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Contribution to a Book

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An umbral calculus approach to poly-Cauchy polynomials with a q parameter

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Abstract

In this paper, we investigate the properties of the poly-Cauchy polynomials with a q parameter which were studied by the third named author, and give various identities with Bernoulli polynomials, Korobov polynomials, Stirling numbers, Frobenius-Euler polynomials, falling and rising factorials by an umbral calculus approach.

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1 Introduction

Let n, k be integers with $n \ge 0$, and let q be a real number with $q \ne 0$. The *poly-Cauchy numbers with a q parameter* of the first kind $c_{n,q}^{(k)}$ and of the second kind $\widehat{c}_{n,q}^{(k)}$ are defined by

$$c_{n,q}^{(k)}(z) = \underbrace{\int_0^1 \dots \int_0^1}_{k} (x_1x_2\dots x_k - z)(x_1x_2\dots x_k - q - z)$$

$$\dots (x_1x_2\dots x_k - (n-1)q - z)dx_1dx_2\dots dx_k$$

and

$$\widehat{c}_{n,q}^{(k)}(z) = \underbrace{\int_0^1 \dots \int_0^1}_{k} (-x_1x_2\dots x_k + z)(-x_1x_2\dots x_k - q + z)$$

$$\dots (-x_1x_2\dots x_k - (n-1)q + z)dx_1dx_2\dots dx_k\,,$$

respectively ([9]). The generating function of the poly-Cauchy polynomials with a q parameter of the first kind $c_{n,q}^{(k)}(z)$ and of the second kind $\hat{c}_{n,q}^{(k)}(z)$ are given by

$$(1+qt)^{-z/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) = \sum_{n=0}^\infty c_{n,q}^{(k)}(z)\frac{t^n}{n!}\,,$$

and

$$(1+qt)^{z/q}\mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right) = \sum_{n=0}^\infty \hat{c}_{n,q}^{(k)}(z)\frac{t^n}{n!}\,,$$

respectively ([9, Theorem 6]), where

$$\mathrm{Lif}_k(z) := \sum_{m=0}^\infty \frac{z^m}{m!(m+1)^k}$$

is the *polylogarithm factorial* function (or simply *polyfactorial* function), which is introduced in [8, 9]. If $q=1$, then $c_{n,1}^{(k)}(z) = c_n^{(k)}(z)$ and $\widehat{c}_{n,1}^{(k)}(z) = \widehat{c}_n^{(k)}(z)$ are the poly-Cauchy polynomials of the first kind and of the second kind, respectively ([2]). Notice that z is replaced by $-z$ in [2]. If $q=1$ and $z=0$, then $c_{n,1}^{(k)}(0) = c_n^{(k)}$ and $\widehat{c}_{n,1}^{(k)}(0) = \widehat{c}_n^{(k)}$ are the poly-Cauchy numbers of the first kind and of the second kind, respectively ([8]). If $q=k=1$ and $z=0$, then $c_{n,1}^{(1)}(0) = c_n$ and $\widehat{c}_{n,1}^{(1)}(0) = \widehat{c}_n$ are the classical Cauchy numbers of the first kind and of the second kind, respectively (see e.g. [1, 15]). The concept about the poly-Cauchy numbers and polynomials have been introduced, and the characteristic and combinatorial properties have been investigated ([2, 8, 9, 10, 11, 12, 13]).

The falling factorial is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^n s(n,l)x^l\,,$$

where $s(n,l)$ is the signed Stirling number of the first kind. The rising factorial is defined by

$$(x)^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n}(-1)^{n-l}s(n,l)x^l \,.$$

Recently, the method of umbral calculus has been introduced to yield various identities in the study of poly-Cauchy numbers ([6, ?]) as well as that of poly-Bernoulli polynomials ([5]). In this paper, we investigate the properties of the poly-Cauchy polynomials with a q parameter of the first kind and of the second kind with umbral calculus viewpoint, give various identities with Bernoulli polynomials, Korobov polynomials, Stirling numbers, Frobenius-Euler polynomials, falling and rising factorials.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\} . \tag{1}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\,\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on P by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \ge 0). \tag{2}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \ge 0), \tag{3}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t) (\ne 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$ for $n,k \ge 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{4}$$

and

$$f(t)=\sum_{k=0}^{\infty}\left\langle f(t)|x^k\right\rangle\frac{t^k}{k!},\quad p(x)=\sum_{k=0}^{\infty}\left\langle t^k|p(x)\right\rangle\frac{x^k}{k!}\tag{5}$$

([16, Theorem 2.2.5]). Thus, by (5), we get

$$t^k p(x)=p^{(k)}(x)=\frac{d^k p(x)}{dx^k}\quad\text{and}\quad e^{yt}p(x)=p(x+y).\tag{6}$$

Sheffer sequences are characterized in the generating function ([16, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t),f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)}e^{y\bar{f}(t)}=\sum_{k=0}^{\infty}\frac{s_k(y)}{k!}t^k\quad(y\in\mathbb{C})\,,$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x)\sim\big(g(t),f(t)\big)$, we have the following equations ([16, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x)=ns_{n-1}(x)\quad(n\geq 0),\tag{7}$$

$$s_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle x^j,\tag{8}$$

$$s_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}s_j(x)p_{n-j}(y)\,,\tag{9}$$

where $p_n(x)=g(t)s_n(x)$.

Assume that $p_n(x)\sim\big(1,f(t)\big)$ and $q_n(x)\sim\big(1,g(t)\big)$. Then the transfer formula ([16, Corollary 3.8.2]) is given by

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)\quad(n\geq 1).$$

For $s_n(x)\sim\big(g(t),f(t)\big)$ and $r_n(x)\sim\big(h(t),l(t)\big)$, assume that

$$s_n(x)=\sum_{m=0}^{n}C_{n,m}r_n(x)\quad(n\geq 0)\,,$$

Then we have ([16, p.132])

$$C_{n,m}=\frac{1}{m!}\left\langle\frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)}l\big(\bar{f}(t)\big)^m\middle|x^n\right\rangle\,.\tag{10}$$

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3 Main results

It is immediate to see that $c_{n,q}^{(k)}(z)$ is the Sheffer sequence for the pair

$$\left(g(t) = \frac{1}{\mathrm{Lif}_k(-t)}, f(t) = \frac{e^{-qt}-1}{q}\right)$$

or

$$c_{n,q}^{(k)}(z) \sim \left(\frac{1}{\mathrm{Lif}_k(-t)}, \frac{e^{-qt}-1}{q}\right), \tag{11}$$

and $\widehat{c}_{n,q}^{(k)}(z)$ is that for the pair

$$\left(g(t) = \frac{1}{\mathrm{Lif}_k(-t)}, f(t) = \frac{e^{qt}-1}{q}\right),$$

or

$$\widehat{c}_{n,q}^{(k)}(z) \sim \left(\frac{1}{\mathrm{Lif}_k(-t)}, \frac{e^{qt}-1}{q}\right) \tag{12}$$

because $\bar{f}(t) = -\ln(1+qt)/q$ and $\bar{f}(t) = \ln(1+qt)/q$, respectively in Lemma 1.

When $x = 0$, $c_{n,q}^{(k)} = c_{n,q}^{(k)}(0)$ (respectively, $\widehat{c}_{n,q}^{(k)} = \widehat{c}_{n,q}^{(k)}(0)$) are called the poly-Cauchy numbers of the first kind (respectively, the poly-Cauchy numbers of the second kind).

3.1 Explicit expressions

It is known that

$$\left(\frac{x}{q}\right)^{(n)} = \frac{x}{q}\left(\frac{x}{q}+1\right)\cdots\left(\frac{x}{q}+n-1\right) \sim (1, 1-e^{-qt}).$$

So,

$$\begin{aligned} n!\delta_{n,k} &= \left\langle (1-e^{-qt})^k \middle| \left(\frac{x}{q}\right)^{(n)} \right\rangle \\ &= \left\langle \left(\frac{e^{-qt}-1}{q}\right)^k \middle| (-q)^n \left(\frac{x}{q}\right)^{(n)} \right\rangle, \end{aligned}$$

yielding that

$$(-q)^n \left(\frac{x}{q}\right)^{(n)} \sim \left(1, \frac{e^{-qt}-1}{q}\right).$$

Similarly, by

$$\left(\frac{x}{q}\right)_n = \frac{x}{q}\left(\frac{x}{q}-1\right)\cdots\left(\frac{x}{q}-n+1\right) \sim (1, 1-e^{-qt})$$

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we get

$$n!\delta_{n,k} = \left\langle (e^{qt}-1)^k \middle| \left(\frac{x}{q}\right)_n \right\rangle = \left\langle \left(\frac{e^{qt}-1}{q}\right)^k \middle| q^n \left(\frac{x}{q}\right)_n \right\rangle ,$$

yielding that

$$q^n \left(\frac{x}{q}\right)_n \sim \left(1, \frac{e^{qt}-1}{q}\right) .$$

First, we shall show the following results by the different methods derived from the umbral calculus, which have been already obtained in [9, Theorem 5].

Theorem 1 *For integers n and k with $n \ge 0$, we have*

$$c_{n,q}^{(k)}(x) = \sum_{m=0}^{n} s(n,m) q^{n-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(m-j+1)^k} ,$$

$$\widehat{c}_{n,q}^{(k)}(x) = \sum_{m=0}^{n} (-1)^m s(n,m) q^{n-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(m-j+1)^k}$$

Proof. Since

$$\frac{1}{\mathrm{Lif}_k(-t)} c_{n,q}^{(k)}(x) = (-q)^n \left(\frac{x}{q}\right)^{(n)} \sim \left(1, \frac{e^{-qt}-1}{q}\right) ,$$

we have

$$\begin{aligned}
c_{n,q}^{(k)}(x) &= (-q)^n \mathrm{Lif}_k(-t) \left(\frac{x}{q}\right)^{(n)} \\
&= (-q)^n \mathrm{Lif}_k(-t) \sum_{m=0}^{n} (-1)^{n-m} s(n,m) \left(\frac{x}{q}\right)^m \\
&= q^n \sum_{m=0}^{n} (-q^{-1})^m s(n,m) \mathrm{Lif}_k(-t) x^m \\
&= q^n \sum_{m=0}^{n} (-q^{-1})^m s(n,m) \sum_{i=0}^{m} \frac{(-1)^i}{i!(i+1)^k} t^i x^m \\
&= q^n \sum_{m=0}^{n} (-q^{-1})^m s(n,m) \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i}}{(i+1)^k} x^{m-i} \\
&= q^n \sum_{m=0}^{n} \sum_{j=0}^{m} (-q^{-1})^m s(n,m) \frac{(-1)^{m-j} \binom{m}{j}}{(m-j+1)^k} x^j \\
&= \sum_{m=0}^{n} s(n,m) q^{n-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(m-j+1)^k} .
\end{aligned}$$

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Since

$$\frac{1}{\mathrm{Lif}_k(-t)}\widehat{c}^{(k)}_{n,q}(x) = (-q)^n \binom{\frac{x}{q}}{n} \sim \left(1, \frac{e^{qt}-1}{q}\right),$$

we have

$$\begin{aligned}
\widehat{c}^{(k)}_{n,q}(x) &= q^n \mathrm{Lif}_k(-t) \left(\frac{x}{q}\right)_n \\
&= q^n \mathrm{Lif}_k(-t) \sum_{m=0}^{n} s(n,m) \left(\frac{x}{q}\right)^m \\
&= \sum_{m=0}^{n} s(n,m) q^{n-m} \mathrm{Lif}_k(-t) x^m \\
&= \sum_{m=0}^{n} s(n,m) q^{n-m} \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i}}{(i+1)^k} x^{m-i} \\
&= \sum_{m=0}^{n} s(n,m) q^{n-m} \sum_{j=0}^{m} \frac{(-1)^{m-j} \binom{m}{j}}{(m-j+1)^k} x^j \\
&= \sum_{m=0}^{n} (-1)^m s(n,m) q^{n-m} \sum_{j=0}^{m} \binom{m}{j} \frac{(-x)^j}{(m-j+1)^k}.
\end{aligned}$$

∎

Proof of Theorem 1 (A different version). As the second different proof, we use the conjugation formula (8).

