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ORTHOGONAL STABILITY OF AN ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN SPACES

CHOONKIL PARK, MADJID ESHAGHI GORDJI, HASSAN AZADI KENARY, AND JUNG RYE LEE*

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of an orthogonally additive-quadratic functional equation in non-Archimedean normed spaces.

1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x+y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonally Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

G. Pinsker [39] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [50] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonally Cauchy functional equation

$$f(x+y) = f(x) + f(y), \qquad x \perp y,$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [18]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [47] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [48] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [47].

Suppose X is a real vector space (algebraic module) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$,

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*Corresponding author.

which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space we mean an orthogonality space having a normed structure.

Some interesting examples are
(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.
(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.
(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [7, 14, 23, 24, 35]).

The stability problem of functional equations originated from the following question of Ulam [52]: *Under what condition does there exist an additive mapping near an approximately additive mapping?* In 1941, Hyers [20] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [41] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$. The result of Th.M. Rassias has provided a lot of influence in the development of what we now call *generalized Hyers-Ulam stability* or *Hyers-Ulam stability* of functional equations. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [10, 11, 21, 25, 46] and references therein for detailed information on stability of functional equations.

R. Ger and J. Sikorska [17] investigated the orthogonal stability of the Cauchy functional equation $f(x+y) = f(x)+f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [49] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [38], [42]–[45]).

The orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ x \perp y$$

was first investigated by F. Vajzović [53] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [15], M. Fochi [16], M.S. Moslehian [31, 32] and Gy. Szabó [51] generalized this result.

In 1897, Hensel [19] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [12, 27, 28, 34]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:
(1) $|r|=0$ if and only if $r=0$;
(2) $|rs|=|r||s|$;
(3) $|r+s|\le max\{|r|,|s|\}$.

Definition 1.2. ([33]) Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||:X\to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(1) $||x||=0$ if and only if $x=0$;
(2) $||rx||=|r|||x||$ ($r\in\mathbb{K}$, $x\in X$);
(3) The strong triangle inequality (ultrametric); namely,

$$||x+y||\le max\{||x||,||y||\},\qquad x,y\in X.$$

Then $(X,||.||)$ is called a non-Archimedean space.

Note that

$$||x_n-x_m||\le max\{||x_{j+1}-x_j||: m\le j\le n-1\}\qquad (n>m).$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Let X be a set. A function $m: X\times X\to[0,\infty]$ is called a *generalized metric* on X if m satisfies
(1) $m(x,y)=0$ if and only if $x=y$;
(2) $m(x,y)=m(y,x)$ for all $x,y\in X$;
(3) $m(x,z)\le m(x,y)+m(y,z)$ for all $x,y,z\in X$.
We recall a fundamental result in fixed point theory.

Theorem 1.4. [4, 13] *Let (X,m) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $\alpha<1$. Then for each given element $x\in X$, either*

$$m(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $m(J^nx,J^{n+1}x)<\infty,\qquad \forall n\ge n_0$;
(2) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y=\{y\in X\mid m(J^{n_0}x,y)<\infty\}$;*
(4) *$m(y,y^*)\le\frac{1}{1-\alpha}m(y,Jy)$ for all $y\in Y$.*

In 1996, G. Isac and Th.M. Rassias [22] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 26, 30, 36, 37, 40]).

In this paper, we prove the Hyers-Ulam stability of the following orthogonally additive-quadratic functional equation

$$2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=\frac{3f(x)}{2}-\frac{f(-x)}{2}+\frac{f(y)}{2}+\frac{f(-y)}{2} \tag{1.1}$$

in non-Archimedean normed spaces by using the fixed point method.

Throughout this paper, assume that $(X,\perp)$ is an orthogonality space and that $(Y,\|.\|_Y)$ is a non-Archimedean Banach space. Assume that $|2|\neq 1$.

2. Hyers-Ulam stability of the orthogonally additive-quadratic functional equation (1.1)

For a given mapping $f:X\to Y$, we define

$$Df(x,y):\ =\ 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-\frac{3f(x)}{2}+\frac{f(-x)}{2}-\frac{f(y)}{2}-\frac{f(-y)}{2}$$

for all all $x,y\in X$ with $x\perp y$, where $\perp$ is the orthogonality in the sense of Rätz.

Let $f:X\to Y$ be an even mapping satisfying $f(0)=0$ and (1.1). Then f is a quadratic mapping, i.e., $2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)=f(x)+f(y)$ holds.

Using the fixed point method and applying some ideas from [17, 21], we prove the orthogonal Hyers-Ulam stability of the additive-quadratic functional equation $Df(x,y)=0$ in non-Archimedean Banach spaces.

Theorem 2.1. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $\alpha<1$ with*

$$\varphi(x,y)\le|4|\alpha\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.1}$$

for all $x,y\in X$ with $x\perp y$. Let $f:X\to Y$ be an even mapping satisfying $f(0)=0$ and

$$\|Df(x,y)\|_Y\le\varphi(x,y) \tag{2.2}$$

for all $x,y\in X$ with $x\perp y$. Then there exists a unique orthogonally quadratic mapping $Q:X\to Y$ such that

$$\|f(x)-Q(x)\|_Y\le\frac{\alpha}{1-\alpha}\varphi(x,0) \tag{2.3}$$

for all $x\in X$.

Proof. Letting $y=0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\|_Y\le\varphi(x,0) \tag{2.4}$$

for all $x \in X$, since $x \perp 0$. Thus

$$\left\| f(x) - \frac{1}{4} f\left(2x\right) \right\|_Y \leq \frac{1}{|4|} \varphi(2x, 0) \leq \frac{|4|\alpha}{|4|} \varphi(x, 0) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$m(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\varphi(x, 0),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, m) is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g\left(2x\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $m(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\|_Y \leq \varphi(x, 0)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{1}{4} g\left(2x\right) - \frac{1}{4} h\left(2x\right) \right\|_Y \leq \alpha\varphi(x, 0)$$

for all $x \in X$. So $m(g, h) = \varepsilon$ implies that $m(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$m(Jg, Jh) \leq \alpha m(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that $m(f, Jf) \leq \alpha$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(2x\right) = 4Q(x) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : m(h, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\|_Y \quad \leq \quad \mu\varphi(x, 0)$$

for all $x \in X$;

(2) $m(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right) = Q(x)$$

for all $x \in X$;

(3) $m(f,Q) \le \frac{1}{1-\alpha} m(f,Jf)$, which implies the inequality

$$m(f,Q) \le \frac{\alpha}{1-\alpha}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|DQ(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{|4|^n} \|Df(2^n x, 2^n y)\|_Y \\ &\le \lim_{n\to\infty} \frac{1}{|4|^n} \varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{|4|^n \alpha^n}{|4|^n} \varphi(x,y) = 0 \end{aligned}$$

for all $x, y \in X$ with $x \perp y$. So $DQ(x,y) = 0$ for all $x, y \in X$ with $x \perp y$. Hence $Q : X \to Y$ is an orthogonally quadratic mapping, as desired. □

Corollary 2.2. *Assume that $(X, \perp)$ is an orthogonality non-Archimedean normed space. Let θ be a positive real number and p a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and*

$$\|Df(x,y)\|_Y \le \theta(\|x\|^p + \|y\|^p) \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{|2|^p \theta}{|4| - |2|^p} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = |2|^{p-2}$ in Theorem 2.1, we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x,y) \le \frac{\alpha}{|4|} \varphi(2x, 2y)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{1}{1-\alpha} \varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.4) that $m(f, Jf) \le 1$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Assume that $(X, \perp)$ is an orthogonality non-Archimedean normed space. Let θ be a positive real number and p a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique orthogonally quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\|_Y \leq \frac{|2|^p \theta}{|2|^p - |4|} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = |2|^{2-p}$ in Theorem 2.3, we get the desired result. $\square$

Let $f : X \to Y$ be an odd mapping satisfying (1.1). Then f is an additive mapping, i.e., $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x)$ holds.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x, y) \leq |2|\alpha\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ with $x \perp y$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\|_Y \leq \frac{\alpha}{|2| - |2|\alpha} \varphi(x, 0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.2), we get

$$\left\|4f\left(\frac{x}{2}\right) - 2f(x)\right\|_Y \leq \varphi(x, 0) \tag{2.8}$$

for all $x \in X$, since $x \perp 0$. Thus

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_Y \leq \frac{1}{|4|}\varphi(2x, 0) \leq \frac{|2|\alpha}{|4|}\varphi(x, 0) \tag{2.9}$$

for all $x \in X$.

Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

It follows from (2.9) that $m(f, Jf) \leq \frac{\alpha}{|2|}$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.6. *Assume that $(X, \perp)$ is an orthogonality non-Archimedean normed space. Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique orthogonally additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\|_Y \leq \frac{|2|^p \theta}{|2|(|2| - |2|^p)} \|x\|^p$$

for all $x \in X$.

Proof. Taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = |2|^{p-1}$ in Theorem 2.5, we get the desired result. $\square$

Theorem 2.7. *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $\alpha < 1$ *with*

$$\varphi(x,y) \le \frac{\alpha}{|2|}\varphi\left(2x, 2y\right)$$

for all $x, y \in X$ *with* $x \perp y$. *Let* $f : X \to Y$ *be an odd mapping satisfying* (2.2). *Then there exists a unique orthogonally additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\|_Y \le \frac{1}{|2| - |2|\alpha}\varphi(x,0)$$

for all $x \in X$.

Proof. Let (S, m) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from (2.8) that $m(f, Jf) \le \frac{1}{|2|}$.

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Corollary 2.8. *Assume that* $(X, \perp)$ *is an orthogonality non-Archimedean normed space. Let* θ *be a positive real number and* p *a real number with* $0 < p < 1$. *Let* $f : X \to Y$ *be an odd mapping satisfying* (2.7). *Then there exists a unique orthogonally additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\|_Y \le \frac{|2|^p\theta}{|2|(|2|^p - |2|)}\|x\|^p$$

for all $x \in X$.

Proof. Taking $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$ with $x \perp y$ and choosing $\alpha = |2|^{1-p}$ in Theorem 2.7, we get the desired result. $\square$

Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (1.1). Let $f_e(x) := \frac{f(x)+f(-x)}{2}$ and $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_e is an even mapping satisfying (1.1) and f_o is an odd mapping satisfying (1.1) such that $f(x) = f_e(x) + f_o(x)$. So we obtain the following.

Theorem 2.9. *Assume that* $(X, \perp)$ *is an orthogonality non-Archimedean normed space. Let* θ *be a positive real number and* p *a positive real number with* $p \neq 1$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (2.7). *Then there exist an orthogonally additive mapping* $A : X \to Y$ *and an orthogonally quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - A(x) - Q(x)\|_Y \le \left(\frac{|2|^p}{|2| \cdot |\ |2| - |2|^p\ |} + \frac{|2|^p}{|\ |4| - |2|^p\ |}\right)\theta\|x\|^p$$

for all $x \in X$.

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STABILITY OF THE LEIBNIZ ADDITIVE-QUADRATIC FUNCTIONAL EQUATION IN QUASI-BETA NORMED SPACE: DIRECT AND FIXED POINT METHODS

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Abstract. In this paper, the authors introduced the Leibniz type additive-quadratic functional equation of the form

$$f(x-t)+f(y-t)+f(z-t)=3f\left(\frac{x+y+z}{3}-t\right)+f\left(\frac{2x-y-z}{3}\right)+f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right)$$

and obtained its general solution and generalized Ulam - Hyers stability of Leibniz AQ - mixed type functional equation in quasi-beta normed space using direct and fixed point methods.

1. INTRODUCTION

The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms was affirmatively answered for Banach spaces by Hyers [9]. It was further generalized via excellent results obtained by a number of authors [2, 6, 18, 21, 23].

Over the last six or seven decades, the above Ulam problem was tackled by numerous authors who provided solutions in various forms of functional equations like

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additive, quadratic, cubic, quartic, mixed type functional equations involving only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 5, 8, 10, 13, 15, 17, 19, 20, 22, 24, 25, 27, 28, 29].

In 2006, K.W. Jun and H.M. Kim [11] introduced the following generalized **additive and quadratic type functional equation**

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \tag{1.1}$$

in the class of function between real vector spaces. For $n = 3$, Pl.Kannappan proved that a function f satisfies the functional equation (1.1) if and only if there exists a symmetric bi-additive function A and additive function B such that $f(x) = B(x,x) + A(x)$ for all x (see [14]). The Hyers-Ulam stability for the equation (1.1) when $n = 3$ was proved by S.M. Jung [12]. The Hyers-Ulam-Rassias stability for the equation (1.1) when $n = 4$ was also investigated by I.S. Chang et al., [4].

Very recently, M. Arunkumar and S. Karthikeyan [3] introduced and established the general solution and generalized Ulam-Hyers stability of $n-$dimensional mixed type additive and quadratic functional equation of the form

$$\begin{aligned} f(-x_1) + f\left(2x_1 - \sum_{i=2}^{n} x_i\right) + f\left(2\sum_{i=2}^{n} x_i\right) + f\left(x_1 + \sum_{i=2}^{n} x_i\right) - f\left(-x_1 - \sum_{i=2}^{n} x_i\right) \\ - f\left(x_1 - \sum_{i=2}^{n} x_i\right) - f\left(-x_1 + \sum_{i=2}^{n} x_i\right) = 3f(x_1) + 3f\left(\sum_{i=2}^{n} x_i\right) \end{aligned} \tag{1.2}$$

in Banach spaces.

Theorem 1.1. ***Leibniz quadratic formula in Euclidean Geometry.*** *Let M be an arbitrary point lying on the plane of the triangle ABC, and G is the centroid (= Gravity center) of ABC, then*

$$|\widehat{MA}|^2 + |\widehat{MB}|^2 + |\widehat{MC}|^2 = 3|\widehat{MG}|^2 + \left(|\widehat{GA}|^2 + |\widehat{GB}|^2 + |\widehat{GC}|^2\right). \tag{1.3}$$

Proof. Let x, y, z, t, g be position vectors of points A, B, C, M, G. Then

$$\widehat{GA} + \widehat{GA} + \widehat{GA} = x - g + y - g + z - g = x + y + z - 3g = 0. \tag{1.4}$$

Hence

$$g = \frac{x+y+z}{3}.$$

Since $\widehat{AG} = \frac{2}{3}\widehat{AA}$, we have

$$g - x = \frac{2}{3}\left(\frac{y+z}{2} - x\right) = \frac{x+y+z}{3}.$$

Thus

$$\widehat{MG} = g - t = \frac{x+y+z}{3} - t, \qquad \widehat{AG} = g - x == \frac{-2x+y+z}{3}$$
$$\widehat{BG} = g - y == \frac{x-2y+z}{3}, \qquad \widehat{CG} = g - z == \frac{x+y-2z}{3},$$

and

$$\widehat{MA} = x - t, \widehat{MB} = y - t, \widehat{MC} = z - t.$$

$$|x-t|^2 + |y-t|^2 + |z-t|^2 = 3\left|\frac{x+y+z}{3} - t\right|^2 + \left|\frac{2x-y-z}{3}\right|^2 + \left|\frac{-x+2y-z}{3}\right|^2 + \left|\frac{-x-y+2z}{3}\right|^2$$

which obviously holds, completing the proof of (1.3). □

The above inequality is transformed into the following Leibniz type additive - quadratic functional equation of the form

$$f(x-t) + f(y-t) + f(z-t) = 3f\left(\frac{x+y+z}{3} - t\right) + f\left(\frac{2x-y-z}{3}\right) + f\left(\frac{-x+2y-z}{3}\right) + f\left(\frac{-x-y+2z}{3}\right) \quad (1.5)$$

having solutions

$$f(x) = ax + bx^2. \quad (1.6)$$

In this paper, the authors obtained its general solution and generalized Ulam - Hyers stability of Leibniz AQ - mixed type functional equation (1.5) in quasi-beta normed space using direct and fixed point methods.

2. GENERAL SOLUTION

In this section, we give the general solution of the Leibniz functional equation (1.5). Throughout this section, we consider X and Y be real vector spaces.

Theorem 2.1. *If an odd function $f : X \to Y$ satisfies the functional equation (1.5) then f is additive.*

Proof. Letting (x, y, z, t) by $(0, 0, 0, 0)$ in (1.5), we get $f(0) = 0$. Replacing (x, y, z, t) by $(2x, x, 0, -t)$ in (1.5), we obtain

$$f(2x+t) + f(t) = 2f(x+t) + f(x) + f(-x) \quad (2.1)$$

for all $x, t \in X$. Using oddness of f in (2.1), we have

$$f(2x+t)+f(t)=2f(x+t) \tag{2.2}$$

for all $x, t \in X$. Interchanging x and t in (2.2), we arrive

$$f(2t+x)+f(x)=2f(x+t) \tag{2.3}$$

for all $x, t \in X$. Replacing t by $t-x$ in (2.2) and using oddness of f, we get

$$f(x+t)-f(x-t)=2f(t) \tag{2.4}$$

for all $x, t \in X$. Again replacing x by $x-t$ in (2.3) and using oddness of f, we get

$$f(x+t)+f(x-t)=2f(x) \tag{2.5}$$

for all $x, t \in X$. Adding (2.4) and (2.5), our result is desired. □

Theorem 2.2. *If an even function $f : X \to Y$ satisfies the functional equation (1.5) then f is quadratic.*

Proof. Letting (x, y, z, t) by $(0, 0, 0, 0)$ in (1.5), we get $f(0) = 0$. Replacing (x, y, z, t) by $(2x, x, 0, -t)$ in (1.5), we obtain

$$f(2x+t)+f(t)=2f(x+t)+f(x)+f(-x) \tag{2.6}$$

for all $x, t \in X$. Using evenness of f in (2.6), we have

$$f(2x+t)+f(t)=2f(x+t)+2f(x) \tag{2.7}$$

for all $x, t \in X$. Replacing t by $t-x$ in (2.7) our result is desired. □

3. DEFINITIONS AND NOTATIONS ON QUASI-BETA NORMED SPACES

In this section, we present here some basic facts concerning quasi-β-Normed spaces and some preliminary results.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.1. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\| \cdot \|$ is a real-valued function on X satisfying the following:

(i) $\| x \| \geq 0$ for all $x \in X$ and $\| x \| = 0$ if and only if $x = 0$.
(ii) $\| \lambda x \| = | \lambda |^{\beta} . \| x \|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\| x + y \| \leq K (\| x \| + \| y \|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called quasi-β-normed space if $\| \cdot \|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\| \cdot \|$.

Definition 3.2. A quasi-β-Banach space is a complete quasi-β-normed space.

Definition 3.3. A qusi-β-norm $\| \cdot \|$ is called a (β, p)-norm $(0 < p \leq 1)$ if

$$\| x + y \|^p \leq \| x \|^p + \| y \|^p$$

for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

For more information one can refer [8, 28] for the concept of quasi-normed spaces and p-Banach space.

4. STABILITY RESULTS: DIRECT METHOD

In this section, we obtain the generalized Ulam-Hyers stability of the Leibniz type function equation in quasi-Beta normed space.

Throughout this section, let us take X is a linear space over $\mathbb{K}$ and Y is a (β, p) Banach space with $p-$norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$.

For notational convenience, we denote for a given mapping $f : X \to Y$ and define the difference operator $Df : X \to Y$ by

$$Df(x, y, z, t) = f(x - t) + f(y - t) + f(z - t) - 3f\left(\frac{x + y + z}{3} - t\right) - f\left(\frac{2x - y - z}{3}\right) + f\left(\frac{-x + 2y - z}{3}\right) + f\left(\frac{-x - y + 2z}{3}\right)$$

for all $x, y, z, t \in X$.

Theorem 4.1. *Let $j = \pm 1$. Let $f_o : X \to Y$ be a mapping for which there exists a function $\alpha : X^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}t) = 0 \tag{4.1}$$

such that the functional inequality

$$\|Df_o(x, y, z, t)\|_Y \leq \alpha(x, y, z, t) \tag{4.2}$$

for all $x, y, z, t \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies (1.5) and the inequality

$$\|f_o(x) - A(x)\|_Y^p \leq \frac{K^{p(n-1)}}{2^{p\beta}} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{2^{pk}} \tag{4.3}$$

for all $x \in X$.

Proof. Replacing (x, y, z, t) by $(2x, x, 0, 0)$ in the functional inequality (4.1), we get

$$\|f_o(2x) - 3f_o(x) - f_o(-x)\|_Y \leq \alpha(2x, x, 0, 0) \tag{4.4}$$

for all $x \in X$. Using oddness of f_o in (4.4), we obtain

$$\|f_o(2x) - 2f_o(x)\|_Y \leq \alpha(2x, x, 0, 0) \tag{4.5}$$

for all $x \in X$. It follows from (4.5) that

$$\left\| \frac{f_o(2x)}{2} - f_o(x) \right\|_Y \leq \frac{1}{2^\beta} \alpha(2x, x, 0, 0) \tag{4.6}$$

for all $x \in X$. Replacing x by $2x$ and dividing by 2 in (4.6), we get

$$\left\| \frac{f_o(2^2x)}{2^2} - \frac{f_o(2x)}{2} \right\|_Y \leq \frac{1}{2^\beta \cdot 2} \alpha(2^2x, 2x, 0, 0) \tag{4.7}$$

for all $x \in X$. From (4.6) and (4.7), we have

$$\begin{aligned} \left\| \frac{f_o(2^2x)}{2^2} - f_o(x) \right\|_Y &\leq K \left[\left\| \frac{f_o(2x)}{2} - f_o(x) \right\|_Y + \left\| \frac{f_o(2^2x)}{2^2} - \frac{f_o(2x)}{2} \right\|_Y \right] \\ &\leq \frac{K}{2^\beta} \left[\alpha(2x, x, 0, 0) + \frac{\alpha(2^2x, 2x, 0, 0)}{2} \right] \end{aligned} \tag{4.8}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$\begin{aligned} \left\| \frac{f_o(2^nx)}{2^n} - f_o(x) \right\|_Y^p &\leq \frac{K^{p(n-1)}}{2^{p\beta}} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+1}x, 2^kx, 0, 0)^p}{2^{pk}} \\ &\leq \frac{K^{p(n-1)}}{2^{p\beta}} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+1}x, 2^kx, 0, 0)^p}{2^{pk}} \end{aligned} \tag{4.9}$$

for all $x \in U$. In order to prove the convergence of the sequence

$$\left\{ \frac{f_o(2^nx)}{2^n} \right\},$$

replacing x by 2^mx and dividing by 2^m in (4.9), for any $m, n > 0$, we deduce

$$\begin{aligned} \left\| \frac{f_o(2^{n+m}x)}{2^{(n+m)}} - \frac{f_o(2^mx)}{2^m} \right\|_Y &= \frac{1}{2^m} \ \left\| \frac{f_o(2^n \cdot 2^mx)}{2^n} - f_o(2^mx) \right\|_Y \\ &\leq \frac{K^{n-1}}{2^\beta} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+m+1}x, 2^{k+m}x, 0, 0)}{2^{k+m}} \\ &\leq \frac{K^{n-1}}{2^\beta} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+m+1}x, 2^{k+m}x, 0, 0)}{2^{k+m}} \\ &\to 0 \quad as \ \ m \to \infty \end{aligned}$$

for all $x \in U$. Thus it follows that a sequence

$$\left\{ \frac{f_o(2^nx)}{2^n} \right\},$$

is a Cauchy in Y and so it converges. Therefore we see that a mapping $A : X \to Y$ defined by

$$A(x) = \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n}$$

is well defined for all $x \in X$. In addition it is clear from (4.1) that the following inequality

$$\begin{aligned}\|DA(x,y,z,t)\|_Y^p &= \lim_{n\to\infty} \frac{1}{2^{pn}} \|Df_o(2^n x, 2^n y, 2^n z, 2^n t)\|_Y^p \\ &\le \lim_{n\to\infty} \frac{1}{2^{pn}} \alpha(2^n x, 2^n y, 2^n z, 2^n t)^p \\ &\to 0 \ as \ n \to \ \infty\end{aligned}$$

holds for all $x, y, z, t \in X$ and so the mapping A is additive. Letting $n \to \infty$ in (4.9) and using the definition of $A(x)$ we see that (4.3) holds for all $x \in U$. To prove uniqueness, we assume now that there is another function $A' : X \to Y$ which satisfies (1.5) and the inequality (4.3) then it follows that $A(2x) = 2A(x), A'(2x) = 2A'(x)$ for all $x \in X$ and all $n \in N$. Thus

$$\begin{aligned}\|A(x) - A'(x)\|_Y^p &= \frac{1}{2^{\beta pn}} \|A(2^n x) - A'(2^n x)\|_Y^p \\ &= \frac{K^p}{2^{\beta pn}} \left\{\|A(2^n x) - f_o(2^n x)\|_Y^p + \|f_o(2^n x) - A'(2^n x)\|_Y^p\right\} \\ &\le \frac{K^p}{2^{\beta n}} \left(\frac{2^p \ K^{p(n-1)}}{2^{p\beta}} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+n+1}x, 2^{k+n}x, 0, 0)^p}{2^{p(k+n)}} \right) \\ &\to 0 \ as \ n \to \infty\end{aligned}$$

for all $x \in X$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.1 concerning the stability of (1.5).

Corollary 4.2. *Let $f_o : X \to Y$ be an odd mapping and there exits real numbers λ and s such that*

$$\|Df_o(x,y,z,t)\|_Y \le \begin{cases} \lambda, & \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s + ||t||^s\right\}, & s < 1 \ or \ s > 1; \\ \lambda \left\{||x||^s ||y||^s ||z||^s ||t||^s + \left\{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right\}\right\}, & s < \frac{1}{4} \ or \ s > \frac{1}{4}; \end{cases} \tag{4.10}$$

for all $x, y, z, t \in U$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f_o(x) - A(x)\|_Y^p \leq \begin{cases} \left(\dfrac{2\lambda K^{(n-1)}}{2^\beta}\right)^p, \\ \left(\dfrac{2(2^s+1)\lambda K^{(n-1)}||x||^s}{2^\beta|2-2^s|}\right)^p, \\ \left(\dfrac{2(2^{4s}+1)\lambda K^{(n-1)}||x||^{4s}}{2^\beta|2-2^{4s}|}\right)^p, \end{cases} \tag{4.11}$$

for all $x \in X$.

Theorem 4.3. *Let $j = \pm 1$. Let $f_e : X \to Y$ be an even mapping for which there exists a function $\alpha : X^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{4^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}z, 2^{nj}t) = 0 \tag{4.12}$$

such that the functional inequality

$$\|Df_e(x, y, z, t)\|_Y \leq \alpha(x, y, z, t) \tag{4.13}$$

for all $x, y, z, t \in X$. Then there exists a unique quadratic mapping $A : X \to Y$ which satisfies (1.5) and the inequality

$$\|f_e(x) - Q(x)\|_Y^p \leq \frac{K^{p(n-1)}}{4^{p\beta}} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{4^{pk}} \tag{4.14}$$

for all $x \in X$.

Proof. Replacing (x, y, z, t) by $(2x, x, 0, 0)$ in the functional inequality (4.12), we get

$$\|f_e(2x) - 3f_e(x) - f_e(-x)\|_Y \leq \alpha(2x, x, 0, 0) \tag{4.15}$$

for all $x \in X$. Using evenness of f_e in (4.15), we obtain

$$\|f_e(2x) - 4f_e(x)\|_Y \leq \alpha(2x, x, 0, 0) \tag{4.16}$$

for all $x \in X$. It follows from (4.16) that

$$\left\|\frac{f_e(2x)}{4} - f_e(x)\right\|_Y \leq \frac{1}{4^\beta}\alpha(2x, x, 0, 0) \tag{4.17}$$

for all $x \in X$. Replacing x by $2x$ and dividing by 2 in (4.17), we get

$$\left\|\frac{f_e(2^2x)}{4^2} - \frac{f_e(2x)}{4}\right\|_Y \leq \frac{1}{4^\beta \cdot 2}\alpha(2^2x, 2x, 0, 0) \tag{4.18}$$

for all $x \in X$. From (4.17) and (4.18), we have

$$\left\|\frac{f_e(2^2x)}{4^2} - f_e(x)\right\|_Y \le K\left[\left\|\frac{f_e(2x)}{4} - f_e(x)\right\|_Y + \left\|\frac{f_e(2^2x)}{4^2} - \frac{f_e(2x)}{4}\right\|_Y\right]$$
$$\le \frac{K}{4^\beta}\left[\alpha(2x,x,0,0) + \frac{\alpha(2^2x,2x,0,0)}{4}\right] \tag{4.19}$$

for all $x \in U$. Proceeding further and using induction on a positive integer n , we get

$$\left\|\frac{f_e(2^nx)}{4^n} - f_e(x)\right\|_Y^p \le \frac{K^{p(n-1)}}{4^{p\beta}}\sum_{k=0}^{n-1}\frac{\alpha(2^{k+1}x,2^kx,0,0)^p}{4^{pk}} \tag{4.20}$$
$$\le \frac{K^{p(n-1)}}{4^{p\beta}}\sum_{k=0}^{\infty}\frac{\alpha(2^{k+1}x,2^kx,0,0)^p}{4^{pk}}$$

for all $x \in U$. In order to prove the convergence of the sequence

$$\left\{\frac{f_e(2^nx)}{4^n}\right\},$$

replacing x by 2^mx and dividing by 4^m in (4.20), for any $m, n > 0$, we deduce

$$\left\|\frac{f_e(2^{n+m}x)}{4^{(n+m)}} - \frac{f_e(2^mx)}{4^m}\right\|_Y = \frac{1}{4^m}\left\|\frac{f_e(2^n\cdot 2^mx)}{4^n} - f_e(2^mx)\right\|_Y$$
$$\le \frac{K^{n-1}}{4^\beta}\sum_{k=0}^{n-1}\frac{\alpha(2^{k+m+1}x,2^{k+m}x,0,0)}{4^{k+m}}$$
$$\le \frac{K^{n-1}}{4^\beta}\sum_{k=0}^{\infty}\frac{\alpha(2^{k+m+1}x,2^{k+m}x,0,0)}{4^{k+m}}$$
$$\to 0 \quad as \ \ m \to \infty$$

for all $x \in U$. Thus it follows that a sequence

$$\left\{\frac{f_e(2^nx)}{4^n}\right\},$$

is a Cauchy in Y and so it converges. Therefore we see that a mapping $Q : X \to Y$ defined by

$$Q(x) = \lim_{n\to\infty}\frac{f_e(2^nx)}{4^n}$$

is well defined for all $x \in X$. To show that Q satisfies (1.5) and it is unique the proof is similar to that of Theorem 4.1.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.5).

Corollary 4.4. *Let $f_e : X \to Y$ be an even mapping and there exits real numbers λ and s such that*

$$\|Df_e(x, y, z, t)\|_Y \leq \begin{cases} \lambda, \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s + ||t||^s\right\}, & s < 2 \text{ or } s > 2; \\ \lambda \left\{||x||^s||y||^s||z||^s||t||^s + \left\{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right\}\right\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \tag{4.21}$$

for all $x, y, z, t \in U$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f_e(x) - Q(x)\|_Y^p \leq \begin{cases} \left(\dfrac{4\lambda K^{(n-1)}}{3 \cdot 4^\beta}\right)^p, \\ \left(\dfrac{4(2^s+1)\lambda K^{(n-1)}||x||^s}{4^\beta|4 - 2^s|}\right)^p, \\ \left(\dfrac{4(2^{4s}+1)\lambda K^{(n-1)}||x||^{4s}}{4^\beta|4 - 2^{4s}|}\right)^p, \end{cases} \tag{4.22}$$

for all $x \in X$.

Now we are ready to prove our main theorem.

Theorem 4.5. *Let $j \in \{-1, 1\}$ and $\alpha : X^4 \to [0, \infty)$ be a function satisfying (4.1) and (4.12) for all $x, y, z, t \in X$. Let $f : X \to Y$ be a function satisfying the inequality*

$$\|Df(x, y, z, t)\|_y \leq \alpha(x, y, z, t) \tag{4.23}$$

for all $x, y, z, t \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\begin{aligned} &\|f(x) - A(x) - Q(x)\|_Y^p \\ &\quad\leq \frac{K^p}{2^p}\left[\frac{K^{p(n-1)}}{2^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^kx, 0, 0)^p}{2^{pk}} + \frac{\alpha(-2^{k+1}x, -2^kx, 0, 0)^p}{2^{pk}}\right)\right. \\ &\qquad\left. + \frac{K^{p(n-1)}}{4^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^kx, 0, 0)^p}{4^{pk}} + \frac{\alpha(-2^{k+1}x, -2^kx, 0, 0)^p}{4^{pk}}\right)\right] \end{aligned} \tag{4.24}$$

for all $x \in X$.

Proof. Let $f_a(x) = \dfrac{f_o(x) - f_o(-x)}{2}$ for all $x \in x$. Then $f_a(0) = 0$ and $f_a(-x) = -f_a(x)$ for all $x \in X$. Hence

$$\|Df_a(x,y,z,t)\|_Y \leq \frac{\alpha(x,y,z,t)}{2} + \frac{\alpha(-x,-y,-z,-t)}{2} \tag{4.25}$$

By Theorem 4.1, we have

$$\|f_a(x) - A(x)\|_Y \leq \frac{1}{2}\frac{K^{p(n-1)}}{2^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{2^{pk}} + \frac{\alpha(-2^{k+1}x, -2^k x, 0, 0)^p}{2^{pk}}\right) \tag{4.26}$$

for all $x \in X$. Also, let $f_q(x) = \dfrac{f_e(x) + f_e(-x)}{2}$ for all $x \in X$. Then $f_q(0) = 0$ and $f_q(-x) = f_q(x)$ for all $x \in x$. Hence

$$\|Df_q(x,y,z,t)\|_Y \leq \frac{\alpha(x,y,z,t)}{2} + \frac{\alpha(-x,-y,-z,t)}{2} \tag{4.27}$$

By Theorem 4.3, we have

$$\|f_q(x) - Q(x)\|_Y \leq \frac{1}{2}\frac{K^{p(n-1)}}{4^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{4^{pk}} + \frac{\alpha(-2^{k+1}x, -2^k x, 0, 0)^p}{4^{pk}}\right) \tag{4.28}$$

for all $x \in X$. Define

$$f(x) = f_a(x) + f_q(x) \tag{4.29}$$

for all $x \in x$. From (5.24),(5.26) and (5.27), we arrive

$$\begin{aligned}
&\|f(x) - A(x) - Q(x)\|_y^p \\
&\quad = \|f_a(x) + f_q(x) - A(x) - Q(x)\|_Y^p \\
&\quad \leq \|f_a(x) - A(x)\|_Y^p + \|f_q(x) - Q(x)\|_Y^p \\
&\quad \leq \frac{K^p}{2^p}\left[\frac{K^{p(n-1)}}{2^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{2^{pk}} + \frac{\alpha(-2^{k+1}x, -2^k x, 0, 0)^p}{2^{pk}}\right)\right. \\
&\qquad \left. + \frac{K^{p(n-1)}}{4^{p\beta}}\sum_{k=0}^{\infty}\left(\frac{\alpha(2^{k+1}x, 2^k x, 0, 0)^p}{4^{pk}} + \frac{\alpha(-2^{k+1}x, -2^k x, 0, 0)^p}{4^{pk}}\right)\right]
\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. $\square$

Using Corollaries 4.2 and 4.4 we have the following Corollary concerning the stability of (1.5).

Corollary 4.6. *Let λ and s be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z,t)\|_Y \leq \begin{cases} \lambda, \\ \lambda\left\{||x||^s+||y||^s+||z||^s+||t||^s\right\}, & s<1 \text{ or } s>1; \\ \lambda\left\{||x||^s||y||^s||z||^s||t||^s+\left\{||x||^{4s}+||y||^{4s}+||z||^{4s}+||t||^{4s}\right\}\right\}, & s<\frac{1}{4} \text{ or } s>\frac{1}{4}; \end{cases} \tag{4.30}$$

for all $x, y, z, t \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x)-A(x)-Q(x)\|_Y^p \leq \begin{cases} \left(\left(\dfrac{2\lambda K^{(n-1)}}{2^\beta}\right)^p+\left(\dfrac{4\lambda K^{(n-1)}}{3\cdot 4^\beta}\right)^p\right), \\ \left(\left(\dfrac{2(2^s+1)\lambda K^{(n-1)}||x||^s}{2^\beta|2-2^s|}\right)^p+\left(\dfrac{4(2^s+1)\lambda K^{(n-1)}||x||^s}{4^\beta|4-2^s|}\right)^p\right), \\ \left(\left(\dfrac{2(2^{4s}+1)\lambda K^{(n-1)}||x||^{4s}}{2^\beta|2-2^{4s}|}\right)^p+\left(\dfrac{4(2^{4s}+1)\lambda K^{(n-1)}||x||^{4s}}{4^\beta|4-2^{4s}|}\right)^p\right) \end{cases} \tag{4.31}$$

for all $x \in X$.

5. STABILITY RESULTS: FIXED METHOD

In this section, the generalized Ulam - Hyers - Rassias stability of the Leibniz AQ - functional equation (1.5) is given by the Fixed point method .

For notational convenience, we denote for a given mapping $f : X \to Y$ and define the difference operator $Df : X \to Y$ by

$$Df(x,y,z,t) = f(x-t)+f(y-t)+f(z-t)-3f\left(\frac{x+y+z}{3}-t\right)-f\left(\frac{2x-y-z}{3}\right) + f\left(\frac{-x+2y-z}{3}\right)+f\left(\frac{-x-y+2z}{3}\right)$$

for all $x, y, z, t \in X$.

Now we will recall the fundamental results in fixed point theory.

Theorem 5.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

$$d(Tx, Ty) \leq Ld(x, y) \tag{A1}$$

for some (Lipschitz constant) $L < 1$. Then,
(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

$(A2)$ $$lim_{n\to\infty} T^n x = x^*,$$

for any starting point $x \in X$;

(iii) One has the following estimation inequalities:

$(A3)$ $$d(T^n x, x^*) \leq \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \geq 0, \forall\ x \in X;$$

$(A4)$ $$d(x, x^*) \leq \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$$

Theorem 5.2. [16]*(The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$$

or

$(B2)$ *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

In this section, let us assume V be a vector space and B Banach space respectively.

Theorem 5.3. *Let $f_o : V \to B$ be a mapping for which there exists a function $\alpha : V^4 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{\alpha(\mu_i^n x,\ \mu_i^n y, \mu_i^n z,\ \mu_i^n t)}{\mu_i^n} = 0 \tag{5.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|Df_o(x, y, z, t)\|_Y \leq \alpha(x, y, z, t) \tag{5.2}$$

for all $x, y, z, t \in V$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \alpha\left(x, \frac{x}{2}, 0, 0\right),$$

has the property

$$\gamma(x) \leq L\ \mu_i\ \gamma\left(\frac{x}{\mu_i}\right) \tag{5.3}$$

for all $x \in V$. Then there exists unique additive function $A : V \to B$ satisfying the functional equation (1.5) and

$$\|\ f_a(x) - A(x)\ \|_Y^p \leq \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p \tag{5.4}$$

holds for all $x \in V$.

Proof. Consider the set $\Omega = \{g/g : V \to B,\ g(0) = 0\}$ and introduce the generalized metric on Ω,

$$d(g, h) = \inf\{M \in (0, \infty) : \|\ g(x) - h(x)\ \|_Y \leq M\gamma(x), x \in V\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\mu_i} g(\mu_i x), \qquad for\ all\ x \in V.$$

Now $g, h \in \Omega$,

$$\begin{aligned} d(g,h) \leq M &\Rightarrow \| g(x) - h(x) \|_Y \leq M\gamma(x), x \in V. \\ &\Rightarrow \left\| \frac{1}{\mu_i} g(\mu_i x) - \frac{1}{\mu_i} h(\mu_i x) \right\|_Y \leq \frac{1}{\mu_i} M\gamma(\mu_i x), x \in V, \\ &\Rightarrow \left\| \frac{1}{\mu_i} g(\mu_i x) - \frac{1}{\mu_i} h(\mu_i x) \right\|_Y \leq L\ M\gamma(x), x \in V, \\ &\Rightarrow \| Tg(x) - Th(x) \|_Y \leq LM\gamma(y), x \in V, \\ &\Rightarrow d(Tg, Th) \leq LM. \end{aligned}$$

This implies $d(Tg, Th) \leq Ld(g, h)$, for all $g, h \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

It follows form (4.6) that,

$$\left\| \frac{f_o(2x)}{2} - f_o(x) \right\|_Y \leq \frac{1}{2^\beta} \alpha(2x, x, 0, 0) \tag{5.5}$$

for all $y \in V$. Using (5.3) for the case $i = 0$ it reduces to

$$\left\| \frac{f_o(2x)}{2} - f_o(x) \right\|_Y \leq L\gamma(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(f_o, Tf_o) \leq L = \frac{1}{2^\beta} \Rightarrow d(f_o, Tf_o) \leq L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (5.5), we get

$$\left\| f_o(x) - 2f_o\left(\frac{x}{2}\right) \right\|_Y \leq \alpha\left(x, \frac{x}{2}, 0, 0\right) \tag{5.6}$$

for all $x \in V$. Using (5.3) for the case $i = 1$ it reduces to

$$\left\| f_o(x) - 2f_o\left(\frac{x}{2}\right) \right\|_Y \leq \gamma(x)$$

for all $X \in V$.

$$\text{i.e.,} \qquad d(f_o, Tf_o) \leq 1 \Rightarrow d(f_o, Tf_o) \leq 1 = L^0 < \infty.$$

In both cases, we have

$$d(f_o, Tf_o) \leq L^{1-i} \tag{5.7}$$

Therefore $(B1\ (i))$ holds.

By $(B1\ (ii))$, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = \lim_{n\to\infty} \frac{f_o(\mu_i^n y)}{\mu_i^n} \qquad \forall\ x \in V. \tag{5.8}$$

In order to prove $A : V \to B$ is Additive. Replacing (x, y, z, t) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t)$ in (5.2) and dividing by μ_i^n, it follows from (5.1) and (5.8), A satisfies (1.5) for all $x, y, z, t \in V$.

By $(B1\ (iii))$, A is the unique fixed point of T in the set $\Delta = \{f_o \in X : d(f_o, A) < \infty\}$, such that

$$\|f_o(x) - A(x)\|_Y \le M\beta(x)$$

for all $x \in V$ and $M > 0$. Finally, by $(B1\ (iV))$, we obtain

$$d(f_o, A) \le \frac{1}{1-L} d(f_o, Tf_o)$$

this implies

$$d(f_o, A) \le \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\| f_o(x) - A(x) \|_Y^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p.$$

for all $x \in V$. This completes the proof of the theorem. □

From Theorem 5.3, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability for the functional equation (1.5).

Corollary 5.4. *Let $f_o : X \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|Df_o(x, y, z, t)\|_Y \le \begin{cases} \lambda \{||x||^s + ||y||^s + ||z||^s + ||t||^s\}, & s < 1 \text{ or } s > 1; \\ \lambda \{||x||^s||y||^s||z||^s||t||^s + \{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\}\}, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \end{cases} \tag{5.9}$$

for all $x, y, z, t \in U$, then there exists a unique additive function $A : X \to Y$ such that

$$\|f_o(x) - A(x)\|_Y^p \le \begin{cases} \left(\dfrac{(2^s+1)\lambda||x||^s}{2^\beta|2-2^s|}\right)^p, \\ \left(\dfrac{(2^{4s}+1)\lambda||x||^{4s}}{2^\beta|2-2^{4s}|}\right)^p, \end{cases} \tag{5.10}$$

for all $x \in X$.

Proof. Setting

$$\alpha(x,y,z,t) = \begin{cases} \lambda\left\{||x||^s + ||y||^s + ||z||^s + ||t||^s\right\}, \\ \lambda\left\{||x||^s||y||^s||z||^s||t||^s + \left\{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right\}\right\} \end{cases}$$

for all $x, y, z, t \in X$.
Then,for $s < 1$ if $i = 0$ and for $s > 1$ if $i = 1$, we get

$$\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t)}{\mu_i^n}$$

$$= \begin{cases} \dfrac{\lambda}{\mu_i^n}\left\{||\mu_i^n x||^s + ||\mu_i^n y||^s + ||\mu_i^n z||^s + ||\mu_i^n t||^s\right\}, \\ \dfrac{\lambda}{\mu_i^n}\left\{||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s\ ||\mu_i^n t||^s \left\{||\mu_i^n x||^{4s} + ||\mu_i^n y||^{4s} + ||\mu_i^n z||^{4s} + ||\mu_i^n w||^{4s}\right\}\right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (5.1) is holds.

But we have $\gamma(x) = \alpha\left(x, \frac{x}{2}, 0, 0\right)$ has the property $\gamma(x) \leq L \cdot \mu_i\ \gamma(\mu_i x)$ for all $x \in X$. Hence

$$\gamma(x) = \frac{1}{2^\beta}\ \alpha\left(x, \frac{x}{2}, 0, 0\right) = \begin{cases} \dfrac{\lambda}{2^\beta}\left(||x||^s + ||\dfrac{x}{2}||^s\right), \\ \dfrac{\lambda}{2^\beta}\left(||x||^{4s} + ||\dfrac{x}{2}||^{4s}\right). \end{cases}$$

Now,

$$\frac{1}{\mu_i}\gamma(\mu_i x) = \begin{cases} \dfrac{\lambda}{2^\beta \mu_i}\left\{\left(||\mu_i x||^s + ||\dfrac{\mu_i x}{2}||^s\right)\right\}, \\ \dfrac{\lambda}{2^\beta \mu_i}\left\{\left(||\mu_i x||^{4s} + ||\dfrac{\mu_i x}{2}||^{4s}\right)\right\}. \end{cases}$$

$$= \begin{cases} \dfrac{\lambda}{2^\beta \mu_i}\mu_i^s\left(\dfrac{1+2^s}{2^s}\right)||x||^s, \\ \dfrac{\lambda}{2^\beta \mu_i}\mu_i^{4s}\left(\dfrac{1+2^{4s}}{2^{4s}}\right)||x||^{4s}. \end{cases}$$

$$= \begin{cases} \mu_i^{s-1}\dfrac{\lambda}{2^\beta}\left(\dfrac{1+2^s}{2^s}\right)||x||^s, \\ \mu_i^{4s-1}\dfrac{\lambda}{2^\beta}\left(\dfrac{1+2^{4s}}{2^{4s}}\right)||x||^{4s}. \end{cases}$$

$$= \begin{cases} \mu_i^{s-1}\gamma(x), \\ \mu_i^{4s-1}\gamma(x). \end{cases}$$

Hence the inequality (5.3) holds either, $L = 2^{s-1}$ for $s < 1$ if $i = 0$ and $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$.

Now from (5.4), we prove the following cases for condition (i).
Case:1 $L = 2^{s-1}$ for $s < 1$ if $i = 0$

$$\begin{aligned}\|f_o(x) - A(x)\|_Y &\leq \frac{\left(2^{(s-1)}\right)^{1-0}}{1-2^{(s-1)}}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{2^\beta}||x||^s\\ &\leq \frac{2^s}{2-2^s}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{2^\beta}||x||^s\\ &\leq \frac{\left(\frac{1+2^s}{2^s}\right)\lambda||x||^s}{2^\beta(2-2^s)}\end{aligned}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned}\|f_o(x) - A(x)\|_Y &\leq \frac{\left(\frac{1}{2^{(s-1)}}\right)^{1-1}}{1-\frac{1}{2^{(s-1)}}}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{2^\beta}||x||^s\\ &\leq \frac{2^s}{2^s-2}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{2^\beta}||x||^s\\ &\leq \frac{(1+2^s)\,\lambda||x||^s}{2^\beta(2^s-2)}\end{aligned}$$

Again, the inequality (5.3) holds either, $L = 2^{4s-1}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{4s-1}}$ for $s > 2$ if $i = 1$.
Now from (5.4), we prove the following cases for condition (ii).
Case:1 $L = 2^{4s-1}$ for $s < 1$ if $i = 0$

$$\begin{aligned}\|f_o(x) - A(x)\|_Y &\leq \frac{\left(2^{(4s-1)}\right)^{1-0}}{1-2^{(4s-1)}}\left\{\frac{1+2^{4s}}{2^{4s}}\right\}\ \frac{\lambda}{2^\beta}||x||^{4s}\\ &\leq \frac{2^{4s}}{2-2^{4s}}\left\{\frac{1+2^{4s}}{2^{4s}}\right\}\ \frac{\lambda}{2^\beta}||x||^{4s}\\ &\leq \frac{(1+2^{4s})\,\lambda||x||^{4s}}{2^\beta(2-2^{4s})}\end{aligned}$$

Case:2 $L = \frac{1}{2^{4s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned}\|f_o(x) - A(x)\|_Y &\leq \frac{\left(\frac{1}{2^{(4s-1)}}\right)^{1-1}}{1-\frac{1}{2^{(4s-1)}}}\left\{\frac{1+2^{4s}}{2^{4s}}\right\}\ \frac{\lambda}{2^\beta}||x||^{4s}\\ &\leq \frac{2^{4s}}{2^{4s}-2}\left\{\frac{1+2^{4s}}{2^{4s}}\right\}\ \frac{\lambda}{2^\beta}||x||^{4s}\\ &\leq \frac{(1+2^{4s})\,\lambda||x||^{4s}}{2^\beta(2^{4s}-2)}\end{aligned}$$

Hence the proof is complete □

Theorem 5.5. *Let $f_e : V \to B$ be a mapping for which there exists a function $\alpha : V^4 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{\alpha(\mu_i^n x,\ \mu_i^n y, \mu_i^n z,\ \mu_i^n t)}{\mu_i^n} = 0 \tag{5.11}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|Df_e(x,y,z,t)\|_Y \le \alpha(x,y,z,t) \tag{5.12}$$

for all $x,y,z,t \in V$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \alpha\left(x, \frac{x}{2}, 0, 0\right),$$

has the property

$$\gamma(x) \le L\ \mu_i^2\ \gamma\left(\frac{x}{\mu_i}\right) \tag{5.13}$$

for all $x \in V$. Then there exists unique quadratic function $Q : V \to B$ satisfying the functional equation (1.5) and

$$\|\ f_a(x) - Q(x)\ \|_Y^p \le \left(\frac{L^{1-i}}{1-L}\right)^p \gamma(x)^p \tag{5.14}$$

holds for all $x \in V$.

Proof. Consider the set $\Omega = \{g/g : V \to B,\ g(0) = 0\}$ and introduce the generalized metric on Ω,

$$d(g,h) = \inf\{M \in (0,\infty) : \|\ g(x) - h(x)\ \|_Y \le M\gamma(x), x \in V\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\mu_i^2} g(\mu_i x), \qquad for\ all\ x \in V.$$

Now $g,h \in X$,

$$\begin{aligned} d(g,h) \le M &\Rightarrow \|\ g(x) - h(x)\ \|_Y \le M\gamma(x), x \in V. \\ &\Rightarrow \left\| \frac{1}{\mu_i^2} g(\mu_i x) - \frac{1}{\mu_i} h(\mu_i x) \right\|_Y \le \frac{1}{\mu_i^2} M\gamma(\mu_i x), x \in V, \\ &\Rightarrow \left\| \frac{1}{\mu_i^2} g(\mu_i x) - \frac{1}{\mu_i} h(\mu_i x) \right\|_Y \le L\ M\gamma(x), x \in V, \\ &\Rightarrow \|\ Tg(x) - Th(x)\ \|_Y \le LM\gamma(y), x \in V, \\ &\Rightarrow d(Tg, Th) \le LM. \end{aligned}$$

This implies $d(Tg,Th) \le Ld(g,h)$, for all $g,h \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

It follows form (4.17) that,

$$\left\|\frac{f_e(2x)}{4} - f_e(x)\right\|_Y \le \frac{1}{4^\beta}\alpha(2x, x, 0, 0) \tag{5.15}$$

for all $y \in V$. Using (5.13) for the case $i = 0$ it reduces to

$$\left\|\frac{f_e(2x)}{4} - f_e(x)\right\|_Y \le L\gamma(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(f_e, Tf_e) \le L = \frac{1}{2^\beta} \Rightarrow d(f_e, Tf_e) \le L = L^1 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (5.15), we get

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\|_Y \le \alpha\left(x, \frac{x}{2}, 0, 0\right) \tag{5.16}$$

for all $x \in V$. Using (5.13) for the case $i = 1$ it reduces to

$$\left\|f_e(x) - 4f_e\left(\frac{x}{2}\right)\right\|_Y \le \gamma(x)$$

for all $X \in V$.

$$\text{i.e.,} \qquad d(f_e, Tf_e) \le 1 \Rightarrow d(f_e, Tf_e) \le 1 = L^0 < \infty.$$

In both cases, we have

$$d(f_e, Tf_e) \le L^{1-i} \tag{5.17}$$

Therefore $(B1\ (i))$ holds.

By $(B1\ (ii))$, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{n\to\infty} \frac{f_e(\mu_i^n y)}{\mu_i^n} \qquad \forall\ x \in V. \tag{5.18}$$

In order to prove $Q : V \to B$ is quadratic. Replacing (x, y, z, t) by $(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t)$ in (5.12) and dividing by μ_i^{2n}, it follows from (5.11) and (5.18), Q satisfies (1.5) for all $x, y, z, t \in V$.

By $(B1\ (iii))$, Q is the unique fixed point of T in the set $\Delta = \{f_e \in X : d(f_e, Q) < \infty\}$, such that

$$\|f_e(x) - Q(x)\| \le M\beta(x)$$

for all $x \in V$ and $M > 0$. Finally, by $(B1\ (iV))$, we obtain

$$d(f_e, A) \le \frac{1}{1-L} d(f_e, Tf_e)$$

this implies

$$d(f_e, A) \le \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\| f_e(x) - Q(x) \|_Y^p \leq \left(\frac{L^{1-i}}{1-L} \right)^p \gamma(x)^p.$$

for all $x \in V$. This completes the proof of the theorem. $\square$

From Theorem 5.5, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability for the functional equation (1.5).

Corollary 5.6. *Let $f_e : X \to V$ be a mapping and there exits real numbers λ and s such that*

$$\|Df_e(x,y,z,t)\|_y \leq \begin{cases} \lambda \{||x||^s + ||y||^s + ||z||^s + ||t||^s\}, & s < 2 \text{ or } s > 2; \\ \lambda \{||x||^s||y||^s||z||^s||t||^s + \{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\}\}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \end{cases} \tag{5.19}$$

for all $x,y,z,t \in U$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\|_Y^p \leq \begin{cases} \left(\frac{(2^s+1)\lambda||x||^s}{2^\beta|4-2^s|} \right)^p, \\ \left(\frac{(2^{4s}+1)\lambda||x||^{4s}}{2^\beta|4-2^{4s}|} \right)^p, \end{cases} \tag{5.20}$$

for all $x \in X$.

Proof. Setting

$$\alpha(x,y,z,t) = \begin{cases} \lambda \{||x||^s + ||y||^s + ||z||^s + ||t||^s\}, \\ \lambda \left\{||x||^s||y||^s||z||^s||t||^s + \left\{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\right\}\right\} \end{cases}$$

for all $x,y,z,t \in X$.
Then,for $s < 1$ if $i = 0$ and for $s > 1$ if $i = 1$, we get

$$\frac{\alpha(\mu_i^n x, \mu_i^n y, \mu_i^n z, \mu_i^n t)}{\mu_i^{2n}}$$
$$= \begin{cases} \frac{\lambda}{\mu_i^{2n}} \{||\mu_i^n x||^s + ||\mu_i^n y||^s + ||\mu_i^n z||^s + ||\mu_i^n t||^s\}, \\ \frac{\lambda}{\mu_i^{2n}} \left\{||\mu_i^n x||^s \ ||\mu_i^n y||^s \ ||\mu_i^n z||^s \ ||\mu_i^n t||^s \left\{||\mu_i^n x||^{4s} + ||\mu_i^n y||^{4s} + ||\mu_i^n z||^{4s} + ||\mu_i^n w||^{4s}\right\} \right\} \end{cases}$$
$$= \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (5.11) is holds.

But we have $\gamma(x) = \alpha\left(x, \frac{x}{2}, 0, 0\right)$ has the property $\gamma(x) \leq L \cdot \mu_i \ \gamma(\mu_i x)$ for all $x \in X$. Hence

$$\gamma(x) = \frac{1}{4^\beta}\ \alpha\left(x, \frac{x}{2}, 0, 0\right) = \begin{cases} \dfrac{\lambda}{4^\beta}\left(||x||^s + ||\dfrac{x}{2}||^s\right), \\ \dfrac{\lambda}{4^\beta}\left(||x||^{4s} + ||\dfrac{x}{2}||^{4s}\right). \end{cases}$$

Now,

$$\begin{aligned} \frac{1}{\mu_i^2}\gamma(\mu_i x) &= \begin{cases} \dfrac{\lambda}{4^\beta \mu_i^2}\left\{\left(||\mu_i x||^s + ||\dfrac{\mu_i x}{2}||^s\right)\right\}, \\ \dfrac{\lambda}{4^\beta \mu_i^2}\left\{\left(||\mu_i x||^{4s} + ||\dfrac{\mu_i x}{2}||^{4s}\right)\right\}. \end{cases} \\ &= \begin{cases} \dfrac{\lambda}{4^\beta \mu_i^2}\mu_i^s\left(\dfrac{1+2^s}{2^s}\right)||x||^s, \\ \dfrac{\lambda}{4^\beta \mu_i^2}\mu_i^{4s}\left(\dfrac{1+2^{4s}}{2^{4s}}\right)||x||^{4s}. \end{cases} \\ &= \begin{cases} \mu_i^{s-2}\dfrac{\lambda}{4^\beta}\left(\dfrac{1+2^s}{2^s}\right)||x||^s, \\ \mu_i^{4s-2}\dfrac{\lambda}{4^\beta}\left(\dfrac{1+2^{4s}}{2^{4s}}\right)||x||^{4s}. \end{cases} \\ &= \begin{cases} \mu_i^{s-2}\gamma(x), \\ \mu_i^{4s-2}\gamma(x). \end{cases} \end{aligned}$$

Hence the inequality (5.13) holds either, $L = 2^{s-2}$ for $s < 2$ if $i = 0$ and $L = \frac{1}{2^{s-2}}$ for $s > 2$ if $i = 1$.
Now from (5.14), we prove the following cases for condition (i).
Case:1 $L = 2^{s-2}$ for $s < 2$ if $i = 0$

$$\begin{aligned} \|f_e(x) - Q(x)\|_Y &\leq \frac{\left(2^{(s-2)}\right)^{1-0}}{1 - 2^{(s-2)}}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{4^\beta}||x||^s \\ &\leq \frac{2^s}{4 - 2^s}\left\{\frac{1+2^s}{2^s}\right\}\ \frac{\lambda}{4^\beta}||x||^s \\ &\leq \frac{(1+2^s)\,\lambda||x||^s}{4^\beta(4-2^s)} \end{aligned}$$

Case:2 $L = \frac{1}{2^{s-1}}$ for $s > 1$ if $i = 1$

$$\begin{aligned}\|f_e(x) - Q(x)\|_Y &\leq \frac{\left(\frac{1}{2^{(s-2)}}\right)^{1-1}}{1 - \frac{1}{2^{(s-2)}}} \left\{\frac{1+2^s}{2^s}\right\} \frac{\lambda}{4^\beta}||x||^s \\ &\leq \frac{2^s}{2^s - 4}\left\{\frac{1+2^s}{2^s}\right\} \frac{\lambda}{4^\beta}||x||^s \\ &\leq \frac{(1+2^s)\,\lambda||x||^s}{4^\beta(2^s-4)}\end{aligned}$$

Again, the inequality (5.13) holds either, $L = 2^{4s-2}$ for $s < \frac{1}{2}$ if $i = 0$ and $L = \frac{1}{2^{4s-2}}$ for $s > \frac{1}{2}$ if $i = 1$.
Now from (5.14), we prove the following cases for condition (ii).
Case:1 $L = 2^{4s-1}$ for $s < \frac{1}{2}$ if $i = 0$

$$\begin{aligned}\|f_e(x) - Q(x)\|_Y &\leq \frac{\left(2^{(4s-2)}\right)^{1-0}}{1 - 2^{(4s-2)}} \left\{\frac{1+2^{4s}}{2^{4s}}\right\} \frac{\lambda}{4^\beta}||x||^{4s} \\ &\leq \frac{2^{4s}}{4 - 2^{4s}}\left\{\frac{1+2^{4s}}{2^{4s}}\right\} \frac{\lambda}{4^\beta}||x||^{4s} \\ &\leq \frac{(1+2^s)\,\lambda||x||^{4s}}{4^\beta(4-2^{4s})}\end{aligned}$$

Case:2 $L = \frac{1}{2^{4s-1}}$ for $s > \frac{1}{2}$ if $i = 1$

$$\begin{aligned}\|f_e(x) - Q(x)\|_Y &\leq \frac{\left(\frac{1}{2^{(4s-2)}}\right)^{1-1}}{1 - \frac{1}{2^{(4s-2)}}} \left\{\frac{1+2^{4s}}{2^{4s}}\right\} \frac{\lambda}{4^\beta}||x||^{4s} \\ &\leq \frac{2^{4s}}{2^{4s} - 4}\left\{\frac{1+2^{4s}}{2^{4s}}\right\} \frac{\lambda}{4^\beta}||x||^{4s} \\ &\leq \frac{(1+2^s)\,\lambda||x||^{4s}}{4^\beta(2^{4s}-4)}\end{aligned}$$

Hence the proof is complete □

Theorem 5.7. *Let $f_o : V \to B$ be a mapping for which there exist a function $\alpha : V^4 \to [0, \infty)$ with the conditions (5.1) and (5.11) where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with*

$$\|Df(x, y, z, t)\|_Y \leq \alpha(x, y, z, t) \tag{5.21}$$

for all $x, y, z, t \in V$. If there exists $L = L(i)$ such that the function

$$x \to \gamma(x) = \alpha\left(x, \frac{x}{2}, 0, 0\right),$$

has the properties (5.3) and (5.13) for all $x \in V$. Then there exists unique additive function $A : V \to B$ and unique quadratic function $Q : V \to B$ satisfying the *functional equation (1.5) and*

$$\| f(x) - A(x) - Q(x) \|_Y^p \leq K^p \left(\frac{L^{1-i}}{1-L}\right)^p [\gamma(x)^p + \gamma(-x)^p] \tag{5.22}$$

holds for all $x \in V$.

Proof. Let $f_a(x) = \dfrac{f_o(x) - f_o(-x)}{2}$ for all $x \in x$. Then $f_a(0) = 0$ and $f_a(-x) = -f_a(x)$ for all $x \in X$. Hence

$$\|Df_a(x,y,z,t)\|_Y \leq \frac{\alpha(x,y,z,t)}{2} + \frac{\alpha(-x,-y,-z,-t)}{2} \tag{5.23}$$

By Theorem 5.3, we have

$$\|f_a(x) - A(x)\|_Y \leq \frac{1}{2}\left(\frac{L^{1-i}}{1-L}\right)[\gamma(x) + \gamma(-x)] \tag{5.24}$$

for all $x \in X$. Also, let $f_q(x) = \dfrac{f_e(x) + f_e(-x)}{2}$ for all $x \in X$. Then $f_q(0) = 0$ and $f_q(-x) = f_q(x)$ for all $x \in x$. Hence

$$\|Df_q(x,y,z,t)\|_Y \leq \frac{\alpha(x,y,z,t)}{2} + \frac{\alpha(-x,-y,-z,t)}{2} \tag{5.25}$$

By Theorem 4.3, we have

$$\|f_q(x) - Q(x)\|_Y \leq \frac{1}{2}\left(\frac{L^{1-i}}{1-L}\right)[\gamma(x) + \gamma(-x)] \tag{5.26}$$

for all $x \in X$. Define

$$f(x) = f_a(x) + f_q(x) \tag{5.27}$$

for all $x \in x$. From (5.24),(5.26) and (5.27), we arrive

$$\begin{aligned}
&\|f(x) - A(x) - Q(x)\|_y^p \\
&\quad = \|f_a(x) + f_q(x) - A(x) - Q(x)\|_Y^p \\
&\quad \leq K^p \|f_a(x) - A(x)\|_Y^p + \|f_q(x) - Q(x)\|_Y^p \\
&\quad \leq K^p \left(\frac{L^{1-i}}{1-L}\right)^p [\gamma(x)^p + \gamma(-x)^p]
\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

Corollary 5.8. *Let λ and s be nonnegative real numbers. Let a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z,t)\|_Y \leq \begin{cases} \lambda, \\ \lambda \{||x||^s + ||y||^s + ||z||^s + ||t||^s\}, & s < 1 \text{ or } s > 1; \\ \lambda \{||x||^s ||y||^s ||z||^s ||t||^s + \{||x||^{4s} + ||y||^{4s} + ||z||^{4s} + ||t||^{4s}\}\}, & s < \frac{1}{4} \text{ or } s > \frac{1}{4}; \end{cases} \tag{5.28}$$

for all $x, y, z, t \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\|_Y^p \leq \begin{cases} \left(\dfrac{1}{2^\beta|2-2^s|} + \dfrac{1}{4^\beta|4-2^s|}\right)^p (2^s+1)^p \lambda^p ||x||^{ps}, \\ \left(\dfrac{1}{2^\beta|2-2^{4s}|} + \dfrac{1}{4^\beta|4-2^{4s}|}\right)^p (2^{4s}+1)^p \lambda^p ||x||^{4ps}, \end{cases} \tag{5.29}$$

for all $x \in X$.

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Random Hybrid Proximal Point Algorithm for Fuzzy Nonlinear Set Valued Inclusions

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Abstract

The main purpose of this paper is to introduced and studied a new class of fuzzy nonlinear set valued random variational inclusions involving random nonlinear (A_t, η_t)-monotone mapping in Hilbert spaces. Using the random hybrid proximal point operator associated with random nonlinear (A_t, η_t)-monotone mapping and random relaxed co-coercive mappings, we proved an existence theorem for the iterative sequences generated by the proposed algorithm.

Keywords: Fuzzy mappings, Hilbert spaces, fuzzy nonlinear set valued random variational inclusions, random relaxed cocoercive mapping, existence theorem, iterative sequences, algorithm.
Mathematics Subject Classification: 47H09, , 47J20, 47J25, 49J40.

1 Introduction

The set valued inclusion problem, which was introduced and studied by De Bella [5], Huang *et al.* [17] is a useful extension of the mathematical analysis. It provides us with unified, natural, novel, innovative and general technique to study a wide range of problem arising in different branches of mathematics, engineering and financial sciences. Ding and Luo [10], Verma [30], Huang [16] and Lan *et al.* [21] introduced the concept of η-subdifferential operators, maximal η-monotone operators, H-monotone operators, A-monotone operators, (H, η)-monotone operators, (A, η)-accretive mappings, (G, η)-monotone operators and defined resolvent operators associated with them respectively. Recently Verma [31] has developed a hybrid version of the Eckstein and Bertsekas [12] proximal point algorithm based on the (A, η)-maximal monotonicity framework [31] and studied convergence of the algorithm.
A fuzzy set introduced in the seminal article written by Zadeh [33] is an existence of a crisp set by enlarging the true valued set $\{0, 1\}$ to the real unit interval $[0, 1]$. Fuzzy set theory is a powerful hand set for modeling, uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various field to all aspects of fuzzyness from theoretical to practical in almost all sciences, technology, networking and industry, in our real world, we mostly perform fuzzy approximations. In 1989 Chang and Zhu [9] introduced the concepts of variational inequalities with fuzzy mappings and extended some results of Lassando [20] in the fuzzy setting. Later, they were developed by Agarwal *et al.* [1], Ahmad *et al.* [2], Ding *et al.* [11], Lee *et al.* [23, 24], Huang [15], Lan *et al.* [21] and Anastassiou *et al.* [4] *etc.*

On the other hand, random variational inequality problems and random quasi variational inequality problems have been considered by Chang [6, 7], Chang and Huang [8], Husain *et al.* [18], Tan [29], Yuan [32], Salahuddin and Ahmad [28], Khan and Salahuddin [19] and Salahuddin [27] *etc.*

Inspired and motivated by recent research works [3, 13, 15, 22, 25, 32, 34], in this paper we proposed a general nonlinear framework for a random hybrid proximal point algorithm using the notion of (A_t, η_t)-monotonicity in fuzzy environment. The existence and convergence analysis for the algorithm of solving a fuzzy nonlinear set valued random variational inclusion problems are explored along with some results on the resolvent operator corresponding to (A_t, η_t)-monotonicity mappings. The results of random sequences $\{x_n(t)\}$ generated by the random algorithm converges linearly to a solution of fuzzy nonlinear set valued random variational inclusion problems as the convergence rate θ is proved.

2 Preliminaries

Let H be a real Hilbert space with $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let $\mathcal{F}(\mathcal{H})$ be a collection of all fuzzy sets over H. A mapping F from H into $\mathcal{F}(\mathcal{H})$ is called a fuzzy mapping on H. If F is a fuzzy mapping on H, then $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on H and $F_x(y)$ is the membership function of y in F_x. Let $S \in \mathcal{F}(H), q \in [0,1]$. Then the set

$$(S)_q = \{u \in H : S(u) \geq q\}$$

is called a q-cut set of S.
In this communication, we denote by (Ω, Σ) a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω and by $\mathcal{B}(H), 2^H, CB(H)$ and $\hat{\mathcal{H}}(\cdot,\cdot)$, the class of Borel σ-field in H, the family of all nonempty subset of H, the family of all nonempty closed bounded subsets of H and the Hausdorff metric on $CB(H)$ respectively. A mapping $x : \Omega \to H$ is said to be measurable if for any $B \in \mathcal{B}(H), \{t \in \Omega : x(t) \in B\} \in \Sigma$.
A mapping $f : \Omega \times H \to H$ is called a random operator if for any $x \in H, f(t,x) = x(t)$ is a measurable. A random operator f is said to be continuous if for any $t \in \Omega$, the mapping $f(t,\cdot) : H \to H$ is continuous. A set valued mapping $T : \Omega \to 2^H$ is said to be measurable if for any $B \in \mathcal{B}(H), T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$. A mapping $u : \Omega \to H$ is called a measurable selection of a set valued measurable mapping $T : \Omega \to 2^H$, if u is a measurable and for any $t \in \Omega, u(t) \in T(t)$. A mapping $T : \Omega \times H \to 2^H$ is called a random set valued mapping if for any $x \in H, T(\cdot, x)$ is measurable. A random set valued mapping $T : \Omega \times H \to CB(H)$ is said to be $\hat{\mathcal{H}}$-continuous if for any $t \in \Omega, T(t,\cdot)$ is continuous in the Hausdorff metric.

Definition 2.1 *A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is called measurable if for any $\alpha \in (0,1], (F(\cdot))_\alpha : \Omega \to 2^H$ is a measurable set valued mapping.*

Definition 2.2 *A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is called a random fuzzy mapping, if for any $x \in H, F(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.*

Let $T, P, Q : \Omega \times H \to \mathcal{F}(H)$ be the three random fuzzy mappings satisfying the following condition (C) :
(C) : there exist three mappings $a, b, c : H \to [0,1]$ such that $(T_{t,x(t)})_{a(x(t))} \in CB(H), (P_{t,x(t)})_{b(x(t))} \in CB(H),\ (Q_{t,x(t)})_{c(x(t))} \in CB(H),\ \forall (t,x) \in \Omega \times H.$
By using the random fuzzy mappings T, P, Q, we can define three random set valued mappings $\tilde{T}, \tilde{P}$ and $\tilde{Q}$ as follows:

$$\tilde{T} : \Omega \times H \to CB(H), x \to (T_{t,x})_{a(x)}\ \forall (t,x) \in \Omega \times H,$$

$$\tilde{P} : \Omega \times H \to CB(H), x \to (P_{t,x})_{b(x)}\ \forall (t,x) \in \Omega \times H,$$

$$\tilde{Q} : \Omega \times H \to CB(H), x \to (Q_{t,x})_{c(x)}\ \forall (t,x) \in \Omega \times H \text{ and } T_{t,x} = T(t, x(t)).$$

In the sequel $\tilde{T}, \tilde{P}$ and $\tilde{Q}$ are called the random set valued mappings induced by random fuzzy mappings T, P and Q, respectively. Let $\eta, N : \Omega \times H \times H \to H$ be two random mappings. Let $f, g, p : \Omega \times H \to H$ be the three random single valued mappings and $M : \Omega \times H \times H \to 2^H$ the random set valued mapping with for each $t \in \Omega, u \in H, M(t, \cdot, u)$ is a maximal η-monotone with Range $(g) \bigcap \mathrm{Dom} M(t, \cdot, u) \neq \emptyset$. we consider the following problem for finding $u, x, y, z : \Omega \to H$ such that for all $t \in \Omega, u(t) \in H, T_{t,u(t)}(x(t)) \geq a(u(t)), P_{t,u(t)}(y(t)) \geq b(u(t)), Q_{t,u(t)}(z(t)) \geq c(u(t))$ and $g(t, u(t)) \bigcap \mathrm{Dom}(M(t, \cdot, z(t))) \neq \emptyset$ for $t \in \Omega$, such that

$$0 \in f_t(x(t)) + N_t(p_t(u(t)), y(t)) + M_t(g_t(u(t)), z(t)). \tag{2.1}$$

The problem (2.1) is called fuzzy nonlinear set valued random variational inclusions. It is known that a number of problems involving the nonmonotone, nonconvex and nonsmooth mapping arising in structural engineering, mechanics, economics and optimization theory can be reduced to study this type of variational inclusions.

3 Random Iterative Algorithm

The following definitions and results are needed to prove the main results.

Lemma 3.1 *[6] Let $T : \Omega \times H \to CB(H)$ be a $\hat{\mathcal{H}}$-continuous random set valued mapping. Then for any measurable mapping $w : \Omega \to H$, the set valued mapping $T(\cdot, w(t)) : \Omega \to CB(H)$ is measurable.*

Lemma 3.2 *[6] Let $P, T : \Omega \to CB(H)$ be the two measurable set valued mappings, $\epsilon \geq 0$ be a constant and $v : \Omega \to H$ be a measurable selection of P. Then there exists a measurable selection $w : \Omega \to H$ of T such that for all $t \in \Omega$,*

$$\|v(t) - w(t)\| \leq (1+\epsilon)\hat{\mathcal{H}}(P(t), T(t)).$$

Definition 3.3 *A random operator $A : \Omega \times H \to H$ is said to be*

(i) randomly monotone, if

$$\langle A_t(u_1(t)) - A_t(u_2(t)), u_1(t) - u_2(t)\rangle \geq 0, \quad \forall u_1(t), u_2(t) \in H, t \in \Omega,$$

(ii) *randomly r_t-strongly monotone, if there exists a measurable mapping $r:\Omega\to(0,\infty)$ such that*

$$\langle A_t(u_1(t))-A_t(u_2(t)),u_1(t)-u_2(t)\rangle\geq r_t\|u_1(t)-u_2(t)\|^2,\forall u_1(t),u_2(t)\in H,t\in\Omega,$$

(iii) *randomly r_t-relaxed monotone, if there exists a measurable mapping $r:\Omega\to(0,\infty)$ such that*

$$\langle A_t(u_1(t))-A_t(u_2(t)),u_1(t)-u_2(t)\rangle\geq -r_t\|u_1(t)-u_2(t)\|^2,$$

$$\forall u_1(t),u_2(t)\in H,t\in\Omega,$$

(iv) *randomly ξ_t-cocoercive if*

$$\langle A_t(u_1(t))-A_t(u_2(t)),u_1(t)-u_2(t)\rangle\geq \xi_t\|u_1(t)-u_2(t)\|^2,$$

$$\forall u_1(t),u_2(t)\in H,t\in\Omega,$$

(v) *randomly (α_t,ξ_t)-relaxed cocoercive, if there exists measurable mappings $\alpha,\xi:\Omega\to(0,\infty)$ such that*

$$\langle A_t(u_1(t))-A_t(u_2(t)),u_1(t)-u_2(t)\rangle\geq -\alpha_t\|A_t(u_1(t))-A_t(u_2(t))\|^2+\xi_t\|u_1(t)-u_2(t)\|^2,$$

$$\forall u_1(t),u_2(t)\in H,t\in\Omega.$$

Definition 3.4 *Let $N:\Omega\times H\times H\to H$ and $p:\Omega\times H\to H$ be the two single valued mappings, and $\tilde{Q}:\Omega\times H\to CB(H)$ the random mapping, then*

(i) *N_t is said to be randomly (α_t,ϵ_t)-p-relaxed cocoercive with respect to first variable of N_t if*

$$\langle N_t(p_t(u(t)),\cdot)-N_t(p_t(v(t)),\cdot),u(t)-v(t)\rangle\geq -\alpha_t\|N_t(p_t(u(t)),\cdot)-N_t(p_t(v(t)),\cdot)\|^2+\epsilon_t\|u(t)-v(t)\|^2$$

$$\forall u(t),v(t)\in H,t\in\Omega.$$

(ii) *N_t is said to be randomly (φ_t,ψ_t)-Q_t-relaxed cocoercive with respect to second variable of N_t if*

$$\langle N_t(\cdot,y_1(t))-N_t(\cdot,y_2(t)),u(t)-v(t)\rangle\geq -\varphi_t\|N_t(\cdot,y_1(t))-N_t(\cdot,y_2(t))\|^2+\psi_t\|u(t)-v(t)\|^2$$

$$\forall y_1(t)\in\tilde{Q}_t(u(t)),y_2(t)\in\tilde{Q}_t(v(t)),u(t),v(t)\in H,t\in\Omega.$$

Definition 3.5 *Let $\eta:\Omega\times H\times H\to H$ be a single valued mapping. The map η_t is called randomly τ_t-Lipschitz continuous if there is a measurable mapping $\tau:\Omega\to(0,\infty)$ such that*

$$\|\eta_t(u(t),v(t))\|\leq\tau_t\|u(t)-v(t)\|,\ \forall u(t),v(t)\in H,t\in\Omega.$$

Definition 3.6 *Let $\eta:\Omega\times H\times H\to H$ be a single valued mapping and let $M:\Omega\times H\to 2^H$ be a random set valued mapping. The random map M_t is said to be*

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(i) *randomly* (r_t, η_t)*-strongly monotone if*

$$\langle u^*(t) - v^*(t), \eta_t(u(t), v(t))\rangle \geq r_t \|u(t) - v(t)\|^2,$$

$$\forall (u(t), u^*(t)), (v(t), v^*(t)) \in \mathit{Graph}(M);$$

(ii) *randomly* η_t*-pseudomonotone if*

$$\langle v^*(t), \eta_t(u(t), v(t))\rangle \geq 0 \Longrightarrow \langle u^*(t), \eta_t(u(t), v(t))\rangle \geq 0$$

$$\forall (u(t), u^*(t)), (v(t), v^*(t)) \in \mathit{Graph}(M);$$

(iii) *randomly* (r_t, η_t)*-relaxed monotone if there exists a measurable mapping* $r : \Omega \to (0, \infty)$ *such that*

$$\langle u^*(t) - v^*(t), \eta_t(u(t), v(t))\rangle \geq -r_t \|u(t) - v(t)\|^2,$$

$$\forall (u(t), u^*(t)), (v(t), v^*(t)) \in \mathit{Graph}(M).$$

Definition 3.7 *A random mapping* $M : \Omega \times H \to 2^H$ *is said to be random maximal* (m_t, η_t)*-relaxed monotone if*

(i) M_t *is random* (M_t, η_t)*-monotone*

(ii) *for* $(u(t), u^*(t)) \in H \times H$ *and*

$$\langle u^*(t) - v^*(t), \eta_t(u(t), v(t))\rangle \geq -m_t \|u(t) - v(t)\|^2, \ \forall (v(t), v^*(t)) \in \mathit{Graph}(M)$$

we have $u^*(t) \in M_t(u(t))$.

Definition 3.8 *Let* $A : \Omega \times H \to H$ *and* $\eta : \Omega \times H \times H \to H$ *be two random single valued mappings, the random mapping* $M : \Omega \times H \to 2^H$ *is said to be randomly* (A_t, η_t)*-monotone if*

(i) M_t *is randomly* (M_t, η_t)*-relaxed monotone,*

(ii) $R(A_t + \rho_t M_t) = H$ *for a measurable mapping* $\rho : \Omega \to (0, 1)$.

Note that alternatively, the random mapping $M : \Omega \times H \to 2^H$ *is said to randomly* (A_t, η_t)*-monotone if*

(i) M_t *is randomly* (M_t, η_t)*-relaxed monotone,*

(ii) $A_t + \rho_t M_t$ *is randomly* η_t*-pseudomonotone for a measurable mapping* $\rho : \Omega \to (0, 1)$.

Proposition 3.9 *Let a random mapping* $A : \Omega \times H \to H$ *be randomly* (r_t, η_t)*-strongly monotone,* $M : \Omega \times H \to 2^H$ *be a randomly* (A_t, η_t)*-monotone mapping, and* $\eta : \Omega \times H \times H \to H$ *be the randomly* τ_t*-Lipschitz continuous, then* M_t *is randomly* (m_t, η_t)*-relaxed monotone and* $(A_t + \rho_t M_t)H = H$ *for* $0 < \rho_t < \frac{r_t}{m_t}$.

Proposition 3.10 *Let a map $A : \Omega \times H \to H$ be the randomly (r_t, η_t)-strongly monotone and $M : \Omega \times H \to 2^H$ be a randomly (A_t, η_t)-monotone mapping. Let $\eta : \Omega \times H \times H \to H$ be the randomly τ_t-Lipschitz continuous. Then $(A_t + \rho_t M_t)$ is randomly maximal η_t-monotone for $0 < \rho_t < \frac{r_t}{m_t}$.*

Proof. Given that A_t is randomly (r_t, η_t)-strongly monotone and M_t is randomly (A_t, η_t)-maximal monotone, then $(A_t + \rho_t M_t)$ is randomly $(r_t - m_t\rho_t, \eta_t)$-strongly monotone. This in turn implies that $(A_t + \rho_t M_t)$ is randomly η_t-pseudomonotone and hence $(A_t + \rho_t M_t)$ is randomly η_t-monotone under given conditions. ■

Proposition 3.11 *Let $A : \Omega \times H \to H$ be a randomly (r_t, η_t)-strongly monotone mapping and $M : \Omega \times H \to 2^H$ be the randomly (A_t, η_t)-monotone mapping. If in addition, $\eta : \Omega \times H \times H \to H$ is randomly τ_t-Lipschitz continuous, then the operator $(A_t + \rho_t M_t)^{-1}$ is randomly single valued for $0 < \rho_t < \frac{r_t}{m_t}$.*

Lemma 3.12 *Let H be a real Hilbert space and $\eta : \Omega \times H \times H \to H$ be a randomly τ_t-Lipschitz continuous nonlinear mapping. Let $A : \Omega \times H \to H$ be a randomly (r_t, ρ_t)-strongly monotone and $M : \Omega \times H \times H \to 2^H$ be randomly (A_t, η_t)-monotone in first argument in M_t. Then the generalized resolvent operator associated with $M_t(\cdot, v(t))$ for a fixed $v(t) \in H$ and defined by*

$$J^{\eta_t, M_t(\cdot, v(t))}_{\rho_t, A_t}(u(t)) = (A_t + \rho_t M_t(\cdot, v(t)))^{-1}(u(t)), \forall u(t) \in H$$

is randomly $(\frac{\tau_t}{r_t - \rho_t m_t})$-Lipschitz continuous.

Definition 3.13 *A random set valued mapping $T : \Omega \times H \to CB(H)$ is said to be random $\hat{\mathcal{H}}$-Lipschitz continuous if there exists a measurable mapping $\lambda_{\hat{\mathcal{H}}_{T_t}} : \Omega \to (0, \infty)$ such that*

$$\hat{\mathcal{H}}(T_t(u_1(t)), T_t(u_2(t))) \leq \lambda_{t, \hat{\mathcal{H}}_t} \|u_1(t) - u_2(t)\|, \ \forall u_1(t), u_2(t) \in H.$$

Lemma 3.14 *The set of measurable mappings $u, x, y, z : \Omega \to H$ is a random solution of problem (2.1) if and only if for all $t \in \Omega, u(t) \in H, x(t) \in \tilde{T}_t(u(t)), y(t) \in \tilde{P}_t(u(t)), z(t) \in \tilde{Q}_t(u(t))$ and*

$$g_t(u(t)) = J^{\eta_t, M_t(\cdot, z(t))}_{\rho_t, A_t}[A_t(g_t(u(t))) - \rho_t(f_t(x(t)) + N_t(p_t(u(t)), y(t)))] \tag{3.1}$$

where $\rho : \Omega \to (0, \infty)$ is a measurable mapping.

Proof. The proof directly follows from the definition of $J^{\eta_t, M_t(\cdot, z(t))}_{\rho_t, A_t}$. ■
Based on Lemma 3.14 and Nadler [26], developed a fuzzy random iterative algorithm for solving the problem (3.1) as follows

Algorithm 3.15 *Suppose that $T, P, Q : \Omega \times H \to \mathcal{F}(H)$ be three fuzzy random mappings satisfying the condition (C). Let $\tilde{T}, \tilde{P}, \tilde{Q} : \Omega \times H \to CB(H)$ be the $\hat{\mathcal{H}}$-continuous random set valued mappings induced by T, P, Q, respectively. Let $A, f, g, p : \Omega \times H \to H$ be the single valued random mappings and $\eta, N : \Omega \times H \times H \to H$ be the two random* *bifunctions. Let* $M : \Omega \times H \times H \to 2^H$ *be a set valued random mapping such that for each fixed* $t \in \Omega, M(t,\cdot,\cdot) : H \times H \to 2^H$ *is randomly* A_t*-monotone mapping with* $Im(g_t) \cap domM_t(\cdot,\cdot) \neq \emptyset$. *For any given measurable mapping* $u_0 : \Omega \to H$, *the set valued random mappings* $\tilde{T}_t(u_0(t)), \tilde{P}_t(u_0(t)), \tilde{Q}_t(u_0(t)) : \Omega \to CB(H)$ *are measurable by Lemma 3.1. Hence there exists measurable selections* $x_0 : \Omega \to H$ *of* $\tilde{T}_t(u_0(t)), y_0 : \Omega \to H$ *of* $\tilde{P}_t(u_0(t))$, *and* $z_0 : \Omega \to H$ *of* $\tilde{Q}_t(u_0(t))$. *By Himmelberg [14], let*

$$u_1(t) = u_0(t) - g_t(u_0(t)) + J^{\eta_t, M_t(\cdot, z_0(t))}_{\rho_t, A_t}[A_t(g_t(u_0(t))) - \rho_t\{f_t(x_0(t)) + N_t(p_t(u_0(t)), y_0(t))\}] + e_0(t).$$

where ρ_t *is same as in Lemma 3.14,* $1 > t > 0$ *is a constant, and* $e_0(t) : \Omega \to H$ *is a measurable function which is a random error to take into account a possible inexact computation of random hybrid proximal point. Then, it is easy to know that* $u_1 : \Omega \to H$ *is a measurable. By Lemma 3.14, there exists a measurable selections* $x_1 : \Omega \to H$ *of* $\tilde{T}_t(u_1(\cdot)), y_1 : \Omega \to H$ *of* $\tilde{P}_t(u_1(\cdot))$ *and* $z_1 : \Omega \to H$ *of* $\tilde{Q}_t(u_1(\cdot))$ *such that for all* $t \in \Omega$,

$$\|x_0(t) - x_1(t)\| \leq (1+1)\hat{\mathcal{H}}(\tilde{T}_t(u_0(t)), \tilde{T}_t(u_1(t))),$$

$$\|y_0(t) - y_1(t)\| \leq (1+1)\hat{\mathcal{H}}(\tilde{P}_t(u_0(t)), \tilde{P}_t(u_1(t))),$$

$$\|z_0(t) - z_1(t)\| \leq (1+1)\hat{\mathcal{H}}(\tilde{Q}_t(u_0(t)), \tilde{Q}_t(u_1(t))).$$

Let

$$u_2(t) = u_1(t) - g_t(u_1(t)) + J^{\eta_t, M_t(\cdot, z_1(t))}_{\rho_t, A_t}[A_t(g_t(u_1(t))) - \rho_t\{f_t(x_1(t)) + N_t(p_t(u_1(t)), y_1(t))\}] + e_1(t).$$

The $u_2(t)$ *is a measurable. Continuing the above process inductively, we can define the following random iterative sequences for fuzzy mappings* $\{u_n(t)\}$, $\{x_n(t)\}$, $\{y_n(t)\}$ *and* $\{z_n(t)\}$ *for solving (2.1) as follows*

$$u_{n+1}(t) = u_n(t) - g_t(u_n(t)) + J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[A_t(g_t(u_n(t))) - \rho_t\{f_t(x_n(t)) + N_t(p_t(u_n(t)), y_n(t))\}] + e_n(t)$$

$$x_n(t) \in \tilde{T}_t(u_n(t)), y_n(t) \in \tilde{P}_t(u_n(t)), z_n(t) \in \tilde{Q}_t(u_n(t)),$$

$$\|x_n(t) - x_{n+1}(t)\| \leq (1 + (1+n)^{-1})\hat{\mathcal{H}}(\tilde{T}_t(u_n(t)), \tilde{T}_t(u_{n+1}(t))),$$

$$\|y_n(t) - y_{n+1}(t)\| \leq (1 + (1+n)^{-1})\hat{\mathcal{H}}(\tilde{P}_t(u_n(t)), \tilde{P}_t(u_{n+1}(t))),$$

$$\|z_n(t) - z_{n+1}(t)\| \leq (1 + (1+n)^{-1})\hat{\mathcal{H}}(\tilde{Q}_t(u_n(t)), \tilde{Q}_t(u_{n+1}(t))),$$

for any $0 < t < 1$ *and* $n = 0, 1, 2, \cdots; e_n(t) : \Omega \to H (n \geq 0)$ *is a random error to take into account a possible inexact computation of the proximal point.*

4 Convergence Results

In this section, we shall give some existence and convergence theorem for fuzzy nonlinear set valued inclusions.

Theorem 4.1 *Let a random mapping $\eta : \Omega \times H \times H \to H$ be randomly (m_t, η_t)-relaxed monotone and Lipschitz continuous with constant τ_t. Let $M : \Omega \times H \times H \to H$ be a random set valued mapping such that for each fixed $t \in \Omega, M(t, \cdot, \cdot) : \Omega \times H \times H \to 2^H$ be the randomly (A_t, η_t)-monotone mapping in the first argument in M_t and $A : \Omega \times H \to H$ be the randomly (r_t, η_t)-strongly monotone and χ_t-Lipschitz continuous with constant χ_t. Let $T, P, Q : \Omega \times H \to \mathcal{F}(H)$ be the fuzzy random mappings satisfies the condition (C) and $\tilde{T}, \tilde{P}, \tilde{Q} : \Omega \times H \to CB(H)$ be the $\hat{\mathcal{H}}$-continuous random set valued mappings induced by T, P, Q, respectively. Suppose that T, P, Q are randomly $\hat{\mathcal{H}}$-Lipschitz continuous with random variables ι_t, υ_t, d_t, respectively. Let $p_t, f_t : \Omega \times H \to H$ be the Lipschitz continuous random mappings with constants s_t, ω_t, respectively. Let $N : \Omega \times H \times H \to H$ be the bilinear random mapping which is Lipschitz continuous with first variable with constant β_t and second variable with γ_t. Assume that $N_t(\cdot, \cdot)$ is randomly (α_t, ϵ_t)-p-relaxed cocoercive with respect to first argument. A random mapping $g : \Omega \times H \to H$ is random strongly monotone with constant ν_t and random Lipschitz continuous with constant ξ_t and Aog is randomly (ς_t, κ_t)-relaxed cocoercive. Let $N_t(\cdot, \cdot)$ be the randomly (φ_t, ψ_t)-Q_t-relaxed cocoercive with respect to the second argument. Let $M : \Omega \times H \times H \to 2^H$ be a set valued mapping such that for each fixed $t \in \Omega, v(t) \in H, M_t(\cdot, v(t)) : H \to 2^H$ be the randomly (A_t, η_t)-monotone random mapping and range $(g_t) \cap domM_t(\cdot, v(t)) \neq \emptyset$. For any $t \in \Omega, u(t), v(t), w(t) \in H$ there exists a random real valued variable $\delta_t > 0$ such that*

$$\|J_{\rho_t, A_t}^{\eta_t, M_t(\cdot, z_n(t))} w(t) - J_{\rho_t, A_t}^{\eta_t, M_t(\cdot, z_{n-1}(t))} w(t)\| \leq \delta_t \|z_n(t) - z_{n-1}(t)\| \tag{4.1}$$

and

$$| \rho - \frac{\triangle}{\Box} | < \frac{\sqrt{\triangle^2 - \Box \ell}}{\Box}$$

$$\triangle > \sqrt{\Box \ell}$$

$$D(t)\tau_t^2 > \tau_t G(t) + m_t(1 - B(t))$$

$$\tau_t > r_t(1 - B(t)) - \tau_t C(t)$$

$$E(t)\tau_t > \sqrt{\tau_t G(t) + m_t(1 - B(t))}$$

where

$$\Box = E^2(t)\tau_t^2 - (\tau_t G(t) + m_t(1 - B(t)))^2$$

$$\triangle = D(t)\tau_t^2 - (\tau_t G(t) + m_t(1 - B(t)))$$

$$\ell = \tau_t^2 - (r_t(1 - B(t)) - \tau_t C(t))^2$$

and

$$\lim_{n\to\infty} \|e_n(t)\| = 0, \sum_{n=1}^{\infty} \|e_n(t) - e_{n-1}(t)\| < \infty, \forall t \in \Omega. \tag{4.2}$$

The random variable iterative sequences $\{u_n(t)\}, \{x_n(t)\}, \{y_n(t)\}$ and $\{z_n(t)\} : \Omega \to H$ generated by Algorithm 3.15, converge strongly to random variables $u^*(t), x^*(t), y^*(t)$ and $z^*(t) : \Omega \to H$ respectively and $(u^*(t), x^*(t), y^*(t), z^*(t))$ is a solution set of problem (2.1).

