mathcal{W}^s((\psi, \phi))$ are exatly the global stable sets of (ϕ, ψ) and (ψ, ϕ).

If $(x_{-1}, x_0) \in \mathcal{W}_+((\psi, \phi))$ or $(x_{-1}, x_0) \in \mathcal{W}_-((\phi, \psi))$, then $T^n((x_{-1}, x_0))$ converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region $I \times I$.

2 Local stability analysis

First, notice the function $f(u, v)$ is decreasing in the first variable and increasing in the second variable. By Theorem 1, for every solutions $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventualy monotonic.

It is clear that Equation (1) has a unique positive equilibrium point $\overline{x} = \frac{B+C}{a+b}$. If we denote

$$f(u, v) = \frac{Buv + Cv^2}{au^2 + buv},$$

then a linearization of Equation (1) is of the form

$$y_{n+1} = sy_n + ty_{n-1},$$

where

$$\begin{aligned} s &= -t=\frac{\partial f}{\partial u}\left(\overline{x},\overline{x}\right)=\left(\frac{-aBu^{2}v-2aCuv^{2}-bCv^{3}}{\left(au^{2}+buv\right)^{2}}\right)\left(\overline{x},\overline{x}\right) \\ &= -\frac{aB\overline{x}^{3}+2aC\overline{x}^{3}+bC\overline{x}^{3}}{\left(a\overline{x}^{2}+b\overline{x}^{2}\right)^{2}}=-\frac{1}{\overline{x}}\cdot\frac{aB+2aC+bC}{\left(a+b\right)^{2}} \\ &= -\frac{aB+2aC+bC}{\left(a+b\right)\left(B+C\right)}. \end{aligned}$$

Lemma 1 *Equation (1) has a unique positive equilibrium point* $\overline{x}=\frac{B+C}{a+b}$.

i) If $(3a+b)\,C<(b-a)\,B$, *then the equilibrium point* $\overline{x}$ *is locally asymptotically stable.*

ii) If $b\leq a$ *or if* $b>a$ *and* $(3a+b)\,C>(b-a)\,B$, *then the equilibrium point* $\overline{x}$ *is a saddle point.*

iii) If $(3a+b)\,C=(b-a)\,B$, *then the equilibrium point* $\overline{x}$ *is non-hyperbolic (with eigenvalues* $\lambda_1=-1$ *and* $\lambda_2=\frac{1}{2}$*).*

Proof. i) Equilibrium point $\overline{x}$ is locally asymptotically stable if

$$|s|<1-t<2\Leftrightarrow|s|<1+s<2\Leftrightarrow-\frac{1}{2}<s<1.$$

Since $s<0$, we have

$$\begin{aligned} s &> -\frac{1}{2}\Leftrightarrow-\frac{aB+2aC+bC}{\left(a+b\right)\left(B+C\right)}>-\frac{1}{2} \\ &\Leftrightarrow 2\left(aB+2aC+bC\right)<\left(a+b\right)\left(B+C\right)=aB+aC+bB+bC \\ &\Leftrightarrow aB+3aC+bC<bB\Leftrightarrow\left(3a+b\right)C<\left(b-a\right)B, \end{aligned}$$

which implies that $b-a>0$. Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $(3a+b)\,C<(b-a)\,B$.

ii) If $|s|>|1-t|$ and $s^2+4t>0$, then the equilibrium point $\overline{x}$ is a saddle point. We obtain

$$s^2+4t>0\Leftrightarrow t^2+4t>0,$$

which is satisfied by $t>0$, and

$$\begin{aligned} |s| &> |1-t|\Leftrightarrow-s>|1+s|\Leftrightarrow s<1+s<-s\Rightarrow s<-\frac{1}{2} \\ &\Leftrightarrow -\frac{aB+2aC+bC}{\left(a+b\right)\left(B+C\right)}<-\frac{1}{2}\Leftrightarrow\left(3a+b\right)C>\left(b-a\right)B. \end{aligned}$$

Notice that, if $b-a\leq0$ then $(3a+b)\,C>(b-a)\,B$ is satisfied, so equiilibrium point $\overline{x}$ is a saddle point if $b\leq a$ or if $b>a$ and $(3a+b)\,C>(b-a)\,B$.

iii) Equilibrium point $\overline{x}$ is non-hyperbolic point if

$$|s|=|1-t|\Leftrightarrow s=-\frac{1}{2}\Leftrightarrow\left(3a+b\right)C=\left(b-a\right)B.$$

Since $s=-t=-\frac{1}{2}$, then the characteristic equation at the equilibrium point is of the form

$$\lambda^2+\frac{1}{2}\lambda-\frac{1}{2}=0,$$

with eigenvalues $\lambda_1=-1$ and $\lambda_2=\frac{1}{2}$. □

3 Periodic solutions

In this section we present results about existence of minimal period-two solutions of Eq. (1).

Theorem 7 *If $b \le a$ or $B \le C$ or $(3a+b)\,C \ge (b-a)\,B$, then Eq. (1) has no minimal period-two solutions.*

b) If $b > a$ and $(3a+b)\,C < (b-a)\,B$, then Equation (1) has minimal period-two solution:

$$\phi, \psi, \phi, \psi, \ldots (\phi \neq \psi \text{ and } \phi > 0 \text{ and } \psi > 0),$$

where

$$\phi = \frac{B-C}{2a}\left(1-\sqrt{D}\right),\ \psi = \frac{B-C}{2a}\left(1+\sqrt{D}\right),\ D = 1 - \frac{4aC}{(b-a)\,(B-C)} > 0.$$

Proof. Notice that $b > a$ and $(3a+b)\,C < (b-a)\,B$ imply that $B > C$.

Suppose that there is a minimal period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ of Eq.(1), where ϕ and ψ are distinct positive real numbers. Then, we have

$$\begin{array}{c} \phi = \frac{B\psi\phi + C\phi^2}{a\psi^2 + b\psi\phi}, \\ \psi = \frac{B\phi\psi + C\psi^2}{a\phi^2 + b\phi\psi}, \end{array} \tag{10}$$

from which we obtain (since that $\phi > 0$ and $\psi > 0$)

$$B\psi + C\phi = a\psi^2 + b\psi\phi, \tag{11}$$

$$B\phi + C\psi = a\phi^2 + b\phi\psi. \tag{12}$$

Subtracting Eq.(12) from Eq.(11), we have

$$B\,(\psi - \phi) + C\,(\phi - \psi) = a\left(\psi^2 - \phi^2\right),$$

i.e.

$$\psi + \phi = \frac{B-C}{a}. \tag{13}$$

Adding Eq.(11) to Eq.(12) we obtain

$$B\,(\psi + \phi) + C\,(\phi + \psi) = a\left(\psi^2 + \varphi^2\right) + 2b\psi\phi$$

i.e.

$$(B+C)\,(\psi + \phi) = a\,(\psi + \phi)^2 + 2\,(b-a)\,\psi\phi. \tag{14}$$

Substituting (13) into (14), we have

$$(B+C)\,\frac{B-C}{a} = a\left(\frac{B-C}{a}\right)^2 + 2\,(b-a)\,\psi\phi$$

i.e.

$$\psi\phi = \frac{C}{a}\frac{B-C}{b-a}. \tag{15}$$

From (13) and (15) we see that Eq.(1) has no minimal period-two solutions if

$$B - C \le 0 \text{ or } b - a \le 0.$$

By (13) and (15) we have that positive ϕ and ψ satisfy the quadratic equation

$$\phi^2 - \frac{B-C}{a}\phi + \frac{C}{a}\frac{B-C}{b-a} = 0, \quad (b > a \text{ and } B > C)$$

with solutions

$$\phi_{\pm} = \frac{B-C}{2a}\left(1 \pm \sqrt{D}\right),$$

where

$$D = 1 - \frac{4aC}{(b-a)(B-C)} = \frac{B(b-a) - C(3a+b)}{(b-a)(B-C)} > 0 \Leftrightarrow (3a+b)C < (b-a)B.$$

Equation (13) implies that

$$\psi_{\pm} = \frac{B-C}{a} - \phi_{\pm} = \phi_{\mp}.$$

□

By substitution

$$\begin{cases} x_{n-1} = u_n, \\ x_n = v_n. \end{cases}$$

Eq.(1) is transformed to the system of equations

$$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = \dfrac{Bu_nv_n + Cu_n^2}{av_n^2 + bu_nv_n} \end{cases}. \tag{16}$$

The map T corresponding to the system (16) is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ h(u,v) \end{pmatrix},$$

where $h(u,v) = \dfrac{Buv + Cu^2}{av^2 + buv}$. The second iteration of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} v \\ h(u,v) \end{pmatrix} = \begin{pmatrix} h(u,v) \\ h(v, h(u,v)) \end{pmatrix} = \begin{pmatrix} h(u,v) \\ k(u,v) \end{pmatrix},$$

where

$$k(u,v) = \frac{Bvh(u,v) + Cu^2}{ah^2(u,v) + bvh(u,v)},$$

and the map T^2 is competitive by Remark 2 . The Jacobian matrix of the map T^2 is

$$J_{T^2}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} \end{pmatrix}.$$

Now we obtain that Jacobian matrix of the map T^2 at the point (ϕ, ψ) is of the form

$$J_{T^2}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \frac{\partial h}{\partial u}\binom{\phi}{\psi} & \frac{\partial h}{\partial v}\binom{\phi}{\psi} \\ \frac{\partial k}{\partial u}\binom{\phi}{\psi} & \frac{\partial k}{\partial v}\binom{\phi}{\psi} \end{pmatrix},$$

where

$$\frac{\partial h}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \frac{aB\psi^2+2aC\phi\psi+bC\phi^2}{\psi\left(a\psi+b\phi\right)^2}, \tag{17}$$

$$\frac{\partial h}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = -\frac{\phi\left(aB\psi^2+2aC\phi\psi+bC\phi^2\right)}{\left(a\psi^2+b\phi\psi\right)^2}, \tag{18}$$

$$\frac{\partial k}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = -\frac{\left(aB\psi^2+2aC\phi\psi+bC\phi^2\right)\left(aB\phi^2+2aC\phi\psi+bC\psi^2\right)}{\phi^2\left(a\psi+b\phi\right)^2\left(a\phi+b\psi\right)^2}. \tag{19}$$

$$\frac{\partial k}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \frac{aB\phi^2+2aC\phi\psi+bC\psi^2}{\phi\left(a\phi+b\psi\right)^2}\left(1+\frac{aB\psi^2+2aC\phi\psi+bC\phi^2}{\psi\left(a\psi+b\phi\right)^2}\right). \tag{20}$$

The corresponding characteristic equation is

$$\lambda^2-p\lambda+q=0,$$

where

$$\begin{aligned} p &= \frac{\partial h}{\partial u}\begin{pmatrix}\phi\\ \psi\end{pmatrix}+\frac{\partial k}{\partial v}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \frac{aB\psi^2+2aC\phi\psi+bC\phi^2}{\psi\left(a\psi+b\phi\right)^2} \\ &\quad +\frac{aB\phi^2+2aC\phi\psi+bC\psi^2}{\phi\left(a\phi+b\psi\right)^2}\left(1+\frac{aB\psi^2+2aC\phi\psi+bC\phi^2}{\psi\left(a\psi+b\phi\right)^2}\right)\end{aligned}$$

i.e.

$$p=a^*+b^*\left(1+a^*\right)=a^*+b^*+a^*b^*,$$

where

$$a^*=\frac{aB\psi^2+2aC\phi\psi+bC\phi^2}{\psi\left(a\psi+b\phi\right)^2},\quad b^*=\frac{aB\phi^2+2aC\phi\psi+bC\psi^2}{\phi\left(a\phi+b\psi\right)^2}.$$

Notice that now is

$$\begin{aligned} &\tfrac{\partial h}{\partial u}\left(\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right)=a^*, && \tfrac{\partial h}{\partial v}\left(\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right)=-\tfrac{\phi}{\psi}a^*, \\ &\tfrac{\partial k}{\partial u}\left(\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right)=-\tfrac{\psi}{\phi}a^*b^*, && \tfrac{\partial k}{\partial v}\left(\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right)=b^*\left(1+a^*\right),\end{aligned}$$

so that

$$\begin{aligned} q &= \det J_{T^2}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=a^*b^*\left(1+a^*\right)-a^{*2}b^*=a^*b^* \\ p &= a^*+b^*+a^*b^*.\end{aligned}$$

Theorem 8 *Suppose that Eq.(1) has the minimal period-two solution. Then, this solution is a saddle point.*

Proof. We need show that

$$|p|>|1+q| \quad\text{and}\quad p^2-4q>0.$$

Indeed,

(i) $p^2-4q>0 \Leftrightarrow \left(a^*+b^*+a^*b^*\right)^2-4a^*b^*>0$

$\Leftrightarrow a^{*2}+b^{*2}+a^{*2}b^{*2}+2a^{*2}b^*+2a^*b^{*2}>2a^*b^*,$

which is sasfied because of $a^{*2}+b^{*2}\geq 2a^*b^*$.

(ii)

$$|p| > |1+q| \Leftrightarrow p > 1+q \Leftrightarrow a^* + b^* > 1$$

$$\begin{aligned}
&\Leftrightarrow \quad \frac{aB\psi^2 + 2aC\phi\psi + bC\phi^2}{\psi\left(a\psi + b\phi\right)^2} + \frac{aB\phi^2 + 2aC\phi\psi + bC\psi^2}{\phi\left(a\phi + b\psi\right)^2} > 1 \\
&\Leftrightarrow \quad \left(aB\psi^2 + 2aC\phi\psi + bC\phi^2\right)\phi\left(a\phi + b\psi\right)^2 + \left(aB\phi^2 + 2aC\phi\psi + bC\psi^2\right)\psi\left(a\psi + b\phi\right)^2 \\
&> \quad \phi\psi\left(a\phi + b\psi\right)^2\left(a\psi + b\phi\right)^2.
\end{aligned}$$

By (11) and (12) we have

$$\begin{aligned}
&|p| > |1+q| \\
&\qquad \Leftrightarrow \quad \left(aB\psi^2 + 2aC\phi\psi + bC\phi^2\right)\phi\left(a\phi + b\psi\right)^2 + \left(aB\phi^2 + 2aC\phi\psi + bC\psi^2\right)\psi\left(a\psi + b\phi\right)^2 \\
&\qquad > \quad \left(B\psi + C\phi\right)\left(a\psi + b\phi\right)\left(B\phi + C\psi\right)\left(a\phi + b\psi\right)
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad L = abBC\left(\phi^4 + \psi^4\right) + 2aC\left(aB + bC\right)\phi\psi\left(\phi^2 + \psi^2\right) + \left(a^2B^2 + 3a^2C^2 + b^2C^2 + 2abBC - b^2B^2\right)\phi^2\psi^2 > 0 \\
&\Leftrightarrow \quad L = abBC\left(\phi^2 + \psi^2\right)^2 + 2aC\left(aB + bC\right)\phi\psi\left(\phi^2 + \psi^2\right) + \left(a^2B^2 + 3a^2C^2 + b^2C^2 - b^2B^2\right)\phi^2\psi^2 > 0.
\end{aligned}$$

Using (13) and (15) we have

$$\phi^2 + \psi^2 = \left(\phi + \psi\right)^2 - 2\phi\psi = \left(B - C\right)\frac{bB - aB - aC - bC}{a^2\left(b - a\right)},$$

which implies

$$\begin{aligned}
L \quad &= \quad abBC\left[\left(B - C\right)\frac{bB - aB - aC - bC}{a^2\left(b - a\right)}\right]^2 + 2aC\left(aB + bC\right)\frac{C\left(B - C\right)^2\left(bB - aB - aC - bC\right)}{a^3\left(b - a\right)^2} \\
&\quad + \left(a^2B^2 + 3a^2C^2 + b^2C^2 - b^2B^2\right)\frac{C^2\left(B - C\right)^2}{a^2\left(b - a\right)^2} \\
&= \quad C\left(B - C\right)^3\left[\left(b - a\right)B - \left(3a + b\right)C\right]\frac{bB - aC}{a^3\left(b - a\right)} > 0,
\end{aligned}$$

which is satisfied because $B - C > 0, b - a > 0$ and $\left(3a + b\right)C < \left(b - a\right)B$. Therefore, $L > 0 \Leftrightarrow |p| > |1+q|$, i.e. the minimal period-two solution of Eq.(1) is a saddle point. □

4 Global results and basins of attraction

In this section we present results about basins of attraction of Eq.(1).

Theorem 9 *If $\left(3a + b\right)C < \left(b - a\right)B$, then Eq.(1) has a unique equilibrium point $\overline{x}$ which is locally asymptotically stable, and two minimal period-two points $\left(\phi, \psi\right)$ and $\left(\psi, \phi\right)$ which are saddle points. Then, the basin of attraction $\mathcal{B}\left(\left(\overline{x}, \overline{x}\right)\right)$ of $\left(\overline{x}, \overline{x}\right)$ is the region between the global stable sets $\mathcal{W}^s\left(\left(\phi, \psi\right)\right)$ and $\mathcal{W}^s\left(\left(\psi, \phi\right)\right)$. The basins of attraction $\mathcal{B}\left(\left(\phi, \psi\right)\right) = \mathcal{W}^s\left(\left(\phi, \psi\right)\right)$ and $\mathcal{B}\left(\left(\psi, \phi\right)\right) = \mathcal{W}^s\left(\left(\psi, \phi\right)\right)$ are exatly the global stable sets of $\left(\phi, \psi\right)$ and $\left(\psi, \phi\right)$. Furthermore,*

i) if $\left(x_{-1}, x_0\right) \in \mathcal{W}_-\left(\left(\phi, \psi\right)\right)$, then $\lim_{n\to\infty} x_{2n} = \infty$ and $\lim_{n\to\infty} x_{2n+1} = 0$;

ii) if $\left(x_{-1}, x_0\right) \in \mathcal{W}_+\left(\psi, \phi\right)$, then $\lim_{n\to\infty} x_{2n} = 0$ and $\lim_{n\to\infty} x_{2n+1} = \infty$.

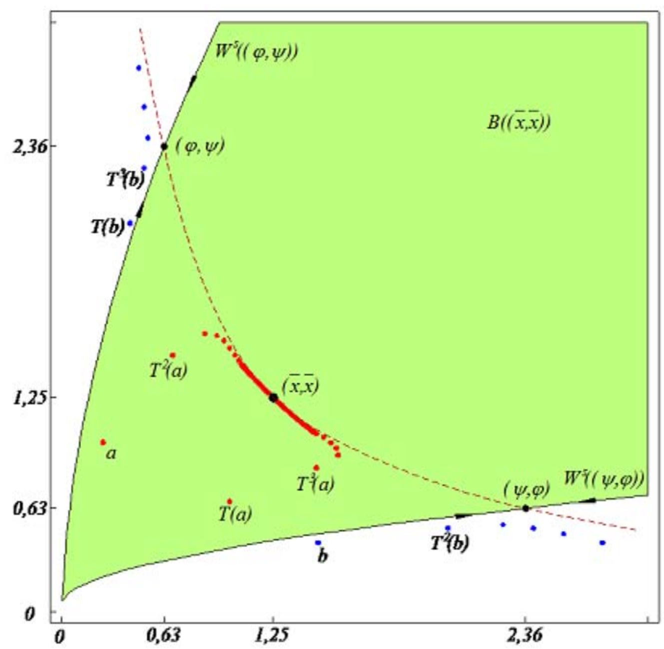

Figure 1: The orbits of solutions with $B=4$, $C=1$, $a=1$, $b=3$.

Proof. Using assumption $(3a+b)\,C<(b-a)\,B$ and its consequence $\frac{B+C}{B-C}\cdot\frac{2a}{a+b}$, it is easy to check that $(\phi,\psi)\preceq_{se}(\overline{x},\overline{x})\preceq_{se}(\psi,\phi)$, where

$$\begin{aligned}\phi &= \frac{B-C}{2a}\left(1-\sqrt{D}\right),\ \psi=\frac{B-C}{2a}\left(1+\sqrt{D}\right),\\ D &= 1-\frac{4aC}{(b-a)\,(B-C)}=\frac{B\,(b-a)-C\,(3a+b)}{(b-a)\,(B-C)},\end{aligned}$$

i.e.

$$\phi\le\overline{x}\le\psi.$$

Since the equilibrium point $(\overline{x},\overline{x})$ is locally asymptotically stable for T it is also locally asymptotically stable for T^2. Equation (1) corresponds to the system of difference equations (16) which can be decomposed into the system of the even-indexed and odd-indexed terms as follows:

$$\begin{cases} u_{2n}=v_{2n-1},\\ u_{2n+1}=v_{2n},\\ v_{2n}=\dfrac{Bu_{2n-1}v_{2n-1}+Cv_{2n-1}^2}{au_{2n-1}^2+bu_{2n-1}v_{2n-1}},\\ v_{2n+1}=\dfrac{Bu_{2n}v_{2n}+Cv_{2n}^2}{au_{2n}^2+bu_{2n}v_{2n}}.\end{cases}\tag{21}$$

The conclusion folows from Lema 1 and from Theorems 6, 7 and 8 and using the facts:

i) if $(u_0,v_0)\in\mathcal{W}_-((\phi,\psi))$, then

$$(u_{2n},v_{2n})=T^{2n}((u_0,v_0))\to(0,\infty)\quad\text{and}\quad(u_{2n+1},v_{2n+1})=T^{2n+1}((u_0,v_0))\to(\infty,0);$$

ii) if $(u_0, v_0) \in \mathcal{W}_+((\psi, \phi))$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (\infty, 0) \quad \text{and} \quad (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (0, \infty).$$

Consequently,

i) if $(x_{-1}, x_0) \in \mathcal{W}_-((\phi, \psi))$, then $T^{2n}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n+1}((x_{-1}, x_0)) \to (\infty, 0)$, i. e.

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = 0;$$

ii) if $(x_{-1}, x_0) \in \mathcal{W}_+((\psi, \phi))$, then $T^{2n+1}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n}((x_{-1}, x_0)) \to (\infty, 0)$, i. e.

$$\lim_{n\to\infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = \infty.$$

(See Figure 1 and Figure 2.) □

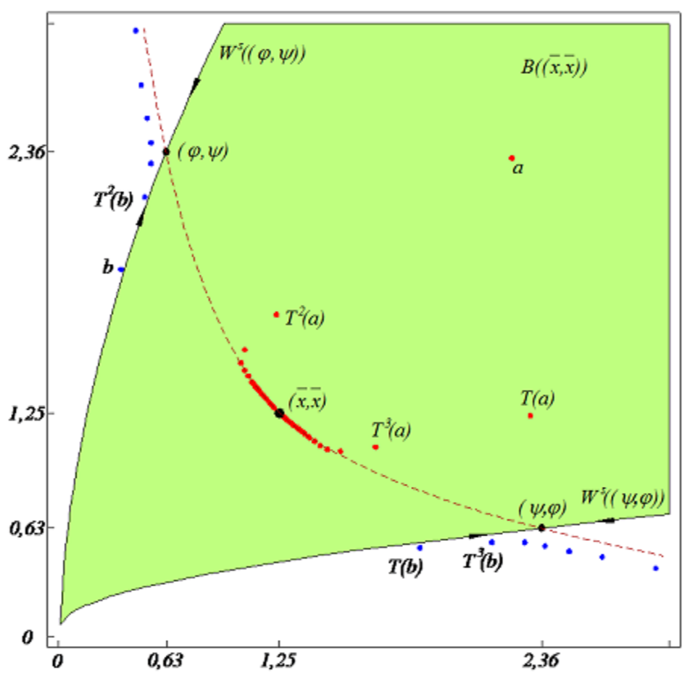

Figure 2: The orbits of solutions with $B = 4$, $C = 1$, $a = 1$, $b = 3$.

Theorem 10 *If $b \le a$ or if $b > a$ and $(3a + b)C > (b - a)B)$, then Eq.(1) has a unique equilibrium point $\overline{x}$ which is saddle point and has no minimal period-two solutions . There exists a set $\mathcal{C}$ which is an invariant subset of the basin of attraction of $(\overline{x}, \overline{x})$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separets $\mathcal{R} = (0, \infty) \times (0, \infty)$ into two connected and invariant components $\mathcal{W}_-((\overline{x}, \overline{x}))$ and $\mathcal{W}_+((\overline{x}, \overline{x}))$ which satisfy:*

i) if $(x_{-1}, x_0) \in \mathcal{W}_-((\overline{x}, \overline{x}))$, then $\lim_{n\to\infty} x_{2n} = \infty$ and $\lim_{n\to\infty} x_{2n+1} = 0$;

ii) if $(x_{-1}, x_0) \in \mathcal{W}_+((\overline{x}, \overline{x}))$, then $\lim_{n\to\infty} x_{2n} = 0$ and $\lim_{n\to\infty} x_{2n+1} = \infty$.

Proof. It is easy to check that $(\overline{x}, \overline{x})$ is a saddle point for the strictly competitive map T^2 as well. The existence of the set $\mathcal{C}$ with stated properties follows from Lema 1 and Theorems 7, 2 and 4. Therefore, using (21), we obtain:

i) if $(u_0, v_0) \in \mathcal{W}_-((\overline{x}, \overline{x}))$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (0, \infty) \quad \text{and} \quad (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (\infty, 0);$$

ii) if $(u_0, v_0) \in \mathcal{W}_+((\overline{x}, \overline{x}))$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (\infty, 0) \quad \text{and} \quad (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (0, \infty).$$

Consequently,

i) if $(x_{-1}, x_0) \in \mathcal{W}_-((\overline{x}, \overline{x}))$, then $T^{2n}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n+1}((x_{-1}, x_0)) \to (\infty, 0)$, that is

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = 0;$$

ii) if $(x_{-1}, x_0) \in \mathcal{W}_+((\overline{x}, \overline{x}))$, then $T^{2n+1}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n}((x_{-1}, x_0)) \to (\infty, 0)$, that is

$$\lim_{n\to\infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = \infty.$$

(See Figure 3.) □

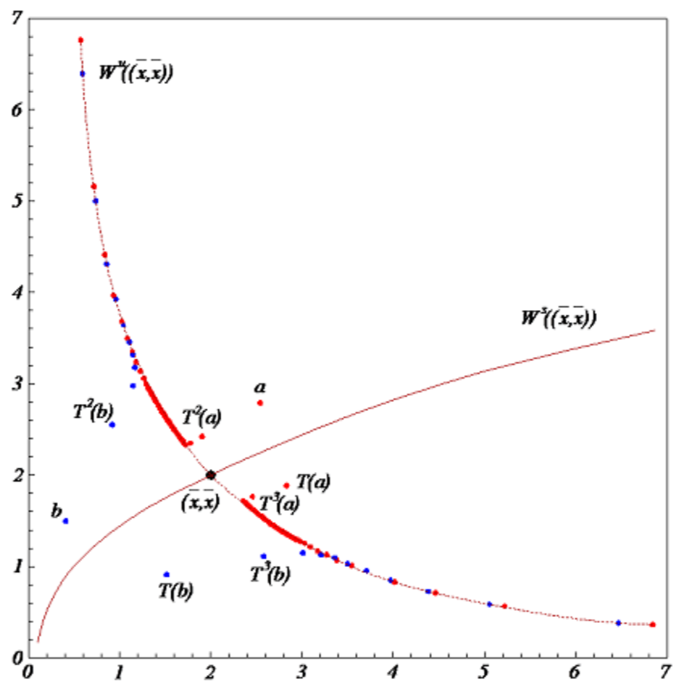

Figure 3: The orbits of solutions with $B = 4$, $C = 1$, $a = 2$, $b = 1$.

Theorem 11 *If $(3a+b)\,C = (b-a)\,B$, then Eq.(1) has a unique equilibrium point $\overline{x}$ which is non-hyperbolic and has no minimal period-two solutions. There exists a set $\mathcal{C}$ which is an invariant subset of the basin of attraction of $\overline{x}$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval and separates $\mathcal{R} = (0,\infty)\times(0,\infty)$ into two connected and invariant components $\mathcal{W}_-\left((\overline{x},\overline{x})\right)$ and $\mathcal{W}_+\left((\overline{x},\overline{x})\right)$ which satisfy:*

i) if $(x_{-1},x_0) \in \mathcal{W}_-\left((\overline{x},\overline{x})\right)$, then $\lim\limits_{n\to\infty} x_{2n} = \infty$ and $\lim\limits_{n\to\infty} x_{2n+1} = 0$;

ii) if $(x_{-1},x_0) \in \mathcal{W}_+\left((\overline{x},\overline{x})\right)$, $\lim\limits_{n\to\infty} x_{2n} = 0$ and $\lim\limits_{n\to\infty} x_{2n+1} = \infty$.

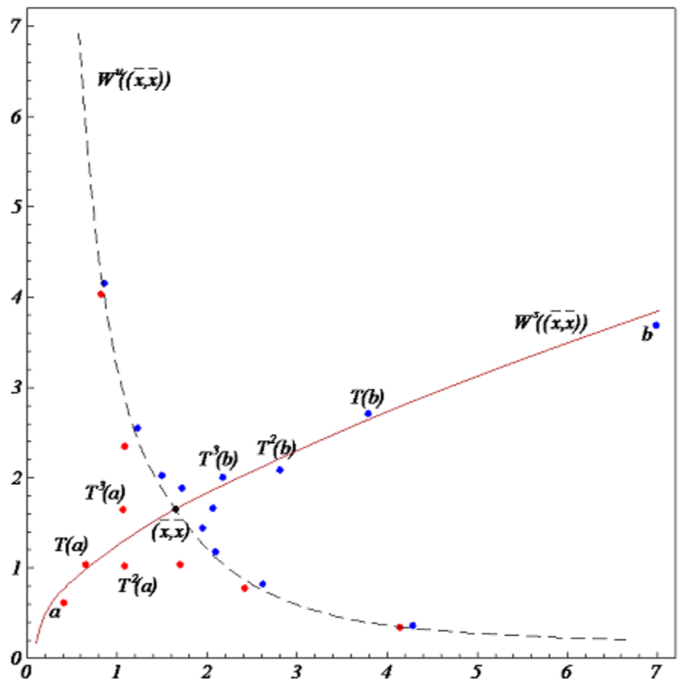

Figure 4: The orbits of solutions with $B=5$, $C=1$, $a=1$, $b=2$.

Proof. By Lema 1, the eigenvalues of the map T at the equilibrium point $(\overline{x},\overline{x})$ are $\lambda_1=-1$, $\lambda_2=\frac{1}{2}$, which means that $\mu_1=\lambda_1^2=1$ and $\mu_2=\lambda_2^2=\frac{1}{4}$ are eigenvalues of the map T^2. Using (17), (18), (19) and (20) we obtain

$$J_{T^2}(\overline{x},\overline{x}) = \left(\begin{array}{cc} R & -R \\ -R^2 & R\,(1+R) \end{array}\right),$$

where $R=\frac{bB-aC}{(a+b)(B+C)}$. The straight-forward calculation shows that the eigenvector corresponding the eigenvalue $\mu_2=\frac{1}{4}$ is of the form

$$\mathbf{v}_2 = \left(\begin{array}{c} v_1 \\ \left(1-\frac{1}{4R}\right)v_1 \end{array}\right),\ v_1 \in \mathbb{R}\setminus\{0\},$$

which shows that eigenvector $\mathbf{v}_2$ is not parallel to coordinate axes. Therefore all conditions of Theorem 2 are satisfied for the map T^2 with $\mathcal{R}=(0,\infty)\times(0,\infty)$. As a consequence of this and using (21), we have:

i) if $(u_0,v_0)\in\mathcal{W}_-\left((\overline{x},\overline{x})\right)$, then $(u_{2n},v_{2n})\to(0,\infty)$ and $(u_{2n+1},v_{2n+1})\to(\infty,0)$;

ii) if $(u_0, v_0) \in \mathcal{W}_+((\overline{x}, \overline{x}))$, then $(u_{2n}, v_{2n}) \to (\infty, 0)$ and $(u_{2n+1}, v_{2n+1}) \to (0, \infty)$.

It means that:

i) if $(x_{-1}, x_0) \in \mathcal{W}_-((\overline{x}, \overline{x}))$, then $T^{2n}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n+1}((x_{-1}, x_0)) \to (\infty, 0)$, i.e.

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = 0;$$

ii) if $(x_{-1}, x_0) \in \mathcal{W}_+((\overline{x}, \overline{x}))$, then $T^{2n+1}((x_{-1}, x_0)) \to (0, \infty)$ and $T^{2n}((x_{-1}, x_0)) \to (\infty, 0)$, i.e.

$$\lim_{n\to\infty} x_{2n} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = \infty.$$

(See Figure 4.) □

Remark 3 As one may notice from the figures all stable manifolds of either saddle point equilibrium points or non-hyperbolic equilibrium points or saddle period-two solutions are asymptotic to the origin, which is the point where Eq.(1) is not defined. These manifolds can not end in any other point on the axes since the union of axes without the origin is an invariant set, Thus the limiting points of the global stable manifolds of either saddle point equilibrium points or saddle period-two solutions have end points at $(0, 0)$ and (∞, ∞).

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Aspects of univalent holomorphic functions involving Sălăgean operator and Ruscheweyh derivative

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Abstract

Making use Sălăgean operator and Ruscheweyh derivative,we introduce a new class of analytic functions $\mathcal{L}(\gamma,\alpha,\beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity and neighborhood property for functions belonging to the class $\mathcal{L}(\gamma,\alpha,\beta)$.

Keywords: Analytic functions, univalent functions, radii of starlikeness and convexity, neighborhood property, Salagean operator, Ruscheweyh operator.
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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, which are analytic and univalent in the open unit disc $U=\{z:z\in\mathbb{C}:|z|<1\}$. $\mathcal{T}$ is a subclass of $\mathcal{A}$ consisting the functions of the form $f(z)=z-\sum_{j=2}^{\infty}|a_j|z^j$. For functions $f,g\in\mathcal{A}$ given by $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$, we define the Hadamard product (or convolution) of f and g by $(f*g)(z)=z+\sum_{j=2}^{\infty}a_jb_jz^j,\ z\in U$.

Definition 1.1 *Sălăgean [5]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator S^n is defined by $S^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}S^0f(z)&=f(z)\\ S^1f(z)&=zf'(z),\ \ldots\\ S^{n+1}f(z)&=z\left(S^nf(z)\right)',\quad z\in U.\end{aligned}$$

Remark 1.1 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $S^nf(z)=z+\sum_{j=2}^{\infty}j^na_jz^j$, $z\in U$.*
If $f\in\mathcal{T}$, $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, then $S^nf(z)=z-\sum_{j=t+1}^{\infty}j^na_jz^j$, $z\in U$.

Definition 1.2 *(Ruscheweyh [4]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}R^0f(z)&=f(z)\\ R^1f(z)&=zf'(z),\ \ldots\\ (n+1)R^{n+1}f(z)&=z\left(R^nf(z)\right)'+nR^nf(z),\quad z\in U.\end{aligned}$$

Remark 1.2 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $R^nf(z)=z+\sum_{j=2}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.*
If $f\in\mathcal{T}$, $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, then $R^nf(z)=z-\sum_{j=t+1}^{\infty}\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.

Definition 1.3 *[1], [2] Let $\gamma\geq 0$, $n\in\mathbb{N}$. Denote by L_γ^n the operator given by $L_\gamma^n:\mathcal{A}\to\mathcal{A}$,*

$$L_\gamma^nf(z)=(1-\gamma)R^nf(z)+\gamma S^nf(z),\qquad z\in U.$$

Remark 1.3 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $L_\gamma^nf(z)=z+\sum_{j=2}^{\infty}\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j$, $z\in U$.*
If $f\in\mathcal{T}$, $f(z)=z-\sum_{j=2}^{\infty}a_jz^j$, then $L_\gamma^nf(z)=z-\sum_{j=t+1}^{\infty}\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j$, $z\in U$.

Following the work of Sh.Najafzadeh and E.Pezeshki [3] we can define the class $\mathcal{L}(\gamma,\alpha,\beta)$ as follows.

Definition 1.4 *For $\gamma \geq 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $\mathcal{L}(\gamma,\alpha,\beta)$ be the subclass of $\mathcal{T}$ consisting of functions that satisfying the inequality*

$$\left|\frac{L_\gamma^{\mu,n} f(z) - 1}{2\nu(L_\gamma^{\mu,n} f(z) - \alpha) - (L_\gamma^{\mu,n} f(z) - 1)}\right| < \beta \tag{1.1}$$

where

$$L_\gamma^{\mu,n} f(z) = (1-\mu)\frac{L_\gamma^n f(z)}{z} + \mu(L_\gamma^n f(z))', \tag{1.2}$$

$0 < \nu \leq 1.$

Remark 1.4 *If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then*
$L_\gamma^{\mu,n} f(z) = 1 - \sum_{j=t+1}^{\infty} [1+\mu(j-1)] \left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^{j-1}$, $z \in U.$

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions is $\mathcal{L}(\gamma,\alpha,\beta)$.

Theorem 2.1 *Let the function $f \in \mathcal{T}$. Then $f \in \mathcal{L}(\gamma,\alpha,\beta)$ if and only if*

$$\sum_{j=t+1}^{\infty} (1+\mu(j-1))[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j < 2\beta\nu(1-\alpha). \tag{2.1}$$

The result is sharp for the function $F(z)$ defined by

$$F(z) = z - \frac{2\beta\nu(1-\alpha)}{(1+\mu(j-1))[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \qquad j \geq t+1.$$

Proof. Suppose f satisfies (2.1). Then for $|z| < 1$, we have

$$\left|L_\gamma^{\mu,n} f(z) - 1\right| - \beta\left|2\nu(L_\gamma^{\mu,n} f(z) - \alpha) - (L_\gamma^{\mu,n} f(z) - 1)\right| =$$

$$\left|-\sum_{j=t+1}^{\infty}(1+\mu(j-1))\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j z^{j-1}\right| -$$

$$\beta\left|2\nu(1-\alpha) - (2\nu-1)\sum_{j=t+1}^{\infty}\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}[1+\mu\,(j-1)]\, a_j z^{j-1}\right| \leq$$

$$\sum_{j=t+1}^{\infty}[1+\mu(j-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_k - 2\beta\nu(1-\alpha) +$$

$$\sum_{j=t+1}^{\infty}\beta(2\nu-1)(1+\mu(j-1))\left\{\gamma\,[1+(j-1)\,\lambda]^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j =$$

$$\sum_{j=t+1}^{\infty}[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} a_j - 2\beta\nu(1-\alpha) < 0$$

Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{L}(\gamma,\alpha,\beta)$. Conversely, assume that

$$\left|\frac{L_\gamma^{\mu,n} f(z) - 1}{2\nu(L_\gamma^{\mu,n} f(z) - \alpha) - L_\gamma^{\mu,n} f(z) - 1)}\right| =$$

$$\left|\frac{-\sum_{j=t+1}^{\infty}[1+\mu(j-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^{j-1}}{2\nu(1-\alpha) - \sum_{j=t+1}^{\infty}[1+\mu(j-1)]\,(2\nu-1)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} a_j z^{j-1}}\right| < \beta,\ z \in U.$$

Since $Re(z) \leq |z|$ for all $z \in U$, we have

$$Re\left\{\frac{\sum_{j=t+1}^{\infty}\left[1+\mu(j-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j-1}}{2\nu(1-\alpha)-\sum_{j=t+1}^{\infty}\left[1+\mu(j-1)\right](2\nu-1)\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j z^{j-1}}\right\}<\beta. \tag{2.2}$$

By choosing choose values of z on the real axis so that $L_{\gamma}^{\mu,n}f(z)$ is real and letting $z \to 1$ through real values, we obtain the desired inequality (2.1). ■

Corollary 2.2 *If $f \in \mathcal{T}$ be in $\mathcal{L}(\gamma,\alpha,\beta)$, then*

$$a_j \leq \frac{2\beta\nu(1-\alpha)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad j \geq t+1, \tag{2.3}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z) = z$ and*

$$f_j(z) = z - \frac{2\beta\nu(1-\alpha)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j, \quad j \geq t+1, \tag{2.4}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma \geq 0$ and $0 < \nu \leq 1$. Then $f(z)$ is in the class $\mathcal{L}(\gamma,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=t}^{\infty}\omega_j f_j(z), \tag{2.5}$$

where $\omega_j \geq 0$ and $\sum_{j=1}^{\infty}\omega_j = 1$.

Proof. Suppose $f(z)$ can be written as in (2.5). Then

$$f(z) = z - \sum_{j=t+1}^{\infty}\omega_j \frac{2\beta\nu(1-\alpha)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j.$$

Now,

$$\sum_{j=t+1}^{\infty}\frac{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}\omega_j$$

$$\frac{2\beta\nu(1-\alpha)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} = \sum_{j=t+1}^{\infty}\omega_j = 1-\omega_1 \leq 1.$$

Thus $f \in \mathcal{L}(\gamma,\alpha,\beta)$.

Conversely, let $f \in \mathcal{L}(\gamma,\alpha,\beta)$. Then by using (2.3), setting

$$\omega_j = \frac{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}a_j, \quad j \geq t+1$$

and $\omega_1 = 1 - \sum_{j=2}^{\infty}\omega_j$, we have $f(z) = \sum_{j=t}^{\infty}\omega_j f_j(z)$. And this completes the proof of Theorem 2.3. ■

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{L}(\gamma,\alpha,\beta)$.

Theorem 3.1 *If $f \in \mathcal{L}(\gamma, \alpha, \beta)$, then*

$$r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1} \leq |f(z)| \tag{3.1}$$

$$\leq r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1}$$

holds if the sequence $\{\sigma_j(\gamma,\beta,\nu)\}_{j=t+1}^{\infty}$ is non-decreasing, and

$$1 - \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t} \leq |f'(z)| \tag{3.2}$$

$$\leq 1 + \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t}$$

holds if the sequence $\{\frac{\sigma_j(\gamma,\beta,\nu)}{j}\}_{j=t+1}^{\infty}$ is non- decreasing, where

$$\sigma_j(\gamma,\beta,\nu) = [1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}.$$

The bounds in (3.1) and (3.2) are sharp, for $f(z)$ given by

$$f(z) = z - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} z^{t+1}, \quad z = \pm r. \tag{3.3}$$

Proof. In view of Theorem 2.1, we have

$$\sum_{j=t+1}^{\infty} a_j \leq \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]}. \tag{3.4}$$

We obtain

$$|z| - |z|^{t+1}\sum_{j=t+1}^{\infty} a_j \leq |f(z)| \leq |z| + |z|^{t+1}\sum_{j=t+1}^{\infty} a_j.$$

Thus

$$r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1} \leq |f(z)| \tag{3.5}$$

$$\leq r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]} r^{t+1}.$$

Hence (3.1) follows from (3.5). Further,

$$\sum_{j=t+1}^{\infty} j a_j \leq \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\frac{(t+n)!}{n!t!}\right]}.$$

Hence (3.2) follows from

$$1 - r^t\sum_{j=t+1}^{\infty} j a_j \leq |f'(z)| \leq 1 + r^t\sum_{j=t+1}^{\infty} j a_j.$$

∎

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{L}(\gamma, \alpha, \beta)$ are given in this section.

Theorem 4.1 *Let the function $f \in \mathcal{T}$ belong to the class $\mathcal{L}(\gamma, \alpha, \beta)$, Then $f(z)$ is close -to-convex of order δ, $0 \leq \delta < 1$ in the disc $|z| < r$, where*

$$r := \inf_{j \geq t+1} \left[\frac{(1-\delta)\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(1-\alpha)} \right]^{\frac{1}{t}}. \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

Proof. For given $f \in \mathcal{T}$ we must show that

$$|f'(z) - 1| < 1 - \delta. \tag{4.2}$$

By a simple calculation we have

$$|f'(z) - 1| \leq \sum_{j=t+1}^{\infty} j a_j |z|^t.$$

The last expression is less than $1-\delta$ if

$$\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j |z|^t < 1.$$

Using the fact that $f \in \mathcal{L}(\gamma, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} a_j \leq 1.$$

(4.2) holds true if

$$\frac{j}{1-\delta}|z|^t \leq \sum_{j=t+1}^{\infty} \frac{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t \leq \sum_{j=t+1}^{\infty} \frac{(1-\delta)\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(1-\alpha)},$$

which completes the proof. ■

Theorem 4.2 *Let $f \in \mathcal{L}(\gamma, \alpha, \beta)$. Then*

1. f is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$ where,

$$r_1 = \inf_{j \geq t+1} \left\{ \frac{(1-\delta)\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)(j-\delta)} \right\}^{\frac{1}{t}}.$$

2. f is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,

$$r_2 = \inf_{j \geq t+1} \left\{ \frac{(1-\delta)\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j(j-1)(1-\alpha)} \right\}^{\frac{1}{t}}.$$

Each of these results is sharp for the extremal function $f(z)$ given by (2.5).

Proof. 1. For $0 \leq \delta < 1$ we need to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{4.3}$$

We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{\sum_{j=t+1}^{\infty}(j-1)a_j\,|z|^t}{1 - \sum_{j=t+1}^{\infty} a_j\,|z|^t}\right|.$$

The last expresion is less than $1-\delta$ if

$$\sum_{j=t+1}^{\infty} \frac{(j-\delta)}{1-\delta} a_j\,|z|^t < 1.$$

Using the fact that $f \in \mathcal{L}(\gamma, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} a_j < 1.$$

(4.3) holds true if

$$\frac{j-\delta}{1-\delta}\,|z|^t < \frac{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t < \frac{(1-\delta)\,[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)\,(j-\delta)},$$

which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$|zf''(z)| < 1 - \delta. \tag{4.4}$$

We have

$$|zf''(z)| \leq \left|\sum_{j=t+1}^{\infty} j(j-1)a_j\,|z|^{t-1}\right| < 1-\delta$$

$$\sum_{j=t+1}^{\infty} \frac{j(j-1)}{1-\delta} a_j\,|z|^{t-1} < 1.$$

Using the fact that $f \in \mathcal{L}(\gamma, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)} a_j < 1.$$

(4.4) holds true if

$$\frac{j\,(j-1)}{1-\delta}\,|z|^{t-1} < \frac{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu(1-\alpha)},$$

or, equivalently,

$$|z|^{t-1} < \frac{(1-\delta)\,[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2\beta\nu j\,(j-1)\,(1-\alpha)},$$

which yields the convexity of the family. ∎

5 Neighborhood Property

In this section we study neighborhood property for functions in the class $\mathcal{L}(\gamma,\alpha,\beta)$.

Definition 5.1 *For functions f belong to $\mathcal{A}$ of the form and $\varepsilon \geq 0$, we define $\eta-\varepsilon$ - neighborhood of f by*

$$N_{\varepsilon}^{\eta}(f)=\{g(z)\in\mathcal{A}:g(z)=z+\sum_{j=2}^{\infty}b_jz^j,\ \ \sum_{j=2}^{\infty}j^{\eta+1}\left|a_j-b_j\right|\leq\varepsilon\},$$

where η is a fixed positive integer.

By using the following lemmas we will investigate the $\eta-\varepsilon$ - neighborhood of function in $\mathcal{L}(\gamma,\alpha,\beta)$.

Lemma 5.1 *Let $-1\leq\beta<1$, if $g(z)=z+\sum_{j=2}^{\infty}b_jz^j$ satisfies*

$$\sum_{j=2}^{\infty}j^{\rho+1}\left|b_j\right|\leq\frac{2\beta\nu(1-\alpha)}{1+\beta(2\nu-1)}$$

then $g(z)\in\mathcal{L}(\gamma,\alpha,\beta)$.

Proof. By using of Theorem 2.1, it is sufficient to show that

$$\frac{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^{\rho}+(1-\gamma)\,\frac{(\rho+j-1)!}{\rho!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}=\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}[1+\beta(2\nu-1)].$$

But

$$\frac{\left[1+\beta(2\nu-1)\right]\left\{\gamma j^{\rho}+(1-\gamma)\,\frac{(\rho+j-1)!}{\rho!(j-1)!}\right\}}{2\beta\nu(1-\alpha)}\leq\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}[1+\beta(2\nu-1)].$$

Therefore it is enough to prove that

$$Q(j,\rho)=\frac{\gamma j^{\rho}+(1-\gamma)\,\frac{(\rho+j-1)!}{\rho!(j-1)!}}{j^{\rho+1}}\leq 1,$$

the result follows because the last inequality holds for all $j\geq t+1$. ∎

Lemma 5.2 *Let $f(z)=z-\sum_{k=2}^{\infty}a_kz^k\in\mathcal{T}$, $\gamma\geq 0$, $0\leq\alpha<1$, $0<\beta\leq 1$ and $\varepsilon\geq 0$. If $\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{L}(\gamma,\alpha,\beta)$, then*

$$\sum_{j=t+1}^{\infty}j^{\rho+1}a_j\leq\frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\rho+1}}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left[\gamma(t+1)^n+(1-\gamma)\,\frac{(t+n)!}{n!t!}\right]},$$

where either $\rho=0$ or $\rho=1$. The result is sharp with the extremal function

$$f(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left[\gamma(t+1)^n+(1-\gamma)\,\frac{(t+n)!}{n!t!}\right]}z^{t+1},\quad z\in U.$$

Proof. Letting $g(z)=\frac{f(z)+\epsilon z}{1+\epsilon}$ we have

$$g(z)=z-\sum_{j=t+1}^{\infty}\frac{a_j}{1+\epsilon}z^j,\quad z\in U.$$

In view of Theorem 2.3, $g(z)=\sum_{j=1}^{\infty}\eta_jg_j(z)$ where $\eta_j\geq 0$, $\sum_{j=1}^{\infty}\eta_j=1$,

$$g_1(z)=z$$

and

$$g_j(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{\left[1+\mu(j-1)\right]\left[1+\beta(2\nu-1)\right]\left\{\gamma j^n+(1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j,\quad j\geq t+1.$$

So we obtain,

$$g(z) = \eta_1 z + \sum_{j=t+1}^{\infty} \eta_j [z - \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j]$$

$$= z - \sum_{j=t+1}^{\infty} \eta_k \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j.$$

Since $\eta_j \geq 0$ and $\sum_{j=2}^{\infty} \eta_j \leq 1$, it follows that

$$\sum_{j=t+1}^{\infty} a_k \leq \sup_{j \geq t+1} j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}}.$$

Since whenever $\rho = 0$ or $\rho = 1$ we conclude

$$W(j,\rho,\gamma,\alpha,\beta,\epsilon) = j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)]\,[1+\beta(2\nu-1)]\left\{\gamma j^n + (1-\gamma)\,\frac{(n+j-1)!}{n!(j-1)!}\right\}},$$

is a decreasing function of j, the result will follow. The proof is complete. ■

Theorem 5.1 *Let $\rho = 0$ or $\rho = 1$ and suppose $0 \leq \beta < 1$ and*

$$-1 \leq \theta < \frac{[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right] - 2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]}$$

$f(z) \in \mathcal{T}$ and $\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{L}(\gamma,\alpha,\beta)$, then the $\eta-\varepsilon$-neighborhood of f is the subset of $\mathcal{L}(\lambda,\alpha,\beta)$, where

$$\varepsilon \leq \frac{2\,(1-\alpha)\left\{\theta\gamma\,[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right] - \beta\gamma\,[1+\theta(2\nu-1)]\,(1+\epsilon)(t+1)^{\eta+1}\right\}}{[1+\theta(2\nu-1)]\,[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]}.$$

The result is sharp.

Proof. For $f(z) = z - \sum_{j=2}^{\infty} |a_j|\, z^j$, let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ be in $N_{\varepsilon}^{\eta}(f)$. So by Lemma 5.2, we have

$$\sum_{j=2}^{\infty} j^{\eta+1}\,|b_j| = \sum_{j=2}^{\infty} j^{\eta+1}\,|a_j - b_j - a_j|$$

$$\leq \varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]}.$$

By using Lemma 5.1, $g(z) \in \mathcal{L}(\gamma,\alpha,\beta)$ if

$$\varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)]\,[1+\beta(2\nu-1)]\left[\gamma(t+1)^n + (1-\gamma)\,\frac{(t+n)!}{n!t!}\right]} \leq \frac{2\theta\nu(1-\alpha)}{1+\theta(2\nu-1)},$$

that is, $\varepsilon \leq \frac{2(1-\alpha)\left\{\theta\gamma[1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(t+1)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]-\beta\gamma[1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1}\right\}}{[1+\theta(2\nu-1)][1+\mu(t-1)][1+\beta(2\nu-1)]\left[\gamma(t+1)^n+(1-\gamma)\frac{(t+n)!}{n!t!}\right]}$ and the proof is complete. ■

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Properties on a subclass of univalent functions defined by using a generalized Sălăgean operator and Ruscheweyh derivative

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Abstract

In this paper we have introduced and studied the subclass $\mathcal{RD}(d,\alpha,\beta)$ of univalent functions defined by the linear operator $RD^n_{\lambda,\gamma}f(z)$ defined by using the Ruscheweyh derivative $R^nf(z)$ and the generalized Sălăgean operator $D^n_\lambda f(z)$, as $RD^n_{\lambda,\alpha}:\mathcal{A}\to\mathcal{A}$, $RD^n_{\lambda,\gamma}f(z)=(1-\gamma)R^nf(z)+\gamma D^n_\lambda f(z)$, $z\in U$, where $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1=\mathcal{A}$. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{RD}(d,\alpha,\beta)$.

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1 Introduction

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$.

Definition 1.1 *(Al Oboudi [8]) For $f\in\mathcal{A}$, $\lambda\geq 0$ and $n\in\mathbb{N}$, the operator D^n_λ is defined by $D^n_\lambda:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned} D^0_\lambda f(z) &= f(z)\\ D^1_\lambda f(z) &= (1-\lambda)f(z)+\lambda zf'(z)=D_\lambda f(z),\ \dots\\ D^{n+1}_\lambda f(z) &= (1-\lambda)D^n_\lambda f(z)+\lambda z\left(D^n_\lambda f(z)\right)'=D_\lambda\left(D^n_\lambda f(z)\right),\quad z\in U. \end{aligned}$$

Remark 1.1 *If $f\in\mathcal{A}$ and $f(z)=z+\sum_{j=2}^\infty a_jz^j$, then $D^n_\lambda f(z)=z+\sum_{j=2}^\infty[1+(j-1)\lambda]^n a_jz^j$, $z\in U$.*

For $\lambda=1$ in the above definition we obtain the Sălăgean differential operator [13].

Definition 1.2 *(Ruscheweyh [12]) For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned} R^0f(z) &= f(z)\\ R^1f(z) &= zf'(z),\ \dots\\ (n+1)R^{n+1}f(z) &= z\left(R^nf(z)\right)'+nR^nf(z),\quad z\in U. \end{aligned}$$

Remark 1.2 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^\infty a_jz^j$, then $R^nf(z)=z+\sum_{j=2}^\infty\frac{(n+j-1)!}{n!(j-1)!}a_jz^j$, $z\in U$.*

Definition 1.3 *[3] Let $\gamma,\lambda\geq 0$, $n\in\mathbb{N}$. Denote by $RD^n_{\lambda,\alpha}$ the operator given by $RD^n_{\lambda,\alpha}:\mathcal{A}\to\mathcal{A}$,*

$$RD^n_{\lambda,\gamma}f(z)=(1-\gamma)R^nf(z)+\gamma D^n_\lambda f(z),\qquad z\in U.$$

Remark 1.3 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^\infty a_jz^j$, then*
$RD^n_{\lambda,\gamma}f(z)=z+\sum_{j=2}^\infty\left\{\gamma[1+(j-1)\lambda]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j$, $z\in U$.

This operator was studied also in [4], [6], [7], [9], [10].

Remark 1.4 *For $\gamma=0$, $RD^n_{\lambda,0}f(z)=R^nf(z)$, where $z\in U$ and for $\gamma=1$, $RD^n_{\lambda,1}f(z)=D^n_\lambda f(z)$, where $z\in U$. For $\lambda=1$, we obtain $RD^n_{1,\gamma}f(z)=L^n_\gamma f(z)$ which was studied in [1], [2] and [5].*

We follow the works of A.R. Juma and H. Ziraz .

Definition 1.4 *Let the function $f\in\mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{RD}(d,\alpha,\beta)$ if it satisfies the following criterion:*

$$\left|\frac{1}{d}\left(\frac{z(RD^n_{\lambda,\gamma}f(z))'+\alpha z^2(RD^n_{\lambda,\gamma}f(z))''}{(1-\alpha)RD^n_{\lambda,\gamma}f(z)+\alpha z(RD^n_{\lambda,\gamma}f(z))'}-1\right)\right|<\beta, \tag{1.1}$$

where $d\in\mathbb{C}-\{0\}$, $0\le\alpha\le1$, $0<\beta\le1$, $z\in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ, $0\le\delta<1$, for these functions.

2 Coefficient Inequality

Theorem 2.1 *Let the function $f\in\mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{RD}(d,\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j\le\beta|d|, \tag{2.1}$$

where $d\in\mathbb{C}-\{0\}$, $0\le\alpha\le1$, $0<\beta\le1$, $z\in U$.

Proof. Let $f(z)\in\mathcal{RD}(d,\alpha,\beta)$. Assume that inequality (2.1) holds true. Then we find that

$$\left|\frac{z(RD^n_{\lambda,\gamma}f(z))'+\alpha z^2(RD^n_{\lambda,\gamma}f(z))''}{(1-\alpha)RD^n_{\lambda,\gamma}f(z)+\alpha z(RD^n_{\lambda,\gamma}f(z))'}-1\right|$$

$$=\left|\frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}\right|$$

$$\le\frac{\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j|z|^{j-1}}<\beta|d|.$$

Choosing values of z on real axis and letting $z\to1^-$, we have

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j\le\beta|d|.$$

Conversely, assume that $f(z)\in\mathcal{RD}(d,\alpha,\beta)$, then we get the following inequality

$$Re\left\{\frac{z(RD^n_{\lambda,\gamma}f(z))'+\alpha z^2(RD^n_{\lambda,\gamma}f(z))''}{(1-\alpha)RD^n_{\lambda,\gamma}f(z)+\alpha z(RD^n_{\lambda,\gamma}f(z))'}-1\right|\}>-\beta|d|$$

$$Re\left\{\frac{z+\sum_{j=2}^{\infty}j(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}-1+\beta|d|\right\}>0$$

$$Re\frac{\beta|d|z+\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}{z+\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j}>0.$$

Since $Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to

$$\frac{\beta|d|r-\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jr^j}{r-\sum_{j=2}^{\infty}(1+\alpha(j-1))\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jr^j}>0.$$

Letting $r\to1^-$ and by the mean value theorem we have desired inequality (2.1).

This completes the proof of Theorem 2.1 ■

Corollary 2.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then*

$$a_j \leq \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}, \quad j \geq 2.$$

3 Distortion Theorems

Theorem 3.1 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$r - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r^2 \leq |f(z)|$$

$$\leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r^2.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}z^2.$$

Proof. Given that $f(z) \in \mathcal{RD}(d,\alpha,\beta)$, from the equation (2.1) and since

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]$$

is non decreasing and positive for $j \geq 2$,then we have

$$(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]\sum_{j=2}^{\infty}a_j \leq$$

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j \leq \beta|d|,$$

which is equivalent to,

$$\sum_{j=2}^{\infty}a_j \leq \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}. \tag{3.1}$$

Using (3.1), we obtain

$$f(z) = z + \sum_{j=2}^{\infty}a_j z^j$$

$$|f(z)| \leq |z| + \sum_{j=2}^{\infty}a_j|z|^j \leq r + \sum_{j=2}^{\infty}a_j r^j \leq r + r^2\sum_{j=2}^{\infty}a_j$$

$$\leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r^2.$$

Similarly,

$$|f(z)| \geq r^2 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r^2.$$

This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$-\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r \leq |f'(z)|$$

$$\leq \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}r.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n+(1-\gamma)\left(n+1\right)\right]}z^2.$$

Proof. From (3.1)

$$f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$$

$$|f'(z)| \leq 1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1} \leq 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \leq 1 + \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right]} r.$$

Similarly,

$$|f'(z)| \geq 1 - \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right]} r.$$

This completes the proof of Theorem 3.2. ■

4 Closure Theorems

Theorem 4.1 *Let the functions f_k, $k = 1, 2, ..., m$, defined by*

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j, \quad a_{j,k} \geq 0, \tag{4.1}$$

be in the class $\mathcal{RD}(d, \alpha, \beta)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{k=1}^{m} \mu_k f_k(z), \quad \mu_k \geq 0,$$

is also in the class $\mathcal{RD}(d, \alpha, \beta)$, where

$$\sum_{k=1}^{m} \mu_k = 1.$$

Proof. We can write

$$h(z) = \sum_{k=1}^{m} \mu_m z + \sum_{k=1}^{m} \sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty} \sum_{k=1}^{m} \mu_k a_{j,k} z^j.$$

Furthermore, since the functions $f_k(z)$, $k = 1, 2, ..., m$, are in the class $\mathcal{RD}(d, \alpha, \beta)$, then from Theorem 2.1 we have

$$\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j - 1 + \beta|d|) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!\,(j-1)!} \right\} a_{j,k} \leq \beta|d|.$$

Thus it is enough to prove that

$$\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j - 1 + \beta|d|) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!\,(j-1)!} \right\} \left(\sum_{k=1}^{m} \mu_k a_{j,k} \right) =$$

$$\sum_{k=1}^{m} \mu_k \sum_{j=2}^{\infty} (1 + \alpha(j-1))(j - 1 + \beta|d|) \left\{ \gamma \left[1 + (j-1)\lambda\right]^n + (1-\gamma) \frac{(n+j-1)!}{n!\,(j-1)!} \right\} a_{j,k}$$

$$\leq \sum_{k=1}^{m} \mu_k \beta|d| = \beta|d|.$$

Hence the proof is complete. ■

Corollary 4.2 *Let the functions f_k, $k = 1, 2$, defined by (4.1) be in the class $\mathcal{RD}(d, \alpha, \beta)$. Then the function $h(z)$ defined by*

$$h(z) = (1 - \zeta) f_1(z) + \zeta f_2(z), \quad 0 \leq \zeta \leq 1,$$

is also in the class $\mathcal{RD}(d, \alpha, \beta)$.

Theorem 4.3 *Let*

$$f_1(z) = z,$$

and

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \geq 2.$$

Then the function $f(z)$ is in the class $\mathcal{RD}(d,\alpha,\beta)$ if and only if it can be expressed in the form:

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z),$$

where $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

Proof. Assume that $f(z)$ can be expressed in the form

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z),$$

$$= z + \sum_{j=2}^{\infty} \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \mu_j z^j.$$

Thus

$$\sum_{j=2}^{\infty} \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} \cdot$$

$$\frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} \mu_j = \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1.$$

Hence $f(z) \in \mathcal{RD}(d,\alpha,\beta)$.

Conversely, assume that $f(z) \in \mathcal{RD}(d,\alpha,\beta)$.

Setting

$$\mu_j = \frac{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|} a_j,$$

since

$$\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j.$$

Thus

$$f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z).$$

Hence the proof is complete. ■

Corollary 4.4 *The extreme points of the class $\mathcal{RD}(d,\alpha,\beta)$ are the functions*

$$f_1(z) = z,$$

and

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \geq 2.$$

5 Inclusion and Neighborhood Results

We define the δ- neighborhood of a function $f(z) \in \mathcal{A}$ by

$$N_\delta(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|a_j - b_j| \le \delta\}. \tag{5.1}$$

In particular, for $e(z) = z$

$$N_\delta(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j|b_j| \le \delta\}. \tag{5.2}$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{RD}^\xi(d,\alpha,\beta)$ if there exists a function $h(z) \in \mathcal{RD}(d,\alpha,\beta)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1 - \xi, \quad z \in U, \quad 0 \le \xi < 1. \tag{5.3}$$

Theorem 5.1 *If*

$$\left\{\gamma\left[1 + (j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\} \ge \left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right], \quad j \ge 2$$

and

$$\delta = \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]},$$

then

$$\mathcal{RD}(d,\alpha,\beta) \subset N_\delta(e).$$

Proof. Let $f \in \mathcal{RD}(d,\alpha,\beta)$. Then in view of assertion (2.1) of Theorem 2.1 and the condition $\left\{\gamma\left[1 + (j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\} \ge \left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]$ for $j \ge 2$, we get

$$(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]\sum_{j=2}^{\infty} a_j \le$$

$$\sum_{j=2}^{\infty}(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!\,(j-1)!}\right\}a_j \le \beta|d|,$$

which implise

$$\sum_{j=2}^{\infty} a_j \le \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]}. \tag{5.4}$$

Applying assertion (2.1) of Theorem 2.1 in conjunction with (5.4), we obtain

$$(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]\sum_{j=2}^{\infty} a_j \le \beta|d|,$$

$$2(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]\sum_{j=2}^{\infty} a_j \le 2\beta|d|$$

$$\sum_{j=2}^{\infty} j a_j \le \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\,(1+\lambda)^n + (1-\gamma)\,(n+1)\right]} = \delta,$$

by virtue of (5.1), we have $f \in N_\delta(e)$.

This completes the proof of the Theorem 5.1. ■

Theorem 5.2 *If $h \in \mathcal{RD}(d,\alpha,\beta)$ and*

$$\xi = 1 - \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right]}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right] - \beta|d|}, \tag{5.5}$$

then

$$N_\delta(h) \subset \mathcal{RD}^\xi(d,\alpha,\beta).$$

Proof. Suppose that $f \in N_\delta(h)$, we then find from (5.1) that

$$\sum_{j=2}^{\infty} j|a_j - b_j| \le \delta,$$

which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty} |a_j - b_j| \le \frac{\delta}{2}. \tag{5.6}$$

Next, since $h \in \mathcal{RD}(d,\alpha,\beta)$ in the view of (5.4), we have

$$\sum_{j=2}^{\infty} b_j \le \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right]}. \tag{5.7}$$

Using 5.6) and (5.7), we get

$$\left|\frac{f(z)}{h(z)} - 1\right| \le \frac{\sum_{j=2}^{\infty}|a_j - b_j|}{1-\sum_{j=2}^{\infty} b_j} \le \frac{\delta}{2\left(1 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[\gamma(1+\lambda)^n+(1-\gamma)(n+1)]}\right)}$$

$$\le \frac{\delta}{2}\frac{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right]}{(1+\alpha)(1+\beta|d|)\left[\gamma\left(1+\lambda\right)^n + \left(1-\gamma\right)\left(n+1\right)\right] - \beta|d|} = 1-\xi,$$

provided that ξ is given by (5.5), thus by condition (5.3), $f \in \mathcal{RD}^\xi(d,\alpha,\beta)$, where ξ is given by (5.5). ∎

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then f is univalent starlike of order δ, $0 \le \delta < 1$, in $|z| < r_1$, where*

$$r_1 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \ge 2.$$

Proof. It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\delta, \quad |z| < r_1.$$

Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} a_j z^{k-1}}|\right| \le \frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j|z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} a_j|z|^{j-1}} \le 1-\delta.$$

It is equivalent to

$$\sum_{j=2}^{\infty}(j-\delta)a_j|z|^{j-1} \le 1-\delta,$$

using Theorem 2.1, we obtain

$$|z| \le \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. ■

Theorem 6.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then f is univalent convex of order δ, $0 \le \delta \le 1$, in $|z| < r_2$, where*

$$r_2 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{k-p}}.$$

The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}z^j, \quad j \ge 2. \tag{6.1}$$

Proof. It suffices to show that

$$\left|\frac{zf''(z)}{f'(z))}\right| \le 1-\delta, \qquad |z| < r_2.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty} ja_j z^{j-1}}\right| \le \frac{\sum_{j=2}^{\infty} j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j|z|^{j-1}}.$$

To prove the theorem, we must show that

$$\frac{\sum_{j=2}^{\infty} j(j-1)a_j|z|^{j-1}}{1-\sum_{j=2}^{\infty} ja_j|z|^{j-1}} \le 1-\delta,$$

$$\sum_{j=2}^{\infty} j(j-\delta)a_j|z|^{j-1} \le 1-\delta,$$

using Theorem 2.1, we obtain

$$|z|^{j-1} \le \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|},$$

or

$$|z| \le \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. ■

Theorem 6.3 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{RD}(d,\alpha,\beta)$. Then f is univalent close-to-convex of order δ, $0 \le \delta < 1$, in $|z| < r_3$, where*

$$r_3 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|} \right\}^{\frac{1}{j-1}}.$$

The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that

$$|f'(z)-1| \leq 1-\delta, \quad |z|<r_3.$$

Then

$$|f'(z)-1| = \left|\sum_{j=2}^{\infty} ja_j z^{j-1}\right| \leq \sum_{j=2}^{\infty} ja_j |z|^{j-1}.$$

Thus $|f'(z)-1| \leq 1-\delta$ if $\sum_{j=2}^{\infty} \frac{ja_j}{1-\delta}|z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if

$$|z|^{j-1} \leq \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}$$

or

$$|z| \leq \left\{\frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma\left[1+(j-1)\lambda\right]^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}}{j\beta|d|}\right\}^{\frac{1}{j-1}}.$$

Hence the proof is complete. ■

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On PPF dependent fixed point theorems and applications

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Abstract

In this paper, we provide an existence result of a fixed point for a mapping whose domain is $C([a,b],E)$ and its range is a subset of E, where E is a Banach space and $C([a,b],E)$ the space of all continuous functions from $[a,b]$ into E. Our proof is different and shorter than the proof given in [5] by using mild assumptions and omitting an ordering on the space. Finally we apply the fixed point obtained to give an existence result for an ordinary differential equation which is known as the periodic boundary value problem without any lower solution or an ordering on the space as assumed in [5].

Keywords : PPF dependent fixed points, Razumikhin classes, Partially ordered sets, Algebraic closedness with respect to a difference.
Mathematics Subject Classification 2000: 54H25, 55M20

1. Introduction and preliminary results

The fixed point theorems for mappings satisfying certain contractive conditions have been continually studied for several decades (see [2,5-11,15] and references contained therein). Bernfeld et al. [3] proved the existence of PPF (past, present and future) dependent fixed points in the Razumikhin class for mappings that have different domains and ranges. After that, Dhage [4] extended the existence of PPF dependent fixed points to PPF common dependent fixed points for mappings satisfying the weaker contractive conditions. In 2007, Drici et al. [5] proved fixed point theorems for mappings with PPF dependence in partially ordered metric spaces. A natural question may arise that "Is it possible to obtain the results obtained in [5] without assuming ordering on the space and furthermore relaxing some assumptions?" The main aim of this paper is to give a positive answer to this question. In the rest of this section, we need the following preliminaries.

Suppose that E is a Banach space with the norm $\|\cdot\|_E$ and I is a closed interval $[a,b]$ in the real line $\mathbb{R}$. Let $E_0 = C(I,E)$ be the set of all continuous

* corresponding author

E-valued mappings on I equipped with the norm $\|\cdot\|_{E_0}$ defined by

$$\|\phi\|_{E_0} = \sup_{t\in I} \|\phi(t)\|_E \quad \forall \phi \in E_0. \tag{1.1}$$

For a fixed element $c \in I$, the Razumikhin class of mappings in E_0 is defined by

$$\mathcal{R}_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_E\}. \tag{1.2}$$

Recall that a point $\phi \in E_0$ is called a PPF dependent fixed point or a fixed point with PPF dependence of a mapping $T : E_0 \to E$ if $T\phi = \phi(c)$ for some $c \in I$.

Remark 1.1 *One can embed the Banach space E into $E_0 = C(I, E)$ by defining $F : E \to E_0$ as $e \to F(e)$ and $F(e)(t) = e$ for all $t \in I$. It is obvious that F is a well-defined mapping and* $\sup_{t\in I} \|F(e)(t)\| = \|e\|_E$ *and so F is an isometry.*

Definition 1.2 Let A be a subset of E. Then

(i) A is said to be topologically closed with respect to the norm topology if for each sequence $\{x_n\}$ in A with $x_n \to x$ as $n \to \infty$ implies $x \in A$.

(ii) A is said to be algebraically closed with respect to the difference if $x-y \in A$ for all $x, y \in A$.

Definition 1.3 A mappings $T : E_0 \to E$ is said to satisfy the condition of Cirić type generalized contraction if there exists a real number $\lambda \in [0, 1)$ satisfying

$$\begin{aligned} \|T\phi - T\alpha\| \quad \le \quad & \lambda \max\Big\{\|\phi-\alpha\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\alpha(c) - T\alpha\|_E, \\ & \frac{1}{2}[\|\phi(c) - T\alpha\|_E + \|\alpha(c) - T\phi\|_E]\Big\}, \end{aligned} \tag{1.3}$$

for all $\phi, \alpha \in E_0$ and for some $c \in I$.

Recently, Dhage [4] proved the existence of PPF fixed points for mappings satisfying the condition of Cirić type generalized contraction (for more details, for example, refer to [13]) assuming topological closedness with respect to the norm topology for a Razumikhin class.

Theorem 1.4 (Dhage, [4]) *Suppose that $T : E_0 \to E$ satisfies the condition of Cirić type generalized contraction. Assume that $\mathcal{R}_c$ is topologically closed with respect to the norm topology and is algebraically closed with respect to the difference, then T has a unique PPF dependent fixed point in $\mathcal{R}_c$.*

Question. Is the result of Theorem 1.4 valid if one omit the assumption of being topological closed of $\mathcal{R}_c$?

The following proposition gives an affirmative answer to the question.

Proposition 1.5 *The Razumikhin class $\mathcal{R}_c$ is topologically closed with respect to the norm topology.*

Proof. Let $\{\phi_n\}$ be a sequence in $\mathcal{R}_c$ converging to ϕ. This implies that

$$\lim_{n\to\infty} \|\phi_n - \phi\|_{E_0} = 0, \quad \text{where } \|\phi_n - \phi\|_{E_0} = \sup_{t\in I} \|\phi_n(t) - \phi(t)\|_E.$$

Therefore

$$\lim_{n\to\infty} \|\phi_n\|_{E_0} = \|\phi\|_{E_0} \ \text{ and } \lim_{n\to\infty} \|\phi_n(t)\|_E = \|\phi(t)\|_E \ \text{ for all } t \in I.$$

Since $\phi_n \in \mathcal{R}_c$ for all $n \in \mathbb{N}$, we obtain that $\|\phi_n\|_{E_0} = \|\phi_n(c)\|_E$. Therefore

$$\lim_{n\to\infty} \|\phi_n(c)\|_E = \|\phi\|_{E_0}.$$

By the uniqueness of the limit, we have $\|\phi\|_{E_0} = \|\phi(c)\|_E$. Hence $\phi \in \mathcal{R}_c$ and thus $\mathcal{R}_c$ is topologically closed with respect to the norm topology. ■

Hence, using Proposition 1.5, we can drop the topological closedness with respect to the norm topology for $\mathcal{R}_c$ in Theorem 1.4.

The following example shows that despite of the closedness of Razumikhin class $\mathcal{R}_c$, the algebraical closedness with respect to the difference of Razumikhin class $\mathcal{R}_c$ may fail.

Example. Let $E_0 = C([0,1],\mathbb{R})$ and $c = 1$. If we take $\phi(x) = x^2$ and $\varphi(x) = x$ then $\phi, \varphi \in \mathcal{R}_c$ while $\phi - \varphi \notin \mathcal{R}_c$.

Proposition 1.6 *If the Razumikhin class $\mathcal{R}_c$ is algebraically closed with respect to the difference, then $\mathcal{R}_c$ is a convex set.*

Proof. Since $\mathcal{R}_c$ is algebraically closed with respect to the difference, we have $\mathcal{R}_c - \mathcal{R}_c \subseteq \mathcal{R}_c$. Using the fact that $-\mathcal{R}_c = \mathcal{R}_c$, we obtain that $\mathcal{R}_c + \mathcal{R}_c \subseteq \mathcal{R}_c$. Since $\lambda\mathcal{R}_c \subseteq \mathcal{R}_c$. for all $\lambda \in [0,1]$, we get that $\lambda\mathcal{R}_c + (1-\lambda)\mathcal{R}_c \subseteq \mathcal{R}_c$. Hence $\mathcal{R}_c$ is a convex set. ■

One can verify that Razumikhin class $\mathcal{R}_c$ is a cone (i.e., $\lambda\phi \in \mathcal{R}_c$, for each $\phi \in \mathcal{R}_c$ and $\lambda \geq 0$). Then by applying the previous theorem, we obtain that Razumikhin class $\mathcal{R}_c$ is a convex cone (also closed).

We need the following famous Banach contraction principle.

Theorem 1.7 *([1]). Let (X,d) be a complete metric space and let T be a contraction on X, i.e., there exists $r \in [0,1)$ such that $d(Tx,Ty) \leq rd(x,y)$ for all $x, y \in X$. Then T has a unique fixed point.*

2. Main results

In this section we investigate a PPF dependent fixed point in E_0 for the mapping $T : E_0 = C(I,E) \to E$, where E is a Banach space with the norm $\|\cdot\|_E$ and I is a closed interval $[a,b]$ of the real line.

In 2007, Drici et al. [5] proved the following fixed point theorems for mappings which are PPF dependent in a partially ordered metric space.

Theorem 2.1 *([5]) Let $(E, d, \leq)$ be a partially ordered complete metric space and $T : E_0 \to E$ is a mapping, where $E_0 = C(I, E)$. Assume that*

(i) T is nondecreasing;

(ii) for all $\phi, \alpha \in E_0$ with $\phi \leq \alpha, d(T\phi, T\alpha) \leq kd_0(\phi, \alpha)$, where $d_0(\phi, \alpha) = \max_{s \in I} d(\phi(s), \alpha(s))$ and $k \in [0, 1)$;

(iii) There exists a lower solution ϕ_0 such that $\phi_0(c) \leq T\phi_0$;

(iv) T is continuous or if $\{\phi_n\}$ is a nondecreasing sequence in E_0 converging to $\phi \in E_0$, then $\phi_n \leq \phi$ for all $n \in \mathbb{N}$.