Since $g(t) = 1/\mathrm{Lif}_k(-t)$ and $\bar{f}(t) = -\ln(1+qt)/q$ for $s_n = c^{(k)}_{n,q}(x)$, by (8) we get

$$c^{(k)}_{n,q}(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left(-\frac{\ln(1+qt)}{q}\right)^j \middle| x^n \right\rangle x^j .$$

Here,

$$\begin{aligned}
&\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left(-\frac{\ln(1+qt)}{q}\right)^j \middle| x^n \right\rangle \\
&= \sum_{m=0}^{n-j} \frac{(-1)^j}{m!(m+1)^k q^{m+j}} \left\langle \left(\ln(1+qt)\right)^{m+j} \middle| x^n \right\rangle \\
&= \sum_{m=0}^{n-j} \frac{(-1)^j}{m!(m+1)^k q^{m+j}} \sum_{l=0}^{n-m-j} \frac{(m+j)!}{(l+m+j)!} s(l+m+j, m+j) \left\langle (qt)^{l+m+j} | x^n \right\rangle \\
&= \sum_{m=0}^{n-j} \frac{(-1)^j}{m!(m+1)^k q^{m+j}} \sum_{l=0}^{n-m-j} \frac{(m+j)!}{(l+m+j)!} q^{l+m+j} s(l+m+j, m+j)(l+m+j)! \delta_{l+m+j,n} \\
&= \sum_{m=0}^{n-j} \frac{(-1)^j (m+j)! q^n}{m!(m+1)^k q^{m+j}} s(n, m+j).
\end{aligned}$$

Therefore,

$$\begin{aligned}c_{n,q}^{(k)}(x) &= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^j\binom{m+j}{m}q^n}{(m+1)^k q^{m+j}}s(n,m+j)\right)x^j\\&=\sum_{j=0}^{n}\left(\sum_{m=j}^{n}\frac{(-1)^j\binom{m}{j}q^n}{(m-j+1)^k q^m}s(n,m)\right)x^j\\&=\sum_{m=0}^{n}s(n,m)q^{n-m}\sum_{j=0}^{m}\binom{m}{j}\frac{(-x)^j}{(m-j+1)^k}\,.\end{aligned}$$

Since $g(t)=1/\mathrm{Lif}_k(-t)$ and $\bar{f}(t)=\ln(1+qt)/q$ for $s_n=\widehat{c}_{n,q}^{(k)}(x)$, by (8) we get

$$\widehat{c}_{n,q}^{(k)}(x)=\sum_{j=0}^{n}\frac{1}{j!}\left\langle \mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right)\left(\frac{\ln(1+qt)}{q}\right)^j\middle| x^n\right\rangle x^j\,.$$

Here,

$$\begin{aligned}&\left\langle \mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right)\left(\frac{\ln(1+qt)}{q}\right)^j\middle| x^n\right\rangle\\&=\sum_{m=0}^{n-j}\frac{(-1)^m}{m!(m+1)^k q^{m+j}}\left\langle\big(\ln(1+qt)\big)^{m+j}\middle| x^n\right\rangle\\&=\sum_{m=0}^{n-j}\frac{(-1)^m}{m!(m+1)^k q^{m+j}}\sum_{l=0}^{n-m-j}\frac{(m+j)!}{(l+m+j)!}q^{l+m+j}s(l+m+j,m+j)(l+m+j)!\delta_{l+m+j,n}\\&=\sum_{m=0}^{n-j}\frac{(-1)^m(m+j)!q^n}{m!(m+1)^k q^{m+j}}s(n,m+j)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{c}_{n,q}^{(k)}(x) &= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^m\binom{m+j}{m}q^n}{(m+1)^k q^{m+j}}s(n,m+j)\right)x^j\\&=\sum_{j=0}^{n}\left(\sum_{m=j}^{n}\frac{(-1)^{m-j}\binom{m}{j}q^n}{(m-j+1)^k q^m}s(n,m)\right)x^j\\&=\sum_{m=0}^{n}(-1)^m s(n,m)q^{n-m}\sum_{j=0}^{m}\binom{m}{j}\frac{(-x)^j}{(m-j+1)^k}\,.\end{aligned}$$

∎

We shall show the formulae for the poly-Cauchy polynomials with a q parameter in terms of Bernoulli polynomials $B_n^{(r)}(x)$ of order r, defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\frac{B_n^{(r)}(x)}{n!}t^n \tag{13}$$

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(see e.g. [16, Section 2.2]). Observe that for any formal power series $g(t) = \sum_{m=0}^{\infty} b_m t^m/m!$

$$p(x) = g(t)x^n, \quad g(at)x^n = a^n p\Big(\frac{x}{a}\Big) \quad (n \ge 0,\ a \ne 0)$$

because t as the differential operator,

$$\begin{aligned} p(x) &= \sum_{m=0}^{\infty} b_m \frac{t^m}{m!} x^n \\ &= \sum_{m=0}^{\infty} b_m \binom{n}{m} x^{n-m} \end{aligned}$$

and

$$\begin{aligned} g(at)x^n &= \sum_{m=0}^{\infty} b_m \frac{(at)^m}{m!} x^n \\ &= \sum_{m=0}^{\infty} b_m a^m \binom{n}{m} x^{n-m} \\ &= a^n \sum_{m=0}^{\infty} b_m \binom{n}{m} \Big(\frac{x}{a}\Big)^{n-m} \\ &= a^n p\Big(\frac{x}{a}\Big). \end{aligned}$$

Theorem 2 *For integers n and k with $n \ge 1$, we have*

$$c_{n,q}^{(k)}(x) = \sum_{j=0}^{n} (-1)^j \sum_{l=0}^{n-j} \frac{\binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j,$$

$$\widehat{c}_{n,q}^{(k)}(x) = (-1)^n \sum_{j=0}^{n} (-1)^j \sum_{l=0}^{n-j} \frac{(-1)^l \binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j.$$

Proof. Since

$$\frac{1}{\mathrm{Lif}_k(-t)} c_{n,q}^{(k)}(x) \sim \left(1, \frac{e^{-qt}-1}{q}\right)$$

and $x^n \sim (1,t)$, for $n \ge 1$ we have

$$\begin{aligned} \frac{1}{\mathrm{Lif}_k(-t)} c_{n,q}^{(k)}(x) &= x \left(\frac{t}{(e^{-qt}-1)/q}\right)^n x^{-1} x^n \\ &= (-1)^n x \left(\frac{-qt}{e^{-qt}-1}\right)^n x^{n-1} \\ &= (-1)^n \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} (-q)^l x^{n-l} \\ &= (-1)^n \sum_{l=0}^{n} \binom{n-1}{l} B_l^{(n)} (-q)^l x^{n-l}. \end{aligned}$$

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Thus,

$$\begin{aligned}
c_{n,q}^{(k)}(x) &= (-1)^n \sum_{l=0}^{n} \binom{n-1}{l} B_l^{(n)} (-q)^l \mathrm{Lif}_k(-t) x^{n-l} \\
&= (-1)^n \sum_{l=0}^{n} \sum_{m=0}^{n-l} \frac{(-1)^m \binom{n-1}{l}\binom{n-l}{m}}{(m+1)^k} (-q)^l B_l^{(n)} x^{n-l-m} \\
&= (-1)^n \sum_{l=0}^{n} \sum_{j=0}^{n-l} \frac{(-1)^{n-j} \binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j \\
&= \sum_{j=0}^{n} (-1)^j \sum_{l=0}^{n-j} \frac{\binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j \,.
\end{aligned}$$

Since

$$\frac{1}{\mathrm{Lif}_k(-t)} \widehat{c}_{n,q}^{(k)}(x) \sim \left(1, \frac{e^{qt}-1}{q}\right)$$

and $x^n \sim (1,t)$, for $n \geq 1$ we have

$$\begin{aligned}
\frac{1}{\mathrm{Lif}_k(-t)} \widehat{c}_{n,q}^{(k)}(x) &= x \left(\frac{t}{(e^{qt}-1)/q}\right)^n x^{-1} x^n \\
&= x \left(\frac{qt}{e^{qt}-1}\right)^n x^{n-1} \\
&= \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} q^l x^{n-l} \\
&= \sum_{l=0}^{n} \binom{n-1}{l} B_l^{(n)} q^l x^{n-l} \,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{c}_{n,q}^{(k)}(x) &= \sum_{l=0}^{n} \binom{n-1}{l} B_l^{(n)} q^l \mathrm{Lif}_k(-t) x^{n-l} \\
&= sum_{l=0}^{n} \sum_{m=0}^{n-l} \frac{(-1)^m \binom{n-1}{l}\binom{n-l}{m}}{(m+1)^k} q^l B_l^{(n)} x^{n-l-m} \\
&= sum_{l=0}^{n} \sum_{j=0}^{n-l} \frac{(-1)^{n-l-j} \binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j \\
&= (-1)^n \sum_{j=0}^{n} (-1)^j \sum_{l=0}^{n-j} \frac{(-1)^l \binom{n-1}{l}\binom{n-l}{j}}{(n-l-j+1)^k} q^l B_l^{(n)} x^j \,.
\end{aligned}$$

∎

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3.2 Sheffer identities

Theorem 3

$$c_{n,q}^{(k)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}(-q)^{n-j}c_{j,q}^{(k)}(x)\left(\frac{y}{q}\right)^{(n-j)},$$
$$\widehat{c}_{n,q}^{(k)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}q^{n-j}\widehat{c}_{j,q}^{(k)}(x)\left(\frac{y}{q}\right)_{n-j}.$$

Proof. Put $s_n=c_{n,q}^{(k)}$ with (11) in (9). Since

$$\begin{aligned}p_n(x)&=\frac{1}{\mathrm{Lif}_k(-t)}c_{n,q}^{(k)}(x)\\&=(-q)^n\left(\frac{x}{q}\right)^{(n)}\sim\left(1,\frac{e^{-qt}-1}{q}\right).\end{aligned}$$

Thus,

$$c_{n,q}^{(k)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}c_{j,q}^{(k)}(x)(-q)^{n-j}\left(\frac{y}{q}\right)^{(n-j)}.$$

Put $s_n=\widehat{c}_{n,q}^{(k)}$ with (12) in (9). Since

$$\begin{aligned}p_n(x)&=\frac{1}{\mathrm{Lif}_k(-t)}\widehat{c}_{n,q}^{(k)}(x)\\&=q^n\left(\frac{x}{q}\right)_n\sim\left(1,\frac{e^{qt}-1}{q}\right).\end{aligned}$$

Thus,

$$\widehat{c}_{n,q}^{(k)}(x+y)=\sum_{j=0}^{n}\binom{n}{j}\widehat{c}_{j,q}^{(k)}(x)q^{n-j}\left(\frac{y}{q}\right)_{n-j}.$$

∎

3.3 Recurrence relations

Theorem 4 *For integers n and k with $n\geq 0$, we have*

$$c_{n,q}^{(k)}(x-q)-c_{n,q}^{(k)}(x)=nqc_{n-1,q}^{(k)}(x),$$
$$\widehat{c}_{n,q}^{(k)}(x+q)-\widehat{c}_{n,q}^{(k)}(x)=nq\widehat{c}_{n-1,q}^{(k)}(x).$$

Proof. Put $s_n=c_{n,q}^{(k)}$ in (7). Then

$$\left(\frac{e^{-qt}-1}{q}\right)c_{n,q}^{(k)}(x)=nc_{n-1,q}^{(k)}(x).$$

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So, we get the first relation.
Put $s_n = \widehat{c}^{(k)}_{n,q}$ in (7). Then

$$\left(\frac{e^{qt}-1}{q}\right)\widehat{c}^{(k)}_{n,q}(x) = n\widehat{c}^{(k)}_{n-1,q}(x)\,.$$

So, we get the second relation. ∎

3.4 Differentiations

The following results ([9, Proposition 2]) can be also obtained by using the umbral calculus.

Theorem 5 *For integers n and k with $n \ge 0$, we have*

$$\frac{d}{dx}c^{(k)}_{n,q}(x) = -n!\sum_{l=0}^{n-1}\frac{(-q)^{n-l-1}}{(n-l)l!}c^{(k)}_{l,q}(x)\,,$$
$$\frac{d}{dx}\widehat{c}^{(k)}_{n,q}(x) = n!\sum_{l=0}^{n-1}\frac{(-q)^{n-l-1}}{(n-l)l!}\widehat{c}^{(k)}_{l,q}(x)\,.$$

We use a formula for $s_n(x)$ in terms of $s_l(x)$.