Proof. From Algorithm 3.15, for any $t \in \Omega$, we have

$$\|u_{n+1}(t) - u_n(t)\| = \|u_n(t) - g_t(u_n(t)) + J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[A_t(g_t(u_n(t))) - \rho_t\{f_t(x_n(t))$$

$$+N_t(p_t(u_n(t)), y_n(t))\}] + e_n(t) - u_{n-1}(t) + g_t(u_{n-1}(t)) - J^{\eta_t, M_t(\cdot, z_{n-1}(t))}_{\rho_t, A_t}[A_t(g_t(u_{n-1}(t)))$$

$$- \rho_t\{f_t(x_{n-1}(t)) + N_t(p_t(u_{n-1}(t)), y_{n-1}(t))\}] - e_{n-1}(t)\|$$

$$\leq \|u_n(t) - u_{n-1}(t) - (g_t(u_n(t)) - g_t(u_{n-1}(t)))\|$$

$$+\|J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[w_n(t)] - J^{\eta_t, M_t(\cdot, z_{n-1}(t))}_{\rho_t, A_t}[w_{n-1}(t)]\| + \|e_n(t) - e_{n-1}(t)\|$$

$$\leq \|u_n(t) - u_{n-1}(t) - (g_t(u_n(t)) - g_t(u_{n-1}(t)))\|$$

$$+\|J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[w_n(t)] - J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[w_{n-1}(t)]\|$$

$$+\|J^{\eta_t, M_t(\cdot, z_n(t))}_{\rho_t, A_t}[w_{n-1}(t)] - J^{\eta_t, M_t(\cdot, z_{n-1}(t))}_{\rho_t, A_t}[w_{n-1}(t)]\| + \|e_n(t) - e_{n-1}(t)\|$$

$$\leq \|u_n(t) - u_{n-1}(t) - (g_t(u_n(t)) - g_t(u_{n-1}(t)))\| + \frac{\tau_t}{r_t - \rho_t m_t}\|w_n(t) - w_{n-1}(t)\|$$

$$+\delta_t\|z_n(t) - z_{n-1}(t)\| + \|e_n(t) - e_{n-1}(t)\| \tag{4.3}$$

where

$$w_n(t) = A_t(g_t(u_n(t))) - \rho_t(f_t(x_n(t)) + N_t(p_t(u_n(t)), y_n(t))).$$

Now

$$\|w_n(t) - w_{n-1}(t)\| = \|A_t(g_t(u_n(t))) - \rho_t(f_t(x_n(t)) + N_t(p_t(u_n(t)), y_n(t)))$$

$$-A_t(g_t(u_{n-1}(t))) + \rho_t(f_t(x_{n-1}(t)) + N_t(p_t(u_{n-1}(t)), y_{n-1}(t)))\|$$

$$= \|A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t)))$$

$$-\rho_t(f_t(x_n(t)) - f_t(x_{n-1}(t)) + N_t(p_t(u_n(t)), y_n(t)) - N_t(p_t(u_{n-1}(t)), y_{n-1}(t)))\|$$

$$= \|u_n(t) - u_{n-1}(t) - (A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t))))\|$$

$$+\|u_n(t) - u_{n-1}(t) - \rho_t(N_t(p_t(u_n(t)), y_n(t)) - N_t(p_t(u_{n-1}(t)), y_{n-1}(t)))\|$$

$$+\rho_t\|f_t(x_n(t)) - f_t(x_{n-1}(t))\|. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$\|u_{n+1}(t) - u_n(t)\| \leq \|u_n(t) - u_{n-1}(t) - (g_t(u_n(t)) - g_t(u_{n-1}(t)))\|$$

$$+\frac{\tau_t}{r_t - \rho_t m_t}[\|u_n(t) - u_{n-1}(t) - (A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t))))\|$$

$$+\|u_n(t) - u_{n-1}(t) - \rho_t(N_t(p_t(u_n(t)), y_n(t)) - N_t(p_t(u_{n-1}(t)), y_{n-1}(t)))\|$$

$$+\rho_t\|f_t(x_n(t)) - f_t(x_{n-1}(t))\|] + \delta_t\|z_n(t) - z_{n-1}(t)\| + \|e_n(t) - e_{n-1}(t)\|. \tag{4.5}$$

Since N_t, g_t, p_t, f_t are random Lipschitz continuous and $\tilde{T}_t, \tilde{P}_t, \tilde{Q}_t$ are randomly $\hat{\mathcal{H}}$-Lipschitz continuous, we have

$$\|g_t(u_n(t)) - g_t(u_{n-1}(t))\| \leq \xi_t \|u_n(t) - u_{n-1}(t)\|, \tag{4.6}$$

$$\|p_t(u_n(t)) - p_t(u_{n-1}(t))\| \leq s_t \|u_n(t) - u_{n-1}(t)\|, \tag{4.7}$$

$$\begin{aligned} \|f_t(x_n(t)) - f_t(x_{n-1}(t))\| &\leq \omega_t \|x_n(t) - x_{n-1}(t)\| \\ &\leq \omega_t \hat{\mathcal{H}}(\tilde{T}_t(u_n(t)), \tilde{T}_t(u_{n-1}(t))) \\ &\leq \omega_t (1 + \frac{1}{n+1}) \iota_t \|u_n(t) - u_{n-1}(t)\|, \end{aligned} \tag{4.8}$$

$$\|z_n(t) - z_{n-1}(t)\| \leq (1 + \frac{1}{n+1}) \hat{\mathcal{H}}(\tilde{P}_t(u_n(t)), \tilde{P}_t(u_{n-1}(t))) \leq (1 + \frac{1}{n+1}) \upsilon_t \|u_n(t) - u_{n-1}(t)\|,$$

$$\|y_n(t) - y_{n-1}(t)\| \leq (1 + \frac{1}{1+n}) \hat{\mathcal{H}}(\tilde{Q}_t(u_n(t)), \tilde{Q}_t(u_{n-1}(t))) \leq (1 + \frac{1}{n+1}) d_t \|u_n(t) - u_{n-1}(t)\|$$

and

$$\begin{aligned} \|N_t(p_t(u_n(t)), y_n(t)) - N_t(p_t(u_{n-1}(t)), y_{n-1}(t))\| &\leq \beta_t \|p_t(u_n(t)) - p_t(u_{n-1}(t))\| \\ &\quad + \gamma_t \|y_n(t) - y_{n-1}(t)\| \\ &\leq \beta_t s_t \|u_n(t) - u_{n-1}(t)\| + \gamma_t (1 + \frac{1}{n+1}) d_t \|u_n(t) - u_{n-1}(t)\| \\ &\leq (\beta_t s_t + \gamma_t (1 + \frac{1}{n+1}) d_t) \|u_n(t) - u_{n-1}(t)\|. \end{aligned} \tag{4.9}$$

Since g_t is random strongly monotone and random Lipschitz continuous, we have

$$\begin{aligned} &\|u_n(t) - u_{n-1}(t) - (g_t(u_n(t)) - g_t(u_{n-1}(t)))\|^2 \leq \|u_n(t) - u_{n-1}(t)\|^2 \\ &-2\langle g_t(u_n(t)) - g_t(u_{n-1}(t)), u_n(t) - u_{n-1}(t)\rangle + \|g_t(u_n(t)) - g_t(u_{n-1}(t))\|^2 \\ &\leq \|u_n(t) - u_{n-1}(t)\|^2 - 2\nu_t \|u_n(t) - u_{n-1}(t)\|^2 + \xi_t^2 \|u_n(t) - u_{n-1}(t)\|^2 \\ &\leq (1 - 2\nu_t + \xi_t^2) \|u_n(t) - u_{n-1}(t)\|^2. \end{aligned} \tag{4.10}$$

Since A_t and g_t are randomly Lipschitz continuous with χ_t and ξ_t respectively, and randomly (ς_t, κ_t)- relaxed cocoercive and from Algorithm 3.15, we obtain

$$\begin{aligned} &\|u_n(t) - u_{n-1}(t) - (A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t))))\|^2 \leq \|u_n(t) - u_{n-1}(t)\|^2 \\ &-2\langle A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t))), u_n(t) - u_{n-1}(t)\rangle + \|A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t)))\|^2 \\ &\leq \|u_n(t) - u_{n-1}(t)\|^2 + \chi_t^2 \xi_t^2 \|u_n(t) - u_{n-1}(t)\|^2 \\ &+2\varsigma_t \|A_t(g_t(u_n(t))) - A_t(g_t(u_{n-1}(t)))\|^2 - 2\kappa_t \|u_n(t) - u_{n-1}(t)\|^2 \\ &\leq \|u_n(t) - u_{n-1}(t)\|^2 + \chi_t^2 \xi_t^2 \|u_n(t) - u_{n-1}(t)\|^2 + 2\varsigma_t \chi_t^2 \xi_t^2 \|u_n(t) - u_{n-1}(t)\|^2 \end{aligned}$$

$$-2\kappa_t\|u_n(t)-u_{n-1}(t)\|^2$$
$$\leq ((1-2\kappa_t)+(2\varsigma_t+1)\chi_t^2\xi_t^2)\|u_n(t)-u_{n-1}(t)\|^2. \tag{4.11}$$

Since $N_t(\cdot,\cdot)$ is randomly (α_t,ϵ_t)-p-relaxed cocoercive with respect to the first argument of N_t. Again $N_t(\cdot,\cdot)$ is randomly (φ_t,ψ_t)-Q_t-relaxed cocoercive with respect to the second argument of N_t; N_t and p_t are randomly Lipschitz continuous, we have

$$\|u_n(t)-u_{n-1}(t)-\rho_t(N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t)))\|^2=\|u_n(t)-u_{n-1}(t)\|^2$$
$$-2\rho_t\langle N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t)),u_n(t)-u_{n-1}(t)\rangle$$
$$+\rho_t^2\|N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t))\|^2$$
$$\leq \|u_n(t)-u_{n-1}(t)\|^2-2\rho_t\langle N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_n(t)),u_n(t)-u_{n-1}(t)\rangle$$
$$-2\rho_t\langle N_t(p_t(u_{n-1}(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t)),u_n(t)-u_{n-1}(t)\rangle$$
$$+\rho_t^2\|N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t))\|^2$$
$$\leq \|u_n(t)-u_{n-1}(t)\|^2-2\rho_t(-\alpha_t\|N_t(p_t(u_n(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_n(t))\|^2$$
$$+\epsilon_t\|u_n(t)-u_{n-1}(t)\|^2)-2\rho_t(-\varphi_t\|N_t(p_t(u_{n-1}(t)),y_n(t))-N_t(p_t(u_{n-1}(t)),y_{n-1}(t))\|^2$$
$$+\psi_t\|u_n(t)-u_{n-1}(t)\|^2)+\rho_t^2(\beta_t\|p_t(u_n(t))-p_t(u_{n-1}(t))\|+\gamma_t\|y_n(t)-y_{n-1}(t)\|)^2$$
$$\leq \|u_n(t)-u_{n-1}(t)\|^2+2\rho_t\alpha_t\beta_t^2s_t^2\|u_n(t)-u_{n-1}(t)\|^2-2\rho_t\epsilon_t\|u_n(t)-u_{n-1}(t)\|^2$$
$$+2\rho_t\varphi_t\gamma_t^2\|y_n(t)-y_{n-1}(t)\|^2-2\rho_t\psi_t\|u_n(t)-u_{n-1}(t)\|^2+\rho_t^2(\beta_ts_t\|u_n(t)-u_{n-1}(t)\|$$
$$+\gamma_t\|y_n(t)-y_{n-1}(t)\|)^2$$
$$\leq \|u_n(t)-u_{n-1}(t)\|^2+2\rho_t\alpha_t\beta_t^2s_t^2\|u_n(t)-u_{n-1}(t)\|^2-2\rho_t\epsilon_t\|u_n(t)-u_{n-1}(t)\|^2$$
$$+2\rho_t\varphi_t\gamma_t^2(1+\frac{1}{n+1})^2d_t^2\|u_n(t)-u_{n-1}(t)\|^2-2\rho_t\psi_t\|u_n(t)-u_{n-1}(t)\|^2$$
$$+\rho_t^2(\beta_ts_t+\gamma_t(1+\frac{1}{n+1})d_t)^2\|u_n(t)-u_{n-1}(t)\|^2$$
$$\leq [1+2\rho_t(\alpha_t\beta_t^2s_t^2-\epsilon_t+\varphi_t\gamma_t^2(1+\frac{1}{n+1})^2d_t^2-\psi_t)+\rho_t^2(\beta_ts_t+\gamma_t(1+\frac{1}{n+1})d_t)^2]$$
$$\|u_n(t)-u_{n-1}(t)\|^2. \tag{4.12}$$

From (4.5),(4.8), (4.9), (4.10), (4.11) and (4.12), we have

$$\|u_{n+1}(t)-u_n(t)\|\leq\sqrt{1-2\nu_t+\xi_t^2}\|u_n(t)-u_{n-1}(t)\|$$
$$+\frac{\tau_t}{r_t-\rho_tm_t}[\sqrt{(1-2\kappa_t)+(2\varsigma_t+1)\chi_t^2\xi_t^2}\|u_n(t)-u_{n-1}(t)\|$$
$$+\sqrt{1+2\rho_t(\alpha_t\beta_t^2s_t^2-\epsilon_t+\varphi_t\gamma_t^2d_t^2(1+\frac{1}{1+n})^2-\psi_t)+\rho_t^2(\beta_ts_t+\gamma_td_t(1+\frac{1}{1+n}))^2}$$
$$\|u_n(t)-u_{n-1}(t)\|+\rho_t\omega_t(1+\frac{1}{1+n})\iota_t\|u_n(t)-u_{n-1}(t)\|]$$

$$+\delta_t(1+\frac{1}{1+n})\upsilon_t\|u_n(t)-u_{n-1}(t)\|+\|e_n(t)-e_{n-1}(t)\|$$

$$\leq [\sqrt{1-2\nu_t+\xi_t^2}+\delta_t(1+\frac{1}{n+1})\upsilon_t+\frac{\tau_t}{r_t-\rho_t m_t}[\sqrt{(1-2\kappa_t)+(2\varsigma_t+1)\chi_t^2\xi_t^2}$$

$$+\sqrt{(1-2\rho_t(-\alpha_t\beta_t^2 s_t^2+\epsilon_t-\varphi_t\gamma_t^2 d_t^2(1+\frac{1}{n+1})^2+\psi_t)+\rho_t^2(\beta_t s_t+\gamma_t d_t(1+\frac{1}{1+n}))^2}$$

$$+\rho_t\omega_t(1+\frac{1}{n+1})\iota_t]\|u_n(t)-u_{n-1}(t)\|+\|e_n(t)-e_{n-1}(t)\|$$

$$\leq [B_n(t)+\frac{\tau_t}{r_t-\rho_t m_t}\{C(t)+\sqrt{1-2\rho_t D_n(t)+\rho_t^2 E_n^2(t)}+G_n(t)\rho_t\}]\|u_n(t)-u_{n-1}(t)\|$$

$$+\|e_n(t)-e_{n-1}(t)\|$$

$$\leq \theta_n(t)\|u_n(t)-u_{n-1}(t)\|+\|e_n(t)-e_{n-1}(t)\| \tag{4.13}$$

where

$$\theta_n(t)=B_n(t)+\frac{\tau_t}{r_t-\rho_t m_t}[C(t)+\sqrt{1-2\rho_t D_n(t)+\rho_t^2 E_n^2(t)}+G_n(t)\rho_t]$$

$$B_n(t)=\sqrt{1-2\nu_t+\xi_t^2}+\delta_t(1+\frac{1}{1+n})\upsilon_t,$$

$$C(t)=\sqrt{(1-2\kappa_t)+(2\varsigma_t+1)\chi_t^2\xi_t^2}$$

$$D_n(t)=-\alpha_t\beta_t^2 s_t^2+\epsilon_t-\varphi_t\gamma_t^2 d_t^2(1+\frac{1}{1+n})^2+\psi_t$$

$$E_n(t)=\beta_t s_t+\gamma_t d_t(1+\frac{1}{1+n})$$

$$G_n(t)=\omega_t\iota_t(1+\frac{1}{1+n}).$$

Letting

$$\theta(t)=B(t)+\frac{\tau_t}{r_t-\rho_t m_t}[C(t)+\sqrt{1-2\rho_t D(t)+\rho_t^2 E^2(t)}+G(t)\rho_t]$$

and

$$B(t)=\sqrt{1-2\nu_t+\xi_t^2}+\delta_t\upsilon_t,$$

$$C(t)=\sqrt{(1-2\kappa_t)+(2\varsigma_t+1)\chi_t^2\xi_t^2},$$

$$D(t)=-\alpha_t\beta_t^2 s_t^2+\epsilon_t-\varphi_t\gamma_t^2 d_t^2+\psi_t,$$

$$E(t)=\beta_t s_t+\gamma_t d_t,$$

$$G_n(t)=\omega_t\iota_t.$$

We have that $\theta_n(t) \to \theta(t)$ as $n \to \infty$. It follows from condition (4.2) and $0 < \theta(t) < 1$, hence there exists $N_0 > 0$ and $\theta^*(t) \in (\theta(t), 1)$ such that $\theta_n(t) < \theta^*(t)$ for all $n \geq N_0$. Therefore, from (4.13), we have

$$\|u_{n+1}(t) - u_n(t)\| \leq \theta^*(t)\|u_n(t) - u_{n-1}(t)\| + \|e_n(t) - e_{n-1}(t)\|, \ \forall n \leq N_0.$$

Without loss of generality, we may assume

$$\|u_{n+1}(t) - u_n(t)\| \leq \theta^*(t)\|u_n(t) - u_{n-1}(t)\| + \|e_n(t) - e_{n-1}(t)\|, \ \forall n \leq 1.$$

Hence, for any $m > n > 0$, we have

$$\|u_m(t) - u_n(t)\| \leq \sum_{i=n}^{m-1} \|u_{i+1}(t) - u_i(t)\|$$

$$\leq \sum_{i=1}^{m-1} \theta_i^*(t)\|u_1(t) - u_0(t)\| + \sum_{i=1}^{m-1}\sum_{j=1}^{i} \theta_{i-j}^*(t)\|e_j(t) - e_{j-1}(t)\|.$$

It follows from condition (4.3) that

$$\|u_m(t) - u_n(t)\| \to 0 \ \text{ as } \ n \to \infty$$

and so $\{u_n(t)\}$ is a Cauchy sequence in H. Let $u_n(t) \to u(t)$ as $n \to \infty$. By the random Lipschitz continuity of $\tilde{T}_t(\cdot), \tilde{P}_t(\cdot)$ and $\tilde{Q}_t(\cdot)$, we obtain

$$\|x_{n+1}(t) - x_n(t)\| \leq (1 + \frac{1}{1+n})\hat{\mathcal{H}}(\tilde{T}_t(u_{n+1}(t)), \tilde{T}_t(u_n(t)))$$

$$\leq \iota_t(1 + \frac{1}{n+1})\|u_{n+1}(t) - u_n(t)\|,$$

$$\|y_{n+1}(t) - y_n(t)\| \leq (1 + \frac{1}{1+n})\hat{\mathcal{H}}(\tilde{Q}_t(u_{n+1}(t)), \tilde{Q}_t(u_n(t)))$$

$$\leq d_t(1 + \frac{1}{n+1})\|u_{n+1}(t) - u_n(t)\|,$$

$$\|z_{n+1}(t) - z_n(t)\| \leq (1 + \frac{1}{1+n})\hat{\mathcal{H}}(\tilde{P}_t(u_{n+1}(t)), \tilde{P}_t(u_n(t)))$$

$$\leq \upsilon_t(1 + \frac{1}{n+1})\|u_{n+1}(t) - u_n(t)\|.$$

It follows that $\{u_n(t)\}, \{x_n(t)\}, \{y_n(t)\}$ and $\{z_n(t)\}$ are also Cauchy sequences in H. We can assume that $u_n(t) \to u^*(t), x_n(t) \to x^*(t), y_n(t) \to y^*(t)$ and $z_n(t) \to z^*(t)$ respectively. Note that $x_n(t) \in \tilde{T}_t(u_n(t))$, we have

$$\begin{aligned} d(x^*(t), \tilde{T}_t(u^*(t))) &\leq \|x^*(t) - x_n(t)\| + d(x_n(t), \tilde{T}_t(u^*(t))) \\ &\leq \|x^*(t) - x_n(t)\| + \hat{\mathcal{H}}(\tilde{T}_t(u_n(t)), \tilde{T}_t(u^*(t))) \\ &\leq \|x^*(t) - x_n(t)\| + \iota_t\|u_n(t) - u^*(t)\| \to 0 \ \text{ as } \ n \to \infty. \qquad (4.14) \end{aligned}$$

Hence $d(x^*(t), \tilde{T}_t(u^*(t))) = 0$ and therefore $x^*(t) \in \tilde{T}_t(u^*(t))$. Similarly we can prove that $y^*(t) \in \tilde{Q}_t(u^*(t))$, and $z^*(t) \in \tilde{P}_t(u^*(t))$. By the random Lipschitz continuity of $\tilde{T}_t(\cdot), \tilde{Q}_t(\cdot)$ and $\tilde{P}_t(\cdot)$ and Lemma 3.14, condition (4.2) and $\lim_{n\to\infty} \|e_n(t)\| = 0$, we have

$$u^*(t) = u^*(t) - g_t(u^*(t)) + J_{\rho_t, A_t}^{\eta_t, M_t(\cdot, z^*(t))}[A_t(g_t(u^*(t))) - \rho_t\{f_t(x^*(t)) + N_t(p_t(u^*(t)), y^*(t))\}].$$

By Lemma 3.14 we know that $(u^*(t), x^*(t), y^*(t), z^*(t))$ is a solution of problem (2.1). This completes the proof. ■

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Hyperbolic expressions of polynomial sequences and parametric number sequences defined by linear recurrence relations of order 2

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Abstract

A sequence of polynomial $\{a_n(x)\}$ is called a function sequence of order 2 if it satisfies the linear recurrence relation of order 2: $a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x)$ with initial conditions $a_0(x)$ and $a_1(x)$. In this paper we derive a parametric form of $a_n(x)$ in terms of e^θ with $q(x) = B$ constant, inspired by Askey's and Ismail's works shown in [2] [6], and [18], respectively. With this method, we give the hyperbolic expressions of Chebyshev polynomials and Gegenbauer-Humbert Polynomials. The applications of the method to construct corresponding hyperbolic form of several well-known identities are also discussed in this paper.

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1 Introduction

In [2, 6, 18], a type of hyperbolic expressions of Fibonacci polynomials and Fibonacci numbers are given using parameterization. We shall extend the idea to polynomial sequences and number sequences defined by linear recurrence relations of order 2.

Many number and polynomial sequences can be defined, characterized, evaluated, and/or classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called a sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n = aa_{n-1} + ba_{n-2}, \quad n \geq 2, \tag{1}$$

for some non-zero constants p and q and initial conditions a_0 and a_1. In Mansour [21], the sequence $\{a_n\}_{n\geq 0}$ defined by (1) is called Horadam's sequence, which was introduced in 1965 by Horadam [14]. [21] also obtained the generating functions for powers of Horadam's sequence. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [8], Hsu [15], Strang [24], Wilf [26], etc.) In [5], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1). For instance, a_n counts the number of ways to tile an n-board (i.e., board of length n) with squares (representing 1s) and dominoes (representing 2s) where each tile, except the initial one has a color. In addition, there are p colors for squares and q colors for dominoes. In particular, Aharonov, Beardon, and Driver (see [1]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$, called the primary solution, can be expressed in terms of Chebyshev polynomial values. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$, where F_n and L_n respectively are Fibonacci numbers and Lucas numbers, and T_n and U_n are Chebyshev polynomials of the first kind and the second kind, respectively (see also in [2, 3]). Some identities drawn from those relations were given by Beardon in [4]. Marr and Vineyard in [22] use the relationship to establish an explicit expression of five-diagonal Toeplitz determinants. In [12], the first two authors presented a new method to construct an explicit formula of $\{a_n\}$ generated by (1). For the sake of the reader's convenience, we cite this result as follows.

Proposition 1.1 *([12]) Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (1), and let α and β be two roots of of quadratic equation $x^2 - ax - b = 0$. Then*

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & if\ \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & if\ \alpha = \beta. \end{cases} \tag{2}$$

If the coefficients of the linear recurrence relation of a function sequence $\{a_n(x)\}$ of order 2 are real or complex-value functions of variable x, i.e.,

$$a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x), \tag{3}$$

we obtain a function sequence of order 2 with initial conditions $a_0(x)$ and $a_1(x)$. In particular, if all of $p(x)$, $q(x)$, $a_0(x)$ and $a_1(x)$ are polynomials, then the corresponding sequence $\{a_n(x)\}$ is a polynomial sequence of order 2. Denote the solutions of

$$t^2 - p(x)t - q(x) = 0$$

by $\alpha(x)$ and $\beta(x)$. Then

$$\alpha(x) = \frac{1}{2}(p(x) + \sqrt{p^2(x) + 4q(x)}), \beta(x) = \frac{1}{2}(p(x) - \sqrt{p^2(x) + 4q(x)}). \tag{4}$$

Similar to Proposition 1.1, we have

Proposition 1.2 *[12] Let $\{a_n\}$ be a sequence of order 2 satisfying the linear recurrence relation (3). Then*

$$a_n(x) = \begin{cases} \left(\frac{a_1(x)-\beta(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\alpha^n(x) - \left(\frac{a_1(x)-\alpha(x)a_0(x)}{\alpha(x)-\beta(x)}\right)\beta^n(x), & if\ \alpha(x) \neq \beta(x); \\ na_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x), & if\ \alpha(x) = \beta(x), \end{cases} \tag{5}$$

where $\alpha(x)$ and $\beta(x)$ are shown in (4).

In this paper, we shall consider the polynomial sequence defined by (3) with $q(x) = B$, a constant, to derive a parametric form of function sequence of order 2 by using the idea shown in [18]. Our construction will focus on four type Chebyshev polynomials and the following Gegenbauer-Humbert polynomial sequences although our method is limited by those function sequences.

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, for example, [8], Gould [10], Lidl, Mullen, and Turnwald[20], the first two of authors with Hsu [11])

$$\Phi(t) \equiv (C - 2xt + yt^2)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n, \tag{6}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (6), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [11])

$$\begin{aligned}
&P_n^{1,1,1}(x) = U_n(x),\ \textit{Chebyshev polynomial of the second kind},\\
&P_n^{1/2,1,1}(x) = \psi_n(x),\ \textit{Legendre polynomial},\\
&P_n^{1,-1,1}(x) = P_{n+1}(x),\ \textit{Pell polynomial},\\
&P_n^{1,-1,1}\left(\frac{x}{2}\right) = F_{n+1}(x),\ \textit{Fibonacci polynomial},\\
&P_n^{1,2,1}\left(\frac{x}{2}\right) = \Phi_{n+1}(x),\ \textit{Fermat polynomial of the first kind},\\
&P_n^{1,2a,2}(x) = D_n(x,a),\ \textit{Dickson polynomial of the second}\\
&\qquad\qquad\qquad\qquad \textit{kind},\ a \neq 0,\ (\textit{see, for example}, [20]),
\end{aligned}$$

where a is a real parameter, and $F_n = F_n(1)$ is the Fibonacci number. In particular, if $y = C = 1$, the corresponding polynomials are called Gegenbauer polynomials (see [8]). More results on the Gegenbauer-Humbert-type polynomials can be found in [16] by Hsu and in [17] by the second author and Hsu, etc.

Similarly, for a class of the generalized Gegenbauer-Humbert polynomial sequences defined by

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{7}$$

for all $n \geq 2$ with initial conditions

$$\begin{aligned}
P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda},\\
P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1},
\end{aligned}$$

the following theorem has been obtained in [12]

Theorem 1.3 *([12]) Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion (6) can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\frac{\left(x+\sqrt{x^2-Cy}\right)^{n+1} - \left(x-\sqrt{x^2-Cy}\right)^{n+1}}{2\sqrt{x^2-Cy}}. \quad (8)$$

We may use recurrence relation (6) to define various polynomials that were defined using different techniques. Comparing recurrence relation (6) with the relations of the generalized Fibonacci and Lucas polynomials shown in Example 4, with the assumption of $P_0^{1,y,C} = 0$ and $P_1^{1,y,C} = 1$, we immediately know

$$P_n^{1,1,1}(x) = 2xP_{n-1}^{1,1,1}(x) - P_{n-2}^{1,1,1}(x) = U_n(2x;0,1)$$

defines the Chebyshev polynomials of the second kind, and

$$P_n^{1,-1,1}(x) = 2xP_{n-1}^{1,-1,1}(x) + P_{n-2}^{1,-1,1}(x) = P_n(2x;0,-1)$$

defines the Pell polynomials.

In addition, in [20], Lidl, Mullen, and Turnwald defined the Dickson polynomials are also the special case of the generalized Gegenbauer-Humbert polynomials, which can be defined uniformly using recurrence relation (6), namely

$$D_n(x;a)) = xD_{n-1}(x;a) - aD_{n-2}(x;a) = P_n^{1,2a,2}(x)$$

with $D_0(x;a) = 2$ and $D_1(x;a) = x$. Thus, the general terms of all of above polynomials can be expressed using (8).

For $\lambda = y = C = 1$, using (8) we obtain the expression of the Chebyshev polynomials of the second kind:

$$U_n(x) = \frac{(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}},$$

where $x^2 \neq 1$. Thus, $U_2(x) = 4x^2 - 1$.

For $\lambda = C = 1$ and $y = -1$, formula (8) gives the expression of a Pell polynomial of degree $n+1$:

$$P_{n+1}(x) = \frac{(x+\sqrt{x^2+1})^{n+1} - (x-\sqrt{x^2+1})^{n+1}}{2\sqrt{x^2+1}}.$$

Thus, $P_2(x) = 2x$.

Similarly, let $\lambda = C = 1$ and $y = -1$, the Fibonacci polynomials are

$$F_{n+1}(x) = \frac{(x+\sqrt{x^2+4})^{n+1} - (x-\sqrt{x^2+4})^{n+1}}{2^{n+1}\sqrt{x^2+4}},$$

and the Fibonacci numbers are

$$F_n = F_n(1) = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\},$$

which has been presented in Example 1.

Finally, for $\lambda = C = 1$ and $y = 2$, we have Fermat polynomials of the first kind:

$$\Phi_{n+1}(x) = \frac{(x+\sqrt{x^2-2})^{n+1} - (x-\sqrt{x^2-2})^{n+1}}{2\sqrt{x^2-2}},$$

where $x^2 \neq 2$. From the expressions of Chebyshev polynomials of the second kind, Pell polynomials, and Fermat polynomials of the first kind, we may get a class of the generalized Gegenbauer-Humbert polynomials with respect to y defined by the following which will be parameterized.

Definition 1.4 *The generalized Gegenbauer-Humbert polynomials with respect to y, denoted by $P_n^{(y)}(x)$ are defined by the expansion*

$$(1 - 2xt + yt^2)^{-1} = \sum_{n \geq 0} P_n^{(y)}(x) t^n,$$

or by

$$P_n^{(y)}(x) = 2xP_{n-1}^{(y)}(x) - yP_{n-2}^{(y)}(x),$$

or equivalently, by

$$P_n^{(y)}(x) = \frac{(x+\sqrt{x^2-y})^{n+1} - (x-\sqrt{x^2-y})^{n+1}}{2\sqrt{x^2-y}}$$

with $P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$, where $x^2 \neq y$. In particular, $P_n^{(-1)}(x)$, $P_n^{(1)}(x)$ and $P_n^{(2)}(x)$ are respectively Pell polynomials, Chebyshev polynomials of the second kind, and Fermat polynomials of the first kind.

In the next section, we shall parameterize the function sequences defined by (3) and number sequences defined by (1) by using the idea of [18]. The application of the parameterization will be applied to construct the corresponding hyperbolic form of several well-known identities.

2 Hyperbolic expressions of parametric polynomial sequences

Suppose $q(x) = b$, a constant, and re-write (5) as

$$\begin{aligned} & a_n(x) \\ = \; & \frac{a_1(x) - \beta(x)a_0(x)}{\alpha(x) - \beta(x)}\alpha^n(x) - \frac{a_1(x) - \alpha(x)a_0(x)}{\alpha(x) - \beta(x)}\beta^n(x) \\ = \; & \frac{a_0(x)(\alpha^{n+1}(x) - \beta^{n+1}(x)) + (a_1(x) - a_0(x)p(x))(\alpha^n(x) - \beta^n(x))}{\alpha(x) - \beta(x)}, \end{aligned} \tag{9}$$

where we assume $\alpha(x) \neq \beta(x)$ due to the reason shown below.

Inspired by [18], we now set

$$(\alpha(x), \beta(x)) = \begin{cases} (\sqrt{b}e^{\theta(x)}, -\sqrt{b}e^{-\theta(x)}), & for\ b > 0, \\ (\sqrt{-b}e^{\theta(x)}, \sqrt{-b}e^{-\theta(x)}), & for\ b < 0, \end{cases} \tag{10}$$

for some real or complex value function $\theta \equiv \theta(x)$. Thus one may have $\alpha(x) \cdot \beta(x) = -b$ and

$$\alpha(x) + \beta(x) = p(x) = \begin{cases} 2\sqrt{b}\sinh(\theta(x)), & for\ b > 0, \\ 2\sqrt{-b}\cosh(\theta(x)), & for\ b < 0, \end{cases} \tag{11}$$

which implies

$$\theta(x) = \begin{cases} \sinh^{-1}\left(\frac{p(x)}{2\sqrt{b}}\right), & for\ b > 0 \\ \cosh^{-1}\left(\frac{p(x)}{2\sqrt{-b}}\right), & for\ b < 0. \end{cases} \tag{12}$$

For $b > 0$, substituting expressions (10) into the last formula of (9) yields

$$a_n(x) = \begin{cases} \frac{b^{(n-1)/2}}{\cosh(\theta)}\Big(a_0(x)\sqrt{b}\cosh((n+1)\theta) \\ \qquad +(a_1(x) - 2a_0(x)\sqrt{b}\sinh(\theta))\sinh(n\theta)\Big), & for\ even\ n, \\ \frac{b^{(n-1)/2}}{\cosh(\theta)}\Big(a_0(x)\sqrt{b}\sinh((n+1)\theta) \\ \qquad +(a_1(x) - 2a_0(x)\sqrt{b}\sinh(\theta))\cosh(n\theta)\Big), & for\ odd\ n, \end{cases} \tag{13}$$

where $\theta = \sinh^{-1}(p(x)/(2\sqrt{b}))$. Still in the case of $b > 0$, substituting (10) into the formula before the last one shown in (9), we obtain an equivalent expression:

$$\begin{aligned} & a_n(x) \\ = & \begin{cases} \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1(x)\sinh n\theta + \sqrt{b}a_0(x)\cosh(n-1)\theta\right), & for\, even\, n; \\ \frac{b^{(n-1)/2}}{\cosh\theta}\left(a_1(x)\cosh n\theta + \sqrt{b}a_0(x)\sinh(n-1)\theta\right), & for\, odd\, n, \end{cases} \end{aligned} \tag{14}$$

where $\theta = \sinh^{-1}(p(x)/(2\sqrt{b}))$.

Similarly, for $b < 0$ we have

$$\begin{aligned} a_n(x) \quad = \quad & \frac{(-b)^{(n-1)/2}}{\sinh(\theta)}\Big(a_0(x)\sqrt{-b}\sinh((n+1)\theta) \\ & +(a_1(x) - 2a_0(x)\sqrt{-b}\cosh(\theta))\sinh(n\theta)\Big), \end{aligned} \tag{15}$$

or equivalently,

$$a_n(x) = \frac{(-b)^{(n-1)/2}}{\sinh\theta}(a_1(x)\sinh n\theta - a_0(x)\sqrt{-b}\sinh(n-1)\theta), \tag{16}$$

where $\theta = \cosh^{-1}(p(x)/(2\sqrt{-b}))$.

We survey the above results as follows.

Theorem 2.1 *Let function sequence $a_n(x)$ be defined by*

$$a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x) \tag{17}$$

with initials $a_0(x)$ and $a_1(x)$, and let function $\theta(x)$ be defined by (12). Then the roots of the characteristic function $t^2 - p(x)t - b$ can be shown as (10), and there hold the hyperbolic expressions of functions $a_n(x)$ shown in (13) and (14) for $b > 0$ and (15) and (16) for $b < 0$.

Let us consider some special cases of Theorem 2.1:

Corollary 2.2 *Suppose $\{a_n(x)\}$ is the function sequence defined by (17) with initials $a_0(x) = 0$ and $a_1(x)$, then*

$$\begin{aligned} a_{2n}(x) &= b^{(2n-1)/2} a_1(x) \frac{\sinh(2n)\theta}{\cosh\theta}; \\ a_{2n+1}(x) &= b^n a_1(x) \frac{\cosh(2n+1)\theta}{\cosh\theta} \end{aligned} \tag{18}$$

for $b > 0$, where $\theta = \sinh^{-1}(p(x)/(2\sqrt{b}))$; and

$$a_n(x) = (-b)^{(n-1)/2} a_1(x) \frac{\sinh n\theta}{\sinh\theta} \tag{19}$$

for $b < 0$, where $\theta = \cosh^{-1}(p(x)/(2\sqrt{-b}))$.

Example 2.1 Let $\{F_n(kx)\}$ be the sequence of the generalized Fibonacci polynomials defined by

$$F_{n+2}(kx) = kxF_{n+1}(kx) + F_n(kx), \quad k \in \mathbb{R}\backslash\{0\},$$

with initials $F_0(kx) = 0$ and $F_1(kx) = 1$. From Corollary 2.2, we have

$$\begin{aligned} F_{2n}(kx) &= F_{2n}(2\sinh\theta) = \frac{\sinh 2n\theta}{\cosh\theta}, \\ F_{2n+1}(kx) &= F_{2n+1}(2\sinh\theta) = \frac{\cosh(2n+1)\theta}{\cosh\theta}, \end{aligned}$$

when $k = 2$ which are (6) and (7) shown in [6]. Obviously, from the above formulas and the identity $\cosh x + \cosh y = 2\cosh((x + y)/2)\cosh((x - y)/2)$, there holds

$$F_{2n+1}(kx) + F_{2n-1}(kx) = 2\cosh(2n\theta),$$

which was given in [6] as (8) when $k = 2$. Identity (9) in [6] is clearly the recurrence relation of $\{F_n(2x)\}$. The expressions of F_{2n} and F_{2n+1} can also be found in [13] with a general complex form

$$F_n(x) = i^{n-1}\frac{\sinh nz}{\sinh z},$$

where $x = 2i\cosh z$.

Corollary 2.3 *Suppose $\{a_n(x)\}$ is the function sequence defined by (17), $a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x)$ ($b > 0$), with initials $a_0(x) = c$, a constant, and $a_1(x) = p(x)$, then*

$$\begin{aligned} a_{2n}(x) &= 2b^n\cosh(2n\theta) + (c-2)b^n\frac{\cosh(2n-1)\theta}{\cosh\theta} \\ a_{2n+1}(x) &= 2b^{n+1/2}\sinh(2n+1)\theta + (c-2)b^{n+1/2}\frac{\sinh 2n\theta}{\cosh\theta}, \end{aligned} \tag{20}$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{b})$. If $\{a_n(x)\}$ is the function sequence defined by (17), $a_n(x) = p(x)a_{n-1}(x) + ba_{n-2}(x)$ ($b < 0$), with initials $a_0(x) = c$, a constant, and $a_1(x) = p(x)$, then

$$a_n(x) = \frac{(-b)^{(n-1)/2}}{\sinh\theta}(2\cosh\theta\sinh n\theta - c\sqrt{-b}\sinh(n-1)\theta), \tag{21}$$

where $\theta(x) = \cosh^{-1}(p(x)/(2\sqrt{-b}))$.

Proof. Substituting $a_0(x) = c$, $a_1(x) = p(x) = 2\sqrt{b}\sinh\theta$ into (14) yields

$$\begin{aligned} a_{2n}(x) &= \frac{b^n}{\cosh\theta}\left[2\sinh\theta\sinh(2n\theta) + c\cosh(2n-1)\theta\right], \\ a_{2n+1}(x) &= \frac{b^n}{\cosh\theta}\left[2\sinh\theta\cosh(2n+1)\theta + c\sinh(2n\theta)\right]. \end{aligned}$$

Then in the above equations using the identities

$$\cosh\theta\cosh(2n\theta) - \sinh\theta\sinh(2n\theta) = \cosh(2n-1)\theta,$$
$$\cosh\theta\sinh(2n+1)\theta - \sinh\theta\cosh(2n+1)\theta = \sinh(2n\theta),$$

respectively, we obtain (20). Similarly, using (16) one may obtain (21).

■

Example 2.2 Since the generalized Lucas polynomials are defined by $L_n(kx) = kxL_{n-1}(kx) + L_{n-2}(kx)$ with the initials $L_0(x) = 2$ and $L_1(x) = kx$, from Corollary 2.3 we have

$$L_{2n}(kx) = L_{2n}(2\sinh\theta) = 2\cosh(2n\theta),$$
$$L_{2n+1}(kx) = L_{2n+1}(2\sinh\theta) = 2\sinh(2n+1)\theta.$$

[13] also presented a general complex form of $L_n(x)$ as

$$L_n(x) = 2i^n \cosh nz,$$

where $x = 2i\cosh z$.

Example 2.3 In 1959, Morgan-Voyce discovered two large families of polynomials, $b_n(x)$ and $B_n(x)$, in his study of electrical ladder networks of resistors [23]. The recurrence relations of the polynomials were presented in [19] as follows.

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x), \quad n \geq 2,$$

where $B_0(x) = 1$ and $B_1(x) = x+2$, while

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x), \quad n \geq 2,$$

where $b_0(x) = 1$ and $b_1(x) = x+1$. It can be found that

$$b_n(x) = B_n(x) - B_{n-1}(x),$$
$$xB_n(x) = b_{n+1}(x) - b_n(x).$$

Using Corollary 2.3, it is easy to obtain the hyperbolic expressions of $B_n(x)$ and $b_n(x)$. From (21) in the corollary and noting $B_1(x) = x+2 = 2\cosh\theta$ and $B_0(x) = 1$, we have

$$B_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \quad x = 2\cosh\theta - 2.$$

Similarly, substituting $b_1(x) = x + 1 = 2\cosh\theta - 1$ and $b_0(x) = 1$ into (16) yields

$$b_n(x) = \frac{\sinh(n+1)\theta - \sinh n\theta}{\sinh\theta} = \frac{\cosh(2n+1)\theta/2}{\cosh\theta/2}, \quad x = 2\cosh\theta - 2.$$

We now consider the generalized Gegenbauer-Humbert polynomial sequences defined by (7) with $\lambda = C = 1$ and denoted by $P_n^{(y)}(x) \equiv P_n^{\lambda,y,C}(x)$. Thus

$$P_n^{(y)}(x) = 2xP_{n-1}^{(y)}(x) - yP_{n-2}^{(y)}(x), \tag{22}$$

$P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$. We use the similar parameterization shown above to present the hyperbolic expression of those generalized Gegenbauer-Humbert polynomial sequences.

Corollary 2.4 *Let $P_n^{(y)}(x)$ be defined by (22) with initials $P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$. If $y < 0$, then*

$$\begin{aligned} P_{2n}^{(y)}(x) &= (-y)^n \frac{\cosh(2n+1)\theta}{\cosh\theta}, \\ P_{2n+1}^{(y)}(x) &= (-y)^{n+1/2} \frac{\sinh(2n+2)\theta}{\cosh\theta}, \end{aligned} \tag{23}$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{-y})$. If $y > 0$, then

$$P_n^{(y)}(x) = y^{n/2} \frac{\sinh(n+1)\theta}{\sinh\theta}, \tag{24}$$

where $\theta(x) = \cosh^{(-1)}(p(x)/(2\sqrt{y})$.

Proof. A similar argument in the proof of (20) with $b = -y$ and $c = 1$ can be used to prove (23):

$$\begin{aligned} P_{2n}^{(y)}(x) &= 2(-y)^n \cosh(2n\theta) - (-y)^n \frac{\cosh(2n-1)\theta}{\cosh\theta} \\ P_{2n+1}^{(y)}(x) &= 2(-y)^{n+1/2} \sinh(2n+1)\theta - (-y)^{n+1/2} \frac{\sinh 2n\theta}{\cosh\theta}, \end{aligned}$$

where $\theta(x) = \sinh^{-1}(p(x)/(2\sqrt{-y})$, which implies (23) due to the identities $\cosh(2n+1)\theta + \cosh(2n-1)\theta = 2\cosh(2n\theta)\cosh\theta$ and $\sinh(2n+2)\theta + \sinh(2n\theta) = 2\sinh(2n+1)\theta\cosh\theta$. To prove (24), we substitute $-b = y$, and $a_1(x) = 2x = 2\sqrt{y}\cosh\theta$, and $a_0(x) = 1$ into (16). Thus

$$\begin{aligned} P_n^{(y)}(x) &= \frac{y^{n/2}}{\sinh\theta}(2\cosh\theta\sinh n\theta - \sinh(n-1)\theta) \\ &= y^{n/2}\frac{\sinh(n+1)\theta}{\sinh\theta}, \end{aligned}$$

where $\theta(x) = \cosh^{-1}(x/\sqrt{y})$ and the identity $\sinh(n+1)\theta + \sinh(n-1)\theta = 2\sinh n\theta\cosh\theta$ is applied in the last step.

■

Example 2.4 Using Corollary 2.4 one may obtain the following hyperbolic expressions of Pell polynomials $P_n(x) = P_n^{(-1)}(x)$ and the Chebyshev polynomials of the second kind $U_n(x) = P_n^{(1)}(x)$:

$$\begin{aligned} P_{2n}(x) &= \frac{\cosh(2n+1)\theta}{\cosh\theta}, \\ P_{2n+1}(x) &= \frac{\sinh(2n+2)\theta}{\cosh\theta}, \end{aligned}$$

where $\theta(x) = \sinh^{-1}(x)$, and

$$U_n(x) = \frac{\sinh(n+1)\theta}{\sinh\theta}, \tag{25}$$

where $\theta(x) = \cosh^{-1}(x)$.

Example 2.5 Finally, we consider the Chebyshev class of polynomials including the polynomials of the first kind, second kind, third kind, and fourth kind, denoted by $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$, respectively, which are defined by

$$a_n(x) = 2xa_{n-1}(x) - a_{n-2}(x), \quad n \geq 2, \tag{26}$$

with $a_0(x) = 1$ and $a_1(x) = x, 2x, 2x-1, 2x+1$ for $a_n(x) = T_n(x), U_n(x), V_n(x)$, and $W_n(x)$, respectively. Noting among those four polynomial sequences only $\{U_n(x)\}$ is in the generalized Gegenbauer-Humbert class, which has been presented in Example 2.3. From (16) there holds

$$T_n(x) = \frac{1}{\sinh\theta}(x\sinh n\theta - \sinh(n-1)\theta),$$

where $x = \cosh\theta$ due to $\theta = \cosh^{-1} x$. By using this substitution and the identity $\sinh(n-1)\theta = \sinh n\theta\cosh\theta - \cosh n\theta\sinh\theta$ we immediately obtain

$$T_n(x) = T_n(\cosh\theta) = \cosh n\theta.$$

Similarly,

$$\begin{aligned}
V_n(x) = V_n(\cosh\theta) &= \frac{\cosh(n+1/2)\theta}{\cosh(\theta/2)},\\
W_n(x) = W_n(\cosh\theta) &= \frac{\sinh(n+1/2)\theta}{\sinh(\theta/2)}.
\end{aligned}$$

A simple transformation $\theta \mapsto i\theta$, $i = \sqrt{-1}$, leads $\cos(i\theta) = \cosh\theta$ and $\sin(i\theta) = -\sinh\theta$. Thus from the trigonometric expressions of $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$ shown below, one may also obtain their corresponding hyperbolic expressions by simply transforming $\theta \mapsto i\theta$, respectively.

$$\begin{aligned}
T_n(\cos\theta) &= \cos n\theta, & U_n(\cos\theta) &= \frac{\sin(n+1)\theta}{\sin\theta},\\
V_n(\cos\theta) &= \frac{\cos(n+1/2)\theta}{\cos(\theta/2)}, & W_n(\cos\theta) &= \frac{\sin(n+1/2)\theta}{\sin(\theta/2)}.
\end{aligned}$$

3 Hyperbolic expressions of parametric number sequences

Suppose $\{a_n\}$ is a number sequence defined by (1), i.e.

$$a_n = aa_{n-1} + ba_{n-2}, \quad n \geq 2, \tag{27}$$

with the given initials a_0 and a_1. From [12] (see Proposition 1.1), the sequence defined by (27) has the expression

$$\begin{aligned} a_n &= \frac{a_0(\alpha^{n+1}-\beta^{n+1})+(a_1-a_0a)(\alpha^n-\beta^n)}{\alpha-\beta} \\ &= \frac{a_1-\beta a_0}{\alpha-\beta}\alpha^n - \frac{a_1-\alpha a_0}{\alpha-\beta}\beta^n, \quad n\geq 2, \end{aligned} \tag{28}$$

where α and β are two distinct roots of characteristic polynomial $t^2 - at - b$. Similar to (10) we denote

$$(\alpha(\theta),\beta(\theta)) = \begin{cases} (\sqrt{b}e^{\theta}, -\sqrt{b}e^{-\theta}) & for\ b>0, \\ (\sqrt{-b}e^{\theta}, \sqrt{-b}e^{-\theta}) & for\ b<0,\ a>0, \\ (-\sqrt{-b}e^{\theta}, -\sqrt{-b}e^{-\theta}) & for\ b<0,\ a<0, \end{cases} \tag{29}$$

for some real or complex number θ. Thus we have

$$a(\theta) = \alpha+\beta = \begin{cases} 2\sqrt{b}\sinh(\theta) & for\ b>0, \\ 2\sqrt{-b}\cosh(\theta) & for\ b<0,\ a>0, \\ -2\sqrt{-b}\cosh(\theta) & for\ b<0,\ a<0, \end{cases} \tag{30}$$

and define a parameter generalization of $\{a_n(\theta)\}$ as

$$a_n(\theta) = \begin{cases} 2\sqrt{b}\sinh(\theta)a_{n-1}(\theta) + ba_{n-2}(\theta) & for\ b>0, \\ 2\sqrt{-b}\cosh(\theta)a_{n-1}(\theta) + ba_{n-2}(\theta) & for\ b<0,\ a>0, \\ -2\sqrt{-b}\cosh(\theta)a_{n-1}(\theta) + ba_{n-2}(\theta) & for\ b<0,\ a<0 \end{cases} \tag{31}$$

with initials $a_0(\theta) = a_0$ and $a_1(\theta) = a_1$ when $a_0 = 0$ or $a_1(\theta) =$ when $a_0 \neq 0$. Obviously, if

$$\theta = \begin{cases} \sinh^{-1}\left(\frac{a}{2\sqrt{b}}\right) & for\ b>0, \\ \cosh^{-1}\left(\frac{a}{2\sqrt{-b}}\right) & for\ b<0,\ a>0, \\ \cosh^{-1}\left(\frac{-a}{2\sqrt{-b}}\right) & for\ b<0,\ a<0, \end{cases} \tag{32}$$

$\{a_n(\theta)\}$ is reduced to $\{a_n\}$.

For $b>0$, substituting expressions (29) into the second expression of a_n in (28), we obtain

$$
\begin{aligned}
& a_n(\theta) \\
= \; & b^{(n-1)/2} \frac{a_1(e^{n\theta} - (-1)^n e^{-n\theta}) + \sqrt{b} a_0 (e^{(n-1)\theta} + (-1)^n e^{-(n-1)\theta})}{e^{\theta} + e^{-\theta}} \qquad (33) \\
= \; & \begin{cases} \frac{b^{(n-1)/2}}{\cosh\theta} \left(a_1 \sinh n\theta + \sqrt{b} a_0 \cosh(n-1)\theta \right), & if\, n\, is\, even, \\ \frac{b^{(n-1)/2}}{\cosh\theta} \left(a_1 \cosh n\theta + \sqrt{b} a_0 \sinh(n-1)\theta \right), & if\, n\, is\, odd, \end{cases} \qquad (34)
\end{aligned}
$$

where $\theta = \sinh^{-1}(a/(2\sqrt{b}))$.

Similarly, for $b < 0$ we have

$$
\begin{aligned}
& a_n \\
= \; & \begin{cases} \frac{(-b)^{(n-1)/2}}{\sinh(\theta)} \big(a_0\sqrt{-b} \sinh((n+1)\theta) \\ \quad +(a_1 - 2a_0\sqrt{-b}\cosh(\theta)) \sinh(n\theta) \big) & for\ a > 0, \\ \frac{(-\sqrt{-b})^{n-1}}{\sinh(\theta)} \big(-a_0\sqrt{-b} \sinh((n+1)\theta) \\ \quad +(a_1 + 2a_0\sqrt{-b}\cosh(\theta)) \sinh(n\theta) \big) & for\ a < 0, \end{cases} \qquad (35)
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
& a_n \\
= \; & \begin{cases} (-b)^{(n-1)/2} \frac{a_1(e^{n\theta} - e^{-n\theta}) - a_0\sqrt{-b}(e^{(n-1)\theta} - e^{-(n-1)\theta})}{e^{\theta} - e^{-\theta}} & for\, a > 0, \\ (-\sqrt{-b})^{n-1} \frac{a_1(e^{n\theta} - e^{-n\theta}) + a_0\sqrt{-b}(e^{(n-1)\theta} - e^{-(n-1)\theta})}{e^{\theta} - e^{-\theta}} & for\, a < 0, \end{cases} \\
= \; & \begin{cases} \frac{(-b)^{(n-1)/2}}{\sinh\theta} (a_1 \sinh n\theta - a_0\sqrt{-b} \sinh(n-1)\theta) & for\, a > 0, \\ \frac{(-\sqrt{-b})^{n-1}}{\sinh\theta} (a_1 \sinh n\theta + a_0\sqrt{-b} \sinh(n-1)\theta) & for\, a < 0, \end{cases} \qquad (36)
\end{aligned}
$$

where $\theta = \cosh^{-1}(a/(2\sqrt{-b}))$ when $a > 0$ and $\cosh^{-1}(-a/(2\sqrt{-b}))$ when $a < 0$.

If the characteristic polynomial $t^2 - at - b$ has the same roots $\alpha = \beta$, then $a = \pm 2\sqrt{-b}$, $\alpha = \beta = \pm\sqrt{-b}$, and

$$
a_n = n a_1 (\pm\sqrt{-b})^{n-1} - (n-1) a_0 (\pm\sqrt{-b})^n. \qquad (37)
$$

We summarize the above results as follows.

Theorem 3.1 *Suppose $\{a_n\}_{n\geq 0}$ is a number sequence defined by (27) with characteristic polynomial $t^2 - at - b$. If the characteristic polynomial has the same roots, then there holds an expression of a_n shown in (37). If the characteristic polynomial has distinct roots, there hold hyperbolic extensions (51) or (52) for $b > 0$ and (36) or (36) for $b < 0$.*

Example 3.1 [18] gave the hyperbolic expression of the generalized Fibonacci number sequence $\{F_n(\theta)\}$ defined by

$$F_n(\theta) = 2\sinh\theta F_{n-1}(\theta) + F_{n-2}(\theta), \quad n \geq 2,$$

with initials $F_0(\theta) = 0$ and $F_1(\theta) = 1$. From Theorem 3.1, one may obtain the same result as that in [18]:

$$\begin{aligned} F_n(\theta) &= \frac{e^{n\theta} - (-1)^n e^{-n\theta}}{e^{\theta} + e^{-\theta}} \\ &= \begin{cases} \frac{\sinh n\theta}{\cosh\theta}, & if\ n\ is\ even; \\ \frac{\cosh n\theta}{\cosh\theta}, & if\ n\ is\ odd, \end{cases} \end{aligned} \tag{38}$$

Similarly, for the generalized Lucas number sequence $\{L_n(\theta)\}$ defined by

$$L_n(\theta) = 2\sinh\theta L_{n-1}(\theta) + L_{n-2}(\theta), \quad n \geq 2,$$

with initials $L_0(\theta) = 2$ and $L_1(\theta) = 2\sinh\theta$, we have

$$L_n(\theta) = e^{n\theta} + (-1)^n e^{-n\theta} = \begin{cases} 2\cosh(n\theta), & if\ n\ is\ even; \\ 2\sinh(n\theta), & if\ n\ is\ odd. \end{cases} \tag{39}$$

Example 3.2 [9] defined the following generalization of Fibonacci numbers and Lucas numbers:

$$f_n = \frac{c^n - d^n}{c - d}, \quad \ell_n = c^n + d^n, \tag{40}$$

where c and d are two roots of $t^2 - st - 1$, $s \in \mathbb{N}$. Denote $\Delta = s^2 + 4$ and $\alpha = \ln c$, where $c = (s + \sqrt{s^2 + 4})/2$. Then the above expressions are equivalent to

$$\frac{1}{2} f_n = \frac{e^{\alpha n} - (-1)^n e^{-\alpha n}}{2\sqrt{\Delta}}, \quad \frac{1}{2}\ell_n = \frac{e^{\alpha n} + (-1)^n e^{-\alpha n}}{2}.$$

It is obvious that by transferring $c \mapsto e^\theta$ and $d \mapsto -e^{-\theta}$ that two expressions in (40) are equivalently (38) and (39), respectively, shown in Example 3.1, which are obtained using Theorem 3.1 with $(a, b, a_0, a_1) = (s, 1, 0, 1)$ and $(s, 1, 2, s)$ for f_n and ℓ_n, respectively. Hence, the corresponding identities regarding f_n and ℓ_n obtained in [9] can be established similarly. However, we may derive more new identities as follows. For instance, there holds

$$\ell_n + sf_n = 2f_{n+1}, \tag{41}$$

which can be proved by substituting $s = e^\theta - e^{-\theta} = 2\sinh\theta$ into the left-hand side. Indeed, for even n, from Example 3.1

$$\ell_n + sf_n = 2\cosh(n\theta) + 2\sinh\theta\frac{\sinh n\theta}{\cosh\theta} = 2\frac{\cosh(n+1)\theta}{\cosh\theta},$$

and similarly, for odd n, $\ell_n + sf_n = 2\sinh(n+1)\theta/\cosh\theta$, which brings (41). When $s = 1$, (41) reduces to the classical identity $L_n + F_n = 2F_{n+1}$.

From the above examples, we find many identities relevant to Fibonacci numbers and Lucas numbers can be proved using hyperbolic identities. Here are more examples.

Example 3.3 In the identity

$$\sinh 2n\theta = 2\sinh n\theta\cosh n\theta$$

substituting (38) and (39), namely, $\sinh 2n\theta = \cosh\theta\ F_{2n}(\theta)$ and

$$\sinh n\theta = \begin{cases} \cosh\theta\ F_n(\theta), & if\ n\ iseven, \\ \frac{1}{2}L_n(\theta), & if\ n\ is\ odd, \end{cases}$$
$$\cosh n\theta = \begin{cases} \frac{1}{2}L_n(\theta), & if\ n\ is\ even, \\ \cosh\theta\ F_n(\theta), & if\ n\ is\ odd, \end{cases}$$

we immediately obtain

$$F_{2n}(\theta) = F_n(\theta)L_n(\theta).$$

Similarly, since $\sinh(m+n)\theta = \cosh\theta\ F_{m+n}(\theta)$ when $m+n$ is even,

$$\sinh m\theta \cosh n\theta = \begin{cases} \frac{1}{2}\cosh\theta\ F_m(\theta)L_n(\theta), & if\ m\ and\ n\ are\ even, \\ \frac{1}{2}\cosh\theta\ F_n(\theta)L_m(\theta), & if\ m\ and\ n\ are\ odd, \end{cases}$$

and

$$\cosh m\theta \sinh n\theta = \begin{cases} \frac{1}{2}\cosh\theta\ F_n(\theta)L_m(\theta), & if\ m\ and\ n\ are\ even, \\ \frac{1}{2}\cosh\theta\ F_m(\theta)L_n(\theta), & if\ m\ and\ n\ are\ odd, \end{cases}$$

from identity

$$\sinh(m+n)\theta = \sinh m\theta \cosh n\theta + \cosh m\theta \sinh n\theta \tag{42}$$

we have

$$2F_{m+n}(\theta) = F_m(\theta)L_n(\theta) + F_n(\theta)L_m(\theta)$$

for even $m+n$.

When $m+n$ is odd, $\sinh(m+n)\theta = L_{m+n}(\theta)/2$, from (42),

$$\sinh m\theta \cosh n\theta = \begin{cases} \cosh^2\theta\ F_m(\theta)F_n(\theta), & if\ m\ is\ even\ and\ n\ is\ odd, \\ \frac{1}{4}L_m(\theta)L_n(\theta), & if\ m\ is\ odd\ and\ n\ is\ even, \end{cases}$$

and

$$\cosh m\theta \sinh n\theta = \begin{cases} \frac{1}{4}L_m(\theta)L_n(\theta), & if\ m\ is\ even\ and\ n\ is\ odd, \\ \cosh^2\theta\ F_m(\theta)F_n(\theta), & if\ m\ is\ odd\ and\ n\ is\ even, \end{cases}$$

we obtain

$$2L_{m+n}(\theta) = F_m(\theta)F_n(\theta) + L_m(\theta)L_n(\theta).$$

More examples can be found in [25].

Our scheme may also extend some well-know identities to their hyperbolic setting.

Example 3.4 [7] considers equation $t^2-at+b=0$ ($b \neq 0$) with distinct roots t_1 and t_2, i.e., $\Delta^2 = a^2-4b \neq 0$, and defines a sequence $\{g_n\}$ by $g_n = ag_{n-1} - bg_{n-2}$ ($n \geq 2$) with initials g_0 and g_1. If the initials are 0 and 1, the corresponding sequence is denoted by $\{r_n\}$. Denote $s_n = t_1^n + t_2^n$ and $\Delta = a^2 - 4b$. Then [7] gives identities

$$r_n = -b^n r_{-n}, \quad s_n = b^n s_{-n}, \tag{43}$$
$$s_n^2 = \Delta r_n^2 + 4b^n, \tag{44}$$
$$s_n s_{n+1} = \Delta r_n r_{n+1} + 2ab^n, , \tag{45}$$
$$2b^n r_{j-n} = r_j s_n - r_n s_j, , \tag{46}$$
$$r_{j+n} = r_n s_j + b^n r_{j-n}., \tag{47}$$
$$\tag{48}$$

We now show all the above identities can be extended to the hyperbolic setting. For $b > 0$, from (36) there holds

$$r_n = (b)^{(n-1)/2} \frac{e^{n\theta} - e^{-n\theta}}{e^\theta - e^{-\theta}} = \frac{(b)^{(n-1)/2}}{\sinh\theta} \sinh n\theta, \tag{49}$$

and similarly,

$$s_n = 2b^{n/2} \cosh n\theta, \tag{50}$$

where $\theta = \cosh^{-1}(a/(2\sqrt{b}))$.

For $b > 0$, substituting expressions (29) into (28), we obtain

$$\begin{aligned} & a_n(\theta) \\ = \quad & b^{(n-1)/2} \frac{a_1(e^{n\theta} - (-1)^n e^{-n\theta}) + \sqrt{b} a_0 (e^{(n-1)\theta} + (-1)^n e^{-(n-1)\theta})}{e^\theta + e^{-\theta}} \end{aligned} \tag{51}$$

$$= \left\{ \begin{array}{ll} \frac{b^{(n-1)/2}}{\cosh\theta} \left(a_1 \sinh n\theta + \sqrt{b} a_0 \cosh(n-1)\theta \right), & if\, n\, is\, even; \\ \frac{b^{(n-1)/2}}{\cosh\theta} \left(a_1 \cosh n\theta + \sqrt{b} a_0 \sinh(n-1)\theta \right), & if\, n\, is\, odd, \end{array} \right. \tag{52}$$

where $\theta = \sinh^{-1}(a/(2\sqrt{b}))$.

Similarly, for $b < 0$ we have

$$\begin{aligned} a_n \quad = \quad & \frac{(-b)^{(n-1)/2}}{\sinh(\theta)} \Big(a_0 \sqrt{-b} \sinh((n+1)\theta) \\ & + (a_1 - 2a_0 \sqrt{-b} \cosh(\theta)) \sinh(n\theta) \Big), \end{aligned} \tag{53}$$

or equivalently,

$$a_n = (-b)^{(n-1)/2}\frac{a_1(e^{n\theta}-e^{-n\theta})-\sqrt{-b}a_0(e^{(n-1)\theta}-e^{-(n-1)\theta})}{e^{\theta}-e^{-\theta}} \quad (54)$$

$$= \frac{(-b)^{(n-1)/2}}{\sinh\theta}(a_1\sinh n\theta - a_0\sqrt{-b}\sinh(n-1)\theta), \quad (55)$$

where $\theta = \cosh^{-1}(a/(2\sqrt{-b}))$.

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On a system of nonlinear differential equations for the model of totally connected traffic *

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Abstract

In the paper the qualitative properties solutions of the system nonlinear equations, describing one-way movement of particles chain on a line with follower velocity defined by some function of distance from the leader, are researched. In the case when the given function is the velocity of the first particle (leader) in the chain, the model is called a model of leader. If the given function is the velocity of the last particle (outsider), the model is called a model of "shepherd".

The sufficient conditions for the existence of the chain with the given constraints on the velocity and acceleration are obtained.

AMS 2000 Mathematics Subject Classification: 34A34, 46E35

Keywords: systems of nonlinear ordinary differential equations; follow-the-leader model; interpretation for traffic

1 Introduction

One of the basic models of traffic flow is a model of follow the leader [1]-[4]. This model reduce to the next differential equations:

$$x_{n+1} - x_n = f(\dot{x}_n), \tag{1}$$

where $x_n(t)$ is a vehicle coordinate,

$$x_n(t) < x_{n+1}(t), \quad n = 1, 2, ... \tag{2}$$

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Flow satisfying (1)-(2) is called *totally connected.* The function f in (1) is a parabola with positive coefficients in classic case,[1]-[2],

$$f(x) = a + bx + cx^2,$$

where a is *static distance,* b is *driver reaction delay* and c is *braking distance coefficient.* Function f by condition $x \geq 0$ is *continuous with several successive derivatives, positive, monotone and convex.* For simplify, we set

$$f(0) = 1.$$

Let us denote the inverse of this function f by g and obtain a system of differential equations

$$\dot{x}_n = g(x_{n+1} - x_n), n = 1, ..., N - 1.$$

2 Follow - the - leader problem statement: the cluster with front wheel drive

We consider a system of ordinary differential equations (ODE)

$$\dot{x}_n = g(x_{n+1} - x_n), n = 1, 2, .., N - 1, \tag{3}$$

where

$$supp(g) = [1, \infty), \tag{4}$$

$$g(1) = 0, \tag{5}$$

g has enough smoothness and,

$$g'(x) > 0, \quad \forall \quad x \geq 1, \tag{6}$$

$$g''(x) \leq 0, \quad \forall \quad x \geq 1. \tag{7}$$

Let the initial conditions are

$$x_1(0) = x_{1,0}, x_2(0) = x_{2,0}, ..., x_{N-1}(0) = x_{N-1,0}$$

such that

$$x_{n+1,0} - x_{n,0} > 1, n = 1, ..., N - 1, \tag{8}$$

and boundary condition is

$$x_N(t) = r(t) \quad \forall \quad t \geq 0. \tag{9}$$

We associate problem (3)-(9) with follow the leader models. For function $r(t)$ let assume the following.

1. The function $\dot{r}(t)$ is absolutely continuous for $t \geq 0$;

2. There is the speed boundaries

$$0 \leq \dot{r}(t) \leq M_1, \quad \forall \quad t \geq 0; \tag{10}$$

3. There is acceleration boundaries

$$|\ddot{r}(t)| \leq M_2 \tag{11}$$

almost everywhere by $\forall \quad t \geq 0$.

Conditions (10)-(11) define functions of the Sobolev class , [8],

$$h(t) \in W^1_\infty(R_+) = \{h \in L_\infty(R_+), \dot{h} \in L^0_\infty(R_+)\},$$

where

$$h(t) = \dot{r}(t) - M_1/2.$$

Main purpose is to investigate properties of the functions cluster $\{x_n\}_{n=1}^{N-1}$, followed the leader $x_N(t)$.

2.1 An elementary case $N = 2$.

We have equation

$$\dot{x} = g(r - x). \tag{12}$$

Lemma 1. *If x is solution of (12),(8)-(9), then $\dot{x} > 0 \quad \forall \quad t > 0$.*

Proof. If $\dot{x} \to 0$, then $g(r - x) \to 0$, and it's equivalent to $r - x \to 1 + 0$. If $r(T) - x(T) = 1$, it is true at a moment of time T then $\dot{x}(T) = 0$ and

$$\ddot{x}(T) = g'(1)(\dot{r}(T) - \dot{x}(T)),$$

whence $\ddot{x}(T) = g'(1)\dot{r}(T) > 0$, which contradicts with (7). So, $r(t) - x(t) - 1$ can't go to null in a finite time.

Lemma 2. *The following inequality is true*

$$||\dot{x}||_{C(R+)} \leq max(||\dot{r}||_{C(R+)}, \dot{x}(0)). \tag{13}$$

Proof. From (12)

$$\ddot{x} = g' \times (\dot{r} - \dot{x}). \tag{14}$$

Suppose $\dot{x}$ reaches local maximum at some point t_0. Then $\ddot{x}(t_0) = 0$, whence and from (14)

$$\dot{r}(t_0) = \dot{x}(t_0). \tag{15}$$

If $\dot{x}$ monotonically increases on R_+, then from (12) $(r - x)(t)$ monotonically increases too, whence $\dot{r}(t) - \dot{x}(t) \geq 0$, ie

$$\dot{r}(t) \geq \dot{x}(t) \geq 0. \tag{16}$$

Analogously if $\dot{x}$ monotonically decreases, then $\dot{r} - \dot{x} \leq 0$, and

$$0 \leq \dot{x}(t) \leq \dot{x}(0). \tag{17}$$

Inequality (13) follows from (15)-(17).

Lemma 3. *The following inequality is true*

$$||\ddot{x}||_{C(R+)} \leq max(||\ddot{r}||_{C(R+)}, \dot{g}(1)||\dot{r}||_{C(R+)}, \dot{g}(1)g(r(0) - x(0))). \tag{18}$$

Proof. From (14) we have

$$\dddot{x} = g'' \times (\dot{r} - \dot{x})^2 + g' \times (\ddot{r} - \ddot{x}). \tag{19}$$

Because $g' > 0$ and $g'' < 0$ for admissible values of arguments, then

$$\dddot{x} = 0 \Longleftrightarrow \ddot{r} - \ddot{x} \geq 0,$$

from where follow

$$\ddot{x}(t) \leq \ddot{r}(t) \tag{20}$$

at those points t, where $\ddot{x}(t)$ has a local extremum. On the other hand from (14) it follows that

$$|\ddot{x}(t)| \leq |\dot{g}(r(t) - x(t))||\dot{r} - \dot{x}|,$$

whence with Lemma 2 and monotonically decreasing $\dot{g}$, we have

$$|\ddot{x}(t)| \leq |\dot{g}(1)|max(||\dot{r}||_{C(R+)}, \dot{x}(0)). \tag{21}$$

Statement of Lemma 3 follows from (20) and (21) .

Lemma 4. *Let suppose*

$$g(r(0) - x(0)) \leq ||\dot{r}||_{C(R+)}, \tag{22}$$

and

$$max(\dot{g}(1)||\dot{r}||_{C(R+)}, \dot{g}(1)g(r(0) - x(0))) \leq \leq ||\ddot{r}||_{C(R+)}. \tag{23}$$

Then the following inequalities are true

$$||\dot{x}||_{C(R+)} \leq ||\dot{r}||_{C(R+)}, \tag{24}$$

$$||\ddot{x}||_{C(R+)} \leq ||\ddot{r}||_{C(R+)}. \tag{25}$$

Proof. It follows from previous lemmas.

Theorem 1. *Solution $x(t)$ of problem (3) - (11) with conditions (22)-(23) and $N = 2$ exists and belongs to the same set of functions(10)-(11) with the leader function $r(t)$.*

2.2 Cluster follows the leader length N.

Applying the obvious considerations of induction can be established that

Theorem 2. *Solution of problem (3)-(11) with conditions*

$$g(x_{n+1}(0) - x_n(0)) \le ||\dot{x}_{n+1}||_{C(R+)}, \tag{26}$$

$$\begin{aligned} max(\dot{x}_{n+1}(1)||\dot{x}_{n+1}||_{C(R+)}, \dot{x}_{n+1}(1)g(x_{n+1}(0) - x_n(0))) \le \\ \le ||\ddot{x}_{n+1}||_{C(R+)}, \end{aligned} \tag{27}$$

$n = 1, ..., N-1$ *exists for any natural* N*. In this case, all links are infinitely differentiable functions, if functions* g *and* r *are those.*

2.3 Uniform movement of the leader

Let us consider some of the specific behaviors of the leader. Suppose that begining from some moment t_0 we have

$$r(t) = r(t_0) + M_1(t - t_0), \quad t \ge t_0 \ge 0. \tag{28}$$

Then if $t > t_0$ is true then we have

$$\ddot{x}_{N-1}(t) = g' \times (\dot{r}(t) - \dot{x}_{N-1}(t)) = f' \times (M_1 - \dot{x}_{N-1}(t)).$$

So far as

$$M_1 - \dot{x}_{N-1} \ge 0,$$

then $\ddot{x}_{N-1} > 0$, $\dot{x}_{N-1}$ monotonically increases and is limited by the constant M_1, i.e.

$$\dot{x}_{N-1} \to M_1, x_{N-1} \to M_1(t - t_0) + C$$

from the top. Thus the movement of the follower also converges to the uniform movement. In general case if

$$r(t) = r(t_0) + M(t - t_0), \quad t \ge t_0, \tag{29}$$

where M isn't necessarily the maximum constant, then

$$\ddot{x}_{N-1}(t) = (a - \dot{x}_{N-1})g'(Mt + M_0). \tag{30}$$

From (30) it follows that if $M > \dot{x}_{N-1}$, then $\dot{x}_{N-1}$ is increasing, and if $M < \dot{x}_{N-1}$, then $\dot{x}_{N-1}$ decreases. Moreover it follows from the concavity of g that

$$0 < g'(x) \le g'(0),$$

which implies that

$$|\dot{x}_{N-1}(t) - M| \to 0$$

monotonically and from equation (29)

$$r(t) - x_{N-1}(t) \to Const.$$

Discoursing by induction, we get

Theorem 3. *If in chain (3)-(9),(26)-(27) of N links the leader $r(t)$ converges in norm $C_1[t_0, \infty)$ to uniform traffic, then next links converge to uniform traffics in this metric too.*

2.4 Generalized cluster traffic

In constraints of statements (3)-(11) function g depends on the numbers of managers, i.e.

$$\dot{x}_n = g_n(x_{n+1} - x_n), n = 1, 2, .., N - 1, \tag{3'}$$

Lemma 5. Let suppose $k = 1, 2, N - 1$

$$g_k(x_{k+1}(0) - x_k(0)) \leq ||\dot{x}_{k+1}||_{C(R+)}, \tag{31}$$

and

$$max(\dot{g}_k(1)||\dot{x}_{k+1}||_{C(R+)}, \dot{g}_k(1)g_k(x_{k+1}(0) - x_k(0))) \leq \\ \leq ||\ddot{x}_k||_{C(R+)}. \tag{32}$$

Then next relations

$$||\dot{x}_k||_{C(R+)} \leq ||\dot{x}_{k+1}||_{C(R+)}, \tag{33}$$

$$||\ddot{x}_k||_{C(R+)} \leq ||\ddot{x}_{k+1}||_{C(R+)}. \tag{34}$$

are true.

Theoreme 4. *Solution of problem (3')-(11), (31)-(32) exist for any natural N. In this case, if functions g_k, $1 \leq k \leq N-1$ and x_N are infinite differentiable, then all links are infinite differentiable functions too.*

2.5 Generalized traffic - cluster with random dynamic dimensions.

Functions f_k, $k = 1, .., N$ are a family of functions depending on a finite number of random variables such as linear or quadratic. In this case, the chains are finite random functions. The conditions (29) - (30) are probability and sufficient conditions of a connected traffic hold with a certain probability, which should be evaluated.

2.6 Cluster with rear wheel drive

We consider the problem (1)-(11)

$$x_{n+1} - x_n = f(\dot{x}_n), \tag{35}$$

$n = 1, ..., N - 1$, where instead (9) we assume

$$x_1(t) = r(t) \ \ \forall \ \ t \geq 0. \tag{9'}$$

From (35) it follows

$$\dot{x}_{n+1}(t) = \dot{x}_n(t) + \ddot{x}_n f'(\dot{x}_n(t)), \tag{36}$$

$$\ddot{x}_{n+1}(t) = \ddot{x}_n(t) + \dddot{x}_n f'(\dot{x}_n(t)) + (\ddot{x}_n)^2 f''(\dot{x}_n(t)). \tag{37}$$

Let us assume $x_1(t) = r(t)$ is admissible operating regime of traffic, which satisfied (10)-(11). If the traffic is not totally accelerating, i.e. monotonically increasing acceleration, what can't be subject to the speed limit, then there will be time t^*, when the acceleration (deceleration) has local (global) maximum. From (37)it follows that since at that moment $\dddot{x}_n(t^*) = 0$, then

$$\ddot{x}_{n+1}(t^*) > \ddot{x}_n(t^*). \tag{38}$$

So, if a moment exists when acceleration x_n peaks, then from (36)it follows that x_{n+1} isn't satisfies admissible conditions.

Theorem 5. *For solution of problem (1)-(9')-(11)*

$$||\ddot{x}_k||_{C(R+)} < ||\ddot{x}_{k+1}||_{C(R+)}, \tag{39}$$

$k = 1, 2, ...,$ *is true. It means gap connected traffic in the link of the chain, where the corresponding rate of acceleration is a maximum.*

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REMOTALITY OF EXPOSED POINTS

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ABSTRACT. In this article, we discuss the problem of remotality of exposed points of bounded sets in certain Banach spaces. Indeed, we present a full characterization of a class of exposed points that are remotal points.

1. INTRODUCTION AND PRELIMINARIES

Let X be a Banach space, and E be a closed bounded convex subset of X. For $x \in X$, let

$$D(x, E) = \sup_{e \in E} \|x - e\|$$

be the maximum distance from x to E. If an $e \in E$ exists such that $D(x, E) = \|x - e\|$, then e is said to be a remotal, or farthest, point in E for x, and we define $F(x, E) = \{e \in E : D(x, E) = \|x - e\|\}$. If $F(x, E) \neq \phi$ for all $x \in X$, then E is said to be a remotal set.

The theory of remotal sets in Banach spaces is not as well as developed as that of proximinal sets; where the minimum distance is required to be attained.
In [3], the authors proposed and discussed the following problem:
Problem 1: When is a boundary point of E a remotal point?
This seems to be a tough question and more general than
Problem 2: When is an extreme point of E a remotal point?
Recall that a point $e \in E$ is said to be an extreme point of the convex set E, if e is not the middle point of any two other points of E. A special type of extreme points are exposed points. A point $e \in E$ is said to be an exposed point of E, if there exists a linear functional $f \in X^*$, the dual space of the normed space X, such that $f(y) < f(e)$ for all $y \in E\backslash\{e\}$. Recall that, in this case, the set $H := \{x \in X : f(x) = f(e)\}$ is called a supporting hyperplane of E at e; see [4].

In [1], it is proved that any normed linear space contains a bounded convex set whose exposed points are not necessarily remotal points. This is why we study here the problem:
Problem 3: When is an exposed point of E a remotal point?
We refer the reader to [3] and [1] for some results on this problem.
The object of this paper is to address problem 3 above, where we give necessary and sufficient conditions for a class of exposed points to be remotal points in certain Banach spaces.
In the sequel, X^* denotes the dual space of the normed space X, $S(m, r)$ denotes the sphere centered at m with radius r and $B(m, r)$ denotes the ball centered

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at m with radius r. If E is a subset of the normed space X, and $x \in X$, then $P(x, E)$ denotes the set of closest elements of E from X. For $1 < p < \infty$, we define the conjugate exponent of p to be the number q that satisfies $1/p + 1/q = 1$. For $1 < p < \infty$, we define the spaces

$$\ell^p := \{(x_i) : x_i \in \mathbb{C}, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

and

$$L^p[a,b] := \{f : [a,b] \to \mathbb{C} : \int_a^b |f(t)|^p \, dt < \infty\}.$$

For $(x_i) \in \ell^p$ and $f \in L^p[a,b]$, the following norms are defined

$$\|(x_i)\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \text{ and } \|f\| = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p}.$$

Recall that $(\ell^p)^* = \ell^q$ and $(L^p[a,b])^* = L^q[a,b]$ where p and q are conjugate exponents.

For $p = \infty$,

$$\ell^\infty := \{(x_i) : x_i \in \mathbb{C}, \sup |x_i| < \infty\}$$

and

$$L^\infty[a,b] := \{f : [a,b] \to \mathbb{C} : \text{ess sup} f < \infty\}.$$

It is known that $(\ell^1)^* = \ell^\infty$ and $(L^1[a,b])^* = L^\infty[a,b]$.
Finally, c_0 is defined to be

$$\{(x_i) : x_i \in \mathbb{C}, x_i \to 0\}.$$

It is known that $(c_0)^* = \ell^\infty$.

We refer the reader to any standard book in functional analysis as a reminder of these concepts; see [2].

2. Basic Results

Definition 2.1. A differentiable strictly convex function defined on $[0, \infty)$ will be called a nice convex function if it satisfies the following properties:

(1)
$$\varphi \geq 0.$$

(2)
$$\varphi(0) = 0.$$

(3)
$$\lim_{x \to \infty} \varphi'(x) = \infty.$$

(4)
$$\varphi'(0) = 0.$$

It follows that if φ is a nice convex function, then φ is strictly increasing. Moreover, since φ is strictly convex and increasing, it is unbounded, hence

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \varphi'(x) = \infty.$$

This observation will be used in the sequel.
Observe that for any $p > 1$, $\varphi(t) = t^p$ is a nice convex function.
Now let X be a Banach space and let X^* be its dual space.

Definition 2.2. The pair (X, X^*) is called a strictly convex pair if there exists a nice convex function φ such that for each $x \in X$, there exists $f_x \in X^*$ with the property

$$f_x(x) = \|f_x\|\,\|x\| = \varphi(\|x\|).$$

It should be noted that the first equality in the above definition always holds, for a certain f, according to the Hahn-Banach theorem. So, in fact, our interest is the second equality.

Example 2.3. The pairs $(\ell^p, \ell^q), 1 < p < \infty$, are strictly convex pairs, with $\varphi(t) = t^p$. Indeed, for $x = (x_n) \in \ell^p$, define

$$f_x(y) = \sum_{n=1}^{\infty} \left(|x_n|^{p-1}\mathrm{sgn}\ x_n\ y_n\right).$$

Then, clearly,

$$f_x \in (\ell^p)^*,\ \|f\| = \|x\|^{p-1} \text{ and } f_x(x) = \|x\|^p = \|f_x\|\,\|x\| = \varphi(\|x\|).$$

Example 2.4. The pairs (c_0, ℓ^1) and (ℓ^1, ℓ^∞) are not strictly convex pairs.

Example 2.5. The pairs $(L^p[0,1], L^q[0,1]),\ 1 < p < \infty$ are strictly convex pairs, but $(L^1[0,1], L^\infty[0,1])$ is not.

Definition 2.6. Let (X, X^*) be a strictly convex pair, φ be the corresponding nice convex function, and let H be a subspace of X. We shall say that H is a $\varphi-$summand subspace of X if there exists a subspace W such that $X = H \oplus W$ in such a way that

$$x = h + w \Rightarrow \varphi\|x\| = \varphi\|h\| + \varphi\|w\|.$$

Example 2.7. If X is a Hilbert space, then (X, X^*) is a strictly convex pair. This can be seen by letting $\varphi(t) = t^2$. In this case, $f_x(y) = < x, y >$. Let H be a nontrivial subspace of X, then H is a $\varphi-$summand of X, with $W = H^\perp$.

Example 2.8. If $1 < p < \infty$, a subspace H of ℓ^p is $p-$summand if, and only if, there exists $J \subset \mathbb{N}$ such that $H = \{(x_n) : x_n = 0,\ \forall n \notin J\}$. By $p-$summand, we mean $\varphi-$summand with $\varphi(t) = t^p$.
Similarly, a subspace H of $L^p[a,b]$ is $p-$summand if, and only if, there exists $E \subset [a,b]$ such that $0 < \mu(E) < 1$ and $H = \{f \in L^p[a,b] : f(t) = 0, \text{ a.e. on } E^c\}$.

Definition 2.9. Let (X, X^*) be a strictly convex pair. An exposed point $e \in E \subset X$ is called a $\varphi-$exposed point if the kernel of the linear functional that supports E uniquely at e is a $\varphi-$summand subspace.

Example 2.10. Let X be a Hilbert space, E be a closed bounded convex subset of X. Then every exposed point of E is a 2-exposed point, since every subspace of a Hilbert space is a 2-summand subspace.

Proposition 2.11. *Let $1 < p < \infty$ and let E be a closed bounded convex subset of ℓ^p. Then, an exposed point e of E is a $p-$exposed point if, and only if, there exists an index j such that $h_j = 0$ for all $h \in H$. Here $h = (h_i)$.*

3. Main Results

Let (X, X^*) be a strictly convex pair, and let φ be the associated nice convex function. Let E be a closed bounded convex subset of X, and e be a $\varphi-$exposed point of E, and H be the supporting hyperplane of E uniquely at e.
Let $x \in X \backslash H$, and denote the minimum distance from x to H by $d(x, H)$, then the ratio

$$R(x, e) = \frac{\varphi\|x - e\|}{d(x, H)}$$

will be called the remotality ratio of E at e with respect to x.

Lemma 3.1. *Let (X, X^*) be a strictly convex space, E be a closed bounded convex subset of E, and e be a $\varphi-$ exposed point of E. If a sphere $S(m, r)$ exists such that*

$$S(m, r) \cap E = \{e\} \text{ and } E \subset B(m, r),$$

and if H is the supporting hyperplane of $S(m, r)$ at e, then

$$\sup_{x \in E} R(x, e) \leq \sup_{x \in S(m,r)} R(x, e).$$

Proof. Without loss of generality, we may assume $e = 0$. Let $x \in E$, and θ be the closest element in $[m] := \{\alpha m : \alpha \in \mathbb{R}\}$ from x. Let x' be the intersection of the array $[\theta, x, -]$ and $S(m, r)$. Clearly, θ is the closest element in $[m]$ from x'. Now, let H be the supporting hyperplane of $S(m, r)$ at $e := 0$. We assert that $\|x\| \leq \|x'\|$.
Since $[m]$ and H are $\varphi-$ summands in X, and φ is strictly convex, then both are proximinal, and if $x = y_1 + z_1$ then $y_1 \in P(x, [m])$ and $z_1 \in P(x, H)$. Similarly, if $x' = y_2 + z_2$, then $y_2 \in P(x', [m])$ and $z_2 \in P(x', H)$. Hence, $y_1 = y_2 = \theta$. Consequently, $\|z_1\| = \|x - \theta\|$ and $\|z_2\| = \|x' - \theta\|$. But, by our choice of x', it can be easily seen that $\|x - \theta\| \leq \|x' - \theta\|$, and hence, $\|z_1\| \leq \|z_2\|$. This implies that $\varphi\|x\| \leq \varphi\|x'\|$. Since φ is increasing, we infer that $\|x\| \leq \|x'\|$.
Moreover, $d(x, H) = d(x', H)$ follows from the fact that $x' \in [\theta, x, -]$. Hence,

$$\|x\| \leq \|x'\| \Rightarrow \frac{\varphi\|x\|}{d(x, H)} \leq \frac{\varphi\|x'\|}{d(x', H)};\ x \in E,\ x' \in S(m, r).$$

Thus, we have shown that for every $x \in E$, there exists $x' \in S(m, r)$ such that $R(x, e) \leq R(x', e)$. This completes the proof of the lemma. □

Lemma 3.2. *Let X be a Banach space and $S(m, r)$ a sphere in X containing* 0. *If* 0 *is a $\varphi-$ exposed point of $S(m, r)$ and H is the hyperplane supporting $S(m, r)$*

uniquely at 0*, then*

$$\sup_{u\in S(m,r)} \frac{\varphi\|u\|}{d(u,H)} < \infty.$$

Proof. Observe first that if $u \in S(m,r)$, then $u = x + \epsilon m$ where $x \in H$, and $0 \le \epsilon \le 2$. Then, $\varphi\|u\| = \varphi\|x\| + \varphi(\epsilon r)$. Now,

$$u - m = x + \epsilon m - m = x + (\epsilon - 1)m \Rightarrow \varphi\|u - m\| = \varphi(r) = \varphi\|x\| + \varphi(|\epsilon - 1|r).$$

Now,

$$\begin{aligned} \frac{\varphi\|u\|}{d(u,H)} &= \frac{\varphi\|x\| + \varphi\|\epsilon m\|}{\|\epsilon m\|} \\ &= \frac{\varphi(r) - \varphi(|\epsilon - 1|r) + \varphi(\epsilon r)}{\epsilon r} := g(\epsilon). \end{aligned}$$

It is clear that the function $g(\epsilon)$ is continuous on $(0,2]$ and that $\lim_{\epsilon\to 0} g(\epsilon) = \varphi'(r)$. Consequently, g is a bounded function. This completes the proof. □

For the proof of the main theorem of this paper, we need the following Lemma. But first, recall from [3] that a nice exposed point of E is an exposed point, where the functional that determines the hyperplane supporting E at e attains its norm. It is worth to remark that exposed points of convex sets in any reflexive space are nice exposed points.

Lemma 3.3. *Let e be a nice exposed point of the convex bounded subset E in a normed space X. If H is the hyperplane that supports E uniquely e, then there exists a sequence of spheres $S(m_k, r_k)$ which lie in the same side of H as E, and such that H is a supporting hyperplane of $S(m_k, r_k)$ for all $k \in \mathbb{N}$.*

Proof. Without loss of generality, assume that $e = 0$, and that $f(y) > 0$ for all $y \in E\backslash\{0\}$. Here $f \in X^*$ is the functional that determines H, and $\|f\| = 1$. If $a \in X$ is such that $f(a) > 0$ and $f(a) = \|f\|$, where such an a exists since f attains its norm, then the spheres $S(ka, kf(a))$ satisfies the required properties. □

Now, we prove the main theorem of the paper.

Theorem 3.4. *Let e be a $\varphi-$nice exposed point of the closed bounded convex subset E of the strictly convex space (X, X^*). Then e is a remotal point of E if, and only if,*

$$\sup_{x\in E} R(x,e) < \infty.$$

Proof. Suppose that e is a remotal point. We assert that $\sup_{x\in E} R(x,e) < \infty$. Again, assume $e = 0$. Being a remotal point, there exists a sphere $S(m,r)$ such that $E \cap S(m,r) = \{0\}$ and $E \subset B(m,r)$. Let H be the supporting hyperplane of $S(m,r)$ uniquely at 0. By Lemma 3.1, it is enough to prove that $\underset{u\in S(m,r)}{R}(u,0) <$ ∞. But this follows from lemma 3.2 $\sup\limits_{u\in S(m,r)} R(u,0) < \infty$.

Conversely, suppose that the remotality ratio $R(x,e)$ is bounded for $x \in E$. To show that e is a remotal point. Suppose on the way of contrary that e is not remotal. Assuming $e = 0$, there exists a sequence of spheres $S(m_k, k)$, by virtue of

Lemma 3.3 such that $0 \in S(m_k, k)$ and $E \backslash B(m_k, k) \neq \phi$, for each $k \in \mathbb{N}$. Observe that all these spheres are still supported by the same hyperplane supporting E at 0. Let $u_k \in E \backslash B(m_k, k)$, hence $\|u_k - m_k\| \geq k$, $\forall k \in \mathbb{N}$. But then, following the same ideas in the beginning of the Lemma 3.2, we find that

$$R(u_k, 0) \geq \frac{\varphi(k) - \varphi(|1-\epsilon_k|k) + \varphi(\epsilon_k k)}{\epsilon_k k}.$$

Here we have two cases:
Case 1: If $0 < \epsilon_k \leq 1$, then

$$\begin{aligned} R(u_k, 0) &\geq \frac{\varphi(k) - \varphi(|1-\epsilon_k|k) + \varphi(\epsilon_k k)}{\epsilon_k k} \\ &= \frac{\varphi(k) - \varphi(k - \epsilon_k k) + \varphi(\epsilon_k k)}{\epsilon_k k} \\ &= \varphi'(c_{\epsilon_k, k}) + \frac{\varphi(\epsilon_k k)}{\epsilon_k k}, \end{aligned}$$

where $k - \epsilon_k k < c_{\epsilon_k, k} < k$, by the mean value theorem.

Case 2: If $1 < \epsilon_k \leq 2$, then

$$\begin{aligned} R(u_k, 0) &\geq \frac{\varphi(k) - \varphi(|1-\epsilon_k|k) + \varphi(\epsilon_k k)}{\epsilon_k k} \\ &= \frac{\varphi(k) - \varphi(\epsilon_k k - k) + \varphi(\epsilon_k k)}{\epsilon_k k} \\ &\geq \frac{\varphi(\epsilon_k k)}{\epsilon_k k}, \end{aligned}$$

where the last inequality is a consequence of the fact that φ is increasing.
Now, since we have infinitely many values of k, we also have infinitely many values of ϵ_k. Consequently, we either have infinitely many values of ϵ_k which are less than or equal to 1, or infinitely many values of ϵ_k which are greater than 1. Let us treat these two cases:

Case I: If there are infinitely many values of ϵ_k which are greater than 1, then there is a corresponding subsequence of the radii, say (k_n), in which $k_n \to \infty$. But then, $R(u_{k_n}, 0)$ is unbounded because $\epsilon_{k_n} k \to \infty$ and

$$R(u_{k_n}, 0) \geq \frac{\varphi(\epsilon_{k_n} k_n)}{\epsilon_{k_n} k_n} \to \infty,$$

where we have used the assumption that

$$\lim_{x \to \infty} \frac{\varphi(x)}{x} = \infty.$$

Case II: If there are infinitely many values of ϵ_k which are less than or equal to 1, then there is a corresponding sequence (k_n) such that $k_n \to \infty$ and

$$R(u_{k_n}, 0) \geq \varphi'(c_{\epsilon_{k_n}, k_n}) + \frac{\varphi(\epsilon_{k_n} k_n)}{\epsilon_{k_n} k_n}, \quad k_n - \epsilon_{k_n} k_n < c_{\epsilon_{k_n}, k_n} < k_n.$$

Now two subcases of this case are available:
Case II-i If the sequence $(\epsilon_{k_n} k_n)$ is bounded. Then $c_{\epsilon_{k_n},k_n} \to \infty$, and hence $R(u_{k_n}, 0) \to \infty$ where we have used the assumption that $\lim_{x\to\infty} \varphi'(x) = \infty$.
Case II-ii If the sequence $(\epsilon_{k_n} k_n)$ is unbounded, then $R(u_{k_n}, 0) \to \infty$ where we have used the fact that

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty.$$

Thus, we have shown that if 0 is not a remotal point of E then the ration $R(u, 0)$ is unbounded, contradicting our assumption. This shows that 0 is a remotal point, and completes the proof. □

4. Miscellaneous Remarks

In this section we a remark and an example in inner product spaces.

Proposition 4.1. *Let H be an inner product space, $S(m, r)$ a sphere in H and e be a $\varphi-$exposed point of $S(m, r)$. Then the ratio $R(u, e) = 2r$ for $u \in S(m, r)$.*

Proof. . Here $\varphi(t) = t^2$. Assuming $e = 0$, for simplicity and following the computations above, we see that

$$\begin{aligned} R(u,0) &= \frac{\varphi(r) - \varphi(|\epsilon - 1|r) + \varphi(\epsilon r)}{\epsilon r} \\ &= \frac{r^2 - \epsilon^2\, r^2 + 2\epsilon r^2 - r^2 + \epsilon^2\, r^2}{\epsilon\, r} \\ &= 2r. \end{aligned}$$

□

The following example was shown in [3] for the purpose of giving an example of an exposed point which is not a remotal point in an inner product space. In the following example, we show that the remotality ratio $R(x, e)$ is unbounded, explaining why e is not a remotal point of E.