Then T has a PPF dependent fixed point in E_0.

Theorem 2.2 *([5]) Assume that the conditions (ii), (iii) and (iv) of Theorem 2.1 hold. Let E_0 be such that for any nondecreasing sequence ϕ_n converging to $\phi^* \in E_0$, $\phi_n \leq \phi^*$. Then T has a PPF dependent fixed point.*

It is difficult to define a partial ordering on the metric space in order to guarantee the assumptions of the aforementioned theorem. It is more difficult when the metric space is uncountable. Hence it is natural to ask whether we can obtain the result of Theorem 2.1 and Theorem 2.2 without having an order on the space with mild assumptions. We will answer the question as follows.

Theorem 2.3 *Let $(E, d, \leq)$ be a complete metric space and $T : E_0 = C(I, E) \to E$ be a mapping. If the following inequality holds*

$$d(T\phi, T\alpha) \leq kd_0(\phi, \alpha) = k \sup_{s \in I} d(\phi(s), \alpha(s)), \quad \forall \phi, \alpha \in E_0, \quad (2.1)$$

then T has a PPF dependent fixed point in E_0.

Proof. We can embed E into E_0 by defining $F : E \to E_0$ as $F(e)(t) = e$ where $e \in E$ and $t \in I$. In other words we suppose an element of E as a constant function of I into E. Moreover

$$d_0(F(e), F(f)) = \sup_{t \in I} d(F(e)(t), F(f)(t)) = d(e, f) \text{ for } f \in E. \quad (2.2)$$

This means that F is also an isometry from E into E_0. Now the composition FoT of F and T is a mapping on E_0 and E_0 is a complete metric space with respect to the metric $d_0(\phi, \varphi) = \sup_{t \in I} d(\phi(t), \varphi(t))$. It is obvious from (2.1) and (2.2) that FoT is a contraction on E_0 and so by using Theorem 1.7, FoT has a unique fixed point $\phi^* \in E_0$. Hence $F(T(\phi^*)) = \phi*$ and so

$$T(\phi^*)(t) = T(\phi^*) = \phi^*(t), \ \forall t \in I.$$

This completes the proof. ■

Remark 2.4 *From the proof of Theorem 2.2 one can inform that the fixed point is unique and if* $x_0 = \phi \in E_0$ *then the sequence* $\{x_n = FoT(x_{n-1})\}_{n\geq 1}$ *is convergent to the fixed point.*

Example 2.5 *Let* $E = \mathbb{R}, [a,b] = [0,1]$, *and* $T : E_0 = C([0,1],\mathbb{R}) \to E = \mathbb{R}$ *be defined by* $T(\phi) = -\frac{1}{2}\phi(1)$ *for all* $\phi \in E_0$. *One can check that* T *cannot fulfill all of the assumptions of Theorem 2.1 and Theorem 2.2 while it satisfies the conditions in Theorem 2.3 and its PPF fixed point is the zero function, i.e.,* $\phi(t) = 0$, *for all* $t \in [0,1]$.

3. Applications

As an application of Theorem 2.3 we consider the following ordinary differential equation which is known as the periodic boundary value problem (in short, PBVP) without any lower solution assumed in [5] and free of considering an ordering on the space;

$$\begin{cases} x^{'}(t) = f(t, x(t), x_t), \qquad (2.3) \\ x_0 = \phi_0 \in C[[-\tau,0],\mathbb{R}] := \mathbf{D}, \\ x(0) = x(J) = \phi_0(0); t \in I = [0,J], \end{cases}$$

where $f \in C[[-\tau,0]\times\mathbb{R}\times\mathbf{D}]$ and $x_t(s) = x(t+s), \quad \forall\ s \in [-\tau,0]$.

Assume that,

$$0 \leq (f(t,x,\phi)+\lambda x)-(f(t,y,\psi)+\lambda y) \leq \mu \sup_{-\tau\leq s\leq 0}|\phi(s)-\psi(s)|, \quad \forall\ 0<\mu<\lambda.$$

Then the (PBVP) has a unique PPF dependent fixed point.

By using the same method as given in [5] and for the sake of the reader we present the sketch of the proof.
Put

$$\begin{cases} x^{'}(t) + \lambda x(t) = \sigma(t), \\ x_0 = \phi_0, \\ x(0) = x(J), \qquad \forall t \in I. \end{cases} \qquad (2.4)$$

and so we have

$$x(t) = x(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\sigma(s)ds, \qquad (2.5)$$

and then using $x(0) = x(J)$, we get

$$x(0)(1-e^{-\lambda J}) = e^{-\lambda J}\int_0^J e^{\lambda s}\sigma(s)ds,$$

and so

$$x(0) = \frac{e^{-\lambda J}}{1-e^{-\lambda J}}\int_0^J e^{\lambda s}\sigma(s)ds. \qquad (2.6)$$

Substituting (2.6) into (2.5), we deduce that

$$x(t) = \int_0^J G(t,s)\sigma(s)ds, \tag{2.7}$$

where

$$\begin{cases} \frac{e^{\lambda(J+s-t)}}{e^{\lambda J-1}}, & 0 \leq s \leq t \leq J, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda J-1}}, & 0 \leq t \leq s \leq J. \end{cases}$$

Now adding $\lambda x(t)$ to both sides of (2.3) we get

$$\begin{cases} x^{'}(t) + \lambda x(t) = f(t, x(t), x_t) + \lambda x(t) \equiv F(t, x(t), x_t), \\ x_0 = \phi_0, x(0) = x(J) = \phi_0(0). \end{cases} \tag{2.8}$$

Define

$$E_1 = \{w := (x_t)_{t\in I} : x_t \in \mathbf{D}, x \in C[[0,1], \mathbb{R}], x(0) = x(J) = \phi_0(0), x_0 = \phi_0 \in \mathbf{D}\},$$

together with

$$d_0(w_1 := (x_t)_{t\in I}, w_2 := (y_t)_{t\in I}) = \sup_{t\in I} \sup_{-\tau\leq s\leq 0} |x_t(s) - y_t(s)|.$$

Now define $S : E_1 \to \mathbb{R}$ as

$$S(w) = S((x_t)_{t\in I}) = \int_0^\tau G(t,s)F(s, x(s), x_s)ds.$$

As proved in [5] S is a contraction and (E_1, d_0) is a complete metric space and so it follows from Theorem 2.3 that S has a unique PPF fixed point and this completes the proof.

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Spline right fractional monotone approximation involving right fractional differential operators

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Abstract

Let $f \in C^{s}([-1,1])$, s$\in \mathbb{N}$ and L^{*} be a linear right fractional differential operator such that $L^{*}(f) \geq 0$ on $[-1,0]$. Then there exists a sequence Q_n, $n \in \mathbb{N}$ of polynomial splines with equally spaced knots of given fixed order such that $L^{*}(Q_n) \geq 0$ on $[-1,0]$. Furthermore f is approximated with rates right fractionally and simultaneously by Q_n in the uniform norm. This constrained right fractional approximation on $[-1,1]$ is given via inequalities invoving a higher modulus of smoothness of $f^{(s)}$.

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1 Introduction

Let $[a,b] \subset \mathbb{R}$ and for $n \geq 1$ consider the partition Δ_n with points $x_{in} = a + i\left(\frac{b-a}{n}\right)$, $i = 0, 1, ..., n$. Hence $\overline{\Delta}_n \equiv \max_{1 \leq i \leq n}(x_{in} - x_{i-1,n}) = \frac{b-a}{n}$.

Let $S_m(\Delta_n)$ be the space of polynomial splines of order $m > 0$ with simple knots at the points x_{in}, $i = 1, ..., n-1$. Then there exists a linear operator $Q_n : Q_n \equiv Q_n(f)$, mapping $B[a,b]$: the space of bounded real valued functions f on $[a,b]$, into $S_m(\Delta_n)$ (see [4], p. 224, Theorem 6.18).

From the same reference [4], p. 227, Corollary 6.21, we get

Corollary 1 *Let* $1 \leq \sigma \leq m$, $n \geq 1$. *Then for all* $f \in C^{\sigma-1}[a,b]$; $r = 0,...,\sigma-1$,

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq C_1 \left(\frac{b-a}{n} \right)^{\sigma-r-1} \omega_{m-\sigma+1} \left(f^{(b-1)}, \frac{b-a}{n} \right), \tag{1}$$

where C_1 *depends only on* m, $C_1 = C_1(m)$.

By denoting $C_2 = C_1 \max_{0 \leq r \leq \sigma-1} (b-a)^{\sigma-r-1}$ we obtain

Lemma 2 *([1]) Let* $1 \leq \sigma \leq m$, $n \geq 1$. *Then for all* $f \in C^{\sigma-1}[a,b]$; $r = 0,...,\sigma-1$,

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C_2}{n^{\sigma-r-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \tag{2}$$

where C_2 *depends only on* m, σ *and* $b-a$. *Here* $\omega_{m-\sigma+1}$ *is the usual modulus of smoothness of order* $m-\sigma+1$.

We are motivated by

Theorem 3 *([1]) Let* h, k, σ, m *be integers,* $0 \leq h \leq k \leq \sigma-1$, $\sigma \leq m$ *and let* $f \in C^{\sigma-1}[a,b]$. *Let* $\alpha_j(x) \in B[a,b]$, $j = h, h+1, ..., k$ *and suppose that* $\alpha_h(x) \geq \alpha > 0$ *or* $\alpha_h(x) \leq \beta < 0$ *for all* $x \in [a,b]$. *Take the linear differential operator*

$$L = \sum_{j=h}^{k} \alpha_j(x) \left[\frac{d^j}{dx^j} \right] \tag{3}$$

and assume, throughout $[a,b]$,

$$L(f) \geq 0. \tag{4}$$

Then, for every integer $n \geq 1$, *there is a polynomial spline function* $Q_n(x)$ *of order* m *with simple knots at* $\left\{ a + i\left(\frac{b-a}{n}\right),\ i = 1, ..., n-1 \right\}$ *such that* $L(Q_n) \geq 0$ *throughout* $[a,b]$ *and*

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C}{n^{\sigma-k-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad 0 \leq r \leq h. \tag{5}$$

Moreover, we find

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C}{n^{\sigma-r-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad h+1 \leq r \leq \sigma-1, \tag{6}$$

where C *is a constant independent of* f *and* n. *It depends only on* m, σ, L, a, b.

Next we specialize on the case of $a = -1$, $b = 1$. That is working on $[-1,1]$.
By Lemma 2 we get

Lemma 4 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}([-1,1])$; $j = 0,1,...,\sigma-1$,*

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_\infty \leq \frac{C_2}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{2}{n} \right), \tag{7}$$

where $C_2 := C_2(m,\sigma) := C_1(m) 2^{\sigma-1}$.

Since

$$\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{2}{n} \right) \leq 2^{m-\sigma+1} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) \tag{8}$$

(see [2], p. 45), we get

Lemma 5 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}([-1,1])$; $j = 0,1,...,\sigma-1$,*

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_\infty \leq \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right), \tag{9}$$

where $C_2^* := C_2^*(m,\sigma) := C_1(m) 2^m$.

We use a lot in this article Lemma 5.

In this article we extend Theorem 3 over $[-1,1]$ to the right fractional level. Indeed here L is replaced by L^*, a linear right Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[-1,0]$. Simultaneous fractional convergence remains true on all of $[-1,1]$.

We make

Definition 6 *([3]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1,1])$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{1-}^{\alpha} f \right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{10}$$

for any $x \in [-1,1]$, where Γ is the gamma function.

We set

$$D_{1-}^{0} f(x) := f(x),$$
$$D_{1-}^{m} f(x) := (-1)^m f^{(m)}(x), \ \forall\, x \in [-1,1]. \tag{11}$$

2 Main Result

Theorem 7 *Let h, k, σ, m be integers, $1 \leq \sigma \leq m$, $n \in \mathbb{N}$, h is even, with $0 \leq h \leq k \leq \sigma - 2$ and let $f \in C^{\sigma-1}([-1,1])$, with $f^{(\sigma-1)}$ having modulus of smoothness $\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \delta\right)$ there, $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[-1,0]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < ... < \alpha_{\sigma-2} < \sigma - 2$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of f of order α_j anchored at 1. Consider the linear right fractional differential operator*

$$L^* := \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j}\right] \tag{12}$$

and suppose, throughout $[-1,0]$, $L^*(f) \geq 0$.

Then, for every integer $n \geq 1$, there exists a polynomial spline function $Q_n(x)$ of order $m > 0$ with simple knots at $\left\{-1 + i\frac{2}{n},\ i = 1, ..., n-1\right\}$ such that $L^*(Q_n) \geq 0$ throughout $[-1,0]$, and

$$\sup_{-1 \leq x \leq 1} \left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \tag{13}$$

$j = h+1, ..., \sigma - 2$.

Set

$$l_j :\equiv \sup_{x \in [-1,1]} \left|\alpha_h^{-1}(x) \alpha_j(x)\right|, \quad h \leq j \leq k. \tag{14}$$

When $j = 1, ..., h$ we derive

$$\sup_{-1 \leq x \leq 1} \left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \cdot$$

$$\left[\left(\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau - \alpha_\tau + 1)}\right) \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h - \alpha_j - \lambda + 1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)}\right]. \tag{15}$$

Finally it holds

$$\sup_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau - \alpha_\tau + 1)} + 1\right]. \tag{16}$$

Proof. Set $\alpha_0 = 0$, thus $\lceil \alpha_0 \rceil = 0$. We have $\lceil \alpha_j \rceil = j$, $j = 1, ..., \sigma - 2$.
Let Q_n *as in Lemma* 5.

We notice that ($x \in [-1,1]$)

$$\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)}(t)\,dt - \int_x^1 (t-x)^{j-\alpha_j-1} Q_n^{(j)}(t)\,dt\right| = \quad (17)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1}\left(f^{(j)}(t) - Q_n^{(j)}(t)\right)dt\right| \le$$

$$\frac{1}{\Gamma(j-\alpha_j)}\int_x^1 (t-x)^{j-\alpha_j-1}\left|f^{(j)}(t) - Q_n^{(j)}(t)\right| dt \overset{(9)}{\le} \quad (18)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left(\int_x^1 (t-x)^{j-\alpha_j-1}\,dt\right)\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right) =$$

$$\frac{1}{\Gamma(j-\alpha_j)}\frac{(1-x)^{j-\alpha_j}}{(j-\alpha_j)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right) = \quad (19)$$

$$\frac{(1-x)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right) \le$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right). \quad (20)$$

Hence

$$\left\|D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} Q_n\right\|_{\infty,[-1,1]} \le \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \quad (21)$$

$j = 0,1,...,\sigma-2$.

We set

$$\rho_n := C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)\,n^{\sigma-j-1}}\right). \quad (22)$$

I. Suppose, throughout $[-1,0]$, $\alpha_h(x) \ge \alpha > 0$. Let $Q_n(x)$, $x \in [-1,1]$, the polynomial spline of order $m > 0$ with simple knots at the points x_{in}, $i = 1,...,n-1$, on $[-1,1]$ ($x_{in} = -1 + i\frac{2}{n}$, $i = 0,1,...,n$, here $\overline{\Delta}_n = \frac{2}{n}$), so that

$$\max_{-1\le x\le 1}\left|D_{1-}^{\alpha_j}\left(f(x) + \rho_n\frac{x^h}{h!}\right) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \le$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \quad (23)$$

$j = 0,1,...,\sigma-2$.

When $j = h+1, ..., \sigma - 2$, then

$$\max_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right), \tag{24}$$

proving (13).

For $j = 1, ..., h$ we find that

$$D_{1-}^{\alpha_j} \left(\frac{x^h}{h!} \right) = (-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)}. \tag{25}$$

Therefore we get from (23)

$$\max_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) + \rho_n \left((-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \leq \tag{26}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right),$$

$j = 1, ..., h$.

Therefore we get for $j = 1, ..., h$, that

$$\max_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \leq \tag{27}$$

$$\rho_n \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) =$$

$$C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right) n^{\sigma-\bar{j}-1}} \right) \cdot$$

$$\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) =$$

$$C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \left[\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)} \frac{1}{n^{\sigma-\bar{j}-1}} \right) \cdot \right. \tag{28}$$

$$\left. \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{1}{n^{\sigma-j-1}} \right] \leq$$

$$C_2^* \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{1}{n} \right) \frac{1}{n^{\sigma-k-1}} \left[\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)} \right) \cdot \right. \tag{29}$$

$$\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\Bigg].$$

Hence for $j=1,...,h$ we derived (15):

$$\max_{-1\leq x\leq 1}\left|\left(D_{1-}^{\alpha_j}f\right)(x)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\leq\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\cdot$$

$$\left[\left(\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}\right)\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\right].\tag{30}$$

When $j=0$ from (23) we obtain

$$\max_{-1\leq x\leq 1}\left|f(x)+\rho_n\frac{x^h}{h!}-Q_n(x)\right|\leq\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right).\tag{31}$$

And

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq\frac{\rho_n}{h!}+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=\tag{32}$$

$$\frac{C_2^*}{h!}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)n^{\sigma-\tau-1}}\right)$$

$$+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)n^{\sigma-\tau-1}}+\frac{1}{n^{\sigma-1}}\right]\leq\tag{33}$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}+1\right].$$

Proving

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}+1\right],\tag{34}$$

So that (16) is established.

Also if $-1\leq x\leq 0$, then

$$\alpha_h^{-1}(x)\,L^*\left(Q_n(x)\right)=\alpha_h^{-1}(x)\,L^*\left(f(x)\right)+\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}+\tag{35}$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\,\alpha_j(x)\left[D_{1-}^{\alpha_j}Q_n(x)-D_{1-}^{\alpha_j}f(x)-\frac{\rho_n}{h!}D_{1-}^{\alpha_j}x^h\right]\overset{(23)}{\geq}$$

$$\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \left(\sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right)\right) =$$

$$\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \rho_n = \rho_n \left[\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1\right] =$$

$$\rho_n \left[\frac{(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right] \geq \rho_n \left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right] \geq 0. \quad (36)$$

Explanation: We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h-\alpha_h < 1$ and $1 \leq h-\alpha_h+1 < 2$. Thus $\Gamma(h-\alpha_h+1) \leq 1$ and $1 - \Gamma(h-\alpha_h+1) \geq 0$. Hence $L^*(Q_n(x)) \geq 0$, $x \in [-1, 0]$.

II. Suppose on $[-1, 0]$ that $\alpha_h(x) \leq \beta < 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be the polynomial spline of order $m > 0$, (as before), so that

$$\max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j}\left(f(x) - \rho_n \frac{x^h}{h!}\right) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \quad (37)$$

$j = 0, 1, ..., \sigma - 2$.

Similarly as before we obtain again inequalities of convergence (13), (15) and (16).

Also if $-1 \leq x \leq 0$, then

$$\alpha_h^{-1}(x) L^*(Q_n(x)) = \alpha_h^{-1}(x) L^*(f(x)) - \rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \quad (38)$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) + \frac{\rho_n}{h!}\left(D_{1-}^{\alpha_j} x^h\right)\right] \overset{(37)}{\leq}$$

$$-\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) = \quad (39)$$

$$\rho_n \left(1 - \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) = \rho_n \left(\frac{\Gamma(h-\alpha_h+1) - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \leq \quad (40)$$

$$\rho_n \left(\frac{1-(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \leq 0,$$

and hence again $L^*(Q_n(x)) \geq 0$, $x \in [-1, 0]$. ■

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On certain differential sandwich theorems involving an extended generalized Sălăgean operator and extended Ruscheweyh operator

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Abstract

In the present paper we introduce sufficient conditions for strong differential subordination and superordination involving the extended operator $DR_\lambda^{m,n}$ and also we obtain sandwich-type results.

Keywords: analytic functions, extended differential operator, strong differential subordination, strong differential superordination.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [16] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [17].

Definition 1.1 *[17] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.1 *[17] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [18].

Definition 1.2 *[18] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, such that $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. In such a case we write $H(z,\zeta)\prec\prec f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.*

Remark 1.2 *[18] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.2 is equivalent to $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$, and $H\left(U\times\overline{U}\right)\subset f\left(U\times\overline{U}\right)$.*

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.3 *We denote by Q^* the set of functions that are analytic and injective on $\overline{U}\times\overline{U}\backslash E(f,\zeta)$, where $E(f,\zeta)=\{y\in\partial U:\lim\limits_{z\to y}f(z,\zeta)=\infty\}$, and are such that $f'_z(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta)=a$ is denoted by $Q^*(a)$.*

For two functions $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ and $g(z,\zeta)=z+\sum_{j=2}^{\infty}b_j(\zeta)z^j$ analytic in $U\times\overline{U}$, the Hadamard product (or convolution) of $f(z,\zeta)$ and $g(z,\zeta)$, written as $(f*g)(z,\zeta)$ is defined by

$$f(z,\zeta)*g(z,\zeta)=(f*g)(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)b_j(\zeta)z^j.$$

Definition 1.4 *([1]) For $f\in\mathcal{A}^*_\zeta$, $\lambda\geq 0$ and $m\in\mathbb{N}$, the extended generalized Sălăgean operator D^m_λ is defined by $D^m_\lambda:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$,*

$$\begin{aligned}
D^0_\lambda f(z,\zeta) &= f(z,\zeta)\\
D^1_\lambda f(z,\zeta) &= (1-\lambda)f(z,\zeta)+\lambda z f'_z(z,\zeta)=D_\lambda f(z,\zeta)\\
&\ldots\\
D^{m+1}_\lambda f(z,\zeta) &= (1-\lambda)D^m_\lambda f(z,\zeta)+\lambda z\left(D^m_\lambda f(z,\zeta)\right)'_z=D_\lambda\left(D^m_\lambda f(z,\zeta)\right),\ \ \textit{for}\ \ z\in U,\ \zeta\in\overline{U}.
\end{aligned}$$

Remark 1.3 *If $f\in\mathcal{A}^*_\zeta$ and $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$, then $D^m_\lambda f(z,\zeta)=z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^m a_j(\zeta)z^j$, for $z\in U$, $\zeta\in\overline{U}$.*

Definition 1.5 *([2]) For $f\in\mathcal{A}^*_\zeta$, $m\in\mathbb{N}$, the extended Ruscheweyh derivative R^m is defined by $R^m:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$,*

$$\begin{aligned}
R^0 f(z,\zeta) &= f(z,\zeta)\\
R^1 f(z,\zeta) &= z f'_z(z,\zeta)\\
&\ldots\\
(m+1)R^{m+1}f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)'_z+mR^m f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.
\end{aligned}$$

Remark 1.4 *If $f\in\mathcal{A}^*_\zeta$, $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$, then $R^m f(z,\zeta)=z+\sum_{j=2}^{\infty}\frac{(m+j-1)!}{m!(j-1)!}a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.*

Extending the results from [10] to the class $\mathcal{A}^*_\zeta$ we obtain:

Definition 1.6 *([11]) Let $\lambda\geq 0$ and $n,m\in\mathbb{N}$. Denote by $DR^{m,n}_\lambda:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$ the operator given by the Hadamard product of the extended generalized Sălăgean operator D^m_λ and the extended Ruscheweyh operator R^n,*

$$DR^{m,n}_\lambda f(z,\zeta)=\left(D^m_\lambda*R^n\right)f(z,\zeta),$$

for any $z\in U$, $\zeta\in\overline{U}$, and each nonnegative integers m,n.

Remark 1.5 *If* $f \in \mathcal{A}_\zeta^*$ *and* $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then*
$DR_\lambda^{m,n} f(z,\zeta) = z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j$, *for* $z \in U$, $\zeta \in \overline{U}$.

Remark 1.6 *For* $m = n$ *we obtain the operator* DR_λ^m *studied in [12], [13], [14], [15], [3], [4], [5].*
For $\lambda = 1$, $m = n$, *we obtain the Hadamard product* SR^n *[6] of the Sălăgean operator* S^n *and Ruscheweyh derivative* R^n, *which was studied in [7], [8], [9].*

Using simple computation one obtains the next result.

Proposition 1.1 *For* $m, n \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have For* $m, n \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have*

$$DR_\lambda^{m+1,n} f(z,\zeta) = (1-\lambda) DR_\lambda^{m,n} f(z,\zeta) + \lambda z \left(DR_\lambda^{m,n} f(z,\zeta)\right)_z' \tag{1.1}$$

and

$$z\left(DR_\lambda^{m,n} f(z,\zeta)\right)_z' = (n+1) DR_\lambda^{m,n+1} f(z,\zeta) - n DR_\lambda^{m,n} f(z,\zeta). \tag{1.2}$$

The main object of the present paper is to find sufficient condition for certain normalized analytic functions to satisfy

$$q_1(z,\zeta) \prec\prec \frac{z DR_\lambda^{m+1,n} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \prec\prec q_2(z,\zeta),$$

where q_1 and q_2 are given convex and univalent functions in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\xi \in \overline{U}$.

In order to prove our strong differential subordination and strong differential superordination results, we make use of the following known results.

Lemma 1.1 *Let the function* q *be univalent in* $U \times \overline{U}$ *and* θ *and* ϕ *be analytic in a domain* D *containing* $q\left(U \times \overline{U}\right)$ *with* $\phi(w) \neq 0$ *when* $w \in q\left(U \times \overline{U}\right)$. *Set* $Q(z,\zeta) = zq_z'(z,\zeta)\phi(q(z,\zeta))$ *and* $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta)$. *Suppose that*

1. Q *is starlike univalent in* $U \times \overline{U}$ *and*

2. $Re\left(\frac{zh_z'(z,\zeta)}{Q(z,\zeta)}\right) > 0$ *for* $z \in U$, $\zeta \in \overline{U}$.

If p *is analytic with* $p(0,\zeta) = q(0,\zeta)$, $p\left(U \times \overline{U}\right) \subseteq D$ *and*

$$\theta(p(z,\zeta)) + zp_z'(z,\zeta)\phi(p(z,\zeta)) \prec\prec \theta(q(z,\zeta)) + zq_z'(z,\zeta)\phi(q(z,\zeta)),$$

then $p(z,\zeta) \prec\prec q(z,\zeta)$ *and* q *is the best dominant.*

Lemma 1.2 *Let the function* q *be convex univalent in* $U \times \overline{U}$ *and* ν *and* ϕ *be analytic in a domain* D *containing* $q\left(U \times \overline{U}\right)$. *Suppose that*

1. $Re\left(\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))}\right) > 0$ *for* $z \in U$, $\zeta \in \overline{U}$ *and*

2. $\psi(z,\zeta) = zq_z'(z,\zeta)\phi(q(z,\zeta))$ *is starlike univalent in* $U \times \overline{U}$.

If $p(z,\zeta) \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$, *with* $p\left(U \times \overline{U}\right) \subseteq D$ *and* $\nu(p(z,\zeta)) + zp_z'(z)\phi(p(z,\zeta))$ *is univalent in* $U \times \overline{U}$ *and*

$$\nu(q(z,\zeta)) + zq_z'(z,\zeta)\phi(q(z,\zeta)) \prec\prec \nu(p(z,\zeta)) + zp_z'(z,\zeta)\phi(p(z,\zeta)),$$

then $q(z,\zeta) \prec\prec p(z,\zeta)$ *and* q *is the best subordinant.*

2 Main results

First, our purpose is to find sufficient conditions for certain normalized analytic functions f such that

$$q_1(z,\zeta) \prec\prec \frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}} \prec\prec q_2(z,\zeta), \quad z\in U, \zeta \in \overline{U}, 0<\delta\le 1,$$

where q_1 and q_2 are given univalent functions.

Theorem 2.1 *Let* $\frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}} \in \mathcal{H}^*\left(U\times\overline{U}\right)$, $z\in U$, $\zeta\in\overline{U}$, $f\in\mathcal{A}^*_\zeta$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $0<\delta\le 1$ *and let the function* $q(z,\zeta)$ *be convex and univalent in* $U\times\overline{U}$ *such that* $q(0,\zeta)=1$. *Assume that*

$$\operatorname{Re}\left(1+\frac{\alpha}{\beta}q(z,\zeta)-\frac{zq'_z(z,\zeta)}{q(z,\zeta)}+\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right)>0, \quad z\in U, \zeta\in\overline{U}, \tag{2.1}$$

for $\alpha,\beta\in\mathbb{C}$,$\beta\ne 0$, $z\in U$, $\zeta\in\overline{U}$, *and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta) := \alpha\frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}}+ \tag{2.2}$$

$$\beta\left[\delta(n+1)-1+(n+2)\frac{DR_\lambda^{m,n+2} f(z,\zeta)}{DR_\lambda^{m,n+1} f(z,\zeta)}-(1+\delta)(n+1)\frac{DR_\lambda^{m,n+1} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}\right].$$

If q *satisfies the following strong differential subordination*

$$\psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta) \prec\prec \alpha q(z,\zeta)+\frac{\beta z q'_z(z,\zeta)}{q(z,\zeta)}, \tag{2.3}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$ *then*

$$\frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}} \prec\prec q(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.4}$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z,\zeta) := \frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}}$, $z\in U$, $\zeta\in\overline{U}$, $z\ne 0, 0<\delta\le 1$, $f\in\mathcal{A}^*_\zeta$. The function p is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$

Differentiating this function, with respect to z,we get
$zp'_z(z,\zeta) = \frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}}\left[\delta+\frac{z\left(DR_\lambda^{m,n+1} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n+1} f(z,\zeta)}-(1+\delta)\frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)}\right]$.

By using the identity (1.2), we obtain

$$\frac{zp'_z(z,\zeta)}{p(z,\zeta)} = \delta(n+1)-1+(n+2)\frac{DR_\lambda^{m,n+2} f(z,\zeta)}{DR_\lambda^{m,n+1} f(z,\zeta)}-$$

$$(1+\delta)(n+1)\frac{DR_\lambda^{m,n+1} f(z,\zeta)}{DR_\lambda^{m,n} f(z,\zeta)}. \tag{2.5}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$, $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta) = zq_z'(z,\zeta)\phi(q(z,\zeta)) = \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)}$,we find that $Q(z,\zeta)$ is starlike univalent in $U \times \overline{U}$.

Let $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta) = \alpha q(z,\zeta) + \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)}$, $z \in U$, $\zeta \in \overline{U}$.

If we derive the function Q, with respect to z, perform calculations, we have $\operatorname{Re}\left(\frac{zh_z'(z,\zeta)}{Q(z,\zeta)}\right) = \operatorname{Re}\left(1 + \frac{\alpha}{\beta}q(z,\zeta) - \frac{zq_z'(z,\zeta)}{q(z,\zeta)} + \frac{zq_{z^2}''(z,\zeta)}{q_z'(z,\zeta)}\right) > 0$.

By using (2.5), we obtain $\alpha p(z,\zeta) + \beta\frac{zp_z'(z,\zeta)}{p(z,\zeta)} =$
$\alpha\frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}} + \beta\left[\delta(n+1) - 1 + (n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m,n+1}f(z,\zeta)} - (1+\delta)(n+1)\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right]$.

By using (2.3), we have $\alpha p(z,\zeta) + \frac{\beta zp_z'(z,\zeta)}{p(z,\zeta)} \prec\prec \alpha q(z,\zeta) + \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)}$.

Therefore, the conditions of Lemma 1.1 are met, so we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.$\frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}} \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and q is the best dominant. ■

Corollary 2.2 *Let $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \le B < A \le 1$, $m,n \in \mathbb{N}$, $\lambda \ge 0$, $z \in U$, $\zeta \in \overline{U}$. Assume that (2.1) holds. If $f \in \mathcal{A}_\zeta^*$ and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta) \prec \alpha\frac{\zeta+Az}{\zeta+Bz} + \beta\frac{(A-B)\zeta z}{(\zeta+Az)(\zeta+Bz)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \ne 0$, $-1 \le B < A \le 1$, $0 < \delta \le 1$ where $\psi_\lambda^{m,n}$ is defined in (2.2), then

$$\frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}} \prec\prec \frac{\zeta+Az}{\zeta+Bz}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ is the best dominant.

Proof. For $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.1 we get the corollary. ■

Theorem 2.3 *Let q be convex and univalent in $U \times \overline{U}$, such that $q(0,\zeta) = 1$, $m,n \in \mathbb{N}$, $\lambda \ge 0$. Assume that*

$$\operatorname{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right) > 0,\ \textit{for}\ \alpha,\beta \in \mathbb{C},\ \beta \ne 0,\ z \in U,\ \zeta \in \overline{U}. \tag{2.6}$$

If $f \in \mathcal{A}_\zeta^*$,$0 < \delta \le 1$, $\frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and $\psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta)$ is univalent in $U \times \overline{U}$, where $\psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta)$ is as defined in (2.2), then

$$\alpha q(z,\zeta) + \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)} \prec\prec \psi_\lambda^{m,n}(\alpha,\beta,\delta;z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.7}$$

implies

$$q(z,\zeta) \prec\prec \frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}}, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.8}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z,\zeta) := \frac{z^\delta DR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^{1+\delta}}$, $z \in U$, $\zeta \in \overline{U}$, $z \ne 0$, $0 < \delta \le 1, f \in \mathcal{A}_\zeta^*$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since q is convex and univalent function, it follows that $\operatorname{Re}\left(\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))}\right) = \operatorname{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

By using (2.7) we obtain

$$\alpha q(z,\zeta) + \beta\frac{zq'_z(z,\zeta)}{q(z,\zeta)} \prec\prec \alpha p(z,\zeta) + \beta\frac{zp'_z(z,\zeta)}{p(z,\zeta)}.$$

Using Lemma 1.2, we have

$$q(z,\zeta) \prec\prec p(z,\zeta) = \frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^{1+\delta}}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant. ■

Corollary 2.4 *Let $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.6) holds. If $f \in \mathcal{A}^*_{\zeta}$, $\frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^{1+\delta}} \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$ and*

$$\alpha\frac{\zeta + Az}{\zeta + Bz} + \beta\frac{(A-B)\zeta z}{(\zeta + Az)(\zeta + Bz)} \prec\prec \psi_{\lambda}^{m,n}(\alpha, \beta, \delta; z, \zeta),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda}^{m,n}$ is defined in (2.2), then

$$\frac{\zeta + Az}{\zeta + Bz} \prec\prec \frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^{1+\delta}}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ is the best subordinant.

Proof. For $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.3 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.3, we state the following sandwich theorem.

Theorem 2.5 *Let q_1 and q_2 be analytic and univalent in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2, \zeta \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$, with $z(q_1)'_z(z,\zeta)$ and $z(q_2)'_z(z,\zeta)$ being starlike univalent. Suppose that q_1 satisfies (2.1) and q_2 satisfies (2.6). If $f \in \mathcal{A}^*_{\zeta}$, $\frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^{1+\delta}} \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$, $0 < \delta \leq 1$ and $\psi_{\lambda}^{m,n}(\alpha, \beta, \delta; z, \zeta)$ is as defined in (2.2) univalent in $U \times \overline{U}$, then*

$$\alpha q_1(z,\zeta) + \frac{\beta z(q_1)'_z(z,\zeta)}{q_1(z,\zeta)} \prec\prec \psi_{\lambda}^{m,n}(\alpha, \beta, \delta; z, \zeta) \prec\prec \alpha q_2(z,\zeta) + \frac{\beta z(q_2)'_z(z,\zeta)}{q_2(z,\zeta)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z,\zeta) \prec\prec \frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^{1+\delta}} \prec\prec q_2(z,\zeta), \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z,\zeta) = \frac{\zeta+A_1 z}{\zeta+B_1 z}$, $q_2(z,\zeta) = \frac{\zeta+A_2 z}{\zeta+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.6 *Let $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.1) and (2.6) hold for $q_1(z,\zeta) = \frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $q_2(z,\zeta) = \frac{\zeta + A_2 z}{\zeta + B_2 z}$, respectively. If $f \in \mathcal{A}^*_\zeta$, $0 < \delta \leq 1$, $\frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and*

$$\alpha \frac{\zeta + A_1 z}{\zeta + B_1 z} + \beta \frac{(A_1 - B_1)\zeta z}{(\zeta + A_1 z)(\zeta + B_1 z)} \prec\prec \psi_\lambda^{m,n}(\alpha,\beta,\delta; z,\zeta)$$

$$\prec\prec \alpha \frac{\zeta + A_2 z}{\zeta + B_2 z} + \beta \frac{(A_2 - B_2)\zeta z}{(\zeta + A_2 z)(\zeta + B_2 z)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (2.2), then

$$\frac{\zeta + A_1 z}{\zeta + B_1 z} \prec\prec \frac{z^\delta DR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^{1+\delta}} \prec\prec \frac{\zeta + A_2 z}{\zeta + B_2 z},$$

hence $\frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $\frac{\zeta + A_2 z}{\zeta + B_2 z}$ are the best subordinant and the best dominant, respectively.

Next, our purpose is to find sufficient conditions for certain normalized analytic functions f such that

$$q_1(z,\zeta) \prec\prec \left(\frac{aDR_\lambda^{m+1,n} f(z,\zeta) + bDR_\lambda^{m,n} f(z,\zeta)}{(a+b)z}\right)^\delta \prec\prec q_2(z,\zeta), \quad z \in U, \zeta \in \overline{U},$$

where q_1 and q_2 are given univalent functions

Theorem 2.7 *Let $\left(\frac{aDR_\lambda^{m+1,n} f(z,\zeta) + bDR_\lambda^{m,n} f(z,\zeta)}{(a+b)z}\right)^\delta \in \mathcal{H}^*\left(U \times \overline{U}\right)$, $f \in \mathcal{A}^*_\zeta$, $z \in U$, $\zeta \in \overline{U}$, $\delta, a, b \in \mathbb{C}, \delta \neq 0, a+b \neq 0$, $m,n \in \mathbb{N}$, $\lambda \geq 0$ and let the function $q(z,\zeta)$ be convex and univalent in $U \times \overline{U}$ such that $q(0,\zeta) = 1$, $z \in U$, $\zeta \in \overline{U}$. Assume that*

$$\operatorname{Re}\left(1 + \frac{\alpha}{\beta} q(z,\zeta) - \frac{zq'_z(z,\zeta)}{q(z,\zeta)} + \frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right) > 0, \tag{2.9}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, $\zeta \in \overline{U}$, and

$$\psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta; z,\zeta) := \alpha \left(\frac{aDR_\lambda^{m+1,n} f(z,\zeta) + bDR_\lambda^{m,n} f(z,\zeta)}{(a+b)z}\right)^\delta + \tag{2.10}$$

$$\frac{\beta\delta\left[aDR_\lambda^{m+2,n} f(z,\zeta) + (b-a) DR_\lambda^{m+1,n} f(z,\zeta) - bDR_\lambda^{m,n} f(z,\zeta)\right]}{\lambda\left(aDR_\lambda^{m+1,n} f(z,\zeta) + bDR_\lambda^{m,n} f(z,\zeta)\right)}.$$

If q satisfies the following strong differential subordination

$$\psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta; z,\zeta) \prec\prec \alpha q(z,\zeta) + \frac{\beta z q'_z(z,\zeta)}{q(z,\zeta)}, \tag{2.11}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, $\zeta \in \overline{U}$, then

$$\left(\frac{aDR_\lambda^{m+1,n} f(z,\zeta) + bDR_\lambda^{m,n} f(z,\zeta)}{(a+b)z}\right)^\delta \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},\ \delta \in \mathbb{C},\ \delta \neq 0, \tag{2.12}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z,\zeta) := \left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}$, $z \in U, \zeta \in \overline{U}$, $z \neq 0$, $\delta, a, b \in \mathbb{C}, \delta \neq 0, a+b \neq 0, f \in \mathcal{A}_{\zeta}^{*}$. The function p is analytic in $U \times \overline{U}$ and $p(0,\zeta) = 1$.

Differentiating this function, with respect to z, we get

$zp_z'(z,\zeta) = \delta\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta-1}\frac{1}{a+b}\cdot$

$\left[\frac{a\left(DR_{\lambda}^{m+1,n}f(z,\zeta)\right)_z'+b\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{z} - \frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{z^2}\right]$.

We have

$$zp_z'(z,\zeta) = \delta\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}\cdot$$
$$\frac{1}{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}\left[az\left(DR_{\lambda}^{m+1,n}f(z,\zeta)\right)_z' + \right.$$
$$\left. bz\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z' - aDR_{\lambda}^{m+1,n}f(z,\zeta) - bDR_{\lambda}^{m,n}f(z,\zeta)\right]. \tag{2.13}$$

By using the identity (1.1) we obtain

$$\frac{zp_z'(z,\zeta)}{p(z,\zeta)} = \frac{\delta\left[aDR_{\lambda}^{m+2,n}f(z,\zeta)+(b-a)DR_{\lambda}^{m+1,n}f(z,\zeta)-bDR_{\lambda}^{m,n}f(z,\zeta)\right]}{\lambda\left(aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)\right)}. \tag{2.14}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta) = zq_z'(z,\zeta)\phi(q(z,\zeta)) = \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)}$,we find that $Q(z,\zeta)$ is starlike univalent in $U \times \overline{U}$.

Let $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta) = \alpha q(z,\zeta) + \frac{\beta zq_z'(z,\zeta)}{q(z,\zeta)}$, $z \in U$, $\zeta \in \overline{U}$.

If we derive the function Q, with respect to z, perform calculations, we have $\operatorname{Re}\left(\frac{zh_z'(z,\zeta)}{Q(z,\zeta)}\right) = \operatorname{Re}\left(1 + \frac{\alpha}{\beta}q(z,\zeta) - \frac{zq_z'(z,\zeta)}{q(z,\zeta)} + \frac{zq_{z^2}''(z,\zeta)}{q_z'(z,\zeta)}\right) > 0$.

By using (2.5), we obtain $\alpha p(z,\zeta) + \beta\frac{zp_z'(z,\zeta)}{p(z,\zeta)} =$

$\alpha\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta} + \frac{\beta\delta\left[aDR_{\lambda}^{m+2,n}f(z,\zeta)+(b-a)DR_{\lambda}^{m+1,n}f(z,\zeta)-bDR_{\lambda}^{m,n}f(z,\zeta)\right]}{\lambda\left(aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)\right)}$.

By using (2.11), we have $\alpha p(z,\zeta) + \beta\frac{zp_z'(z,\zeta)}{p(z,\zeta)} \prec\prec \alpha q(z,\zeta) + \beta\frac{zq_z'(z,\zeta)}{p(z,\zeta)}$.

From Lemma 1.1, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}$ $\prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, $\delta \in \mathbb{C}, \delta \neq 0$ and q is the best dominant. ■

Corollary 2.8 *Let* $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $z \in U$, $\zeta \in \overline{U}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.9) holds. If* $f \in \mathcal{A}_{\zeta}^{*}$ *and*

$$\psi_{\lambda}^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta) \prec\prec \alpha\frac{\zeta+Az}{\zeta+Bz} + \beta\frac{(A-B)\zeta z}{(\zeta+Az)(\zeta+Bz)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.10), then*

$$\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta} \prec\prec \frac{\zeta+Az}{\zeta+Bz}, \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best dominant.*

Proof. For $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, in Theorem 2.7 we get the corollary. ■

Theorem 2.9 *Let q be convex and univalent in $U\times\overline{U}$ such that $q(0,\zeta)=1$. Assume that*

$$\operatorname{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right)>0,\ \textit{for}\ \alpha,\beta\in\mathbb{C},\ \beta\neq 0. \tag{2.15}$$

If $f\in\mathcal{A}_{\zeta}^{*}$, $\delta,a,b\in\mathbb{C},\delta\neq 0,a+b\neq 0$, $\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}\in\mathcal{H}^{*}[q(0,\zeta),1,\zeta]\cap Q^{*}$ and $\psi_{\lambda}^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta)$ is univalent in $U\times\overline{U}$, where $\psi_{\lambda}^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta)$ is as defined in (2.10), then

$$\alpha q(z,\zeta)+\frac{\beta z q_z'(z,\zeta)}{q(z,\zeta)}\prec\prec\psi_{\lambda}^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta) \tag{2.16}$$

implies

$$q(z,\zeta)\prec\prec\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta},\quad \delta\in\mathbb{C},\ \delta\neq 0,\ z\in U,\ \zeta\in\overline{U}, \tag{2.17}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z,\zeta):=\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}$, $z\in U$, $\zeta\in\overline{U}$, $z\neq 0$, $a,b\in\mathbb{C}$, $a+b\neq 0$, $\delta\in\mathbb{C}$, $\delta\neq 0$, $f\in\mathcal{A}_{\zeta}^{*}$. The function p is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$.

By setting $\nu(w):=\alpha w$ and $\phi(w):=\frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since q is convex and univalent function, it follows that $\operatorname{Re}\left(\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))}\right)=\operatorname{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right)>0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$.

By using (2.7) we obtain

$$\alpha q(z,\zeta)+\beta\frac{zq_z'(z,\zeta)}{q(z,\zeta)}\prec\prec\alpha p(z,\zeta)+\beta\frac{zp_z'(z,\zeta)}{p(z,\zeta)}.$$

From Lemma 1.2, we have

$$q(z,\zeta)\prec\prec p(z,\zeta)=\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta},\quad z\in U,\ \zeta\in\overline{U},\ \delta\in\mathbb{C},\ \delta\neq 0,$$

and q is the best subordinant. ■

Corollary 2.10 *Let $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, $z\in U$, $\zeta\in\overline{U}$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. Assume that (2.15) holds. If $f\in\mathcal{A}_{\zeta}^{*}$, $\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta}\in\mathcal{H}^{*}[q(0,\zeta),1,\zeta]\cap Q^{*}$, $\delta\in\mathbb{C}$, $\delta\neq 0$ and*

$$\alpha\frac{\zeta+Az}{\zeta+Bz}+\beta\frac{(A-B)\zeta z}{(\zeta+Az)(\zeta+Bz)}\prec\prec\psi_{\lambda}^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\le B<A\le 1$, where $\psi_{\lambda}^{m,n}$ is defined in (2.10), then

$$\frac{\zeta+Az}{\zeta+Bz}\prec\prec\left(\frac{aDR_{\lambda}^{m+1,n}f(z,\zeta)+bDR_{\lambda}^{m,n}f(z,\zeta)}{(a+b)z}\right)^{\delta},\delta\in\mathbb{C},\ \delta\neq 0,\ a,b\in\mathbb{C},\ a+b\neq 0$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best subordinant.*

Proof. For $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.9 we get the corollary. ■
Combining Theorem 2.7 and Theorem 2.9, we state the following sandwich theorem.

Theorem 2.11 *Let q_1 and q_2 be convex and univalent in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$. Suppose that q_1 satisfies (2.9) and q_2 satisfies (2.15). If $f \in \mathcal{A}^*_\zeta$, $\left(\frac{aDR_\lambda^{m+1,n}f(z,\zeta)+bDR_\lambda^{m,n}f(z,\zeta)}{(a+b)z}\right)^\delta \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $a,b \in \mathbb{C}$, $a+b \neq 0$ and $\psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta)$ is as defined in (2.10) univalent in $U \times \overline{U}$, then*

$$\alpha q_1(z,\zeta) + \frac{\beta z (q_1)'_z(z,\zeta)}{q_1(z,\zeta)} \prec\prec \psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta) \prec\prec \alpha q_2(z,\zeta) + \frac{\beta z (q_2)'_z(z,\zeta)}{q_2(z,\zeta)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z,\zeta) \prec\prec \left(\frac{aDR_\lambda^{m+1,n}f(z,\zeta)+bDR_\lambda^{m,n}f(z,\zeta)}{(a+b)z}\right)^\delta \prec\prec q_2(z,\zeta), \quad z \in U, \zeta \in \overline{U},\ \delta \in \mathbb{C},\ \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z,\zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$, $q_2(z,\zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, where $-1 \le B_2 < B_1 < A_1 < A_2 \le 1$, we have the following corollary.

Corollary 2.12 *Let $m,n \in \mathbb{N}$, $\lambda \ge 0$. Assume that (2.9) and (2.15) hold for $q_1(z,\zeta) = \frac{\zeta+A_1z}{\zeta+B_1z}$ and $q_2(z,\zeta) = \frac{\zeta+A_2z}{\zeta+B_2z}$, respectively. If $f \in \mathcal{A}^*_\zeta$, $\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^\delta \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and*

$$\alpha\frac{\zeta+A_1z}{\zeta+B_1z} + \beta\frac{(A_1-B_1)\zeta z}{(\zeta+A_1z)(\zeta+B_1z)} \prec\prec \psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z,\zeta)$$

$$\prec\prec \alpha\frac{\zeta+A_2z}{\zeta+B_2z} + \beta\frac{(A_2-B_2)\zeta z}{(\zeta+A_2z)(\zeta+B_2z)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, where $\psi_\lambda^{m,n}$ is defined in (2.10), then

$$\frac{\zeta+A_1z}{\zeta+B_1z} \prec\prec \left(\frac{aDR_\lambda^{m+1,n}f(z,\zeta)+bDR_\lambda^{m,n}f(z,\zeta)}{(a+b)z}\right)^\delta \prec\prec \frac{\zeta+A_2z}{\zeta+B_2z}, \quad z \in U,\ \zeta \in \overline{U},$$

$\delta \in \mathbb{C}$, $\delta \neq 0$, $a,b \in \mathbb{C}$, $a+b \neq 0$, hence $\frac{\zeta+A_1z}{\zeta+B_1z}$ and $\frac{\zeta+A_2z}{\zeta+B_2z}$ are the best subordinant and the best dominant, respectively.

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HERMITE-HADAMARD TYPE INEQUALITIES FOR $(h-(\alpha,m))$-CONVEX FUNCTIONS

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Abstract. In this paper, we define $(h-(\alpha,m))$-convex functions that is a new class of convex functions as a generalization of $(\alpha,m)-$convexity and $h-$convexity. We also prove some Hadamard's type inequalities.

1. INTRODUCTION

Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$. The inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard's inequality for convex functions, [9].

In [6], Toader defined $m-$convexity as the following.

Definition 1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $(-f)$ is $m-$convex.

In [3], Dragomir proved the following theorem.

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be an $m-$convex function with $m\in(0,1]$ and $0\le a<b$. If $f\in L_1[a,b]$, then the following inequalities hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ &\le \frac{1}{2}\left[\frac{f(a)+mf\left(\frac{a}{m}\right)}{2}+m\frac{f\left(\frac{b}{m}\right)+mf\left(\frac{b}{m^2}\right)}{2}\right]. \end{aligned} \tag{1.2}$$

In [10], definition of $(\alpha,m)-$convexity was introduced by Miheşan as following.

Definition 2. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $(\alpha,m)-$convex, where $(\alpha,m)\in[0,1]^2$, if we have*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

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Denote by $K_m^{\alpha}(b)$ the class of all $(\alpha, m)-$convex functions on $[0,b]$ for which $f(0) \leq 0$. If we take $(\alpha, m) = \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$, it can be easily seen that $(\alpha, m)-$convexity reduces to increasing, $\alpha-$starshaped, starshaped, $m-$convex, convex and $\alpha-$convex functions, respectively.

In [16], Set et al. proved the following Hadamard type inequalities for $(\alpha, m)-$convex functions.

Theorem 2. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha, m)-$convex function with $(\alpha, m) \in (0,1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a,b] \cap L_1\left[\frac{a}{m}, \frac{b}{m}\right]$, then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^b \frac{f(x) + m(2^{\alpha}-1) f\left(\frac{x}{m}\right)}{2^{\alpha}} dx\right]. \tag{1.3}$$

Theorem 3. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha, m)-$convex function with $(\alpha, m) \in (0,1]^2$. If $0 \leq a < b < \infty$ and $f \in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \min\left\{\frac{f(a) + \alpha m f\left(\frac{b}{m}\right)}{\alpha+1}, \frac{f(b) + \alpha m f\left(\frac{a}{m}\right)}{\alpha+1}\right\}. \tag{1.4}$$

Theorem 4. *Let $f : [0,\infty) \to \mathbb{R}$ be an $(\alpha, m)-$convex function with $(\alpha, m) \in (0,1]^2$. If $f \in L_1[ma,b]$ where $0 \leq a < b$, then one has the inequality:*

$$\begin{aligned} & \frac{1}{mb-a}\int_a^{mb} f(x)\,dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)\,dx \\ \leq\ & \frac{\alpha m+1}{\alpha+1}\left[f(a)+f(b)\right]. \end{aligned} \tag{1.5}$$

In [1], Bakula et al. proved the following theorem.

Theorem 5. *Let $f, g : [0,\infty) \to [0,\infty)$ be such that fg is in $L^1([a,b])$, where $0 \leq a < b < \infty$. If f is $(\alpha_1, m_1)-$ convex and g is $(\alpha_2, m_2)-$ convex on $[a,b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \min\{N_1, N_2\}, \tag{1.6}$$

where

$$\begin{aligned} N_1 &= \frac{f(a)g(a)}{\alpha_1+\alpha_2+1} + m_2\left[\frac{1}{\alpha_1+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f(a) g\left(\frac{b}{m_2}\right) \\ & +m_1\left[\frac{1}{\alpha_2+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{b}{m_1}\right) g(a) \\ & +m_1 m_2\left[1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right) \end{aligned}$$

and

$$\begin{aligned} N_2 &= \frac{f(b)g(b)}{\alpha_1+\alpha_2+1} + m_2\left[\frac{1}{\alpha_1+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f(b) g\left(\frac{a}{m_2}\right) \\ & +m_1\left[\frac{1}{\alpha_2+1} - \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g(b) \\ & +m_1 m_2\left[1 - \frac{1}{\alpha_1+1} - \frac{1}{\alpha_2+1} + \frac{1}{\alpha_1+\alpha_2+1}\right] f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right). \end{aligned}$$

For the recent results based on the above definition see the papers [1], [2], [11]-[14] and [16].

In [8], Hudzik and Maligranda considered among others the class of functions which are $s-$convex in the second sense.

Definition 3. *A function $f:\mathbb{R}^+\to\mathbb{R}$, where $\mathbb{R}^+=[0,\infty)$, is said to be $s-$convex in the second sense if*

$$f(\alpha x+\beta y)\le \alpha^s f(x)+\beta^s f(y)$$

for all $x,y\in[0,\infty)$, $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ and for some fixed $s\in(0,1]$. This class of $s-$convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s=1$, $s-$convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [4], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense.

Theorem 6. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the second sense, where $s\in(0,1)$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[a,b]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}. \tag{1.7}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.7).

In [7], Godunova and Levin introduced the following class of functions.

Definition 4. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to belong to the class of $Q(I)$ if it is non-negative and for all $x,y\in I$ and $\lambda\in(0,1)$ satisfies the inequality;*

$$f(\lambda x+(1-\lambda)y)\le\frac{f(x)}{\lambda}+\frac{f(y)}{1-\lambda}.$$

In [5], Dragomir et al. defined the following new class of functions.

Definition 5. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is $P-$function or that f belongs to the class of $P(I)$, if it is non-negative and for all $x,y\in I$ and $\lambda\in[0,1]$, satisfies the following inequality;*

$$f(\lambda x+(1-\lambda)y)\le f(x)+f(y).$$

In [5], Dragomir et al. proved the following inequalities of Hadamard type for class of $Q(I)-$ functions and $P-$ functions.

Theorem 7. *Let $f\in Q(I)$, $a,b\in I$ with $a<b$ and $f\in L_1[a,b]$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{4}{b-a}\int_a^b f(x)dx$$

and

$$\frac{1}{b-a}\int_a^b p(x)\,f(x)dx\le\frac{f(a)+f(b)}{2}$$

where $p(x)=\frac{(b-x)(x-a)}{(b-a)^2}$, $x\in[a,b]$.

Theorem 8. *Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a)+f(b)]. \tag{1.8}$$

In [15], Varošanec defined the following class of functions.

I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 6. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$,$\alpha \in (0,1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.9}$$

If inequality (1.9) is reversed, then f is said to be $h-$concave, i.e. $f \in SV(h, I)$.

Obviously, if $h(\alpha) = \alpha$, then all non-negative convex functions belong to $SX(h, I)$ and all non-negative concave functions belong to $SV(h, I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

In [17], Sarıkaya et al. proved a variant of Hadamard inequality which holds for $h-$convex functions.

Theorem 9. *Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a, b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq [f(a)+f(b)]\int_0^1 h(\alpha)\,d\alpha. \tag{1.10}$$

In [12], Özdemir et al. defined $(h, m)-$convexity and obtained Hermite-Hadamard-type inequalities as following .

Theorem 10. *Let $f : [0, \infty) \to \mathbb{R}$ be an $(h, m)-$convex function with $m \in (0, 1]$, $t \in [0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx \\ \leq\ & \min\left\{ f(a)\int_0^1 h(t)dt + mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,dt\ ,\right. \\ &\left. f(b)\int_0^1 h(t)dt + mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,dt \right\}. \end{aligned} \tag{1.11}$$

Theorem 11. *Let $f : [0, \infty) \to \mathbb{R}$ be an $(h, m)-$convex function with $m \in (0, 1]$, $t \in [0, 1]$. If $0 \leq a < b < \infty$ and $f \in L_1[ma, b]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)\,dx + \frac{1}{b-ma}\int_{ma}^b f(x)\,dx\right] \\ \leq\ & [f(a)+f(b)]\int_0^1 h(t)\,dt. \end{aligned} \tag{1.12}$$

Theorem 12. *Let $f:[0,\infty)\to\mathbb{R}$ be an $(h,m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds:*

(1.13)
$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\le \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx\\
&\le h\left(\frac{1}{2}\right)\left[f(a)+mf\left(\frac{a}{m}\right)+mf\left(\frac{b}{m}\right)+m^2f\left(\frac{b}{m^2}\right)\right]\int_0^1 h(t)dt.
\end{aligned}
$$

The aim of this paper is to define a new class of convex function and then establish new Hermite-Hadamard-type inequalities.

2. MAIN RESULTS

In the beginning we give a new definition $(h-(\alpha,m))-$convex function.

I and J are intervals on $\mathbb{R}$, $(0,1)\subseteq J$ and functions h and f are real non-negative functions defined on J and I, respectively.

Definition 7. *Let $h:J\subset\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$. We say that $f:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ is an $(h-(\alpha,m))-$convex function, or that f belongs to the class $SX((h-(\alpha,m)),b)$, if f is non-negative, we have*

$$
f(tx+m(1-t)y)\le h(t^\alpha)f(x)+mh(1-t^\alpha)f(y) \tag{2.1}
$$

for all $x,y\in[0,b]$, $(\alpha,m)\in[0,1]^2$ and $t\in[0,1]$.

If the inequality (2.1) is reversed, then f is said to be $(h-(\alpha,m),b)-$concave function on $[a,b]$.

Obviously, if we choose $h(t)=t$, then we have non-negative $(\alpha,m)-$convex functions. If we choose $\alpha=m=1$, then we have $h-$convex functions. If we choose $\alpha=1$, then we have $(h-m)-$convex functions. If we choose $\alpha=m=1$ and $h(t)=\left\{t,1,\frac{1}{t},t^s\right\}$, then we obtain non-negative convex functions, $p-$functions, Godunova-Levin functions and $s-$convex functions in the second sense, respectively.

The following theorems were obtained by using the $(h-(\alpha,m))-$convex functions.

Theorem 13. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$ and $f:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ be an $(h-(\alpha,m))-$convex function with $(\alpha,m)\in[0,1]\times(0,1]$ and $t\in[0,1]$. If $f\in L_1[ma,b]$, $h\in L_1[0,1]$, one has the following inequality:*

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)dx \\
&\le \min\left\{f(a)\int_0^1 h(t^\alpha)dt+mf\left(\frac{b}{m}\right)\int_0^1 h\left(1-t^\alpha\right)dt,\right.\\
&\qquad\left. f(b)\int_0^1 h(t^\alpha)dt+mf\left(\frac{a}{m}\right)\int_0^1 h\left(1-t^\alpha\right)dt\right\}.
\end{aligned} \tag{2.2}
$$

Proof. Since f is $(h-(\alpha,m))-$convex function, $t\in[0,1]$ and $(\alpha,m)\in[0,1]\times(0,1]$, then

$$
f(tx+m(1-t)y)\le h(t^\alpha)f(x)+mh(1-t^\alpha)f(y)
$$

for all $x, y \geq 0$. It follows that

$$f(ta+(1-t)b) \leq h(t^{\alpha})f(a)+mh(1-t^{\alpha})f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a) \leq h(t^{\alpha})f(b)+mh(1-t^{\alpha})f\left(\frac{a}{m}\right).$$

Integrating the above inequalities with respect to t over $[0,1]$, we have

$$\int_0^1 f(ta+(1-t)b)dt \leq f(a)\int_0^1 h(t^{\alpha})dt+mf\left(\frac{b}{m}\right)\int_0^1 h(1-t^{\alpha})dt$$

and

$$\int_0^1 f(tb+(1-t)a)dt \leq f(b)\int_0^1 h(t^{\alpha})dt+mf\left(\frac{a}{m}\right)\int_0^1 h(1-t^{\alpha})dt.$$

However,

$$\int_0^1 f(ta+(1-t)b)dt=\int_0^1 f(tb+(1-t)a)dt=\frac{1}{b-a}\int_a^b f(x)dx,$$

we get

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)dx \\ \leq \quad & \min\left\{f(a)\int_0^1 h(t^{\alpha})dt+mf\left(\frac{b}{m}\right)\int_0^1 h\left(1-t^{\alpha}\right)dt,\right. \\ & \left. f(b)\int_0^1 h(t^{\alpha})dt+mf\left(\frac{a}{m}\right)\int_0^1 h\left(1-t^{\alpha}\right)dt\right\} \end{aligned}$$

which is the required result. The proof is completed. □

Remark 1. *In Theorem 13, if we take $h(t)=t$, then the inequality (2.2) reduces to inequality (1.4).*

Remark 2. *In Theorem 13, if we take $\alpha=m=1$, then the inequality (2.2) reduces to the right hand side of inequality (1.10).*

Remark 3. *In Theorem 13, if we take $\alpha=1$, then the inequality (2.2) reduces to inequality (1.11).*

Remark 4. *In Theorem 13, if we take $\alpha=m=1$ and $h(t)=\{t,1,t^s\}$, then the inequality (2.2) reduces to the right hand side of inequality in (1.1), (1.8) and (1.7) which are Hermite-Hadamard-type for non-negative convex functions, $p-$functions and $s-$convex functions in the second sense, respectively.*

Theorem 14. *Let $h: J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$ and $f: [0,b] \subseteq [0,\infty) \to \mathbb{R}$ be an $(h-(\alpha,m))-$convex function with $(\alpha,m) \in [0,1] \times (0,1]$ and $t \in [0,1]$. If $f \in L_1[ma,b]$, $h \in L_1[0,1]$, one has the following inequality:*

(2.3)

$$\begin{aligned} & \frac{1}{mb-a}\int_a^{mb} f(x)\,dx+\frac{1}{b-ma}\int_{ma}^b f(x)\,dx \\ \leq \quad & [f(a)+f(b)]\left[\int_0^1 h(t^{\alpha})dt+m\int_0^1 h(1-t^{\alpha})dt\right] \end{aligned}$$

for all $0 \le ma \le a \le mb < b < \infty$.

Proof. Since f is an $(h-(\alpha,m))-$convex function, we can write

$$\begin{aligned} f(ta+m(1-t)b) &\le h(t^{\alpha})f(a)+mh(1-t^{\alpha})f\left(b\right), \\ f(tb+m(1-t)a) &\le h(t^{\alpha})f(b)+mh(1-t^{\alpha})f\left(a\right), \\ f((1-t)a+m\left(1-(1-t)\right)b) &\le h\left((1-t)^{\alpha}\right)f(a)+mh\left(1-(1-t)^{\alpha}\right)f\left(b\right) \end{aligned}$$

and

$$f((1-t)b+m\left(1-(1-t)\right)a) \le h\left((1-t)^{\alpha}\right)f(b)+mh\left(1-(1-t)^{\alpha}\right)f\left(a\right)$$

for all $t\in[0,1]$ and $(\alpha,m)\in[0,1]\times(0,1]$.

Adding the above inequalities with each other, we get

$$\begin{aligned} & f(ta+m(1-t)b)+f(tb+m(1-t)a)+f((1-t)a+mtb)+f((1-t)b+mta) \\ \le\ & h(t^{\alpha})\left[f(a)+f\left(b\right)\right]+mh(1-t^{\alpha})\left[f(a)+f\left(b\right)\right] \\ & +h\left((1-t)^{\alpha}\right)\left[f(a)+f\left(b\right)\right]+mh\left(1-(1-t)^{\alpha}\right)\left[f(a)+f\left(b\right)\right]. \end{aligned}$$

Integrating the above inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} & \frac{2}{mb-a}\int_a^{mb} f(x)dx+\frac{2}{b-ma}\int_{ma}^{b} f(x)dx \\ \le\ & \left[f(a)+f\left(b\right)\right]\left[\int_0^1\left(h(t^{\alpha})+h\left((1-t)^{\alpha}\right)\right)dt+m\int_0^1\left(h(1-t^{\alpha})+h(1-(1-t)^{\alpha})\right)dt\right]. \end{aligned}$$

Since $\int_0^1 h(t^{\alpha})dt=\int_0^1 h((1-t)^{\alpha})dt$ and $\int_0^1 h(1-t^{\alpha})dt=\int_0^1 h(1-(1-t)^{\alpha})dt$, we obtain the desired result. □

Remark 5. *In Theorem 14, if we take* $h\left(t\right)=t$, *then the inequality (2.3) reduces to inequality (1.5).*

Remark 6. *In Theorem 14, if we take* $\alpha=m=1$, *then the inequality (2.3) reduces to the right hand side of inequality (1.10).*

Remark 7. *In Theorem 14, if we take* $\alpha=1$, *then the inequality (2.3) reduces to inequality (1.12).*

Remark 8. *In Theorem 14, if we take* $\alpha=m=1$ *and* $h\left(t\right)=\{t,1,t^{s}\}$, *then the inequality (2.3) reduces to the right hand side of inequality in (1.1), (1.8) and (1.7) which are Hermite-Hadamard-type for non-negative convex functions,* $p-$*functions and* $s-$*convex functions in the second sense, respectively.*

Theorem 15. *Let* $h:J\subseteq\mathbb{R}\to\mathbb{R}$ *be a non-negative function,* $h\neq0$ *and* $f:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ *be an* $(h-(\alpha,m))-$*convex function with* $(\alpha,m)\in[0,1]\times(0,1]$

and $t \in [0,1]$. *If* $f \in L_1[a,b]$, $h \in L_1[0,1]$, *one has the following inequalities:*

(2.4)

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \left(h\left(\frac{1}{2^\alpha}\right)f(x)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{x}{m}\right)\right)dx$$

$$\leq \left[h\left(\frac{1}{2^\alpha}\right)f(a)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{a}{m}\right)\right]\int_0^1 h(t^\alpha)dt$$

$$+m\left[h\left(\frac{1}{2^\alpha}\right)f\left(\frac{b}{m}\right)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{b}{m^2}\right)\right]\int_0^1 h(1-t^\alpha)dt$$

for all $0 \leq a < b < \infty$.

Proof. Since f is an $(h-(\alpha,m))$ −convex function, we can write

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)f(x)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{y}{m}\right)$$

for all $x,y \in [0,\infty)$. If we choose $x = ta+(1-t)b$, $y = tb+(1-t)a$, we get

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)f(ta+(1-t)b)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{tb+(1-t)a}{m}\right) \tag{2.5}$$

for all $t \in [0,1]$ and $(\alpha,m) \in [0,1]\times(0,1]$. Thus by integrating with respect to t over $[0,1]$, we have

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)\int_0^1 f(ta+(1-t)b)\,dt+mh\left(1-\frac{1}{2^\alpha}\right)\int_0^1 f\left(\frac{tb+(1-t)a}{m}\right)dt.$$

Taking into account that

$$\int_0^1 f(ta+(1-t)b)\,dt = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 f\left(\frac{tb+(1-t)a}{m}\right)dt = \frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx.$$

Then we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \left(h\left(\frac{1}{2^\alpha}\right)f(x)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{x}{m}\right)\right)dx$$

which is the first inequality in (2.4).

To prove the second inequality in (2.4), we use the right side of (2.5) with $(h-(\alpha,m))$ −convexity of f, we have

$$h\left(\frac{1}{2^\alpha}\right)f(ta+(1-t)b)+mh\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{(1-t)a+tb}{m}\right)$$

$$\leq h\left(\frac{1}{2^\alpha}\right)\left[h(t^\alpha)f(a)+mh(1-t^\alpha)f\left(\frac{b}{m}\right)\right]$$

$$+mh\left(1-\frac{1}{2^\alpha}\right)\left[h((1-t)^\alpha)f\left(\frac{a}{m}\right)+mh(1-(1-t)^\alpha)f\left(\frac{b}{m^2}\right)\right]$$

for all $t\in[0,1]$ and $(\alpha,m)\in[0,1]\times(0,1]$. Thus by integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 h\left(\frac{1}{2^\alpha}\right) f\left(ta+(1-t)b\right)+mh\left(1-\frac{1}{2^\alpha}\right) f\left(\frac{(1-t)\,a+tb}{m}\right)\\
\leq\ & h\left(\frac{1}{2^\alpha}\right)\left[f(a)\int_0^1 h(t^\alpha)dt+mf\left(\frac{b}{m}\right)\int_0^1 h(1-t^\alpha)dt\right]\\
&+mh\left(1-\frac{1}{2^\alpha}\right)\left[f\left(\frac{a}{m}\right)\int_0^1 h\left((1-t)^\alpha\right)dt+mf\left(\frac{b}{m^2}\right)\int_0^1 h(1-(1-t)^\alpha)dt\right]
\end{aligned}$$

which is equal to second inequality in (2.4). □

Remark 9. *In Theorem 15, if we take $h(t)=t$, then we have*

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \frac{f(x)+m\left(2^\alpha-1\right)f\left(\frac{x}{m}\right)}{2^\alpha}dx\\
&\leq \frac{1}{2^\alpha}\left[\frac{f(a)+m\left(2^\alpha-1\right)f\left(\frac{a}{m}\right)}{\alpha+1}+\alpha m\frac{f\left(\frac{b}{m}\right)+m\left(2^\alpha-1\right)f\left(\frac{b}{m^2}\right)}{\alpha+1}\right].
\end{aligned}$$

The left hand side of this inequality is in (1.3).

Remark 10. *In Theorem 15, if we take $\alpha=1$ and $h(t)=t$, then the inequality (2.4) reduces to inequality (1.2).*

Remark 11. *In Theorem 15, if we take $\alpha=1$, then the inequality (2.4) reduces to inequality (1.13).*

Remark 12. *In Theorem 15, if we take $\alpha=m=1$ and $h(t)=\{t,1,t^s\}$, then the inequality (2.4) reduces to inequalities in (1.1), (1.8) and (1.7) which are Hermite-Hadamard-type for non-negative convex functions, $p-$functions and $s-$convex functions in the second sense, respectively.*

Theorem 16. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function, $h\neq 0$ and $f,g:[0,b]\subseteq[0,\infty)\to\mathbb{R}$ be such that $fg\in L_1[a,b]$, $h_1h_2\in L_1[0,1]$ where $0\leq a<b<\infty$. If f is $(h_1-(\alpha_1,m_1))-$convex and g is $(h_2-(\alpha_2,m_2))-$convex function on $[0,b]$, one has the following inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\leq\min\{H_1,H_2\} \tag{2.6}$$

where

$$\begin{aligned}
H_1 =\ & f(a)g(a)\int_0^1 h_1(t^{\alpha_1})h_2(t^{\alpha_2})dt+m_2f(a)g\left(\frac{b}{m_2}\right)\int_0^1 h_1(t^{\alpha_1})h_2(1-t^{\alpha_2})dt\\
&+m_1f\left(\frac{b}{m_1}\right)g(a)\int_0^1 h_2(t^{\alpha_2})h_1(1-t^{\alpha_1})dt\\
&+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 h_1(1-t^{\alpha_1})h_2(1-t^{\alpha_2})dt,
\end{aligned}$$

and

$$\begin{aligned} H_2 &= f(b)g(b)\int_0^1 h_1(t^{\alpha_1})h_2(t^{\alpha_2})dt + m_2 f(b)g\left(\frac{a}{m_2}\right)\int_0^1 h_1(t^{\alpha_1})h_2(1-t^{\alpha_2})dt \\ &\quad + m_1 f\left(\frac{a}{m_1}\right)g(b)\int_0^1 h_2(t^{\alpha_2})h_1(1-t^{\alpha_1})dt \\ &\quad + m_1 m_2 f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right)\int_0^1 h_1(1-t^{\alpha_1})h_2(1-t^{\alpha_2})dt \end{aligned}$$

for some fixed $(\alpha_1, m_1), (\alpha_2, m_2) \in [0,1] \times (0,1]$ *and* $t \in [0,1]$.

Proof. Since $f \in SX(h_1 - (\alpha_1, m_1), b)$ and $g \in SX(h_2 - (\alpha_2, m_2), b)$, we have

$$f(ta+(1-t)b) \le h_1(t^{\alpha_1})f(a) + m_1 h_1(1-t^{\alpha_1})f\left(\frac{b}{m_1}\right)$$

and

$$g(ta+(1-t)b) \le h_2(t^{\alpha_2})g(a) + m_2 h_2(1-t^{\alpha_2})g\left(\frac{b}{m_2}\right)$$

for all $t \in [0,1]$. Since f and g are non-negative,

$$\begin{aligned} & f(ta+(1-t)b)g(ta+(1-t)b) \\ \le\ & h_1(t^{\alpha_1})h_2(t^{\alpha_2})f(a)g(a) + m_2 f(a)g\left(\frac{b}{m_2}\right)h_1(t^{\alpha_1})h_2(1-t^{\alpha_2}) \\ & + m_1 f\left(\frac{b}{m_1}\right)g(a)\,h_2(t^{\alpha_2})h_1(1-t^{\alpha_1}) + m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)h_1(1-t^{\alpha_1}))h_2(1-t^{\alpha_2}). \end{aligned}$$

Then integrating the resulting inequality with respect to t over $[0,1]$,

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \\ \le\ & f(a)g(a)\int_0^1 h_1(t^{\alpha_1})h_2(t^{\alpha_2})dt + m_2 f(a)g\left(\frac{b}{m_2}\right)\int_0^1 h_1(t^{\alpha_1})h_2(1-t^{\alpha_2})dt \\ & + m_1 f\left(\frac{b}{m_1}\right)g(a)\int_0^1 h_2(t^{\alpha_2})h_1(1-t^{\alpha_1})dt \\ & + m_1 m_2 f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right)\int_0^1 h_1(1-t^{\alpha_1})h_2(1-t^{\alpha_2})dt. \end{aligned}$$

Analogously we have

$$
\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\
\leq\ & f(b)g(b)\int_0^1 h_1(t^{\alpha_1})h_2(t^{\alpha_2})dt + m_2 f(b) g\left(\frac{a}{m_2}\right)\int_0^1 h_1(t^{\alpha_1})h_2(1-t^{\alpha_2})dt \\
& + m_1 f\left(\frac{a}{m_1}\right) g(b)\int_0^1 h_2(t^{\alpha_2})h_1(1-t^{\alpha_1})dt \\
& + m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right)\int_0^1 h_1(1-t^{\alpha_1})h_2(1-t^{\alpha_2})dt
\end{aligned}
$$

which completes the proof. □

Remark 13. *In Theorem 16, if we take $h_1(t) = h_2(t) = t$, then the inequality (2.6) reduces to inequality (1.6).*

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Voronovskaya Type Asymptotic Expansions for Generalized Discrete Singular Operators

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Abstract. Here we give asymptotic expansions for the generalized discrete versions of unitary Picard, Gauss-Weierstrass, and Poisson-Cauchy singular operators. These are of Voronovskaya type expansions and they are connected to the approximation properties of these operators.

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1 Background

In [2] p.307-310, the authors studied the smooth general singular integral operators $\Theta_{r,\xi}(f;x)$ defined as follows. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, they defined

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1, ..., r, \\ 1 - \sum\limits_{i=1}^{r} (-1)^{r-i}\binom{r}{i} i^{-n}, & j = 0 \end{cases} \tag{1}$$

that is $\sum\limits_{j=0}^{r} \alpha_j = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable. For each $\xi > 0$, μ_ξ is a probability Borel measure on $\mathbb{R}$.