Lemma 2 *For $s_n(x) \sim \big(g(t), f(t)\big)$,*

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t)|x^{n-l}\right\rangle s_l(x)\,.$$

Proof of Theorem 5. Since $\bar{f}(t) = -\ln(1+qt)/q$ for $s_n = c^{(k)}_{n,q}$, by Lemma 2

$$\begin{aligned}\frac{d}{dx}c^{(k)}_{n,q}(x) &= \sum_{l=0}^{n-1}\binom{n}{l}\left\langle -\frac{\ln(1+qt)}{q}\middle| x^{n-l}\right\rangle c^{(k)}_{l,q}(x)\\
&= -\frac{1}{q}\sum_{l=0}^{n-1}\binom{n}{l}\left\langle \sum_{j=1}^{\infty}\frac{(-1)^{j-1}q^j t^j}{j}\middle| x^{n-l}\right\rangle c^{(k)}_{l,q}(x)\\
&= -\frac{1}{q}\sum_{l=0}^{n-1}\binom{n}{l}\frac{(-1)^{n-l-1}q^{n-l}}{n-l}(n-l)!c^{(k)}_{l,q}(x)\\
&= -n!\sum_{l=0}^{n-1}\frac{(-q)^{n-l-1}}{(n-l)l!}c^{(k)}_{l,q}(x)\,.\end{aligned}$$

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Since $\bar{f}(t) = \ln(1+qt)/q$ for $s_n = \widehat{c}_{n,q}^{(k)}$, by Lemma 2

$$\begin{aligned}\frac{d}{dx}\widehat{c}_{n,q}^{(k)}(x) &= \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \frac{\ln(1+qt)}{q}\middle| x^{n-l}\right\rangle \widehat{c}_{l,q}^{(k)}(x)\\ &= \frac{1}{q}\sum_{l=0}^{n-1}\binom{n}{l}\left\langle \sum_{j=1}^{\infty}\frac{(-1)^{j-1}q^j t^j}{j}\middle| x^{n-l}\right\rangle \widehat{c}_{l,q}^{(k)}(x)\\ &= \frac{1}{q}\sum_{l=0}^{n-1}\binom{n}{l}\frac{(-1)^{n-l-1}q^{n-l}}{n-l}(n-l)!\widehat{c}_{l,q}^{(k)}(x)\\ &= n!\sum_{l=0}^{n-1}\frac{(-q)^{n-l-1}}{(n-l)l!}\widehat{c}_{l,q}^{(k)}(x)\,.\end{aligned}$$

∎

3.5 Recurrence relations including Cauchy numbers

Theorem 6

$$c_{n,q}^{(k)}(x) = -xc_{n-1,q}^{(k)}(x+q) + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}q^l c_l\big(c_{n-l}^{(k-1)}(x+q) - c_{n-l}^{(k)}(x+q)\big)\,,$$

$$\widehat{c}_{n,q}^{(k)}(x) = x\widehat{c}_{n-1,q}^{(k)}(x-q) + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}q^l c_l\big(\widehat{c}_{n-l}^{(k-1)}(x-q) - \widehat{c}_{n-l}^{(k)}(x-q)\big)\,.$$

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Proof. By (2),

$$
\begin{aligned}
c_{n,q}^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} c_{l,q}^{(k)}(y)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle (1+qt)^{-y/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) \middle| x^n \right\rangle \\
&= \left\langle (1+qt)^{-y/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) \middle| x\cdot x^{n-1} \right\rangle \\
&= \left\langle \partial_t\left((1+qt)^{-y/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t(1+qt)^{-y/q}\right)\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle (1+qt)^{-y/q}\left(\partial_t\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\right) \middle| x^{n-1} \right\rangle \\
&= -y\left\langle (1+qt)^{-(y+q)/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) \middle| x^{n-1} \right\rangle \\
&\quad + q\left\langle (1+qt)^{-(y+q)/q}\frac{\mathrm{Lif}_{k-1}\left(\frac{\ln(1+qt)}{q}\right) - \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)}{qt} \middle| \frac{qt}{\ln(1+qt)}x^{n-1} \right\rangle .
\end{aligned}
$$

Since the generating function of the classical Cauchy numbers of the first kind c_n is given by

$$
\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n\frac{t^n}{n!}
$$

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(see e.g. [1, 8, 15]), we have

$$\begin{aligned}
c_{n,q}^{(k)}(y) &= -yc_{n-1,q}^{(k)}(y+q) \\
&\quad + q\left\langle (1+qt)^{-(y+q)/q}\frac{\mathrm{Lif}_{k-1}\left(\frac{\ln(1+qt)}{q}\right)-\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)}{qt}\middle| \sum_{l=0}^{n-1}\binom{n-1}{l}q^l c_l x^{n-l-1}\right\rangle \\
&= -yc_{n-1,q}^{(k)}(y+q) + q\sum_{l=0}^{n-1}\binom{n-1}{l}q^l c_l \\
&\quad \times \left\langle (1+qt)^{-(y+q)/q}\frac{\mathrm{Lif}_{k-1}\left(\frac{\ln(1+qt)}{q}\right)-\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)}{qt}\middle| t\left(\frac{x^{n-l}}{n-l}\right)\right\rangle \\
&= -yc_{n-1,q}^{(k)}(y+q) + \sum_{l=0}^{n-1}\binom{n-1}{l}\frac{1}{n-l}q^l c_l \\
&\quad \times \left\langle (1+qt)^{-(y+q)/q}\left(\mathrm{Lif}_{k-1}\left(\frac{\ln(1+qt)}{q}\right)-\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\right)\middle| x^{n-l}\right\rangle \\
&= -yc_{n-1,q}^{(k)}(y+q) + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}q^l c_l\left(c_{n-l}^{(k-1)}(y+q)-c_{n-l}^{(k)}(y+q)\right).
\end{aligned}$$

Thus, we get the first relation.

Similarly, by (2),

$$
\begin{aligned}
\widehat{c}_{n,q}^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty} \widehat{c}_{l,q}^{(k)}(y)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle (1+qt)^{y/q}\mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right) \middle| x^n \right\rangle \\
&= \left\langle (1+qt)^{y/q}\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right) \middle| x\cdot x^{n-1} \right\rangle \\
&= \left\langle \partial_t\left((1+qt)^{y/q}\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right)\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \big(\partial_t(1+qt)^{y/q}\big)\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right) \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle (1+qt)^{y/q}\left(\partial_t\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right)\right) \middle| x^{n-1} \right\rangle \\
&= y\left\langle (1+qt)^{(y-q)/q}\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right) \middle| x^{n-1} \right\rangle \\
&\quad + q\left\langle (1+qt)^{(y-q)/q}\frac{\mathrm{Lif}_{k-1}\big(\frac{-\ln(1+qt)}{q}\big)-\mathrm{Lif}_k\big(\frac{-\ln(1+qt)}{q}\big)}{qt} \middle| \frac{qt}{\ln(1+qt)}x^{n-1} \right\rangle .
\end{aligned}
$$

Since

$$
\frac{qt}{\ln(1+qt)}x^{n-1} = \sum_{l=0}^{n-1}\binom{n-1}{l}q^l c_l x^{n-l-1},
$$

we have

$$
\begin{aligned}
\widehat{c}_{n,q}^{(k)}(y) &= y\widehat{c}_{n-1,q}^{(k)}(y-q) + q\sum_{l=0}^{n-1}\binom{n-1}{l}q^l c_l \\
&\quad \times \left\langle (1+qt)^{(y-q)/q}\frac{\mathrm{Lif}_{k-1}\big(\frac{-\ln(1+qt)}{q}\big)-\mathrm{Lif}_k\big(\frac{-\ln(1+qt)}{q}\big)}{qt} \middle| t\left(\frac{x^{n-l}}{n-l}\right) \right\rangle \\
&= y\widehat{c}_{n-1,q}^{(k)}(y-q) + \sum_{l=0}^{n-1}\binom{n-1}{l}\frac{1}{n-l}q^l c_l \\
&\quad \times \left\langle (1+qt)^{(y-q)/q}\left(\mathrm{Lif}_{k-1}\big(\frac{-\ln(1+qt)}{q}\big)-\mathrm{Lif}_k\big(\frac{-\ln(1+qt)}{q}\big)\right) \middle| x^{n-l} \right\rangle \\
&= y\widehat{c}_{n-1,q}^{(k)}(y-q) + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}q^l c_l\big(\widehat{c}_{n-l}^{(k-1)}(y-q) - \widehat{c}_{n-l}^{(k)}(y-q)\big) .
\end{aligned}
$$

Thus, we get the second relation. ∎

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3.6 More recurrence relations

Theorem 7 *For integers n and k with $n \geq 1$, we have*

$$\sum_{m=1}^{n}(-q)^{m-1}(m-1)!\binom{n}{m}c_{n-m,q}^{(k)} = \sum_{m=1}^{n}(-q)^{m-1}(m-1)!\binom{n-1}{m-1}c_{n-m,q}^{(k-1)},$$

$$\sum_{m=1}^{n}(-q)^{m-1}(m-1)!\binom{n}{m}\widehat{c}_{n-m,q}^{(k)} = \sum_{m=1}^{n}(-q)^{m-1}(m-1)!\binom{n-1}{m-1}\widehat{c}_{n-m,q}^{(k-1)}.$$

Proof. We shall compute

$$\left\langle \frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| x^n \right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle \frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| x^n \right\rangle \\
&= \frac{1}{q}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| \ln(1+qt)x^n \right\rangle \\
&= \frac{1}{q}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| \sum_{m=1}^{\infty}\frac{(-1)^{m-1}(qt)^m}{m}x^n \right\rangle \\
&= \frac{1}{q}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| \sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^m\binom{n}{m}x^{n-m} \right\rangle \\
&= \sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^{m-1}\binom{n}{m}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| x^{n-m} \right\rangle \\
&= \sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^{m-1}\binom{n}{m}\left\langle \sum_{i=0}^{\infty}c_{i,q}^{(k)}\frac{t^i}{i!}\middle| x^{n-m} \right\rangle \\
&= \sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^{m-1}\binom{n}{m}c_{n-m,q}^{(k)}.
\end{aligned}$$

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On the other hand,

$$\begin{aligned}
&\left\langle \frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle| x^n\right\rangle \\
&= \left\langle \int_0^t \left(\frac{\ln(1+qs)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qs)}{q}\right)\right)' ds \middle| x^n\right\rangle \\
&= \left\langle \int_0^t \frac{\mathrm{Lif}_{k-1}\left(\frac{\ln(1+qs)}{q}\right)}{1+qs} ds \middle| x^n\right\rangle \\
&= \left\langle \int_0^t \left(\sum_{i=0}^{\infty}(-qs)^i\right)\left(\sum_{j=0}^{\infty} c_{j,q}^{(k-1)}\frac{s^j}{j!}\right) ds \middle| x^n\right\rangle \\
&= \left\langle \int_0^t \left(\sum_{r=0}^{\infty}\sum_{j=0}^{r}(-q)^{r-j} c_{j,q}^{(k-1)}\frac{s^r}{j!}\right) ds \middle| x^n\right\rangle \\
&= \left\langle \sum_{r=0}^{\infty}\sum_{j=0}^{r}(-q)^{r-j} c_{j,q}^{(k-1)}\frac{t^{r+1}}{j!(r+1)} \middle| x^n\right\rangle \\
&= \sum_{j=0}^{n-1}(-q)^{n-j-1} c_{j,q}^{(k-1)}\frac{n!}{j!n} \\
&= (n-1)!\sum_{m=0}^{n-1}\frac{(-q)^{n-m-1}}{m!}c_{m,q}^{(k-1)}.
\end{aligned}$$

Thus, for $n \geq 1$, we obtain

$$\sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^{m-1}\binom{n}{m}c_{n-m,q}^{(k)} = (n-1)!\sum_{m=1}^{n}\frac{(-q)^{m-1}}{(n-m)!}c_{n-m,q}^{(k-1)}.$$

Similarly, we shall compute

$$\left\langle -\frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)\middle| x^n\right\rangle$$

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in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&-\frac{1}{q}\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\middle|\ln(1+qt)x^n\right\rangle\\
&=\sum_{m=1}^{n}(-1)^m(m-1)!q^{m-1}\binom{n}{m}\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\middle|x^{n-m}\right\rangle\\
&=\sum_{m=1}^{n}(-1)^m(m-1)!q^{m-1}\binom{n}{m}\left\langle \sum_{i=0}^{\infty}\widehat{c}_{i,q}^{(k)}\frac{t^i}{i!}\middle|x^{n-m}\right\rangle\\
&=\sum_{m=1}^{n}(-1)^m(m-1)!q^{m-1}\binom{n}{m}\widehat{c}_{n-m,q}^{(k)}\,.
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \int_0^t\Big(\frac{-\ln(1+qs)}{q}\mathrm{Lif}_k\Big(\frac{-\ln(1+qs)}{q}\Big)\Big)'ds\middle|x^n\right\rangle\\
&=\left\langle \int_0^t\frac{-\mathrm{Lif}_{k-1}\big(\frac{-\ln(1+qs)}{q}\big)}{1+qs}ds\middle|x^n\right\rangle\\
&=\left\langle \int_0^t(-1)\left(\sum_{i=0}^{\infty}(-qs)^i\right)\left(\sum_{j=0}^{\infty}\widehat{c}_{j,q}^{(k-1)}\frac{s^j}{j!}\right)ds\middle|x^n\right\rangle\\
&=\left\langle -\sum_{r=0}^{\infty}\sum_{j=0}^{r}(-q)^{r-j}\widehat{c}_{j,q}^{(k-1)}\frac{t^{r+1}}{j!(r+1)}\middle|x^n\right\rangle\\
&=-\sum_{j=0}^{n-1}(-q)^{n-j-1}\widehat{c}_{j,q}^{(k-1)}\frac{n!}{j!n}\\
&=-(n-1)!\sum_{m=1}^{n}\frac{(-q)^{m-1}}{(n-m)!}\widehat{c}_{n-m,q}^{(k-1)}\,.
\end{aligned}$$

Thus, for $n\ge 1$, we obtain

$$\sum_{m=1}^{n}(-1)^{m-1}(m-1)!q^{m-1}\binom{n}{m}\widehat{c}_{n-m,q}^{(k)}=(n-1)!\sum_{m=1}^{n}\frac{(-q)^{m-1}}{(n-m)!}\widehat{c}_{n-m,q}^{(k-1)}\,.$$

∎

3.7 Some relations with Korobov polynomials

The *Korobov polynomials* of the first kind $K_{n,q}(x)$ $(q\neq 0)$ ([14]) are given by

$$\frac{qt(1+t)^x}{(1+t)^q-1}=\sum_{j=0}^{\infty}K_{j,q}(x)\frac{t^j}{j!}\,. \tag{14}$$

Theorem 8 *For integers n and k with $n \ge 0$, we have*

$$\begin{aligned}&c_{n,q}^{(k)}(x)\\&=\sum_{i=0}^{n}\sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}(-1)^{n-i-l-m}\binom{n}{l}\binom{n-l}{i}q^{n-l-m}\frac{(n-i-l)!}{m!(n-i-l+1)}c_{l,q}c_{m,q}^{(k-1)}K_{i,\frac{1}{q}}\left(-\frac{x}{q}\right)\,,\\&\widehat{c}_{n,q}^{(k)}(x)\\&=\sum_{i=0}^{n}\sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}(-1)^{n-i-l-m}\binom{n}{l}\binom{n-l}{i}q^{n-l-m}\frac{(n-i-l)!}{m!(n-i-l+1)}\widehat{c}_{l,q}\widehat{c}_{m,q}^{(k-1)}K_{i,-\frac{1}{q}}\left(\frac{x}{q}\right)\,.\end{aligned}$$