Example 4.2. Let $X = \mathbb{R}^2$ endowed with the standard norm, and let

$$E_0 = \left\{ \left(\pm\frac{1}{n}, \frac{1}{n^3} \right) : n \in \mathbb{N} \right\}.$$

Let E be the closed convex hull of E_0, then clearly 0 is a 2-exposed point of E. It was shown that 0 is not a remotal point of E, [3].
Easy computations show that

$$R\left((\frac{1}{n}, \frac{1}{n^3}), (0,0) \right) = n + \frac{1}{n^3},$$

and hence

$$R\left((\frac{1}{n}, \frac{1}{n^3}), (0,0) \right) \to \infty.$$

We conclude our paper with the problem:
Problem Describe exposed points which are necessarily remotal points.

In this paper, we have answered the question for $\varphi-$nice exposed points in strictly convex spaces (X, X^*).

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The dual reciprocity boundary element method for two-dimensional Burgers' equations with inverse multiquadric approximation scheme

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Abstract

The two-dimensional Burgers' equation is a mathematical model to describe various kinds of phenomena such as turbulence and viscous fluid. In this paper, the dual reciprocity boundary element method (DRBEM) is used for solving this problem. In DRBEM, the fundamental solution of the Laplace equation is applied for the integral equation formulation and hence a domain integral arises in the boundary integral equation. Further, the time derivative is approximated by the forward divided difference of it, and the domain integral also appears from these approximations. The domain integral is transformed into boundary integral by using the dual reciprocity method (DRM). This method is applied on some test experiments and the numerical results have been compared with the exact solutions and the solutions in [1, 25]. Root-mean-square error (RMSE) of the solutions show the efficiency and the accuracy of the method.

Keywords: Nonlinear two-dimensional Burgers' equation; Dual reciprocity boundary element method; Radial basis function.

2010 Mathematics Subject Classification: 35K55; 65M99; 33E99.

1 Introduction

The nonlinear coupled Burgers' equation is a special form of incompressible Navier-Stokes equation without having pressure term and continuity equation. Burgers' equation is a fundamental partial differential equation (PDE) from fluid mechanics. It is used in various areas of applied mathematics and physics, such as modeling of gas dynamics and turbulence, heat conduction, and acoustic waves [2, 5, 15, 18].

The exact solution of the Burgers' equations can be obtained for simple geometry using the Hopf-Cole transformation [8, 11]. Using the Hopf-Cole transformation, the exact solution of the Burgers' equations was given by Fletcher [9]. The numerical solutions were obtained by Jain and Hola [12] using two algorithms based on cubic spline function

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technique, Fletcher [10] who discussed the comparison of a number of different numerical approaches, Wubs and Goede [23] using an explicit-implicit method, Bahadir [1] using a fully implicit finite-difference scheme, Zhu et al. [25] using the discrete Adomian decomposition method and Young et al. [24] using the Eulerian-Lagrangian method.

Boundary element method (BEM) is attractive and important computational techniques for solving problems in applied sciences and engineering. The main idea in this method is to convert the original PDE to an equivalent boundary integral equation by using Green's theorem and a fundamental solution. Consequently the main advantage in this method over the classical domain methods such as finite element method (FEM) and finite difference method (FDM), is that only boundary discretization is required due to dimension reduction [6]. But there are some difficulties in extending the method to applications such as nonhomogeneous, nonlinear and time dependent problems. The main drawback in these cases is the need to discretize the domain into a series of internal cells to deal with the terms taken to the boundary by application of the fundamental solution. This additional discretization destroys some of the attraction of the method. Several methods have been suggested for the resolution of these problems that in these methods, the DRM is the most efficient method. This method was introduced by Brebbia and Nardini [4] and Partridge and Brebbia [16]. The main idea behind this approach is to expand the inhomogeneous, nonlinear and time dependent terms in terms of its values at the nodes which lie in domain and boundary. These terms are approximated by interpolation in terms of some well-known functions $\phi(r)$, called radial basis functions (RBFs), where r is the distance between a source point and the field point. These functions are a powerful tool for scattered data interpolation problem [17, 22].
By applying the DRM, the problem will be reduced to a boundary only formulation, thus we do not have any domain integration in the boundary integral equation. The DRBEM is used by Chino and Tosaka [7] for the one-dimensional time independent Burgers' equation. Kakuda and Tosaka [13] adopted the generalized BEM to treat the Burgers' equations.

The organization of this paper is as follows. In Section 2, we describe the DRBEM for the nonlinear two-dimensional Burgers' equations. The results of three numerical experiments are presented in Section 3 and are compared with the analytical solutions and the results in [1, 25]. Finally, a brief discussion and conclusion is presented in Section 4.

2 The dual reciprocity boundary element method

Consider the coupled two-dimensional Burgers' equations:

$$\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{R}(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= \frac{1}{R}(v_{xx} + v_{yy}), \end{aligned} \tag{1}$$

with the initial conditions:

$$\begin{aligned} u(x, y, 0) &= f_1(x, y), \qquad (x, y) \in \Omega, \\ v(x, y, 0) &= f_2(x, y), \qquad (x, y) \in \Omega, \end{aligned} \tag{2}$$

and the boundary conditions:

$$u(x, y, t) = g_1(x, y, t), \qquad (x, y) \in \Gamma,$$

$$v(x,y,t) = g_2(x,y,t), \qquad (x,y) \in \Gamma, \tag{3}$$

where $\Omega = \{(x,y)|a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$ and Γ is its boundary. $u(x,y,t)$ and $v(x,y,t)$ are the two unknown variables which can be regarded as the velocities in fluid-related problems. $f_1(x,y)$, $f_2(x,y)$, $g_1(x,y,t)$ and $g_2(x,y,t)$ are all known functions. R is the Reynolds number.

In order to implement the dual reciprocity method, we consider the time derivative and the nonlinear terms in Eqs. (1), with $b_1(x,y,t)$ and $b_2(x,y,t)$ in the following forms:

$$R(u_t + uu_x + vu_y) = b_1(x,y,t),$$

$$R(v_t + uv_x + vv_y) = b_2(x,y,t).$$

Thus, Eqs. (1) convert to the following system:

$$\begin{aligned} \nabla^2 u &= b_1(x,y,t), \\ \nabla^2 v &= b_2(x,y,t), \end{aligned} \tag{4}$$

where $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$. Now, we approximate $b_1(x,y,t)$ and $b_2(x,y,t)$ as a linear combination of interpolation functions for each of them. Therefore, we choose $N+L$ collocation points where N is the number of boundary points and L is the number of internal points. The collocation points are denoted by (x_i, y_i) for $i = 1, 2, \ldots, N+L$.
The approximation of b_1 and b_2 can be written over domain Ω in the following forms:

$$\begin{aligned} b_1(x,y,t) &= \sum_{i=1}^{N+L} \varphi_i(x,y)\alpha_i(t), \\ b_2(x,y,t) &= \sum_{i=1}^{N+L} \varphi_i(x,y)\beta_i(t), \end{aligned} \tag{5}$$

where the interpolation function, φ_i is a radial basis function (RBF). In this work, we use the inverse multiquadric (IMQ) approximation scheme

$$\varphi_i(x,y) = ({r_i}^2 + \varepsilon^2)^{-2},$$

where $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}$ and ε is a shape parameter. Toutip [21] used a linear function $\varphi_i(r) = 1 + r_i$ in the DRBEM.
Now, if the function ψ_i be the particular solution of Laplace's equation

$$\nabla^2 \psi_i = \varphi_i,$$

then, Eqs. (5) convert to the following expressions

$$\begin{aligned} b_1(x,y,t) &= \sum_{i=1}^{N+L} \nabla^2 \psi_i(x,y)\alpha_i(t), \\ b_2(x,y,t) &= \sum_{i=1}^{N+L} \nabla^2 \psi_i(x,y)\beta_i(t). \end{aligned} \tag{6}$$

For IMQ-RBF, the function ψ_i is given as follows:

$$\psi_i(x, y) = \frac{1}{2\varepsilon^2}\ln(r_i).$$

The above function is a combination of logarithmic RBF and multiquadic (MQ) RBF. Initially, this combination of RBFs used by Mazarei and Aminataei [14] for the solution of Possions' equation.
Substituting Eqs. (6) into Eqs. (4), and writing the weight residual formulation of Eq. (4) with using the second Green's theorem [19], lead to:

$$\delta_k u_k + \int_\Gamma \frac{\partial u_k^*}{\partial n} u d\Gamma - \int_\Gamma u_k^* \frac{\partial u}{\partial n} d\Gamma = \sum_{i=1}^{N+L} [\delta_k \psi_{ki} + \int_\Gamma \frac{\partial u_k^*}{\partial n}\psi_i d\Gamma - \int_\Gamma u_k^* \frac{\partial \psi_i}{\partial n} d\Gamma]\alpha_i(t),$$

$$\delta_k v_k + \int_\Gamma \frac{\partial u_k^*}{\partial n} v d\Gamma - \int_\Gamma u_k^* \frac{\partial v}{\partial n} d\Gamma = \sum_{i=1}^{N+L} [\delta_k \psi_{ki} + \int_\Gamma \frac{\partial u_k^*}{\partial n}\psi_i d\Gamma - \int_\Gamma u_k^* \frac{\partial \psi_i}{\partial n} d\Gamma]\beta_i(t),$$

for $k = 1, 2, \ldots, N + L$, where $u_k^* = -\frac{1}{2\pi}\ln(r_k)$, $\delta_k = \frac{\theta_k}{2\pi}$; θ_k is the interior angle at the point k, and $\psi_{ki} = \psi_i(x_k, y_k)$. The term $\frac{\partial \psi_i}{\partial n}$ is the normal derivative of ψ_i and can be written as

$$\hat{q}_i = \frac{\partial \psi_i}{\partial n} = \frac{\partial \psi_i}{\partial x}\cdot\frac{\partial x}{\partial n} + \frac{\partial \psi_i}{\partial y}\cdot\frac{\partial y}{\partial n}.$$

At this step, the boundary Γ is discretized into N elements, thus we rewrite the above equations in the following expressions

$$\begin{aligned} \delta_k u_k + \sum_{i=1}^{N} H_{ki} u_i - \sum_{i=1}^{N} G_{ki} {q_1}_i &= \sum_{i=1}^{N+L} S_{ki}\alpha_i(t), \\ \delta_k v_k + \sum_{i=1}^{N} H_{ki} v_i - \sum_{i=1}^{N} G_{ki} {q_2}_i &= \sum_{i=1}^{N+L} S_{ki}\beta_i(t), \end{aligned} \tag{7}$$

for $k = 1, 2, \ldots, N + L$, where ${q_1}_i = \frac{\partial u}{\partial n}(x_i, y_i, t)$, ${q_2}_i = \frac{\partial v}{\partial n}(x_i, y_i, t)$,

$$S_{ki} = \delta_k \psi_{ki} + \sum_{i=1}^{N} H_{ki}\psi_i - \sum_{i=1}^{N} G_{ki}\hat{q}_i,$$

and the definition of the terms of H_{ki} and G_{ki} are defined as in [21].

From Eqs. (5), we obtain

$$\begin{aligned} \alpha_i(t) &= \sum_{j=1}^{N+L} F_{ij} b_1(x_j, y_j, t) = \sum_{j=1}^{N+L} F_{ij} {b_1}_j(t), \\ \beta_i(t) &= \sum_{j=1}^{N+L} F_{ij} b_2(x_j, y_j, t) = \sum_{j=1}^{N+L} F_{ij} {b_2}_j(t), \end{aligned} \tag{8}$$

where $F = \Phi^{-1}$, Φ is a $(N+L)\times(N+L)$ matrix that $\Phi(k,i) = \varphi_i(x_k, y_k)$. Substituting Eqs. (8) into the right hand side of Eqs. (7), lead to:

$$\begin{aligned}
\sum_{i=1}^{N+L} S_{ki}\alpha_i(t) &= \sum_{i=1}^{N+L} S_{ki}\sum_{j=1}^{N+L} F_{ij}b_{1j}(t) = \sum_{j=1}^{N+L} P_{kj}b_{1j}(t),\\
\sum_{i=1}^{N+L} S_{ki}\beta_i(t) &= \sum_{i=1}^{N+L} S_{ki}\sum_{j=1}^{N+L} F_{ij}b_{2j}(t) = \sum_{j=1}^{N+L} P_{kj}b_{2j}(t),
\end{aligned} \tag{9}$$

where

$$P_{kj} = \sum_{i=1}^{N+L} S_{ki}F_{ij}.$$

By combining Eqs. (7) and (9), we have

$$\begin{aligned}
\delta_k u_k(t) + \sum_{i=1}^{N} H_{ki}u_i(t) - \sum_{i=1}^{N} G_{ki}q_{1i}(t) &= \sum_{j=1}^{N+L} P_{kj}b_{1j}(t),\\
\delta_k v_k(t) + \sum_{i=1}^{N} H_{ki}v_i(t) - \sum_{i=1}^{N} G_{ki}q_{2i} &= \sum_{j=1}^{N+L} P_{kj}b_{2j}(t),
\end{aligned} \tag{10}$$

for $k = 1, 2, \ldots, N+L$. we note that

$$b_{1j}(t) = R(u_t(x_j, y_j, t) + u(x_j, y_j, t)u_x(x_j, y_j, t) + v(x_j, y_j, t)u_y(x_j, y_j, t)),$$

$$b_{2j}(t) = R(v_t(x_j, y_j, t) + u(x_j, y_j, t)v_x(x_j, y_j, t) + v(x_j, y_j, t)v_y(x_j, y_j, t)).$$

For the time derivatives, we use forward difference method to approximate the time derivatives $u_t(x_j, y_j, t)$ and $v_t(x_j, y_j, t)$. Thus, we obtain

$$u_t(x_j, y_j, t) = \frac{u_j^{n+1} - u_j^n}{\Delta t}, \quad v_t(x_j, y_j, t) = \frac{v_j^{n+1} - v_j^n}{\Delta t}, \tag{11}$$

where $u_j^n = u(x_j, y_j, n\triangle t)$ and $v_j^n = v(x_j, y_j, n\triangle t)$. Also, we approximate u_x, u_y, v_x and v_y as described in [21]. Therefore, we obtain

$$\begin{aligned}
u_x(x_j, y_j, t) &= \sum_{i=1}^{N+L} \hat{L}_i(x_j, y_j)u_i(t), \quad u_y(x_j, y_j, t) = \sum_{i=1}^{N+L} \check{L}_i(x_j, y_j)u_i(t),\\
v_x(x_j, y_j, t) &= \sum_{i=1}^{N+L} \hat{L}_i(x_j, y_j)v_i(t), \quad v_y(x_j, y_j, t) = \sum_{i=1}^{N+L} \check{L}_i(x_j, y_j)v_i(t),
\end{aligned}$$

where

$$\hat{L}_i(x, y) = \sum_{i=1}^{N+L} F_{ij}\frac{\partial \varphi_i}{\partial x}(x, y), \quad \check{L}_i(x, y) = \sum_{i=1}^{N+L} F_{ij}\frac{\partial \varphi_i}{\partial y}(x, y).$$

Substituting the above approximations in Eqs. (10), we obtain the following expressions:

$$\delta_k u_k^{n+1} + \sum_{i=1}^{N} H_{ki} u_i^{n+1} - \sum_{i=1}^{N} G_{ki} {q_1}_i^{n+1} = \sum_{j=1}^{N+L} P_{kj} [\lambda u_j^{n+1} - \lambda u_j^n + \tilde{u}_j \sum_{i=1}^{N+L} u_i^{n+1} \hat{L}_{ji} + \tilde{v}_j \sum_{i=1}^{N+L} u_i^{n+1} \check{L}_{ji}], \quad (12)$$

$$\delta_k v_k^{n+1} + \sum_{i=1}^{N} H_{ki} v_i^{n+1} - \sum_{i=1}^{N} G_{ki} {q_2}_i^{n+1} = \sum_{j=1}^{N+L} P_{kj} [\lambda v_j^{n+1} - \lambda v_j^n + \tilde{u}_j \sum_{i=1}^{N+L} v_i^{n+1} \hat{L}_{ji} + \tilde{v}_j \sum_{i=1}^{N+L} v_i^{n+1} \check{L}_{ji}], \quad (13)$$

for $k = 1, 2, \ldots, N+L$, where $\lambda = \frac{R}{\Delta t}$, $\hat{L}_{ji} = \hat{L}_i(x_j, y_j)$ and $\check{L}_{ji} = \hat{L}_i(x_j, y_j)$. $\tilde{u}_j$ and $\tilde{v}_j$ are given by the known approximations of $u_j(t)$ and $v_j(t)$, respectively, as described in the below. Using the boundary conditions (2), we have

$$u_j^n = g_1(x_j, y_j, n\Delta t), \quad v_j^n = g_2(x_j, y_j, n\Delta t), \quad j = 1, 2, \ldots, N,$$

in each time step.

At first time step, when $n = 0$, the initial conditions (2) give $u_j^0 = f_1(x_j, y_j)$ and $v_j^0 = f_2(x_j, y_j)$. In each time step, at first, we put $\tilde{u}_j = u_j^n$ and $\tilde{v}_j = v_j^n$. Having these, Eqs. (12) and (13) are solved as a system of linear algebraic equations for unknowns u_j^{n+1} and v_j^{n+1} for $j = N+1, \ldots, N+L$ and ${q_1}_j^{n+1}$ and ${q_2}_j^{n+1}$ for $j = 1, \ldots, N$. Recompute $\tilde{u}_j = u_j^{n+1}$ and $\tilde{v}_j = v_j^{n+1}$, where u_j^{n+1} and v_j^{n+1} are obtained from solving Eqs. (12) and (13). We iterate between calculating $\tilde{u}_j$ and $\tilde{v}_j$ and solving the approximation values of the unknowns, until the solutions of u_j^{n+1} and v_j^{n+1} satisfy the condition of the iteration method in each time step. Here, we use the following criteria for stopping the iterations in each time step,

$$\max_{L \leqslant j \leqslant N+L} |u_j^{n+1,l} - u_j^{n+1,l-1}| \leqslant \zeta,$$

and

$$\max_{L \leqslant j \leqslant N+L} |v_j^{n+1,l} - v_j^{n+1,l-1}| \leqslant \zeta,$$

where ζ is a fixed number. Also, $u_j^{n+1,l}$ and $v_j^{n+1,l}$ are the values of the u_j^{n+1} and v_j^{n+1} at the $l-th$ iteration. When this condition is satisfied, we put

$$u_j^{n+1} = u_j^{n+1,l}, \quad v_j^{n+1} = v_j^{n+1,l},$$

and go ahead to the next time step. This iteration method is namely called as predictor-corrector method.

3 The numerical experiments

Three experiments are studied to investigate the robustness and the accuracy of the proposed method. We compare the numerical results of the two-dimensional Burgers' equations by using this scheme with the analytical solutions and solutions in [1]. The RMSE which is defined by

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(u_{num}(X_i) - u_{exa}(X_i))^2}{N}},$$

is used to measure the accuracy of our scheme wherein X_i is the collocation points.

We perform the computations associated with our experiments in Maple 16 on a PC with a CPU of 2.4 GHZ.

Experiment 1. In this experiment, we consider the two-dimensional Burgers' equations (1) with exact solutions

$$\begin{aligned} u(x,y,t) &= \frac{3}{4} - \frac{1}{4[1+\exp(-4x+4y-t)/(32\mu)]}, \\ v(x,y,t) &= \frac{3}{4} + \frac{1}{4[1+\exp(-4x+4y-t)/(32\mu)]}. \end{aligned} \tag{14}$$

Above solutions obtained using a Hopf-Cole transformation in [9]. The initial conditions are obtained from (14) at $t = 0$, and the boundary conditions in (3) can be obtained from the exact solutions. In this experiment, the Reynolds number $R = 80$, time step size $\Delta t = 10^{-4}$, shape parameter $\varepsilon = 1.5$ and $\zeta = 10^{-18}$ are used. The computational domain for this problem is $\Omega = \{(x,y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. The numerical computation were performed using 13 internal points and 12 boundary points. Tables 1 and 2 give the numerical and exact solutions of u and v at internal points at time levels $t = 0.01$, 0.1 and $t = 0.3$.

Table 1
Comparison of numerical solutions with the exact solutions of u at $t = 0.01$, 0.1 and $t = 0.3$ with $R = 80$ of experiment 1.

Points	$t = 0.01$		$t = 0.1$		$t = 0.3$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.1,0.1)	0.62359	0.62344	0.61058	0.60946	0.57821	0.58021
(0.5,0.1)	0.50424	0.50439	0.50252	0.50352	0.50278	0.50214
(0.9,0.1)	0.50055	0.50008	0.50442	0.50006	0.50873	0.50004
(0.3,0.3)	0.62391	0.62344	0.61352	0.60946	0.58623	0.58021
(0.7,0.3)	0.50411	0.50439	0.50183	0.50352	0.50334	0.50214
(0.1,0.5)	0.74527	0.74539	0.74356	0.74426	0.74417	0.74067
(0.5,0.5)	0.62403	0.62344	0.61488	0.60946	0.59220	0.58021
(0.9,0.5)	0.50390	0.50439	0.49917	0.50352	0.49488	0.50214
(0.3,0.7)	0.74518	0.74539	0.74275	0.74426	0.73881	0.74067
(0.7,0.7)	0.62394	0.62344	0.61418	0.60946	0.59092	0.58021
(0.1,0.9)	0.74996	0.74991	0.74975	0.74989	0.74624	0.74982
(0.5,0.9)	0.74511	0.74539	0.74284	0.74426	0.73990	0.74067
(0.9,0.9)	0.62381	0.62344	0.61310	0.60946	0.58856	0.58021

Table 2
Comparison of numerical solutions with the exact solutions of v at $t = 0.01$, 0.1 and $t = 0.3$ with $R = 80$ of experiment 1.

Points	$t = 0.01$		$t = 0.1$		$t = 0.3$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.1,0.1)	0.87658	0.87656	0.89148	0.89054	0.92685	0.91979
(0.5,0.1)	0.99572	0.99561	0.99726	0.99648	1.00091	0.99786
(0.9,0.1)	0.99947	0.99992	0.99588	0.99994	0.99318	0.99996
(0.3,0.3)	0.87607	0.87656	0.88668	0.89054	0.91727	0.91979
(0.7,0.3)	0.99589	0.99561	0.99800	0.99648	0.99674	0.99786
(0.1,0.5)	0.75470	0.75461	0.75619	0.75574	0.75505	0.75933
(0.5,0.5)	0.87596	0.87656	0.88480	0.89054	0.90690	0.91979
(0.9,0.5)	0.99614	0.99561	1.00115	0.99648	1.00420	0.99786
(0.3,0.7)	0.75482	0.75461	0.75712	0.75574	0.75977	0.75933
(0.7,0.7)	0.87609	0.87656	0.88633	0.89054	0.90973	0.91979
(0.1,0.9)	0.75001	0.75009	0.75009	0.75011	0.75474	0.75018
(0.5,0.9)	0.75497	0.75461	0.75788	0.75574	0.75917	0.75933
(0.9,0.9)	0.87604	0.87656	0.88580	0.89054	0.90979	0.91979

Table 3
Comparison of absolute errors of $u(x, y, t)$ between the numerical solution using our method and the solution in $[1, 25]$ at $t = 0.01$ for $R = 100$ of experiment 1.

Points	Proposed method	Bahadir [1]	Zhu et al. [25]	Exact
(0.1,0.1)	1.76859E-4	7.24132E-5	5.91368E-5	0.62305
(0.5,0.1)	6.50996E-5	2.42869E-5	4.84030E-6	0.50162
(0.9,0.1)	5.75592E-4	8.39751E-6	3.41000E-8	0.50001
(0.3,0.3)	7.88296E-4	8.25331E-5	5.91368E-5	0.62305
(0.7,0.3)	3.92464E-4	8.25331E-5	4.84030E-6	0.50162
(0.1,0.5)	2.76094E-4	8.25331E-5	1.64290E-6	0.74827
(0.5,0.5)	9.79140E-4	7.32522E-5	5.91368E-5	0.62305

Table 4
Comparison of absolute errors of $v(x, y, t)$ between the numerical solution using our method and the solution in $[1, 25]$ at $t = 0.01$ for $R = 100$ of experiment 1.

Points	Proposed method	Bahadir [1]	Zhu et al. [25]	Exact
(0.1,0.1)	8.72333E-6	8.35601E-5	5.91368E-5	0.87695
(0.5,0.1)	2.10136E-5	5.13642E-5	4.84030E-6	0.99838
(0.9,0.1)	5.49827E-4	7.03298E-6	3.41000E-8	0.99999
(0.3,0.3)	8.10210E-4	6.15201E-5	5.91368E-5	0.87695
(0.7,0.3)	3.86695E-4	5.41000E-5	4.84030E-6	0.99838
(0.1,0.5)	2.40453E-4	7.35192E-5	1.64290E-6	0.75173
(0.5,0.5)	9.86737E-4	8.51040E-5	5.91368E-5	0.87695

We compare the absolute error of our scheme with the absolute errors of Bahadir method [1] and Zhu et al. method [25] in Tables 3 and 4. In $[1, 25]$, points are uniformly distributed and their number is 400 whereas in our scheme, points are scattered and their number is 25. Tables 5 and 6 show RMSEs of u and v at $t = 0.05$, 0.1 and $t = 0.2$ for different Reynolds numbers, respectively. We also plot the graphs of the numerical and exact solutions of u and v at internal points at time level $t = 0.05$ for $R = 100$ in Fig. 1.

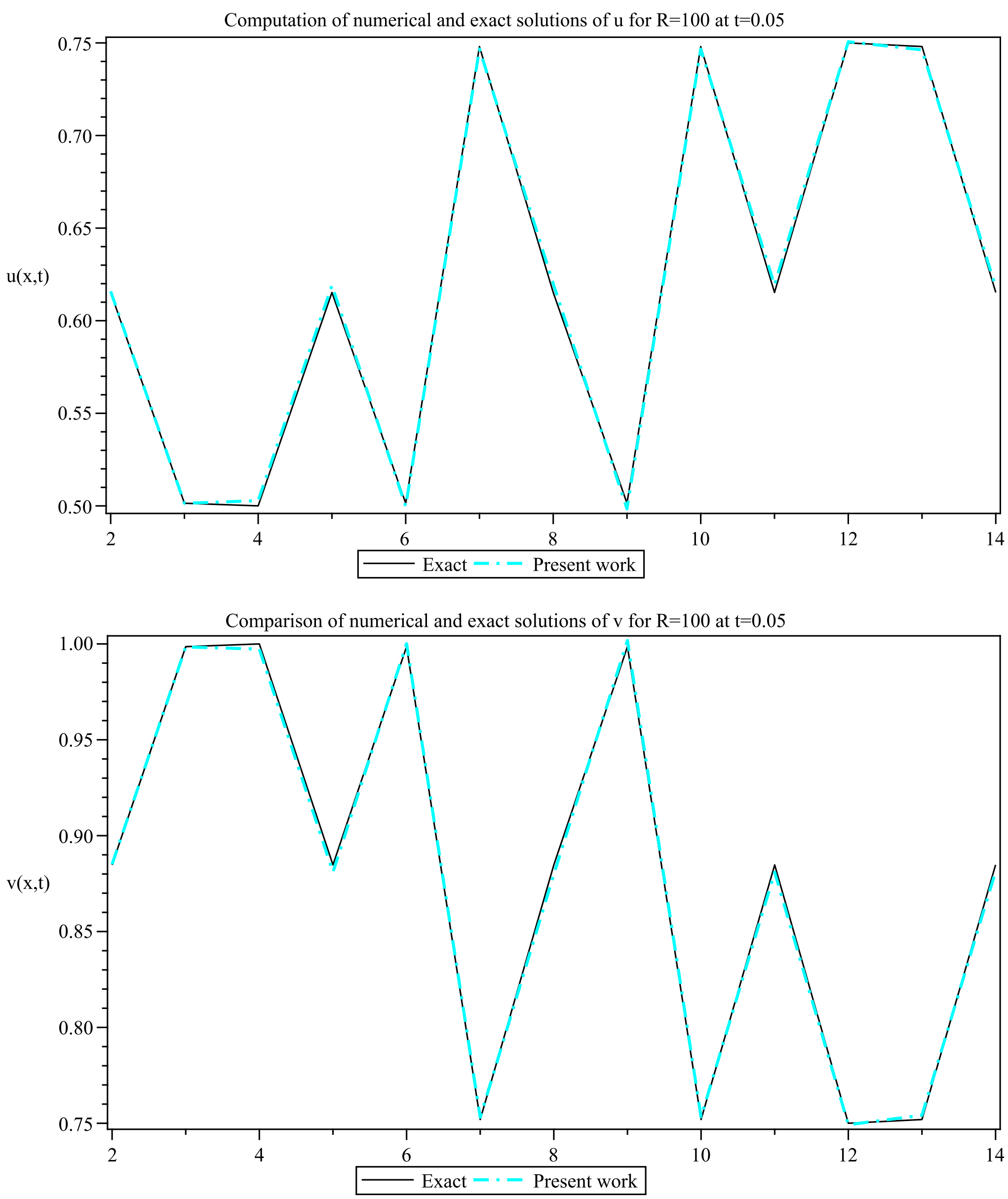

Figure 1:
Comparison of numerical and exact solutions of u and v for $R = 100$ at time level $t = 0.05$ of experiment 1.

Table 5
RMSE of u at different times for different Reynolds numbers of experiment 1.

Reynolds number	t=0.05	t=0.1	t=0.2
50	6.39710×10^{-4}	1.26354×10^{-3}	3.29974×10^{-3}
80	1.70407×10^{-3}	3.15403×10^{-3}	5.25154×10^{-3}
100	2.60059×10^{-3}	4.91042×10^{-3}	8.74585×10^{-3}

Table 6
RMSE of v at different times for different Reynolds numbers of experiment 1.

Reynolds number	t=0.05	t=0.1	t=0.2
50	1.35422×10^{-4}	1.25054×10^{-3}	3.29974×10^{-3}
80	1.76704×10^{-3}	3.24367×10^{-3}	5.34885×10^{-3}
100	2.66745×10^{-3}	5.02683×10^{-3}	8.98673×10^{-3}

Table 7
Comparison of numerical solutions with the exact solutions of u at $t = 0.01$, 0.2 and $t = 0.4$ of experiment 2.

Points	$t = 0.01$		$t = 0.2$		$t = 0.4$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.125,0.125)	0.24760	0.24755	0.21872	0.21739	0.22270	0.22059
(0.125,0.250)	0.37264	0.37257	0.35425	0.35326	0.39152	0.40441
(0.125,0.375)	0.49758	0.49760	0.48721	0.48913	0.55193	0.58824
(0.250,0.125)	0.37009	0.37007	0.29956	0.29891	0.25950	0.25735
(0.250,0.250)	0.49511	0.49510	0.43510	0.43478	0.43837	0.44118
(0.250,0.375)	0.62003	0.62012	0.56762	0.57065	0.59603	0.62500
(0.375,0.125)	0.49259	0.49260	0.37977	0.38043	0.28571	0.29412
(0.375,0.250)	0.61760	0.61762	0.51538	0.51630	0.47267	0.47794
(0.375,0.375)	0.74257	0.74265	0.64956	0.65217	0.64687	0.66176

Experiment 2. In this experiment, we consider the two-dimensional Burgers' equations (1) with the initial conditions (2) at $t = 0$ are given by

$$f_1(x, y) = x + y, \qquad f_2(x, y) = x - y.$$

The exact solutions are given by [3]

$$u(x, y, t) = \frac{x + y - 2xt}{1 - 2t^2}, \qquad v(x, y, t) = \frac{x - y - 2yt}{1 - 2t^2},$$

and the boundary functions $g_1(x, y, t)$ and $g_2(x, y, t)$ can be obtained from the exact solutions. In this experiment, we consider $\Delta t = 10^{-4}$, $\varepsilon = 1.5$, $\zeta = 10^{-18}$ and $\Omega = \{(x, y)|0 \leqslant x \leqslant 0.5, 0 \leqslant y \leqslant 0.5\}$. The numerical computations were performed using 25 points that distributed uniformly. The numerical solutions compared with the exact solutions at internal points at time levels $t = 0.01$, 0.2 and $t = 0.4$ for arbitrary Reynolds number R are listed in Tables 7 and 8.

Table 8
Comparison of numerical solutions with the exact solutions of v at $t = 0.01$, 0.2 and $t = 0.4$ of experiment 2.

Points	$t = 0.01$		$t = 0.2$		$t = 0.4$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.125,0.125)	-0.00248	-0.00250	-0.05405	-0.05435	-0.14888	-0.14706
(0.125,0.250)	-0.13000	-0.13003	-0.24421	-0.24457	-0.47729	-0.47794
(0.125,0.375)	-0.25756	-0.25755	-0.43538	-0.43478	-0.80380	-0.80882
(0.250,0.125)	0.12252	0.12252	0.08167	0.08152	0.03891	0.03677
(0.250,0.250)	-0.00500	-0.00500	-0.10862	-0.10870	-0.29433	-0.29412
(0.250,0.375)	-0.13257	-0.13253	-0.30054	-0.29891	-0.63517	-0.62500
(0.375,0.125)	0.24756	0.24755	0.21726	0.21739	0.21796	0.22059
(0.375,0.250)	0.12004	0.12002	0.02716	0.02717	-0.11364	-0.11029
(0.375,0.375)	-0.00752	-0.00750	-0.16435	-0.16304	-0.45597	-0.44118

Table 9
Comparison of numerical solutions with the exact solutions of u at $t = 1$, 1.5 and $t = 2$ with $R = 1000$ of experiment 3.

Points	$t = 1$		$t = 1.5$		$t = 2$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.25,0.25)	0.00205	0.00000	0.00272	0.00000	0.00322	0.00000
(0.25,0.50)	0.00244	0.00000	0.00320	0.00000	0.00376	0.00000
(0.25,0.75)	0.00366	0.00000	0.00481	0.00000	0.00564	0.00000
(0.50,0.25)	0.00658	0.00637	0.00647	0.00614	0.00637	0.00592
(0.50,0.50)	0.01110	0.01141	0.01060	0.01089	0.01020	0.01040
(0.50,0.75)	0.00961	0.00637	0.01056	0.00614	0.00113	0.00592
(0.75,0.25)	0.00033	0.00000	0.00045	0.00000	0.00055	0.00000
(0.75,0.50)	0.00015	0.00000	0.00023	0.00000	0.00031	0.00000
(0.75,0.75)	0.00274	0.00000	0.00381	0.00000	0.00476	0.00000

Experiment 3. In the following experiment, we consider the two-dimensional Burgers' equation with the initial conditions:

$$u(x, y, 0) = \frac{-4\pi\cos(2\pi x)\sin(\pi y)}{R(2 + \sin(2\pi x) + \sin(\pi y)},$$

$$v(x, y, 0) = \frac{-2\pi\sin(2\pi x)\cos(\pi y)}{R(2 + \sin(2\pi x) + \sin(\pi y)},$$

and the exact solutions are as follows [20]:

$$u(x, y, t) = \frac{-4\pi e^{\frac{-5\pi^2 t}{R}}\cos(2\pi x)\sin(\pi y)}{R(2 + e^{\frac{-5\pi^2 t}{R}}\sin(2\pi x) + \sin(\pi y)},$$

$$v(x, y, t) = \frac{-2\pi e^{\frac{-5\pi^2 t}{R}}\sin(2\pi x)\cos(\pi y)}{R(2 + e^{\frac{-5\pi^2 t}{R}}\sin(2\pi x) + \sin(\pi y)}.$$

The boundary conditions are taken from the exact solutions and the computational domain is $\Omega = \{(x, y)|0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$. The numerical computations were performed using $\Delta t = 10^{-3}$, $\varepsilon = 1.5$, $\zeta = 10^{-18}, R = 1000$ and 25 points that distributed uniformly. Tables 9 and 10 show the numerical solutions and the exact solutions of u and v at time levels $t = 1$, 1.5 and $t = 2$.

Table 10
Comparison of numerical solutions with the exact solutions of v at $t = 1, 1.5$ and $t = 2$ with $R = 1000$ of experiment 3.

Points	$t = 1$		$t = 1.5$		$t = 2$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
(0.25,0.25)	-0.00208	-0.00211	-0.00202	-0.00206	-0.00197	-0.00201
(0.25,0.50)	-0.00007	0.00000	-0.00114	0.00000	-0.00017	0.00000
(0.25,0.75)	0.00196	0.00000	0.00182	0.00000	0.00167	0.00201
(0.50,0.25)	-0.00008	0.00000	-0.00012	0.00000	-0.00015	0.00000
(0.50,0.50)	0.00001	0.00000	0.00002	0.00000	0.00003	0.00000
(0.50,0.75)	0.00008	0.00000	0.00012	0.00000	0.00015	0.00000
(0.75,0.25)	0.00212	0.00211	0.00207	0.00206	0.00202	0.00201
(0.75,0.50)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
(0.75,0.75)	-0.00214	-0.00211	-0.00211	-0.00206	-0.00208	-0.00201

4 conclusions

In this paper, we apply DRBEM with IMQ-RBF for solving the nonlinear two-dimensional Burgers' equations. The numerical results which are given in the previous section show that the proposed method is a reliable tool for Burgers' equations. We may improve the solutions of such problems by linearization and using optimization value of shape parameter. The results have very close relation to the shape parameter ε. The choice of the shape parameter is still a pendent question. Advantage of the presented scheme is that we could use the scattered points for interpolation of nonhomogeneous, nonlinear and time dependent terms in DRM. Therewith, we would like to emphasize that, the scheme introduced in this paper can be studied for any other nonlinear PDEs.

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ON ASYMPTOTICALLY ALMOST AUTOMORPHIC C-SEMIGROUPS

G. M. N'GUÉRÉKATA

Abstract. We introduce the concepts of complete trajectory, rest point and translation invariant set in the context of C-semigroups and prove that the principal part of an asymptotically almost automorphic C-semigroup is a complete trajectory and describe some of their properties.

1. Introduction

It is well-known that the concepts of C_0-semigroups and abstract dynamical systems are equivalent (see for instance [7] Theorem 2.7.2). We studied for the first time (topological and dynamical) properties of asymptotically almost automorphic C_0-semigroups in [7] Section 2.7. In this paper, we prove that some of the properties can be extended to C-semigroups, a generalization of C_0-semigroups introduced by Da Prato ([2]). C-semigroups have the advantage to be applied to many differential and integral equations that may be written as abstract Cauchy problems on a Banach space when C_0-semigroups cannot be used directly. For instance backward heat equations, Shrödinger equations on L^p, with $p \neq 2$, the Laplace equation, etc...See for instance [4, 9] and references therein for recent developments.

In this paper, $\mathbb{X}$ will denote a Banach space with norm $\|\cdot\|$. For a given linear operator $A : \mathbb{X} \to \mathbb{X}$, $D(A), R(A)$ will represent respectively the domain and the range of A. $C_0(\mathbb{R}^+, \mathbb{X})$ will denote the space of all continuous functions $f : \mathbb{R}^+ \to \mathbb{X}$ such that $\lim_{t\to\infty} \|f(t)\| = 0$.

2. Asymptotically Almost automorphic functions

Definition 2.1. (S. Bochner)

Let $f : \mathbb{R} \mapsto \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that:

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

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is well-defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{X})$ the set of all such functions.

Remark 2.2. Clearly when the convergence above is uniform in $t \in \mathbb{R}$, f is almost periodic. Thus the class of almost automorphic functions is larger than the one of almost periodic functions.

Remark 2.3. The function g is measurable, but not continuous in general. As one can see with the example below, almost automorphic functions may not be uniformly continuous. But if the function g in the above definition is continuous, then f is uniformly continuous ([8].)

Example The function $\psi(t) := sin(\frac{1}{2+cost+cos\sqrt{2}t})$ is almost automorphic. But since it is not uniformly continuous, it is not almost periodic.

Denote by $AA(\mathbb{X})$, the set of all almost automorphic functions $f : \mathbb{R} \to \mathbb{X}$. With the sup norm $\sup_{t\in\mathbb{R}}\|f(t)\|$, this space turns out to be a Banach space.

Definition 2.4. A bounded continuous function $f : \mathbb{R}^+ \to \mathbb{X}$ is said to be asymptotically almost automorphic, if there exists $g \in AA(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$ such that $f(t) = g(t) + h(t)$ for every $t \geq 0$.

Denote by $AAA(\mathbb{X})$ the linear space of all functions $f : \mathbb{R}^+ \to \mathbb{X}$ which are asymptotically almost automorphic. Then it turns out to be a Banach space when equipped with the norm

$$|f| = \sup_{t\in\mathbb{R}} \|g(t)\| + \sup_{t\geq 0} \|h(t)\|.$$

Moreover $AAA(\mathbb{X}) = AA(\mathbb{X}) \oplus C_0(\mathbb{R}^+; \mathbb{X})$.

Remark 2.5. Note that $AAA(\mathbb{X})$ can also be equipped with the equivalent norm $\|f\| := \sup_{t\in\mathbb{R}^+} \|f(t)\|$; (cf. Lemma 1.8 [3]). Moreover the range of any asymptotically almost automorphic function is relatively compact (cf. Lemma 1.9 [3]).

Remark 2.6. If $f \in AAA(\mathbb{X})$ with $f = g + h$ then $\overline{\{g(t) : t \in \mathbb{R}\}} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$ (Lemma 1.7 [3]).

$$|f| = \sup_{t\in\mathbb{R}} \|g(t)\| + \sup_{t\geq 0} \|h(t)\|.$$

Moreover $AAA(\mathbb{X}) = AA(\mathbb{X}) \oplus C_0(\mathbb{R}^+; \mathbb{X})$.

Remark 2.7. Note that $AAA(\mathbb{X})$ can also be equipped with the equivalent norm $\|f\| := \sup_{t\in\mathbb{R}^+} \|f(t)\|$; (cf. Lemma 1.8 [3]). Moreover the range of any asymptotically almost automorphic function is relatively compact (cf. Lemma 1.9 [3]).

3. C-semigroups

Definition 3.1. Let $\mathcal{S}$ be a Banach space and C be an injective operator in $L(X)$. A family of bounded linear operators $\mathcal{S} := (S(t))_{t\geq 0}$ is called an exponentially bounded C-semigroup if the following are satisfied:

- (i) $S(0) = C$,
- (ii) $S(t+s)C = S(t)S(s)$; $\forall t, s \geq 0$,
- (iii) $S(\cdot)x : [0, \infty) \to X$ is continuous for any $x \in X$,
- (iv) There exists $M \geq 0$ and $\delta \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\delta t}$ for $t \geq 0$.

Remark 3.2. $C = I$, then $\mathcal{S}$ is a C_0-semigroup.

We define an operator A as follows:

$$D(A) := \{x \in X / \lim_{h\to 0^+} \frac{S(t)x - Cx}{h} \in R(C)\}$$
$$Ax := C^{-1} \lim_{h\to 0^+} \frac{S(t)x - Cx}{h}, \ \forall x \in D(A)\}.$$

This operator is called the *generator* of $\mathcal{S}$. It is well-known that A is closed, but not necessarily densely defined.

Lemma 3.3. *Let C be an injective linear operator and $\mathcal{S} := (S(t))_{t\geq 0}$ be a C-semigroup with generator A. Then the following properties hold:*

- *(i) $S(t)S(s) = S(s)S(t)$, for all $t, s, \geq 0$,*
- *(ii) If $x \in D(A)$, then $S(t)x \in D(A)$, $AS(t)x = S(t)Ax$, and*
- *(iii) $\int_0^t S(\xi)Axd\xi = S(t)x - Cx$, $\forall t \geq 0$,*
- *(iv) $\int_0^t S(\xi)xd\xi \in D(A)$ and $A\int_0^t S(\xi)xd\xi = S(t)x - Cx$, $\forall x \in X$, and $t \geq 0$,*
- *(v) A is closed and satisfies $C^{-1}AC = A$,*
- *(vi) $R(C) \subset \overline{D(D)}$.*

3.1. Complete trajectories.

In what follows we assume that $\mathbb{X} = D(C) = R(C)$. Let $\mathcal{S} := (S(t))_{t\geq 0}$ be a C-semigroup. Then C and C^{-1} will commute with $S(t)$ on $\mathbb{X}$.

Definition 3.4. Let $x \in \mathbb{X}$. The set

$$\gamma^+(x) := \{S(t)x/t \in \mathbb{R}^+\}$$

is called the trajectory (or orbit) of $S(t)x$.

Definition 3.5. A function $\varphi : \mathbb{R} \to \mathbb{X}$ is said to be a complete trajectory of $\mathcal{S}$ if $C\varphi(t) = S(t-a)\varphi(a)$ for all $a \in \mathbb{R}$ and all $t \geq a$.

Theorem 3.6. *If $S(t)x \in AAA(\mathbb{X})$ for some $x \in \mathbb{X}$, then the principal term of $S(t)x$ is a complete trajectory of $\mathcal{S}$.*

Proof. Let $S(t)x = f(t)+h(t)$, $t \in \mathbb{R}^+$ where $f \in AA(\mathbb{X})$ and $h \in C_0(\mathbb{R}^+, \mathbb{X})$. Then there exists $(n_k) \subset (n) = \mathbb{N}$ such that

$$g(t) := \lim_{k \to infty} f(t+n_k)$$

exists for each $t \in \mathbb{R}$ and

$$\lim_{k\to\infty} g(t-n_k) = f(t)$$

for each $t \in \mathbb{R}$.

Define $C\varphi(t) := S(t)x$; then $C\varphi(0) = S(0)x = Cx$. Therefore $\varphi(0) = x$. Let $y = C^{-1}x$. Fix $a \in \mathbb{R}$ and choose k large enough such that $a + n_k \geq 0$. If $s \geq 0$, we have

$$C\varphi(a+s+n_k) = S(a+s+n_k)x = S(a+s+n_k)Cy = S(s)S(a+n_k)y$$

$$= S(s)S(a+n_k)C^{-1}x = S(s)C^{-1}S(a+n_k)x = S(s)\varphi(a+n_k).$$

Therefore for $s \geq 0$ and $a+n_k \geq 0$, we get

$$f(a+s+n_k) + h((a+s+n_k) = S(a+s+n_k)x = S(s)\varphi(a+n_k).$$

Since

$$\lim_{k\to\infty} f(a+s+n_k) = g(a+s)$$

and

$$\lim_{k\to\infty} h(a+s+n_k) = 0,$$

then

$$\lim_{k\to\infty} \varphi(a+s+n_k) = \lim_{k\to\infty} C^{-1}S(s)\varphi(a+n_k) = C^{-1}g(a+s).$$

It is also clear that

$$\lim_{k\to\infty} \varphi(a+n_k) = C^{-1}g(a).$$

Therefore in view of the continuity of $S(s)$ we obtain

$$\lim_{k\to\infty} S(s)\varphi(a+n_k) = S(s)C^{-1}g(a).$$

It follows immediately that

$$S(s)C^{-1}g(a) = g(a+s), \forall a \in \mathbb{R}, \ \ \forall s \geq 0.$$

On the other hand, since

$$\lim_{k\to\infty} g(t-n_k) = f(t)$$

for each $t \in \mathbb{R}$ and

$$g(a+s-n_k) = S(s)C^{-1}g(a-n_k), \ \forall a \in \mathbb{R}, \ \forall s \geq 0,$$

it follows that

$$\lim_{k\to\infty} g(a+s-n_k) = S(s)C^{-1}f(a), \ \forall a \in \mathbb{R}, \ \forall s \geq 0.$$

Therefore

$$f(a+s) = S(s)C^{-1}f(a),\ \forall a \in \mathbb{R},\ \forall s \geq 0.$$

Finally let's put $s = t - a$ with $t \geq 0$. Then we obtain

$$Cf(t) = S(t-a)f(a),\ \forall a \in \mathbb{R},\ \forall s \geq 0,$$

which proves that f is a complete trajectory.

□

3.2. **ω-limit sets.**

Definition 3.7. Given $x \in X$ and f the principal term of $S(t)x$, the set

$$\omega^+(x) := \{y \in X / \exists 0 \leq t_n \to \infty,\ \lim_{n\to\infty} S(t_n)x = Cy\}$$

will be called the ω-limit set of $S(t)x$, and the set

$$\omega_f^+(x) := \{y \in X / \exists 0 \leq t_n \to \infty,\ \lim_{n\to\infty} f(t_n) = y\}$$

is the ω-limit set of f.

We now describe some topological properties of the above ω-limit sets.

Theorem 3.8. $\omega^+(x) \neq \emptyset$

Proof. Since $f \in AA(X)$, there exists $(n_k) \subset (n) = \mathbb{N}$ such that

$$\lim_{k\to\infty} f(n_k) = g(0).$$

But we have

$$\lim_{k\to\infty} S(n_k)x = \lim_{k\to\infty} f(n_k).$$

Therefore

$$\lim_{k\to\infty} S(n_k)x = g(0).$$

Now take $\xi = C^{-1}g(0)$. Then $\xi \in \omega^+(x)$. This completes the proof.

□

Theorem 3.9.

$$\omega^+(x) = \omega_f^+(x)$$

Proof. This follows immediately from the fact that

$$\lim_{t\to\infty} S(t)x = \lim_{t\to\infty} f(t).$$

□

Let's now recall this definition

Definition 3.10. A set $\mathcal{A} \subset X$ is said to be invariant under $\mathcal{S}$ if $S(t)y \in C\mathcal{A}$ for every $y \in \mathcal{A}$ and $t \in \mathbb{R}^+$.

We can prove the following

Theorem 3.11. *$\omega^+(x)$ is invariant under $\mathcal{S}$.*

Proof. Let $y \in \omega^+(x)$. Then there exists $0 \le t_n \to \infty$ such that $\lim_{n\to\infty} S(t_n)x = Cy$. Fix $t \in \mathbb{R}^+$ and consider $s_n := t + t_n$, $n = 1, 2, ...$ Obviously $\lim_{n\to\infty} s_n = \infty$. Since

$$S(s_n)Cx = S(t)S(t_n)x,\ n = 1, 2, ...,$$

in using continuity of $S(t)$, we get

$$\lim_{n\to\infty} S(s_n)Cx = \lim_{n\to\infty} S(t)S(t_k)x = S(t)Cy = CS(t)y.$$

which proves that

$$S(t)y \in C\omega^+(x).$$

The proof is complete. □

Theorem 3.12. *$\omega^+(x)$ is a closed subset of X.*

Proof. It suffices to prove that $\overline{\omega^+(x)} \subset \omega^+(x)$. Let $y \in \overline{\omega^+(x)}$. Then there exists a sequence $y_m \in \omega^+(x)$ such that $\lim_{m\to\infty} y_m = y$. Now for each y_m, there exists $0 \le t_{m,n} \to \infty$ such that

$$\lim_{n\to\infty} S(t_{m,n})x = Cy_m.$$

Now define recursively a sequence t_{k,n_k} as follows. Choose

$t_{1,n_1} > 1$ such that $\|Cy_1 - S(t_{1,n_1})x\| < \frac{1}{2}$,

$t_{2,n_2} > \max(2, t_{1,n_1})$ such that $\|Cy_2 - S(t_{2,n_2})x\| < \frac{1}{2^2}$,

$t_{k,n_k} > \max(k, t_{k-1,n_{k-1}})$ such that $\|Cy_k - S(t_{k,n_k})x\| < \frac{1}{2^k}$.

Let $s_k := t_{k,n_k}$, $k = 1, 2, ...$ It is clear that $s_k \ge 0$ and $\lim_{k\to\infty} s_k = \infty$.

Also we have

$$\|S(s_k)x - Cy\| \le \|S(s_k) - Cy_k\| + \|Cy_k - Cy\| < \frac{1}{2^k} + \|C\|_{L(X)}\|y_k - y\|.$$

Since $\lim_{k\to\infty} y_k = y$, then

$$\lim_{k\to\infty} S(s_k)x = Cy,$$

which proves that $y \in \omega^+(x)$. The proof is complete.

□

Theorem 3.13. *If $\gamma^+(x)$ is relatively compact, then $\omega^+(x)$ is compact.*

Proof. It is clear that

$$\omega^+(x) = \omega_f^+(x) \subset \overline{\gamma^+(x)}.$$

The conclusion follows since $\omega^+(x)$ is closed. □

Theorem 3.14. $\lim_{t\to\infty} \inf_{y\in\omega^+(x)} \|S(t)x - Cy\| = 0$.

Proof. Let $\nu(t) := \inf_{y\in\omega^+(x)} \|S(t)x - Cy\|$. We need to prove that $\lim_{t\to\infty} \nu(t) = 0$. Suppose that $\lim_{t\to\infty} \nu(t) \neq 0$. Then there exists $\epsilon > 0$ such that for every $n = 1, 2, ...$, there exists $t'_n \geq n$ such that $\nu(t'_n) \geq \epsilon$. In other words

$$\exists t'_n \geq n,\ \|S(t'_n)x - Cy\| \geq \epsilon,\ \ \forall y \in \omega^+(x),\ \forall n = 1, 2, ...$$

Since $\gamma_f(x)$ is relatively compact, there exists a subsequence $(t_n) \subset (t'_n)$ such that $(f(t_n))_n$ is convergent, say to $\overline{y}$.

Since $t_n \to \infty$ as $n \to \infty$, we get

$$\lim_{n\to\infty} S(t_n)x = \lim_{n\to\infty} f(t_n) = \overline{y}$$

.

Take $\xi = C^{-1}\overline{y}$. Then $\xi \in \omega^+(x)$, which is a contradiction. The theorem is proved.

□

Definition 3.15. A point $x \in X$ is called a rest point for $\mathcal{S}$ if $S(t)x = Cx$ for every $t \geq 0$.

Theorem 3.16. *If x is a rest point of $\mathcal{S}$, then $\omega^+(x) = \{x\}$.*

Proof. Since $S(t)x = Cx$ for all $t \geq 0$, then for all (t_n) with $0 \leq t_n \to \infty$, we get

$$\lim_{t\to\infty} S(t_n)x = Cx.$$

Thus $x \in \omega^+(x)$.

Conversely let $y \in \omega^+(x)$. There exists $0 \leq t_n \to \infty$ such that

$$\lim_{t\to\infty} S(t_n)x = Cy.$$

But $S(t_n)x = Cx$ for every $n = 1, 2, ...$ Therefore $Cy = Cx$, so $y = x$, which completes the proof. □

Remark 3.17. We recover some of the results in [7] Section 2.7 when $C = I$, that is in the context of strongly continuous semigroups..

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On Some Problems in Multivariate Interpolation

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Abstract

It is well known that a space of polynomials of degree $N-1$ interpolate at every N points on the real or complex line. In this article we ask how many spaces of dimension N are needed so that for every N points on the plane, at least one of these spaces admits unique interpolation. We also propose some "ideal" extensions of this problem and present what meager knowledge we have about possible answers to these questions. At the very least, we hope that the reader will find the questions interesting, challenging and contributes to their resolution.

1 Introduction

Throughout this article the letter $\Bbbk$ will stand for either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. An N-dimensional space F of functions from a topological space Z containing at least N elements into $\Bbbk$, is called *Haar* if any non-zero function $f \in F$ has at most $(N-1)$ zeroes. It is easily seen from linear algebra that being Haar is equivalent to any one of the following properties:

(i) For every choice of scalars $(a_1, \ldots, a_N)$ and any choice of distinct points $\mathcal{Z}_N := \{z_1, \ldots, z_N\} \subset Z$, there exists unique $f \in F$ such that

$$f(z_j) = a_1, \quad j = 1, \ldots, N.$$

(ii) For every choice of distinct points $\mathcal{Z}_N = \{z_1, \ldots, z_N\} \subset Z$ and for every function g on Z, there exists unique $f \in F$ such that

$$f(z_j) = g(z_j), \quad j = 1, \ldots, N.$$

(iii) For every choice of basis $(f_1, \ldots, f_N)$ for F and for any choice of distinct points $\mathcal{Z}_N := \{z_1, \ldots, z_N\} \subset Z$, the (Vandermonde) determinant

$$\det(f_k(z_j)) \neq 0.$$

The property of being Haar is of interest in approximation theory (cf. [12, 13]). The properties (i) and (ii) describe the unique solvability of the interpolation problem from the space F. Also the property of being Haar is equivalent to the following best approximation property:

(iv) Let F be a space of continuous functions on Z. Then F is Haar if and only if for every compact $K \subset Z$ and and every continuous $g \in C(K)$ there exists unique best approximation $f^* \in F$ to G; that is for every $g \in C(K)$ there exists unique $f^* \in F$ such that

$$\|g - f^*\|_{C(K)} = \inf\{\|g - f\|_{C(K)} : f \in F\}.$$

Here is, yet another, description of Haar property:

Definition 1.1. An ideal I in the algebra $C(Z)$ is called a radical ideal if $g^m \in I$ for some $m \in \mathbb{N}$ implies $g \in I$.

Now let $\mathcal{Z}_N := \{z_1, \dots, z_N\} \subset Z$ and let

$$I(\mathcal{Z}_n) := \{g \in C(Z) : g(z_j) = 0 \text{ for all } j = 0, \dots, N\}.$$

Then $I(\mathcal{Z}_N)$ is a radical ideal in the ring $C(Z)$,

$$\dim(C(Z)/I(\mathcal{Z}_N)) = N$$

and

(v) The Haar property is equivalent to the decomposition

$$\mathbb{k}[x] = C(Z) \oplus I(\mathcal{Z}_N).$$

for every set of distinct points $\mathcal{Z}_N = \{z_1, \dots, z_N\} \subset Z$.

In this article we will be interested in Haar spaces and its multidimensional analogues consisting of polynomials. Thus it pays to introduce symbols $\mathbb{k}[x]$, $\mathbb{k}[x, y]$ and $\mathbb{k}[x_1, \dots, x_d]$ to denote the algebra of polynomials in one, two and d variables with coefficients in the field $\mathbb{k}$.

A non-zero polynomial $p \in \mathbb{k}[x]$ of degree at most $(N-1)$ has at most $(N-1)$ zeroes. Hence the N-dimensional space $\mathcal{P}_{N-1} \subset \mathbb{k}[x]$ of such polynomials is Haar. In fact, over the complex field, the space $\mathcal{P}_{N-1}$ is the unique space in $\mathbb{C}[x]$ that has this property (cf. [17]). Furthermore

Theorem 1.2 ([17])**.** *The space $\mathcal{P}_{N-1}$ is the "universal ideal complement", that is $\mathcal{P}_{N-1}$ complements every ideal $I \subset \mathbb{k}[x]$:*

$$\mathbb{k}[x] = \mathcal{P}_{N-1} \oplus I \tag{1.1}$$

such that

$$\dim(\mathbb{k}[x]/I) = N \tag{1.2}$$

and it is a unique space in $\mathbb{k}[x]$ that has this property.

In terms of approximation theory this states that $\mathcal{P}_n$ is the unique "extended Tchebushev system" (cf. [12]) in $\mathbb{k}[x]$, i.e., it is the unique space where every Hermite interpolation problem is solvable.

So what happens in two or more variables?

2 Description of the problems

For $d, N > 1$, every N-dimensional subspace $F \subset \mathbb{C}[x_1, \ldots, x_d]$ contains a non-constant polynomial $f \in F$. The set of zeroes of f is infinite (cf. [6, p. 458, Proposition 2]); in particular there is a set $\mathcal{Z}_N := \{\mathbf{z}_1, \ldots, \mathbf{z}_N\} \subset \mathbb{C}^d$ of N distinct points such that f vanishes on $\mathcal{Z}_n$ and F is not Haar. The analogous result in the real case relies on an ingenious and extremely cute "Mairhuber argument (cf. [16])":

Let $F = \operatorname{span}[f_1, f_2, \ldots, f_N] \subset \Bbbk[x_1, \ldots, x_d]$. And let $\mathcal{Z}_N := \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$ be distinct points in $\mathbb{R}^d$ with $d \geq 2$. Position two points $\mathbf{z}_1, \mathbf{z}_2$ on diametrically opposite ends of the unit circle and points $\mathbf{z}_3, \ldots, \mathbf{z}_N$ outside the circle. If the space F is Haar, that implies that the determinant

$$D(\mathcal{Z}_N) = \det(f_k(\mathbf{z}_j)) \neq 0$$

for any $\mathcal{Z}_N$. As we rotate the diameter, the points $\mathbf{z}_1$ and $\mathbf{z}_2$ switch positions and hence $D(\mathcal{Z}_N)$ changes sign. By the intermediate value theorem, there exists a pair $\mathbf{z}_1, \mathbf{z}_2$ such that $D(\mathcal{Z}_N) = 0$; by (iii) F is not Haar. In particular for $d, N > 1$ and for every N-dimensional subspace $F \subset \mathbb{R}[x, y]$ there exists a (radical) ideal I with $\dim(\mathbb{R}[x, y]/I) = N$ such that

$$I \cap F \neq \{0\}.$$

This phenomenon is known as "the loss of Haar"; which brings us to the main topic of this article.

Problem 2.1. For a given $d, N \geq 1$ what is the least number $\nu_r(\Bbbk) = \nu_r^N(\Bbbk^d)$ of N-dimensional subspaces $F_1, \ldots, F_{\nu_r(\Bbbk)} \subset \Bbbk[x_1, \ldots, x_d]$ such that every radical ideal I of codimension N, (i.e., $\dim(\Bbbk[x_1, \ldots, x_d]/I) = N$) complements one of the subspaces $F_1, \ldots, F_{\nu_r(\Bbbk)}$? And what are those subspaces?

The subscript r in $\nu_r^{\Bbbk}(n)$ is short for radical ideals, since these are the type of ideals we are attempting to complement. The problem of this type is just as interesting and as open for other types of ideals:

Problem 2.2. For a given $d, N \geq 1$ what is the least number $\nu(\Bbbk) = \nu^N(\Bbbk^d)$ of N-dimensional subspaces $F_1, \ldots, F_{\nu(\Bbbk)} \subset \Bbbk[x_1, \ldots, x_d]$ such that every ideal I of codimension N complements one of the subspaces $F_1, \ldots, F_{\nu(\Bbbk)}$? And what are those subspaces?

Problem 2.3. For a given $d, N \geq 1$ what is the least number $\nu_p(\Bbbk) = \nu_p^N(\Bbbk^d)$ of N-dimensional subspaces $F_1, \ldots, F_{\nu_p(\Bbbk)} \subset \Bbbk[x_1, \ldots, x_d]$ such that every primary ideal I of codimension N complements one of the subspaces $F_1, \ldots, F_{\nu_p(\Bbbk)}$? And what are those subspaces?

(Recall that an ideal $I \subset \Bbbk[x, y]$ is primary if $pq \in I$ implies $p \in I$ or $q^m \in I$ for some integer m).

In addition to approximation theory (cf. [15]), these problems are closely related to important problems in combinatorics (Young tableaux (cf. [10]), algebraic geometry (subspace arraignments cf. [1, 2]) and topology of configuration spaces (cf. [3, 5, 11, 18, 19]).

For $N > 2$ all three of these problems are wide open and, for $d > 2$, we do not even know a reasonable conjecture for the numbers $\nu^{\mathbb{k}}$, $\nu_r^{\mathbb{k}}$ and $\nu_p^{\mathbb{k}}$.

As will be explained in the last section, a working conjecture for $d = 2$ is: $\nu_r^N\left(\mathbb{k}^2\right) = N$.

The fact that there exist finitely many N-dimensional subspaces $F_1, \ldots, F_m \subset \mathbb{k}\left[x_1, \ldots, x_d\right]$ that complement every ideal of codimension N was first proved in [9]. The introduction of Groebner bases provided a simple proof (cf. [7]).

Definition 2.4. A subspace $F \subset \mathbb{k}\left[x_1, \ldots, x_d\right]$ is called D-invariant if for every $f \in F$ all partial derivatives $\frac{\partial}{\partial x_i} f \in F$.

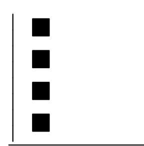
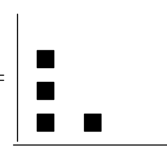
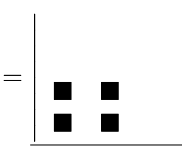
Theorem 2.5. *For every ideal I of codimension N there exists a D-invariant subspace $F \subset \mathbb{F}[\mathbf{x}]$ spanned by monomials, such that*

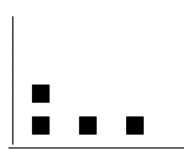
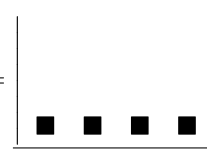
$$F \oplus I = \mathbb{k}\left[x_1, \ldots, x_d\right].$$

A moment of reflection on D-invariance and the monomial nature of this space, leads to the conclusion that every such space consist of polynomials of degree at most $N-1$ and, since there are only finitely many monomials of degree at most $N-1$, hence there are only finitely many such spaces.

Corollary 2.6. *There exist finitely many N-dimensional subspaces $F_1, \ldots, F_m$ of $\mathbb{k}\left[x_1, \ldots, x_d\right]$ that complement every ideal of codimension N.*

It is convenient to use the Young tableaux to visualize such subspaces. For instance for $d = 2$ and $N = 4$ the five subspaces in question are illustrated by tables (staircases):

$\Gamma_1 =$  $\Gamma_2 =$  $\Gamma_3 =$ 

$\Gamma_4 =$  $\Gamma_5 =$ 

These five tables represent all possible D-invariant subspaces of dimension 4 spanned by monomials. Thinking of the vertical axes as the number of mono-

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mials in y, we can write all five gammas as

$$\begin{aligned}
\Gamma_1 &= \left[1, y, y^2, y^3\right], \\
\Gamma_2 &= \left[1, y, y^2, x\right], \\
\Gamma_3 &= [1, y, x, xy], \\
\Gamma_4 &= \left[1, y, x, x^2\right], \\
\Gamma_5 &= \left[1, x, x^2, x^3\right].
\end{aligned}$$

Now the spaces $F_j := \operatorname{span} \Gamma_j$ represent the five subspaces.

Clearly no four of those subspaces can serve the same purpose, for an ideal generated by, say $\langle x^4, y \rangle$ does not complement the first four subspaces.

Observe that the space $F := \operatorname{span}\{1\}$ complement every ideal of codimension 1, hence provide a universal ideal complement for all maximal ideals (ideals of codimension 1).

In the next section we will prove that, for $N = 2$, $\nu^2\left(\mathbb{k}^d\right) = \nu_r^2\left(\mathbb{k}^d\right) = \nu_p^2\left(\mathbb{k}^d\right) = d$. The main result of this paper is the modest equality for $d = 2$, $N = 3$ presented in Section 4:

$$\nu^3\left(\mathbb{k}^2\right) = \nu_p^2\left(\mathbb{k}^2\right) = 3.$$

Unfortunately the proof is computational.

3 Interpolation at two points

Theorem 3.1. *For all $d \geq 1$ we have*

$$\nu^2\left(\mathbb{k}^d\right) = \nu_r^2\left(\mathbb{k}^d\right) = \nu_p^2\left(\mathbb{k}^d\right) = d.$$

Proof. For $i = 1, \ldots, d$ define spaces

$$F_i := \operatorname{span}\{1, x_i\} \subset \mathbb{k}[x_1, \ldots, x_d].$$

These are all D-invariant two-dimensional spaces spanned by monomials. Hence, by Theorem 2.5, every ideal of codimension 2 complements one of these spaces.

To prove that $\nu_r^2\left(\mathbb{C}^d\right) \geq d$ we start with $m < d$ spaces

$$F_1 := \operatorname{span}\{f_{1,1}, f_{1,2}\}, \ldots, F_m := \operatorname{span}\{f_{m,1}, f_{m,2}\} \tag{3.1}$$

and show the existence of two distinct points

$$\mathbf{z}_1 := (z_{1,1}, \ldots, z_{1,d}) \text{ and } \mathbf{z}_2 := (z_{2,1}, \ldots, z_{2,d})$$

in $\mathbb{C}^d$ such that the ideal

$$I_{\mathbf{z}_1,\mathbf{z}_2} := \{f \in \mathbb{C}[x_1, \ldots, x_d] : f(\mathbf{z}_1) = f(\mathbf{z}_2) = 0\}$$

has a non-trivial intersection $F_i \cap I_{\mathbf{z}_1,\mathbf{z}_2} \neq \{0\}$ for all $i = 1, \ldots, m$.

Suppose not. That is suppose that for any $\mathbf{z}_1 \neq \mathbf{z}_2$ in $\mathbb{C}^2$ the intersection $F_i \cap I_{\mathbf{z}_1,\mathbf{z}_2} = \{0\}$ for some i. This means that $m < d$ polynomials in $2d$ variables

$$\varphi_i\left(z_{1,1}, \ldots, z_{1,d}, z_{2,1}, \ldots, z_{2,d}\right) := \det \begin{pmatrix} f_{i,1}\left(\mathbf{z}_1\right) & f_{i,2}\left(\mathbf{z}_1\right) \\ f_{i,1}\left(\mathbf{z}_2\right) & f_{i,2}\left(\mathbf{z}_1\right) \end{pmatrix} \tag{3.2}$$

vanish simultaneously if and only if $\mathbf{z}_1 = \mathbf{z}_2$. Hence

$$\mathcal{Z}\left(\langle\varphi_1, \ldots, \varphi_m\rangle\right) = \mathcal{W} := \left\{\left(z_{1,1}, \ldots, z_{1,d}, z_{2,1}, \ldots, z_{2,d}\right) \in \mathbb{C}^{2d} : z_{1,i} = z_{2,i}\right\}$$

for all $i = 1, \ldots, d$. Therefore

$$\mathcal{W} := \{\left(z_{1,1}, \ldots, z_{1,d}, z_{1,1}, \ldots, z_{1,d}\right) \in \mathbb{C}^{2d} : \left(z_{1,1}, \ldots, z_{1,d}\right) \in \mathbb{C}^d$$

is a d-dimensional space while the variety $\mathcal{Z}\left(\langle\varphi_1, \ldots, \varphi_m\rangle\right)$ is defined as the zero locus of $m < d$ polynomials in $2d$ variables, hence (cf. [6, p. 463, Exercise 2])

$$\dim \mathcal{Z}\left(\langle\varphi_1, \ldots, \varphi_m\rangle\right) \geq 2d - m.$$

Thus $d \geq 2d - m$ which contradict the assumption $m < d$.

As is the case with the Mairhuber argument, in the real case the proof that $\nu_r^2\left(\mathbb{R}^d\right) \geq d$ is completely different and relies on a topological argument. Once again, let F_j, $j = 1, \ldots, m$ be $m < d$ subspaces of $\Bbbk\left[x_1, \ldots, x_d\right]$ with bases as in (3.1). Since the product

$$\prod_{j=1}^{m} f_{j,1} \tag{3.3}$$

is a nonzero polynomial, hence there exists a point $\mathbf{z}_0 \in \mathbb{R}^d$ such that $f_{j,1}\left(\mathbf{z}_0\right) \neq 0$ for all $j = 1, \ldots, m$ and thus there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^d$ of $\mathbf{z}_0$ such that the polynomial (3.3) does not vanish in $\mathcal{U}$. In particular the rational functions:

$$\psi_j := \frac{f_{j,2}}{f_{j,1}}, \quad j = 1, \ldots, m \tag{3.4}$$

are continuous on $\mathcal{U}$. Now we let $S^{d-1} \subset \mathcal{U}$ be a $(d-1)$-dimensional sphere centered at $\mathbf{z}_0$. Then the mapping $\Psi : S^{d-1} \to \mathbb{R}^m$ defined by

$$\Psi(\mathbf{z}) = \left(\psi_1(\mathbf{z}), \ldots, \psi_m(\mathbf{z})\right) \tag{3.5}$$

is a continuous mapping and since $m \leq d-1$, by the Borsuk's antipodal theorem, there exist two distinct points $\mathbf{z}_1, \mathbf{z}_2 \in S^{d-1}$ such that $\Psi\left(\mathbf{z}_1\right) = \Psi\left(\mathbf{z}_2\right)$. From (3.5) and (3.4) it follows that

$$f_{j,2}\left(\mathbf{z}_1\right) f_{j,1}\left(\mathbf{z}_2\right) - f_{j,1}\left(\mathbf{z}_1\right) f_{j,2}\left(\mathbf{z}_2\right) = 0.$$

Therefore all m determinants (3.2) vanish and none of the spaces F_j, $j = 1, \ldots, m$ complement the radical ideal

$$I_{\mathbf{z}_1,\mathbf{z}_2} := \{f \in \mathbb{R}\left[x_1, \ldots, x_d\right] : f\left(\mathbf{z}_1\right) = f\left(\mathbf{z}_2\right) = 0\}$$

of codimension 2.

It remains to prove that $\nu_p^2\left(\Bbbk^d\right) \geq d$. To this end, for every $i = 1, \ldots, m < d$ choose $f_i \in F_i$ such $f_i(\mathbf{0}) = 0$ and consider a system of linear equations

$$\sum_{k=1}^{d} a_k \left(\frac{\partial}{\partial x_k} f_i(\mathbf{0})\right) = 0, \quad i = 1, \ldots, m.$$

Since $m < d$ this system has a non-trivial solution $(a_1^*, \ldots, a_d^*)$. Now consider the ideal

$$I := \left\{f \in \Bbbk\left[x_1, \ldots, x_d\right] : f(\mathbf{0}) = 0, \sum_{k=1}^{d} a_k^* \frac{\partial}{\partial x_k} f(\mathbf{0}) = 0\right\}.$$

This is a primary ideal (cf. [8]) and from our choice of a_k^* it follows that $F_i \cap I \neq \{0\}$ for all $i = 1, \ldots, m$. $\square$

4 Main result

Theorem 4.1. *For $d = 2$, we have $\nu^3\left(\Bbbk^2\right) = \nu_p^2\left(\Bbbk^2\right) = 3$, i.e., for any two three-dimensional $F, G \subset \Bbbk[x, y]$ spaces there exists a primary ideal $I \subset \Bbbk[x, y]$ of codimension three such that*

$$I \cap F \neq \{0\} \text{ and } I \cap G \neq \{0\}. \tag{4.1}$$

Proof. It follows from Theorem 2.5 that one of the three three-dimensional spaces:

$$\operatorname{span}\left\{1, x, x^2\right\}, \quad \operatorname{span}\left\{1, y, y^2\right\}, \quad \operatorname{span}\{1, x, y\}$$

complement every ideal of codimension 3. Hence $\nu^{\Bbbk,2}(3) \leq 3$ and, in particular, $\nu_p^{\Bbbk,2}(3) \leq 3$. Next we will show that no two subspaces will do. Let $X := ax + by$ and $Y = cx + dy$ be two non-zero directions in $\Bbbk^2$ and let D_X and D_Y denote the partial derivatives in the directions X and Y respectively. It is not hard to see (cf. [8]) that the set of polynomials $p \in \Bbbk[x, y]$ that are annihilated by the following three functionals

$$\lambda_0(f) = f(0), \quad \lambda_1(p) = \left(D_X p\right)(0), \quad \lambda_2(p) = \left(r D_X^2 p + D_Y p\right)(0)$$

that depend on parameters $(a, b, c, d, r) \in \Bbbk^5$ is an ideal of codimension three and, in fact a primary ideal

$$I = I(a, b, c, d, r) := \{f \in \Bbbk[x, y] : \lambda_0(f) = \lambda_1(f) = \lambda_2(f) = 0\} \tag{4.2}$$

with the associated zero-locus $\mathcal{Z}(I) = \{0\}$. To prove the theorem we need to prove that for any two three-dimensional $F, G \subset \Bbbk[x, y]$ there exist $(a, b, c, d, r) \in \Bbbk^5$ and non-trivial polynomials $f \in F$ and $g \in G$ such that $\lambda_i(f) = \lambda_i(g) = 0$ for all $i = 0, 1, 2$. Since $\lambda_i(h) = 0$ for all monomials of degree greater than 2 we can assume without loss of generality, that the spaces F and G consist of polynomials of degree at most 2. (It is this assumption that will allow us to reduce the prove to a manageable computation). To further simplify the computations, we assume without loss of generality that

$$F = \operatorname{span}\{f_0, f_1, f_2\} \text{ and } G = \{g_0, g_1, g_2\}$$

with $f_k(0) = g_k(0) = 0$ for $i = 1, 2$. To prove (4.1) we have to guarantee the existence of non-trivial solutions to the linear equation

$$\lambda_k (A_{11} f_1 + A_{12} f_2) = 0, \quad \lambda_k (A_{21} g_1 + A_{22} g_2) = 0, \quad i = 1, 2$$

or, what amounts to the same thing, we need to prove the existence of non-trivial (a, b, c, d, r) such that

$$\det (\lambda_k (f_m)) = \det (\lambda_k (g_m)) = 0, \quad m, k = 1, 2. \tag{4.3}$$

To this end let

$$\begin{aligned} f_k &= u_{k,1} x + u_{k,2} y + u_{k,3} x^2 + u_{k,4}, xy + u_{k,5} y^2, \\ g_k &= v_{k,1} x + v_{k,2} y + v_{k,3} x^2 + v_{k,4}, xy + v_{k,5} y^2. \end{aligned}$$

An easy computation shows that

$$(\lambda_i (f_k)) = \begin{pmatrix} a u_{1,1} + b u_{1,2} & r \left(a^2 u_{1,3} + 2ab u_{1,4} + b^2 u_{1,5}\right) + c u_{1,1} + d u_{1,2} \\ a u_{2,1} + b u_{2,2} & r \left(a^2 u_{2,3} + 2ab u_{2,4} + b^2 u_{2,5}\right) + c u_{2,1} + d u_{2,2} \end{pmatrix}$$

and

$$(\lambda_i (g_k)) = \begin{pmatrix} a v_{1,1} + b v_{1,2} & r \left(a^2 v_{1,3} + 2ab v_{1,4} + b^2 v_{1,5}\right) + c v_{1,1} + d v_{1,2} \\ a v_{2,1} + b v_{2,2} & r \left(a^2 v_{2,3} + 2ab v_{2,4} + b^2 v_{2,5}\right) + c v_{2,1} + d v_{2,2} \end{pmatrix}.$$

Case 1. Set $r = 0$. Then the two determinants are

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} u_{1,1} & u_{2,1} \\ u_{2,1} & u_{2,2} \end{vmatrix}, \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} v_{1,1} & v_{2,1} \\ v_{2,1} & v_{2,2} \end{vmatrix}.$$

If the two determinants depending on the linear terms of f_i and g_i are both zero then we set $(a, b, c, d) = (1, 0, 0, 1)$ that solves the equations (4.3).

Case 2. Suppose that the linear terms in f_1 and f_2 are linearly independent. Then, after an easy algebraic manipulation, we can set

$$\begin{pmatrix} u_{1,1} & u_{2,1} \\ u_{2,1} & u_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and letting $r = 1$ the first determinant becomes

$$u_{2,3} a^3 + (2u_{2,4} - u_{1,3})\, a^2 b + (u_{2,5} - 2u_{1,4})\, ab^2 + ad + (-u_{1,5})\, b^3 - bc.$$

Now two minor computational miracles occur.

The first is choosing $b = 1$, and

$$c = au_{2,5} - 2au_{1,4} - a^2u_{1,3} + 2a^2u_{2,4} + a^3u_{2,3} + ad - u_{1,5}$$

not only verifies the first of the equations (3.3) but also changes the second equation into an equation of the form

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = 0 \tag{4.4}$$

where the coefficients depend on $u_{k,m}$ and $v_{k,k}$ but not d.

Choosing $a = 1$ and

$$d = bu_{1,3} - 2bu_{2,4} + 2b^2u_{1,4} - b^2u_{2,5} + b^3u_{1,5} + bc - u_{2,3}$$

to verify the first equation, the second equation becomes

$$\delta b^3 + \gamma b^2 + \beta b + \alpha = 0$$

with the same coefficients as (4.4) written in the reverse order. And this is the second miracle.

Subcase 1: $\alpha \neq 0$, $\delta \neq 0$. Then the cubic equation always has a non-zero solution in $\Bbbk$.

Subcase 2: $\delta = 0$, then the first equation is satisfied with $a = 0$, $b = 1$, $c = 1$, $d = 1$.

Subcase 3: $\alpha = 0$, then the second equation is satisfied with $a = 1$, $b = 0$, $c = 1$, $d = 1$.

□

For the record:

$$\begin{aligned}
\alpha &= (v_{1,1}v_{2,3} - v_{2,1}v_{1,3} - v_{1,1}u_{2,3}v_{2,2} + u_{2,3}v_{1,2}v_{2,1}) \\
\beta &= 2v_{1,1}v_{2,4} + v_{1,2}v_{2,3} - 2v_{2,1}v_{1,4} - v_{1,3}v_{2,2} \\
&\quad + u_{1,3}v_{1,1}v_{2,2} - u_{1,3}v_{1,2}v_{2,1} - 2v_{1,1}u_{2,4}v_{2,2} + 2v_{1,2}v_{2,1}u_{2,4} \\
\gamma &= v_{1,1}v_{2,5} + 2v_{1,2}v_{2,4} - v_{2,1}v_{1,5} - 2v_{2,2}v_{1,4} \\
&\quad + 2v_{1,1}u_{1,4}v_{2,2} - 2u_{1,4}v_{1,2}v_{2,1} - v_{1,1}v_{2,2}u_{2,5} + v_{1,2}v_{2,1}u_{2,5} \\
\delta &= (v_{1,2}v_{2,5} - v_{2,2}v_{1,5} + v_{1,1}u_{1,5}v_{2,2} - v_{1,2}v_{2,1}u_{1,5})
\end{aligned}$$

As a corollary, we obtain the following.

Corollary 4.2. *For all $N \geq 3$ the numbers $\nu^N\left(\Bbbk^2\right) \geq 3$.*

Proof. Let F_1 and F_2 be two N-dimensional subspaces of $\Bbbk[x, y]$ and let $N > 3$. Let $z_1, \ldots, z_{N-3} \in \Bbbk^2$ be distinct points different from 0. For $i = 1, 2$ let

$$F_i' := \{f \in F_i : f(z_j) = 0, \quad j = 1, \ldots, N-3\}.$$

Then F_1' and F_2' are two subspaces of dimension ≥ 3 and by the previous theorem there exists an ideal $I(a, b, c, d, r)$ defined by functionals (3.1) such that such that

$$F_i' \cap I(a, b, c, d, r) \neq \{0\}. \tag{4.5}$$

The ideal

$$J := I(a,b,c,d,r) \cap \{f \in \mathbb{k}[x,y] : f(z_j) = 0, \quad j = 1,\dots,N-3\}$$

is an ideal of codimension N and from (4.5) we conclude that $J \cap F_i \neq \{0\}$ for $i = 1, \dot{2}$. □

5 Additional remarks

As we mentioned in Section 2, Problems 2.1, 2.2 and 2.3 are closely related to some interesting questions in algebraic geometry and combinatorics. In this section we will outline this relationship assuming that a reader has but a brief exposure to the subject. Let $\mathbb{k}[\mathbf{x}] = \mathbb{k}[x_1,\dots,x_d]$ stands for polynomials in d variables with coefficients in $\mathbb{k}$. With every ideal $J \subset \mathbb{k}[x_1,\dots,x_d]$ we associate an affine variety

$$\mathcal{Z}(J) = \left\{\mathbf{z} = (z_1, z_2, \dots, z_d) \in \mathbb{k}^d : f(\mathbf{z}) = 0 \text{ for all } f \in J\right\}.$$

A set $\mathcal{W} \subset \mathbb{k}^d$ is an affine variety if there exists an ideal $J \subset \mathbb{k}[x_1,\dots,x_d]$ such that

$$\mathcal{W} = \mathcal{Z}(J).$$

An important characteristic of an affine variety $\mathcal{W}$ is an "arithmetic rank of $\mathcal{W}$" defined to be a minimal number of polynomials that generate an ideal J with $\mathcal{W} = \mathcal{Z}(J)$. Likewise an arithmetic rank of an ideal $K \subset \mathbb{k}[x_1,\dots,x_d]$ is the minimal number of polynomials that generate an ideal J with

$$\mathcal{Z}(K) = \mathcal{Z}(J).$$

There is a relationship between our interpolation problem and arithmetic rank. Consider a subset $\mathcal{U} \subset \mathbb{k}^N$ consisting of distinct N-tuple of points in $\mathbb{k}$. We claim that the complement to this set $\mathcal{W} := \mathcal{U}^c$ is an affine algebraic set in $\mathbb{k}^N$, i.e., a zero-locus of some polynomials. Indeed let

$$\mathcal{W}_{i,j} = \left\{(z_0,\dots,z_n) \in \mathbb{k}^N : z_i = z_j\right\}.$$

Then $\mathcal{W}_{i,j}$ is just a linear subspace of $\mathbb{k}^N$ and $\mathcal{W} = \cup_{i<j}\mathcal{W}_{i,j}$ is a union of affine varieties hence itself an affine variety (cf. [6], p.). The space $\mathcal{P}_{N-1} \subset \mathbb{k}[x]$ having a Haar propetry means that, for every sequence of distinct points $\mathcal{Z}_N := \{z_1,\dots,z_N\} \subset \mathbb{k}$, the Vandermonde determinant

$$h(z_0,\dots,z_n) := \det\left(z_j^k\right)$$

is equal to zero if and only if $z_i = z_j$ for some $i \neq j$. Observe that V is a polynomial in N variables and the zero set (affine variety) of this polynomial:

$$\mathcal{Z}(h) := \left\{(z_1,\dots,z_N) \in \mathbb{k}^N : h(z_1,\dots,z_N) = 0\right\} = \mathcal{W}.$$

Hence the arithmetic rank of $\mathcal{W}$ is 1.

The two variables analogue lead to consider the set $\mathcal{U}$ of distinct points

$$(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n) \in \left(\Bbbk^2\right)^N.$$

Letting $\mathbf{z}_j := (z_{1,j}, z_{2,j})$ we see that the complement $\mathcal{W}$ of $\mathcal{U}$ in $\Bbbk^{2N}$ is again an affine variety $\mathcal{W} = \cup_{i<j}\mathcal{W}_{i,j}$ where $\mathcal{W}_{i,j}$ are codimension 2 subspaces of $\Bbbk^{2n}$ defined by linear equations $\mathbf{z}_i = \mathbf{z}_j$, i.e.,

$$\mathcal{W}_{i,j} = \{((z_{1,1}, z_{2,1}), \ldots, (z_{1,n}, z_{2,n})) \in \Bbbk^{2N} : z_{1,i} - z_{1,j} = z_{2,i} - z_{2,j} = 0\}.$$

The variety $\mathcal{W}$ is an important variety called subspace arrangement and is an object of intense study in algebraic geometry and combinatorics ([10, 1, 2]).

What is an arithmetic rank of $\mathcal{W}$? Let, as in Problem 2.1, $d = 2$ and $\nu_r^N\left(\mathbb{C}^2\right)$ be the minimal number of N-dimensional subspaces such that every radical ideal I of codimension N complements one of these subspaces and let $F_1, \ldots, F_{\nu_r^N(\mathbb{C}^2)}$ be just such subspaces. Let $(f_{1,k}, \ldots, f_{n,k})$ be a basis in F_k. Then the determinants

$$h_k\left(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\right) := \det\left(\left(f_{j,k}\left(\mathbf{z}_m\right)\right)_{j,m=1}^n\right), \quad k = 1, \ldots, i \tag{5.1}$$

form a set of $\nu_r^N\left(\mathbb{C}^2\right)$ polynomials in $2n$ variables that do not simultaneously vanish on $\mathcal{U}$, hence vanish simultaneously if (and only if) $(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n) \in \mathcal{W}$. In other words the polynomials $\left\{h_k, k = k = 1, \ldots, \nu_r^N\left(\mathbb{C}^2\right)\right\}$ generate an ideal with the variety $\mathcal{W}$ and hence the arithmetic rank of $\mathcal{W}$ is $\leq \nu_r^N\left(\mathbb{C}^2\right)$.

Without going into further details of commutative algebra (it will take us too far off the track of this article) let us just mention that it follows from the work of Burch [4] and Haiman [10] that the arithmetic rank of $\mathcal{W}$ in $\mathbb{C}^{2N}$ is $\leq N$ and that the N generators of an ideal with the variety $\mathcal{W}$ are indeed alternating polynomials in $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ just like our determinants (5.1). Therefore it is reasonable to conjecture (as was done by Kyungyong Lee [14]) that $\nu_r^N\left(\mathbb{C}^2\right) = N$. All that is needed is to prove that the following conjecture holds:

Conjecture 5.1. *$\nu_r^N\left(\mathbb{C}^2\right) < \nu_r^{N+1}\left(\mathbb{C}^2\right)$.*

Embarrassingly, we do not have a proof for it even for $N = 2$. There is a number of partial results suggesting that, for $d = 2$, $\nu_r^3\left(\Bbbk^2\right) = 3$ yet the proof thus far has been escaping us. To add insult to injury, we can not even prove that $\nu_r^N\left(\mathbb{C}^2\right) \leq \nu_r^{N+1}\left(\mathbb{C}^2\right)$.

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Large family of pseudorandom sequences of k symbols constructed by using multiplicative character

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Abstract

In a series of papers C. Mauduit and A. Sárközy introduced and studied the measures of finite sequences of k symbols. In this paper we construct a new family of pseudorandom sequences of k symbols by using multiplicative character, and study the properties of these sequences.

Keywords: pseudorandom sequence; k symbol; f-well-distribution measure; f-correlation measure; character sum.

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§1. Introduction

In 2002 C. Mauduit and A. Sárközy [5] initiated to study plentiful finite sequences of k symbols

$$E_N = (e_1, e_2, \cdots, e_N) \in \mathcal{A}^N,$$

where $\mathcal{A} = \{a_1, a_2, \cdots, a_k\}$ $(k \in \mathbb{N}, k \geq 2)$ is a finite set of k symbols. Write

$$x(E_N, a, M, u, v) = |\{j : 0 \leq j \leq M-1, e_{u+jv} = a\}|,$$

and for $w = (a_{i_1}, \cdots, a_{i_l}) \in \mathcal{A}^l$, $D = (d_1, \cdots, d_l)$ with non-negative integers $d_1 < \cdots < d_l$,

$$g(E_N, w, M, D) = |\{n : 1 \leq n \leq M, (e_{n+d_1}, \cdots, e_{n+d_l}) = w\}|.$$

Then we get the following definition of pseudorandom measures.

Definition 1.1. The f-well-distribution measure of E_N is defined as

$$\delta(E_N) = \max_{a,M,u,v} \left| x(E_N, a, M, u, v) - \frac{M}{k} \right|,$$

where the maximum is taken over all $a \in \mathcal{A}$ and u, v, M with $1 \leq u \leq u + (M-1)v \leq N$.

Definition 1.2. The f-correlation measure of order l of E_N is defined as

$$\gamma_l(E_N) = \max_{w,M,D} \left| g(E_N, w, M, D) - \frac{M}{k^l} \right|,$$

where the maximum is taken over all $w \in \mathcal{A}^l$, and $D = (d_1, \cdots, d_l)$ and M such that $0 \leq d_1 < \cdots < d_l \leq N - M$.

We hope that both $\delta(E_N)$ and $\gamma_l(E_N)$ (at least for small l) are "small" in terms of N (in particular, both are $o(N)$ as $N \to \infty$, and ideally it is $N^{1/2+\varepsilon}$). If both $\delta(E_N)$ and $\gamma_l(E_N)$ are "small", we say that E_N is a "good" pseudorandom sequence. Many pseudorandom sequences of k symbols have been studied (see [1], [2], [5], [6]). For example, in [1] and [2] R. Ahlswede, C. Mauduit and A. Sárközy proved the following:

Proposition 1.1. *Assume that $k \in \mathbb{N}$, $k \geq 2$, p is a prime number, χ is a character modulo p of order k, $f(x) \in \mathbb{F}_p[x]$ has degree $h(>0)$, $f(x)$ has no multiple zero in $\overline{\mathbb{F}}_p$. Define the sequence $E_p = (e_1, \cdots, e_p)$ on the k letter alphabets of the k-th roots of unity by*

$$e_n = \begin{cases} \chi(f(n)), & \text{for } (f(n), p) = 1, \\ +1, & \text{for } p \mid f(n). \end{cases}$$

Then

(i) we have $\delta(E_p) < 11hp^{1/2} \log p$.

(ii) if $l \in \mathbb{N}$ is such that the triple (r, t, p) is k-admissible for all $1 \leq r \leq h$, $1 \leq t \leq l(k-1)$, then $\gamma_l(E_p) < 10lhkp^{1/2} \log p$.

Proposition 1.2.

(i) If $k, r, t \in \mathbb{N}, 1 \leq t \leq k$, p is a prime and $r < p$, then the triple (r, t, p) is k-admissible.

(ii) If $k, r, t \in \mathbb{N}$, p is a prime and

$$(4t)^r < p,$$

then (r, t, p) is k-admissible.

(iii) If $k \in \mathbb{N}, k \geq 2$, the prime factorization of k is $k = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$ (where $q_1, \ldots, q_s$ are distinct primes and $\alpha_1, \ldots, \alpha_s \in \mathbb{N}$), and p is a prime such that each of $q_1, \ldots, q_s$ is a primitive root modulo p, then for every pair $r, t \in \mathbb{N}$ with $r, t < p$, the triple (r, t, p) is k-admissible.

In this paper we further give large family of pseudorandom sequences of k symbols, and study the pseudorandom properties by using the estimate for character sums and the methods in [3]. The main results are the following:

Theorem 1.1. *Assume that $k \in \mathbb{N}$, $k \geq 2$, p is a prime number, χ is a character modulo p of order k, $f(x) \in \mathbb{F}_p[x]$ has degree $h(>0)$. Define the sequence $E_{p-1} = (e_1, \cdots, e_{p-1})$ on the k letter alphabets of the k-th roots of unity by*

$$e_n = \begin{cases} \chi(f(n) + \overline{n}), & \text{for } (f(n) + \overline{n}, p) = 1, \\ +1, & \text{for } p \mid f(n) + \overline{n}, \end{cases}$$

where $\overline{n}$ is the inverse of n modulo p such that $n\overline{n} \equiv 1(\mathrm{mod}\ p)$ and $1 \leq \overline{n} \leq p-1$. Then

(i) $\delta(E_{p-1}) < 9(h+k)p^{1/2}\log p + h.$

(ii) If $f(x)+f(-x) \equiv 0(\mathrm{mod}\ p)$ has no solutions, then

$$\gamma_2(E_{p-1}) < 18k(h+k)p^{1/2}\log p + 2h.$$

(iii) On the other hand, if $xf(x)+1 \equiv 0(\mathrm{mod}\ p)$ has no solutions, then

$$\gamma_l(E_{p-1}) < 9lk(k+h)p^{1/2}\log p + lh.$$

From Theorem 1.1 we can get the following corollaries.