They defined for $x \in \mathbb{R}$ the integrals

$$\Theta_{r,n,\xi}(f;x) := \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x + jt) \right) d\mu_\xi(t). \tag{2}$$

and they assumed $\Theta_{r,n,\xi}(f;x) \in \mathbb{R}$, $\forall x \in \mathbb{R}$. They also stated the fact that the operators $\Theta_{r,n,\xi}$ are not in general positive. They gave their main result as

Theorem 1 *Suppose* $\xi^{-n} \int_{-\infty}^{\infty} |t|^n d\mu_\xi(t) \leq \rho$, $\forall \xi \in (0,1]$, $\rho > 0$ *and* $c_{k,\xi} := \int_{-\infty}^{\infty} t^k d\mu_\xi(t)$, $k = 1, \ldots, n-1$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be such that* $f^{(n)}$ *exists,* $n \in \mathbb{N}$, *and is bounded and let* $\xi \to 0+$, $0 < \gamma \leq 1$. *Then*

$$\Theta_{r,n,\xi}(f;x) - f(x) = \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c_{k,\xi} + o(\xi^{n-\gamma}). \tag{3}$$

When $n = 1$ *the sum collapses.*

They also covered the following special cases:

Corollary 2 *(n=1 case) Let* f *such that* f'*exists and it is bounded. Let* $\xi \to 0+$, $0 < \gamma \leq 1$. *Here suppose* $\xi^{-1} \int_{-\infty}^{\infty} |t| d\mu_\xi(t) \leq \rho$, $\forall \xi \in (0,1]$, $\rho > 0$. *Then*

$$\Theta_{r,1,\xi}(f;x) - f(x) = o\left(\xi^{1-\gamma}\right). \tag{4}$$

Corollary 3 *(n=2 case) Let* f *such that* f'' *exists and it is bounded. Let* $\xi \to 0+$, $0 < \gamma \leq 1$. *Here suppose* $\xi^{-2} \int_{-\infty}^{\infty} t^2 d\mu_\xi(t) \leq \rho$, $\forall \xi \in (0,1]$, $\rho > 0$. *Then*

$$\Theta_{r,2,\xi}(f;x) - f(x) = f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + o\left(\xi^{2-\gamma}\right). \tag{5}$$

Corollary 4 *(n=3 case) Let* f *such that* $f^{(3)}$ *exists and it is bounded. Let* $\xi \to 0+$, $0 < \gamma \leq 1$, *with* $\xi^{-3} \int_{-\infty}^{\infty} |t|^3 d\mu_\xi(t) \leq \rho$, $\forall \xi \in (0,1]$, $\rho > 0$. *Then*

$$\Theta_{r,3,\xi}(f;x) - f(x) = f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} + o(\xi^{3-\gamma}). \tag{6}$$

Corollary 5 *(n=4 case) Let* f *such that* $f^{(4)}$ *exists and it is bounded. Let* $\xi \to 0+$, $0 < \gamma \leq 1$, *with* $\xi^{-4} \int_{-\infty}^{\infty} t^4 d\mu_\xi(t) \leq \rho$, $\forall \xi \in (0,1]$, $\rho > 0$. *Then*

$$\begin{aligned} \Theta_{r,4,\xi}(f;x) - f(x) &= f'(x) \left(\sum_{j=1}^{r} \alpha_j j \right) c_{1,\xi} + \frac{f''(x)}{2} \left(\sum_{j=1}^{r} \alpha_j j^2 \right) c_{2,\xi} \qquad (7) \\ &\quad + \frac{f'''(x)}{6} \left(\sum_{j=1}^{r} \alpha_j j^3 \right) c_{3,\xi} + o(\xi^{4-\gamma}). \end{aligned}$$

On the other hand, in [3], the authors defined important special cases of $\Theta_{r,\xi}$ operators for discrete probability measures μ_ξ as follows :

Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$, $0 < \xi \leq 1$, $x \in \mathbb{R}$.

i) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{8}$$

they defined the generalized discrete Picard operators as

$$P^*_{r,n,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}. \tag{9}$$

ii) When

$$\mu_\xi(\nu) = \frac{e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}, \tag{10}$$

they defined the generalized discrete Gauss-Weierstrass operators as

$$W^*_{r,n,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) e^{\frac{-\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}}}. \tag{11}$$

iii) Let $\alpha \in \mathbb{N}$, and $\beta > \frac{1}{\alpha}$. When

$$\mu_\xi(\nu) = \frac{\left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}, \tag{12}$$

they defined the generalized discrete Poisson-Cauchy operators as

$$\Theta^*_{r,n,\xi}(f;x) := \frac{\sum\limits_{\nu=-\infty}^{\infty} \left(\sum\limits_{j=0}^{r} \alpha_j f(x+j\nu) \right) \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{13}$$

They observed that for c constant they have

$$P^*_{r,n,\xi}(c;x) = W^*_{r,n,\xi}(c;x) = \Theta^*_{r,n,\xi}(c;x) = c. \tag{14}$$

They assumed that the operators $P_{r,n,\xi}^{*}(f;x)$, $W_{r,n,\xi}^{*}(f;x)$, and $\Theta_{r,n,\xi}^{*}(f;x) \in \mathbb{R}$, for $x \in \mathbb{R}$. This is the case when $\|f\|_{\infty,\mathbb{R}} < \infty$.

In [3], for $k = 1, ..., n$, the authors defined the sums

$$c_{k,\xi}^{*} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-|\nu|}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-|\nu|}{\xi}}}, \tag{15}$$

$$p_{k,\xi}^{*} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^{k} e^{\frac{-\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{\frac{-\nu^{2}}{\xi}}}, \tag{16}$$

and for $\alpha \in \mathbb{N}$, $\beta > \frac{n+r+1}{2\alpha}$, they introduced

$$q_{k,\xi}^{*} := \frac{\sum\limits_{\nu=-\infty}^{\infty} \nu^{k} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum\limits_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}. \tag{17}$$

Furthermore, they proved that these sums $c_{k,\xi}^{*}$, $p_{k,\xi}^{*}$, and $q_{k,\xi}^{*}$ are finite.

2 Main Results

First, we present our results for the generalized discrete Picard operators.

Proposition 6 *Let $n \in \mathbb{N}$. Then, there exists $K_1 > 0$ such that*

$$\frac{\xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}} < K_1 < \infty \tag{18}$$

for all $\xi \in (0,1]$.

Proof. Since $\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6}$ (Euler, 1741), we have that

$$\begin{aligned} 1 &< \sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}} \\ &= 1 + 2\sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} \\ &< 1 + 2\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \\ &= 1 + \frac{\pi^2}{3} < \infty. \end{aligned} \tag{19}$$

Thus

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}} < 1. \tag{20}$$

Therefore, by (20), we obtain

$$\begin{aligned} \frac{\xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{|\nu|}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}} &< \xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{|\nu|}{\xi}} \\ &= \sum_{\nu=-\infty}^{\infty} \left(\frac{|\nu|}{\xi}\right)^n e^{-\frac{|\nu|}{\xi}} \\ &= \sum_{\nu=-\infty}^{\infty} \left(\left(\frac{|\nu|}{\xi}\right)^n e^{-\frac{|\nu|}{2\xi}}\right) e^{-\frac{|\nu|}{2\xi}} \\ &: = R_1. \end{aligned} \tag{21}$$

For $\nu \geq 1$, we have

$$\left(\frac{|\nu|}{\xi}\right)^n e^{-\frac{|\nu|}{2\xi}} = \left(\frac{\nu}{\xi}\right)^n e^{-\frac{\nu}{2\xi}} = \frac{z^n}{e^{\frac{z}{2}}} \tag{22}$$

where $z := \frac{\nu}{\xi} \geq 1$. Since

$$e^{\frac{z}{2}} = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \geq \frac{z^n}{2^n n!}, \tag{23}$$

then

$$2^n n! \geq \frac{z^n}{e^{\frac{z}{2}}}. \tag{24}$$

Thus, by (21), (22), and (24), we get

$$\begin{aligned} R_1 &= 2\sum_{\nu=1}^{\infty} \left(\left(\frac{\nu}{\xi}\right)^n e^{-\frac{\nu}{2\xi}} \right) e^{-\frac{\nu}{2\xi}} \\ &\leq 2^{n+1} n! \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{2\xi}} \\ &\leq 2^{n+1} n! \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{2}}. \end{aligned} \tag{25}$$

Define the function $g(\nu) = e^{-\frac{\nu}{2}}$ for $\nu \in [1, \infty)$. Thus, we have $g'(\nu) = -\frac{1}{2} e^{-\frac{\nu}{2}}$. We see that $g'(\nu) < 0$ for all $\nu \in [1, \infty)$. Then, by [4], we obtain

$$\begin{aligned} \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{2}} &\leq g(1) + \int_1^{\infty} e^{-\frac{\nu}{2}} d\nu \\ &= e^{-\frac{1}{2}} + 2e^{-\frac{1}{2}} \\ &= 3e^{-\frac{1}{2}}. \end{aligned} \tag{26}$$

Hence, by (21), (25), and (26), we have

$$\begin{aligned} R_1 &\leq 3e^{-\frac{1}{2}} 2^{n+1} n! \\ &: = K_1 < \infty \end{aligned} \tag{27}$$

for all $\xi \in (0, 1]$. ∎

Now, we present our main result for $P^*_{r,n,\xi}$.

Theorem 7 *Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(n)}$ exists, $n \in \mathbb{N}$, and is bounded. Let $\xi \to 0^+$ and $0 < \gamma \leq 1$. Then*

$$P^*_{r,n,\xi}(f; x) - f(x) = \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!} \left(\sum_{j=1}^{r} \alpha_j j^k \right) c^*_{k,\xi} + o(\xi^{n-\gamma}). \tag{28}$$

When $n = 1$, the sum on R.H.S. collapses.

Proof. By (15), *Theorem* 1, and *Proposition* 6. ∎

For $n = 1$, we have

Corollary 8 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f' exists, and is bounded. Let $\xi\rightarrow 0^{+}$ and $0<\gamma\leq 1$. Then*

$$P_{r,1,\xi}^{*}(f;x)-f(x)=o(\xi^{1-\gamma}). \tag{29}$$

Proof. By *Theorem* 7. ■

For $n=2$, we get

Corollary 9 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f'' exists, and is bounded. Let $\xi\rightarrow 0^{+}$ and $0<\gamma\leq 1$. Then*

$$P_{r,2,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)c_{1,\xi}^{*}+o(\xi^{2-\gamma}). \tag{30}$$

Proof. By *Theorem* 7. ■

For $n=3$, we obtain

Corollary 10 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f''' exists, and is bounded. Let $\xi\rightarrow 0^{+}$ and $0<\gamma\leq 1$. Then*

$$P_{r,3,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)c_{1,\xi}^{*}+\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)c_{2,\xi}^{*}+o(\xi^{3-\gamma}). \tag{31}$$

Proof. By *Theorem* 7. ■

For $n=4$, we derive

Corollary 11 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that $f^{(4)}$ exists, and is bounded. Let $\xi\rightarrow 0^{+}$ and $0<\gamma\leq 1$. Then*

$$\begin{aligned} P_{r,4,\xi}^{*}(f;x)-f(x) &= f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)c_{1,\xi}^{*} \qquad (32)\\ &+\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)c_{2,\xi}^{*}+\frac{f'''(x)}{6}\left(\sum_{j=1}^{r}\alpha_{j}j^{3}\right)c_{3,\xi}^{*}+o(\xi^{4-\gamma}). \end{aligned}$$

Proof. By *Theorem* 7. ■

Next, we present our results for the generalized discrete Gauss-Weierstrass operators.

Proposition 12 *Let $n\in\mathbb{N}$. Then, there exists $K_{2}>0$ such that*

$$\frac{\xi^{-n}\sum\limits_{\nu=-\infty}^{\infty}|\nu|^{n}e^{-\frac{\nu^{2}}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty}e^{-\frac{\nu^{2}}{\xi}}}<K_{2}<\infty \tag{33}$$

for all $\xi\in(0,1]$.

Proof. Since

$$|\nu| \leq \nu^2$$

for all $\nu \in \mathbb{Z}$, we have

$$\frac{|\nu|}{\xi} \leq \frac{\nu^2}{\xi}$$

for all $\xi \in (0,1]$. Thus

$$e^{-\frac{|\nu|}{\xi}} \geq e^{-\frac{\nu^2}{\xi}}. \tag{34}$$

Additionally, since $\sum\limits_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6}$ (Euler, 1741), we notice that

$$\begin{aligned} 1 &< \sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}} \qquad (35) \\ &= 1 + 2\sum_{\nu=1}^{\infty} e^{-\frac{\nu^2}{\xi}} \\ &< 1 + 2\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \\ &= 1 + \frac{\pi^2}{3} < \infty \end{aligned}$$

which yields that

$$\frac{1}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} < 1. \tag{36}$$

Therefore, we get

$$\begin{aligned} \frac{\xi^{-n} \sum\limits_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{\nu^2}{\xi}}}{\sum\limits_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} &\leq \xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{\nu^2}{\xi}} \qquad (37) \\ &\leq \xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n e^{-\frac{|\nu|}{\xi}} \\ &= \sum_{\nu=-\infty}^{\infty} \left(\frac{|\nu|}{\xi}\right)^n e^{-\frac{|\nu|}{\xi}} \\ &: = R_1. \end{aligned}$$

Hence, by (27), we obtain the desired result. ■

Now, we present our main result for $W^*_{r,n,\xi}$.

Theorem 13 *Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(n)}$ exists, $n\in\mathbb{N}$, and is bounded. Let $\xi\to 0^{+}$ and $0<\gamma\leq 1$. Then*

$$W_{r,n,\xi}^{*}(f;x)-f(x)=\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}\left(\sum_{j=1}^{r}\alpha_{j}j^{k}\right)p_{k,\xi}^{*}+o(\xi^{n-\gamma}). \tag{38}$$

When $n=1$, the sum on R.H.S. collapses.

Proof. By (16), *Theorem* 1, and *Proposition* 12. ■

For $n=1$, we have

Corollary 14 *Let $f:\mathbb{R}\to\mathbb{R}$ be such that f' exists, and is bounded. Let $\xi\to 0^{+}$ and $0<\gamma\leq 1$. Then*

$$W_{r,1,\xi}^{*}(f;x)-f(x)=o(\xi^{1-\gamma}). \tag{39}$$

Proof. By *Theorem* 13. ■

For $n=2$, we get

Corollary 15 *Let $f:\mathbb{R}\to\mathbb{R}$ be such that f'' exists, and is bounded. Let $\xi\to 0^{+}$ and $0<\gamma\leq 1$. Then*

$$W_{r,2,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)p_{1,\xi}^{*}+o(\xi^{2-\gamma}). \tag{40}$$

Proof. By *Theorem* 13. ■

For $n=3$, we obtain

Corollary 16 *Let $f:\mathbb{R}\to\mathbb{R}$ be such that f''' exists, and is bounded. Let $\xi\to 0^{+}$ and $0<\gamma\leq 1$. Then*

$$W_{r,3,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)p_{1,\xi}^{*}+\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)p_{2,\xi}^{*}+o(\xi^{3-\gamma}). \tag{41}$$

Proof. By *Theorem* 13. ■

For $n=4$, we derive

Corollary 17 *Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f^{(4)}$ exists, and is bounded. Let $\xi\to 0^{+}$ and $0<\gamma\leq 1$. Then*

$$\begin{aligned}W_{r,4,\xi}^{*}(f;x)-f(x) &= f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)p_{1,\xi}^{*} \qquad (42)\\ &\quad+\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)p_{2,\xi}^{*}+\frac{f'''(x)}{6}\left(\sum_{j=1}^{r}\alpha_{j}j^{3}\right)p_{3,\xi}^{*}+o(\xi^{4-\gamma}).\end{aligned}$$

Proof. By *Theorem* 13. ∎

Finally, we present our results for the generalized discrete Poisson-Cauchy operators.

Proposition 18 *Let $\alpha, n \in \mathbb{N}$ and $\beta > \frac{n+1}{2\alpha}$. Then, there exists $K_3 > 0$ such that*

$$\frac{\xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} < K_3 < \infty \tag{43}$$

for all $\xi \in (0, 1]$.

Proof. By [1], we have

$$\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta} < \infty.$$

Then, we observe that

$$\begin{aligned} \infty &> \sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta} \qquad (44) \\ &= \xi^{-2\alpha\beta} + 2\sum_{\nu=1}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta} \\ &\geq \xi^{-2\alpha\beta}. \end{aligned}$$

Therefore, we have

$$\frac{1}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} \leq \xi^{2\alpha\beta}. \tag{45}$$

Hence, we get

$$\begin{aligned} \frac{\xi^{-n} \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}} &\leq \xi^{2\alpha\beta - n} \sum_{\nu=-\infty}^{\infty} |\nu|^n \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta} \qquad (46) \\ &\leq \xi^{2\alpha\beta - n} \sum_{\nu=-\infty}^{\infty} |\nu|^{n-2\alpha\beta} \\ &\leq \sum_{\nu=-\infty}^{\infty} |\nu|^{n-2\alpha\beta} \\ &= 2\sum_{\nu=1}^{\infty} \left(\frac{1}{\nu}\right)^{2\alpha\beta - n} < \infty, \end{aligned}$$

for all $\xi \in (0, 1]$ since $2\alpha\beta - n > 1$. ∎

Now, we present our main result for $\Theta^*_{r,n,\xi}$.

Theorem 19 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that $f^{(n)}$ exists, $n\in\mathbb{N}$, and is bounded. Let $\xi\rightarrow 0^{+}$, $0<\gamma\leq 1$, and $\beta>\frac{n+r+1}{2\alpha}$. Then*

$$\Theta_{r,n,\xi}^{*}(f;x)-f(x)=\sum_{k=1}^{n-1}\frac{f^{(k)}(x)}{k!}\left(\sum_{j=1}^{r}\alpha_{j}j^{k}\right)q_{k,\xi}^{*}+o(\xi^{n-\gamma}). \tag{47}$$

When $n=1$, the sum on R.H.S. collapses.

Proof. By (17), *Theorem* 1, and *Proposition* 18. ■

For $n=1$, we have

Corollary 20 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f' exists, and is bounded. Let $\beta>\frac{r+2}{2\alpha}$, $\xi\rightarrow 0^{+}$, and $0<\gamma\leq 1$. Then*

$$\Theta_{r,1,\xi}^{*}(f;x)-f(x)=o(\xi^{1-\gamma}). \tag{48}$$

Proof. By *Theorem* 19. ■

For $n=2$, we get

Corollary 21 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f'' exists, and is bounded. Let $\beta>\frac{r+3}{2\alpha}$, $\xi\rightarrow 0^{+}$, and $0<\gamma\leq 1$. Then*

$$\Theta_{r,2,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)q_{1,\xi}^{*}+o(\xi^{2-\gamma}). \tag{49}$$

Proof. By *Theorem* 19. ■

For $n=3$, we obtain

Corollary 22 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that f''' exists, and is bounded. Let $\beta>\frac{r+4}{2\alpha}$, $\xi\rightarrow 0^{+}$, and $0<\gamma\leq 1$. Then*

$$\Theta_{r,3,\xi}^{*}(f;x)-f(x)=f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)q_{1,\xi}^{*}+\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)q_{2,\xi}^{*}+o(\xi^{3-\gamma}). \tag{50}$$

Proof. By *Theorem* 19. ■

For $n=4$, we derive

Corollary 23 *Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be such that $f^{(4)}$ exists, and is bounded. Let $\beta>\frac{r+5}{2\alpha}$, $\xi\rightarrow 0^{+}$, and $0<\gamma\leq 1$. Then*

$$\begin{aligned}\Theta_{r,4,\xi}^{*}(f;x)-f(x) &= f'(x)\left(\sum_{j=1}^{r}\alpha_{j}j\right)q_{1,\xi}^{*} \\ &\quad +\frac{f''(x)}{2}\left(\sum_{j=1}^{r}\alpha_{j}j^{2}\right)q_{2,\xi}^{*}+\frac{f'''(x)}{6}\left(\sum_{j=1}^{r}\alpha_{j}j^{3}\right)q_{3,\xi}^{*}+o(\xi^{4-\gamma}).\end{aligned} \tag{51}$$

Proof. By *Theorem* 19. ■

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SIMPLE AQ AND SIMPLE CQ FUNCTIONAL EQUATIONS

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Abstract. In this paper, the authors established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x),$$

$$g(2x) = 12g(x) + 4g(-x),$$

via Banach space using direct and fixed point method.

1. INTRODUCTION

In mathematics, a functional equation is any equation that specifies a function in implicit form. Often, the equation relates the value of a function (or functions) at some point with its values at other points. For instance, properties of functions can be determined by considering the types of functional equations they satisfy.

But the theory of functional equations is relatively young. The beginning of a theory of functional equations is connected with the work of an excellent specialist in this field, Hungarian mathematician J. Aczel. The stability problem for functional equations first was planed in 1940 by Ulam [41]:

When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?.

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If the problem accepts a solution, we say that the equation is stable. This phenomenon is called $Ulam-Hyers\ stability$ and has been extensively investigated for different functional equations.

Let $(G, .)$ be a groupoid and let $(Y, .)$ be a groupoid with the metric ρ. The following definition of stability of the equation of additive homomorphism from G to Y

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

is formulated.

Definition 1.1. Equation (1.1) is stable in Hyers-Ulam sense, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every function $f : G \to Y$ fulfilling

$$\rho\left(f(x+y), f(x) + f(y)\right) \leq \delta, \qquad x, y \in G$$

there exists a solution g of (1.1) satisfying

$$\rho\left(f(x), g(x)\right) \leq \epsilon, \qquad x \in G.$$

The study of stability problems for functional equations concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [22]. It was further generalized and excellent results were obtained by number of authors [2, 17, 33, 36, 39]. Its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 13, 14, 15, 16, 23, 26, 27, 37]. The generalized Ulam-Hyers stability of various types of functions equations in various spaces were discussed in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 20, 19, 24, 29, 30, 31, 32, 34, 35, 38, 40, 42, 43, 44].

In this paper, the authors established the general solution and generalized Ulam - Hyers stability of the simple additive-quadratic and simple cubic-quartic functional equations

$$f(2x) = 3f(x) + f(-x), \tag{1.2}$$

and

$$g(2x) = 12g(x) + 4g(-x), \tag{1.3}$$

having solutions

$$f(x) = ax + bx^2, \tag{1.4}$$

and

$$g(x) = cx^3 + dx^4, \tag{1.5}$$

respectively.

In Section 2 the the general solution of (1.2) and (1.3) are respectively provided.

In Section 3 the generalized Ulam - Hyers stability of (1.2) and (1.3) are respectively proved via Banach spaces a direct method.

In Section 4 the generalized Ulam - Hyers stability of (1.2) and (1.3) are respectively given via Banach spaces with the help of fixed point method.

2. GENERAL SOLUTION OF (1.2) AND (1.3)

In this section, the general solution of the functional equations (1.2) and (1.3) are respectively given. For this, let us consider $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces.

2.1. **GENERAL SOLUTION OF (1.2).** Using oddness and evenness of f, the following lemmas are trivial. Hence, we omit the proofs.

Lemma 2.1. *An odd mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2), then f is additive.*

Lemma 2.2. *An even mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2), then f is quadratic.*

2.2. **GENERAL SOLUTION OF (1.3).** Using oddness and evenness of g, the following lemmas are trivial. Hence, we omit the proofs.

Lemma 2.3. *An odd mapping $g : \mathcal{X} \to \mathcal{Y}$ satisfying (1.3), then g is cubic.*

Lemma 2.4. *An even mapping $g : \mathcal{X} \to \mathcal{Y}$ satisfying (1.3), then g is quartic.*

3. STABILITY RESULTS IN BANACH SPACE: DIRECT METHOD

In this section, the generalized Ulam - Hyers stability of the functional equations (1.2) and (1.3) are respectively is provided. Also throughout this section, let us consider $\mathcal{X}$ and $\mathcal{Y}$ to be a normed space and a Banach space, respectively.

3.1. **STABILITY RESULTS OF (1.2).**

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\alpha : \mathcal{X} \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\alpha\left(2^{nj}x\right)}{2^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha\left(2^{nj}x\right)}{2^{nj}} = 0 \tag{3.1}$$

for all $x \in \mathcal{X}$. Let $f_a : \mathcal{X} \to \mathcal{Y}$ be an odd function satisfying the inequality

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\| \leq \alpha\left(x\right) \tag{3.2}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.2) such that

$$\|f_a(x) - A(x)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj}x)}{2^{kj}} \tag{3.3}$$

for all $x \in \mathcal{X}$. The mapping $A(x)$ is defined by

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^{nj}x)}{2^{nj}} \tag{3.4}$$

for all $x \in \mathcal{X}$.

Proof. Assume $j = 1$. Using oddness of f_a in (3.2), it follows that

$$\left\| f_a(x) - \frac{f_a(2x)}{2} \right\| \le \frac{\alpha\,(x)}{2} \tag{3.5}$$

for all $x \in \mathcal{X}$. Now replacing x by $2x$ and dividing by 2 in (3.9), we get

$$\left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \le \frac{\alpha\,(2x)}{2^2} \tag{3.6}$$

for all $x \in \mathcal{X}$. From (3.9) and (3.6), we obtain

$$\begin{aligned} \left\| f_a(x) - \frac{f_a(2^2x)}{2^2} \right\| &\le \left\| f_a(x) - \frac{f_a(2x)}{2} \right\| + \left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \\ &\le \frac{1}{2}\left[\alpha(x) + \frac{\alpha(2x)}{2}\right] \end{aligned} \tag{3.7}$$

for all $x \in \mathcal{X}$. In general for any positive integer n, we get

$$\left\| f_a(x) - \frac{f_a(2^nx)}{2^n} \right\| \le \frac{1}{2}\sum_{k=0}^{n-1} \frac{\alpha(2^kx)}{2^k} \le \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha(2^kx)}{2^k} \tag{3.8}$$

for all $x \in \mathcal{X}$. In order to prove the convergence of the sequence

$$\left\{ \frac{f_a(2^nx)}{2^n} \right\},$$

replace x by 2^mx and dividing by 2^m in (3.8), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f_a(2^mx)}{2^m} - \frac{f_a(2^{n+m}x)}{2^{(n+m)}} \right\| &= \frac{1}{2^m} \left\| f_a(2^mx) - \frac{f_a(2^n \cdot 2^mx)}{2^n} \right\| \\ &\le \frac{1}{2}\sum_{k=0}^{\infty} \frac{\alpha(2^{k+m}x)}{2^{k+m}} \to 0 \quad as \ \ m \to \infty \end{aligned}$$

for all $x \in \mathcal{X}$. Hence the sequence $\left\{ \frac{f_a(2^nx)}{2^n} \right\}$ is a Cauchy sequence. Since $\mathcal{Y}$ is complete, there exists a mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$A(x) = \lim_{n\to\infty} \frac{f_a(2^nx)}{2^n}, \ \ \forall \ x \in \mathcal{X}.$$

Letting $n \to \infty$ in (3.8), we see that (3.3) holds for all $x \in \mathcal{X}$. To prove that A satisfies (1.2), replacing x by $2^n x$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n}\left\| f_a(2^n \cdot 2x) - 3f_a(2^n x) - f_a(-2^n x)\right\| \leq \frac{1}{2^n}\alpha(2^n x)$$

for all $x \in \mathcal{X}$. Letting $n \to \infty$ in the above inequality and using the definition of $A(x)$ and (3.1), we see that

$$A(2x) = 3A(x) + A(-x).$$

Hence A satisfies (1.2) for all $x \in \mathcal{X}$. To prove that A is unique, let $B(x)$ be another additive mapping satisfying (1.2) and (3.3), then

$$\begin{aligned}\|A(x) - B(x)\| &= \frac{1}{2^n}\|A(2^n x) - B(2^n x)\| \\ &\leq \frac{1}{2^n}\{\|A(2^n x) - f_a(2^n x)\| + \|f_a(2^n x) - B(2^n x)\|\} \\ &\leq \sum_{k=0}^{\infty}\frac{\alpha(2^{k+n}x)}{2^{(k+n)}} \to 0 \quad as \;\; n \to \infty\end{aligned}$$

for all $x \in \mathcal{X}$. Hence A is unique. Thus the theorem holds for $j = 1$.

Replacing x by $\frac{x}{2}$ in (3.9), we arrive

$$\left\| 2f_a\left(\frac{2x}{2}\right) - f_a(x)\right\| \leq \alpha\left(\frac{x}{2}\right) \tag{3.9}$$

for all $x \in \mathcal{X}$. The rest of the proof is similar to that of case $j = 1$. Thus, for $j = -1$ also the theorem is true. Hence the proof is complete. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.2).

Corollary 3.2. *Let λ and r be nonnegative real numbers. Let an odd function $f_a : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\| \leq \left\{\begin{array}{l}\lambda, \\ \lambda||x||^r, \quad r \neq 1;\end{array}\right. \tag{3.10}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f_a(x) - A(x)\| \leq \left\{\begin{array}{l}|\lambda|, \\ \dfrac{\lambda||x||^r}{|2 - 2^r|},\end{array}\right. \tag{3.11}$$

for all $x \in \mathcal{X}$.

Now, we will provide an example to illustrate that the functional equation (1.2) is not stable for $r = 1$ in condition (ii) of Corollary 3.2.

Example 3.3. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\alpha(x) = \begin{cases} \mu x, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n}, \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|f_a(2x) - 3f_a(x) - f_a(-x)| \leq 10\ \mu|x| \tag{3.12}$$

for all $x \in \mathbb{R}$. Then there do not exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_a(x) - A(x)| \leq \kappa|x|, \qquad for\ all \quad x \in \mathbb{R}. \tag{3.13}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\ \mu.$$

Therefore, we see that f_a is bounded. We are going to prove that f_a satisfies (3.12).

If $x = 0$ then (3.12) is trivial. If $|x| \geq \frac{1}{2}$ then the left hand side of (3.12) is less than 10μ. Now suppose that $0 < |x| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x| < \frac{1}{2^{k-1}}, \tag{3.14}$$

so that $2^{k-1}x < \frac{1}{2}$ and consequently

$$2^{k-1}(x), 2^{k-1}(-x), 2^{k-1}(2x) \in (-1, 1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(x), 2^n(-x), 2^n(2x) \in (-1, 1)$$

and

$$\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x)) = 0$$

for $n = 0, 1, \dots, k-1$. From the definition of f_a and (3.14), we obtain that

$$\begin{aligned}\Big|f_a(2x) - 3f_a(x) - f_a(-x)\Big| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x))\Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x))\Big| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n} 5\mu = 5\ \mu \times \frac{2}{2^k} = 10\ \mu|x|.\end{aligned}$$

Thus f_a satisfies (3.12) for all $x \in \mathbb{R}$ with $0 < |x| < \dfrac{1}{2}$.

We claim that the additive functional equation (1.2) is not stable for $r = 1$ in condition (ii) of Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.13). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_a(x)| \leq (\kappa + |c|)\,|x|. \tag{3.15}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \dots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\, x$$

which contradicts (3.15). Therefore the additive functional equation (1.2) is not stable in sense of Ulam, Hyers and Rassias if $r = 1$, assumed in the inequality condition (ii) of (3.11). □

Theorem 3.4. *Let $j \in \{-1, 1\}$ and $\alpha : \mathcal{X} \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\alpha\left(2^{nj} x\right)}{4^{nj}} \quad \textit{converges in } \mathbb{R} \quad \textit{and} \quad \lim_{n\to\infty} \frac{\alpha\left(2^{nj} x\right)}{4^{nj}} = 0 \tag{3.16}$$

for all $x \in \mathcal{X}$. Let $f_q : \mathcal{X} \to \mathcal{Y}$ be an even function satisfying the inequality

$$\|f_q(2x) - 3f_q(x) - f_q(-x)\| \leq \alpha\left(x\right) \tag{3.17}$$

for all $x \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q_2 : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.2) such that

$$\|f_q(x) - Q_2(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\alpha(2^{kj} x)}{4^{kj}} \tag{3.18}$$

for all $x \in \mathcal{X}$. *The mapping* $Q_2(x)$ *is defined by*

$$Q_2(x) = \lim_{n\to\infty} \frac{f_q(2^{nj}x)}{4^{nj}} \tag{3.19}$$

for all $x \in \mathcal{X}$.

Proof. Assume $j = 1$. Using evenness of f_q in (3.17), it follows that

$$\left\| f_q(x) - \frac{f_q(2x)}{4} \right\| \leq \frac{\alpha(x)}{4} \tag{3.20}$$

for all $x \in \mathcal{X}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1.2).

Corollary 3.5. *Let* λ *and* r *be nonnegative real numbers. Let an even function* $f_q : \mathcal{X} \to \mathcal{Y}$ *satisfies the inequality*

$$\|f_q(2x) - 3f_q(x) - f_q(-x)\| \leq \begin{cases} \lambda, \\ \lambda||x||^r, \quad r \neq 2; \end{cases} \tag{3.21}$$

for all $x \in \mathcal{X}$. *Then there exists a unique quadratic function* $Q_2 : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f_q(x) - Q_2(x)\| \leq \begin{cases} \dfrac{\lambda}{|3|}, \\ \dfrac{\lambda||x||^r}{|4-2^r|}, \end{cases} \tag{3.22}$$

for all $x \in \mathcal{X}$.

Now, we will provide an example to illustrate that the functional equation (1.2) is not stable for $r = 2$ in condition (ii) of Corollary 3.5.

Example 3.6. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\alpha(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{4^n}, \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|f_q(2x) - 3f_q(x) - f_q(-x)| \leq \frac{20\mu}{3}|x|^2 \tag{3.23}$$

for all $x \in \mathbb{R}$. Then there do not exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|f_q(x) - Q_2(x)| \leq \kappa |x|^2, \qquad for\ all \quad x \in \mathbb{R}. \tag{3.24}$$

Proof. Now

$$|f_q(x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n x)|}{|4^n|} = \sum_{n=0}^{\infty} \frac{\mu}{4^n} = \frac{4\mu}{3}.$$

Therefore, we see that f_q is bounded. We are going to prove that f_q satisfies (3.23).

If $x = 0$ then (3.23) is trivial. If $|x|^2 \geq \frac{1}{24}$ then the left hand side of (3.23) is less than $\frac{20\mu}{3}$. Now suppose that $0 < |x|^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$\frac{1}{4^{k+1}} \leq |x|^2 < \frac{1}{4^k}, \tag{3.25}$$

so that $4^{k-1}x^2 < \frac{1}{4}$ and consequently

$$2^{k-1}(x), 2^{k-1}(-x), 2^{k-1}(2x) \in (-1, 1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(x), 2^n(-x), 2^n(2x) \in (-1, 1)$$

and

$$\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x)) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_q and (3.25), we obtain that

$$\begin{aligned}
\Big|f_q(2x) - 3f_q(x) - f_q(-x)\Big| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x))\Big| \\
&\leq \sum_{n=k}^{\infty} \frac{1}{2^n}\Big|\alpha(2^n(2x)) - 3\alpha(2^n(x)) - \alpha(2^n(-x))\Big| \\
&\leq \sum_{n=k}^{\infty} \frac{1}{4^n} 5\mu = 5\ \mu \times \frac{4}{3 \cdot 2^k} = \frac{20\mu}{3}|x|.
\end{aligned}$$

Thus f_q satisfies (3.23) for all $x \in \mathbb{R}$ with $0 < |x|^2 < \frac{1}{4}$.

We claim that the quadratic functional equation (1.2) is not stable for $r = 2$ in condition (ii) of Corollary 3.5. Suppose on the contrary that there exist a quadratic mapping $Q_2 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.24). Since f_q is bounded and continuous for all $x \in \mathbb{R}$, Q_2 is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.4, Q_2 must have the form $Q_2(x) = cx^2$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|f_q(x)| \leq (\kappa + |c|)\,|x|^2. \tag{3.26}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\alpha(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x > (\kappa + |c|)\, x^2$$

which contradicts (3.26). Therefore the quadratic functional equation (1.2) is not stable in sense of Ulam, Hyers and Rassias if $r = 2$, assumed in the inequality condition (ii) of (3.22). $\square$

Now, we are ready to prove our main theorem.

Theorem 3.7. *Let $j \in \{-1, 1\}$ and $\alpha : \mathcal{X} \to [0, \infty)$ be a function with conditions (3.1) and (3.16) for all $x \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a function satisfying the inequality*

$$\|f(2x) - 3f(x) - f(-x)\| \leq \alpha\,(x) \tag{3.27}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.2) such that

$$\begin{aligned} \|f(x) - A(x) - Q_2(x)\| \leq \frac{1}{2} \Bigg[& \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x)}{2^{kj}} \right) \\ & + \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{4^{kj}} + \frac{\alpha(-2^{kj}x)}{4^{kj}} \right) \Bigg] \end{aligned} \tag{3.28}$$

for all $x \in \mathcal{X}$. The mapping $A(x)$ and $Q_2(x)$ are defined in (3.4) and (3.19) respectively for all $x \in \mathcal{X}$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in \mathcal{X}$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in \mathcal{X}$. Hence

$$\|f_o(2x) - 3f_o(x) - f_o(-x)\| \leq \frac{\alpha(x)}{2} + \frac{\alpha(-x)}{2} \tag{3.29}$$

for all $x \in \mathcal{X}$. By Theorem 3.1, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x)}{2^{kj}} \right) \tag{3.30}$$

for all $x \in \mathcal{X}$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in \mathcal{X}$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in \mathcal{X}$. Hence

$$\|f_e(2x) - 3f_e(x) - f_e(-x)\| \leq \frac{\alpha(x)}{2} + \frac{\alpha(-x)}{2} \tag{3.31}$$

for all $x \in \mathcal{X}$. By Theorem 3.4, we have

$$\|f_e(x) - Q_2(x)\| \le \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{4^{kj}} + \frac{\alpha(-2^{kj}x)}{4^{kj}} \right) \tag{3.32}$$

for all $x \in \mathcal{X}$. Define

$$f(x) = f_e(x) + f_o(x) \tag{3.33}$$

for all $x \in \mathcal{X}$. From (3.30),(3.32) and (3.33), we arrive

$$\begin{aligned} \|f(x) - A(x) - Q_2(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q_2(x)\| \\ &\le \|f_o(x) - A(x)\| + \|f_e(x) - Q_2(x)\| \\ &\le \frac{1}{2} \left[\frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{2^{kj}} + \frac{\alpha(-2^{kj}x)}{2^{kj}} \right) \right. \\ &\qquad \left. + \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\alpha(2^{kj}x)}{4^{kj}} + \frac{\alpha(-2^{kj}x)}{4^{kj}} \right) \right] \end{aligned}$$

for all $x \in \mathcal{X}$. Hence the theorem is proved. □

Using Corollaries 3.2 and 3.5, we have the following corollary concerning the stability of (1.2).

Corollary 3.8. *Let λ and r be nonnegative real numbers. Let a function $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|f(2x) - 3f(x) - f(-x)\| \le \begin{cases} \lambda, \\ \lambda||x||^r, \quad r \ne 1, 2; \end{cases} \tag{3.34}$$

for all $x \in \mathcal{X}$. Then there exists a unique additive function $A : \mathcal{X} \to \mathcal{Y}$ and a unique quadratic function $Q_2 : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x) - Q_2(x)\| \le \begin{cases} \lambda \left(|1| + \dfrac{1}{|3|} \right), \\ \lambda \left(\dfrac{||x||^r}{|2 - 2^r|} + \dfrac{||x||^r}{|4 - 2^r|} \right), \end{cases} \tag{3.35}$$

for all $x \in \mathcal{X}$.

3.2. STABILITY RESULTS OF (1.3).

Theorem 3.9. *Let $j \in \{-1, 1\}$ and $\beta : \mathcal{X} \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\beta(2^{nj}x)}{8^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \to \infty} \frac{\beta(2^{nj}x)}{8^{nj}} = 0 \tag{3.36}$$

for all $x \in \mathcal{X}$. Let $g_c : \mathcal{X} \to \mathcal{Y}$ be an odd function satisfying the inequality

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\| \leq \beta\,(x) \tag{3.37}$$

for all $x \in \mathcal{X}$. Then there exists a unique cubic mapping $C : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.3) such that

$$\|g_c(x) - C(x)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{8^{kj}} \tag{3.38}$$

for all $x \in \mathcal{X}$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty} \frac{g_c(2^{nj}x)}{8^{nj}} \tag{3.39}$$

for all $x \in \mathcal{X}$.

Proof. Assume $j = 1$. Using oddness of g_c in (3.37), it follows that

$$\left\| g_c(x) - \frac{g_c(2x)}{8} \right\| \leq \frac{\beta\,(x)}{8} \tag{3.40}$$

for all $x \in \mathcal{X}$. The rest of the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.9 concerning the stability of (1.3).

Corollary 3.10. *Let μ and r be nonnegative real numbers. Let an odd function $g_c : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\| \leq \begin{cases} \mu, \\ \mu||x||^r, \quad r \neq 3; \end{cases} \tag{3.41}$$

for all $x \in \mathcal{X}$. Then there exists a unique cubic function $C : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_c(x) - C(x)\| \leq \begin{cases} \dfrac{\mu}{|7|}, \\ \dfrac{\mu||x||^r}{|8 - 2^r|}, \end{cases} \tag{3.42}$$

for all $x \in \mathcal{X}$.

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $r = 3$ in condition (ii) of Corollary 3.10.

Example 3.11. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\beta(x) = \begin{cases} \mu x^3, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g_c : \mathbb{R} \to \mathbb{R}$ by

$$g_c(x) = \sum_{n=0}^{\infty} \frac{\beta(2^n x)}{8^n}, \qquad \textit{for all} \quad x \in \mathbb{R}.$$

Then g_c satisfies the functional inequality

$$|g_c(2x) - 12g_c(x) - 4g_c(-x)| \leq \frac{17\mu \times 8^3}{7}|x|^3 \tag{3.43}$$

for all $x \in \mathbb{R}$. Then there do not exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|g_c(x) - C(x)| \leq \kappa|x|^3, \qquad \textit{for all} \quad x \in \mathbb{R}. \tag{3.44}$$

Proof. Now

$$|g_c(x)| \leq \sum_{n=0}^{\infty} \frac{|\beta(2^n x)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{\mu}{8^n} = \frac{8\mu}{7}.$$

Therefore, we see that g_c is bounded. We are going to prove that g_c satisfies (3.43).

If $x = 0$ then (3.43) is trivial. If $|x|^3 \geq \frac{1}{8}$ then the left hand side of (3.43) is less than $\frac{8 \times 17\mu}{7}$. Now suppose that $0 < |x|^3 < \frac{1}{8}$. Then there exists a positive integer k such that

$$\frac{1}{8^{k+2}} \leq |x| < \frac{1}{8^{k+1}}, \tag{3.45}$$

so that $8^{k-1}x < \frac{1}{8}$ and consequently

$$2^{k-1}(x), 2^{k-1}(-x), 2^{k-1}(2x) \in (-1, 1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(x), 2^n(-x), 2^n(2x) \in (-1, 1)$$

and

$$\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x)) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of g_c and (3.45), we obtain that

$$\begin{aligned} \left|g_c(2x) - 12g_c(x) - 4g_c(-x)\right| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n}\left|\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x))\right| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n}\left|\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x))\right| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{8^n} 17\mu = 17\ \mu \times \frac{8}{7 \cdot 8^k} = \frac{17\mu \times 8^3}{7}|x|^3. \end{aligned}$$

Thus g_c satisfies (3.43) for all $x \in \mathbb{R}$ with $0 < |x|^3 < \frac{1}{8}$.

We claim that the cubic functional equation (1.3) is not stable for $r = 3$ in condition (ii) of Corollary 3.10. Suppose on the contrary that there exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.44). Since g_c is bounded and continuous for all $x \in \mathbb{R}$, C is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.9, C must have the form $C(x) = cx^3$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|g_c(x)| \leq (\kappa + |c|)\, |x|^3. \tag{3.46}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$g_c(x) = \sum_{n=0}^{\infty} \frac{\beta(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x^3)}{2^n} = m\mu x^3 > (\kappa + |c|)\, x^3$$

which contradicts (3.46). Therefore the cubic functional equation (1.3) is not stable in sense of Ulam, Hyers and Rassias if $r = 3$, assumed in the inequality condition (ii) of (3.42). □

Theorem 3.12. *Let $j \in \{-1, 1\}$ and $\beta : \mathcal{X} \to [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} \frac{\beta\left(2^{nj} x\right)}{16^{nj}} \quad \text{converges in } \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta\left(2^{nj} x\right)}{16^{nj}} = 0 \tag{3.47}$$

for all $x \in \mathcal{X}$. Let $g_q : \mathcal{X} \to \mathcal{Y}$ be an even function satisfying the inequality

$$\|g_q(2x) - 12 g_q(x) - 4 g_q(-x)\| \leq \beta\left(x\right) \tag{3.48}$$

for all $x \in \mathcal{X}$. Then there exists a unique quartic mapping $Q_4 : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.3) such that

$$\|g_q(x) - Q_4(x)\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj} x)}{16^{kj}} \tag{3.49}$$

for all $x \in \mathcal{X}$. The mapping $Q_4(x)$ is defined by

$$Q_4(x) = \lim_{n\to\infty} \frac{g_q(2^{nj} x)}{16^{nj}} \tag{3.50}$$

for all $x \in \mathcal{X}$.

Proof. Assume $j = 1$. Using evenness of g_q in (3.48), it follows that

$$\left\| g_q(x) - \frac{g_q(2x)}{16} \right\| \leq \frac{\beta\left(x\right)}{16} \tag{3.51}$$

for all $x \in \mathcal{X}$. The rest of the proof similar to the Theorem 3.1 □

The following corollary is an immediate consequence of Theorem 3.12 concerning the stability of (1.3).

Corollary 3.13. *Let μ and r be nonnegative real numbers. Let an even function $g_q : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|g_q(2x) - 12g_q(x) - 4g_q(-x)\| \leq \begin{cases} \mu, \\ \mu||x||^r, \quad r \neq 4; \end{cases} \tag{3.52}$$

for all $x \in \mathcal{X}$. Then there exists a unique quartic function $Q_4 : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g_q(x) - Q_4(x)\| \leq \begin{cases} \dfrac{\mu}{|15|}, \\ \dfrac{\mu||x||^r}{|4-2^r|}, \end{cases} \tag{3.53}$$

for all $x \in \mathcal{X}$.

Now, we will provide an example to illustrate that the functional equation (1.3) is not stable for $r = 4$ in condition (ii) of Corollary 3.13.

Example 3.14. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\beta(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g_q : \mathbb{R} \to \mathbb{R}$ by

$$g_q(x) = \sum_{n=0}^{\infty} \frac{\beta(2^n x)}{16^n}, \qquad for\ all \quad x \in \mathbb{R}.$$

Then g_q satisfies the functional inequality

$$|g_q(2x) - 12g_q(x) - 4g_q(-x)| \leq \frac{17 \times 16\mu}{15}\ |x|^4 \tag{3.54}$$

for all $x \in \mathbb{R}$. Then there do not exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ such that

$$|g_q(x) - Q_4(x)| \leq \kappa|x|^4, \qquad for\ all \quad x \in \mathbb{R}. \tag{3.55}$$

Proof. Now

$$|g_q(x)| \leq \sum_{n=0}^{\infty} \frac{|\beta(2^n x)|}{|16^n|} = \sum_{n=0}^{\infty} \frac{\mu}{16^n} = \frac{16\mu}{15}.$$

Therefore, we see that g_q is bounded. We are going to prove that g_q satisfies (3.54).

If $x = 0$ then (3.54) is trivial. If $|x|^4 \geq \dfrac{1}{16}$ then the left hand side of (3.54) is less than $\dfrac{16 \times 17\mu}{15}$. Now suppose that $0 < |x|^4 < \dfrac{1}{16}$. Then there exists a positive integer k such that

$$\frac{1}{16^k} \leq |x|^4 < \frac{1}{16^{k-1}}, \tag{3.56}$$

so that $2^{k-1}x < \dfrac{1}{2}$ and consequently

$$2^{k-1}(x), 2^{k-1}(-x), 2^{k-1}(2x) \in (-1, 1).$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$2^n(x), 2^n(-x), 2^n(2x) \in (-1, 1)$$

and

$$\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x)) = 0$$

for $n = 0, 1, \ldots, k-1$. From the definition of g_q and (3.56), we obtain that

$$\begin{aligned}\left|g_q(2x) - 12g_q(x) - 4g_q(-x)\right| &\leq \sum_{n=0}^{\infty} \frac{1}{2^n}\left|\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x))\right| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{2^n}\left|\beta(2^n(2x)) - 12\beta(2^n(x)) - 4\beta(2^n(-x))\right| \\ &\leq \sum_{n=k}^{\infty} \frac{1}{16^n} 17\mu = 17\ \mu \times \frac{16}{15 \cdot 16^k} = \frac{17 \times 16\mu}{15}|x|^4.\end{aligned}$$

Thus g_q satisfies (3.54) for all $x \in \mathbb{R}$ with $0 < |x|^4 < \dfrac{1}{16}$.

We claim that the quartic functional equation (1.3) is not stable for $r = 4$ in condition (ii) of Corollary 3.13. Suppose on the contrary that there exist a quartic mapping $Q_4 : \mathbb{R} \to \mathbb{R}$ and a constant $\kappa > 0$ satisfying (3.55). Since g_q is bounded and continuous for all $x \in \mathbb{R}$, Q_4 is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.12, Q_4 must have the form $Q_4(x) = cx^4$ for any x in $\mathbb{R}$. Thus, we obtain that

$$|g_q(x)| \leq (\kappa + |c|)\,|x|^4. \tag{3.57}$$

But we can choose a positive integer m with $m\mu > \kappa + |c|$.

If $x \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$g_q(x) = \sum_{n=0}^{\infty} \frac{\beta(2^n x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n x)}{2^n} = m\mu x^4 > (\kappa + |c|)\, x^4$$

which contradicts (3.57). Therefore the quartic functional equation (1.3) is not stable in sense of Ulam, Hyers and Rassias if $r = 4$, assumed in the inequality condition (ii) of (3.53). □

Now we are ready to prove our main theorem.

Theorem 3.15. *Let $j \in \{-1, 1\}$ and $\beta : \mathcal{X} \to [0, \infty)$ be a function with conditions (3.36) and (3.47) for all $x \in \mathcal{X}$. Let $g : \mathcal{X} \to \mathcal{Y}$ be a function satisfying the inequality*

$$\|g(2x) - 12g(x) - 4g(-x)\| \leq \beta(x) \tag{3.58}$$

for all $x \in \mathcal{X}$. Then there exists a unique cubic mapping $C : \mathcal{X} \to \mathcal{Y}$ and a unique quartic mapping $Q_4 : \mathcal{X} \to \mathcal{Y}$ which satisfying (1.3) such that

$$\begin{aligned}\|g(x) - C(x) - Q_4(x)\| \leq \frac{1}{2}\Bigg[&\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}x)}{8^{kj}} + \frac{\beta(-2^{kj}x)}{8^{kj}}\right) \\ &+\frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}x)}{16^{kj}} + \frac{\beta(-2^{kj}x)}{16^{kj}}\right)\Bigg]\end{aligned} \tag{3.59}$$

for all $x \in \mathcal{X}$. The mapping $C(x)$ and $Q_4(x)$ are defined in (3.39) and (3.50) respectively for all $x \in \mathcal{X}$.

Proof. The proof of the Theorem is similar to the Theorem 3.7. □

Using Corollaries 3.10 and 3.13, we have the following corollary concerning the stability of (1.3).

Corollary 3.16. *Let μ and r be nonnegative real numbers. Let a function $g : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$\|g(2x) - 12g(x) - 4g(-x)\| \leq \begin{cases} \mu, \\ \mu||x||^r, \quad r \neq 3, 4; \end{cases} \tag{3.60}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique cubic function $C : \mathcal{X} \to \mathcal{Y}$ and a unique quartic function $Q_4 : \mathcal{X} \to \mathcal{Y}$ such that

$$\|g(x) - C(x) - Q_4(x)\| \leq \begin{cases} \mu\left(\dfrac{1}{|7|} + \dfrac{1}{|16|}\right), \\ \mu\left(\dfrac{||x||^r}{|8-2^r|} + \dfrac{||x||^r}{|16-2^r|}\right), \end{cases} \tag{3.61}$$

for all $x \in \mathcal{X}$.

4. STABILITY RESULTS OF (1.2) AND (1.3): FIXED POINT METHOD

In this section, we apply a fixed point method for achieving stability of the functional equations (1.2) and (1.3) are respectively present.

Now, first we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii)The fixed point for each given element x^* is globally attractive, that is
(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point $x \in X$;*
(iii) One has the following estimation inequalities:
(A3) $d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X;$

(A4) $d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$

Theorem 4.2. [28] *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty\ \forall\ n \geq 0,$$

or there exists a natural number n_0 such that
(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T
(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Hereafter throughout this section, let us consider $\mathcal{E}$ and $\mathcal{F}$ to be a normed space and a Banach space, respectively.

4.1. FIXED POINT STABILITY RESULTS OF (1.2).

Theorem 4.3. *Let $f_a : \mathcal{E} \to \mathcal{F}$ be a odd mapping for which there exist a function $\zeta : \mathcal{E} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\kappa_i^k} \zeta(\kappa_i^k x) = 0 \tag{4.1}$$

where

$$\kappa_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 1, \end{cases} \tag{4.2}$$

such that the functional inequality

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\| \leq \zeta(x) \tag{4.3}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

has the property

$$\psi(x) = \frac{L}{\kappa_i}\ \psi\left(\kappa_i x\right). \tag{4.4}$$

for all $x \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.2) and

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\psi(x) \tag{4.5}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{E} \to \mathcal{F},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p,q) = \inf\{K \in (0,\infty) : \| p(x) - q(x) \| \leq K\psi(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\kappa_i} p(\kappa_i x),$$

for all $x \in \mathcal{E}$. Now $p, q \in \Gamma$,

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \| p(x) - q(x) \| \leq K\psi(x), x \in \mathcal{E}, \\ \Rightarrow & \left\| \frac{1}{\kappa_i} p(\kappa_i x) - \frac{1}{\kappa_i} q(\kappa_i x) \right\| \leq \frac{1}{\kappa_i} K\psi(\kappa_i x), x \in \mathcal{E}, \\ \Rightarrow & \left\| \frac{1}{\kappa_i} p(\kappa_i x) - \frac{1}{\kappa_i} q(\kappa_i x) \right\| \leq LK\psi(x), x \in \mathcal{E}, \\ \Rightarrow & \| \Upsilon p(x) - \Upsilon q(x) \| \leq LK\psi(x), x \in \mathcal{E}, \\ \Rightarrow & d(p,q) \leq LK. \end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

Using oddness of f_a in (4.3), we arrive

$$\|f_a(2x) - 2f(x)\| \leq \zeta(x) \tag{4.6}$$

for all $x \in \mathcal{E}$. It follows from (4.6) that

$$\left\| \frac{f_a(2x)}{2} - f_a(x) \right\| \leq \frac{\zeta(x)}{2} \tag{4.7}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 0$ it reduces to

$$\left\| \frac{f_a(2x)}{2} - f_a(x) \right\| \leq \ L\psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon f_a, f_a) \leq L \Rightarrow d(\Upsilon f_a, f_a) \leq L = L^1 < \infty. \tag{4.8}$$

Again replacing $x = \frac{x}{2}$ in (4.6), we get

$$\left\| f_a(x) - 2f_a\left(\frac{x}{2}\right) \right\| \leq \zeta\left(\frac{x}{2}\right) \tag{4.9}$$

for all $x \in \mathcal{E}$. Using (4.4) for the case $i = 1$ it reduces to

$$\left\| f_a(x) - 2f_a\left(\frac{x}{2}\right) \right\| \leq \ \psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(f_a, \Upsilon f_a) \leq 1 \Rightarrow d(f_a, \Upsilon f_a) \leq 1 = L^0 < \infty. \tag{4.10}$$

From (4.8) and (4.10), we arrive

$$d(f_a, \Upsilon f_a) \leq L^{1-i}.$$

Therefore (FP1) holds.

By (FP2), it follows that there exists a fixed point A of Υ in Γ such that

$$A(x) = \lim_{k\to\infty} \frac{f_a(\kappa_i^k x)}{\kappa_i^k}, \qquad \forall\ x \in \mathcal{E}. \tag{4.11}$$

To order to prove $A : \mathcal{E} \to \mathcal{F}$ is additive. Replacing x by $\kappa_i^k x$ in (4.3) and dividing by κ_i^k, it follows from (4.1) that

$$\frac{1}{\kappa_i^k}\left\| f_a(\kappa_i^k 2x) - 3f_a(\kappa_i^k x) - f_a(-\kappa_i^k x) \right\| \leq \frac{1}{\kappa_i^k}\zeta(\kappa_i^k x)$$

for all $x \in \mathcal{E}$. Letting $k \to \infty$ in the above inequality and using the definition of $A(x)$, we see that

$$A(2x) = 3A(x) + A(-x)$$

i.e., A satisfies the functional equation (1.2) for all $x \in \mathcal{E}$.

By (FP3), A is the unique fixed point of Υ in the set

$$\Delta = \{A \in \Gamma : d(f_a, A) < \infty\},$$

such that

$$\|f_a(x) - A(x)\| \leq K\psi(x)$$

for all $x \in \mathcal{E}$ and $K > 0$. Finally by (FP4), we obtain

$$d(f_a, A) \leq \frac{1}{1-L} d(f_a, \Upsilon f_a)$$

this implies

$$d(f_a, A) \leq \frac{L^{1-i}}{1-L}$$

which yields

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\psi(x)$$

this completes the proof of the theorem. $\square$

The following corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.2).

Corollary 4.4. *Let $f_a : \mathcal{E} \to \mathcal{F}$ be an odd mapping and there exists real numbers ρ and r such that*

$$\|f_a(2x) - 3f_a(x) - f_a(-x)\| \leq \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \neq 1; \end{cases} \tag{4.12}$$

for all $x \in \mathcal{E}$. Then there exists a unique additive function $A : \mathcal{E} \to \mathcal{F}$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} (i) & |\rho|, \\ (ii) & \dfrac{\rho||x||^r}{|2-2^r|}, \end{cases} \tag{4.13}$$

for all $x \in \mathcal{E}$.

Proof. Setting

$$\zeta(x) = \begin{cases} \rho, \\ \rho||x||^r, \end{cases}$$

for all $x \in \mathcal{E}$. Now,

$$\frac{1}{\kappa_i^k}\zeta(\kappa_i^k x) = \begin{cases} \dfrac{\rho}{\kappa_i^k}, \\ \dfrac{\rho}{\kappa_i^k}||\kappa_i^k x_i||^r, \end{cases} = \begin{cases} \to 0 \text{ as } k \to \infty, \\ \to 0 \text{ as } k \to \infty. \end{cases}$$

Thus, (4.1) is holds.

But we have $\psi(x) = \zeta\left(\frac{x}{2}\right)$ has the property $\psi(x) = \dfrac{L}{\kappa_i}\ \psi(\kappa_i x)$ for all $x \in \mathcal{E}$. Hence

$$\psi(x) = \zeta\left(\frac{x}{2}\right) = \begin{cases} \rho \\ \dfrac{\rho}{2^r}||x||^r. \end{cases}$$

Now,

$$\frac{1}{\kappa_i}\psi(\kappa_i x) = \begin{cases} \dfrac{\rho}{\kappa_i}, \\ \dfrac{\rho}{\kappa_i}||\kappa_i x||^r, \end{cases} = \begin{cases} \dfrac{\rho}{\kappa_i}, \\ \dfrac{\rho}{\kappa_i}\kappa_i^r||x||^r, \end{cases} = \begin{cases} \kappa_i^{-1}\rho, \\ \kappa_i^{r-1}\rho||x||^r, \end{cases} = \begin{cases} \kappa_i^{-1}\psi(x), \\ \kappa_i^{r-1}\psi(x). \end{cases}$$

Hence the inequality (4.4) holds either, $L = 2^{-1}$ if $i = 0$ and $L = \frac{1}{2^{-1}}$ if $i = 1$. Now from (4.5), we prove the following cases for condition (i).

Case:1 $L = 2^{-1}$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{(2^{-1})^{1-0}}{1-2^{-1}}\psi(x) = \rho.$$

Case:2 $L = \frac{1}{2^{-1}}$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{\left(\frac{1}{2^{-1}}\right)^{1-1}}{1 - \frac{1}{2^{-1}}}\psi(x) = -\rho.$$

Also the inequality (4.4) holds either, $L = 2^{r-1}$ for $r < 1$ if $i = 0$ and $L = \frac{1}{2^{r-1}}$ for $r > 1$ if $i = 1$. Now from (4.5), we prove the following cases for condition (ii).

Case:3 $L = 2^{r-1}$ for $r < 1$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{\left(2^{(r-1)}\right)^{1-0}}{1 - 2^{(r-1)}}\psi(x) = \frac{1}{2 - 2^r}\rho||x||^r = \frac{\rho||x||^r}{2 - 2^r}.$$

Case:4 $L = \frac{1}{2^{r-1}}$ for $r > 1$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{\left(\frac{1}{2^{(r-1)}}\right)^{1-1}}{1 - \frac{1}{2^{(r-1)}}}\psi(x) = \frac{1}{2^r - 2}\rho||x||^s = \frac{\rho||x||^r}{2^r - 2}.$$

Hence the proof is complete. □

Theorem 4.5. *Let $f_q : \mathcal{E} \to \mathcal{F}$ be a even mapping for which there exist a function $\zeta : \mathcal{E} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\kappa_i^{2k}}\zeta(\kappa_i^k x) = 0 \tag{4.14}$$

where κ_i is defined in (4.2) such that the functional inequality

$$\|f_q(2x) - 3f_q(x) - f_q(-x)\| \leq \zeta(x) \tag{4.15}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

has the property

$$\psi(x) = \frac{L}{\kappa_i^2}\ \psi\left(\kappa_i x\right). \tag{4.16}$$

for all $x \in \mathcal{E}$. Then there exists a unique quadratic mapping $Q_2 : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.2) and

$$\|f_q(x) - Q_2(x)\| \leq \frac{L^{1-i}}{1 - L}\psi(x) \tag{4.17}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{E} \to \mathcal{F},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p, q) = \inf\{K \in (0, \infty) :\| p(x) - q(x) \|\leq K\psi(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\kappa_i^2} p(\kappa_i x),$$

for all $x \in \mathcal{E}$. Now $p, q \in \Gamma$,

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \parallel p(x) - q(x) \parallel \leq K\psi(x), x \in \mathcal{E}, \\ & \Rightarrow \left\| \frac{1}{\kappa_i^2} p(\kappa_i x) - \frac{1}{\kappa_i^2} q(\kappa_i x) \right\| \leq \frac{1}{\kappa_i^2} K\psi(\kappa_i x), x \in \mathcal{E}, \\ & \Rightarrow \left\| \frac{1}{\kappa_i^2} p(\kappa_i x) - \frac{1}{\kappa_i^2} q(\kappa_i x) \right\| \leq LK\psi(x), x \in \mathcal{E}, \\ & \Rightarrow \parallel \Upsilon p(x) - \Upsilon q(x) \parallel \leq LK\psi(x), x \in \mathcal{E}, \\ & \Rightarrow d(p,q) \leq LK. \end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

Using evenness of f_q in (4.15), we arrive

$$\|f_q(2x) - 4f(x)\| \leq \zeta(x) \tag{4.18}$$

for all $x \in \mathcal{E}$. It follows from (4.18) that

$$\left\| \frac{f_q(2x)}{4} - f_q(x) \right\| \leq \frac{\zeta(x)}{4} \tag{4.19}$$

for all $x \in \mathcal{E}$. Using (4.16) for the case $i = 0$ it reduces to

$$\left\| \frac{f_q(2x)}{4} - f_q(x) \right\| \leq \; L\psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon f_q, f_q) \leq L \Rightarrow d(\Upsilon f_q, f_q) \leq L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (4.18), we get

$$\left\| f_q(x) - 4f_q\left(\frac{x}{2}\right) \right\| \leq \zeta\left(\frac{x}{2}\right) \tag{4.20}$$

for all $x \in \mathcal{E}$. Using (4.16) for the case $i = 1$ it reduces to

$$\left\| f_q(x) - 4f_q\left(\frac{x}{2}\right) \right\| \leq \; \psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(f_q, \Upsilon f_q) \leq 1 \Rightarrow d(f_q, \Upsilon f_q) \leq 1 = L^0 < \infty.$$

From the above two cases, we arrive

$$d(f_q, \Upsilon f_q) \leq L^{1-i}$$

Therefore (FP1) holds. The rest of the proof is similar to that of Theorem 4.3. $\square$

The following corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.2).

Corollary 4.6. *Let $f_q : \mathcal{E} \to \mathcal{F}$ be an even mapping and there exists real numbers ρ and r such that*

$$\|f(2x) - 3f(x) - f(-x)\| \leq \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \neq 2; \end{cases} \tag{4.21}$$

for all $x \in \mathcal{E}$. Then there exists a unique quadratic function $Q_2 : \mathcal{E} \to \mathcal{F}$ such that

$$\|f_q(x) - Q_2(x)\| \leq \begin{cases} (i) & \dfrac{\rho}{|3|}, \\ (ii) & \dfrac{\rho||x||^r}{|4-2^r|}, \end{cases} \tag{4.22}$$

for all $x \in \mathcal{E}$.

Proof. The proof of the corollary is similar lines to the of Corollary 4.4. □

Now, we are ready to prove the main fixed point stability results.

Theorem 4.7. *Let $f : \mathcal{E} \to \mathcal{F}$ be a mapping for which there exist a function $\zeta : E \to [0, \infty)$ with the conditions (4.1) and (4.14) where κ_i is defined (4.2) such that the functional inequality*

$$\|f(2x) - 3f(x) - f(-x)\| \leq \zeta(x) \tag{4.23}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

with the properties (4.4) and (4.16) for all $x \in \mathcal{E}$. Then there exists a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation and a unique quadratic mapping $Q : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.2) and

$$\|f(x) - A(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}(\psi(x) + \psi(-x)) \tag{4.24}$$

for all $x \in \mathcal{E}$.

Proof. It follows from (3.29) and Theorem 4.3, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\psi(x) + \psi(-x)\right) \tag{4.25}$$

for all $x \in \mathcal{E}$. Also, it follows from (3.31) and Theorem 4.5, we have

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\psi(x) + \psi(-x)\right) \tag{4.26}$$

for all $x \in \mathcal{E}$. Define

$$f(x) = f_e(x) + f_o(x) \tag{4.27}$$

for all $x \in \mathcal{E}$. From (4.25),(4.26) and (4.27), we arrive

$$\begin{aligned}\|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q(x)\| \\ &\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \\ &\leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left[(\psi(x) + \psi(-x)) + (\psi(x) + \psi(-x))\right] \\ &\leq \frac{L^{1-i}}{1-L}\left(\psi(x) + \psi(-x)\right)\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

The following corollary is an immediate consequence of Theorem 4.7, using Corollaries 4.4 and 4.6 concerning the stability of (1.2).

Corollary 4.8. *Let $f : \mathcal{E} \to \mathcal{F}$ be a mapping and there exists real numbers ρ and r such that*

$$\|f(2x) - 3f(x) - f(-x)\| \leq \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \neq 1, 2; \end{cases} \tag{4.28}$$

for all $x \in \mathcal{E}$. Then there exists a unique additive function $A : \mathcal{E} \to \mathcal{F}$ and a unique quadratic function $Q_2 : \mathcal{E} \to \mathcal{F}$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} |\rho|\left(1 + \dfrac{1}{3}\right), \\ \rho||x||^r\left(\dfrac{1}{|2-2^r|} + \dfrac{1}{|4-2^r|}\right), \end{cases} \tag{4.29}$$

for all $x \in \mathcal{E}$.

4.2. FIXED POINT STABILITY RESULTS OF (1.3).

Theorem 4.9. *Let $g_c : \mathcal{E} \to \mathcal{F}$ be a odd mapping for which there exist a function $\zeta : \mathcal{E} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\kappa_i^{3k}}\zeta(\kappa_i^k x) = 0 \tag{4.30}$$

where

$$\kappa_i = \begin{cases} 2 & if \quad i = 0; \\ \frac{1}{2} & if \quad i = 1, \end{cases} \tag{4.31}$$

such that the functional inequality

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\| \leq \zeta(x) \tag{4.32}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

has the property

$$\psi(x) = \frac{1}{\kappa_i^3}\ \psi\left(\kappa_i x\right). \tag{4.33}$$

for all $x \in \mathcal{E}$. Then there exists a unique cubic mapping $C : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.3) and

$$\|g_c(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\psi(x) \tag{4.34}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{E} \to \mathcal{F},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(x) - q(x) \|\leq K\psi(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\kappa_i^3} p(\kappa_i x),$$

for all $x \in \mathcal{E}$. Now $p, q \in \Gamma$,

$$\begin{aligned} d(p,q) \leq K \Rightarrow & \| p(x) - q(x) \|\leq K\psi(x), x \in \mathcal{E}, \\ \Rightarrow & \left\| \frac{1}{\kappa_i} p(\kappa_i x) - \frac{1}{\kappa_i^3} q(\kappa_i x) \right\| \leq \frac{1}{\kappa_i^3} K\psi(\kappa_i x), x \in \mathcal{E}, \\ \Rightarrow & \left\| \frac{1}{\kappa_i^3} p(\kappa_i x) - \frac{1}{\kappa_i^3} q(\kappa_i x) \right\| \leq LK\psi(x), x \in \mathcal{E}, \\ \Rightarrow & \| \Upsilon p(x) - \Upsilon q(x) \|\leq LK\psi(x), x \in \mathcal{E}, \\ \Rightarrow & d(p,q) \leq LK. \end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

Using oddness of g_c in (4.32), we arrive

$$\|g_c(2x) - 8f(x)\| \leq \zeta(x) \tag{4.35}$$

for all $x \in \mathcal{E}$. It follows from (4.35) that

$$\left\| \frac{g_c(2x)}{8} - g_c(x) \right\| \leq \frac{\zeta(x)}{8} \tag{4.36}$$

for all $x \in \mathcal{E}$. Using (4.33) for the case $i = 0$ it reduces to

$$\left\| \frac{g_c(2x)}{8} - g_c(x) \right\| \leq\ L\psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon g_c, g_c) \leq L \Rightarrow d(\Upsilon g_c, g_c) \leq L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (4.35), we get

$$\left\| g_c(x) - 8g_c\left(\frac{x}{2}\right) \right\| \leq \xi\left(\frac{x}{2}\right) \tag{4.37}$$

for all $x \in \mathcal{E}$. Using (4.33) for the case $i = 1$ it reduces to

$$\left\| g_c(x) - 8g_c\left(\frac{x}{2}\right) \right\| \leq \ \psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(g_c, \Upsilon g_c) \leq 1 \Rightarrow d(g_c, \Upsilon g_c) \leq 1 = L^0 < \infty.$$

From the above two cases, we arrive

$$d(g_c, \Upsilon g_c) \leq L^{1-i}.$$

Therefore (FP1) holds. The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.9 concerning the stability of (1.3).

Corollary 4.10. *Let $g_c : \mathcal{E} \to \mathcal{F}$ be an odd mapping and there exists real numbers ρ and r such that*

$$\|g_c(2x) - 12g_c(x) - 4g_c(-x)\| \leq \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \neq 3; \end{cases} \tag{4.38}$$

for all $x \in \mathcal{E}$. Then there exists a unique cubic function $C : \mathcal{E} \to \mathcal{F}$ such that

$$\|g_c(x) - C(x)\| \leq \begin{cases} (i) & \dfrac{\rho}{|7|}, \\ (ii) & \dfrac{\rho||x||^r}{|8 - 2^r|}, \end{cases} \tag{4.39}$$

for all $x \in \mathcal{E}$.

Proof. The proof of the corollary is similar lines to the of Corollary 4.4. □

Theorem 4.11. *Let $g_q : \mathcal{E} \to \mathcal{F}$ be a even mapping for which there exist a function $\zeta : \mathcal{E} \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{1}{\kappa_i^{4k}} \zeta(\kappa_i^k x) = 0 \tag{4.40}$$

where κ_i is defined in (4.31) such that the functional inequality

$$\|g_q(2x) - 12g_q(x) - 4g_q(-x)\| \leq \zeta(x) \tag{4.41}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

has the property

$$\psi(x) = \frac{L}{\kappa_i^4}\ \psi\left(\kappa_i x\right). \tag{4.42}$$

for all $x \in \mathcal{E}$. Then there exists a unique quartic mapping $Q_4 : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.3) and

$$\|g_q(x) - Q_4(x)\| \leq \frac{L^{1-i}}{1-L}\psi(x) \tag{4.43}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\Gamma = \{p/p : \mathcal{E} \to \mathcal{F},\ p(0) = 0\}$$

and introduce the generalized metric on Γ,

$$d(p,q) = \inf\{K \in (0,\infty) :\| p(x) - q(x) \|\leq K\psi(x), x \in \mathcal{E}\}.$$

It is easy to see that (Γ, d) is complete.

Define $\Upsilon : \Gamma \to \Gamma$ by

$$\Upsilon p(x) = \frac{1}{\kappa_i^4} p(\kappa_i x),$$

for all $x \in \mathcal{E}$. Now $p, q \in \Gamma$,

$$\begin{aligned}
d(p,q) \leq K \Rightarrow & \| p(x) - q(x) \|\leq K\psi(x), x \in \mathcal{E}.\\
\Rightarrow & \left\| \frac{1}{\kappa_i^4}p(\kappa_i x) - \frac{1}{\kappa_i^4}q(\kappa_i x)\right\| \leq \frac{1}{\kappa_i^4}K\psi(\kappa_i x), x \in \mathcal{E},\\
\Rightarrow & \left\| \frac{1}{\kappa_i^4}p(\kappa_i x) - \frac{1}{\kappa_i^4}q(\kappa_i x)\right\| \leq LK\psi(x), x \in \mathcal{E},\\
\Rightarrow & \| \Upsilon p(x) - \Upsilon q(x) \|\leq LK\psi(x), x \in \mathcal{E},\\
\Rightarrow & d(p,q) \leq LK.
\end{aligned}$$

This implies $d(\Upsilon p, \Upsilon q) \leq Ld(p,q)$, for all $p, q \in \Gamma$. i.e., T is a strictly contractive mapping on Γ with Lipschitz constant L.

Using evenness of g_q in (4.41), we arrive

$$\|g_q(2x) - 16f(x)\| \leq \zeta(x) \tag{4.44}$$

for all $x \in \mathcal{E}$. It follows from (4.44) that

$$\left\|\frac{g_q(2x)}{16} - g_q(x)\right\| \leq \frac{\zeta(x)}{16} \tag{4.45}$$

for all $x \in \mathcal{E}$. Using (4.42) for the case $i = 0$ it reduces to

$$\left\|\frac{g_q(2x)}{16} - g_q(x)\right\| \leq\ L\psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(\Upsilon g_q, g_q) \le L \Rightarrow d(\Upsilon g_q, g_q) \le L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (4.44), we get

$$\left\| g_q(x) - 16 g_q\left(\frac{x}{2}\right) \right\| \le \xi\left(\frac{x}{2}\right) \tag{4.46}$$

for all $x \in \mathcal{E}$. Using (4.42) for the case $i = 1$ it reduces to

$$\left\| g_q(x) - 16 g_q\left(\frac{x}{2}\right) \right\| \le \ \psi(x)$$

for all $x \in \mathcal{E}$,

$$\text{i.e.,} \quad d(g_q, \Upsilon g_q) \le 1 \Rightarrow d(g_q, \Upsilon g_q) \le 1 = L^0 < \infty.$$

From the above two cases, we arrive

$$d(g_q, \Upsilon g_q) \le L^{1-i}.$$

Therefore (FP1) holds. The rest of the proof is similar to that of Theorem 4.3. □

The following corollary is an immediate consequence of Theorem 4.11 concerning the stability of (1.3).

Corollary 4.12. *Let $g_q : \mathcal{E} \to \mathcal{F}$ be an even mapping and there exists real numbers ρ and r such that*

$$\|g_q(2x) - 12 g_q(x) - 4 g_q f(-x)\| \le \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \ne 4; \end{cases} \tag{4.47}$$

for all $x \in \mathcal{E}$. Then there exists a unique quartic function $Q_4 : \mathcal{E} \to \mathcal{F}$ such that

$$\|g_q(x) - Q_2(x)\| \le \begin{cases} (i) & \dfrac{\rho}{|15|}, \\ (ii) & \dfrac{\rho||x||^r}{|16 - 2^r|}, \end{cases} \tag{4.48}$$

for all $x \in \mathcal{E}$.

Proof. The proof of the corollary is similar lines to the of Corollary 4.4. □

Now, we are ready to prove the main fixed point stability results.

Theorem 4.13. *Let $g : \mathcal{E} \to \mathcal{F}$ be a mapping for which there exist a function $\zeta : \mathcal{E} \to [0, \infty)$ with the conditions (4.30) and (4.40) where κ_i is defined (4.31) such that the functional inequality*

$$\|g(2x) - 12 g(x) - 4 g(-x)\| \le \zeta(x) \tag{4.49}$$

for all $x \in \mathcal{E}$. If there exists $L = L(i) < 1$ such that the function

$$x \to \psi(x) = \zeta\left(\frac{x}{2}\right),$$

with the properties (4.33) and (4.42) for all $x \in \mathcal{E}$. Then there exists a unique cubic mapping $C : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation and a unique quartic mapping $Q_4 : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.3) and

$$\|g(x) - C(x) - Q4(x)\| \leq \frac{L^{1-i}}{1-L}(\psi(x) + \psi(-x)) \tag{4.50}$$

for all $x \in \mathcal{E}$.

Proof. The proof of the Theorem is similar to the Theorem 4.7. □

The following Corollary is an immediate consequence of Theorem 4.13, using Corollaries 4.10 and 4.12 concerning the stability of (1.3).

Corollary 4.14. *Let $g : \mathcal{E} \to \mathcal{F}$ be a mapping and there exists real numbers ρ and r such that*

$$\|g(2x) - 12g(x) - 4g(-x)\| \leq \begin{cases} (i) & \rho, \\ (ii) & \rho||x||^r, \quad r \neq 2, 4; \end{cases} \tag{4.51}$$

for all $x \in \mathcal{E}$. Then there exists a unique cubic function $C : \mathcal{E} \to \mathcal{F}$ and a unique quartic function $Q_4 : \mathcal{E} \to \mathcal{F}$ such that

$$\|g(x) - C(x) - Q_4(x)\| \leq \begin{cases} |\rho| \left(\dfrac{1}{7} + \dfrac{1}{15}\right), \\ \rho||x||^r \left(\dfrac{1}{|8-2^r|} + \dfrac{1}{|16-2^r|}\right), \end{cases} \tag{4.52}$$

for all $x \in \mathcal{E}$.