Proof. By the definition of $c_{n,q}^{(k)}(x)$,

$$\begin{aligned}c_{n,q}^{(k)}(y)&=\left\langle(1+qt)^{-y/q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle|x^n\right\rangle\\&=\left\langle\frac{(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle|\frac{q\big((1+qt)^{1/q}-1\big)}{\ln(1+qt)}x^n\right\rangle\,.\end{aligned}$$

Since

$$\frac{q\big((1+qt)^{1/q}-1\big)}{\ln(1+qt)}=\sum_{l=0}^{\infty}c_{l,q}\frac{t^l}{l!}\,,$$

we get

$$\begin{aligned}c_{n,q}^{(k)}(y)&=\left\langle\frac{(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\frac{\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\middle|\sum_{l=0}^{\infty}\binom{n}{l}c_{l,q}x^{n-l}\right\rangle\\&=\sum_{l=0}^{\infty}\binom{n}{l}c_{l,q}\left\langle\frac{(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\int_0^t\left(\frac{\ln(1+qs)}{q}\mathrm{Lif}_k\left(\frac{\ln(1+qs)}{q}\right)\right)'ds\middle|x^{n-l}\right\rangle\\&=\sum_{l=0}^{\infty}\binom{n}{l}c_{l,q}\left\langle\frac{(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\sum_{r=0}^{\infty}\sum_{m=0}^{r}(-q)^{r-m}c_{m,q}^{(k-1)}\frac{t^{r+1}}{m!(r+1)}\middle|x^{n-l}\right\rangle\\&=\sum_{l=0}^{\infty}\binom{n}{l}c_{l,q}\left\langle\frac{t(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\middle|\sum_{r=0}^{\infty}\sum_{m=0}^{r}(-q)^{r-m}c_{m,q}^{(k-1)}\frac{t^{r}}{m!(r+1)}x^{n-l}\right\rangle\\&=\sum_{l=0}^{\infty}\binom{n}{l}c_{l,q}\sum_{r=0}^{n-l}\sum_{m=0}^{r}(-q)^{r-m}c_{m,q}^{(k-1)}\frac{\binom{n-l}{r}r!}{m!(r+1)}\left\langle\frac{t(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}\middle|x^{n-l-r}\right\rangle\,.\end{aligned}$$

Replacing t by qt, q by $1/q$ and x by $-y/q$ in (14), we have

$$\frac{t(1+qt)^{-y/q}}{(1+qt)^{1/q}-1}=\sum_{j=0}^{\infty}K_{j,\frac{1}{q}}\left(-\frac{y}{q}\right)\frac{(qt)^j}{j!}\,.$$

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Hence,

$$c_{n,q}^{(k)}(y) = \sum_{l=0}^{\infty} \binom{n}{l} c_{l,q} \sum_{r=0}^{n-l} \sum_{m=0}^{r} (-q)^{r-m} c_{m,q}^{(k-1)} \frac{\binom{n-l}{r} r!}{m!(r+1)} \left\langle \sum_{j=0}^{\infty} K_{j,\frac{1}{q}} \left(-\frac{y}{q}\right) \frac{(qt)^j}{j!} \middle| x^{n-l-r} \right\rangle .$$

Since

$$\left\langle \sum_{j=0}^{\infty} K_{j,\frac{1}{q}} \left(-\frac{y}{q}\right) \frac{(qt)^j}{j!} \middle| x^{n-l-r} \right\rangle = K_{n-l-r,\frac{1}{q}} \left(-\frac{y}{q}\right) q^{n-l-r} ,$$

we have

$$\begin{aligned} c_{n,q}^{(k)}(y) &= \sum_{l=0}^{n} \sum_{r=0}^{n-l} \sum_{m=0}^{r} (-1)^{r-m} \binom{n}{l} \binom{n-l}{r} q^{n-l-m} \frac{r!}{m!(r+1)} c_{l,q} c_{m,q}^{(k-1)} K_{n-l-r,\frac{1}{q}} \left(-\frac{y}{q}\right) \\ &= \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{r=m}^{n-l} (-1)^{r-m} \binom{n}{l} \binom{n-l}{r} q^{n-l-m} \frac{r!}{m!(r+1)} c_{l,q} c_{m,q}^{(k-1)} K_{n-l-r,\frac{1}{q}} \left(-\frac{y}{q}\right) \\ &= \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{i=0}^{n-l-m} (-1)^{n-l-m-i} \binom{n}{l} \binom{n-l}{i} q^{n-l-m} \frac{(n-l-i)!}{m!(n-l-i+1)} c_{l,q} c_{m,q}^{(k-1)} K_{i,\frac{1}{q}} \left(-\frac{y}{q}\right) \\ &= \sum_{i=0}^{n} \sum_{l=0}^{n-i} \sum_{m=0}^{n-i-l} (-1)^{n-i-l-m} \binom{n}{l} \binom{n-l}{i} q^{n-l-m} \frac{(n-i-l)!}{m!(n-i-l+1)} c_{l,q} c_{m,q}^{(k-1)} K_{i,\frac{1}{q}} \left(-\frac{y}{q}\right) . \end{aligned}$$

Thus, we obtain the first relation.
Similarly, by the definition of $c_{n,q}^{(k)}(x)$,

$$\begin{aligned} \widehat{c}_{n,q}^{(k)}(y) &= \left\langle (1+qt)^{y/q} \mathrm{Lif}_k \left(-\frac{\ln(1+qt)}{q} \right) \middle| x^n \right\rangle \\ &= \left\langle \frac{(1+qt)^{y/q}}{(1+qt)^{-1/q} - 1} \frac{-\ln(1+qt)}{q} \mathrm{Lif}_k \left(\frac{-\ln(1+qt)}{q} \right) \middle| \frac{q\big(1-(1+qt)^{-1/q}\big)}{\ln(1+qt)} x^n \right\rangle . \end{aligned}$$

Since

$$\frac{q\big(1-(1+qt)^{-1/q}\big)}{\ln(1+qt)} = \sum_{l=0}^{\infty} \widehat{c}_{l,q} \frac{t^l}{l!} ,$$

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we get

$$\begin{aligned}
\widehat{c}_{n,q}^{(k)}(y) &= \sum_{l=0}^{\infty}\binom{n}{l}\widehat{c}_{l,q}\left\langle \frac{(1+qt)^{y/q}}{(1+qt)^{-1/q}-1}\frac{-\ln(1+qt)}{q}\mathrm{Lif}_k\left(\frac{-\ln(1+qt)}{q}\right)\middle| x^{n-l}\right\rangle \\
&= \sum_{l=0}^{\infty}\binom{n}{l}\widehat{c}_{l,q}\left\langle \frac{(1+qt)^{y/q}}{(1+qt)^{-1/q}-1}\int_0^t\left(\frac{-\ln(1+qs)}{q}\mathrm{Lif}_k\left(\frac{-\ln(1+qs)}{q}\right)\right)' ds\middle| x^{n-l}\right\rangle \\
&= \sum_{l=0}^{\infty}\binom{n}{l}\widehat{c}_{l,q}\left\langle \frac{-t(1+qt)^{y/q}}{(1+qt)^{-1/q}-1}\middle| \sum_{r=0}^{\infty}\sum_{m=0}^{r}(-q)^{r-m}\widehat{c}_{m,q}^{(k-1)}\frac{t^r}{m!(r+1)}x^{n-l}\right\rangle \\
&= \sum_{l=0}^{\infty}\binom{n}{l}\widehat{c}_{l,q}\sum_{r=0}^{n-l}\sum_{m=0}^{r}(-q)^{r-m}\widehat{c}_{m,q}^{(k-1)}\frac{\binom{n-l}{r}r!}{m!(r+1)}\left\langle \frac{-t(1+qt)^{y/q}}{(1+qt)^{-1/q}-1}\middle| x^{n-l-r}\right\rangle .
\end{aligned}$$

Replacing t by qt, q by $-1/q$ and x by y/q in (14), we have

$$\frac{-t(1+qt)^{y/q}}{(1+qt)^{-1/q}-1} = \sum_{j=0}^{\infty}K_{j,-\frac{1}{q}}\left(\frac{y}{q}\right)\frac{(qt)^j}{j!} .$$

Hence,

$$\begin{aligned}
c_{n,q}^{(k)}(y) &= \sum_{l=0}^{\infty}\binom{n}{l}\widehat{c}_{l,q}\sum_{r=0}^{n-l}\sum_{m=0}^{r}(-q)^{r-m}\widehat{c}_{m,q}^{(k-1)}\frac{\binom{n-l}{r}r!}{m!(r+1)}K_{n-l-r,-\frac{1}{q}}\left(\frac{y}{q}\right)q^{n-l-r} \\
&= \sum_{l=0}^{n}\sum_{r=0}^{n-l}\sum_{m=0}^{r}(-1)^{r-m}\binom{n}{l}\binom{n-l}{r}q^{n-l-m}\frac{r!}{m!(r+1)}\widehat{c}_{l,q}\widehat{c}_{m,q}^{(k-1)}K_{n-l-r,-\frac{1}{q}}\left(\frac{y}{q}\right) \\
&= \sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{r=m}^{n-l}(-1)^{r-m}\binom{n}{l}\binom{n-l}{r}q^{n-l-m}\frac{r!}{m!(r+1)}\widehat{c}_{l,q}\widehat{c}_{m,q}^{(k-1)}K_{n-l-r,-\frac{1}{q}}\left(\frac{y}{q}\right) \\
&= \sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{i=0}^{n-l-m}(-1)^{n-l-m-i}\binom{n}{l}\binom{n-l}{i}q^{n-l-m}\frac{(n-l-i)!}{m!(n-l-i+1)}\widehat{c}_{l,q}\widehat{c}_{m,q}^{(k-1)}K_{i,-\frac{1}{q}}\left(\frac{y}{q}\right) \\
&= \sum_{i=0}^{n}\sum_{l=0}^{n-i}\sum_{m=0}^{n-i-l}(-1)^{n-i-l-m}\binom{n}{l}\binom{n-l}{i}q^{n-l-m}\frac{(n-i-l)!}{m!(n-i-l+1)}\widehat{c}_{l,q}\widehat{c}_{m,q}^{(k-1)}K_{i,-\frac{1}{q}}\left(\frac{y}{q}\right) .
\end{aligned}$$

Thus, we obtain the second relation. ∎

3.8 Some relations including Stiring numbers of the first kind

Theorem 9 *For integers n and k with $n \ge 0$, we have*

$$c_{n+1,q}^{(k)}(x) = -xc_{n,q}^{(k)}(x+q) + q^n\sum_{j=0}^{n}\sum_{m=j}^{n}\frac{(-1)^j q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)(x+q)^j ,$$

$$\widehat{c}_{n+1,q}^{(k)}(x) = x\widehat{c}_{n,q}^{(k)}(x-q) - q^n\sum_{j=0}^{n}\sum_{m=j}^{n}\frac{(-1)^{m-j} q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)(x-q)^j .$$

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We use the following recurrence formula for Sheffer sequences ([16, Corollary 3.7.2]).