Corollary 1.1. *Let $p > 2$ be a prime with $p \equiv \pm 3(\mathrm{mod}\ 8)$, and $f_1(x) = h(x)^2 - 2 \in \mathbb{F}_p[x]$, where $h(x) = a_0 + a_2x^2 + a_4x^4 + \cdots \in \mathbb{F}_p[x]$. Define $E'_{p-1} = (e'_1, \ldots, e'_{p-1})$ by*

$$e'_n = \begin{cases} \chi(f_1(n)+\overline{n}), & \text{for } (f_1(n)+\overline{n}, p) = 1, \\ +1, & \text{for } p \mid f_1(n)+\overline{n}. \end{cases}$$

Then

$$\delta(E'_{p-1}) < 9(\deg(f_1)+k)p^{1/2}\log p + \deg(f_1),$$
$$\gamma_2(E'_{p-1}) < 18k(\deg(f_1)+k)p^{1/2}\log p + 2\deg(f_1).$$

Corollary 1.2. *Let $p > 2$ be a prime with $p \equiv \pm 5(\mathrm{mod}\ 12)$, and $f_2(x) = xh(x)^2 + 4h(x)$, where $h(x) \in \mathbb{F}_p[x]$ is any polynomial. Define $E''_{p-1} = (e''_1, \ldots, e''_{p-1})$ by*

$$e''_n = \begin{cases} \chi(f_2(n)+\overline{n}), & \text{for } (f_2(n)+\overline{n}, p) = 1, \\ +1, & \text{for } p \mid f_2(n)+\overline{n}. \end{cases}$$

Then

$$\delta(E''_{p-1}) < 9(\deg(f_2)+k)p^{1/2}\log p + \deg(f_2),$$
$$\gamma_l(E''_{p-1}) < 9lk(\deg(f_2)+k)p^{1/2}\log p + l\deg(f_2).$$

§2. Some lemmas

Lemma 2.1. *Suppose that p is a prime number, χ is a non-principal character modulo p of order k, $f(x) \in \mathbb{F}_p[x]$ has a factorization $f(x) = b(x-x_1)^{d_1}\cdots(x-x_s)^{d_s}$ (where $x_i \neq x_j$ for $i \neq j$) in $\overline{\mathbb{F}}_p$ with $(k, d_1, \cdots, d_s) = 1$. Let X, Y be real numbers with $0 < Y \leq p$. Then we have*

$$\left|\sum_{X<n\leq X+Y} \chi\left(f(n)\right)\right| < 9\deg(f)p^{1/2}\log p.$$

Proof. This is Theorem 2 of [4]. □

Lemma 2.2. *The assertion of Lemma 2.1 also holds if assumption* $(k, d_1, \cdots, d_s) = 1$ *is replaced by*

$$(k, d_1, \cdots, d_s) < k$$

Proof. This is Lemma 2 of Theorem 2 in [1]. □

§3. The proof of the theorem

(i) Let a be a k-th root of unity, $u, v, M \in \mathbb{N}$ and

$$1 \leq u \leq u + (M-1)v \leq p-1.$$

Now using the notation we have

$$\begin{aligned} x\left(E_{p-1}, a, M, u, v\right) &= \left|\{j : 0 \leq j \leq M-1, e_{u+iv} = a\}\right| = \sum_{\substack{0 \leq j \leq M-1 \\ e_{u+jv} = a}} 1 \\ &\leq \sum_{\substack{0 \leq j \leq M-1 \\ \chi(f(u+jv) + \overline{u+jv}) = a}} 1 + \deg(f). \end{aligned}$$

Define

$$S(a, m) = \frac{1}{k} \sum_{t=1}^{k} \left(\overline{a}\chi(m)\right)^t,$$

then

$$S(a, m) = \begin{cases} 1, & \text{if } \ \chi(m) = a, \\ 0, & \text{if } \ \chi(m) \neq a. \end{cases}$$

And hence we derive

$$\begin{aligned} x\left(E_{p-1}, a, M, u, v\right) &\leq \left| \sum_{j=0}^{M-1} S(a, f(u+iv) + \overline{u+iv}) \right| + \deg(f) \\ &= \left| \sum_{j=0}^{M-1} \frac{1}{k} \sum_{t=1}^{k} \left(\overline{a}\chi(f(u+iv) + \overline{u+iv})\right)^t \right| + \deg(f) \\ &\leq \frac{M}{k} + \left| \frac{1}{k} \sum_{t=1}^{k-1} \overline{a}^t \sum_{j=0}^{M-1} \chi^t\left(f(u+iv) + \overline{u+iv}\right) \right| + \deg(f). \end{aligned}$$

Noting that χ is k-th non-principal character. Then

$$\begin{aligned} \chi^t\left(f(u+iv) + \overline{u+iv}\right) &= \chi^t((u+jv)^k)\chi^t\left(f(u+iv) + \overline{u+iv}\right) \\ &= \chi^t\left((u+jv)^k f(u+jv) + (u+jv)^{k-1}\right). \end{aligned}$$

And we define

$$F(j) = (u + jv)^k f(u + jv) + (u + jv)^{k-1}.$$

It is easy to show that $j = -u\overline{v}$ is $(k-1)$-th root of $F(j)$. By Lemma 2.1 we have

$$\left|\sum_{j=0}^{M-1} \chi^t \left(f(u + iv) + \overline{u + iv}\right)\right| < 9(k + \deg(f))p^{1/2} \log p.$$

Hence

$$\left|\frac{1}{k}\sum_{t=1}^{k-1} \overline{a}^t \sum_{j=0}^{M-1} \chi^t \left(f(u + iv) + \overline{u + iv}\right)\right| < 9(k + \deg(f))p^{1/2} \log p. \tag{3.1}$$

Therefore

$$\left|x\left(E_{p-1}, a, M, u, v\right) - \frac{M}{k}\right| < 9(k + \deg(f))p^{1/2} \log p + \deg(f).$$

Then

$$\delta\left(E_{p-1}\right) = \max_{a,M,u,v}\left|x\left(E_{p-1}, a, M, u, v\right) - \frac{M}{k}\right| < 9(k + h)p^{1/2} \log p + h.$$

(ii) Next we consider the correlation measure of E_{p-1} under the condition of $l = 2$. First we suppose that the congruence $f(x) + f(-x) \equiv 0 \,(\bmod\ p)$ has no solution. For $0 \le d_1 \le d_2 \le p - 1 - M$,we can get

$$g\left(E_{p-1}, w, M, D\right) = \left|\left\{n : 1 \le n \le M, \left(e_{n+d_1}, e_{n+d_2}\right) = \left(b_1, b_2\right)\right\}\right|.$$

Here we have

$$e_{n+d_1} = \chi\left(f\left(n + d_1\right) + \overline{n + d_1}\right), \qquad e_{n+d_2} = \chi\left(f\left(n + d_2\right) + \overline{n + d_2}\right),$$

except for the values of n such that

$$f\left(n + d_i\right) + \overline{n + d_i} \equiv 0 \,(\bmod\ p), \qquad 1 \le i \le 2.$$

For fixed i, this congruence may have at most $\deg(f)$ solutions and we know that there must be at most 2 values about i. Thus the total number of solutions of the above-mentioned formula is $\le 2\deg(f)$. For all n, we have

$$\prod_{i=1}^{2} S(b_i, f(n + d_i) + \overline{n + d_i}) = \begin{cases} 1, & \text{if } e_{n+d_1} = b_1, e_{n+d_2} = b_2, \\ 0, & \text{otherwise.} \end{cases}$$

So that we can get

$$g(E_{p-1}, w, M, D) \le \left|\sum_{1\le n\le M} \prod_{i=1}^{2} S(b_i, f(n + d_i) + \overline{n + d_i})\right| + 2\deg(f),$$

and

$$\sum_{1\le n\le M}\prod_{i=1}^{2} S(b_i, f(n+d_i)+\overline{n+d_i}) = \sum_{n=1}^{M}\prod_{i=1}^{2}\left(\frac{1}{k}\sum_{t_i=1}^{k}\left(\overline{b_i}\chi\left(f(n+d_i)+\overline{n+d_i}\right)\right)^{t_i}\right)$$

$$= \frac{1}{k^2}\sum_{t_1=0}^{k-1}\sum_{t_2=0}^{k-1}\overline{b_1}^{t_1}\overline{b_2}^{t_2}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\left(f(n+d_2)+\overline{n+d_2}\right)^{t_2}\right)$$

$$= \frac{M}{k^2}+\frac{1}{k^2}\sum_{t_2=1}^{k-1}\overline{b_2}^{t_2}\sum_{n=1}^{M}\chi\left(\left(f(n+d_2)+\overline{n+d_2}\right)^{t_2}\right)$$

$$+\frac{1}{k^2}\sum_{t_1=1}^{k-1}\overline{b_1}^{t_1}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\right)$$

$$+\frac{1}{k^2}\sum_{1\le t_1\le k-1}\sum_{1\le t_2\le k-1}\overline{b_1}^{t_1}\overline{b_2}^{t_2}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\left(f(n+d_2)+\overline{n+d_2}\right)^{t_2}\right).$$

It follows from (3.1) that

$$\left|\frac{1}{k^2}\sum_{t_i=1}^{k-1}\overline{b_i}^{t_i}\sum_{n=1}^{M}\chi\left(\left(f(n+d_i)+\overline{n+d_i}\right)^{t_i}\right)\right| < \frac{1}{k}9(k+h)p^{1/2}\log p.$$

Therefore

$$\left|g(E_{p-1},w,M,D)-\frac{M}{k^2}\right|$$

$$\le \left|\frac{1}{k^2}\sum_{t_2=1}^{k-1}\overline{b_2}^{t_2}\sum_{n=1}^{M}\chi\left(\left(f(n+d_2)+\overline{n+d_2}\right)^{t_2}\right)\right|$$

$$+\left|\frac{1}{k^2}\sum_{t_1=1}^{k-1}\overline{b_1}^{t_1}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\right)\right|$$

$$+\frac{1}{k^2}\sum_{1\le t_1\le k-1}\sum_{1\le t_2\le k-1}\left|\sum_{n=1}^{M}\chi((f(n+d_1)+\overline{n+d_1})^{t_1}(f(n+d_2)+\overline{n+d_2})^{t_2})\right|$$

$$+2\deg(f).$$

Noting that χ is k-th non-principal character. Then

$$\chi\left(f(n+d_1)+\overline{n+d_1}\right) = \chi\left((n+d_1)^k f(n+d_1)+(n+d_1)^{k-1}\right),$$
$$\chi\left(f(n+d_2)+\overline{n+d_2}\right) = \chi\left((n+d_2)^k f(n+d_2)+(n+d_2)^{k-1}\right).$$

Let

$$G(n)=\left[(n+d_1)^k f(n+d_1)+(n+d_1)^{k-1}\right]^{t_1}\left[(n+d_2)^k f(n+d_2)+(n+d_2)^{k-1}\right]^{t_2}.$$

It is obvious that $-d_1$, $-d_2$ are the zeros of $G(n)$. If the multiplicities of $-d_1$ and $-d_2$ can both be divided by k, then we get

$$(d_2-d_1)f(d_2-d_1)+1\equiv 0(\bmod\ p), \qquad (d_1-d_2)f(d_1-d_2)+1\equiv 0(\bmod\ p).$$

And we will obtain

$$f(d_2-d_1)+f(d_1-d_2)\equiv 0(\bmod\ p),$$

which is impossible. Then we can see that $n=-d_1$ is $(k-1)t_1$-th root of $G(n)$, or $n=-d_2$ is $(k-1)t_2$-th root of $G(n)$. That is to say, $G(n)$ has at least one zero whose multiplicity is not divisible by k. Then from Lemma 2.1 and Lemma 2.2 we have

$$\left|\sum_{n=1}^{M}\chi(G(n))\right|<9(2h+2k)(k-1)p^{1/2}\log p.$$

Hence

$$\begin{aligned}\gamma_2(E_{p-1}) &= \max_{a,M,u,v}\left|x\left(E_{p-1},w,M,D\right)-\frac{M}{k^2}\right|\\ &< 18(h+k)(k-1)p^{1/2}\log p+\frac{2}{k}9(k+h)p^{1/2}\log p+2h\\ &< 18k(h+k))p^{1/2}\log p+2h.\end{aligned}$$

(iii) The final step is to estimate $\gamma_l(E_{p-1})$. We suppose that the congruence $xf(x)+1\equiv 0(\bmod\ p)$ has no solution. Since we will get the result after following a similar method of the proof of (ii), we know that

$$\prod_{i=1}^{l}S(b_i,f(n+d_i)+\overline{n+d_i})=\begin{cases}1, & \text{if }\ e_{n+d_1}=b_1,\cdots,e_{n+d_l}=b_l,\\ 0, & \text{otherwise.}\end{cases}$$

So we obtain

$$\sum_{1\le n\le M}\prod_{i=1}^{l}S(b_i,f(n+d_i)+\overline{n+d_i})=\sum_{n=1}^{M}\prod_{i=1}^{l}\left(\frac{1}{k}\sum_{t_i=1}^{k}\left(\overline{b_i}\chi\left(f(n+d_i)+\overline{n+d_i}\right)\right)^{t_i}\right)$$
$$=\frac{1}{k^l}\sum_{t_1=0}^{k-1}\cdots\sum_{t_l=0}^{k-1}\overline{b_1}^{t_1}\cdots\overline{b_l}^{t_l}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\cdots\left(f(n+d_l)+\overline{n+d_l}\right)^{t_l}\right).$$

Let us split this sum in two parts: $\sum_1$ denotes the contribution of the terms with $t_1=\cdots=t_l=0$, e.g., $\sum_1=\frac{M}{k^l}$; and $\sum_2$ is the contribution of the terms with $(t_1,\cdots,t_l)\neq(0,\cdots,0)$. Then we have

$$\sum\nolimits_2=\frac{1}{k^l}\underset{(t_1,\cdots,t_l)\neq(0,\cdots,0)}{\sum_{1\le t_1\le k-1}\cdots\sum_{1\le t_l\le k-1}}\overline{b_1}^{t_1}\cdots\overline{b_l}^{t_l}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\cdots\left(f(n+d_l)+\overline{n+d_l}\right)^{t_l}\right).$$

On account of

$$\chi\left(f(n+d_i)+\overline{n+d_i}\right)=\chi\left((n+d_i)^kf(n+d_i)+(n+d_i)^{k-1}\right),$$

we can define

$$F(n)=\left[(n+d_1)^kf(n+d_1)+(n+d_1)^{k-1}\right]^{t_1}\cdots\left[(n+d_l)^kf(n+d_l)+(n+d_l)^{k-1}\right]^{t_l}.$$

Since $(t_1, \cdots, t_l) \neq (0, \cdots, 0)$, we know that there is at least one $t_i \neq 0$. Noting that $xf(x)+1 \equiv 0(\bmod\ p)$ has no solution, then $F(n)$ has one zero $-d_i$ whose multiplicity is $(k-1)t_i$, where $1 \le t_i \le k-1$. Applying Lemma 2.2 we get

$$\left|\sum_{n=1}^{M} \chi(F(n))\right| < 9\deg(F)p^{1/2}\log p.$$

Then

$$\begin{aligned}
&g(E_{p-1}, w, M, D) \le \left|\sum_{1\le n\le M}\prod_{i=1}^{l} S(b_i, f(n+d_i)+\overline{n+d_i})\right| + l\deg(f)\\
&\le \left|\frac{1}{k^l}\sum_{\substack{1\le t_1\le k-1}}\cdots\sum_{\substack{1\le t_l\le k-1\\(t_1,\cdots,t_l)\neq(0,\cdots,0)}} \overline{b_1}^{t_1}\cdots\overline{b_l}^{t_l}\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\cdots\left(f(n+d_l)+\overline{n+d_l}\right)^{t_l}\right)\right|\\
&\quad + \frac{M}{k^l} + l\deg(f)\\
&\le \frac{1}{k^l}\sum_{\substack{1\le t_1\le k-1}}\cdots\sum_{\substack{1\le t_l\le k-1\\(t_1,\cdots,t_l)\neq(0,\cdots,0)}}\left|\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\cdots\left(f(n+d_l)+\overline{n+d_l}\right)^{t_l}\right)\right|\\
&\quad + \frac{M}{k^l} + lh.
\end{aligned}$$

Now we can obtain

$$\begin{aligned}
&\left|g(E_{p-1}, w, M, D) - \frac{M}{k^l}\right|\\
&\le \frac{1}{k^l}\sum_{\substack{0\le t_1\le k-1}}\cdots\sum_{\substack{0\le t_l\le k-1\\(t_1,\cdots,t_l)\neq(0,\cdots,0)}}\left|\sum_{n=1}^{M}\chi\left(\left(f(n+d_1)+\overline{n+d_1}\right)^{t_1}\cdots\left(f(n+d_l)+\overline{n+d_l}\right)^{t_l}\right)\right| + lh\\
&< 9kl(k+h)p^{1/2}\log p + lh.
\end{aligned}$$

Therefore

$$\gamma_l(E_{p-1}) < 9lk(k+h)p^{1/2}\log p + lh.$$

§4. The proof of the corollaries

Proof of corollary 1.1. Noting that

$$f_1(x) = (a_0 + a_2x^2 + a_4x^4 + \cdots)^2 - 2,$$

we have

$$f_1(x) + f_1(-x) = 2(a_0 + a_2x^2 + a_4x^4 + \cdots)^2 - 4.$$

Since 2 is a quadratic nonresidue modulo p for $p \equiv \pm 3(\bmod\ 8)$, the congruence $f_1(x)+f_1(-x) \equiv 0(\bmod\ p)$ has no solution. Then from Theorem 1.1 we get

$$\delta(E'_{p-1}) < 9(\deg(f_1)+k)p^{1/2}\log p + \deg(f_1),$$
$$\gamma_2(E'_{p-1}) < 18k(\deg(f_1)+k)p^{1/2}\log p + 2\deg(f_1).$$

This proves Corollary 1.1.

Proof of corollary 1.2. We have

$$xf_2(x)+1 = x^2h(x)^2 + 4xh(x) + 1 = (xh(x)+2)^2 - 3,$$

Since 3 is a quadratic nonresidue modulo p for $p \equiv \pm 5(\bmod\ 12)$, we know that the congruence $xf_2(x)+1 \equiv 0(\bmod\ p)$ has no solution. So from Theorem 1.1 we have

$$\delta(E''_{p-1}) < 9(\deg(f_2)+k)p^{1/2}\log p + \deg(f_2),$$
$$\gamma_l(E''_{p-1}) < 9lk(\deg(f_2)+k)p^{1/2}\log p + l\deg(f_2).$$

This completes the proof of Corollary 1.2.

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DIFFERENCE SEQUENCE SPACES OF FUZZY REAL NUMBERS

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ABSTRACT. In this paper, we introduce a difference sequence space of fuzzy real numbers defined by a sequence of modulus functions. Also we study some topological properties and inclusion relations in this space.

1. INTRODUCTION

Fuzzy set theory, compared to other mathematical theories, is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [31] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [17] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. For more details about sequence spaces and sequence spaces of fuzzy numbers see ([1], [8], [18], [19], [20], [23], [24], [28]) and references therein.

The concept of statistical convergence was introduced by Fast [13] and also independently by Buck [4] and Schoenberg [26] for real and complex sequences. Further this concept was studied by Fridy [12], Connor [6] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence is closely related to the concept convergence appears to have been restricted to real or complex sequences, but in Nanda [22], Savaş [25], Basarir et al. [2], Tripathy et al. [27], Kumar et al. [14] extended the idea to apply to sequences of Fuzzy numbers.

The concept of statistical pre-Cauchy sequence was given by Connor et al. [7] for scalar sequences. It is shown that statistically convergent sequences are statistically pre-cauchy sequence any bounded statistically pre-Cauchy sequence with a nowhere dense set of limit points is statistically convergent.

The notion of difference sequence spaces was introduced by Kızmaz [15], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [10] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [5] and Et et al. [11]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [30] who studied the spaces $l_\infty(\Delta_\nu)$, $c(\Delta_\nu)$ and $c_0(\Delta_\nu)$. Recently, Esi et al. [9] and Tripathy et al. [29] have

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introduced a new type of generalized difference operators and unified those as follows. Let ν, m be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_\nu^m) = \{x = (x_k) \in w : (\Delta_\nu^m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_\nu^m x = (\Delta_\nu^m x_k) = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+\nu})$ and $\Delta_\nu^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \left(\begin{array}{c} m \\ i \end{array} \right) x_{k+\nu i}.$$

Taking $\nu = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [10]. Taking $m = \nu = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [15].

2. Definitions and Preliminaries

Definition 2.1. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [16] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{ x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Also, it was shown in [16] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Definition 2.2. A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions:

(1) X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$;
(2) X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1 - \lambda) y) \geq \min[X(x), X(y)]$;
(3) X is upper semi-continuous; i.e., if for each $\epsilon > 0, X^{-1}([0, a + \epsilon))$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}^n$;
(4) The closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex }\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (non separable) metric space.
For $0 < \alpha \leq 1$, the α-level set, $X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a nonempty compact convex, subset of $\mathbb{R}^n$, as is the support X^0. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of α-level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda[X]^\alpha$$

for each $0 \leq \alpha \leq 1$. Define for each $1 \leq q < \infty$

$$d_q(X, Y) = \Big\{\int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha\Big\}^{1/q}$$

and $d_\infty(X, Y) = \sup\limits_{0\leq\alpha\leq1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X, Y) = \lim\limits_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(L(\mathbb{R}^n), d_\infty)$ is a complete, separable and locally compact metric space.

Definition 2.3. A metric $\overline{d}$ on $L(\mathbb{R}^n)$ is said to be translation invariant if $\overline{d}(X + Z,\ Y + Z) = \overline{d}(X, Y)$ for all $X, Y, Z \in L(\mathbb{R}^n)$.

Definition 2.4. A sequence $X = (X_k)$ of fuzzy real numbers is said to be Δ-bounded if the set $\{\Delta X_k : k \in N\}$ of fuzzy real numbers is bounded.

Definition 2.5. A sequence $X = (X_k)$ of fuzzy real numbers is said to be Δ-convergent to a fuzzy real number X_0, written as $\lim\limits_{k\to\infty} \Delta X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer k_0 such that $\overline{d}(\Delta X_k, X_0) < \epsilon$ for all $k > k_0$.

Definition 2.6. A sequence $X = (X_k)$ of fuzzy real numbers is said to be Δ^m_ν-convergent to a fuzzy real number X_0, written as $\lim\limits_{k\to\infty} \Delta^m_\nu X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer k_0 such that $\overline{d}(\Delta^m_\nu X_k, X_0) < \epsilon$ for all $k > k_0$.

We need following lemmas in the present paper:
Lemma 2.1. (Basarir and Mursaleen [3]) If $\overline{d}$ is a translation invariant metric. Then
(i) $\overline{d}(X + Y, \overline{0}) \leq \overline{d}(X, \overline{0}) + \overline{d}(Y, \overline{0})$
(ii) $\overline{d}(\lambda X, \overline{0}) \leq |\lambda|\overline{d}(X, \overline{0})$, $|\lambda| > 1$.
Lemma 2.2. (Maddox [21]) Let a_k, b_k for all k be sequences of complex numbers and (p_k) be a bounded sequence of positive real numbers, then

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$$

and

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$$

where $C = \max(1, 2^{H-1})$, $H = \sup p_k$ and λ is any complex number.
Lemma 2.3. (Maddox [21]) Let $a_k \geq 0, b_k \geq 0$ for all k be sequences of complex numbers and $1 \leq p_k \leq \sup p_k < \infty$, then

$$\Big(\sum_k |a_k + b_k|^{p_k}\Big)^{\frac{1}{M}} \leq \Big(\sum_k |a_k|^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k |b_k|^{p_k}\Big)^{\frac{1}{M}}$$

where $M=\max(1,H),\ H=\sup p_k$.
Let $(E_k,\overline{d_k})$ be a sequence of fuzzy linear metric spaces under the translation invariant metrices $\overline{d_k}'s$ such that $E_{k+1}\subseteq E_k$ for each $k\in\mathbb{N}$ where $X_k=\Big(\Big(X_{k,s}\Big)_{s=1}^{\infty}\Big)\in E_k$ for each $k\in\mathbb{N}$. We define $W(E)=\{X=(X_k):X_k\in E_k$ for each $k\in\mathbb{N}\}$. It is easy to verify that the space $W(E)$ is a linear space of fuzzy real numbers under coordinatewise addition and scalar multiplication. For $X=(X_k)\in W(E)$ and $\lambda=(\lambda_k)$ be a sequence of real numbers, we define $\lambda X=(\lambda_k X_k)$.
Let $F=(f_k)$ be a sequence of modulus functions, $p=(p_k)$ is a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. In the present paper we define the following sequence space:

$$W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)=\Big\{X=(X_k)\in W(E):\frac{1}{n}\sum_{s=1}^{n}\Big[f_k\Big(\sup_k\overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s},L_k\Big)\Big)\Big]^{p_k}\to 0\ \text{ as } n\to\infty\Big\}$$

where

$$\Delta_\nu^m X_{k,s}=\sum_{i=0}^{m}(-1)^i\binom{m}{i}X_{k+\nu i,s}.$$

The main purpose of this paper is to study difference sequence spaces of fuzzy real numbers in more general settings defined by a sequence of modulus functions and a multiplier sequence $u=(u_k)$. We also make an effort to study some topological properties and interesting inclusion relations in the third section of this paper. In the section fourth of this paper we have studied statistical convergence and some of their properties.

3. Main Results

Theorem 3.1. *Let $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$ is a linear space over the field $\mathbb{R}$ of real numbers.*

Proof. Let $X=(X_k)$ and $Y=(Y_k)\in W(E)$ and $\alpha,\beta\in\mathbb{R}$. Then it is easy to prove

$$\frac{1}{n}\sum_{s=1}^{n}\Big[f_k\Big(\sup_k\overline{d_k}\Big(u_k\Delta_\nu^m\Big(\alpha X_{k,s}+\beta Y_{k,s}\Big),L_k\Big)\Big)\Big]^{p_k}\to 0\ \text{ as } n\to\infty,$$

by using lemma (2.1) (2.2) (2.3), the subadditivity property of modulus functions and the result $f(\lambda x)\le(1+[|\lambda|])f(x)$. Therefore $\alpha X+\beta Y\in W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$. Hence $W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$ is a linear space. □

Theorem 3.2. *Let $(E_k,\overline{d_k})$ be a sequence of complete metric spaces and (p_k) be a bounded sequence of positive real numbers such that $\inf p_k>0$. Then the sequence space $W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$ is a complete metric space with respect to the metric*

$$g(X,Y)=\sum_{i=1}^{m}f_k\Big(\sup_k\overline{d_k}\Big(X_{k,i},Y_{k,i}\Big)\Big)+\sup_n\Big[\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k\overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s},u_k\Delta_\nu^m Y_{k,s}\Big)\Big)\Big)^{p_k}\Big]^{\frac{1}{M}}.$$

Proof. Let $(X^{(q)})$ be a cauchy sequence in $W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$ where $X^{(q)}=\Big(\Big(X_{k,s}^{(q)}\Big)_{s=1}^{\infty}\Big)_{k=1}^{\infty}\in W^{\mathcal{F}}(\Delta_\nu^m,F,u,p)$ for each $q\in\mathbb{N}$. Then

$$g(X^{(q)},X^{(r)})\to 0\ \text{ as } q,r\to\infty.$$

This means

$$\sum_{i=1}^{m}f_k\Big(\sup_k\overline{d_k}\Big(X_{k,i}^{(q)},X_{k,i}^{(r)}\Big)\Big)+\sup_n\Big[\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k\overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)},u_k\Delta_\nu^m X_{k,s}^{(r)}\Big)\Big)\Big)^{p_k}\Big]^{\frac{1}{M}}$$

$$\to 0 \text{ as } q, r \to \infty,$$

which implies that

$$\sum_{i=1}^{m} f_k\Big(\sup_k \overline{d_k}\Big(X_{k,i}^{(q)}, X_{k,i}^{(r)}\Big)\Big) \to 0 \text{ as } q, r \to \infty \tag{3.1}$$

and

$$\sup_n \Big[\tfrac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}^{(r)}\Big)\Big)\Big)^{p_k}\Big]^{\frac{1}{M}} \to 0 \text{ as } q, r \to \infty. \tag{3.2}$$

Now from equation (3.1), we have

$$f_k\Big(\sup_k \overline{d_k}\Big(X_{k,i}^{(q)}, X_{k,i}^{(r)}\Big)\Big) \to 0 \text{ as } q, r \to \infty \text{ for each } i = 1, 2, ...m.$$

But (f_k) is a sequence of modulus functions, so we have

$$\sup_k \overline{d_k}\Big(X_{k,i}^{(q)}, X_{k,i}^{(r)}\Big) \to 0 \text{ as } q, r \to \infty \text{ for each } i = 1, 2, ...m.$$

Therefore $\{X_{k,i}^{(q)}\}$ is a cauchy sequence in E_k for each $i = 1, 2, ...m$ and for all k.
Again from equation (3.2), since (f_k) is a sequence of modulus functions, we have

$$\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}^{(r)}\Big) \to 0 \text{ as } q, r \to \infty \text{ for each } s = 1, 2, ...n.$$

Thus $(u_k\Delta_\nu^m X_{k,s}^{(q)})$ is a cauchy sequence in E_k for each $s = 1, 2, ...n$ and for each $k \in \mathbb{N}$. But given that each E_k is complete. So let $X_{k,i}^{(q)} \to X_{k,i}$ as $q \to \infty$ for each $i = 1, 2, ...m$ and for all k and $u_k\Delta_\nu^m X_{k,s}^{(q)} \to u_k\Delta_\nu^m X_{k,s}$ as $q \to \infty$ for each s=1,2,.......n and for all k. Therefore by using equations (3.1) and (3.2), we get

$$\sum_{i=1}^{m} f_k\Big(\sup_k \overline{d_k}\Big(X_{k,i}^{(q)}, X_{k,i}\Big)\Big) \to 0 \text{ as } q \to \infty$$

and

$$\sup_n \Big[\tfrac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}^{(r)}\Big)\Big)\Big)^{p_k}\Big]^{\frac{1}{M}} \to 0 \text{ as } q \to \infty. \tag{3.3}$$

i.e.

$$g(X^{(q)}, X) \to 0 \text{ as } q \to \infty.$$

Now, we shall show that $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$. From equation (3.3), we have
$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}^{(r)}\Big)\Big)\Big)^{p_k} \to 0$ as $q \to \infty$ for all $n \in \mathbb{N}$.
i.e. given $\epsilon > 0$, there exists $q_0 \in \mathbb{N}$ such that
$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}\Big)\Big)\Big)^{p_k} < \dfrac{\epsilon}{3}$ for all $q > q_0$ and for all $n \in \mathbb{N}$.
Since $X^{(q)} \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$, we can find $L^{(q)}$ such that
$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, L_k^{(q)}\Big)\Big)\Big)^{p_k} < \dfrac{\epsilon}{3}$ for all $n > n_0$ where $L_k^{(q)} \in E_k$.
Similarly, for $X^{(r)} \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$, we can find $L^{(r)}$ such that
$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(r)}, L_k^{(r)}\Big)\Big)\Big)^{p_k} < \dfrac{\epsilon}{3}$ for all $n > n_1$ where $L_k^{(r)} \in E_k$.

Consider $n_2 = \max(q_0, n_0, n_1)$. Then

$$
\begin{aligned}
(3.4)\qquad f_k\Big(\sup_k \overline{d_k}\Big(L_k^{(q)}, L_k^{(r)}\Big)\Big) &= \frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(L_k^{(q)}, L_k^{(r)}\Big)\Big)\Big)^{p_k} \\
&\leq C\,\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}^{(q)}, L_k^{(q)}\Big)\Big)\Big)^{p_k} \\
&+ C\,\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}^{(r)}\Big)\Big)\Big)^{p_k} \\
&+ C\,\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(r)}, L_k^{(r)}\Big)\Big)\Big)^{p_k} \\
&< \epsilon, \ \text{ for all } q, r \geq n_2.
\end{aligned}
$$

Choose $\epsilon = f(\epsilon_1), \epsilon_1 > 0$ and using the fact that sequence of modulus function is monotone, we get

$$\overline{d_k}(L_k^{(q)}, L_k^{(r)}) < \epsilon_1 \ \text{ for all } q, r \geq n_2.$$

i.e. $L_k^{(q)}$ is a cauchy sequence in E_k. But given that E_k is complete. So $L_k^{(q)} \to L_k$ as $q \to \infty$. From equation (3.4) we get

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(L_k^{(q)}, L_k\Big)\Big)\Big)^{p_k} < \epsilon, \ \ \forall\, q \geq n_2.$$

Hence we have

$$
\begin{aligned}
\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k} &\leq C\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, u_k\Delta_\nu^m X_{k,s}\Big)\Big)\Big)^{p_k} \\
&+ C\,\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}^{(q)}, L_k^{(q)}\Big)\Big)\Big)^{p_k} \\
&+ C\,\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(L_k^{(q)}, L_k^{(r)}\Big)\Big)\Big)^{p_k} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \epsilon \\
&= 5\,\frac{\epsilon}{3}, \ \text{ for all } n \geq n_2.
\end{aligned}
$$

which implies that $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$ and hence $W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$ is a complete metric space. □

Theorem 3.3. *Let (p_k) and (t_k) be two sequences of positive real numbers such that $0 < p_k \leq t_k$ for all $k \in \mathbb{N}$ and the sequence $\left(\frac{t_k}{p_k}\right)$ be bounded. Then $W^{\mathcal{F}}(\Delta_\nu^m, F, u, t) \subset W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$.*

Proof. Let $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, t)$ which implies

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{t_k} \to 0 \text{ as } n \to \infty.$$

Consider $\mu_k = \Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{t_k}$ and $\lambda_k = \left(\frac{p_k}{t_k}\right)$ be such that $0 < \lambda \leq \lambda_k \leq 1$.

Define

$$c_k = \begin{cases} \mu_k, & \text{if } \mu_k \geq 1 \\ 0, & \text{if } \mu_k < 1 \end{cases}$$

and

$$d_k = \begin{cases} 0, & \text{if } \mu_k \geq 1 \\ \mu_k, & \text{if } \mu_k < 1 \end{cases}$$

Then we have $\mu_k = c_k + d_k$ and $\mu_k^{\lambda_k} = c_k^{\lambda_k} + d_k^{\lambda_k}$. Thus it follows that $c_k^{\lambda_k} \leq c_k \leq \mu_k$ and $d_k^{\lambda_k} \leq d_k^{\lambda}$.
Therefore

$$\begin{aligned} \frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k} &\leq \frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{t_k} + \frac{1}{n}\sum_{s=1}^{n} d_k^{\lambda} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

which implies that $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$. □

Theorem 3.4. *Let $F = (f_k)$ and $G = (g_k)$ be two sequence of modulus functions. Then we have*
(i) $W^{\mathcal{F}}(\Delta_\nu^m, F, u, p) \cap W^{\mathcal{F}}(\Delta_\nu^m, G, u, p) \subseteq W^{\mathcal{F}}(\Delta_\nu^m, F+G, u, p)$
(ii) $W^{\mathcal{F}}(\Delta_\nu^m, F, u, p) = W^{\mathcal{F}}(\Delta_\nu^m, G, u, p)$ *if* $0 < \inf \frac{F(x)}{G(x)} \leq \sup \frac{F(x)}{G(x)} < \infty$.

Proof. The proof is easy so we omit it. □

4. Δ_ν^m- Statistical Convergence

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The natural density of a subset K of $\mathbb{N}$ is defined by $\delta(k) = \lim_{n\to\infty} \frac{1}{n}\big|\{k \leq n : k \in K\}\big|$, where $\big|\{k \leq n : k \in K\}\big|$ denotes the number of elements of K not exceeding n. We shall be concerned with the integer sets having density zero.
If $X = (X_k)$ is a sequence that satisfies a property P for all k except a set of natural density zero, then we say that (X_k) satisfies P for almost all k and we write it by a.a.k.

Definition 4.1. The sequence $X = \Big(\Big(\Big(X_{k,s}\Big)_{s=1}^{\infty}\Big)_k\Big)$ of fuzzy real numbers is said to be Δ_ν^m-statistically convergent to a fuzzy real number $L = (L_1, L_2, L_3, ...)$ where $L_k \in E_k$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\}\Big| = 0.$$

Let $S^{\mathcal{F}}(\Delta_\nu^m, u)$ denotes the set of all Δ_ν^m-statistically convergent sequences of real numbers.

Definition 4.2. The sequence $X = \Big(\Big(\Big(X_{k,s}\Big)_{s=1}^{\infty}\Big)_k\Big)$ of fuzzy real numbers is said to be Δ_ν^m-statistically Cauchy sequence, if for every $\epsilon > 0$, there exists positive integer s_o (depends upon ϵ only) such that

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, u_k\Delta_\nu^m X_{k,s_o}) \geq \epsilon\Big\}\Big| = 0.$$

Definition 4.3. The sequence $X = (((X_{k,s})_{s=1}^{\infty})_k)$ of fuzzy real numbers is said to be Δ_ν^m-statistically pre-Cauchy sequence, if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^2}\Big|\Big\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}) \geq \epsilon\Big\}\Big| = 0.$$

Remark 4.1. If a sequence is Δ_ν^m-convergent, then it is Δ_ν^m-statistical convergent. But the converse may not be true. This is clear from the following example.

Example 4.1. Let $E_k = L(\mathbb{R}), u_k = 1$ for each $k \in \mathbb{N}, m = \nu = 1$. Consider the sequence X, when $k = 10^n$

$$X_k(t) = \begin{cases} \frac{k}{k-1}(t + 2 - \frac{1}{k}), & \text{if } \frac{1-2k}{k} \le t \le -1 \\ \frac{k}{k+1}(\frac{1}{k} - t), & \text{if } -1 \le t \le \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

and when $k \neq 10^n$

$$X_k(t) = \begin{cases} t - 5, & \text{if } 5 \le t \le 6 \\ 7 - t, & \text{if } 6 \le t \le 7 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} \left[\frac{1-2k+k\alpha-\alpha}{k}, \frac{1-k\alpha-\alpha}{k}\right], & \text{when } k = 10^n \\ [5+\alpha, 7-\alpha], & \text{otherwise} \end{cases}$$

i.e.

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{1-9k+2k\alpha-\alpha}{k}, \frac{1-2k\alpha-5k-\alpha}{k}], & \text{when } k = 10^n \\ [\frac{5k+2k\alpha+4+3\alpha}{k+1}, \frac{9k-2k\alpha+8-\alpha}{k+1}], & \text{when } k+1 = 10^n \\ [-2+2\alpha \ , \ 2-2\alpha], & \text{otherwise.} \end{cases}$$

Clearly $\Delta X_k \to L$ statistically, where $L = [-2+2\alpha \ , \ 2-2\alpha]$ but (ΔX_k) is not a convergent sequence.

Theorem 4.1. *Let $F = (f_k)$ be a sequence of modulus functions and $0 < h = \inf p_k \le p_k \le \sup p_k = H$. Then $W^{\mathcal{F}}(\Delta_\nu^m, F, u, p) \subsetneq S^{\mathcal{F}}(\Delta_\nu^m, u)$.*

Proof. Let $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$ and $\epsilon > 0$ be given. Then

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$= \frac{1}{n}\sum_{\substack{s=1 \\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k)\ge\epsilon}}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$+ \frac{1}{n}\sum_{\substack{s=1 \\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) < \epsilon}}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$\ge \frac{1}{n}\sum_{\substack{s=1 \\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \ge \epsilon}}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$\ge \min(f(\epsilon)^h, f(\epsilon)^H)\ \frac{1}{n}\Big|\Big\{s \le n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \ge \epsilon\Big\}\Big|$$

which implies that X is Δ_ν^m-statistical convergent.

Remark 4.2. The inclusion is strict. Clear from the following example.

Example 4.2. Let $F(x) = f_k(x) = x, p_k = 1, u_k = 1$ for all k, $m = \nu = 1, E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$. Consider the sequence (X_k), when $k = 5^n$

$$X_k(t) = \begin{cases} k(t + \frac{1}{k}), & \text{if } \frac{-1}{k} \leq t \leq 0 \\ k(\frac{1}{k} - t), & \text{if } 0 \leq t \leq \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

and when $k \neq 5^n$

$$X_k(t) = \begin{cases} t - 5, & \text{if } 5 \leq t \leq 6 \\ 7 - t, & \text{if } 6 \leq t \leq 7 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [\frac{\alpha-1}{k}, \frac{1-\alpha}{k}], & \text{when } k = 5^n \\ [5 + \alpha, 7 - \alpha], & \text{otherwise} \end{cases}$$

i.e.

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{\alpha-1-7k+\alpha k}{k}, \frac{1-5k-\alpha k-\alpha}{k}], & \text{when } k = 5^n \\ [\frac{k\alpha+2\alpha+5k+4}{k+1}, \frac{7k-k\alpha+8-2\alpha}{k+1}], & \text{when } k + 1 = 5^n \\ [-2 + 2\alpha \ , \ 2 - 2\alpha], & \text{otherwise.} \end{cases}$$

Then $\Delta X_k \to L$ statistically, where $L = [-2+2\alpha \ , \ 2-2\alpha]$ but $(\Delta X_k) \notin W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$.

Theorem 4.2. *If $F = (f_k)$ is a sequence of bounded modulus functions, then $S^{\mathcal{F}}(\Delta_\nu^m, u) \subseteq W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$.*

Proof. Let $\epsilon > 0$ be given and (f_k) be a sequence of bounded modulus functions,there exists an integer K such that $f_k(x) < K$ for all $x \geq 0$ and for all $k \in \mathbb{N}$. Let $X = (X_k)$ is Δ_ν^m-statistically convergent sequence. Consider

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$= \frac{1}{n} \sum_{\substack{s=1 \\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon}}^{n} \Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$+ \frac{1}{n} \sum_{\substack{s=1 \\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) < \epsilon}}^{n} \Big(f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,s}, L_k\Big)\Big)\Big)^{p_k}$$

$$\leq \max(k^h, k^H)\ \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\}\Big| + \max(f(\epsilon)^h, f(\epsilon)^H)$$

$$\to 0 \text{ as } n \to \infty.$$

Therefore $X \in W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$ which implies that $S^{\mathcal{F}}(\Delta_\nu^m, u) \subseteq W^{\mathcal{F}}(\Delta_\nu^m, F, u, p)$.

Theorem 4.3. *If the sequence $X = (X_k)$ is Δ_ν^m-statistically convergent, then X is Δ_ν^m-statistically Cauchy .*

Proof. Let X is Δ_ν^m-statistically convergent sequence and let $\epsilon > 0$ be given. Then we have,

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\}\Big| = 0,$$

i.e.

$$\sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) < \epsilon, \quad a.a.s.$$

In particular choose $s_1 \in \mathbb{N}$ such that $\sup\limits_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) < \epsilon$. Thus

$$\begin{aligned} \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, u_k\Delta_\nu^m X_{k,s_1}) &\leq \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \\ &+ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s_1}, L_k) \\ &< \epsilon + \epsilon = 2\,\epsilon \ \ a.a.s. \end{aligned}$$

which implies that X is a Δ_ν^m-statistically Cauchy sequence.

Theorem 4.4. *If $X = \left(\left(\left(X_{k,s}\right)_{s=1}^\infty\right)_k\right)$ is a sequence for which there is a Δ_ν^m-statistically convergent sequence $Y = \left(\left(\left(Y_{k,s}\right)_{s=1}^\infty\right)_k\right)$ such that $u_k\Delta_\nu^m X_{k,s} = u_k\Delta_\nu^m Y_{k,s}$ a.a.s. Then the sequence X is also Δ_ν^m-statistically convergent sequence.*

Proof. Let $u_k\Delta_\nu^m X_{k,s} = u_k\Delta_\nu^m Y_{k,s}$ $a.a.s.$ and Y is Δ_ν^m-statistically convergent sequence. Let $\epsilon > 0$ be given. Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\} \subseteq &\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m Y_{k,s}, L_k) \geq \epsilon\Big\} \\ &\cup \Big\{s \leq n : u_k\Delta_\nu^m X_{k,s} \nsim u_k\Delta_\nu^m Y_{k,s}\Big\}. \end{aligned}$$

Since Y is Δ_ν^m-statistically convergent sequence, which implies the set $\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m Y_{k,s}, L_k) \geq \epsilon\}$ contains a fixed number of elements say $s_0 = s_0(\epsilon)$, then

$$\begin{aligned} \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\}\Big| &\leq \frac{s_o}{n} + \frac{1}{n}\Big|\Big\{s \leq n : u_k\Delta_\nu^m X_{k,s} \nsim u_k\Delta_\nu^m Y_{k,s}\Big\}\Big| \\ &\to 0 \text{ as } n \to \infty \ (\textit{because } \ u_k\Delta_\nu^m X_{k,s} = u_k\Delta_\nu^m Y_{k,s}), \end{aligned}$$

which implies that X is a Δ_ν^m-statistically convergent sequence.

Theorem 4.5. *If X is a sequence of fuzzy real numbers such that X is Δ_ν^m-statistically convergent sequence. Then X is Δ_ν^m-statistically bounded sequence.*

Proof. Suppose X is Δ_ν^m-statistically convergent sequence. Then given $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{s \leq n : \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) \geq \epsilon\Big\}\Big| = 0$$

Since L is a fuzzy number, so we have $\sup\limits_k \overline{d_k}(L_k, \overline{0}) < T$ (say). Then we have

$$\begin{aligned} \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, \overline{0}) &\leq \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,s}, L_k) + \sup_k \overline{d_k}(L_k, \overline{0}) \\ &\leq \epsilon + T \ \ a.a.k., \end{aligned}$$

which implies that X is a Δ_ν^m-statistically bounded sequence.

Remark 4.3. In general the converse is not true. This we shall prove in the following example.

Example 4.3. Let $F = f_k(x) = x, p_k = 1, u_k = 1$ for each $k \in \mathbb{N}$, $m = \nu = 1, E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$. Consider the sequence (X_k) as, when $k = 10^n$

$$X_k(t) = \begin{cases} (kt+1), & \text{if } \frac{-1}{k} \le t \le 0 \\ (1-kt), & \text{if } 0 \le t \le \frac{1}{k} \\ 0, & \text{otherwise} \end{cases}$$

and when $k \neq 10^n$ and k is odd

$$X_k(t) = \begin{cases} t+7, & \text{if } -7 \le t \le -6 \\ -t-5, & \text{if } -6 \le t \le -5 \\ 0, & \text{otherwise} \end{cases}$$

and when $k \neq 10^n$ and k is even

$$X_k(t) = \begin{cases} t-5, & \text{if } 5 \le t \le 6 \\ 7-t, & \text{if } 6 \le t \le 7 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [\frac{\alpha-1}{k}, \frac{1-\alpha}{k}], & \text{when } k = 10^n \\ [-7+\alpha, -5-\alpha], & \text{when } k \neq 10^n \text{ and } k \text{ is odd} \\ [5+\alpha, 7-\alpha], & \text{when } k \neq 10^n \text{ and } k \text{ is even} \end{cases}$$

$$\text{i.e. } [u_k\ \Delta_v^m\ X_k]^\alpha = [\Delta X_k]^\alpha = \begin{cases} [\frac{\alpha-1+\alpha k+\ 5k}{k}, \frac{1+7k-\alpha-\alpha k}{k}], & \text{when } k = 10^n \\ [\frac{-7k+2\alpha+k\alpha-8}{k+1}, \frac{-5k-k\alpha-4-2\alpha}{k+1}], & \text{when } k+1 = 10^n \\ [-14+2\alpha\ ,\ -10-2\alpha], & \text{when } k \neq 10^n \text{ and } k \text{ is odd} \\ [10+2\alpha\ ,\ 14-2\alpha], & \text{when } k \neq 10^n \text{ and } k \text{ is even,} \end{cases}$$

which implies that X is a Δ_ν^m-statistically bounded sequence, but not Δ_v^m-statistically convergent sequence.

Remark 4.4. A sequence X is a Δ_ν^m-statistically pre-Cauchy sequence, but not Δ_ν^m-statistically convergent sequence.

Example 4.4. Let $F = f_k(x) = x, p_k = 1, u_k = 1$ for each $k \in \mathbb{N}$, $m = \nu = 1, E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$. Consider the sequence (X_k) as, when k is odd

$$X_k(t) = \begin{cases} t+7, & \text{if } -7 \le t \le -6 \\ -t-5, & \text{if } -6 \le t \le -5 \\ 0, & \text{otherwise} \end{cases}$$

and when k is even

$$X_k(t) = \begin{cases} t-5, & \text{if } 5 \le t \le 6 \\ 7-t, & \text{if } 6 \le t \le 7 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [-7+\alpha, -\alpha-5], & \text{when } k \text{ is odd} \\ [5+\alpha, 7-\alpha], & \text{when } k \text{ is even} \end{cases}$$

i.e.

$$[u_k\ \Delta_v^m\ X_k]^\alpha = \begin{cases} [2\ (-7+\alpha)\ ,\ 2(-\alpha-5)], & \text{when } k \text{ is odd} \\ [2\ (5+\alpha)\ ,\ 2(7-\alpha)], & \text{when } k \text{ is even,} \end{cases}$$

which implies that the sequence X is a Δ_ν^m-statistically pre-Cauchy sequence, but not Δ_ν^m-statistically convergent sequence.

Theorem 4.6. *Let X be a sequence of fuzzy real numbers such that $(u_k\Delta_\nu^m X_{k,s})$ is bounded. Then X is a Δ_ν^m-statistically pre-cauchy sequence if and only if*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)=0$$

for bounded sequence (f_k) of modulus functions.

Proof. Suppose $\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\le n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)=0$. Given $\epsilon>0$, and for any $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\tfrac{1}{n^2}\textstyle\sum_{i,j\le n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\ \Delta_\nu^m\ X_{k,j}\Big)\Big)\\
&=\frac{1}{n^2}\sum_{\substack{i,j\le n\\ \sup_k \overline{d_k}(u_k\Delta_\nu^m X_{k,i},u_k\Delta_\nu^m X_{k,j})<\epsilon}} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&+\frac{1}{n^2}\sum_{\substack{i,j\le n\\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i},u_k\Delta_\nu^m X_{k,j}\Big)\ge\epsilon}} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&\ge\frac{1}{n^2}\sum_{\substack{i,j\le n\\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i},u_k\Delta_\nu^m X_{k,j}\Big)\ge\epsilon}} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&\ge f(\epsilon)\ \frac{1}{n^2}\Big|\Big\{(i,j): i,j\le n, \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\ge\epsilon\Big\}\Big|
\end{aligned}$$

and thus X is a Δ_ν^m-statistically pre-Cauchy sequence.
Conversely, Let X is a Δ_ν^m-statistically pre-Cauchy sequence and $\epsilon>0$ be given. Choose $\delta>0$ such that $f(\delta)<\frac{\epsilon}{2}$. Since f_k is a sequence of bounded modulus functions so there exists an integer D such that

$$f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)<D$$

Now for each $n\in\mathbb{N}$, consider

$$\begin{aligned}
&\tfrac{1}{n^2}\textstyle\sum_{i,j\le n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&=\frac{1}{n^2}\sum_{\substack{i,j\le n\\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i},u_k\Delta_\nu^m X_{k,j}\Big)<\delta}} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&+\frac{1}{n^2}\sum_{\substack{i,j\le n\\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i},u_k\Delta_\nu^m X_{k,j}\Big)\ge\delta}} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big)\\
&\le f(\delta)+D\frac{1}{n^2}\Big|\Big\{(i,j): i,j\le n, \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\ge\delta\Big\}\Big|\\
&\le \frac{\epsilon}{2}+D\frac{1}{n^2}\Big|\Big\{(i,j): i,j\le n, \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\ge\delta\Big\}\Big|.
\end{aligned}$$

Since X is a Δ_ν^m-statistically pre-Cauchy sequence, so that

$$\frac{1}{n^2}\left|\left\{(i,j) : i,j \leq n,\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big) \geq \delta\right\}\right| \to 0 \text{ as } n \to \infty.$$

Thus there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n^2}\left|\left\{(i,j) : i,j \leq n,\ \sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k Delta_\nu^m X_{k,j}\Big) \geq \delta\right\}\right| < \frac{\epsilon}{2D} \text{ for all } n \geq n_0.$$

i.e.

$$\frac{1}{n^2}\sum_{i,j\leq n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big) \leq \epsilon,\ n \geq n_0.$$

Hence, we have

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i,j\leq n} f_k\Big(\sup_k \overline{d_k}\Big(u_k\Delta_\nu^m X_{k,i}, u_k\Delta_\nu^m X_{k,j}\Big)\Big) = 0.$$

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Existence of periodic solutions for a class of nonlinear discrete systems*

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Abstract

This paper is concerned with the existence of positive periodic solutions to nonlinear discrete systems of the type

$$x_i(n) = \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), x_2(k), \dots, x_m(k)), \quad i = 1, 2, \dots, m,$$

which arises in some epidemic model. Our main results are proved by using the method of sub-super solutions and Schauder's fixed point theorem.

Keywords: periodic solutions, nonlinear discrete systems, sub-super solutions, Schauder's fixed point.

2000 Mathematics Subject Classification: 39A23, 34C25.

1 Introduction

Since the work of Cooke and Kaplan [7], there has been of great interest for many authors to study the the following delay integral equation.

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \tag{1.1}$$

which is a kind of model for the spread of some infectious disease. Especially, the existence of bounded solutions for equation (1.1) and its variants has been extensively studied. There is a large literature on this topic. For example, we refer the reader to [1–4, 8–16] and references therein for some recent developments.

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In [5, 6], the authors investigated the following integral system:

$$x(t) = \int_{t-\tau_1}^{t} f(s, x(s), y(s))ds \quad y(t) = \int_{t-\tau_2}^{t} g(s, x(s), y(s))ds. \tag{1.2}$$

Stimulated by [5, 6], in this paper, we will study the following discrete systems

$$x_i(n) = \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), x_2(k), \ldots, x_m(k)), \quad i = 1, 2, \ldots, m, \tag{1.3}$$

where n belongs to the set of integers, and m, τ_i are fixed positive integers. More specifically, we aim to extend the main result in [5] to discrete case with m variables.

2 Main results

Throughout the rest of this paper, we denote

$$\mathbb{N}_n^m = \{n, n+1, \ldots, n+m-1\},$$

where n, m are positive integers. Moreover, we denote

$$\prod_i E_i = E_1 \times E_2 \times \cdots \times E_m, \quad \prod_{j \neq i} E_j = E_1 \times E_2 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_m$$

where E_i $(i, j = 1, 2, \ldots, m)$ are some sets.

Next, we will study the existence of solutions for the system (1.3). Throughout the rest of this paper, we assume the following two conditions hold:

(H1) $f_i : \mathbb{Z} \times \prod_j I_j \to \mathbb{R} (i = 1, 2, \ldots, m)$ are continuous nonnegative functions with respect to the last m variables, where I_j $(j = 1, 2, \ldots, m)$ are subintervals of $[0, +\infty)$. Moreover, f_i $(i = 1, 2, \ldots, m)$ are T-periodic (T is a fixed positive integer) with respect to the first variable.

(H2) For all $i \in \{1, 2, \ldots, m\}$ and $(k, x_1, \ldots, x_m) \in \mathbb{Z} \times \prod_j I_j$, there holds

$$f_i(k, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m) = 0.$$

It follows from (H2) that $\underbrace{(0, \ldots, 0)}_{m}$ is a trivial solution of the system (1.3). In this following, we will study the existence of nontrivial T-periodic solution for the system (1.3).

Let E be the real Banach space of all T-periodic functions $x : \mathbb{Z} \to \mathbb{R}$ with the norm $\|x\| = \max_{k \in \mathbb{N}_1^T} |x(k)|$. If $x, y \in E$, with $x(k) \leq y(k)$, $\forall k \in \mathbb{Z}$, we denote $[x, y]_E$ be the following set

$$[x, y]_E = \{z \in E : x(k) \leq z(k) \leq y(k), \forall k \in \mathbb{Z}\}.$$

Next, by using the method of sub-super solutions and Schauder's fixed point theorem, we establish a theorem on nontrivial solutions to the system (1.3).

Theorem 2.1. *Assume that the following assumptions hold:*

(i) there exists a pair $(x_{i(0)})$*-*$(x_i{}^{(0)})$ *of sub-super solutions of* (1.3), *i.e.,* $x_{i(0)}, x_i{}^{(0)} : \mathbb{Z} \to I_i$ $(i = 1, 2, \dots, m)$ *are T-periodic functions such that* $x_{i(0)}(k) \le x_i{}^{(0)}(k)$ *for all* $k \in \mathbb{Z}(i = 1, 2, \dots, m)$, *and*

$$\begin{aligned} x_{i(0)}(n) &\le \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \dots, x_{i(0)}(k), \dots, x_m(k)) \\ &\le \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \dots, x_i{}^{(0)}(k), \dots, x_m(k)) \le x_i{}^{(0)}(n), \end{aligned}$$

for all $n \in \mathbb{Z}$ *and* $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \prod_{j \ne i} [x_{j(0)}, x_j{}^{(0)}]_E$, $i = 1, 2, \dots, m$;

(ii) f_i *is nondecreasing with respect to* $x_i \in \left[\min_{k\in\mathbb{Z}} x_{i(0)}(k), \max_{k\in\mathbb{Z}} x_i{}^{(0)}(k)\right]$ *for every fixed* $(k, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{Z} \times \prod_{j\ne i} I_j$, $i = 1, 2, \dots, m$. *Then the system* (1.3) *has at least one solution* $(x_i) \in \prod_j [x_{j(0)}, x_j{}^{(0)}]_E$.

Proof. We define a subset B of the Banach space $\underbrace{E \times \cdots \times E}_{m}$ by

$$B = \{(x_i) \in \underbrace{E \times \cdots \times E}_{m} : x_{i(0)}(k) \le x_i(k) \le x_i{}^{(0)}(k), \forall k \in \mathbb{Z}\}.$$

It is easy to see that B is convex, closed and bounded. In addition, we define a mapping $F : B \to \underbrace{E \times \cdots \times E}_{m}$ by

$$\begin{aligned} F(x_1, \dots, x_m)(n) &= (\sum_{k=n-\tau_1}^{n} f_1(k, x_1(k), \dots, x_m(k)), \dots, \sum_{k=n-\tau_m}^{n} f_m(k, x_1(k), \dots, x_m(k))) \\ &:= (F_1(x_1, \dots, x_m)(n), \dots, F_m(x_1, \dots, x_m)(n)), \quad n \in \mathbb{Z}. \end{aligned}$$

For every $i \in \{1, 2, \dots, m\}$, $n \in \mathbb{Z}$ and $(x_i) \in B$, by (i) and (ii), we have

$$\begin{aligned} F_i(x_1, \dots, x_m)(n) &= \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \dots, x_i(k), \dots, x_m(k)) \\ &\le \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \dots, x_i{}^{(0)}(k), \dots, x_m(k)) \\ &\le x_i{}^{(0)}(n). \end{aligned}$$

Similarly, we can get

$$F_i(x_1, \dots, x_m)(n) \ge x_{i(0)}(n),$$

for every $i \in \{1,2,\ldots,m\}$, $n \in \mathbb{Z}$ and $(x_i) \in B$. Thus, we conclude that $F(B) \subset B$.

By a direct calculation and the continuity of f_i, we can show that every F_i is continuous. Thus, F is a continuous mapping. By the above proof, we have

$$0 \leq |F_i(x_1,\ldots,x_m)(n)| \leq \|x_i{}^{(0)}\|,$$

for all $i \in \{1,2,\ldots,m\}$, $n \in \mathbb{Z}$ and $\{x_i\} \in B$, which yields that every $F_i(B)$ is precompact in E. Then, we conclude that

$$F(B) = F_1(B) \times \cdots F_m(B)$$

is also precompact. Then, by Schauder's fixed point theorem, there exists a fixed point of F in B, which is just a solution of the system (1.3). □

Theorem 2.2. *Suppose that*

(i) For $i \in \{1,2,\ldots,m\}$, $I_i = [0, M_i]$, (M_i is positive constants)

$$f_i(k, x_1, \ldots, x_m) \leq \frac{M_i}{\tau_i}, \quad \forall (k, x_1, x_2, \ldots, x_m) \in \mathbb{Z} \times \prod_j [0, M_j].$$

(ii) For $i \in \{1,2,\ldots,m\}$,

$$\liminf_{x_i \to 0^+} \frac{f_i(k, x_1, \ldots, x_m)}{x_i} = a_i(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

uniformly for $(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z} \times \prod_{j \neq i} [0, M_j]$, where a_i is a continuous function satisfying

$$\min_{n \in \mathbb{N}_1^T} \sum_{k=n-\tau_i}^{n} a_i(k, x_1(k), \ldots, x_{i-1}(k), x_{i+1}(k), \ldots, x_m(k)) \geq m_i > 1$$

for all $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \prod_{j \neq i} [0, M_j]_E$, and m_i is a constant independent of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$.

(iii) For $i \in \{1,2,\ldots,m\}$, f_i is nondecreasing with respect to $x_i \in [0, M_i]$ for any fixed $(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z} \times \prod_{j \neq i} [0, M_j]$.

Then the system (1.3) *has at least one T-periodic solution with positive infinimum.*

Proof. It suffices to construct sub-super solutions for the system (1.3). Let $\epsilon \in (0,1)$ satisfying

$$m_i - \epsilon \tau_i \geq 1, \quad i = 1, 2, \ldots, m.$$

Then, by (ii), for every $i \in \{1,2,\ldots,m\}$, there exists $\delta_i \in (0, M_i)$ such that

$$f_i(k, x_1, \ldots, x_m) \geq (a_i(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) - \epsilon) x_i,$$

for all $x_i \in [0, \delta_i]$ and $(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z} \times \prod_{j \neq i} [0, M_j]$.

Let $x_{i(0)} \equiv \delta_i,\ {x_i}^{(0)} \equiv M_i$ for every $i \in \{1, 2, \ldots, m\}$. Then, for all $x_j \in [\delta_j, M_j]$ $(j \neq i)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}
x_{i(0)}(n) &= \delta_i \\
&\leq (m_i - \epsilon\tau_i) x_{i(0)}(n) \\
&\leq \sum_{k=n-\tau_i}^{n} (a_i(k, x_1(k), \ldots, x_{i-1}(k), x_{i+1}(k), \ldots, x_m(k)) - \epsilon) x_{i(0)}(n) \\
&\leq \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \ldots, x_{i-1}(k), x_{i(0)}(k), x_{i+1}(k), \ldots, x_m(k)) \\
&\leq \sum_{k=n-\tau_i}^{n} f_i(k, x_1(k), \ldots, x_{i-1}(k), {x_i}^{(0)}(k), x_{i+1}(k), \ldots, x_m(k)) \\
&\leq \sum_{k=n-\tau_i}^{n} \frac{M_i}{\tau_i} \\
&= {x_i}^{(0)}.
\end{aligned}$$

This completes the proof. $\square$

In the above two theorems, we only discuss the existence of T-periodic solutions for the system (1.3). Next, we present an uniqueness theorem.

Theorem 2.3. *For every $i \in \{1, 2, \ldots, m\}$, suppose that $I_i = [0, +\infty)$, f_i is nondecreasing with respect to every x_j in I_j $(j = 1, \ldots, m)$, and*

$$f_i(k, \alpha x_1, \ldots, \alpha x_m) > \alpha f_i(k, x_1, \ldots, x_m),$$

for all $\alpha \in (0, 1)$, $k \in \mathbb{Z}$ and $x_i \in (0, +\infty)$, $i = 1, \ldots, m$. Then the system (1.3) *has at most one T-periodic solution (x_i) satisfying $x_i(k) > 0,\ \forall k \in \mathbb{Z}$.*

Proof. Let (x_i^1) and (x_i^2) be two distinct T-periodic solution of (1.3) with

$$x_i^1(k) > 0,\ x_i^2(k) > 0,\ \forall k \in \mathbb{Z},\ i = 1, 2, \ldots, m.$$

Without loss for generality, we can assume that there exists $k_1 \in \mathbb{Z}$ such that $x_1^1(k_1) > x_1^2(k_1)$. Letting

$$\mu = \min\left\{ \frac{x_i^2(k)}{x_i^1(k)}, k \in \mathbb{Z}, i = 1, 2, \ldots, m \right\},$$

we have $0 < \mu < 1$ and

$$x_i^2(k) \geq \mu x_i^1(k), \quad k \in \mathbb{Z},\ i = 1, 2, \ldots, m.$$

Moreover, there exist $k_0 \in \mathbb{Z}$ and $j_0 \in \{1, 2, \ldots, m\}$ such that

$$x_{j_0}^2(k_0) = \mu x_{j_0}^1(k_0).$$

On the other hand, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} x_{j_0}^2(n) &= \sum_{k=n-\tau_{j_0}}^{n} f_{j_0}(k, \cdots, x_{j_0}^2(k), \cdots) \\ &\geq \sum_{k=n-\tau_{j_0}}^{n} f_{j_0}(k, \cdots, \mu x_{j_0}^1(k), \cdots) \\ &> \mu \sum_{k=n-\tau_{j_0}}^{n} f_{j_0}(k, \cdots, x_{j_0}^1(k), \cdots) \\ &= x_{j_0}^1(n), \end{aligned}$$

which is a contradiction. This completes the proof. □

Next, we give two examples, which do not aim at generality but illustrate how our theorems can be used.

Example 2.4. For every $i \in \{1, 2, \ldots, m\}$, let p_i be a positive constant, $I_i = [0, \frac{\pi}{2p_i}]$, and

$$f_i(k, x_1, \ldots, x_i, \ldots, x_m) = b_i(k) \sin(p_i x_i) c_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m),$$

where $b_i : \mathbb{Z} \to (0, +\infty)$ is a T-periodic function, and $c_i : \prod_{j \neq i} [0, \frac{\pi}{2p_j}] \to (0, +\infty)$ is a continuous function satisfying

$$\frac{1}{p_i \tau_i} < b_i(k) c_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \leq \frac{\pi}{2p_i \tau_i},$$

for all $(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z} \times \prod_{j \neq i} [0, \frac{\pi}{2p_j}]$. It is easy to verify that (H1) and (H2) hold. In addition, by a direct calculations, one can show that (i)-(iii) of Theorem 2.2 hold with $M_i = \frac{\pi}{2p_i}$, and

$$a_i(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) = p_i b_i(k) c_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m).$$

Thus, the system (1.3) has at least one T-periodic solution with positive infinimum.

Example 2.5. For every $i \in \{1, 2, \ldots, m\}(m \geq 2)$, let $I_i = [0, +\infty)$, and

$$f_i(k, x_1, \ldots, x_m) = b_i(k) \prod_{j=1}^{m} \frac{\sqrt[m]{x_j}}{l_j + x_j}$$

where $b_i : \mathbb{Z} \to (0, +\infty)$ is a T-periodic function, and l_j $(j = 1, 2, \ldots, m)$ are positive constants. Moreover, suppose that there exists a constant $a > 0$ such that

$$\prod_{j=1}^{m}(l_j + a) < \sum_{k=n-\tau_i}^{n} b_i(k) \leq \prod_{j=1}^{m}\left(l_j + \frac{\min\{l_1, l_2, \ldots, l_m\}}{m-1}\right),$$

for all $n \in \mathbb{Z}$ and $i = 1, 2, \ldots, m$.

It is easy to see that (H1) and (H2) hold. Let

$$x_{i(0)} \equiv a, \quad {x_i}^{(0)} \equiv \frac{\min\{l_1, l_2, \ldots, l_m\}}{m-1}, \quad i = 1, 2, \ldots, m.$$

Noting

$$\frac{\partial f_i}{\partial x_i} = b_i(k) \prod_{j \neq i} \frac{\sqrt[m]{x_j}}{l_j + x_j} \frac{\sqrt[m]{x_i}(l_i + (1-m)x_i)}{m x_i (l_i + x_i)^2}, \quad i = 1, 2, \ldots, m,$$

We conclude that every f_i is nondecreasing with respect to $x_i \in \left[x_{i(0)}, {x_i}^{(0)}\right]$ for every fixed $(k, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z} \times \prod_{j \neq i} \left[x_{i(0)}, {x_i}^{(0)}\right]$.

Moreover, for all $x_j \in [x_{j(0)}, {x_j}^{(0)}]$ $(j \neq i)$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}
x_{i(0)}(n) &= x_{i(0)} \\
&= \prod_{j=1}^{m}(l_j + a) \prod_{j=1}^{m} \frac{\sqrt[m]{x_{j(0)}}}{l_j + x_{j(0)}} \\
&< \sum_{k=n-\tau_i}^{n} b_i(k) \prod_{j=1}^{m} \frac{\sqrt[m]{x_{j(0)}}}{l_j + x_{j(0)}} \\
&\leq \sum_{k=n-\tau_i}^{n} b_i(k) \prod_{j \neq i} \frac{\sqrt[m]{x_j(k)}}{l_j + x_j(k)} \frac{\sqrt[m]{x_{i(0)}(k)}}{l_i + x_{i(0)}(k)} \\
&\leq \sum_{k=n-\tau_i}^{n} b_i(k) \prod_{j \neq i} \frac{\sqrt[m]{x_j(k)}}{l_j + x_j(k)} \frac{\sqrt[m]{{x_i}^{(0)}(k)}}{l_i + {x_i}^{(0)}(k)} \\
&\leq \sum_{k=n-\tau_i}^{n} b_i(k) \prod_{j=1}^{m} \frac{\sqrt[m]{{x_j}^{(0)}}}{l_j + {x_j}^{(0)}} \\
&\leq \prod_{j=1}^{m}\left(l_j + \frac{\min\{l_1, l_2, \ldots, l_m\}}{m-1}\right) \prod_{j=1}^{m} \frac{\sqrt[m]{{x_j}^{(0)}}}{l_j + {x_j}^{(0)}} \\
&= {x_i}^{(0)} = {x_i}^{(0)}(n),
\end{aligned}$$

which means that $(x_{i(0)})$, $({x_i}^{(0)})$ is a pair of the sub-super solutions for the system (1.3).

In addition, for all $\alpha \in (0, 1)$, $k \in \mathbb{Z}$ and $x_i \in (0, +\infty)$ $(i = 1, \ldots, m)$, there holds

$$f_i(k, \alpha x_1, \ldots, \alpha x_m) = b_i(k) \prod_{j=1}^{m} \frac{\sqrt[m]{\alpha x_j}}{l_j + \alpha x_j} > \alpha b_i(k) \prod_{j=1}^{m} \frac{\sqrt[m]{x_j}}{l_j + x_j} = \alpha f_i(k, x_1, \ldots, x_m).$$

Then, combining Theorem 2.1 and Theorem 2.3, we know that the system (1.3) has a unique solution (x_i) such that $x_i(k) > 0,\ k \in \mathbb{Z},\ i = 1, 2, \ldots, m$.

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MECHANICAL MODELS WITH INTERNAL BODY FORCES

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Abstract. The method of additional conditions or MAC allows to create new mathematical models in mechanics and physics. This method was used to put some additional terms into the classical statements of the problems using the test problem. The only requirement was to include the solution of the test problem into new equations. This approach seems to be too formal. Therefore this paper suggests a mechanical method to put additional terms into the traditionally accepted theories. The additional terms in the equations of motion in continuum mechanics appear as a result of the application of the constitutive laws for the body forces and body moments. The theories of the string, beam, membrane, plate and elasticity are described in the paper including the internal body forces. The displacements potentials in elasticity with internal body forces are introduced similar to the Galerkin potential.

1. Introduction

The statement of the problems in the modern continuum mechanics includes the constitutive law for stresses and does not consider the constitutive law for the internal body forces and body moments. An elastic or fluid body with the given displacement of its one point create the infinite stresses acting near that point in the body [3], [4], [7], [8],[9], [10], [11], [21], [22]. The body forces are considered as the external forces like gravitational, electromagnetic forces [5]. Then the linearized theories must accept the solutions with nonphysical singularities in displacements and temperature. The introduction of the internal body forces allows to improve the solutions of the problems at least in the sense of excluding the nonphysical singularities.

2. Internal body forces and moments

Consider a real solid and let us take some control volume, which includes a fixed number of particles. The control volume is surrounded by a control surface. The particles which are inside the control surface are internal particles and they belong to the control volume. The particles which are outside the control surface are the external particles and they do not belong to control volume. All other particles belong to the boundary particles of the control volume.

There are interactions between particles. The resultant of the forces applied to all internal particles of the control volume from the external particles is the internal body force. The principle moment of the forces and moments applied to all internal forces from the external particles is the internal body moment. The

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forces and moments applied to the boundary particles of the control volume from the external particles are the surface forces and moments.

Continuum mechanics considers the limit as the control volume tends to zero. Then there are two real possibilities. The first one is when the limit of the control volume will come to a point in an empty space. Then there are no body forces and moments for a small enough volume. The second case is when the limiting point belongs to some particle and the control volume finally consists of one particle inside a control surface and there are no any particles belonging to the control surface. Then there are the body force and body moment and there are no surface forces and moments. It means that the continuum mechanics gives just a mathematical model to the real solid, but it is not unique model.

Continuum mechanics accepts stresses and the constitutive law for stresses. But the constitutive laws for the internal body forces and moments are ignored. There are two other possibilities. The first one is to ignore the stresses and to consider just the internal body forces and moments and the constitutive laws for them. The second possibility is to accept the constitutive law for stresses together with the constitutive laws for the internal body forces and moments.

This paper will use the following constitutive law for the internal body forces in elastic solid, which does not move as a rigid body:

$$\mathbf{f} = -\alpha_1 \mathbf{u} - \alpha_2 \nabla^2 \mathbf{u} - \alpha_3 \nabla^4 \mathbf{u}, \tag{2.1}$$

where $\mathbf{u}$ is the displacement vector, $\alpha_1, \alpha_2, \alpha_3$ are the material constants, which we suppose to be nonnegative, ∇ is the gradient.

The constitutive law for the body moment is used in the beam and plate theories in this paper.

The equation (2.1) will change in general the values and the number of speeds of harmonic waves in continuum medium. But the dynamical problems in solids will not be considered in this paper.

The constitutive law for the internal body forces in fluid mechanics can be taken in the similar form

$$\mathbf{f} = -\beta_1 \mathbf{v} - \beta_2 \nabla^2 \mathbf{v} - \beta_3 \nabla^4 \mathbf{v}, \tag{2.2}$$

where $\mathbf{v}$ is the velocity vector, $\beta_1, \beta_2, \beta_3$ are the material constants, which are supposed to be nonnegative, ∇ is the gradient.

3. String with internal body forces

3.1. **Statement of the problem.** Many books and papers consider the statement of the string problem, for example [3], [12], [19], [23], [28]. The equation of one-dimensional motion of the string is taken in the form

$$T_0 \frac{\partial^2 u}{\partial x^2} - \alpha_1 u - \alpha_2 \frac{\partial^2 u}{\partial x^2} - \alpha_3 \frac{\partial^4 u}{\partial x^4} = \rho \frac{\partial^2 u}{\partial t^2} - q(x,t), \tag{3.1}$$

where T_0 is the tension applied to the string, $x-$ is a Cartesian coordinate of a cross-section, $0 \leq x \leq L$, $L-$ is the length of the string, $\rho-$ is the density of mass per unit length, $u-$ is the transversal displacement of a cross-section, $t-$ is time, $q(x,t)-$ is the density of the transversal external body forces per unit length. The density of the transversal internal body forces per unit length is taken in the form of Eq. (2.1).

3.2. **Example of string without internal body forces.** Consider a simple particular example to show that the theory of the string with internal body forces has solution, where the classical problem does not have any one.

Let us take the steady state problem without any given distributed external forces and the length of the string is infinite. Then the classical equation is

$$\frac{d^2u}{dx^2} = 0. \tag{3.2}$$

If the boundary conditions are

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{3.3}$$

then it is easy to see, that the solution of the stated problem Eqs. (3.2), (3.3) does not exist.

3.3. **Example 1 of string with internal body forces.** Consider now the same steady state problem for the string with the internal body forces. The differential equation of the problem at $\alpha_3 = 0$ is

$$T_0\frac{d^2u}{dx^2} - \alpha_1 u - \alpha_2\frac{d^2u}{dx^2} = 0. \tag{3.4}$$

The boundary conditions are the Eq. (3.3). The solution of the problem (3.3), (3.4) with internal body force exists and equals

$$u = u_0 \exp(\lambda x), \tag{3.5}$$

where

$$\lambda = -\sqrt{\frac{\alpha_1}{T_0 - \alpha_2}}. \tag{3.6}$$

The above solution Eq. (3.5) exists if

$$T_0 > \alpha_2. \tag{3.7}$$

3.4. **Example 2 of string with internal body forces.** If we consider more general problem with internal body force and $\alpha_3 \neq 0$, then the differential equation of the problem will take the form

$$T_0\frac{d^2u}{dx^2} - \alpha_1 u - \alpha_2\frac{d^2u}{dx^2} - \alpha_3\frac{d^4u}{dx^4} = 0. \tag{3.8}$$

The boundary conditions are taken the Eqs. (3.2), (3.3):

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{3.9}$$

$$\frac{d^2u}{dx^2}(0) = 0, \frac{d^2u}{dx^2}(\infty) = 0. \tag{3.10}$$

The boundary conditions Eqs. (3.10) are obtained as follows - we require that the equation Eq. (3.2) without body forces should be satisfied at the boundary.

The solution of the problem with internal body forces Eqs. (3.8), (3.9), (3.10) exists and equals

$$u = \frac{u_0}{\lambda_2^2 - \lambda_1^2}(\lambda_2^2 \exp \lambda_1 x - \lambda_1^2 \exp \lambda_2 x), \tag{3.11}$$

where

$$\lambda_{1,2} = -\sqrt{\frac{T_0 - \alpha_2 \pm \sqrt{(T_0 - \alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}, \tag{3.12}$$

where two inequalities should be fulfilled. The first inequality is the Eq. (3.7) and the second one is

$$(T_0 - \alpha_2)^2 - 4\alpha_1\alpha_3 > 0. \tag{3.13}$$

If the left hand-side of the Eq. (3.13) equals to zero, then the solution will take the form

$$u = u_0 \left(1 + \frac{\lambda x}{2}\right) \exp(-\lambda x), \tag{3.14}$$

where

$$\lambda = \sqrt{\frac{2\alpha_1}{T_0 - \alpha_2}}. \tag{3.15}$$

The considered example of the string problem shows that the introduced internal body forces allow to obtain solutions in the cases, where the classical problem does not have any solution.

4. Beam with internal body forces

4.1. **Statement of the problem.** Consider an elastic beam [12]. The equation of motion of the beam with internal body forces and body moments could be written in the form

$$(EI + \alpha_3)\frac{\partial^4 u}{\partial x^4} - (T + \alpha_4 - \alpha_2)\frac{\partial^2 u}{\partial x^2} + \alpha_1 u - q(x,t) + \rho\frac{\partial^2 u}{\partial t^2} = 0, \tag{4.1}$$

where EI is the bending stiffness of the beam, $x-$ is the Cartesian coordinate of a cross-section, $0 \leq x \leq L$, $L-$ is the length of the beam, $\rho-$ is the density of mass per unit length, $u-$ is the transversal displacement of a cross-section, $t-$ is time, $q(x,t)-$ is the density of the transversal external body forces per unit length, $T-$ is the tension. The internal transversal body forces are taken in the following form

$$f = -\alpha_1 u - \alpha_2\frac{\partial^2 u}{\partial x^2} - \alpha_3\frac{\partial^4 u}{\partial x^4}. \tag{4.2}$$

The internal body moments could be taken in the following form

$$m = -\alpha_4\theta - \alpha_5\nabla^2\theta - \alpha_6\nabla^4\theta, \tag{4.3}$$

where

$$\theta = \frac{\partial u}{\partial x} \tag{4.4}$$

and it is included into the angular momentum equation for an infinitesimal cross-sectional element of the beam

$$N = \frac{\partial M}{\partial x} - m, \tag{4.5}$$

where $M-$ is the bending moment, $N-$ is the transversal shear force, $\alpha_1, \alpha_2, \alpha_3-$ are materials constants.

4.2. **Example of beam without internal body forces and moments.** Consider a simple particular example to show that the theory of the beam with internal body forces has solution, where the classical problem does not have any one.

Let us take the steady state problem without any given distributed external forces and without the tension T, the length of the beam is infinite. Then the classical equation is

$$\frac{d^4u}{dx^4} = 0. \tag{4.6}$$

If the boundary conditions are

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{4.7}$$

$$\frac{d^2u}{dx^2}(0) = 0, \frac{d^2u}{dx^2}(\infty) = 0. \tag{4.8}$$

Then it is easy to see, that the solution of the stated problem Eqs. (4.6), (4.7), (4.8) does not exist.

4.3. **Example of beam with internal body forces and moments.** If we consider the beam problem with internal body forces, where $\alpha_5 = 0, \alpha_6 = 0$ then the differential equation of the problem will take the form

$$(EI + \alpha_3)\frac{d^4u}{dx^4} + (\alpha_2 - \alpha_4)\frac{d^2u}{dx^2} + \alpha_1 u = 0. \tag{4.9}$$

The boundary conditions are taken the Eqs. (4.7), (4.8).

The solution of the problem with internal body forces Eqs. (4.7), (4.8), (4.9) exists and equals

$$u = \frac{u_0}{\lambda_2^2 - \lambda_1^2}(\lambda_2^2 \exp \lambda_1 x - \lambda_1^2 \exp \lambda_2 x), \tag{4.10}$$

where

$$\lambda_{1,2} = -\sqrt{\frac{\alpha_4 - \alpha_2 \pm \sqrt{(\alpha_4 - \alpha_2)^2 - 4\alpha_1(\alpha_3 + EI)}}{2(EI + \alpha_3)}}, \tag{4.11}$$

where the following two inequalities should be fulfilled

$$\alpha_4 > \alpha_2, \tag{4.12}$$

$$(\alpha_4 - \alpha_2)^2 - 4\alpha_1(EI + \alpha_3) > 0. \tag{4.13}$$

If the left-hand side of the Eq. (4.13) equals to zero, then the solution will take the form

$$u = u_0\left(1 + \frac{\lambda x}{2}\right)\exp(-\lambda x), \tag{4.14}$$

where

$$\lambda = \sqrt{\frac{2\alpha_1}{\alpha_4 - \alpha_2}}. \tag{4.15}$$

The considered example of the beam problem shows that the introduced internal body forces and the internal body moments allow to obtain solutions in the cases, where the classical problem does not have any solution.

5. Membrane with internal body forces

5.1. **Statement of the problem.** Let us consider an elastic membrane. The equation of motion of the membrane is described in [2], [16], [19], [23], [25], [27] and [28]. This membrane equation with internal body forces is

$$T_0\nabla^2 u - \alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u + q(x,y,t) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{5.1}$$

where the membrane lies in the plane (x, y) in its natural state, T_0 is its tension per a unit of length, $u(x, y, t)$ is the transversal displacement of the point (x, y) of the initially plane membrane, ρ is the density of mass per unit area, t is time, $q(x, y, t)$ is the density of the transversal external body forces per unit area. The tension T_0 is constant in this statement of the problem. The internal transversal body forces are taken in the following form

$$f = -\alpha_1 u - \alpha_2\nabla^2 u - \alpha_3\nabla^4 u. \tag{5.2}$$

5.2. **Example of membrane without body forces.** Consider a simple particular example to show that the theory of the membrane with internal body forces has solution, where the classical problem does not have any one.

Let us take the steady state problem without any given distributed external forces and the external boundary of the membrane lies at infinity. It means that for any external boundary point is required $\sqrt{x^2+y^2} \to \infty$. Then the classical equation is

$$\nabla^2 u = 0. \tag{5.3}$$

If the boundary conditions are

$$u(0) = u_0 \neq 0, u(\infty) = 0, \tag{5.4}$$

then it is easy to see, that the solution of the stated problem Eqs. (5.3), (5.4) does not exist. The given problem is symmetric in this case and solution should depend on r only. The polar coordinates are taken with the origin at a given point. Then the equation Eq. (5.3) will take the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0. \tag{5.5}$$

The general solution of the Eq. (5.5) is

$$u = A + B\ln r, \tag{5.6}$$

where A, B are arbitrary constants. These constants cannot be found using both boundary conditions (5.4). Then the required solution does not exist.

If we accept the singularity at the origin then the solution, which satisfies the condition at infinity, is

$$u = 0. \tag{5.7}$$

This solution does not satisfy the real situation with membranes [18]), but it satisfies the condition at infinity.

The nonlinear membrane equation was considered in [29], [30]. Unfortunately the experimental solutions are not the solutions of the Zhilin's membrane equation.

5.3. **Example 1 of membrane with internal body forces.** Consider now the same steady state problem for the membrane with the internal body forces. The differential equation of the problem at $\alpha_3 = 0$ is

$$T_0 \nabla^2 u - \alpha_1 u - \alpha_2 \nabla^2 u = 0. \tag{5.8}$$

The solution of the equation Eq. (5.8) is considered in the form

$$u = u(r). \tag{5.9}$$

The boundary conditions are the Eq. (5.4).

Then the Eq. (5.8) will take the form

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - s^2 u = 0, \tag{5.10}$$

where

$$s = \sqrt{\frac{\alpha_1}{T_0 - \alpha_2}}, \tag{5.11}$$

and it is required that

$$T_0 - \alpha_2 > 0. \tag{5.12}$$

The general solution of the Eq. (5.10) is

$$u(r) = C_1 I_0(sr) + C_2 K_0(sr), \tag{5.13}$$

where I_0, K_0 are the Macdonald functions, C_1, C_2 are arbitrary constants. The functions I_0, K_0 have the following limit values:

$$I_0(0) = 1, I_0(\infty) = \infty, K_0(0) = \infty, K_0(\infty) = 0. \tag{5.14}$$

It means that the general solution (5.13) cannot satisfy the boundary conditions (5.4) and the solution of the stated problem (5.10), (5.4) does not exists.

The singularity at the origin under the applied force is often accepted in the classical theories. If we do it then the function I_0 will be excluded and the solution of the problem will take the form

$$u(r) = C_2 K_0(sr), \tag{5.15}$$

where the constant C_2 should be obtained from balance of forces applied to the membrane.

We see in this particular problem that the internal body forces introduced into the classical problem exclude the singularity at infinite but the singularity at the origin remains.

This model of membrane uses the Bessel equation. Another model was developed in [17] where the Airy equation [26] was a tool to describe the membrane behavior.

5.4. **Example 2 of membrane with internal body forces.** If we consider the more general steady state membrane problem with the internal body forces and $\alpha_3 \neq 0$, then the differential equation of the problem will take the form

$$T_0 \nabla^2 u - \alpha_1 u - \alpha_2 \nabla^2 u - \alpha_3 \nabla^4 u = 0. \tag{5.16}$$

We are looking for a solution of the Eq. (5.16) $u = u(r)$, which satisfies the boundary conditions Eq. (5.4). We will see that these boundary conditions Eq. (5.4) are sufficient to obtain the solution of the given problem. The Eq. (5.16) could be written in the following form

$$(\nabla^2 - s_1^2)(\nabla^2 - s_2^2) u = 0, \tag{5.17}$$

where

$$s_1 = |\lambda_1|, s_2 = |\lambda_2|, \tag{5.18}$$

and λ_1, λ_2 are given according to the Eq. (3.12). It is supposed that the inequalities (3.7), (3.13) are fulfilled also.

The general solution of the Eq. (5.17) is the sum of two functions:

$$u = u_1 + u_2, \tag{5.19}$$

where u_1, u_2 are the general solutions of the equations

$$\nabla^2 u_1 - s_1^2 u_1 = 0, \tag{5.20}$$

$$\nabla^2 u_2 - s_2^2 u_2 = 0. \tag{5.21}$$

The Eqs. (5.20), (5.21) are the same equations as the Eq. (5.10). Then the general solution of the Eq. (5.17) will be

$$u = C_1 I_0(s_1 r) + C_2 K_0(s_1 r) + C_3 I_0(s_2 r) + C_4 K_0(s_2 r), \tag{5.22}$$

where I_0, K_0 are the Macdonald functions and C_1, C_2, C_3, C_4 are the arbitrary constants.

If $r \to \infty$ then

$$I_0(s_1 r) = \frac{\exp(s_1 r)}{\sqrt{2\pi s_1 r}} \left[1 + O\left(\frac{1}{s_1 r}\right)\right] \tag{5.23}$$

and

$$I_0(s_2 r) = \frac{\exp(s_2 r)}{\sqrt{2\pi s_2 r}} \left[1 + O\left(\frac{1}{s_2 r}\right)\right]. \tag{5.24}$$

The behavior of these functions Eqs. (5.23), (5.24) shows that the condition at infinity Eq. (5.4) will be satisfied only in the case

$$C_1 = 0, C_3 = 0. \tag{5.25}$$

The function K_0 has the property

$$\lim_{r\to\infty} K_0(s_1 r) = \lim_{r\to\infty} K_0(s_2 r) = 0. \tag{5.26}$$

Consider now the functions $K_0(s_1 r), K_0(s_2 r)$ near the origin. We have

$$K_0(s_1 r) = -I_0(s_1 r)\left[\ln\left(\frac{s_1 r}{2}\right) + C\right] + \sum_{k=0}^{\infty} \frac{\Phi(k)}{(k!)^2}\left(\frac{s_1 r}{2}\right)^{2k}, \tag{5.27}$$

where

$$\Phi(k) = \sum_{s=1}^{k} \frac{1}{s}, \Phi(0) = 0, \tag{5.28}$$

$$I_0(s_1 r) = \sum_{\nu=0}^{\infty} \frac{1}{(\nu!)^2}\left(\frac{s_1 r}{2}\right)^{2\nu}, \tag{5.29}$$

the Euler constant is

$$C = 0.5772\ldots. \tag{5.30}$$

If r is small then the function Eq. (5.27) will be

$$K_0(s_1 r) = -\ln\frac{s_1 r}{2} + O(r^2 \ln r). \tag{5.31}$$

It could be obtained similarly that

$$K_0(s_2r) = -\ln\frac{s_2r}{2} + O(r^2\ln r). \tag{5.32}$$

Using the Eqs. (5.25), (5.31), (5.32) the general solution Eq. (5.22) will take the form

$$u(r) = -C_3\ln\frac{s_1r}{2} - C_4\ln\frac{s_2r}{2} + O(r^2\ln r) \tag{5.33}$$

or

$$u(r) = -(C_3+C_4)\ln r - C_3\ln\frac{s_1}{2} - C_4\ln\frac{s_2}{2} + O(r^2\ln r). \tag{5.34}$$

The logarithmic singularity in Eq. (5.34) will be excluded if we take

$$C_4 = -C_3. \tag{5.35}$$

Then the Eq. (5.33) will be transformed to the form

$$u(r) = C_3\ln\frac{s_2}{s_1} + O(r^2\ln r). \tag{5.36}$$

The constant C_3 could be obtained if we satisfy the first boundary condition in the Eq. (5.4) and we find

$$u(r) = \frac{u_0}{\ln\frac{s_2}{s_1}}\left[K_0(s_1r) - K_0(s_2r)\right]. \tag{5.37}$$

This example shows that the solution does not have a singularity at the origin and at the infinity and that corresponds to the real situation with real membrane. As we have seen this is impossible in the classical theory.

6. Plate with internal body forces

6.1. **Statement of the problem.** There are many books, where the different plates problems are taken into consideration, for example [6], [14], [15], [23], [25], [27], [28] and many other papers and manuscripts. Let us consider an elastic plate with constant flexural rigidity and with internal body forces and the internal body moments. The governing equations in cartesian coordinates are

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + f = \rho h\frac{\partial^2 w}{\partial t^2}, \tag{6.1}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - m_x, \tag{6.2}$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - m_y, \tag{6.3}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{6.4}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \tag{6.5}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}, \tag{6.6}$$

where t is time variable, x, y, z are Cartesian coordinates, x, y in plane, ρ is the density, h is the plate thickness, E, G are Young modulus and shear modulus, ν is the Poisson ratio, u, v, w are the displacements in x, y, z directions, M_x, M_y, M_{xy} are the bending and twisting moments per unit length, Q_x, Q_y are the transverse shear forces per unit length, q is the transverse loading per unit area, f is the transverse internal body force per unit area, m_x, m_y are the internal body moments per unit area.

The flexural rigidity is

$$D = \frac{Eh^3}{12(1-\nu^2)}. \tag{6.7}$$

The internal body force f is taken in the form

$$f = -\alpha_1 w - \alpha_2 \nabla^2 w - \alpha_3 \nabla^4 w, \tag{6.8}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the material constants.

The internal body moments m_x, m_y are taken in the form

$$m_x = -\alpha_4 \theta_x - \alpha_5 \nabla^2 \theta_x - \alpha_6 \nabla^4 \theta_x, \tag{6.9}$$

$$m_y = -\alpha_4 \theta_y - \alpha_5 \nabla^2 \theta_y - \alpha_6 \nabla^4 \theta_y, \tag{6.10}$$

where $\alpha_4, \alpha_5, \alpha_6$ are the material constants and

$$\theta_x = \frac{\partial w}{\partial x}, \theta_y = \frac{\partial w}{\partial y}. \tag{6.11}$$

If we substitute the Eqs. (6.2), (6.3), (6.4), (6.5) and (6.6) into the Eq. (6.1) then we get

$$D\nabla^4 w - \frac{\partial m_x}{\partial x} - \frac{\partial m_y}{\partial y} + q + f = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{6.12}$$

If the Eqs. (6.8), (6.9), (6.10), (6.11) are used for the expressions of the internal body forces and body moments then the equation governing the transverse motion of the plate will take the form

$$\alpha_6 \nabla^6 w + (\alpha_5 - \alpha_3 - D)\nabla^4 w + (\alpha_4 - \alpha_2)\nabla^2 w - \alpha_1 w + q = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{6.13}$$

6.2. **Example of plate without internal body forces.** Consider a simple particular example to show that the theory of the plate with internal body forces has solution, where the classical problem does not have any one.

Let us take the steady state plate problem without any given distributed external forces and the external boundary of the plate lies at infinity. It means that for any external boundary point is required $\sqrt{x^2 + y^2} \to \infty$. Then the classical equation is

$$\nabla^4 w = 0. \tag{6.14}$$

Consider symmetric problem, where the solution w is a function on r only, where r is the distance of a given point to the origin.

If the boundary conditions are

$$w(0) = w_0 \neq 0, w(\infty) = 0 \tag{6.15}$$

and

$$\frac{dw}{dr}(0) = 0, \frac{dw}{dr}(\infty) = 0, \tag{6.16}$$

then it is easy to see, that the solution of the stated problem Eqs. (6.14), (6.15), (6.16) does not exist. To show that consider the equation Eq. (6.14). It will take the form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)=0. \tag{6.17}$$

The general solution of the Eq. (6.17) is

$$w=A_1+A_2\ln r+A_3r^2+A_4r^2\ln r, \tag{6.18}$$

where A_1, A_2, A_3, A_4 are arbitrary constants. These constants cannot be found using both boundary conditions Eqs. (6.15) and (6.16). Then the required solution does not exist.

If we accept the singularity at the origin then the solution, which satisfies the conditions at infinity, is

$$w=0. \tag{6.19}$$

This solution does not satisfy the real situation with plates, but it satisfies the conditions at infinity.