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α-η Dominated Mappings and Related Common Fixed Point Results in Closed Ball

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Abstract: Recently Salimi et al. [Fixed Point Theory Appl., 2013:151] modified the notion of α-admissible mappings. In this paper, the concept of α-η dominated mappings is introduced and several general common fixed point results for two, three and four mappings in a closed ball in 0-complete partial metric spaces are established.

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1 Introduction and Preliminaries

Let $T : X \longrightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain. It is possible that $T : X \longrightarrow X$ is not a contraction but $T : Y \longrightarrow X$ is a contraction, where Y is a subset of X. One can obtain fixed point results for such mapping by using suitable conditions. Recently Arshad et al. [7] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions in closed ball in a complete partial metric space(see also [5, 6, 16, 20, 27, 28, 29]).

Matthews [17] introduced the concept of a partial metric space. In partial metric spaces, Matthews excluded an additional condition of metric spaces that is $d(x, x) = 0$ for all x. As it is better to prove a result with weaker conditions, so one should prove new results in partial metric spaces. Partial metric spaces have applications in theoretical computer science (see [15, 17]). Romaguera [23] intoduced 0-complete partial metric space which is a genralization of complete partial metric space.

The existence of fixed points of α-admissible mappings in complete metric spaces has been studied by several researchers (see [4, 14, 15, 25, 26] and references therein). In this paper we discuss common fixed point results for α-η dominated mappings in a closed ball in 0-complete partial metric space. The results given in this paper improve and extend several recent results in [7, 29].

Consistent with [12, 17, 23], the following definitions and results will be needed in the sequel.

Definition 1.1. [17] Let X be a nonempty set. If for any $x, y, z \in X$, mapping $p : X \times X \to R^+$ satisfies

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,

(P_2) $p(x,x) \leq p(x,y)$,

(P_3) $p(x,y) = p(y,x)$,

(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Then it is said to be a partial metric on X and the pair (X,p) is called a partial metric space.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$. Also $\overline{B_p(x,r)} = \{y \in X : p(x,y) \leq p(x,x) + r\}$ is a closed ball in (X,p).

It is clear that if $p(x,y) = 0$, then from P_1 and P_2, $x = y$. But if $x = y$, then $p(x,y)$ may not be 0.

Definition 1.2. [17] Let (X,p) be a partial metric space, then,

(a) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.

(b) [23] A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim\limits_{n,m\to\infty} p(x_n,x_m) = 0$. The space (X,p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $\lim_{n\to\infty} p(x,x_n) = p(x,x) = 0$.

(c) [23] If (X,p) is complete, then it is 0-complete.

Romaguera [23] has given an example which proves that converse assertion of (c) does not hold. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

We require the following lemma for subsequent use:

Lemma 1.3. [12] Let X be a non empty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.

2 Common fixed point results

Definition 2.1: Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. We say that T is α-η dominated mapping if $x \in X$ such that $\alpha(x,Tx) \geq \eta(x,Tx)$. If $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,z) \geq \eta(y,z)$ implies that $\alpha(x,z) \geq \eta(x,z)$ then we say that T is triangle α-η dominated mapping. Also, if we take $\eta(x,y) = 1$ then T is called α-dominated mapping and triangle α-dominated mapping respectively and if we take $\alpha(x,y) = 1$ then T is called η-subdominated mapping and triangle η-subdominated mapping respectively.

In the following we present common fixed point theorems for the pair of α-η-dominated contractive mappings in a closed ball.

Theorem 2.2. Let (X,p) be a 0-complete partial metric space. Suppose there exist two functions, $\alpha,\eta : X\times X\to[0,+\infty)$ such that S and T are α-η dominated mapping. Let $x_0,x,y\in X$, $r>0$. If there exist some k,t such that, $k+2t\in[0,1)$ and the following conditions hold:

$$p(Sx,Ty)\leq kp(x,y)+t[p(x,Sx)+p(y,Ty)], \tag{2.1}$$

for all $x,y\in\overline{B(x_0,r)}$ such that $\alpha(x,y)\geq\eta(x,y)$ or $\alpha(y,x)\geq\eta(y,x)$ and

$$p(x_0,Sx_0)\leq(1-\lambda)[r+p(x_0,x_0)], \tag{2.2}$$

where $\lambda=\dfrac{k+t}{1-t}$.

Then there exists a point x^* in $\overline{B_p(x_0,r)}$ such that $p(x^*,x^*)=0$. Moreover, if for any sequence $\{x_n\}$ in $\overline{B_p(x_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B_p(x_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq\eta(x_n,u)$ for all $n\in N\cup\{0\}$, then x^* is a common fixed point of S and T.

Proof. Let x_1 in X be such that $x_1=Sx_0$ and $x_2=Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1}=Sx_{2i},\ \text{and}\ x_{2i+2}=Tx_{2i+1},\ \text{where} i=0,1,2,\ldots.$$

As S is α-η dominated mapping then $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$. As T is α-η dominated mapping then $\alpha(x_1,x_2)\geq\eta(x_1,x_2)$. Continuing in this way we obtain $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$. First, we show that $x_n\in\overline{B_p(x_0,r)}$ for all $n\in N$. Using inequality (2.2) and the fact that $\lambda=\dfrac{k+t}{1-t}$, we have,

$$p(x_0,Sx_0)\leq r+p(x_0,x_0).$$

It implies that $x_1\in\overline{B_p(x_0,r)}$. Let $x_2,\cdots,x_j\in\overline{B_p(x_0,r)}$ for some $j\in N$. If $j=2i+1$, then as $x_1,x_2,...,x_j\in\overline{B_p(x_0,r)}$ and $\alpha(x_{2i},x_{2i+1})\geq\eta(x_{2i},x_{2i+1})$,where $i=0,1,2,\ldots\frac{j-1}{2}$. So using inequality (2.1), we obtain,

$$\begin{aligned}p(x_{2i+1},x_{2i+2}) &= p(Sx_{2i},Tx_{2i+1})\leq kp(x_{2i},x_{2i+1})\\ &\quad +t[p(x_{2i},Sx_{2i})+p(x_{2i+1},Tx_{2i+1})],\end{aligned}$$

which implies that,

$$\begin{aligned}p(x_{2i+1},x_{2i+2}) &\leq \lambda p(x_{2i},x_{2i+1})\\ &\leq \lambda^2p(x_{2i-1},x_{2i})\leq...\leq\lambda^{2i+1}p(x_0,x_1).\end{aligned} \tag{2.3}$$

If $j = 2i+2$, then as $x_1, x_2, ..., x_j \in \overline{B_p(x_0, r)}$ and $\alpha(x_{2i+1}, x_{2i+2}) \geq \eta(x_{2i+1}, x_{2i+2})$, where $i = 0, 1, 2, \ldots, \frac{j-2}{2}$. We obtain,

$$p(x_{2i+2}, x_{2i+3}) \leq \lambda^{2i+2} p(x_0, x_1). \tag{2.4}$$

Thus from inequalities (2.3) and (2.4) , we have,

$$p(x_j, x_{j+1}) \leq \lambda^j p(x_0, x_1) \text{ for some } j \in N. \tag{2.5}$$

Now

$$\begin{aligned}
p(x_0, x_{j+1}) &\leq p(x_0, x_1) + ... + p(x_j, x_{j+1}) \\
&\leq p(x_0, x_1) + ... + \lambda^j p(x_0, x_1), \qquad \text{(by 2.5)} \\
p(x_0, x_{j+1}) &\leq p(x_0, x_1)[1 + ... + \lambda^{j-1} + \lambda^j] \\
&\leq (1-\lambda)[r + p(x_0, x_0)] \frac{(1-\lambda^{j+1})}{1-\lambda} \\
&\leq r + p(x_0, x_0),
\end{aligned}$$

gives $x_{j+1} \in \overline{B_p(x_0, r)}$. Hence $x_n \in \overline{B_p(x_0, r)}$. Also $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, then

$$p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1), \text{ for all } n \in N. \tag{2.6}$$

So we have,

$$\begin{aligned}
p(x_{n+k}, x_n) &\leq p(x_{n+k}, x_{n+k-1}) + ... + p(x_{n+1}, x_n) \\
&\leq \lambda^{n+k-1} p(x_0, x_1) + ... + \lambda^n p(x_0, x_1), \quad \text{(by 2.6)} \\
p(x_{n+k}, x_n) &\leq \lambda^n p(x_0, x_1)[\lambda^{k-1} + \lambda^{k-2} + ... + 1] \\
&\leq \lambda^n p(x_0, x_1) \frac{(1-\lambda^k)}{1-\lambda} \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Notice that the sequence $\{x_n\}$ is a 0-Cauchy sequence in $(\overline{B_p(x_0, r)}, p)$. As $(\overline{B_p(x_0, r)}, p)$ is complete, therefore by Definition 1.2, there exists a point $x^* \in \overline{B_p(x_0, r)}$ with

$$\lim_{n \to \infty} p(x_n, x^*) = p(x^*, x^*) = 0. \tag{2.7}$$

Now,

$$p(x^*, Sx^*) \leq p(x^*, x_{2n+2}) + p(x_{2n+2}, Sx^*) - p(x_{2n+2}, x_{2n+2}).$$

On taking limit as $n \to \infty$ and using the fact that $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ when $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x^*$, we have,

$$\begin{aligned}
p(x^*, Sx^*) \leq \; & \lim_{n \to \infty} [p(x^*, x_{2n+2}) + kp(x_{2n+1}, x^*) \\
& + t\{p(x_{2n+1}, x_{2n+2}) + p(x^*, Sx^*)\}],
\end{aligned}$$

by inequalities (2.6) and (2.7), we obtain,

$$(1-t)\, p(x^*, Sx^*) \leq 0$$

and $x^* = Sx^*$. Similarly, by using,

$$p(x^*, Tx^*) \leq p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) - p(x_{2n+1}, x_{2n+1}),$$

we can show that $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B_p(x_0, r)}$.
■

If we take $T = S$ for all $x,\ y \in X$ in Theorem 2.2, we obtain following result.

Corollary 2.3. Let (X, p) be a 0-complete partial metric space. Suppose there exist two functions, $\alpha, \eta : X \times X \to [0, +\infty)$ such that S is α-η dominated mapping . Let $x_0, x, y \in X$, $r > 0$. If there exist some k, t such that, $k+2t \in [0, 1)$ and the following conditions hold:

$$p(Sx, Sy) \leq kp(x, y) + t[p(x, Sx) + p(y, Sy)],$$

for all $x, y \in \overline{B(x_0, r)}$ such that $\alpha(x, y) \geq \eta(x, y)$ and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$.

If for any sequence $\{x_n\}$ in $\overline{B_p(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_p(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \eta(x_n, u)$ for all $n \in N \cup \{0\}$. Then there exists a point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$.

If $\eta(x, y) = 1$ for all $x,\ y \in X$ in Theorem 2.2, we obtain following result.

Corollary 2.4. Let (X, p) be a 0-complete partial metric space. Suppose there exists, $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mapping. Let $x_0, x, y \in X$, $r > 0$. If there exist some k, t such that, $k + 2t \in [0, 1)$ and the following conditions hold:

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all $x, y \in \overline{B(x_0, r)}$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$.

If for any sequence $\{x_n\}$ in $\overline{B_p(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_p(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \geq 1$ for all $n \in N \cup \{0\}$. Then there exists a point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$.

If $\alpha(x, y) = 1$ for all $x,\ y \in X$ in Theorem 2.2, we obtain following result.

Corollary 2.5. Let (X, p) be a 0-complete partial metric space. Suppose there exists, $\eta : X \times X \to [0,+\infty)$ such that S and T are η-subdominated mapping. Let $x_0, x, y \in X$, $r > 0$. If there exist some k, t such that $k + 2t \in [0, 1)$ and $\eta(x, y) \leq 1$ or $\eta(y, x) \leq 1$ implies that

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all $x, y \in \overline{B(x_0, r)}$ and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$.

If for any sequence $\{x_n\}$ in $\overline{B_p(x_0, r)}$ such that $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_p(x_0, r)}$ as $n \to +\infty$ then $\eta(x_n, u) \leq 1$ for all $n \in N \cup \{0\}$. Then there exists a point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$.

Corollary 2.6. Let (X, d) be a complete metric space. Suppose there exists, $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mapping. Let

$$\alpha(x, y)d(Sx, Ty) \leq kd(x, y) + t[d(x, Sx) + d(y, Ty)]$$

holds for all $x, y \in X$ and $k + 2t \in [0, 1)$.

If for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then S and T have a common fixed point.

Theorem 2.7. Adding the following conditions to the hypotheses of Theorem 2.2.

(i) Let S and T are triangle α-η dominated mapping.

(ii) If for any two points x, y in $\overline{B_p(x_0, r)}$ there exists a point $z_0 \in \overline{B_p(x_0, r)}$ such that $\alpha(x, z_0) \geq \eta(x, z_0)$, $\alpha(y, z_0) \geq \eta(y, z_0)$.

(iii) For all $z \in \overline{B_p(x_0, r)}$ such that $\alpha(z, Sx_0) \geq \eta(z, Sx_0)$ implies

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz).$$

Then S and T have a unique common fixed point x^* and $p(x^*, x^*) = 0$.
Proof. Let y be another point in $\overline{B_p(x_0, r)}$ such that $y = Sy = Ty$. Now,

$$\begin{aligned} p(y, y) &= p(Sy, Ty) \\ &\leq kp(y, y) + t\{p(y, Ty) + p(y, Sy)\} \\ (1 - k - 2t)p(y, y) &\leq 0. \end{aligned}$$

This implies that,

$$p(y, y) = 0. \tag{2.9}$$

Now if $\alpha(x^*, y) \geq \eta(x^*, y)$, then,

$$\begin{aligned} p(x^*, y) &= p(Sx^*, Ty) \\ &\leq kp(x^*, y) + t[p(x^*, Sx^*) + p(y, Ty)] \\ (1-k)p(x^*, y) &\leq 0. \ \text{(by 2.7 and 2.9)} \end{aligned}$$

This shows that $x^* = y$. Now if $\alpha(x^*, y) \ngeq \eta(x^*, y)$, then there exists a point $z_0 \in \overline{B_p(x_0, r)}$ such that $\alpha(x^*, z_0) \geq \eta(x^*, z_0)$ and $\alpha(y, z_0) \geq \eta(y, z_0)$. Choose a point z_1 in X such that $z_1 = Tz_0$ and $z_2 = Sz_1$. Continuing this process, we construct a sequence z_n of points in X such that,

$$z_{2i+1} = Tz_{2i}, \text{ and } z_{2i+2} = Sz_{2i+1}, \text{ where} i = 0, 1, 2, \ldots.$$

As T is α-η dominated mapping then $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$. As S is α-η dominated mapping then $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$. Continuing in this way we obtain $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$ for all $n \in N \cup \{0\}$. As S and T are triangle α-η dominated mapping , so $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$ and $\alpha(z_1, z_2) \geq \eta(z_1, z_2)$ implies that $\alpha(z_0, z_2) \geq \eta(z_0, z_2)$. Continuing in this way we obtain $\alpha(z_0, z_{n+1}) \geq \eta(z_0, z_{n+1})$. Now $\alpha(x^*, z_0) \geq \eta(x^*, z_0)$ and $\alpha(z_0, z_{n+1}) \geq \eta(z_0, z_{n+1})$ implies that $\alpha(x^*, z_{n+1}) \geq \eta(x^*, z_{n+1})$. Also $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ and $\alpha(x^*, z_{n+1}) \geq \eta(x^*, z_{n+1})$ implies that $\alpha(x_n, z_{n+1}) \geq \eta(x_n, z_{n+1})$. Now we will show that $z_n \in \overline{B_p(x_0, r)}$ for all $n \in N$. Now,

$$\begin{aligned} p(Sx_0, Tz_0) &\leq kp(x_0, z_0) + t[p(x_0, x_1) + p(z_0, Tz_0)] \\ &\leq kp(x_0, z_0) + t[p(x_0, z_0) + p(x_1, Tz_0)], \quad \text{(by (iii))} \\ p(Sx_0, Tz_0) &\leq \lambda p(x_0, z_0) \text{ and} \end{aligned} \tag{2.10}$$

$$\begin{aligned} p(x_0, z_1) &\leq p(x_0, x_1) + p(x_1, z_1) - p(x_1, x_1) \\ &\leq (1-\lambda)[r + p(x_0, x_0)] + \lambda p(x_0, z_0), \quad \text{(by 2.2 and 2.10)} \\ p(x_0, z_1) &\leq (1-\lambda)[r + p(x_0, x_0)] + \lambda[r + p(x_0, x_0)] \quad (\text{as } z_0 \in \overline{B_p(x_0, r)}) \\ &= r + p(x_0, x_0), \end{aligned}$$

implies that $z_1 \in \overline{B_p(x_0, r)}$. Let $z_2, z_3, ..., z_j \in \overline{B_p(x_0, r)}$ for some $j \in N$. If j is even, then, we have,

$$\begin{aligned} p(x_1, Tz_j) &= kp(x_0, z_j) + t[p(x_0, x_1) + p(z_j, Tz_j)] \\ &\leq kp(x_0, z_j) + t[p(x_0, z_j) + p(x_1, Tz_j)], \ \text{(by (iii))} \\ p(x_1, Tz_j) &\leq \lambda p(x_0, z_j) \leq \lambda[r + p(x_0, x_0)]. \quad (\text{as } z_j \in \overline{B_p(x_0, r)}) \end{aligned} \tag{2.11}$$

Now,

$$\begin{aligned} p(x_0, Tz_j) &\leq p(x_0, x_1) + p(x_1, Tz_j) - p(x_1, x_1) \\ &\leq (1-\lambda)[r + p(x_0, x_0)] + \lambda[r + p(x_0, x_0)], \ \text{(by 2.11)} \\ p(x_0, z_{j+1}) &\leq r + p(x_0, x_0) \end{aligned} \tag{2.12}$$

Now, if j is odd then, following similar arguments as we have used to prove inequality (2.5), we have,

$$p(z_j, z_{j+1}) \leq \lambda^j p(z_0, z_1) \text{ for some } j \in N. \tag{2.13}$$

Now, we have,

$$\begin{aligned}
p(x_2, z_{j+1}) &= p(Tx_1, Sz_j) \leq kp(x_1, z_j) + t[p(x_1, Tx_1) + p(z_j, Sz_j)] \\
&\leq kp(x_1, z_j) + t[\lambda p(x_0, x_1) + \lambda p(z_{j-1}, Tz_{j-1})], \text{ (by 2.6 and 2.13)} \\
p(x_2, z_{j+1}) &\leq kp(x_1, z_j) + t\lambda[p(x_0, z_{j-1}) + p(x_1, z_j)], \text{ (by (iii))} \\
p(x_2, z_{j+1}) &\leq (k + t\lambda)p(x_1, Tz_{j-1}) + t\lambda[r + p(x_0, x_0)], \text{ (as } z_{j-1} \in \overline{B_p(x_0, r)}) \\
p(x_2, z_{j+1}) &\leq [(k + t\lambda)\lambda + t\lambda][r + p(x_0, x_0)], \text{ (by 2.11, as } j - 1 \text{ is even)} \\
p(x_2, z_{j+1}) &\leq \lambda^2[r + p(x_0, x_0)] \qquad (2.14)
\end{aligned}$$

Now,

$$\begin{aligned}
p(x_0, z_{j+1}) &\leq p(x_0, x_1) + p(x_1, x_2) + p(x_2, z_{j+1}) \\
&\leq p(x_0, x_1) + \lambda p(x_0, x_1) + \lambda^2[r + p(x_0, x_0)], \text{ (by 2.6 and 2.14)} \\
p(x_0, z_{j+1}) &\leq r + p(x_0, x_0). \qquad (2.15)
\end{aligned}$$

Therefore, from inequalities (2.12) and (2.15), $z_{j+1} \in \overline{B_p(x_0, r)}$ in both cases. Hence $z_n \in \overline{B_p(x_0, r)}$ for all $n \in N$. Thus, inequality (2.13) becomes

$$p(z_n, z_{n+1}) \leq \lambda^n p(z_0, z_1) \longrightarrow 0 \text{ as } n \to \infty. \tag{2.16}$$

As $\alpha(y, z_0) \geq \eta(y, z_0)$ and $\alpha(z_0, z_{n+1}) \geq \eta(z_0, z_{n+1})$ implies that $\alpha(y, z_{n+1}) \geq \eta(y, z_{n+1})$. Also $\alpha(x^*, z_{n+1}) \geq \eta(x^*, z_{n+1})$. Then, for $i \in N$,

$$\begin{aligned}
p(Tx^*, Sz_{2i-1}) &\leq kp(x^*, z_{2i-1}) + t[p(x^*, Tx^*) + p(z_{2i-1}, Sz_{2i-1})] \\
&= kp(Sx^*, Tz_{2i-2}) + tp(z_{2i-1}, z_{2i}), \\
p(x^*, Sz_{2i-1}) &\leq k^2 p(x^*, z_{2i-2}) + ktp(z_{2i-2}, z_{2i-1}) + tp(z_{2i-1}, z_{2i}) \\
&\;\;\vdots \\
&\leq k^{2i} p(x^*, z_0) + k^{2i-1} tp(z_0, z_1) + \cdots \\
&\quad + ktp(z_{2i-2}, z_{2i-1}) + tp(z_{2i-1}, z_{2i}).
\end{aligned}$$

On taking limit as $i \to \infty$ and by inequality (2.16), we have,

$$p(x^*, Sz_{2i-1}) = 0. \tag{2.17}$$

Similarly,

$$p(Sz_{2i-1}, y) \longrightarrow 0 \text{ as } i \to \infty. \tag{2.18}$$

Now by using inequality (2.17) and (2.18), we have

$$p(x^*, y) \leq p(x^*, Sz_{2i-1}) + p(Sz_{2i-1}, y) \longrightarrow 0 \text{ as } i \to \infty.$$

So, $x^* = y$. Hence $x^*$is a unique common fixed point of T and S in $\overline{B_p(x_0, r)}$. ■

Example 2.8. Let $X = [0, +\infty) \cap Q$ and $p : X \times X \to R^+$ be the 0-complete partial metric on X defined by $p(x, y) = \max\{x, y\}$. Define $\alpha(x, y) = 2x - y$, $\eta(x, y) = x - y$ and

$$Sx = \begin{cases} \dfrac{x}{16} \text{ if } x \in [0,1] \cap Q \\ x - \dfrac{1}{6} \text{ if } x \in (1, \infty) \cap Q \end{cases} \quad \text{and } Tx = \begin{cases} \dfrac{5x}{17} \text{ if } x \in [0,1] \cap Q \\ x - \dfrac{1}{7} \text{ if } x \in (1, \infty) \cap Q \end{cases}.$$

Clearly, S and T are α-η dominated mappings. Take, $k = \frac{1}{5}$, $t = \frac{1}{6}$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, then $\overline{B_p(x_0, r)} = [0, 1] \cap Q$. We have, $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, $\lambda = \dfrac{k+t}{1-t} = \dfrac{11}{25}$ with

$$(1 - \lambda)[r + p(x_0, x_0)] = \frac{14}{25}$$

and

$$p(x_0, Sx_0) = p(\frac{1}{2}, \frac{1}{32}) = \frac{1}{2} < (1 - \lambda)[r + p(x_0, x_0)].$$

Also if, $x = y = 2 \in (1, \infty) \cap Q$, then,

$$\begin{aligned} p(S2, T2) &= \max\{2 - \frac{1}{6}, 2 - \frac{1}{7}\} = \frac{13}{7} \\ &\geq \frac{1}{5}\max\{2, 2\} + \frac{1}{6}[\max\{2, 2 - \frac{1}{6}\} + \max\{2, 2 - \frac{1}{7}\}] = \frac{16}{15} \end{aligned}$$

So the contractive condition does not hold on X.

$$\begin{aligned} \text{Now if, } x, y &\in \overline{B_p(x_0, r)}, \text{then } p(Sx, Ty) = \max\{\frac{x}{16}, \frac{5y}{17}\} \\ &\leq \frac{1}{5}\max\{x, y\} + \frac{1}{6}[\max\{x, \frac{x}{16}\} + \max\{y, \frac{5y}{17}\}] \\ &= kp(x, y) + t[p(x, Sx) + p(y, Ty)] \end{aligned}$$

Also, $\alpha(z, Sx_0) \geq \eta(z, Sx_0)$ implies

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz)$$

for all $z \in \overline{B_p(x_0, r)}$.

Therefore, all the conditions of Theorem 2.7 are satisfied. Moreover, 0 is the common fixed point of S and T and $p(0, 0) = 0$.

In Theorem 2.7, the condition “If for any sequence $\{x_n\}$ in $\overline{B_p(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_p(x_0, r)}$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \eta(x_n, u)$ for all $n \in N \cup \{0\}$”, the condition (i), (ii), (iii) are imposed to restrict the condition (2.1) only for α-η dominated mapping and for those x, y in $\overline{B_p(x_0, r)}$ for which $\alpha(x, y) \geq \eta(x, y)$ or $\alpha(y, x) \geq \eta(y, x)$.

However, the following result relax these restrictions but impose the condition (2.1) for all elements in $\overline{B_p(x_0, r)}$.

Theorem 2.9. Let (X, p) be a 0-complete partial metric space, $x_0 \in X$, $r > 0$ and $S, T : X \to X$ be two mappings. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all x, y in $\overline{B_p(x_0, r)}$ and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$. Then there exists a unique point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$. Moreover, S and T have no fixed point other than x^*.

Proof. By following similar arguments of Theorem 2.1, we can obtain a unique point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^* = Tx^*$. Let $y = Ty$. Then y is the fixed point of T and it will not be a fixed point of S. Now,

$$\begin{aligned} p(x^*, y) &= p(Sx^*, Ty) \leq kp(x^*, y) + t[p(x^*, Sx^*) + p(y, Ty)] \\ p(x^*, y) &\leq \frac{t}{1-k} p(y, y) \leq p(y, y). \quad \text{(by 2.7)} \end{aligned}$$

A contradiction, as $p(y, y) \leq p(x^*, y)$. Hence $x^* = y$. Thus T has no fixed point other than x^*. Similarly S has no fixed point other than x^*.
■

In Theorem 2.7, the conditions (iii) and (2.2) are imposed to restrict the condition (2.1) only for x, y in $\overline{B_p(x_0, r)}$ and Example 2.8 explains the utility of these restrictions. However, the following result relax the conditions (iii) and (2.2) but impose the condition (2.1) for all elements $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$ or $\alpha(y, x) \geq \eta(y, x)$.

Theorem 2.10. Let (X, p) be a 0-complete partial metric space. Suppose there exist two functions, $\alpha, \eta : X \times X \to [0, +\infty)$ such that S and T are triangle α-η dominated mapping. If there exist some k, t such that, $k + 2t \in [0, 1)$ and the following conditions hold:

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$ or $\alpha(y, x) \geq \eta(y, x)$.

If for any sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in X$ as $n \to +\infty$ then $\alpha(x_n, u) \geq \eta(x_n, u)$ for all $n \in N \cup \{0\}$. Also for any two points x, y in X there exists a point $z_0 \in X$ such that $\alpha(x, z_0) \geq \eta(x, z_0)$, $\alpha(y, z_0) \geq \eta(y, z_0)$. Then there exists a point x^* in X such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$.

Now we apply our Theorem 2.7 to obtain unique common fixed point of three mappings in closed ball in 0-complete partial metric space.

Theorem 2.11. Let (X,p) be a partial metric space, $S,T,f:X\to X$ such that $SX\cup TX\subset fX$. Suppose there exist two functions, $\alpha,\eta:X\times X\to[0,+\infty)$ such that S T and f are α-η dominated mapping and $\alpha(fx,Sx)\geq\eta(fx,Sx)$ and $\alpha(fx,Tx)\geq\eta(fx,Tx)$. Suppose for $k+2t\in[0,1)$, $x_0\in X$, $r>0$, $\overline{B_p(fx_0,r)}\subseteq fX$ and for all $fx,fy\in\overline{B(fx_0,r)}$, $\alpha(fx,fy)\geq\eta(fx,fy)$ implies that

$$p\left(Sx,Ty\right)\leq kp(fx,fy)+t[p(fx,Sx)+p(fy,Ty)], \tag{2.19}$$

and

$$p(fx_0,Sx_0)\leq(1-\lambda)[r+p(x_0,x_0)], \tag{2.20}$$

where $\lambda=\dfrac{k+t}{1-t}$ and

$$p(fx_0,Sx_0)+p(fy,Ty)\leq p(fx_0,fy)+p(Sx_0,Ty), \tag{2.21}$$

for all $fy\in\overline{B_p(fx_0,r)}$ such that $\alpha(fy,Sx_0)\geq\eta(fy,Sx_0)$.

If for any sequence $\{x_n\}$ in $\overline{B_p(fx_0,r)}$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in N\cup\{0\}$ and $x_n\to u\in\overline{B_p(fx_0,r)}$ as $n\to+\infty$ then $\alpha(x_n,u)\geq\eta(x_n,u)$ for all $n\in N\cup\{0\}$ and for any two points x,y in $\overline{B_p(fx_0,r)}$ there exists a point $z_0\in\overline{B_p(fx_0,r)}$ such that $\alpha(x,z_0)\geq\eta(x,z_0)$, $\alpha(y,z_0)\geq\eta(y,z_0)$. If fX is 0-complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in $\overline{B_p(fx_0,r)}$. Also $p(fz,fz)=0$.

Proof. By Lemma 1.3, there exists $E\subset X$ such that $fE=fX$ and $f:E\to X$ is one-to-one. Now since $SX\cup TX\subset fX$, we define two mappings $g,h:fE\to fE$ by $g(fx)=Sx$ and $h(fx)=Tx$ respectively. Since f is one-to-one on E, then g,h are well-defined. As $\alpha(fx,Sx)\geq\eta(fx,Sx)$ implies that $\alpha(fx,g(fx))\geq\eta(fx,g(fx))$ and $\alpha(fx,Tx)\geq\eta(fx,Tx)$ implies that $\alpha(fx,h(fx))\geq\eta(fx,h(fx))$ therefore g and h are dominated maps. Now $fx_0\in\overline{B_p(fx_0,r)}\subseteq fX$, then $fx_0\in fX$. Let $y_0=fx_0$, choose a point y_1 in fX such that $y_1=h(y_0)$. As $\alpha(y_0,h(y_0))\geq\eta(y_0,h(y_0))$ so $\alpha(y_0,y_1)\geq\eta(y_0,y_1)$ and let $y_2=g(y_1)$. Now $\alpha(y_1,g(y_1))\geq\eta(y_1,g(y_1))$ gives $\alpha(y_1,y_2)\geq\eta(y_1,y_2)$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1}=h(y_{2i})\text{ and }y_{2i+2}=g(y_{2i+1}),\ \text{ where }i=0,1,2,...,$$

$\alpha(y_n,y_{n+1})\geq\eta(y_n,y_{n+1})$. Following similar arguments of Theorem 2.2, $y_n\in\overline{B_p(fx_0,r)}$. Also by inequalities (2.20) and (2.21) we obtain,

$$p(fx_0,g(fx_0))+p(fy,h(fy))\leq p(fx_0,fy)+p(g(fx_0),h(fy)),$$

for all $fy\in\overline{B_p(fx_0,r)}$ such that $\alpha(fy,Sx_0)\geq\eta(fy,Sx_0)$ and

$$p(fx_0,g(fx_0))\leq(1-\lambda)[r+p(x_0,x_0)].$$

By using inequality (2.19), for $fx, fy \in \overline{B(fx_0, r)}$, $\alpha(fx, fy) \geq \eta(fx, fy)$ implies that

$$p\left(g(fx), h(fy)\right) \leq kp(fx, fy) + t[p(fx, g(fx)) + p(fy, h(fy))].$$

As fX is a 0-complete space, all conditions of Theorem 2.7 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B_p(fx_0, r)}$ of g and h. Also $p(fz, fz) = 0$. ■

The rest of the proof is similar to the proof given in Theorem 4 [7], so we leave it. Hence we obtain a unique common fixed point of S, T and f.

Unique common fixed point results of three and four mappings in 0-complete partial metric space in a closed ball are given below which can be proved with the help of Theorem 2.5, by using the technique given in Theorem 7 [7]

Theorem 2.12. Let (X, p) be a partial metric space, $x_0 \in X$, $r > 0$ and S, T and f are self mappings on X such that $SX \cup TX \subset fX$, $\overline{B_p(fx_0, r)} \subseteq fX$. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p\left(Sx, Ty\right) \leq kp(fx, fy) + t[p(fx, Sx) + p(fy, Ty)],$$

for all $fx, fy \in \overline{B_p(fx_0, r)}$ and

$$p(fx_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k + t}{1 - t}$. If fX is 0-complete subspace of X and (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point fz in $\overline{B_p(fx_0, r)}$. Also $p(fz, fz) = 0$.

Theorem 2.13. Let (X, p) be a partial metric space, $x_0 \in X$, $r > 0$ and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$ and $\overline{B_p(fx_0, r)} \subseteq fX$. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p\left(Sx, Ty\right) \leq kp(fx, gy) + t[p(fx, Sx) + p(gy, Ty)],$$

for all $fx, fy \in \overline{B_p(fx_0, r)}$ and

$$p(fx_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k + t}{1 - t}$. If fX is 0-complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B_p(fx_0, r)}$. Also $p(fz, fz) = 0$.

Corollary 2.14. (Theorem 2.9 of [29]) Let (X, p) be a partial metric space and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$. Assume that for $r > 0$ and x_0 be an arbitrary point in X, the following conditions hold:

$$p\left(Sx, Ty\right) \leq kp(fx, gy)$$

for all elements $fx,\, gy \in \overline{B(fx_0, r)} \subseteq fX$; $0 \leq k < 1$ and

$$p(fx_0, Sx_0) \leq (1-k)[r + p(fx_0, fx_0)].$$

If fX is 0-complete subspace of X then there exists $fz \in X$ such that $p(fz, fz) = 0$. Also if (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Further S and T have no fixed point other than x^*.

Corollary 2.15. (Theorem 7 of [7]) Let (X, p) be a partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$ and $\overline{B(fx_0, r)} \subseteq fX$. Assume that the following condition holds:

$$p\,(Sx, Ty) \leq k[p(fx, Sx) + p(gy, Ty)]$$

for all $fx, fy \in \overline{B(fx_0, r)}$, where $0 \leq k < 1/2$, and

$$p(fx_0, Sx_0) \leq (1-\lambda)[r + p(fx_0, fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If fX is complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B(fx_0, r)}$. Also $p(fz, fz) = 0$.

The study of existence of fixed points in partially ordered sets has been initiated by Ran and Reurings [22] with applications to matrix equations. Agarwal et al. [2], Ciric et al. [10] and Nashine et al. [18] presented some new results for nonlinear contractions in partially ordered metric spaces and noted that their theorems can be used to investigate a large class of problems. [1, 3, 8, 9, 19, 21, 24] gave some fixed point theorems in ordered partial metric spaces.

Recall that if $(X, \preceq)$ is a preordered set and $T : X \to X$ is such that for $x \in X$, with $Sx \preceq x$, then the mapping T is said to be dominated. Define the set ∇ by

$$\nabla = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

Let $(X, \preceq, p)$ be a preordered 0-complete partial metric space. From Theorem 2.2 to Theorem 2.7, Theorem 2.10 and Theorem 2.11, we derive following important results in preordered 0-complete partial metric spaces.

Theorem 2.16. Let $(X, \preceq, p)$ be a preordered 0-complete partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all (x, y) in $(\overline{B_p(x_0, r)} \times \overline{B_p(x_0, r)}) \cap \nabla$ and

$$p(x_0, Sx_0) \leq (1-\lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B_p(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B_p(x_0,r)}$ such that $x^* = Sx^* = Tx^*$ and $p(x^*,x^*) = 0$. Also, x^* is unique, if for any two points x, y in $\overline{B_p(x_0,r)}$ there exists a point $z_0 \in \overline{B_p(x_0,r)}$ such that $z_0 \preceq x$ and $z_0 \preceq y$ and for all $z \in \overline{B_p(x_0,r)}$ such that $z \preceq Sx_0$, we have

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz)$$

Corollary 2.17. Let $(X, \preceq, p)$ be a preordered 0-complete partial metric space, $x_0 \in X$ and $S, T : X \to X$ be two dominated mappings. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p(Sx, Ty) \leq kp(x, y) + t[p(x, Sx) + p(y, Ty)],$$

for all (x, y) in ∇. If, for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in X such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$. Also, x^* is unique, if for any two points x, y in X there exists a point $z_0 \in X$ such that $z_0 \preceq x$ and $z_0 \preceq y$.

In Theorem 2.13, the condition "for all $z \in \overline{B_p(x_0,r)}$ such that $z \preceq Sx_0$, we have

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz)\text{"}$$

is imposed to obtain unique fixed point of a contractive mapping satisfying conditions (2.1). However, the following result relax this restriction but impose the conditions (2.1) and (2.2) for $t = 0$.

Corollary 2.18. (Theorem 2.2 of [29]) Let $(X, \preceq, p)$ be an ordered 0-complete partial metric space, $S, T : X \to X$ be dominated maps and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with

$$p(Sx, Ty) \leq kp(x, y) \text{ for all comparable elements } x, y \text{ in } \overline{B_p(x_0,r)}$$

$$\text{and } p(x_0, Sx_0) \leq (1-k)[r + p(x_0, x_0)].$$

If for a non-increasing sequence $\{x_n\}$ in $\overline{B_p(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists $x^* \in \overline{B_p(x_0,r)}$ such that $p(x^*, x^*) = 0$ and $x^* = Sx^* = Tx^*$. Also if, for any two points x, y in $\overline{B_p(x_0,r)}$ there exists a point $z \in \overline{B_p(x_0,r)}$ such that $z \preceq x$ and $z \preceq y$, then x^* is a unique common fixed point in $\overline{B_p(x_0,r)}$.

Corollary 2.19. (Theorem 1 of [7]) Let $(X, \preceq, p)$ be an ordered complete partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings.

Suppose for $t \in [0, \frac{1}{2})$, the following conditions hold:

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)],$$

for all (x, y) in $(\overline{B_p(x_0, r)} \times \overline{B_p(x_0, r)}) \cap \nabla$ and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{t}{1-t}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B_p(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B_p(x_0, r)}$ such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$. Also, x^* is unique, if for any two points x, y in $\overline{B_p(x_0, r)}$ there exists a point $z_0 \in \overline{B_p(x_0, r)}$ such that $z_0 \preceq x$ and $z_0 \preceq y$ and

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz)$$

for all $z \in \overline{B_p(x_0, r)}$ such that $z \preceq Sx_0$.

Theorem 2.20. Let $(X, \preceq, p)$ be a preordered partial metric space, $x_0 \in X$, $r > 0$ and S, T be self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$, $Tx \preceq fx$, $Sx \preceq fx$ and $\overline{B_p(fx_0, r)} \subseteq fX$. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$p(Sx, Ty) \leq kp(fx, fy) + t[p(fx, Sx) + p(fy, Ty)],$$

for all $(fx, fy) \in (\overline{B_p(fx_0, r)} \times \overline{B_p(fx_0, r)}) \cap \nabla$ and

$$p(fx_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)],$$

where $\lambda = \dfrac{k+t}{1-t}$ and

$$p(fx_0, Sx_0) + p(fy, Ty) \leq p(fx_0, fy) + p(Sx_0, Ty),$$

for all $fy \in \overline{B_p(fx_0, r)}$ such that $fy \preceq Sx_0$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B_p(fx_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and for any two points z and x in $\overline{B_p(fx_0, r)}$ there exists a point $y \in \overline{B_p(fx_0, r)}$ such that $y \preceq z$ and $y \preceq x$. If fX is 0-complete subspace of X and (S, f) and (T, f) are weakly compatible, then S, T and f have a unique common fixed point fz in $\overline{B_p(fx_0, r)}$. Also $p(fz, fz) = 0$.

Corollary 2.21. (Theorem 2.8 of [29]) Let $(X, \preceq, p)$ be an ordered partial metric space and S, T self mapping and f be a domonited mapping on X such that $SX \cup TX \subset fX$ and $Tx, Sx \preceq fx$. Assume that for $r > 0$ and x_0 be an arbitrary point in X, the following conditions hold:

$$p(Sx, Ty) \leq kp(fx, fy)$$

for all comparable elements $fx,\ fy \in \overline{B(fx_0,r)} \subseteq fX$; $0 \leq k < 1$ and

$$p(fx_0,Tx_0) \leq (1-k)[r + p(fx_0,fx_0)].$$

If for a non-increasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$, also for any two points z and x in $\overline{B(fx_0,r)}$ there exists a point $y \in \overline{B(fx_0,r)}$ such that $y \preceq z$ and $y \preceq x$ that is every pair of elements in $\overline{B(fx_0,r)}$ has a lower bound in $\overline{B(fx_0,r)}$. If fX is 0-complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in $\overline{B(fx_0,r)}$. Also $p(fz,fz)=0$.

Corollary 2.22. (Theorem 4 of [7]) Let $(X,\preceq,p)$ be a ordered partial metric space, $x_0,x,y \in X$, $r>0$ and S,T self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$, $\overline{B(fx_0,r)} \subseteq fX$ and $(Tx,fx),(Sx,fx) \in \nabla$. Assume that the following conditions hold:

$$p\,(Sx,Ty) \leq k[p(fx,Sx)+p(fy,Ty)]$$

for all $(fx,\ fy) \in (\overline{B(fx_0,r)} \times \overline{B(fx_0,r)}) \cap \nabla$; where $0 \leq k < 1/2$,

$$p(fx_0,Sx_0)+p(fy,Ty) \leq p(fx_0,fy)+p(Sx_0,Ty)$$

for all $fy \in \overline{B(fx_0,r)}$ such that $fy \preceq Sx_0$,

$$p(fx_0,Tx_0) \leq (1-\lambda)[r+p(fx_0,fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(fx_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and for any two points z and x in $\overline{B(fx_0,r)}$ there exists a point $y \in \overline{B(fx_0,r)}$ such that $y \preceq z$ and $y \preceq x$. If fX is complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in $\overline{B(fx_0,r)}$. Also $p(fz,fz)=0$.

Theorem 2.23. Let $(X,\preceq,p)$ be a preordered partial metric space, $x_0 \in X$, $r>0$ and S,T be self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$ and $Tx \preceq fx$, $Sx \preceq fx$. Suppose for $k+2t \in [0,1)$, the following conditions hold:

$$p\,(Sx,Ty) \leq kp(fx,fy)+t[p(fx,Sx)+p(fy,Ty)],$$

for all $(fx,fy) \in \nabla$. If, for a nonincreasing sequence $\{x_n\}$ in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ and for any two points z and x in X there exists a point $y \in X$ such that $y \preceq z$ and $y \preceq x$. If fX is 0-complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in $\overline{B_p(fx_0,r)}$. Also $p(fz,fz)=0$.

Remark 2.24. We can obtain the preordered complete metric version of all theorems, which are still not present in the literature.

Competing interests

The author declares that he has no competing interests.

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TABLE OF CONTENTS, JOURNAL OF CONCRETE AND APPLICABLE MATHEMATICS, VOL. 13, NO.'S 1-2, 2015

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QUALITATIVE ANALYSIS OF A SOLOW MODEL ON TIME SCALES

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Abstract. In this paper, we further analyze the Solow model on time scales. This model was recently introduced by the authors and it combines the continuous and discrete Solow models and extends them to different time scales. Assuming constant labor force growth in the Solow model, we establish a comparison theorem. Then, under the more realistic assumption that the labor force growth rate is a monotonically decreasing function, we discuss the comparison theorem as well as stability and monotonicity of the solutions of the Solow model. The economic meanings are also indicated in some remarks.

1. Introduction

The neoclassical growth model, developed by Solow [17] and Swan [18], had a great impact on how the economists think about economic growth. Since then, it has stimulated an enormous amount of work [2, 12, 20]. Since differential equation systems are usually more easily handled than difference systems from the analytical point of view, some of the economic models have used continuous timing [1, 9, 13–15] while others are given in difference models because some people think economic data are collected at discrete intervals and transformation of capital into investments depends on the length of time lag, etc. [8, 10, 21].

Hence, in economic modeling, either continuous timing or discrete timing is present, and there is not a common view among economists on which representation of time is better for economic models [14]. Meanwhile, many results concerning differential equations may carry over quite easily to corresponding results for difference equations, while other results seems to be completely different in nature from their continuous counterparts [6].

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The blanket assumption that economic processes are either solely continuous or solely discrete, while convenient for traditional mathematical approaches, may sometimes be inappropriate, because in reality many economic phenomena do feature both continuous and discrete elements. In biology, a familiar example is a "seasonal breeding population in which generations do not overlap" [6, 19]. A similar typical example in economics is the "seasonally changing investment and revenue in which seasons play an important effect on this kind of economic activity". In addition, option pricing and stock dynamics in finance [11] and the frequency and duration of market trading in economics [19] also contain this hybrid continuous-discrete processes.

Therefore, there is a great need to find a more flexible mathematical framework to accurately model the dynamical blend of such systems, so that they are precisely described and better understood. To meet this requirement, an emerging, progressive and modern area of mathematics, known as "dynamic equations on time scales", has been introduced. This calculus has the capacity to act as the framework to effectively describe the above phenomena and to make advances in their associate fields, see e.g., [3–5, 19].

This theory was introduced by Stefan Hilger in 1988 in his Ph.D. thesis [16] in order to unify continuous and discrete analysis, and has been developed by many mathematicians. A time scale $\mathbb{T}$ is defined as any nonempty closed subset of $\mathbb{R}$. In the time scales setting, once a result is established, special cases include the result for the differential equation when the time scale is the set of all real numbers $\mathbb{R}$ and the result for the difference equation when the time scale is the set of all integers $\mathbb{Z}$. The induction principle plays an important rôle in the proofs of some of our results, so we give it here.

Theorem 1.1 (See [6, Theorem 1.7])**.** *Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t): \ t \in [t_0, \infty)\}$$

is a family of statements satisfying:

A. The statement $S(t_0)$ is true.
B. If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.
C. If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(r)$ is true for all $r \in U \cap (t, \infty)$.
D. If $t \in [t_0, \infty)$ is left-dense and $S(r)$ is true for all $r \in [t_0, t)$, then $S(\sigma(t))$ is true.

Then $S(t)$ is true for all $t \in [t_0, \infty)$.

For other notations and a systematic introduction to time scales theory, we refer the reader to [6, 7].

2. The Solow Model on Time Scales

In this section, we will first recall some elements of the Solow model on general time scales as introduced by the authors in [5]. In the original Solow model [1, 17], the key elements are the production function, i.e., how the inputs of capital K and labor L are transformed into outputs, and how capital and labor force change over time. Here we still assume the following:

1. The production function F satisfies:
   (a) $F(\lambda K, \lambda L) = \lambda F(K, L)$ for any $\lambda, K, L \in \mathbb{R}^+$ (constant return to scales);
   (b) $F(K, 0) = F(0, L) = 0$ for any $K, L \in \mathbb{R}^+$;
   (c) $\dfrac{\partial F}{\partial K} > 0$, $\dfrac{\partial F}{\partial L} > 0$, $\dfrac{\partial^2 F}{\partial K^2} < 0$, $\dfrac{\partial^2 F}{\partial L^2} < 0$;
   (d) $\lim_{K\to 0^+} \dfrac{\partial F}{\partial K} = \lim_{L\to 0^+} \dfrac{\partial F}{\partial L} = \infty$, $\lim_{K\to\infty} \dfrac{\partial F}{\partial K} = \lim_{L\to\infty} \dfrac{\partial F}{\partial L} = 0$ (Inada conditions).
2. The capital stock changes are equal to the gross investment $I = sF(K, L)$ minus the capital depreciation δK, where s and δ are the savings rate and the depreciation factor of goods, respectively.
3. The labor force L changes at a constant rate n.

The three assumptions give, for any $t \in \mathbb{T}$

$$\begin{cases} Y(t) &= F(K(t), L(t)), \\ K^\Delta(t) &= I(t) - \delta K(t), \\ I(t) &= sY(t), \\ L^\Delta(t) &= nL(t). \end{cases} \tag{2.1}$$

From (2.1), we obtain

$$K^\Delta(t) = sY(t) - \delta K(t) = sF(K(t), L(t)) - \delta K(t). \tag{2.2}$$

Define

$$k(t) := \frac{K(t)}{L(t)} \quad \text{and} \quad y(t) := \frac{Y(t)}{L(t)},$$

which are regarded as the capital stock per worker and the production per worker, respectively. Let

$$f(k) := F\left(\frac{K}{L}, 1\right) = F(k, 1)$$

be the production function in intensive form. Then condition 1 changes to

$$\begin{cases} f(0) = 0; \\ f'(k) > 0 \quad \text{and} \quad f''(k) < 0 \quad \text{for all} \quad k \in \mathbb{R}^+ \\ \lim_{k \to 0^+} f'(k) = \infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0. \end{cases} \tag{2.3}$$

Applying the time scales quotient rule [6, Theorem 1.20], we use (2.2) to find

$$\begin{aligned} k^\Delta(t) = \left(\frac{K}{L}\right)^\Delta (t) &= \frac{K^\Delta(t)L(t) - K(t)L^\Delta(t)}{L(t)L^\sigma(t)} \\ &= \frac{K^\Delta(t)}{L^\sigma(t)} - \frac{K(t)L^\Delta(t)}{L(t)L^\sigma(t)} \\ &= \frac{K^\Delta(t)}{L(t)(1+\mu(t)n)} - \frac{K(t)n}{L(t)(1+\mu(t)n)} \\ &= \frac{sF(K(t), L(t)) - \delta K(t)}{L(t)(1+\mu(t)n)} - \frac{K(t)n}{L(t)(1+\mu(t)n)} \\ &= \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{1+\mu(t)n} k(t), \end{aligned}$$

i.e.,

$$k^\Delta(t) = \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{1+\mu(t)n} k(t), \tag{2.4}$$

which describes the Solow model on time scales. When $\mathbb{T} = \mathbb{R}$, equation (2.4) is the continuous Solow model in [1], whereas when $\mathbb{T} = \mathbb{Z}$, equation (2.4) is the discrete Solow model discussed in [10].

Equation (2.4) has a nontrivial equilibrium, denoted by $\hat{k}_n$, which is the unique positive solution of the equation

$$sf(k) = (\delta + n)k.$$

For $n = 0$, we denote by $\hat{k}_0$ the nontrivial steady state of equation (2.4). Obviously, we have

$$\lim_{n \to 0^+} \hat{k}_n = \hat{k}_0,$$

and $\hat{k}_n$ increases to $\hat{k}_0$ as n decreases to zero.

Next we will discuss some sufficient conditions for the existence and uniqueness of solutions of initial value problems for equation (2.4). Some comparison theorems between two solutions with different initial conditions will be given.

Theorem 2.1. *Assume* (2.3). *For* $t_0 \in \mathbb{T}$ *and* $k_0 \in \mathbb{R}^+$*, the initial value problem*

$$\begin{cases} k^{\Delta}(t) &= \dfrac{s}{1+\mu(t)n} f(k(t)) - \dfrac{\delta+n}{1+\mu(t)n} k(t), \\ k(t_0) &= k_0, \end{cases} \tag{2.5}$$

has a unique solution on $\mathbb{T}^+_{t_0} = \{t \in \mathbb{T} : t \geq t_0\}$.

Proof. Let

$$u(t,k) = \frac{s}{1+\mu(t)n} f(k) - \frac{\delta+n}{1+\mu(t)n} k.$$

Then $u(\cdot, k)$ is rd-continuous and regressive on $\mathbb{T}$, and

$$\begin{aligned} |u(t,k_1) - u(t,k_2)| &= \left| \frac{\partial u}{\partial k}(t,\xi) \right| |k_1 - k_2| \\ &= \left| \frac{s}{1+\mu(t)n} f'(\xi) - \frac{\delta+n}{1+\mu(t)n} \right| |k_1 - k_2| \\ &\leq (sf'(t_0) + \delta + n)|k_1 - k_2|, \end{aligned}$$

where $\xi \in (k_1, k_2)$. With the theorem of global existence and uniqueness in [6, Theorem 8.20], we can deduce that the solution of the problem (2.5) exists uniquely. □

Hence in the following, with condition (2.3), we always have the existence and uniqueness of solutions for initial value problems (2.5).

Theorem 2.2. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* k_1 *and* k_2 *be solutions of equation* (2.4) *on* $\mathbb{T}^+_{t_0}$ *with initial conditions* $k_1(t_0) = k_{01}$ *and* $k_2(t_0) = k_{02}$*, respectively. If* $0 < k_{01} < k_{02}$*, then*

$$k_1 < k_2 \quad on \quad \mathbb{T}^+_{t_0}.$$

Proof. We use the induction principle Theorem 1.1.

A. If $t = t_0$, then the result is obvious from the hypothesis.

B. If $t \in \mathbb{T}^+_{t_0}$ is right-scattered and $k_1(t) < k_2(t)$, then

$$\begin{aligned} k_1(\sigma(t)) - k_2(\sigma(t)) &= k_1(t) - k_2(t) + \mu(t)(k_1^{\Delta}(t) - k_2^{\Delta}(t)) \\ &= k_1(t) - k_2(t) + \frac{\mu(t)}{1+\mu(t)n} [sf(k_1(t)) - (\delta+n)k_1(t)] \\ &\quad - \frac{\mu(t)}{1+\mu(t)n} [sf(k_2(t)) - (\delta+n)k_2(t)] \\ &= k_1(t) - k_2(t) + \frac{s\mu(t)}{1+\mu(t)n} [f(k_1(t)) - f(k_2(t))] \end{aligned}$$

$$
\begin{aligned}
&-\frac{(\delta+n)\mu(t)}{1+\mu(t)n}\left[k_1(t)-k_2(t)\right] \\
=\ & \frac{1-\mu(t)\delta}{1+\mu(t)n}\left[k_1(t)-k_2(t)\right]+\frac{\mu(t)s}{1+\mu(t)n}\left[f(k_1(t))-f(k_2(t))\right] \\
<\ & 0,
\end{aligned}
$$

so

$$k_1(\sigma(t)) < k_2(\sigma(t)).$$

C. If t is right-dense and $k_1(t) < k_2(t)$, then there exists a right neighborhood $\mathring{U}^+(t)\cap\mathbb{T}$ of t such that $k_1(r) < k_2(r)$ for any $r \in \mathring{U}^+(t)\cap\mathbb{T}$. For if such a neighborhood does not exist, then there must exist a decreasing series $\{t_n\} \subset \mathring{U}^+(t)\cap\mathbb{T}$ and $\lim_{n\to\infty} t_n = t$, such that

$$k_1(t_n) \geq k_2(t_n)$$

and taking limit on both sides, we obtain $k_1(t) \geq k_2(t)$, a contradiction.

D. If t is left-dense and $k_1(r) < k_2(r)$ for any $r \in [t_0, t)\cap\mathbb{T}$, then from the continuity of the solutions, $k_1(t) \leq k_2(t)$. Uniqueness of solutions of initial value problems yields $k_1(t) < k_2(t)$.

Now an application of Theorem 1.1 concludes the proof. □

Note that Theorem 2.2 implies that the solution for the initial value problem (2.5) is always positive provided $k(t_0) > 0$.

Corollary 2.3. *Assume* (2.3) *and let $\delta > 0$ be such that $-\delta \in \mathcal{R}^+$. Then all solutions of equation* (2.4) *converge to the nontrivial steady state $\hat{k}_n$ monotonically, and the equilibrium point $\hat{k}_n$ is asymptotically stable and hence is a global attractor.*

Proof. If $k(t_0) < \hat{k}_n$, then Theorem 2.2 implies that

$$k(t) < \hat{k}_n \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Hence

$$k^\Delta(t) = \frac{s}{1+\mu(t)n}f(k(t)) - \frac{\delta+n}{1+\mu(t)n}k(t) > 0 \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

This means that k is increasing to the equilibrium point. Similar arguments apply to the case with $k(t_0) > \hat{k}_n$, which concludes the proof. □

Remark 2.4. Corollary 2.3 means that for two countries or districts with constant population growth rates, the one with the smaller population growth rate has a bigger capital per worker in the long run.

3. Improved Solow Model on Time Scales

In Section 2, we assumed that the labor force L grows at a constant rate n on the time scale, i.e.,

$$L^{\Delta}(t) = nL(t), \tag{3.1}$$

which implies that the labor force grows exponentially, that is,

$$L(t) = L_0 e_n(t, t_0),$$

where L_0 is the initial labor level at $t_0 \in \mathbb{T}$. With the properties of the exponential function on time scales and the fact that $n > 0$, we have

$$\lim_{t\to\infty} L(t) = \infty.$$

This means the labor force approaches infinity when t goes to infinity, which is unrealistic, because in reality the environment has a carrying capacity. So the simple growth model of labor in equation (3.1) can provide an adequate approximation to such growth only for an initial period, but does not accommodate growth reductions due to competition for environmental resources such as food, habitat and the policy factor etc. [1].

Since the 1950s, developing countries have recognized that the high population growth rate has seriously hampered the economic growth and adopted the population control policy. As a result, the population growth rates of many countries decreased fast in the last 40 years, such as in China. Also due to the aging of the population and, consequently, a dramatic increase in the number of deaths, the population growth rate decreased below zero in some developed countries, and is projected to decrease to zero during the next few decades in the developing countries [1].

So to incorporate the numerical upper bound on the growth size, on the reference of [10], we revise Condition 3 from Section 2 as follows.

3.′ The labor force L satisfies the following properties:

(a) The population is strictly increasing and bounded, i.e.,

$$L > 0,\ L^{\Delta} > 0 \quad \text{on} \quad \mathbb{T}_{t_0}^{+} \quad \text{and} \quad \lim_{t\to\infty} L(t) = L_{\infty} < \infty.$$

(b) The population growth rate is decreasing to 0, i.e.,

$$\text{If} \quad n = \frac{L^{\Delta}}{L}, \quad \text{then} \quad \lim_{t\to\infty} n(t) = 0 \quad \text{and} \quad n^{\Delta} < 0 \quad \text{on} \quad \mathbb{T}_{t_0}^{+}.$$

Hence equation (2.4) takes the form

$$k^{\Delta}(t) = \frac{s}{1+\mu(t)n(t)} f(k(t)) - \frac{\delta + n(t)}{1+\mu(t)n(t)} k(t). \tag{3.2}$$

Note that this is a nonautonomous dynamic equation on a time scale. Next we give the theorem of existence and uniqueness for solutions of initial value problems for (3.2).

Theorem 3.1. *Assume* (2.3). *For* $t_0 \in \mathbb{T}$ *and* $k_0 \in \mathbb{R}^+$, *the initial value problem*

$$\begin{cases} k^\Delta(t) &= \dfrac{s}{1+\mu(t)n(t)} f(k(t)) - \dfrac{\delta + n(t)}{1+\mu(t)n(t)} k(t) \\ k(t_0) &= k_0, \end{cases} \tag{3.3}$$

has a unique solution on $\mathbb{T}^+_{t_0}$.

Proof. Following the same way as in the proof of Theorem 2.1, we let

$$u(t,k) = \frac{s}{1+\mu(t)n(t)} f(k) - \frac{\delta + n(t)}{1+\mu(t)n(t)} k.$$

So $u(\cdot, k)$ is rd-continuous and regressive, and

$$\left| \frac{\partial u}{\partial k}(t, \xi) \right| \le s f'(k_0) + \delta + n(t_0).$$

From the theorem of global existence and uniqueness in [6], the solution of the problem (3.3) exists uniquely. □

Theorem 3.2. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* k_1 *and* k_2 *be solutions of equation* (3.2) *with initial conditions* $k_1(t_0) = k_{01}$ *and* $k_2(t_0) = k_{02}$, *respectively. If* $0 < k_{01} < k_{02}$, *then*

$$k_1 < k_2 \quad \text{on} \quad \mathbb{T}^+_{t_0}.$$

Proof. The proof is similar to the proof of Theorem 2.2. □

Remark 3.3. Theorem 3.2 means that if two economies have the same fundamentals, then the one with the bigger initial capital per worker will always have the bigger capital per worker for ever on any time scale. The result in Theorem 3.2 includes the results in [1] and [10] as special cases.

Theorem 3.4. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* k_1 *and* k_2 *be solutions of the dynamic equations on the same time scale*

$$k^\Delta(t) = \frac{s}{1+\mu(t)n_1(t)} f(k(t)) - \frac{\delta + n_1(t)}{1+\mu(t)n_1(t)} k(t) =: u(k(t), t) \tag{3.4}$$

and

$$k^\Delta(t) = \frac{s}{1+\mu(t)n_2(t)} f(k(t)) - \frac{\delta + n_2(t)}{1+\mu(t)n_2(t)} k(t) =: v(k(t), t), \tag{3.5}$$

respectively, with the same initial condition $k_1(t_0) = k_2(t_0)$. *If* $n_1 < n_2$ *on* $\mathbb{T}^+_{t_0}$, *then*

$$k_1 \geq k_2 \quad \text{on} \quad \mathbb{T}^+_{t_0}.$$

Proof. From $n_1(t) < n_2(t)$, we have $u(k(t), t) > v(k(t), t)$ for all $t \in \mathbb{T}^+_{t_0}$. Let $z := k_1 - k_2$. Obviously, we have $z(t_0) = k_1(t_0) - k_2(t_0) = 0$ and

$$z^\Delta(t_0) = k_1^\Delta(t_0) - k_2^\Delta(t_0) = u(k_1(t_0), t_0) - v(k_2(t_0), t_0) > 0.$$

So z is right-increasing at t_0, i.e., if t_0 is right-scattered, then we have $z(\sigma(t_0)) > z(t_0) = 0$; if t_0 is right-dense, then there exists a nonempty neighborhood $\mathring{U}^+(t_0) \cap \mathbb{T}$ of t_0 such that $z(t) > 0$ for any $t \in \mathring{U}^+(t_0) \cap \mathbb{T}$. We now show that $z \geq 0$ holds on $\mathbb{T}^+_{t_0}$. If this is not the case, then there must be a point $t_1 > t_0$, $t_1 \in \mathbb{T}$ such that $z(t_1) < 0$ and $z(t) \geq 0$ when $t \in (t_0, t_1) \cap \mathbb{T}$. If t_1 is left-dense, then continuity of z gives that $z(t_1) \geq 0$, which contradicts the assumption. Hence t_1 is left-scattered. Let $\rho(t_1) = t_2$. Then $z(t_2) \geq 0$, i.e., $k_1(t_2) \geq k_2(t_2)$. Let k_2' be the solution of equation (3.5) satisfying the initial condition $k_2'(t_2) = k_1(t_2)$. From the discussion in the beginning of this proof, we obtain that $k_1 - k_2'$ is also right-increasing at t_2, i.e.,

$$k_1(t) > k_2'(t) \quad \text{for} \quad t \in \mathring{U}^+(t_2) \cap \mathbb{T}, \tag{3.6}$$

where $\mathring{U}^+(t_2) \cap \mathbb{T}$ is a nonempty right neighborhood of t_2 (at least including t_1). Taking into account that $k_2(t_2) \leq k_2'(t_2)$, Theorem 3.2 gives

$$k_2(t) \leq k_2'(t) \quad \text{for all} \quad t \in \mathbb{T}^+_{t_2}. \tag{3.7}$$

From (3.6) and (3.7), we have

$$k_1(t) > k_2(t) \quad \text{for} \quad t \in \mathring{U}^+(t_2) \cap \mathbb{T},$$

and thus $k_1(t_1) > k_2(t_1)$, which contradicts the fact $z(t_1) < 0$. This concludes the proof. □

Remark 3.5. Theorem 3.4 implies that, on any economic domain, for two economies with the same initial capital per worker, the economy with the smaller population growth rate will always have the bigger capital per worker on any time scale. The result here also includes the results in [1, 13] and [10] as special cases.

Theorem 3.6. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *If* k *solves* (3.2), *then* $\lim_{t\to\infty} k(t) = \hat{k}_0$.

Proof. We want to prove that for any $\varepsilon > 0$, there exists $T > 0$, such that if $t > T$, $t \in \mathbb{T}$, we have $|k(t) - \hat{k}_0| < \varepsilon$. Now let $\varepsilon > 0$. Since

$$\lim_{n \to 0^+} \hat{k}_n = \hat{k}_0,$$

we know that there exists $\overline{n} > 0$ such that

$$|\hat{k}_n - \hat{k}_0| < \frac{\varepsilon}{3} \quad \text{for all} \quad n \in (0, \overline{n}).$$

Let $t_1 \in \mathbb{T}_{t_0}^+$ such that $n_{t_1} = n(t_1) < \overline{n}$, i and let $k_{n_{t_1}}$ and k_0 be the solutions of

$$k^\Delta(t) = \frac{s}{1 + \mu(t)n_{t_1}} f(k(t)) - \frac{\delta + n_{t_1}}{1 + \mu(t)n_{t_1}} k(t)$$

and

$$k^\Delta(t) = sf(k(t)) - \delta k(t),$$

respectively, with the initial conditions

$$k_{n_{t_1}}(t_1) = k_0(t_1) = k(t_1).$$

Then Theorem 3.4 implies that

$$k_{n_{t_1}}(t) \leq k(t) \leq k_0(t) \quad \text{for all} \quad t \in \mathbb{T}_{t_1}^+.$$

Since $\lim\limits_{t \to \infty} k_0(t) = \hat{k}_0$, there exists $T_1 > 0$ such that

$$|k_0(t) - \hat{k}_0| < \frac{\varepsilon}{3} \quad \text{for all} \quad t > T_1.$$

Moreover, since $\lim\limits_{t \to \infty} k_{n_{t_1}}(t) = \hat{k}_{n_{t_1}}$, there exists $T_2 > 0$ such that

$$|k_{n_{t_1}}(t) - \hat{k}_{n_{t_1}}| < \frac{\varepsilon}{3} \quad \text{for all} \quad t > T_2.$$

Hence for $t > T := \max\{T_1, T_2, t_1\}$, we have

$$\hat{k}_0 - \frac{2}{3}\varepsilon < \hat{k}_{n_{t_1}} - \frac{\varepsilon}{3} < k_{n_{t_1}}(t) \leq k(t) \leq k_0(t) < \hat{k}_0 + \frac{\varepsilon}{3},$$

which implies that $|k(t) - \hat{k}_0| < \varepsilon$ for any $t \in \mathbb{T}_T^+$. $\square$

Remark 3.7. Theorem 3.6 says that for any economic domain $\mathbb{T}$, the population growth rate $n(t)$ has no influence on the level of per worker output in the long run. That is, provided that the economy possesses a population growth rate strictly decreasing to zero, the capital per worker always converges to the positive steady state of the Solow model on a time scale with a population growth rate of zero.

Theorem 3.8. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Then the solution* k *of* (3.2) *with* $k(t_0) = k_0$ *is asymptotically stable.*

Proof. To prove the Lyapunov stability of k in equation (3.2) with initial condition $k(t_0) = k_0$, we have to show that for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any solution q of equation (3.2) with initial condition $q(t_0) = q_0$ and such that $|k(t_0) - q(t_0)| < \eta$, we have

$$|k(t) - q(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Let φ_1 and φ_2 be the solutions of equation (3.2) with initial conditions $\varphi_1(t_0) = \dfrac{3}{2}k(t_0)$ and $\varphi_2(t_0) = \dfrac{1}{2}k(t_0)$, respectively. From Theorem 3.6, we have

$$\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = \hat{k}_0 = \lim_{t\to\infty} k(t).$$

Thus, for any $\varepsilon > 0$, there exists $t_1 > t_0$, $t_1 \in \mathbb{T}^+_{t_0}$, such that

$$|\varphi_1(t) - k(t)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\varphi_2(t) - k(t)| < \frac{\varepsilon}{2} \quad \text{for all} \quad t \in \mathbb{T}^+_{t_1}.$$

Let q solve (3.2) with the initial condition $q_0 \in \left(\dfrac{1}{2}k(t_0), \dfrac{3}{2}k(t_0)\right)$. From Theorem 3.2, we have

$$\varphi_1(t) < q(t) < \varphi_2(t) \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Thus

$$|q(t) - k(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}^+_{t_1}.$$

Next we choose η such that for any solution q with initial value q_0, $|q_0 - k_0| < \eta$ implies $|k - q| < \varepsilon$ on $[t_0, t_1] \cap \mathbb{T}$. Following the proof of the theorem of continuous dependence on initial conditions, making use of the finite covering theorem, we can obtain that for any $\varepsilon > 0$, there exists $\eta < k_0/2$ such that $|q_0 - k_0| < \eta$ implies $|k(t) - q(t)| < \varepsilon$ for all $t \in [t_0, t_1] \cap \mathbb{T}$. From Theorem 3.6, for any solutions k and q of equation (3.2), we have that

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} q(t) = \hat{k}_0,$$

and then

$$\lim_{t\to\infty} |q(t) - k(t)| = 0.$$

So the solution of equation (3.2) is asymptotically stable. □

Remark 3.9. Theorem 3.8 says that under the same fundamentals, if two economies operating on the same time domain have nearly the same initial capital per worker, the following capitals per worker will take on similar behavior.

Next we will present the monotonicity of the solutions of (3.2).

Theorem 3.10. *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* $t_0 \in \mathbb{T}$ *and* k, $k_{n_{t_0}}$, k_0 *be solutions of the dynamic equation* (3.2),

$$k^{\Delta}(t) = \frac{s}{1+\mu(t_0)n_{t_0}} f(k(t)) - \frac{\delta + n_{t_0}}{1+\mu(t_0)n_{t_0}} k(t), \tag{3.8}$$

and

$$k^{\Delta}(t) = sf(k(t)) - \delta k(t), \tag{3.9}$$

respectively, with the initial values

$$k(t_0) = k_{n_{t_0}}(t_0) = k_0(t_0).$$

Then

1. $k_{n_{t_0}} \le k \le k_0$ *on* $\mathbb{T}^+_{t_0}$*;*
2. *If* $k(t_0) \le \hat{k}_{n_0}$, *then* k *is strictly increasing on* $\mathbb{T}^+_{t_0}$*;*
3. *If* $\hat{k}_{n_0} < k(t_0) \le \hat{k}_0$, *then there exists* $\widetilde{t} \in \mathbb{T}$ *such that* k *is decreasing on* $[t_0, \widetilde{t}] \cap \mathbb{T}$ *and is increasing on* $\mathbb{T}^+_{\widetilde{t}}$*;*
4. *If* $\hat{k}_0 < k(t_0)$, *then* k *is increasing on* $\mathbb{T}^+_{t_0}$, *or there exists* $\widetilde{t} \in \mathbb{T}$ *such that* k *is decreasing on* $[t_0, \widetilde{t}] \cap \mathbb{T}$ *and is increasing on* $\mathbb{T}^+_{\widetilde{t}}$.

Here $\hat{k}_{n_0}$ *and* $\hat{k}_0$ *are the equilibria of* (3.8) *and* (3.9), *respectively.*

Proof. 1. For $t > t_0$, $t \in \mathbb{T}$, we have $n(t_0) > n(t) > 0$. So from Theorem 3.4, we obtain the result easily.

2. We want to prove the statement $S(t)$ given by $k^{\Delta}(t) > 0$ is true for any $t \in \mathbb{T}^+_{t_0}$. To do this, we use Theorem 1.1.

   A. Since $k(t_0) < \hat{k}_{n_0}$, we have $k^{\Delta}(t_0) > 0$. So $S(t)$ holds at $t = t_0$.

   B. If t is right-scattered and $k^{\Delta}(t) > 0$, then

$$\begin{aligned}
k(\sigma(t)) &= k(t) + \mu(t)k^{\Delta}(t) \\
&= k(t) + \mu(t)\frac{sf(k(t)) - (\delta + n(t))k(t)}{1+\mu(t)n(t)} \\
&= \frac{(1-\mu(t)\delta)k(t) + s\mu(t)f(k(t))}{1+\mu(t)n(t)} \\
&< \frac{(1-\mu(t)\delta)k(\sigma(t)) + s\mu(t)f(k(\sigma(t)))}{1+\mu(t)n(\sigma(t))} \\
&= \frac{[1+\mu(t)n(\sigma(t))]k(\sigma(t))}{1+\mu(t)n(\sigma(t))} \\
&\quad + \frac{\mu(t)[sf(k(\sigma(t))) - (\delta + n(\sigma(t)))k(\sigma(t))]}{1+\mu(t)n(\sigma(t))} \\
&= k(\sigma(t)) + \frac{\mu(t)}{1+\mu(t)n(\sigma(t))}[1+\mu(\sigma(t))n(\sigma(t))]k^{\Delta}(\sigma(t)),
\end{aligned}$$

so $k^\Delta(\sigma(t)) > 0$.

C. If t is right-dense and $k^\Delta(t) > 0$, then there exists a neighborhood $\mathring{U}^+(t) \cap \mathbb{T}$ such that $k^\Delta(r) > 0$ for any $r \in \mathring{U}^+(t) \cap \mathbb{T}$. To prove this, we assume that there does not exist such a neighborhood. Then there must exist a decreasing sequence $\{t_n\} \subset \mathring{U}^+(t) \cap \mathbb{T}$ such that $\lim_{n\to\infty} t_n = t$ and $k^\Delta(t_n) \leq 0$. From the properties of f, taking limit on both sides, we obtain $k^\Delta(t) \leq 0$, which is a contradiction.

D. Assume that t is left-dense and $k^\Delta(r) > 0$ for any $r \in [t_0, t) \cap \mathbb{T}$. From continuity, we can get $k^\Delta(t) \geq 0$. If $k^\Delta(t) = 0$, then for any $r \in [t_0, t) \cap \mathbb{T}$, from the chain rule in [6], we have

$$\begin{aligned}
[(1+\mu n)k^\Delta]^\Delta(r) &= [s(f \circ k) - (\delta + n)k]^\Delta(r) \\
&= sf'(k(r))k^\Delta(r) - n^\Delta(r)k(r) \\
&\quad -(\delta + n^\sigma(r))k^\Delta(r).
\end{aligned}$$

Taking limit on both sides when $r \to t^-$, we obtain

$$[(1+\mu n)k^\Delta)]^\Delta(t) = -n^\Delta(t)k(t) > 0.$$

So since t is left-dense and from the continuity, we have

$$(1+\mu(t)n(t))k^\Delta(t) > (1+\mu(r)n(r))k^\Delta(r) > 0$$

for all $r \in \mathring{U}^-(t) \cap \mathbb{T}$. Hence $k^\Delta(t) > 0$.

3. If $\hat{k}_{n_0} < k(t_0) \leq \hat{k}_0$, then

$$\begin{aligned}
k^\Delta(t_0) &= \frac{s}{1+\mu(t_0)n(t_0)} f(k(t_0)) - \frac{\delta + n(t_0)}{1+\mu(t_0)n(t_0)} k(t_0) \\
&= \frac{s}{1+\mu(t_0)n(t_0)} f(k_{n_{t_0}}(t_0)) - \frac{\delta + n(t_0)}{1+\mu(t_0)n(t_0)} k_{n_{t_0}}(t_0) \\
&= k^\Delta_{n_{t_0}}(t_0) < 0.
\end{aligned}$$

Hence k is right-decreasing at t_0, i.e., if t_0 is right-scattered, then $k(\sigma(t_0)) < k(t_0)$; if t_0 is right-dense, then there exists a nonempty neighborhood $\mathring{U}^+(t_0) \cap \mathbb{T}$ of t_0 such that $k(t) < k(t_0)$ for any $t \in \mathring{U}^+(t_0) \cap \mathbb{T}$. If $k^\Delta \leq 0$ is true on $\mathbb{T}^+_{t_0}$, then k is decreasing on $\mathbb{T}^+_{t_0}$. Considering $\lim_{t\to\infty} k(t) = \hat{k}_0$ in Theorem 3.6, we have

$$\hat{k}_0 \leq k(t) < k(t_0) \leq \hat{k}_0 \quad \text{for} \quad t \in \mathbb{T}^+_{t_0},$$

which is a contradiction. So there must exist $\widetilde{t} \in \mathbb{T}^+_{t_0}$ such that $k^\Delta(\widetilde{t}) > 0$, and for simplicity we assume $\widetilde{t}$ is the first point that verifies the inequality. So it must be proved that $k^\Delta(t) > 0$ for all $t \in \mathbb{T}^+_{\widetilde{t}}$, which is similar to the proof of Statement 2.

4. Following the same proof as in Statement 3, we can obtain the monotonicity.

This completes the proof. □

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SINGULARITIES OF THE CONTINUOUS WAVELET TRANSFORM IN $L^p(\mathbb{R}^n)$

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Abstract. The continuous wavelet transform for functions f in $L^p(\mathbb{R}^n)$ where $1 \leq p < \infty$ is defined with respect to a radially symmetric admissible function so that the singularities of f are the singularities of the continuous wavelet transform.

Key words and phrases: The space $L^p(\mathbb{R}^n)$, the continuous wavelet transform, inversion formula, admissibility condition.

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1. Introduction

The local regularity of functions f in the Hilbert space $L^2(\mathbb{R}^n)$ by means of the continuous wavelet transform has been studied in [3], where the main point of this result is the use of the inversion formula given for functions in $L^2(\mathbb{R}^n)$, and where the continuous wavelet transform is defined with respect to a radially symmetric admissible function h in $C_0^\infty(\mathbb{R}^n)$.