Lemma 3 *If $s_n(x) \sim \big(g(t), f(t)\big)$, then*

$$s_{n+1} = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)\,.$$

Proof of Theorem 9. Consider the Sheffer sequence $s_n = c_{n,q}^{(k)}$ in Lemma 3. By $f(t) = (e^{-qt}-1)/q$ and $g(t) = 1/\mathrm{Lif}_k(-t)$, we get $1/f'(t) = -e^{qt}$ and

$$\frac{g'(t)}{g(t)} = \frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\,.$$

Thus,

$$c_{n+1,q}^{(k)}(x) = e^{qt}\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}c_{n,q}^{(k)}(x) - x c_{n,q}^{(k)}(x+q)\,.$$

We obtain

$$\begin{aligned}
\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}c_{n,q}^{(k)}(x) &= \mathrm{Lif}'_k(-t)(-q)^n\left(\frac{x}{q}\right)^{(n)}\\
&= (-q)^n\mathrm{Lif}'_k(-t)\sum_{m=0}^n(-1)^{n-m}s(n,m)\left(\frac{x}{q}\right)^{(n)}\\
&= q^n\sum_{m=0}^n(-q^{-1})^m s(n,m)\mathrm{Lif}'_k(-t)x^m\\
&= q^n\sum_{m=0}^n(-q^{-1})^m s(n,m)\sum_{r=0}^m\frac{(-1)^r}{r!(r+1)^k}t^r x^m\\
&= q^n\sum_{m=0}^n\sum_{r=0}^m\frac{(-1)^{m+r}q^{-m}\binom{m}{r}}{(r+1)^k}s(n,m)x^{m-r}\\
&= q^n\sum_{m=0}^n\sum_{j=0}^m\frac{(-1)^j q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)x^j\,.
\end{aligned}$$

Therefore, we have

$$c_{n+1,q}^{(k)}(x) = -x c_{n,q}^{(k)}(x+q) \quad + q^n\sum_{j=0}^n\sum_{m=j}^n\frac{(-1)^j q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)(x+q)^j\,.$$

Similarly, consider the Sheffer sequence $s_n = \widehat{c}_{n,q}^{(k)}$ in Lemma 3. By $f(t) = (e^{qt}-1)/q$ and $g(t) = 1/\mathrm{Lif}_k(-t)$, we get

$$\widehat{c}_{n+1,q}^{(k)}(x) = -e^{-qt}\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\widehat{c}_{n,q}^{(k)}(x) + x\widehat{c}_{n,q}^{(k)}(x-q)\,.$$

We obtain

$$\begin{aligned}
\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\widehat{c}^{(k)}_{n,q}(x) &= \mathrm{Lif}'_k(-t)q^n\left(\frac{x}{q}\right)_n \\
&= q^n\mathrm{Lif}'_k(-t)\sum_{m=0}^{n}s(n,m)\left(\frac{x}{q}\right)^m \\
&= q^n\sum_{m=0}^{n}q^{-m}s(n,m)\mathrm{Lif}'_k(-t)x^m \\
&= q^n\sum_{m=0}^{n}q^{-m}s(n,m)\sum_{r=0}^{m}\frac{(-1)^r\binom{m}{r}}{(r+1)^k}x^{m-r} \\
&= q^n\sum_{m=0}^{n}q^{-m}s(n,m)\sum_{j=0}^{m}\frac{(-1)^{m-j}\binom{m}{j}}{(m-j+1)^k}x^{j} \\
&= q^n\sum_{m=0}^{n}\sum_{j=0}^{m}\frac{(-1)^{m-j}q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)x^{j} .
\end{aligned}$$

Therefore, we have

$$\widehat{c}^{(k)}_{n+1,q}(x) = x\widehat{c}^{(k)}_{n,q}(x-q) - q^n\sum_{j=0}^{n}\sum_{m=j}^{n}\frac{(-1)^{m-j}q^{-m}\binom{m}{j}}{(m-j+1)^k}s(n,m)(x-q)^{j} .$$

∎

3.9 Some relations with Bernoulli polynomials

By applying Lemma 1 about (13), for nonnegative integer r, we have

$$B^{(r)}_n \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right) . \tag{15}$$

Theorem 10 *For integers n and k with $n \geq 0$, we have*

$$\begin{aligned}
c^{(k)}_{n,q}(x) &= (-1)^m\sum_{m=0}^{n}\left(\sum_{l=0}^{n-m}\sum_{i=0}^{n-m-l}\binom{n}{l}\binom{n-l}{i+m}q^i s(i+m,m)\widehat{\mathbb{C}}^{(s)}_{l,q}c^{(k)}_{n-m-l-i,q}\right)B^{(s)}_m(x) , \\
\widehat{c}^{(k)}_{n,q}(x) &= \sum_{m=0}^{n}\left(\sum_{l=0}^{n-m}\sum_{i=0}^{n-m-l}\binom{n}{l}\binom{n-l}{i+m}q^i s(i+m,m)\mathbb{C}^{(s)}_{l,q}\widehat{c}^{(k)}_{n-m-l-i,q}\right)B^{(s)}_m(x) .
\end{aligned}$$

Proof. Put $c^{(k)}_{n,q}(x) = \sum_{m=0}^{n} C_{n,m}B^{(s)}_m(x)$ for (11) and (15). Then by (10) we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left(\frac{e^{-\ln(1+qt)/q}-1}{-\ln(1+qt)/q}\right)^s\left(-\frac{\ln(1+qt)}{q}\right)^m\middle| x^n\right\rangle \\
&= \frac{(-q^{-1})^m}{m!}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left(\frac{q\big(1-(1+qt)^{-1/q}\big)}{\ln(1+qt)}\right)^s\big(\ln(1+qt)\big)^m\middle| x^n\right\rangle .
\end{aligned}$$

Define $\widehat{\mathbb{C}}_{l,q}^{(r)}$ $(r > 0)$ by

$$\left(\frac{q\big(1-(1+qt)^{-1/q}\big)}{\ln(1+qt)} \right)^r = \sum_{l=0}^{\infty} \widehat{\mathbb{C}}_{l,q}^{(r)} \frac{t^l}{l!} \,,$$

so that $\widehat{\mathbb{C}}_{l,q}^{(1)} = \widehat{c}_{l,q} = \widehat{c}_{l,q}^{(1)}$. Then,

$$\begin{aligned} C_{n,m} &= \frac{(-q^{-1})^m}{m!} \left\langle \mathrm{Lif}_k\Big(\frac{\ln(1+qt)}{q}\Big)\big(\ln(1+qt)\big)^m \middle| \sum_{l=0}^{n} \widehat{\mathbb{C}}_{l,q}^{(s)} \frac{t^l}{l!} x^n \right\rangle \\ &= \frac{(-q^{-1})^m}{m!} \sum_{l=0}^{n} \widehat{\mathbb{C}}_{l,q}^{(s)} \binom{n}{l} \left\langle \mathrm{Lif}_k\Big(\frac{\ln(1+qt)}{q}\Big)\big(\ln(1+qt)\big)^m \middle| x^{n-l} \right\rangle \\ &= \frac{(-q^{-1})^m}{m!} \sum_{l=0}^{n-m} \widehat{\mathbb{C}}_{l,q}^{(s)} \binom{n}{l} \\ &\quad \times \left\langle \mathrm{Lif}_k\Big(\frac{\ln(1+qt)}{q}\Big) \middle| \sum_{i=0}^{n-m-l} \frac{m!}{(i+m)!} s(i+m,m)(qt)^{i+m} x^{n-l} \right\rangle \\ &= \frac{(-q^{-1})^m}{m!} \sum_{l=0}^{n-m} \widehat{\mathbb{C}}_{l,q}^{(s)} \binom{n}{l} \sum_{i=0}^{n-m-l} \frac{m!}{(i+m)!} s(i+m,m) q^{i+m} (n-l)_{i+m} \\ &\quad \times \left\langle \mathrm{Lif}_k\Big(\frac{\ln(1+qt)}{q}\Big) \middle| x^{n-m-l-i} \right\rangle . \end{aligned}$$

Since

$$\left\langle \mathrm{Lif}_k\Big(\frac{\ln(1+qt)}{q}\Big) \middle| x^{n-m-l-i} \right\rangle = c_{n-m-l-i,q}^{(k)} \,,$$

we have

$$C_{n,m} = (-1)^m \sum_{l=0}^{n-m} \sum_{i=0}^{n-m-l} \binom{n}{l} \binom{n-l}{i+m} q^i s(i+m,m) \widehat{\mathbb{C}}_{l,q}^{(s)} c_{n-m-l-i,q}^{(k)} .$$

Thus,

$$c_{n,q}^{(k)}(x) = (-1)^m \sum_{m=0}^{n} \left(\sum_{l=0}^{n-m} \sum_{i=0}^{n-m-l} \binom{n}{l} \binom{n-l}{i+m} q^i s(i+m,m) \widehat{\mathbb{C}}_{l,q}^{(s)} c_{n-m-l-i,q}^{(k)} \right) B_m^{(s)}(x) \,.$$

Similarly, put $\widehat{c}_{n,q}^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x)$ for (12) and (15). Then by (10) we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big) \left(\frac{e^{\ln(1+qt)/q}-1}{\ln(1+qt)/q}\right)^s \left(\frac{\ln(1+qt)}{q}\right)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!q^m} \left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big) \left(\frac{q\big((1+qt)^{1/q}-1\big)}{\ln(1+qt)}\right)^s \big(\ln(1+qt)\big)^m \middle| x^n \right\rangle . \end{aligned}$$

Define $\mathbb{C}_{l,q}^{(r)}$ $(r>0)$ by

$$\left(\frac{q\big((1+qt)^{1/q}-1\big)}{\ln(1+qt)}\right)^r=\sum_{l=0}^{\infty}\mathbb{C}_{l,q}^{(r)}\frac{t^l}{l!},$$

so that $\mathbb{C}_{l,q}^{(1)}=c_{l,q}=c_{l,q}^{(1)}$. Then,

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!q^m}\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\big(\ln(1+qt)\big)^m\middle|\sum_{l=0}^{n}\mathbb{C}_{l,q}^{(s)}\frac{t^l}{l!}x^n\right\rangle\\
&=\frac{1}{m!q^m}\sum_{l=0}^{n}\mathbb{C}_{l,q}^{(s)}\binom{n}{l}\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\big(\ln(1+qt)\big)^m\middle|x^{n-l}\right\rangle\\
&=\frac{1}{m!q^m}\sum_{l=0}^{n-m}\mathbb{C}_{l,q}^{(s)}\binom{n}{l}\\
&\quad\times\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\middle|\sum_{i=0}^{n-m-l}\frac{m!}{(i+m)!}s(i+m,m)(qt)^{i+m}x^{n-l}\right\rangle\\
&=\frac{1}{m!q^m}\sum_{l=0}^{n-m}\mathbb{C}_{l,q}^{(s)}\binom{n}{l}\sum_{i=0}^{n-m-l}\frac{m!}{(i+m)!}s(i+m,m)q^{i+m}(n-l)_{i+m}\\
&\quad\times\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\middle|x^{n-m-l-i}\right\rangle.
\end{aligned}$$

Since

$$\left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\middle|x^{n-m-l-i}\right\rangle=\widehat{c}_{n-m-l-i,q}^{(k)},$$

we have

$$C_{n,m}=\sum_{l=0}^{n-m}\sum_{i=0}^{n-m-l}\binom{n}{l}\binom{n-l}{i+m}q^i s(i+m,m)\mathbb{C}_{l,q}^{(s)}\widehat{c}_{n-m-l-i,q}^{(k)}.$$

Thus,

$$\widehat{c}_{n,q}^{(k)}(x)=\sum_{m=0}^{n}\left(\sum_{l=0}^{n-m}\sum_{i=0}^{n-m-l}\binom{n}{l}\binom{n-l}{i+m}q^i s(i+m,m)\widehat{\mathbb{C}}_{l,q}^{(s)}\widehat{c}_{n-m-l-i,q}^{(k)}\right)B_m^{(s)}(x).$$

∎

3.10 Some relations with Frobenius-Euler polynomials

For $\lambda\in\mathbb{C}$ with $\lambda\neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

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(see e.g. [4]). Hence, by Lemma 1 we have

$$H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r, t\right). \tag{16}$$

Theorem 11 *For integers n and k with $n \ge 0$, we have*

$$\begin{aligned} c_{n,q}^{(k)}(x) &= \left(\frac{\lambda}{\lambda-1}\right)^r \sum_{m=0}^{n}(-1)^m \\ &\quad \times \left(\sum_{l=0}^{n-m}\sum_{i=0}^{r} q^l(-\lambda^{-1})^i\binom{r}{i}\binom{n}{l+m}s(l+m,n)c_{n-l-m,q}^{(k)}(i)\right)H_m^{(r)}(x|\lambda), \\ \widehat{c}_{n,q}^{(k)}(x) &= \left(\frac{\lambda}{\lambda-1}\right)^r \sum_{m=0}^{n} \\ &\quad \times \left(\sum_{l=0}^{n-m}\sum_{i=0}^{r} q^l(-\lambda^{-1})^i\binom{r}{i}\binom{n}{l+m}s(l+m,n)\widehat{c}_{n-l-m,q}^{(k)}(i)\right)H_m^{(r)}(x|\lambda). \end{aligned}$$

Proof. Put $c_{n,q}^{(k)}(x) = \sum_{m=0}^{n} C_{n,m}H_m^{(r)}(x|\lambda)$ for (11) and (16). Then by (10) we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left(\frac{e^{-\ln(1+qt)/q}-\lambda}{1-\lambda}\right)^r\left(-\frac{\ln(1+qt)}{q}\right)^m \middle| x^n\right\rangle \\ &= \frac{(-q^{-1})^m}{m!(1-\lambda)^r}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\left((1+qt)^{-1/q}-\lambda\right)^r \middle| \big(\ln(1+qt)\big)^m x^n\right\rangle. \end{aligned}$$

Since

$$\big(\ln(1+qt)\big)^m = \sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)(qt)^{l+m}, \tag{17}$$

we have

$$\begin{aligned} C_{n,m} &= \frac{(-q^{-1})^m}{m!(1-\lambda)^r}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)q^{l+m}(n)_{l+m} \\ &\quad \times \left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)\sum_{i=0}^{r}\binom{r}{i}(-\lambda)^{r-i}(1+qt)^{-i/q} \middle| x^{n-l-m}\right\rangle \\ &= \frac{(-q^{-1})^m}{m!(1-\lambda)^r}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)q^{l+m}(n)_{l+m} \\ &\quad \times \sum_{i=0}^{r}\binom{r}{i}(-\lambda)^{r-i}\left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)(1+qt)^{-i/q} \middle| x^{n-l-m}\right\rangle. \end{aligned}$$