6.3. **Example of plate with internal body forces.** Consider now the same steady state problem for the plate with the internal body forces and without the internal body moments. The differential equation of the problem at $\alpha_2=0, \alpha_3=0$ is

$$D\nabla^4 w+\alpha_1 w=0. \tag{6.20}$$

The solution of the equation Eq. (6.20) with the boundary conditions (6.15), (6.16) is considered in [25] as a H. Herz problem for an infinite plate on elastic support under a transversal force applied to one point of a plate. The coefficient α_1 in H. Herz problem belongs to the external to the plate elastic support. We consider the plate without any external elastic support but with the internal body forces. The solution of the H. Herz problem is as follows.

The displacements are

$$w=-\frac{Pl^2}{2\pi D}\mathbf{kei}(x), \tag{6.21}$$

where

$$l^4=\frac{D}{\alpha_1}, x=\frac{r}{l} \tag{6.22}$$

and $\mathbf{kei}(x)$ is the Kelvin function. P is the applied external force.

If x is small then

$$\mathbf{kei}(x)=-\left(\frac{x^2}{4}\right)\ln x-\frac{\pi}{4}+(1+\ln 2-C)\frac{x^2}{4}+\ldots, \tag{6.23}$$

where $C=0.5772\ldots$ is the Euler constant.

If x is large then

$$\mathbf{kei}(x)\sim\frac{\exp\left(-\frac{x}{\sqrt{2}}\right)}{\sqrt{\frac{2x}{\pi}}}\sin\left(\frac{x}{\sqrt{2}}+\frac{\pi}{8}\right). \tag{6.24}$$

The solution (6.21) could be accepted because it gives the finite displacement under the applied force. But the form of the solution at large x (6.24) is not applicable in case of the plate without an elastic support.

The solution (6.21) creates also the infinite bending stresses under the applied force because the bending moments for small r are

$$M_r \sim \frac{P(1+\nu)}{4\pi} \ln \frac{2l}{r}, \tag{6.25}$$

$$M_t \sim \frac{P(1+\nu)}{4\pi} \ln \frac{2l}{r}. \tag{6.26}$$

6.4. **Example 1 of plate with internal body forces and moments.** Consider now the same steady state plate problem but the internal body moments are also included. the differential equation of the problem at

$$\alpha_2 = 0, \alpha_3 = 0, \alpha_5 = 0, \alpha_6 = 0 \tag{6.27}$$

is

$$D\nabla^4 w - \alpha_4 \nabla^2 w + \alpha_1 w = 0. \tag{6.28}$$

The Eq. (6.28) will be the same Eq. (5.16) if we replace the parameters D, α_4 through the parameters $\alpha_3, T_0 - \alpha_2$ respectively. Then we take the solution (5.37), where

$$s_1 = \sqrt{\frac{\alpha_4 + \sqrt{\alpha_4^2 - 4\alpha_1 D}}{2D}}, \tag{6.29}$$

$$s_2 = \sqrt{\frac{\alpha_4 - \sqrt{\alpha_4^2 - 4\alpha_1 D}}{2D}}. \tag{6.30}$$

The solution now could be written in the form

$$w = \frac{w_0}{\ln \frac{s_2}{s_1}} \left[K_0(s_1 r) - K_0(s_2 r) \right]. \tag{6.31}$$

The solution Eq. (6.31) satisfies the boundary conditions Eqs. (6.15) as was stated above. To satisfy the boundary conditions Eqs. (6.16) we have to consider the derivative of the function w Eq. (6.31).

$$\frac{dw}{dr} = \frac{w_0}{\ln \frac{s_2}{s_1}} \left[\frac{dK_0(s_1 r)}{dr} - \frac{dK_0(s_2 r)}{dr} \right] = \frac{w_0}{\ln \frac{s_2}{s_1}} \left[s_2 K_1(s_2 r) - s_1 K_1(s_1 r) \right] \tag{6.32}$$

or

$$\frac{dw}{dr} = \frac{w_0}{\ln \frac{s_2}{s_1}} \left\{ -s_1 I_1(s_1 r) \left[\ln\left(\frac{s_1 r}{2}\right) + C \right] + s_2 I_1(s_2 r) \left[\ln\left(\frac{s_2 r}{2}\right) + C \right] \right\} + \tag{6.33}$$

(6.34)
$$+\frac{w_0}{\ln \frac{s_2}{s_1}} \left\{ -\frac{1}{r} \left[I_0(s_1 r) - I_0(s_2 r) \right] + \sum_{k=1}^{\infty} \frac{1}{(k-1)!k!} \left[s_1 \left(\frac{s_1 r}{2}\right)^{2k-1} - s_2 \left(\frac{s_2 r}{2}\right)^{2k-1} \right] \right\},$$

where

$$\frac{dI_0(x)}{dx} = I_1(x), \frac{dK_0(x)}{dx} = -K_1(x). \tag{6.35}$$

The Eq. (6.32) shows that

$$\lim_{r\to\infty}\frac{dw}{dr}=0 \tag{6.36}$$

and the second condition of the Eqs. (6.16) is satisfied. The Eqs. (6.33), (6.34) show that

$$\lim_{r\to 0}\frac{dw}{dr}=0 \tag{6.37}$$

and the first condition of the Eqs. (6.16) is fulfilled.

The logarithmic singularity in bending moments at $r=0$ remains in the case of the solution Eq. (6.31). We can show that if the moments per unit length are considered.

$$M_r=-D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right), \tag{6.38}$$

$$M_\theta=-D\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right), \tag{6.39}$$

$$M_{r\theta}=0. \tag{6.40}$$

Consider now the expressions $\frac{1}{r}\frac{dw}{dr}$ and $\frac{d^2w}{dr^2}$ at small r. We will get

$$\frac{1}{r}\frac{dw}{dr}\sim O(1) \tag{6.41}$$

and

$$\frac{d^2w}{dr^2}\sim\frac{w_0}{\ln\frac{s_2}{s_1}}(s_1^2-s_2^2)\ln r. \tag{6.42}$$

Then the moments Eqs. (6.38), (6.39) are

$$M_r\sim D\frac{w_0}{\ln\frac{s_2}{s_1}}(s_1^2-s_2^2)\ln r, \tag{6.43}$$

and

$$M_\theta\sim\nu D\frac{w_0}{\ln\frac{s_2}{s_1}}(s_1^2-s_2^2)\ln r \tag{6.44}$$

at small r. The singularity in bending stresses near the applied force could be excluded using the more general constitutive law for the internal body moments. It will be shown in the next example.

6.5. **Example 2 of plate with internal body forces and moments.** Let us take the following equation Eq. (6.13) without any external pressure q and inertial term also.

$$\alpha_6\nabla^6w+(\alpha_5-\alpha_3-D)\nabla^4w+(\alpha_4-\alpha_2)\nabla^2w-\alpha_1w=0. \tag{6.45}$$

The boundary conditions are Eqs. (6.15), (6.16).

The characteristic algebraic equation corresponding to the Eq. (6.45) is

$$\lambda^3+r\lambda^2+s\lambda+t=0, \tag{6.46}$$

where

$$r=\frac{\alpha_5-\alpha_3-D}{\alpha_6},s=\frac{\alpha_4-\alpha_2}{\alpha_6},t=-\frac{\alpha_1}{\alpha_6}. \tag{6.47}$$

Let the parameters in Eq. (6.47) satisfy the inequalities

$$r < 0, s > 0, t < 0. \tag{6.48}$$

We suppose that the Eq. (6.46) has three real roots. This will be the case if the discriminant of the Eq. (6.46) is negative:

$$\left(\frac{p}{3}\right)^3 - \left(\frac{q}{2}\right)^2 < 0, \tag{6.49}$$

where

$$p = \frac{3s - r^2}{3}, \tag{6.50}$$

$$q = \frac{2r^3}{27} - \frac{rs}{3} + t. \tag{6.51}$$

It follows from the Routh-Hurwitz theorem that all three roots of the Eq. (6.46) will be positive if the additional inequality is true

$$t - sr > 0. \tag{6.52}$$

If the Eq. (6.46) has three positive roots $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ then the Eq. (6.45) could be written in the form

$$(\nabla^2 - \lambda_1)(\nabla^2 - \lambda_2)(\nabla^2 - \lambda_3)w = 0. \tag{6.53}$$

The general solution of the Eq. (6.53) is the sum of three functions:

$$w = w_1 + w_2 + w_3, \tag{6.54}$$

where w_1, w_2, w_3 are the general solutions of the equations

$$\nabla^2 w_1 - \lambda_1 w_1 = 0, \tag{6.55}$$

$$\nabla^2 w_2 - \lambda_2 w_2 = 0, \tag{6.56}$$

$$\nabla^2 w_3 - \lambda_3 w_3 = 0. \tag{6.57}$$

The Eqs. (6.55), (6.56), (6.57) are the same equations as the Eq. (5.10). Then the general solution of the Eq. (6.53) will be

$$w = C_1 I_0(\sqrt{\lambda_1}r) + C_2 K_0(\sqrt{\lambda_1}r) + C_3 I_0(\sqrt{\lambda_2}r) + C_4 K_0(\sqrt{\lambda_2}r) + C_5 I_0(\sqrt{\lambda_3}r) + C_6 K_0(\sqrt{\lambda_3}r), \tag{6.58}$$

where I_0, K_0 are the Macdonald functions and $C_1, C_2, C_3, C_4, C_5, C_6$ are the arbitrary constants.

If $r \to \infty$ then

$$I_0(\sqrt{\lambda_1}r) = \frac{\exp(\sqrt{\lambda_1}r)}{\sqrt{2\pi\sqrt{\lambda_1}r}}\left[1 + O\left(\frac{1}{\sqrt{\lambda_1}r}\right)\right], \tag{6.59}$$

$$I_0(\sqrt{\lambda_2}r) = \frac{\exp(\sqrt{\lambda_2}r)}{\sqrt{2\pi\sqrt{\lambda_2}r}}\left[1 + O\left(\frac{1}{\sqrt{\lambda_2}r}\right)\right] \tag{6.60}$$

and

$$I_0(\sqrt{\lambda_3}r) = \frac{\exp(\sqrt{\lambda_3}r)}{\sqrt{2\pi\sqrt{\lambda_3}r}}\left[1 + O\left(\frac{1}{\sqrt{\lambda_3}r}\right)\right]. \tag{6.61}$$

The behavior of these functions Eqs. (6.59), (6.60), (6.61) shows that the condition at infinity Eq. (6.15) will be satisfied only in the case

$$C_1 = 0, C_3 = 0, C_5 = 0. \tag{6.62}$$

The functions K_0 tend to infinity as r tends to infinity. I we substitute the Eq. (6.62) into the Eq. (6.58) then it will be

$$w = C_2 K_0(\sqrt{\lambda_1} r) + C_4 K_0(\sqrt{\lambda_2} r) + C_6 K_0(\sqrt{\lambda_3} r). \tag{6.63}$$

The function Eq. (6.63) allows to find all constants C_2, C_4, C_6 satisfying the boundary conditions at $r = 0$ and excluding the singularity of the bending moments M_r, M_θ. We obtain the following system of linear algebraic equations

$$C_2 + C_4 + C_6 = 0, \tag{6.64}$$

$$\ln\frac{\sqrt{\lambda_1}}{2} C_2 + \ln\frac{\sqrt{\lambda_2}}{2} C_4 + \ln\frac{\sqrt{\lambda_3}}{2} C_6 = -w_0, \tag{6.65}$$

$$\lambda_1 C_2 + \lambda_2 C_4 + \lambda_3 C_6 = 0. \tag{6.66}$$

The solution of the system of Eqs. (6.64), (6.65), (6.66) is

$$C_2 = w_0 \frac{\lambda_3 \ln\frac{\sqrt{\lambda_2}}{2} - \lambda_2 \ln\frac{\sqrt{\lambda_3}}{2}}{\lambda_1 \ln\frac{\lambda_3}{\lambda_2} + \lambda_2 \ln\sqrt{\frac{\lambda_1}{\lambda_3}} + \lambda_3 \ln\sqrt{\frac{\lambda_2}{\lambda_1}}}, \tag{6.67}$$

$$C_4 = w_0 \frac{\lambda_1 \ln\frac{\sqrt{\lambda_3}}{2} - \lambda_3 \ln\frac{\sqrt{\lambda_1}}{2}}{\lambda_1 \ln\frac{\lambda_3}{\lambda_2} + \lambda_2 \ln\sqrt{\frac{\lambda_1}{\lambda_3}} + \lambda_3 \ln\sqrt{\frac{\lambda_2}{\lambda_1}}}, \tag{6.68}$$

$$C_6 = w_0 \frac{\lambda_2 \ln\frac{\sqrt{\lambda_1}}{2} - \lambda_1 \ln\frac{\sqrt{\lambda_2}}{2}}{\lambda_1 \ln\frac{\lambda_3}{\lambda_2} + \lambda_2 \ln\sqrt{\frac{\lambda_1}{\lambda_3}} + \lambda_3 \ln\sqrt{\frac{\lambda_2}{\lambda_1}}}. \tag{6.69}$$

The solution of the stated problem is given in the Eq. (6.63), where the constants C_2, C_4, C_6 are presented in the Eqs. (6.67), (6.68), (6.69).

This example shows that the solution does not have a singularity at the origin and at infinity for the bending stresses and displacements and that corresponds to the real situation with real plate. As we have seen this is impossible in the classical theory.

7. Elasticity with internal body forces

7.1. **Statement of the problem.** There are many books, where the different elasticity problems are taken into consideration, for example [1], [3], [4], [5], [7], [12], [17], [24] and many other papers and manuscripts.The differential equations of the stated problem are the equations of the linear isotropic elasticity in 3D domain [13]. We have

$$\varrho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \varrho_0 \mathbf{B} + (\lambda + \mu)\nabla e + \mu \nabla^2 \mathbf{u}, \tag{7.1}$$

where dilatation e equals

$$e = \mathtt{div}\mathbf{u} \tag{7.2}$$

and $\mathbf{u}$ is the displacement vector, ϱ_0 is the density, $\varrho_0\mathbf{B}$ the external body force per unit volume, λ and μ are Lame's coefficients or Lame's constants, ∇ is the gradient, ∇^2 is the Laplacian.

Let us consider a linear isotropic elastic body with internal body forces. The governing equations are taken in case of the steady state problem without external forces

$$\nabla\text{div}\mathbf{u} + (1-2\nu)\nabla^2\mathbf{u} - \alpha_1\mathbf{u} - \alpha_2\nabla^2\mathbf{u} - \alpha_3\nabla^4\mathbf{u} = 0, \tag{7.3}$$

where ν is the Poisson ratio and the internal body force is taken in the form of the Eq. (2.1).

The system of differential Eqs. (7.3) has the fourth order therefore the second boundary condition should be given at the boundary surface with respect to the classical case. It seems to be possible to apply the following boundary conditions at the boundary surface: given

- displacements and Eq. (7.3) without internal body forces
- stresses and Eq. (7.3) without internal body forces
- displacements and stresses
- displacements and stresses as a function of displacements
- stresses and displacements as a function of stresses
- stresses as a function of displacements and Eq. (7.3) without internal body forces
- conditions obtained in the variational formulation of the problem
- other possible conditions.

We will not consider here the question of applicable boundary conditions in details.

7.2. **Example of elasticity without internal body forces.** Let us take an example of classical linear isotropic elastic problem considered in more details in [18]. An elastic body occupies the unbounded cylinder $0 \le r \le R$, where R is the finite radius of the cylinder. Let the displacement field of the body is in cylindrical coordinates r, φ, z:

$$u_r = u_r(r,\varphi),\ u_\varphi = u_\varphi(r,\varphi),\ u_z = u_z(r). \tag{7.4}$$

Then the component u_z satisfies the equation

$$\frac{d^2u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} = 0. \tag{7.5}$$

it could be considered separately from the components u_r, u_φ if the boundary conditions allow that. We can suppose for simplicity that $u_r \equiv 0, u_\varphi \equiv 0$. Let us apply the boundary conditions for u_z

$$u_z(0) = u_0 \neq 0,\ u_z(R) = 0. \tag{7.6}$$

The problem Eqs. (7.4), (7.5) coincides with the classical membrane problem Eqs. (5.3), (5.4) and the conclusions obtained in membrane problem should be repeated here: a continuous solution of the stated problem does not exist for any finite or infinite radius of the cylinder.

7.3. **Example 1 of elasticity with internal body forces.** Consider the problem for an elastic cylinder presented in the previous section but the internal body forces are included with $\alpha_2 = 0, \alpha_3 = 0$. The equations of motion Eq. (7.3) in cylindrical coordinates will take the form

$$(\lambda+\mu)\frac{\partial e}{\partial r} + \mu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \varphi^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi} - \frac{u_r}{r^2}\right) - \alpha_1 u_r = 0, \tag{7.7}$$

$$\frac{(\lambda+\mu)}{r}\frac{\partial e}{\partial \varphi} + \mu\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2} + \frac{\partial^2 u_\varphi}{\partial z^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi}{r^2}\right) - \alpha_1 u_\varphi = 0, \tag{7.8}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \varphi^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) - \alpha_1 u_z = 0, \tag{7.9}$$

where r, φ, z are cylindrical coordinates, λ, μ are the Lame parameters, u_r, u_φ, u_z are components of the displacement vector in cylindrical coordinates,

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}. \tag{7.10}$$

Consider the following displacement field of the body in cylindrical coordinates r, φ, z:

$$u_r = 0,\ u_\varphi = 0, u_z = u_z(r). \tag{7.11}$$

Then the component u_z satisfies the equation

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} - \frac{\alpha_1}{\mu}u_z = 0. \tag{7.12}$$

The boundary conditions for u_z are the Eqs. (7.6).

The problem Eqs. (7.12), (7.6) coincides with the problem in Example 1 of membrane with internal body forces Eqs. (5.4), (5.8) and we can write the solution of stated problem in case of infinite radius R in the form of Eq. (5.15)

$$u_z = C_2 K_0(sr), \tag{7.13}$$

where

$$s = \sqrt{\frac{\alpha_1}{\mu}} \tag{7.14}$$

and the constant C_2 could be obtained using the balance of forces applied to the cylinder.

The solution (7.13) does not satisfy the first boundary condition in Eq. (7.6) and it has singularity at the origin. We can obtain the continuous solution of the stated cylinder problem and this solution will satisfy both boundary conditions Eq. (7.6). We use in this case the more general internal body force and describe that solution in the next section.

7.4. **Example 2 of elasticity with internal body forces.** Consider now the same problem as in the previous example 1 but the internal body force has all three nonzero coefficients. The Eq. (7.3) is taken into consideration. The infinite cylinder of radius R is considered. The radius $R = \infty$ for simplicity. The distribution of displacements is given

$$u_r = 0, u_\varphi = 0, u_z = u_z(r). \tag{7.15}$$

Then the differential equation for u_z will take the form

$$\alpha_3\nabla^4 u_z + (\alpha_2 - \mu)\nabla^2 u_z + \alpha_1 u_z = 0, \tag{7.16}$$

where operator ∇ has the following expression

$$\nabla = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{7.17}$$

If we suppose that $\alpha_2 - \mu < 0$ then the Eq. (7.16) coincides with the Eq. (6.28). Using the boundary conditions Eq. (7.6) which coincide with the Eqs. (5.4) used above to obtain the solution Eq. (6.31). This solution will be in the considered case

$$u_z(r) = \frac{u_0}{\ln\frac{s_2}{s_1}}\left[K_0(s_1 r) - K_0(s_2 r)\right], \tag{7.18}$$

where

$$s_1 = \sqrt{\frac{\mu - \alpha_2 + \sqrt{(\mu-\alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}, \tag{7.19}$$

$$s_2 = \sqrt{\frac{\mu - \alpha_2 - \sqrt{(\mu-\alpha_2)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}}. \tag{7.20}$$

The solution Eq. (7.18) satisfies also the conditions

$$\frac{du_z}{dr}(0) = 0, \frac{du_z}{dr}(\infty) = 0. \tag{7.21}$$

7.5. **Galerkin type of displacement potential.** The solution of the Eq. (7.3) can be obtained using similar methods as in the classical theory without internal body forces. Consider for example the displacement potentials method following [5].

7.6. **Example 1 of Galerkin potential.** Consider the equation (7.3), where $\alpha_2 = 0, \alpha_3 = 0$

$$\nabla\mathrm{div}\mathbf{u} + (1-2\nu)\nabla^2\mathbf{u} - \alpha_1\mathbf{u} = 0. \tag{7.22}$$

Thew vector displacement potential $\mathbf{F}$ is introduced in the form

$$2\mu\mathbf{u} = \left[2(1-\nu)\nabla^2 - \nabla\mathrm{div} - \alpha_1\right]\mathbf{F}. \tag{7.23}$$

If the expression for $\mathbf{u}$ in Eq. (7.23) is substituted into the Eq. (7.22) then the following differential equation with respect to the potential $\mathbf{F}$ will be obtained

$$\left[2(1-2\nu)(1-\nu)\nabla^4 - \alpha_1(3-4\nu)\nabla^2 + \alpha_1^2\right]\mathbf{F} = 0. \tag{7.24}$$

The Eq. (7.24) can be written in the form

$$\left[\nabla^2 - \frac{\alpha_1}{1-2\nu}\right]\left[\nabla^2 - \frac{\alpha_1}{2(1-\nu)}\right]\mathbf{F} = 0. \tag{7.25}$$

Then $\mathbf{F}$ could be presented as the sum of two functions

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}, \tag{7.26}$$

where the functions $\mathbf{F_1}, \mathbf{F_2}$ satisfy the equations

$$\left[\nabla^2 - \frac{\alpha_1}{1-2\nu}\right]\mathbf{F_1} = 0, \tag{7.27}$$

$$\left[\nabla^2 - \frac{\alpha_1}{2(1-\nu)}\right] \mathbf{F_2} = 0. \tag{7.28}$$

7.7. **Example 2 of Galerkin type potential.** Consider the equation (7.3), where $\alpha_2 = 0$

$$\nabla \mathrm{div}\mathbf{u} + (1-2\nu)\nabla^2\mathbf{u} - \alpha_1\mathbf{u} - \alpha_3\nabla^4\mathbf{u} = 0. \tag{7.29}$$

Thew vector displacement potential $\mathbf{F}$ is introduced in the form

$$2\mu\mathbf{u} = \left[2(1-\nu)\nabla^2 - \nabla\mathrm{div} - \alpha_1 - \alpha_3\nabla^4\right]\mathbf{F}. \tag{7.30}$$

If the expression for $\mathbf{u}$ in Eq. (7.30) is substituted into the Eq. (7.29) then the following differential equation with respect to the potential $\mathbf{F}$ will be obtained

$$\left\{\alpha_3^2\nabla^8 - (3-4\nu)\alpha_3\nabla^6 + 2\left[\alpha_1\alpha_3(1-2\nu)(1-\nu)\right]\nabla^4 - (3-4\nu)\alpha_1\nabla^2 + \alpha_1^2\right\}\mathbf{F} = 0. \tag{7.31}$$

The Eq. (7.31) can be written in the form

$$\left[(\alpha_3\nabla^4 + \alpha_1)^2 - (3-4\nu)(\alpha_3\nabla^4 + \alpha_1)\nabla^2 + 2(1-2\nu)(1-\nu)\nabla^4\right]\mathbf{F} = 0 \tag{7.32}$$

The Eq. (7.31) can be transformed to the equation

$$\left[\alpha_3\nabla^4 - 2(1-\nu)\nabla^2 + \alpha_1\right]\left[\alpha_3\nabla^4 - (1-2\nu)\nabla^2 + \alpha_1\right]\mathbf{F} = 0. \tag{7.33}$$

The Eq. (7.33) can be written also in this form

$$(\nabla^2 - s_1)(\nabla^2 - s_2)(\nabla^2 - s_3)(\nabla^2 - s_4)\mathbf{F} = 0, \tag{7.34}$$

where

$$s_1 = \frac{1-\nu+\sqrt{(1-\nu)^2 - \alpha_1\alpha_3}}{\alpha_3}, s_2 = \frac{1-\nu-\sqrt{(1-\nu)^2 - \alpha_1\alpha_3}}{\alpha_3}, \tag{7.35}$$

$$s_3 = \frac{1-2\nu+\sqrt{(1-2\nu)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}, s_4 = \frac{1-2\nu-\sqrt{(1-2\nu)^2 - 4\alpha_1\alpha_3}}{2\alpha_3}. \tag{7.36}$$

Then $\mathbf{F}$ could be presented as the sum of four functions

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \mathbf{F_4}, \tag{7.37}$$

where the functions $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}$ satisfy the equations

$$\left(\nabla^2 - s_1\right)\mathbf{F_1} = 0, \tag{7.38}$$

$$\left(\nabla^2 - s_2\right)\mathbf{F_2} = 0. \tag{7.39}$$

$$\left(\nabla^2 - s_3\right)\mathbf{F_3} = 0, \tag{7.40}$$

$$\left(\nabla^2 - s_4\right)\mathbf{F_4} = 0. \tag{7.41}$$

8. Conclusion

An introduction into the continuum theory with internal body forces and moments is given. The models of string, beam, membrane, plate, linear isotropic elasticity are considered with the internal body forces. The examples show that the singularities which are usual one in classical continuum theory could be easily eliminated in the presented theory.

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A New Comprehensive Class of Analytic Functions Defined by Ruscheweyh Derivative and Multiplier Transformations

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Abstract

Let $\mathcal{A}(p,n)$ denote the class of normalized analytic functions $f(z)$ in the open unit disc $f(z) = z^p + \sum\limits_{k=p+n}^{\infty} a_k z^k$, $p,n \in \mathbb{N} := \{1,2,3,\ldots\}$. We consider in this paper the operator
$RI_p^{\gamma}(m,\lambda,l)f(z) := (1-\gamma)\, D^m f(z) + \gamma I_p(m,\lambda,l)f(z)$ where
$I_p(m,\lambda,l)f(z) = z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m a_k z^k$ and
$(m+1)D^{m+1}f(z) = z(D^m f(z))' + mD^m f(z)$, $m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$ is the Ruscheweyh operator. By making use of the above mentioned differential operator, a new subclass of $p-$valent functions in the open unit disc is introduced. The new subclass is denoted by $\mathcal{AL}_p^{\gamma}(m,n,\mu,\alpha,\lambda,l)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

Keywords: Analytic function, $p-$valent starlike function, $p-$valent convex function, multiplier transformations, Ruscheweyh derivative.

2000 Mathematical Subject Classification: 30C45

1 Introduction and definitions

Let $\mathcal{A}(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \qquad p,n \in \mathbb{N} := \{1,2,3,\ldots\} \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. In particular we set $\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$. Let $\mathcal{H}(U)$ the space of holomorphic functions in U, $n \in \mathbb{N}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U. By $\mathcal{S}_n^*(p,\alpha)$ we denote a subclass of $\mathcal{A}(p,n)$ consisting of $p-$valently starlike univalent functions of order α in U, $0 \leq \alpha < p$ which satisfies $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, $z \in U$. Further, a function f belonging to $\mathcal{S}$ is said to be $p-$valently convex of order α in U, if and only if $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha$, $z \in U$, for some α, $(0 \leq \alpha < p)$. We denote by $\mathcal{K}_n(p,\alpha)$ the class of functions in $\mathcal{S}$ which are $p-$valently convex of order α in U and denote by $\mathcal{R}(p,\alpha)$ the class of functions in $\mathcal{A}(p,n)$ which satisfy $\mathrm{Re}\, f'(z) > \alpha$, $z \in U$.

It is well known that $\mathcal{K}_n(p,\alpha) \subset \mathcal{S}_n^*(p,\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Definition 1.1 [4] *Let $f \in \mathcal{A}(p,n)$. For $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$, we define the multiplier transformations $I_p(m,\lambda,l)$ on $\mathcal{A}(p,n)$ by the following infinite series*

$$I_p(m,\lambda,l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m a_k z^k. \tag{1.2}$$

It follows from (1.2) that

$$I_p(0,\lambda,l)f(z) = f(z)$$
$$(p+l)I_p(2,\lambda,l)f(z) = [p(1-\lambda)+l]I_p(1,\lambda,l)f(z) + \lambda z(I_p(1,\lambda,l)f(z))'$$
$$I_p(m_1,\lambda,l)(I_p(m_2,\lambda,l)f(z)) = I_p(m_2,\lambda,l)(I_p(m_1,\lambda,l)f(z)).$$

For $p=1$, $l=0$, $\lambda \geq 0$, the operator $D_\lambda^m := I_1(m,\lambda,0)$ was introduced and studied by Al-Oboudi [3] which reduces to the Sălăgean differential operator [11] for $\lambda = 1$. The operator $I_l^m := I_1(m,1,l)$ was studied recently by Cho and Srivastava [6] and Cho and Kim [7]. The operator $I_m := I_1(m,1,1)$ was studied by Uralegaddi and Somanatha [13], the operator $D_\lambda^\delta := I_1(\delta,\lambda,0)$, $\delta \geq 0$ was introduced by Acu and Owa [1] and the operator $I_p(m,l) := I_p(m,1,l)$ was investigated recently by Kumar, Taneja and Ravichandran [12].
If f is given by (1.1) then we have $I_p(m,\lambda,l)f(z) = (f * \varphi_{p,\lambda,l}^m)(z)$, where $\varphi_{p,\lambda,l}^m(z) = z^p + \sum_{k=p+n}^{\infty}\left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m z^k$.

Definition 1.2 [10] *Ruscheweyh has defined the operator $D^m : \mathcal{A}(p,n) \to \mathcal{A}(p,n)$,*

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z), ..., \\ (m+1)D^{m+1}f(z) &= z\left[D^m f(z)\right]' + mD^m f(z), \quad z \in U. \end{aligned}$$

To prove our main theorem we shall need the following lemma.

Lemma 1.3 [9] *Let u be analytic in U with $u(0)=1$ and suppose that*

$$\operatorname{Re}\left(1+\frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}, \quad z \in U. \tag{1.3}$$

Then $\operatorname{Re} u(z) > \alpha$ for $z \in U$ and $1/2 \leq \alpha < 1$.

2 Main results

Definition 2.1 *For a function $f \in \mathcal{A}(p,n)$ we define the differential operator*

$$RI_p^\gamma(m,\lambda,l)f(z) := (1-\gamma)\,D^m f(z) + \gamma I_p(m,\lambda,l)f(z) \tag{2.1}$$

where $m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $\gamma \geq 0$, $l \geq 0$.

Remark 2.2 *For $p=1$, $l=0$, $\lambda=1$ the above defined operator was introduced in [2].*

Definition 2.3 *We say that a function $f \in \mathcal{A}(p,n)$ is in the class $\mathcal{AL}_p^\gamma(m,n,\mu,\alpha,\lambda,l)$, $n,m \in \mathbb{N}$, $\mu \geq 0$, $\alpha \in [0,p)$, $\gamma \geq 0$ if*

$$\left|\frac{RI_p^\gamma(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{RI_p^\gamma(m,\lambda,l)f(z)}\right)^\mu - p\right| < p-\alpha, \qquad z \in U. \tag{2.2}$$

Remark 2.4 *The family $\mathcal{AL}_p^\gamma(m,n,\mu,\alpha,\lambda,l)$ is a new comprehensive class of analytic functions which includes various new subclasses of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{AL}_p^1(m,n,\mu,\alpha,\lambda,l)$ was studied in [5] $\mathcal{AL}_1^1(0,1,1,\alpha,1,0) \equiv \mathcal{S}_1^*(1,\alpha)$, $\mathcal{AL}_1^1(1,1,1,\alpha,1,0) \equiv \mathcal{K}_1(1,\alpha)$ and $\mathcal{AL}_1^1(0,1,0,\alpha,1,0) \equiv \mathcal{R}(1,\alpha)$. Another interesting subclass is the special case $\mathcal{AL}_1^1(0,1,2,\alpha,1,l) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathcal{AL}_1^1(0,1,\mu,\alpha,1,0) \equiv \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [3].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{AL}_p^{\gamma}(m,n,\mu,\alpha,\lambda,l)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2.5 *For the function $f\in\mathcal{A}(p,n)$, $n,m\in\mathbb{N}$, $\mu\geq 0$, $1/2\leq\alpha<1$ if*

$$(2.3)\quad \frac{(m+2)RI_p^{\gamma}(m+2,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}-\mu(m+1)\frac{RI_p^{\gamma}(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}+\gamma\left(\frac{p+l}{\lambda}-m-2\right)\frac{I_p(m+2,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}+$$

$$+\gamma\mu\left(\frac{p+l}{\lambda}-m-1\right)\frac{I_p(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}-\gamma\left[\frac{p(1-\lambda)+l}{\lambda}-m-1\right]\frac{I_p(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}+$$

$$+\gamma\mu\left[\frac{p(1-\lambda)+l}{\lambda}-m\right]\frac{I_p(m,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}+(m+p)(\mu-1)\prec 1+\beta z,\quad z\in U,$$

where $\beta=\frac{3\alpha-1}{2\alpha}$, then $f\in\mathcal{AL}_p^{\gamma}(m,n,\mu,\alpha,\lambda,l)$.

Proof. If we consider

$$u(z)=\frac{RI_p^{\gamma}(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{RI_p^{\gamma}(m,\lambda,l)f(z)}\right)^{\mu}$$

then $u(z)$ is analytic in U with $u(0)=1$. Taking into account the relation

$$(p+l)I_p(m+1,\lambda,l)f(z)=[p(1-\lambda)+l]\,I_p(m,\lambda,l)f(z)+\lambda z\left(I_p(m,\lambda,l)f(z)\right)'$$

a simple differentiation yields

$$\frac{zu'(z)}{u(z)}=\frac{(m+2)RI_p^{\gamma}(m+2,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}-\mu(m+1)\frac{RI_p^{\gamma}(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}+$$

$$+\gamma\left(\frac{p+l}{\lambda}-m-2\right)\frac{I_p(m+2,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}+\gamma\mu\left(\frac{p+l}{\lambda}-m-1\right)\frac{I_p(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}-$$

$$-\gamma\left[\frac{p(1-\lambda)+l}{\lambda}-m-1\right]\frac{I_p(m+1,\lambda,l)f(z)}{RI_p^{\gamma}(m+1,\lambda,l)f(z)}+\gamma\mu\left[\frac{p(1-\lambda)+l}{\lambda}-m\right]\frac{I_p(m,\lambda,l)f(z)}{RI_p^{\gamma}(m,\lambda,l)f(z)}+(m+p)(\mu-1)-1.$$

Using (2.3) we get

$$\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right)>\frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1.3 we deduce that

$$\mathrm{Re}\left\{\frac{RI_p^{\gamma}(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{RI_p^{\gamma}(m,\lambda,l)f(z)}\right)^{\mu}\right\}>\alpha.$$

Therefore, $f\in\mathcal{AL}_p^{\gamma}(m,n,\mu,\alpha,\lambda,l)$, by Definition 2.3. ■

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 2.6 *If $f\in\mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right\}>-\frac{1}{2}$, $z\in U$, then $f\in\mathcal{AL}_1^1(1,1,1,\frac{1}{2},1,0)$ or $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2}$, $z\in U$. That is, f is convex of order $\frac{1}{2}$.*

Corollary 2.7 *If $f\in\mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right\}>-\frac{1}{2}$, $z\in U$, then $f\in\mathcal{AL}_1^1(1,1,0,\frac{1}{2},1,0)$, that is $\mathrm{Re}\{f'(z)+zf''(z)\}>\frac{1}{2}$, $z\in U$.*

Corollary 2.8 *If $f\in\mathcal{A}(1,1)$ and $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2}$, $z\in U$, then $\mathrm{Re}\,f'(z)>\frac{1}{2}$, $z\in U$. In another words, if the function f is convex of order $\frac{1}{2}$ then $f\in\mathcal{AL}_1^1(0,1,0,\frac{1}{2},1,0)\equiv\mathcal{R}\left(1,\frac{1}{2}\right)$.*

Corollary 2.9 *If $f\in\mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}>-\frac{3}{2}$, $z\in U$, then $f\in\mathcal{AL}_1^1(0,1,1,\frac{1}{2},1,0)$. In another words f is starlike of order $\frac{1}{2}$.*

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THE NUMERICAL SOLUTION OF NON-LINEAR NON-LOCAL PROBLEMS FOR ELLIPTIC EQUATIONS

AYDIN Y. ALIYEV

Abstract. A non-local problem for an elliptic equation in a rectangular domain was investigated. A rectangular grid for the corresponding difference problem was constructed and the error of the approximate solutions of non-local problems was estimated.

Various application problems (heat conductivity [1], [2], [3], fluid mechanics [4], the theory of elasticity and shells [5], etc.) are reduced to non-local boundary value problems. Non-local boundary conditions are especially difficult for justification of classical finite difference schemes due to the complexity of the structure of the matrices obtained from systems of equations. This difficulty manifests itself especially in the justification of numerical methods in the case of non-linear equations. In this paper we consider the non-local boundary value problem for a quasi-linear equation. We found the numerical solutions of stated problem using the finite difference method, and estimated the error of the approximate solutions of non-local problems.

1. Introduction

Let $\Omega = \{0 < x < a,\ 0 < y < b\}$. Denote by $\Gamma^1 = \{0 \le x \le a,\ y = b\}$, $\Gamma^2 = \{x = 0,\ 0 < y < b\}$, $\Gamma^3 = \{0 \le x \le a,\ y = 0\}$, $\Gamma^4 = \{x = a,\ 0 < y < b\}$, $\Gamma^l = \{x = l,\ 0 < y < b,\ 0 < l < b\}$, $\Gamma = \bigcup_{i=1}^{4} \Gamma^i$, $\sigma = \Gamma^1 \cup \Gamma^3$, $\bar\Omega = \Omega \cup \Gamma$.

Suppose that $f(x, y, z, p, q)$ is a given continuous function determined $\forall (x, y) \in \bar\Omega$ and for all z, p, q. We'll assume that the partial derivatives of f'_z, f'_p, f'_q exists and satisfies

$$f'_z \ge 0, \tag{1}$$

$$|f'_p|,\ |f'_q| \le M < \infty. \tag{2}$$

Let $L[u] \equiv \Delta u - f(x, y, u, u_x, u_y)$. Assume that $\varphi,\ \psi$ are the given continuous functions of their domain definitions.

We need to find a function $u(x, y)$ continuous in $\bar\Omega$,twice continuously differentiable in Ω, satisfying the equation

$$L[u] = 0 \tag{3}$$

and the boundary conditions

$$u|_\sigma = \varphi, \tag{4}$$

$$l[u] = u(l, y) - \alpha(y)u(a, y) = \psi(y), \quad 0 < y < b, \tag{5}$$

$$\alpha(y) \ge 1,\ \ 0 < y < b, \tag{6}$$

Key words and phrases. Non-local, estimated error, difference problem, difference operator, non-linear.

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$$l^{(1)}[u] = \left(\frac{\partial u}{\partial x} + \beta(y)\frac{\partial u}{\partial y} + \delta(y)u\right)\Big|_{\Gamma^2} = \gamma(y), \quad \delta(y) \le 0. \tag{7}$$

Let $h_1 = a/N_1$, $h_2 = b/N_2$. We construct a grid area with lines $x = x_i$, $y = y_j$, $i = 0, 1, ..., N_1$, $j = 0, 1, ..., N_2$ and let $x_k < l \le x_{k+1}$.

We introduce the denotation

$$\Omega_h = \{(x_i, y_j) : \ i = 1, 2, ..., N_1 - 1, \ \ j = 1, 2, ..., N_2 - 1\},$$

$$\Gamma_h^1 = \{(x_i, b) : \ i = 1, 2, ..., N_1\}, \ \Gamma_h^2 = \{(0, y_j) : \ j = 1, 2, ..., N_2 - 1\},$$

$$\Gamma_h^3 = \{(x_i, 0) : \ i = 1, 2, ..., N_1\}, \ \Gamma_h^4 = \{(a, y_j) : \ j = 1, 2, ..., N_2 - 1\},$$

$$\sigma_h = \Gamma_h^1 \cup \Gamma_h^3, \quad \Gamma_h = \bigcup_{i=1}^{4} \Gamma_h^i, \ \ \bar{\Omega}_h = \Omega_h \cup \Gamma_h.$$

We approximate the operators L and l difference operators L_h, l_h defined as follows:

$$L_h[u_{ij}] \equiv \Delta_h[u_{ij}] - f(x_i, y_j, u_{ij}, D_{h_1 x^o}[u_{ij}], D_{h_2 y^o}[u_{ij}]), \tag{8}$$

$$l_h[u_{N_1 j}] \equiv \frac{l - x_k}{h_1} u_{k+1j} + \frac{x_{k+1} - l}{h_1} u_k - \alpha_j u_{N_1 j}, \tag{9}$$

where

$$\begin{cases} \Delta_h[u_{ij}] = u_{\bar{x}x} + u_{\bar{y}y}, \quad u_{\bar{x}x} = \dfrac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h_1^2}, \\ u_{\bar{y}y} = \dfrac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h_2^2}, \ \ D_{h_1 \overset{\circ}{x}}[u_{ij}] = \dfrac{u_{i+1j} - u_{i-1j}}{2h_1}, \\ D_{h_2 \overset{\circ}{y}}[u_{ij}] = \dfrac{u_{ij+1} - u_{ij-1}}{2h_2}. \end{cases} \tag{10}$$

We formulate a difference problem corresponding to the stated problem to find a function U that is defined in $\bar{\Omega}_h$ such that

$$L_h[U_{ij}] = 0 \ \text{ in } \ \Omega_h, \tag{11}$$

$$l_h[U_{N_1 j}] = \psi_j \ \text{ in } \ \Gamma_h^4, \tag{12}$$

$$U_{ij} = \varphi_{ij} \ \text{ in } \ \sigma_h, \tag{13}$$

$$l_h^{(1)}[U_{0j}] = \frac{U_{1j} - U_{0j}}{h_1} + \beta_j^+ \frac{U_{0j+1} - U_{0j}}{h_2} +$$

$$+\beta_j^- \frac{U_{0j} - U_{0j-1}}{h_2} + \delta_j U_{0j} = \gamma_j \quad \text{in} \quad \Gamma_h^2, \tag{14}$$

where

$$\beta_j^+ = \frac{\beta_j + |\beta_j|}{2} \ge 0, \qquad \beta_j^- = \frac{\beta_j - |\beta_j|}{2} \le 0.$$

We'll assume that the domain $\bar{\Omega}_h$ is connected and the satisfies inequality

$$Mh < 2\theta, \tag{15}$$

where $h = \max\{h_1, h_2\}$, $0 < \theta < 1$ – a some fixed number.

2. Results

Consider the linear difference operator

$$\Lambda_h[U_{ij}] = \begin{cases} \Lambda_h'[U_{ij}] \ \text{ in } \ \Omega_h, \\ l_h[U_{N_1j}] \ \text{ in } \ \Gamma_h^4, \\ l_h^{(1)}[U_{0j}] \ \text{ in } \ \Gamma_h^2, \end{cases} \tag{16}$$

where

$$\Lambda_h'[U_{ij}] = \Delta_h[U_{ij}] + \xi_{ij} D_{h_1 \overset{\circ}{x}}[U_{ij}] + \eta_{ij} D_{h_2 \overset{\circ}{y}}[U_{ij}] - \mu_{ij} U_{ij},$$

$$|\xi_{ij}|, \ |\eta_{ij}| \leq M, \tag{17}$$

$$\mu_{ij} \geq 0. \tag{18}$$

Due to the standard scheme the following lemma is proved.

Lemma 1. *Let $V \neq const$ be a function defined in $\bar{\Omega}_h$, and satisfying $\Lambda_h[V] \geq 0$ ($\Lambda_h[V] \leq 0$). Then V it may take the greatest positive (least negative) value only at the nodal points of the σ_h.*

Let U be an approximate solution of the problem (11)-(14).

Theorem 1. *Let the current solution u of (3)-(7) has limited third derivatives in Ω and second derivatives are continuous in $\bar{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ of the approximate solution satisfies the equation*

$$\varepsilon_{ij} = O(h).$$

Proof. On the basis of Taylor's formula, we have

$$\begin{cases} \Lambda_h'[\varepsilon_{ij}] = O(h) \ \text{ in } \ \Omega_h, \\ l_h[\varepsilon_{N_1j}] = O(h^2) \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij} = 0, \ \text{ in } \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}] = O(h) \ \text{ in } \ \Gamma_h^2. \end{cases} \tag{19}$$

We represent the solution of (19) as

$$\varepsilon_{ij} = \varepsilon_{ij}^1 + \varepsilon_{ij}^2, \tag{20}$$

where

$$\begin{cases} \Lambda_h'[\varepsilon_{ij}^1] = O(h) \ \text{ in } \ \Omega_h, \\ \varepsilon_{N_1j}^1 = 0 \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij}^1 = 0, \ \text{ in } \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}^1] = O(h) \ \text{ in } \ \Gamma_h^2. \end{cases} \tag{21}$$

$$\begin{cases} \Lambda_h'[\varepsilon_{ij}^2] = 0 \ \text{ in } \ \Omega_h, \\ l_h[\varepsilon_{N_1j}^2] = -l_h[\varepsilon_{N_1j}^1] + O(h^2) \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij}^2 = 0, \ \text{ in } \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}^2] = 0 \ \text{ in } \ \Gamma_h^2. \end{cases} \tag{22}$$

First, we estimate the system (21). Consider the function

$$g(x,y) = \frac{1}{K}(e^{\nu_0 a} - e^{\nu_0 x}),$$

where

$$\nu_0 = \frac{M}{\theta} arcth\left(\frac{3\theta - \theta^2}{2}\right), \quad k = \mu_0\nu_0, \quad \mu_0 = \min\left\{1, \frac{M}{2}(1-\theta)\right\}.$$

It is easy to verify, that

$$\left\{ \begin{array}{l} \Lambda'_h[g_{ij}] \le -1 \ \text{ in } \ \Omega_h, \\ l_h^{(1)}[g_{0j}] \le -1 \ \text{ in } \ \Gamma_h^2. \end{array} \right. \tag{23}$$

On the basis of (21), (23) and Lemma 1 we get that the function

$$G_{ij}^{\pm} = c \cdot h \cdot g_{ij} \pm \varepsilon_{ij}^1$$

is positive on $\bar{\Omega}_h$ (for the selected finite constant C).

From this inequality it follows that

$$\max_{\bar{\Omega}_h} |\varepsilon_{ij}^1| \le C_1 h, \quad C_1 = const > 0. \tag{24}$$

Denote by $w = \max\limits_{\Gamma_h^4} |\varepsilon_{N_1 j}^2|$ and let the $\bar{\omega}_{ij}$ – be the solution of

$$\Lambda'_h[\bar{\omega}_{ij}] = 0 \ \text{ in } \ \Omega_h,$$
$$\bar{\omega}_{N_1 j} = w \ \text{ in } \ \Gamma_h^4,$$
$$\bar{\omega}_{ij} = 0 \ \text{ in } \ \sigma_h,$$
$$l_h^{(1)}[\bar{\omega}_{0j}] = 0 \ \text{ in } \ \Gamma_h^2.$$

Lemma 1 implies that

$$|\varepsilon_{ij}^2| \le \bar{\omega}_{ij} \ \text{ in } \ \bar{\Omega}_h, \tag{25}$$

$$\bar{\omega}_{ij} \le \tau_i w, \quad 0 < \tau_i < 1 \ \text{ in } \ \Omega_h. \tag{26}$$

On the other hand

$$l_h[\varepsilon_{N_1 j}^2] = -l_h[\varepsilon_{N_1 j}^1] + O(h^2) \ \text{ in } \ \Gamma_h^4.$$

Hence, respectively to (25), (26) we have

$$\alpha_j |\varepsilon_{N_1 j}^2| \le \frac{l - x_k}{h_1} |\varepsilon_{k+1j}^2| + \frac{x_{k+1} - l}{h_1} |\varepsilon_{kj}^2| + \frac{l - x_k}{h_1} |\varepsilon_{k+1j}^1| + \frac{x_{k+1} - l}{h_1} |\varepsilon_{kj}^1| + C_2 h^2$$

or

$$\alpha_j w \le \tau w + C_1 h + C_2 h,$$

where

$$\tau = \max\{\tau_{k+1}, \tau_k\}.$$

Hence we have

$$w \le \frac{C_3 h}{\alpha_j - \kappa_i} \le C_4 h, \tag{27}$$

where

$$C_4 = \frac{C_3}{\min\limits_j (\alpha_j - \tau)}.$$

Then from (25)-(27) we have

$$\max_{\bar{\Omega}_h} |\varepsilon_{ij}^2| \le C_5 h, \quad C_5 = \max_i \tau_i C_4. \tag{28}$$

Based on (20), (24) and (28) we have

$$\max_{\bar{\Omega}_h} |\varepsilon_{ij}| \le C_6 h, \tag{29}$$

where $C_6 = C_1 + C_5$.

Theorem 1 is proved.

Below we show that by imposing additional conditions on the function $\beta(y), \delta(y)$ the order of accuracy with in h^2 can be improved.

As can be seen from the above, it is sufficient to increase the order of approximation of the operator $l_h^{(1)}$.

Assume, that $h_1 = wh^2$ $(0 < w \le 1)$ and $\beta(y)$, $\delta(y)$ satisfy one of the following conditions

$$|\beta(y)| < w, \tag{30}$$

$$|\beta(y)| \ge w,\ \delta'(y) \le 0, \tag{31}$$

$$|\beta(y)| \le -w,\ \delta'(y) \ge 0. \tag{32}$$

Consider the operators

$$l_{1h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1}U_{0j-1}}{2h_2} + \delta_j U_{0j}, \tag{33}$$

$$l_{2h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1}U_{0j}}{h_2} + \delta_j U_{0j}, \tag{34}$$

$$l_{3h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j} - U_{0j-1}}{h_2} + \delta_j U_{0j}. \tag{35}$$

Let

$$\left|\frac{\partial^p u_{0j}}{\partial x^p}\right|_{(0,j)},\ \left|\frac{\partial^p u_{0j}}{\partial y^p}\right|_{(0,j)} \le M_j^{(p)},\quad (p \ge 1).$$

Taking into account (3), (7), (33) and applying the Taylor formula is easy to see that

$$\left|\tilde{l}_{1h}^{(1)} u_{0j} - (l^{(1)}u)_{(0,j)}\right| \le c^{(1)} h_2^2, \tag{36}$$

where

$$\tilde{l}_{1h}^{(1)} u_{0j} \cong l_{1h}^{(1)} u_{0j} + \frac{h_1}{2}\frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} -$$

$$-\frac{h_1}{2} f\left\{0, y_j, u_{0j}, D_{h_1x}[u_{0j}], D_{h_2y}[u_{0j}]\right\},$$

$$D_{h_1x}[u_{ij}] = \frac{u_{i+1j-u_{ij}}}{h_1},\qquad D_{h_2y}[u_{ij}] = \frac{u_{ij+1} - u_{ij}}{h_2},$$

$$C^{(1)} = \max_j \left\{\frac{2(w^2 + w + \beta) + h_1 M}{12} M_j^{(3)} + \frac{w^2 M}{4} M_j^{(2)}\right\}.$$

Indeed, from (33) we have:

$$l_{1h}^{(1)} u_{0j} = (l^{(1)}u)_{(0,j)} + \frac{h_1}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_{(0,j)} + R_j^{(1)},$$

$$R_j^{(1)} = \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x^3}\bigg|_{(\xi_0^{(1)},j)} + \frac{h_2^2}{12}\left[\frac{\partial^3 u}{\partial y^3}\bigg|_{(0,\eta_j^{(1)})} + \frac{\partial^3 u}{\partial y^3}\bigg|_{(0,\eta_j^{(2)})}\right]\beta_j.$$

From (3) we have:

$$\frac{\partial^2 u}{\partial x^2}\bigg|_{(0,j)} = -\frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} + f\left(0, y_j, u_{0j}, D_{h_1x}[u_{0j}], D_{h_2\overset{\circ}{y}}[u_{0j}]\right) -$$

$$-\frac{h_2}{6}\left[\frac{\partial^3 u}{\partial y^3}\bigg|_{(0,\eta_j^{(3)})} - \frac{\partial^3 u}{\partial y^3}\bigg|_{(0,\eta_j^{(4)})}\right] + f'_p(0, y_j, u_{0j}, p_j, q_j)\,\frac{\partial^2 u}{\partial x^2}\bigg|_{\left(\xi_0^{(2)},j\right)}\frac{h_1}{2} +$$

$$+f'_q(0, y_j, u_{0j}, p_j, q_j)\left[\frac{\partial^3 u}{\partial y^3}\bigg|_{\left(0,\eta_j^{(3)}\right)} + \frac{\partial^3 u}{\partial y^3}\bigg|_{\left(0,\eta_j^{(4)}\right)}\right]\frac{h_2^2}{12}.$$

Taking into account this $l_{1h}^{(1)}u_{ij}$, we get:

$$l_{1h}^{(1)}u_{0j} = (l^{(1)}u)_{(0,j)} - \tfrac{h_1}{2}\tfrac{u_{0j+1}-2u_{0j}+u_{0j-1}}{h_2^2} + \\ + \tfrac{h_1}{2}f'(0, y_j, u_{0j}, D_{h_1x}[u_{0j}], D_{h_2\overset{\circ}{y}}[u_{0j}]) + \tilde{R}_j^{(1)},$$

where

$$\tilde{R}_j^{(1)} = \frac{h_1^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_{(\xi_i^{(1)},j)} + \frac{h_2^2}{12}\left[\left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(1)}\right)} + \left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(2)}\right)}\right]\beta_j -$$

$$-\frac{h_1 h_2}{12}\left[\left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(3)}\right)} - \left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(4)}\right)}\right] + \frac{h_1^2}{4}f'_p(0, y_j, u_{0j}, p_j, q_j)\left.\frac{\partial^2 u}{\partial x^2}\right|_{\left(\xi_0^{(2)},j\right)} +$$

$$+\frac{h_1 h_2^2}{24}f'_q(0, y_j, u_{0j}, p_j, q_j)\left[\left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(3)}\right)} + \left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0,\eta_j^{(4)}\right)}\right].$$

Hence we find that

$$\tilde{l}_{1h}^{(1)}u_{0j} = (l^{(1)}u)_{(0,j)} + \tilde{R}_j^{(1)},$$

consequently,

$$\left|\tilde{l}_{1h}^{(1)}u_{0j} - (l^{(1)}u)_{(0,j)}\right| \le \left|\tilde{R}_j^{(1)}\right|.$$

And this implies (36).

Now we prove that

$$\left|\tilde{l}_{2h}^{(1)}u_{0j} - (l^{(1)}u)_{(0,j)}\right| \le C^{(2)}h_2^2, \tag{37}$$

where

$$\tilde{l}_{2h}^{(1)}u_{0j} \equiv l_{2h}^{(1)}u_{0j} + \frac{\beta_j h_2 - h_1}{2\beta_j}D_{h_1h_2xy}[u_{0j}] + \frac{\delta_j}{\beta_j}(\beta_j h_2 - h_1)D_{h_2y}[u_{0j}] +$$

$$+\frac{\delta'_j}{2\beta_j}(\beta_j h_2 - h_1)u_{0j} - \frac{\gamma'_j}{2\beta_j}(\beta_j h_2 - h_1) - \frac{h_1}{2}f(0, y_j, u_{0j}, D_{h_1x}[u_{0j}], D_{h_2y}[u_{0j}]),$$

$$C^{(2)} = \max_j\left\{\left[\frac{\beta_j + w^2}{6} + \frac{(\beta_j - w)(1-w)}{4\beta_j} + \frac{h_1 M}{12}\right]M_i^{(3)} + \right.$$

$$\left. + \left[\frac{\left|\delta_j + \beta'_j\right|(\beta_j - w)}{4\beta_j} + \frac{w^2 M}{4}\right]M_j^{(2)}\right\},$$

$$D_{h_1h_2xy}[u_{0j}] = D_{h_1x}\left\{D_{h_2y}[u_{0j}]\right\}.$$

Suppose that $\beta(y) \ne 0$. Then from (7) we have:

$$\frac{\partial^2 u(0,u)}{\partial y^2} = -\frac{1}{\beta(y)}\frac{\partial^2 u(0,y)}{\partial x \partial y} - \frac{\delta'(y)}{\beta(y)}u(0,y) - \frac{\delta(y) + \beta'_j}{\beta(y)}\frac{\partial u(0,y)}{\partial y} + \frac{\gamma'(y)}{\beta(y)}. \tag{38}$$

Obviously

$$l_{2h}^{(1)}u_{0j} = (l_3 u)_{(0,j)} + \frac{h_1}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{(0,j)} + \frac{h_2}{2}\beta_j\left.\frac{\partial^2 u}{\partial y^2}\right|_{(0,j)} + R_j^{(2)},$$

where

$$R_j^{(2)} = \frac{h_1^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_{(\xi_0^{(1)},j)} + \beta_j\frac{h_2^2}{6}\left.\frac{\partial^3 u}{\partial y^3}\right|_{(0,\eta_j^{(1)})}.$$

From (3) we get:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(0,j)} = -\left.\frac{\partial^2 u}{\partial y^2}\right|_{(0,j)} + f\left(0, y_j, u_{0j}, \left.\frac{\partial u}{\partial x}\right|_{(0,j)}, \left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right).$$

Then

$$l_{2h}^{(1)} u_{0j} = (l^{(1)}u)_{(0,j)} + \frac{1}{2}(\beta_j h_2 - h_1) \left.\frac{\partial^2 u}{\partial y^2}\right|_{(0,j)} +$$
$$+\frac{h_1}{2} f\left(0, y_j, u_{0j}, \left.\frac{\partial u}{\partial x}\right|_{(0,j)}, \left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right) + R_j^{(2)}.$$

Taking into account (38)

$$l_{2h}^{(1)} u_{0j} = (l^{(1)}u)_{(0,j)} + \frac{1}{2}(\beta_j h_2 - h_1)\times$$
$$\times\left[-\frac{1}{\beta_j} \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(0,j)} - \frac{\delta_j'}{\beta_j} u_{0j} - \frac{\delta_j + \beta_j'}{\beta_j} \left.\frac{\partial u}{\partial y}\right|_{(0,j)} + \frac{\gamma_j}{\beta_j}\right] +$$
$$+\frac{h_1}{2} f\left(0, y_j, u_{0j}, \left.\frac{\partial u}{\partial x}\right|_{(0,j)}, \left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right) + R_j^{(2)} =$$
$$= (l^{(1)}u)_{(0,j)} - \frac{1}{2\beta_j}(\beta_j h_2 - h_1) \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(0,j)} -$$
$$-\frac{\delta_j + \beta_j}{2\beta_j}(\beta_j h_2 - h_1) \left.\frac{\partial u}{\partial y}\right|_{(0,j)} - \frac{\delta_j'}{2\beta_j}(\beta_j h_2 - h_1) u_{0j} + \frac{\gamma_j'}{2\beta_j}(\beta_j h_2 - h_1) +$$
$$+\frac{h_1}{2} f\left(0, y_j, u_{0j}, \left.\frac{\partial u}{\partial x}\right|_{(0,j)}, \left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right) + R_j^{(2)} = (l^{(1)}u)_{(0,j)} -$$
$$-\frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[u_{0j}] - -\frac{\delta_j + \beta_j'}{\beta_j}(\beta_j h_2 - h_1) D_{h_2 y}[u_{0j}] -$$
$$-\frac{\delta_j'}{2\beta_j}(\beta_j h_2 - h_1) u_{0j} + \frac{\gamma_j'}{2\beta_j}(\beta_j h_2 - h_1) +$$
$$+\frac{h_1}{2} f\left(0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 y}[u_{0j}]\right) + \tilde{R}_j^{(2)},$$

where

$$\tilde{R}_j^{(2)} = R_j^{(2)} - \frac{\beta_j h_2 - h_1}{4\beta_j}\left[\frac{\partial^3 u}{\partial x^2 \partial y} h_1 - \frac{\partial^3 u}{\partial x \partial y^2} h_2\right] -$$
$$-\frac{\delta_j + \beta_j'}{4\beta_j}(\beta_j h_2 - h_1) h_2 \left.\frac{\partial^2 u}{\partial y^2}\right|_{\left(0, \eta_j^{(2)}\right)} + \frac{h_1^2}{4} f_p'(0, y_j, u_{0j}, p_j, q_j) \left.\frac{\partial^2 u}{\partial x^2}\right|_{\left(\xi_0^{(2)}, j\right)} +$$
$$+\frac{h_1 h_2^2}{24} f_q'(0, y_j, u_{0j}, p_j, q_j)\left[\left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0, \eta_j^{(2)}\right)} + \left.\frac{\partial^3 u}{\partial y^3}\right|_{\left(0, \eta_j^{(3)}\right)}\right].$$

Then

$$\left|\tilde{l}_{2h}^{(1)} u_{0j} - (l_3 u)_{(0,j)}\right| \le \left|\tilde{R}_j^{(2)}\right|.$$

This implies (37).

Finally, we prove that

$$\left|\tilde{l}_{3h}^{(1)} u_{0j} - (l_3 u)_{(0,j)}\right| \le C^{(2)} h_2^2, \tag{39}$$

where

$$C^{(3)}=\left(\frac{\left|w^{2}+\beta\right|}{6}+\frac{|w+\beta|\,(w+1)}{2\,|\beta|}+\frac{h_{1}M}{12}\right)M_{3}+$$
$$+\left(\frac{|w+\beta|\,|\delta+\beta'|}{4\,|\beta|}+\frac{w^{2}M}{4}\right)M_{2},$$
$$\tilde{l}_{3h}^{(1)}u_{0j}\equiv l_{3h}^{(1)}u_{0j}-\frac{\beta_{j}h_{2}-h_{1}}{2\beta_{j}}D_{h_{1}h_{2}x\overline{y}}[u_{0j}]-\frac{\delta_{j}}{\beta_{j}}(\beta_{j}h_{2}+h_{1})D_{h_{2}\overline{y}}[u_{0j}]-$$
$$-\frac{\delta_{j}'}{2\beta_{j}}(\beta_{j}h_{2}+h_{1})u_{0j}+\frac{\gamma_{j}'}{2\beta_{j}}(\beta_{j}h_{2}-h_{1})+\frac{h_{1}}{2}f(0,y_{j},u_{0j},D_{h_{1}x}[u_{0j}],D_{h_{2}y}[u_{0j}]),$$
$$D_{h_{1}h_{2}xy}[u_{0j}]=D_{h_{1}x}\left\{D_{h_{2}\overline{y}}[u_{0j}]\right\}.$$

Indeed,

$$l_{3h}^{(1)}u_{0j}=(l^{(1)}u)_{(0,j)}-\frac{h_{1}+h_{2}\beta_{j}}{2}\left.\frac{\partial^{2}u}{\partial y^{2}}\right|_{(0,j)}-$$
$$-\frac{h_{1}}{2}f\left(0,y_{j},u_{0j},\left.\frac{\partial u}{\partial x}\right|_{(0,j)},\left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right)+R_{j}^{(3)},$$
$$R_{j}^{(3)}=\frac{h_{1}^{2}}{6}\left.\frac{\partial^{3}u}{\partial x^{3}}\right|_{(\xi_{0}^{(1)},j)}+\beta_{j}\frac{h_{2}^{2}}{6}\left.\frac{\partial^{3}u}{\partial y^{3}}\right|_{(0,\eta_{j}^{(1)})}.$$

Taking into account (38)

$$l_{3h}^{(1)}u_{0j}=(l^{(1)}u)_{(0,j)}+\frac{h_{1}+h_{2}\beta_{j}}{2\beta_{j}}\frac{\partial^{2}u}{\partial x\partial y}+$$
$$+\frac{h_{1}+h_{2}\beta_{j}}{2}\frac{\delta_{j}'}{\beta_{j}}u_{0j}+\frac{h_{1}+h_{2}\beta_{j}}{2}\frac{\delta_{j}+\beta_{j}'}{\beta_{j}}\left.\frac{\partial u}{\partial y}\right|_{(0,j)}-$$
$$-\frac{h_{1}+h_{2}\beta_{j}}{2}\frac{\gamma_{j}'}{\beta_{j}}-\frac{h_{1}}{2}f\left(0,y_{j},u_{0j},\left.\frac{\partial u}{\partial x}\right|_{(0,j)},\left.\frac{\partial u}{\partial y}\right|_{(0,j)}\right)+R_{j}^{(3)},$$
$$l_{3h}^{(1)}u_{0j}=(l^{(1)}u)_{(0,j)}+\frac{h_{1}+h_{2}\beta_{j}}{2\beta_{j}}D_{h_{1}h_{2}x\overline{y}}[u_{0j}]+\frac{(h_{1}+h_{2}\beta_{j})(\delta_{j}+\beta')}{2\beta_{j}}D_{h_{2}\overline{y}}[u_{0j}]+$$
$$+\frac{(h_{1}+h_{2}\beta_{j})\delta_{j}'}{2\beta_{j}}u_{0j}-\frac{(h_{1}+h_{2}\beta_{j})\gamma_{j}'}{2\beta_{j}}-\frac{h_{1}}{2}f\left(0,y_{j},u_{0j},D_{h_{1}x}[u_{0j}],D_{h_{2}\overline{y}}[u_{0j}]\right)+\tilde{R}_{j}^{(3)},$$

where

$$\tilde{R}_{j}^{(3)}=R_{j}^{(3)}+\frac{h_{1}+h_{2}\beta_{j}}{2\beta_{j}}\left[\frac{\partial^{3}u}{\partial x^{2}\partial y}h_{1}+\frac{\partial^{3}u}{\partial x\partial y^{2}}h_{2}\right]+$$
$$+\frac{(h_{1}+h_{2}\beta_{j})(\delta_{j}+\beta_{j}')}{4\beta_{j}}h^{2}\left.\frac{\partial^{2}u}{\partial y^{2}}\right|_{\left(0,\eta_{j}^{(2)}\right)}-\frac{h_{1}^{2}}{4}f_{p}'(0,y_{j},u_{0j},p_{j},q_{j})\left.\frac{\partial^{2}u}{\partial x^{2}}\right|_{\left(\xi_{0}^{(2)},j\right)}-$$
$$-\frac{h_{1}h_{2}^{2}}{24}f_{q}'(0,y_{j},u_{0j},p_{j},q_{j})\left[\left.\frac{\partial^{3}u}{\partial y^{3}}\right|_{\left(0,\eta_{j}^{(3)}\right)}+\left.\frac{\partial^{3}u}{\partial y^{3}}\right|_{\left(0,\eta_{j}^{(4)}\right)}\right].$$

Consequently,

$$\left|\tilde{l}_{3h}^{(1)}u_{0j}-(l^{(1)}u)_{(0,j)}\right|\leq\left|\tilde{R}_{j}^{(3)}\right|,$$

which was required to prove.

We now state the difference problem corresponding to the problem (3)-(7).

It is required to find a discrete function $U_{ij}^{(k)}$ $(k = 1, 2, 3)$ determined in $\bar{\Omega}_h$ satisfying the properties (11) - (13), and one of the following conditions

$$\tilde{l}_{kh}^{(1)} U_{0j} = \gamma_j \quad (j = \overline{1, N_2 - 1}, \quad k = 1, 2, 3) \tag{40}$$

respectively, when one of the conditions (30), (31) and (32) is satisfied. The solution of the difference scheme (11) - (13), with one of the conditions (40) will be taken as an approximate solution of the problem (3) - (7) at the points $\bar{\Omega}_h$.

Consider the following linear difference operators:

$$\Lambda_h^{(k)}[U_{ij}] = \begin{cases} \tilde{L}_h[U_{ij}], \\ l_h[U_{N_1 j}], \\ \bar{l}_{kh}^{(1)}[U_{0j}], \ \ (k = 1, 2, 3), \end{cases}$$

where

$$\tilde{L}_h[U_{ij}] \equiv \Delta_h[U_{ij}] + \xi_{ij} D_{h_1 \overset{\circ}{x}}[U_{ij}] + \eta_{ij} D_{h_2 \overset{\circ}{y}}[U_{ij}] - \mu_{ij} U_{ij},$$

$$\bar{l}_{1h}^{(1)}[U_{0j}] \equiv l_{1h}^{(1)}[U_{0j}] + \frac{h_1}{2} \frac{U_{0j+1} - 2U_{0j} + U_{0j-1}}{h_2^2} -$$

$$-\frac{h_1}{2} \left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j D_{h_2 \overset{\circ}{y}}[U_{0j}]} - \mu_{0j} U_{0j} \right],$$

$$\bar{l}_{2h}^{(1)}[U_{0j}] \equiv l_{2h}^{(1)}[U_{0j}] + \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[U_{0j}] +$$

$$+ \frac{\delta_j}{\beta_j} (\beta_j h_2 - h_1) D_{h_2 y}[U_{0j}] + \frac{\delta_j'}{2\beta_j} (\beta_j h_2 - h_1) U_{0j} -$$

$$-\frac{h_1}{2} \left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j D_{h_2 y}[U_{0j}]} - \mu_{0j} U_{0j} \right],$$

$$\bar{l}_{3h}^{(1)}[U_{0j}] \equiv l_{3h}^{(1)}[U_{0j}] - \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[U_{0j}] -$$

$$- \frac{\delta_j}{\beta_j} (\beta_j h_2 + h_1) D_{h_2 y}[U_{0j}] + \frac{\delta_j'}{2\beta_j} (\beta_j h_2 + h_1) U_{0j} +$$

$$+\frac{h_1}{2} \left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j D_{h_2 \overline{y}}[U_{0j}]} - \mu_{0j} U_{0j} \right].$$

We assume that if (30) is satisfied, then

$$M h_2 < 2(1 - \sup |\beta(x)|), \tag{41}$$

and if the (31), (32) are satisfied, then

$$\overline{M} h_2 < 1, \tag{42}$$

where

$$\overline{M} = \max \left\{ \frac{M}{1 + (\sup |\beta|)^{-1}}, \frac{M + \sup \left(\frac{|\beta| + 1}{|\beta|} |\beta' + \delta| \right)}{\inf |\beta| + (\sup |\beta|)^{-1}} \right\}.$$

Lemma 2. *Let $V \neq const$ be a function defined in $\bar{\Omega}_h$, that satisfies the inequality $\Lambda_h^{(k)}[V_{ij}] \geq 0$ $(\Lambda_h^{(k)}[V_{ij}] \leq 0)$ $k = 1, 2, 3$. Then V may take the greatest positive (least negative) value only at the points σ_h.*

Proof. It's obvious that

$$\bar{l}_{1h}^{(1)}[U_{ij}] \equiv A_{1j}^{(1)} U_{1j} + A_{2j}^{(1)} U_{0j-1} + A_{3j}^{(1)} U_{0j+1} - A_{0j}^{(1)} U_{0j},$$

$$\bar{l}_{2h}^{(1)}[U_{ij}] \equiv A_{1j}^{(2)}U_{1j} + A_{2j}^{(2)}U_{1j+1} + A_{3j}^{(2)}U_{0j+1} - A_{0j}^{(2)}U_{0j},$$
$$\bar{l}_{3h}^{(1)}[U_{ij}] \equiv A_{1j}^{(3)}U_{1j} + A_{2j}^{(3)}U_{0j-1} + A_{3j}^{(3)}U_{0j-1} - A_{0j}^{(3)}U_{0j},$$

where

$$A_{0j}^{(1)} = \frac{h_1}{h_2^2} + \frac{1}{h_1} - \delta_j + \frac{\xi_j}{2} + \frac{h_1}{2}\mu_j,\ A_{1j}^{(1)} = \frac{1}{h_1}\left(1 - \frac{h_1}{2}\xi_j\right),$$
$$A_{2j}^{(1)} = \frac{1}{2h_2}\left(\frac{h_1}{h_2} - \beta_j - \frac{h_1}{2}\eta_j\right),\ A_{3j}^{(1)} = \frac{1}{2h_2}\left(\frac{h_1}{h_2} + \beta_j + \frac{h_1}{2}\eta_j\right),$$
$$A_{0j}^{(2)} = \beta_j\left(\frac{1}{h_1} + \frac{1}{h_2}\right) - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}(\beta_j h_2 - h_1) -$$
$$-\frac{\delta_j'}{2\beta_j}(\beta_j h_2 - h_1) - \frac{\xi_j}{2} - \frac{h_1}{2h_2}\eta_j - \frac{h_1}{2}\mu_j,$$
$$A_{1j}^{(2)} = \frac{\beta_j}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} - \frac{\xi_j}{2},\quad A_{2j}^{(2)} = \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1},$$
$$A_{3j}^{(2)} = \frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2\beta_j}(\beta_j h_2 - h_1) - \frac{h_1}{2h_2}\eta_j,$$
$$A_{0j}^{(3)} = \frac{1}{h_1} - \frac{\beta_j}{h_2} - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\delta_j(\beta_j h_2 + h_1)}{h_2\beta_j} +$$
$$+\frac{\delta_j'}{2\beta_j}(\beta_j h_2 + h_1) + \frac{\xi_j}{2} - \frac{h_1}{2h_2}\eta_j + \frac{h_1}{2}\mu_j,$$
$$A_{1j}^{(3)} = \frac{1}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\xi_j}{2},$$
$$A_{2j}^{(3)} = -\frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\delta_j}{h_2\beta_j}(\beta_j h_2 + h_1) - \frac{h_1}{2h_2}\eta_j,$$
$$A_{3j}^{(3)} = \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2}.$$

All these coefficients are positive and satisfy the following conditions:

$$A_{0j}^{(1)} - A_{1j}^{(1)} - A_{2j}^{(1)} - A_{3j}^{(1)} = -\delta_j + \frac{h_1}{2}\mu_j \geq 0,$$
$$A_{0j}^{(2)} - A_{1j}^{(2)} - A_{2j}^{(2)} - A_{3j}^{(2)} = -\delta_j - \frac{h_1}{2}\mu_j - \frac{\delta_j'}{2\beta_j}(\beta_j h_2 - h_1) \geq 0,$$
$$A_{0j}^{(3)} - A_{1j}^{(3)} - A_{2j}^{(3)} - A_{3j}^{(3)} = -\delta_j + \frac{\delta_j'}{2\beta_j}(\beta_j h_2 + h_1) + \frac{h_1}{2}\mu_j \geq 0.$$

Taking into account these properties of the coefficients, applying Lemma 1 we obtain Lemma 2.

Corollary. *Lemma 2 implies that the solution of (11)-(13) (40) is unique.*

Theorem 2. *Let u the exact solution of the problem (3)-(7) limited the fourth derivatives and continued in the third derivative $\overline{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$, where U_{ij}- the approximate solution of (11)-(13), (40), the estimate $\varepsilon = O(h^2)$.*

Proof. With the help of Taylor's formula for the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ we have:

$$\begin{cases} \tilde{L}_h[\varepsilon_{ij}] = O(h^2) & \text{in } \Omega_h, \\ l_h[\varepsilon_{N_1 j}] = O(h^2) & \text{in } \Gamma_h^4, \\ \varepsilon_{ij} = 0 & \text{in } \sigma_h, \\ \bar{l}_{kh}^{(1)}[\varepsilon_{0j}] = O(h^2),\ k = 1,2,3 & \text{in } \Gamma_h^2. \end{cases} \tag{43}$$

As in the proof of Theorem 1, we represent the solution of the system (43) of the form

$$\varepsilon_{ij} = \varepsilon^1_{ij} + \varepsilon^2_{ij},$$

where

$$\begin{cases} \tilde{L}_h[\varepsilon^1_{ij}] = O(h^2) & \text{in} \quad \Omega_h, \\ \varepsilon^1_{N_1 j} = 0 & \text{in} \quad \Gamma^4_h, \\ \varepsilon^1_{ij} = 0 & \text{in} \quad \sigma_h, \\ \bar{l}^{(1)}_{kh}[\varepsilon^1_{ij}] = O(h^2), \ k = 1,2,3 & \text{in} \quad \Gamma^2_h, \end{cases} \tag{44}$$

$$\begin{cases} \tilde{L}_h[\varepsilon^2_{ij}] = 0 & \text{in} \quad \Omega_h, \\ l_h[\varepsilon^2 N_1 j] = -\,l_h[\varepsilon^1_{N_1 j}] + O(h^2) & \text{in} \quad \Gamma^4_h, \\ \varepsilon^2_{ij} = 0 & \text{in} \quad \sigma_h, \\ \bar{l}^{(1)}_{kh}[\varepsilon^2_{0j}] = 0, \ k = 1,2,3 & \text{in} \quad \Gamma^2_h. \end{cases} \tag{45}$$

An estimate of $\max\limits_{\overline{\Omega}_h} \left|\varepsilon^1_h\right| \le c_7 h^2$ for the solutions system of (44) is obtained on the basis of Lemma 2, due to scheme of proof of Theorem 1 by the majorant function

$$g(x,y) = \frac{1}{k}(e^{v_0 a} - e^{v_0 x}),$$

and the parameters k and ν_0 are selected as follows:

$$k = \mu_0 v_0, \ \ \mu_0 = \min\left\{\alpha^0, M\beta^0\right\},$$

$$\alpha^0 = \begin{cases} \sup|\beta| & \text{if} \quad |\beta| < 1, \\ \frac{1-\theta}{2} & \text{if} \quad |\beta| \ge 1, \end{cases}$$

$$\beta^0 = \begin{cases} \sup|\beta| & \text{if} \quad |\beta| < 1, \\ 1-\theta & \text{if} \quad |\beta| \ge 1, \end{cases}$$

$$v_0 = \frac{2M}{\bar{\delta}} \ \operatorname{arcth}\left(\frac{2\bar{\delta} - \bar{\delta}^2}{2}\right),$$

$$\bar{\delta} = \begin{cases} 1 - \sup|\beta| & \text{if} \quad |\beta| < 1, \\ \theta & \text{if} \quad |\beta| \ge 1. \end{cases}$$

An estimate of $\max\limits_{\overline{\Omega}_h} \left|\varepsilon^2_h\right| \le c_8 h^2$ for the solutions of the system (45) is obtained by the same way as the estimate of the solution of system (22) in the proof of Theorem 1.

Theorem 2 is proved.

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SOME GENERATING RELATIONS FOR GENERALIZED EXTENDED HYPERGEOMETRIC FUNCTIONS INVOLVING GENERALIZED FRACTIONAL DERIVATIVE OPERATOR

RAKESH K.PARMAR

ABSTRACT. Very recently, Lee et al.[10] have established generalization of the extended beta function, hypergeometric function and confluent hypergeometric function introduced by earlier researchers in this area. The aim of this research paper is to obtain some linear and bilinear generating relations for generalized extended Gauss, Appell and Lauricella hypergeometric functions in one, two and three variables by defining the further generalization of the extended fractional derivative operator. Some properties and Mellin transform of the generalized extended fractional derivative operator are also obtained.

1. INTRODUCTION

Several extensions of well known special functions have been obtained recently by several authors (see, for example [1, 2, 3, 4, 5]). Especially, Chaudhry *et al.*[4] introduced the following extension of classical Beta function :

$$B_p(x,y) = B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left[-\frac{p}{t(1-t)}\right] dt$$

$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0) \tag{1}$$

and proved that this extension has connection with Macdonald, error and Whittaker's functions.

It is obvious, that $B_0(x,y) = B(x,y;0) = B(x,y)$

More recently, Chaudhry *et al.*[6] considered the extension of Gauss hypergeometric functions as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n,c-b;p)}{B(b,c-b)}(a)_n \frac{z^n}{n!}$$

$$(p \geq 0, \mid z \mid < 1, \Re(c) > \Re(b) > 0) \tag{2}$$

and $(\alpha)_k$, denotes Pochhammer's symbol or ascending factorial,defined by

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)...(\alpha+k-1) & ,k \geq 1 \\ 1 & ,k = 0, \alpha \neq 0 \end{cases}$$

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They obtained the corresponding Euler type integral representation :

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} exp\left[-\frac{p}{t(1-t)}\right] dt$$

$$(p \geq 0 \text{ and } \mid arg(1-z) \mid < \pi, \Re(c) > \Re(b) > 0) \tag{3}$$

Clearly, $F_0(a,b;c;z) = \ {}_2F_1(a,b;c;z)$.

Very recently, Lee *et al.*[10] introduced further generalization of extended Beta function and extended Gauss's hypergeometric function as:

$$B_{p;k}(x,y) = B(x,y;p;k) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left[-\frac{p}{t^k(1-t)^k}\right] dt$$

$$(\Re(p) > 0, \Re(k) > 0, \Re(x) > 0, \Re(y) > 0) \tag{4}$$

$$F_p(a,b;c;z;k) = F_{p;k}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_{p;k}(b+n,c-b)}{B(b,c-b)} (a)_n \frac{z^n}{n!}$$

$$(p \geq 0, \Re(k) > 0; \mid z \mid < 1; \Re(c) > \Re(b) > 0) \tag{5}$$

They called these functions as generalized extended beta function (GEBF) and generalized extended hypergeometric functions (GEGHF) and obtained the Euler type integral representation :

$$F_{p;k}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} exp\left[-\frac{p}{t^k(1-t)^k}\right] dt$$

$$(p > 0, p = 0, \Re(k) > 0 \text{and} \mid arg(1-z) \mid < \pi, \Re(c) > \Re(b) > 0) \tag{6}$$

Clearly, it is seen that for $k = 1$, it gives the Chaudhry *et al.*[6] results and for $p = 0$, it reduces to original functions.

They also obtained the various integral representations, some properties, differentiation formulas,transformations formulas, recurrence relations , summation formulas,Beta distribution and Mellin transforms of these functions.

Very recently, using the well-known Riemann-Liouville integral representation for fractional derivative

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} dt \tag{7}$$

which is valid for $Re(\mu) < 0$, where the integration path is a line from 0 to z in the complex $t-$ plane and where the case $m-1 < Re(\mu) < m (m = 1,2,3,...)$ yields

$$D_z^{\mu} f(z) = \frac{d^m}{dz^m} D_z^{\mu-m} f(z) = \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^z f(t)(z-t)^{-\mu+m-1} dt \right\}$$

Ozarslan and Özergin [9] defined the following extended Riemann-Liouville fractional derivative by adding a new parameter. Explicitly, they considered

$$D_z^{\mu,p} f(z) = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} exp\left[\frac{-pz^2}{t(z-t)}\right] dt \tag{8}$$

with $\Re(\mu) < 0, \Re(p) > 0$ and for $m-1 < \Re(\mu) < m(m = 1, 2, 3, ...)$

$$D_z^{\mu,p} f(z) = \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int_0^z f(t)(z-t)^{-\mu+m-1} exp\left[\frac{-pz^2}{t(z-t)}\right] dt \right\}$$

The path of integration is a line from 0 to z in the complex $t-$ plane. It is easy to see that the case $p = 0$ gives the classical Riemann-Liouville fractional derivative operator. Using this definition, they calculated the extended fractional derivatives for some elementary functions.

Furthermore, they also defined the extended Appell$'s$ hypergeometric functions of two variables $F_1(a, b, c; d; x, y; p)$ and $F_2(a, b, c; d, e; x, y; p)$, and Lauricella's hypergeometric function of three variables as :

$$F_1(a, b, c; d; x, y; p) = \sum_{n,m=0}^{\infty} \frac{B(a+m+n, d-a; p)}{B(a, d-a)} (b)_n (c)_m \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(max\{\mid x \mid, \mid y \mid\} < 1; \Re(p) \geqq 0) \tag{9}$$

$$F_2(a, b, c; d, e; x, y; p) = \sum_{n,m=0}^{\infty} \frac{(a)_{m+n} B(b+n, d-b; p) B(c+m, e-c; p)}{B(b, d-b) B(c, e-c)} \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(\mid x \mid + \mid y \mid < 1; \Re(p) \geqq 0) \tag{10}$$

and

$$F_{D,p}^3(a, b, c, d; e; x, y, z) = \sum_{m,n,r=0}^{\infty} \frac{B_p(a+m+n+r, e-a)(b)_m (c)_n (d)_r}{B(a, e-a)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!}$$

$$(\sqrt{\mid x \mid} + \sqrt{\mid y \mid} + \sqrt{\mid z \mid} < 1; \Re(p) \geqq 0) \tag{11}$$

Here again, the case p =0 gives the familiar functions.

They also obtained their integral representation and showed the connection between these functions and the extended Riemann-Liouville fractional derivative operator.

The aim of this paper is to present further generalization of extended fractional derivative operator to obtain some linear and bilinear generating relations for hypergeometric functions and some properties and Mellin transform are also determined for this operator.The plan of this paper is as follow:

Firstly, in section 2, further generalization of the extended Appell's hypergeometric functions of two variables $F_1(a, b, c; d; x, y; p; k)andF_2(a, b, c; d, e; x, y; p; k)$ and extended Lauricella's hypergeometric function of three variables $F_{D,p;k}^3(a, b, c, d; e; x, y, z)$

are defined and integral representations of generalized extended Appell's hypergeometric functions are obtained.In section 3, further generalization of extended fractional derivative operator is defined to obtain the generalized extended fractional derivative for some elementary functions and generating relations are calculated in terms of generalized extended Appell's hypergeometric functions and Lauricella's hypergeometric function. In section 4, some results related to Mellin transforms and extended fractional derivative operator are given. Finally, in section 4, some generating relations for generalized extended hypergeometric function are obtained via further generalized fractional derivative operator as explained in [7].

2. The Generalized Extended Appell's functions and Lauricella's Hypergeometric function

In this section, generalization of the extended Appell's hypergeometric functions of two variables, $F_1(a,b,c;d;x,y;p;k)$ and $F_2(a,b,c;d,e;x,y;p;k)$, and extended Lauricella's hypergeometric function of three variables $F^3_{D,p;k}$ $(a,b,c,d;e;x,y,z)$ are considered as:

$$F_1(a,b,c;d;x,y;p;k) = \sum_{n,m=0}^{\infty} \frac{B_{p;k}(a+m+n,d-a)}{B(a,d-a)}(b)_n(c)_m \frac{x^n}{n!}\frac{y^m}{m!}$$

$$(max\{\mid x\mid,\mid y\mid\}<1;\Re(p)\geqq 0) \tag{12}$$

$$F_2(a,b,c;d,e;x,y;p;k) = \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}B_{p;k}(b+n,d-b)B_{p;k}(c+m,e-c)}{B(b,d-b)B(c;e-c)}\frac{x^n}{n!}\frac{y^m}{m!}$$

$$(\mid x\mid+\mid y\mid<1;\Re(p)\geqq 0) \tag{13}$$

and

$$F^3_{D,p;k}(a,b,c,d;e;x,y,z) = \sum_{m,n,r=0}^{\infty} \frac{B_{p;k}(a+m+n+r,e-a)(b)_m(c)_n(d)_r}{B(a,e-a)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!}$$

$$(\sqrt{\mid x\mid}+\sqrt{\mid y\mid}+\sqrt{\mid z\mid}<1;\Re(p)\geqq 0) \tag{14}$$

respectively.

It is easily seen that the case k =0 gives the Özarslan and Özergin [9] results and p=0 gives the original functions.

2.1. Integral Representation of Generalized Extended Appell's functions.

In this section integral representation of generalized extended Appell's functions of two variables is presented:

Theorem 2.1. *For the generalized extended Appell's functions $F_1(a,b,c;d;x,y;p;k)$, following integral representation holds true:*

$$F_1(a,b,c;d;x,y;p;k) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)}\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}exp\left[-\frac{p}{t^k(1-t)^k}\right]dt$$

$(p\geq 0,\Re(k)>0\ and\ \mid arg(1-x)\mid<\pi,\mid arg(1-y)\mid<\pi,\Re(d)>\Re(a)>0,\Re(b)>0,\Re(c)>0)$

Proof. Let $\mid x \mid < 1, \mid y \mid < 1, \Re(b) > 0$ and $\Re(c) > 0$. Expressing $(1-xt)^{-b}$ and $(1-yt)^{-c}$ as Binomial series, and considering that the series involved are uniformly convergent and the integral involved is absolutely convergent , so we have to right to interchange the order of summation and integration to obtain:

$\int\limits_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}exp\left[-\frac{p}{t^k(1-t)^k}\right]dt$

$= \int\limits_0^1 t^{a-1}(1-t)^{d-a-1}exp\left[-\frac{p}{t^k(1-t)^k}\right]\sum\limits_{n=0}^{\infty}(b)_n\frac{(xt)^n}{n!}\sum\limits_{m=0}^{\infty}(c)_m\frac{(yt)^m}{m!}dt$

$= \sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty}(b)_n(c)_m\frac{x^n}{n!}\frac{y^m}{m!}\int\limits_0^1 t^{a+m+n-1}(1-t)^{d-a-1}exp\left[-\frac{p}{t^k(1-t)^k}\right]dt$

Finally by (4) and (12), we get

$$\int\limits_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}exp\left[-\frac{p}{t^k(1-t)^k}\right]dt$$
$$= \frac{\Gamma(a)\Gamma(d-a)}{\Gamma(d)}F_1(a,b,c;d;x,y:p;k)$$

Here the demonstration of the integral representation is completed by applying the principle of analytic continuation. Since the integral on the right hand side is analytic in the cut planes $\mid arg(1-x) \mid < \pi, \mid arg(1-y) \mid < \pi$.

□

Theorem 2.2. *For the function* $F_2(a,b,c;d,e;x,y:p;k)$*, the following integral representation holds true:*

$$F_2(a,b,c;d,e;x,y:p;k) = \frac{1}{B(b,d-b)B(c,e-c)}\int\limits_0^1\int\limits_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}$$
$$.exp\left[-\frac{p}{t^k(1-t)^k} - \frac{p}{s^k(1-s)^k}\right]dtds$$

$(p > 0; p = 0, \Re(k) > 0$ *and* $\mid x \mid + \mid y \mid < 1; \Re(d) > \Re(b) > 0, \Re(e) > \Re(c) > 0, \Re(a) > 0)$

Proof. Suppose $\mid x \mid + \mid y \mid < 1; \Re(a) > 0$. Using binomial series of $(1-xt-ys)^{-a}$ and the summation formula

$\sum\limits_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!} = \sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty}f(m+n)\frac{x^n}{n!}\frac{y^m}{m!}$, we have

$\int\limits_0^1\int\limits_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}exp\left[-\frac{p}{t^k(1-t)^k} - \frac{p}{s^k(1-s)^k}\right]dtds$

$= \int\limits_0^1\int\limits_0^1 t^{b-1}(1-t)^{d-b-1}exp\left[-\frac{p}{t^k(1-t)^k}\right]s^{c-1}(1-s)^{e-c-1}exp\left[-\frac{p}{s^k(1-s)^k}\right]\sum\limits_{N=0}^{\infty}(a)_N\frac{(xt+ys)^N}{N!}dtds$

we get

$\int\limits_0^1\int\limits_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a}exp\left[-\frac{p}{t^k(1-t)^k} - \frac{p}{s^k(1-s)^k}\right]dtds$

$= \frac{1}{B(b,d-b)B(c,e-c)}\int\limits_0^1\int\limits_0^1 t^{b-1}(1-t)^{d-b-1}exp\left[-\frac{p}{t^k(1-t)^k}\right]s^{c-1}(1-s)^{e-c-1}exp\left[-\frac{p}{s^k(1-s)^k}\right]$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(a)_{m+n}\tfrac{(xt)^n}{n!}\tfrac{(ys)^m}{m!}dtds$$

Since the series involved are uniformly convergent and the integral involved is absolutely convergent , so we have a right to interchange the order of summation and integration to obtain

$$\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} exp\left[-\frac{p}{t^k(1-t)^k}-\frac{p}{s^k(1-s)^k}\right]dtds$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(a)_{m+n}\frac{x^n}{n!}\frac{y^m}{m!}\int_0^1 t^{b+n-1}(1-t)^{d-b-1}exp\left[-\frac{p}{t^k(1-t)^k}\right]dt$$
$$\int_0^1 s^{c+m-1}(1-s)^{e-c-1}exp\left[-\frac{p}{s^k(1-s)^k}\right]ds$$

Finally by (4) and (13), we get

$$\int_0^1\int_0^1 \frac{t^{b-1}(1-t)^{d-b-1}s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} exp\left[-\frac{p}{t^k(1-t)^k}-\frac{p}{s^k(1-s)^k}\right]dtds$$
$$=B(b,d-b)B(c;e-c)F_2(a,b,c;d,e;x,y;p;k)$$

□

3. Generalized Extended Riemann-Liouville Fractional Derivative Operator

The investigations of various authors in the field of fractional calculus and its applications in different areas of science and engineering is well presented in [8]. The use of fractional derivative in the generating function theory is explained by Srivastava and Manocha [7]. In this section, following generalization of the extended Riemann-Liouville fractional derivative is considered :

$$D_z^{\mu,p;k}\{f(z)\}=\frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}exp\left(\frac{-pz^{2k}}{t^k(z-t)^k}\right)dt$$

$$(\Re(\mu)<0,\Re(p)>0,\Re(k)>0) \tag{15}$$

and for $m-1<Re(\mu)<m(m=1,2,3,\ldots)$

$$D_z^{\mu,p;k}\{f(z)\}=\frac{d^m}{dz^m}D_z^{\mu-m;k}\{f(z)\}$$
$$=\tfrac{d^m}{dz^m}\left\{\tfrac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}exp\left(-\tfrac{pz^{2k}}{t^k(z-t)^k}\right)dt\right\}$$

where the path of integration is a line from 0 to z in the complex t-plane.

For the case $k=1$, we obtain Özarslan et al.[9] result and for $p=0$ we obtain the classical Riemann-Liouville fractional derivative operator.

3.1. **Generalized Extended Fractional Derivative of Some Elementary function.** In this section, fractional derivatives of some elementary functions are calculated and also determines the extended fractional integral of an analytic function.