In this paper we extend this local property for functions f in the space $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$. In this case we will use the inversion formula for the continuous wavelet transform for functions in $L^p(\mathbb{R}^n)$ given in [5]. For this purpose, we will consider two symmetric functions h_1 and h_2 instead of one h, one for the decomposition and the other one for the inversion formula, in such a way that the admissibility condition will depend of h_1 and h_2.

2. Notations and definitions

In this section we will give the definition of and admissible function in order to define the continuous wavelet transform for a function in the space $L^p(\mathbb{R}^n)$.

Definition 1. *For h in $L^2(\mathbb{R}^n)$, the dilation operator $J_a : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and the translation operator $T_b : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are defined respectively as:*

1) $(J_a h)(x) = a^{-\frac{n}{2}} h(a^{-1}x)$, where $a > 0$ and $x \in \mathbb{R}^n$.

2) $(T_b h)(x) = h(x - b)$, where $x, b \in \mathbb{R}^n$.

Now, the admissibility condition is given.

Definition 2. *A radially symmetric function h in $L^2(\mathbb{R}^n)$ is admissible if*

$$C_h \equiv \int_{\mathbb{R}^+} |\eta(k)|^2 \frac{1}{k} dk < \infty, \quad \text{where} \quad \widehat{h}(y) = \eta(|y|).$$

In this case, $\widehat{h}$ is the Fourier transform of h.

Thus, the continuous wavelet transform with respect to an admissible function now is given.

Definition 3. *Let f be in $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, and let h be a radially symmetric admissible function in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. The continuous wavelet transform of f with respect to h is defined as*

$$(L_h f)(a, b) = \left[(J_a h)^{\sim} * f\right](b), \tag{1}$$

where $*$ means convolution and $h^{\sim}(x) = h(-x)$.

Remark 1. *According to Definition 3, and since $J_a h \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, it follows from Young's Inequality that $(L_h f)(a,b) \in L^p(\mathbb{R}^n)$ and*

$$|(L_h f)(a,b)|_p \leq \|f\|_p \, \|J_a h\|_1$$

Remark 2. *The inversion formula of the continuous wavelet transform for f in $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ can be obtained from Theorem 1 given by Weisz in* [5]. *In this case,*

$$f = \lim_{S \to 0, U \to \infty} \frac{1}{C_{h_1,h_2}} \int_S^T \int_{\mathbb{R}^n} (L_{h_1} f)(a,b) T_b J_a h_2 \frac{1}{a^{n+1}} \, da \, db, \tag{2}$$

where the convergence is in the $L_p(\mathbb{R}^n)$ norm, and where h_1 and h_2 are radially symmetric admissible functions in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and where

$$0 < C_{h_1,h_2} \equiv \int_{\mathbb{R}^+} \left| \eta_1(k) \overline{\eta_2(k)} \right| \frac{1}{k} dk < \infty. \tag{3}$$

In this case, $\widehat{h}_1(y) = \eta_1(|y|_j)$ and $\widehat{h}_2(y) = \eta_2(|y|_j)$, where $j = 1$ or $j = \infty$.

We should remark also that for the special case $p = 2$, the inversion formula (2) and the condition (3) are given in [1]. Also, for the case that $f \in L^p(\mathbb{R})$ the inversion formula for the continuous wavelet transform is given in [4].

Lemma 1. *If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $h \in C_0^\infty(\mathbb{R}^n)$ is a radially symmetric admissible function, then for $a > 0$, $(L_h f)(a,b)$ is of class C^∞, and*

$$\partial_b^\alpha (L_h f)(a,b) = \int_{\mathbb{R}^n} f(x) \frac{1}{a^{\frac{n}{2}}} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (\partial^\alpha h) \left(\frac{x-b}{a} \right) dx. \tag{4}$$

for any multi-index $\alpha \in \mathbb{R}^n$.

Proof. Since $J_a h \in C_0^\infty(\mathbb{R}^n)$, it follows that $(J_a h)^\sim * f \in C^\infty$ and

$$\partial_b^\alpha \left[(J_a h)^\sim * f \right](b) = \left[\partial_b^\alpha (J_a h)^\sim * f \right](b) \quad \text{for any multi-index } \alpha \in \mathbb{R}^n. \tag{5}$$

Then from (1), $(L_h f)(a,b)$ is of class C^∞, and

$$\partial_b^\alpha (L_h f)(a,b) = \left[\partial_b^\alpha (J_a h)^\sim * f \right](b) = \int_{\mathbb{R}^n} \partial_b^\alpha \left[\frac{1}{a^{\frac{n}{2}}} h \left(\frac{x-b}{a} \right) \right] f(x) dx$$
$$= \int_{\mathbb{R}^n} \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} \frac{1}{a^{\frac{n}{2}}} h^\alpha \left(\frac{x-b}{a} \right) f(x) dx.$$

This proves Lemma 1. $\square$

Note that from (4) and (1),

$$\partial_b^\alpha (L_h f)(a,b) = \frac{(-1)^{|\alpha|}}{a^{|\alpha|}} (L_{\partial^\alpha h} f)(a,b) \tag{6}$$

3. Partial result

In this section we will prove first that if f in $L^p(\mathbb{R}^n)$ is of class C^∞ in a neighborhood of b_0, then we have the existence of the limit of

$$\frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial_b^\alpha (L_h f)(a,b)$$

as $(a,b) \to (0,b_1)$ for any b_1 in a neighborhood of b_0. That is, we have the following result.

Theorem 1. *Suppose that $h \in C_0^\infty(\mathbb{R})$ is a non-zero radially symmetric admissible function where $\widehat{h}(0) = 0$. If f in $L^p(\mathbb{R}^n)$ is of class C^∞ in a neighborhood of $x = b_0$ in $\mathbb{R}^n$ where $1 \leq p < \infty$, then for each multi-index $\alpha \in \mathbb{R}^n$,*

$$\lim_{(a,b)\to(0,b_1)} \frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial_b^\alpha (L_h f)(a,b)$$

exists for each b_1 in a neighborhood of $b_0 \in \mathbb{R}^n$.

Proof. First, note that from Lemma 1 the function $\mathcal{W}_h^\alpha f$ is continuous at any $(a_1, b_1) \in \mathbb{R}^+ \times \mathbb{R}^n$ for any multi-index $\alpha \in \mathbb{R}^n$.

Suppose now that f in $L^p(\mathbb{R}^n)$ is of class C^∞ in a neighborhood of $x = b_0 \in \mathbb{R}^n$ containing the closed ball $\overline{B_\Delta(b_0)}$, where $\Delta > 0$. Take b_1 in the open ball $B_{\frac{\Delta}{2}}(b_0)$, and choose $b \in B_{\frac{\Delta}{2}}(b_0)$.

Now, since $f \in L^p(\mathbb{R}^n)$ and $h \in C_0^\infty(\mathbb{R})$, it follows from (4) that

$$\partial_b^\alpha (L_h f)(a,b) = \int_{\mathbb{R}^n} f(x)\frac{1}{a^{\frac{n}{2}}}(-1)^{|\alpha|}\partial_x^\alpha \left[h\left(\frac{x-b}{a}\right)\right] dx.$$

Moreover, since $h \in C_0^\infty(\mathbb{R}^n)$, there exists $L > 0$ such that *supp*$h \subset \overline{B_L(0)}$. Then *supp* $h^\alpha \subset \overline{B_L(0)}$ also.

Then

$$\partial_b^\alpha (L_h f)(a,b) = \int_{B_{aL}(b)} f(x) \frac{1}{a^{\frac{n}{2}}} (-1)^{|\alpha|} \partial_x^\alpha \left[h\left(\frac{x-b}{a}\right)\right] dx.$$

Now consider a such that $a \in (0, \frac{\Delta}{2L})$. Then for $b \in B_{\frac{\Delta}{2}}(b_0)$ we have $B_{aL}(b) \subset B_\Delta(b_0)$. Then by hypothesis f is of class C^∞ on $B_{aL}(b)$. Therefore we may integrate by parts to obtain

$$\partial_b^\alpha (L_h f)(a,b) = \int_{B_{aL}(b)} \partial_x^\alpha f(x) \frac{1}{a^{\frac{n}{2}}} h\left(\frac{x-b}{a}\right) dx.$$

Hence, for $x = b + ay$,

$$\partial_b^\alpha (L_h f)(a,b) = \int_{B_L(0)} a^{\frac{n}{2}} \partial_b^\alpha f(b+ay) h(y)\, dy.$$

Since f is C^∞ at the points in the region of integration, for $y \in B_L(0)$ we have from Taylor's formula with integral remainder given by

$$\partial_b^\alpha f(b+ay) = \partial^\alpha f(b) + \int_0^1 \sum_{|\beta|=1} \frac{1}{\beta!} \partial_b^{\beta+\alpha} f(b+tay)\, a\, y^\beta\, dt,$$

that

$$\begin{aligned}
&\frac{1}{a}\frac{1}{a^{\frac{n}{2}}} \partial_b^\alpha (L_h f)(a,b) = \frac{1}{a}\frac{1}{a^{\frac{n}{2}}} \int_{B_L(0)} a^{\frac{n}{2}} \partial_b^\alpha f(b+ay) h(y) dy \\
&= \frac{1}{a} \int_{B_L(0)} \left[\partial^\alpha f(b) + \int_0^1 \sum_{|\beta|=1} \partial_b^{\beta+\alpha} f(b+tay)\, a y^\beta\, dt \right] h(y)\, dy \\
&= \frac{1}{a} \partial^\alpha f(b) \int_{B_L(0)} h(y) dy + \int_{B_L(0)} \int_0^1 \sum_{|\beta|=1} \partial_b^{\beta+\alpha} f(b+tay)\, y^\beta\, h(y)\, dt\, dy.
\end{aligned}$$

Since $\widehat{h}(0) = 0$, then $\displaystyle\int_{B_L(0)} h(y) dy = 0$, and since f is of class C^∞ in a neighborhood of $x = b_0$, and since $\partial^{\beta+\alpha} f$ is continuous near b_1, it follows that for b and b_1 in $B_{\frac{\Delta}{2}}(b_0)$,

$$\begin{aligned}\lim_{(a,b)\to(0,b_1)} \frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial_b^{\alpha}(L_h f)(a,b) &= \int_{B_L(0)}\int_0^1 \sum_{|\beta|=1} \partial_b^{\beta+\alpha} f(b_1)\, y^{\beta}\, \overline{h}(y)\, dt\, dy \\ &= \sum_{|\beta|=1}\left[\partial_b^{\beta+\alpha} f(b_1) \int_{B_L(0)} y^{\beta}\, \overline{h}(y)\, dy\right].\end{aligned}$$

Note that since $h \in C_0^{\infty}(\mathbb{R}^n)$, we have

$$\int_{B_L(0)} y^{\beta}\, \overline{h}(y)\, dy \;<\; \infty.$$

Thus, $\displaystyle\lim_{(a,b)\to(0,b_1)} \frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial_b^{\alpha}(L_h f)(a,b)$ exists for each multi-index $\alpha \in \mathbb{R}^n$. $\square$

Now, let us prove the converse of Theorem 1, which is our main result.

4. Main result

Theorem 2. *Suppose $h_1 \in C_0^{\infty}(\mathbb{R}^n)$ and $h_2 \in C_0(\mathbb{R}^n)$ are non-zero radially symmetric admissible functions and satisfy condition (3). Consider f in $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. If for each multi-index $\alpha \in \mathbb{R}^n$ we have the existence of the limit of $(\mathcal{W}_{h_1}^{\alpha} f)(a,b) \equiv \frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial_b^{\alpha}(L_h f)(a,b)$ as $(a,b) \to (0,b_1)$ for each b_1 in an open neighborhood of $x = b_0 \in \mathbb{R}^n$, then f is of class C^{∞} in an open neighborhood of $b_0 \in \mathbb{R}^n$ for any multi-index $\alpha \in \mathbb{R}^n$.*

Proof. Suppose then that for any multi-index $\alpha \in \mathbb{R}^n$

$$F_{h_1}^{\alpha}(b_1) \equiv \lim_{(a,b)\to(0,b_1)} (\mathcal{W}_{h_1}^{\alpha} f)(a,b)$$

exists for each b_1 in an open neighborhood containing the closed ball $\overline{B_r(b_0)}$, where $r > 0$.

Now, for fixed x in $\overline{B_r(b_0)}$, let

$$(\mathcal{I}_{h_1,h_2}^{\alpha} f)(a,x,y) = \begin{cases} h_2(-y)\,(\mathcal{W}_{h_1}^{\alpha} f)(a, x+ay) & \text{if} \quad a > 0 \\ h_2(-y)\, F_{h_1}^{\alpha}(x) \quad \text{if} \quad a = 0 \end{cases}$$

Then we have the following notes.

Note 1. *For x in $\overline{B_r(b_0)}$, the function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is well-defined for all $a \geq 0$ and all y in $\mathbb{R}^n$.*

Note 2. *For $a > 0$, the function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is given by*

$$(\mathcal{I}^{\alpha}_{h_1,h_2}f)(a,x,y) = h_2(-y)\frac{1}{a}\frac{1}{\sqrt{a^n}}\frac{(-1)^{|\alpha|}}{a^{|\alpha|}}\left[(J_a h_1^{\alpha})^{\sim} * f\right](x+ay), \tag{7}$$

and since $h_1 \in C_0^{\infty}$, it comes from the definition of $(\mathcal{W}^{\alpha}_{h_1}f)(a,b)$ and from (6), that for fixed $y \in \mathbb{R}^n$ and $a > 0$, the function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is infinitely differentiable in the variable x.

Then we have the following three results, where the proofs are given in the Appendix.

Lemma 2. *The function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is continuous on $[0,\infty) \times \overline{B_r(b_0)} \times \mathbb{R}^n$.*

Lemma 3. *For fixed x in $\overline{B_r(b_0)}$, the function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is in $L^1([0,\infty) \times \mathbb{R}^n)$.*

Lemma 4. *For x in the open ball $B_r(b_0)$, let*

$$w(x) = \int_0^{\infty}\int_{\mathbb{R}^n}(\mathcal{I}_{h_1,h_2}f)(a,x,y)dyda,$$

and let

$$(I^{\alpha}_{h_1,h_2}f)(x) = \int_0^{\infty}\int_{\mathbb{R}^n}(\mathcal{I}^{\alpha}_{h_1,h_2}f)(a,x,y)dyda.$$

Then

$$\partial^{\alpha}w(x) = (I^{\alpha}_{h_1,h_2}f)(x) \quad \textit{for each multi-index}\ \alpha \in \mathbb{R}^n. \tag{8}$$

Back to the proof of Theorem 2, note that from Lemma 4, the function w is of class C^{∞} on $B_r(b_0)$. Thus if we define

$$u_c(x) = \int_{\frac{1}{c}}^{c}\int_{\mathbb{R}^n} h_2(-y)\frac{1}{a}\frac{1}{\sqrt{a^n}}(L_{h_1}f)(a,x+ay)dyda$$

for any x in $\mathbb{R}^n$ and $c > 0$, we have from Lemma 4 that for $x \in B_r(b_0)$,

$$\lim_{c\to+\infty} u_c(x) = w(x).$$

That is, $u_c \to w$ pointwise on $B_r(b_0)$ as $c \to +\infty$.

On the other hand, by (2), we have $u_c \to C_{h_1,h_2} f$ in the $L^p(\mathbb{R}^n)$ norm. Then $f = (C_{h_1,h_2})^{-1} w$ almost everywhere on $B_r(b_0)$.

Finally, since from (8) the function w is C^∞ on $B_r(b_0)$, it follows that f is of class C^∞ on $B_r(b_0)$.

This completes the proof of Theorem 2. □

5. Appendix

Proof of Lemma 2. Let (a_1, x_1, y_1) be any point in $[0, \infty) \times \overline{B_r(b_0)} \times \mathbb{R}^n$.

Note that if $a_1 > 0$, then from (7), the function $\mathcal{I}^\alpha_{h_1,h_2} f$ is continuous at (a_1, x_1, y_1).

Now, if $a_1 = 0$, then by hypothesis of Theorem 2,

$$\lim_{(a,x,y)\to(0,x_1,y_1)} (\mathcal{I}^\alpha_{h_1,h_2} f)(a, x, y) = \lim_{(a,x,y)\to(0,x_1,y_1)} \left[h_2(-y)(\mathcal{W}^\alpha_{h_1} f)(a, x + ay)\right]$$

$$= h_2(-y_1) \lim_{(a,x,y)\to(0,x_1,y_1)} (\mathcal{W}^\alpha_{h_1} f)(a, x + ay) = h_2(-y_1) \lim_{(a,b)\to(0,x_1)} (\mathcal{W}^\alpha_{h_1} f)(a, b)$$

$$= h_2(-y_1) F^\alpha_{h_1}(x_1) = (\mathcal{I}^\alpha_{h_1,h_2} f)(0, x_1, y_1).$$

This completes the proof of Lemma 2. □

Proof of Lemma 3. Note that for $a > 0$, and from the definition of $(\mathcal{W}^\alpha_{h_1} f)(a, b)$ and then from (6) that

$$\begin{aligned}(\mathcal{I}^\alpha_{h_1,h_2} f)(a, x, y) &= h_2(-y)(\mathcal{W}^\alpha_{h_1} f)(a, x + ay) \\ &= h_2(-y)\frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\partial^\alpha_x (L_{h_1} f)(a, x + ay) \qquad (9) \\ &= h_2(-y)\frac{1}{a}\frac{1}{a^{\frac{n}{2}}}\frac{(-1)^{|\alpha|}}{a^{|\alpha|}}\left(L_{h_1^\alpha} f\right)(a, x + ay).\end{aligned}$$

Now, since $f \in L^p(\mathbb{R}^n)$ and $h_1 \in C_0^\infty(\mathbb{R}^n)$, we can choose $1 \leq q < \infty$ so that $\frac{1}{p} + \frac{1}{q} = 1$ and hence, $h \in L^q(\mathbb{R}^n)$. Thus, from Hölder's inequality,

$$\left|(\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y)\right| \leq |h_2(-y)|\, a^{-n-1-|\alpha|+\frac{n}{q}} \|f\|_p \, \|h_1^\alpha\|_q \tag{10}$$

Now, let

$$(\mathcal{G}^\alpha_{h_1,h_2} f)(a,y) = \begin{cases} \left|(\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y)\right| & \text{if} \quad 0 \leq a \leq 1 \\ |h_2(-y)|\, a^{-n-1-|\alpha|+\frac{n}{q}} \|f\|_p \, \|h_1^\alpha\|_q & \text{if} \quad a > 1 \end{cases}$$

Then

$$\left|(\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y)\right| \leq (\mathcal{G}^\alpha_{h_1,h_2} f)(a,y) \tag{11}$$

for all $(a,x,y) \in [0,\infty) \times \overline{B_r(b_0)} \times \mathbb{R}^n$.

Hence,

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \left|(\mathcal{G}^\alpha_{h_1,h_2} f)(a,y)\right| dyda \\
&= \int_0^1 \int_{\mathbb{R}^n} \left|(\mathcal{G}^\alpha_{h_1,h_2} f)(a,y)\right| dyda + \int_1^\infty \int_{\mathbb{R}^n} \left|(\mathcal{G}^\alpha_{h_1,h_2} f)(a,y)\right| dyda \\
&= \int_0^1 \int_{\mathbb{R}^n} \left|(\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y)\right| dyda + \int_1^\infty \int_{\mathbb{R}^n} |h_2(-y)|\, a^{-n-1-|\alpha|+\frac{n}{q}} \, \|f\|_p \, \|h_1^\alpha\|_q \, dyda.
\end{aligned}$$

Suppose now that $supp\, h_2 \subset \overline{B_d(0)}$ for some $d > 0$. Then

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \left|(\mathcal{G}^\alpha_{h_1,h_2} f)(a,y)\right| dyda \\
&= \int_0^1 \int_{\overline{B_d(0)}} \left|(\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y)\right| dyda \\
&+ \|f\|_p \, \|h_1^\alpha\|_q \left(\int_{\overline{B_d(0)}} |h_2(-y)| dy \right) \left(\int_1^\infty a^{-n-1-|\alpha|+\frac{n}{q}} da \right).
\end{aligned} \tag{12}$$

Since from Lemma 2 the function $\mathcal{I}^{\alpha}_{h_1,h_2}f$ is continuous on $[0,\infty)\times\overline{B_d(0)}$, and $\displaystyle\int_1^{\infty} a^{-n-1-|\alpha|+\frac{n}{q}}da < \infty$ for any multi-index $\alpha \in \mathbb{R}^n$, where $\frac{1}{p}+\frac{1}{q}=1$ with $1 \le q < \infty$, it follows that

$$\mathcal{G}^{\alpha}_{h_1,h_2}f \in L^1([0,\infty)\times\mathbb{R}^n). \tag{13}$$

Hence, $(\mathcal{I}^{\alpha}_{h_1,h_2}f)(\cdot,x,\cdot) \in L^1([0,\infty)\times\mathbb{R}^n)$.

This completes the proof of Lemma 3. $\square$

Proof of Lemma 4. First note that from (9),

$$\partial^{\alpha}_x(\mathcal{I}_{h_1,h_2}f)(a,x,y) = (\mathcal{I}^{\alpha}_{h_1,h_2}f)(a,x,y) \tag{14}$$

for any multi-index $\alpha \in \mathbb{R}^n$.

Now note that since from Note 1 for $\alpha = 0$ the function $\mathcal{I}_{h_1,h_2}f$ is well-defined on $[0,\infty)\times\overline{B_r(b_0)}\times\mathbb{R}^n$, from Lemma 3 the function $(\mathcal{I}_{h_1,h_2}f)(\cdot,x,\cdot)$ is integrable for each $x \in \overline{B_r(b_0)}$, from Note 2 we have that $\partial_x(\mathcal{I}_{h_1,h_2}f)$ exists, and since from (11) and (13) there is $\mathcal{G}_{h_1,h_2}f \in L^1([0,\infty)\times\mathbb{R}^n)$ such that $|\partial^{\alpha}_x(\mathcal{I}_{h_1,h_2}f)(a,x,y)| \le (\mathcal{G}_{h_1,h_2}f)(a,y)$ for all $[0,\infty)\times\overline{B_r(b_0)}\times\mathbb{R}^n$, it follows from (Theorem 2.27, [2]) that

$$w(x) = \int_0^{\infty}\int_{\mathbb{R}^n}(\mathcal{I}_{h_1,h_2}f)(a,x,y)dyda$$

is differentiable and

$$\partial w(x) = \int_0^{\infty}\int_{\mathbb{R}^n}\partial_x(\mathcal{I}_{h_1,h_2}f)(a,x,y)dyda$$

Hence, for any multi-index $\alpha \in \mathbb{R}^n$ and from (14)

$$\partial^\alpha w(x) = \int_0^\infty \int_{\mathbb{R}^n} \partial_x^\alpha (\mathcal{I}_{h_1,h_2} f)(a,x,y) dy da = \int_0^\infty \int_{\mathbb{R}^n} (\mathcal{I}^\alpha_{h_1,h_2} f)(a,x,y) dy da$$

$$= (I^\alpha_{h_1,h_2} f)(x)$$

This completes the proof of Lemma 4. □

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SOME ESTIMATION OF THE STRUVE TRANSFORM IN A QUOTIENT SPACE OF BOEHMIANS

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Abstract

In this paper, we investigate the Struve transformation on certain space of Boehmians. The transform under consideration is defined and some of its properties are also illustrated .

Keywords: Struve transform; Modified Struve, Generalized function; Boehmian.

1 Introduction

In some physical situations, differential equations are sometimes governed by boundary conditions that are not enough smooth but are generalized functions. It is of importance to extend the classical integral transforms to generalized functions especially that possess a convolution property of Fourier convolution type. As a youngest space of generalized functions, and more particularly of distributions, the space of Boehmians was initiated by the concept of regular operators [**6**]. Regular operators form a subalgebra of the field of Mikusinski operators and include only such functions whose support is bounded from the left. The space of Boehmians was initiated to contain all regular operators, all distributions and some objects which are neither operators nor distributions.

In literature, several integral transforms have been extended to various soaces of Boehmians by many authors. We reacall, Roopkumar in [**19, 20**] ; Karunakaran and Vembu in [**18**] ; Karunakaran and Roopkumar in [**17**] ; Mikusinski and Zayed in [**16**] ; Al-Omari in [**6**] ; Al-Omari and Kilicman in [**5, 9, 15**] ; Al-Omari et. al. in [**11**] ; Loonker et. al. in [**25, 26**] and many others.

The Struve $\mathcal{H}_v$ - transform as an example of a symmetric Watson transform is defined by [**1, 2**]

$$\left(S_v^{tr} g\right)(x) = \int_0^\infty \sqrt{xy}\mathcal{H}_v(xy)\, g(y)\, dy, x \in \mathbb{R}_+, \tag{1}$$

where $\mathcal{H}_v$ denotes the Struve function of order v with a power series form

$$\mathcal{H}_v(x) = \sum_0^\infty \frac{(-1)^m}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+d+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2m+\alpha+1}, \tag{2}$$

Γ is the gamma function.

When $\operatorname{Re} v > \frac{1}{2}$, the Struve function can be written in an integral representation; giving

$$\mathcal{H}_v(x) = \frac{2\left(\frac{x}{2}\right)^v}{\sqrt{\pi}\Gamma\left(v+v+\frac{1}{2}\right)} \int_0^\infty \sin(x\cos t)\sin^{2v}(t)\,dt. \tag{3}$$

Struve functions have many applications in physics and applied mathematics. In optics, they occur as the normalized line spread function and, in fluid dynamics they occur as acoustics for impedance calculations [**3**].

The Struve transform have been investigated on the space $\boldsymbol{l}_{v,q}(\mathbb{R}_+)$ of those measurable functions defined on $\mathbb{R}_+$ such that [**1**]

$$\|f\|_{\mu,q} = \left(\int_0^\infty |x^\mu f(x)|^q \frac{dx}{x}\right)^{\frac{1}{q}} < \infty. \tag{4}$$

In the strip, $-2 < \operatorname{Re} v < 0$, $v = \frac{1}{2}, q = 2$; we have $\boldsymbol{l}_{\frac{1}{2},2} = \boldsymbol{l}_2(\mathbb{R}_+)$. Hence, the Struve transform is bounded on $\boldsymbol{l}_2(\mathbb{R}_+)$ and for $\operatorname{Re} v \neq -1$, we have [**1**, p.p.209]

$$\left\|S_v^{tr} g\right\|_{\boldsymbol{l}_2(\mathbb{R}_+)} \leq C\|g\|_{\boldsymbol{l}_2(\mathbb{R}_+)}. \tag{5}$$

The Bessel-Struve transform was defined in [**24**] and the modified Struve transform was defined in [**4**].

2 Definitions and Notations

The construction of Boehmians is similar to the construction of the field of quotients and in some cases, it just gives the field of quotients. On the other hand, the construction is possible where there are zero divisors, such as space $\mathcal{C}$ (the space of continous functions) with the operations of pointwise additions and convolution.

Let $\mathcal{G}$ be a linear space and $\mathcal{S}$ be a subspace of $\mathcal{G}$. We assume to each pair of elements $f \in \mathcal{G}$ and $\omega \in \mathcal{S}$, is assigned the product $f * g$ such that the following conditions are satisfied :

(1) *If $\omega, \psi \in \mathcal{S}$, then $\omega \star \psi \in \mathcal{S}$ and $\omega \star \psi = \psi \star \omega$.*

(2) *If $f \in \mathcal{G}$ and $\omega, \psi \in \mathcal{S}$, then $(f * \omega) * \psi = f * (\omega \star \psi)$.*

(3) *If $f, g \in \mathcal{G}, \omega \in \mathcal{S}$ and $\lambda \in \mathbb{R}$, then*

$$(f+g) * \omega = f * \omega + g * \omega \text{ and } \lambda(f * \omega) = (\lambda f) * \omega.$$

Let Δ be a family of sequences from $\mathcal{S}$, such that :

Δ_1 *If $f, g \in \mathcal{G}, \{\delta_n\} \in \Delta$ and $f * \delta_n = g * \delta_n$, then $f = g, \forall n \in \mathbb{N}$.*

Δ_2 *If $\{\omega_n\}, \{\delta_n\} \in \Delta$, then $\{\omega_n \star \psi_n\} \in \Delta$.*

Elements of Δ will be called delta sequences. Consider the class $\mathcal{A}$ of pair of sequences defined by

$$\mathcal{A} = (\{f_n\}, \{\omega_n\}) : \{f_n\} \subseteq \mathcal{G}^{\mathbb{N}}, \{\omega_n\} \in \Delta,$$

for each $n \in \mathbb{N}$.

An element $(\{f_n\}, \{\omega_n\}) \in \mathcal{A}$ *is called a quotient of sequences, denoted by,* $\left[\dfrac{\{f_n\}}{\{\omega_n\}}\right]$ *if* $f_n * \omega_m = f_m * \omega_n, \forall n, m \in \mathbb{N}$.

Two quotients of sequences $\dfrac{\{f_n\}}{\{\omega_n\}}$ *and* $\dfrac{\{g_n\}}{\{\psi_n\}}$ *are said to be equivalent,* $\dfrac{\{f_n\}}{\{\omega_n\}} \sim \dfrac{\{g_n\}}{\{\psi_n\}}$, *if* $f_n * \psi_m = g_m * \omega_n, \forall n, m \in \mathbb{N}$.

The relation $\sim$ *is an equivalent relation on* $\mathcal{A}$ *and hence, splits* $\mathcal{A}$ *into equivalence classes. The equivalence class containing* $\dfrac{\{f_n\}}{\{\omega_n\}}$ *is denoted by* $\left[\dfrac{\{f_n\}}{\{\omega_n\}}\right]$. *These equivalence classes are called* ***Boehmians****; or* ***usual Boehmians****; and the space of all Boehmians is denoted by* $\mathcal{B}$.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way : $\left[\dfrac{\{f_n\}}{\{\omega_n\}}\right] + \left[\dfrac{\{g_n\}}{\{\psi_n\}}\right] = \left[\dfrac{\{f_n * \psi_n + g_n * \omega_n\}}{\{\omega_n * \psi_n\}}\right]$ *and* $\alpha \left[\dfrac{\{f_n\}}{\{\omega_n\}}\right] = \left[\alpha \dfrac{\{f_n\}}{\{\omega_n\}}\right] = \left[\dfrac{\{\alpha f_n\}}{\{\omega_n\}}\right]$, $\alpha \in \mathbb{C}$, *space of complex numbers.*

The operation $*$ *and the differentiation are defined by :* $\left[\dfrac{\{f_n\}}{\{\omega_n\}}\right] * \left[\dfrac{\{g_n\}}{\{\psi_n\}}\right] = \left[\dfrac{\{f_n * g_n\}}{\{\omega_n * \psi_n\}}\right]$ *and* $\mathcal{D}^\alpha \left[\dfrac{\{f_n\}}{\{\omega_n\}}\right] = \left[\dfrac{\{\mathcal{D}^\alpha f_n\}}{\{\omega_n\}}\right]$.

Many a time, $\mathcal{G}$ *is equipped with a notion of convergence. The relationship between the notion of convergence and* $*$ *are given by:*

(4) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\mathcal{G}$ *and,* $\omega \in \mathcal{S}$ *is any fixed element, then*

$$f_n * \omega \to f * \omega \text{ in } \mathcal{G} \text{ as } n \to \infty.$$

(5) *If* $f_n \to f$ *as* $n \to \infty$ *in* $\mathcal{G}$ *and* $\{\omega_n\} \in \Delta$, *then*

$$f_n * \omega_n \to f \text{ in } \mathcal{G} \text{ as } n \to \infty.$$

The operation $*$ *can be extended to* $\mathcal{B} \times \mathcal{S}$ *by : If* $\left[\dfrac{\{f_n\}}{\{\omega_n\}}\right] \in \mathcal{B}$ *and* $\omega \in \mathcal{S}$, *then*

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] * \omega = \left[\frac{\{f_n * \omega\}}{\{\omega_n\}}\right].$$

In $\mathcal{B}$, two types of convergence, δ - convergence and Δ - convergence, are defined as follows:

δ **- convergence:** *A sequence of Boehmians* $\{\beta_n\}$ *in* $\mathcal{B}$ *is said to be* δ *convergent to a Boehmian* β *in* $\mathcal{B}$, *denoted by* $\beta_n \xrightarrow{\delta} \beta$, *if there exists a delta sequence* $\{\omega_n\}$ *such that*

$$(\beta_n * \omega_k), (\beta * \omega_k) \in \mathcal{G}, \forall k, n \in \mathbb{N},$$

and

$$(\beta_n * \omega_k) \to (\beta * \omega_k) \text{ as } n \to \infty, \textit{in } \mathcal{G}, \textit{ for every } k \in \mathbb{N}.$$

The following is equivalent for the statement of δ - convergence :

$\beta_n \xrightarrow{\delta} \beta$ as $n \to \infty$ *in* $\mathcal{B}$ *if and only if there is* $\{f_{n,k}\}, \{f_k\} \in \mathcal{G}$ *and* $\{\omega_k\} \in \Delta$ *such that* $\beta_n = \left[\frac{\{f_{n,k}\}}{\{\omega_k\}}\right], \beta = \left[\frac{\{f_k\}}{\{\omega_k\}}\right]$ *and for each* $k \in \mathbb{N}, f_{n,k} \to f_k$ *as* $n \to \infty$ *in* $\mathcal{G}$.

Δ **- convergence:** *A sequence of Boehmians* $\{\beta_n\}$ *in* $\mathcal{B}$ *is said to be* Δ *- convergent to a Boehmian* β *in* $\mathcal{B}$*, denoted by* $\beta_n \xrightarrow{\Delta} \beta$*, if there exists a* $\{\omega_n\} \in \Delta$ *such that* $(\beta_n - \beta) * \omega_n \in \mathcal{G}, \forall n \in \mathbb{N}$*, and* $(\beta_n - \beta) * \omega_n \to 0$ *as* $n \to \infty$ *in* $\mathcal{G}$.

See; [**5**] [**9**], [**11**], [**13**], [**27**], [**28**], [**29**] for further investigation of the abstract construction of Boehmians.

3 Basic Theorem and Notations

Let $\boldsymbol{\kappa}(\mathbb{R}_+)$ denote the space of test functions of compact support defined on $\mathbb{R}_+$ [**14**, **10**]. Then, the following definitions are needful tour next investigations.

Definition 1 *Let* $\phi \in \boldsymbol{l}_2(\mathbb{R}_+), \varphi \in \boldsymbol{\kappa}(\mathbb{R}_+)$*; then denote by* $\bullet$ *the Mellin - type convolution product of first kind given by* [**14**, **12**]

$$(\phi \bullet \varphi)(y) = \int_0^\infty \eta^{-1}\phi\left(y\eta^{-1}\right)\varphi(\eta)\,d\eta. \tag{6}$$

Properties of $\bullet$ may be introduced as :
$(i)\ (\phi \bullet \psi)(t) = (\psi \bullet \phi)(t)$;
$(ii)\ ((\phi + \psi) \bullet \varphi)(t) = (\phi \bullet \varphi)(t) + (\psi \bullet \varphi)(t)$;
$(iii)\ (\alpha\phi \bullet \psi)(t) = \alpha(\psi \bullet \phi)(t), \alpha$ is complex number ;
$(vi)\ ((\phi \bullet \psi) \bullet \varphi)(t) = (\phi \bullet (\psi \bullet \varphi))(t)$.

Definition 2 *Let* $\phi \in \boldsymbol{l}_2(\mathbb{R}_+)$ *and* $\varphi \in \boldsymbol{\kappa}(\mathbb{R}_+)$*; then we define a product* $\times$*, between* ϕ *and* φ*, by*

$$(\phi \times \varphi)(x) = \int_0^\infty \phi(x\eta)\varphi(\eta)\,d\eta. \tag{7}$$

Theorem 3 *Let* $-2 < \operatorname{Re} v < 0$ *and* $\varphi \in \boldsymbol{\kappa}(\mathbb{R}_+)$*; then* $S_v^{tr}(\phi \bullet \varphi) = S_v^{tr}\phi \times \varphi$*, for every* $\phi \in \boldsymbol{l}_2(\mathbb{R}_+)$.

Proof Under the Hypothesis of the theorem we by Definition 1.2. and (1) write

$$\begin{aligned} S_v^{tr}(\phi \bullet \varphi)(x) &= \int_0^\infty (\phi \bullet \varphi)(y)\sqrt{xy}\mathcal{H}_v(xy)\,dy \\ &= \int_0^\infty \left(\int_0^\infty \eta^{-1}\phi\left(y\eta^{-1}\right)\varphi(\eta)\,d\eta\right)\sqrt{xy}\mathcal{H}_v(xy)\,dy \\ &= \int_0^\infty \eta^{-1}\varphi(\eta)\int_0^\infty \phi\left(y\eta^{-1}\right)\sqrt{xy}\mathcal{H}_v(xy)\,dyd\eta \end{aligned} \tag{8}$$

By change of variables $y\eta^{-1}=\theta$ and Fubini's theorem, (8) then gives

$$\begin{aligned} S_v^{tr}(\phi\bullet\varphi)(x) &= \int_0^\infty\left(\int_0^\infty \phi(\theta)\sqrt{x\theta\eta}\mathcal{H}_v(x\theta\eta)\,d\theta\right)\varphi(\eta)\,d\eta \\ &= \int_0^\infty\left(S_v^{tr}\phi\right)(x\eta)\,\varphi(\eta)\,d\eta \\ &= \left(S_v^{tr}\phi\times\varphi\right)(x). \end{aligned}$$

This completes the proof of the theorem.

4 Space of Boehmians

Let us now construct spaces of Boehmians where our Struve transform is defined.

Let Δ be the set of delta sequences (approximating identities) satisfying the following properties :

(1) : $\{\delta_n\}\in\boldsymbol{\kappa}(\mathbb{R}_+)$;

(2) : $\int_0^\infty \delta_n(x)\,dx=1, n\in\mathbb{N}$;

(3) : $\int_0^\infty |\delta_n(x)|\,dx<\infty, n\in\mathbb{N}$;

(4) : $\operatorname{supp}\delta_n(x)\subseteq[a,b_n], 0<a_n<b_n$ and $a_n,b_n\to 0$ as $n\to\infty$, where

$$\operatorname{supp}\delta_n(x)=\{x\in\mathbb{R}_+:\delta_n(x)\neq 0,\forall n\in\mathbb{N}\}.$$

We first consider the space $\mathcal{B}(\boldsymbol{l}_2,(\boldsymbol{\kappa},\bullet),\times)$ for our construction.

Theorem 4 *Let $\phi_1,\phi_2\in\boldsymbol{l}_2(\mathbb{R}_+),\varphi_1,\varphi_2\in\boldsymbol{\kappa}(\mathbb{R}_+)$; then we have the following identities*

(*i*) $(\phi_1+\phi_2)\times\varphi_1=\phi_1\times\varphi_1+\phi_2\times\varphi_1$.

(*ii*) $\varphi_1\bullet\varphi_2=\varphi_2\bullet\varphi_1$ *in* $\boldsymbol{\kappa}(\mathbb{R}_+)$.

(*iii*) *Let* $\phi_n\to\phi$ *in* $\boldsymbol{l}_2(\mathbb{R}_+)$; *then for every* $\varphi\in\boldsymbol{\kappa}(\mathbb{R}_+),\phi_n\times\varphi\to\phi\times\varphi$ *as* $n\to\infty$.

Proof of part (*i*) and (*iii*) follows from simple integration.

Part (*ii*) follows from the properties of $\bullet$ enumerated in [**14**].

Theorem 5 *Let $\phi\in\boldsymbol{l}_2(\mathbb{R}_+)$ and $\varphi_1,\varphi_2\in\boldsymbol{\kappa}(\mathbb{R}_+)$; then $\varphi\times(\varphi_1\bullet\varphi_2)=\varphi\times(\phi\times\varphi_1)\times\varphi_2$.*

Proof Under the hypothesis of the theorem and Fubini's theorem we write

$$\begin{aligned} (\phi\times(\varphi_1\bullet\varphi_2))(x) &= \int_0^\infty \phi(x\eta)(\varphi_1\bullet\varphi_2)(\eta)\,d\eta \\ &= \int_0^\infty \phi(x\eta)\int_0^\infty y^{-1}\varphi_1\left(\eta y^{-1}\right)\varphi_2(y)\,dyd\eta \\ &= \int_0^\infty \varphi_2(y)\,y^{-1}\int_0^\infty \phi(x\eta)\,\varphi_1\left(\eta y^{-1}\right)dyd\eta. \qquad (9) \end{aligned}$$

Change of variables on (9) implies

$$\begin{aligned}(\phi\times(\varphi_1\bullet\varphi_2))(x) &= \int_0^\infty\int_0^\infty \phi(xy\xi)\varphi_1(\xi)\,d\xi\varphi_2(y)\,dy\\ &= \int_0^\infty(\phi\times\varphi_1)(xy)\varphi_2(y)\,dy\\ &= ((\phi\times\varphi_1)\times\varphi_2)(x).\end{aligned}$$

This completes the proof of the theorem.

As final in this construction of $\mathcal{B}(\boldsymbol{l}_2,(\boldsymbol{\kappa},\bullet),\times)$, we prove the following theorem.

Theorem 6 *Let* $\phi\in\boldsymbol{l}_2(\mathbb{R}_+)$ *and* $\{\delta_n\}$ *be a delta sequence*; *then* $\phi\times\delta_n\to\phi$ as $n\to\infty$.

Proof By properties delta sequences we have

$$\begin{aligned}\|\phi\times\delta_n-\phi\|^2_{\boldsymbol{l}_2(\mathbb{R}_+)} &= \int_0^\infty |(\phi\times\delta_n)(x)-\phi(x)|^2\,dx\\ &= \int_0^\infty\left|\int_0^\infty \phi(x\eta)\delta_n(\eta)-\int_0^\infty\phi(x)\delta_n(\eta)\,d\eta\right|^2 dx\\ &\le \int_{a_n}^{b_n}\int_0^\infty|\phi(x\eta)-\phi(x)|^2|\delta_n(\eta)|\,dxd\eta\\ &\qquad\text{(By Jensen's inequality)}\\ &= M(b_n-a_n)\int_0^\infty|\phi(x\eta)-\phi(x)|^2\,dx\to0\text{ as }n\to\infty,\end{aligned}$$

where $K_n=[a_n,b_n]$, is a compact set containing the support of $\delta_n,\forall n\in\mathbb{N}$. But

$$0\le\|\phi\times\delta_n\|_{\boldsymbol{l}_2(\mathbb{R}_+)}-\|\phi\|_{\boldsymbol{l}_2(\mathbb{R}_+)}\le\|\phi\times\delta_n-\phi\|_{\boldsymbol{l}_2(\mathbb{R}_+)}\to0\text{ as }n\to\infty.$$

Hence, we have obtained

$$\|\phi\times\delta_n\|_{\boldsymbol{l}_2(\mathbb{R}_+)}-\|\phi\|_{\boldsymbol{l}_2(\mathbb{R}_+)}\to0\text{ as }n\to\infty.$$

This completes the proof of the theorem.

This our space is constructed and regarded as a space of Boehmians.

The sum of two Boehmians and multiplication by a scalar can be defined in a natural way :

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right]+\left[\frac{\{g_n\}}{\{\psi_n\}}\right]=\left[\frac{\{f_n\times\psi_n+g_n\times\omega_n\}}{\{\omega_n\bullet\psi_n\}}\right]\tag{10}$$

and

$$\alpha\left[\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\alpha\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\frac{\{\alpha f_n\}}{\{\omega_n\}}\right],$$

$\alpha\in\mathbb{C}$, the space of complex numbers.

The operation $\times$ and the differentiation are defined by :

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] \times \left[\frac{\{g_n\}}{\{\psi_n\}}\right] = \left[\frac{\{f_n \times g_n\}}{\{\omega_n \bullet \psi_n\}}\right] \text{ and } \mathcal{D}^{\alpha}\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\mathcal{D}^{\alpha} f_n\}}{\{\omega_n\}}\right]. \tag{11}$$

Similarly, the space $\mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$ can be established.

Sum and multiplication by a scalar in $\mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$ can be defined as :

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] + \left[\frac{\{g_n\}}{\{\psi_n\}}\right] = \left[\frac{\{f_n \bullet \psi_n + g_n \bullet \omega_n\}}{\{\omega_n \bullet \psi_n\}}\right] \tag{12}$$

and

$$\alpha\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\alpha\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\alpha f_n\}}{\{\omega_n\}}\right],$$

$\alpha \in \mathbb{C}$, the space of complex numbers.

The product $\bullet$ and the differentiation in $\mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$ are defined as :

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] \bullet \left[\frac{\{g_n\}}{\{\psi_n\}}\right] = \left[\frac{\{f_n \bullet g_n\}}{\{\omega_n \bullet \psi_n\}}\right] \text{ and } \mathcal{D}^{\alpha}\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\mathcal{D}^{\alpha} f_n\}}{\{\omega_n\}}\right]. \tag{13}$$

Definition 7 *Let* $\left[\frac{\{\phi_n\}}{\{\delta_n\}}\right] \in \mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$; *then, in view of Theorem* 3, *for* $-2 <$ $\operatorname{Re} v < 0$, *we define the Struve transform of* $\left[\frac{\{\phi_n\}}{\{\delta_n\}}\right]$ *as*

$$S_v^{tr}\left[\frac{\{\phi_n\}}{\{\delta_n\}}\right] = \left[\frac{S_v^{tr}\{\phi_n\}}{\{\delta_n\}}\right] \tag{14}$$

which belongs to the space $\mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \times\right)$.

Definition (7) is well - defined by Theorem 3. For more details, let $\left[\frac{\{\psi_n\}}{\{\mu_n\}}\right] = \left[\frac{\{\xi_n\}}{\{\epsilon_n\}}\right] \in \mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$; then

$$\psi_n \bullet \epsilon_m = \xi_m \bullet \mu_n. \tag{15}$$

Applying Definition 7 and Theorem 3, Equation 15 then gives

$$S_v^{tr}\psi_n \times \epsilon_m = S_v^{tr}\xi_m \times \mu_n. \tag{16}$$

Hence, from (16), we see that $\frac{\{S_v^{tr}\psi_n\}}{\{\mu_n\}} \sim \frac{\{S_v^{tr}\xi_n\}}{\{\epsilon_n\}}$ in $\mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \times\right)$. Therefore

$$\left[\frac{\{S_v^{tr}\psi_n\}}{\{\mu_n\}}\right] = \left[\frac{\{S_v^{tr}\xi_n\}}{\{\epsilon_n\}}\right].$$

This completes the proof of the theorem.

Theorem 8 *Let* $\beta_1 = \left[\frac{\{\psi_n\}}{\{\mu_n\}}\right] \in \mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$ *and* $\beta_2 = \left[\frac{\{\xi_n\}}{\{\epsilon_n\}}\right] \in \mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$; *then*

$$S_v^{tr}(\beta_1 \bullet \beta_2) = S_v^{tr}\beta_1 \times \beta_2.$$

Proof Assume the requirements of the theorem are satisfied for some β_1 and $\beta_2 \in \mathcal{B}\left(\boldsymbol{l}_2, (\boldsymbol{\kappa}, \bullet), \bullet\right)$; then

$$\begin{aligned} S_v^{tr}(\beta \bullet \beta_2) &= S_v^{tr}\left(\left[\frac{\{\psi_n\} \bullet \{\xi_n\}}{\{\mu_n\} \bullet \{\epsilon_n\}}\right]\right) \\ &\qquad \text{(By Equation 13)} \\ &= \left[\frac{S_v^{tr}(\{\psi_n\} \bullet \{\xi_n\})}{\{\mu_n\} \bullet \{\epsilon_n\}}\right] \\ &\qquad \text{(By Equation 14)} \\ &= \left[\frac{\{S_v^{tr}\psi_n\} \times \{\xi_n\}}{\{\mu_n\} \bullet \{\epsilon_n\}}\right] \\ &\qquad \text{(By Theorem 3)} \\ &= \left[\frac{\{S_v^{tr}\psi_n\}}{\{\mu_n\}}\right] \times \left[\frac{\{\xi_n\}}{\{\epsilon_n\}}\right]. \quad \text{(By Equation 11)} \end{aligned}$$

Therefore, we have obtained

$$S_v^{tr}(\beta_1 \bullet \beta_2) = S_v^{tr}(\beta) \times \beta_2.$$

This completes the proof of the theorem.

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ON THE EXTENDED FOURIER TRANSFORM OF GENERALIZED FUNCTIONS

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Abstract

Integral transforms of strong Boehmians have not yet been reported in the literature. Perhaps, because the strong Boehm ian space is not widely known nor used [10]. In this article, we derive certain space of Boehmians (the space of stronge Boehmians) which handle the Fourier transform with a different performance. The Fourier transform of a strong Boehmian has been established as a one-to-one mapping which is continuous with respect to δ and Δ convergence. The inverse Fourier transform is also defined and some of its properties are obtained. Related theorems are discussed in some detail.

Keywords: Fourier Transform; Distribution space; Boehmian space.

1 Introduction.

The Fourier transform of a function $f \in \mathcal{L}^1(\mathcal{R})$ of one variable is defined by

$$\tilde{f}(\varsigma) := \mathcal{F}f(\varsigma) := \int_{\mathcal{R}} f(\sigma) e^{i\varsigma\sigma} d\sigma, \tag{1}$$

If, in addition, $\tilde{f} \in \mathcal{L}^1(\mathcal{R})$, then it is possible to recover the function f by means of the inversion formula

$$f(\sigma) := \mathcal{F}^{-1}\left(\tilde{f}\right)(\sigma) := \frac{1}{2\pi}\int_{\mathcal{R}} \tilde{f}(\varsigma) e^{-i\sigma\varsigma} d\varsigma, \tag{2}$$

almost everywhere.

Following, are elementary properties of the Fourier transform on $\mathcal{L}^1(\mathcal{R})$:

(i) If $f \in \mathcal{L}^1(\mathcal{R})$ then its Fourier transform exists, continuous, and

$$\left|\tilde{f}(\varsigma)\right| \leq \|f\|_1 . \tag{3}$$

(ii) If $f \in \mathcal{L}^1(\mathcal{R})$ and $g \in \mathcal{L}^1(\mathcal{R})$ then the Parseval formula can be read as

$$\int_{\mathcal{R}} f(\sigma)\tilde{g}(\sigma) d\sigma = \int_{\mathcal{R}} \tilde{f}(\varsigma) g(\varsigma) d\varsigma. \tag{4}$$

(iii) If $f \in \mathcal{L}^1(\mathcal{R})$ and $g \in \mathcal{L}^1(\mathcal{R})$ and $\bullet$ is the usual convolution product then $f \bullet g$ exists for almost all $\varsigma \in \mathcal{R}$ and is a member of $\mathcal{L}^1(\mathcal{R})$; moreover,

$$(f \bullet g)^{\sim} = \tilde{f}\tilde{g} \text{ and } (fg)^{\sim} = (2\pi)^{-1}\tilde{f} \bullet \tilde{g}. \tag{5}$$

The space $\mathcal{S}(\mathcal{R})$, of smooth functions of rapid descent, is defined as the space of complex valued $C^{\infty}-$functions on $\mathcal{R}$ such that for every choice of non-negative integers m, n,

$$\gamma_{m,n}(\psi) := \sup_{\sigma \in \mathcal{R}} \left| \sigma^m \mathcal{D}^k \psi(\sigma) \right| < \infty. \tag{6}$$

A sequence (ψ_n) is said to converge to zero in $\mathcal{S}(\mathcal{R})$ if $\gamma_{m,n}(\psi_n) \to 0$ as $n \to \infty$ for all $m, n \in \mathcal{N}$, $(\mathcal{N} = \{1, 2, 3, ...\})$.

Theorem 1.1. The Fourier transform and its inverse are continuous isomorphisms from $\mathcal{S}(\mathcal{R})$ into $\mathcal{S}(\mathcal{R})$. (see [17, p.185]).

Aided by above theorem, the distributional Fourier transform $\acute{\mathcal{F}}$ on $\mathcal{S}'$, the dual of $\mathcal{S}$ of tempered distributions, is discribed to be the adjoint mapping of $\mathcal{F}$ such that

$$\left\langle \acute{\mathcal{F}} f, \psi \right\rangle = \langle f, \mathcal{F}\psi \rangle, \tag{7}$$

for every $\psi \in \mathcal{S}$.

For convenience, notations $\mathcal{F}f$ and $\tilde{f}$ are used interchangeably in this note. In [11], Mikusinski discusses some basic properties of the space of integrable Boehmians and further shows that the Fourier transform of an integrable Boehmian is always a continuous function and has all basic properties of the Fourier transform in the space of Lebesgue integrable functions. On the other hand, the author in [13], establishes that the Fourier transform of a tempered Boehmian is a distribution.

In an earlier paper [2], we have extended the Fourier transform to certain space of ultradifferentiable Boehmians. In the present paper we discuss certain class of Boehmians for the Fourier transform with careful attention. An avenue towards this end is to discuss the space of Boehmians and derive its properties, while the Fourier transform of Boehmians may be considered satisfactory and well recognized. The inverse Fourier transform is established. Further work on this theory is obtained.

This paper is organized as follows: The Fourier transform is reviewed in Section 1. Section 2 presents a general construction of strong Boehmian space. The construction of general Boehmian spaces is presented in Section 3. The convolution theorem and the space $\mathcal{B}$ of Boehmians are given in Section 4. The extended Fourier transform and its properties are established in Section 5.

2 The Space $\mathcal{S}_t(\mathcal{S}, \mathcal{G}, \Delta, \bullet)$ of Strong Boehmians of Rapid Descent.

Many authors have extended several integral transforms to Boehmian spaces but none to the space of strong Boehmians. Perhaps, because the space of strong

Boehmians is not widely known nor used. General construction of strong Boehmians was briefly presented in [10]. The space of strong Boehmians is identified with a subspace of the space of general Boehmians. Denote by $\mathcal{R}$ the field of real numbers then $\mathcal{D}(\mathcal{R})$ will denote the Schwartz space of test functions of compact support over $\mathcal{R}$. By $\mathcal{S}(\mathcal{H})$ we denote the space of rapidly decreasing functions (the space of rapid descent) on $\mathcal{H}$, $\mathcal{H} = \mathcal{R}_{+1} \times \mathcal{R}, \mathcal{R}_{+1} = [1, \infty)$. The dual space of $\mathcal{S}(\mathcal{H})$, $\mathcal{S}'(\mathcal{H})$, consists of tempered distributions(distributions of slow growth); see, for example, [8,14,17].

The usual convolution product $\bullet$ between two functions, $\psi \in \mathcal{S}(\mathcal{H})$ and $\epsilon \in \mathcal{D}(\mathcal{R})$, is defined by

$$(\psi \bullet \epsilon)(\varsigma) = \int_{\mathcal{R}} \psi(\alpha, \sigma)\, \epsilon(\varsigma - \sigma)\, d\sigma. \tag{8}$$

where $\alpha \in \mathcal{R}_{+1}$.

Let $\mathcal{G}(\mathcal{R})$ be the subset of $\mathcal{D}(\mathcal{R})$ of test functions where $\int_{\mathcal{R}} \epsilon(\sigma)\, d\sigma = 1$.

Let $\psi \in \mathcal{S}(\mathcal{H})$ and $\epsilon \in \mathcal{G}(\mathcal{R})$, then the pair of functions (ψ, ϵ), or $\frac{\psi}{\epsilon}$, is said to be a quotient of functions if and only if

$$\psi(\alpha, \sigma) \bullet d_\beta \epsilon(\sigma) = \psi(\beta, \sigma) \bullet d_\alpha \epsilon(\sigma), \tag{9}$$

$\alpha, \beta \in \mathcal{R}_{+1}$, and

$$d_r \epsilon(\sigma) = r\epsilon(r\sigma). \tag{10}$$

Two quotients of functions $\frac{\psi}{\epsilon}$ and $\frac{\kappa}{\tau}$ are said to be equivalent, $\frac{\psi}{\epsilon} \sim \frac{\kappa}{\tau}$, if and only if

$$\psi(\alpha, \sigma) \bullet d_\beta \tau(\sigma) = \kappa(\beta, \sigma) \bullet d_\alpha \epsilon(\sigma), \tag{11}$$

$\alpha, \beta \in \mathcal{R}_{+1}$.

Denote by $\mathcal{A} = \left\{ \frac{\psi}{\epsilon} : \forall \psi \in \mathcal{S}(\mathcal{H}), \epsilon \in \mathcal{G}(\mathcal{R}) \right\}$ then the equivalence class containing $\frac{\psi}{\epsilon}$ is said to be a **strong Boehmian**. The space of all such Boehmians is denoted by $\mathcal{S}_t(\mathcal{S}, \mathcal{G}, \bullet)$. A typical element in $\mathcal{S}_t(\mathcal{S}, \mathcal{G}, \bullet)$ is denoted by $\left[\frac{\psi}{\epsilon}\right]$. Following, are auxilliary results which are useful in the sequel [10, p.p.886].

(r_1) If $\epsilon, \tau \in \mathcal{G}(\mathcal{R})$ then $\epsilon \bullet \tau \in \mathcal{G}(\mathcal{R})$.

(r_2) If $\psi \in \mathcal{S}(\mathcal{H})$ and $\epsilon \in \mathcal{G}(\mathcal{R})$ then $\psi \bullet \epsilon \in \mathcal{S}(\mathcal{H})$.

(r_3) If $\frac{\psi}{\epsilon} \in \mathcal{A}$ and $\tau \in \mathcal{G}(\mathcal{R})$ then $\frac{\psi \bullet \tau}{\epsilon \bullet \tau} \in \mathcal{A}$ and $\frac{\psi}{\epsilon} \sim \frac{\psi \bullet \tau}{\epsilon \bullet \tau}$.

(r_4) If $\epsilon \in \mathcal{G}(\mathcal{R})$ then $d_r \epsilon \in \mathcal{G}(\mathcal{R}), r \geq 1, d_r$ has the usual meaning in Equ.(10).

Addition and scalar multiplications of strong Boehmians are respectively defined by usual way

$$\left[\frac{\psi}{\epsilon}\right] + \left[\frac{\kappa}{\tau}\right] = \left[\frac{\psi \bullet \tau + \kappa \bullet \epsilon}{\epsilon \bullet \tau}\right] \text{ and } \lambda \left[\frac{\psi}{\epsilon}\right] = \left[\frac{\lambda \psi}{\epsilon}\right].$$

Differentiation in $\mathcal{S}_t(\mathcal{S}, \mathcal{G}, \bullet)$ and the operation $\bullet$ are respectively defined by

$$\mathcal{D}^k \left[\frac{\psi}{\epsilon}\right] = \left[\frac{\mathcal{D}^k \psi}{\epsilon}\right] \text{ and } \left[\frac{\psi}{\epsilon}\right] \bullet \tau = \left[\frac{\psi \bullet \tau}{\epsilon}\right].$$

In $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$, convergence is defined as follows: *A sequence* (β_n) *of strong Boehmians is said to be* $\mathcal{G}$ ***convergent*** *to a Boehmian* β *in* $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ *if for some* $\psi_n,\psi\in\mathcal{S}(\mathcal{H})$ *and* $\epsilon\in\mathcal{G}(\mathcal{R})$ *such that* $\beta_n=\left[\frac{\psi_n}{\epsilon}\right], x=\left[\frac{\psi}{\epsilon}\right]$, *we have* $\psi_n\to\psi$ *as* $n\to\infty$ *in* $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$.

Denote by $\Delta=\{(\epsilon_n)\in\mathcal{G}:\operatorname{supp}\epsilon_n\to 0 \text{ as } n\to\infty\}$ then Δ is a family of delta sequences on $\mathcal{R}$ which correspond to the Dirac delta distribution. Since the strong Boehmian space can be identified with a subspace of general Boehmians, it seems more convenient to consider the Boehmian subspace $\mathcal{S}_t(\mathcal{S},\mathcal{G},\Delta,\bullet)$ of $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$, in short $\mathcal{S}_t$, rather than $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ where the former is obtained through injecting $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ with the set Δ of delta sequences. Concept of quotients, Equ.(9), and equivalence classes, Equ.(11), are retained as in $\mathcal{S}_t$. That is, $\frac{\psi_n}{\epsilon_n}$ is quotient in $\mathcal{S}_t$ if

$$\psi_n(\alpha,\sigma)\bullet d_\beta\epsilon_m(\sigma)=\psi_m(\beta,\sigma)\bullet d_\alpha\epsilon_n(\sigma),$$

and $\frac{\psi_n}{\epsilon_n}\sim\frac{\kappa_n}{\tau_n}$ if

$$\psi_n(\alpha,\sigma)\bullet d_\beta\tau_m(\sigma)=\kappa_m(\beta,\sigma)\bullet d_\alpha\epsilon_n(\sigma), \alpha,\beta\in\mathcal{R}_{+1}.$$

The sum of two Boehmians and multiplication by a scalar in $\mathcal{S}_t$ are defined by

$$\left[\frac{\psi_n}{\epsilon_n}\right]+\left[\frac{\kappa_n}{\tau_n}\right]=\left[\frac{\psi_n\bullet\tau_n+\kappa_n\bullet\epsilon_n}{\epsilon_n\bullet\tau_n}\right] \text{ and } a\left[\frac{\psi_n}{\epsilon_n}\right]=\left[\frac{a\psi_n}{\epsilon_n}\right], a\in\mathbb{C},$$

where $\mathbb{C}$ denotes the field of complex numbers.

The operation $\bullet$ and the differentiation are respectively defined by

$$\left[\frac{f_n}{\epsilon_n}\right]\bullet\left[\frac{\kappa_n}{\tau_n}\right]=\left[\frac{f_n\bullet\kappa_n}{\epsilon_n\bullet\tau_n}\right] \text{ and } \mathcal{D}^\alpha\left[\frac{\psi_n}{\epsilon_n}\right]=\left[\frac{\mathcal{D}^\alpha\psi_n}{\epsilon_n}\right].$$

Definition 2.1. *If* $\left[\frac{\psi_n}{\epsilon_n}\right]\in\mathcal{S}_t$ *and* $\phi\in\mathcal{S}$ *then* $\left[\frac{\psi_n}{\epsilon_n}\right]\bullet\phi=\left[\frac{\psi_n\bullet\phi}{\epsilon_n}\right]$.

In $\mathcal{S}_t$, two types of convergence, δ convergence and Δ convergence, are defined as follows:

Definition 2.2. *A sequence of Boehmians* (β_n) *in* $\mathcal{S}_t$ *is said to be* δ *convergent to a Boehmian* β *in* $\mathcal{S}_t$, *denoted by* $\beta_n\xrightarrow{\delta}\beta$, *if there exists a delta sequence* (ϵ_n) *such that* $(\beta_n\bullet\epsilon_n),(\beta\bullet\epsilon_n)\in\mathcal{S},\forall k,n\in\mathcal{N}$, *and*

$$(\beta_n\bullet\epsilon_k)\to(\beta\bullet\epsilon_k) \text{ as } n\to\infty, \text{ in } \mathcal{S}, \text{ for every } k\in\mathcal{N}.$$

The following lemma is equivalent to the statement of δ *convergence*

Lemma 2.3. $\beta_n\xrightarrow{\delta}\beta$ $(n\to\infty)$ *in* $\mathcal{S}_t$ *if and only if there is* $\psi_{n,k},\psi_k\in\mathcal{S}$ *and* $(\epsilon_k)\in\Delta$ *such that* $\beta_n=\left[\frac{\psi_{n,k}}{\epsilon_k}\right],\beta=\left[\frac{\psi_k}{\epsilon_k}\right]$ *and for each* $k\in\mathcal{N}$,

$$\psi_{n,k}\to\psi_k \text{ as } n\to\infty \text{ in } \mathcal{S}.$$

Definition 2.4. *A sequence of Boehmians* (β_n) *in* $\mathcal{S}_t$ *is said to be* Δ *convergent to a Boehmian* β *in* $\mathcal{S}_t$, *denoted by* $\beta_n\xrightarrow{\Delta}\beta$, *if there exists a* $(\epsilon_n)\in\Delta$ *such that* $(\beta_n-\beta)\bullet\epsilon_n\in\mathcal{S},\forall n\in\mathcal{N}$, *and* $(\beta_n-\beta)\bullet\epsilon_n\to 0$ *as* $n\to\infty$ *in* $\mathcal{S}$.

3 Boehmian Spaces, their General Construction.

The idea of construction of Boehmians was initiated by the concept of regular operators introduced by Boehme [9]. Regular operators form a subalgebra of the field of Mikusinski operators and they include only such functions whose support is bounded from the left. In a concrete case, the space of Boehmians contains all regular operators, all distributions and some objects which are neither operators nor distributions [4]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(**i**) A nonempty set $\mathcal{L}$;

(**ii**) A commutative semigroup $(\mathcal{M}, *)$;

(**iii**) An operation $\star : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ such that:

$$\psi \star (\epsilon_1 * \epsilon_2) = (\psi \star \epsilon_1) \star \epsilon_2 \text{ for each } \psi \in \mathcal{L} \text{ and } \epsilon_1, \epsilon_2, \in \mathcal{M};$$

(**vi**) A collection $\Delta \subset \mathcal{M}^{\mathcal{N}}$ such that:

(a) If $\psi, \kappa \in \mathcal{L}, (\epsilon_n) \in \Delta, \psi \star \epsilon_n = \kappa \star \epsilon_n$ for all n, then $\psi = \kappa$;

(b) If $(\epsilon_n), (\tau_n) \in \Delta$, then $(\epsilon_n * \tau_n) \in \Delta$. Δ is the set of all delta sequences.

Let $\mathcal{A} = \{(\psi_n, \epsilon_n) : \psi_n \in \mathcal{L}, (\epsilon_n) \in \Delta, \psi_n \star \epsilon_m = \psi_m \star \epsilon_n, m, n \in \mathcal{N}\}$. If (ψ_n, ϵ_n), $(\kappa_n, \tau_n) \in \mathcal{A}, \psi_n \star \tau_m = \kappa_m \star \epsilon_n, \forall m, n \in \mathcal{N}$, then we say $(\psi_n, \epsilon_n) \sim (\kappa_n, \tau_n)$. The relation $\sim$ is an equivalence relation in $\mathcal{A}$. The space of equivalence classes in $\mathcal{A}$ is denoted by $\mathcal{B}$. Elements of $\mathcal{B}$ are general Boehmians.

Between $\mathcal{L}$ and $\mathcal{B}$ there is a canonical embedding expressed as $\psi \to \left[\frac{\psi \star \epsilon_n}{\epsilon_n}\right]$.

The operation $\star$ can be extended to $\mathcal{B} \times \mathcal{L} \to \mathcal{B}$ by $\left[\frac{\psi_n}{\epsilon_n}\right] \star \psi = \left[\frac{\psi_n \star \psi}{\epsilon_n}\right]$.

(**i**) *A sequence (β_n) in $\mathcal{B}$ is said to be δ* ***convergent*** *to β in $\mathcal{B}$, denoted by $\beta_n \xrightarrow{\delta} \beta$, if there exists a delta sequence (ϵ_n) such that $(\beta_n \star \epsilon_n)$, $(\beta \star \epsilon_n) \in \mathcal{L}$, $\forall k, n \in \mathcal{N}$, and $(\beta_n \star \epsilon_k) \to (\beta \star \epsilon_k)$ as $n \to \infty$, in $\mathcal{L}$, for every $k \in \mathcal{N}$.*

(**ii**)*A sequence (β_n) in $\mathcal{B}$ is said to be Δ* ***convergent*** *to β in $\mathcal{B}$, denoted by $\beta_n \xrightarrow{\Delta} \beta$, if there exists a $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \star \epsilon_n \in \mathcal{L}, \forall n \in \mathcal{N}$, and $(\beta_n - \beta) \star \epsilon_n \to 0$ as $n \to \infty$ in $\mathcal{L}$.*

The following is equivalent for the statement of δ convergence: $\beta_n \xrightarrow{\delta} \beta \ (n \to \infty)$ *in $\mathcal{B}$ if and only if there is $\psi_{n,k}, \psi_k \in \mathcal{L}$ and $(\epsilon_k) \in \Delta$ such that $\beta_n = \left[\frac{\psi_{n,k}}{\epsilon_k}\right]$, $\beta = \left[\frac{\psi_k}{\epsilon_k}\right]$ and for each $k \in \mathcal{N}, \psi_{n,k} \to \psi_k$ as $n \to \infty$ in $\mathcal{L}$.* For further discussion; see, for example, $[1-7, 9, 11-13, 15-16]$.

4 The Space $\mathcal{B}\left(\mathcal{S}, \tilde{\mathcal{G}}, \tilde{\Delta}, \star\right)$ of General Boehmians.

In the present section, we establish necessary and sufficient results required for construting the desired space $\mathcal{B}\left(\mathcal{S}, \tilde{\mathcal{G}}, \tilde{\Delta}, \star\right)$ of general Boehmians.

Theorem 4.1. (The Convolution Theorem) *Let $\psi \in \mathcal{S}(\mathcal{H})$ and $\varphi \in \mathcal{D}(\mathcal{R})$ then*

$$\mathcal{F}(\psi(\alpha, \sigma) \bullet d_\beta \varphi(\sigma))(\varsigma) = \mathcal{F}\psi(\alpha, \varsigma)\, \mathcal{F}(d_\beta \varphi)(\varsigma) \tag{12}$$

where $\mathcal{H} = \mathcal{R}_{+1} \times \mathcal{R}$.

Proof For every $\psi \in \mathcal{S}(\mathcal{H})$ and $\varphi \in \mathcal{D}(\mathcal{R})$, Equ.(8) implies that

$$\mathcal{F}(\psi(\alpha,\sigma) \bullet d_\beta\varphi(\sigma))(\varsigma) = \int_{\mathcal{R}} d_\beta\varphi(\sigma) \int_{\mathcal{R}} \psi(\alpha,\sigma - t) e^{i\varsigma\sigma} dt d\sigma.$$

The substitution $\sigma = t + y$ completes the proof of the theorem. Since delta sequences (approximate identities) correspond to the dirac delta distribution it can be easily established that :

Lemma 4.2. (**i**) $\mathcal{F}\epsilon_n(\varsigma) \to 1$ *as* $n \to \infty$, $\forall (\epsilon_n) \in \Delta$.

(**ii**) $\mathcal{F}(d_\beta\epsilon_n)(\varsigma) \to 1$ *as* $n \to \infty, \forall\beta \in \mathcal{R}_{+1}, (\epsilon_n) \in \Delta$.
(**iii**) $\operatorname{supp}(\epsilon_n \bullet \tau_n) \subset \operatorname{supp}\epsilon_n + \operatorname{supp}\tau_n, \forall n \in \mathcal{N}$.
Let $\tilde{\Delta} = \{(\tilde{\epsilon}_n) : (\epsilon_n) \in \Delta\}$ and $\tilde{\mathcal{G}} = \{\mathcal{F}\varphi : \varphi \in \mathcal{G}\}$.
We define an operation $\star$ by

$$\psi(\alpha,\varsigma) \star \mathcal{F}\varphi(\varsigma) = \psi(\alpha,\varsigma)\mathcal{F}(d_\beta\varphi)(\varsigma), \tag{13}$$

for every $\psi \in \mathcal{S}(\mathcal{H})$ and $\mathcal{F}\varphi \in \tilde{\mathcal{G}}$.

In particular, if $\psi \in \mathcal{S}(\mathcal{R})$ and $\mathcal{F}\varphi \in \tilde{\mathcal{G}}$, then $\star$ can be interpreted to mean

$$\psi(\varsigma) \star \mathcal{F}\varphi(\varsigma) = \psi(\varsigma)\mathcal{F}\varphi(\varsigma) \tag{14}$$

Lemma 4.3 . *Let* $\psi \in \mathcal{S}(\mathcal{H})$ *and* $\mathcal{F}\varphi \in \tilde{\mathcal{G}}$ *then* $\psi(\alpha,\varsigma) \star \mathcal{F}\varphi(\varsigma) \in \mathcal{S}(\mathcal{H})$.

Proof Since $d_\beta\varphi \in \mathcal{G}$, by (r_4), it follows $\mathcal{F}(d_\beta\varphi) \in \tilde{\mathcal{G}}$. But the fact $\mathcal{G} \subset \mathcal{D}(\mathcal{R}) \subset \mathcal{S}(\mathcal{R})$ and $\mathcal{F}$ maps $\mathcal{S}(\mathcal{R})$ into $\mathcal{S}(\mathcal{R})$, by Theorem 1.1, implies

$$\psi(\alpha,\varsigma) \star \mathcal{F}\varphi(\varsigma) \in \mathcal{S}(\mathcal{H}).$$

This completes the proof of the lemma.

Lemma 4.4. *Let* $(\tilde{\tau}_n), (\tilde{\epsilon}_n) \in \tilde{\Delta}$ *then* $(\tilde{\tau}_n \star \tilde{\epsilon}_n) \in \tilde{\Delta}$ *and* $\tilde{\tau}_n \star \tilde{\epsilon}_m = \tilde{\epsilon}_m \star \tilde{\tau}_n$, *for every* $n, m \in \mathcal{N}$.

Proof Let $(\tilde{\tau}_n), (\tilde{\epsilon}_n) \in \tilde{\Delta}$ then, using Equ. (16) and Equ.(5) we get $\tilde{\tau}_n(\varsigma) \star \tilde{\epsilon}_n(\varsigma) = \mathcal{F}(\tau_n \bullet \epsilon_n)$. But $\tau_n \bullet \epsilon_n \in \Delta$. Hence $\tilde{\tau}_n \star \tilde{\epsilon}_n \in \tilde{\Delta}$.

Next, in Δ we have $\tau_n \bullet \epsilon_m = \epsilon_m \bullet \tau_n$. Applying the Fourier transform on both sides and using Equ.(5) yields

$$\tilde{\tau}_n(\varsigma)\tilde{\epsilon}_m(\varsigma) = \tilde{\epsilon}_m(\varsigma)\tilde{\tau}_n(\varsigma). \tag{15}$$

From Equ.(14), Equ.(15) yields $\tilde{\tau}_n(\varsigma) \star \tilde{\epsilon}_m(\varsigma) = \tilde{\epsilon}_m(\varsigma) \star \tilde{\tau}_n(\varsigma)$. The proof is therefore completed.

Lemma 4.5. *Let* $\psi(\alpha,\varsigma), \kappa(\beta,\varsigma) \in \mathcal{S}(\mathcal{H})$ *and* $(\tilde{\epsilon}_n) \in \tilde{\Delta}$ *be such that* $\psi(\alpha,\varsigma) \star \tilde{\epsilon}_n(\varsigma) = \kappa(\beta,\varsigma) \star \tilde{\epsilon}_n(\varsigma)$ *then* $\psi = \kappa$ *for every* $\varsigma \in \mathcal{R}, \alpha, \beta \in \mathcal{R}_{+1}$.

Proof From the hypothesis of the lemma and Equ.(13) we write

$$(\psi(\alpha,\varsigma)-\kappa(\beta,\varsigma))\star\tilde{\epsilon}_n(\varsigma)=0$$

for all $n\in\mathcal{N}$. Upon allowing $n\to\infty$ and using Part (i) of Lemma 4.2., we get $\psi(\alpha,\varsigma)=\kappa(\beta,\varsigma)$ for all $\alpha,\beta\in\mathcal{R}_{+1}$ and $\varsigma\in\mathcal{R}$.

This completes the proof of the Lemma.

Lemma 4.6. *Let $\psi,\kappa\in\mathcal{S}(\mathcal{H})$ and $(\tilde{\epsilon}_n)\in\tilde{\Delta}$ then the following holds*

(1) $(\psi(\alpha,\varsigma)+\kappa(\beta,\varsigma))\star\tilde{\epsilon}_n=\psi(\alpha,\varsigma)\star\tilde{\epsilon}_n+\kappa(\beta,\varsigma)\star\tilde{\epsilon}_n$, *for all* $n\in\mathcal{N}$.
(2) $k\psi(\alpha,\varsigma)\star\tilde{\epsilon}_n=\psi(\alpha,\varsigma)\star k\tilde{\epsilon}_n=k(\psi(\alpha,\varsigma)\star\tilde{\epsilon}_n)$, *for all* $n\in\mathcal{N},k\in\mathcal{R}$.
Proof is straightforward consequence from definitions.

Lemma 4.7. *Let $(\tilde{\epsilon}_n),(\tilde{\tau}_n)\in\tilde{\Delta}$ and $\psi(\alpha,\varsigma)\in\mathcal{S}(\mathcal{H})$ then we have*

$$(\psi(\alpha,\varsigma)\star\tilde{\epsilon}_n(\varsigma))\star\tilde{\tau}_n(\varsigma)=\psi(\alpha,\varsigma)\star(\tilde{\epsilon}_n(\varsigma))\star(\tilde{\tau}_n(\varsigma)).$$

Proof of this lemma can easily be established from Equ.(13) and Equ.(14).

Lemma 4.8. *Let $\psi_n\to\psi\in\mathcal{S}(\mathcal{H})$ as $n\to\infty$ and $\mathcal{F}\varphi\in\tilde{\mathcal{G}}$ then*

$$\psi_n(\alpha,\varsigma)\star\mathcal{F}\varphi(\varsigma)\to\psi(\alpha,\varsigma)\star\mathcal{F}\varphi(\varsigma)$$

as $n\to\infty$.