Since

$$\mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right)(1+qt)^{-i/q}=\sum_{j=0}^{\infty}c_{j,q}^{(k)}(i)\frac{t^j}{j!}\,,$$

we have

$$\begin{aligned}C_{n,m}&=\frac{(-q^{-1})^m}{m!(1-\lambda)^r}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)q^{l+m}(n)_{l+m}\sum_{i=0}^{r}\binom{i=0}{r}\binom{r}{i}(-\lambda)^{r-i}c_{n-l-m,q}^{(k)}(i)\\&=(-1)^{m+r}\left(\frac{\lambda}{1-\lambda}\right)^r\sum_{l=0}^{n-m}\sum_{i=0}^{r}q^l(-\lambda^{-1})^i\binom{r}{i}\binom{n}{l+m}s(l+m,n)c_{n-l-m,q}^{(k)}(i)\,.\end{aligned}$$

Thus,

$$\begin{aligned}c_{n,q}^{(k)}(x)&=\left(\frac{\lambda}{\lambda-1}\right)^r\sum_{m=0}^{n}(-1)^m\\&\quad\times\left(\sum_{l=0}^{n-m}\sum_{i=0}^{r}q^l(-\lambda^{-1})^i\binom{r}{i}\binom{n}{l+m}s(l+m,n)c_{n-l-m,q}^{(k)}(i)\right)H_m^{(r)}(x|\lambda)\,.\end{aligned}$$

Similarly, put $\widehat{c}_{n,q}^{(k)}(x)=\sum_{m=0}^{n}C_{n,m}H_m^{(r)}(x|\lambda)$ for (12) and (16). Then by (10) we get

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle \mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)\left(\frac{e^{\ln(1+qt)/q}-\lambda}{1-\lambda}\right)^r\left(\frac{\ln(1+qt)}{q}\right)^m\middle|x^n\right\rangle\\&=\frac{q^{-m}}{m!(1-\lambda)^r}\left\langle \mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)\left((1+qt)^{1/q}-\lambda\right)^r\middle|\left(\ln(1+qt)\right)^m x^n\right\rangle\,.\end{aligned}$$

By (17) we have

$$\begin{aligned}C_{n,m}&=\frac{q^{-m}}{m!(1-\lambda)^r}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)q^{l+m}(n)_{l+m}\\&\quad\times\left\langle \mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)\sum_{i=0}^{r}\binom{r}{i}(-\lambda)^{r-i}(1+qt)^{i/q}\middle|x^{n-l-m}\right\rangle\\&=\frac{q^{-m}}{m!(1-\lambda)^r}\sum_{l=0}^{n-m}\frac{m!}{(l+m)!}s(l+m,m)q^{l+m}(n)_{l+m}\\&\quad\times\sum_{i=0}^{r}\binom{r}{i}(-\lambda)^{r-i}\left\langle \mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)(1+qt)^{i/q}\middle|x^{n-l-m}\right\rangle\,.\end{aligned}$$

Since

$$\mathrm{Lif}_k\left(-\frac{\ln(1+qt)}{q}\right)(1+qt)^{i/q}=\sum_{j=0}^{\infty}\widehat{c}_{j,q}^{(k)}(i)\frac{t^j}{j!}\,,$$

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we have

$$C_{n,m} = \left(\frac{\lambda}{\lambda-1}\right)^r \sum_{l=0}^{n-m} \sum_{i=0}^{r} q^l (-\lambda^{-1})^i \binom{r}{i} \binom{n}{l+m} s(l+m,n) \widehat{c}_{n-l-m,q}^{(k)}(i) .$$

Thus,

$$\begin{aligned} \widehat{c}_{n,q}^{(k)}(x) &= \left(\frac{\lambda}{\lambda-1}\right)^r \sum_{m=0}^{n} (-1)^m \\ &\quad \times \left(\sum_{l=0}^{n-m} \sum_{i=0}^{r} q^l (-\lambda^{-1})^i \binom{r}{i} \binom{n}{l+m} s(l+m,n) \widehat{c}_{n-l-m,q}^{(k)}(i) \right) H_m^{(r)}(x|\lambda) . \end{aligned}$$

∎

3.11 Some relations with falling and rising factorials

Theorem 12 *For integers n and k with $n \ge 0$, we have*

$$\begin{aligned} c_{n,q}^{(k)}(x) &= \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} c_{n,i}^{(k)}(-i) \right) x^{(m)} , \\ \widehat{c}_{n,q}^{(k)}(x) &= \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} \widehat{c}_{n,i}^{(k)}(m-i) \right) x_m . \end{aligned}$$

Proof. Put $c_{n,q}^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} x^{(m)}$ for (11) and $x^{(n)} \sim (1, 1-e^{-t})$. Then by (10) we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) (1 - e^{\ln(1+qt)/q})^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) \left(1 - (1+qt)^{1/q}\right)^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \left\langle \mathrm{Lif}_k\left(\frac{\ln(1+qt)}{q}\right) (1+qt)^{i/q} \middle| x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} (-1)^i c_{n,q}^{(k)}(-i) . \end{aligned}$$

Thus,

$$c_{n,q}^{(k)}(x) = \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} c_{n,i}^{(k)}(-i) \right) x^{(m)} .$$

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Similarly, put $\widehat{c}_{n,q}^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} x_m$ for (12) and $x_n \sim (1, e^t - 1)$. Then by (10) we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)(e^{\ln(1+qt)/q} - 1)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)\big((1+qt)^{1/q} - 1\big)^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \left\langle \mathrm{Lif}_k\Big(-\frac{\ln(1+qt)}{q}\Big)(1+qt)^{i/q} \middle| x^n \right\rangle \\
&= \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \widehat{c}_{n,q}^{(k)}(-i) \\
&= \frac{1}{m!} \sum_{i=0}^{m} \binom{m}{i} (-1)^{i} \widehat{c}_{n,q}^{(k)}(m-i)\,.
\end{aligned}$$

Thus,

$$\widehat{c}_{n,q}^{(k)}(x) = \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} \widehat{c}_{n,i}^{(k)}(m-i) \right) x_m\,.$$

∎

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TRIPLED FIXED POINT THEOREMS FOR MIXED MONOTONE CHATTERJEA TYPE CONTRACTIVE OPERATORS

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ABSTRACT. Starting from the papers [Berinde, V., Borcut, M., *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal., **74** (2011), 4889-4897], [Borcut, M., Berinde, V., *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces* , Appl. Math. Comput., **218 (10)** (2012), 5929-5936] and [Borcut, M., *Tripled coincidente point theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 7339-7346.], we present new results on the existence and uniqueness of tripled fixed points for nonlinear mappings in partially ordered complete metric spaces satisfying more general contractive conditions.

1. INTRODUCTION

In some very recent papers, Berinde and Borcut [6], Borcut and Berinde [7], Borcut [8] have introduced and studied the concept of *tripled fixed point*, respectively *tripled coincidence point* for nonlinear contractive mappings $F : X^3 \to X$, in partially ordered complete metric spaces and obtained existence as well as existence and uniqueness theorems of tripled fixed points, respectively of tripled coincidence points, for some classes of contractive type mappings.

The presented theorems in [6], [7], [8], extend several existing results in the literature: [14], [18], [15]. For the sake of completeness, we recall the main concepts and results from [6] which are needed for the present paper.

Let$(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space X^3 the following partial order: for $(x, y, z), (u, v, w) \in X^3$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 1. [6] *Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$ a mapping. We say that F has the mixed monotone property if $F(x, y, z)$ is nondecreasing in x and z, and is nonincreasing in y, that is, for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2. [6] *An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if*

$$F(x, y, z) = x, F(y, x, y) = y, \text{ and } F(z, y, x) = z.$$

Let (X,d) be a metric space. The mapping $\overline{d}: X \times X \times X \to X$, given by

$$\overline{d}\left[(x,y,z),(u,v,w)\right] = d(x,u) + d(y,v) + d(z,w),$$

defines a metric on $X \times X \times X$, which will be denoted for convenience by d, too.

Definition 3. *Let X, Y, Z be nonempty sets and $F : X \times X \times X \to Y$, $G : Y \times Y \times Y \to Z$. We define the symmetric composition (or, the s-composition, for short) of F and G, $G * F : X \times X \times X \to Z$, by*

$$(G * F)(x,y,z) = G\left(F(x,y,z), F(y,x,y), F(z,y,x)\right)(x,y,z \in X).$$

For each nonempty set X, denote by P_x the projection mapping

$$P_X : X \times X \times X \to X, P(x,y,z) = x \text{ for } x,y,z \in X.$$

The symmetric composition has the following properties:

Proposition 1. *(Associativity). If $F : X \times X \times X \to Y$, $G : Y \times Y \times Y \to Z$ and*

$$H : Z \times Z \times Z \to W, \text{ then } (H * G) * F = H * (G * F).$$

Proposition 2. *(Identity Element). If $F : X \times X \times X \to Y$, then*

$$F * P_X = P_Y * F = F.$$

Proposition 3. *(Mixed Monotonicity). If $(X,\leq)$, $(Y,\leq)$, $(Z,\leq)$ are partially ordered sets and the mappings $F : X \times X \times X \to Y, G : Y \times Y \times Y \to Z$ are mixed monotone, then $G * F$ is mixed monotone.*

Proposition 4. *If $(X,\leq)$ is a partially ordered set and F is mixed monotone, then $F^n = F * F^{n-1} = F^{n-1} * F$ is mixed monotone, for every $n \geq 1$.*

The first main result in [6] is given by the following theorem.

Theorem 1. [6] *Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist the constants $j,k,l \in [0,1)$ with $j + k + l < 1$ for which*

$$d\left(F(x,y,z), F(u,v,w)\right) \leq jd(x,u) + kd(y,v) + ld(z,w), \tag{1.1}$$

$\forall x \geq u, y \leq v, z \geq w$. If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0), y_0 \geq F(y_0,x_0,y_0) \text{ and } z_0 \leq F(z_0,y_0,x_0),$$

then there exist $x,y,z \in X$ such that

$$x = F(x,y,z), y = F(y,x,y) \text{ and } z = F(z,y,x).$$

Remark 1. *If we take $j = k = l = \frac{\alpha}{3}$ in Theorem 1, then the contraction condition (1.1) can be written in a slightly simplified form*

$$d\left(F(x,y,z), F(u,v,w)\right) \leq \frac{\alpha}{3}[d(x,u) + d(y,v) + d(z,w)]. \tag{1.2}$$

Theorem 2. [6] *By adding to the hypotheses of Theorem 1 the condition: for every $(x,y,z), (x_1,y_1,z_1) \in X^3$, there exists a $(u,v,w) \in X^3$ that is comparable to (x,y,z) and (x_1,y_1,z_1), then the tripled fixed point of F is unique.*

Theorem 3. [6] *In addition to the hypotheses of Theorem 1, suppose that $x_0, y_0, z_0 \in X$ are comparable. Then $x = y = z$.*

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2. Main results

Based on the notions and results presented in the first section, we will prove new existence and uniqueness theorems for operators which verify a Chatterjea contraction type condition, adapted to the case X^3.

Theorem 4. *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ such that*

$$d\left(F\left(x,y,z\right),F\left(u,v,w\right)\right) \leq \frac{k}{8}[d\left(x,F(u,v,w)\right) + d\left(y,F(v,u,v)\right) + \tag{2.1}$$
$$+d\left(z,F(w,v,u)\right) + d\left(u,F(x,y,z)\right) + d\left(v,F(y,x,y)\right) + d\left(w,F(z,y,x)\right)].$$

Also suppose either
(a) F is continuous or
(b) X has the following property:

$$(i)\ \ \textit{if a nondecreasing sequence } \{x_n\} \to x, \textit{ then } x_n \leq x \textit{ for all } n,$$
$$(ii)\ \ \textit{if a nonincreasing sequence } \{y_n\} \to y, \textit{ then } y_n \geq y \textit{ for all } n.$$

If there exist $x_0, y_0, z_0 \in X$ such that,

$$x_0 \leq F\left(x_0,y_0,z_0\right), y_0 \geq F\left(y_0,x_0,y_0\right) \textit{ and } z_0 \leq F\left(z_0,y_0,x_0\right), \tag{2.2}$$

then there exist $x, y, z \in X$ such that,

$$x = F\left(x,y,z\right), y = F\left(y,x,y\right) \textit{ and } z = F\left(z,y,x\right).$$

Proof. Let the sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ be defined by

$$x_{n+1} = F(x_n, y_n, z_n) = F^{n+1}(x_0, y_0, z_0), y_{n+1} = F(y_n, x_n, y_n) = F^{n+1}(y_0, x_0, y_0),$$
$$z_{n+1} = F(z_n, y_n, x_n) = F^{n+1}(z_0, y_0, x_0), (n = 0, 1, ...).$$

Since F^n is mixed monotone for every n (by Proposition 4), it follows by (2.2) that $\{x_n\}$ and $\{z_n\}$ are nondecreasing and $\{y_n\}$ is nonincreasing. Indeed, due to the mixed monotone property of F, it is easy to show that

$$x_2 = F\left(x_1,y_1,z_1\right) \geq F\left(x_0,y_0,z_0\right) = x_1$$
$$y_2 = F\left(y_1,x_1,y_1\right) \leq F\left(y_0,x_0,y_0\right) = y_1$$
$$z_2 = F\left(z_1,y_1,x_1\right) \geq F\left(z_0,y_0,x_0\right) = z_1$$

and thus we obtain three sequences satisfying the following conditions

$$x_0 \leq x_1 \leq ... \leq x_n \leq ...,$$
$$y_0 \geq y_1 \geq ... \geq y_n \geq ...,$$
$$z_0 \leq z_1 \leq ... \leq z_n \leq ....$$