Theorem 3.1. *Let* $\Re(\lambda) > -1, \Re(\mu) < 0, \Re(p) > 0$ *and* $\Re(k) > 0$. *Then*

$$D_z^{\mu,p;k}\{z^\lambda\} = \frac{B_{p;k}(\lambda+1,-\mu)}{\Gamma(-\mu)} z^{\lambda-\mu}$$

Proof. With the help of the representation (15) for the generalized extended fractional derivative and generalized beta function (4), we get

$$\begin{aligned} D_z^{\mu,p;k}\{z^\lambda\} &= \tfrac{1}{\Gamma(-\mu)} \int_0^z t^\lambda (z-t)^{-\mu-1} exp\left(\tfrac{-pz^{2k}}{t^k(z-t)^k}\right) dt \\ &= \tfrac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^\lambda (1-u)^{-\mu-1} exp\left(\tfrac{-pz^{2k}}{u^k z^k (z-uz)^k}\right) du \\ &= \tfrac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^\lambda (1-u)^{-\mu-1} exp\left(\tfrac{-p}{u^k(1-u)^k}\right) du \\ &= \tfrac{B_{p;k}(\lambda+1,-\mu)}{\Gamma(-\mu)} z^{\lambda-\mu} \end{aligned}$$

□

Theorem 3.2. *Let* $\Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\mu) < 0, \Re(p) > 0, \Re(k) > 0$ *and* $\mid z \mid < 1$. *Then*

$$D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{p;k}(\alpha,\lambda;\mu;z)$$

Proof. By making use of (15) for the generalized extended fractional derivative,we have by direct calculation

$$\begin{aligned} D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-z)^{-\alpha}\} &= \tfrac{1}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{-\alpha} exp\left(\tfrac{-pz^{2k}}{t^k(z-t)^k}\right)(z-t)^{\mu-\lambda-1} dt \\ &= \tfrac{z^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{-\alpha}\left(1-\tfrac{t}{z}\right)^{\mu-\lambda-1} exp\left(\tfrac{-pz^{2k}}{t^k(z-t)^k}\right) dt \\ &= \tfrac{z^{\mu-\lambda-1}z^\lambda}{\Gamma(\mu-\lambda)} \int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1} exp\left(\tfrac{-p}{u^k(1-u)^k}\right) du. \end{aligned}$$

Using definition (6) , we get

$$\begin{aligned} D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-z)^{-\alpha}\} &= \tfrac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda,\mu-\lambda) F_{p;k}(\alpha,\lambda;\mu;z) \\ &= \tfrac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{p;k}(\alpha,\lambda;\mu;z). \end{aligned}$$

□

Theorem 3.3. *Let* $\Re(\mu) > \Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \Re(k) > 0; \mid az \mid < 1$ *and* $\mid bz \mid < 1$. *Then*

$$D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda,\alpha,\beta;\mu;az,bz;p;k)$$

Proof. Using the definition (15) and Theorem (2.1), we get

$$\begin{aligned} &D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\} \\ &= \tfrac{1}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta} exp\left(\tfrac{-pz^{2k}}{t^k(z-t)^k}\right)(z-t)^{\mu-\lambda-1} dt \end{aligned}$$

$$= \frac{z^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta}\left(1-\frac{t}{z}\right)^{\mu-\lambda-1} exp\left(\frac{-pz^{2k}}{t^k(z-t)^k}\right) dt$$
$$= \frac{z^{\mu-\lambda-1}z^{\lambda}}{\Gamma(\mu-\lambda)} \int_0^1 u^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1} exp\left(\frac{-p}{u^k(1-u)^k}\right) du$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}F_1(\lambda,\alpha,\beta;\mu;az,bz;p;k)$$

□

Theorem 3.4. *More generally, letting* $\Re(\mu) > \Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(p) > 0, \Re(k) > 0, \mid az \mid < 1, \mid bz \mid < 1 and \ \mid cz \mid < 1,$ *we have*

$$D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F^3_{D,p;k}(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz)$$

Proof. Using Theorem 3.1 and definition (14), we obtain

$$D_z^{\lambda-\mu,p;k}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\}$$
$$= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} \sum_{n,n,r=0}^{\infty} \frac{(\alpha)_m(\beta)_n(\gamma)_r}{m!n!r!} a^m b^n c^r B_{p;k}(\lambda+m+n+r,\mu-\lambda) z^{m+n+r}$$
$$= \frac{B(\lambda,\mu-\lambda)}{\Gamma(\mu-\lambda)} z^{\mu-1} \sum_{m,n,r=0}^{\infty} \frac{B_{p;k}(\lambda+m+n+r,\mu-\lambda)}{B(\lambda,\mu-\lambda)} \frac{(\alpha)_m(\beta)_n(\gamma)_r}{m!n!r!}(az)^m(bz)^n(cz)^r$$
$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F^3_{D,p;k}(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz).$$

□

Theorem 3.5. *For* $\Re(\mu) > \Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(p) > 0, \Re(k) > 0; \mid \frac{x}{1-z} \mid < 1 and \ \mid x \mid + \mid z \mid < 1,$ *we have*

$$D_z^{\lambda-\mu,p;k}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p;k}\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right)\right\} = \frac{1}{B(\beta,\gamma-\beta)\Gamma(\mu-\lambda)} z^{\mu-1}F_2(\alpha,\beta,\lambda;\gamma,\mu;x,z;p;k)$$

Proof. Using Theorem 3.1 and (13), we get

$$D_z^{\lambda-\mu,p;k}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p;k}\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right)\right\}$$
$$= D_z^{\lambda-\mu,p;k}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n B_{p;k}(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)} D_z^{\lambda-\mu,p;k}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n B_{p;k}(\beta+n,\gamma-\beta)\frac{x^n}{n!}(1-z)^{-\alpha-n}\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)} \sum_{m,n=0}^{\infty} B_{p;k}(\beta+n,\gamma-\beta)\frac{x^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!} D_z^{\lambda-\mu,p;k}\left\{z^{\lambda-1+m}\right\}$$
$$= \frac{1}{B(\beta,\gamma-\beta)} \sum_{m,n=0}^{\infty} B_{p;k}(\beta+n,\gamma-\beta)\frac{x^n}{n!}\frac{(\alpha)_{n+m}}{m!}\frac{B_{p;k}(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)} z^{\mu+m-1}$$
$$= \frac{1}{B(\beta,\gamma-\beta)\Gamma(\mu-\lambda)} z^{\mu-1}F_2(\alpha,\beta,\lambda;\gamma,\mu;x,z;p;k)$$

□

Theorem 3.6. *Let f (z) be an analytic function in the disc* $\mid z \mid < \rho$ *and has the power series expansion* $f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then*

$$D_z^{\mu,p;k}\{z^{\lambda-1}f(z)\} = \sum_{n=0}^{\infty} a_n D_z^{\mu,p;k}[z^{\lambda+n-1}]$$
$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_{p;k}(\lambda+n,-\mu) z^n$$

provided that $\Re(\lambda) > 0, \Re(\mu) < 0, \Re(p) > 0, \Re(k) > 0 and \ \mid z \mid < \rho.$

Proof. By making use of (15) for the generalized extended fractional derivative,we have

$$D_z^{\mu,p;k}\{z^{\lambda-1}f(z)\} = D_z^{\mu,p;k}\left\{z^{\lambda-1}\sum_{n=0}^{\infty} a_n z^n\right\}$$
$$= \frac{1}{\Gamma(-\mu)}\int_0^z t^{\lambda-1}\sum_{n=0}^{\infty} a_n t^n (z-t)^{-\mu-1} exp\left(\frac{-pz^{2k}}{t^k(z-t)^k}\right)dt$$
$$= \frac{1}{\Gamma(-\mu)}\int_0^1 (z\xi)^{\lambda-1} z^{-\mu-1}(1-\xi)^{-\mu-1} exp\left(\frac{-p}{\xi^k(1-\xi)^k}\right)\sum_{n=0}^{\infty} a_n (z\xi)^n z d\xi$$
$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\int_0^1 (\xi)^{\lambda-1}(1-\xi)^{-\mu-1} exp\left(\frac{-p}{\xi^k(1-\xi)^k}\right)\sum_{n=0}^{\infty} a_n (z\xi)^n d\xi$$

Since the series $\sum_{n=0}^{\infty} a_n z^n \xi^n$ is uniformly convergent in the disc $|\, z \,| < \rho$ for $0 \leq \xi \leq 1$ and the integral involved is convergent for the given constraints. So we have a right to change the order of integration and summation to obtain

$$D_z^{\mu,p;k}\{z^{\lambda-1}f(z)\} = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n (z)^n \int_0^1 (\xi)^{\lambda+n-1}(1-\xi)^{-\mu-1} exp\left(\frac{-p}{\xi^k(1-\xi)^k}\right)d\xi$$
$$= \sum_{n=0}^{\infty} a_n \frac{z^{\lambda+n-1-\mu}}{\Gamma(-\mu)} B_{p;k}(\lambda+n,-\mu)$$
$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty} a_n B_{p;k}(\lambda+n,-\mu) z^n$$

□

4. Mellin Transforms of the Generalized Extended Riemann-Liouville Fractional Derivative Operator

In this section,Mellin transforms of the generalized extended fractional derivatives is obtained and an application is also presented.

Theorem 4.1. *Let the generalized extended Riemann-Liouville fractional derivative be defined by (15). Then we have for* $\Re(\lambda) > -1, \Re(\mu) < 0, \Re(s) > 0, \Re(p) > 0, \Re(k) > 0$,

$$\mathcal{M}\left\{D_z^{\mu,p;k}(z^\lambda) : s\right\} = \frac{\Gamma(s)}{\Gamma(-\mu)} B(\lambda+ks+1, ks-\mu) z^{\lambda-\mu}$$

Proof. Making use the definition of the Mellin transform, we have

$$\mathcal{M}\left\{D_z^{\mu,p;k}(z^\lambda) : s\right\} = \int_0^{\infty} p^{s-1} D_z^{\mu,p;k}(z^\lambda) dp$$
$$= \frac{1}{\Gamma(-\mu)}\int_0^{\infty} p^{s-1}\int_0^z t^\lambda (z-t)^{-\mu-1} exp\left(\frac{-pz^{2k}}{t^k(z-t)^k}\right) dtdp$$
$$= \frac{z^{-\mu-1}}{\Gamma(-\mu)}\int_0^{\infty} p^{s-1}\int_0^z t^\lambda \left(1-\frac{t}{z}\right)^{-\mu-1} exp\left(\frac{-pz^{2k}}{t^k(z-t)^k}\right) dtdp$$
$$= \frac{z^{-\mu-1}}{\Gamma(-\mu)}\int_0^{\infty} p^{s-1}\int_0^1 u^\lambda z^\lambda (1-u)^{-\mu-1} exp\left(\frac{-p}{u^k(1-u)^k}\right) zdudp.$$

Since, uniform convergence of the inegral guarantees that the order of the integrals can be changed.We, therefore, have

$$\mathcal{M}\left\{D_z^{\mu,p;k}(z^\lambda) : s\right\} = \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda (1-u)^{-\mu-1}\int_0^{\infty} p^{s-1} exp\left(\frac{-p}{u^k(1-u)^k}\right) dpdu$$

Making the substitution $t = \frac{p}{u(1-u)}$, we get

$$\mathcal{M}\left\{D_z^{\mu,p;k}(z^\lambda) : s\right\} = \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda (1-u)^{-\mu-1}\left[u^{ks}(1-u)^{ks}\Gamma(s)\right] du$$

$$= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\Gamma(s)\int_0^1 u^{\lambda+ks}(1-u)^{ks-\mu-1}du$$
$$= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\Gamma(s)B(\lambda+ks+1, ks-\mu)$$

□

Theorem 4.2. *Let the generalized extended Riemann-Liouville fractional derivative is defined by (15). Then we have for* $\Re(\mu) < 0, \Re(s) > 0, \Re(\alpha) > 0, \Re(p) > 0, \Re(k) > 0$ *and* $|z| < 1$,

$$\mathcal{M}\left\{D_z^{\mu,p;k}((1-z)^{-\alpha}) : s\right\} = \frac{\Gamma(s)z^{-\mu}B(sk+1, sk-\mu)}{\Gamma(-\mu)}F(\alpha, ks+1; 2ks-\mu+1; z)$$

Proof. Letting $\Re(\mu) < 0, \Re(s) > 0, \Re(\alpha) > 0, \Re(p) > 0, \Re(k) > 0$ and $|z| < 1$ and then using Theorem 4.1 with $\lambda = n$ and writing $(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}z^n$, we have

$$\mathcal{M}\left\{D_z^{\mu,p;k}((1-z)^{-\alpha}) : s\right\} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathcal{M}\left\{D_z^{\mu,p;k}(z^n) : s\right\}$$
$$= \frac{\Gamma(s)}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}B(n+ks+1, ks-\mu)z^{n-\mu}$$
$$= \frac{\Gamma(s)z^{-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}B(n+ks+1, ks-\mu)\frac{(\alpha)_n z^n}{n!}$$
$$= \frac{\Gamma(s)z^{-\mu}}{\Gamma(-\mu)}B(ks+1, sk-\mu) \quad F(\alpha, ks+1; 2ks-\mu+1; z)$$

□

5. Generating functions

In this section, linear and bilinear generating relations for the generalized extended hypergeometric functions are obtained by the methods described in H. M. Srivastava, H. L. Manocha [7].The main results are as follow:

Theorem 5.1. *For the generalized extended hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_{p;k}(\lambda+n, \alpha; \beta; x)t^n = (1-t)^{-\lambda}F_{p;k}\left(\lambda, \alpha; \beta; \frac{x}{1-t}\right)$$

$(|x| < \min(1, |1-t|)$ *and* $\Re(\lambda) > 0, \Re(\beta) > \Re(\alpha) > 0, \Re(p) > 0, \Re(k) > 0)$

Proof. Writing the elementary identity

$$[(1-x)-t]^{-\lambda} = (1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda}$$

in the following form, we have

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-\lambda}\left(\frac{t}{1-x}\right)^n = (1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda} \quad (|t| < |1-x|)$$

Multiplying both sides of the above equality by $x^{\alpha-1}$ and applying the definition of generalized extended fractional derivative operator $D_x^{\alpha-\beta,p;k}$ on both sides, we can write

$$D_x^{\alpha-\beta,p;k}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-\lambda}\left(\frac{t}{1-x}\right)^n x^{\alpha-1}\right\} = (1-t)^{-\lambda}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}\left(1-\frac{x}{1-t}\right)^{-\lambda}\right\}$$

Interchanging the order, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}(1-x)^{-\lambda-n}\right\}t^n = (1-t)^{-\lambda}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}\left(1-\frac{x}{1-t}\right)^{-\lambda}\right\}$$

Applying Theorem 3.2, we get the desired result.

□

Theorem 5.2. *For the generalized extended hypergeometric functions, we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_{p;k}(\rho-n,\alpha;\beta;x)t^n=(1-t)^{-\lambda}F_1\left(\alpha,\rho,\lambda;\beta;x,\frac{-xt}{1-t};p;k\right)$$

$(\Re(\beta)>\Re(\alpha)>0,\Re(\rho)>0,\Re(\lambda)>0,\Re(p)>0,\Re(k)>0;\mid t\mid<\frac{1}{1+|x|})$

Proof. Considering the identity,

$[1-(1-x)t]^{-\lambda}=(1-t)^{-\lambda}\left[1+\frac{xt}{1-t}\right]^{-\lambda}$

and writing in the form, we have, for $\mid t\mid<\mid 1-x\mid$ that

$\sum\limits_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^n t^n=(1-t)^{-\lambda}\left[1-\frac{-xt}{1-t}\right]^{-\lambda}$

Multiplying both sides of the above equality by $x^{\alpha-1}(1-x)^{-\rho}$ and applying the generalized extended fractional derivative operator $D_x^{\alpha-\beta,p;k}$ on both sides, we get

$D_x^{\alpha-\beta,p;k}\left\{\sum\limits_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^{\alpha-1}(1-x)^{-\rho+n}t^n\right\}=(1-t)^{-\lambda}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}(1-x)^{-\rho}\left(1-\frac{-xt}{1-t}\right)^{-\lambda}\right\}$

Interchanging the order, which is valid for $Re(\alpha)>0$ and $\mid xt\mid<\mid 1-t\mid$, we get

$\sum\limits_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}(1-x)^{-\rho+n}\right\}t^n=(1-t)^{-\lambda}D_x^{\alpha-\beta,p;k}\left\{x^{\alpha-1}(1-x)^{-\rho}\left(1-\frac{-xt}{1-t}\right)^{-\lambda}\right\}$

Applying Theorem 3.2 and Theorem 3.3, we get the desired result.

□

Theorem 5.3. *For the generalized extended hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_{p;k}(\gamma,-n;\delta;y)F_{p;k}(\lambda+n,\alpha;\beta;x)t^n=(1-t)^{-\lambda}F_2\left(\lambda,\alpha,\gamma;\beta,\delta;\frac{x}{1-t},\frac{-yt}{1-t};p;k\right)$$

$(\Re(\delta)>\Re(\gamma)>0,\Re(\alpha),\Re(\lambda),\Re(\beta),\Re(p),\Re(k)>0;\mid t\mid<\frac{1-|x|}{1+|y|}$ *and* $\mid x\mid<1)$

Proof. Replacing $t\to(1-y)t$ in (5.1), multiplying the resulting equality by $y^{\gamma-1}$ and then applying the generalized extended fractional derivative operator $D_y^{\gamma-\delta,p;k}$, we get

$D_y^{\gamma-\delta,p;k}\left\{\sum\limits_{n=0}^{\infty}\frac{(\lambda)_n}{n!}y^{\gamma-1}F_{p;k}(\lambda+n,\alpha;\beta;x)(1-y)^n t^n\right\}$

$=D_y^{\gamma-\delta,p;k}\left\{(1-(1-y)t)^{-\lambda}y^{\gamma-1}F_{p;k}\left(\lambda,\alpha;\beta;\frac{x}{1-(1-y)t}\right)\right\}$

Interchanging the order, which is valid for $\mid x\mid<1$, $\mid\frac{1-y}{1-x}t\mid<1$ and $\mid\frac{x}{1-t}\mid+\mid\frac{yt}{1-t}\mid<1$, we can write that

$\sum\limits_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_y^{\gamma-\delta,p;k}\left\{y^{\gamma-1}(1-y)^n\right\}F_{p;k}(\lambda+n,\alpha;\beta;x)t^n$

$=(1-t)^{-\lambda}D_y^{\gamma-\delta,p;k}\left\{y^{\gamma-1}\left(1-\frac{-yt}{1-t}\right)^{-\lambda}F_{p;k}\left(\lambda,\alpha;\beta;\frac{\frac{x}{1-t}}{1-\frac{-yt}{1-t}}\right)\right\}$

Using Theorem 3.2 and Theorem 3.5, we get the result.

□

6. Concluding Remarks and Observations

In this present investigation, generalization of the extended fractional derivative operator related to a generalized extended Beta function, which was used in order to obtain some linear and bilinear generating relations involving the extended hypergeometric functions [9] have introduced and studied . Also the generalized extended fractional derivative operator is applied to derive generating relations for the generalized extended Gauss, Appell and Lauricella hypergeometric functions in one, two and three variables. Many other properties and relationships involving (for example) Mellin transforms and the generalized extended fractional derivative operator are also given.

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AN EQUIVALENT REFORMULATION OF ABSOLUTE WEIGHTED MEAN METHODS

MEHMET ALI SARIGOL

Abstract. We proved an equivalent definition of absolute summability of a numerical series by a weighted mean method in terms of ordinary convergence of another series as in results of Hardy [2] and Moricz and Rhoades [4].

1. Introdution.

Consider a series

$$\sum_{v=0}^{\infty} a_v \tag{1}$$

of complex numbers, with partial sums s_n and, let (p_n) be a sequence of positive numbers with $P_n = p_0 + p_1 + ... + p_n \to \infty$ as n$\to \infty$. The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v, \ n = 0, 1, ... \tag{2}$$

defines the sequence of the weighted means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series (1) is said to be summable $\left(\overline{N}, p_n\right)$ to L and absolute summable $\left|\overline{N}, p_n\right|$ if (see [1])

$$T_n \to s \ \ as \ \ n \to \infty \ \ and \ \ \sum_{v=0}^{\infty} \left|\overline{\Delta} T_n\right| < \infty, \tag{3}$$

where $\overline{\Delta} T_n = T_n - T_{n-1}$, respectively. Also, we recall a weighted mean matrix $\overline{N}$ is an infinite lower matrix with entries $a_{nv} = p_v / P_n$, and zero otherwise. For $p_n = 1$, the summabilities $\left(\overline{N}, p_n\right)$ and $\left|\overline{N}, p_n\right|$ are reduced to $(C, 1)$ and $|C, 1|$.

In [2] Hardy introduced a new sequence defined by

$$\sigma_n = \sum_{v=n}^{\infty} \frac{a_v}{v+1}, \ n = 0, 1, ... \tag{4}$$

and proved that an equivalent definition of summability (C,1) of a numerical series in terms of ordinary convergence of another series in (4) as follows.

Theorem 1.1. The series (1) is summable $(C, 1)$ to a finte number L if and only if the series

$$\sum_{n=0}^{\infty} \sigma_n$$

converges to the same limit L.

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Establishing the following theorem, Moricz and Rhoades [4] (see, also, [5]) studied the same problem for the summability $\left(\overline{N},p_n\right)$ method, which also includes result of Hardy.

Theorem 1.2. Let (p_n) be positive numbers such that the following conditions are satisfied:

$$P_n \to \infty \text{ and } \frac{p_n}{P_n} \to 0 \text{ as } n \to \infty,$$

$$\frac{p_{n-1}p_{n+1}}{p_nP_n} + P_n \sum_{v=n}^{\infty} \frac{1}{P_{v+1}} \left| \frac{p_v}{p_{v+1}} - \frac{p_{v+1}P_v}{p_{v+2}P_{v+2}} \right| = O\left(1\right)$$

and

$$\frac{p_n}{p_{n+1}} + \frac{1}{P_n} \sum_{v=n}^{\infty} P_{v+1} \left| \frac{p_{v+1}}{p_v} - \frac{p_{v-1}P_{v-1}}{p_vP_{v+1}} \right| = O\left(1\right),$$

and with the agreement that $p_{-1} = P_{-1} = 0$. Then, the series (1) is summable $\left(\overline{N},p_n\right)$ to a finite number L if and only if

$$\sum_{n=0}^{\infty} b_n$$

converges to the limit L, where

$$b_n = \sum_{v=n}^{\infty} \frac{p_n}{P_v} a_v, \ n = 0, 1, \dots . \tag{5}$$

2. Main Resuls

Note that $\left|\overline{N},p_n\right|$ implies $\left(\overline{N},p_n\right)$ but not conversely, and these methods are different. So it is natural to ask for the equivalent reformulation of $\left|\overline{N},p_n\right|$. In this paper we give an affirmative answer establishing the following theorem.

Theorem 2.1. Let (p_n) be a sequence of positive numbers such that the following conditions are satisfied:

$$i-) \ \ P_n \to \infty, \qquad ii-) \ \ \frac{p_n}{p_{n+1}} = O\left(1\right), \tag{6}$$

$$iii-) \ \ \frac{1}{p_n} = O(1), \qquad iv-) \ \ \frac{P_np_{n+1}}{p_nP_{n+1}} = O(1). \tag{7}$$

Then, the series (1) is summable $\left|\overline{N},p_n\right|$ if and only if the series $\sum b_n$ is absolutely convergent, in this case,

$$\lim_n T_n = \sum_{n=0}^{\infty} b_n. \tag{8}$$

It turns out from the proof of theorem 2.1 that the necessity part is valid under conditions ($7iii$) and ($7iv$), while the sufficient part is valid under the conditions ($6i$) and ($6ii$).

Let us consider a few special cases.

Corollary 2.2. The series (1) is summable $|C,1|$ if and only if $\sum \sigma_n$ is absolutely convergent , and in this case,

$$\lim_n \frac{1}{n+1} \sum_{v=0}^{n} s_v = \sum_{n=0}^{\infty} \sigma_n.$$

where is defined by (4).

If $\overline{N} = H$, the harmonic summability, determined by $p_n = 1/(n+1),\ n = 0, 1, ...,$ then $P_n \simeq \log(n+1)$.The conditions (6i), (6ii) and (7iv) are satisfied but (7iii). So Theorem 2.1 implies the following.

Corollary 2.3. If

$$\sum_{n=0}^{\infty} \left| \sum_{v=n}^{\infty} \frac{a_v}{(n+1)\,P_v} \right| < \infty$$

then

$$\sum_{n=1}^{\infty} \left| \sum_{v=1}^{n} \frac{P_{v-1} a_v}{(n+1)\log(n+1)\log(n+2)} \right| < \infty$$

and

$$lim_n \frac{1}{P_n} \sum_{v=0}^{n} \frac{s_v}{v+1} = \sum_{n=0}^{\infty} \sum_{v=n}^{\infty} \frac{a_v}{(n+1)\,P_v}.$$

Finally, if $p_n = n+1, n = 0, 1, ...,$ then $P_n = (n+1)(n+2)/2$ and one observes that the conditions of Theorem 2.1 are satisfied. Hence, by (5) and (8), we have the following.

Corollary 2.4.

$$\sum_{n=0}^{\infty} \left| \sum_{v=1}^{n} \frac{(v+1)(v+2)a_v}{(n+1)\,(n+2)(n+3)} \right| < \infty \text{ iff } \sum_{n=0}^{\infty} \left| \sum_{v=n}^{\infty} \frac{(n+1)a_v}{(v+1)\,(v+2)} \right| < \infty,$$

in this case,

$$lim_n \frac{1}{(n+1)(n+2)} \sum_{v=0}^{n} (v+1)s_v = \sum_{n=0}^{\infty} \sum_{v=n}^{\infty} \frac{(n+1)a_v}{(v+1)\,(v+2)}$$

Proof Theorem 2.1. Before the proof, we recall that an infinite matrix $A = (a_{nv})$ is absolutely regular if given any absolutely convergent series of complex numbers with sum L, the series

$$A_n(a) = \sum_{v=0}^{\infty} a_{nv} a_v;\ n = 0, 1, ...$$

all converget and if the series $\sum A_n(a)$ is absolutely convergent with sum L. As is well known (see, [3], p.189), a matrix A is absolutely regular if and only if

$$i-) \sup_v \sum_{n=0}^{\infty} |a_{nv}| < \infty, \qquad ii-) \sum_{n=0}^{\infty} a_{nv} = 1 \ (v = 0, 1, ...) \tag{9}$$

We now turn to the proof of the theorem.

Sufficiency. Suppose that the series $\sum_{n=0}^{\infty} b_n$ is absolutely convergent and converges to a finite number L. Then, it follows from (5) that, for

$$a_n = P_n \left(\frac{b_n}{p_n} - \frac{b_{n+1}}{p_{n+1}} \right),$$

and so

$$a_0 = \overline{\Delta} T_0 = b_0,$$

$$a_0 = \overline{\Delta} T_0 = b_0,\ \overline{\Delta} T_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad (P_{-1} = 0) \tag{10}$$

$$= \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n} P_{v-1}P_v \left(\frac{b_v}{p_v} - \frac{b_{v+1}}{p_{v+1}} \right)$$

$$= \frac{p_n}{P_nP_{n-1}} \left\{ \sum_{v=1}^{n} P_{v-1} \left(1 + \frac{p_{v-1}}{p_v} \right) b_v - P_{n-1}P_n \frac{b_{n+1}}{p_{n+1}} \right\}$$

Hence we can write

$$\overline{\Delta} T_n = \sum_{v=0}^{\infty} a_{nv} b_v, \ n = 0, 1, ...,$$

where

$$a_{nv} = \begin{cases} a_{00} = 1, \quad a_{n0} = 0, \ n \geq 1 \\ \frac{p_n P_{v-1}}{P_n P_{n-1}} \left(1 + \frac{p_{v-1}}{p_v} \right), \ 1 \leq v \leq n \\ -\frac{p_{n+1}}{P_{n-1}}, \qquad v = n+1, \\ 0, \qquad v > n. \end{cases}$$

Therefore the series (1) is summable$\left|\overline{N}, p_n\right|$ and $\lim_n T_n = L$ if and only if A is absolutely regular. On the other hand, it is easily seen that, for $v = 1, 2, \dots$,

$$\begin{aligned} \sum_{n=1}^{\infty} |a_{nv}| &= \frac{p_{v-1}}{p_v} + P_{v-1} \left(1 + \frac{p_{v-1}}{p_v} \right) \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}} \\ &= 1 + \frac{2p_{v-1}}{p_v} \end{aligned}$$

which is bounded by (6). Also, (9ii) is satisfied. Hence A is absolutely regular, whence result.

Necessity. Assume that the series (1) is summable$\left|\overline{N}, p_n\right|$ and $\lim_n T_n = L$. By inversion of (10), we get, for $n \geq 1$,

$$a_0 = \overline{\Delta} T_0 = T_0; \ a_n = \frac{P_n}{p_n} \overline{\Delta} T_n - \frac{P_{n-2}}{p_{n-1}} \overline{\Delta} T_{n-1}$$

which gives us

$$\begin{aligned} b_n &= \sum_{v=n}^{\infty} \frac{p_n}{P_v} a_v = p_n \lim_m \sum_{v=n}^{m} \frac{1}{P_v} \left(\frac{P_v}{p_v} \overline{\Delta} T_v - \frac{P_{v-2}}{p_{v-1}} \overline{\Delta} T_{v-1} \right) \\ &= p_n \lim_m \left\{ \frac{\overline{\Delta} T_m}{p_m} + \sum_{v=n}^{m} \left(\frac{1}{P_v} - \frac{P_{v-1}}{p_v P_{v+1}} \right) \overline{\Delta} T_v + \frac{P_{n-2}}{p_{n-1} P_n} \overline{\Delta} T_{n-1} \right\}. \end{aligned}$$

On the other hand, by (7iii), we have $\frac{\overline{\Delta} T_m}{p_m} \to 0 \ as \ m \to \infty$, and so

$$\begin{aligned} b_n &= p_n \left\{ \sum_{v=n}^{\infty} \left(\frac{1}{P_v} - \frac{P_{v-1}}{p_v P_{v+1}} \right) \overline{\Delta} T_v + \frac{P_{n-2}}{p_{n-1} P_n} \overline{\Delta} T_{n-1} \right\} \\ &= \sum_{v=0}^{\infty} b_{nv} \overline{\Delta} T_v \end{aligned}$$

where

$$b_{nv} = \begin{cases} 0, \ v < n-1 \\ \frac{p_n P_{n-2}}{p_{n-1} P_n}, \ v = n-1 \\ p_n \left(\frac{1}{P_v} - \frac{P_{v-1}}{p_v P_{v+1}} \right), \ v \geq n \end{cases}$$

Therefore the series $\sum b_n$ is absolutely convergent and converges to a fiite number L if and only if B absolutely regular. But, it easily is seen from the definition of matrix B that

$$\begin{aligned}\sum_{n=0}^{\infty}|b_{nv}| &= \frac{p_{v+1}P_{v-1}}{p_vP_{v+1}}+\left|\frac{1}{P_v}-\frac{P_{v-1}}{p_vP_{v+1}}\right|\sum_{n=0}^{v}p_n \\ &= 1+\frac{2P_vp_{v+1}}{P_{v+1}p_v}-\frac{2p_{v+1}}{P_{v+1}}\end{aligned}$$

which is bounded by ($7iv$). Also ($9ii$) holds. Hence the matrix B is absolutely regular which completes the proof.

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On the effectiveness of the exponential Ruscheweyh differential operator product sets in $\mathbb{C}^n$

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Abstract

In the present paper, the convergence properties of exponential Ruscheweyh differential operator product set of polynomials of several complex variables in hyperelliptical regions are studied. These new results extend and improve a lot of known works from the one complex variable case to the case of several complex variables in hyperelliptical regions.

Mathematics Subject Classification(2000): 32A05, 32A15, 32A99.
Keywords: **Ruscheweyh differential operator, Basic sets of polynomials, Hyperelliptical regions**

1 Introduction

The problem of the derived sets of any finite order for a given basic set of polynomials in one complex variable has been studied by many authors we may mention, for instance, Mikhail [1], Makar [2] and Newns [3]. For the two complex variables case, we mention Kumuyi et al [4] and Abul-Ez et al [5]. In all the above studies, only simple basic sets are considered. Recently, in [6] the author studied this problem in a new region which is called hyperelliptical regions. Also, more recently in [7, 8] the authors studied this problem in Clifford setting. The purpose of this paper is to prove, under some conditions, that the set of exponential Ruscheweyh differential operator product of polynomials of several complex variables is a basic set. Then, acting by the exponential Ruscheweyh differential operator product on basic sets in hyperelliptical regions we establish that the effectiveness property is preserved. Notice that the Ruscheweyh differential operator has been used in [9]. The rest of this paper is organized as follows: In Section 2, we recall some definitions and notations of holomorphic functions of several complex variables in hyperelliptical regions and basic series of basic sets of polynomials of several complex variables in hyperelliptical regions ([6, 10]). In Section 3, we present basic properties of the exponential Ruscheweyh differential

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operator product set. Section 4 is given to establish the effectiveness of the exponential Ruscheweyh differential operator product set of basic set of polynomials of several complex variables in an closed hyperellipse. The effectiveness of the exponential Ruscheweyh differential operator product set of basic set of polynomials of several complex variables in an open hyperellipse and in the regions $D(\overline{E}_{[\mathbf{r}]})$, which means unspecified domain containing the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$, are obtained in Section 5.

2 Notation and preliminaries

To avoid lengthy scripts, the following notations are adopted throughout this work (see [6, 10, 11]).

$$\begin{aligned}
&\mathbf{m} = (m_1, m_2, ..., m_n); \quad <\mathbf{m}>= m_1 + m_2 + ... + m_n;\\
&\mathbf{h} = (h_1, h_2, ..., h_n); \qquad <\mathbf{h}>= h_1 + h_2 + ... + h_n;\\
&\mathbf{z} = (z_1, z_2, ..., z_n); \quad \mathbf{z^m} = z_1^{m_1}.z_2^{m_2}.....z_n^{m_n}; \quad \mathbf{0} = (0, 0, ..., 0);\\
&|<\mathbf{z}>|^2 = |z_1|^2 + |z_2|^2 + ... + |z_n|^2; \quad \mathbf{t^m} = t_1^{m_1}.t_2^{m_2}.....t_n^{m_n};\\
&\mathbf{r} = (r_1, r_2, ..., r_n); \quad [\mathbf{r}^*] = [\mathbf{r}] \text{ if } r_s = r \ \ \forall \ \ s \in I; \ \ I = \{1, 2, 3, ..., n\};\\
&\alpha([\mathbf{r}], [\mathbf{R}]) = \max\Big\{r_1 \ \Pi_{s=2}^n \ R_s; r_\nu \ \Pi_{s=1}^n \ R_s \ \Pi_{s=\nu+1}^n \ R_s; r_n \ \Pi_{s=1}^{n-1} \ R_s\Big\};\\
&\text{where } \ \mathbf{R} = (R_1, R_2, ..., R_n), \ \ \nu = \{2, 3, 4, .., n-1\}, \quad s \in I.
\end{aligned}$$

In these notations, $m_1, m_2, ..., m_k$ and $h_1, h_2, ..., h_k$ are non-negative integers while $t_1, t_2, ..., t_n$ are non-negative numbers, $0 < t_s < 1$, $|\mathbf{t}| = (\sum_{s=1}^n \ t_s^2)^{(\frac{1}{2})} = 1$. Also, square brackets are used here in functional notation to express the fact that the function is either a function of several complex variables or one related to such function. In the space of several complex variables $\mathbb{C}^n$; an open hyperelliptical region $\sum_{s=1}^n \ \frac{|z_s|^2}{r_s^2} < 1$ is here denoted by $E_{[\mathbf{r}]}$ and its closure $\sum_{s=1}^n \ \frac{|z_s|^2}{r_s^2} \leq 1$ by $\overline{E}_{[\mathbf{r}]}$, where r_s; $s \in I$, are positive numbers. In terms of the introduced notations, these regions satisfy the following inequalities:

$$\begin{aligned}
E_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| < 1\},\\
\overline{E}_{[\mathbf{r}]} &= \{\mathbf{w} : |\mathbf{w}| \leq 1\},
\end{aligned} \tag{2.1}$$

where $\mathbf{w} = (w_1, w_2, ..., w_k)$, $w_s = \frac{z_s}{r_s}$; $s \in I$.

Suppose now that the function $f(\mathbf{z})$, given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \ a_{\mathbf{m}} \ \mathbf{z^m}, \tag{2.2}$$

is regular in $\overline{E}_{[\mathbf{r}]}$ and

$$M[f; \ [\mathbf{r}]] = \sup_{\overline{E}_{[\mathbf{r}]}} \ |f(\mathbf{z})|.$$

2

From (2.1), we easily see that $\{|z_s| \leq r_s \ t_s : \ |\mathbf{t}| = 1\} \subset \overline{E}_{[\mathbf{r}]}$, where **t** is the vector $(t_1, t_2, ..., t_n)$. Hence it follows that

$$|a_{\mathbf{m}}| \leq \ \sigma_{\mathbf{m}} \ \frac{M[f;[\rho]]}{\Pi_{s=1}^{k}(\rho_s)^{m_s}}, \tag{2.3}$$

for all $0 < \rho_s < r_s; \ \ s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|\mathbf{t}|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{< \mathbf{m} >\}^{\frac{<\mathbf{m}>}{2}}}{\Pi_{s=1}^{n} \ m_s^{\frac{m_s}{2}}} \qquad (see[10]), \tag{2.4}$$

and $1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{<\mathbf{m}>}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0; \ s \in I$.

Thus, it follows that

$$\limsup_{<\mathbf{m}>\to\infty} \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \ \Pi_{s=1}^{k} \ (\rho_s)^{<\mathbf{m}>-m_s}} \right\}^{\frac{1}{<\mathbf{m}}>} \leq \ \frac{1}{\Pi_{s=1}^{k} \ \rho_s}. \tag{2.5}$$

Since ρ_s can be chosen arbitrary near to $r_s; s \in I$, we conclude that

$$\limsup_{<\mathbf{m}>\to\infty} \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \ \Pi_{s=1}^{n} \ (r_s)^{<\mathbf{m}>-m_s}} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \ \frac{1}{\Pi_{s=1}^{n} \ r_s}. \tag{2.6}$$

Then, it can be easily proved that the function $f(\mathbf{z})$ is regular in the open hyperelliptical $E_{[\mathbf{r}]}$. The numbers r_s, given in (2.6), is thus conveniently called the radii of regularity of the function $f(\mathbf{z})$.

Definition 2.1. [6, 10, 11] A set of polynomials

$$\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0[\mathbf{z}], P_1[\mathbf{z}], P_2[\mathbf{z}], ..., P_n[\mathbf{z}], ...\},$$

is said to be basic when every polynomial in the complex variables z_s, $s \in I$, can be uniquely expressed as a finite linear combination of the elements of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to [11], the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and and only if there exists a unique row-finite matrix $\overline{P}$ such that

$$\overline{P}P = P\overline{P} = \mathbf{I}, \tag{2.7}$$

where $P = [P_{\mathbf{m};\mathbf{h}}]$ is the matrix of coefficients, $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and **I** is the unit matrix.

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \ \sum_{\mathbf{h}} P_{\mathbf{m};\mathbf{h}} \ \mathbf{z}^{\mathbf{h}}, \tag{2.8}$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \ \sum_{\mathbf{h}} \overline{P}_{\mathbf{m};\mathbf{h}} \ \mathbf{z}^{\mathbf{h}}, \tag{2.9}$$

3

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m};\mathbf{h}}\ P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m};\mathbf{h}}\ \overline{P}_{\mathbf{h}}[\mathbf{z}]. \tag{2.10}$$

Hence, for the function $f(\mathbf{z})$ given in (2.2) we get

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}], \tag{2.11}$$

where

$$\Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h};\mathbf{m}}\ a_{\mathbf{h}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h};\mathbf{m}} \frac{f^{\mathbf{h}}(\mathbf{0})}{\mathbf{h}!}, \tag{2.12}$$

and $h! = h(h-1)(h-2)...3.2.1$. The series $\sum_{\mathbf{m}}^{\infty} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}]$ is the associated basic series of $f(\mathbf{z})$.

Definition 2.2. [6, 10, 11]. The associated basic series $\sum_{\mathbf{m}}^{\infty} \Pi_{\mathbf{m}}\ P_{\mathbf{m}}[\mathbf{z}]$ is said to represent $f(\mathbf{z})$ in

(i) $\overline{E}_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $\overline{E}_{[\mathbf{r}]}$,

(ii) $E_{[\mathbf{r}]}$ when it converges uniformly to $f(\mathbf{z})$ in $E_{[\mathbf{r}]}$,

(iii) $D(\overline{E}_{[\mathbf{r}]})$ when it converges uniformly to $f(\mathbf{z})$ in some hyperelliptical surrounding the hyperelliptical $\overline{E}_{[\mathbf{r}]}$, not necessarily the former hyperelliptical.

Definition 2.3. [6, 10, 11] The set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be simple set, when the polynomial $P_{\mathbf{m}}[\mathbf{z}]$ is of degree $< \mathbf{m} >$, that is to say

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=0}^{(\mathbf{m})} P_{\mathbf{m};\mathbf{h}}\ \mathbf{z}^{\mathbf{h}}. \tag{2.13}$$

If the coefficient $P_{\mathbf{m},\mathbf{m}}$ of $z_1^{m_1} z_2^{m_2} ... z_s^{m_s}$ in (2.13) is unity, then the simple set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be absolutely monic.

Definition 2.4. [6, 10, 11] Let $N_{\mathbf{m}} = N_{m_1,m_2,...,m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m};\mathbf{h}}$ in the representation (2.9). A basic set satisfying the condition

$$\lim_{<\mathbf{m}>} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1, \tag{2.14}$$

is called a Cannon set and if

$$\lim_{<\mathbf{m}>} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = a > 1,$$

then the set is called a general basic set.

Now, let $\mathcal{D}_{\mathbf{m}} = \mathcal{D}_{m_1,m_2,...,m_n}$ be the degree of the polynomial of the highest degree in the representation (2.9), that is to say, if $\mathcal{D}_{\mathbf{h}} = \mathcal{D}_{h_1,h_2,...,h_n}$ is the degree of the polynomial $P_{\mathbf{m}}$, then $\mathcal{D}_{\mathbf{h}} < \mathcal{D}_{\mathbf{m}}\ \forall\ h_s < m_s$. Since the elements of the basic set are linearly independent, then $N_{\mathbf{m}} \leq 1+2+3+...+(\mathcal{D}_{\mathbf{m}}+1) \leq \lambda_1\ \mathcal{D}^2_{\mathbf{m}}$, where λ_1 is a constant. Therefore, the conditions (2.14) for a basic set to be a Cannon set implies the following condition (see [6, 10]):

$$\lim_{<\mathbf{m}>\to\infty} \{\mathcal{D}_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1. \tag{2.15}$$

For any function $f(\mathbf{z})$ of several complex variables, there is formally an associated basic series $\sum_{\mathbf{h}=\mathbf{0}}^{\infty} \Pi_{\mathbf{h}}\ P_{\mathbf{h}}[\mathbf{z}]$. When this associated series converges uniformity to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain. In other words, as in the classical terminology of Whittaker (see [12]), the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective in that domain. The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also according to the domain in which they are represented.
To study the convergence properties of such basic sets of polynomials in hyperelliptical regions (c.f.[6, 10]), we consider the following notations for Cannon sums:

$$\Omega[P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}] = \sigma_{\mathbf{m}}\ \Pi_{s=1}^{n}\{r_s\}^{<\mathbf{m}>-m_s} \sum_{\mathbf{h}} |\overline{P}_{\mathbf{m},\mathbf{h}}|\ M(P_{\mathbf{m}}, E_{[\mathbf{r}]}). \tag{2.16}$$

Also, the Cannon function for the basic sets of polynomials in hyperelliptical regions was defined as follows:

$$\Omega[P, \overline{E}_{[\mathbf{r}]}] = \limsup_{<\mathbf{m}>\to\infty} \{\Omega[P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}]\}^{\frac{1}{<\mathbf{m}>}}. \tag{2.17}$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperelliptical regions, we have from [10], the following results.

Theorem 2.1. *The necessary and sufficient condition for the Cannon basic set $\{P_{\boldsymbol{m}}[\boldsymbol{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{E}_{[\boldsymbol{r}]}$ is that*

$$\Omega[P, \overline{E}_{[\boldsymbol{r}]}] = \prod_{s=1}^{n} r_s.$$

Theorem 2.2. *The necessary and sufficient condition for the Cannon basic set $\{P_{\boldsymbol{m}}[\boldsymbol{z}]\}$ of polynomials of several complex variables to be effective in the open hyperellipse $E_{[\boldsymbol{r}]}$ is that*

$$\Omega[P, E_{[\boldsymbol{R}]}] < \alpha([\boldsymbol{r}], [\boldsymbol{R}]).$$

Theorem 2.3. *The Cannon basic set $\{P_{\boldsymbol{m}}[\boldsymbol{z}]\}$ of polynomials of several complex variables will be effective in $D(\overline{E}_{[\boldsymbol{r}]})$, if and only if*

$$\Omega[P, D(\overline{E}_{[\boldsymbol{r}]})] = \prod_{s=1}^{n} r_s.$$

Consider the Ruschewey differential operator product D^n acting on the monomials $\mathbf{z}^{\mathbf{m}}$, such that

$$D^n\ \mathbf{z}^{\mathbf{m}} = \begin{cases} [\prod_{s=1}^{n} D_{z_s}^{n_s}]\ \mathbf{z}^{\mathbf{m}}, & \mathbf{m} \neq \mathbf{0} \\ 1, & \mathbf{m} = \mathbf{0}, \end{cases} \tag{2.18}$$

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where
$$D_{z_s}^{n_s} z_s^{m_s} = \frac{z_s}{n_s!}\left(z_s^{n_s+m_s-1}\right)^{(n_s)},$$

the derivatives are repeated n_s_times, $s \in I$. Special cases of this operator D^n was introduced in [9].

3 Basic properly of exponential Ruscheweyh differential operator product set

Now, we define the exponential Ruscheweyh differential operator product $E^n = \exp(D^n)$ acting on the monomials $\mathbf{z}^{\mathbf{m}}$ as

Definition 3.1. Let E^n act on $\mathbf{z}^{\mathbf{m}}$ as follows
$$E^n \mathbf{z}^{\mathbf{m}} = \begin{cases} \exp\left(\prod_{s=1}^{n} \frac{(m_s)_{n_s}}{n_s!}\right)\mathbf{z}^{\mathbf{m}}, & \mathbf{m} \neq \mathbf{0} \\ e, & \mathbf{m} = \mathbf{0}. \end{cases} \tag{3.1}$$

Inserting the operator E^n in (2.10), we obtain the following relation
$$\begin{cases} \exp\left(\prod_{s=1}^{n} \frac{(m_s)_{n_s}}{n_s!}\right)\mathbf{z}^{\mathbf{m}} = \sum \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}^{*}(\mathbf{z}), & \mathbf{m} \neq \mathbf{0} \\ e = \sum \overline{P}_{\mathbf{0},\mathbf{h}} P_{\mathbf{h}}^{*}(\mathbf{z}), & \mathbf{m} = \mathbf{0}, \end{cases} \tag{3.2}$$

where $(m)_n = m(m+1)...(m+n-1)$ is the Pochhammer symbol and
$$\begin{aligned} P_{\mathbf{m}}^{*}(\mathbf{z}) &= E^n P_{\mathbf{m}}(\mathbf{z}) = P_{\mathbf{0},\mathbf{h}}(\mathbf{z}) + \sum P_{\mathbf{m},\mathbf{h}} \exp(D^n)\,\mathbf{z}^{\mathbf{h}} \\ &= \sum_{\mathbf{h}} \gamma_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}} \end{aligned} \tag{3.3}$$

and
$$\gamma_{\mathbf{n},\mathbf{h}} = \begin{cases} \exp\left(\prod_{s=1}^{n} \frac{(h_s)_{n_s}}{n_s!}\right), & \mathbf{h} \neq \mathbf{0} \\ e, & \mathbf{h} = \mathbf{0}. \end{cases}$$

The set $\{P_{\mathbf{m}}^{*}(\mathbf{z})\}$ is called the exponential Ruscheweyh differential operator product set of several complex variables.

Now, it is natural to ask the question: if the parent set $\{P_{\mathbf{m}}(\mathbf{z})\}$ is basic would $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ be also basic?

The answer this question is affirmative as follows
$$P_{\mathbf{m}}^{(*)}(\mathbf{z}) = E^n P_{\mathbf{m}}(\mathbf{z}) = \sum_{\mathbf{h}} \gamma_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}} = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}}^{*} \mathbf{z}^{\mathbf{h}}.$$

The matrix of coefficients $P^{(*)}$ of this set $P^{(*)} = \gamma_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}}$.

Also, the matrix of operators $\overline{P}^{(*)}$ follows from the representation

$$\mathbf{z}^{\mathbf{m}} = \frac{1}{\gamma_{\mathbf{n},\mathbf{h}}} \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}^{*}(\mathbf{z}) = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}}^{(*)} P_{\mathbf{h}}^{*}(\mathbf{z}),$$

that is to say

$$\overline{P}^{(*)} = \left(\frac{1}{\gamma_{\mathbf{n},\mathbf{m}}}\right) \overline{P}_{\mathbf{m},\mathbf{h}}. \tag{3.4}$$

Therefore

$$\begin{aligned} P^{(*)}\ \overline{P}^{(*)} &= \left(\sum P_{\mathbf{m},\mathbf{h}}^{(*)} \overline{P}_{\mathbf{h},\mathbf{k}}^{(*)}\right) \\ &= \left(\sum \gamma_{\mathbf{n},\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \frac{1}{\gamma_{\mathbf{n},\mathbf{h}}} \overline{P}_{\mathbf{h},\mathbf{k}}\right) = P\overline{P} = \mathbf{I}. \end{aligned} \tag{3.5}$$

Similarly, we find that,

$$\overline{P}^{(*)} P^{*} = \left(\frac{\gamma_{\mathbf{n},\mathbf{k}}}{\gamma_{\mathbf{n},\mathbf{m}}} \delta_{\mathbf{k}}^{\mathbf{m}}\right) = \mathbf{I},$$

where $\delta_{\mathbf{k}}^{\mathbf{m}}$ is the Kronneker symbol. Thus the basic property of the exponential Ruscheweyh differential operator product set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ is well defined from the parent set. Hence a representation of the monomial $\mathbf{z}^{\mathbf{m}}$ by the set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ of polynomials is possible.

4 Effectiveness of exponential Ruscheweyh differential operator product set of polynomials in closed hyperellipse

In this section, we give the answer of the following question: Let the set $\{P_{\mathbf{m}}(\mathbf{z})\}$ be effective in closed hyperellipse $\overline{E}_{[\mathbf{r}]}$. Does the set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ still effective in the same region?

Let $\{P_{\mathbf{m}}(\mathbf{z})\}$ be a basic set of polynomials of several complex variables and $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ be exponential Ruscheweyh differential operator product set associated to $\{P_{\mathbf{m}}(\mathbf{z})\}$. Let $\Omega\left[P_{\mathbf{m}}^{(*)}, \overline{E}_{[\mathbf{r}]}\right]$ be the Cannon sum of the set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ for the hyperellipse $\overline{E}_{[\mathbf{r}]}$, then

$$\begin{aligned} \Omega\left[P_{\mathbf{m}}^{(*)}, \overline{E}_{[\mathbf{r}]}\right] &= \sigma_{\mathbf{m}} \prod_{s=1}^{k} \{r_s\}^{<\mathbf{m}>-m_s} \sum_{\mathbf{h}} \left|\overline{P}_{\mathbf{m},\mathbf{h}}^{(*)}\right| M\left(P_{\mathbf{h}}^{*}, \overline{E}_{[\mathbf{r}]}\right) \\ &= \frac{\sigma_{\mathbf{m}}}{\gamma_{\mathbf{n},\mathbf{m}}} \prod_{s=1}^{k} \{r_s\}^{<\mathbf{m}>-m_s} \sum_{\mathbf{h}} \left|\overline{P}_{\mathbf{m},\mathbf{h}}\right| M\left(P_{\mathbf{h}}, \overline{E}_{[\mathbf{r}]}\right), \end{aligned} \tag{4.1}$$

where

$$M\left(P_{\mathbf{h}}^{(*)}, \overline{E}_{[\mathbf{r}]}\right) = \max_{\overline{E}_{[\mathbf{r}]}} \left|\mathbf{P}_{\mathbf{h}}^{(*)}(\mathbf{z})\right|.$$

Now, we let, $\mathcal{D}_{\mathbf{m}}$ be the degree of the polynomial of the highest in the representation (2.10). Hence by Cauchy's inequality, we see that

$$\begin{aligned}
M\left(P_{\mathbf{m}}^{(*)}, \overline{E}_{[\mathbf{r}]}\right) &= \max_{\overline{E}_{[\mathbf{r}]}} \left|P_{\mathbf{h}}^{(*)}(\mathbf{z})\right| \leq \sum_{\mathbf{h}} \left|P_{\mathbf{m},\mathbf{h}}^{(*)}\right| \frac{\prod_{s=1}^{n}\{r_s\}^{h_s}}{\sigma_h} \\
&= \sum_{\mathbf{h}} \gamma_{\mathbf{n},\mathbf{h}} \left|P_{\mathbf{m},\mathbf{h}}\right| \frac{\prod_{s=1}^{n}\{r_s\}^{h_s}}{\sigma_{\mathbf{h}}} \leq M\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) \sum_{h} \gamma_{\mathbf{n},\mathbf{h}} \\
&= M\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) \left[1 + \sum_{\mathbf{h}\geq 1} \gamma_{\mathbf{n},\mathbf{h}}\right] \\
&= M\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) \left[1 + \sum_{h\geq 1} \exp\left(\prod_{s=1}^{n} \frac{(h_s)_{n_s}}{n_s!}\right)\right] \\
&\leq K\, N_{\mathbf{m}}\, \mathcal{D}_{\mathbf{m}}^{n} M\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) \leq K_1 \mathcal{D}_{\mathbf{m}}^{n+2} M\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right),
\end{aligned} \tag{4.2}$$

where K_1 is a constant and the power n here because we differentiated n_s times. then the relation between the Cannon sums of the two sets $\{P_{\mathbf{m}}(\mathbf{z})\}$ and $\{P_{\mathbf{m}}^{(\star)}(\mathbf{z})\}$ can be obtained from the relations (4.1) and (4.2) as follows

$$\Omega\left[P_{\mathbf{m}}^{(*)}, \overline{E}_{[\mathbf{r}]}\right] \leq \frac{K_1 \mathcal{D}_{\mathbf{m}}^{n+2}}{\gamma_{\mathbf{n},\mathbf{m}}} \Omega\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) = K_2 \Omega\left[P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right]$$

where $K_2 = \frac{K_1 \mathcal{D}_{\mathbf{m}}^{n+2}}{\gamma_{\mathbf{n},\mathbf{m}}}$. Consider condition (2.15), we find that

$$\Omega\left[P^{(*)}, \overline{E}_{[\mathbf{r}]}\right] \leq \limsup_{<\mathbf{m}>\to\infty} \{\Omega[P_{\mathbf{m}}^{(*)}, \overline{E}_{[\mathbf{r}]}]\}^{\frac{1}{\langle\mathbf{m}\rangle}} \leq \prod_{s=1}^{n} r_s,$$

but

$$\Omega\left[P^{(*)}, \overline{E}_{[\mathbf{r}]}\right] \geq \prod_{s=1}^{n} r_s.$$

Then,

$$\Omega\left[P^{(*)}, \overline{E}_{[\mathbf{r}]}\right] = \prod_{s=1}^{n} r_s,$$

Therefore, according to (2.15) and using Theorem 2.1, we deduce that the effectiveness of the original set $\{P_{\mathbf{m}}(\mathbf{z})\}$ in $\overline{E}_{[\mathbf{r}]}$ implies the effectiveness of exponential Ruscheweyh differential operator product set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ in $\overline{E}_{[\mathbf{r}]}$. Hence, we obtain the following theorem:

Theorem 4.1. *If the Cannon basic set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ of polynomials in the several complex variables $z_s,\ s \in I$, for which the condition (2.15) is satisfied, is effective in the closed hyperellipse $\overline{E}_{[\boldsymbol{r}]}$, then the exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\boldsymbol{m}}(\boldsymbol{z})\}$ of polynomials associated with the set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ will be effective in $\overline{E}_{[\boldsymbol{r}]}$.*

If the condition (2.15) is not satisfied, then the set $\{P^{(*)}_{\mathbf{m}}(\mathbf{z})\}$ can not be effective in $\overline{E}_{[\mathbf{r}]}$. To ensure that, we give the following example:

Example 4.1. *Consider the set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ of polynomials of several complex variable $z_{s},\ s \in I$, is given by*

$$\begin{cases} P_{\boldsymbol{m}}(\boldsymbol{z}) = \sigma_{\boldsymbol{m}} \prod_{s=1}^{n} z_s^{m_s} + \sigma_{a\boldsymbol{m}} \prod_{s=1}^{n} z_s^{a\boldsymbol{m}_s}, \boldsymbol{m} \neq \boldsymbol{0}, \\ P_{\boldsymbol{m}}(\boldsymbol{z}) = \sigma_{\boldsymbol{m}} \prod_{s=1}^{n} z_s^{m_s} \qquad , otherwise, \end{cases}$$

where $a = b^{(<\boldsymbol{m}>)},\ b > 1$, then

$$z_s^{m_s} = \boldsymbol{z}^{\boldsymbol{m}} = \frac{1}{\sigma_{\boldsymbol{m}}}\left[P_{\boldsymbol{m}}(\boldsymbol{z}) - P_{a\boldsymbol{m}}(\boldsymbol{z})\right],$$

and the Cannon sum $\Omega\left[P^{(*)}_{\boldsymbol{m}}, \overline{E}_{[\boldsymbol{r}]}\right]$ will given by

$$\Omega\left[P^{(*)}_{\boldsymbol{m}}, \overline{E}_{[\boldsymbol{r}]}\right] = \prod_{s=1}^{n}\left[r_s^{\langle m\rangle} + 2r_s^{\langle m\rangle+(a-1)m_s}\right].$$

It turns out that

$$\Omega\left[P^{(*)}, \overline{E}_{[\boldsymbol{I}]}\right] \leq \limsup_{\langle \boldsymbol{m}\rangle\to\infty}\{\Omega\left[P_{\boldsymbol{m}}, \overline{E}_{[\boldsymbol{I}]}\right]\}^{\frac{1}{\langle \boldsymbol{m}\rangle}} = 1.$$

That is mean that the set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ is effective in $\overline{E}_{[\boldsymbol{I}]}$ for $r_s = 1,\ s \in I$.

Now, construct exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\mathbf{m}}(\mathbf{z})\}$ as follows

$$\begin{cases} P^{(*)}_{\mathbf{m}}(\mathbf{z}) = \sigma_{\mathbf{m}}\gamma_{\mathbf{n},\mathbf{m}}\mathbf{z}^{\mathbf{m}} + \sigma_{a\mathbf{m}}\gamma_{\mathbf{n},a\mathbf{m}} \prod_{s=1}^{k} z_s^{am_s}, \mathbf{m} \neq \mathbf{0}, \\ P^{(*)}_{\mathbf{m}}(\mathbf{z}) = \sigma_{\mathbf{m}}\gamma_{\mathbf{n},\mathbf{m}}\mathbf{z}^{\mathbf{m}}, \qquad otherwise. \end{cases}$$

Hence, it follows that

$$\mathbf{z}^{\mathbf{m}} = \frac{1}{\sigma_{\mathbf{m}}\gamma_{\mathbf{n},\mathbf{m}}}\left[P^{(*)}_{\mathbf{m}}(\mathbf{z}) - P^{(*)}_{a\mathbf{m}}(\mathbf{z})\right]$$

and the Cannon sum $\Omega[P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}]$ will given by

$$\begin{aligned}\Omega[P, \overline{E}_{[\mathbf{r}]}] &= \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_s\}^{\langle\mathbf{m}\rangle - m_s} \sum_{h} \left|\overline{P}^{(*)}_{\mathbf{m},\mathbf{h}}\right| M\left(P^{(*)}_{\mathbf{h}}, \overline{E}_{[\,\mathbf{r}]}\right) \\ &= \frac{1}{\gamma_{\mathbf{n},\mathbf{m}}} \left[\gamma_{\mathbf{n},\mathbf{m}} \prod_{s=1}^{n} r_s{}^{\langle\mathbf{m}\rangle} + 2\gamma_{\mathbf{n},a\mathbf{m}} \prod_{s=1}^{n} r_s^{\langle\mathbf{m}\rangle + (a-1)m_s}\right] \\ &= \gamma_{\mathbf{n},a\mathbf{m}} \prod_{s=1}^{n} r_s{}^{\langle m\rangle} + \zeta\,(a) \prod_{s=1}^{n} r_s^{\langle m\rangle + (a-1)m_s},\end{aligned}$$

where $\zeta\,(a) > 1$ is a constant depending only on a and

$$\Omega[P, \overline{E}_{[\mathbf{1}]}] = \limsup_{<\mathbf{m}>\to\infty} \{1 + \zeta\,(a)\}^{\frac{1}{<\mathbf{m}>}} > 1.$$

That is to say that the exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\mathbf{m}}\,(\mathbf{z})\}$ is not effective in $\overline{E}_{[\mathbf{1}]}$ for $r_s = 1,\ s \in I$, although the original set $\{P_{\mathbf{m}}\,(\mathbf{z})\}$ is effective in $\overline{E}_{[\mathbf{1}]}$. The reason for this, obviously, that the condition (2.15) is not satisfied by the set as $\{P_{\mathbf{m}}\,(\mathbf{z})\}$ required.

5 Effectiveness of exponential Ruscheweyh differential operator product set of polynomials in open hyperellipse and the region $D(\overline{E}_{[\mathbf{r}]})$.

In this section, we establish the effectiveness property for the exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\mathbf{m}}\,(\mathbf{z})\}$ in open Hyperllipse and the Region $D\left(\overline{E}_{[\mathbf{r}]}\right)$.

Suppose that the Cannon sum $\{P_{\mathbf{m}}\,(\mathbf{z})\}$ is effective in $E_{[\mathbf{R}]}$. Then from the properties of Cannon functions, it follows from Theorem 1.1 in [10], that

$$\Omega\Big[P, E_{[\mathbf{R}]}\Big] < \ \alpha([\mathbf{r}], [\mathbf{R}]),\ \text{for all } 0 < R_s < r_s, s \in I. \tag{5.1}$$

Constructing the sets of numbers $\{r_i^{(s)}, s \in I\}$, (cf. [10]) in such a way that $0 < r_0^{(s)} < r_s,\ s \in I$ and

$$\frac{r_0^{(s)}}{r_0^{(j)}} = \frac{r_s}{r_j},\ j, s \in I, \tag{5.2}$$

$$r_{i+1}^{(s)} = \frac{1}{2}\left(r_s + r_i^{(s)}\right); s \in I; i \geq 0. \tag{5.3}$$

It follows, easily, from (5.2) and (5.3) that

$$\frac{r_i^{(s)}}{r_i^{(j)}} = \frac{r_s}{r_j},\ j, s \in I;\ i \geq 0. \tag{5.4}$$

Therefore it follows that

$$R_s < r_i^{(s)} < r_s; s \in I; i \geq 0. \tag{5.5}$$

Now, since the set $\{P_{\mathbf{m}}(\mathbf{z})\}$ accord to (5.1), in view of (2.10) and (2.13), then corresponding to the numbers $r_i^{(s)}; s \in I$, there exists a constant $K \geq 1$ such that

$$\sigma_{\mathbf{m}} \prod_{s=1}^{n} \left\{ r_i^{(s)} \right\}^{\langle \mathbf{m} \rangle - m_s} G\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) < K \left\{ r_{i+1}^{(1)} \prod_{s=1}^{n} r_i^{(s)} \right\}^{\langle \mathbf{m} \rangle},$$

form which we get, in view of (5.4), the following inequality

$$\begin{aligned} G\left(P_{\mathbf{m}}, \overline{E}_{[\mathbf{r}]}\right) &< \frac{K}{\sigma_{\mathbf{m}}} \left\{ \frac{r_{i+1}^{(1)}}{r_i^{(1)}} \right\}^{\langle \mathbf{m} \rangle} \prod_{s=1}^{k} \left\{ r_i^{(s)} \right\}^{m_s} \\ &= \frac{K}{\sigma_{\mathbf{m}}} \prod_{s=1}^{n} \left\{ \frac{r_{i+1}^{(1)}}{r_i^{(1)}} r_i^{(s)} \right\}^{m_s} \\ &= \frac{K}{\sigma_{\mathbf{m}}} \prod_{s=1}^{n} \left\{ \frac{r_{i+1}^{(s)}}{r_i^{(s)}} r_i^{(s)} \right\}^{m_s} \\ &= \frac{K}{\alpha_m} \prod_{s=1}^{nk} \left\{ r_{i+1}^{(s)} \right\}^{m_s}; \ (m_s \geq 0; s \in I). \end{aligned}$$

Now, for the numbers $R_s, \ r_s, s \in I$, we have at least one of the following cases:

1. $\frac{R_1}{R_s} \leq \frac{r_1}{r_s}, \ s \in I$ or
2. $\frac{R_v}{R_s} \leq \frac{r_v}{r_s}; \ s \in I$, $v = 2$ or 3 or ....or $n-1$ or
3. $\frac{R_n}{R_s} \leq \frac{r_n}{r_s}; \ s \in I.$

Suppose now, that the relation 1 is satisfied, then from the construction of the sets $\left\{ r_i^{(s)}; \ s \in I \right\}$, we see that

$$\frac{R_1}{R_s} \leq \frac{r_1}{r_s} = \frac{r_{i+1}^{(1)}}{r_{i+1}^{(s)}}; \ s \in I. \tag{5.6}$$

Thus in view of (5.5) and (5.6) the Cannon sum of the set $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ for the hyperellipse $E_{[\mathbf{R}]}$, leads to

$$
\begin{aligned}
\Omega\Big[P_{\mathbf{m}}, E_{[\mathbf{R}]}\Big] &= \frac{\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{R_s\}^{\langle\mathbf{m}\rangle - m_s}}{\gamma_{\mathbf{n},\mathbf{m}}}\sum_{\mathbf{h}}\left|\overline{P}^{(*)}_{\mathbf{m},\mathbf{h}}\right| M\left(P^{(*)}_{\mathbf{h}}, \overline{E}_{[\mathbf{R}]}\right)\\
&< L\frac{\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{R_s\}^{\langle\mathbf{m}\rangle - m_s}}{\gamma_{\mathbf{n},\mathbf{m}}}\sum_{\mathbf{h}}\left|\overline{P}_{\mathbf{m},\mathbf{h}}\right| M\left(P_{\mathbf{h}}, \overline{E}_{[\mathbf{R}]}\right)\\
&= L\frac{\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{R_s\}^{\langle\mathbf{m}\rangle - m_s}}{\gamma_{\mathbf{n},\mathbf{m}}} G\left(P_{\mathbf{m}}, \overline{E}_{[r_i]}\right)\\
&< \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{k}\{R_s\}^{\langle\mathbf{m}\rangle - m_s}\left\{r^{(s)}_{i+1}\right\}^{m_s}\\
&= \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{R_1}{R_s}\right\}^{m_s}\prod_{s=2}^{n}\{R_s\}^{\langle\mathbf{m}\rangle}\\
&\leq \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{r_1}{r_s}\right\}^{m_s}\prod_{s=2}^{n}\{R_s\}^{\langle\mathbf{m}\rangle}\\
&= \frac{KL}{\alpha_{n,m}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{r^{(1)}_{i+1}}{r^{(s)}_{i+1}}\right\}^{m_s}\prod_{s=2}^{n}\{R_s\}^{\langle m\rangle}\\
&= \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\left\{\{r^{(1)}_{i+1}\}\prod_{s=2}^{n}R_s\right\}^{\langle\mathbf{m}\rangle},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\Omega\Big[P, E_{[\mathbf{R}]}\Big] &= \limsup_{\langle\mathbf{m}\rangle\to\infty}\{\Omega[P_{\mathbf{m}}, E_{[\mathbf{R}]}]\}^{\frac{1}{\langle\mathbf{m}\rangle}}\\
&\leq r^{(1)}_{i+1}\prod_{s=2}^{n}R_s < r_1\prod_{s=2}^{n}R_s,
\end{aligned}
\tag{5.7}
$$

where

$$
L = 1 + \sum_{(h)\geq 1}\exp\left(\prod_{s=1}^{n}\frac{(h_s)_{n_s}}{n_s!}\right)\prod_{s=1}^{n}\left\{\frac{R^{(s)}_i}{r^{(s)}_i}\right\}^{h_s}\ \forall 0 < R_s < r_s; s\in I.
$$

Also, if the relation 2 is satisfied for $v = 2$ or 3 or ....or $n-1$, then we have

$$
\frac{R_v}{R_s} \leq \frac{r_v}{r_s} = \frac{r^{(v)}_{i+1}}{r^{(s)}_{i+1}}. \tag{5.8}
$$

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Thus (5.5) and (5.8) leads to

$$\begin{aligned}
\Omega\Big[P^*_{\mathbf{m}}, E_{[\mathbf{R}]}\Big] &< \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{R_s\}^{\langle\mathbf{m}\rangle-m_s}\left\{r^{(s)}_{i+1}\right\}^{m_s} \\
&= \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{R_v}{R_s}\right\}^{m_s}\prod_{s=1,s\neq v}^{k}\{R_s\}^{\langle m\rangle} \\
&\leq \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{r_v}{r_s}\right\}^{m_s}\prod_{s=1,s\neq v}^{n}\{R_s\}^{\langle m\rangle} \\
&= \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\prod_{s=1}^{n}\{r^{(s)}_{i+1}\}^{m_s}\left\{\frac{r^{(v)}_{i+1}}{r^{(s)}_{i+1}}\right\}^{m_s}\prod_{s=1,s\neq v}^{n}\{R_s\}^{\langle m\rangle} \\
&= \frac{KL}{\gamma_{\mathbf{n},\mathbf{m}}}\left\{r^{(v)}_{i+1}\prod_{s=1,s\neq v}^{n}\{R_s\}\right\}^{\langle m\rangle}
\end{aligned}$$

Therefore,

$$\Omega\Big[P^{(*)}, E_{[\mathbf{R}]}\Big] \leq r^{(v)}_{i+1}\prod_{s=1,s\neq v}^{n} R_s < r_v\prod_{s=1,s\neq v}^{n} R_s, \tag{5.9}$$

where $v = 2$ or 3 or ....or $n-1$. Similarly if the relation 3 is satisfied, we proceed as above to show

$$\Omega\Big[P^{(*)}, E_{[\mathbf{R}]}\Big] < r_v\prod_{s=1}^{n-1}R_s. \tag{5.10}$$

Thus, it follows in view of (5.7), (5.9) and (5.10) that

$$\Omega\Big[P^{(*)}, E_{[\mathbf{R}]}\Big] < \alpha\left([r],[R]\right). \tag{5.11}$$

Therefore, according to (5.11) and using Theorem 2.2, the exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\mathbf{m}}(\mathbf{z})\}$ is effective in the open hyperellipse $E_{[\mathbf{r}]}$ when the original set $\{P_{\mathbf{m}}(\mathbf{z})\}$ is effective in $E_{[\mathbf{r}]}$.

Hence, we obtain the following theorem:

Theorem 5.1. *If the Cannon basic set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ of polynomials in the several complex variables z_s, $s \in I$, is effective in the open hyperellipse $E_{[\boldsymbol{r}]}$, then the exponential Ruscheweyh differential operator product set $\{P^{(*)}_{\boldsymbol{m}}(\boldsymbol{z})\}$ of polynomials associated with the set $\{P_{\boldsymbol{m}}(\boldsymbol{z})\}$ will be effective in $E_{[\boldsymbol{r}]}$.*

Now, using a similar proof as done to Theorem 5.1, the following relation follows

$$\Omega\Big[P^{(*)}, D(\overline{E}_{[\mathbf{r}]})\Big] = \prod_{s=1}^{n}r_s \text{ when } \Omega[P, D(\overline{E}_{[\mathbf{r}]})] = \prod_{s=1}^{n}r_s.$$

Therefore, by using Theorem 2.3, we obtain the following theorem:

Theorem 5.2. *If the Cannon basic set* $\{P_{\boldsymbol{m}}(z)\}$ *of polynomials in the several complex variables* $z_s,\ s \in I$, *is effective in the region* $D(\overline{E}_{[\boldsymbol{r}]})$, *then the exponential Ruscheweyh differential operator product set* $\{P_{\boldsymbol{m}}^{(*)}(z)\}$ *of polynomials associated with the set* $\{P_{\boldsymbol{m}}(z)\}$ *will be effective in* $D(\overline{E}_{[\boldsymbol{r}]})$.

To get the results concerning the effectiveness in the hyperspherical regions $\overline{S}_r$ (cf. [6, 11]) as special cases from the results concerning the effectiveness in the hyperelliptical regions $\overline{E}_{[\mathbf{r}]}$, put $r = r_s;\ s \in I$, in Theorem 4.1, Theorem 5.1 and Theorem 5.2 we can arrive to the following result

Corollary 5.1. *The effectiveness of the sets* $\{P_{\boldsymbol{m}}(z)\};\ s \in I$ *in the equiellipse*

i $\overline{E}_{[\boldsymbol{r}^*]}$ *yields the effectiveness of the set* $\{P_{\boldsymbol{m}}^{(*)}(z)\}$ *in the hyperspherical* $\overline{S}_r$.

ii $E_{[\boldsymbol{r}^*]}$ *yields the effectiveness of the set* $\{P_{\boldsymbol{m}}^{(*)}(z)\}$ *in the hyperspherical* S_r.

iii $D(\overline{E}_{[\boldsymbol{r}^*]})$ *yields the effectiveness of the set* $\{P_{\boldsymbol{m}}^{(*)}(z)\}$ *in the region* $D(\overline{S}_r)$.

Remark 5.1. It is worthy ensure that all results obtained in this work are also true for the exponential Ruscheweyh differential operator sum set $\{P_{\mathbf{m}}^{(\maltese)}(\mathbf{z})\}$ of polynomials of several complex variables in hyperelliptical regions and hyperspherical when the Ruschewey differential operator sum D^n acting on the monomials $\mathbf{z}^{\mathbf{m}}$, in the form

$$D^n\ \mathbf{z}^{\mathbf{m}} = \begin{cases} [\sum_{s=1}^{n} D_{z_s}^{n_s}]\ \mathbf{z}^{\mathbf{m}}, & \mathbf{m} \neq \mathbf{0} \\ 1, & \mathbf{m} = \mathbf{0}, \end{cases}$$

Remark 5.2. Similar results for the sets $\{P_{\mathbf{m}}^{(*)}(\mathbf{z})\}$ and $\{P_{\mathbf{m}}^{(\maltese)}(\mathbf{z})\}$ in hyperelliptical regions can be obtained when the original set $\{P_{\mathbf{m}}(\mathbf{z})\}$ is general basic set.

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Normality, Regularity and compactness of sb^*-closed sets in Topological spaces

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Abstract

Abstract: In this paper, we introduce and study the concept of normality, Regularity and compactness of sb* - closed set in topological spaces and some of the properties are discuseed.

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Keywords:sb*-closed set, dimension (X), Inductive dimension (X), sb*-compact space.

1. Introduction

Levine[9] introduced the concept of g- closed sets and studied their properties. Regular open sets and strongly regular open sets have been introduced and investigated by stone and Tang[17] respectively. Brouwer[2] introduced the dimension theory in a topological spaces. This dimension function coincides with small inductive dimension. The development of the theory of covering dimension for normal spaces is due to Aleksandrov[1], Dowker[3,4,6], Hemmingsen[8] and Morita[10,11,12]. They obtained the important characterization of dimensions interms of extension of mapping. Ostrand[13] has shown that covering dimension can be based on locally finite open coverings for all topological spaces and

obtained other interesting results for the covering dimension of general spaces. Dowker[5] introduced the class of totally normal spaces.The authors[14,15,16] introduced and studied the properties of sb* - closed sets, sb* - continuity, sb* irresolute maps and homeomorphisms in topological spaces.
In this paper, we introduce and study the concept of normality, regularity and compactness of sb* - closed sets in topological spaces.

2. Preliminaries

In this section, we begin by recalling some basic definitions.

Let (X, τ) be a topological space and A be a subset of X. The closure of A and interior of A are denoted by $cl(A)$ and $int(A)$ respectively.

Definition 2.1[9]:A subset A of a topological space (X, τ) is called a g- closed set if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X.

Definition 2.2[14]: A subset A of a topological space (X, τ) is called a strongly b^*- closed set (briefly sb^*- closed) if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is b open in X.

Definition 2.3[15]: Let X and Y be topological spaces. A map f: X $\rightarrow$ Y is called strongly b* - continuous (sb*- continuous) if the inverse image of every open set in Y is sb* - open in X.

Definition 2.4[16]: Let X and Y be topological spaces. A map f: (X,τ) $\rightarrow$ (Y, σ) is said to be sb* - Irresolute if the inverse image of every sb* - closed set in Y is sb* - closed set in X.

Definition 2.5[12]: The covering dimension of a topological space is defined in terms of the order of open refinements of finite open coverings of the space. The order of a family $\{A_i\}_{i \in \wedge}$ of subsets, not all empty, of some set is largest integer n for which there exists a subset of M of $\wedge$ with n+1 elements such that $\cap_{i \in M} A_i$ is non - empty, or is ∞ if there is no such largest integer. A family of empty subsets has order -1.

Definition 2.6[4]: The small inductive dimension of a space X, ind X is defined inductively as follows. A space X satisfies ind X = -1 if and only if X is empty. If n is a non - negative integer, then indX$\leq$ n means that for each point x $\in$ X and each open set G such that x $\in$ G there exists an open set U such that x$\in$ U $\subset$ G and indbd(U) $\leq$ n-1. If ind X =n, it is true that ind X $\leq$ n, but it is not true that ind X $\leq$ n-1. If there exists no integer n for which indX $\leq$ n then indX = ∞.

Definition 2.7[7]: If X =ϕ, then Dind X = -1. Assuming that the inequality Dind X $\leq$n-1 is defined, it is said that Dind X $\leq$ n if for any finite open covering u = $\{U_1, U_2, ..., U_k\}$ there is a family v = $\{V_1, V_2, ..., V_k\}$of disjoint open sets such that v refines u and Dind(X - $U_{i=1}^{m} V_i$) $\leq$ n-1.

3. Properties of sb* - closed maps

In this section, we study some properties of sb* - closed maps.

Theorem 3.1: If f: X $\rightarrow$ Y is continuous and sb* - closed and A is a sb* -closed set of X then f(A) is sb* - closed.
Proof : Let f(A) $\subseteq$ O, where O is an open set of Y. Since f is continuous, $f^{-1}(O)$ is an open set containing A. Hence cl(int(A)) $\subseteq$ $f^{-1}(O)$ as A is sb* - closed set. Since f is sb* - closed, f(cl(int(A))) is sb* - closed set contained in an open set O, which implies that cl(int(f(cl(A))))$\subseteq$ O and hence cl(int(f(A))) $\subseteq$ O. So f(A) is sb*-closed in Y.

Theorem 3.2: If a map f : X $\rightarrow$ Y is sb* - closed and continuous and A is sb* - closed set of X, then f_A : A $\rightarrow$ Y is continuous and sb* -closed.
Proof: Let F be a closed set of A. Then F is sb* - closed set of X. From theorem 3.1, it follows that f_A(F) = f(F) is sb*-closed set of Y. Hence f_A is sb* - closed and continuous.

Theorem 3.3: If f: X $\rightarrow$ Y is sb* -closed and A = $f^{-1}(B)$ for some closed set B of Y then f_A: A $\rightarrow$ Y is sb* - closed.
Proof: Let F be a closed set in A. Then there is a closed set H in X such that F = A $\cap$ H. Then f_A(F) = f(A$\cap$H) = f(H) $\cap$ f(B). Since f is sb* - closed , f(H) is sb* - closed in Y. So f(H) $\cap$ B is sb* -closed in Y. Since the intersection of a closed and sb* - closed set is sb* - closed, f_A is sb* - closed.

Theorem 3.4: If a map f: (X, τ) $\rightarrow$ $(Y, \sigma\ )$ is sb* - closed and A is closed set of X, then f_A: (A, τ_A) $\rightarrow$ (Y, σ) is sb* - closed.
Proof: Let F be a closed set of A. Then F = A $\cap$ E for some closed set E of X and so F is closed set of (X, τ). Since f is sb* - closed, f(F) is sb* - closed set in (Y, σ). But f(F) = f_A(F) and therefore f_A :(A, τ_A) $\rightarrow$ (Y, σ) is sb* - closed.

Theorem 3.5: For any bijection map f: (X, τ) $\rightarrow$ (Y, σ), the following statments are equivalent.
(i) f^{-1} : (Y, σ) $\rightarrow$ (X, τ) is sb* - continuous.
(ii) f is sb* - open map.
(iii) f is sb* - closed map.
Proof: (i) $\Rightarrow$ (ii) : Let U be an open set of (X, τ). By assumption, $(f^{-1})^{-1}$(U) = f(U) is sb* - open in (Y, σ) and so f is sb* - open.
(ii) $\Rightarrow$ (iii) : Let F be a closed set of (X, τ). Then F^c is an open set of (X, τ). By assumption, f(F^c) is sb* open in (Y, σ).That is f(F^c)= $(f(F))^c$ is sb* open in (Y, σ) and therefore f(F) is sb* - closed in (Y, σ). Hence f is sb* - closed.
(iii) $\Rightarrow$ (i) : Let F be a closed set of (X, τ). By assumption, f(F) is sb* - closed in (Y, σ). But f(F) = $(f^{-1})^{-1}$(F) and therefore f^{-1} is sb* - continuous.

4.Normality,Regularity and compactness of sb* - closed set

In this section, we study Normality,Regularity and compactness of sb* - closed sets and also we discuss their properties.

Theorem 4.1: If a map f: X $\rightarrow$ Y is continuous, sb* - closed from a normal space X onto a space Y, then Y is normal.
Proof: Let A and B be disjoint closed sets of Y. Then $f^{-1}(A)$, f^{-1}(B) are disjoint closed sets of X. Since X is normal, there are disjoint open sets U, V in X such that f^{-1}(A)$\subseteq$ U and f^{-1}(B)$\subseteq$ V. Since f is sb* - closed, there are open sets G, H in Y such that A $\subseteq$ G, B $\subseteq$ H and f^{-1}(G)$\subseteq$ U and f^{-1}(H)$\subseteq$ V. Since U, V are disjoint, int (G), int (H) are disjoint open sets. Since G is sb* - open, A is closed and A $\subseteq$ G, A $\subseteq$ cl(int(G)). Similarly B $\subseteq$ cl(int(H)). Hence Y is normal.

Theorem 4.2: If f: (X, τ) $\rightarrow$ (Y, σ) is an open, continuous, sb* - closed surjection, where X is regular then Y is regular.
Proof: Let U be an open set in Y and p $\in$ U. Since f is surjection there exists a point x $\in$ X such that f(x) = p. Since X is regular and f is continuous, there is an open set V in X such that x $\in$ V $\subseteq$cl(V) $\subseteq$ f^{-1}(U). Here p $\in$ f(V) $\subset$ f(cl(V)) $\subset$ U. Since f is sb* - closed, f(cl(V)) is sb* - closed set contained in the open set U. By hypothesis, cl(f(cl(V))) = f(cl(V)) and cl(f(V)) = cl(f(cl(V))). Therefore p $\in$ f(V) $\subset$ cl(f(V)) $\subset$ U and f(V) is open, since f is open. Hence Y is regular.

Theorem 4.3: If A is sb* - closed set of a space X, then ind A $\leq$ ind X.

Proof: Let A is sb* -closed set of X, then indA $\leq$ n. By the induction proof, the result holds trivially if n = -1. By assumption that for every sb* - closed set A of X, ind X $\leq$n-1. $\Rightarrow$ ind A$\leq$n-1.
Let X be a space with ind X $\leq$ n. Let A be a sb* - closed set of X. Let E be a closed set of A and G be an open set of A such that E $\subset$ G. Then there exists a closed set F of X and an open set H of X such that E = A $\cap$ F and G = A $\cap$ H. Since E is closed in A and A is sb* - closed set in X, E is sb* -closed in X. Since E $\subset$ H and H is open , cl(E) $\subseteq$ H. Since indX $\leq$ n, there exists an open set V of X such that cl(E) $\subset$ V $\subset$ H and indbd(V) $\leq$ n-1. Let U = V $\cap$ A is an open set of A such that E $\subset$ V$\cap$ A $\subset$ G and bd_A(V $\cap$ A) $\subseteq$ bd(V) $\cap$ A is sb* - closed set of bd(V). By the induction hypothesis inbd$_A$(V) $\leq$ n-1. Hence ind A $\leq$ n. Therefore indA $\leq$ind X.

Theorem 4.4: If A is a sb* - closed set of a space X then dim A $\leq$ dim X.
Proof: Let A is a sb* - closed set of X. If dim X = 0 then dim A $\leq$ 0. Hence dim A $\leq$ dim X.
Suppose that dim X = n , where n is the largest integer greater than or equal to -1. ie., dim X $\leq$ 0. If n = -1, dim X = -1 which implies that X = ϕ and hence a sb* - closed set A = ϕ. Therefore dim A also equal to -1 and thus dim A $\leq$ dim X.

Next suppose that dim X = n, where n $\geq$-1. Let A be a sb* - closed set of X. Let $\{U_1, U_2, U_3, ...U_k\}$ be a finite open covering of A. Then for i = 1,2,3,.....k, there exists open sets V_i of X such that U_i = A $\cap V_i$. Since A is sb* - closed and $U_{i=1}^k V_i$ is an open set containing A. cl(int (A)) $\subset U_{i=1}^k V_i$. Since cl(int(A)) is a closed set,dimcl(int(A)) $\leq$ n. So the open cover $\{cl(int(A)) \cap V_i, i = 1, 2, 3, ..., k\}$, cl(int(A)) has a refinement cl(int(A))$\cap$ W_i, i = 1,2,3,....,k of order atmost n+1, where each W_i is open in X and cl(int$(A)) \cap W_i \subset cl(int(A)) \cap V_i$ for each i. Then $\{A \cap W_i, i = 1, 2, 3, ..., k\}$ is an open cover of A refining $\{U_i, i = 1, 2, 3, ..., k\}$ and of order not exceeding n+1. Hence dim A $\leq$ n which implies that dim A $\leq$ dimX.

Theorem 4.5: If A is a sb* - closed set of a space X then Dind A $\leq$ DindX.

Proof: Let X be a topological space such that Dind X = n and A is a sb* - closed set of X. We know that cl(int(A))$\subset U_{i=1}^k V_i$ and DindA $\leq$ n.similarly cl(int(A)) is a closed set , Dind cl(int(A)) $\leq$ n. Hence for every open cover cl(int(A))$\cap V_i$, i = 1,2,3,....k there is a disjoint family W_j , j = 1,2,3,....k of open sets cl(int(A)) refining cl(int(A))$\cap V_i, i =$

$1, 2, 3, ..., k$ such that Dind(cl(int(A))-$U_{j=1}^{k} W_j$ ≤n-1. But A- $U_{j=1}^{k} W_j \subset clint((A)) - U_{j=1}^{k} W_j$ and A- $U_{j=1}^{k} W_j = A \cap (cl(int(A) - U_{j=1}^{k} W_{j)})$ is a sb* - closed set as the intersection of sb* - closed set and a closed set is sb* - closed set. By induction hypothesis, Dind(A-$U_{j=1}^{k} W_j) \leq n-1$. Also $W_j \cap$A, j = 1,2,3,.....,k is a disjoint family of open sets of A refining $\{U_1, U_2, U_3, ...U_k\}$. Thus Dind A ≤ n and hence DindA ≤Dind X.

Defintion 4.6: Let (X, τ) be a topological space and Let B be a subset of X. A collection $\{A_i : i \in \wedge\}$of sb* - open sets of X is called a sb* - open cover of B if B⊆ $\cup\{A_i : i \in \wedge\}$.

Definition 4.7 : A topological space (X, τ) is sb* compact, if every sb* - open cover of X has a finite subcover.

Definition 4.8: A subset A of a topological space X is said to be sb* compact relative to X if, for every collection $\{A_i : i \in \wedge\}$ of sb* - open subsets of X such that B⊆ $\cap\{A_i : i \in \wedge\}$ there exists a finite subset $\wedge_0$ of $\wedge$ such that B $\subseteq \cup\{A_i : i \in \wedge_0\}$.

Defintion 4.9 : A subset B of a topologicalo space X is said to be sb* compact space if B is sb* compact as a subspace of X.

Theorem 4.10: Every sb* - closed subset of a sb* compact space is sb* compact relative to X.
Proof: Let A be a sb* - closed subset of sb* compact space X. Then $\{X - A\{$ is sb* - open in X. Let S be a cover of A. Then S $\cup\{X - A\}$ is a sb* - open cover of X. Since X is sb* - compact space, it contains a finite subcover of X, $(A_{i_1}, A_{i_2}, ...A_{i_k}\} \cup \{X - A\}$, $A_{i_k} \in S$. Then $(A_{i_1} \cup A_{i_2} \cup .... \cup A_{i_k}\}$ is a finite subcollection of S that covers A. This proves that A is sb* compact relative to X.

Theorem 4.11: A sb* - continuous image of a sb* compact space is compact.
Proof: Let f: X → Y be a sb* continuous map from a sb* compact space X onto a topological space Y. Let $\{A_i : i \in \wedge\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \wedge\}$ is a sb* - open cover of X. Since X is sb* compact, $\{f^{-1}(A_i) : i \in \wedge\}$ has a finite subcover, namely $\{f^{-1}(A_{i1}), f^{-1}(A_{i2}), ..., f^{-1}(A_{in})\}$. Then $\{A_{i_1}, A_{i_2}, ...A_{i_k}\}$ is a cover of Y. Thus Y is compact.

Theorem 4.12: A space X is sb* compact if and only if every family of sb* - closed set in X with empty intersection has a finite sub family with empty intersection.

Proof: Suppose X is compact and $\{A_i : i \in \wedge\}$ is a family of sb* - closed sets in X such that $\cap\{A_i : i \in \wedge\} = \phi$. Then $\cup\{X - A_i : i \in \wedge\}$ is a sb* - open cover of X. Since X is sb* compact, this cover has a finite sub cover for X. This implies that $\cap_{k=1}^{n} A_{ik} = \phi$.
Conversely, suppose that every family of sb* - closed sets in X which has empty intersection has a finite sub family with empty intersection. Let $\{U_i : i \in \wedge\}$ be a sb* -open cover of X. Then $\cup\{U_i : i \in \wedge\} =$ X. This implies that $\cap\{X - U_i : i \in \wedge\} = \phi$. Since X -$U_i$ is sb* - closed for each $i \in \wedge$. By assumption, there is a finite sub family, $(X - A_{i_1}, X - A_{i_2}, ...X - A_{i_k}\}$ with empty intersection. Therefore $\cup_{i=1}^{n} U_{i_k} = X$. Hence X is sb* - compact.

Theorem 4.13: Let f: X $\rightarrow$ Y be a sb* -irresolute surjection and X be a sb* compact. Then Y is compact.
Proof: Let f: X $\rightarrow$ Y be a sb* irresolute surjection and X be a sb* compact space X onto a topological space Y. Let $\{A_i : i \in \wedge\}$ be a sb* -open cover of Y. Then $\{f^{-1}(A_i : i \ in \wedge\}$ is a sb* - open cover of X. Since X is sb* compact, $\{f^{-1}(A_i : i \in \wedge\}$ has a finite subcover, namely $\{f^{-1}(A_{i1}), f^{-1}(A_{i2}), ..., f^{-1}(A_{in})\}$. Then $\{A_{i_1}, A_{i_2}, ...A_{i_k}\}$ is a finite subcover of Y. Thus Y is sb* compact.

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New results on harmonious labeling

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Abstract

In this paper, I present some new classes of harmonious graphs and I have given partial answers to some of the open problems listed in [7].

Keywords: Harmonious, sequential and indexable labelings.

Mathematical Subject Classifications: 05C78

1.Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notation and terminologies of graph theory as in [3]. Most graph labeling methods trace their origin to one introduced by Rosa[19] in 1967, or one given by Graham and Sloane[11] in 1980. Harmonious graphs naturally arose in the study by Graham and Sloane [11] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph G of order p and size q to be harmonious if there is an injective function, called a harmonious labeling,

$$f : V(G) \to \mathbb{Z}_q ,$$

where $\mathbb{Z}_q$ is the group of integers modulo q, such that the induced function

$$f^* : E(G) \to \mathbb{Z}_q$$

defined by

$$f(xy) = f(x) + f(y)\text{, for all edge } xy \in E(G)$$

is bijection. The image of $f\,(=\mathrm{Im}(f))$ is called the corresponding set of vertex labels. This definition extends to the case when G is a tree or in general for a graph G with $p = q + 1$ by allowing exactly two vertices to have the same label. Graham and Sloane [11] proved that if a harmonious graph has an even number of edges q and the degree of every vertex is divisible by 2^k then q is divisible by 2^{k+1}. This necessary condition called the harmonious parity condition. There are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. Youssef [24] has shown that if G is harmonious then nG and $G^{(n)}$, the graph consisting of n copies of G with one vertex in common, are harmonious for all odd n.

Chang, Hsu, and Rogers [2] and Grace [10] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling f of a graph G with q vertices to be strongly c-harmonious if the vertex labels are from $\left\{0,1,\cdots,q-1\right\}$ and the edge labels induced by $f(x)+f(y)$ for each edge xy are $c, c+1, \cdots, c+q-1$. Grace called such a labeling sequential. In case of a tree, Grace allows the vertex labels to range from 0 to q with. Strongly 1-harmonious is called strongly harmonious. By taking the edge labels of sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph.

Acharya and Hegde [1] call a graph G with p vertices and q edges (k,d)-indexable if there is an injective function from $V(G)$ to $\left\{0,1,2,\ldots,p-1\right\}$ such that the set of edge labels induced by adding the vertex labels is a subset of $\left\{k, k+d, k+2d, \ldots, k+(q-1)d\right\}$. When the set of edges is $\left\{k, k+d, k+2d, \ldots, k+(q-1)d\right\}$, the graph is said to be strongly (k,d)-indexable. A $(k,1)$-indexable is more simply called k-indexable and strongly 1-indexable graphs are simply called strongly indexable. Hegde and Shetty [12] also proved that if G is strongly k-indexable Eulerian graph with q edges then, one has $q \equiv 0,3 \pmod 4$ if k is even, and $q \equiv 0,1 \pmod 4$ if k is odd. They further showed how strongly k-indexable graphs can be used to construct polygons of equal internal angles with sides of different lengths.

Germina [9] has proved the following: fans $P_n + K_1$ are strongly indexable if and only if $1 \le n \le 6$; $P_n + K_2$ is strongly indexable if and only if $n = 1,2$; the only strongly indexable complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$. Also, $K_n \times P_m$ is a strongly indexable if and only if $n = 3$ and $m \ge 1$.

In 1970 Kotzig and Rosa [15] defined an edge-magic total labeling of a graph G as a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, |V(G)| + |E(G)|\}$ such that for all edges xy, $f(x) + f(y) + f(xy)$ is constant. Enomoto, Llado, Nakamigawa, and Ringel [4] call an edge-magic total labeling super edge-magic if the set of vertex labels is $\{1, 2, \cdots, |V(G)|\}$.

The reference [7] surveys the current state of knowledge for all variations of graph labelings appearing in this paper. We present some new classes of harmonious graphs and we present partial answers to some of the open problems listed in [7].

2. Main results

Grace[10] showed that an odd cycle with one or more pendant edges at each vertex is harmonious and conjectured that an even cycle with one pendant edge attached at each vertex, is harmonious. This conjecture has been proved by Liu and Zhang [16]. In their 1980 paper Graham and Sloane [11] proved that $C_n \times P_m$ is harmonious when n is odd and they used a computer software to show $C_4 \times P_2$, the cube, is not harmonious. In 1992 Gallian, Prout, and Winters [8] proved that $C_n \times P_2$ is harmonious when $n \neq 4$. In 1992, Jungreis and Reid [14] showed that $C_4 \times P_m$ is harmonious when $m \geq 3$. However we generalize the above results for odd cycles.

Theorem 1. The graph G obtained from $C_n \times P_m$ by adding p pendant edges to every vertex of the outer cycle is harmonious for all $n \geq 3, m \geq 1$ and $p \geq 0$.

Proof. Let $V(C_n) = \{u_1, u_2, ..., u_n\}$ and $V(P_m) = \{v_1, v_2, ..., v_m\}$ and let the pendant vertices at each vertex of the outer cycle be $w_1^i, w_2^i, ..., w_p^i$, $1 \leq i \leq n$. Put $q = |E(G)| = (2m + p - 1)n$, and for abbreviation, we write (i, j) instead of (u_i, v_j).