Proof Let $\psi_n\to\psi$ in $\mathcal{S}(\mathcal{H})$ and $\mathcal{F}\varphi\in\tilde{\mathcal{G}}$ then, from Equ.(6) and Equ.(13),

$$\begin{aligned}\left|\varsigma^m\mathcal{D}_\varsigma^k(\psi_n\star\mathcal{F}\varphi-\psi\star\mathcal{F}\varphi)\right| &= \left|\varsigma^m\mathcal{D}_\varsigma^k((\psi_n-\psi)\star\mathcal{F}\varphi)\right|\\ &= \left|\varsigma^m\mathcal{D}_\varsigma^k((\psi_n-\psi)(\alpha,\varsigma)\mathcal{F}(d_\beta\varphi)(\varsigma))\right|\\ &\le M\left|\varsigma^m\mathcal{D}_\varsigma^k(\psi_n(\alpha,\varsigma)-\psi(\alpha,\varsigma))\right|\\ &\to 0 \text{ as } n\to\infty,\ 0<M\in\mathcal{R}. \end{aligned} \tag{16}$$

Hence, from Equ.(16), we get $\gamma_{m,k}((\psi_n-\psi)\star\mathcal{F}\varphi)\to 0$ as $n\to\infty$ for every $m,k\in\mathcal{N}$. The proof is, thus, completed.

Lemma 4.9. *Let $\psi_n\to\psi$ in $\mathcal{S}(\mathcal{H})$ as $n\to\infty$ then $\psi_n(\alpha,\varsigma)\star\tilde{\epsilon}_n(\varsigma)\to\psi(\alpha,\varsigma)$* as *$n\to\infty$ for every $(\tilde{\epsilon}_n)\in\tilde{\Delta},\alpha\in\mathcal{R}_{+1}$.*

Proof of this Lemma is an obvious consequence from Lemma 4.2. The desired Boehmian space $\mathcal{B}\left(\mathcal{S},\tilde{\mathcal{G}},\tilde{\Delta},\star\right)$ is therefore described and, in short, denoted by $\mathcal{B}$. We define addition and scalar multiplication in $\mathcal{B}\left(\mathcal{S},\tilde{\mathcal{G}},\tilde{\Delta},\star\right)$ by

$$\left[\frac{\psi_n}{\tilde{\epsilon}_n}\right]+\left[\frac{\kappa_n}{\tilde{\tau}_n}\right]=\left[\frac{\psi_n\star\tilde{\tau}_n+\kappa_n\star\tilde{\epsilon}_n}{\tilde{\epsilon}_n\star\tilde{\tau}_n}\right]$$

and

$$k\left[\frac{\psi_n}{\tilde{\epsilon}_n}\right]=\left[\frac{k\psi_n}{\tilde{\epsilon}_n}\right], k\in\mathcal{R}.$$

The operation $\star$ and differentiation of Boehmians in $\mathcal{B}$ are defined by

$$\left[\frac{\psi_n}{\tilde{\epsilon}_n}\right]\star\left[\frac{\kappa_n}{\tilde{\tau}_n}\right]=\left[\frac{\psi_n\star\kappa_n}{\tilde{\epsilon}_n\star\tilde{\tau}_n}\right] \text{ and } \frac{d^k}{d\varsigma^k}\left[\frac{\psi_n}{\tilde{\epsilon}_n}\right]=\left[\frac{\frac{d^k}{d\varsigma^k}\psi_n}{\tilde{\epsilon}_n}\right].$$

Theorem 4.9. *The mapping $\mathcal{S}_t\to\mathcal{B}\left(\mathcal{S},\tilde{\mathcal{G}},\tilde{\Delta},\star\right)$ defined by $i(\psi)=\left[\frac{\psi\star\tilde{\epsilon}_n}{\tilde{\epsilon}_n}\right]$ is a continuous imbedding mapping.*

Proof Let $i(\psi)=i(\kappa)$ then $\left[\frac{\psi\star\tilde{\epsilon}_n}{\tilde{\epsilon}_n}\right]=\left[\frac{\kappa\star\tilde{\tau}_n}{\tilde{\tau}_n}\right]$. Using the concept of equivalence classes in $\mathcal{B}\left(\mathcal{S},\tilde{\mathcal{G}},\tilde{\Delta},\star\right)$ we get $(\psi\star\tilde{\epsilon}_n)\star\tilde{\tau}_m=(\kappa\star\tilde{\tau}_m)\star\tilde{\epsilon}_n$ and, in particular, $(\psi\star\tilde{\epsilon}_n)\star\tilde{\tau}_n=(\kappa\star\tilde{\tau}_n)\star\tilde{\epsilon}_n$. Considering the limit as $n\to\infty$ and using Lemma 4.2 we establish that $\psi=\kappa$. To derive continuity of the mapping i, let $\psi(\alpha,\varsigma)\to 0$ as $n\to\infty$ then $\psi(\alpha,\varsigma)\star\tilde{\epsilon}_n(\varsigma)\to 0$ as $n\to\infty$. Hence $\left[\frac{\psi\star\tilde{\epsilon}_n}{\tilde{\epsilon}_n}\right]\to 0$ as $n\to\infty$. This completes the proof of the theorem.

5 The Extended Fourier Transform for Boehmians.

Let $\beta\in\mathcal{S}_t$ be such that $\beta=\left[\frac{\psi_n}{\epsilon_n}\right]$ then we define its extended Fourier transform by

$$\tilde{\beta}:=\mathcal{F}\beta:=\left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right]. \tag{17}$$

in $\mathcal{B}$.

Theorem 5.1. *$\mathcal{F}\beta:\mathcal{S}_t\to\mathcal{B}$ is a well-defined mapping.*

Proof Let $\beta_1=\beta_2\in\mathcal{S}_t$ be such that $\beta_1=\left[\frac{\psi_n}{\epsilon_n}\right]$ and $\beta_2=\left[\frac{\kappa_n}{\tau_n}\right]$ then $\psi_n(\alpha,\sigma)\bullet d_\beta\tau_m(\sigma)=\kappa_m(\beta,\sigma)\bullet d_\alpha\epsilon_n(\sigma)$. Applying the Fourier transform on both sides and employing Theorem 4.1 yield

$$\mathcal{F}\psi_n(\alpha,\varsigma)\,\mathcal{F}(d_\beta\tau_m)(\varsigma)=\mathcal{F}\kappa_m(\beta,\varsigma)\,\mathcal{F}(d_\alpha\epsilon_n)(\varsigma). \tag{18}$$

Hence, from Equ.(13), Equ.(18) becomes $\mathcal{F}\psi_n(\alpha,\varsigma)\star\tilde{\tau}_m(\varsigma)=\mathcal{F}\kappa_m(\beta,\varsigma)\star\tilde{\epsilon}_n(\varsigma)$. This implies

$$\frac{\mathcal{F}\psi_n(\alpha,\varsigma)}{\tilde{\epsilon}_n(\varsigma)}\sim\frac{\mathcal{F}\kappa_n(\beta,\varsigma)}{\tilde{\tau}_n(\varsigma)}.$$

Therefore, $\left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right]=\left[\frac{\mathcal{F}\kappa_n}{\tilde{\tau}_n}\right]$. The proof is completed.

Theorem 5.2. *The mapping $\mathcal{F}\beta:\mathcal{S}_t\to\mathcal{B}$ is one-one.*

Proof Let $\mathcal{F}\beta_1 = \mathcal{F}\beta_2$, in $\mathcal{B}$, where $\mathcal{F}\beta_1 = \left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right]$ and $\mathcal{F}\beta_2 = \left[\frac{\mathcal{F}\kappa_n}{\tilde{\tau}_n}\right]$. Then

$$\mathcal{F}\psi_n(\alpha,\varsigma) \star \tilde{\tau}_m(\varsigma) = \mathcal{F}\kappa_m(\beta,\varsigma) \star \tilde{\epsilon}_n(\varsigma).$$

Upon employing Equ.(13) we get

$$\mathcal{F}\psi_n(\alpha,\varsigma)\,\mathcal{F}(d_\beta\tau_m)(\varsigma) = \mathcal{F}\kappa_m(\beta,\varsigma)\,\mathcal{F}(d_\alpha\epsilon_n)(\varsigma).$$

By Theorem 4.1,

$$\mathcal{F}(\psi_n(\alpha,\sigma) \bullet d_\beta\tau_m(\sigma))(\varsigma) = \mathcal{F}(\kappa_m(\beta,\sigma) \bullet d_\alpha\epsilon_n(\sigma))(\varsigma).$$

Since $\mathcal{F}$is one-one mapping, we get

$$\psi_n(\alpha,\sigma) \bullet d_\beta\tau_m(\sigma) = \kappa_m(\beta,\sigma) \bullet d_\alpha\epsilon_n(\sigma).$$

Hence

$$\frac{\psi_n(\alpha,\sigma)}{\epsilon_n(\sigma)} \sim \frac{\kappa_n(\beta,\sigma)}{\tau_n(\sigma)}$$

That is, $\beta_1 = \beta_2$. Hence, the theorem is proved.

With this analysis, the inverse Fourier transform $\mathcal{F}^{-1}\tilde{\beta}$ can be recovered from $\tilde{\beta} \in \mathcal{B}$ by

$$\mathcal{F}^{-1}\tilde{\beta} = \left[\frac{\psi_n}{\epsilon_n}\right] \tag{19}$$

in $\mathcal{S}$, where $\tilde{\beta} = \left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right]$.

Theorem 5.3. *The mapping $\mathcal{F}^{-1}\tilde{\beta} : \mathcal{B} \to \mathcal{S}_t$ is well-defined.*

Proof is analogous to that of Theorem 5.1.

Theorem 5.4. *The mapping $\mathcal{F}^{-1}\tilde{\beta} : \mathcal{B} \to \mathcal{S}_t$ is one-one.*

Proof Let $\tilde{\beta}_1 = \tilde{\beta}_2 \in \mathcal{B}$ such that $\tilde{\beta}_1 = \left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right]$ and $\tilde{\beta}_2 = \left[\frac{\mathcal{F}\kappa_n}{\tilde{\tau}_n}\right]$ then

$$\mathcal{F}\psi_n(\alpha,\varsigma) \star \tilde{\tau}_m(\varsigma) = \mathcal{F}\kappa_m(\alpha,\varsigma) \star \tilde{\epsilon}_n(\varsigma).$$

By aid of Equ.(13), we have

$$\mathcal{F}\psi_n(\alpha,\varsigma)\,\mathcal{F}(d_\beta\tau_m)(\varsigma) = \mathcal{F}\kappa_m(\beta,\varsigma)\,\mathcal{F}(d_\alpha\epsilon_n)(\varsigma).$$

Now, Theorem 4.1 implies

$$\mathcal{F}(\psi_n(\alpha,\sigma) \bullet d_\beta\tau_m(\sigma))(\varsigma) = \mathcal{F}(\kappa_m(\beta,\sigma) \bullet d_\alpha\tau_n(\sigma))(\varsigma).$$

Therefore,

$$\psi_n(\alpha,\sigma) \bullet d_\beta\tau_m(\sigma) = \kappa_m(\beta,\sigma) \bullet d_\alpha\epsilon_n(\sigma).$$

That is

$$\left[\frac{\psi_n}{\epsilon_n}\right] = \left[\frac{\kappa_n}{\tau_n}\right].$$

Hence, $\mathcal{F}^{-1}\tilde{\beta}_1 = \mathcal{F}^{-1}\tilde{\beta}_2$. This completes the Proof of the theorem.

Theorem 5.5. *$\mathcal{F}\beta : \mathcal{S}_t \to \mathcal{B}$ and $\mathcal{F}^{-1}\tilde{\beta} : \mathcal{B} \to \mathcal{S}_t$ are linear.*

Proof Let $\beta_1, \beta_2 \in \mathcal{S}$ and $k \in \mathcal{R}$ be such that $\beta_1 = \left[\frac{\psi_n}{\epsilon_n}\right], \beta_2 = \left[\frac{\kappa_n}{\tau_n}\right]$. Addition of Boehmians implies

$$\beta_1 + \beta_2 = \left[\frac{\psi_n \bullet d_\beta \tau_n + \kappa_n \bullet d_\alpha \epsilon_n}{\epsilon_n \bullet \tau_n}\right] \tag{20}$$

Employing Equ.(17) on Equ.(20) , we get

$$\mathcal{F}(\beta_1 + \beta_2) = \left[\frac{\mathcal{F}\left(\psi_n \bullet d_\beta \tau_n + \kappa_n \bullet d_\alpha \epsilon_n\right)}{\mathcal{F}(\epsilon_n \bullet \tau_n)}\right].$$

Thus, by Theorem 4.1 and linearity of the Fourier transform we get

$$\mathcal{F}(\beta_1 + \beta_2) = \left[\frac{\mathcal{F}\psi_n(\alpha,\varsigma)\mathcal{F}\left(d_\beta \tau_n\right)(\varsigma) + \mathcal{F}\kappa_n(\beta,\varsigma)\mathcal{F}(d_\alpha \epsilon_n)(\varsigma)}{\mathcal{F}\epsilon_n(\varsigma)\mathcal{F}\tau_n(\varsigma)}\right]$$

Equ.(13) implies

$$\begin{aligned}\mathcal{F}(\beta_1 + \beta_2) &= \left[\frac{\mathcal{F}\psi_n(\alpha,\varsigma)\star\tilde{\tau}_n(\varsigma) + \mathcal{F}\kappa_n(\beta,\varsigma)\star\tilde{\epsilon}_n(\varsigma)}{\tilde{\epsilon}_n \star \tilde{\tau}_n(\varsigma)}\right] \\ &= \left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right] + \left[\frac{\mathcal{F}\kappa_n}{\tilde{\tau}_n}\right]\end{aligned}$$

Further, if $k \in \mathcal{R}$ then

$$k(\mathcal{F}\beta_1) = k\left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right] = \left[\frac{k\mathcal{F}\psi_n}{\tilde{\epsilon}_n}\right].$$

Hence, $\mathcal{F}\beta : \mathcal{S}_t \to \mathcal{B}$ is linear. Proof of linearity of $\mathcal{F}^{-1}\tilde{\beta} : \mathcal{B} \to \mathcal{S}_t$ is analogous. This completes the proof of the theorem.

Theorem 5.6. *$\mathcal{F}\beta : \mathcal{S}_t \to \mathcal{B}$ is continuous with respect to δ convergence.*

Proof Let $\beta_n \xrightarrow{\delta} \beta$ in $\mathcal{S}_t$ as $n \to \infty$ then, there are $\psi_{n,k}$ and ψ_k in $\mathcal{S}(\mathcal{H})$ such that

$$\beta_n = \left[\frac{\psi_{n,k}}{\epsilon_k}\right] \text{ and } \beta = \left[\frac{\psi_k}{\epsilon_k}\right]$$

where $\psi_{n,k} \to \psi_k$ as $n \to \infty$ for every $k \in \mathcal{N}$. The continuity condition of the Fourier transform implies $\mathcal{F}\psi_{n,k} \to \mathcal{F}\psi_k$ as $n \to \infty$, in $\mathcal{S}(\mathcal{H})$. Thus, $\left[\frac{\mathcal{F}\psi_{n,k}}{\tilde{\epsilon}_n}\right] \to \left[\frac{\mathcal{F}\psi_k}{\tilde{\epsilon}_k}\right]$ as $n \to \infty$. Hence the theorem.

Theorem 5.7. *$\mathcal{F}\beta : \mathcal{S}_t \to \mathcal{B}$ is continuous with respect to Δ convergence.*

Proof Let $\beta_n \xrightarrow{\Delta} \beta$ in $\mathcal{S}_t$ as $n \to \infty$. Then, we find $\psi_n \in \mathcal{S}(\mathcal{H})$ and $(\epsilon_n) \in \Delta$ such that $(\beta_n - \beta) \bullet \epsilon_n = \left[\frac{\psi_n \bullet \epsilon_k}{\epsilon_k}\right]$ and $\psi_n \to 0$ as $n \to \infty$. Hence, we have $\mathcal{F}((\beta_n - \beta) \bullet \epsilon_n) = \left[\frac{\mathcal{F}(\psi_n \bullet \epsilon_k)}{\tilde{\epsilon}_k}\right] := \mathcal{F}\psi_n \to 0$ as $n \to \infty$ in $\mathcal{S}(\mathcal{H})$. Therefore

$$\mathcal{F}((\beta_n - \beta) \bullet \epsilon_n) = (\mathcal{F}\beta_n - \mathcal{F}\beta) \star \tilde{\epsilon}_n \to 0 \text{ as } n \to \infty.$$

Hence, $\mathcal{F}\beta_n \xrightarrow{\Delta} \mathcal{F}\beta$ as $n \to \infty$.

Comparative Study: By observing the structure of the Boehmian spaces $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ and $\mathcal{S}_t$, it worths noting that, those spaces can be satisfactory interchanged in the preceding analysis. In this case, the extended Fourier transform of a Boehmian in $\mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ is a well defined mapping satisfying its desired properties:

Definition 5.8. *Let* $\left[\frac{\psi(n,.)}{\epsilon}\right] \in \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ *then its Fourier transform* $\hat{\mathcal{F}}$ *is defined by*

$$\hat{\mathcal{F}}\left[\frac{\psi}{\epsilon}\right] = \left[\frac{\mathcal{F}\omega_n(.)}{\tilde{\varepsilon}_n(.)}\right] \tag{21}$$

where $\omega_n(.) = \psi(n,.)$ *and* $\varepsilon_n(.) = d_n\epsilon(.)$.

From the context, the righthand side of Equ.(21) is meaningful in the sense of general Boehmians.

Theorem 5.9. $\hat{\mathcal{F}} : \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet) \to \mathcal{B}(\mathcal{S},\mathcal{D},\Delta,\star)$ *is well-defined.*

Proof Let $\left[\frac{\psi(n,.)}{\epsilon}\right] = \left[\frac{\varphi(n,.)}{\sigma}\right]$ then $\psi(n,.) \bullet d_m\sigma(.) = \varphi(m,.) \bullet d_n\epsilon(.)$. Employing the Fourier transform and Theorem 4.1. we get $\mathcal{F}\psi(n,.)\,\mathcal{F}(d_m\sigma(.)) = \mathcal{F}\varphi(m,.)\,\mathcal{F}(d_n\epsilon(.))$.

Hence,

$$\mathcal{F}\omega_n(.)\,\tilde{\sigma}_m = \mathcal{F}\varphi_m(.)\,\tilde{\varepsilon}_n,$$

where $\omega_n(.) = \psi(n,.), d_n\sigma(.) = \sigma_n(.) \in \Delta, \varepsilon_n(.) = d_n\epsilon(.) \in \Delta$ and $\varphi(m,.) = \varphi_m(.)$. Equ.(15) implies $\mathcal{F}\omega_n(.) \star \tilde{\sigma}_m = \mathcal{F}\varphi_m(.) \star \tilde{\varepsilon}_n$. Therefore

$$\hat{\mathcal{F}}\left[\frac{\psi(n,.)}{\epsilon}\right] = \hat{\mathcal{F}}\left[\frac{\varphi(n,.)}{\sigma}\right]$$

The theorem is thus completed.

Theorem 5.10. $\hat{\mathcal{F}} : \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet) \to \mathcal{B}(\mathcal{S},\mathcal{D},\Delta,\star)$ *is one-one mapping.*

See Theorem 5.2. for similar proof.

Theorem 5.11. *The mapping* $\hat{\mathcal{F}} : \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet) \to \mathcal{B}(\mathcal{S},\mathcal{D},\Delta,\star)$ *is linear.*

Proof is straightforward consequence from definitions.

Theorem 5.12. *The mapping* $\hat{\mathcal{F}} : \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet) \to \mathcal{B}(\mathcal{S},\mathcal{D},\Delta,\star)$ *is continuous with respect to* G *convergence.*

Proof Assume for each $\beta \in \mathcal{S}_t(\mathcal{S},\mathcal{G},\bullet)$ and $\psi_n, \psi \in \mathcal{S}(\mathcal{H}), \epsilon \in \mathcal{G}(\mathcal{R})$ such that $\beta_n = \left[\frac{\psi_n}{\epsilon}\right], \beta = \left[\frac{\psi}{\epsilon}\right]$, $\psi_n \to \psi$ as $n \to \infty$. Then, continuity of the Fourier transform yields $\mathcal{F}\psi_n \to \mathcal{F}\psi$. Hence, $\left[\frac{\mathcal{F}\psi_n}{\tilde{\epsilon}}\right] \to \left[\frac{\mathcal{F}\psi}{\tilde{\epsilon}}\right]$ as $n \to \infty$. This completes the proof of the theorem. Rest of results can be treated accordingly.

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Intrinsic Supermoothness

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Abstract

The phenomenon, known as"supersmoothness" was first observed for bivariate splines and attributed to the polynomial nature of splines. We generalize this phenomenon to smooth functions and higher order derivatives. Moreover, we show that locally supersmoothness characterizes non-smooth curves.

AMS classification: Primary 26B05, Secondary 26B35

Key words and phrases: supersmoothness, piecewise bivariate function, polynomial splines, smooth curve

1 Introduction

In this short article we address supersmoothness: a phenomenon where under certain circumstances continuity of a function of two variables implies its differentiability at a point or, consequently, differentiability of a bivariate function implies its higher order differentiability at a point. Supersmoothness was first observed for bivariate splines by Farin in [2]. He considered a triangle Δ partitioned into three subtriangles Δ_1, Δ_2 and Δ_3 as shown in Figure 1. A spline F on this triangulation of Δ is a function of two variables such that for each $i = 1, 2, 3$, the restriction $F\,|_{\Delta_i} = f_i$ a polynomial. Farin proved that if the spline F is differentiable of order n, then it has all $(n+1)$st order partial derivatives at the origin $\mathbf{0} := (0, 0)$. That is for all $n \geq 1$:

$$F \in C^n(\Delta) \Rightarrow F \in C^{n+1}(\mathbf{0}). \tag{1}$$

Supersmoothness of splines was observed for trivariate splines in [1], and studied in general in [4] and in [5]. This phenomenon has been attributed to the polynomial nature of splines.

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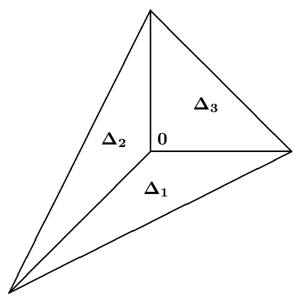

Figure 1: First example

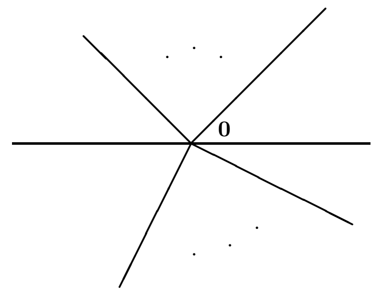

Figure 2: Collinearity matters

In the next section we will demonstrate that basic supersmoothness is a rather general property of non-smooth curves, not just polynomials. Loosely speaking: if we want to continuously glue two smooth bivariate functions along a curve with a "corner" at a point P, the resulting continuous function must be differentiable at P, as if to compensate for the singularity of the curve. Moreover, locally, supersmoothness characterizes non-smooth curves.

In Section 3 we address another peculiarity of supersmoothness. We first show that property (1) holds for all smooth functions defined over a partition of $\mathbb{R}^2$ by $n+2$ non-collinear rays emanating from the origin, $n \geq 0$. The assumption that the rays are not collinear is significant. If just two of the rays are parallel the phenomenon of automatic supersmoothness disappears alltogether. This can be seen on the following simple example. Consider the partition of $\mathbb{R}^2$ by the x-axis. For any $n \geq 0$, let $f(x, y)$ be equal to y^{n+1} on the upper half plane and zero on the lower one. We now can add any n rays emanating from the origin but not along the x-axis. This will form a partition of $\mathbb{R}^2$ by $n+2$ rays as in Figure 2. Then f has all derivatives of order n, yet $f \notin C^{n+1}(\mathbf{0})$.

2 Gluing functions along a curve

In this section we will show that a version of supersmoothness occurs when we glue two differentiable functions along a curve with sharp corners as in Figure 3. Namely, we will show that the resulting piecewise function is differentiable at every sharp corner of the curve. To some extent this property of supersmoothness characterizes curves with sharp corners.

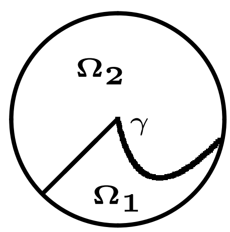

Figure 3: Curves for gluing

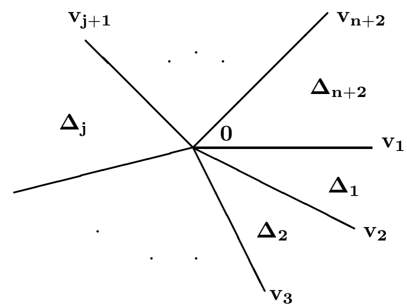

Figure 4: Supersmoothness of higher derivatives

In contrast with most of the research on curves in both analysis and differential geometry, we are interested in non-smooth curves. While regularity of a curve is defined globally, non-smoothness has to be localized to a point $P := (x_0, y_0)$.

We limit our considerations to non-self-intersecting curves and adopt the following, intuitively clear version of "local smoothness". Let γ be the trace of a continuous non-self-intersecting curve $\gamma(t) : [a, b] \to \mathbb{R}^2$, also known as a Jordan arc. Without loss of generality assume that $\gamma(0) = P$ and $a < 0 < b$.

We shall say that γ is smooth at P if γ can be represented as a graph of a continuously differentiable function in some neighborhood of P. More precisely,

Definition 1 *The trace of a Jordan arc γ is smooth at a point P if there exist open intervals I, J and a function $f \in C^1(I)$ such that*

$$P = (x_0, f(x_0)) \in I \times J, \quad \textit{and} \quad \gamma \cap (I \times J) = \{(x, f(x)), \quad x \in I\}.$$

Different (weaker) versions of the following result can be found in many textbooks. We provide the proof for completeness.

Theorem 2 *The trace of a Jordan arc γ is smooth at P if and only if there exists a neighborhood U of P and a function h continuously differentiable* *on* U *such that*

$$h(x,y)=0 \quad \textit{if} \quad (x,y)\in\gamma\cap U, \quad \textit{and} \quad \nabla h(P)\neq \mathbf{0}. \tag{2}$$

Proof. If γ is smooth at P we use the neighborhood $I\times J$ and the C^1-continuous function f from Definition 1 to construct $h(x,y):=y-f(x)$. Clearly, h satisfies all the desirable properties. Conversely, without loss of generality, assume $P=(0,0)$ and let h be a C^1-continuous function on some neighborhood U of P, such that h vanishes on γ, and $h_y(P)\neq 0$. Then by the Implicit Function Theorem, there exist open intervals I_1 and J_1 and a C^1-continuous function f such that

$$h(x,y)=0, \quad (x,y)\in\ I_1\times J_1 \ \text{ iff } \ y=f(x), \quad x\in I_1, \quad y\in J_1.$$

We can assume that $I_1\times J_1\subset U$, which implies that

$$\gamma\cap(I_1\times J_1)\subseteq\{(x,f(x)), \quad x\in I_1\}.$$

We now need to show that there exist perhaps other intervals $I\subseteq I_1$ and $J\subseteq J_1$ such that γ coincides with the graph of f in $I\times J$. To this end, we consider the inverse image $\gamma^{-1}(I_1\times J_1)$, which is an open set in $[a,b]$ containing zero. Thus, there exists $c>0$ such that $\gamma(t):=(u(t),v(t))$ maps $(-c,c)$ into $\gamma\cap(I_1\times J_1)$. We observe that if $u(c/2)=0$, then $v(c/2)$ also vanishes since f is a function passing through $(0,0)$. Then we have $\gamma(c/2)=\gamma(0)$ which contradicts the assumption that γ has no self-intersections. Thus, neither $u(c/2)$ nor $u(-c/2)$ vanish. Without loss of generality we can assume $u(c/2)>0$. Then $u(-c/2)$ must be negative. Otherwise either γ has a self-intersection or f is not a function. Since $\gamma(t)$ is continuous, its trace from $t=-c/2$ to $t=c/2$ must coincide with the graph of f from $u(-c/2)<0$ to $u(c/2)>0$. Thus, for $I:=\big(u(-c/2),u(c/2)\big)$, and $J:=J_1$, the function f satisfies Definition 1. ■

As a corollary we obtain the promised result on supersmoothness:

Theorem 3 *Let* $\gamma\subset\mathbb{R}^2$ *be the trace of a Jordan arc that divides the open disk* Ω *into two subsets* Ω_1 *and* Ω_2 *as in Figure 3. Further assume that* γ *is not smooth at* $P\in\gamma$. *Let* f_1,f_2 *be* C^1 *functions on* Ω *continuously glued along* γ, *that is, let*

$$F(x,y):=\begin{cases} f_1(x,y) & \textit{if} \quad (x,y)\in\Omega_1\\ f_2(x,y) & \textit{if} \quad (x,y)\in\Omega_2\end{cases} \tag{3}$$

be a continuous function on Ω. *Then the piecewise function* F *is differentiable at* P, *that is,*

$$\nabla f_1(P)=\nabla f_2(P). \tag{4}$$

Proof. Consider a C^1 function $h = f_1 - f_2$. The fact that f_1 and f_2 are continuously glued along γ means that $h(\gamma) = 0$ and by Theorem 2

$$0 = \nabla h(P) = \nabla f_1(P) - \nabla f_2(P).$$

Thus, $\nabla f_1(P) = \nabla f_2(P)$, and the proof is complete. ■

A partial converse of Theorem 3 holds true in the following sense:

Theorem 4 *Let $\gamma \subset \mathbb{R}^2$ be the trace of a Jordan arc that divides the open disk Ω into two subsets Ω_1 and Ω_2. Assume that γ is smooth at a point $P \in \gamma$. Then there exists a neighborhood U of P and and two differentiable functions $f_1, f_2 \in C^1(U)$ such that the function*

$$F(x,y) := \begin{cases} f_1(x,y) & \text{if} \quad (x,y) \in \Omega_1 \cap U \\ f_2(x,y) & \text{if} \quad (x,y) \in \Omega_2 \cap U \end{cases} \tag{5}$$

is not differentiable at P.

Proof. Let U and h be chosen as in Theorem 2, i.e., satisfying conditions (2). Let $f_1(x,y) := h(x,y)$ and $f_2(x,y) \equiv 0$. Then, since $h(\gamma \cap U) = 0$, the function F defined by (5) is continuous and not differentiable at P because $\nabla f_2(P) = \mathbf{0} \neq \nabla f_1(P)$. ■

Theorem 4 provides only a partial converse of Theorem 3 because the function F is defined locally, in a neighborhood U of P, and not on all of Ω. We believe that the global version of this theorem also holds and end this section with a conjecture.

Conjecture 5 *Let $\gamma \subset \mathbb{R}^2$ be a continuous curve that divides an open disk Ω centered at P into two subsets Ω_1 and Ω_2. Then γ is smooth at P if and only if we can glue two continuously differentiable functions along the curve as in (5) so that the resulting piecewise function F is not differentiable at P.*

3 Supersmoothness of higher derivatives.

Consider two non-collinear rays v_1 and v_2 emanating from the origin in $\mathbb{R}^2$. The curve formed by these two rays is not smooth and partitions the open unit disk Ω into two sectors Δ_1 and Δ_2. It follows from the results of the previous section that two differentiable functions f_1 and f_2 continuously glued along the boundary of the sectors as in (5) produce a piecewise function F_2 differentiable at the origin:

$$F_2 \in C(\Omega) \;\Rightarrow\; F_2 \in C^1(\mathbf{0}). \tag{6}$$

Farin's observation (1) shows that for three pairwise non-collinear rays emanating from the origin and a piecewise function F_3 consisting of three differentiable pieces as in Figure 1 the following holds:

$$F_3 \in C^1(\Omega) \;\Rightarrow\; F_3 \in C^2(\mathbf{0}).$$

However, as it was pointed out in the introduction, for three non-collinear rays amplification (6) may not hold, that is, in general

$$F_3 \in C(\Omega) \quad \not\Rightarrow \; F_3 \in C^1(\mathbf{0}).$$

In this section we extend this pattern. For a fixed $n \geq 0$, we partition the open disk Ω into $n+2$ sectors $\Delta_1, \ldots, \Delta_{n+2}$, by pairwise non-collinear vectors (rays) $v_1, \ldots, v_{n+2}$, positioned clockwise as in Figure 4. Then we create a piecewise function F_{n+2} by gluing $n+2$ functions $f_1, \ldots, f_{n+2} \in C^n(\Omega)$ along the rays. Thus for $1 \leq j \leq n+1$, the sector Δ_j is formed by v_j and v_{j+1}, and the sector Δ_{n+2} is formed by v_{n+2} and v_1. We will show that similarly to (1) the following holds:

$$F_{n+2} \in C^n(\Omega) \quad \Rightarrow \; F_{n+2} \in C^{n+1}(\mathbf{0}); \tag{7}$$

yet the weaker assumption $F_{n+2} \in C^{n-1}(\Omega)$ may not imply the associated conclusion that $F_{n+2} \in C^n(\mathbf{0})$.

We start with a simple lemma that shows that two differentiable functions continuously glued along a ray v must be differentiable in the direction of v. We use D_v to denote the directional derivative in the direction of v.

Lemma 6 *Let $v = (a, b)$ be a unit vector in $\mathbb{R}^2$. Let f and g be continuously differentiable functions in an ε-neighborhood of the origin in $\mathbb{R}^2$ such that*

$$f(ta, tb) = g(ta, tb), \quad \textit{for all} \;\; t \in [0, \varepsilon). \tag{8}$$

Then

$$D_v f(ta, tb) = D_v g(ta, tb), \quad \textit{for all} \;\; t \in [0, \varepsilon).$$

Proof. It suffices to prove the result for $t = 0$. We obtain

$$\begin{aligned} D_v f(\mathbf{0}) &= \lim_{t\to 0} \frac{f(ta, tb) - f(\mathbf{0})}{t} = \lim_{t\to 0+} \frac{f(ta, tb) - f(\mathbf{0})}{t} \\ &\overset{\text{by }(8)}{=} \lim_{t\to 0+} \frac{g(ta, tb) - g(\mathbf{0})}{t} = \lim_{t\to 0} \frac{g(ta, tb) - g(\mathbf{0})}{t} = D_v g(\mathbf{0}), \end{aligned}$$

where the second and the fourth equalities follow from the continuity of $D_v f$ and $D_v g$, respectively. ■

We are now ready to prove statement (7). For brevity, we use $F := F_{n+2}$.

Theorem 7 *Let functions $f_1, \ldots, f_{n+2}$, be n times continuously differentiable on Ω and let F be defined piecewise on each sector Δ_j by $F|_{\Delta_j} := f_j$, $j = 1, \ldots, n+2$. If $F \in C^n(\Omega)$ then F has all derivatives of order $n+1$ at the origin; that is, $F \in C^{n+1}(\mathbf{0})$, $n \geq 0$.*

Proof. If $n = 0$, the proof is given in Theorem 3. Let $n \geq 1$. We will show that for two neighboring functions, say f_j and f_{j+1}, all partial derivatives of order $n+1$ coincide at the origin. Then for every $k = 0, \ldots, n$,

$$D_x^k D_y^{n-k} f_1(\mathbf{0}) = D_x^k D_y^{n-k} f_2(\mathbf{0}) = \ldots = D_x^k D_y^{n-k} f_{n+2}(\mathbf{0}),$$

which would prove the theorem. Without loss of generality we consider the neighboring functions f_1 and f_2. It is clearly enough to prove that

$$D_{v_2}^k D_{v_1}^{n-k} f_1(\mathbf{0}) = D_{v_2}^k D_{v_1}^{n-k} f_2(\mathbf{0}), \quad \text{for every} \quad k = 0, \ldots, n. \tag{9}$$

Observe that for $k \geq 1$, the assumption $F \in C^n(\Omega)$ implies that the functions $D_{v_2}^{k-1} D_{v_1}^{n-k} f_1$ and $D_{v_2}^{k-1} D_{v_1}^{n-k} f_2$ are continuously glued along the ray v_2. Hence, by Lemma 6 we obtain

$$D_{v_2}\big(D_{v_2}^{k-1} D_{v_1}^{n-k} f_1\big)(\mathbf{0}) = D_{v_2}\big(D_{v_2}^{k-1} D_{v_1}^{n-k} f_2\big)(\mathbf{0})$$

which implies (9) for $k \geq 1$. Hence it remains to prove that

$$D_{v_1}^n f_1(\mathbf{0}) = D_{v_1}^n f_2(\mathbf{0}). \tag{10}$$

Since all the vectors v_j are pairwise non-collinear we can find constants α_j and β_j such that $v_1 = \alpha_j v_2 + \beta_j v_j$ for all $j = 3, \ldots, n+2$. Then

$$\begin{aligned} D_{v_1}^n &= (\alpha_3 D_{v_2} + \beta_3 D_{v_3}) \ldots (\alpha_{n+2} D_{v_2} + \beta_{n+2} D_{v_{n+2}}) \\ &= D_{v_2} p(D_{v_2}, \ldots, D_{v_{n+2}}) + \gamma \prod_{j=3}^{n+2} D_{v_j} \end{aligned} \tag{11}$$

for some constant γ and some homogeneous polynomial p of order $n-1$. Since, by the assumption, $p(D_{v_2}, \ldots, D_{v_{n+2}}) f_1$ and $p(D_{v_2}, \ldots, D_{v_{n+2}}) f_2$ coincide along the ray v_2, by Lemma 6

$$D_{v_2} p(D_{v_2}, \ldots, D_{v_{n+2}}) f_1(\mathbf{0}) = D_{v_2} p(D_{v_2}, \ldots, D_{v_{n+2}}) f_2(\mathbf{0}). \tag{12}$$

Similarly, for every $k = 3, \ldots, n+2$, the functions

$$\prod_{j=3, j\neq k}^{n+2} D_{v_j} f_{k-1} \quad \text{and} \quad \prod_{j=3, j\neq k}^{n+2} D_{v_j} f_k$$

coincide along the ray v_k. Hence, by Lemma 6, for every $k = 3, \ldots, n+2$,

$$\prod_{j=3}^{n+2} D_{v_j} f_{k-1}(\mathbf{0}) = D_{v_k} \prod_{\substack{j=3 \\ j\neq k}}^{n+2} D_{v_j} f_{k-1}(\mathbf{0}) = D_{v_k} \prod_{\substack{j=3 \\ j\neq k}}^{n+2} D_{v_j} f_k(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_k(\mathbf{0}).$$

Thus, we obtain the following chain of equalities

$$\prod_{j=3}^{n+2} D_{v_j} f_2(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_3(\mathbf{0}) = \cdots = \prod_{j=3}^{n+2} D_{v_j} f_{n+2}(\mathbf{0}) = \prod_{j=3}^{n+2} D_{v_j} f_1(\mathbf{0}). \tag{13}$$

The last equality follows from Lemma 6 since f_1 and f_{n+2} share a common edge v_1. Thus

$$\begin{aligned} D^n_{v_1} f_2(\mathbf{0}) &\overset{\text{by (11)}}{=} D_{v_2} p(D_{v_2}, \ldots, D_{v_{n+2}}) f_2(\mathbf{0}) + \gamma \prod_{j=3}^{n+2} D_{v_j} f_2(\mathbf{0}) \\ &\overset{\text{by (12,13)}}{=} D_{v_2} p(D_{v_2}, \ldots, D_{v_{n+2}}) f_1(\mathbf{0}) + \gamma \prod_{j=3}^{n+2} D_{v_j} f_1(\mathbf{0}) \overset{\text{by (11)}}{=} D^n_{v_1} f_1(\mathbf{0}). \end{aligned}$$

which completes the proof of (9), and consequently proves the theorem. ■

The next result is a direct consequence of applying Theorem 7 to the derivatives of the piecewise function.

Corollary 8 *Let functions $f_1, \ldots, f_{n+2}$, be m times continuously differentiable on Ω, with $m \geq n$, and let F_{n+2} be defined piecewise on each sector Δ_j by $F_{n+2}\,|_{\Delta_j} := f_j$, $j = 1, \ldots, n+2$. If $F_{n+2} \in C^m(\Omega)$ then F_{n+2} has all derivatives of order $m+1$ at the origin, that is, $F_{n+2} \in C^{m+1}(\mathbf{0})$, $m \geq n \geq 0$.*

We finish this section and this article by constructing polynomials (hence smooth functions) $f_1, \ldots, f_{n+2}$, $n \geq 1$, such that the spline F_{n+2} defined by $F_{n+2}\,|_{\Delta_j} = f_j$ is in $C^{n-1}(\Omega)$ yet $F_{n+2} \notin C^n(\mathbf{0})$. We note that if $n = 0$, it is immediately obvious that $f_1 \equiv 0$ and $f_2 \equiv 1$ do not join continuously at the origin. The following observation is the key to the construction:

Lemma 9 *Given $n \geq 1$, consider the polynomial*

$$g(x,y) := \sum_{i=1}^{n+1} c_i (y + a_i x)^n.$$

Then the system of equations with the unknowns $(c_1, \ldots, c_{n+1})$:

$$\frac{\partial^k}{\partial x^j \partial y^{k-j}}\, g(x,0) = 0, \quad \textit{for all} \;\; 0 \leq j \leq k \leq n-1,$$

has a non-trivial solution.

Proof. Indeed for $0 \le k \le n-1$ and $0 \le j \le k$ we have

$$\begin{aligned}\sum_{i=1}^{n+1} c_i \frac{\partial^k}{\partial x^j \partial y^{k-j}} (y + a_i x)^n \Big|_{y=0} &= \frac{n!}{(n-k)!} \sum_{i=1}^{n+1} c_i a_i^j (y + a_i x)^{n-k} \Big|_{y=0} \\ &= \frac{n!}{(n-k)!} x^{n-k} \sum_{i=1}^{n+1} c_i a_i^{n-k+j} = 0.\end{aligned}$$

With $s := n-k+j$, the system of n equations with $n+1$ unknowns

$$\sum_{i=1}^{n+1} c_i a_i^s = 0, \quad s = 0, \dots, n-1,$$

has a non-trivial solution. ■

Now we can proceed with our construction. As in Figure 4, choose $n+2$ consecutive positioned clockwise rays v_i emanating from the origin whose equations are given by the following lines l_i

$$l_1 : \; y = 0, \quad l_2 : \; y + a_2 x = 0, \quad \dots \quad , \quad l_{n+2} \; : y + a_{n+2} x = 0.$$

Note that without loss of generality we assume that v_1 goes along the positive direction of the x-axis. Define $f_1 :\equiv 0$ to be the function between v_1 and v_2. Let the function between v_k and v_{k+1} be defined as follows:

$$f_k := \sum_{i=2}^{k} c_i l_i^n, \quad \text{for each} \;\; 2 \le k \le n+2,$$

with the convention $v_{n+3} := v_1$. We next define:

$$g_k(x,y) := f_{k+1}(x,y) - f_k(x,y) = c_{k+1}(y + a_{k+1}x)^n, \quad \text{for all} \;\; 2 \le k \le n+1.$$

All partial derivatives of g_k of order $n-1$ or less vanish for $y = -a_{k+1}x$, that is, at the line l_{k+1}. It remains to choose the coefficients $c_2, \dots, c_{n+2}$ in such a way that f_{n+2} is glued smoothly to $f_1 \equiv 0$ at l_1, that is, so that all derivatives of order $n-1$ or less of the polynomial

$$f_{n+2} = \sum_{i=2}^{n+2} c_i l_i^n$$

vanish at $y = 0$. By Lemma 9 this leads to a system of n equations with $n+1$ unknowns $(c_2, \dots, c_{n+2})$ that has a nontrivial solution.

Hence there exists a non-zero homogeneous polynomial f_{n+2} of order n between l_{n+2} and l_1 which is C^{n-1}-smoothly glued to f_{n+1} across l_{n+1} and C^{n-1}-smoothly glued to $f_1 \equiv 0$ across l_1. Finally f_{n+2} is a nonzero homogeneous polynomial of order n. Thus there exists a partial derivative of f_{n+2} of order n which is a non-zero constant. In particular, its value at the origin is not zero, yet the same derivative of $f_1 \equiv 0$ is zero. The resulting piecewise function F_{n+2} does not have a derivative of order n at the origin.

Remark 10 *The existence of the spline F_{n+2} implicitly constructed above also follows from Theorem 9.3 in [3]. Indeed this theorem shows that the dimension of polynomial splines of degree n and smoothness $n-1$ defined over the union of $n+2$ sectors is strictly greater than $\binom{n+2}{2}$. The latter is the dimension of bivariate polynomials. Thus, there exists a spline that does not have a derivative of order n at the origin. We decided to provide a development here that would be accessible to an audience not familiar with spline theory.*

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Approximation by Kantorovich and Quadrature type quasi-interpolation neural network operators

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Abstract

Here we present multivariate basic approximation by Kantorovich and Quadrature type quasi-interpolation neural network operators with respect to supremum norm. This is done with rates using the first multivariate modulus of continuity. We approximate continuous and bounded functions on $\mathbb{R}^N$. When they are also uniformly continuous we have pointwise and uniform convergences.

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Keywords and Phrases: sigmoidal and hyperbolic tangent activation functions, multivariate quasi-interpolation neural network approximation, Kantorovich and quadrature type operators.

1 Background

We consider here the sigmoidal function of logarithmic type

$$s_i(x_i) = \frac{1}{1+e^{-x_i}}, \quad x_i \in \mathbb{R},\ i = 1, ..., N;\ x := (x_1, ..., x_N) \in \mathbb{R}^N,$$

each has the properties $\lim_{x_i \to +\infty} s_i(x_i) = 1$ and $\lim_{x_i \to -\infty} s_i(x_i) = 0$, $i = 1, ..., N$.

These functions play the role of activation functions in the hidden layer of neural networks, also have applications in biology, demography, etc.

As in [7], we consider

$$\Phi_i(x_i) := \frac{1}{2}\left(s_i(x_i+1) - s_i(x_i-1)\right), \quad x_i \in \mathbb{R},\ i = 1, ..., N.$$

We notice the following properties:

i) $\Phi_i(x_i) > 0, \ \forall\, x_i \in \mathbb{R}$,

ii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R}$,

iii) $\sum_{k_i=-\infty}^{\infty} \Phi_i(nx_i - k_i) = 1, \ \forall\, x_i \in \mathbb{R};\ n \in \mathbb{N}$,

iv) $\int_{-\infty}^{\infty} \Phi_i(x_i)\, dx_i = 1$,

v) Φ_i is a density function,

vi) Φ_i is even: $\Phi_i(-x_i) = \Phi_i(x_i), \ \ x_i \geq 0$, for $i = 1, ..., N$.

We see that ([5])

$$\Phi_i(x_i) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{\left(1+e^{x_i-1}\right)\left(1+e^{-x_i-1}\right)}, \quad i = 1, ..., N.$$

vii) Φ_i is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$, $i = 1, ..., N$.

Let $0 < \beta < 1$, $n \in \mathbb{N}$. Then as in [5] we get

viii)

$$\sum_{\left\{\begin{array}{c} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{array}\right.}^{\infty} \Phi_i(nx_i - k_i) = \sum_{\left\{\begin{array}{c} k_i = -\infty \\ : |nx_i - k_i| > n^{1-\beta} \end{array}\right.}^{\infty} \Phi_i(|nx_i - k_i|)$$

$$\leq 3.1992 e^{-n^{(1-\beta)}}, \quad i = 1, ..., N.$$

We use here the complete multivariate activation function ([4])

$$\Phi(x_1, ..., x_N) := \Phi(x) := \prod_{i=1}^{N} \Phi_i(x_i), \quad x \in \mathbb{R}^N. \tag{1}$$

It has the properties ([4]):

(i)' $\Phi(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

We see that

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1 - k_1, x_2 - k_2, ..., x_N - k_N) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} \Phi_i(x_i - k_i) = \prod_{i=1}^{N} \left(\sum_{k_i=-\infty}^{\infty} \Phi_i(x_i - k_i)\right) = 1. \tag{2}$$

That is

(ii)'

$$\sum_{k=-\infty}^{\infty} \Phi(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(x_1-k_1,...,x_N-k_N) = 1, \tag{3}$$

$k := (k_1,...,k_n)$, $\forall\ x \in \mathbb{R}^N$.

(iii)'

$$\sum_{k=-\infty}^{\infty} \Phi(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Phi(nx_1-k_1,...,nx_N-k_N) = 1, \tag{4}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)'

$$\int_{\mathbb{R}^N} \Phi(x)\,dx = 1, \tag{5}$$

that is Φ is a multivariate density function.

Here $\|x\|_\infty := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$ upon the multivariate context.

For $0<\beta<1$ and $n \in \mathbb{N}$, fixed $x \in \mathbb{R}^N$, we have proved ([4])

(v)'

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \Phi(nx-k) \leq 3.1992e^{-n^{(1-\beta)}}. \tag{6}$$

Let $f \in C_B(\mathbb{R}^N)$ (bounded and continuous functions on $\mathbb{R}^N$, $N \in \mathbb{N}$). We define the multivariate Kantorovich type neural network operators ($n \in \mathbb{N}$, $\forall$ $x \in \mathbb{R}^N$)

$$K_n(f,x) := K_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) \Phi(nx-k) := \tag{7}$$

$$\sum_{k_1=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right) \left(\prod_{i=1}^{N} \Phi_i(nx_i-k_i)\right).$$

We observe that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N =$$

$$\int_0^{\frac{1}{n}} ... \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, ..., t_N + \frac{k_N}{n}\right) dt_1 ... dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \quad (8)$$

Thus it holds

$$K_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) \Phi(nx - k). \quad (9)$$

Again for $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$, we define the multivariate neural network operators of quadrature type $Q_n(f, x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,...,r_N} \geq 0$, such that

$$\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} ... \sum_{r_N=0}^{\theta_N} w_{r_1,...,r_N} = 1; \quad k \in \mathbb{Z}^N$$

and

$$\delta_{nk}(f) := \delta_{n,k_1,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) \quad (10)$$

$$:= \sum_{r_1=0}^{\theta_1} ... \sum_{r_N=0}^{\theta_N} w_{r_1,...,r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \quad (11)$$

where $\frac{r}{\theta} = \left(\frac{r_1}{\theta_1}, ..., \frac{r_N}{\theta_N}\right)$.

We define

$$Q_n(f, x) := Q_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) \Phi(nx - k) \quad (12)$$

$$:= \sum_{k_1=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,...,k_N}(f) \left(\prod_{i=1}^{N} \Phi_i(nx_i - k_i)\right), \quad \forall\, x \in \mathbb{R}^N.$$

We consider also here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$ (see also [2])

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}}. \quad (13)$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$.

This function plays the role of an activation function in the hidden layer of neural networks.

We further consider ([2])

$$\Psi(x) := \frac{1}{4}(\tanh(x+1) - \tanh(x-1)) > 0, \quad \forall\, x \in \mathbb{R}. \quad (14)$$

We easily see that $\Psi(-x) = \Psi(x)$, that is Ψ is even on $\mathbb{R}$. Obviously Ψ is differentiable, thus continuous.

Proposition 1 *([2]) $\Psi(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $\Psi(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x \to -\infty} \Psi(x) = 0 = \lim_{x \to \infty} \Psi(x)$.

Infact Ψ has the bell shape with horizontal asymptote the x-axis. So the maximum of Ψ is zero, $\Psi(0) = 0.3809297$.

Theorem 2 *([2]) We have that $\sum_{i=-\infty}^{\infty} \Psi(x-i) = 1, \ \forall\ x \in \mathbb{R}$.*

Thus

$$\sum_{i=-\infty}^{\infty} \Psi(nx-i) = 1, \quad \forall\ n \in \mathbb{N},\ \forall\ x \in \mathbb{R}.$$

Also it holds

$$\sum_{i=-\infty}^{\infty} \Psi(x+i) = 1, \quad \forall x \in \mathbb{R}.$$

Theorem 3 *([2]) It holds $\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$.*

So $\Psi(x)$ is a density function on $\mathbb{R}$.

Theorem 4 *([2]) Let $0 < \alpha < 1$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Psi(nx-k) \leq e^4 \cdot e^{-2n^{(1-\alpha)}}.$$

In this article we also use the complete multivariate activation function

$$\Theta(x_1, ..., x_N) := \Theta(x) := \prod_{i=1}^{N} \Psi(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{15}$$

It has the properties (see [3])

(i) $\Theta(x) > 0, \ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} \Theta(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \Theta(x_1-k_1, ..., x_N-k_N) = 1, \tag{16}$$

where $k := (k_1, ..., k_N)$, $\forall\ x \in \mathbb{R}^N$.

(iii)

$$\sum_{k=-\infty}^{\infty} \Theta(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \Theta(nx_1-k_1, \dots, nx_N-k_N) = 1, \tag{17}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$.

(iv)

$$\int_{\mathbb{R}^N} \Theta(x)\, dx = 1, \tag{18}$$

that is Θ is a multivariate density function.

By [3] we get

(v)

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \Theta(nx-k) \le e^4 \cdot e^{-2n^{(1-\beta)}}, \tag{19}$$

$0 < \beta < 1$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$.

We also define the following Kantorovich type neural network operators, $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, $n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, similarly to (7):

$$L_n(f,x) := L_n(f,x_1,\dots,x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \right) \Theta(nx-k) := \tag{20}$$

$$\sum_{k_1=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,\dots,t_N)\, dt_1 \dots dt_N \right) \left(\prod_{i=1}^{N} \Psi(nx_i-k_i) \right).$$

Similarly to (9) it holds

$$L_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right) dt \right) \Theta(nx-k). \tag{21}$$

Finally we define, similarly to (12), (for any $x \in \mathbb{R}^N$) the following quadrature type neural network operators

$$T_n(f,x) := T_n(f,x_1,\dots,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, \Theta(nx-k) \tag{22}$$

$$:= \sum_{k_1=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,\dots,k_N}(f) \left(\prod_{i=1}^{N} \Psi(nx_i-k_i) \right),$$

where $\delta_{nk}(f)$ is as in (10), (11).

For $f \in C_B(\mathbb{R}^N)$ we define the first multivariate modulus of continuity

$$\omega_1(f,h) := \sup_{\substack{x,y \in \mathbb{R}^N \\ \|x-y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{23}$$

Given that $f \in C_U(\mathbb{R}^N)$ (uniformly continuous functions on $\mathbb{R}^N$) we get that $\lim_{h \to 0} \omega_1(f,h) = 0$, the same definition for ω_1.

In this article we study the pointwise and uniform convergence of operators K_n, Q_n, L_n and T_n to the unit operator I with rates. We are inspired by [1], [2], [3], [4], [5], [6], [7].

2 Main Results

We present

Theorem 5 *Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. Then*

1)

$$|K_n(f,x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + (6.3984)\|f\|_\infty e^{-n^{(1-\beta)}} =: \rho_1 \tag{24}$$

2)

$$\|K_n(f) - f\|_\infty \leq \rho_1. \tag{25}$$

Proof. We have that

$$K_n(f,x) - f(x) =$$

$$\sum_{k=-\infty}^{\infty}\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right)dt\right)\Phi(nx-k) - f(x)\sum_{k=-\infty}^{\infty}\Phi(nx-k) =$$

$$\sum_{k=-\infty}^{\infty}\left[\left(n^N \int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right) - f(x)\right]\Phi(nx-k) = \tag{26}$$

$$\sum_{k=-\infty}^{\infty}\left[n^N \int_0^{\frac{1}{n}}\left(f\left(t+\frac{k}{n}\right) - f(x)\right)dt\right]\Phi(nx-k).$$

Hence

$$|K_n(f,x) - f(x)| \leq \sum_{k=-\infty}^{\infty}\left[n^N \int_0^{\frac{1}{n}}\left|f\left(t+\frac{k}{n}\right) - f(x)\right|dt\right]\Phi(nx-k) \tag{27}$$

$$= \sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \end{cases}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| dt \right] \Phi(nx - k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| dt \right] \Phi(nx - k) \leq$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + 2\|f\|_{\infty} \sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} \Phi(nx - k) \overset{(6)}{\leq} \tag{28}$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + (6.3984)\|f\|_{\infty} e^{-n^{(1-\beta)}},$$

proving the claim. ■

We continue with

Theorem 6 *Let $f \in C_B(\mathbb{R}^N)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $n, N \in \mathbb{N}$. Then*

1)

$$|Q_n(f, x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + (6.3984)\|f\|_{\infty} e^{-n^{(1-\beta)}} = \rho_1, \tag{29}$$

2)

$$\|Q_n(f) - f\|_{\infty} \leq \rho_1. \tag{30}$$

Proof. We notice that

$$Q_n(f, x) - f(x) = \sum_{k=-\infty}^{\infty} \delta_{nk}(f) \Phi(nx - k) - f(x) \sum_{k=-\infty}^{\infty} \Phi(nx - k)$$

$$= \sum_{k=-\infty}^{\infty} (\delta_{nk}(f) - f(x)) \Phi(nx - k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right) \right) \Phi(nx - k). \tag{31}$$

Hence it holds

$$|Q_n(f, x) - f(x)| \leq \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right| \right) \Phi(nx - k)$$

$$= \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right| \right) \Phi(nx-k) + \tag{32}$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left| f\left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right| \right) \Phi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2\|f\|_\infty \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \Phi(nx-k) \overset{(6)}{\leq} \tag{33}$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + (6.3984)\|f\|_\infty e^{-n^{(1-\beta)}},$$

proving the claim. ■

We further state

Theorem 7 *Same assumptions as in Theorem 5. Then*

1)

$$|L_n(f,x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \|f\|_\infty 2e^4 e^{-2n^{(1-\beta)}} =: \rho_2, \tag{34}$$

and

2)

$$\|L_n(f) - f\|_\infty \leq \rho_2. \tag{35}$$

Proof. As in Theorem 5, using (19). ■

Theorem 8 *Same assumptions as in Theorem 5. Then*

1)

$$|T_n(f,x) - f(x)| \leq \rho_2, \tag{36}$$

and

2)

$$\|T_n(f) - f\|_\infty \leq \rho_2. \tag{37}$$

Proof. As in Theorem 6, using (19) ■

Conclusion 9 *When* $f \in \left(C_B\left(\mathbb{R}^N\right) \cap C_U\left(\mathbb{R}^N\right)\right)$*, then* $K_n(f,x) \to f(x)$*,* $Q_n(f,x) \to f(x)$*,* $L_n(f,x) \to f(x)$*,* $T_n(f,x) \to f(x)$*, pointwise, as* $n \to \infty$*, and* $K_n(f) \to f$*,* $Q_n(f) \to f$*,* $L_n(f) \to f$*,* $T_n(f) \to f$*, uniformly, as* $n \to \infty$*, all at the speed of* $\frac{1}{n^\beta}$*,* $0 < \beta < 1$*.*

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Asymptotics for Szegö polynomials of Polya type

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Abstract

We characterize limits for Szegö polynomials of fixed degree k with respect to measures formed by convolving $m < k$ point masses with convex combinations of Fejér kernels. The polynomial limit as is shown to be the same for all kernels in this class, which includes the Poisson and Fejér kernels. The moments of such kernels are convex functions for positive values of the index. Our result generalizes recently-proposed methods for frequency estimation.

Keywords: Szegö polynomial, orthogonal polynomial, frequency estimation, Poisson kernel, Fejér kernel.

1. Introduction

Szegö polynomials are a class of monic, orthogonal polynomials with respect to measures on the unit circle, and have been studied extensively. See [5, 9, 12, 13] for background. In frequency analysis, the measures are often estimates of the spectral measure of a random process such as sinusoids in noise. In [1, 2] measures are formed by convolving the Poisson and Fejér kernels, respectively, with the sum of point masses $\sum_{j=1}^{m} \alpha_j \delta_{\theta_j}$, where δ_{θ_j} is the point mass measure at θ_j, the α_j are positive, the θ_j are distinct. (We identify the unit circle with the interval $[-\pi, \pi)$.) Both are examples of ap-

Table 1: Poisson and Fejér kernels

kernel	density	moments $\widehat{\psi}_h(\ell)$	h
Poisson:	$\psi_h(\theta) = \dfrac{1-r^2}{\lvert e^{i\theta}-r\rvert^2}$	$r^{\lvert\ell\rvert}$	$1-r$
Fejér:	$\psi_h(\theta) = \dfrac{1}{n}\left[\dfrac{\sin(n\theta/2)}{\sin(\theta/2)}\right]^2$	$(1-\dfrac{\lvert\ell\rvert}{n})^+$	$\dfrac{1}{n}$

proximate identities ψ_h with $h \to 0$ as either $r \to 1$ or $n \to \infty$, respectively, as indicated in Table 1, where $x^+ = max\{x, 0\}$. In [1] it is shown that the case where ψ_h is the Poisson kernel arises as the spectral estimate in the "R-process" technique of frequency estimation proposed in [8], where, in an effort to deal with the non-uniqueness of polynomial limits, the moments of the periodogram are multiplied by those of the Poisson kernel. In the "V-process" proposed in [11], the moments of the periodogram were multiplied by those of the *wrapped Gaussian*, resulting in a different polynomial limit. Our results here and the remarks in the introductions of [1, 2] will give the limit polynomial when the moments of the periodogram are multiplied by those of any one of a class of kernels.

We denote by $P_k(z, \mu)$ the the Szegö polynomial of degree k with respect to the measure μ. For any kernel ψ_h, we have the weak-star limit

$$\lim_{h\to 0} \psi_h * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j} = \sum_{j=1}^{m} \alpha_j \delta_{\theta_j} \tag{1}$$

(See, e.g., [7] for discussion of properties of kernels, or *approximate identities*.) However, for $m < k$ $P_k(z, \sum_{j=1}^m \alpha_j \delta_{\theta_j})$ is not uniquely defined; furthermore, the associated Szegö polynomials of fixed degree k do not necessarily converge. That is, $\mu_h \to \sum_{j=1}^{m} \alpha_j \delta_{\theta_j}$ does not guarantee existence of the limit $\lim_{h\to 0} P_k(z, \mu_h)$ even if μ_h converges strongly. (See [1] for an example.) On the other hand, any limit polynomial will have m zeros at the "signal" locations θ_j (the terminology owing to the fact that $\sum_{j=1}^{m} \alpha_j \delta_{\theta_j}$ is the spectral measure of a purely sinusoidal process). The limit behavior of the remaining $k-m$ "extraneous" zeros and the problem of distinguishing them from the signal zeroes have been subjects of study in the literature.

In [1, 2] it is shown that $P_k(z, \psi_h * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j})$ do converge; in fact to the same limit. In each case, the extraneous factor is the Szegö polynomial of degree $k-m$ with respect to a related measure.

In the present paper we prove a conjecture of the authors in [3] and characterize the limit polynomial for a class of kernels which includes the Poisson and Fejèr, and whose moments $\widehat{\psi}_h(\ell)$ are convex functions of ℓ for $\ell > 0$.

2. Background

Given a measure, μ, on the unit circle, the k^{th} Szegö polynomial $P_k(z, \mu)$, with respect μ, is the polynomial in the complex variable z which attains the minimum

$$\min_{p\in\mathbf{\Lambda}_k} \int_{-\pi}^{\pi} |p(e^{i\theta})|^2 \, d\mu(\theta) = \int_{-\pi}^{\pi} |P_k(e^{i\theta}, \mu)|^2 \, d\mu(\theta), \tag{2}$$

where $\mathbf{\Lambda}_k$ is the set of monic polynomials of degree k.

The *prediction error power*, which we denote $\rho_k(\mu)$ is a multiple of the minimum attained in (2):

$$\rho_k(\mu) := \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_k(e^{i\theta}, \mu)|^2 \, d\mu. \tag{3}$$

The $P_k(z, \mu)$ are also characterized by the orthogonality property

$$\int_{-\pi}^{\pi} P_k(e^{i\theta}, \mu) \, \overline{p}(e^{i\theta}) \, d\mu(\theta) = 0 \tag{4}$$

for any any polynomial p of degree less than k.

The *reflection coefficients* of the $P_k(z, \mu)$ are the constant terms

$$R_k(\mu) := P_k(0, \mu). \tag{5}$$

We denote by $P_k^*(z, \mu)$ the *reverse polynomial*: $P_k^*(z, \mu) = z^k \overline{P_k}(1/z, \mu)$. The zeros of P^* are obtained from those of P by reflection in the unit circle. Equivalently, the coefficients are reversed and conjugated. Note that we have $|P_k^*(e^{i\theta}, \mu)| = |P_k(e^{i\theta}, \mu)|$, and that $P_k^*(0, \mu) = 1$.

3. Kernels of Polya type

A positive function f satisfies the *Polya criterion* if it is non-increasing, even, and if $f(x)$ is convex for $x > 0$. Such functions are the characteristic functions of positive measures. (See, e.g., [4], p482.)

In [3] the authors define sequences, densities, and kernels of *Polya type*: If $\widehat{\phi}(n) = a_n = f(n)$ for some function f satisfying the Polya criterion, then $\{a_n\}$ and ψ is a sequence and density, respectively, of *Polya type*; $\{\psi_h\}$ is a kernel of Polya type if ψ_h is a density of Polya type for each value of h. We see that the Poisson and Fejér kernels are kernels of Polya type.

Let ϕ_n denote the Fejér kernel as in Table 1. Then $\psi(\theta) = \sum_{n=1}^{\infty} a_n \phi_n(\theta)$ is a density of Polya type if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n = 1$. Now suppose that $A_N := \{a_{N,n}\}$ is a sequence of sequences with

$$\sum_{n=1}^{\infty} a_{N,n} = 1 \quad \text{for all} \quad N = 1, 2, 3, ... \tag{6}$$

$$\lim_{N\to\infty} a_{N,n} = 0 \quad \text{for all} \quad n = 1, 2, 3, ... \tag{7}$$

It is easy to see that

$$\psi_{A_N}(\theta) = \sum_{n=1}^{\infty} a_{N,n} \phi_n(\theta) \tag{8}$$

satisfies the properties of an approximate identity with $N \to \infty$. Thus $\psi_{A_N}(\theta)$ is a kernel of Polya type, and we have the weak-star limit

$$\psi_{A_N} * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j} \to \sum_{j=1}^{m} \alpha_j \delta_{\theta_j}. \tag{9}$$

We define kernels ψ_{A_h} for continuous parameter $h \to 0$ similarly. For example, it is not too hard to show (see [3]) that the Poisson kernel is given by $a_{r,n} = (1-r)^2 n r^{n-1}$, while the Fejér kernel corresponds to $A_N = \{0, 0, 0, ..., 1, 0, 0, ...\}$, with 1 in the N-th position.

The following was conjectured in [3]. We will assume that A_N is indexed with discrete parameter $N \to \infty$. The continuous case $h \to 0$ is similar.

Theorem 3.1. *Suppose $A_N \to 0$ and let ψ_{A_N} be as defined in (8), where $\phi_n(\theta)$ is the Fejèr kernel and the θ_j are distinct in (1). Then for each $k \geq m$*

$$\lim_{N\to\infty} P_k(z, \psi_{A_N} * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j}) = P_{k-m}(z, \nu) \prod_{j=1}^{m} (z - e^{i\theta_j}) \tag{10}$$

where

$$\frac{d\nu}{d\theta} = \sum_{j=1}^{m} \alpha_j \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2. \tag{11}$$

Remarks: Thus the extraneous limit factor is the degree $k-m$ Szegö polynomial with respect to the measure ν, and the limit is the same as in the case of the Fejér and Poisson kernels, which are special cases. It is also follows that when the moments of the periodogram (or any spectral estimate which converges weak-star to point masses as in (1) are multiplied by those of a kernel of the form 20, where A_N is any sequence satisfying 6 and 7, and limits are taken first with the periodogram index approaching infinity, the limit of the Szegö polynomials is given by 10 and 11. The justification for this is given in the introduction of [1].

The proof of Theorem 3.1, is an extension of Theorem 2.1 of [2]. We first extend supporting results of [2]. We will use the notations

$$\mu_N := \psi_{A_N} * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j}, \tag{12}$$

and

$$\zeta := e^{i\theta}$$

where the latter represents an arbitrary point on the unit circle.

Lemma 3.1. *Under the hypotheses of Theorem 3.1, the following hold.*

i *The sequence $\{P_k(z,\mu_N)\}$ as $N\to\infty$ has a limit point, and all limit points are of the form $Q(z)\prod_{j=1}^{m}(z-e^{i\theta_j})$ where Q is a monic polynomial of degree $k-m$.*

ii *The reflection coefficients $R_k(\mu_N)$ are bounded away from the unit circle uniformly in N.*

iii *$P_k(z,\mu_N)$ and the m zeroes which approach the θ_j do so at the rate $O(1/N)$.*

Proof: To establish i note that the $P_k(z,\mu_N)$ are monic polynomials with zeros contained in the unit circle, so $\{P_k(z,\mu_N)\}$ has limit points, and by Proposition 2.1 of [1], all of these are of the above form.

We establish ii using the arguments of Corollary 2.1 of [2]. For each n, we have the following representations of $\phi_n(\theta) * \sum_{j=1}^m \alpha_j \delta_{\theta_j}$ ([2], (12) and (16)).

$$
\begin{aligned}
(\phi_n * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j})' &= \\
|g_n(\zeta)|^2 &:= \frac{1}{n} \sum_{j=1}^{m} \alpha_j |\zeta^{(n-1)} + e^{i\theta_j}\zeta^{(n-2)} + \cdots + e^{i(n-1)\theta_j}|^2 \quad (13)\\
(\phi_n * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j})' &= \frac{2}{n} \frac{\sum_{j=1}^m \alpha_j (1-\cos n(\theta-\theta_j)) \prod_{p\neq j}^m |\zeta - e^{i\theta_p}|^2}{\Pi_{j=1}^m |\zeta - e^{i\theta_j}|^2} \quad (14)
\end{aligned}
$$

where g is a polynomial of degree $n-1$ with all zeros outside the circle. Using (13) we can express the convolution (12) as

$$
\begin{aligned}
\mu_N'(\theta) &= \left(\sum_{n=1}^{\infty} a_{N,n}\phi_n(\theta)\right) * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j} \\
&= \sum_{n=1}^{\infty} a_{N,n} \left(\phi_n(\theta) * \sum_{j=1}^{m} \alpha_j \delta_{\theta_j}\right) \\
&= \sum_{n=1}^{\infty} a_{N,n} |g_n(\zeta)|^2 \qquad (15)
\end{aligned}
$$

The sum in (15), regarded as a function of the complex variable z on the plane, is a convex combination of subharmonic functions, so it is subharmonic. Using (15) with (3), we then write

$$
\begin{aligned}
\rho_k(\mu_N) &= \frac{1}{2\pi} \int |P_k(\zeta,\mu_N)|^2 \, d\mu_N \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_k^*(\zeta,\mu_N)|^2 \sum_{n=1}^{\infty} a_{N,n} |g_n(\zeta)|^2 \, d\theta,
\end{aligned}
$$

$$
\begin{aligned}
&\geq |P_k^*(0,\mu_N)|^2 \sum_{n=1}^{\infty} a_{N,n} |g_n(0)|^2 \\
&= \sum_{j=1}^{m} \alpha_j \sum_{n=1}^{\infty} \frac{1}{n} a_{N,n} \ , \qquad (16)
\end{aligned}
$$

On the other hand, using (14) we can express (12) as

$$
\mu_N'(\theta) = 2 \sum_{n=1}^{\infty} \frac{1}{n} a_{N,n} \frac{\sum_{j=1}^m \alpha_j (1-\cos n(\theta-\theta_j)) \prod_{p\neq j}^m |\zeta - e^{i\theta_p}|^2}{\Pi_{j=1}^m |\zeta - e^{i\theta_j}|^2} \qquad (17)
$$

Now using (3) with (17), we replace P_k with $\Pi_{j=1}^m(\zeta - e^{i\theta_j})$ in (2) to obtain

$$
\begin{aligned}
\rho_k(\mu_N) \ &\leq \ \frac{1}{\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{m} |\zeta - e^{i\theta_j}|^2 \\
&\times \ \sum_{n=1}^{\infty} \frac{1}{n} a_{N,n} \frac{\sum_{j=1}^m \alpha_j (1-\cos n(\theta-\theta_j)) \prod_{p\neq j}^m |\zeta - e^{i\theta_p}|^2}{\Pi_{j=1}^m |\zeta - e^{i\theta_j}|^2} \ d\theta \\
&\leq \ 4^m \sum_{j=1}^{m} \alpha_j \sum_{n=1}^{\infty} \frac{1}{n} a_{N,n} \qquad (18)
\end{aligned}
$$

The estimates (16) and (18), and the relation ([6]) $|R_{k+1}(\mu_N)|^2 = 1 - \rho_k(\mu_N)/\rho_{k+1}(\mu_N)$, now give $|R_k| \leq \sqrt{1 - \frac{1}{4^m}}$ to establish *ii*.

To establish *iii*, we use (8) and $\widehat{\phi}_n(\theta)$ from Table PFtable1 to write

$$
\begin{aligned}
\widehat{\psi}_{A_N}(\ell) \ &= \ \sum_{n=1}^{\infty} a_{N,n} \widehat{\phi}_n(\theta) \\
&= \ \sum_{n=1}^{\infty} a_{N,n} (1 - \frac{|\ell|}{n})^+ \qquad (19) \\
&= \ \sum_{n=1}^{N} a_{N,n} (1 - \frac{|\ell|}{n})^+ + \sum_{n=N+1}^{\infty} a_{N,n} (1 - \frac{|\ell|}{n})^+ .
\end{aligned}
$$

We see from the form of 20 that $\widehat{\psi}_{A_N}(\ell)$ can be expressed as a power series in $1/N$, so *iii* follows from [1], Corollaries 2.1 and 2.2.

Without loss of generality we will write

$$P_k(z,\mu_N) = Q_N(z)\Pi_{j=1}^m (z - w_j^{(N)}) \ , \tag{20}$$

where $Q_N \to Q$ and $w_j^{(N)} \to e^{i\theta_j}$ for $j = 1, 2, 3, ..., m$, and

$$|w_j^{(N)} - e^{i\theta_j}| \leq \frac{L_j}{N} \tag{21}$$

for some constants, L_j, as in (24), [2].

Proof of Theorem 3.1: The argument is an extension of Theorem 2.1, [2]. We show that the limit factor Q in Lemma 3.1 has the orthogonality property which characterizes $P_{k-m}(z,\nu)$.

We can use this orthogonality property with (17) and (20) to write

$$\int Q_N(\zeta)\overline{T(\zeta)}\prod_{j=1}^m (\zeta - w_j^{(N)}) \times \sum_n \frac{1}{n} a_{N,n} \left(\frac{\sum_{j=1}^m \alpha_j (1-\cos n(\theta-\theta_j)) \prod_{p\neq j}^m |\zeta - e^{i\theta_p}|^2}{\Pi_{j=1}^m |\zeta - e^{i\theta_j}|^2} \right) \, d\theta = 0$$

for any polynomial $T(z)$ of degree $k-1$ or less. In particular, if $T(z) = t(z)\prod_{j=1}^m (z - e^{i\theta_j})$ with $t(z)$ an arbitrary polynomial of degree $k-m-1$ or less we have

$$\int Q_N(\zeta) \ \ \overline{t(\zeta)} \ \ \frac{\prod_{j=1}^m (\zeta - w_j^{(N)})}{\prod_{j=1}^m (\zeta - e^{i\theta_j})} \times \sum_n \frac{1}{n} a_{N,n} \left(\sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p\neq j}^m |\zeta - e^{i\theta_p}|^2 \right) \times \ d\theta = 0. \tag{22}$$

Re-arranging factors in (22) gives

$$\int Q_N(\zeta) \ \ \overline{t(\zeta)} \ \ \prod_{j=1}^m (\zeta - w_j^{(N)})$$

$$\begin{aligned} \times \quad & \sum_n \frac{1}{n} a_{N,n} \left(\sum_{j=1}^{m} \alpha_j \frac{(1 - \cos n(\theta - \theta_j))}{\zeta - e^{i\theta_j}} \right) \\ \times \quad & d\theta = 0. \end{aligned} \tag{23}$$

Let N be large enough so that the intervals $(\theta_j - \frac{1}{N}, \theta_j + \frac{1}{N})$ are disjoint, and consider the integral of the integrand in (22) separately over the union of these intervals and its complement. Let $I_j(N) = (\theta_j - \frac{1}{N}, \theta_j + \frac{1}{N})$ and let $\mathcal{X}_N$ be the indicator function for the set $[-\pi, \pi) \setminus \cup_{j=1}^m I_j(N)$. Then (22) gives

$$\int \gamma(N, \zeta) \mathcal{X}_N(\theta)\ d\theta + \int_{\cup_{j=1}^m I_j(N)} \gamma(N, \zeta)\ d\theta\ = 0, \tag{24}$$

Where $\gamma(N, \theta)$ is the integrand in (22). We show that both integrals in (24) converge to zero.

Claim: For each j,

$$\sum_n a_{N,n} \left| (\zeta - w_j^{(N)}) \frac{1 - \cos n(\theta - \theta_j)}{\zeta - e^{i\theta_j}} \right| \leq M_j \tag{25}$$

for constants M_j.