Now, for $n \in \mathbb{N}$, denote

$$D_{x_{n+1}} = d\left(x_{n+1},x_n\right), D_{y_{n+1}} = d\left(y_{n+1},y_n\right), D_{z_{n+1}} = d\left(z_{n+1},z_n\right)$$

and

$$D_{n+1} = D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}.$$

Using (2.1), we get

$$D_{x_{n+1}} = d\left(x_{n+1},x_n\right) = d(F(x_n,y_n,z_n), F(x_{n-1},y_{n-1},z_{n-1}))$$

$$\leq \frac{k}{8}\left[d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(z_n, F_{z_{n-1}}\right)\right.$$
$$\left. + d\left(x_{n-1}, F_{x_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(z_{n-1}, F_{z_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right.$$
$$\left. + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[d\left(x_{n-1}, x_n\right) + d\left(y_{n-1}, y_n\right) + d\left(z_{n-1}, z_n\right)\right.$$
$$\left. + d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right) + d\left(z_n, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right],$$

and therefore

$$D_{x_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{2.3}$$

Similarly, we obtain for the sequences $\left\{D_{y_{n+1}}\right\}, \left\{D_{z_{n+1}}\right\}$

$$D_{y_{n+1}} = d\left(y_{n+1}, y_n\right) = d(F(y_n, x_n, y_n), F(y_{n-1}, x_{n-1}, y_{n-1}))$$
$$\leq \frac{k}{8}\left[d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right)\right.$$
$$\left. + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x^n}\right) + d\left(y_{n-1}, F_{y_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(y_n, y_n\right) + d\left(x_n, x_n\right) + d\left(y_n, y_n\right)\right.$$
$$\left. + d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[2d\left(y_{n-1}, y_{n+1}\right) + d\left(x_{n-1}, x_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[2d\left(y_{n-1}, y_n\right) + d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right) + 2d\left(y_n, y_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right],$$

and so

$$D_{y_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + 2D_{y_n} + D_{x_{n+1}} + 2D_{y_{n+1}}\right] \tag{2.4}$$

and

$$D_{z_{n+1}} = d\left(z_{n+1}, z_n\right) = d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}))$$
$$\leq \frac{k}{8}\left[d\left(z_n, F_{z_{n-1}}\right) + d\left(y_n, F_{y_{n-1}}\right) + d\left(x_n, F_{x_{n-1}}\right)\right.$$
$$\left. + d\left(z_{n-1}, F_{z_n}\right) + d\left(y_{n-1}, F_{y_n}\right) + d\left(x_{n-1}, F_{x_n}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_n, x_n\right) + d\left(y_n, y_n\right) + d\left(z_n, z_n\right)\right.$$
$$\left. + d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$= \frac{k}{8}\left[d\left(x_{n-1}, x_{n+1}\right) + d\left(y_{n-1}, y_{n+1}\right) + d\left(z_{n-1}, z_{n+1}\right)\right]$$
$$\leq \frac{k}{8}\left[d\left(x_{n-1}, x_n\right) + d\left(y_{n-1}, y_n\right) + d\left(z_{n-1}, z_n\right)\right.$$
$$\left. + d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right) + d\left(z_n, z_{n+1}\right)\right]$$

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$$= \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right],$$

and therefore

$$D_{z_{n+1}} \leq \frac{k}{8}\left[D_{x_n} + D_{y_n} + D_{z_n} + D_{x_{n+1}} + D_{y_{n+1}} + D_{z_{n+1}}\right]. \tag{2.5}$$

By using relations (2.3), (2.4) and (2.5), we get

$$D_{n+1} \leq \frac{k}{8}\left[3D_{x_n} + 4D_{y_n} + 2D_{z_n} + 3D_{x_{n+1}} + 4D_{y_{n+1}} + 2D_{z_{n+1}}\right]$$
$$\leq \frac{k}{8}\left[4D_{x_n} + 4D_{y_n} + 4D_{z_n} + 4D_{x_{n+1}} + 4D_{y_{n+1}} + 4D_{z_{n+1}}\right]$$
$$\leq \frac{k}{2}\left[D_n + D_{n+1}\right].$$

Therefore, for all $n \geq 1$, we have

$$D_{n+1} \leq \alpha \cdot D_n \leq ... \leq \alpha^n \cdot D_1, \text{ where } \alpha = \frac{k}{2-k} \in [0,1), \text{ when } k \in [0,1).$$

Because $D_{x_{n+1}} \leq D_{n+1}, D_{y_{n+1}} \leq D_{n+1}$ and $D_{z_{n+1}} \leq D_{n+1}$, then we have

$$D_{x_{n+1}} \leq \alpha^n \cdot D_1, D_{y_{n+1}} \leq \alpha^n \cdot D_1 \text{ and } D_{z_{n+1}} \leq \alpha^n \cdot D_1 \tag{2.6}$$

This implies that $\{x_n\}, \{y_n\}, \{z_n\}$ are Cauchy sequences in X. Indeed, let $m \geq n$, then

$$d(x_m, x_n) \leq D_{x_m} + D_{x_{m-1}} + ... + D_{x_{n+1}} \leq$$
$$\leq \left[\alpha^{m-1} + \alpha^{m-2} + ... + \alpha^n\right] \cdot D_1 = \frac{\alpha^n - \alpha^m}{1-\alpha} \cdot D_1 < \frac{\alpha^n}{1-\alpha} \cdot D_1.$$

Similarly, we can verify that $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences. Since X is a complete metric space, there exist $x, y, z \in X$ such that,

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z.$$

Finally, we claim that

$$x = F(x,y,z), y = F(y,x,y) \text{ and } z = F(z,y,x).$$

Assume the first assumption (a) holds. This means F is continuous at (x,y,z), and hence, for a given $\frac{\epsilon}{2} > 0$, there exists a $\delta > 0$ such that,

$$d((x,y,z),(u,v,w)) = d(x,u) + d(y,v) + d(z,w) < \delta$$
$$\Rightarrow d(F(x,y,z), F(u,v,w)) < \frac{\epsilon}{2}.$$

Since

$$\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z,$$

for $\eta = \min\left(\frac{\epsilon}{2}, \frac{\delta}{2}\right)$, there exist n_0, m_0, p_0 such that, for $n \geq n_0, m \geq m_0, p \geq p_0$,

$$d(x_n, x) < \eta, d(y_n, y) < \eta, d(z_n, z) < \eta.$$

Now, for $n \in \mathbb{N}, n \geq max\{n_0, m_0, p_0\}$, we have

$$d(F(x,y,z), x) \leq d(F(x,y,z), x_{n+1}) + d(x_{n+1}, x)$$
$$= d(F(x,y,z), F(x_n, y_n, z_n)) + d(x_{n+1}, x) < \frac{\epsilon}{2} + \eta \leq \epsilon,$$

and this implies that $x = F(x,y,z)$. Similarly, we can show that

$$y = F(y,x,y) \text{ and } z = F(z,y,x).$$

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Suppose now that (b) holds. Since $\{x_n\}, \{z_n\}$ are non-decreasing and $x_n \to x$, $z_n \to z$, and also $\{y_n\}$ is non-increasing and $y_n \to y$, from (b) we have $x_n \leq x, y_n \geq y$ and $z_n \leq z$, for all n. Then by triangle inequality and (2.1), we get

$$
\begin{aligned}
d(x, F(x,y,z)) &\leq d(x, x_{n+1}) + d(x_{n+1}, F(x,y,z)) \\
&= d(x, x_{n+1}) + d(F(x_n, y_n, z_n), F(x,y,z)) \\
&\leq d(x, x_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\
&\quad + d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))],
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
d(y, F(y,x,y)) \leq d(y, y_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + 2d(y_n, y_{n+1}) \\
+ d(x, F(x,y,z)) + 2d(y, F(y,x,y))],
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
d(z, F(z,y,x)) \leq d(z, z_{n+1}) + \frac{k}{8}[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \\
+ d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x))].
\end{aligned} \tag{2.9}
$$

By summing (2.7), (2.8), (2.9) we obtain

$$
\begin{aligned}
& d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \\
& \leq \frac{2}{2-k}[d(x, x_{n+1}) + d(y, y_{n+1}) + d(z, z_{n+1})] \\
& + \frac{k}{4(2-k)}[3d(x_n, x_{n+1}) + 4d(y_n, y_{n+1}) + 2d(z_n, z_{n+1})],
\end{aligned}
$$

and let $n \to \infty$ one obtains

$$
d(x, F(x,y,z)) + d(y, F(y,x,y)) + d(z, F(z,y,x)) \leq 0,
$$

that is, $x = F(x,y,z), y = F(y,x,y), z = F(z,y,x)$. □

3. Uniqueness of tripled fixed points

In [6], [7] and [8] the authors also considered some additional conditions to ensure the uniqueness of the tripled fixed point and also appropriate conditions to ensure that for such a tripled fixed point (x,y,z) we have all components equal: $x = y = z$.

Similarly, one can prove that the tripled fixed point ensured by Theorem 4 is in fact unique, provided that the product space $X \times X \times X$ endowed with the partial order mentioned earlier possesses an additional property.

Theorem 5. *If, in addition to the hypotheses of Theorem 4, the condition: for every $(x,y,z), (x_1,y_1,z_1) \in X \times X \times X$, there exists a $(u,v,w) \in X \times X \times X$ that is comparable to (x,y,z) and (x_1,y_1,z_1), is satisfied, then the tripled fixed point of F is unique.*

Proof. If $(x^*, y^*, z^*) \in X \times X \times X$ is another tripled fixed point of F, then we show that

$$
d((x,y,z), (x^*, y^*, z^*)) = 0,
$$

where

$$
\lim_{x\to\infty} x_n = x, \lim_{x\to\infty} y_n = y, \lim_{x\to\infty} z_n = z.
$$

We consider two cases.

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Case 1. If (x,y,z) is comparable to (x^*,y^*,z^*) with respect to the ordering in $X\times X\times X$ then, for every $n=0,1,2,\ldots$, the triple

$$\left(F^n(x,y,z),F^n(y,x,y),F^n(z,y,x)\right)=(x,y,z) \text{ is comparable to}$$
$$\left(F^n(x^*,y^*,z^*),F^n(y^*,x^*,y^*),F^n(z^*,y^*,x^*)\right)=(x^*,y^*,z^*).$$

Also, using the process for obtaining (2.6), we get

$$\begin{aligned}
&d\left((x,y,z),(x^*,y^*,z^*)\right)=d(x,x^*)+d(y,y^*)+d(z,z^*)\\
&=d\left(F^n(x,y,z),F^n(x^*,y^*,z^*)\right)+d\left(F^n(y,x,y),F^n(y^*,x^*,y^*)\right)\\
&\quad+d\left(F^n(z,y,x),F^n(z^*,y^*,x^*)\right)\\
&\le\alpha^n\left[d(x,x^*)+d(y,y^*)+d(z,z^*)\right]=\alpha^n d\left((x,y,z),(y^*,x^*,z^*)\right),\alpha\in[0,1).
\end{aligned}$$

By letting $n\to\infty$, this implies that $d\left((x,y,z),(y^*,x^*,z^*)\right)=0$.

Case 2 : If (x,y,z) are not comparable to (x^*,y^*,z^*), then there exists an upper bound or a lower bound $(u,v,w)\in X\times X\times X$ of (x,y,z) and (x^*,y^*,z^*). Then, for all $n=1,2,...,$

$$\left(F^n(u,v,w),F^n(v,u,v),F^n(w,v,u)\right) \text{ is comparable to}$$
$$\left(F^n(x,y,z),F^n(y,x,y),F^n(z,y,x)\right)=(x,y,z) \text{ and to}$$
$$\left(F^n(x^*,y^*,z^*),F^n(y^*,x^*,y^*),F^n(z^*,y^*,x^*)\right)=(x^*,y^*,z^*).$$

We have,

$$\begin{aligned}
d\left(\begin{pmatrix}x\\y\\z\end{pmatrix},\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix}\right)&=d\left(\begin{pmatrix}F^n(x,y,z)\\F^n(y,x,y)\\F^n(z,y,x)\end{pmatrix},\begin{pmatrix}F^n(x^*,y^*,z^*)\\F^n(y^*,x^*,y^*)\\F^n(z^*,y^*,x^*)\end{pmatrix}\right)\\
&\le d\left(\begin{pmatrix}F^n(x,y,z)\\F^n(y,x,y)\\F^n(z,y,x)\end{pmatrix},\begin{pmatrix}F^n(u,v,w)\\F^n(v,u,v)\\F^n(w,v,u)\end{pmatrix}\right)\\
&\quad+d\left(\begin{pmatrix}F^n(u,v,w)\\F^n(v,u,v)\\F^n(w,v,u)\end{pmatrix},\begin{pmatrix}F^n(x^*,y^*,z^*)\\F^n(y^*,x^*,y^*)\\F^n(z^*,y^*,x^*)\end{pmatrix}\right)\\
&\le\alpha^n\left\{\left[d(x,u)+d(y,v)+d(z,w)\right]+\left[d(u,x^*)+d(v,y^*)+d(w,z^*)\right]\right\}\to0
\end{aligned}$$
$$\text{as } n\to\infty, \text{ and so } d\left(\begin{pmatrix}x\\y\\z\end{pmatrix},\begin{pmatrix}x^*\\y^*\\z^*\end{pmatrix}\right)=0.$$

□

Theorem 6. *In addition to the hypotheses of Theorem 4, suppose that $x_0,y_0,z_0\in X$ are comparable. Then $x=y=z$.*

Proof. Recall that $x_0,y_0,z_0,\in X$ are such that

$$x_0\le F(x_0,y_0,z_0),y_0\ge F(y_0,x_0,y_0),z_0\le F(z_0,y_0,x_0).$$

Now, if $x_0\le y_0$, and $z_0\le y_0$ we claim that, for all $n\in\mathbb{N}$, $x_n\le y_n$ and $z_n\le y_n$. Indeed, by the mixed monotone property of F,

$$x_1=F(x_0,y_0,z_0)\le F(y_0,x_0,y_0)=y_1$$

and

$$z_1=F(z_0,y_0,x_0)\le F(y_0,x_0,y_0)=y_1.$$

Assume that $x_n \le y_n$ and $z_n \le y_n$ for some n. Then

$$x_{n+1} = F^{n+1}(x_0, y_0, z_0) = F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0))$$
$$= F(x_n, y_n, z_n) \le F(y_n, x_n, y_n) = y_{n+1},$$

and similarly for z_n. Now,

$$d(x, y) \le d(x, x_{n+1}) + d(y, x_{n+1}) \le d(x, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y, y_{n+1})$$
$$= d(x, F^{n+1}(x_0, y_0, z_0)) + d\left[F(F^n(x_0, y_0, z_0), F^n(y_0, x_0, y_0), F^n(z_0, y_0, x_0)),\right.$$
$$\left., F(F^n(y_0, x_0, y_0), F^n(x_0, y_0, x_0), F^n(y_0, x_0, y_0))\right] + d(y, y_{n+1}) \to 0$$

as $n \to \infty$.