Define a labeling function,

$$f : V(G) \to \mathbb{Z}_q$$

as follows

For $1 \leq j \leq m$

$$f(i,j) = \begin{cases} (j-1)n + i - j(\operatorname{mod} n), & j(\operatorname{mod} n) \leq i \leq n \\ nj + i - j(\operatorname{mod} n), & 1 \leq i \leq j(\operatorname{mod} n) - 1 \end{cases}$$

For $1 \leq k \leq p$,

$$f(w_k^i) = \begin{cases} (m+k-1)n + i - (m+1)(\bmod n), & (m+1)(\bmod n) \le i \le n \\ (m+k)n + i - (m+1)(\bmod n), & 1 \le i \le m(\bmod n) \end{cases}$$

It is not difficult to verify that f is a harmonious labeling. □

Figure 1 shows the harmonious labeling of the graph $C_7 \times P_2$ with 3 pendant edges at each vertex of the outer cycle.

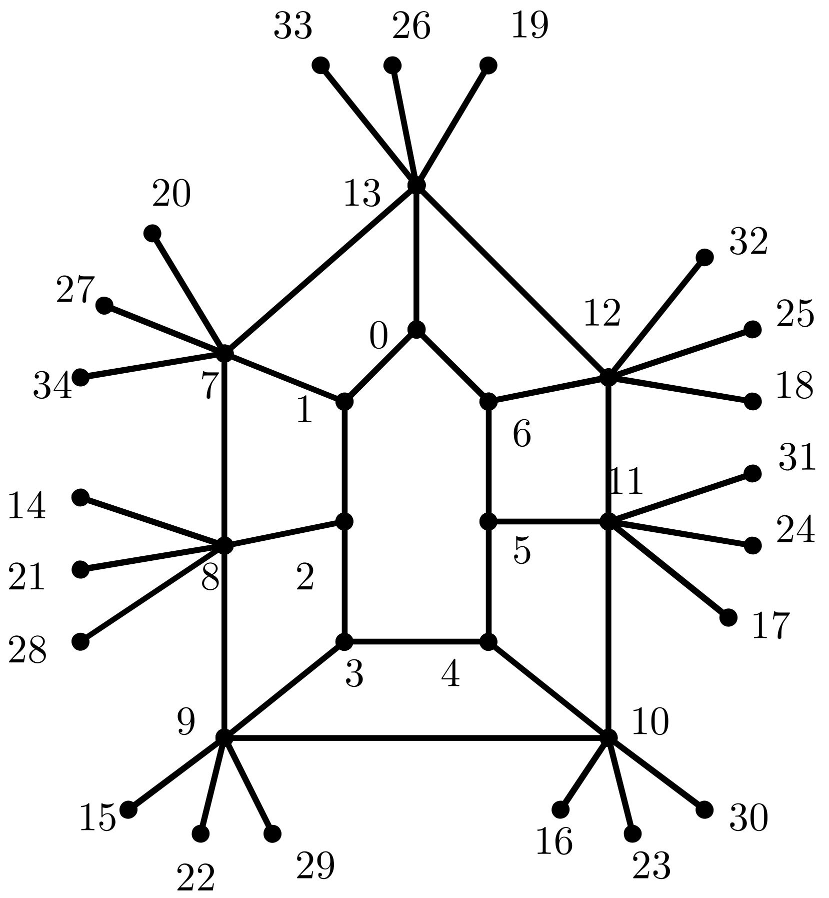

Figure 1

The following two results concern the harmoniousness of the disjoint union a complete graph and a star.

Theorem 2. $K_3 \cup S_n$ is harmonious for all $n \geq 1$.

Proof. A harmonious labeling of the graph is described as in Figure 2.

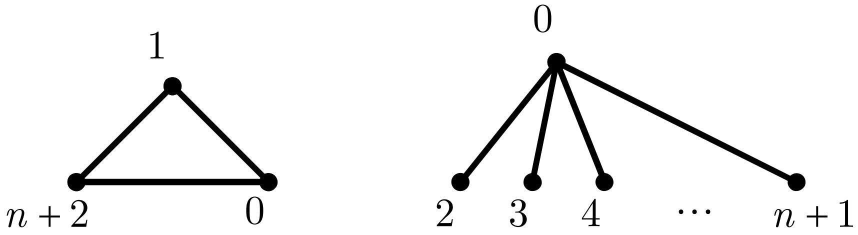

Figure 2 □

Theorem 3. $K_4 \cup S_n$ is harmonious if and only if $n \equiv 0 (\bmod 6)$

Proof. Let $q = \left|E(K_4 \cup S_n)\right| = n + 6$. Suppose that the graph has a harmonious labeling f where the label assigned to each vertex as indicated in Figure 3

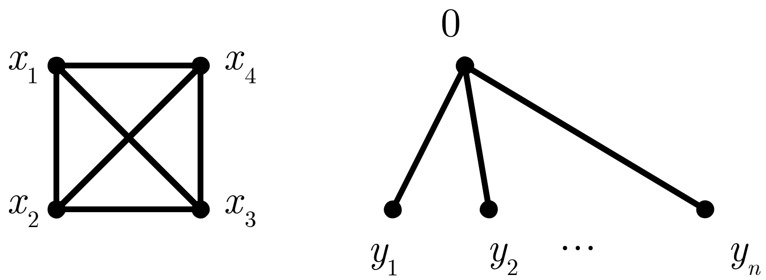

Figure 3

Let $t \in \mathbb{Z}_q$ such that $t \notin \operatorname{Im}(f)$. Then K_4 must give the remaining six edge labels , which are : $0, x_1, x_2, x_3, x_4, t$. Adding the edge labels on K_4, we get

$$2\sum_{i=1}^{4} x_i \equiv t (\bmod q) \quad \cdots (1)$$

If the edge labels 0 and t are produced by two independent edges, we get $t \equiv 0(\bmod q)$ which is absurd. Then we may assume that $x_1 + x_2 \equiv t(\bmod q)$ and $x_1 + x_3 \equiv 0(\bmod q)$, and

$$x_2 + x_4 \equiv x_3(\bmod q) \quad \cdots(2)$$

(since otherwise $x_1 \equiv x_2(\bmod q)$ which is absurd). Then we have also,

$$\begin{aligned} x_1 + x_4 &\equiv x_2(\bmod q) \quad \cdots(3) \\ x_3 + x_4 &\equiv x_1(\bmod q) \quad \cdots(4) \\ x_2 + x_3 &\equiv x_4(\bmod q) \quad \cdots(5) \end{aligned}$$

Therefore, we have $2x_2 \equiv 0(\bmod q)$ (by adding equations (2) and (5)) and substituting from $x_1 + x_2 \equiv t(\bmod q)$ and equation (4) into equation (1), we get $t + 2x_1 \equiv 0(\bmod q)$, also we have $x_1 \equiv t + x_2(\bmod q)$ or $2x_1 \equiv 2t(\bmod q)$, that is $3t \equiv 0(\bmod q)$. Also equation (3) gives $t + x_4 \equiv 0(\bmod q)$ and adding equations (2) and (4) we get $x_2 + 2x_4 \equiv x_1(\bmod q)$ or $2x_4 \equiv t(\bmod q)$ or $3x_4 \equiv 0(\bmod q)$. That is we have

$$\begin{aligned} 2x_2 &\equiv 0(\bmod q) \\ 3t &\equiv 0(\bmod q) \\ 3x_4 &\equiv 0(\bmod q) \end{aligned}$$

If q is odd, we get $x_2 \equiv 0(\bmod q)$ which is absurd. If q is even and is not divisible by 3, we get $x_4 \equiv t \equiv 0(\bmod q)$ which is absurd too.

Conversely, Let $n \equiv 0(\bmod 6)$. Define a bijection

$$f : V(K_4 \cup S_n) \to \mathbb{Z}_q - \{t\}$$

such as

$$f(x_1) = 5\frac{q}{6} \quad (or\ \frac{q}{6})$$
$$f(x_2) = \frac{q}{2}$$
$$f(x_3) = \frac{q}{6} \quad (or\ 5\frac{q}{6})$$
$$f(x_4) = 2\frac{q}{3} \quad (or\ \frac{q}{3})$$

and $t = \frac{q}{3} \quad (or\ 2\frac{q}{3})$. It is easy to verify that f is a harmonious labeling of $K_4 \cup S_n$. □

Graham and Sloane [11] showed that all paths $P_n, n \geq 2$ are harmonious and Grace [10] showed that $P_n^2, n \geq 3$ is harmonious while Seoud, Abdel Maqsoud and Sheehan [21] showed that $P_n^3, n \geq 4$ is harmonious and conjectured that P_n^k is not harmonious if $k \geq 4$ and $n \geq k+1$. The same conjecture was made by Fu and Wu [6]. However, the following example disprove such a conjecture.

Example 1 P_8^4 is harmonious

Let $q = \left|E(P_8^4)\right| = 22$ and the vertices of P_n be $v_1, v_2, \ldots, v_8$ such that for $1 \leq i < j \leq 8$, $v_i v_j \in E(P_8^4)$ if and only if $\left|i - j\right| \leq 4$.

Define a labeling function

$$f : V(P_8^4) \to \mathbb{Z}_{22}$$

As follows

$$f(v_1) = 0, \quad f(v_2) = 1, \quad f(v_3) = 5, \quad f(v_4) = 10,$$
$$f(v_5) = 21, \quad f(v_6) = 15, \quad f(v_7) = 19, \quad f(v_8) = 20.$$

Since,

$$\begin{aligned} f^*(E(P_8^4)) = &\{f(v_i) + f(v_j) : 1 \le i < j \le 8, \quad |i-j| = 1\} \cup \\ &\{f(v_i) + f(v_j) : 1 \le i < j \le 8, \quad |i-j| = 2\} \cup \\ &\{f(v_i) + f(v_j) : 1 \le i < j \le 8, \quad |i-j| = 3\} \cup \\ &\{f(v_i) + f(v_j) : 1 \le i < j \le 8, \quad |i-j| = 4\} \\ = &\{1, 6, 15, 9, 14, 12, 17\} \cup \{5, 11, 4, 3, 18, 13\} \cup \\ &\{10, 22, 20, 7, 19\} \cup \{21, 16, 2, 8\}. \end{aligned}$$

So, f is a harmonious labeling of P_8^4.

Remark The number 8 in the previous example is the least number n for which P_n^4 is harmonious, where $n \ge 5$. Since $P_5^4 = K_5$ is not harmonious [11] and the graphs P_6^4 and P_7^4 are not harmonious as the maximum number of edges in harmonious graphs of order 6 and 7 are 13 and 17 respectively [11].

We mention that the harmoniousness of the square of cycles is still an open problem. Let $n \ge 4$, from the harmonious parity condition, if C_n^2 is harmonious, then $n \equiv 0 (\mathrm{mod}\, 4)$. We conjecture that this necessary condition is also sufficient in this case. We have $C_4^2 = K_4$ is harmonious by [11] and Figure 4 below gives a harmonious labeling of C_8^2. But we could not go any further at this moment.

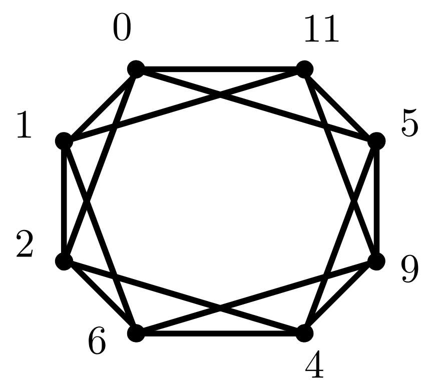

Figure 4

Conjecture 1. C_n^2 is harmonious if and only if $n \equiv 0 (\mathrm{mod}\, 4)$, where $n \geq 4$.

Liu and Zhang [17] have shown that mK_n is not harmonious for n odd and $m \equiv 2(\mathrm{mod}\, 4)$, and is harmonious for $n = 3$ and m odd. They conjecture that mK_3 is not harmonious when $m \equiv 0(\mathrm{mod}\, 4)$. We point out this conjecture was settled by Seoud, Abdel Maqsoud and Sheehan [21] who proved that mC_n is not harmonious if m or n is even and by noticing that $K_3 = C_3$

Theorem 4. Let T be a tree of order n. If $T + K_1$ is strongly indexable, then $T + S_m$ is harmonious for all $m \geq 1$.

Proof. Let $V(K_1) = \{v_0\}$, $V(S_m) = \{v_0, v_1, v_2, \cdots, v_m\}$ where v_0 is the center vertex of S_m and $q = |E(T + S_m)| = (m+2)n + m - 1$. Suppose g is a strongly indexable labeling of $T + K_1$. Define a labeling function

$$f : V(T + S_m) \to \{0, 1, \cdots, q-1\}$$

as follows

$$f\big|_{V(T)} = g\big|_{V(T)}$$
$$f(v_0) = g(v_0)$$
$$f(v_i) = (n+1)i + n - 1, \quad 1 \le i \le m$$

Since,

$$f^*(E(T+S_m)) = \{1,2,\cdots,2n-1\} \cup \{2n,2n+1,\cdots,3n; \\ 3n+1,3n+2,\cdots,4n+1;\cdots;(m+1)n+m-1, \\ (m+1)n+m,\cdots,(m+2)n+m-1\}.$$

Then f is a strongly harmonious labeling of $T+S_m$ and hence the graph is harmonious. □

Selvaraju and Sethuraman [20] and [22] have shown that $P_n + P_2$ is harmonious and they ask whether $P_n + P_m$ or $P_n + S_m$ is harmonious. Lu [18] showed that $P_3 + S_m$ is harmonious. As, $P_n + K_1$ is strongly indexable if and only if $1 \le n \le 6$, by Germina [9]. The following result gives a partial answer to the question of Selvaraju and Sethuraman.

Corollary 5. $P_n + S_m$ is harmonious for all $1 \le n \le 6$ and $m \ge 1$.

Also as $S_n + K_1 = K_{1,1,n}$ is strongly indexable [9] , then we have the following

Corollary 6. $S_n + S_m$ is harmonious for all $m, n \ge 1$.

Sparklers $Sp_{m,n}$ is the graph obtained by joining an end vertex of a path P_m to the center of a star S_n [7]. The following is another corollary on the above theorem.

Corollary 7. As $Sp_{5,n} + K_1$ is strongly indexable as indicated in Figure 5, then $Sp_{5,n} + S_m$ is harmonious.

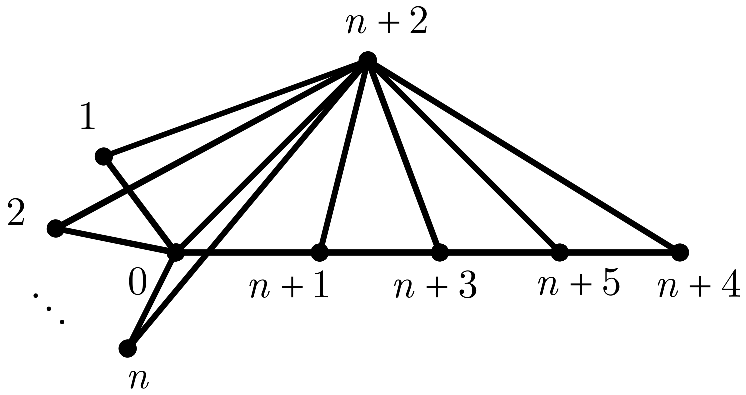

Figure 5

Yang, Lu, and Zeng [23] showed that all graphs of the form $C_{2n} \cup C_{2j+1}$ are harmonious except for $(n, j) = (2,1)$. Figueroa-Centeno, Ichishima, Muntaner-Batle, and Oshima [5] proved that $C_3 \cup C_n$ is super edge-magic if and only $n \geq 6$ and n is even ; $C_4 \cup C_n$ is super edge-magic if and only if $n \geq 5$ and n is odd and $C_5 \cup C_n$ is super edge-magic if and only if $n \geq 4$ and n is even is harmonious if and only if. Figueroa-Centeno et al. [5] conjectured that $C_m \cup C_n$ is super edge-magic if and only if $m + n \geq 9$ and $m + n$ is odd. In 2002 Hegde and Shetty [13] showed that a graph has a strongly k-indexable labeling if and only if it has a super edge-magic labeling. For a (p, q) graph with $p = q + 1$ or $p = q$, the notions of sequential labelings and strongly k-indexable labelings coincide. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled [7].

Comment From the above statements either the Conjecture of Figueroa-Centeno et al. [5] is true or otherwise we obtain a graph which is harmonious but not sequentially labeled which represents an achievement.

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MAPPING PROPERTIES OF MIXED FRACTIONAL INTEGRO-DIFFERENTIATION IN HÖLDER SPACES

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August 3, 2013

Abstract

We study mixed Riemann-Liouville fractional integrals and mixed fractional derivative in Marchaud form of function of two variables in Hölder spaces of different orders in each variables. We consider Hölder spaces defined both by first order differences in each variable and also by the mixed second order difference, the main interest being in the evaluation of the latter for the mixed fractional integral in both the cases where the density of the integral belongs to the Hölder class defined by usual or mixed differences. The obtained results extend the well known theorem of Hardy-Littlewood for one-dimensuianl fractional integrals to the case of mixed Hölderness.

1. Introduction

The mapping properties of the one-dimensional fractional Riemann-Liouville operator

$$\left(I_{a+}^{\alpha}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{x}\frac{\varphi(t)dt}{(x-t)^{1-\alpha}},\quad x>a, \tag{1.1}$$

are well studied both in weighted Hölder spaces or in generalized Hölder spaces. A non-weighted statement on action of the fractional integral operator from H_0^{λ} into $H_0^{\lambda+\alpha}$ is due to Hardy and Littlewood ([1], see [11], Theorems 3.1 and 3.2), and it is known that the operator I_{a+}^{α} with $0<\alpha<1$ establishes an isomorphism between the Hölder spaces $H_0^{\lambda}([a,b])$ and $H_0^{\lambda+\alpha}([a,b])$ of function vanishing at the point $x=a$, if $\lambda+\alpha<1$. The weighted results with power weights were obtained in [9], [10](see their presentation in [11], Theorems 3.3, 3.4 and 13.13). For weighted generalized Hölder spaces $H_0^{\omega}(\rho)$ of function φ with a given dominant of continuity modulus of $\rho\varphi$, mapping properties in the case of power weight were studied in [7], [8], [12] (see also their presentation in [11], Section 13.6). Different proofs were suggested in [3], [4], where the case of complex fractional orders was also considered, the shortest proof being given in [3].

The case of weights more general than power ones, including in particular power-logarithmic type weights, in the spaces $H_0^{\omega}(\rho)$ was considered in [13], where operators more general than just fractional integrals were treated. We refer also to paper [2] where the mapping properties of fractional integration operators were reconsidered in terms of the Matuszewska-Orlich indices of the characteristic ω defining the generalized Hölder space H^{ω}. Finally, we mention also the papers [5], [6], where fractional integrals were studied in spaces of Nikolsky type.

In the multidimensional case, statements on mapping properties in generalized Hölder spaces are known [14] for the Riesz fractional integrals (see also this presentation in [11], Theorem 25.5).

Mixed Riemann-Liouville fractional integrals of order (α,β):

$$\left(I_{0+,0+}^{\alpha,\beta}\varphi\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y\frac{\varphi(t,\tau)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau, \tag{1.2}$$

and mixed fractional differentiation operators in the form Marchaud of order (α,β):

$$\left(\mathbf{D}_{a+,c+}^{\alpha,\beta}f\right)(x,y)=\frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left[\frac{f(x,y)}{x^{\alpha}y^{\beta}}+\frac{\beta}{x^{\alpha}}\int\limits_0^y\frac{f(x,y)-f(x,\tau)}{(y-\tau)^{1+\beta}}d\tau+\right.$$

$$\left.+\frac{\alpha}{y^{\beta}}\int\limits_0^x\frac{f(x,y)-f(t,y)}{(x-t)^{1+\alpha}}dt+\alpha\beta\int\limits_0^x\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{x-t,y-\tau}\,f\right)(t,\tau)}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}dtd\tau\right], \tag{1.3}$$

where $x>0,\ y>0$,were not studied either in the usual Hölder spaces, or in the Hölder spaces defined by mixed differences. Meanwhile, there arise "points of interest" related to the investigation of the above mixed differences of fractional integrals (1.2) and differentials (1.3). For operators (1.2) and (1.3) in Hölder spaces of mixed order there arise some questions to be answered in relation to the usage of these or Those differences in the definition of Hölder spaces. Such mapping properties in Hölder spaces of mixed order were not studied. This paper is aimed to fill in this gap. We deal with non-weighted spaces.

We consider the operators (1.2) and (1.3) in the rectangle

$$Q=\{(x,y):\ \ 0<x<a,\ \ 0<y<d\}.$$

2. Preliminaries

2.1. Notation and some properties of Hölder spaces

For a continuous function $\varphi(x,y)$ on $\mathbf{R}^2$ we introduce the notation

$$\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y) = \varphi(x+h,y)-\varphi(x,y), \quad \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y) = \varphi(x,y+\eta)-\varphi(x,y),$$

$$\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y) = \varphi(x+h,y+\eta)-\varphi(x+h,y)-\varphi(x,y+\eta)+\varphi(x,y),$$

so that

$$\varphi(x+h,y+\eta) = \left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y) + \left(\overset{1,0}{\Delta}_h \varphi\right)(x,y) + \\ + \left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y) + \varphi(x,y). \tag{2.1}$$

everywhere in the sequel by C_1, C_2, C_3, C etc we denote positive constants which may different values in different occurrences, and even in the same line.

We introduce two types of Hölder spaces by the following definitions.

Definition 2.1. **I**. Let $\lambda,\gamma \in (0,1]$. We say that $\varphi \in H^{\lambda,\gamma}(Q)$, if

$$|\varphi(x_1,y_1)-\varphi(x_2,y_2)| \le C_1|x_1-x_2|^\lambda + |y_1-y_2|^\gamma \tag{2.2}$$

for all $(x_1,y_1),\ (x_2,y_2) \in Q$. Condition (2.2) is equivalent to the couple of the separate conditions

$$\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right| \le C_1|h|^\lambda, \quad \left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right| \le C_2|\eta|^\gamma$$

uniform with respect to another variable. By $H_0^{\lambda,\gamma}(Q)$ we define a subspace of functions $f \in H^{\lambda,\gamma}(Q)$, vanishing at the boundaries $x=0$ and $y=0$ of Q.

II. Let $\lambda = 0$ and/or $\gamma = 0$. We put $H^{0,0}(Q) = L^\infty(Q)$ and

$$H^{\lambda,0}(Q) = \{\varphi \in L^\infty(Q): \quad \left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right| \le C_1|h|^\lambda\}, \quad \lambda \in (0,1],$$

$$H^{0,\gamma}(Q) = \{\varphi \in L^\infty(Q): \quad \left|\left(\overset{0,1}{\Delta}_h \varphi\right)(x,y)\right| \le C_2|h|^\gamma\}, \quad \gamma \in (0,1].$$

Definition 2.2. We say that $\varphi(x,y) \in \tilde{H}^{\lambda,\gamma}(Q)$, where $\lambda,\gamma \in (0,1]$, if

$$\varphi \in H^{\lambda,\gamma}(Q) \quad \text{and} \quad \left|\left(\overset{1,1}{\Delta}_{h,\eta} \varphi\right)(x,y)\right| \le C_3|h|^\lambda|\eta|^\gamma. \tag{2.3}$$

we say that $\varphi(x,y) \in \tilde{H}_0^{\lambda,\gamma}(Q)$, if $\varphi(x,y) \in \tilde{H}^{\lambda,\gamma}(Q)$ and $\varphi(x,y)\Big|_{x=0,y=0} = 0$.

These spaces become Banach spaces under the standard definition of the norms:

$$\left\|\varphi\right\|_{H^{\lambda,\gamma}} := \left\|\varphi\right\|_{C(Q)} + \sup_{x,x+h\in[0,b]} \sup_{y\in[0,d]} \frac{\left|\left(\overset{1,0}{\Delta}_h \varphi\right)(x,y)\right|}{|h|^\lambda} + \sup_{x\in[0,b]} \sup_{y,y+\eta\in[0,d]} \frac{\left|\left(\overset{0,1}{\Delta}_\eta \varphi\right)(x,y)\right|}{|\eta|^\gamma},$$

$$\left\|\varphi\right\|_{\tilde{H}^{\lambda,\gamma}} := \left\|\varphi\right\|_{H^{\lambda,\gamma}} + \sup_{x,x+h\in[0,b]} \sup_{y,y+\eta\in[0,d]} \frac{\left|\left(\overset{1,1}{\Delta}_{h,\eta}\,\varphi\right)(x,y)\right|}{|h|^{\lambda}|\eta|^{\gamma}}.$$

note that

$$\varphi \in H^{\lambda,\gamma} \Rightarrow \left|\left(\overset{1,1}{\Delta}_{h,\eta}\,\varphi\right)(x,y)\right| \leq C_{\theta}|h|^{\theta\lambda}|\eta|^{(1-\theta)\gamma} \tag{2.4}$$

for any $\theta \in [0,1]$, where $C_{\theta} = 2C_1^{\theta}C_2^{1-\theta}$, so that

$$\tilde{H}^{\lambda,\gamma}(Q) \hookrightarrow H^{\lambda,\gamma}(Q) \hookrightarrow \bigcap_{0\leq\theta\leq 1} \tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q), \tag{2.5}$$

where $\hookrightarrow$ stands for the continuous embedding, and the norm for $\bigcap\limits_{0\leq\theta\leq 1} \tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$ is introduced as the maximum in θ of norms for $\tilde{H}^{\theta\lambda,(1-\theta)\gamma}(Q)$. Since $\theta \in [0,1]$ is arbitrary, it isn't hard to see that the inequality in (2.4) is equivalent (up to the constant factor C) to

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\,\varphi\right)(x,y)\right| \leq C\min\{|h|^{\lambda}|,\ \eta|^{\gamma}\} \tag{2.6}$$

2.2. A one-dimensional statements

The following statements are known, being fist proved in [1], see also the presentations of these proofs in [11], p.57 and 190. We use the schemes of the proofs to make the presentation easier for the two-dimensional case.

Theorem 2.3. Let $\varphi(x) \in H^{\lambda}([0,b]),\ 0 < \lambda < 1,\ 0 < \alpha < 1$ and $\lambda + \alpha < 1$. Then for the fractional operator $(I^{\alpha}_{0+}f)(x)$ representation

$$\left(I^{\alpha}_{0+}\varphi\right)(x) = \frac{\varphi(0)}{\Gamma(1+\alpha)}x^{\alpha} + \psi(x), \tag{2.7}$$

holds, where $\psi(x) \in H^{\alpha+\lambda}$ and

$$|\psi(x)| \leq Cx^{\lambda+\alpha}. \tag{2.8}$$

The proof of the theorem is the same as in [11], pp. 54-55.

Lemma 2.4. If $f(x) \in H^{\lambda+\alpha}([0,b])$ and $0 < \lambda,\ 0 < \alpha + \lambda < 1$, then

$$z(x) = \frac{f(x) - f(0)}{|x|^{\alpha}} \in H^{\lambda}([0,\ b]), \quad \text{and} \quad \left\|z\right\|_{H^{\lambda}} \leq C\left\|f\right\|_{H^{\lambda+\alpha}},$$

where C doesn't depend from $f(x)$.

Proof. Let $h > 0;\ x,\ x+h \in [0,b]$. We consider the difference

$$|z(x+h) - z(x)| \leq \frac{|f(x+h) - f(x)|}{(x+h)^{\alpha}} + |f(x) - f(0)|\frac{(x+h)^{\alpha} - x^{\alpha}}{x^{\alpha}(x+h)^{\alpha}}.$$

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Since $f \in H^{\lambda+\alpha}$, we have

$$|f(x+h) - f(x)| \le C_1 h^{\lambda+\alpha}, \quad |f(x) - f(0)| \le C_2 x^{\lambda+\alpha}. \tag{2.9}$$

Using these inequalities we obtain

$$|z(x+h) - z(x)| \le C_1 \frac{h^{\lambda+\alpha}}{(x+h)^\alpha} + C_2 x^\lambda \frac{(x+h)^\alpha - x^\alpha}{(x+h)^\alpha} = Z_1 + Z_2.$$

For Z_1, we have

$$Z_1 = C_1 \left(\frac{h}{x+h}\right)^\alpha h^\lambda \le C h^\lambda.$$

Let's estimate Z_2, here we shall consider two cases: $x \le h$ and $x > h$. In the first case, we use inequality $|\sigma_1^\mu - \sigma_2^\mu| \le |\sigma_1 - \sigma_2|^\mu$, $(\sigma_1 \ne \sigma_2)$ and obtain

$$Z_2 \le x^\lambda \frac{h^\alpha}{(x+h)^\alpha} \le C h^\lambda.$$

In second case, using $(1+t)^\alpha - 1 \le \alpha t, \ \ t > 0$ we have

$$Z_2 = C_2 \frac{x^\lambda}{(x+h)^\alpha} \left| \left(1 + \frac{h}{x}\right)^\alpha - 1 \right| \le C h x^{\lambda-1} \le C h^\lambda,$$

which completes the proof.

The Marchaud fractional differentiation operator has a form:

$$\left(\mathbf{D}_{0+}^\alpha f\right)(x) = \frac{f(x)}{x^\alpha \Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt, \tag{2.10}$$

where $0 < \alpha < 1$.

Theorem 2.5. If $f(x) \in H^{\lambda+\alpha}([a,b]),\ 0 < \alpha + \lambda < 1$, that

$$\left(\mathbf{D}_{0+}^\alpha f\right)(x) = \frac{f(0)}{x^\alpha \Gamma(1-\alpha)} + \chi(x), \tag{2.11}$$

where $\chi(x) \in H^\lambda([0,\ b])$ and $\chi(0) = 0$, thus $\left\|\chi\right\|_{H^\lambda} \le C \left\|f\right\|_{H^{\lambda+\alpha}}$.

Proof. We present $\left(\mathbf{D}_{0+}^\alpha f\right)(x)$ as

$$\left(\mathbf{D}_{0+}^\alpha f\right)(x) = \frac{f(0)}{x^\alpha \Gamma(1-\alpha)} + \frac{f(x) - f(0)}{\Gamma(1-\alpha) x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt,$$

receive equality (2.11), where

$$\chi(x) = \chi_1(x) + \chi_2(x) = \frac{f(x) - f(0)}{\Gamma(1-\alpha) x^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt.$$

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Here $\chi_1(x) \in H^\lambda([0,\ b])$ by Lemma 2.4. It is enough to show $\chi_2(x) \in H^\lambda([0,\ b])$.

Let $h > 0,\ \ x,\ x+h \in [0,b]$. Let's consider the difference

$$\chi_2(x+h) - \chi_2(x) = \int_0^x \frac{f(x+h)-f(x)}{(x+h-t)^{1+\alpha}}dt + \int_x^{x+h} \frac{f(x+h)-f(t)}{(x+h-t)^{1+\alpha}}dt +$$

$$+ \int_0^x [f(x)-f(t)]\left[(x+h-t)^{-\alpha-1} - (x-t)^{-1-\alpha}\right]dt = I_1 + I_2 + I_3.$$

Since $f \in H^{\lambda+\alpha}([0,b])$, then we have for I_1

$$|I_1| \le Ch^{\lambda+\alpha} \int_0^x (t+h)^{-1-\alpha}dt \le C_1 h^\lambda.$$

Let's estimate I_2. We have

$$|I_2| \le C \int_x^{x+h} (x+h-t)^{\lambda-1}dt = C_2 h^\lambda.$$

For I_3, we have

$$|I_3| \le C \int_0^x (x-t)^{\lambda+\alpha}\left|(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right|dt =$$

$$= Ch^\lambda \int_0^{\frac{x}{h}} t^\lambda \left|(1+t)^{-1-\alpha} - t^{-1-\alpha}\right|dt \le C_3 h^\lambda,$$

where

$$C_3 = C\int_0^\infty t^\lambda \left|(1+t)^{-1-\alpha} - t^{-1-\alpha}\right|dt < \infty.$$

Finally, it remains to note that $\chi_2(0) = 0$, since

$$|\chi_2(x)| \le C\int_0^x t^{\lambda-1}dt.$$

3. Mapping properties of the mixed fractional integration operator in the Hölder spaces

Lemma 3.1. Let $\varphi(x,y)\in H^{\lambda,\gamma}(Q),\ \ 0\le\lambda,\gamma\le 1,\ \ 0<\alpha,\ \beta<1$. Then for the mixed fractional integral operator (1.2) the representation

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)=\frac{\varphi(0,0)x^{\alpha}y^{\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)}+\frac{\psi_1(x)y^{\beta}}{\Gamma(1+\beta)}+\frac{x^{\alpha}\psi_2(y)}{\Gamma(1+\alpha)}+\psi(x,y) \quad (3.1)$$

holds, where

$$\psi_1(x)=\frac{1}{\Gamma(\alpha)}\int\limits_0^x\frac{\varphi(t,0)-\varphi(0,0)}{(x-t)^{1-\alpha}}dt,\quad \psi_2(y)=\frac{1}{\Gamma(\beta)}\int\limits_0^y\frac{\varphi(0,\tau)-\varphi(0,0)}{(y-\tau)^{1-\beta}}d\tau,$$

$$\psi(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{t,\tau}\,\varphi\right)(0,0)}{(x-t)^{1-\alpha}(y-\tau)^{1-\beta}}dtd\tau,$$

and

$$|\psi_1(x)|\le C_1x^{\lambda+\alpha},\quad |\psi_2(y)|\le C_2y^{\gamma+\beta}, \quad (3.2)$$

$$|\psi(x,y)|\le C\min_{\theta\in[0,1]}x^{\alpha+\theta\lambda}y^{\beta+(1-\theta)\gamma}=Cx^{\alpha}y^{\beta}\min\{x^{\lambda},y^{\gamma}\}. \quad (3.3)$$

Proof. Representation (3.1) itself is easily obtained by means of (2.1). Since $\varphi\in H^{\lambda,\gamma}(Q)$, inequalities (3.2) are obvious. Estimate (3.3) is obtained by means of (2.4) and (2.6).

Theorem 3.2. Let $0\le\lambda,\gamma<1$. The operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from $H^{\lambda,\gamma}_0(Q)$ to $H^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha<1$ and $\gamma+\beta<1$.

Proof. Sice $\varphi(x,y)\in H^{\lambda,\gamma}_0(Q)$, by (3.1) we have

$$\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)=\psi(x,y).$$

We denote

$$g(t,\tau)=\left(\overset{1,1}{\Delta}_{t,\tau}\,\varphi\right)(0,0) \quad (3.4)$$

for brevity. Note that

$$\left(\overset{1,1}{\Delta}_{t,\tau}\,\varphi\right)(0,0)=\varphi(t,\tau)$$

for $\varphi\in H^{\lambda,\gamma}_0$, but we prefer to keep the notation for $g(t,\tau)$ via the mixed difference as in (3.4). By (2.4) we have

$$|g(t,\tau)|\le Ct^{\theta\lambda}\tau^{(1-\theta)\gamma}\le C\min\{t^{\lambda},\tau^{\gamma}\}. \quad (3.5)$$

For $h>0,\ x,x+h\in Q_1=[0,b]$, we consider the difference

$$\psi(x+h,y)-\psi(x,y)=\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(1+\alpha)\Gamma(\beta)}\int\limits_0^y\frac{g(x,y-\tau)}{\tau^{1-\beta}}+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^h\int\limits_0^y \frac{g(x+t,y-\tau)-g(x,y-\tau)}{(h-t)^{1-\alpha}\tau^{1-\beta}}dtd\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^x\int\limits_0^y [g(x-t,y-\tau)-g(x,y-\tau)]\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]\tau^{\beta-1}dtd\tau=$$

$$=\Delta_1+\Delta_2+\Delta_3. \tag{3.6}$$

We make use of (3.5) with $\theta=1$ and obtain

$$|\Delta_1|\le C|(x+h)^\alpha-x^\alpha|x^\lambda\le Ch^{\alpha+\lambda}.$$

For Δ_2 in view of (2.4), we have

$$|g(x-t,y-\tau)-g(x,y-\tau)|=\left|\left(\overset{1,1}{\Delta}_{-t,y-\tau}\ \varphi\right)(x,0)\right|\le C|t|^\lambda, \tag{3.7}$$

and then

$$\Delta_2\le Ch^{\lambda+\alpha}.$$

For Δ_3 by (3.7) and (2.4) we have

$$\Delta_3\le C\int\limits_0^x t^\lambda|t^{\alpha-1}-(t+h)^{\alpha-1}|dt\le C_0h^{\lambda+\alpha},\quad C_0=\int\limits_0^\infty t^\lambda|t^{\alpha-1}-(t+1)^{\alpha-1}|dt<\infty.$$

Gathering the estates for $\Delta_1,\Delta_2,\Delta_3$ we obtain

$$|\psi(x+h,y)-\psi(x,y)|\le Ch^{\lambda+\alpha}.$$

Rearranging symmetrically representation (3.6), we can similarly obtain that

$$|\psi(x,y+\eta)-\psi(x,y)|\le C\eta^{\gamma+\beta},$$

which proves the theorem.

Theorem 3.3. The mixed fractional integral operator $I^{\alpha,\beta}_{0+,0+}$ is bounded from the space $\tilde{H}^{\lambda,\gamma}_0(Q),\ 0\le\lambda,\gamma\le 1$ into the space $\tilde{H}^{\lambda+\alpha,\gamma+\beta}_0(Q)$, if $\lambda+\alpha\le 1$ and $\gamma+\beta\le 1$.

Proof. Let $\varphi\in\tilde{H}^{\lambda,\gamma}_0(Q)$. By Theorem 3.2 and embedding (2.5), for $f(x,y)=\left(I^{\alpha,\beta}_{0+,0+}\varphi\right)(x,y)$ it satisfies to estimate the difference $\left(\overset{1,1}{\Delta}_{h,\eta}\ f\right)(x,y)$. Since $\varphi(x,y)\Big|_{x=0,y=0}=0$, according to (3.1) we have $f(x,y)=\psi(x,y)$, where $\psi(x,y)$ is the function from (3.1). The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in

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each variable not losing the corresponding information in another variable. The suggested splitting runs as follows

$$\left(\overset{1,1}{\Delta}_{h,\eta}\, f\right)(x,y)=\left(\overset{1,1}{\Delta}_{h,\eta}\, \psi\right)(x,y)=\sum_{k=1}^{9}T_k:=$$

$$:=\frac{g(x,y)}{\Gamma(1+\alpha)\Gamma(1+\beta)}\left[(x+h)^{\alpha}-x^{\alpha}\right]\left[(y+\eta)^{\beta}-y^{\beta}\right]+$$

$$+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)}\int\limits_{-h}^{0}\frac{g(x-t,y)-g(x,y)}{(t+h)^{1-\alpha}}dt+$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(1+\alpha)\Gamma(\beta)}\int\limits_{-\eta}^{0}\frac{g(x,y-\tau)-g(x,y)}{(\tau+\eta)^{1-\beta}}d\tau+$$

$$+\frac{(y+\eta)^{\beta}-y^{\beta}}{\Gamma(\alpha)\Gamma(1+\beta)}\int\limits_{0}^{x}\left[g(x-t,y)-g(x,y)\right]\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dt+$$

$$+\frac{(x+h)^{\alpha}-x^{\alpha}}{\Gamma(1+\alpha)\Gamma(\beta)}\int\limits_{0}^{y}\left[g(x,y-\tau)-g(x,y)\right]\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]d\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{-h}^{0}\int\limits_{-\eta}^{0}\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)}{(h+t)^{1-\alpha}(\eta+\tau)^{1-\beta}}dtd\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{-h}^{0}\int\limits_{0}^{y}\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)}{(h+t)^{1-\alpha}}\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]dtd\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{0}^{x}\int\limits_{-\eta}^{0}\frac{\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)}{(\eta+\tau)^{1-\beta}}\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]dtd\tau+$$

$$+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_{0}^{x}\int\limits_{0}^{y}\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)\left[(t+h)^{\alpha-1}-t^{\alpha-1}\right]\left[(\tau+\eta)^{\beta-1}-\tau^{\beta-1}\right]dtd\tau,$$

where $h>0,\ \eta>0;\ x,\ x+h\in Q_1;\ y,y+\eta\in Q_2$ and $g(x,y)$ is the function from (3.4). The validity of this representation may be checked directly.

Since $\varphi\in\tilde{H}^{\lambda,\gamma}$, we have $|g(x,y)|=|\left(\overset{1,1}{\Delta}_{x,y}\,\varphi\right)(0,0)|\leq Cx^{\lambda}y^{\gamma}$ and then

$$|T_1|\leq Cx^{\lambda}y^{\gamma}\left|(x+h)^{\alpha}-x^{\alpha}\right|\left|(y+\eta)^{\beta}-y^{\beta}\right|,$$

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$$|T_2| \le C y^\gamma \left|(y+\eta)^\beta - y^\beta\right| \int\limits_{-h}^{0} \frac{|t|^\lambda}{(t+h)^{1-\alpha}} dt,$$

$$|T_3| \le C x^\lambda \left|(x+h)^\alpha - x^\alpha\right| \int\limits_{-\eta}^{0} \frac{|\tau|^\gamma}{(\tau+\eta)^{1-\beta}} d\tau,$$

$$|T_4| \le C y^\gamma \left|(y+\eta)^\beta - y)^\beta\right| \int\limits_{0}^{x} |t|^\lambda \left|(t+h)^{\alpha-1} - t^{\alpha-1}\right| dt,$$

$$|T_5| \le C x^\lambda \left|(x+h)^\alpha - x^\alpha\right| \int\limits_{0}^{y} |\tau|^\gamma \left|(\tau+\eta)^{\beta-1} - \tau^{\beta-1}\right| d\tau.$$

For $T_6 - T_9$ we similarly, make use of

$$\left|\left(\overset{1,1}{\Delta}_{-t,-\tau}\, g\right)(x,y)\right| = \left|\left(\overset{1,1}{\Delta}_{-t,-\tau}\, \varphi\right)(x,y)\right| \le C|t|^\lambda |\eta|^\gamma.$$

and obtain

$$|T_6| \le C \int\limits_{-h}^{0}\int\limits_{-\eta}^{0} \frac{|t|^\lambda |\tau|^\gamma}{(h+t)^{1-\alpha}(\eta+\tau)^{1-\beta}} dt d\tau,$$

$$|T_7| \le C \int\limits_{-h}^{0}\int\limits_{0}^{y} \frac{|t|^\lambda |\tau|^\gamma}{(h+t)^{1-\alpha}} \left|(\eta+\tau)^{\beta-1} - \tau^{\beta-1}\right| dt d\tau,$$

$$|T_8| \le \int\limits_{0}^{x}\int\limits_{-\eta}^{0} \frac{|t|^\lambda |\tau|^\gamma}{(\eta+\tau)^{1-\beta}} \left|(h+t)^{\alpha-1} - t^{\alpha-1}\right| dt d\tau,$$

$$|T_9| \le \int\limits_{0}^{x}\int\limits_{0}^{y} |t|^\lambda |\tau|^\gamma \left|(h+t)^{\alpha-1} - t^{\alpha-1}\right| \left|(\eta+\tau)^{\beta-1} - \tau^{\beta-1}\right| dt d\tau,$$

after which every term is estimated in the standard way, and we get

$$\left|\left(\overset{1,1}{\Delta}_{h,\eta}\, f\right)(x,y)\right| \le C_3 h^{\lambda+\alpha} \eta^{\gamma+\beta}.$$

This completes the proof.

4. Mapping properties of the mixed fractional differentiation operator in the Hölder spaces

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Lemma 4.1. Let $f(x,y) \in \widetilde{H}^{\lambda+\alpha,\gamma+\beta}(Q)$, $\;0<\alpha+\lambda<1,\; 0<\beta+\gamma<1$. Then for the mixed fractional differential operator (1.3) the representation

$$\left(\mathbf{D}^{\alpha,\beta}_{0+,0+}f\right)(x,y) = \frac{\Gamma^{-1}(1-\alpha)}{\Gamma(1-\beta)}\left[\frac{f(0,0)}{x^\alpha y^\beta}+\frac{\chi_1(x)}{y^\beta}+\frac{\chi_2(y)}{x^\alpha}+\chi(x,y)\right], \quad (4.1)$$

holds, where

$$\chi_1(x) = \frac{f(x,0)-f(0,0)}{x^\alpha} + \alpha\int\limits_0^x \frac{f(t,0)-f(0,0)}{(x-t)^{1+\alpha}}dt,$$

$$\chi_2(y) = \frac{f(0,y)-f(0,0)}{y^\beta} + \beta\int\limits_0^y \frac{f(0,\tau)-f(0,0)}{(y-\tau)^{1+\beta}}d\tau,$$

$$\chi(x,y) = \frac{\left(\overset{1,1}{\Delta}_{x,\,y}\, f\right)(0,0)}{x^\alpha y^\beta} + \frac{\alpha}{y^\beta}\int\limits_0^x \frac{\left(\overset{1,1}{\Delta}_{x-t,\,y}\, f\right)(t,0)}{(x-t)^{1+\alpha}}dt +$$

$$+\frac{\beta}{x^\alpha}\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{x,y-\tau}\, f\right)(0,\tau)}{(y-\tau)^{1+\beta}}d\tau + \alpha\beta\int\limits_0^x\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{x-t,\,y-\tau}\, f\right)(t,\tau)}{(x-t)^{1+\alpha}(y-\tau)^{1+\beta}}dtd\tau$$

and

$$|\chi_1(x)| \le C_1 x^\lambda, \quad |\chi_2(y)| \le C_2 y^\gamma, \quad (4.2)$$

$$|\chi(x,y)| \le C_3 x^\lambda y^\gamma. \quad (4.3)$$

Proof. Representation (4.1) itself is easily obtained by means of (2.1). Since $f \in H^{\lambda+\alpha,\gamma+\beta}(Q)$, inequalities (4.2) are obvious. Estimate (4.3) is obtained by means of (2.4), i.e.

$$\chi(x,y) \le C\Bigg[x^\lambda y^\gamma + \alpha y^\gamma\int\limits_0^x (x-t)^{\lambda-1}dt + \beta x^\lambda\int\limits_0^y (y-\tau)^{\gamma-1}d\tau +$$

$$+\alpha\beta\int\limits_0^x\int\limits_0^y (x-t)^{\lambda-1}(y-\tau)^{\gamma-1}dtd\tau\Bigg].$$

It is easy to receive

$$\chi(x,y) \le Cx^\lambda y^\gamma\left[1+\int\limits_0^1 s^{\lambda-1}ds + \int\limits_0^1 \xi^{\gamma-1}d\xi + \int\limits_0^1\int\limits_0^1 s^{\lambda-1}\xi^{\gamma-1}dsd\xi\right] \le C_3 x^\lambda y^\gamma.$$

Theorem 4.2. Let $f(x) \in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$, $0<\lambda+\alpha<1$, $0<\gamma+\beta<1$. Then the operator $\mathbf{D}^{\alpha,\beta}_{0+,0+}$ continuously maps $\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$ into $\widetilde{H}_0^{\lambda,\gamma}(Q)$.

Proof. Since $f(x,y) \in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$, by (4.1) we have

$$\varphi(x,y) = \left(\mathbf{D}^{\alpha,\beta}_{0+,0+} f\right)(x,y) = \chi(x,y).$$

Let $h > 0;\ \ x, x+h \in [0,\ b]$. We consider the difference

$$\chi(x+h,y) - \chi(x,y) = \sum_{k=0}^{10} \Phi_k := \frac{1}{y^\beta} \frac{\left(\overset{1,1}{\Delta}_{h,\ y}\ f\right)(0,0)}{(x+h)^\alpha} +$$

$$+\frac{1}{y^\beta}\left(\overset{1,1}{\Delta}_{x,\ y}\ f\right)(0,0)\left[(x+h)^{-\alpha} - x^{-\alpha}\right] + \frac{\alpha}{y^\beta}\int\limits_0^x \frac{\left(\overset{1,1}{\Delta}_{h,\ y}\ f\right)(x,0)}{(x+h-t)^{1+\alpha}} dt +$$

$$+\frac{\alpha}{y^\beta}\int\limits_x^{x+h} \frac{\left(\overset{1,1}{\Delta}_{x+h-t,\ y}\ f\right)(t,0)}{(x+h-t)^{1+\alpha}} dt + \frac{\beta}{(x+h)^\alpha}\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{h,y-\tau}\ f\right)(0,\tau)}{(y-\tau)^{1+\beta}} d\tau +$$

$$+\frac{\alpha}{y^\beta}\int\limits_0^x \left(\overset{1,1}{\Delta}_{x-t,\ y}\ f\right)(t,0)\left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right] dt +$$

$$+\beta\left[(x+h)^{-\alpha} - x^{-\alpha}\right]\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{x,y-\tau}\ f\right)(0,\tau)}{(y-\tau)^{1+\beta}} d\tau +$$

$$+\alpha\beta\int\limits_0^x\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{h,\ y-\tau}\ f\right)(x,\tau)}{(x+h-t)^{1+\alpha}(y-\tau)^{1+\beta}} dt d\tau +$$

$$+\alpha\beta\int\limits_x^{x+h}\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{x+h-t,\ y-\tau}\ f\right)(t,\tau)}{(x+h-t)^{1+\alpha}(y-\tau)^{1+\beta}} dt d\tau +$$

$$+\alpha\beta\int\limits_0^x\int\limits_0^y \frac{\left(\overset{1,1}{\Delta}_{x-t,\ y-\tau}\ f\right)(t,\tau)}{(y-\tau)^{1+\beta}}\left[(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right] dt d\tau. \quad (4.4)$$

Since $f \in \widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}$, we have

$$|\Phi_1| \le C y^\gamma \frac{h^{\lambda+\alpha}}{(x+h)^\alpha} \le C_1 \frac{h^{\lambda+\alpha}}{(x+h)^\alpha},$$

$$|\Phi_2| \le C y^\gamma x^{\lambda+\alpha}\left|(x+h)^{-\alpha} - x^{-\alpha}\right| \le C_2 \frac{x^\lambda}{(x+h)^\alpha}\left[(x+h)^\alpha - x^\alpha\right],$$

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$$|\Phi_3| \leq C\alpha\, y^\gamma h^{\lambda+\alpha} \int\limits_0^x \frac{dt}{(x+h-t)^{1+\alpha}} \leq C_3 h^{\lambda+\alpha} \int\limits_0^x \frac{dt}{(x+h-t)^{1+\alpha}},$$

$$|\Phi_4| \leq C\alpha\, y^\gamma \int\limits_x^{x+h} (x+h-t)^{\lambda-1} dt \leq C_4 \int\limits_x^{x+h} (x+h-t)^{\lambda-1} dt,$$

$$|\Phi_5| \leq C \frac{h^{\lambda+\alpha}\beta}{(x+h)^\alpha} \int\limits_0^y (y-\tau)^{\gamma-1} d\tau \leq C_5 \frac{h^{\lambda+\alpha}}{(x+h)^\alpha},$$

$$|\Phi_6| \leq C\alpha\, y^\gamma \int\limits_0^x (x-t)^{\lambda+\alpha} \left|(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right| dt \leq$$

$$\leq C_6 \int\limits_0^x (x-t)^{\lambda+\alpha} \left|(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right| dt,$$

$$|\Phi_7| \leq C\beta x^{\lambda+\alpha} \left|(x+h)^{-\alpha} - x^{-\alpha}\right| \int\limits_0^y \frac{d\tau}{(y-\tau)^{1-\gamma}} \leq C_7 \frac{x^\lambda}{(x+h)^\alpha} \left[(x+h)^\alpha - x^\alpha\right],$$

$$|\Phi_8| \leq C\alpha\beta h^{\lambda+\alpha} \int\limits_0^x \frac{dt}{(x+h-t)^{1+\alpha}} \int\limits_0^y (y-\tau)^{\gamma-1} d\tau \leq C_8 h^{\lambda+\alpha} \int\limits_0^x \frac{dt}{(x+h-t)^{1+\alpha}},$$

$$|\Phi_9| \leq C\alpha\beta \int\limits_x^{x+h} (x+h-t)^{\lambda-1} dt \int\limits_0^y (y-\tau)^{\gamma-1} d\tau \leq C_9 \int\limits_x^{x+h} (x+h-t)^{\lambda-1} dt$$

$$|\Phi_{10}| \leq C\alpha\beta \int\limits_0^x (x-t)^{\lambda+\alpha} \left|(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right| dt \int\limits_0^y (y-\tau)^{1+\beta} d\tau \leq$$

$$\leq C_{10} \int\limits_0^x (x-t)^{\lambda+\alpha} \left|(x+h-t)^{-1-\alpha} - (x-t)^{-1-\alpha}\right| dt,$$

where

$$\int\limits_0^y (y-\tau)^{\gamma-1} d\tau < \infty.$$

Using estimations $Z_1,\ Z_2$ of the proof of Lemma 2.4 and estimations $I_i,\ i = 1,2,3$ of the proof of the Theorem 2.5, it is easily possible to receive an estimation

$$|\chi(x+h,y) - \chi(x,y)| \leq Ch^\lambda.$$

Rearranging symmetrically representation (4.4), we can similarly obtain that

$$|\chi(x,y+h) - \chi(x,y)| \leq Ch^\gamma.$$

The main moment in the estimations is to find the corresponding splitting which allows to derive the best information in each variable not losing the corresponding information in another variable.

Let $h,\ \eta > 0;\ \ x, x+h \in [0,b],\ \ y, y+\eta \in [0,d]$. We consider the difference

$$\left(\overset{1,1}{\Delta}_{h,\,\eta}\,\chi\right)(x,y) = \sum_{k=1}^{25} P_k :=$$

$$:= \frac{\left(\overset{1,1}{\Delta}_{h,\,\eta}\,f\right)(x,y)}{(x+h)^{\alpha}(y+\eta)^{\beta}} + \frac{\left(\overset{1,1}{\Delta}_{h,\,y}\,f\right)(x,0)}{(x+h)^{\alpha}}\,\frac{\left[(y+\eta)^{\beta}-y^{\beta}\right]}{(y+\eta)^{\beta}y^{\beta}} +$$

$$+\frac{\left(\overset{1,1}{\Delta}_{x,\,\eta}\,f\right)(0,y)}{(y+\eta)^{\beta}}\,\frac{\left[(x+h)^{\alpha}-x^{\alpha}\right]}{(x+h)^{\alpha}x^{\alpha}} +$$

$$+\left(\overset{1,1}{\Delta}_{x,\,y}\,f\right)(x,y)\,\frac{\left[(x+h)^{\alpha}-x^{\alpha}\right]}{(x+h)^{\alpha}x^{\alpha}}\,\frac{\left[(y+\eta)^{\beta}-y^{\beta}\right]}{(y+\eta)^{\beta}y^{\beta}} +$$

$$+\frac{\beta}{(x+h)^{\alpha}}\int\limits_{y}^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,\,y+\eta-\tau}\,f\right)(x,\tau)}{(y+\eta-\tau)^{1+\beta}}d\tau + \frac{\beta}{(x+h)^{\alpha}}\int\limits_{0}^{y}\frac{\left(\overset{1,1}{\Delta}_{h,\,\eta}\,f\right)(x,y)}{(y+\eta-\tau)^{1+\beta}}d\tau +$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int\limits_{y}^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x,\,y+\eta-\tau}\,f\right)(0,\tau)}{(y+\eta-\tau)^{1+\beta}}d\tau +$$

$$+\frac{\beta}{(x+h)^{\alpha}}\int\limits_{0}^{y}\left(\overset{1,1}{\Delta}_{h,\,y-\tau}\,f\right)(x,\tau)\left[(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right]d\tau +$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int\limits_{0}^{y}\frac{\left(\overset{1,1}{\Delta}_{x,\,\eta}\,f\right)(0,y)}{(y+\eta-\tau)^{1+\beta}}d\tau +$$

$$+\beta\left[x^{-\alpha}-(x+h)^{-\alpha}\right]\int\limits_{0}^{y}\left(\overset{1,1}{\Delta}_{x,\,y-\tau}\,f\right)(0,\tau)\left[(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right]d\tau +$$

$$+\frac{\alpha}{(y+\eta)^{\beta}}\int\limits_{x}^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\,\eta}\,f\right)(t,y)}{(x+h-t)^{1+\alpha}}dt + \frac{\alpha}{(y+\eta)^{\beta}}\int\limits_{0}^{x}\frac{\left(\overset{1,1}{\Delta}_{h,\,\eta}\,f\right)(x,y)}{(x+h-t)^{1+\alpha}}dt +$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int\limits_{x}^{x+h}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\,y}\,f\right)(t,0)}{(x+h-t)^{1+\alpha}}dt +$$

$$+\frac{\alpha}{(y+\eta)^{\beta}}\int\limits_0^x\left(\overset{1,1}{\Delta}_{x-t,\,\eta}\,f\right)(t,0)\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dt+$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int\limits_0^x\frac{\left(\overset{1,1}{\Delta}_{h,\,y}\,f\right)(x,0)}{(x+h-t)^{1+\alpha}}dt+$$

$$+\alpha\left[y^{-\beta}-(y+\eta)^{-\beta}\right]\int\limits_0^x\left(\overset{1,1}{\Delta}_{x-t,\,y}\,f\right)(t,0)\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dt+$$

$$+\int\limits_0^x\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{h,\,\eta}\,f\right)(x,y)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int\limits_0^x\int\limits_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{h,\,y+\eta-\tau}\,f\right)(x,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+$$

$$+\int\limits_0^x\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{h,\,y-\tau}\,f\right)(x,\tau)}{(x+h-t)^{1+\alpha}}\left[(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right]dtd\tau+$$

$$+\int\limits_x^{x+h}\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\,\eta}\,f\right)(t,y)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+\int\limits_x^{x+h}\int\limits_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\,y+\eta-\tau}\,f\right)(t,\tau)dtd\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}}+$$

$$+\int\limits_x^{x+h}\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{x+h-t,\,y-\tau}\,f\right)(t,\tau)}{(x+h-t)^{1+\alpha}}\left[(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right]dtd\tau+$$

$$+\int\limits_0^x\int\limits_0^y\frac{\left(\overset{1,1}{\Delta}_{x-t,\,\eta}\,f\right)(t,y)}{(y+\eta-\tau)^{1+\beta}}\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dtd\tau+$$

$$+\int\limits_0^x\int\limits_y^{y+\eta}\frac{\left(\overset{1,1}{\Delta}_{x-t,\,y+\eta-\tau}\,f\right)(t,\tau)}{(y+\eta-\tau)^{1+\beta}}\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]dtd\tau+$$

$$+\int\limits_0^x\int\limits_0^y\left(\overset{1,1}{\Delta}_{x-t,\,y-\tau}\,f\right)(t,\tau)\left[(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right]\times$$

$$\times\left[(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right]dtd\tau.$$

The validity of this representation may be checked directly.
Since $f(x,y)\in\widetilde{H}_0^{\lambda+\alpha,\gamma+\beta}(Q)$, we have

$$|P_1|\le C\frac{h^{\lambda+\alpha}\eta^{\gamma+\beta}}{(x+h)^{\alpha}(y+\eta)^{\beta}},$$

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$$|P_2| \le C\frac{h^{\lambda+\alpha}y^{\gamma}}{(x+h)^{\alpha}}\frac{\left|(y+\eta)^{\beta}-y^{\beta}\right|}{(y+\eta)^{\beta}},$$

$$|P_3| \le C\frac{x^{\lambda}\eta^{\gamma+\beta}}{(y+\eta)^{\beta}}\frac{|(x+h)^{\alpha}-x^{\alpha}|}{(x+h)^{\alpha}},$$

$$|P_4| \le Cx^{\lambda}y^{\gamma}\frac{|(x+h)^{\alpha}-x^{\alpha}|}{(x+h)^{\alpha}}\frac{\left|(y+\eta)^{\beta}-y^{\beta}\right|}{(y+\eta)^{\beta}},$$

$$|P_5| \le C\frac{h^{\lambda+\alpha}}{(x+h)^{\alpha}}\int\limits_{y}^{y+\eta}(y+\eta-\tau)^{\gamma-1}d\tau,$$

$$|P_6| \le C\frac{h^{\lambda+\alpha}\eta^{\gamma+\beta}}{(x+h)^{\alpha}}\int\limits_{0}^{y}\frac{d\tau}{(y+\eta-\tau)^{1+\beta}},$$

$$|P_7| \le Cx^{\lambda+\alpha}\left|x^{-\alpha}-(x+h)^{-\alpha}\right|\int\limits_{y}^{y+\eta}(y+\eta-\tau)^{\gamma-1}d\tau,$$

$$|P_8| \le C\frac{h^{\lambda+\alpha}}{(x+h)^{\alpha}}\int\limits_{0}^{y}(y-\tau)^{\gamma+\beta-1}\left|(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right|d\tau,$$

$$|P_9| \le x^{\lambda+\alpha}\eta^{\gamma+\beta}\left|x^{-\alpha}-(x+h)^{-\alpha}\right|\int\limits_{0}^{y}\frac{d\tau}{(y+\eta-\tau)^{1+\beta}},$$

$$|P_{10}| \le Cx^{\lambda+\alpha}\left|x^{-\alpha}-(x+h)^{-\alpha}\right|\int\limits_{0}^{y}\frac{\left|(y-\tau)^{-1-\beta}-(y+\eta-\tau)^{-1-\beta}\right|}{(y-\tau)^{-\gamma-\beta}}d\tau,$$

$$|P_{11}| \le C\frac{\eta^{\gamma+\beta}}{(y+\eta)^{\beta}}\int\limits_{x}^{x+h}(x+h-t)^{\lambda-1}dt,$$

$$|P_{12}| \le C\frac{h^{\lambda+\alpha}\eta^{\gamma+\beta}}{(y+\eta)^{\beta}}\int\limits_{0}^{x}\frac{dt}{(x+h-t)^{1+\alpha}},$$

$$|P_{13}| \le y^{\gamma+\beta}\left|y^{-\beta}-(y+\eta)^{-\beta}\right|\int\limits_{x}^{x+h}(x+h-t)^{\lambda-1}dt,$$

$$|P_{14}| \le C\frac{\eta^{\gamma+\beta}}{(y+\eta)^{\beta}}\int\limits_{0}^{x}(x-t)^{\lambda+\alpha}\left|(x-t)^{-1-\alpha}-(x+h-t)^{-1-\alpha}\right|dt,$$

$$|P_{15}| \le Ch^{\lambda+\alpha}y^{\gamma+\beta}\left|y^{-\beta}-(y+\eta)^{-\beta}\right|\int\limits_{0}^{x}\frac{dt}{(x+h-t)^{1+\alpha}}dt,$$

$$|P_{16}| \leq y^{\gamma+\beta} \left| y^{-\beta} - (y+\eta)^{-\beta} \right| \int\limits_0^x \frac{\left|(x-t)^{-1-\alpha} - (x+h-t)^{-1-\alpha}\right|}{(x-t)^{-\lambda-\alpha}} dt,$$

$$|P_{17}| \leq C h^{\lambda+\alpha} \eta^{\gamma+\beta} \int\limits_0^x \int\limits_0^y \frac{dt d\tau}{(x+h-t)^{1+\alpha}(y+\eta-\tau)^{1+\beta}},$$

$$|P_{18}| \leq C h^{\lambda+\alpha} \int\limits_0^x \int\limits_y^{y+\eta} \frac{(y+\eta-\tau)^{\gamma-1} dt d\tau}{(x+h-t)^{1+\alpha}},$$

$$|P_{19}| \leq C h^{\lambda+\alpha} \int\limits_0^x \int\limits_0^y \frac{(y-\tau)^{\gamma+\beta}}{(x+h-t)^{1+\alpha}} \left|(y-\tau)^{-1-\beta} - (y+\eta-\tau)^{-1-\beta}\right| dt d\tau,$$

$$|P_{20}| \leq C \eta^{\gamma+\beta} \int\limits_x^{x+h} \int\limits_0^y \frac{(x+h-t)^{\lambda-1} dt d\tau}{(y+\eta-\tau)^{1+\beta}}$$

$$|P_{21}| \leq C \int\limits_x^{x+h} \int\limits_y^{y+\eta} (x+h-t)^{\lambda-1}(y+\eta-\tau)^{\gamma-1} dt d\tau,$$

$$|P_{22}| \leq C \int\limits_x^{x+h} \int\limits_0^y \frac{(y-\tau)^{\gamma+\beta}}{(x+h-t)^{1-\lambda}} \left|(y-\tau)^{-1-\beta} - (y+\eta-\tau)^{-1-\beta}\right| dt d\tau,$$

$$|P_{23}| \leq C \eta^{\gamma+\beta} \int\limits_0^x \int\limits_0^y \frac{(x-t)^{\lambda+\alpha}}{(y+\eta-\tau)^{1+\beta}} \left|(x-t)^{-1-\alpha} - (x+h-t)^{-1-\alpha}\right| dt d\tau,$$

$$|P_{24}| \leq C \int\limits_0^x \int\limits_y^{y+\eta} (x-t)^{\lambda+\alpha}(y+\eta-\tau)^{\gamma-1} \left|(x-t)^{-1-\alpha} - (x+h-t)^{-1-\alpha}\right| dt d\tau,$$

$$|P_{25}| \leq C \int\limits_0^x \int\limits_0^y (x-t)^{\lambda+\alpha}(y-\tau)^{\gamma+\beta} \left|(x-t)^{-1-\alpha} - (x+h-t)^{-1-\alpha}\right| \times$$
$$\times \left|(y-\tau)^{-1-\beta} - (y+\eta-\tau)^{-1-\beta}\right| dt d\tau,$$

after which every term is estimated in the standard way, and we get

$$\left| \left(\overset{1,1}{\Delta}_{h,\,\eta}\, \varphi \right)(x,y) \right| \leq C_3 h^\lambda \eta^\gamma.$$

This completes the proof.

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Some fixed point theorems of set-valued increasing operators*

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Abstract

In this paper, we study two kinds of set-valued increasing operators in partially ordered Banach spaces and partially ordered topological spaces respectively. We obtain three fixed point theorems, which generalize and improve some earlier results.

Keywords: set-valued, increasing operator, partially ordered, fixed point, weakly compact set.

1 Introduction

The fixed point theory for various set-valued operators has been of great interest for many authors. Recently, there is a larger literature on fixed point theory of set-valued operators. We refer the reader to [1–8, 10–13] and references therein for some contributions on this topic. Especially, several authors have studied the fixed point theory for set-valued increasing operators in partially ordered spaces (see, e.g., [2, 5, 6, 8, 10–12] and references therein).

In this paper, we will make further study on the fixed point theory of set-valued increasing operators in partially ordered spaces. More specifically, we will consider two kinds of set-valued increasing operators $A = CB$ and $A = \sum_{i=1}^{m} C_i B_i$, where C, C_i are single-valued increasing operators and B, B_i are set-valued increasing operators. For some earlier

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works on these operators, we refer the reader to [10–12]. As one will see, our main results are generalizations and improvements of [11, 12].

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ iff $y - x \in P$. For $D \subset E$, the weak closure of D is denoted by $\overline{D}^W$ and the complement set of D is denoted by $\mathcal{C}D$. $\overline{co}(D)$ denotes the closed convex hull of D. If $\{x_n\} \subset D$ converges weakly to $x \in E$ then we write $x_n \xrightarrow{W} x$.

Definition 1.1. [10] *Let X, Y be partially ordered sets, M be a subset of X and $A : M \to 2^Y$ be a set-valued operator. The operator A is called a set-valued increasing operator if for any $x \in M$, $y \in M$, $x \leq y$ and any $u \in Ax$, then there exists $v \in Ay$ such that $u \leq v$.*

Definition 1.2. [11] *Let X be an additive group with an ordering structure. X is called an ordered additive group if x, y, z, $w \in X$ and $x \leq y$, $z \leq w$ imply $x + z \leq y + w$.*

Remark 1.3. Let S_1, S_2 are two nonempty sets in X. We define $S_1 + S_2$ as follows:

$$S = S_1 + S_2 = \{x_1 + x_2 \in X | x_1 \in S_1, x_2 \in S_2\}.$$

Since X is an ordered additive group, we have $S \subset X$.

Definition 1.4. [10] *Let X be a Hausdorff topological space with a partially ordered structure. X is said to be a partially topological space if for any two directed sequences $\{x_\tau | \tau \in T\}$ and $\{y_\tau | \tau \in T\}$ in X, $x_\tau \leq y_\tau (\forall \tau \in T)$, $\{x_\tau\}$ is a net converging to $\overline{x}$, $\{y_\tau\}$ is a net converging to $\overline{y}$ imply $\overline{x} \leq \overline{y}$.*

Lemma 1.5. [6] *Let (E, P) be a partially ordered Banach space, W be a nonempty subset of E and $y \in E$. If $z \leq y$ (or $y \leq z$) for all $z \in W$, then for all $z \in \overline{co}(W)$, $z \leq y$ (or $y \leq z$).*

Lemma 1.6. [9] *Let X be a Banach space. Suppose that $M \subset X$ is closed and convex. If $\{x_n\}$ is a sequence in M with $x_n \xrightarrow{W} x$, then $x \in M$.*

Lemma 1.7. [12] *If X is a partially ordered topological space, then for any $\alpha \in X$, $\{y \in X | y \geq \alpha\}$ is a closed set in X.*

2 Main results

Theorem 2.1. *Let X be a partially ordered set, D be a nonempty subset of X and (Y, P) be a partially ordered Banach space. U is a convex closed set in Y. If the operator $A : D \to 2^X$ satisfies the following conditions* *(i) There exists a set-valued increasing operator $B : D \to 2^Y$ with $B(D) = \bigcup\limits_{x\in D} Bx \subset U$ and an increasing operator $C : U \to D$ such that $A = CB$.*

(ii) There exists $x_0 \in D$ and $u \in Ax_0$ such that $x_0 \leq u$.

(iii) Any totally ordered subset of $B(D)$ is a relatively weakly compact subset in Y.

(iv) For any $x \in D$, Bx is a weakly compact set in Y.

Then A has a fixed point in D, i.e. there exists $x^* \in D$ such that $x^* \in Ax^*$.

Proof. Set $G = \{x \in D|$ there exists $u \in Ax$, such that $x \leq u\}$. From condition (ii) we have $x_0 \in G$, so G is nonempty. Suppose that N is any totally ordered set of G. In what follows, we now show that N has an upper bound in G. Since B is a set-valued increasing operator, for any $x \in N$, $y \in N$, $x \leq y$ and any $u \in Bx$, there exists $v \in By$ such that $u \leq v$, so there exists a totally ordered set $D_1 \subset B(N)$ in Y and for any $x \in N$, $D_1 \bigcap Bx \neq \emptyset$. By the hypothesis (iii), we get $\overline{D}_1^W$ is a weakly compact set. Then, it follows from the Krein-Smulian theorem that $\overline{co}(\overline{D}_1^W)$ is also weakly compact. So $\overline{co}(D_1) \subset \overline{co}(\overline{D}_1^W)$ implies $\overline{co}(D_1)$ is a weakly compact set.

For any $y \in D_1$, set $T(y) = \{z \in Y|z \geq y\}$. Since P is a convex closed set, $T(y)$ is also a convex closed set. Let $J(y) = \{z \in \overline{co}(D_1)|z \geq y\} = \overline{co}(D_1)\bigcap T(y)$, then $J(y)$ is a convex closed set, thus $J(y)$ is a weakly closed set. Obviously, $J(y) \neq \emptyset$ for $y \in J(y)$. For $y_1, y_2, \cdots, y_n \in D_1$, we assume $y^* = \max\{y_i|i = 1, 2, \cdots, n\}$. Since D_1 is a totally ordered set, y^* makes sense and $y_i \leq y^*$ which implies $y^* \in \bigcap\limits_{i=1}^{n} J(y_i)$, then we get

$$\bigcap_{i=1}^{n} J(y_i) \neq \emptyset. \tag{2.1}$$

Now we claim $\bigcap\limits_{y\in D_1} J(y) \neq \emptyset$. If we assume otherwise, then we get $\overline{co}(D_1) \subset \bigcup\limits_{y\in D_1} \mathcal{C}J(y)$. Evidently, $\{\mathcal{C}J(y)|y \in D_1\}$ is an open cover of $\overline{co}(D_1)$ in weak topology. As $\overline{co}(D_1)$ is a weakly compact set, $\overline{co}(D_1)$ has a finite subcover, that is, there exists $y_1', y_2', \cdots, y_m' \in D_1$ such that $\overline{co}(D_1) \subset \bigcup\limits_{i=1}^{m} \mathcal{C}J(y_i')$. Note that $J(y_i') \subset \overline{co}(D_1)$, we have

$$\bigcap_{i=1}^{m} J(y_i') \subset \overline{co}(D_1) \subset \bigcup_{i=1}^{m} \mathcal{C}J(y_i').$$

Then $\bigcap\limits_{i=1}^{m} J(y_i') \subset \bigcup\limits_{i=1}^{m} \mathcal{C}J(y_i')$ implies $\bigcap\limits_{i=1}^{m} J(y_i') = \emptyset$ contradicting (2.1). Hence, our claim holds, i.e. there exists $\overline{y} \in \bigcap\limits_{y\in D_1} J(y)$. This means that for any $y \in D_1$

$$y \leq \overline{y} \in \bigcap_{y\in D_1} J(y) \subset \overline{co}(D_1). \tag{2.2}$$

By $B(D) \subset U$ and the fact that U is a convex closed subset of Y, we have

$$\overline{y} \in \overline{co}(D_1) \subset \overline{co}(B(N)) \subset \overline{co}(B(D)) \subset \overline{co}(U) = U.$$

Then $\overline{x} = C\overline{y} \in D$ is well defined. In order to show that $\overline{x}$ is an upper bound of N in G, we will divide it into two steps.

Step 1. $\overline{x}$ is an upper bound of N.

In fact, for any $x_1 \in N$ there exists $x_2 \in N$ such that $x_1 \leq x_2$. Since B is a set-value increasing operator, for any $y \in Bx_1$, there exists $y^{'} \in Bx_2 \bigcap D_1$ such that $y \leq y^{'} \leq \overline{y}$. Moreover, from monotonicity of C, we know

$$Cy \leq C\overline{y} = \overline{x}. \tag{2.3}$$

As a result of $x_1 \in G$, there exists $u_0 \in Ax_1$ such that $x_1 \leq u_0$. Since $u_0 \in Ax_1$, there exists $y_0 \in Bx_1$ such that $u_0 = Cy_0$, then by (2.3) we have

$$u_0 = Cy_0 \leq C\overline{y} = \overline{x}.$$

Therefore, we get $x_1 \leq \overline{x}$, i.e. $\overline{x}$ is an upper bound of N.

Step 2. $\overline{x} \in G$.

As B is a set-valued increasing operator, for any $x \in N$, $x \leq \overline{x}$, and any $y \in D_1 \bigcap Bx$, there exists $v_y \in B\overline{x}$ such that $y \leq v_y$. From the hypothesis (iv), we know $B\overline{x}$ is a weakly compact set which implies that there exists a subset $\{v_{y_k}\}$ of the following set

$$\{v_y | y \leq v_y, v_y \in B\overline{x}, y \in D_1 \bigcap Bx\}$$

such that $\{v_{y_k}\}$ converges weakly to some $v \in B\overline{x}$. Since $y \leq v_y$, i.e. $v_y - y \in P$, we have $v_{y_k} - y \in P$. By Lemma 1.6 with $v_{y_k} - y \xrightarrow{W} v - y$, we can get $v - y \in P$. Thus for all $y \in D_1$, $y \leq v$. By Lemma 1.5 with $\overline{y} \in \overline{co}(D_1)$, we have $\overline{y} \leq v$. Furthermore, as C is an increasing operator, we can obtain $\overline{x} = C\overline{y} \leq Cv = v^{'}$, where $v^{'} \in CB\overline{x} = A\overline{x}$. We have proved that for $\overline{x} \in D$, there exists $v' \in A\overline{x}$ such that $\overline{x} \leq v'$. Thus, $\overline{x} \in G$.

The two steps show that any totally ordered subset of G has an upper bound in G. It follows from Zorn's lemma that G has a maximal element denoted by x^*. Since $x^* \in G$, there exists $u^* \in Ax^*$ such that $x^* \leq u^*$. As C is an increasing operator and B is a set-value increasing operator, we know A is also a set-value increasing operator. So there exists $v^* \in Au^*$ such that $u^* \leq v^*$ which implies $u^* \in G$. Since x^* is a maximal element, we get $x^* = u^* \in Ax^*$, that is, x^* is a fixed point of A in D. $\square$

Remark 2.2. In the case of B being a single-valued operator, the condition (iv) is obviously true. Thus, Theorem 2.1 generalizes [6, Theorem 1]. But, here we use a different approach.

Theorem 2.3. *Let X be an ordered additive group, D be a nonempty subset in X, (Y_i, P_i) $(i = 1, 2)$ be partially ordered Banach spaces, U_1 and U_2 be convex closed subsets of Y_1 and Y_2 respectively. If the operator $A : D \to 2^X$ satisfies the following conditions*

(I) There exists set-valued increasing operators $B_i : D \to 2^{Y_i}$ with $B_i(D) = \bigcup_{x \in D} B_i x \subset U_i$ (i=1,2) and increasing operators $C_i : U_i \to D$ $(i = 1, 2)$ such that $A = C_1B_1 + C_2B_2$.

(II) There exists $x_0 \in D$ and $u \in Ax_0$ such that $x_0 \le u$.

(III) Any totally ordered subset of $B_i(D)$ is a relatively weakly compact subset in Y_i.

(IV) For any $x \in D$, $B_i x$ are weakly compact sets in Y_i .

Then A has a fixed point in D, that is, there exists $x^* \in D$ such that $x^* \in Ax^*$.

Proof. Set $K = \{x \in D|$ there exists $u \in Ax$ such that $x \le u\}$. By the condition (II), we know $x_0 \in K$, so K is nonempty. Suppose that N is any totally ordered set of K. We want to show that N has an upper bound in K. Since B_1 is a set-value increasing operator, for any $x \in N$, $y \in N$, $x \le y$ and any $u_1 \in B_1x$, there exists $v_1 \in B_1y$ such that $u_1 \le v_1$. Thus there exists a totally ordered set $S_1 \subset B_1(N)$ in Y_1 and for any $x \in N$, $S_1 \bigcap B_1x \neq \emptyset$. Similarly, there exists a totally ordered set $S_2 \subset B_2(N)$ in Y_2 and for any $x \in N$, $S_2 \bigcap B_2x \neq \emptyset$. From the condition (III), we know $\overline{S}_1^W$ and $\overline{S}_2^W$ are weakly compact sets. Then, it follows from the Krein-Smulian Theorem that $\overline{co}(\overline{S}_1^W)$ and $\overline{co}(\overline{S}_2^W)$ are also weakly compact. Moreover, $\overline{co}(\overline{S}_1) \subset \overline{co}(\overline{S}_1^W)$ and $\overline{co}(\overline{S}_2) \subset \overline{co}(\overline{S}_2^W)$ imply that $\overline{co}(S_1)$ and $\overline{co}(S_2)$ are weakly compact sets.

For any $p \in S_1$ and $q \in S_2$, set $T_1(p) = \{y_1 \in Y_1 | y_1 \ge p\}$ and $T_2(q) = \{y_2 \in Y_2 | y_2 \ge q\}$ respectively. Since P_1, P_2 are convex closed sets, $T_1(p)$, $T_2(q)$ are also convex closed sets. Let $J_1(p) = \overline{co}(S_1) \bigcap T_1(p)$, $J_2(q) = \overline{co}(S_2) \bigcap T_2(q)$, then $J_1(p)$, $J_2(q)$ are convex closed sets. So $J_1(p)$, $J_2(q)$ are weakly closed sets. Obviously, $J_1(p) \neq \emptyset$ and $J_2(q) \neq \emptyset$ for $p \in J_1(p)$ and $q \in J_2(q)$. For $p_1, p_2, \cdots, p_n \in S_1$, we set $p^* = \max\{p_i | i = 1, 2, \cdots, n\}$. Since S_1 is a totally ordered set, p^* makes sense and $p_i \le p^*$, which implies $p^* \in \bigcap_{i=1}^{n} J_1(p_i)$, then we get

$$\bigcap_{i=1}^{n} J_1(p_i) \neq \emptyset. \tag{2.4}$$

Now we claim that $\bigcap_{p \in S_1} J_1(p) \neq \emptyset$. If we assume otherwise, then we get $\overline{co}(S_1) \subset \bigcup_{p \in S_1} \mathcal{C}J_1(p)$, this means that $\{\mathcal{C}J_1(p) | p \in S_1\}$ is an open cover of $\overline{co}(S_1)$ in weak topology.

As $\overline{co}(S_1)$ is a weakly compact set, $\overline{co}(S_1)$ has a finite subcover, i.e. there exists $p_1^{'}$, $p_2^{'}$,..., $p_m^{'} \in S_1$ such that $\overline{co}(S_1) \subset \bigcup\limits_{i=1}^{m} \mathcal{C}J_1(p_i^{'})$. Note that $J_1(p_i^{'}) \subset \overline{co}(S_1)$, we obtain

$$\bigcap_{i=1}^{m} J_1(p_i^{'}) \subset \overline{co}(S_1) \subset \bigcup_{i=1}^{m} \mathcal{C}J_1(p_i^{'}).$$

Hence $\bigcap\limits_{i=1}^{m} J_1(p_i^{'}) = \emptyset$ which contradicts the previous result (2.4). This means our claim holds, so there exists $\overline{p} \in \bigcap\limits_{p \in S_1} J_1(p)$. Again for every $p \in S_1$, $p \leq \overline{p} \in \bigcap\limits_{p \in S_1} J_1(p) \subset \overline{co}(S_1)$. Using the same method we can prove that there exists $\overline{q} \in \bigcap\limits_{q \in S_2} J_2(q)$ and for every $q \in S_2$, $q \leq \overline{q} \in \bigcap\limits_{q \in S_2} J_2(q) \subset \overline{co}(S_2)$.

By the fact that U_1 and U_2 are convex closed sets in Y_1 and Y_2 respectively, we get

$$\overline{p} \in \overline{co}(S_1) \subset \overline{co}(B_1(N)) \subset \overline{co}(B_1(D)) \subset \overline{co}(U_1) = U_1,$$

$$\overline{q} \in \overline{co}(S_2) \subset \overline{co}(B_2(N)) \subset \overline{co}(B_2(D)) \subset \overline{co}(U_2) = U_2.$$

Then $C_1\overline{p}$, $C_2\overline{q}$ are well defined. Setting $\overline{x} = C_1\overline{p} + C_2\overline{q}$, we have $\overline{x} \in D$. In order to show that $\overline{x}$ is an upper bound of N in K, we will divide it into two steps.

Step 1. $\overline{x}$ is an upper bound of N.

Indeed, for any $x_1 \in N$ there exists $x_2 \in N$ such that $x_1 \leq x_2$. Since B_1 is a set-value increasing operator , for any $z_1 \in B_1x_1$ there exists $z_2 \in B_1x_2 \bigcap S_1$ such that $z_1 \leq z_2 \leq \overline{p}$. Besides, by monotonicity of C_1, we have

$$C_1z_1 \leq C_1\overline{p}. \tag{2.5}$$

As B_2 is a set-valued increasing operator, for any $w_1 \in B_2x_1$ there exists $w_2 \in B_2x_2 \bigcap S_2$ such that $w_1 \leq w_2 \leq \overline{q}$, then

$$C_2w_1 \leq C_2\overline{q}. \tag{2.6}$$

Since X is an ordered additive group, by (2.5) and (2.6), we get

$$C_1z_1 + C_2w_1 \leq C_1\overline{p} + C_2\overline{q} = \overline{x}. \tag{2.7}$$

As result of $x_1 \in K$, there exists $u_0 \in Ax_1 = C_1B_1x_1 + C_2B_2x_1$ such that $x_1 \leq u_0$, where $u_0 = C_1z_0 + C_2w_0$ for some $z_0 \in B_1x_1$ and $w_0 \in B_2x_1$. By (2.7), then we obtain

$$x_1 \leq u_0 \leq \overline{x},$$

i.e. $\overline{x}$ is an upper bound of N.

Step 2. $\overline{x} \in K$.

Since B_1 is a set-valued increasing operator, for any $x \in N$, $x \leq \overline{x}$, and any $y \in S_1 \bigcap B_1 x$, there exists $u_y \in B_1 \overline{x}$ such that $y \leq u_y$. From the condition (IV), we know $B_1 \overline{x}$ is a weakly compact set which implies that there exists a subset $\{u_{y_k}\}$ of the following set

$$\{u_y | y \leq u_y, u_y \in B_1 \overline{x}, y \in S_1 \bigcap B_1 x\}$$

such that $\{u_{y_k}\}$ converges weakly to some $u^{'} \in B_1 \overline{x}$. So we have $u_{y_k} - y \in P_1$ and $u_{y_k} - y \xrightarrow{W} u^{'} - y$. By Lemma 1.6, we can get $u^{'} - y \in P_1$. Thus

$$\forall y \in S_1, y \leq u^{'}. \tag{2.8}$$

In a similar way, we can obtain that for any $z \in S_2 \bigcap B_2 x$, there exists $v_z \in B_2 \overline{x}$ such that $z \leq v_z$. Then $B_2 \overline{x}$ is a weakly compact set implies that there exists a subset $\{v_{z_i}\}$ of the following set

$$\{v_z | z \leq v_z, v_z \in B_2 \overline{x}, z \in S_2 \bigcap B_2 x\}$$

such that $\{v_{z_i}\}$ converges weakly to some $v^{'} \in B_2 \overline{x}$. By Lemma 1.6 we have

$$\forall z \in S_2, z \leq v^{'}. \tag{2.9}$$

By (2.8), (2.9), Lemma 1.5 with $\overline{p} \in \overline{co}(S_1)$ and $\overline{q} \in \overline{co}(S_2)$, we get

$$\overline{p} \leq u^{'}, \overline{q} \leq v^{'}.$$

Since C_1, C_2 are increasing operators, $C_1 \overline{p} \leq C_1 u'$, $C_2 \overline{q} \leq C_2 v^{'}$. From the hypothesis (I), since X is an ordered additive group,

$$\overline{x} = C_1 \overline{p} + C_2 \overline{q} \leq C_1 u' + C_2 v^{'} \in C_1 B_1 \overline{x} + C_2 B_2 \overline{x} = A \overline{x}.$$

Consequently, $\overline{x} \in K$.

From the two steps, we have showed that any totally ordered subset of K has an upper bound in K. It follows from Zorn's lemma that K has a maximal element denoted by x^*. Since $x^* \in K$, there exists $u^* \in Ax^*$ such that $x^* \leq u^*$. Again as A is a set-value increasing operator, there exists $v^* \in Au^*$ such that $u^* \leq v^*$. By the definition of K, $u^* \in K$. But x^* is a maximal element which implies $x^* = u^* \in Ax^*$, i.e. x^* is a fixed point of A in D. $\square$

From Theorem 2.3, we can obtain the following corollary:

Corollary 2.4. *If in Theorem 2.3 we substitute the operator $A = C_1B_1 + C_2B_2$ by the set-valued increasing operator $A = \sum\limits_{i=1}^{m} C_iB_i$, we can also obtain a fixed point for the operator A.*

Theorem 2.5. *Let X be an ordered additive group, D be a nonempty subset in X, and $Y_i(i = 1, 2, \cdots, m)$ be partially ordered topological spaces. If the operator $A : D \to 2^X$ satisfies the following conditions*

(a) There exists $x_0 \in D$ and $u \in Ax_0$ such that $x_0 \leq u$.

(b)There exists set-valued increasing operators $B_i : D \to 2^{Y_i}$ and increasing operators $C_i : \overline{B_i(D)} \to X(i = 1, 2, \cdots, n)$ such that $A = \sum\limits_{i=1}^{m} C_iB_i$.

(c) Any totally ordered subset of $B_i(D)$ is a relatively compact set.

(d)For any $x \in D$, B_ix are compact sets in Y_i.

Then A has a fixed point x^* in D, i.e. $x^* \in Ax^*$.

Proof. Set $R = \{x \in D|$ there exists $u \in Ax$ such that $x \leq u\}$. By the condition (a), we have $x_0 \in R$, so $R \neq \emptyset$. Let N be any totally ordered subset of R. We want to show that N has an upper bound in R.

Let $i(1 \leq i \leq m)$ be fixed. Since $B_i : D \to 2^{Y_i}$ is a set-value increasing operator, for any $x \in N$, $y \in N$, $x \leq y$ and any $u_i \in B_ix$, there exists $v_i \in B_iy$ such that $u_i \leq v_i$. Thus there exists $S_i \subset B_i(N)$ where S_i is a totally ordered set in Y_i and for any $x \in N$, $S_i \bigcap B_ix \neq \emptyset$. From the hypothesis (c), $\overline{S}_i$ is a compact set in Y_i. For any $p_i \in S_i$, set

$$U(p_i) = \{z \in \overline{S}_i|z \geq p_i\} = \overline{S}_i \bigcap \{z \in Y_i|z \geq p_i\}.$$

Since Y_i is a partially ordered topological space, by Lemma 1.7, we know $U(p_i)$ is a closed set in Y_i. Now we consider the closed subset family $\{U(p_i)|p_i \in S_i\}$ of $\overline{S}_i$ where $\{U(p_{i,j})|p_{i,j} \in S_i, j = 1, 2, \cdots, n\}$ are finite members given arbitrarily. Set

$$p_i^* = \max\{p_{i,j}|j = 1, 2, \cdots, n\}.$$

Since S_i is a totally ordered set, p_i^* makes sense and $p_{i,j} \leq p_i^*, j = 1, 2, \cdots, n$ which implies $p_i^* \in \bigcap\limits_{j=1}^{n} U(p_{i,j})$, so $\bigcap\limits_{j=1}^{n} U(p_{i,j})$ is nonempty. Note that $\overline{S}_i$ is a compact set, by virtue of finite intersection property of compact sets, we have

$$\bigcap_{p_i \in S_i} U(p_i) \neq \emptyset.$$

Let $\overline{p}_i \in \bigcap\limits_{p_i \in S_i} U(p_i)$. Then for any $p_i \in S_i$, $p_i \leq \overline{p}_i \in \bigcap\limits_{p_i \in S_i} U(p_i) \subset \overline{S}_i$, thus there exists $\{p_{i,\alpha}|\alpha \in \Lambda\} \subset S_i$ such that $\{p_{i,\alpha}|\alpha \in \Lambda\}$ is a net converging to $\overline{p}_i \in \overline{S}_i$. Since

$\overline{p}_i \in \overline{S}_i \subset \overline{B_i(N)} \subset \overline{B_i(D)}$, $C_i\overline{p}_i$ is well defined. Set $\overline{x} = \sum_{i=1}^{m} C_i\overline{p}_i$. In what follows, we want to show that $\overline{x}$ is an upper bound of N in R.

First, for any $x_1 \in N$, there exists $x_2 \in N$ such that $x_1 \leq x_2$. Since B_i is a set-value increasing operator, for any $y_{i,1} \in B_i x_1$ there exists $y_{i,2} \in B_i x_2 \bigcap S_i$ such that $y_{i,1} \leq y_{i,2} \leq \overline{p}_i$. Again by monotonicity of C_i, we get

$$C_i y_{i,1} \leq C_i\overline{p}_i. \tag{2.10}$$

As $x_1 \in R$, there exists some $u_0 \in Ax_1$ such that

$$x_1 \leq u_0.$$

Since $u_0 \in Ax_1 = \sum_{i=1}^{m} C_i B_i x_1$, there exists $d_i \in B_i x_1$ such that $u_0 = \sum_{i=1}^{m} C_i d_i$. By (2.10), for X is an ordered additive group, we have

$$x_1 \leq u_0 = \sum_{i=1}^{m} C_i d_i \leq \sum_{i=1}^{m} C_i\overline{p}_i = \overline{x}.$$

Therefore, $\overline{x}$ is an upper bound of N.

Second, for any $x \in N$, $x \leq \overline{x}$ and any $y \in S_i \bigcap B_i x$, there exists $u_y \in B_i\overline{x}$ such that $y \leq u_y$. By the condition (d), we know $B_i\overline{x}$ is a compact set, so there exists a subset $\{u_{y_\tau}\}$ of the following set

$$\{u_y | y \leq u_y, u_y \in B_i\overline{x}, y \in S_i \bigcap B_i x\}$$

such that $\{u_{y_\tau}\}$ is a net converging to some $u_i \in B_i\overline{x}$. As Y_i are partially ordered topological spaces, by Definition 1.4, we get $y \leq u_i$. At this time, we have $p_{i,\beta} \leq u_i$ where $\{p_{i,\beta}\}$ is a subsequence of $\{p_{i,\alpha} | \alpha \in \Lambda\} \subset S_i$. Since $\overline{S}_i$ is a compact set and $\{p_{i,\alpha} | \alpha \in \Lambda\}$ is a net converging to $\overline{p}_i$, then the subsequence $\{p_{i,\beta}\}$ is also a net converging to $\overline{p}_i$. Since Y_i are partially ordered topological spaces with $p_{i,\beta} \leq u_i$, we know $\overline{p}_i \leq u_i$. Again by monotonicity of C_i, we get $C_i\overline{p}_i \leq C_i u_i$. Set $\overline{u} = \sum_{i=1}^{m} C_i u_i$. The fact X is ordered additive group implies

$$\overline{x} = \sum_{i=1}^{m} C_i\overline{p}_i \leq \sum_{i=1}^{m} C_i u_i = \overline{u}$$

and

$$\overline{u} \in \sum_{i=1}^{m} C_i B_i\overline{x} = A\overline{x}.$$

with $u_i \in B_i\overline{x} \subset Y_i$ and $C_i u_i \in C_i B_i\overline{x} \subset X$. Consequently, $\overline{x} \in R$.

This shows that $\overline{x}$ is an upper bound of N in R. It follows from Zorn's lemma that R has a maximal element denoted by x^*. Since $x \in R$, there exists $u^* \in Ax^*$ such that $x^* \leq u^*$. Again by the fact that A is a set-valued increasing operator, there exists $y^* \in Au^*$ such that $u^* \leq y^*$, which means $u^* \in R$. Since x^* is a maximal element of R, $x^* = u^* \in Ax^*$, i.e. x^* is a fixed point of A. $\square$

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DYNAMICS AND APPROXIMATIONS FOR 2D GENERALIZED NAVIER-STOKES EQUATION WITH PIECEWISE DISTRIBUTED CONTROLS

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Abstract. We study the dynamics of a piecewise (in time) distributed optimal control problem for Generalized Navier-Stokes equation. The long-time behavior of solutions for an optimal distributed control problem associated with the tracking of the velocity of the Generalized Navier Stockes equations is studied. The existence of a solution of optimal control problem is proved also optimality system is derived. The long-time decay properties for the optimal solutions is established. We also study the dynamics of semidiscrete and fully discrete approximations of this problem. Some computational results are presented, which reinforces the theoretical results derived.

1. Introduction

The control of viscous flows is very crucial to many technological and scientific applications. We are motivated to study the asymptotic behaviors and dynamics of solutions for the controlled Generalized Navier-Stokes equation.