Proof of Claim: Without loss of generality, assume that $\theta_j = 0$. In the present context, (29) of [2] becomes

$$|\zeta - w_j^{(N)}| \leq \min\{2, |\theta| + \frac{L_j}{N}\}. \tag{26}$$

Now from (21), (26) and the bound

$$|\zeta - 1| > |\theta|/2 \quad \text{for} \quad |\theta| < \pi \tag{27}$$

it follows that

$$\sum_n a_{N,n} \left| (\zeta - w_j^{(N)}) \frac{1 - \cos n\theta}{\zeta - 1} \right| \leq 2 \sum_n a_{N,n} (|\theta| + \frac{L_j}{N}) \left| \frac{1 - \cos n\theta}{\theta} \right|. \tag{28}$$

First consider $0 < |\theta| \leq \frac{1}{N}$. Since $\frac{1-\cos\theta}{|\theta|} < 1$, $\frac{1-\cos n\theta}{|\theta|} < n$, so considering the partial sum on the right-hand side of (28) we have

$$\begin{aligned}\sum_{n=1}^{N} a_{N,n}(|\theta| + \frac{L_j}{N}) \left|\frac{1-\cos n\theta}{\theta}\right| &\leq \sum_{n=1}^{N} a_{N,n}(1+L_j)(\frac{1}{N})\, n \\ &< (1+L_j)\sum_{n=1}^{N} a_{N,n}. \end{aligned} \tag{29}$$

Letting $N \to \infty$ we see that the left-hand-side of (25) is less than $2(1+L_j)$, establishing the claim for $0 < |\theta| \leq 1/N$.

For $|\theta| > 1/N$, (21) and (27) give

$$\begin{aligned}\left|\frac{\zeta - w_j^{(N)}}{\zeta - 1}\right| \leq 1 + \frac{|w_j^{(N)} - 1|}{|\zeta - 1|} &< 1 + \frac{2L_j}{N|\theta|} \\ &< 1 + 2L_j. \end{aligned}$$

Thus the left-hand-side of (25) is less than $\sum_n a_{N,n}(1+2L_j)(2) = 2(1+2L_j)$. This establishes the claim.

Using (25) we see that the integrand in (23) is bounded uniformly in N, so that the second integral in (24) converges to zero as $N \to \infty$. It follows that the first integral in (24) also converges to zero. That is,

$$\lim_{N\to\infty} \int \gamma(N,\zeta)\mathcal{X}_N \, d\theta = 0 \tag{30}$$

By expanding the terms in the factor $(1 - \cos n\theta)$, and keeping in mind that $\sum aN = 1$, we can express $\gamma(N,\zeta)$ as

$$\begin{aligned}\gamma(N,\theta) &= Q_N(\zeta)\overline{t(\zeta)} \prod_{j=1}^{m} \frac{(\zeta - w_j^{(N)})}{(\zeta - e^{i\theta_j})} \sum_{j=1}^{m} \alpha_j \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2 \\ &- Q_N(\zeta)\overline{t(\zeta)} \prod_{j=1}^{m} \frac{(\zeta - w_j^{(N)})}{(\zeta - e^{i\theta_j})} \end{aligned}$$

$$\times \quad \sum_n \frac{1}{n} a_{N,n} \sum_{j=1}^{m} \alpha_j \cos n(\theta-\theta_j) \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2, \tag{31}$$

The first term on the right remains unchanged from the situation in [2], and thus, by $\mathcal{L}_1$ convergence to unity of the factor $\prod_{j=1}^{m} \frac{(\zeta-w_j^{(N)})}{(\zeta-e^{i\theta_j})}\mathcal{X}_N$, given in Lemma 2.3 of [2], as $N\to\infty$ its integral on $[-\pi,\pi)\setminus\cup_{j=1}^{m} I_j(N)$ converges to

$$\int_{-\pi}^{\pi} Q(\zeta)\overline{t(\zeta)} \sum_{j=1}^{m} \alpha_j \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2 \, d\theta \ . \tag{32}$$

The theorem can then be established by showing that this integral is zero; indeed, in that case from the fact that t is arbitrary and the orthogonality condition (4) it would follow that Q is the desired Szegö polynomial. To show that (refforthchar) is zero, in light of (30), it suffices to show that the integral of the second term in (31) on $[-\pi,\pi)\setminus\cup_{j=1}^{m} I_j(N)$ converges to zero.

Using (25) and Lemma 2.3, [2], we see that the second term is a product of a bounded factor and a factor which converges to 1 in $\mathcal{L}_1$. Thus, by Lemma 2.4, [2], the limit of its integral can be expressed.

$$\lim_{N\to\infty} \int Q(\zeta)\overline{t(\zeta)} \sum_n a_{N,n} \sum_{j=1}^{m} \alpha_j \cos n(\theta-\theta_j) \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2 \, d\theta \, . \tag{33}$$

Exchanging summation and integration gives

$$\lim_{N\to\infty} \sum_n a_{N,n} \int Q(\zeta)\overline{t(\zeta)} \sum_{j=1}^{m} \alpha_j \cos n(\theta-\theta_j) \prod_{p\neq j}^{m} |\zeta - e^{i\theta_p}|^2 \, d\theta \, . \tag{34}$$

The integral in 34 appears in [2], Equation (41); we see that the integral is a constant times the real part of the n^{th} Fourier coefficient of a continuous function. Thus, the sum in (34) is a convex sum of the real parts of these coefficients, which approach zero as $n\to\infty$ by Riemann-Lebesgue. on the other hand, by (7), we can make the partial sum for $n < N$ as small as we wish by choosing N large enough. Thus we can conclude that the limit (34) is zero to finish the proof.

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On one nonclassical problem for a multidimensional parabolic equation

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In this work we study the problem with the nonlocal integral condition of the first type for the multidimensional heat equation. Applying the connection between nonlocal and inverse problems we investigate the associated inverse problem. The existence and uniqueness of the generalized solution are shown using Galerkin method.
Keywords: Integral condition, inverse problem, parabolic equation

1 Introduction

Mathematical modeling of some physical processes leads to the problems with nonlocal conditions for partial differential equations. These are conditions connecting either values of unknown solution and its derivatives on boundary of the process or values of unknown solution within the region of the process.
A special class of nonlocal problems includes integral conditions. Integral conditions are naturally generalization of discrete conditions. These conditions commonly arise in situations where the boundary of the process is inaccessible (heat conduction, processes in liquid plasma, dynamics of ground waters, thermo-elasticity and some technological process).

Problems with nonlocal conditions for parabolic and hyperbolic equations have been intensively studied in the previous years. The first papers devoted to second-order partial differential equations with nonlocal integral conditions go back to Cannon [1] and Kamynin [2]. Later their results were extended [3], [4], [5], [6].

In this paper we consider a problem with the nonlocal condition in the form of the first order integral operator. The existence of the solution is proved for a special case.

2 Preliminary notes

Consider the equation

$$u_t - \triangle u + c(x)u = f(x,t) \tag{1}$$

in the cylinder $Q_T = \{(x,t) : x \in \Omega, t \in (0,T)\}$, where $\Omega \in R^n$ is a bounded domain with a smooth boundary $\partial\Omega$, $S_T = \{(x,t) : x \in \partial\Omega, t \in (0,T)\}$ is a lateral surface of Q_T, with the initial condition

$$u(x,0) = \varphi(x), \quad x \in \overline{\Omega}, \tag{2}$$

and the nonlocal condition

$$\int_\Omega K(x,t)u(x,t)\,dx = E(t), \quad t \in (0,T), \tag{3}$$

where functions $\varphi(x)$, $K(x,t)$, $E(t)$ are known.

Let the term $f(x,t)$ of the equation (1) can be represented as a product of functions $f_1(x) \cdot f_2(t)$. As the equation (1) is linear, $f(x,t)$ is a product of functions of different variables, the coefficient $c(x)$ depends on a single variable, so the trace of the derivative $\dfrac{\partial u}{\partial n}$ on the lateral surface of the cylinder Q_T also can be represented as a product of functions

$$\left.\frac{\partial u}{\partial n}\right|_{S_T} = q(t)g(x). \tag{4}$$

Let $\psi(x) = \left.\dfrac{\partial u(x,0)}{\partial n}\right|_{S_T}$, $r(t) = \dfrac{q(t)}{q(0)}$. Then the condition (4) becomes

$$\left.\frac{\partial u}{\partial n}\right|_{S_T} = r(t)\psi(x). \tag{5}$$

We shall consider the problem (1)–(3), (5). Since the function $r(t)$ is unknown, this problem can be represented as the inverse problem with respect to the pair of functions (u,r). The condition (3) plays role of a condition of an integral overdetermination [7], [8].
Assume that

$$K(x,t) \in C^1(\overline{\Omega} \times [0,T]),\ E(t) \in W_2^1(0,T),\ \varphi(x) \in C^1(\overline{\Omega}); \tag{6}$$

$$f(x,t) \in L_2(Q_T),\ c(x) \in C(\overline{\Omega}). \tag{7}$$

Let a pair (u,r) be a solution of the problem (1)– (3), (5). Multiply the equation (1) by the function $K(x,t)$ and integrate it over Ω. It follows from integration by parts that

$$E_t + \int_\Omega (-K_t + cK)u\,dx + \int_\Omega K_x u_x\,dx - \int_\Omega Kf\,dx = r(t)\int_{\partial\Omega} K\psi(x)\,ds.$$

Hence

$$r(t) = \frac{1}{h(t)} \left[E_t + \int_\Omega \overline{K} u\, dx + \int_\Omega K_x u_x\, dx - \int_\Omega K f\, dx \right], \tag{8}$$

where $h(t) = \displaystyle\int_{\partial\Omega} K(x,t)\psi(x)\, ds, \quad \overline{K} = -K_t + cK.$

Lemma 2.1 *Let the conditions (6), (7) hold and*

$$E(0) = \int_\Omega K(x,0)\varphi(x)\, dx. \tag{9}$$

Then problem (1)–(3), (5) is equivalent to the problem (1), (2), (5), (8).

Proof. As we have shown a solution of the problem (1)–(3), (5) ia also a solution of (1), (2), (5), (8). It remains to prove the converse.

Assume that the pair (u, r) ia a solution of (1), (2), (5), (8). From the condition (8) it follows that

$$r(t) \int_{\partial\Omega} K(x,t)\psi(x)\, ds = E_t + \int_\Omega (-K_t + cK) u\, dx + \int_\Omega K_x u_x\, dx - \int_\Omega K f\, dx.$$

Using the condition (5) we obtain

$$E_t = \int_\Omega K_t u\, dx + \int_\Omega K \triangle u\, dx - \int_\Omega K c u\, dx + \int_\Omega K f\, dx. \tag{10}$$

On the other hand, the equation (1) implies the equality

$$\int_\Omega K u_t\, dx = \int_\Omega K \triangle u\, dx + \int_\Omega K f\, dx - \int_\Omega c K u\, dx. \tag{11}$$

Comparing (11) and (10) we see that $E_t = \displaystyle\int_\Omega K_t u\, dx + \int_\Omega K u_t\, dx$. Therefore we have the following Cauchy problem for $\displaystyle\int_\Omega K u\, dx$:

$$\frac{d}{dt} \int_\Omega K u\, dx = E_t, \tag{12}$$

$$\int_\Omega K(x,0) u(x,0)\, dx = E(0), \tag{13}$$

where the condition (13) follows from the compatibility condition (9).

We note that the Cauchy problem (12)–(13) has the unique solution $\displaystyle\int_\Omega K u\, dx = E(t)$ [9]. Lemma is proved.

Definition 2.2 *Following [10], we define as $V_2(Q_T)$ a Hilbert space which consists of all elements of $W_2^{1,0}(Q_T)$ with the norm:*

$$|u|^2 = ess \sup_{0 \le t \le T} \int_\Omega u^2\,dx + \int_{Q_T} u_x^2\,dx\,dt.$$

We define $\widehat{W}_2^1(Q_T) = \{u(x,t) \in W_2^1(Q_T),\ u(x,T) = 0\}$.

We shall inroduce a generalized solution of the problem (1), (2), (5), (8) using the standard method [10]. To this end we assume that (u, r) is a solution of (1), (2), (5), (8), multiply (1) by $\eta(x,t) \in \widehat{W}_2^1(Q_T)$ and integrate the obtained identity over Q_T. It follows from (2), (5) and integration by parts that

$$\int_{Q_T} (-u\eta_t + u_x\eta_x + cu\eta)\,dx\,dt = \int_{S_T} r\psi\eta\,ds\,dt +$$

$$+ \int_{Q_T} f\eta\,dx\,dt + \int_\Omega \varphi\eta(x,0)\,dx. \tag{14}$$

Definition 2.3 *A pair of functions (u, r) is said to be a solution of the problem (1), (2), (5), (8), if $u \in V_2(Q_T)$, $r(t) \in L_2(0,T)$ and $\forall\, \eta(x,t) \in \widehat{W}_2^1(Q_T)$ functions $u(x,t), r(t)$ satisfy the integral identity (14), equality (8) and $r(0) = 1$.*

3 Main result

Theorem 3.1 *Let the conditions of Lemma 2.1 hold, and*

$$0 < c_0 \le |c(x)| \le c_1, \quad \int_{\partial\Omega} K(x,t)\psi(x)\,ds \ge h_1 > 0, \tag{15}$$

$$\int_\Omega \overline{K}^2\,dx \le K_0, \quad \int_\Omega K_x^2\,dx \le K_1,\ 0 < t < T, \quad K_0,\ K_1 > 0; \tag{16}$$

$$E_t(0) + \int_\Omega \overline{K}(x,0)\varphi(x)\,dx + \int_\Omega K_x(x,0)\varphi_x(x)\,dx -$$

$$- \int_\Omega K(x,0)f(x,0)\,dx = \int_{\partial\Omega} K(x,0)\psi(x)\,ds. \tag{17}$$

Then there exists a unique solution of the problem (1), (2), (5), (8).

The proof of the theorem 3.1 is organized as follows. First we prove that a solution of the problem satisfies an integral identity, obtain a number of inequalities and apply Grownwall's lemma to get an energy estimate. To prove existence we construct approximations of the generalized solution by Faedo-Galerkin method. Then we obtain a priori estimates to guarantee convergence of approximations. Finally, we show that the limit of approximations is the required solution.

Lemma 3.2 *Let the function $u(x,t) \in V_2(Q_T)$ be a solution of the problem (1), (2), (5) with given function $r(t) \in L_2(0,T)$. Then*

$$\frac{1}{2}\int_\Omega u^2\,dx + \int_{Q_\tau} u_x^2\,dx\,dt = \frac{1}{2}\int_\Omega \varphi^2\,dx - \int_{Q_\tau} cu^2\,dx\,dt+$$

$$+\int_{S_\tau} ur(t)\psi(x)\,ds\,dt + \int_{Q_\tau} fu\,dx\,dt \quad \textit{for} \quad \tau\in[0,T]. \tag{18}$$

Proof. Let function $r(t) \in L_2(0,T)$. Assume that function $u \in W_2^1(Q_T)$ is a solution of (1), (2), (5) and hence, $u(x,t)$ satisfies the integral identity (14) for all $\eta(x,t) \in W_2^1(Q_T)$, $\eta(x,T)=0$. We take $\eta(x,t)$ as

$$\eta(x,t) = \begin{cases} u(x,t) & 0\le t\le\tau, \\ 0 & \tau<t\le T, \end{cases} \qquad 0\le\tau\le T.$$

From integration by parts in (14) it follows that $u(x,t) \in W_2^1(Q_T)$ satisfies the integral identity (18). We shall prove that a function $u(x,t) \in V_2(Q_T)$ also satisfies this identity. Consider a sequence of the functions $u^m \in W_2^1(Q_T)$ which satisfies (14) and hence, (18):

$$\frac{1}{2}\int_\Omega (u^m)^2(x,\tau)\,dx + \int_{Q_\tau}(u_x^m)^2\,dx\,dt + \int_{Q_\tau} c(u^m)^2\,dx\,dt = \frac{1}{2}\int_\Omega\varphi^2\,dx+$$

$$+\int_{S_\tau} r\psi u^m\,ds\,dt + \int_{Q_\tau} fu^m\,dx\,dt. \tag{19}$$

Functions of $W_2^1(Q_T)$ form a dense set in $V_2^{1,0}(Q_T)$ [10] and hence, in $V_2(Q_T)$. Therefore, there exists a function $u^* \in V_2(Q_T)$ such that $|u^m - u^*|_{Q_T} \to 0$ as $m\to\infty$, that is $ess \sup\limits_{0\le t\le T}\int_\Omega (u^m-u^*)^2\,dx + \int_{Q_T}(u_x^m-u_x^*)^2\,dx\,dt \to 0$
So, the functional sequences $\{u^m\}$, $\{u_x^m\}$ converge strongly to the u^*, u_x^* in $L_2(\Omega)$ and $L_2(Q_T)$ respectively.

The next step is to prove that the sequence u^m converges strongly to the u^* in $L_2(Q_T)$. To this end consider functions $\varphi_k(x)$, $k=1,2,\ldots$ which form an orthogonal basis in $L_2(\Omega)$. Then for an arbitrary $\varepsilon>0$ we can find such number N_ε that every function $u(x) \in W_2^1(\Omega)$ satisfies inequality ([10], p. 529):

$$\|u\|^2_{L_2(\Omega)} \le 2\sum_{k=1}^{N_\varepsilon}(u,\varphi_k)^2 + 2\varepsilon^2\|u\|^2_{W_2^1(\Omega)}, \tag{20}$$

where N_ε does not depend on u. Functions u^m-u^* belong to $W_2^1(\Omega)$ for every $t\in[0,T]$, $m=1,2,\ldots$ and therefore, (20) with an appropriate choice of ε gives us $\|u^m-u^*\|^2_{L_2(\Omega)} \le 2\left(\sum\limits_{k=1}^{N_\varepsilon}(u^m-u^*,\varphi_k)^2\right)^{1/2} + \|u_x^m-u_x^*\|^2_{L_2(\Omega)}$. Hence,

$$\|u^m-u^*\|^2_{L_2(Q_T)} \le 2\int_0^T\sum_{k=1}^{N_\varepsilon}(u^m-u^*,\varphi_k)^2\,dt + \|u_x^m-u_x^*\|^2_{L_2(Q_T)}. \tag{21}$$

Passing to the limit in (21) as $m \to \infty$ and using strong convergence $u^m(x,t)$ in $L_2(\Omega)$ and u^m_x in $L_2(Q_T)$ we obtain that the sequence u^m also converges strongly to u^* in $L_2(Q_T)$.

Once again consider the equality (19) for functions $u^m(x,t)$ and prove that

$$\lim_{m\to\infty} \int_{S_\tau} r(t)\psi(x)u^m \, ds\, dt = \int_{S_\tau} r(t)\psi(x)u^* \, ds\, dt. \tag{22}$$

Note, that a function $u \in W_2^1(\Omega)$, where domain Ω has a smooth boundary, satisfies the inequality ([11], p. 77):

$$\int_{\partial\Omega} u^2 \, ds \le \int_\Omega (\varepsilon u_x^2 + a(\varepsilon)u^2) \, dx, \tag{23}$$

where $\varepsilon > 0$ is an arbitrary, value of $a(\varepsilon)$ is defined by Ω.
Estimate $\left|\int_{S_\tau} r\psi u^m \, ds\, dt - \int_{S_\tau} r\psi u^* \, ds\, dt\right|$, using the Cauchy inequality and (23):

$$\left|\int_{S_\tau} r\psi u^m \, ds\, dt - \int_{S_\tau} r\psi u^* \, ds\, dt\right|^2 = \left|\int_{S_\tau} r\psi(u^m - u^*) \, ds\, dt\right|^2 \le$$

$$\le \left(\int_{S_\tau} r^2\psi^2 \, ds\, dt\right)\left(\int_{S_\tau} (u^m - u^*)^2 \, ds\, dt\right) \le \left(\int_0^\tau r^2 \, dt\right)\left(\int_{\partial\Omega} \psi^2 \, ds\right) \times$$

$$\times \left(\varepsilon \int_{Q_\tau} (u_x^m - u_x^*)^2 \, dx\, dt + a(\varepsilon)\int_{Q_\tau} (u^m - u^*)^2 \, dx\, dt\right)$$

Then, using strong convergence u^m to u^* in $W_2^{1,0}(Q_T)$, we see that

$$\left|\int_{S_\tau} r\psi u^m \, ds\, dt - \int_{S_\tau} r\psi u^* \, ds\, dt\right| \to 0 \quad \text{as} \quad m \to \infty.$$

Note, that

$$\left|\int_{Q_\tau} c(x)(u^m)^2 \, dx\, dt\right| \le c_1\|u^m\|^2_{L_2(Q_T)}, \tag{24}$$

$$\left|\int_{Q_\tau} f(x,t)u^m \, dx\, dt\right| \le \|f(x,t)\|_{L_2(Q_T)} \|u^m\|_{L_2(Q_T)}. \tag{25}$$

It follows from (22), (24), (25) and strong convergence of $u^m(x,t)$ to $u^*(x,t)$ that we can pass to the limit as $m \to \infty$ in (19). As the result, we obtain (18) for the function $u \in V_2(Q_T)$.

Lemma 3.3 *Let the function $u(x,t)$ be a solution of the problem (1), (2), (5) with a given function $r(t) \in L_2(0,T)$. Then*

$$|u|_{Q_T} \le Const. \tag{26}$$

Proof. By Lemma 3.2 the solution $u(x,t)$ of the problem (1), (2), (5) satisfies the integral identity

$$\frac{1}{2}\int_\Omega u^2 dx + \int_{Q_\tau} u_x^2 dxdt = \frac{1}{2}\int_\Omega \varphi^2\, dx - \int_{Q_\tau} cu^2 dxdt + \int_{S_\tau} ur\psi\, dsdt + \int_{Q_\tau} fu\, dxdt.$$

Using the ε-inequality for the surface integral, the elementary inequality for the last term and (15), we obtain

$$\frac{1}{2}\int_\Omega u^2\, dx + \int_{Q_\tau} u_x^2\, dx\, dt \le \frac{1}{2}\int_\Omega \varphi^2\, dx + c_1\int_{Q_\tau} u^2\, dx\, dt + \frac{1}{2}\int_{Q_\tau} f^2\, dx\, dt +$$

$$+\frac{1}{2\varepsilon}\int_{S_\tau} u^2\, ds\, dt + \frac{\varepsilon}{2}\int_{S_\tau} r^2\psi^2\, ds\, dt + \frac{1}{2}\int_{Q_\tau} u^2\, dx\, dt. \tag{27}$$

Using inequality (23) for the 4th term of (27) we get

$$\frac{1}{2}\int_\Omega u^2\, dx + \frac{1}{2}\int_{Q_\tau} u_x^2\, dx\, dt \le \frac{1}{2}\int_\Omega \varphi^2\, dx + \frac{1}{2}\int_{Q_\tau} f^2\, dx\, dt +$$

$$+\frac{1}{2}\left(2c_1 + 1 + a(\varepsilon)\varepsilon^{-1}\right)\int_{Q_\tau} (z^m)^2 dxdt + \frac{G\varepsilon}{2}\int_0^\tau (p^m)^2 dt, \text{ where } G = \int_{\partial\Omega} \psi^2\, ds.$$

Denote $F(\tau) = \int_\Omega \varphi^2\, dx + \int_{Q_\tau} f^2\, dx\, dt + G\varepsilon\int_0^\tau (p^m)^2\, dt$, $M = 2c_1 + 1 + a(\varepsilon)\varepsilon^{-1}$. Then

$$\int_\Omega u^2\, dx + \int_{Q_\tau} u_x^2\, dx\, dt \le M\int_{Q_\tau} u^2\, dx\, dt + F(\tau). \tag{28}$$

Hence, $\int_\Omega u^2(x,\tau)\, dx \le M\int_{Q_\tau} u^2\, dx\, dt + F(\tau)$. Applying Gronwall's inequality to the last expression, we obtain $\int_0^\tau\int_\Omega u^2\, dx\, dt \le \frac{F(\tau)}{M}\left(e^{M\tau} - 1\right)$. Then from the (28) it follows that $\int_\Omega u^2\, dx + \int_{Q_\tau} u_x^2\, dx\, dt \le F(\tau)e^{M\tau}$. As $\tau \in [0,T]$, then $|u|_{V_2(Q_T)} \le Const$. Lemma 3.3 is proved.

4 Proof of the Theorem 3.1

Existence. Let the solution of the problem (1), (2), (5), (8) be the sequence $\{(u^m, r^m)\}$, $\quad m = 1, 2, \ldots$. defined as

$$r^m(t) = \frac{1}{h(t)}\left[E_t + \int_\Omega \overline{K}u^{m-1}\, dx + \int_\Omega K_x u_x^{m-1}\, dx - \int_\Omega Kf\, dx\right], \tag{29}$$

$$\int_{Q_T} (-u^m\eta_t + u_x^m\eta_x + cu^m\eta)\, dx\, dt = \int_{S_T} r^m(t)\psi(x)\eta\, ds\, dt +$$

$$+\int_{Q_T} f\eta\, dx\, dt + \int_\Omega \varphi\eta(x,0)\, dx, \tag{30}$$

$\forall\, \eta(x,t) \in W_2^1(Q_T)$, $\eta(x,T) = 0$, $\forall\, m = 1, 2, \ldots,$ and $u^0 = 0$.

First, we prove that the pair (u^m, r^m) is well-defined, that is, for every function $r^m(t)$, defined by (29), there exists a solution $u^m(x,t)$ of the problem (1), (2), (5) which satisfies (30). We shall use Galerkin method. Let us temporarily omit the superscript of the functions $u^m(x,t)$ and $r^m(t)$.

Let $\varphi_k(x)$ be a basis in $W_2^1(\Omega)$, and $(\varphi_k, \varphi_l) = \delta_k^l$. We define approximations $u^N = \sum_{k=1}^{N} c_k^N(t)\varphi_k(x)$, where the functions $c_k^N(t)$ are defined by

$$\frac{d}{dt}(u^N, \varphi_k) + (u^N_{x_i}, \varphi_{k\, x_j}) + (cu^N, \varphi_k) = (f, \varphi_k) + \int_{\partial\Omega} r(t)\psi(x)\varphi_k\, ds, \tag{31}$$

$$c_k^N(0) = (\varphi, \varphi_k), \quad k = 1, \ldots, N. \tag{32}$$

The equalities (31) form the system of N ordinary differential equations:

$$\frac{d}{dt}c_k^N(t) + Ac_k^N(t) = F_k(t) \tag{33}$$

with bounded coefficients and summable on $[0,T]$ free terms of the equation as follows from the conditions (6), (7). Therefore, (33) with (32) define values of $c_k^N(t)$ ([9], p.27).

We multiply each equation of (31) by its own c_k^N, sum obtained equalities from $k = 1$ to $k = N$, integrate with respect to t from 0 to t_1, $t_1 < T$ and obtain

$$\frac{1}{2}\int_\Omega (u^N)^2\, dx + \int_{Q_{t_1}} (u^N_x)^2\, dx\, dt = \frac{1}{2}\int_\Omega \varphi^2(x)\, dx - \int_{Q_{t_1}} c(u^N)^2\, dx\, dt +$$

$$+\int_{S_{t_1}} u^N r(t)\psi(x)\, ds\, dt + \int_{Q_{t_1}} fu^N\, dx\, dt.$$

Then by Lemma 2.1 we have the estimation (26) for the function u

$$|u^N|_{Q_T} \le Const. \tag{34}$$

We shall prove that the sequence $\{u^N\}$ has a subsequence in $L_2(\Omega)$ which converges uniformly with respect to t on $[0,T]$. Thereby, we shall show that $l_{N,k}(t) = (u^N(x,t), \varphi_k(x))$ converge uniformly on $[0,T]$.

From (34) it follows that the functions $l_{N,k}(t)$ are uniformly bounded. Let us show that they are equicontinuous on $[0,T]$ for a fixed k and an arbitrary $N \ge k$, that is $|l_{N,k}(t + \triangle t) - l_{N,k}(t)| \to 0$ when $\triangle t \to 0$.
From (31) it follows that

$$\int_t^{t+\triangle t} \frac{d}{dt} l_{N,k}(t)\, dt + \int_t^{t+\triangle t} (u^N_{x_i}, \varphi_{k\, x_j})\, dt + \int_t^{t+\triangle t} (cu^N, \varphi_k)\, dt =$$

$$= \int_t^{t+\triangle t} (f, \varphi_k)\, dt + \int_t^{t+\triangle t} \int_{\partial\Omega} r(t)\psi(x)\varphi_k\, ds\, dt.$$

Therefore,

$$|l_{N,k}(t+\triangle t) - l_{N,k}(t)| \le \int_t^{t+\triangle t} (\left|(u^N_{x_i},\, \varphi_{k\, x_j})\right| + \left|(cu^N,\, \varphi_k)\right| + |(f, \varphi_k)| +$$

$$+ \left| \int_{\partial\Omega} r(t)\psi(x)\varphi_k\, ds \right|)\, dt.$$

Applying the Cauchy inequality, (6), (7), (34) we obtain

$$\int_t^{t+\triangle t} \int_\Omega |u^N_{x_i}\varphi_{k\, x_j}|\, dx\, dt \le \triangle t Const \left\|\varphi_{kx}\right\|_{L_2(\Omega)},$$

$$\int_t^{t+\triangle t} \int_\Omega |cu^N\varphi_k|\, dx\, dt \le \triangle t Const \left\|\varphi_{kx}\right\|_{L_2(\Omega)};$$

$$\int_t^{t+\triangle t} |r(t)| \left| \int_{\partial\Omega} \psi(x)\varphi_k\, ds \right| dt \to 0 \quad \text{as} \quad \triangle t \to 0;$$

$$\int_t^{t+\triangle t} \int_\Omega |f\varphi_k|\, dx\, dt \le \triangle t |f|_{Q_{t,t+\triangle t}} \left\|\varphi_k\right\|_{L_2(\Omega)}.$$

Therefore $|l_{N,k}(t+\triangle t) - l_{N,k}(t)| \le \varepsilon(\triangle t) \left\|\varphi_k\right\|_{W_2^1(\Omega)}$, where $\varepsilon(\triangle t) \to 0$ as $\triangle t \to 0$ and $\varepsilon(\triangle t)$ does not depend on N. It means that $l_{N,k}$, $N = k,\, k{+}1,\, \ldots,$ are uniformly continuous with respect to t.

Then we choose a subsequence N_m for which the functions $l_{N_m,k}(t)$ converge uniformly on $[0,T]$ to some continuous function $l_k(t)$ for every $k = 1, 2, \ldots$.

Define the function $u = \sum\limits_{k=1}^{\infty} l_k(t)\varphi_k(x)$. Let us show that the sequence u^{N_m} weakly converges to $u(x,t)$ in $L_2(\Omega)$ uniformly with respect to $t \in [0,T]$. To this end we consider an arbitrary function $\varphi(x) \in L_2(\Omega)$. We shall show that $(u^{N_m} - u, \varphi) \to 0$ as $N_m \to \infty$. We have

$$(u^{N_m} - u, \varphi) = \left(u^{N_m} - u, \sum_{k=1}^{\infty} (\varphi, \varphi_k)\varphi_k \right) = \sum_{k=1}^{s} (\varphi, \varphi_k)(u^{N_m} - u, \varphi_k) +$$

$$+ \left(u^{N_m} - u, \sum_{k=s+1}^{\infty} (\varphi, \varphi_k)\varphi_k \right). \tag{35}$$

Estimate the second term on the right-hand side part of(35):

$$\left| \left(u^{N_m} - u, \sum_{k=s+1}^{\infty} (\varphi, \varphi_k)\varphi_k \right) \right|^2 \le \|u^{N_m} - u\|^2 \left\| \sum_{k=s+1}^{\infty} (\varphi, \varphi_k)\varphi_k \right\|^2 =$$

$$= \|u^{N_m} - u\|^2 \left(\sum_{k=s+1}^{\infty} (\varphi, \varphi_k)^2 \right) \le M_1 R(s),$$

where $R(s)$ is the remainder of the convergent Fourier series, M_1 does not depend on N_m. We choose such s that $M_1R(s)$ is less than $\varepsilon_1 > 0$.
Consider the first term on the right-hand side part of (35):

$$\sum_{k=1}^{s} (\varphi, \varphi_k)(u^{N_m} - u, \varphi_k) = \sum_{k=1}^{s} (\varphi, \varphi_k)(l_{N_m,k} - l_k).$$

From the uniform convergence of $l_{N_m,k}$ to l_k in $[0,T]$ it follows that for fixed s the sum $\sum_{k=1}^{s} (\varphi, \varphi_k)(l_{N_m,k} - l_k)$ can be made less than an abitrary small number uniformly for every $t \in [0,T]$. Therefore, $|(u^{N_m} - u, \varphi)| < \varepsilon_1$ for every $t \in [0,T]$. It means that the sequence $\{u^{N_m}\}$ weakly converges $u(x,t)$ in $L_2(\Omega)$ uniformly for t in $[0,T]$.

Therefore, the sequence $\{u^{N_m}\}$ has a subsequence which weakly converges to the function u in $L_2(Q_T)$ with $u_x^{N_m}$. Applying the property of weak convergence for the limit function also we have that $|u|_{Q_T} \le Const$. Hence, $u \in V_2(Q_T)$.

We need only to show that this limit function is a required generalized solution. To show that (14) is valid we multiply (31) by $d_k(t) \in C^1[0,T]$, $d_k(T) = 0$, take the sum with respect to k from 1 to $N' \le N$ and integrate the result with respect to t from 0 to T. We have

$$\int_0^T (-(u^N, \Phi_t^{N'}) + (u_{x_j}^N, \Phi_{x_i}^{N'}) + (cu^N, \Phi^{N'}))\, dt = \int_0^T (f, \Phi^{N'})\, dt +$$

$$+(\varphi(x), \Phi^{N'}(x,0)) + \int_0^T \int_{\partial\Omega} r(t)\psi(x)\Phi^{N'}\, ds\, dt, \tag{36}$$

where $\Phi^{N'}(x,t) = \sum_{k=1}^{N'} d_k(t)\varphi_k(x)$.

Since the functions $\Phi^{N'}$, $\Phi_t^{N'}$, $\Phi_{x_i}^{N'} \in L_2(Q_T)$ and subsequence u^{N_m} weakly converges in $L_2(Q_T)$, so it is possible to pass to the limit in (36) for the chosen subsequence N_m with a fixed N' . This gives us (36) for $u(x,t) \in V_2(Q_T)$:

$$\int_0^T (-(u, \Phi_t^{N'}) + (u_{x_j}, \Phi_{x_i}^{N'}) + (cu, \Phi^{N'}))\, dt = \int_0^T (f, \Phi^{N'})\, dt +$$

$$+(\varphi(x), \Phi^{N'}(x,0)) + \int_0^T \int_{\partial\Omega} r(t)\psi(x)\Phi^{N'}\, ds\, dt, \tag{37}$$

As the set $\Phi = \bigcup_{N'=1}^{\infty} \Phi^{N'}$ is dense in $\widehat{W}_2^1(Q_T)$ [12], it follows that the limit relation is fulfillednfor every function $\Phi(x,t) \in \widehat{W}_2^1(Q_T)$ and hence, u is the solution of the problem (1), (2), (5). Thus, the pair (u^m, r^m) is defined correctly.

We shall prove that the sequence $\{(u^m, r^m)\}$ is fundamental. Assume that $z^m = u^{m+1} - u^m, \quad p^m = r^{m+1} - r^m$. Then from (29), (30) we obtain

$$p^m(t) = \frac{1}{h(t)}\left[\int_\Omega \overline{K} z^{m-1}\,dx + \int_\Omega K_x z_x^{m-1}\,dx\right] \tag{38}$$

and $$\int_{Q_T} (-z^m \eta_t + z_x^m \eta_x + c z^m \eta)\,dx\,dt = \int_{S_T} p^m(t)\psi(x)\eta\,ds\,dt, \tag{39}$$

$\forall\, \eta(x,t) \in W_2^1(Q_T)$, $\eta(x,T) = 0$, $\forall\, m = 1, 2, \ldots$.
Consider $\|p^m\|^2_{L_2(0,T)}$. From (38) it follows that

$$\int_0^\tau (p^m)^2\,dt = \int_0^\tau \frac{1}{h^2(t)}\left(\int_\Omega \overline{K} z^{m-1}\,dx + \int_\Omega K_x z_x^{m-1}\,dx\right)^2 dt.$$

Using the Cauchy inequality we obtain

$$\int_0^\tau (p^m)^2\,dt \le \frac{2}{h_1^2}\int_0^\tau \left[\left(\int_\Omega \overline{K}^2\,dx\right)\left(\int_\Omega (z^{m-1})^2\,dx\right) + \right.$$

$$\left. + \left(\int_\Omega K_x^2\,dx\right)\left(\int_\Omega (z_x^{m-1})^2\,dx\right)\right] dt \le M^2 |z^{m-1}|^2_{V_2(Q_\tau)}, \tag{40}$$

where positive constant M depends on K, h, T. From (39) and lemma 3.2 the following equality is valid

$$\frac{1}{2}\int_\Omega (z^m)^2\,dx + \int_{Q_\tau} (z_x^m)^2\,dx\,dt = -\int_{Q_\tau} c(z^m)^2\,dx\,dt + \int_{S_\tau} z^m p^m \psi\,ds\,dt. \tag{41}$$

From (41) and arguments of the proof of lemma 3.3 we obtain that

$$|z^m|_{V_2(Q_\tau)} \le \sqrt{\varepsilon G}\exp\left[(c_1 + a(\varepsilon)(2\varepsilon)^{-1})\tau\right]\|p^m\|_{L_2(0,\tau)}. \tag{42}$$

Therefore, from (40), (42) it follows that

$$|z^m|_{V_2(Q_\tau)} \le \sqrt{\varepsilon}\sqrt{G}M\exp\left[(c_1 + a(\varepsilon)(2\varepsilon)^{-1})\tau\right]|z^{m-1}|_{V_2(Q_\tau)},$$

$$\|p^m\|_{L_2(0,\tau)} \le \sqrt{\varepsilon}\sqrt{G}M\exp\left[(c_1 + a(\varepsilon)(2\varepsilon)^{-1})\tau\right]\|p^{m-1}\|_{L_2(0,\tau)}.$$

Choose $\varepsilon_0 > 0$ and $0 < \sigma \le 1$ such that $0 < \sqrt{\varepsilon_0}\sqrt{G}M \le \dfrac{1}{2}$ and
$\exp\left[(c_1 + a(\varepsilon_0)(2\varepsilon_0)^{-1})\sigma\right] \le \dfrac{3}{2}$. Then $|z^m|_{V_2(Q_\sigma)} \le \dfrac{3}{4}|z^{m-1}|_{V_2(Q_\sigma)}$,

$$\|p^m\|_{L_2(0,\sigma)} \le \frac{3}{4}\|p^{m-1}\|_{L_2(0,\sigma)}, \quad \forall m = 1, 2, \ldots, \quad \text{where} \quad Q_\tau = \Omega \times (0, \sigma].$$

Hence, the sequence $\{(u^m, r^m)\}$ is fundamental in $V_2(Q_\sigma) \times L_2(0,\sigma)$. It means that there exists the unique pair of functions $(u, r) \in V_2(Q_\sigma) \times L_2(0,\sigma)$

such that $u^m \to u$ in $V_2(Q_\sigma)$ and $r^m \to r$ in $L_2(0,\sigma)$.

Letting $m \to \infty$ in (29), (30), we obtain that the pair (u, r) is a local solution of the problem (1), (2), (5), (8) in the cylinder Q_σ. Taking into account the compatibility condition (17) we can see that $r(0) = 1$.

Uniqueness. Suppose there exist two different pairs of solution (u_1, r_1), $(u_1, r_1) \in V_2(Q_\sigma) \times L_2(0,\sigma)$. From the section 2 it follows that

$$|u_1 - u_2|_{V_2(Q_\sigma)} \leq \frac{3}{4}|u_1 - u_2|_{V_2(Q_\sigma)}, \quad \|r_1 - r_2\|_{L_2(0,\sigma)} \leq \frac{3}{4}\|r_1 - r_2\|_{L_2(0,\sigma)}.$$

Hence, $u_1 \equiv u_2$ in $V_2(Q_\sigma)$ and $r_1 \equiv r_2$ in $L_2(0,\sigma)$ and the solution of the problem (1), (2), (5), (8) is unique in the cylinder Q_σ.

Now we are to prove the existence and uniqueness of the solution in the whole cylinder Q_T. Let $t \geq \sigma$ $u(x,\sigma) = \varphi(x,\sigma)$, where $\varphi(x,\sigma)$ be a value of the local solution $u(x,t)$ when $t = \sigma$.

Repeating described calculations for $\sigma \leq t \leq 2\sigma$ we obtain

$$|z^m|_{V_2(Q_{\sigma,\,2\sigma})} \leq \frac{3}{4}|z^{m-1}|_{V_2(Q_{\sigma,\,2\sigma})},$$

$$\|p^m\|_{L_2(\sigma,\,2\sigma)} \leq \frac{3}{4}\|p^{m-1}\|_{L_2(\sigma,\,2\sigma)}, \quad \forall m = 1,\, 2,\, \ldots.$$

From these inequalities and sections 1,2 it follows that there exist the unique solution of the problem (1), (2),(5), (8) as $\sigma \leq t \leq 2\sigma$.

Repeating these arguments, for a finite number of steps we conclude that there exists the unique solution (u, r) of the problem (1), (2), (5), (8) in the whole cylinder Q_T.

This ends the proof.

5 Conclusion

In this research we consider the problem for the multidimensional parabolic equation with the first order nonlocal integral condition which cannot be reduce to the second order nonlocal condition as it can be done in one-dimensional case. We establish the connection between nonlocal and inverse problems and applying this useful result to prove existence and uniqueness of the solution of the nonlocal problem.

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Multivariate Fuzzy-Random Perturbed Neural Network Approximation

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Abstract

In this paper we study the rate of multivariate pointwise and uniform convergences in the q-mean to the Fuzzy-Random unit operator of perturbed multivariate normalized Fuzzy-Random neural network operators of Stancu, Kantorovich and Quadrature types. These multivariate Fuzzy-Ranfom operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given via multivariate Probabilistic-Jackson type inequalities using the multivariate Fuzzy-Random modulus of continuity of the involved multivariate Fuzzy-Random function. Also some interesting results in multivariate Fuzzy-Random Analysis are given of independent merit.

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1 Fuzzy Random Analysis Basics

We begin with

Definition 1 *(see [16]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [16]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [16], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall\, u, v, w \in \mathbb{R}_{\mathcal{F}}, \qquad (1) \\ D(k \odot u, k \odot v) &= |k|\, D(u,v), \quad \forall\, u, v \in \mathbb{R}_{\mathcal{F}},\ \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall\, u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in \mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0,1]$.

We need

Lemma 2 *([5]) For any $a, b \in \mathbb{R} : a \cdot b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leq |a - b| \cdot D(u, \widetilde{o}), \qquad (2)$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{o} := \chi_{\{0\}}$.

Lemma 3 *([5])*

(i) If we denote $\widetilde{o} := \chi_{\{0\}}$, *then* $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ *is the neutral element with respect to* $\oplus$, *i.e.,* $u \oplus \widetilde{o} = \widetilde{o} \oplus u = u$, $\forall\ u \in \mathbb{R}_{\mathcal{F}}$.

(ii) With respect to $\widetilde{o}$, *none of* $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \widetilde{o}$ *has opposite in* $\mathbb{R}_{\mathcal{F}}$.

(iii) Let $a, b \in \mathbb{R} : a \cdot b \geq 0$, *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a+b) \odot u = a \odot u \oplus b \odot u$. *For general* $a, b \in \mathbb{R}$, *the above property is false.*

(iv) For any $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) For any $\lambda, \mu \in \mathbb{R}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) If we denote $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall\ u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.,*

$$\|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o}, \quad \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$
$$\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \quad \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v). \tag{3}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is not a linear space over $\mathbb{R}$; and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space.

As in Remark 4.4 ([5]) one can show easily that a sequence of operators of the form

$$L_n(f)(x) := \sum_{k=0}^{n*} f(x_{k_n}) \odot w_{n,k}(x), \quad n \in \mathbb{N}, \tag{4}$$

($\sum^{*}$ denotes the fuzzy summation) where $f : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$, $x_{k_n} \in \mathbb{R}^d$, $d \in \mathbb{N}$, $w_{n,k}(x)$ real valued weights, are linear over $\mathbb{R}^d$, i.e.,

$$L_n(\lambda \odot f \oplus \mu \odot g)(x) = \lambda \odot L_n(f)(x) \oplus \mu \odot L_n(g)(x), \tag{5}$$

$\forall \lambda, \mu \in \mathbb{R}$, any $x \in \mathbb{R}^d$; $f, g : \mathbb{R}^d \to \mathbb{R}_{\mathcal{F}}$. (Proof based on Lemma 3 (iv).)

We further need

Definition 4 *(see also [14], Definition 13.16, p. 654) Let* $(X, \mathcal{B}, P)$ *be a probability space. A fuzzy-random variable is a* $\mathcal{B}$*-measurable mapping* $g : X \to \mathbb{R}_{\mathcal{F}}$ *(i.e., for any open set* $U \subseteq \mathbb{R}_{\mathcal{F}}$, *in the topology of* $\mathbb{R}_{\mathcal{F}}$ *generated by the metric* D, *we have*

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{6}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. *Let* $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ *and* $0 < q < +\infty$. *We say* $g_n(s) \overset{\text{"}q\text{-mean"}}{\underset{n \to +\infty}{\rightarrow}} g(s)$ *if*

$$\lim_{n \to +\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{7}$$

Remark 5 *(see [14], p. 654) If* $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, *let us denote* $F : X \to \mathbb{R}_+ \cup \{0\}$ *by* $F(s) = D(f(s), g(s))$, $s \in X$. *Here,* F *is* $\mathcal{B}$*-measurable, because*

$F = G \circ H$, where $G(u,v) = D(u,v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.

Definition 6 *(see [14], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t,s)$, $t \in T$, $s \in X$.*

Remark 7 *(see [14], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t,s) = f(t)$, $\forall\, t \in T$, $s \in X$.*

Remark 8 *(see [14], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.*

Remark 9 *(see [14], p. 655) Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,*

$$(f \oplus g)(t,s) = f(t,s) \oplus g(t,s),$$
$$(k \odot f)(t,s) = k \odot f(t,s), \quad t \in T, s \in X.$$

Definition 10 *(see also Definition 13.18, pp. 655-656, [14]) For a fuzzy-random function $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$
$$\sup\left\{ \left(\int_X D^q(f(x,s), f(y,s))\, P(ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d, \ \ \|x-y\|_{l_1} \le \delta \right\}, \quad (8)$$

$0 < \delta$, $1 \le q < \infty$.

Definition 11 *([9]) Here $1 \le q < +\infty$. Let $f : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over $\mathbb{R}^d$, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:whenever $\|x-y\|_{l_1} \le \delta$, $x, y \in \mathbb{R}^d$, implies that*

$$\int_X (D(f(x,s), f(y,s)))^q\, P(ds) \le \varepsilon. \quad (9)$$

We denote it as $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Proposition 12 *([9]) Let $f \in C_{FR}^{U_q}(\mathbb{R}^d)$. Then $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$, any $\delta > 0$.*

Proposition 13 *([9]) Let $f, g : \mathbb{R}^d \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})} (f, \delta)_{L^q} = \Omega_1^{(\mathcal{F})} (f, 0)_{L^q} = 0$, *iff* $f \in C_{FR}^{U_q} (\mathbb{R}^d)$.

(iii) $\Omega_1^{(\mathcal{F})} (f, \delta_1 + \delta_2)_{L^q} \leq \Omega_1^{(\mathcal{F})} (f, \delta_1)_{L^q} + \Omega_1^{(\mathcal{F})} (f, \delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1^{(\mathcal{F})} (f, n\delta)_{L^q} \leq n\Omega_1^{(\mathcal{F})} (f, \delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1^{(\mathcal{F})} (f, \lambda\delta)_{L^q} \leq \lceil \lambda \rceil \Omega_1^{(\mathcal{F})} (f, \delta)_{L^q} \leq (\lambda + 1) \Omega_1^{(\mathcal{F})} (f, \delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, *where* $\lceil \cdot \rceil$ *is the ceilling of the number.*

(vi) $\Omega_1^{(\mathcal{F})} (f \oplus g, \delta)_{L^q} \leq \Omega_1^{(\mathcal{F})} (f, \delta)_{L^q} + \Omega_1^{(\mathcal{F})} (g, \delta)_{L^q}$, $\delta > 0$. *Here* $f \oplus g$ *is a fuzzy random function.*

(vii) $\Omega_1^{(\mathcal{F})} (f, \cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$, *for* $f \in C_{FR}^{U_q} (\mathbb{R}^d)$.

We give

Definition 14 *([7]) Let* $f(t,s)$ *be a stochastic process from* $\mathbb{R}^d \times (X, \mathcal{B}, P)$ *into* $\mathbb{R}$, $d \in \mathbb{N}$, *where* $(X, \mathcal{B}, P)$ *is a probability space. We define the q-mean multivariate first moduli of continuity of* f *by*

$$\Omega_1 (f, \delta)_{L^q} :=$$

$$\sup \left\{ \left(\int_X |f(x,s) - f(y,s)|^q P(ds) \right)^{\frac{1}{q}} : x, y \in \mathbb{R}^d, \ \|x - y\|_{l_1} \leq \delta \right\}, \quad (10)$$

$\delta > 0$, $1 \leq q < \infty$.

For more see [7].

We mention

Proposition 15 *([9]) Assume that* $\Omega_1^{(\mathcal{F})} (f, \delta)_{L^q}$ *is finite,* $\delta > 0$, $1 \leq q < \infty$. *Then*

$$\Omega_1^{(\mathcal{F})} (f, \delta)_{L^q} \geq \sup_{r \in [0,1]} \max \left\{ \Omega_1 \left(f_-^{(r)}, \delta \right)_{L^q}, \Omega_1 \left(f_+^{(r)}, \delta \right)_{L^q} \right\}. \quad (11)$$

The reverse direction "$\leq$*" "is not possible.*

Remark 16 *([9]) For each* $s \in X$ *we define the usual first modulus of continuity of* $f(\cdot, s)$ *by*

$$\omega_1^{(\mathcal{F})} (f(\cdot, s), \delta) := \sup_{\substack{x,y \in \mathbb{R}^d \\ \|x-y\|_{l_1} \leq \delta}} D(f(x,s), f(y,s)), \quad \delta > 0. \quad (12)$$

Therefore

$$D^q (f(x,s), f(y,s)) \leq \left(\omega_1^{(\mathcal{F})} (f(\cdot, s), \delta) \right)^q, \quad (13)$$

$\forall\ s \in X$ *and* $x, y \in \mathbb{R}^d : \|x - y\|_{l_1} \leq \delta$, $\delta > 0$.

Hence it holds

$$\left(\int_X D^q\left(f\left(x,s\right),f\left(y,s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}} \leq \left(\int_X \left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}, \tag{14}$$

$\forall\ x,y \in \mathbb{R}^d : \|x-y\|_{l_1} \leq \delta.$

We have that

$$\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q} \leq \left(\int_X \left(\omega_1^{(\mathcal{F})}\left(f\left(\cdot,s\right),\delta\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}, \tag{15}$$

under the assumption that the right hand side of (15) is finite.

The reverse "$\geq$" of the last (15) is not true.

Also we have

Proposition 17 *([6]) (i) Let $Y\left(t,\omega\right)$ be a real valued stochastic process such that Y is continuous in $t \in [a,b]$. Then Y is jointly measurable in $\left(t,\omega\right)$.*

(ii) Further assume that the expectation $\left(E\left|Y\right|\right)(t) \in C\left([a,b]\right)$, or more generally $\int_a^b \left(E\left|Y\right|\right)(t)\,dt$ makes sense and is finite. Then

$$E\left(\int_a^b Y\left(t,\omega\right)dt\right) = \int_a^b \left(EY\right)(t)\,dt. \tag{16}$$

According to [13], p. 94 we have the following

Definition 18 *Let $\left(Y,\mathcal{T}\right)$ be a topological space, with its σ-algebra of Borel sets $\mathcal{B} := \mathcal{B}\left(Y,\mathcal{T}\right)$ generated by $\mathcal{T}$. If $\left(X,\mathcal{S}\right)$ is a measurable space, a function $f : X \to Y$ is called measurable iff $f^{-1}\left(B\right) \in \mathcal{S}$ for all $B \in \mathcal{B}$.*

By Theorem 4.1.6 of [13], p. 89 f as above is measurable iff

$$f^{-1}\left(C\right) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We would need

Theorem 19 *(see [13], p. 95) Let $\left(X,\mathcal{S}\right)$ be a measurable space and $\left(Y,d\right)$ be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n\left(x\right) \to f\left(x\right)$ in Y. Then f is measurable. I.e., $\lim_{n\to\infty} f_n = f$ is measurable.*

We need also

Proposition 20 *Let f,g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then*

(i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.

(ii) $f \oplus g$ is a fuzzy random variable.

For the definition of general fuzzy integral we follow [15] next.

Definition 21 *Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega \tag{17}$$

is measurable, see [15].

Theorem 22 *([15]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \leq r \leq 1 \right\}, \tag{18}$$

the following are equivalent
(1) F is measurable,
(2) $\forall\ r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are measurable.

Following [15], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\} \tag{19}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 23 *([15]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{ \left(F_-^{(r)}(w), F_+^{(r)}(w) \right) | 0 \leq r \leq 1 \right\}$$

is called integrable if for each $r \in [0,1]$, $F_\pm^{(r)}$ are integrable, or equivalently, if $F_\pm^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\}.$$

By [15], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 24 *([15]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu; \tag{20}$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)\, d\mu. \tag{21}$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu. \tag{22}$$

Above μ could be the Lebesgue measure, in this article the multivariate Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{23}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\, r \in [0, 1]. \tag{24}$$

Next we state Fubini's theorem for fuzzy number-valued functions and fuzzy number-valued integrals, see [15].

Theorem 25 *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two complete σ-finite measure spaces, and let $(\Omega_1 \times \Omega_2, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2)$ be their product measure space. If a fuzzy number-valued function $F : \Omega_1 \times \Omega_2 \to \mathbb{R}_{\mathcal{F}}$ is $\mu_1 \times \mu_2$-integrable, then*

(1) the fuzzy-number-valued function $F(\cdot, \omega_2) : \omega_1 \to F(\omega_1, \omega_2)$ is μ_1-integrable for $\omega_2 \in \Omega_2$, μ_2-a.e.,

(2) the fuzzy-number-valued function $\omega_2 \to \int_{\Omega_1} F(\omega_1, \omega_2)\, d\mu_1(\omega_1)$ is μ_2-integrable,

and

(3)

$$\int_{\Omega_1 \times \Omega_2} F d(\mu_1 \times \mu_2) = \int_{\Omega_2} \left(\int_{\Omega_1} F(\omega_1, \omega_2)\, d\mu_1(\omega_1)\right) d\mu_2(\omega_2) = \tag{25}$$

$$\int_{\Omega_1} \left(\int_{\Omega_2} F(\omega_1, \omega_2)\, d\mu_2(\omega_2)\right) d\mu_1(\omega_1).$$

We further mention

Theorem 26 *Let* $f : \prod_{i=1}^{N} [a_i, b_i] \to L_{\mathcal{F}}(X, \mathcal{B}, P)$, *be a fuzzy-random function. We assume that* $f\left(\overrightarrow{t}, s\right)$ *is fuzzy continuous in* $\overrightarrow{t} \in \prod_{i=1}^{N} [a_i, b_i]$, *for any* $s \in X$. *Then* $\int_{\prod_{i=1}^{N}[a_i,b_i]} f\left(\overrightarrow{t}, s\right) d\overrightarrow{t}$ *exists in* $\mathbb{R}_{\mathcal{F}}$ *and it is a fuzzy-random variable in* $s \in X$.

Proof. By definition of fuzzy integral we notice here that

$$\int_{\prod_{i=1}^{N}[a_i,b_i]} f\left(\overrightarrow{t}, s\right) d\overrightarrow{t} =$$

$$\left\{\left(\int_{\prod_{i=1}^{N}[a_i,b_i]} f_{-}^{(r)}\left(\overrightarrow{t}, s\right) d\overrightarrow{t}, \int_{\prod_{i=1}^{N}[a_i,b_i]} f_{+}^{(r)}\left(\overrightarrow{t}, s\right) d\overrightarrow{t}\right) | 0 \leq r \leq 1\right\}, \quad (26)$$

where $d\overrightarrow{t}$ is the multivariate Lebesgue measure on $\prod_{i=1}^{N} [a_i, b_i]$. Because $f\left(\overrightarrow{t}, s\right)$ is fuzzy continuous in $\overrightarrow{t}$, we get that $f_{\pm}^{(r)}\left(\overrightarrow{t}, s\right)$, $0 \leq r \leq 1$, are real valued continuous in $\overrightarrow{t}$, for each $s \in X$. Hence the real integrals $\int_{\prod_{i=1}^{N}[a_i,b_i]} f_{\pm}^{(r)}\left(\overrightarrow{t}, s\right) d\overrightarrow{t}$ are multivariate Riemann integrals that exist, for each $s \in X$.

Thus $\int_{\prod_{i=1}^{N}[a_i,b_i]} f\left(\overrightarrow{t}, s\right) d\overrightarrow{t} \in \mathbb{R}_{\mathcal{F}}$, i.e. it exists.

By Theorem 19 and the definition of multivariate Riemann integral we get that $\int_{\prod_{i=1}^{N}[a_i,b_i]} f_{\pm}^{(r)}\left(\overrightarrow{t}, s\right) d\overrightarrow{t}$ are P-measurable functions in $s \in X$.

Taking into account (24) and Theorem 22 we derive that $\int_{\prod_{i=1}^{N}[a_i,b_i]} f\left(\overrightarrow{t}, s\right) d\overrightarrow{t}$ is a fuzzy-random variable in $s \in X$. ■

2 Main Results

We are morivated by [9], [11], [12].

Here the activation function $b : \mathbb{R}^d \to \mathbb{R}_+$, $d \in \mathbb{N}$, is of compact support $B := \prod_{j=1}^{d} [-T_j, T_j]$, $T_j > 0$, $j = 1, ..., d$. That is $b(x) > 0$ for any $x \in B$, and clearly b may have jump discontinuities. Also the shape of the graph of b is immaterial.

Typically in neural networks approximation we take b to be a d-dimensional bell-shaped function (i.e. per coordinate is a centered bell-shaped function), or a product of univariate centered bell-shaped functions, or a product of sigmoid functions, in our case all them of compact support B.

Example 27 *Take* $b(x) = \beta(x_1)\beta(x_2)...\beta(x_d)$, *where* β *is any of the following functions,* $i = 1, ..., d$:

(*i*) $\beta(x_j)$ *is the characteristic function on* $[-1, 1]$,

(*ii*) $\beta(x_j)$ *is the hat function over* $[-1, 1]$, *that is,*

$$\beta(x_j) = \begin{cases} 1 + x_j, & -1 \leq x_j \leq 0, \\ 1 - x_j, & 0 < x_j \leq 1, \\ 0, & \textit{elsewhere}, \end{cases} \tag{27}$$

(*iii*) *the truncated sigmoids*

$$\beta(x_j) = \begin{cases} \frac{1}{1+e^{-x_j}} \ \textit{or} \ \tanh x_j \ \textit{or} \ \operatorname{erf}(x_j), \ \ \textit{for } x_j \in [-T_j, T_j], \textit{ with large } T_j > 0, \\ 0, \ \ x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

(*iv*) *the truncated Gompertz function*

$$\beta(x_j) = \begin{cases} e^{-\alpha e^{-\beta x_j}}, \ \ x_j \in [-T_j, T_j]; \ \alpha, \beta > 0; \ \textit{large } T_j > 0, \\ 0, \ \ x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

The Gompertz functions are also sigmoid functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

Thus the general activation function b we will be using here includes all kinds of activation functions in neural network approximations.

Here we consider functions $f \in C_{FR}^{U_q}(\mathbb{R}^d)$.

Let here the parameters: $0 < \alpha < 1$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$; $\overline{r} = (r_1, ..., r_d) \in \mathbb{N}^d$, $i = (i_1, ..., i_d) \in \mathbb{N}^d$, with $i_j = 1, 2, ..., r_j$, $j = 1, ..., d$; also let $w_i = w_{i_1,...,i_d} \geq 0$, such that $\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} = 1$, in brief written as $\sum_{i=1}^{\overline{r}} w_i = 1$. We further consider the parameters $k = (k_1, ..., k_d) \in \mathbb{Z}^d$; $\mu_i = (\mu_{i_1}, ..., \mu_{i_d}) \in \mathbb{R}_+^d$, $\nu_i = (\nu_{i_1}, ..., \nu_{i_d}) \in \mathbb{R}_+^d$; and $\lambda_i = \lambda_{i_1,...,i_d}$, $\rho_i = \rho_{i_1,...,i_d} \geq 0$. Call $\nu_i^{\min} = \min\{\nu_{i_1}, ..., \nu_{i_d}\}$.

In this article we study in q-mean ($1 \leq q < \infty$) the pointwise and uniform convergences with rates over $\mathbb{R}^d$, to the fuzzy-random unit operator, of the following fuzzy-random normalized one hidden layer multivariate perturbed neural network operators, where $s \in X$, $(X, \mathcal{B}, P)$ a probability space, $n \in \mathbb{N}$,

(i) the Stancu type

$$\left(H_n^{FR}(f)\right)(x, s) = \left(H_n^{FR}(f)\right)(x_1, ..., x_d, s) = \tag{28}$$

$$\frac{\sum_{k=-n^2}^{n^2*} \left(\sum_{i=1}^{\overline{r}*} w_i \odot f\left(\frac{k+\mu_i}{n+\nu_i}, s\right)\right) \odot b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\frac{\sum_{k_1=-n^2}^{n^2 *} \ldots \sum_{k_d=-n^2}^{n^2 *} \left(\sum_{i_1=1}^{r_1 *} \ldots \sum_{i_d=1}^{r_d *} w_{i_1,\ldots,i_d} \odot f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, \ldots, \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}, s\right)\right) \odot}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

(ii) the Kantorovich type

$$\left(K_n^{FR}(f)\right)(x,s) = \tag{29}$$

$$\frac{\sum_{k=-n^2}^{n^2 *} \left(\sum_{i=1}^{\overline{r} *} w_i (n+\rho_i)^d \odot \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right) dt \right) \odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} =$$

$$\sum_{k_1=-n^2}^{n^2 *} \ldots \sum_{k_d=-n^2}^{n^2 *} \left(\sum_{i_1=1}^{r_1 *} \ldots \sum_{i_d=1}^{r_d *} w_{i_1,\ldots,i_d} \left(n+\rho_{i_1,\ldots,i_d}\right)^d \odot \right.$$

$$\frac{\left. \int \ldots \int_0^{\frac{1}{n+\rho_{i_1,\ldots,i_d}}} \ldots \int f\left(t_1+\frac{k_1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}, \ldots, t_d+\frac{k_d+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}, s\right) dt_1 \ldots dt_d \right) \odot}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)} \tag{30}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii) the quadrature type

$$\left(M_n^{FR}(f)\right)(x,s) = \frac{\sum_{k=-n^2}^{n^2 *} \left(\sum_{i=1}^{\overline{r} *} w_i \odot f\left(\frac{k}{n}+\frac{i}{n\overline{r}}, s\right)\right) \odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} = \tag{31}$$

$$\frac{\sum_{k_1=-n^2}^{n^2 *} \ldots \sum_{k_d=-n^2}^{n^2 *} \left(\sum_{i_1=1}^{r_1 *} \ldots \sum_{i_d=1}^{r_d *} w_{i_1,\ldots,i_d} \odot f\left(\frac{k_1}{n}+\frac{i_1}{nr_1}, \ldots, \frac{k_d}{n}+\frac{i_d}{nr_d}, s\right)\right) \odot}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

Similar operators defined for d-dimensional bell-shaped activation functions and sample coefficients $f\left(\frac{k}{n}\right) = f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right)$ were studied initially in [9].

In this article we assume that

$$n \geq \max_{j \in \{1,\ldots,d\}} \left\{ T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right\}, \tag{32}$$

see also [3], p. 91.

So, by (32) we can rewrite the operators as follows ($[\cdot]$ is the integral part and $\lceil\cdot\rceil$ the ceiling of a number).

We denote by $T=(T_1,...,T_d)$, $[nx+Tn^{\alpha}]=([nx_1+T_1n^{\alpha}],...,[nx_d+T_dn^{\alpha}])$, $\lceil nx-Tn^{\alpha}\rceil=(\lceil nx_1-T_1n^{\alpha}\rceil,...,\lceil nx_d-T_dn^{\alpha}\rceil)$. It holds

(i)

$$\left(H_n^{FR}(f)\right)(x,s)=\left(H_n^{FR}(f)\right)(x_1,...,x_d,s)= \tag{33}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k+\mu_i}{n+\nu_i},s\right)\right)\odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}=$$

$$\frac{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]*}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]*}\left(\sum_{i_1=1}^{r_1*}...\sum_{i_d=1}^{r_d*}w_{i_1,...,i_d}\odot f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}},...,\frac{k_d+\mu_{i_d}}{n+\nu_{i_d}},s\right)\right)\odot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

(ii)

$$\left(K_n^{FR}(f)\right)(x,s)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\left(n+\rho_i\right)^d\odot\int_0^{\frac{1}{n+\rho_i}}f\left(t+\frac{k+\lambda_i}{n+\rho_i},s\right)dt\right)\odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{34}$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]*}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]*}\left(\sum_{i_1=1}^{r_1*}...\sum_{i_d=1}^{r_d*}w_{i_1,...,i_d}\left(n+\rho_{i_1,...,i_d}\right)^d\odot\right.$$

$$\frac{\left.\int...\int_0^{\frac{1}{n+\rho_{i_1,...,i_d}}}...\int f\left(t_1+\frac{k_1+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}},...,t_d+\frac{k_d+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}},s\right)dt_1...dt_d\right)\odot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}...\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)} \tag{35}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii)

$$\left(M_n^{FR}(f)\right)(x,s)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k}{n}+\frac{i}{n\overline{r}},s\right)\right)\odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{36}$$

$$\frac{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]*} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]*} \left(\sum_{i_1=1}^{r_1*} \cdots \sum_{i_d=1}^{r_d*} w_{i_1,\ldots,i_d} \odot f\left(\frac{k_1}{n}+\frac{i_1}{nr_1},\ldots,\frac{k_d}{n}+\frac{i_d}{nr_d},s \right) \right) \odot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

So if $\left|n^{1-\alpha}\left(x_j-\frac{k_j}{n}\right)\right| \leq T_j$, all $j=1,\ldots,d$, we get that

$$\left\|x-\frac{k}{n}\right\|_{l_1} \leq \frac{\|T\|_{l_1}}{n^{1-\alpha}}. \tag{37}$$

For convinience we call

$$V(x) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) =$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right). \tag{38}$$

We make

Remark 28 *Here always k is as in*

$$-n^2 \leq nx_j - T_jn^{\alpha} \leq k_j \leq nx_j + T_jn^{\alpha} \leq n^2, \tag{39}$$

I) We observe that

$$\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{l_1} \leq \left\|\frac{k}{n+\nu_i}-x\right\|_{l_1} + \left\|\frac{\mu_i}{n+\nu_i}\right\|_{l_1}$$

$$\leq \left\|\frac{k}{n+\nu_i}-x\right\|_{l_1} + \frac{\|\mu_i\|_{l_1}}{n+\nu_i^{\min}}. \tag{40}$$

Next see that

$$\left\|\frac{k}{n+\nu_i}-x\right\|_{l_1} \leq \left\|\frac{k}{n+\nu_i}-\frac{k}{n}\right\|_{l_1} + \left\|\frac{k}{n}-x\right\|_{l_1} \leq \frac{\|k\nu_i\|_{l_1}}{n\left(n+\nu_i^{\min}\right)} + \frac{\|T\|_{l_1}}{n^{1-\alpha}} \tag{41}$$

$$\leq \frac{\|k\|_{l_2}\|\nu_i\|_{l_2}}{n\left(n+\nu_i^{\min}\right)} + \frac{\|T\|_{l_1}}{n^{1-\alpha}} =: (*). \tag{42}$$

We notice for $j=1,\ldots,d$ we get

$$|k_j| \leq n|x_j| + T_jn^{\alpha}.$$

Hence

$$\|k\|_{l_2} \leq \|n|x| + Tn^{\alpha}\|_{l_2} \leq n\|x\|_{l_2} + \|T\|_{l_2} n^{\alpha}, \tag{43}$$

where $|x| = (|x_1|, ..., |x_d|)$.

Thus

$$(*) \leq \frac{\left(n\|x\|_{l_2} + \|T\|_{l_2} n^{\alpha}\right)\|\nu_i\|_{l_2}}{n\left(n+\nu_i^{\min}\right)} + \frac{\|T\|_{l_1}}{n^{1-\alpha}}. \tag{44}$$

So we get

$$\left\|\frac{k}{n+\nu_i} - x\right\|_{l_1} \leq \frac{\left(n\|x\|_{l_2} + \|T\|_{l_2} n^{\alpha}\right)\|\nu_i\|_{l_2}}{n\left(n+\nu_i^{\min}\right)} + \frac{\|T\|_{l_1}}{n^{1-\alpha}} = \tag{45}$$

$$\left(\frac{\|\nu_i\|_{l_2}\|x\|_{l_2}}{n+\nu_i^{\min}}\right) + \frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{n^{1-\alpha}\left(n+\nu_i^{\min}\right)} + \frac{\|T\|_{l_1}}{n^{1-\alpha}}.$$

Hence it holds

$$\left\|\frac{k+\mu_i}{n+\nu_i} - x\right\|_{l_1} \leq$$

$$\left(\frac{\|\nu_i\|_{l_2}\|x\|_{l_2} + \|\mu_i\|_{l_1}}{n+\nu_i^{\min}}\right) + \frac{1}{n^{1-\alpha}}\left(\|T\|_{l_1} + \frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{\left(n+\nu_i^{\min}\right)}\right). \tag{46}$$

II) We also have for

$$0 \leq t_j \leq \frac{1}{n+\rho_i}, \quad j = 1, ..., d, \tag{47}$$

that

$$\left\|t + \frac{k+\lambda_i}{n+\rho_i} - x\right\|_{l_1} \leq \|t\|_{l_1} + \left\|\frac{k+\lambda_i}{n+\rho_i} - x\right\|_{l_1} \leq$$

$$\frac{d}{n+\rho_i} + \left\|\frac{k+\lambda_i}{n+\rho_i} - x\right\|_{l_1} \leq \tag{48}$$

$$\frac{d}{n+\rho_i} + \left\|\frac{\lambda_i}{n+\rho_i}\right\|_{l_1} + \left\|\frac{k}{n+\rho_i} - x\right\|_{l_1} =$$

$$\frac{d}{n+\rho_i} + \frac{d\lambda_i}{n+\rho_i} + \left\|\frac{k}{n+\rho_i} - x\right\|_{l_1} =$$

$$\frac{d(1+\lambda_i)}{n+\rho_i} + \left\|\frac{k}{n+\rho_i} - x\right\|_{l_1} \leq \tag{49}$$

$$\frac{d(1+\lambda_i)}{n+\rho_i} + \left\|\frac{k}{n+\rho_i} - \frac{k}{n}\right\|_{l_1} + \left\|\frac{k}{n} - x\right\|_{l_1} \leq$$

$$\frac{d(1+\lambda_i)}{n+\rho_i} + \frac{\rho_i}{n(n+\rho_i)}\|k\|_{l_1} + \frac{\|T\|_{l_1}}{n^{1-\alpha}} \leq \tag{50}$$

$$\frac{d(1+\lambda_i)}{n+\rho_i}+\frac{\rho_i}{n(n+\rho_i)}\left(n\|x\|_{l_1}+\|T\|_{l_1}n^{\alpha}\right)+\frac{\|T\|_{l_1}}{n^{1-\alpha}}=$$

$$\frac{d(1+\lambda_i)}{n+\rho_i}+\frac{\rho_i\|x\|_{l_1}}{(n+\rho_i)}+\frac{\rho_i\|T\|_{l_1}}{n^{1-\alpha}(n+\rho_i)}+\frac{\|T\|_{l_1}}{n^{1-\alpha}}= \tag{51}$$

$$\left(\frac{\rho_i\|x\|_{l_1}+d(\lambda_i+1)}{n+\rho_i}\right)+\frac{\|T\|_{l_1}}{n^{1-\alpha}}\left(1+\frac{\rho_i}{n+\rho_i}\right). \tag{52}$$

We have found that

$$\left\|t+\frac{k+\lambda_i}{n+\rho_i}-x\right\|_{l_1}\leq$$

$$\left(\frac{\rho_i\|x\|_{l_1}+d(\lambda_i+1)}{n+\rho_i}\right)+\frac{\|T\|_{l_1}}{n^{1-\alpha}}\left(1+\frac{\rho_i}{n+\rho_i}\right). \tag{53}$$

III) We observe that

$$\left\|\frac{k}{n}+\frac{i}{n\overline{r}}-x\right\|_{l_1}\leq\left\|\frac{k}{n}-x\right\|_{l_1}+\frac{1}{n}\left\|\frac{i}{\overline{r}}\right\|_{l_1}\leq\frac{\|T\|_{l_1}}{n^{1-\alpha}}+\frac{d}{n}. \tag{54}$$

Convergence results follow.

Theorem 29 *Let $f\in C_{FR}^{U_q}(\mathbb{R}^d)$, $1\leq q<+\infty$, $x\in\mathbb{R}^d$ and $n\in\mathbb{N}$ such that $n\geq\max_{j\in\{1,...,d\}}\left(T_j+|x_j|,T_j^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $T_j>0$. Then*

$$\left(\int_X D^q\left(\left(H_n^{\mathcal{FR}}(f)\right)(x,s),f(x,s)\right)P(ds)\right)^{\frac{1}{q}}\leq$$

$$\sum_{i=1}^{\overline{r}}w_i\Omega_1^{(\mathcal{F})}\left(f,\left(\frac{\|\nu_i\|_{l_2}\|x\|_{l_2}+\|\mu_i\|_{l_1}}{n+\nu_i^{\min}}\right)+\frac{1}{n^{1-\alpha}}\left(\|T\|_{l_1}+\frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{\left(n+\nu_i^{\min}\right)}\right)\right)_{L^q}= \tag{55}$$

$$\sum_{i_1=1}^{r_1}...\sum_{i_d=1}^{r_d}w_{i_1,...,i_d}\cdot$$

$$\Omega_1^{(\mathcal{F})}\left(f,\left(\frac{\|\nu_i\|_{l_2}\|x\|_{l_2}+\|\mu_i\|_{l_1}}{n+\nu_i^{\min}}\right)+\frac{1}{n^{1-\alpha}}\left(\|T\|_{l_1}+\frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{\left(n+\nu_i^{\min}\right)}\right)\right)_{L^q}, \tag{56}$$

where $i=(i_1,...,i_d)$.

As $n\to\infty$, we get that

$$\left(H_n^{\mathcal{FR}}(f)\right)(x,s)\overset{\text{"q-mean"}}{\to}f(x,s)$$

with rates at the speed $\frac{1}{n^{1-\alpha}}$.

Proof. We may write

$$\left(H_n^{\mathcal{F}R}(f)\right)(x,s)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k+\mu_i}{n+\nu_i},s\right)\right)\odot\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}. \tag{57}$$

We observe that

$$D\left(\left(H_n^{\mathcal{F}R}(f)\right)(x,s),f(x,s)\right)=$$

$$D\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k+\mu_i}{n+\nu_i},s\right)\right)\odot\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},f(x,s)\odot 1\right)= \tag{58}$$

$$D\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k+\mu_i}{n+\nu_i},s\right)\right)\odot\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},\right.$$

$$\left.\left(\sum_{i=1}^{\overline{r}*}w_i\odot f(x,s)\right)\odot\frac{V(x)}{V(x)}\right)= \tag{59}$$

$$D\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k+\mu_i}{n+\nu_i},s\right)\right)\odot\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f(x,s)\right)\odot\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right)\overset{(1)}{\leq}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{\overline{r}}w_iD\left(f\left(\frac{k+\mu_i}{n+\nu_i},s\right),f(x,s)\right)\right). \tag{60}$$

Thus it holds so far that

$$D\left(\left(H_n^{\mathcal{F}R}(f)\right)(x,s),f(x,s)\right)\leq$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{\overline{r}}w_iD\left(f\left(\frac{k+\mu_i}{n+\nu_i},s\right),f(x,s)\right)\right). \tag{61}$$

Hence

$$\left(\int_X D^q\left(\left(H_n^{\mathcal{F}R}(f)\right)(x,s),f(x,s)\right)P(ds)\right)^{\frac{1}{q}}\leq$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{\overline{r}}w_i\left(\int_X D^q\left(f\left(\frac{k+\mu_i}{n+\nu_i},s\right),f(x,s)\right)P(ds)\right)^{\frac{1}{q}}\right)\leq \tag{62}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{l_1}\right)_{L^q}\right) \overset{(46)}{\leq} \tag{63}$$

$$\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left(\frac{\|\nu_i\|_{l_2}\|x\|_{l_2}+\|\mu_i\|_{l_1}}{n+\nu_i^{\min}}\right)+\frac{1}{n^{1-\alpha}}\left(\|T\|_{l_1}+\frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{\left(n+\nu_i^{\min}\right)}\right)\right)_{L^q},$$

proving the claim. ■

Corollary 30 *(to Theorem 29) Let* $x \in \prod_{j=1}^{d}\left[-\gamma_j, \gamma_j\right] \subset \mathbb{R}^d$, $\gamma_j > 0$, $\gamma := (\gamma_1, ..., \gamma_d)$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max_{j\in\{1,\ldots,d\}}\{T_j+\gamma_j, T_j^{-\frac{1}{\alpha}}\}$. *Then*

$$\left\|\left(\int_X D^q\left(\left(H_n^{\mathcal{FR}}(f)\right)(x,s), f(x,s)\right) P(ds)\right)^{\frac{1}{q}}\right\|_{\infty, \prod_{j=1}^{d}\left[-\gamma_j,\gamma_j\right]} \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left(\frac{\|\nu_i\|_{l_2}\|\gamma\|_{l_2}+\|\mu_i\|_{l_1}}{n+\nu_i^{\min}}\right)+\frac{1}{n^{1-\alpha}}\left(\|T\|_{l_1}+\frac{\|\nu_i\|_{l_2}\|T\|_{l_2}}{\left(n+\nu_i^{\min}\right)}\right)\right)_{L^q}. \tag{64}$$

We continue with

Theorem 31 *All assumptions as in Theorem 29. Additionally assume that* $f(t,s)$ *is fuzzy continuous in* $t \in \mathbb{R}^d$, *for each* $s \in X$. *Then*

$$\left(\int_X D^q\left(\left(K_n^{\mathcal{FR}}(f)\right)(x,s), f(x,s)\right) P(ds)\right)^{\frac{1}{q}} \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left(\frac{\rho_i\|x\|_{l_1}+d(\lambda_i+1)}{n+\rho_i}\right)+\frac{\|T\|_{l_1}}{n^{1-\alpha}}\left(1+\frac{\rho_i}{n+\rho_i}\right)\right)_{L^q} = \tag{65}$$

$$\sum_{i_1=1}^{r_1} \ldots \sum_{i_d=1}^{r_d} w_{i_1,\ldots,i_d} \Omega_1^{(\mathcal{F})}\left(f, \left(\frac{\rho_i\|x\|_{l_1}+d(\lambda_i+1)}{n+\rho_i}\right)+\frac{\|T\|_{l_1}}{n^{1-\alpha}}\left(1+\frac{\rho_i}{n+\rho_i}\right)\right)_{L^q}.$$

As $n \to \infty$, *we get*

$$\left(K_n^{\mathcal{FR}}(f)\right)(x,s) \overset{"q\text{-}mean"}{\rightarrow} f(x,s)$$

with rates at the speed $\frac{1}{n^{1-\alpha}}$.