Several treatments of similar optimal control problems can be found in literature. Indeed, the optimal control with the systems governed by Navier-Stokes, Boussinesq and MHD equations was studying by L. Hou and Y. Yan [8], by H. Chun Lee and B. Chun Shin [4] and in [9], respectively. The existence of solutions of Generalized Navier-Stokes equation in Besov spaces was studied by Wu [11] and by Cheskidev and Dai [3].

We formulate here a controllability problem for the Generalized Navier-Stokes equation: find a (u, f) such that the functional

$$J_{(0;+\infty)}(u,f) = \frac{\alpha}{2}\int_0^{+\infty}\int_\Omega |u - U|^2\,dxdt + \frac{\beta}{2}\int_0^{+\infty}\int_\Omega |f-F|^2\,dxdt \tag{1.1}$$

is minimized subject to the 2-D Generalized Navier-Stokes equation:

$$\frac{\partial u}{\partial t} + \nu(-\triangle)^r u + (u.\nabla)u + \nabla p = f \quad \text{in } \Omega\times(0,\infty) \tag{1.2}$$

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$$\nabla . u = 0 \quad \text{in } \Omega \times (0, \infty) \tag{1.3}$$

$$u = 0, \ \triangle u = 0, ..., \triangle^{r-1} u = 0 \text{ on } \ \partial\Omega \times (0, \infty) \tag{1.4}$$

and

$$u(. \ , 0) = u_0 \quad \text{in } \Omega \tag{1.5}$$

where $r \geq 1$ is an integer and $\mathbf{n}$ is an outward normal vector of Ω, also $\nu > 0$ is the kinematic viscosity. Here $\alpha, \beta > 0$ are given constants, Ω is a bounded, sufficiently smooth domain in $\mathbb{R}^2$ with $\partial\Omega$ denoting its boundary; U and F are a given desired velocity field, a given desired body force, respectively. Also, f is a distributed control (body force), u and p denote the velocity field and the pressure field, respectively.

We choose the fixed body force F as

$$F := \partial_t U + \nu(-\triangle)^r U + (U.\nabla)\, U + \nabla P \tag{1.6}$$

We make the following regularity assumptions on the prescribed data U and F :

$$\begin{cases} U \in L^\infty\left(0, \infty; \mathbf{H}^2(\Omega) \cap \mathbf{V}^r\right) \\ F \in L^\infty\left(0, \infty; \mathbf{L}^2(\Omega)\right). \end{cases} \tag{A1}$$

Thus one application of the optimal control problem is to match a steady state flows field through the control of external forces. Observe that U is not an optimal solution because U in general does not satisfy the initial conditions. For technical reasons, we will need the following assumption

$$|||\nabla U|||^2 > \tfrac{\nu^2 \lambda_1}{8}\left(1 - 4\lambda_1^{2r-2}\right) \tag{A2}$$

Our plan of the paper is as follows: Section 2 is devoted to preliminary material. In Section 3 we construct a quasi-optimal control solution and some preliminary estimates for all solutions of the Generalized Navier-Stokes equation. In Section 4 we prove the existence of an optimal solution on the finite time interval. In Section 5 and Section 6 we will analyze semidiscrete and fully discrete approximations, respectively. Finally, in Section 7 the results of some computational experiments are presented.

2. Notation and formulation of the optimal control problem

Throughout this work, C denotes a generic constant depending only on the physical domain Ω, the viscosity constant ν. We will use the standard notations for the function spaces $L^p(\Omega)$ with the norm denoted by $\|.\|_{L^p(\Omega)}$ and the Sobolev spaces $H^m(\Omega)$ with the norm denoted by $\|.\|_m$. We simply denote by $\|.\|$ the norm of $L^2(\Omega)$. The space $H_0^m(\Omega)$ is consisting of functions in $H^m(\Omega)$ which vanish on boundary $\partial\Omega$. The vector valued counterparts of these spaces are denoted by $\mathbf{L}^p(\Omega), \mathbf{H}^m(\Omega)$ and $\mathbf{H}_0^m(\Omega)$.

We now introduce the solenoidal spaces

$\mathbf{W}^r = \{u \in \mathbf{H}^{r-1}(\Omega),\ \nabla.u = 0 \quad \text{and} \ \ u.\mathbf{n}|_{\partial\Omega} = 0\}$

$\mathbf{V}^r = \{u \in \mathbf{H}_0^r(\Omega),\ \nabla.u = 0 \quad \text{and} \ \ \triangle u = ... = \triangle^{r-1} u = 0 \ \text{ on } \partial\Omega\}$

We identify the dual space of $\mathbf{W}^r$ with $\mathbf{W}^r$ itself under the $\mathbf{L}^2(\Omega)$ inner product and the dual space of $\mathbf{V}^r$ is denoted by $(\mathbf{V}^r)^*$. We have

$$\mathbf{V}^r \subset (\mathbf{V}^r)^*,$$

where the injections are continuous and each space is dense in the following one. Next, we introduce the temporal-spatial function spaces $L^r(0, T; \mathbf{H}^m(\Omega))$ defined on $Q_T = \Omega \times (0, T)$ equipped with the norm

$$\|u\|_{L^p(0,T;\mathbf{H}^m)} = \left(\int_0^T \|u(t)\|_m^p \, dt\right)^{1/p}, \text{ where } p \in [1, \infty).$$

We simply denote Q_∞ by Q. The solenoidal temporal-spatial function space

$$\mathcal{H}^r(Q_T) = \{u \in L^2(0, T; \mathbf{V}^r); \ \ \partial_t u \in L^2(0, T; (\mathbf{V}^r)^*)\}$$

that associated norm is given by

$$\|v\|_{\mathcal{H}^r}^2 = \|v\|_{L^2(0,T;\mathbf{V}^r)}^2 + \|\partial_t v\|_{L^2(0,T;(\mathbf{V}^r)^*)}^2.$$

We denote by $|||.|||$ the simplified norm notations of $\|.\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$. This norm will be applied solely to U, ∇U and $\triangle U$.

For a function u in a temporal-spatial space, we often use the notation $u(t) := u(., t)$ to stand for the restriction of u at time t as a function defined over the spatial domain Ω.

We introduce some standard continuous linear, bilinear and trilinear forms:

$$\begin{aligned}
k(u, p) &= -\int_\Omega p\nabla.\varphi dx \qquad \forall \varphi \in \mathbf{H}^r(\Omega) \quad \forall p \in L_0^2(\Omega)\\
a_{2k}(u, \varphi) &= \nu \int_\Omega ((-\triangle)^k u).((-\triangle)^k \varphi) dx, \qquad k \in \mathbb{N}^*,\ \forall u, \varphi \in \mathbf{H}^{2k}(\Omega),\\
a_{(2k+1)}(u, \varphi) &= \nu \int_\Omega \nabla((-\triangle)^k u) : \nabla((-\triangle)^k \varphi) dx, \qquad k \in \mathbb{N},\ \forall u, \varphi \in \mathbf{H}^{2k+1}(\Omega),\\
\mathbf{c}(u, v, w) &= \int_\Omega (u.\nabla)v.w dx \quad \forall u,\ v,\ w \in \mathbf{H}^r(\Omega)
\end{aligned}$$

where the colon notation : denotes the inner product on $\mathbb{R}^{2\times2}$. Also, we denote by $\langle .,. \rangle$ the duality pairing between a Banach space and its dual. Note that for all

$u,\ v,\ w \in \mathbf{H}^1(\Omega)$, $\mathbf{c}$ have the following continuity properties (see [10])

$$|\mathbf{c}(u,v,w)| \leq 2^{1/4}.\, \|u\|^{1/2} . \|\nabla u\|^{1/2} . \|\nabla v\| . \|w\|^{1/2} . \|\nabla w\|^{1/2} . \tag{2.1}$$

Also the trilinear form $\mathbf{c}$ have followings properties

$$\mathbf{c}(u,v,w) = -\mathbf{c}(u,w,v) \text{ and } \mathbf{c}(u,v,v) = 0 \text{ for all } u,\ v,\ w \in \mathbf{H}^1(\Omega). \tag{2.2}$$

Let $\lambda_1 > 0$ be the greatest real number satisfying the Poincaré inequality, $\forall \varphi \in \mathbf{H}^r(\Omega)$

$$\lambda_1 \|\varphi\| \leq \|\nabla \varphi\| . \tag{2.3}$$

Let $\Pi : \mathbf{L}^2(\Omega) \to \mathbf{W}^r$ be the Leray operator (i.e., the orthogonal projection with respect to the $\mathbf{L}^2(\Omega)-$norm), it is well known (see [5] and [6]) that there are constants $\gamma_1 > 0$ and $\gamma_2 > 0$ depending only on Ω such that

$$\gamma_1 \|\Pi\Delta\varphi\| \leq \|\Delta\varphi\| \leq \gamma_2 \|\Pi\Delta\varphi\| , \quad \forall \varphi \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^r_0(\Omega).$$

So that $\|\Pi\Delta.\|$ is equivalent to the $\mathbf{H}^2(\Omega)$-norm on $\mathbf{H}^2(\Omega) \cap \mathbf{H}^r(\Omega)$

Definition 2.1. Given $T \in (0,\infty)$, $u_0 \in \mathbf{W}^r$ and $f \in L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$, u is said to be a solution of the Generalized Navier-Stokes equation on $(0,T)$ if and only if $u \in \mathcal{H}^r\left(Q_T\right)$ and u satisfies

$$\begin{aligned} &\langle \partial_t u(t), \varphi\rangle + a_r\left(u(t),\varphi\right) + \mathbf{c}\left(u(t),u(t),\varphi\right) \\ &\qquad + k(\varphi, p(t)) = \langle f(t), \varphi\rangle \quad \forall \varphi \in \mathbf{V}^r \text{ a.e. } t \in (0,\infty), \end{aligned} \tag{2.4}$$

$$k(u(t), r) = 0 \quad \forall r \in L^2_0(\Omega) \tag{2.5}$$

and

$$\lim_{t\to 0^+} u(t) = u_0 \quad \text{in } \mathbf{W}^r. \tag{2.6}$$

We point out that $u \in \mathcal{H}^r\left(Q_T\right)$ implies $u \in C\left([0,T];\mathbf{W}^r\right)$. Hence, (2.6) makes sense.

Now for $T=\infty$, we define a solution for the Generalized Navier-Stokes equation as follows.

Definition 2.2. Given $u_0 \in \mathbf{W}^r$ and $f \in L^2_{loc}\left(0,T;\mathbf{L}^2(\Omega)\right)$, u is said to be a solution of the Generalized Navier-Stokes equation on $(0,\infty)$ if and only if $u \in L^2_{loc}\left(0,\infty;\mathbf{V}^r\right) \cap L^\infty\left(0,\infty;\mathbf{W}^r\right)$, $\partial_t u \in L^2_{loc}\left(0,\infty;(\mathbf{V}^r)^*\right)$ and u satisfies (2.4) $-$ (2.6)with $T=\infty$.

We define the admissible elements as follows with X_T and Y_T denoting respectively the functional spaces as follows:

$$\begin{aligned}
X_T &= \mathcal{H}^r(Q_T) \quad \text{for } T \in (0,\infty) \\
X_\infty &= \left\{u \in L^2_{loc}(0,\infty;\mathbf{V}^r) \cap L^\infty(0,\infty;\mathbf{W}^r); \partial_t u \in L^2_{loc}(0,\infty;(\mathbf{V}^r)^*)\right\} \\
Y_T &= L^2(0,T;(\mathbf{V}^r)^*) \quad \text{for } T \in (0,\infty), \\
Y_\infty &= L^2_{loc}(0,\infty;(\mathbf{V}^r)^*).
\end{aligned}$$

Definition 2.3. For a given $T \in (0,\infty]$, a pair $(u,f) \in X_T \times Y_T$ is called an *admissible element* if $J_T(u,f) < \infty$ and (u,f) satisfies $(2.4) - (2.6)$. The set of all admissible elements are denoted by $\mathcal{U}_{ad}(T)$.

Now for each $T \in (0,\infty]$, we state the optimal control problem on $(0,T)$ as follows:

$$\begin{aligned}
\text{find a } (u,f) &\in \mathcal{U}_{ad}(T) \text{ such that} \\
J_T(u,f) &\leq J_T(\omega,h) \quad \forall(\omega,h) \in \mathcal{U}_{ad}(T).
\end{aligned} \tag{2.7}$$

We point out that in general, the initial state u_0 is at a certain distance away from the desired flow, or $u_0 \neq U(t)$ for all t, the cost functional generally has a positive minimum. We give following Lemma which we are proved in [2].

Lemma 2.4. *For all $u \in \mathbf{V}^r$, we have*

$$||\bigwedge{}^r u||_{L^2} \geq \lambda_1^{2r-1}||\nabla u||_{L^2} \tag{2.8}$$

where λ_1 is a constant that appears in the Poincarré inequality and $\bigwedge = (-\triangle)$.

The use of the Lemma 2.4, the Schwarz inequality and r integrations by parts give $\forall u \in \mathbf{V}^r$,

$$a_r^\nu(u,u) \geq \nu\lambda_1^{2r-2)}\|\nabla u\|^2. \tag{2.9}$$

Also

$$a_{2k}^\nu(v(t),-\Pi\Delta v(t)) = \left\langle \nu(-\Delta)^{2k}v(t),-\Pi\Delta v(t)\right\rangle = \nu\left\|\Pi\nabla(-\Delta)^k v(t)\right\|^2$$

and

$$a_{(2k+1)}^\nu(v(t),-\Pi\Delta v(t)) = \left\langle -\nu\Delta(-\Delta)^{2k}v(t),-\Pi\Delta v(t)\right\rangle = \nu\left\|\Pi(-\Delta)^{k+1}v(t)\right\|^2.$$

Throughout this paper we denote by

$$v = u - U \text{ and } g = f - F$$

unless we specify them. Then $(2.4) - (2.6)$ are equivalent to

$$v \in X_T \cap L^2(0,\infty;\mathbf{V^r}), \qquad g \in Y_T \cap L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$$

(2.10)

$$\begin{aligned}&\langle \partial_t v(t),\varphi\rangle + a_r(v(t),\varphi) + \mathbf{c}(v(t),v(t),\varphi)\\ &\quad + \mathbf{c}(v(t),U(t),\varphi) + \mathbf{c}(U(t),v(t),\varphi) = \langle g(t),\varphi\rangle, \quad \forall \varphi \in \mathbf{V}^r \text{ a.e. } t \in (0,\infty)\end{aligned}$$

and

$$\lim_{t\to 0^+} v(t) = u_0 - U_0 \text{ in } \mathbf{W}^r \tag{2.11}$$

3. Preliminary estimates for the dynamics

3.1. **A quasi optimizer.** To estimate the dynamics of the optimal control solution, we need to find a sharp bound for the value of $\inf_{(u,f)\in\mathcal{U}_{ad}(T)} J_T(u,f)$. It is important that this bound is uniform in T. We now construct a quasi-optimizer $(\widetilde{u},\widetilde{f}) \in \mathcal{U}_{ad}(\infty)$ for $J_\infty(.,.)$. We can in turn derive some preliminary estimates for the optimal solutions. By a quasi-optimizer we mean an element $(\widetilde{u},\widetilde{f}) \in \mathcal{U}_{ad}(\infty)$ satisfying $\|\widetilde{u}(t) - U(t)\| \to 0$ as $t \to \infty$. The following Theorem asserts the existence of such an element.

Theorem 3.1. *Assume that the assumptions* ($A1$) *and* ($A2$) *hold. Then there exists a pair* $\left(\widetilde{u},\widetilde{f}\right) \in \mathcal{U}_{ad}(\infty)$ *satisfying* $\forall t \geq 0$

$$\|\widetilde{u}(t) - U(t)\|^2 \leq \|u_0 - U_0\|^2 e^{-\epsilon t} \tag{3.1}$$

and $\forall T \in (0,\infty]$

$$J_T(\widetilde{u},\widetilde{f}) \leq \frac{\alpha\|u_0 - U_0\|^2}{2\epsilon}\left(1 - e^{-\epsilon T}\right) \tag{3.2}$$

with

$$\epsilon := 2\nu\lambda_1^{2r-2} - \tfrac{\nu}{2} - \tfrac{4}{\nu\lambda_1}\,|||\nabla U|||^2 \tag{3.3}$$

Remark 3.2. It follows from Theorem 3.1 that $\lim\limits_{T\to\infty} \min\limits_{(\widetilde{u},\widetilde{f})\in\mathcal{U}_{ad}(T)} J_T(\widetilde{u},\widetilde{f}) = 0$. We see that a quasi optimizer $(\widetilde{u},\widetilde{f})$ has been created in the sense that $\|\widetilde{u}(t) - U(t)\| \to 0$ as $t \to \infty$ and $J_\infty(\widetilde{u},\widetilde{f})$ is bounded. In fact, $\|\widetilde{u}(t) - U(t)\| \to 0$ exponentially as $t \to \infty$. The true optimizer is expected to have the property $\|\widetilde{u}(t) - U(t)\| \to 0$ as $t \to \infty$ and at the same time, minimize the work involved to realize and maintain the optimizer flow.

3.2. **Estimate for the dynamics of admissible elements.** In this section, we will derive some estimates for the dynamics of all solutions of (1.2) – (1.4). These estimates in turn will allow us to derive preliminary estimates for the dynamics of the optimal solutions. First we consider the $L^\infty\left(0,T;\mathbf{L}^2(\Omega)\right)$ estimates in terms of the initial data and the functional values.

Theorem 3.3. *Let $T \in (0, \infty]$. Assume that the assumptions $(A1)$ and $(A2)$ hold. If $(u, f) \in \mathcal{U}_{ad}(T)$, then $\forall t \in [0, T]$,*

$$\|u(t) - U(t)\|^2 \leq \|u_0 - U_0\|^2 + \frac{2}{\sqrt{\alpha\beta}} J_T(u, f). \tag{3.4}$$

If in addition,

$$J_T(u, f) \leq J_T(\widetilde{u}, \widetilde{f})$$

then

$$\|u(t) - U(t)\|^2 \leq K_0 \|u_0 - U_0\|^2 \tag{3.5}$$

where ϵ and $(\widetilde{u}, \widetilde{f})$ are defined in Theorem 3.1 and $K_0 = \left(1 + \frac{1}{2\epsilon}\sqrt{\frac{\alpha}{\beta}}\right)$.

Proof. Setting $\varphi = v$ in (2.10) and applying the Schwarz and the Young inequalities we find

$$\tfrac{d}{dt}\|v(t)\|^2 + \epsilon\|v(t)\|^2 \leq \tfrac{1}{\sqrt{\alpha\beta}}(\alpha\|(v)(t)\|^2 + \beta\|g(t)\|^2) \tag{3.6}$$

Multiplying both sides of this inequality by $e^{\epsilon t}$ and then integrating in t over $(0, t)$, lead us to

$$\begin{aligned} \|v(t)\|^2 &\leq \|v(0)\|^2 e^{-\epsilon t} + \frac{1}{\sqrt{\alpha\beta}} \int_0^t \left(\alpha \|v(s)\|^2 + \beta \|g(s)\|^2\right) e^{\epsilon(s-t)} ds \\ &\leq \|v(0)\|^2 e^{-\epsilon t} + \frac{2}{\sqrt{\alpha\beta}} J_T(u, f). \end{aligned}$$

This yields the inequality (3.4). Moreover combining the condition $J_T(u, f) \leq J_T(\widetilde{u}, \widetilde{f})$ with the inequality (3.4) and the Theorem 3.1 we find the inequality (3.5). □

Now, using the uniform Gronwall's inequality we derive $L^\infty(0, T; \mathbf{H}^r)$ estimates.

Theorem 3.4. *Let $T \in (0, \infty]$ and $(u, f) \in \mathcal{U}_{ad}(T)$. Assume that the assumptions $(A1)$ and $(A2)$ hold and assume further that $J_T(u, f) \leq J_T(\widetilde{u}, \widetilde{f})$. Then for each $\varepsilon > 0$, we have*

$$u - U \in L^2(0, T; \mathbf{H}^r(\Omega)) \cap L^\infty(\varepsilon, T; \mathbf{H}^r(\Omega)) \cap C([\varepsilon, T]; \mathbf{H}^r(\Omega)),$$

with

$$\int_0^T \|\nabla u(s) - \nabla U(s)\|^2 ds \leq K_1 \|u_0 - U_0\|^2 \tag{3.7}$$

and

$$\|\nabla u(t) - \nabla U(t)\|^2 \leq K_2 \|u_0 - U_0\|^2, \forall t \geq \varepsilon, \tag{3.8}$$

where

$$K_1 = \frac{\lambda_1}{\varepsilon}\left(1 + \frac{1}{\varepsilon}\sqrt{\frac{\alpha}{\beta}}\right)$$

and

$$K_2 = 2CK_0\left(\frac{1}{\nu^3} + \frac{2}{\theta\nu}\right)\|u_0 - U_0\|^2$$

and

$$K_3 = 2\left(C_5 + C_6\right).$$

The following preliminary estimates for the optimal solutions is an immediate consequence of Theorems 3.3 and 3.4

Theorem 3.5. *Assume that the assumptions* $(A1)$ *and* $(A2)$ *hold. Let* $T \in (0, \infty]$ *and* $(\widehat{u}, \widehat{f}) \in \mathcal{U}_{ad}(T)$ *be an optimal solution for* (2.7). *Then*

$$\|\widehat{u}\,(t) - U(t)\|^2 \leq K_0 \|u_0 - U_0\|^2 \tag{3.9}$$

$$\int_0^T \|\nabla\widehat{u}(s) - \nabla U(s)\|^2 \, ds \leq K_1 \|u_0 - U_0\|^2 \tag{3.10}$$

and

$$\|\nabla\widehat{u}(t) - \nabla U(t)\|^2 \leq K_2(\varepsilon) \|u_0 - U_0\|^2 \tag{3.11}$$

$\forall t \geq \varepsilon$, *where all constants are as defined in Theorem 3.3 and Theorem 3.4.*

3.3. **Existence of solution and dynamics of optimal controls.** The existence results are similar to the results from Generalized MHD equations [2], in both case, finite time interval and infinite time interval. The following Theorem gives the results.

Theorem 3.6.

- *Let* $T \in (0, \infty)$. *Then there exists an optimal solution* $(\widehat{u}, \widehat{f}) \in \mathcal{U}_{ad}(T)$ *for the problem* (2.7), *i.e. there exists at least an element* $\widehat{f} \in L^2\left(0, T; \mathbf{L}^2(\Omega)\right)$ *and* $\widehat{u} \in C\left([0, T]; \mathbf{W}^r\right) \cap L^2\left(0, T; \mathbf{V}^r\right)$ *such that the functional* $J_T(u, f)$ *attains its minimum at* $(\widehat{u}, \widehat{f})$ *and* $\widehat{u}$ *satisfies* $(2.4)-(2.6)$ *with* $\widehat{f} = f$.
- *There exists an optimal solution* $(\widehat{u}, \widehat{f}) \in \mathcal{U}_{ad}(T)$ *for* (2.7) *with* $T = \infty$.

For many feedback control models, the controlled flow exponentially decays to the desired flow. For our optimal control system, Theorem 3.4 and Theorem 3.5 gave some preliminary results as $\|u\,(t) - U(t)\|$ stays bounded.

Lemma 3.7. *Let* $T \in (0, \infty)$. *Assume that* $(u, f) \in \mathcal{U}_{ad}(T)$ *and* $\lambda_1 > 1$. *If* $\|(u, b)\ (t) - U(t)\| > 0$ *for all* $t \in (t_1, t_2) \subset [0, T]$, *then*

$$\|u\,(t_2) - U(t_2)\| \leq \|u\,(t_1) - U(t_1)\| + K_4\sqrt{t_2 - t_1}\left(J_T(u, f)\right)^{1/2}$$

with $K_4 = \left(\frac{1}{\alpha} \left(\frac{2}{\nu}\right)^2 \left(|||\nabla U|||^2 \right)^2 + \frac{1}{\beta} \right)^{1/2}$.

If in addition, the assumptions $(A1)$ *and* $(A2)$ *hold and* $J_T(u,f) \le J_T(\widetilde{u}, \widetilde{f})$, *where* $(\widetilde{u}, \widetilde{f})$ *is defined in Theorem 3.1, then*

$$\|u(t_2) - U(t_2)\| \le \|u(t_1) - U(t_1)\| + K_4 \sqrt{t_2 - t_1} \, \|u_0 - U_0\| \sqrt{\frac{\alpha}{2\epsilon}}.$$

Proof. By setting $\varphi = v(t)$ in (2.10) we obtain

$$\|v(t)\| \tfrac{d}{dt} \|v(t)\| + \epsilon_1 \|v(t)\|^2 \le C_0 . \|v(t)\|^2 + \|g(t)\| . \|v(t)\|$$

where

$\epsilon_1 = \nu\lambda_1 \left(\lambda_1^{2(2k-1)} - 1 \right)$ and $C_0 = \frac{1}{\nu} |||\nabla U|||^2$.

If $\|v(t)\| > 0$ for all $t \in (t_1, t_2)$, then we may divide this inequality by $\|v(t)\|$, multiplying by $e^{\epsilon_1 t}$ and then integrating over (t_1, t_2), we are led to

$$\|v(t_2)\| e^{\epsilon_1 t_2} \le \|v(t_1)\| e^{\epsilon_1 t_1} + \left(\tfrac{1}{\alpha} C_0^2 + \tfrac{1}{\beta} \right)^{1/2} \int_{t_1}^{t_2} \left(\alpha \|v(t)\|^2 + \beta \|g(t)\|^2 \right)^{1/2} e^{\epsilon_1 t} dt$$

we have

$$\begin{aligned} \|v(t_2)\| \le & \|v(t_1)\| e^{-\epsilon_1 (t_2 - t_1)} \\ & + \left(\tfrac{1}{\alpha} C_0^2 + \tfrac{1}{\beta} \right)^{1/2} \left(\int_{t_1}^{t_2} (\alpha \|v(t)\|^2 + \beta \|g(t)\|^2) dt \right)^{1/2} \left(\int_{t_1}^{t_2} e^{-2\varepsilon_1 (t_2 - t)} dt \right)^{1/2}, \end{aligned}$$

with $e^{-\epsilon_1 (t_2 - t_1)} < 1$

$$\begin{aligned} \|v(t_2)\| & \le \|v(t_1)\| + \left(\frac{1}{\alpha} C_0^2 + \frac{1}{\beta} \right)^{1/2} (J_T(u,f))^{1/2} . \left(\int_{t_1}^{t_2} e^{-2\varepsilon_1 (t_2 - t)} dt \right)^{1/2} \\ & \le \|v(t_1)\| + \sqrt{t_2 - t_1} \left(\frac{1}{\alpha} C_0^2 + \frac{1}{\beta} \right)^{1/2} (J_T(u,f))^{1/2}, \end{aligned}$$

where we have used the fact that $1 - e^{-y} \le y$ for $y \ge 0$. Hence, we have shown (3.12) and (3.12) simply follows from the bound (3.2) so that applying the mean value theorem to the last factor we have the result. □

We give the asymptotic decay property of $\|u(t) - U(t)\|$ as $t \to \infty$ for any $(u,f) \in \mathcal{U}_{ad}(\infty)$.

Theorem 3.8. *Assume that* $(u,f) \in \mathcal{U}_{ad}(T)$. *Then*

$$\lim_{t \to \infty} \|u(t) - U(t)\| = 0. \tag{3.12}$$

4. Semidiscrete approximations of the piecewise optimal control problem

We semidiscretize the functional $J_{(t_n,t_{n+1})}(u,f)$ by the right-endpoint rectangle rule
$\displaystyle\int_{t_n}^{t_{n+1}}\varphi(t)dt \approx (t_{n+1}-t_n)\,\varphi(t_{n+1}) = \delta\varphi(t_{n+1})$ so that the semidiscretized functional becomes

$$J^{n+1}(u,f)=\frac{\delta\alpha}{2}\left\|u^{(n)}-U^{n+1}\right\|^2+\frac{\delta\beta}{2}\left\|f\ -F^{n+1}\right\|^2,\qquad \forall u\in\mathbf{V}^r,\forall f\ \in\mathbf{L}^2\left(\Omega\right),$$

where $U^{n+1}=U^{n+1}\left(x\right)=U\left(x,t_{n+1}\right)$ and $F^{n+1}=F^{n+1}\left(x\right)=F\left(x,t_{n+1}\right)$ with $t_n=\delta n$ for $n=0,\ 1,\ 2,\ \ldots$ For convenience, we define

$$\mathcal{L}^{n+1}(u,f)=\frac{\alpha}{2}\left\|u-U^{n+1}\right\|^2+\frac{\beta}{2}\left\|f-F^{n+1}\right\|^2,$$

so that the minimization of the functional $J^{n+1}(u,f)$ is equivalent to the minimization of the functional $\mathcal{L}^{n+1}(u,f)$. Using the techniques of [7] concerning optimal control problems for the steady-state Navier-Stokes equations, we can show the existence of a solution $(\widehat{u},\widehat{p},\widehat{f})^{n+1}$ for the $(n+1)th$ optimal control problem. The remainder of this Section will be devoted to the study of $\widehat{u}^n$ as $n\to\infty$. We now study the behavior of the semidiscrete solutions $\widehat{u}^n$ as $n\to\infty$. By finite difference approximation formula

$$\partial_t U(x,t)=\frac{1}{\Delta t}\left(U(x,t+\Delta t)-U(x,t)\right)-\ \partial_{tt}U(x,t+\alpha\Delta t).\Delta t,$$

where $\alpha\stackrel{\texttt{def}}{=}\alpha(x,t)$ with $|\alpha|<1$, we have that

$$\begin{aligned}\frac{1}{\Delta t}\left\langle U^{n+1},\varphi\right\rangle+a_r\left(U^{n+1},\varphi\right)+\ \mathbf{c}\left(U^{n+1},U^{n+1},\varphi\right)&=\frac{1}{\Delta t}\left\langle U^n,\varphi\right\rangle\\ +\left\langle f^{n+1},\varphi\right\rangle-\left\langle\tau^{n+1},\varphi\right\rangle,\ \forall\varphi\in\mathbf{V}^r&\end{aligned}\tag{4.1}$$

and

$$k\left(U^{n+1},r\right)=0,\qquad\forall r\in L_0^2\left(\Omega\right)\tag{4.2}$$

where

$$\tau^{n+1}=\Delta t.\partial_{tt}U(x,t_n+\alpha(x_n,t_n)\Delta t)\Delta t).\tag{4.3}$$

Lemma 4.1. *Assume that hypotheses (A1)-(A2) and*

$$\begin{aligned}&\partial_t U\in C\left([0,\infty)\,;\mathbf{H}^1\right)\\ &\partial_{tt}U\in L^\infty\left(0,\infty;\mathbf{L}^2\left(\Omega\right)\right)\cap C\left([0,\infty)\,;\mathbf{L}^2\left(\Omega\right)\right)\end{aligned}\tag{A4}$$

hold. Assume further that $\left(\widehat{u},\widehat{p},\widehat{f}\right)^{n+1}$ *is a solution of the* $(n+1)th$ *semidiscrete optimal control problem for* $n=1,2,\ldots$. *Then*

$$\mathcal{L}^{n+1}\left((\widehat{u},\widehat{f})^{n+1}\right)\leq\frac{\alpha}{2}\left(\frac{\|\widehat{u}^n-U^n\|^2}{1+C_5\Delta t}+\frac{C_6\,(\Delta t)^3}{1+C_5\Delta t}\right)\tag{4.4}$$

where

$$C_5=C_5(\nu,\Omega)\overset{def}{=}\frac{\epsilon\lambda_1}{2}\qquad \textit{and}\;\; C_6=C_6(\nu,\Omega,U)\overset{def}{=}\frac{2\,|\|\partial_{tt}U\||^2}{\epsilon\lambda_1}.\tag{4.5}$$

Proof. Let $(\widetilde{u},\widetilde{p})^{n+1}$ be a solution of the equations

$$\begin{aligned}\frac{1}{\Delta t}\langle\widetilde{u}^{n+1},\varphi\rangle+a_r\left(\widetilde{u}^{n+1},\varphi\right)+\mathbf{c}\left(\widetilde{u}^{n+1},\widetilde{u}^{n+1},\varphi\right)&\\+k\left(\varphi,\widetilde{p}^{n+1}\right)=\frac{1}{\Delta t}\langle\widehat{u}^n,\varphi\rangle+\left\langle f^{n+1},\varphi\right\rangle,\quad\forall\varphi\in\mathbf{V}^r&\end{aligned}\tag{4.6}$$

$$k\left(\widetilde{u}^{n+1},r\right)=0,\quad\forall r\in L_0^2\left(\Omega\right)\tag{4.7}$$

(The existence of such a $(\widetilde{u},\widetilde{p})^{n+1}$ can be proved by using the techniques for proving the existence of a solution for the steady-state Navier-Stokes equations). Set $\widetilde{f}^{n+1}=F^{n+1}$; then we see that $\left(\widetilde{u},\widetilde{f},\widetilde{p}\right)^{n+1}$ satisfies the semidiscrete Navier-Stokes equations (4.6) – (4.7). Let $\widetilde{v}^{n+1}=\widetilde{u}^{n+1}-U^{n+1}$, $\widehat{v}^n=\widehat{u}^n-U^n$ and $\widetilde{q}^{n+1}=\widetilde{p}^{n+1}$. Then by subtracting (4.1) – (4.2) from (4.6) – (4.7), we obtain

$$\begin{aligned}&\frac{1}{\Delta t}\left\langle\widetilde{v}^{n+1},\varphi\right\rangle+a_r\left(\widetilde{v}^{n+1},\varphi\right)+\widetilde{\mathbf{c}}\left(\widetilde{v}^{n+1},\widetilde{v}^{n+1},\varphi\right)+\widetilde{\mathbf{c}}\left(U^{n+1},\widetilde{v}^{n+1},\varphi\right)\\&\quad+\widetilde{\mathbf{c}}\left(\widetilde{v}^{n+1},U^{n+1},\varphi\right)+k\left(\varphi,\widetilde{p}^{n+1}\right)=\frac{1}{\Delta t}\left\langle\widehat{v}^n,\varphi\right\rangle-\left\langle\tau^{n+1},\varphi\right\rangle,\quad\forall\varphi\in\mathbf{H}_0^r\left(\Omega\right)\end{aligned}\tag{4.8}$$

and

$$k\left(\widetilde{v}^{n+1},r\right)=0,\qquad\forall r\in L_0^2\left(\Omega\right).\tag{4.9}$$

Setting $\varphi=\widetilde{v}^{n+1}$ in (4.8), we have by Young‘s inequality

$$\begin{aligned}\frac{1}{2\Delta t}\left(\left\|\widetilde{v}^{n+1}\right\|^2-\|\widehat{v}^n\|^2+\left\|\widehat{v}^{n+1}-\widetilde{v}^n\right\|^2\right)&\\+\frac{\epsilon}{2}\left\|\nabla\widetilde{v}^{n+1}\right\|^2\leq\frac{\epsilon\lambda_1}{4}\left\|\widetilde{v}^{n+1}\right\|^2+\frac{1}{\epsilon\lambda_1}\left\|\tau^{n+1}\right\|^2.&\end{aligned}\tag{4.10}$$

Dropping the term $\left\|\widehat{v}^{n+1}-\widetilde{v}^n\right\|^2$, applying Poincaré inequality and rearranging, we have

$$\frac{1}{2\Delta t}\left(\left\|\widetilde{v}^{n+1}\right\|^2-\|\widehat{v}^n\|^2\right)+\frac{\epsilon\lambda_1}{4}\left\|\widetilde{v}^{n+1}\right\|^2\leq\frac{1}{\epsilon\lambda_1}\left\|\tau^{n+1}\right\|^2,\tag{4.11}$$

so that using the estimates

$$\left\|\tau^{n+1}\right\|\leq\Delta t|\,\|\partial_{tt}U\|\,|\quad\text{ and }\quad|\,\|\partial_{tt}U\|\,|=\|\partial_{tt}U\|_{\mathbf{L}^\infty(\Omega)},$$

we are led to

$$(1+C_5\Delta t)\left\|\widetilde{v}^{n+1}\right\|^2\leq\|\widehat{v}^n\|^2+C_6\,(\Delta t)^3,$$

where C_5 and C_6 are defined by (4.5). Hence, we arrive at

$$\mathcal{L}^{n+1}\left((\widetilde{u},\widetilde{f})^{n+1}\right)=\frac{\alpha}{2}\left\|\widetilde{v}^{n+1}\right\|^{2}\leq\frac{\alpha}{2}\left(\frac{\left\|\widehat{v}^{n}\right\|^{2}}{(1+C_5\Delta t)}+\frac{C_6}{(1+C_5\Delta t)}(\Delta t)^{3}\right),$$

$(\widehat{u},\widehat{p},\widehat{f})^{n+1}$ being a solution for the $(n+1)th$ optimal control problem, the desired estimate follows trivially from this last inequality. □

Theorem 4.2. *Assume that the hypotheses (A1)-(A2) and (A4) hold and* $0<\Delta t\leq 1$. *Then there are positive constants* ξ_1 *and* ρ_1 *such that*

$$\left\|\widehat{u}^{n+1}-U^{n+1}\right\|^{2}\leq(1-\xi_1\Delta t)\left\|\widehat{u}^{n}-U^{n}\right\|^{2}+\ C_6(\Delta t)^{3}$$

with $1-\xi_1\Delta t>0$ *and*

$$\left\|\widehat{u}^{n}-U^{n}\right\|^{2}\leq\left\|u_0-U_0\right\|^{2}.e^{-\xi_1 t_n}+\ \rho_1(\Delta t)^{2} \tag{4.12}$$

where

$$\xi_1=\frac{C_5-\sqrt{\alpha/\beta}}{(1+C_5\Delta t)^{2}}\qquad \textit{and}\qquad \rho_1=\frac{C_6}{\xi_1}. \tag{4.13}$$

In the semidiscretization of the Navier-Stokes equations we used the first-order backward Euler scheme. Therefore, the appearance of the term $O(\Delta t)$ in the last estimate is expected. If we use higher-order approximation scheme, we expect to obtain improved estimates. However, the analysis in the context of semidiscrete piecewise optimal control with more sophisticated schemes becomes complicated.

The proof of Theorem 4.2 gives a rough estimate of

$$\left\|\nabla\widehat{u}^{n}-\nabla U^{n}\right\|=\left\|\nabla\widehat{v}^{n}\right\|.$$

Proposition 4.3. *Assume that the conditions of Theorem 4.2 hold. Then*

$$\begin{aligned}\Delta t\|\nabla\widehat{u}^{n}-\nabla U^{n}\|^{2}\leq{}&\frac{2}{\epsilon}\left(1+\sqrt{\frac{\alpha}{\beta}}\right)e^{-\xi_1\delta}\left\|u_0-U_0\right\|^{2}e^{-\xi_1 t_n}\\&+\frac{2}{\epsilon}\left(1+\sqrt{\frac{\alpha}{\beta}}\right)(\rho_1+C_6)(\Delta t)^{2}.\end{aligned} \tag{4.14}$$

We now derive an improved bound for the eventual error in $\mathbf{H_{0n}}$ norm. We first observe the following direct consequence of (4.14).

Lemma 4.4. *Assume that the conditions of Theorem 4.2 hold. Then for any constant* $\sigma>0$, *there exist constants* $\epsilon_0=\epsilon_0(\Omega,\nu;\sigma)>0$ *and* $\widetilde{t}=\widetilde{t}(\Omega,\nu,u_0,U_0;\sigma)>0$ *such that*

$$\Delta t\left\|\nabla\widehat{v}^{n}\right\|^{2}\leq\sigma,\quad\forall t_n\geq\widetilde{t},\ \forall\Delta t\in(0,\epsilon_0). \tag{4.15}$$

We also need a stronger version of Proposition 4.3.

Proposition 4.5. *Assume that the conditions of Theorem 4.2 hold. then for each* $n \geq 1$,

$$\mathcal{L}^{n+1}\left((\widehat{u}, \widehat{f})^{n+1}\right) \leq \frac{\alpha}{2}\left(\frac{\|u_0 - U_0\|^2 e^{-\xi_1 t_n} + \rho_1 (\Delta t)^2 + C_6 (\Delta t)^3}{1 + C_5 \Delta t}\right). \tag{4.16}$$

Moreover, for all $n_2 \geq n_1 \geq 1$,

$$\frac{\epsilon \Delta t}{2} \sum_{n=n_1+1}^{n_2} \|\nabla \widehat{v}^n\|^2 \leq \|\widehat{v}^{n_1}\|^2 + C_7 (t_{n_2} - t_{n_1}) \left(\|u_0 - U_0\|^2 e^{-\xi_1 t_{n_1}} + (\Delta t)^2\right) \tag{4.17}$$

where

$$C_7 := C_7(\nu, \Omega, U) = \sqrt{\frac{\alpha}{\beta}}\left(1 + \rho_1 + C_6\sqrt{\frac{\beta}{\alpha}}\right).$$

Proof. Combining (4.4) and (4.12) yields (4.16). By using (4.16)together with (4.12), we obtain that

$$\left\|\widehat{v}^{n+1}\right\|^2 - \|\widehat{v}^n\|^2 + \frac{\epsilon \Delta t}{2}\left\|\nabla \widehat{v}^{n+1}\right\|^2 \leq \sqrt{\frac{\alpha}{\beta}} \|u_0 - U_0\|^2 e^{-\xi_1 t_n} \Delta t + (\Delta t)^3 \sqrt{\frac{\alpha}{\beta}}\left(\rho_1 + C_6\sqrt{\frac{\beta}{\alpha}}\right).$$

Summing up n over $n_1 \leq n \leq n_2 - 1$, we have (4.17). □

Theorem 4.6. *Assume that the hypotheses (A1)-(A4) hold. Then there exist constants* $\epsilon_0 = \epsilon_0(\Omega, \nu; \sigma) > 0$ *and* $\widetilde{t} = \widetilde{t}(\Omega, \nu, u_0; \sigma) > 0$ *such that*

$$\begin{aligned} \|\nabla \widehat{u}^n - \nabla U^n\|^2 &\leq C_8\left(\left(\tfrac{1}{\tau} + 1 + \tau\right)\|u_0 - U_0\|^2 e^{-\xi_1(t_n - \tau)} + (\Delta t)^2\right) \\ &\quad . \exp\left\{C_9(1 + \tau)\left(\|u_0 - U_0\|^4 e^{-2\xi_1(t_n - \tau)} + (\Delta t)^4\right)\right\}, \end{aligned} \tag{4.18}$$

$\forall\ \Delta t \in (0, \epsilon_0)$ *and* $\forall\ t_n \geq \widetilde{t}$, *where* ξ_1 *is as in Theorem 4.2 and* C_8, C_9 *are constants depending only on* Ω, ν, U *and* B.

5. Fully discrete approximations of the piecewise optimal control problem

Let $\mathbf{X}_h \subset \mathbf{H}_0^r(\Omega)$ and $S_h \subset L_0^2(\Omega)$ be two families of the finite-dimensional subspaces. First, we have the approximation properties: there exist an integer $k \geq 1$ and a constant $C' > 0$, independent of h, u and p such that for $1 \leq m \leq k$

$$\begin{aligned} \inf_{u_h \in \mathbf{X}_h} \|u - u_h\|_1 &\leq C' h^m \|u\|_{m+1} \quad \forall u \in \mathbf{H}^{m+1}(\Omega) \cap \mathbf{H}_0^r(\Omega), \\ \inf_{p_h \in S_h} \|p - p_h\|_0 &\leq C' h^m \|p\|_m \qquad \forall p \in H^m(\Omega) \cap L_0^2(\Omega). \end{aligned}$$

Next, we assume the *inf-sup condition*, or *Ladyzhenskaya-Babuska-Brezzi condition* there exists a constant C'', independent of h, such that

$$\inf_{0 \neq p_h \in S_h} \sup_{0 \neq u_h \in \mathbf{V}_{h,0\mathbf{r}}} \frac{k(u_h, p_h)}{\|u_h\|_1 \|p_h\|_0} \geq C''. \tag{5.1}$$

This condition assures the stability of finite-element discretizations of the Navier Stokes equations. For each $n \geq 0$, we define the affine space $Y_h^{n+1} \stackrel{\text{def}}{=} \{f_h = y_h + F_h^{n+1} : y_h \in \mathbf{X}_h\}$, for the approximate distributed controls, where F_h^{n+1} is the L^2 projection of F^{n+1} onto $\mathbf{X}_h$. In order to preserve the antisymmetric property of the trilinear form $\mathbf{c}(.,.,.)$, we introduce the form

$$\overline{\mathbf{c}}\,(u,v,w) = \tfrac{1}{2}\{\mathbf{c}\,(u,v,w) - \mathbf{c}\,(u,w,v)\} \tag{5.2}$$

It can be easily verified that

$$\overline{\mathbf{c}}\,(u,v,w) = \mathbf{c}\,(u,v,w)\,, \qquad \overline{\mathbf{c}}\,(u,v,w) = -\overline{\mathbf{c}}\,(u,w,v) \quad \text{and} \quad \overline{\mathbf{c}}\,(u,v,v) = 0$$

on all $\mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$. We also have

$$|\overline{\mathbf{c}}\,(u,v,w)\,| \leq \overline{C}_0\,\|\nabla u\|\,.\,\|v\|_{L^\infty}\,.\,\|\nabla w\|\,, \tag{5.3}$$

$$|\overline{\mathbf{c}}\,(u,v,w)\,| \leq \overline{C}_1\,\|\nabla u\|\,.\,\|\nabla v\|\,.\,\|\nabla w\| \tag{5.4}$$

$$|\overline{\mathbf{c}}\,(u,v,w)\,| \leq \overline{C}_2\,\|u\|_2\,.\,\|v\|\,.\,\|\nabla w\| \tag{5.5}$$

for all $u \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ and $v, w \in \mathbf{H}_0^1(\Omega)$, where $\overline{C}_0$, $\overline{C}_1$ and $\overline{C}_2$ are positive reals. We define the fully discrete approximations of the piecewise optimal control problem.

- Set $\Delta t = \delta$.
- Define $\widehat{u}_h^0 = u_{0,h}$ where $u_{0,h}$ is the $\mathbf{L}^2(\Omega)$-projection (or interpolation) of u_0 onto $\mathbf{X}_h$.
- *The* (n+1)*th fully discrete optimal control problem*:
  for $n = 0, 1, 2, \ldots$, find $\left(\widehat{u}, \widehat{p}, \widehat{f}\right)^{n+1} \in \mathbf{X}_h \times S_h \times Z_h^{n+1}$ such that the functional

$$\mathcal{L}_h^{n+1}(u_h^{n+1}, f_h^{n+1}) \stackrel{\text{def}}{=} \frac{\alpha}{2}\left\|u_h^{n+1} - U^{n+1}\right\|^2 + \frac{\beta}{2}\left\|f_h^{n+1} - F^{n+1}\right\|^2 \quad \forall u_h^{n+1} \in \mathbf{X}_h,\ \forall f_h^{n+1} \in Z_h^{n+1}$$

is minimized subject to the fully discrete Generalized Navier Stokes equations

$$\begin{aligned}\frac{1}{\Delta t}\left\langle u_h^{n+1}, \psi_h\right\rangle &+ a_r\left(u_h^{n+1}, \varphi_h\right) + \overline{\mathbf{c}}\left(u_h^{n+1}, v_h^{n+1}, \varphi_h\right) \\ &+ k\left(\varphi_h, p_h^{n+1}\right) = \frac{1}{\Delta t}\langle \widehat{u}_h^n, \varphi_h\rangle + \left\langle f_h^{n+1}, \varphi_h\right\rangle, \ \forall \varphi_h \in \mathbf{X}_h\end{aligned} \tag{5.6}$$

and

$$k\left(u_h^{n+1}, r_h\right) = 0, \qquad \forall r_h \in S_h. \tag{5.7}$$

Using the techniques of [7] concerning finite element approximations of optimal control problems for the steady-state Navier-Stokes equations, we can show the existence of a solution $\widehat{u}_h^{n+1}, \widehat{p}_h^{n+1}, \widehat{f}_h^{n+1}$ for the $(n+1)th$ fully discrete optimal control problem. We now study the behavior of the fully discrete solutions $\widehat{u}_h^n$ as $n \to \infty$. For every t, we introduce an auxiliary element $U_h(t), P_h(t) \in \mathbf{X}_h \times S_h$ determined by

$$a_r\left(U_h(t), \varphi_h\right) + k\left(\varphi_h, P_h(t)\right) = a_r\left(U(t), \varphi_h\right) \quad \forall \varphi_h \in \mathbf{X}_h \tag{5.8}$$

and

$$k\left(U_h(t), r_h\right) = 0 \quad \forall r_h \in S_h. \tag{5.9}$$

The existence of such a $(U_h(t), P_h(t))$ follows from the well-known results for the finite element approximations of the steady-state Navier-Stokes equations. Furthermore, under the assumption that there is a $k \geq 1$ such that

$$U \in C\left([0,\infty); \mathbf{H}^{k+1}(\Omega)\right) \cap L^\infty\left(0,\infty; \mathbf{H}^{k+1}(\Omega)\right). \tag{A6}$$

The following error estimates hold:

$$\|U_h(t) - U(t)\|_1 + \|P_h(t)\| \leq \overline{C}_3 h^k \|U(t)\|_{k+1} \leq \overline{C}_3 h^k \|U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))} \tag{5.10}$$

and

$$\|U_h(t) - U(t)\| \leq \overline{C}_4 h^{k+1} \|U(t)\|_{k+1} \leq \overline{C}_4 h^{k+1} \|U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))} \tag{5.11}$$

where $\overline{C}_3$ and $\overline{C}_4$ are constant depending on Ω only; see, e.g. [8], By differentiating (5.8), (5.9) with respect t, we see that $(\partial_t U_h(t), \partial_t P_h(t))$ satisfies a system of equations similar to (5.8), (5.9) so that under the assumption

$$\partial_t U \in C\left([0,\infty); \mathbf{H}^{k+1}(\Omega)\right) \cap L^\infty\left(0,\infty; \mathbf{H}^{k+1}(\Omega)\right), \tag{A7}$$

we have the error estimates

$$\|\partial_t U_h(t) - \partial_t U(t)\|_1 + \|\partial_t P_h(t)\| \leq \overline{C}_3 h^k \|\partial_t U(t)\|_{k+1} \leq \overline{C}_3 h^k \|\partial_t U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))} \tag{5.12}$$

and

$$\|\partial_t U_h(t) - \partial_t U(t)\| \leq \overline{C}_4 h^{k+1} \|\partial_t U(t)\|_{k+1} \leq \overline{C}_4 h^{k+1} \|\partial_t U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))} \quad \forall s \in [0,2]. \tag{5.13}$$

By differentiating (5.8), (5.9) twice with respect t, we see that $(\partial_{tt} U_h(t), \partial_{tt} P_h(t))$ also satisfies a system of equations similar to (5.8), (5.9) so that under the assumption

$$\partial_{tt} U \in C\left([0,\infty); \mathbf{H}^{k+1}(\Omega)\right) \cap L^\infty\left(0,\infty; \mathbf{H}^{k+1}(\Omega)\right) \tag{A8}$$

we have the error estimates

(5.14)
$$\|\partial_{tt}U_h(t)-\partial_{tt}U(t)\|_1+\|\partial_{tt}P_h(t)\|\le \overline{C}_3h^k\,\|\partial_{tt}U(t)\|_1\le \overline{C}_3h^k\,\|\partial_{tt}U\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))}$$

and

(5.15)
$$\|\partial_{tt}U_h(t)-\partial_{tt}U(t)\|\le \overline{C}_4h^s\,\|\partial_{tt}U(t)\|_s\le \overline{C}_4h^s\,\|\partial_{tt}U\|_{L^\infty(0,\infty;\mathbf{H}^s(\Omega))}\quad \forall s\in[0,1]$$

in particular,

$$\|\partial_{tt}U_h(t)-\partial_{tt}U(t)\|\le \overline{C}_4\,\|\partial_{tt}U(t)\|_s\le \overline{C}_4\,\|\partial_{tt}U\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))}\,. \tag{5.16}$$

Note that the regularity assumption (A8) for $\partial_{tt}U$ is weaker than the assumption (A6) for U or (A7) for ∂_tU. The proof of the following Lemma is same as [8].

Lemma 5.1. *Assume that hypotheses (A1), (A2), (A4), (A6), (A7) and (A8) hold. Assume that further*

$$\|U\|_{L^\infty(0,\infty;\mathbf{L}^4(\Omega))}<\tfrac{\bar{\epsilon}}{\overline{C}_0} \tag{A9}$$

For each integer $n\ge 0$, let $(\widehat{u}_h^{n+1},\widehat{p}_h^{n+1},\widehat{f}_h^{n+1})$ be a solution the $(n+1)$th fully discrete optimal control problem. Then there exists an $h_0>0$ and constants $\overline{K}_1$, $\overline{K}_2$ and $\overline{K}_3$ such that for all $h\le h_0$ and all n,

(5.17)
$$\mathcal{L}_h^{n+1}(\widehat{u}_h^{n+1},\widehat{f}_h^{n+1})\le \alpha\left(\frac{\|\widehat{u}_h^{n+1}-U_h^{n+1}\|^2}{1+\lambda_1\overline{K}_1\Delta t}+\frac{\overline{K}_2h^{2k+2}\Delta t}{1+\lambda_1\overline{K}_1\Delta t}+\frac{\overline{K}_3(\Delta t)^3}{1+\lambda_1\overline{K}_1\Delta t}\right)+\alpha\overline{C}_4^2h^{2k+2}\|U\|^2_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}$$

where

$$h_0\overset{def}{=}\min\left\{\left(\frac{\bar{\varepsilon}-\overline{C}_0\,\|U\|_{L^\infty(0,\infty;\mathbf{L}^4(\Omega))}}{\overline{C}_1\overline{C}_3\,\|U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}}\right)^{1/k},1\right\} \tag{5.18}$$

$$\overline{K}_1\overset{def}{=}\frac{1}{2}\left(\bar{\varepsilon}-\overline{C}_1\overline{C}_3h^k\,\|U\|_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}-\overline{C}_0\,\|U\|_{L^\infty(0,\infty;\mathbf{L}^4(\Omega))}\right) \tag{5.19}$$

(5.20)
$$\overline{K}_2\overset{def}{=}\frac{4}{\overline{K}_1}\Big(4\overline{C}_2^2\overline{C}_4^2h^{2k+2}\left\|U^{n+1}\right\|_2^2\|U\|^2_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}+\overline{C}_1^2\overline{C}_3^4h^{4k}\|U\|^4_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}+\frac{\overline{C}_4^2h^{2k}}{\lambda_1}\|\partial_tU\|^2_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))}\Big)$$

$$\overline{K}_3\overset{def}{=}\frac{2}{\lambda_1}(\overline{C}_4^2+1)\,\|\partial_{tt}U\|^2_{L^\infty(0,\infty;\mathbf{H}^{k+1}(\Omega))} \tag{5.21}$$

with the constants $\overline{C}_0,\overline{C}_1,\overline{C}_2,\overline{C}_3$ and $\overline{C}_4$ defined by (5.3), (5.4), (5.5), (5.10) *and* (5.11)*, respectively .*

Theorem 5.2. *Assume that the hypotheses of Lemma 5.1 hold. Assume further that* $u_0 \in \boldsymbol{H}^{k+1}(\Omega)$ *and*

$$\text{(A10)} \qquad \frac{\alpha}{\beta} < \frac{(\lambda_1 \overline{K}_1)^2}{8}.$$

where $\overline{K}_1$ *is defined by* (5.7)*. Let* h_0 *be defined by* (5.18)*. Then there are positive constants* δ_0, $\overline{K}_4$, $\overline{K}_5$, $\overline{K}_6$ *and* $\overline{\kappa}$ *such that for all* $h \leq h_0$ *and all* $\Delta t \leq \delta_0$,

$$\text{(5.22)} \qquad \|\widehat{u}_h^{n+1} - U_h^{n+1}\|^2 \leq (1 - \overline{K}_4 \Delta t)\|\widehat{u}_h^n - U_h^n\|^2 \overline{K}_5 (\Delta t)^3 + \overline{K}_6 h^{2k+2}(\Delta t)$$

and

$$\text{(5.23)} \qquad \|\widehat{u}_h^n - U_h^n\|^2 \leq 3e^{-\overline{K}_4 t_n}\|\widehat{u}_0 - U^0\|^2 + \overline{\kappa}[(\Delta t)^2 + h^{2k+2}].$$

As a consequence of Theorem 5.2 and the triangle inequality

$$\|\widehat{u}(t_n) - \widehat{u}_h^n\|^2 \leq 2\,\|\widehat{u}(t_n) - U^n\|^2 + 2\,\|U^n - \widehat{u}_h^n\|^2$$

we obtain an estimate for the difference between the continuous and fully discrete solutions of the piecewise optimal control problem.

Remark 5.3. In order to solve the $(n+1)$th fully discrete optimal control problem for each n, we need to introduce a Lagrange multiplier $(\widehat{\xi}_h^{n+1}, \widehat{\pi}_h^{n+1})$ to convert the $(n+1)$th fully discrete optimal control problem into a discrete optimality system of equations (similar to the semi discrete case).

6. **Computational example**

Thanks to GNU licence, we have implemented the following algorithm.
(a) initialization:

- Chose a (sufficiently small) $\delta > 0$ and set $\Delta t = \delta$. Choose h (sufficiently small).
- Define $u_h^0 = U_h^0$ where U_h^0 is the $\mathbf{L}^2(\Omega)$ projection of U^0 on to $\mathbf{X}_h$.

(b) solving the $(n+1)th$ fully discrete optimal control problem:
For $n = 0, 1, 2, ...$, find a $(u_h^{n+1}, p_h^{n+1}, \xi_h^{n+1}, \pi_h^{n+1}) \in \mathbf{X}_h \times \mathbf{S}_h \times \mathbf{X}_h \times \mathbf{S}_h$ such that

$$\text{(6.1)} \qquad \begin{aligned} &\frac{1}{\Delta t}\left\langle u_h^{n+1}, \varphi_h \right\rangle + a_r\left(u_h^{n+1}, \varphi_h\right) + \overline{\mathbf{c}}\left(u_h^{n+1}, u_h^{n+1}, \varphi_h\right) + k\left(\varphi_h, p_h^{n+1}\right) \\ &\qquad = \left\langle F_h^{n+1} - \beta^{-1}\xi_h^{n+1}, \varphi_h \right\rangle + \frac{1}{\Delta t}\left\langle u_h^n, \varphi_h \right\rangle, \qquad \forall \varphi_h \in \mathbf{X}_h, \end{aligned}$$

$$\text{(6.2)} \qquad k\left(u_h^{n+1}, q_h\right) = 0, \qquad \forall q_h \in S_h,$$

$$\text{(6.3)} \qquad \begin{aligned} &\frac{1}{\Delta t}\left\langle \xi_h^{n+1}, \varphi_h \right\rangle + a_r\left(\xi_h^{n+1}, \varphi_h\right) + k\left(\varphi_h, \pi_h^{n+1}\right) + \overline{\mathbf{c}}\left(\varphi_h, u_h^{n+1}, \xi_h^{n+1}\right) \\ &\qquad + \overline{\mathbf{c}}\left(u_h^{n+1}, \varphi_h, \xi_h^{n+1}\right) = \alpha\left\langle u_h^{n+1} - U^{n+1}, \varphi_h \right\rangle, \quad \forall \varphi_h \in \mathbf{X}_h, \end{aligned}$$

$$k\left(\xi_h^{n+1}, r_h\right) = 0, \qquad \forall r_h \in S_h, \tag{6.4}$$

(c) Set

$$f_h^{n+1} = F_h^{n+1} - \beta^{-1}\xi_h^{n+1}$$

We use a gradient method to implement this algorithm. The finite elements are chosen to be the Taylor-Hood elements; i.e., the finite element space $\mathbf{V}_h$ is chosen to be piecewise biquadratic elements (for u_h and ξ_h) and S_h is chosen to be piecewise linear elements (for p_h and π_h). Newton s method is used to solve the finite-dimensional nonlinear system of equations. We choose the domain $\Omega = (0,1) \times (0,1)$. The desired velocity is given by $U(x,t) = (U_1(x,y), U_2(x,y))$ where

$$U_1 = \tfrac{d}{dy}\phi(t,x)\phi(t,y) \qquad U_2 = -\tfrac{d}{dx}\phi(t,x)\phi(t,y)$$

with

$$\phi(t,z) = (1-z)^2(1-\cos(2k\pi z t)), \qquad z \in [0,1].$$

The integer parameter k involved in U adjusts the number of eddies of circulation presented in the desired flow, thus determines the complexity of the desired flow. We choose the kinematic viscosity $\nu = 1/Re = 0.01$, the time step $\Delta t = 0.1$, $h = 1/16$, $\alpha = 10$ and $\beta = 0.1$

For the initial velocity we choose

$$U_1^0 = (\cos(2\pi x) - 1)\sin(2\pi y) \qquad \text{and} \qquad U_2^0 = \sin(2\pi x)(1 - \cos(2\pi y))$$

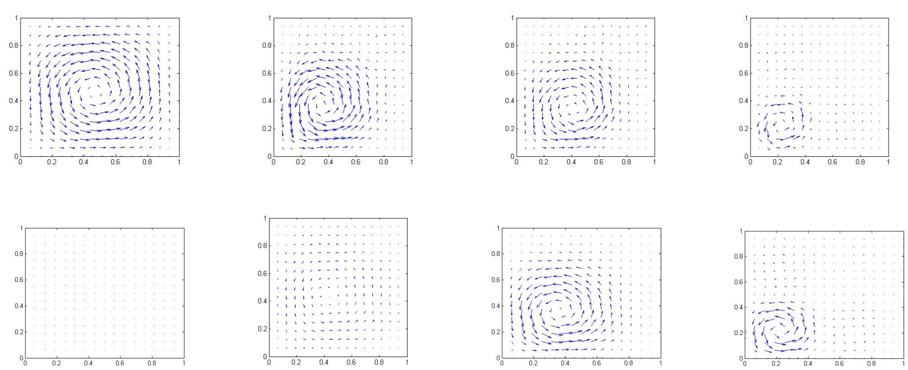

Fig. 1: *Controlled (first row) and target (second row) at* $t = 0.0$, $t = 0.15$, $t = 0.5$ *and* $t = 1$.

In our numerical computations, we observed that the graphics for the decreasing of the error $\|u - U\|$ does'nt change enough when we pass from the case $r = 1$ to the case $r = 2$.

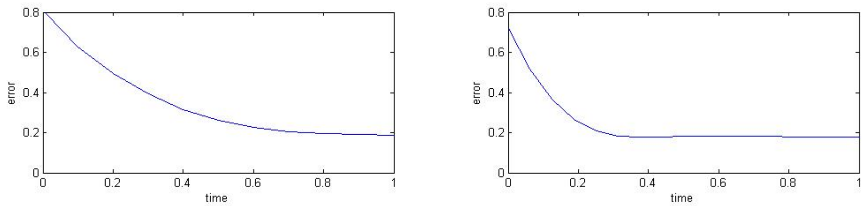

Fig. 2: *The error graphics for $\beta = 0.1$ and $\beta = 0.001$, respectively.*

More over the quickness of the decreasing of the error $\|u - U\|$ between the controlled flow u and the target flow U depends on β. Indeed the more β becomes small, more the decreasing is rapid.

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Some Applications on Generating Functions

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In this paper, we calculate the generating functions by using the concepts of symmetric functions. Although the methods cited in previous works are in principle constructive, we are concerned here only with the question of manipulating combinatorial objects, known as symmetric operators. The proposed generalized symmetric functions can be used to find explicit formulas of the Fibonacci numbers, and of the Tchebychev polynomials of first and second kinds. Moreover, we give new results for the product of Hadamard.

1 Introduction

By studying the Fibonacci sequence ($F_{n+2} = F_{n+1} + F_n$ with $F_0 = F_1 = 1$), we note its close connection with the equation $x^2 = x + 1$, whose roots are the golden numbers Φ_1 and Φ_2. It is also noticed that the eigenvalues of the symmetric matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

represent the two golden numbers Φ_1 and Φ_2 of Fibonacci sequence [3]. Consequently, we obtain the following Vieta's formulas

$$\sigma_1 = \lambda_1 + \lambda_2 = 1 \text{ and } \sigma_2 = \lambda_1\lambda_2 = -1 \tag{2}$$

where σ_1, σ_2 are called elementary symmetric functions of real roots λ_1, λ_2, respectively. So, the eigenvectors of matrix M are multiples of

$$\overrightarrow{v_1} = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{v_2} = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \tag{3}$$

If we assume that $|\lambda_1| > |\lambda_2|$, then for any positive integer n, we have [3]

$$M^n = \begin{pmatrix} S_n(\lambda_1 + \lambda_2) & \sigma_1 S_{n-1}(\lambda_1 + \lambda_2) \\ S_{n-1}(\lambda_1 + \lambda_2) & \sigma_2 S_{n-2}(\lambda_1 + \lambda_2) \end{pmatrix} \tag{4}$$

where $S_n(\lambda_1 + \lambda_2) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$.

In this paper, we are interested in the use of symmetric functions to generate the well-knwon Fibonacci numbers and Tchebychev polynomiales of first and

1

second kinds. In this framework, some necessary preliminaries and definitions are given in Section 2. In Section 3, we propose a new theorem which allows the determination of the generating functions. The proposed theorem is based on symmetric functions and a new proposition on the symmetric operators. In Section 4, some applications are given for the generating functions of Fibonacci numbers and Tchebychev polynomials. The products of Hadamard are given in Section 5.

2 Preliminaries

2.1 Definition of symmetric functions in several variables

Consider an equation of degree n of the form

$$(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)=0 \tag{5}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ being real roots. If we expand the left hand side, we obtain

$$x^n-\sigma_1 x^{n-1}+\sigma_2 x^{n-2}-\sigma_3 x^{n-3}+\cdots+(-1)^n\sigma_n=0 \tag{6}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are homogeneous and symmetrical polynomials in $\lambda_1, \lambda_2, \ldots, \lambda_n$. To be more accurate, these polynomials can be denoted as $\sigma_i(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $i=1,2,\ldots,n$, or simply as $\sigma_i^{(n)}$.

The general formula of the polynomials $\sigma_i^{(n)}$ are given by [9]

$$\sigma_i^{(n)}=\sum_{m_1+m_2+\cdots+m_n=i}\lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_n^{m_n} \tag{7}$$

with $m_1, m_2, ..., m_n = 0$ or 1.

The polynomials $\sigma_i^{(n)}$ can be considered as the sum of all distinct products that can be formed by monomial polynomials C_n^i. It is noticed that $\sigma_i^{(n)}=0$ for $i>n$.

2.2 Symmetric functions

Let A and B be two alphabets, we denote by $S_n(A-B)$ the coefficients of the rational sequence of poles A and zeros B as follows [2]

$$\sum_{n=0}^{\infty}S_n(A-B)z^n=\frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)} \tag{8}$$

Equation (8) can be rewritten in the following form

$$\sum_{n=0}^{\infty}S_n(A-B)z^n=\left(\sum_{n=0}^{\infty}S_n(A)z^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-B)z^n\right) \tag{9}$$

with

$$S_n(A-B)=\sum_{j=0}^{n} S_{n-j}(-B)S_j(A) \tag{10}$$

The polynomial whose roots are B is written as

$$S_n(x-B)=\sum_{j=0}^{n} S_{n-j}(-B)z^n,\ \text{with card}(B)=n \tag{11}$$

On the other hand, if A has cardinality equal to 1, i.e., $A=\{x\}$, then equality (8) can be rewritten as follows [1]

$$\sum_{n=0}^{\infty} S_n(x-B)z^n=\frac{\prod_{b\in B}(1-bz)}{(1-xz)}=1+\cdots+S_{n-1}(x-B)z^{n-1}+\frac{S_n(x-B)}{(1-xz)}z^n \tag{12}$$

where $S_{n+k}(x-B)=x^k S_n(x-B)$ for all $k\geq 0$.

The summation is actually limited to a finite number of terms since $S_{-k}(\cdot)=0$ for all $k>0$. In particular, we have

$$\prod_{b\in B}(x-b)=S_n(x-B)=S_0(-B)x^n+S_1(-B)x^{n\ -1}+S_2(-B)x^{n\ -2}+\cdots \tag{13}$$

where $S_k(-B)$ are the coefficients of the polynomials $S_n(x-B)$ for $0\leq k\leq n$. This coefficients are zero for $k>n$.

For example, if all $b\in B$ are equal, i.e., $B=nb$, then we have

$$S_n(x-nb)=(x-b)^n \tag{14}$$

By choosing $b=1$, i.e., $B=\left\{\underbrace{1,1,...1}_{n}\right\}$, we obtain

$$S_k(-n)=(-1)^k\binom{n}{k}\ \text{and}\ S_k(n)=\binom{n+k-1}{k} \tag{15}$$

By combining (10) and (15), we obtain the following expression

$$S_n(A-nx)=S_n(A)-\binom{n}{1}S_{n-1}(A)x+\binom{n}{2}S_{n-2}(A)x^2-\cdots+(-1)^n\binom{n}{n}x^n \tag{16}$$

For any pair (x,y) we can associate the divided difference ∂_{xy} defined by [8]

$$\partial xy(f)=\frac{f(x,y,z,\ldots)-f(y,x,z,\ldots)}{x-y} \tag{17}$$

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3 The major formulas

In this section, we provide some definitions and a new propostion which will be useful for the next theorem.

Definition 1 *The inverse of the sequence* $\sum_{n=0}^{\infty} S_n(A)z^n$ *is the sequence* $\sum_{n=0}^{\infty} S_n(-A)z^n$, *that is*

$$\sum_{n=0}^{\infty} S_n(A)z^n = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)z^n} \tag{18}$$

Definition 2 *The symmetric operator* π_{xy}^n *is defined by [7]*

$$\pi_{xy}^n f(x) = \frac{x^n f(x) - y^n f(y)}{x-y} \tag{19}$$

Proposition 1 *Given an alphabet* $E_2 = \{e_1, e_2\}$, *then for any positive integer* k, *the operator* $\pi_{e_1e_2}^k$ *satisfied the following formula*

$$\pi_{e_1e_2}^k f(e_1) = f(e_1)S_{k-1}(e_1+e_2) + e_2^k \partial_{e_1e_2}(f) \tag{20}$$

Proof. From (19) we have

$$\begin{aligned}
\pi_{e_1e_2}^k f(e_1) &= \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \\
&= \frac{e_1^k f(e_1) - e_2^k f(e_1) + e_2^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \\
&= \frac{f(e_1)\left[e_1^k - e_2^k\right] + e_2^k \left[f\ (e_1) - f(e_2)\right]}{e_1 - e_2}
\end{aligned}$$

Using the formulas (4) and (17) we obtain

$$\pi_{e_1e_2}^k f(e_1) = f(e_1)S_{k-1}(e_1+e_2) + e_2^k \partial_{e_1e_2}(f)$$

This completes the proof of proposition 1. ■

Theorem 2 *Given two alphabets* $E_2 = \{e_1, e_2\}$ *and* $A = \{a_1, a_2, ...\}$, *then*

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_n(A)S_{k+n-1}(e_1+e_2)z^n \\
&= \frac{\sum\limits_{n=0}^{k-1} S_n(-A)e_1^n e_2^n S_{k-n-1}(e_1+e_2)z^n - e_1^k e_2^k z^{k+1} \sum\limits_{n=0}^{\infty} S_{n+k+1}(-A)S_n(e_1+e_2)z^n}{\left(\sum\limits_{n\ =0}^{\infty} S_n(A)e_1^n z^n\right)\left(\sum\limits_{n\ =0}^{\infty} S_n(A)e_1^n z^n\right)}
\end{aligned} \tag{21}$$

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Proof. Let $f(e_1) = \sum_{n=0}^{\infty} e_1^n S_n(A) z^n$, then the left hand side of formula (21) can be written as

$$\begin{aligned}
\pi_{e_1e_2} f(e_1) &= \pi_{e_1e_2}\left(\sum_{n\ =0}^{\infty} S_n(A) e_1^n z^n\right) \\
&= \frac{e_1^k \sum\limits_{n=0}^{\infty} S_n(A) e_1^n z^n - e_2^k \sum\limits_{n=0}^{\infty} S_n(A_2) e_1^n z^n}{e_1 - e_2} \\
&= \sum_{n=0}^{\infty} S_n(A)\left(\frac{e_1^{n+k} - e_2^{n+k}}{e_1 - e_2}\right) z^n \\
&= \sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(e_1 + e_2) z^n
\end{aligned}$$

and the right hand side of this formula can be written as

$$\begin{aligned}
& S_{k-1}(e_1+e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f\ (e_1) \\
= \ & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n} + e_2^k \partial_{e_1 e_2} \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n} \\
= \ & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n} - \frac{\sum\limits_{n=0}^{\infty} S_n(-A) S_{n-1}(e_1+e_2) z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)} \\
= \ & \frac{\sum\limits_{j=0}^{\infty} S_n(-A)\left[e_2^n S_{k-1}(e_1+e_2) - e_2^k S_{n-1}(e_1+e_2)\right] z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)} \\
= \ & \frac{\sum\limits_{j=0}^{k-1} S_n(-A)\left[e_2^n S_{k-1}(e_1+e_2) - e_2^k S_{n-1}(e_1+e_2)\right] z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)} \\
& + \frac{\sum\limits_{j=k+1}^{\infty} S_n(-A)\left[e_2^n S_{k-1}(e_1+e_2) - e_2^k S_{n-1}(e_1+e_2)\right] z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)} \\
= \ & \frac{\sum\limits_{n=0}^{k-1} S_n(-A) e_1^n e_2^n S_{k-n-1}(e_1+e_2) z^n - e_1^k e_2^k z^{k+1} \sum\limits_{n=0}^{\infty} S_{n+k+1}(-A) S_n(e_1+e_2) z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^n z^n\right)}
\end{aligned}$$

This completes the proof.of Theorem 2. ■

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4 Applications to the generating functions

In this section, we attempt to give results for some well-known generating functions. In fact, we will use Theorem 2 to derive Fibonacci numbers and Tchebychev polynomials of second kind. Moreover, the generating functions for some special cases of Fibonacci numbers and Tchebychev polynomials are given.

Then Theorem 2 can be written

Corollary 3 *If $A_2 = \{a_1, a_2\}$ and $k = 1$ then*

$$\sum_{n=0}^{\infty} S_n(A_2) S_n(e_1 + e_2) z^n = \frac{1 - e_1 e_2 a_1 a_2 z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A_2) e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A_2) e_2^n z^n\right)} \tag{22}$$

Case 1: For $a_1 = 1$ and $a_2 = 0$, one can apply Corollary 3 to arrive at [3]

$$\sum_{n=0}^{\infty} S_n(e_1 + [-e_2]) z^n = \frac{1}{(1 - e_1 z)(1 - e_2 z)} \tag{23}$$

In (23) replace e_2 by $(-e_2)$, and choose e_1, e_2 such that: $e_1 - e_2 = 1,\ e_1 e_2 = 1$ to obtain

$$\sum_{n=0}^{\infty} S_n(e_1 + [-e_2]) z^n = \frac{1}{1 - z - z^2}, \text{ with } F_n = S_n(e_1 + [-e_2]) \tag{24}$$

where F_n are Fibonacci numbers.

Also, if we replace e_1 by $(2e_1)$, e_2 by $(-2e_2)$ with the condition $4e_1e_2 = -1$, then there follows that

$$\sum_{n=0}^{\infty} S_n(2e_1 + [-2e_2]) z^n = \frac{1}{1 - 2(e_1 - e_2)z + z^2}, \text{ with } U_n(e_1 - e_2) = S_n(2e_1 + [-2e_2]) \tag{25}$$

where U_n are the Tchebychev polynomials of second kind.

By using the previous formula (25), we can deduce that

$$\sum_{n=0}^{\infty} \left[S_n(2e_1 + [-2e_2]) - (e_1 - e_2) S_{n-1}(2e_1 + [-2e_2])\right] z^n = \frac{1 - (e_1 - e_2)z}{1 - 2(e_1 - e_2)z + z^2} \tag{26}$$

Then the Tchebychev polynomials of first kind can be derived directly as follows [3]

$$T_n(e_1 - e_2) = [S_n(2e_1 + [-2e_2]) - (e_1 - e_2) S_{n-1}(2e_1 + [-2e_2])] \tag{27}$$

Case 2: For $a_1 = 1$, $a_2 = x$, and $e_1 = 1$, $e_2 = y$, in an application of Corollary 3 yields the following result [6]

$$\sum_{n=0}^{\infty} [1 + x + \cdots + x^n] [1 + y + \cdots + y^n] z^n = \frac{1 - xyz^2}{[(1-z)(1-xz)(1-yz)(1-xyz)]} \tag{28}$$

Case 3: By replacing e_2 by $(-e_2)$ and a_2 by $(-a_2)$, we obtain

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{1 - e_1e_2a_1a_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)} \tag{29}$$

This case consists of three related parts.

Firstly, by making the following restrictions: $a_1 - a_2 = 1$, $a_1a_2 = 1$, and $e_1 - e_2 = 1$, $e_1e_2 = 1$ in (29) we gives

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4} = \sum_{n=0}^{\infty} F_n^2 z^n \tag{30}$$

This corresponds to the square of Fibonacci numbers [5] given by

$$F_n^2 = S_n(a_1 + [-a_2])S_n(e_1 + [-e_2]) \tag{31}$$

Secondly, by making the following restrictions:$e_1 - e_2 = 1$, $e_1e_2 = 1$, $a_1a_2 = -1$, and by replacing $(a_1 - a_2)$ by $2(a_1 - a_2)$ in (29), we get the identity of Foata [5], involving the product of Fibonacci numbers with Tchebychev polynomial of second kind as follows

$$\frac{1 + z^2}{1 - 2(a_1 - a_2)z + (3 - 4(a_1 - a_2)^2)z^2 + 2(a_1 - a_2)z^3 + z^4} = \sum_{n=0}^{\infty} F_n U_n(a_1 - a_2)z^n \tag{32}$$

In the last case, choose a_i and e_i such that $e_1e_2 = -1$, $a_1a_2 = -1$, and by replace $(a_1 - a_2)$ by $2(a_1 - a_2)$, and $(e_1 - e_2)$ by $2(e_1 - e_2)$ in (29), to obtain the identity of Foata [5], involving the square of Tchebychev polynomials of second kind given by

$$\sum_{n=0}^{\infty} U_n(e_1 - e_2)U_n(a_1 - a_2)z^n$$
$$= \frac{1 - z^2}{1 - 4(e_1 - e_2)(a_1 - a_2)z + (4(a_1 - a_2)^2 + 4(e_1 - e_2)^2 - 2)z^2 - 4(e_1 - e_2)(a_1 - a_2)z^3 + z^4} \tag{33}$$

Notice that, under the same restrictions and by using (25) and (27), and the fact that

$$S_{n-1}(2a_1 + [-2a_2]) = \frac{(2a_1)^n - (-2a_2)^n}{2a_1 + 2a_2} \tag{34}$$

we obtain the identity of Foata [4], involving the product of Tchebychev polynomials of second kind with Tchebychev polynomials of first kind:

$$\begin{aligned}&\sum_{n=0}^{\infty} U_n(e_1-e_2)T_n(a_1-a_2)z^n\\ =\ &\frac{1-2(e_1-e_2)(a_1-a_2)z+(2(a_1-a_2)^2-1)z^2}{1-4(e_1-e_2)(a_1-a_2)Z+(4(a_1-a_2)^2+4(e_1-e_2)^2-2)z^2-4(e_1-e_2)(a_1-a_2)z^3+z^4}\end{aligned} \tag{35}$$

and also the identity of Foata [5], involving the square of Tchebychev polynomials of first kind:

$$\begin{aligned}&\sum_{n=0}^{\infty} T_n(e_1-e_2)T_n(a_1-a_2)z^n\\ =\ &\frac{1-3(e_1-e_2)(a_1-a_2)z+(2(a_1-a_2)^2+2(e_1-e_2)^2-1)z^2-(e_1-e_2)(a_1-a_2)z^3}{1-4(e_1-e_2)(a_1-a_2)Z+(4(a_1-a_2)^2+4(e_1-e_2)^2-2)z^2-4(e_1-e_2)(a_1-a_2)z^3+z^4}\end{aligned} \tag{36}$$

5 The product of Hadamard

In this section, we show the efficiency of the proposed method by determining the product of Hadamard. In fact, by taking $A = \Phi$ in (8), we obtain

$$\sum_{n=0}^{\infty} S_n(-B)z^n = \prod_{b\in B}(1-bz) \tag{37}$$

For the special case where $a_1 = a_2 = 1$ in (37), we have

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2} \tag{38}$$

By replacing z by $e_1 z$ in (38), we get

$$\sum_{n=0}^{\infty}(n+1)e_1^n z^n = \frac{1}{(1-e_1 z)^2} \tag{39}$$

Use Corollary 3 with the action of the operator $\pi_{e_1 e_2}$ on both sides of the identity (39) to obtain

$$\sum_{n=0}^{\infty}(n+1)S_n(e_1+e_2)z^n = \frac{1-e_1e_2z^2}{(1-e_1z)^2(1-e_2z)^2} \tag{40}$$

By taking $e_1 = 1$ and $e_2 = 1$, we have

$$\sum_{n=0}^{\infty}(n+1)^2 z^n = \frac{1+z}{(1-z)^3}. \tag{41}$$

On the other hand, using formula (22) with the action of the operator $\pi_{e_1e_2}$ on both sides of (41), and by replacing z by e_1z leads to

$$\sum_{n=0}^{\infty}(n+1)^2S_n(e_1+e_2)z^n=\pi_{e_1e_2}\frac{1}{(1-e_1z)^3}+z\pi_{e_1e_2}\frac{e_1}{(1-e_1z)^3} \tag{42}$$

Using formulas (15), (19) and (21),it follows that

$$\pi_{e_1e_2}\frac{1}{(1-e_1z)^3}=\frac{1-e_1e_2z^2\sum\limits_{n=0}^{1}(-1)^{n+2}\binom{3}{n+2}S_n(e_1+e_2)z^n}{(1-e_1z)^3(1-e_2z)^3} \tag{43}$$

$$\pi_{e_1e_2}\frac{e_1}{(1-e_1z)^3}=\frac{\left[\sum\limits_{n=0}^{1}(-1)^n\binom{3}{n}e_1^ne_2^nS_{1-n}(E_2)z^n-e_1^2e_2^2z^3\sum\limits_{n=0}^{1}(-1)^{n+3}\binom{3}{n+3}S_n(E_2)z^n\right]}{(1-e_1z)^3(1-e_2z)^3} \tag{44}$$

Notice that, for $e_1=1$ and $e_2=1$, we have

$$\sum_{n=0}^{\infty}(n+1)^3z^n=\frac{\left[\begin{array}{c}\left[1-z^2\sum\limits_{n=0}^{1}(-1)^{n+2}\binom{3}{n+2}\binom{n+1}{n}z^n\right]+\\ z\left[\sum\limits_{n=0}^{1}(-1)^n\binom{3}{n}\binom{2-n}{1-n}z^n-z^3\sum\limits_{n=0}^{0}(-1)^{n+3}\binom{3}{n+3}\binom{n+1}{n}z^n\right]\end{array}\right]}{(1-z)^6} \tag{45}$$

which gives after simplification

$$\sum_{n=0}^{\infty}(n+1)^3z^n=\frac{1+4z+z^2}{(1-z)^4} \tag{46}$$

Using the same procedure, we deduce, for instance, the following identities

$$\sum_{n=0}^{\infty}(n+1)^4z^n=\frac{1+11z+11z^2+z^3}{(1-z)^5} \tag{47}$$

$$\sum_{n=0}^{\infty}(n+1)^5z^n=\frac{1+26z+66z^2+26z^3+z^4}{(1-z)^6} \tag{48}$$

$$\sum_{n=0}^{\infty}(n+1)^6z^n=\frac{1+57z+302z^2+302z^3+57z^4+z^5}{(1-z)^7} \tag{49}$$

$$\sum_{n=0}^{\infty}(n+1)^7z^n=\frac{1+120z+1191z^2+2416z^3+1191z^4+120z^{5+}z^6}{(1-z)^8} \tag{50}$$

$$\sum_{j=0}^{\infty}(n+1)^8z^j=\frac{1+247z+4293z^2+15619z^3+15619z^4+4293z^5+247z^6+z^7}{(1-z)^9} \tag{51}$$

$$\sum_{n=0}^{\infty}(n+1)^9 z^j = \frac{1+502z+14608z^2+88234z^3+156190z^4+88234z^5+14608z^6+502z^7+z^8}{(1-z)^{10}} \tag{52}$$

$$\sum_{n=0}^{\infty}(n+1)^{10} z^j = \frac{1+1013z+47840z^2+455192z^3+1310354z^4+1310354z^5+455192z^6}{(1-z)^{11}} + \frac{47840z^7+1013z^8+z^9}{(1-z)^{11}} \tag{53}$$

6 Conclusion

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric fonctions. The obtained results agree with the results obtained in some previous works.

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New Expansions for Two Trigonometric Functions

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Abstract

We introduce a new type expansions for the functions $\sin(\pi x)$ and $\cot(\pi x)$, $0 < x < 1$. In particular, the $\sin(\pi x)$ is expressed as an infinite product (different from the Euler's product for the sine function), while the $\cot(\pi x)$ is expressed as an infinite series of terms involving the logarithmic function. The resulting formulas lead to some product expansions for e^{γ}, φ (the golden ratio), as well as $e^{\lambda\pi}$, where λ takes some specific real, algebraic values.

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1 Introduction

In a recent paper [4] a product type expansion for the Gamma function $\Gamma(x)$ was obtained:

$$\Gamma(x) = \sqrt{2e\pi}e^{-x} \prod_{k=0}^{\infty} \left[\prod_{j=0}^{k} (x+j)^{(x+j)\binom{k}{j}(-1)^j} \right]^{\frac{1}{k+1}}, \qquad x > 0. \tag{1.1}$$

In the same paper [4] it was shown that the Psi (or Digamma) function,

$$\Psi(x) := \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{1.2}$$

admits the following representation

$$\Psi(x) = \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \ln(x+j), \qquad x > 0. \tag{1.3}$$

The expression (1.3) has been also derived by J. Guillera and J. Sondow (see [3]), with the help of the so-called Lerch transcendent. In [4], expressions (1.1) and (1.3) are derived by a fundamentally different approach, that is they result as a solution of an appropriate difference equation. Expression (1.1) for the $\Gamma(x)$, $x > 0$, is obtained as a solution of the difference equation

$$\ln \Gamma(x+1) - \ln \Gamma(x) = \ln x, \qquad x > 0, \tag{1.4}$$

while expression (1.3) for the $\Psi(x)$, $x > 0$, is obtained as a solution of the difference equation

$$\Psi(x+1) - \Psi(x) = \frac{1}{x}, \qquad x > 0 \tag{1.5}$$

2 An expansion for the function $\sin(\pi x)$, $0 < x < 1$

Making use of the well known reflection formula for the Gamma function (see [1], Th. 2.12)

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \qquad 0 < x < 1, \tag{2.1}$$

and taking into consideration (1.1), the following product type expansion for $\sin(\pi x)$ is obtained

$$\sin(\pi x) = \frac{1}{2}\prod_{k=0}^{\infty}\left[\prod_{j=0}^{k}\left\{(x+j)^{(x+j)}(1-x+j)^{(1-x+j)}\right\}^{\binom{k}{j}(-1)^{j+1}}\right]^{\frac{1}{k+1}}, \tag{2.2}$$

i.e.
$\sin(\pi x) =$

$$\frac{1}{2}\cdot\frac{1}{x^x(1-x)^{1-x}}\cdot\left[\frac{(x+1)^{x+1}(2-x)^{2-x}}{x^x(1-x)^{1-x}}\right]^{\frac{1}{2}}\cdot\left[\frac{(x+1)^{2(x+1)}(2-x)^{2(2-x)}}{x^x(x+2)^{x+2}(1-x)^{1-x}(3-x)^{3-x}}\right]^{\frac{1}{3}}\cdot$$

$$\left[\frac{(x+1)^{3(x+1)}(x+3)^{x+3}(2-x)^{3(2-x)}(4-x)^{4-x}}{x^x(x+2)^{3(x+2)}(1-x)^{1-x}(3-x)^{3(3-x)}}\right]^{\frac{1}{4}}\cdots. \tag{2.3}$$

This product formula for $\sin(\pi x)$, $0 < x < 1$, which expresses $\sin(\pi x)$ in terms of x alone, is very different from the well known Euler's product expansion of the sine function and, as far we know, is new.

3 An expansion for the function $\cot(\pi x)$, $0 < x < 1$

With the aid of the reflection formula for the Psi function (see [2]) we have

$$\Psi(1-x) - \Psi(x) = \pi \cot(\pi x) \tag{3.1}$$

and using (3.1), the following expression for the $\cot(\pi x)$ is obtained:

$$\cot(\pi x) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{k+1} \left[\ln \prod_{j=0}^{k} \left\{ \frac{1-x-j}{x+j} \right\}^{\binom{k}{j}(-1)^j} \right], \tag{3.2}$$

i.e.

$$\pi \cot(\pi x) = \ln\left(\frac{1-x}{x}\right) + \frac{1}{2}\ln\left(\frac{(1-x)(1+x)}{(2-x)x}\right) + \frac{1}{3}\ln\left(\frac{(1-x)(3-x)(1+x)^2}{(2-x)^2 x(2+x)}\right) +$$

$$\frac{1}{4}\ln\left(\frac{(1-x)(3-x)^3(1+x)^3(3+x)}{(2-x)^3(4-x)x(2+x)^3}\right) + \dots . \tag{3.3}$$

In the next paragraph we show some rather interesting applications of the expansions, just derived.

4 Applications

1. Setting $x = 1$ in (1.3) and recalling that $\Psi(1) = -\gamma$ (see [2]), where γ is the Euler's constant, an expression for e^{γ} is obtained, i.e.

$$e^{\gamma} = \left(\frac{2}{1}\right)^{1/2} \left(\frac{2^2}{1\cdot 3}\right)^{1/3} \left(\frac{2^3\cdot 4}{1\cdot 3^3}\right)^{1/4} \left(\frac{2^4\cdot 4^4}{1\cdot 3^6\cdot 5}\right)^{1/5} \cdots . \tag{4.1}$$

This expression was first derived by J. Ser [5] and subsequently rederived by J. Sondow.

2. Let φ be the golden ratio, namely $\varphi = \frac{1+\sqrt{5}}{2} = \frac{1}{2}\csc\left(\frac{\pi}{10}\right)$. Applying (2.2)–(2.3) at $x = \frac{1}{10}$, the following product for e^{φ} is obtained:

$$\varphi = \left(1^1\cdot 9^9\right)^{1/10} \left(\frac{1^1\cdot 9^9}{11^{11}\cdot 19^{19}}\right)^{1/20} \left(\frac{1^1\cdot 9^9\cdot 21^{21}\cdot 29^{29}}{11^{22}\cdot 19^{38}}\right)^{1/30} \cdot$$

$$\left(\frac{1^9\cdot 9^9\cdot 21^{63}\cdot 29^{87}}{11^{33}\cdot 19^{57}\cdot 31^{31}\cdot 39^{39}}\right)^{1/40} \cdots . \tag{4.2}$$

Knowing that φ can also be expressed as $\varphi = 2\sin\left(\frac{3\pi}{10}\right)$, another product expression can be obtained if we set $x = \frac{3}{10}$ in (2.2)–(2.3):

$$\varphi = \left(\frac{1}{3^3 \cdot 7^7}\right)^{1/10} \left(\frac{13^{13} \cdot 17^{17}}{3^3 \cdot 7^7}\right)^{1/20} \left(\frac{13^{26} \cdot 17^{34}}{3^3 \cdot 7^7 \cdot 23^{23} \cdot 27^{27}}\right)^{1/30} \cdot \left(\frac{13^{39} \cdot 17^{51} \cdot 33^{33} \cdot 37^{37}}{3^3 \cdot 7^7 \cdot 23^{69} \cdot 27^{81}}\right)^{1/40} \cdots . \tag{4.3}$$

3. It may be of interest to notice that (3.2) can be used to find fancy product expansions of numbers of the form $e^{\lambda\pi}$, where λ is a real algebraic number of a certain kind. We present some examples.
(i) By setting $x = \frac{1}{4}$ in (3.2)–(3.3) and recalling that $\cot(\frac{\pi}{4}) = 1$, one easily obtains an expression for e^{π}:

$$e^{\pi} = \left(\frac{3}{1}\right)^{1/1} \left(\frac{3 \cdot 5}{1 \cdot 7}\right)^{1/2} \left(\frac{3 \cdot 5^2 \cdot 11}{1 \cdot 7^2 \cdot 9}\right)^{1/3} \left(\frac{3 \cdot 5^3 \cdot 11^3 \cdot 13}{1 \cdot 7^3 \cdot 9^3 \cdot 15}\right)^{1/4} \cdots . \tag{4.4}$$

This product expansion for e^{π} has also been derived by J. Guillera and J. Sondow in [3].
(ii) By setting $x = \frac{1}{3}$ in (3.2)–(3.3) one obtains

$$e^{\frac{\pi}{\sqrt{3}}} = \left(\frac{2}{1}\right)^{1/1} \left(\frac{2 \cdot 4}{1 \cdot 5}\right)^{1/2} \left(\frac{2 \cdot 4^2 \cdot 8}{1 \cdot 5^2 \cdot 7}\right)^{1/3} \left(\frac{2 \cdot 4^3 \cdot 8^3 \cdot 10}{1 \cdot 5^3 \cdot 7^3 \cdot 11}\right)^{1/4} \cdots , \tag{4.5}$$

while for $x = \frac{1}{6}$ we obtain

$$e^{\pi\sqrt{3}} = \left(\frac{5}{1}\right)^{1/1} \left(\frac{5 \cdot 7}{1 \cdot 11}\right)^{1/2} \left(\frac{5 \cdot 7^2 \cdot 11}{1 \cdot 11^2 \cdot 13}\right)^{1/3} \left(\frac{5 \cdot 7^3 \cdot 17^3 \cdot 19}{1 \cdot 11^3 \cdot 13^3 \cdot 22}\right)^{1/4} \cdots . \tag{4.6}$$

(iii) The formula $\varphi = \frac{1}{2}\csc\left(\frac{\pi}{10}\right)$ also implies $\cot\left(\frac{\pi}{10}\right) = \sqrt{4\varphi^2 - 1} = \sqrt{4\varphi + 3}$ (since $\varphi^2 = \varphi + 1$). Making use of (3.2)–(3.3), at $x = \frac{1}{10}$, we obtain the following expression, which involves e, π, and φ:

$$e^{\pi\sqrt{4\varphi+3}} = \left(\frac{9}{1}\right)^{1/1} \left(\frac{9 \cdot 11}{1 \cdot 19}\right)^{1/2} \left(\frac{9 \cdot 11^2 \cdot 29}{1 \cdot 19^2 \cdot 21}\right)^{1/3} \left(\frac{9 \cdot 11^3 \cdot 29^3 \cdot 31}{1 \cdot 19^3 \cdot 21^3 \cdot 39}\right)^{1/4} \cdots . \tag{4.7}$$

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