Proof. By Remark 5 the function $F(t,s) := D\left(f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right), f(x,s)\right)$ is $\mathcal{B}$-measurable over X with respect to s. Also F is continuous in $t \in \mathbb{R}^d$, similar reasoning as in explanation of Remark 5, for each $s \in X$.

Thus for $F(t,s)$, by joint measurability theorem of Caratheodory, see [1], p. 156, we get that $F(t,s)$ is jointly measurable over $\mathbb{R}^d \times X$. Hence in what follow we can apply Fubini;s theorem.

We may rewrite

$$\left(K_n^{\mathcal{F}R}(f)\right)(x,s) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*} \left(\sum_{i=1}^{\overline{r}*} w_i (n+\rho_i)^d \odot \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right) dt\right)$$

$$\odot \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}. \tag{67}$$

We observe that

$$D\left(\left(K_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right) =$$

$$D\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*} \left(\sum_{i=1}^{\overline{r}*} w_i (n+\rho_i)^d \odot \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right) dt\right)\right.$$

$$\odot \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*} \left(\sum_{i=1}^{\overline{r}*} w_i (n+\rho_i)^d \odot \int_0^{\frac{1}{n+\rho_i}} f(x,s)\, dt\right) \odot \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right) \overset{(1)}{\leq} \tag{68}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{\overline{r}*} w_i (n+\rho_i)^d D\left(\int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right) dt, \int_0^{\frac{1}{n+\rho_i}} f(x,s)\, dt\right)\right) \overset{(21)}{\leq}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} D\left(f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right), f(x,s)\right) dt\right). \tag{69}$$

That is it holds

$$D\left(\left(K_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right) \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} D\left(f\left(t+\frac{k+\lambda_i}{n+\rho_i}, s\right), f(x,s)\right) dt\right). \tag{70}$$

Hence (let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$)

$$\left(\int_X D^q \left(\left(K_n^{\mathcal{F}R} (f) \right) (x,s), f(x,s) \right) P(ds) \right)^{\frac{1}{q}} \leq \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \left(\int_X \left(\int_0^{\frac{1}{n+\rho_i}} D\left(f\left(t + \frac{k+\lambda_i}{n+\rho_i}, s \right), f(x,s) \right) dt \right)^q P(ds) \right)^{\frac{1}{q}} \right) \tag{71}$$

(by Hölder's inequality)

$$\leq \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \right.$$

$$\left. \left(\int_X \frac{1}{(n+\rho_i)^{\frac{dq}{p}}} \left(\int_0^{\frac{1}{n+\rho_i}} D^q \left(f\left(t + \frac{k+\lambda_i}{n+\rho_i}, s \right), f(x,s) \right) dt \right) P(ds) \right)^{\frac{1}{q}} \right)$$

$$\overset{\text{(by Fubini's theorem)}}{=} \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\overline{r}} w_i \frac{(n+\rho_i)^d}{(n+\rho_i)^{\frac{d}{p}}} \right.$$

$$\left. \left(\int_0^{\frac{1}{n+\rho_i}} \left(\int_X D^q \left(f\left(t + \frac{k+\lambda_i}{n+\rho_i}, s \right), f(x,s) \right) P(ds) \right) dt \right)^{\frac{1}{q}} \right) \overset{(8)}{\leq}$$

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\overline{r}} w_i \frac{(n+\rho_i)^d}{(n+\rho_i)^{\frac{d}{p}}} \right.$$

$$\left. \left(\int_0^{\frac{1}{n+\rho_i}} \Omega_1^{(\mathcal{F})} \left(f, \left\| t + \frac{k+\lambda_i}{n+\rho_i} - x \right\|_{l_1} \right)_{L^q}^q dt \right)^{\frac{1}{q}} \right) \overset{(53)}{\leq} \tag{73}$$

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\overline{r}} w_i \frac{(n+\rho_i)^d}{(n+\rho_i)^{\frac{d}{p}}} \right.$$

$$\left. \frac{1}{(n+\rho_i)^{\frac{d}{p}}} \Omega_1^{(\mathcal{F})} \left(f, \left(\frac{\rho_i \|x\|_{l_1} + d(\lambda_i + 1)}{n+\rho_i} \right) + \frac{\|T\|_{l_1}}{n^{1-\alpha}} \left(1 + \frac{\rho_i}{n+\rho_i} \right) \right)_{L^q} \right) =$$

$$\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})} \left(f, \left(\frac{\rho_i \|x\|_{l_1} + d(\lambda_i + 1)}{n+\rho_i} \right) + \frac{\|T\|_{l_1}}{n^{1-\alpha}} \left(1 + \frac{\rho_i}{n+\rho_i} \right) \right)_{L^q}, \tag{74}$$

proving the claim. ■

The case of $q = 1$ goes through in a similar and simpler way.

Corollary 32 *(to Theorem 31) Let $x \in \prod_{j=1}^{d}\left[-\gamma_j, \gamma_j\right] \subset \mathbb{R}^d$, $\gamma_j > 0$, $\gamma := (\gamma_1, ..., \gamma_d)$ and $n \in \mathbb{N}$ such that $n \geq \max_{j\in\{1,...,d\}}\{T_j + \gamma_j, T_j^{-\frac{1}{\alpha}}\}$. Then*

$$\left\|\left(\int_X D^q\left(\left(K_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right)P(ds)\right)^{\frac{1}{q}}\right\|_{\infty, \prod_{j=1}^{d}\left[-\gamma_j,\gamma_j\right]} \leq \tag{75}$$

$$\sum_{i=1}^{\overline{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left(\frac{\rho_i\|\gamma\|_{l_1} + d(\lambda_i+1)}{n+\rho_i}\right) + \frac{\|T\|_{l_1}}{n^{1-\alpha}}\left(1 + \frac{\rho_i}{n+\rho_i}\right)\right)_{L^q}.$$

We finish with

Theorem 33 *All as in Theorem 29. Then*

$$\left(\int_X D^q\left(\left(M_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right)P(ds)\right)^{\frac{1}{q}} \leq$$

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{\|T\|_{l_1}}{n^{1-\alpha}} + \frac{d}{n}\right)_{L^q}. \tag{76}$$

As $n \to \infty$, we get that

$$\left(M_n^{\mathcal{F}R}(f)\right)(x,s) \overset{"q\text{-}mean"}{\to} f(x,s)$$

with rates at the speed $\frac{1}{n^{1-\alpha}}$.

Proof. We may rewrite

$$\left(M_n^{\mathcal{F}R}(f)\right)(x,s) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*} w_i \odot f\left(\frac{k}{n} + \frac{i}{n\overline{r}}, s\right)\right) \odot \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}. \tag{77}$$

We observe that

$$D\left(\left(M_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right) =$$

$$D\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*} w_i \odot f\left(\frac{k}{n} + \frac{i}{n\overline{r}}, s\right)\right) \odot \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)},\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]*}\left(\sum_{i=1}^{\overline{r}*} w_i \odot f(x,s)\right) \odot \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}\right) \overset{(1)}{\leq}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{\overline{r}} w_i D\left(f\left(\frac{k}{n} + \frac{i}{n\overline{r}}, s\right), f(x,s)\right)\right). \tag{78}$$

Hence it holds

$$\left(\int_X D^q\left(\left(M_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right) P(ds)\right)^{\frac{1}{q}} \leq$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$\left(\sum_{i=1}^{\bar{r}} w_i \left(\int_X D^q\left(f\left(\frac{k}{n}+\frac{i}{n\bar{r}}, s\right), f(x,s)\right) P(ds)\right)^{\frac{1}{q}}\right) \leq \tag{79}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{\bar{r}} w_i \Omega_1^{(\mathcal{F})}\left(f, \left\|\frac{k}{n}+\frac{i}{n\bar{r}}-x\right\|_{l_1}\right)_{L^q}\right) \overset{(54)}{\leq}$$

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{\|T\|_{l_1}}{n^{1-\alpha}}+\frac{d}{n}\right)_{L^q}, \tag{80}$$

proving the claim. ■

Corollary 34 *(to Theorem 33) Here all as in Corollary 30. Then*

$$\left\|\left(\int_X D^q\left(\left(M_n^{\mathcal{F}R}(f)\right)(x,s), f(x,s)\right) P(ds)\right)^{\frac{1}{q}}\right\|_{\infty, \prod_{j=1}^{d}[-\gamma_j,\gamma_j]} \leq$$

$$\Omega_1^{(\mathcal{F})}\left(f, \frac{\|T\|_{l_1}}{n^{1-\alpha}}+\frac{d}{n}\right)_{L^q}. \tag{81}$$

Comment 35 *All the convergence fuzzy random results of this article can have real analogous results for real valued random functions (stochastic processes) using* $\Omega_1(f,\delta)_{L^q}$*, see (10).*

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NOTES ON THE GENERALIZED HANKEL INTEGRAL TRANSFORM AND ITS EXTENSION TO A CLASS OF BOEHMIANS

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Abstract

In this paper we establish certain spaces of Boehmians for the generalized Hankel integral transform $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}$. We estimate the generalization $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ of $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}$ and explore some of its properties. Continuity of $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ with respect to δ and Δ - convergence is also discussed in some detail.

Keywords: $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}$ Transform; $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ Transform; Generalized Function; Boehmian Space.

1 INTRODUCTION

Classical integral transforms have found their application in various areas of mathematics, mathematical physics, engineering and some other differential equations that arise in certain physical problems. As some physical situations are governed by differential equations whose boundary conditions are not enough smooth but are generalized functions; it was of great importance to extend some classical integral transforms to generalized functions. Among those integral transforms that have applications in the space of Boehmians we recall, but are not limited, some such as : Fourier transform [**17**] ; Radon transform [**16, 19**] ; Stieltjes tranform [**20**] ; Hartley - Hilbert and Fourier - Hilbert transforms [**15**] ; Hartley transform [**6, 13**] ; diffraction Fresnel transform [**5**] ; Fresnel wavelet transform [**9**] ; Hilbert transform [**18**] ; Fourier sine (cosine) transform [**11**] ; Ridgelet transform [**1**] ; Hankel [**24**] ; Wavelet [**25**] , and many others to mention but a few.

In addition to this sequence of integrals, this paper investigates the integral transform [**2**]

$$\left(\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f\right)(x) = x^{\sigma} \int_0^{\infty} \jmath_{\mu}\left(\lambda\left(xt\right)^{\frac{1}{k}}\right) t^{w} f\left(t\right) \mathrm{d}t, \left(x>0\right), \tag{1}$$

with $\eta \in \mathbb{C}\left(\operatorname{Re}\left(\eta\right)>-1\right), \sigma \in \mathbb{C}, w \in \mathbb{C}, k>0$ and $\lambda>0, \jmath_{\mu}$ is the Bessel function of the first kind of order $\mu, f \in C_0$ is a continuous function of compact support on $(0,\infty)$ for the range of parameters that indicated.

The substitution $\sigma = w = \frac{1}{2}, \lambda = k = 1$ gives the classical Hankel integral [**10**]

$$(\boldsymbol{l}_\eta f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} \jmath_\eta (xt) f(t) \, \mathtt{d}t, \tag{2}$$

$\eta \in \mathbb{C}, \operatorname{Re}(\eta) > -1; x > 0$, whereas, the substitution $\lambda = k, \sigma = w = \dfrac{1}{k} - \dfrac{1}{2}$ is in fact a modified Hankel transform

$$(\boldsymbol{l}_{\eta,k} f)(x) = \int_0^\infty (xt)^{\frac{1}{k}-\frac{1}{2}} \jmath_\eta \left(k (xt)^{\frac{1}{k}}\right) f(t) \, \mathtt{d}t, \tag{3}$$

$\eta \in \mathbb{C}, \operatorname{Re}(\eta) > -1; k > 0; x > 0$.

Let f be a complex-valued function defined on $(0, \infty)$. Then, the generalized Hankel transform (1) has thoroughly been investigated on the space $\boldsymbol{l}_{v,r}$ of those complex-valued Lebesgue measurable functions defined on $(0, \infty)$ such that

$$\|f\|_{v,r} = \left(\int_0^\infty |x^v f(x)|^r \frac{\mathtt{d}x}{x} \right)^{\frac{1}{r}},$$

where $1 \leq r < \infty$ and $\|f\|_{v,\infty} = ess\sup_{x>0} x^v |f(x)|, v \in (-\infty, \infty)$.

By $\gamma(\tau)$ we denote the maximum value function given by [**1**, (16)]

$$\gamma(\tau) = \max\left(\frac{1}{r}, \frac{1}{r'}\right), \tag{4}$$

where $\dfrac{1}{r} + \dfrac{1}{r'} = 1$ and $1 \leq \infty$.

By $\curlyvee$ we denote the Mellin - type convolution product of first kind [**10**, **12**]

$$(f \curlyvee g)(t) = \int_0^\infty y^{-1} f\left(ty^{-1}\right) g(y) \, \mathtt{d}y, \tag{5}$$

and, by $\mathcal{D}$ we denote the Schwartz' space of test functions of compact support defined on $(0, \infty)$.

Properties of the product $\curlyvee$ are enumerated as [**3**, **4**]

(i) $(f \curlyvee g)(t) = (g \curlyvee f)(t)$;
(ii) $((f + g) \curlyvee h)(t) = (f \curlyvee h)(t) + (g \curlyvee h)(t)$;
(iii) $(\alpha f \curlyvee g)(t) = \alpha (g \curlyvee f)(t)$, α is complex number ;
(vi) $((f \curlyvee g) \curlyvee h)(t) = (f \curlyvee (g \curlyvee h))(t)$.

For a complete investigation, we recall the following theorem [**2**, Theorem 4 (9)]
.

Theorem 1 Let $1 \leq r \leq \infty$ and $\gamma(\tau)$ be given as in (4). If $1 < r < \infty$ and $\gamma(r) \leq k(v - \operatorname{Re}(w) - 1) + \frac{3}{2} < \operatorname{Re}(\eta) + \frac{3}{2}$; then for all $s \geq r$ such that

$$s' \geq \left(k(v - \operatorname{Re}(w) - 1) + \frac{3}{2}\right)^{-1}$$

and $\dfrac{1}{s} + \dfrac{1}{s'} = 1$, the operator $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}$ belongs to $\left(\boldsymbol{l}_{v,r}, \boldsymbol{l}_{1-v+\operatorname{Re}(w-\sigma),s}\right)$ and is further a one-to-one mapping from $\boldsymbol{l}_{v,r}$ onto $\boldsymbol{l}_{1-v+\operatorname{Re}(w-\sigma),s}$.

2 NECESSARY THEOREMS

Theorem 2 Let f, g be $\boldsymbol{l}_1$ functions defined on $(0, \infty)$ and $x > 0$. Let $\times$ denote the product defined by the integral equation

$$(f \times g)(x) = \int_0^\infty y^{\mathrm{Re}(w-\sigma)} f(xy)\, g(y)\, \mathtt{d}y. \tag{6}$$

Then, we have

$$(\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}(f \curlyvee g))(x) = (\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f \times g)(x),$$

whenever the integrals exist.

Proof Let $f, g \in \boldsymbol{l}_1$. Then, the Fubini's theorem gives

$$\begin{aligned}
(\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}(f \curlyvee g))(x) &= x^\sigma \int_0^\infty \jmath_\mu\left(\lambda (xt)^{\frac{1}{k}}\right) t^w (f \curlyvee g)(t)\, \mathtt{d}t \\
\text{i.e} \quad &= x^\sigma \int_0^\infty \jmath_\mu\left(\lambda (xt)^{\frac{1}{k}}\right) t^w \int_0^\infty f\left(ty^{-1}\right) g(y)\, y^{-1} \mathtt{d}y\mathtt{d}t.
\end{aligned}$$

Hence, change of variables yields

$$\begin{aligned}
(\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}(f \curlyvee g))(x) &= \int_0^\infty (xy)^\sigma \int_0^\infty \jmath_\mu\left(\lambda ((xy)\, z)^{\frac{1}{k}}\right) z^w f(z)\, \mathtt{d}z y^{w-\sigma} g(y)\, \mathtt{d}y \\
\text{i.e} \quad &= \int_0^\infty y^{(w-\sigma)} (\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f)(xy)\, g(y)\, \mathtt{d}y.
\end{aligned}$$

Hence, the theorem is completely proved.

Theorem 3 Let f, g and h be $\boldsymbol{l}_1$ functions defined on $(0, \infty)$; then we have $f \times (g \curlyvee h) = (f \times g) \times h$.

Proof By using definitions (5) and (6), we obtain

$$\begin{aligned}
(f \times (g \curlyvee h))(x) &= \int_0^\infty y^{\mathrm{Re}(w-\sigma)} f(xy)\, (g \curlyvee h)(y)\, \mathtt{d}y \\
\text{i.e} \quad &= \int_0^\infty y^{\mathrm{Re}(w-\sigma)} \left(\int_0^\infty t^{-1} g\left(yt^{-1}\right) h(t)\, \mathtt{d}t\right) \mathtt{d}y \\
\text{i.e} \quad &= \int_0^\infty t^{-1} h(t) \int_0^\infty y^{\mathrm{Re}(w-\sigma)} f(xy)\, g\left(yt^{-1}\right) \mathtt{d}y\mathtt{d}t.
\end{aligned}$$

As earlier, change of variables and Fubini's theorem yield

$$\begin{aligned}
(f \times (g \curlyvee h))(x) &= \int_0^\infty h(t) \int_0^\infty (tz)^{\mathrm{Re}(w-\sigma)} f(x(tz))\, g(z)\, \mathtt{d}z\mathtt{d}t \\
\text{i.e} \quad &= \int_0^\infty t^{w-\sigma} h(t) \int_0^\infty z^{\mathrm{Re}(w-\sigma)} f((xt)\, z)\, g(z)\, \mathtt{d}z\mathtt{d}t \\
\text{i.e} \quad &= \int_0^\infty t^{\mathrm{Re}(w-\sigma)} (f \times g)(xt)\, h(t)\, \mathtt{d}t.
\end{aligned}$$

That is,

$$(f \times (g \curlyvee h))(x) = ((f \times g) \times h)(x).$$

This completes the proof of the theorem.

For more convenience, let us consider the case where $r = s$ and $v = \dfrac{\mathrm{Re}\,(w-\sigma)+1}{z}$. Therefore, we state and prove the requirements of construction.

Theorem 4 Let $f \in \boldsymbol{l}_{v,r}$ and $g \in \mathcal{D}$. Then, $f \times g \in \boldsymbol{l}_{v,r}$.

Proof Let $f \in \boldsymbol{l}_{v,r}$ and $g \in \mathcal{D}$ be given; then we have

$$\int_0^\infty |x^v (f \times g)(x)|^r \frac{\mathtt{d}x}{x} = \int_0^\infty \left| x^v \int_0^\infty y^{\mathrm{Re}(w-\sigma)} f(xy)\, g(y)\, \mathtt{d}y \right|^r \frac{\mathtt{d}x}{x}\ .$$

By Jensen's inequality, we get

$$\begin{aligned}
\int_0^\infty |x^v (f \times g)(x)|^r \frac{\mathtt{d}x}{x} &\leq \int_0^\infty \int_0^\infty \left| x^v y^{\mathrm{Re}(w-\sigma)} f(xy)\, g(y) \right|^r \mathtt{d}y \frac{\mathtt{d}x}{x} \\
\text{i.e} \qquad &\leq \int_0^\infty \int_0^\infty \left| x^v y^{\mathrm{Re}(w-\sigma)} f(xy) \right|^r |g(y)|\, \mathtt{d}y \frac{\mathtt{d}x}{x} \\
&\quad \left(\text{since } \frac{\mathtt{d}x}{x} \text{ is a measure on} \right. \\
&\quad \left. (0,\infty) \text{ we can apply Jensen's inequality}\right) \\
\text{i.e} \qquad &\leq \int_0^\infty \left| y^{\mathrm{Re}(w-\sigma)-v\ r} g(y) \right| \left(\int_0^\infty |(xy)^v f(xy)|^r \frac{\mathtt{d}x}{x} \right) \mathtt{d}y \qquad (7)
\end{aligned}$$

Applying change of variables for (**7**) and using the fact that $f \in \boldsymbol{l}_{v,r}$ imply

$$\begin{aligned}
\int_0^\infty |x^v (f \times g)(x)|^r \frac{\mathtt{d}x}{x} &\leq \int_0^\infty \left| y^{(\mathrm{Re}(w-\sigma)-v)r+1} g(y) \right| \mathtt{d}y \int_0^\infty |z^v f(z)|^r \frac{\mathtt{d}z}{z} \\
&\leq M \int_0^\infty |z^v f(z)|^r \frac{\mathtt{d}z}{z}\ ,
\end{aligned}$$

where

$$M = \int_K \left| y^{(\mathrm{Re}(w-\sigma)-v)r+1} g(y) \right| \mathtt{d}y.$$

Therefore,

$$\|f \times g\|_{v,r} < M\, \|f\|_{v,r}\ . \qquad (8)$$

The proof of the theorem is therefore completed.

2.1 CONSTRUCTED SPACES OF BOEHMIANS

Let us denote by Δ the set of all sequences $(\delta_n) \in \mathcal{D}$ such that

$$\int_0^\infty \delta_n(x)\, \mathtt{d}x = 1, \qquad (9)$$

$$\int_0^\infty |\delta_n(x)|\, \mathrm{d}x < A, A \in \mathbb{R}, A > 0, \tag{10}$$

$$\operatorname{supp} \delta_n(x) \to 0 \text{ as } n \to \infty, \tag{11}$$

then every sequence (δ_n) satisfying the equations $(9)-(11)$ is called a delta sequence.

Following theorems are needful for establishing the space $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\times}^{\mathcal{D},\Delta,\curlyvee}$ of Boehmians.

Theorem 5 Let $f \in \boldsymbol{l}_{v,r}$ and $\phi, \psi \in \mathcal{D}$; then $f \times (\phi \curlyvee \psi) = (f \times \phi) \times \psi$.

Similar proof is already given to Theorem 3. Hence, we prefer we omit the details.

Theorem 6 Let $f_1, f_2 \in \boldsymbol{l}_{v,r}$ and $\phi_1, \phi_2 \in \mathcal{D}$; then we have

(i) $(f_1 + f_2) \times \phi_1 = f_1 \times \phi_1 + f_2 \times \phi_1$,

(ii)Let $f_n \to f$ as $n \to \infty$; then for every $\phi \in \mathcal{D}$ we have $f_n \times \phi \to f \times \phi$ as $n \to \infty$.

Proof of this theorem is straightforward. It follows from simple integration.

Theorem 7 Let $f \in \boldsymbol{l}_{v,r}$ and $(\delta_n) \in \Delta$; then $f \times \delta_n \to f$ as $n \to \infty$ in $\boldsymbol{l}_{v,r}$.

Proof Let $f \in \boldsymbol{l}_{v,r}$ and $(\delta_n) \in \Delta$. Since the space $\mathcal{D}$ is dense in $\boldsymbol{l}_{v,r}$ we can easily choose $\alpha \in \mathcal{D}$ such that

$$\|f - \alpha\| < \epsilon, \tag{12}$$

$\epsilon > 0$.

By using (8) and (12) we get that

$$\|(f - \alpha) \times \delta_n\| \le M \|f - \alpha\| < M\epsilon. \tag{13}$$

Now, for each fixed x and $y \in (0, \infty)$, we define a function g such that

$$g(y) = y^{\operatorname{Re}(w-\sigma)} x^{v-1} \alpha(xy).$$

Therefore, $g(y) \in \mathcal{D}$ and $g(1) = x^{v-1}\alpha(x)$. Hence, it is uniformly continuous on $(0, \infty)$. Thus, for every choice of $0 < \epsilon$, there can be found $\delta > 0$ such that $|g(y) - g(x)| < \epsilon$ whenever $|y - x| \le \delta$.

Since $\operatorname{supp} \delta_n \to 0$ as $n \to \infty$ there is an integer $n_1 \in \mathbb{N}$ such that

$$\operatorname{supp} \delta_n \subseteq [-\delta, \delta], \forall n \ge n_1.$$

In addition, let $[a, b]$ be a bounded set such that $\operatorname{supp} \alpha \subseteq [a, b]$. Hence, $\alpha(x) = 0, \forall x \notin [a - \delta, b + \delta]$.

Moreover, the fact that $|(g(y)-g(1))|<\epsilon$ and Jensen's inequality imply

$$\begin{aligned}
\|(\alpha\times\delta_n-\alpha)(x)\|_{v,r}^r &\le \int_0^\infty x^v\left|\int_0^\infty\left(y^{\mathrm{Re}(w-\sigma)}\alpha(xy)\,\delta_n(y)\,\mathtt{d}y-\alpha(x)\,\delta_n(t)\right)\mathtt{d}y\right|^r\frac{\mathtt{d}x}{x}\\
\text{i.e} &\le \int_0^\infty x^v\int_0^\infty|(g(y)-g(1))|^r\,|\delta_n(y)|\,\mathtt{d}y\mathtt{d}x\\
\text{i.e} &\le \int_{a-\delta}^{b+\delta}\epsilon^r\left(\int_0^\infty|\delta_n(y)|\,dy\right)\mathtt{d}x\\
\text{i.e} &\le \epsilon^r(b-a+2\delta)\,N, \qquad (14)
\end{aligned}$$

where

$$N=\int_0^\infty|\delta_n(y)|\,\mathtt{d}y.$$

Hence, by using $(12),(13)$ and (14), we, for large values of n, write

$$\begin{aligned}
\|f\times\delta_n-f\|_{v,r}^r &\le \|(f-\alpha)\times\delta_n\|_{v,r}^r+\|\alpha\times\delta_n-\alpha\|_{v,r}^r+\|\alpha-f\|_{v,r}^r\\
&< M\epsilon+\epsilon^r(b-a+2\delta)\,N+\epsilon\\
&= \left(M+\epsilon^{r-1}(b-a+2\delta)\,N+1\right)\epsilon.
\end{aligned}$$

Hence, $f\times\delta_n\to f$ as $n\to\infty$.

The Boehmian space $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is therefore well defined.

A typical element in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is denoted by the equivalence class of quotients as

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right],$$

where $\{f_n\}\in\boldsymbol{l}_{v,r}$ and $\{\omega_n\}\in\Delta$.

Definition 8 (i)The sum of two Boehmians $\left[\frac{\{f_n\}}{\{\omega_n\}}\right],\left[\frac{\{g_n\}}{\{\psi_n\}}\right]\in\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is defined by

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right]+\left[\frac{\{g_n\}}{\{\psi_n\}}\right]=\left[\frac{\{f_n\times\psi_n+g_n\times\omega_n\}}{\{\omega_n\curlyvee\psi_n\}}\right],$$

(ii)The multiplication of a Boehmian by a scalar in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is defined by

$$\alpha\left[\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\alpha\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\frac{\{\alpha f_n\}}{\{\omega_n\}}\right],$$

$\alpha\in\mathbb{C}$, the space of complex numbers.

(iii)The operation $\times$ and the differentiation in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ are defined by

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\times\left[\frac{\{g_n\}}{\{\psi_n\}}\right]=\left[\frac{\{f_n\times g_n\}}{\{\omega_n\curlyvee\psi_n\}}\right]\text{ and }\mathcal{D}^\alpha\left[\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\frac{\{\mathcal{D}^\alpha f_n\}}{\{\omega_n\}}\right].$$

Similarly, the reader can establish the space $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$.

Definitions for $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$ can be stated as follows.

Definition 9 (i) The sum and scalar multiplication in $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\curlyvee}^{\mathcal{D},\Delta,\curlyvee}$ are defined as

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] + \left[\frac{\{g_n\}}{\{\psi_n\}}\right] = \left[\frac{\{f_n \curlyvee \psi_n + g_n \curlyvee \omega_n\}}{\{\omega_n \curlyvee \psi_n\}}\right]$$

and

$$\alpha\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\alpha\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\alpha f_n\}}{\{\omega_n\}}\right],$$

(ii)The operation $\curlyvee$ and the differentiation in $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\curlyvee}^{\mathcal{D},\Delta,\curlyvee}$ are also given as

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right] \curlyvee \left[\frac{\{g_n\}}{\{\psi_n\}}\right] = \left[\frac{\{f_n \curlyvee g_n\}}{\{\omega_n \curlyvee \psi_n\}}\right] \text{ and } \mathcal{D}^{\alpha}\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\mathcal{D}^{\alpha} f_n\}}{\{\omega_n\}}\right].$$

Definition 10 Let $\left[\frac{\{f_n\}}{\{\omega_n\}}\right] \in \boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\curlyvee}^{\mathcal{D},\Delta,\curlyvee}$; then, by vertue of Theorem 2, we define the generalized Hankel transform $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ of $\left[\frac{\{f_n\}}{\{\omega_n\}}\right]$ as

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\right) = \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n\}}{\{\omega_n\}}\right] \tag{15}$$

which belongs to the space $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\times}^{\mathcal{D},\Delta,\curlyvee}$.

Theorem 11 The mapping in (15) is well - defined.

Let $\left[\frac{\{f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{g_n\}}{\{\epsilon_n\}}\right] \in \boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\curlyvee}^{\mathcal{D},\Delta,\curlyvee}$; then we have

$$f_n \curlyvee \epsilon_m = g_m \curlyvee \omega_n. \tag{16}$$

Employing the mapping (15) and Theorem 2 imply that

$$\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n \times \epsilon_m = \boldsymbol{l}_{\eta;\sigma,w;k,\lambda} g_m \times \omega_n. \tag{17}$$

Hence, $\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n\}}{\{\omega_n\}} \sim \frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} g_n\}}{\{\epsilon_n\}}$ in $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\times}^{\mathcal{D},\Delta,\curlyvee}$.

Therefore,

$$\left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n\}}{\{\omega_n\}}\right] = \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} g_n\}}{\{\epsilon_n\}}\right].$$

This completes the proof of the theorem.

Lemma 12 $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ is an isomorphism from $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\curlyvee}^{\mathcal{D},\Delta,\curlyvee}$ into $\boldsymbol{\kappa}_{\boldsymbol{l}_{v,r},\times}^{\mathcal{D},\Delta,\curlyvee}$.

Proof Let us first establish that $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ is one-to-one.

Given $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\right)=\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{g_n\}}{\{\epsilon_n\}}\right]\right)$. Then, it follows that $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\times\epsilon_m=\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}g_m\times\omega_n$. Therefore, Theorem 2 implies

$$\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(f_n\curlyvee\epsilon_m\right)=\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(g_m\curlyvee\omega_n\right).$$

Employing the transform $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}$ implies $f_n\curlyvee\epsilon_m=g_m\curlyvee\omega_n$. That is,

$$\left[\frac{\{f_n\}}{\{\omega_n\}}\right]=\left[\frac{\{g_n\}}{\{\epsilon_n\}}\right].$$

Now we establish that $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ is onto.

Let $\left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\}}{\{\omega_n\}}\right]\in\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ be arbitrary; then $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\times\omega_m=\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_m\times\omega_n$ for every choice of $m,n\in\mathbb{N}$. Hence, $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(f_n\curlyvee\omega_m\right)=\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(f_m\curlyvee\omega_n\right)$. That is,

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\right)=\left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\}}{\{\omega_n\}}\right].$$

This complete the proof of the lemma.

The product $\curlyvee$ can also be extended to $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$ in the sense that

$$\begin{aligned}
\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\curlyvee\phi\right) &= \boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\curlyvee\phi\}}{\{\omega_n\}}\right]\right)\\
\text{i.e} &= \left[\frac{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(f_n\curlyvee\phi\right)}{\{\omega_n\}}\right]\\
&\qquad\qquad\text{((By Equation 15))}\\
\text{i.e} &= \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\times\phi\}}{\{\omega_n\}}\right]\\
&\qquad\qquad\text{((By Theorem 2))}\\
\text{i.e} &= \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}f_n\}}{\{\omega_n\}}\right]\times\phi
\end{aligned}$$

Hence, by Equation 15, we obtain

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\curlyvee\phi\right)=\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_n\}}{\{\omega_n\}}\right]\right)\times\phi.$$

Theorem 13 The mapping $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}:\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}\to\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is continuous with respect to δ and Δ - convergence.

Proof We show first that $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}:\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}\to\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$ is continuous with respect to δ - convergence.

Let $\beta_n\xrightarrow{\delta}\beta$ in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$ as $n\to\infty$. Then, there are $f_{n,k}$ and f_k in $\boldsymbol{l}_{v,r}$ such that

$$\beta_n=\left[\frac{\{f_{n,k}\}}{\{\omega_k\}}\right]\text{ and }\beta=\left[\frac{\{f_k\}}{\{\omega_k\}}\right]$$

and $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbb{N}$. Therefore $\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_{n,k} \to \boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_k$ as $n \to \infty$ in $\boldsymbol{l}_{v,r}$. Therefore,

$$\left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_{n,k}\}}{\{\omega_k\}}\right] \to \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_k\}}{\{\omega_k\}}\right]$$

as $n \to \infty$ in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}$.

That is

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_{n,k}\}}{\{\omega_k\}}\right]\right) \to \boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left(\left[\frac{\{f_k\}}{\{\omega_k\}}\right]\right) \text{ as } n \to \infty \text{ in } \boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\times}.$$

Now, we establish the continuity of $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}$ with respect to Δ - convergence.

Let $\{\beta_n\}, \beta \in \boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$ be given such that $\beta_n \xrightarrow{\Delta} \beta$ in $\boldsymbol{\kappa}^{\mathcal{D},\Delta,\curlyvee}_{\boldsymbol{l}_{v,r},\curlyvee}$ as $n \to \infty$. Then, we find $\{f_n\} \in \boldsymbol{l}_{v,r}$ and $\{\omega_n\} \in \Delta$ such that $(\beta_n - \beta) \curlyvee \omega_n = \left[\frac{\{f_n \curlyvee \omega_k\}}{\{\omega_k\}}\right] \simeq f_n$ and $f_n \to 0$ as $n \to \infty$. Therefore

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left((\beta_n - \beta) \curlyvee \omega_n\right) = \left[\frac{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda}\left(f_n \curlyvee \omega_k\right)}{\omega_k}\right].$$

Hence, we have

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left((\beta_n - \beta) \curlyvee \omega_n\right) = \left[\frac{\{\boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n \times \omega_k\}}{\omega_k}\right] \simeq \boldsymbol{l}_{\eta;\sigma,w;k,\lambda} f_n \to 0$$

as $n \to \infty$ in $\boldsymbol{l}_{v,r}$. Therefore

$$\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\left((\beta_n - \beta) \curlyvee \omega_n\right) = \left(\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\beta_n - \boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\beta\right) \times \omega_n \to 0 \text{ as } n \to \infty.$$

Hence, $\boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\beta_n \xrightarrow{\Delta} \boldsymbol{l}^{ge}_{\eta;\sigma,w;k,\lambda}\beta$ as $n \to \infty$.

The proof of this theorem is completed.

Conflict of Interests The author declares that there is no conflict of interests regarding the publication of this paper.

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ELASTIC STRING WITH INTERNAL BODY FORCES

IGOR NEYGEBAUER

ABSTRACT. The internal body forces are included into the model of an elastic string. That allows to consider the steady state problems for infinite string. The body forces are considered as nonlocal forces. Variational formulation of the string problem is given using the principle of virtual work. Some analytical solutions of the stated boundary value problems are presented.

1. INTRODUCTION

The models of an elastic string are considered in many papers and books [1], [2], [3],[4], [5], [6], [7], [8], [9], [14], [16]. These models do not use any constitutive law for the internal body forces and they cannot give a solution of the steady state problem for the infinite string. There are exist MAC models in mechanics [10], [11], [12]. The MAC model for an elastic string is given in [13], where an additional term in form of a local internal body force was added to the string equation. That string model can consider infinite strings with zero displacements at infinity. This paper consider the nonlocal internal body forces and that allows to use arbitrary boundary conditions at infinity.

2. INTERNAL BODY FORCES

The internal body forces could be considered from two points of view. The first one is that these forces are physical forces with their specific law. The second one is that the additional body forces are some corrections to the stated physical model to include the specific solutions into the MAC model. We will not go into detailed physical considerations of the deformations of a string like [13]. The constitutive law for the internal body forces of a string is taken in the following form:

$$f = f_1 + f_2, \tag{2.1}$$

where $0 < x < L$, L is the finite length of the string, $L < \infty$, the first force f_1 is the resultant force of interactions between a given internal point of a string and other internal points.

$$f_1 = \frac{1}{L}\int_0^L \left\{\alpha_1\left[u(x) - u(x')\right] + \alpha_2\left[\frac{\partial^2 u}{\partial x^2}(x) - \frac{\partial^2 u}{\partial x^2}(x')\right]\right\} dx' + \tag{2.2}$$

$$+\frac{1}{L}\int_0^L \left\{\alpha_3\left[\frac{\partial^4 u}{\partial x^4}(x) - \frac{\partial^4 u}{\partial x^4}(x')\right]\right\} dx' = \tag{2.3}$$

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$$= \alpha_1 u(x) + \alpha_2 \frac{\partial^2 u}{\partial x^2}(x) + \alpha_3 \frac{\partial^4 u}{\partial x^4}(x) - C_1, \tag{2.4}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the material constants and the constant C_1 equals

$$C_1 = \frac{1}{L}\int_0^L \left[\alpha_1 u(x') + \alpha_2 \frac{\partial^2 u}{\partial x^2}(x') + \alpha_3 \frac{\partial^4 u}{\partial x^4}(x')\right] dx'. \tag{2.5}$$

The local internal body force f_1 was considered in [13] in the form

$$f_1 = \alpha_1 u(x) + \alpha_2 \frac{\partial^2 u}{\partial x^2}(x) + \alpha_3 \frac{\partial^4 u}{\partial x^4}(x). \tag{2.6}$$

The second force f_2 is the resultant force of interactions between a given internal point of a string and the boundary points.

$$f_2 = \beta_1 [u(x) - u(0)] + \beta_2 \left[\frac{\partial^2 u}{\partial x^2}(x) - \frac{\partial^2 u}{\partial x^2}(0)\right] + \tag{2.7}$$

$$+\beta_3 \left[\frac{\partial^4 u}{\partial x^4}(x) - \frac{\partial^4 u}{\partial x^4}(0)\right] + \beta_1 [u(x) - u(L)] + \tag{2.8}$$

$$+\beta_2 \left[\frac{\partial^2 u}{\partial x^2}(x) - \frac{\partial^2 u}{\partial x^2}(L)\right] + \beta_3 \left[\frac{\partial^4 u}{\partial x^4}(x) - \frac{\partial^4 u}{\partial x^4}(L)\right] = \tag{2.9}$$

$$= 2\beta_1 u(x) + 2\beta_2 \frac{\partial^2 u}{\partial x^2}(x) + 2\beta_3 \frac{\partial^4 u}{\partial x^4}(x) - C_2, \tag{2.10}$$

where $\beta_1, \beta_2, \beta_3$ are the material constants and the constant C_2 equals

$$C_2 = \beta_1 [u(0) + u(L)] + \beta_2 \left[\frac{\partial^2 u}{\partial x^2}(0) + \frac{\partial^2 u}{\partial x^2}(L)\right] + \beta_3 \left[\frac{\partial^4 u}{\partial x^4}(0) + \frac{\partial^4 u}{\partial x^4}(L)\right]. \tag{2.11}$$

The force f_2 was taken as $f_2 = 0$ in [13]. Then the equation 2.1 will take the form

$$f = \gamma_1 u(x) + \gamma_2 \frac{\partial^2 u}{\partial x^2}(x) + \gamma_3 \frac{\partial^4 u}{\partial x^4}(x) - C, \tag{2.12}$$

where $C = C_1 + C_2$ and

$$\gamma_1 = \alpha_1 + 2\beta_1, \gamma_2 = \alpha_2 + 2\beta_2, \gamma_3 = \alpha_3 + 2\beta_3. \tag{2.13}$$

If the length of the string is infinite then $u(\infty)$ in the equation (2.11) will be replaced by the parameter p which could obtained satisfying the boundary condition.

3. Statement of the problem

Many books and papers consider the statement of the string problem, for example [3], [14], [15], [17]. The equation of one-dimensional motion of the string is taken in the form

$$T_0 \frac{\partial^2 u}{\partial x^2} - f = \rho \frac{\partial^2 u}{\partial t^2} - q(x,t), \tag{3.1}$$

where T_0 is the constant tension applied to the string, $x-$ is a Cartesian coordinate of a cross-section, $0 \le x \le L$, $L-$ is the length of the string, $\rho-$ is the density of mass per unit length, $u-$ is the transversal displacement of a cross-section, $t-$ is time, $q(x,t)-$ is the density of the transversal external body forces per unit length. The density of the transversal internal body forces per unit length is taken in the form of Eq. (2.12).

The following boundary conditions could be taken into consideration:

$$u(0) = u_0, u(L) = u_L, \tag{3.2}$$

$$\frac{\partial^2 u}{\partial x^2}(0) = 0, \frac{\partial^2 u}{\partial x^2}(L) = 0. \tag{3.3}$$

The transversal forces at the ends of the string is given by

$$P(0) = -T_0 u'(0), P(L) = T_0 u'(L). \tag{3.4}$$

4. Principle of virtual work and boundary conditions

Consider a string with $\beta_3 = 0$. That means that the non-local internal body forces are included into the string equation.

4.1. Case $\alpha_3 = 0$. Consider the steady state problem given by the eq. (3.1) at $\alpha_3 = 0$

$$T_0 \frac{\partial^2 u}{\partial x^2} - \alpha_1 u - \alpha_2 \frac{\partial^2 u}{\partial x^2} + C + q = 0. \tag{4.1}$$

Let $\delta u(x)$ is a virtual displacement of the string and $P(0), P(L)$ are the transversal forces acting at the end points of the string. Then multiplying the Eq. (4.1) by δu and integrating the result from 0 to L we will get

$$\int_0^L \left(T_0 \frac{\partial^2 u}{\partial x^2} - \alpha_1 u - \alpha_2 \frac{\partial^2 u}{\partial x^2} + C + q \right) \delta u(x) dx = 0. \tag{4.2}$$

We can use the integration by parts and transform the integral

$$\int_0^L (T_0 - \alpha_2) u'' \delta u dx = (T_0 - \alpha_2) u' \delta u|_0^L - \int_0^L (T_0 - \alpha_2) u' \delta u' dx. \tag{4.3}$$

If we substitute the eq. (4.3) into the eq. (4.2) then we obtain

$$\int_0^L \left[(T_0 - \alpha_2) u' \delta u' + \alpha_1 u \delta u - C \delta u - q \delta u \right] dx - (T_0 - \alpha_2) u' \delta u|_0^L \tag{4.4}$$

or

$$\int_0^L \frac{1}{2} \delta \left[(T_0 - \alpha_2) u'^2 + \alpha_1 u^2 \right] dx = \int_0^L (C + q) \delta u dx + (T_0 - \alpha_2) u' \delta u|_0^L. \tag{4.5}$$

The last term in the eq. (4.5) equals zero according to the prescribed boundary conditions eq. (3.2). Then the following variational problem is obtained:

$$\delta \int_0^L \frac{1}{2} \left[(T_0 - \alpha_2) u'^2 + \alpha_1 u^2 - 2qu \right] dx - 2 \int_0^L C \delta u dx = 0. \tag{4.6}$$

The external transversal forces at the ends of a string should be calculated as

$$P(L) = T_0 u'(L), P(0) = -T_0 u'(0). \tag{4.7}$$

If we accept the existence of the potential energy and the potential energy of a string is introduced in the form

$$\Pi = \frac{1}{2} \int_0^L \left[(T_0 - \alpha_2) u'^2 + \alpha_1 u^2 \right] dx \tag{4.8}$$

then the principle of virtual work could be applied:

$$\delta \Pi = \delta' W, \tag{4.9}$$

where the variation of the potential energy is

$$\delta\Pi = \frac{1}{2}\delta\int_0^L \left[\alpha_1 u^2 + (T_0 - \alpha_2)u'^2\right] dx. \tag{4.10}$$

The virtual work of the external forces q, P and of a part of internal forces is

$$\delta' W = \int_0^L (C+q)\delta u dx + P\delta u|_0^L, \tag{4.11}$$

where P is the transversal force applied at the ends of the string. If we substitute the eqs. (4.10), (4.11) into the eq. (4.9) then the following equation will be available

$$\delta\int \frac{1}{2}\left[\alpha_1 u^2 + (T_0 - \alpha_2)u'^2\right] dx = \int_0^L (C+q)\delta u dx + P\delta u|_0^L. \tag{4.12}$$

Subtracting eq. (4.12) from the eq. (4.5) we will get

$$[(T_0 - \alpha_2)u' - P]\,\delta u|_0^L = 0. \tag{4.13}$$

The eq. (4.13) shows that the following boundary conditions at the ends of the string could be applied:

$$P(L) = (T_0 - \alpha_2)u'|_L, P(0) = -(T_0 - \alpha_2)u'|_0 \tag{4.14}$$

or

$$u \tag{4.15}$$

is prescribed.

4.2. **Case $\alpha_3 \neq 0$.** Consider the steady state problem given by the eq. (3.1)

$$T_0\frac{\partial^2 u}{\partial x^2} - \alpha_1 u - \alpha_2\frac{\partial^2 u}{\partial x^2} - \alpha_3\frac{\partial^4 u}{\partial x^4} + C + q = 0. \tag{4.16}$$

Let $\delta u(x)$ is a virtual displacement of the string and $P(0), P(L)$ are the transversal forces acting at the end points of the string. Then multiplying the Eq. (4.16) by δu and integrating the result from 0 to L we will get

$$\int_0^L \left(T_0\frac{\partial^2 u}{\partial x^2} - \alpha_1 u - \alpha_2\frac{\partial^2 u}{\partial x^2} - \alpha_3\frac{\partial^4 u}{\partial x^4} + C + q\right)\delta u(x) dx = 0. \tag{4.17}$$

We can use the integration by parts and transform the integrals

$$\int_0^L \alpha_3 u''''\delta u dx = [\alpha_3 u'''\delta u]_0^L - \int_0^L \alpha_3 u'''\delta u' dx = \tag{4.18}$$

$$= \alpha_3 u'''\delta u|_0^L - \alpha_3 u''\delta u'|_0^L + \int_0^L \alpha_3 u''\delta u'' dx, \tag{4.19}$$

$$\int_0^L (T_0 - \alpha_2)u''\delta u dx = (T_0 - \alpha_2)u'\delta u|_0^L - \int_0^L (T_0 - \alpha_2)u'\delta u' dx. \tag{4.20}$$

If we substitute the eqs. (4.18), (4.19), (4.20) into the eq. (4.17) then we obtain

$$\int_0^L \left[(T_0 - \alpha_2)\, u'\delta u' + \alpha_1 u\delta u + \alpha_3 u''\delta u'' - (C+q)\delta u\right] dx - \tag{4.21}$$

$$-(T_0 - \alpha_2)u'\delta u|_0^L + \alpha_3 u'''\delta u|_0^L - \alpha_3 u''\delta u'|_0^L = 0 \tag{4.22}$$

or

$$\int_0^L \frac{1}{2}\delta\left[(T_0-\alpha_2)u'^2+\alpha_1u^2+\alpha_3u''^2\right]dx = \tag{4.23}$$

$$= \int_0^L (C+q)\delta u dx + \left[(T_0-\alpha_2)u'-\alpha_3u'''\right]\delta u|_0^L + \alpha_3u''\delta u'|_0^L. \tag{4.24}$$

The last two terms of the eq. (4.24) equal zero because of the boundary conditions eqs. (3.2) and (3.3). Then the following variational problem is obtained:

$$\delta\int_0^L \frac{1}{2}\left[(T_0-\alpha_2)u'^2+\alpha_1u^2+\alpha_3u''^2-2qu\right]dx - 2\int_0^L C\delta u dx = 0. \tag{4.25}$$

The external transversal forces at the ends of a string should be calculated according to the eqs. (4.7).

If we accept the existence of the potential energy and the potential energy of a string is introduced in the form

$$\Pi = \frac{1}{2}\int_0^L \left[(T_0-\alpha_2)u'^2+\alpha_1u^2+\alpha_3u''^2\right]dx \tag{4.26}$$

then the principle of virtual work could be applied:

$$\delta\Pi = \delta' W, \tag{4.27}$$

where the variation of the potential energy is

$$\delta\Pi = \frac{1}{2}\delta\int_0^L \left[\alpha_1u^2+(T_0-\alpha_2)u'^2+\alpha_3u''^2\right]dx. \tag{4.28}$$

The virtual work of the external forces q, P and of the part of the internal forces is

$$\delta' W = \int_0^L (C+q)\delta u dx + P\delta u|_0^L, \tag{4.29}$$

where P is the transversal force applied at the ends of the string. If we substitute the eqs. (4.28), (4.29) into the eq. (4.27) then the following equation will be available

$$\delta\int \frac{1}{2}\left[\alpha_1u^2+(T_0-\alpha_2)u'^2+\alpha_3u''^2\right]dx = \int_0^L (C+q)\delta u dx + P\delta u|_0^L. \tag{4.30}$$

Subtracting eq. (4.30) from the eq. (4.23), (4.24) we will get

$$\left[(T_0-\alpha_2)u'-\alpha_3u'''-P\right]\delta u|_0^L + \alpha_3u''\delta u'|_0^L = 0. \tag{4.31}$$

The eq. (4.31) shows that the following boundary conditions at the ends of the string could be applied

- either $u''=0$ or u' is prescribed,
- either $P=-\alpha_3u'''+(T_0-\alpha_2)u'$ or u is prescribed.

5. Examples of string with local internal body forces

5.1. **Example of string without internal body forces.** Consider a simple particular example to show that the theory of the string with internal body forces has solution, where the classical problem does not have any one.

Let us take the steady state problem without any given distributed external forces and the length of the string is infinite. Then the classical equation is

$$\frac{d^2u}{dx^2} = 0. \tag{5.1}$$

If the boundary conditions are

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{5.2}$$

then it is easy to see, that the solution of the stated problem Eqs. (5.1), (5.2) does not exist.

5.2. **Example 1 of string with internal body forces.** Consider now the same steady state problem for the string with the internal body forces. The differential equation of the problem at $\alpha_3 = 0$ is

$$T_0\frac{d^2u}{dx^2} - \alpha_1 u - \alpha_2\frac{d^2u}{dx^2} = 0. \tag{5.3}$$

The boundary conditions are the Eq. (5.2). The solution of the problem (5.2), (5.3) with internal body force exists and equals

$$u = u_0 \exp(\lambda x), \tag{5.4}$$

where

$$\lambda = -\sqrt{\frac{\alpha_1}{T_0 - \alpha_2}}. \tag{5.5}$$

The above solution Eq. (5.4) exists if

$$T_0 > \alpha_2. \tag{5.6}$$

The first of the eqs. (3.4) gives the following value of the transversal force applied at the origin

$$P(0) = -T_0u'(0) = -T_0u_0\lambda = \frac{T_0u_0\sqrt{\alpha_1}}{\sqrt{T_0 - \alpha_2}}. \tag{5.7}$$

The second of the eqs. (4.14) gives the external transversal force applied at the origin

$$P = -(T_0 - \alpha_2)u'(0) = u_0\sqrt{\alpha_1(T_0 - \alpha_2)}. \tag{5.8}$$

5.3. **Example 2 of string with internal body forces.** If we consider more general problem with internal body force and $\alpha_3 \neq 0$, then the differential equation of the problem will take the form

$$T_0\frac{d^2u}{dx^2} - \alpha_1 u - \alpha_2\frac{d^2u}{dx^2} - \alpha_3\frac{d^4u}{dx^4} = 0. \tag{5.9}$$

The boundary conditions are taken the Eqs. (5.1), (5.2):

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{5.10}$$

$$\frac{d^2u}{dx^2}(0) = 0, \frac{d^2u}{dx^2}(\infty) = 0. \tag{5.11}$$

The boundary conditions Eqs. (5.11) are obtained as follows - we require that the equation Eq. (5.1) without body forces should be satisfied at the boundary or we can use the variational conditions.

The solution of the problem with internal body forces Eqs. (5.9), (5.10), (5.11) exists and equals

$$u = \frac{u_0}{\lambda_2^2 - \lambda_1^2}(\lambda_2^2 \exp \lambda_1 x - \lambda_1^2 \exp \lambda_2 x), \tag{5.12}$$

where

$$\lambda_{1,2} = -\sqrt{\frac{T_0 - \alpha_2 \pm \sqrt{(T_0 - \alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}, \tag{5.13}$$

where two inequalities should be fulfilled. The first inequality is the Eq. (5.6) and the second one is

$$(T_0 - \alpha_2)^2 - 4\alpha_1\alpha_3 > 0. \tag{5.14}$$

The external transversal force applied at the origin according to the variational approach is

$$P(0) = -(T_0 - \alpha_2)u'(0) + \alpha_3 u'''(0) = \frac{u_0\sqrt{\alpha_1}\left(\sqrt{\alpha_1\alpha_3} + T_0 - \alpha_2\right)}{\sqrt{T_0 - \alpha_2 + 2\sqrt{\alpha_1\alpha_3}}}. \tag{5.15}$$

The eq. (3.4) gives another expression of P:

$$P(0) = -T_0 u'(0) = \frac{T_0 u_0 \sqrt{\alpha_1}}{\sqrt{T_0 - \alpha_2 + 2\sqrt{\alpha_1\alpha_3}}}. \tag{5.16}$$

If the left hand-side of the Eq. (5.14) equals to zero, then the solution will take the form

$$u = u_0\left(1 + \frac{\lambda x}{2}\right)\exp(-\lambda x), \tag{5.17}$$

where

$$\lambda = \sqrt{\frac{2\alpha_1}{T_0 - \alpha_2}}. \tag{5.18}$$

Then the transversal end forces are

$$P(0) = -(T_0 - \alpha_2)u'(0) + \alpha_3 u'''(0) = u_0\sqrt{\frac{\alpha_1(T_0 - \alpha_2)}{2}}\left[1 + \frac{2\alpha_1\alpha_3}{(T_0 - \alpha_2)^2}\right] \tag{5.19}$$

and

$$P(0) = -T_0 u'(0) = \frac{T_0 u_0 \sqrt{\alpha_1}}{2(T_0 - \alpha_2)} \tag{5.20}$$

correspondingly.

The considered example of the string problem shows that the introduced internal body forces allow to obtain solutions in the cases, where the classical problem does not have any solution.

6. Examples of the string with nonlocal internal body forces

6.1. Example 1 of string with finite length. Consider the problem

$$T_0u'' - (\alpha_1 + 2\beta_1)u + \frac{1}{L}\int_0^L \alpha_1 u dx + \beta_1(u_0 + u_L) = 0, \tag{6.1}$$

$$u(0) = u_0, u(L) = u_L. \tag{6.2}$$

Let

$$\gamma_1 = \alpha_1 + 2\beta_1, \tag{6.3}$$

and the constant is

$$C = \frac{1}{L}\int_0^L \alpha_1 u dx + \beta_1(u_0 + u_L). \tag{6.4}$$

Then the equation(6.1) will take the form

$$T_0u'' - \gamma_1 u + C = 0. \tag{6.5}$$

The general solution of the equation (6.5) is

$$u = A_1e^{kx} + A_2e^{-kx} + \frac{C}{\gamma_1}, \tag{6.6}$$

where A_1, A_2 are arbitrary constants and

$$k = \sqrt{\frac{\gamma_1}{T_0}}. \tag{6.7}$$

If the equation (6.6) is substituted into the equations (6.2) and (6.4) then the following system of the three linear algebraic equations with respect to C, A_1, A_2 will be obtained.

$$\alpha_1(e^{kL} - 1)A_1 - \alpha_1(e^{-kL} - 1)A_2 - 2\beta_1 kLC = -\beta_1 kL(u_0 + u_L), \tag{6.8}$$

$$A_1 + A_2 + \frac{1}{\gamma_1}C = u_0, \tag{6.9}$$

$$e^{kL}A_1 + e^{-kL}A_2 + \frac{1}{\gamma_1}C = u_L. \tag{6.10}$$

This system of equations could be easily solved and the solution of the problem will be the equation (6.6).

6.2. Example 2 of the string with infinite length. Consider the problem given in example 1. Let $L \to 0$. It is supposed that the solution should be bounded. Then the equation (6.6) shows that

$$A_1 = 0. \tag{6.11}$$

Then the equations (6.8), (6.9), (6.10) will take the form

$$C = \frac{\gamma_1}{2}(u_0 + u_\infty), \tag{6.12}$$

$$u_0 = A_2 + \frac{C}{\gamma_1}, \tag{6.13}$$

$$u_\infty = \frac{C}{\gamma_1}. \tag{6.14}$$

Substituting the equation (6.12) into the equation (6.14) and simplifying the result the following consequence follows

$$u_0 = u_\infty. \tag{6.15}$$

The equation (6.15) the contradicts to the real situation when we can apply the boundary conditions independent on both ends of a string. Therefore the statement of the problem should be improved. Let

$$T_0 u'' - \gamma_1 u + C = 0, \tag{6.16}$$

$$u(0) = u_0, u(\infty) = u_\infty, \tag{6.17}$$

where

$$C = \frac{1}{L}\int_0^L \alpha_1 u dx + \beta_1(u_0 + p), L \to \infty. \tag{6.18}$$

p is a parameter which will be defined later. Following the solution of the examples 1 and 2 the equations (6.12), (6.13), (6.14) will take the form

$$C = \frac{\gamma_1}{2}(u_0 + p), \tag{6.19}$$

$$u_0 = A_2 + \frac{C}{\gamma_1}, \tag{6.20}$$

$$u_\infty = \frac{C}{\gamma_1}. \tag{6.21}$$

The equations (6.19), (6.20), (6.21) give

$$A_2 = u_0 - u_\infty, \tag{6.22}$$

$$p = 2u_\infty - u_0. \tag{6.23}$$

Then the solution of the given problem is

$$u = (u_0 - u_\infty)e^{-kx} + u_\infty. \tag{6.24}$$

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On a Characterization of Hilbert Spaces through Minimality of Orthogonal Projections and Related Topics.

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Abstract

In the first part we present a new characterization of finite dimensional Hilbert spaces in terms of minimal projections. If a finite dimensional Banach space $X = (\mathbb{R}^n, ||\cdot||)$ has the property that for every subspace V in X, the orthogonal projection from X onto V has the minimal norm, then the space X has to be isometric to the Hilbert space.

In the second part we estimate the norms of minimal projections onto subspaces of X as operators from l_p^n to l_2^n. In the case of hyperplanes the obtained bound is proved to be optimal.

1 Introduction

In this paper we will consider a finite dimensional Banach spaces $X = (\mathbb{R}^n, ||\cdot||)$. The symbols $S(X)$ and $B(X)$ will, respectively, denote the unit sphere and the unit ball of X. We will use $\langle\cdot,\cdot\rangle$ to denote the standard inner product on $\mathbb{R}^n$ and identify the dual space X^* with $(\mathbb{R}^n, ||\cdot||_*)$ where

$$\|x\|_* := \sup\{\langle x, y\rangle : y \in S(X)\}. \tag{1}$$

If V is a linear subspace of X then the set of all projections from X onto V is denoted by $P(X, V)$. A projection Q from X onto V is called minimal if

$$||Q|| = \lambda(V, X) = \inf\{\|P\| : P \in P(X, V)\}. \tag{2}$$

If X is a Hilbert space (i.e., $\|x\| = \|x\|_2 := \sqrt{\langle x, x\rangle}$) then, for any subspace V of X, the orthogonal projection P_V onto V is the unique projection of norm one. In particular, the projection P_V is *minimal*, i.e., it has the least norm among all projections onto V. The main result of section 1 is the reverse statement:

Theorem 1. *If for every subspace $V \subset X = (\mathbb{R}^n, ||\cdot||)$ the orthogonal projection is minimal then Y is isometric to a Hilbert space.*

Since the minimal projection onto every one-dimensional subspace of any Banach space has norm 1, the theorem will directly follow from the following statement:

Theorem 2. *If for every one-dimensional subspace $V \subset X = (\mathbb{R}^n, ||\cdot||)$ the orthogonal projection onto Y has norm 1 then X is isometric to a Hilbert space.*

Is is worth mentioning that there are many other characterizations of Hilbert spaces through projections (see a survey paper [7] for detailed information) and the relation between minimal projections and other class of projections (orthogonal or radial) is also being investigated (see [2], [6] and [10]). The results of Section 1 can be viewed in terms of characterizing minimal projections in the original norm $||\cdot||$ through minimality in ℓ_2^n norm. This leads to section 2 where we investigate the minimal projections equipped with $p \to 2$ norm (that is we will consider the projections as operators from l_p^n to l_2^n.). We can define a $p \to 2$ norm of a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$||L||_{p\to 2} = \sup_{||x||_p=1} ||Lx||_2. \tag{3}$$

We obtain the theorem (see Theorem 11 and Theorem 12):

Theorem 3. *For $p > 2$, the norms of all minimal projections as operators from l_p^n to l_2^n are bounded by $n^{\frac{1}{2}-\frac{1}{p}}$. Additionally, for the class of minimal projections onto hyperplanes, this estimate is optimal (when n is even) and the bound is attained for the hyperplane $T := ker1 = \{x : \sum_{i=1}^n x_i = 0\}$.*

It is interesting to note that obtaining the sharp estimate for the norms of minimal projections onto hyperplanes in ℓ_p^n is an open problem (except for the cases $p = 1, 2, \infty$, see [10]) although it is conjectured that the hyperplane $T := ker1 = \{x : \sum_{i=1}^n x_i = 0\}$ is the worst possible case ([11]).

2 Characterization of Hilbert Spaces.

The purpose of this section is to establish a characterization of finite dimensional Hilbert spaces in terms of minimal projections.

Before we prove the Theorem 2 we will need to define a few preliminaries (see Chapter 2 in [1]). By Hahn-Banach theorem, for every $x \in S(X)$ there exists a functional $f \in S(X^*)$ such that $f(x) = 1$. A point $x \in S(X)$

is called *smooth* if such functional is unique. The space X is called smooth if every point in $S(X)$ is smooth. A well-known characterization of smooth points is the following

Theorem 4. *A point $x \in S(X)$ is smooth if and only if the norm $||\cdot||$ is Gateaux differentiable at x, i.e., for every $y \in S(x)$ the following limit exists*

$$\lim_{\lambda\to 0}\frac{\|x+\lambda y\|-\|x\|}{\lambda} := \rho'(x,y). \tag{4}$$

A point $x \in S(X)$ is called an exposed point if there exists a functional $f \in S(X^*)$ such that $f(x) = 1$ and for every $y \in S(X)$, $y \neq x$, $f(y) < 1$.

Finally, the space X is called strictly convex if, for any $x, y \in S(X)$, $x \neq y$ and all $t \in (0,1)$ we have

$$\|tx + (1-t)y\| < 1. \tag{5}$$

In a strictly convex space every point $x \in S(x)$ is an exposed point (see remark after Theorem 1, Chapter 2 in [1]).

We are now ready to address the proof of Theorem 2. We will start with the following

Lemma 5. *If $X = (\mathbb{R}^n, ||\cdot||)$ the orthogonal projection onto every one-dimensional subspace of X is minimal then X is strictly convex and smooth.*

Proof. If every orthogonal projection is minimal then, in particular, every orthogonal projection onto a one-dimensional subspace is minimal hence is of norm one. Now, assume that X is not strictly convex. Then there exists two distinct vectors $u, v \in S(X)$ such that $w(t) = tu + (1-t)v \in S(X)$ for all $t \in [0,1]$. Since

$$\langle w(t), u-v\rangle = t\,\|u-v\|_2^2 + (\langle u,v\rangle - \|v\|_2^2), \tag{6}$$

we can choose $t \in (0,1)$ such that

$$\langle w(t), u-v\rangle \neq 0. \tag{7}$$

Since

$$w(s) = su + (1-s)v = w(t) + (s-t)(u-v) \in S(X), \tag{8}$$

we obtain

$$w(t) + \alpha(u-v) \in S(X) \tag{9}$$

for all α in some interval $(-\varepsilon,\varepsilon)$.

We have

$$P_{w(t)}(w(t)+\alpha(u-v)) = \left\langle \frac{w(t)}{\|w(t)\|_2}, w(t)+\alpha(u-v)\right\rangle \frac{w(t)}{\|w(t)\|_2} \tag{10}$$

$$= \frac{\|w(t)\|_2^2 + \alpha\,\langle w(t), u-v\rangle}{\|w(t)\|_2^2} w(t) \tag{11}$$

and since $w(t), w(t)+\alpha(u-v) \in S(X)$ we conclude that for all $\alpha \in (-\varepsilon,\varepsilon)$

$$\left\|P_{w(t)}\right\| \geq \left\|P_{w(t)}(w(t)+\alpha(u-v))\right\| = \frac{\|w(t)\|_2^2 + \alpha\,\langle w(t), u-v\rangle}{\|w(t)\|_2^2}. \tag{12}$$

By (7) we can choose $\alpha \in (-\varepsilon, \varepsilon)$ such that $\alpha \langle w(t), u - v \rangle > 0$, hence $\|P_{w(t)}\| > 1$ contradicting the assumptions of the lemma.

To prove that X is uniformly smooth, we observe that for every orthogonal projection P on X, the adjoint P^* is an orthogonal projection on $X^* = (\mathbb{R}^n, ||\cdot||_*)$ and $\|P\| = \|P^*\|$. Thus, in particular, the orthogonal projection onto every one-dimensional space in X^* has norm 1 and X^* is strictly convex. This implies that X is smooth (see Theorem 2, Chapter 2 in [1]). $\square$

Theorem 6. *Let $X = (\mathbb{R}^n, || \cdot ||)$ be a normed space. The following are equivalent:*

(i) X is a Hilbert space, i.e., $\|\cdot\| = \|\cdot\|_2$.

(ii) For every subspace $V \subset X$, the orthogonal projection onto V is minimal.

(iii) For every one-dimensional subspace $V \subset X$, the orthogonal projection onto V is minimal, i.e., has norm one.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious. We will prove that (iii) implies (i). Observe that if V is a one-dimensional subspace of X spanned by a vector $u \in S(x)$ then every projection Q onto V has the form $Qx = f(x)u$ for some $f \in X^*$ with $f(u) = 1$. Additionally $\|Q\| = \|f\|$. If Q has norm one then $f \in S(X^*)$. Since, by the previous lemma, every point $u \in S(X)$ is smooth there exists the unique functional $f_u \in S(X^*)$ such that $f_u(u) = 1$ and thus there exists the unique projection onto V of norm one: $Qx = f_u(x)u$. By the previous lemma, u is an exposed point:

$$(u + \ker f_u) \cap S(X) = \{u\}, \tag{13}$$

i.e., $u + \ker f_u$ is the unique hyperplane tangent to the sphere $S(X)$. Finally, since Q is the orthogonal projection, we have

$$f_u(x)u = \left\langle \frac{u}{\|u\|_2}, x \right\rangle \frac{u}{\|u\|_2} = \left\langle \frac{u}{\|u\|_2^2}, x \right\rangle u \tag{14}$$

and

$$f_u = \frac{u}{\|u\|_2^2} \ . \tag{15}$$

The above arguments means that $\ker f_u = u^\perp := \{x \in X : \langle u, x \rangle = 0\}$ and hence at every point u on the sphere $S(X)$ the (unique) tangent hyperplane at this point is perpendicular to the vector u. This, combined with the smoothness of $S(X)$, gives a convincing geometric argument that $S(X)$ must be the Euclidean sphere. In a sequel we present a formal proof of this. Since the norm $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$ is smooth, it is Gateaux differentiable at every $x \neq 0$ and for every $u \in S(X)$

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = f_{\frac{x}{\|x\|}}(y) = \frac{\langle x, y \rangle \|x\|}{\|x\|_2^2} \tag{16}$$

Since the norm is a Gateaux differentiable Lipschitz function on a finite-dimensional space, it is Frechet differentiable away from zero (see [5]) and its Gateuax derivative in the directions of unit basis vectors give the gradient of $||\cdot||$, i.e.,

$$\nabla(||x||) = \left(\frac{||x||}{||x||_2^2}\right)\cdot x. \tag{17}$$

On the other hand

$$\nabla\left(\frac{||x||}{||x||_2}\right) = \frac{||x||_2\cdot\nabla(||x||) - ||x||\cdot\nabla(||x||_2)}{||x||_2^2} \tag{18}$$

$$= \frac{||x||_2\cdot\nabla(||x||) - ||x||\cdot\frac{x}{||x||_2}}{||x||_2^2} \tag{19}$$

$$= \frac{||x||_2^2\cdot\nabla(||x||) - ||x||\cdot x}{||x||_2^3}. \tag{20}$$

As a result, using (17) we obtain

$$\nabla\left(\frac{||x||}{||x||_2}\right) = 0. \tag{21}$$

Hence

$$\|x\| = C\,\|x\|_2 \tag{22}$$

which is what we set out to prove. ☐

Remark 7. *It is interesting to note that the part (iii) in the statement of the theorem 6 cannot be replaced by*

(iii') For every two-dimensional subspace $U\subset X$ the orthogonal projection onto U is minimal;

or by

(iii") For every subspace of $U\subset X$ of codimension 1, the orthogonal projection onto U is minimal.

Indeed, A. Komisarski [4], constructed an example of a three dimensional Banach space X, not isometric to a Hilbert space, such that all minimal projections onto every two-dimensional supspaces are unique, orthogonal and have the same norm $\lambda > 1$.

3 Minimizing the norms of projections as operators from l_p^n to l_2^n.

In this section we will investigate the projections as operators from l_p^n to l_2^n. For a fixed subspace V, we will minimize the norm of all such projections onto V. The goal will be to find the maximal norms of all minimal projections.

Definition 8. *For any linear subspace V of $\mathbb{R}^n$ we define*

$$\lambda_{p\to 2}(V,\mathbb{R}^n) = \inf\{||P||_{p\to 2} : P\in P(\mathbb{R}^n, V)\}, \tag{23}$$

and

$$\lambda_{p\to 2}^N = \sup\{\lambda_{p\to 2}(V,\mathbb{R}^n) : \dim V = N\}. \tag{24}$$

Observe that

Remark 9. *Geometrically speaking, the norm of a projection*

$$P : (\mathbb{R}^n, ||\cdot||_1) \to (V, ||\cdot||_2) \tag{25}$$

is the smallest constant C such that

$$P(B_{||\cdot||_1}) \subset C \cdot \left(B_{||\cdot||_2} \cap V\right). \tag{26}$$

As a result,

Remark 10. *Fix $p \leq 2$. For any linear subspace $V \neq 0$ of $\mathbb{R}^n$, the orthogonal projection $P_V : \ell_p^n \to (V, ||\cdot||_2)$ has norm one. Hence*

$$\lambda_{p\to 2}(V, \mathbb{R}^n) = 1 \tag{27}$$

and

$$\lambda_{p\to 2}^N = 1 \tag{28}$$

The main result of this section is the following theorem

Theorem 11. *Fix $p > 2$ and consider $\mathbb{R}^n$. For $N \geq 1$ we have*

$$\lambda_{p\to 2}^N \leq n^{\frac{1}{2}-\frac{1}{p}}. \tag{29}$$

Proof. Take q such that $\frac{1}{p}+\frac{1}{q}=1$. Considering the dual projections we can see that

$$\begin{aligned}\lambda_{p\to 2}(V, \mathbb{R}^n) &= \inf\{||P||_{p\to 2} : P \in P(\mathbb{R}^n, V)\} &(30)\\ &= \inf\{||P^*||_{2\to q} : P^* \in P(\mathbb{R}^n, V)\} = \lambda_{2\to q}(V, \mathbb{R}^n). &(31)\end{aligned}$$

Hence

$$\lambda_{p\to 2}(V, \mathbb{R}^n) \leq ||P_V||_{2\to q}, \tag{32}$$

where P_V is the orthogonal projection onto V. Noting that $P_V(B(\ell_p^n)) = B(\ell_p^n) \cap V$ and using Remark 9 we can observe that

$$||P_V||_{2\to q} = ||Id/_V||_{2\to q} \tag{33}$$

and

$$\lambda_{p\to 2}(V, \mathbb{R}^n) \leq ||Id/_V||_{2\to q}. \tag{34}$$

For a fixed N, the union of all subspaces of dimension N gives the whole space $\mathbb{R}^n$. As a result

$$\lambda_{p\to 2}^N = \sup_{\dim V = N} \lambda_{p\to 2}(V, \mathbb{R}^n) \leq \sup_{\dim V = N} ||Id/_V||_{2\to q} = ||Id||_{2\to q}. \tag{35}$$

Once can easily compute the last quantity using the classical inequality between power means (see [3]). For $q < 2$ we have

$$\left(\frac{\sum_{i=1}^n |x_i|^q}{n}\right)^{1/q} \leq \left(\frac{\sum_{i=1}^n |x_i|^2}{n}\right)^{1/2}, \tag{36}$$

with "=" if and only if $|x_i| = c$, for all i. Using the above inequality we get

$$||Id(x)||_q = ||x||_q = \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \tag{37}$$

$$\leq n^{1/q-1/2}\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = n^{\frac{1}{2}-\frac{1}{p}}\,||x||_2. \tag{38}$$

□

We will show now that the estimate in the above theorem is essentially the optimal.

Theorem 12. *Fix $p > 2$ and consider $\mathbb{R}^n$, where n is an even integer. Then*

$$\lambda^{n-1}_{p\to 2} = n^{\frac{1}{2}-\frac{1}{p}}. \tag{39}$$

Proof. Consider the hyperplane $T := \{x : \sum_{i=1}^{n} x_i = 0\}$. Any projection onto T is given by

$$P_z x = x - (\sum_{i=1}^{n} x_i)\cdot z, \tag{40}$$

for some z such that $\sum_{i=1}^{n} z_i = 1$. Let

$$Qx = x - \frac{1}{n}(\sum_{i=1}^{n} x_i)\cdot(1, ..., 1). \tag{41}$$

Using standard averaging argument (see [9] and [12]) we will show that, for any z, we have $||Q||_{p\to 2} \leq ||P_z||_{p\to 2}$. Fix any permutation $\sigma \in S_n$, we will denote $z_\sigma = (z_{\sigma(1)}, ..., z_{\sigma(n)})$. One can easily see that $||P_z||_{p\to 2} = ||P_{z_\sigma}||_{p\to 2}$ and

$$Q = \frac{1}{n!}\sum_{\sigma\in S_n} P_{z_\sigma}. \tag{42}$$

Hence $||Q||_{p\to 2} \leq ||P_z||_{p\to 2}$ and $\lambda_{p\to 2}(T, \mathbb{R}^n) = ||Q||_{p\to 2}$. Assume $n = 2k$, take $x \in S(\ell_p^n)$ such that $x_{2i} = -\frac{1}{n^{1/p}}$ and $x_{2i-1} = \frac{1}{n^{1/p}}$, for $i = 1, ..., k$. Then $||Q(x)||_2 = n^{\frac{1}{2}-\frac{1}{p}}$. As a result

$$n^{\frac{1}{2}-\frac{1}{p}} = ||Q||_{p\to 2} = \lambda_{p\to 2}(T, \mathbb{R}^n) \leq \lambda^{n-1}_{p\to 2} \leq n^{\frac{1}{2}-\frac{1}{p}}, \tag{43}$$

which completes the proof. □

Essentially the above theorem shows that $T := ker1 = \{x : \sum_{i=1}^{n} x_i = 0\}$ is the maximal hyperplane for minimal projections considered as a operators between ℓ_p^n and ℓ_2^n spaces.

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TABLE OF CONTENTS, JOURNAL OF CONCRETE AND APPLICABLE MATHEMATICS, VOL. 13, NO.'S 3-4, 2015