This implies that $d(x, y) = 0$ and hence we have $x = y$.

Similarly, we obtain that $d(x, z) = 0$ and $d(y, z) = 0$. The other cases for x_0, y_0, z_0 are similar. $\square$

4. Example and final remarks

Let $X = [0, 1]$ be endowed with the usual metric $d(x, y) = |x - y|$ and let $F : X^3 \to X$ be given by $F(x, y, z) = \dfrac{1}{20}$, for $(x, y, z) \in \left[0, \dfrac{4}{5}\right] \times [0, 1]^2$ and $F(x, y, z) = \dfrac{11}{80}$, for $(x, y, z) \in \left[\dfrac{4}{5}, 1\right] \times [0, 1]^2$.

Then F satisfies Chatterjea's contractive condition (2.1) with $k = \dfrac{14}{15} < 1$ but does not satisfy the Banach type contractive condition (1.1).

Let us first prove the first part of the assertion above. It suffices to completely cover the following limit case.

Case 1. $x \in \left[\dfrac{4}{5}, 1\right]$, and $u, y, z, v, w \in \left[0, \dfrac{4}{5}\right)$

In this case $F(x, y, z) = \dfrac{11}{80}$, $F(u, v, w) = \dfrac{1}{20}$ and so condition (2.1) reduces to

(4.1)
$$\left|\frac{11}{80} - \frac{1}{20}\right| \le \frac{k}{8}\left[\left|x - \frac{1}{20}\right| + \left|y - \frac{1}{20}\right| + \left|z - \frac{1}{20}\right| + \left|u - \frac{11}{80}\right| + \left|v - \frac{1}{20}\right| + \left|w - \frac{1}{20}\right|\right].$$

For $x \in \left[\dfrac{4}{5}, 1\right]$, we have

$$\left|x - \frac{1}{20}\right| \ge \left|\frac{4}{5} - \frac{1}{20}\right| = \frac{3}{4}$$

and hence the minimum value of the right hand side of (4.1) is greater or equal to $\dfrac{k}{8} \cdot \dfrac{3}{4}$.

Therefore, in order to have (4.1) satisfied for all $x \in \left[\dfrac{4}{5}, 1\right]$ and $u, y, z, v, w \in \left[0, \dfrac{4}{5}\right)$, with $x \ge u$, $y \le v$, $z \ge w$, i.e.,

$$\left|\frac{1}{20} - \frac{11}{80}\right| \le \frac{k}{8} \cdot \frac{3}{4},$$

it suffices to take k such that $\dfrac{14}{15} \le k < 1$.

Note that for the remaining cases to be discussed, the right hand side of (2.1) will be greater than the value obtained in Case 1.

For example, in the **Case 2.** $x, v \in \left[\frac{4}{5}, 1\right]$ and $u, y, z, w \in \left[0, \frac{4}{5}\right)$, the minimum value of the right hand side of (2.1) will be greater or equal to $\frac{k}{8} \cdot \frac{6}{4}$.

Note also that in the cases $x, u \in \left[\frac{4}{5}, 1\right]$ or $x, u \in \left[0, \frac{4}{5}\right)$, the left hand side of (2.1) is always zero and so (1.2) is satisfied for all values of $y, z, v, w \in [0, 1]$.

This proves that, indeed, F satisfies (2.1) with $k = \frac{14}{15} < 1$.

F is not continuous but X satisfies property (b) in Theorem 4. Moreover, by taking $x_0 = 0$, $y_0 = \frac{1}{5}$ and $z_0 = \frac{1}{8}$, one can check that (2.2) is fulfilled. Thus, all assumptions in Theorem 4 are satisfied and hence F does admit tripled fixed points. By Theorem 5 we actually conclude that F has a unique tripled fixed point, $\left(\frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$.

Now let us show that F does not satisfy (1.1).

Assume the contrary, that is, that F does satisfy (1.1) and take $\epsilon > 0$ such that $u = \frac{4}{5} - \epsilon \in \left[0, \frac{4}{5}\right)$, $x = \frac{4}{5}$ and $y = z$, $v = w \in [0, 1]$ arbitrary in (1.1) to obtain

$$\frac{7}{80} \leq i \cdot \epsilon, \ \epsilon > 0. \tag{4.2}$$

Now letting $\epsilon \to 0$ in (4.2) we reach to a contradiction. This proves that, indeed, F does not satisfy (1.1).

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SOFT BOOLEAN ALGEBRA AND ITS PROPERTIES

RIDVAN ŞAHIN AND AHMET KÜÇÜK

ABSTRACT. Molodtsov [21] introduced the concept of soft theory which can be used as a generic mathematical tool for dealing with uncertainty. In this paper, we apply the notion of the soft set theory of Molodtsov to the theory of Boolean algebras which is a well-known algebraic structure. We introduce the concepts of soft filter and soft ideal on the soft Boolean algebra as well as notions of a soft Boolean algebra and soft Boolean homomorphism, and investigate basic properties as intersection, union and product of the soft Boolean algebras. Also we give several illustrative examples.

1. INDRODUCTION

In 1999, Molodtsov [21] initiated the theory of soft sets as a new mathematical tool to deal with uncertainties while modelling the problems in engineering, physics, computer science, economics, social sciences, and medical science. Maji et al. [18] showed the applications of soft set theory in decision making problem by defining several operations on soft set. In theoretical aspects, Maji et al. [19] introduced several operators for soft set theory such as equality of two soft sets, subset and superset of a soft set, complement of a soft set, null soft set, and absolute soft set. Recently, some new operations in soft set theory has been given by Irfan Ali et al. in [2], also see [23]. Later, the properties and applications of soft set theory have been studied by many authors (e.g. [3, 5, 14, 15, 17, 20, 22, 25, 26]).

At present, studies on the soft set theory is progressing rapidly on algebraic structures. Aktas and Cagman [1] defined a basic version of soft group theory. Sezgin et al. [24] introduced the concepts of normalistic soft group and normalistic soft group homomorphism. Feng et al. [4] studied soft semi rings. Jun et al. [9, 11] applied soft sets in the theories of BCK/BCI-algebras. Kazancı et al. [13] introduced soft BCH-algebras and studied their basic properties. Several other studies on soft BCH-algebras have been discussed in [7, 8, 10]. Jun et al. [12] applied the notion of the soft sets to the theory of Hilbert algebras.

In this paper, we apply the notion of the soft set theory of Molodtsov to the theory of Boolean algebras. We introduce the concepts of soft filter and soft ideal on the soft Boolean algebra as well as concept of a soft Boolean algebra. We also investigate basic properties as intersection, union and product of the soft Boolean algebras, and define soft Boolean homomorphism and obtain some properties. Also we give several illustrative examples.

This paper is organized as follows. In the next two sections, we give some important concepts of Boolean algebra and basic definitions of soft set theory. In

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Section 4, we present the definition of soft Boolean algebra and some properties of soft Boolean algebra. Finally, we summary the paper in Section 5.

2. Basic Results on Boolean Algebras

In the middle of the 19th century, George Boole introduced the concept of Boolean algebras by attempting to formalize propositional logic. Boolean algebra now plays a central role in mathematical logic, probability theory and computer design. In this section, we give some basic notions in Boolean algebra. For more details on Boolean algebras, we refer the reader to [6, 16].

Definition 1. *A Boolean algebra is a tuple* $(K, \wedge, \vee, \neg, \bar{0}, \bar{1})$ *(briefly* K*), where* K *is a set with two distinguished elements* $\bar{0}, \bar{1} \in K$*,* $\neg : K \to K$ *is a unary operation and* $\wedge, \vee : K \times K \to K$ *are binary operations (called meet and join, respectively) such that*

(1) $\wedge, \vee$ *are associative,*
(2) $\wedge, \vee$ *are commutative,*
(3) $\wedge$ *and* $\vee$ *are distributive, i.e.* $\forall x, y, z \in B : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*,*
(4) $\forall x \in K : x \wedge \neg x = \bar{0}$ *and* $x \vee \neg x = \bar{1}$*.*

Let K be a Boolean algebra and $x, y \in K$. Then K carries a natural partial order. In other words, an relation "$\leq$ " defined $x \leq y$ if $x = x \wedge y$ or $x \vee y = y$, is an ordering relation on K, where $x \vee y$ and $x \wedge y$ are least upper bound and greatest lower bound of $\{x, y\}$, respectively. The element $\bar{0}$ in Boolean algebra K is called to be zero element if $x \vee \bar{0} = \bar{0} \vee x = x$ and $x \wedge \bar{0} = \bar{0} \wedge x = \bar{0}$ for any $x \in K$. Similarly, the element $\bar{1}$ in Boolean algebra K is called to be unit element if $x \vee \bar{1} = \bar{1} \vee x = \bar{1}$ and $x \wedge \bar{1} = \bar{1} \wedge x = x$ for any $x \in K$.

Remark 1. *Let* K *be a Boolean algebra,* X *be any set and* $P(X)$ *its power set. Then*

(1) $(P(X), \cap, \cup, -, \varnothing, X)$*, where "$-$" is the complement operation of sets, is a Boolean algebra of sets.*
(2) *For* x *and* y *in the Boolean algebra* K*,* $x \nleq y$ *if* $x \leq y$ *does not hold.* $x < y$ *(*x *is strictly smaller than* y*) if* $x \leq y$ *but* $x \neq y$*.*

Notation 1. *Throughout this article, we assume that "$\leq$" is a natural partial order defined on natural integer* $\mathbb{N}$*.*

Definition 2. *Let* K *be a Boolean algebra. Then*

(1) $a \in K$ *is an atom of* K*, if* $\bar{0} < a$ *but there is no* x *in* K *satisfying* $\bar{0} < x < a$*.* K *is atomless if it has no atoms and atomic if for each positive element* x *(i.e.,* $x \neq \bar{0}$*) of* K*, there is some atom* a *such that* $a \leq x$*.*
(2) *For any* $S \subseteq K$ *with* $S \neq \varnothing$*,* K *is complete iff both* $inf(S)$ *and* $sup(S)$ *exist for every nonempty subset* S *of* K*.*

Definition 3. *Let* K *be a Boolean algebra.*

(1) *A nonempty subset* M *of* K *is said to be a subalgebra of* K *if* $x, y \in M$ *implies* $x \vee y$*,* $x \wedge y$ *and* $\neg x \in M$*. Moreover, note that every Boolean algebra is itself a subalgebra.*

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(2) *A filter on K is a subset F of K such that:*
(i) $\bar{0} \notin F, \bar{1} \in F$; (ii) if $x \in F$ and $y \in F$, then $x \wedge y \in F$; (iii) if $x, y \in K$, $x \in F$ and $x \leq y$, then $y \in F$.

(3) *An ideal on K is a subset I of K such that:*
(i) $\bar{0} \in I, \bar{1} \notin I$; (ii) if $x \in I$ and $y \in I$, then $x \vee y \in I$; (iii) if $x, y \in K$, $x \in I$ and $y \leq x$, then $y \in I$.

Definition 4. *Let K and L be two Boolean algebras. A mapping $\phi : K \to L$ is called a (Boolean) homomorphism if it preserves the operations:*

$$\begin{cases} \phi(a \wedge b) = \phi(a) \wedge \phi(b) \\ \phi(a \vee b) = \phi(a) \vee \phi(b) \\ \phi(\neg a) = \neg \phi(a) \end{cases}$$

for all $a, b \in K$. If ϕ is bijective, then it is called a (Boolean) isomorphism. If there is a Boolean isomorphism $\phi : K \to L$, then K and L are said to be isomorphic, and denoted by $K \simeq L$.

3. Basic Results on Soft Sets

In this paper, U is an initial universe set, $P(U)$ its power set and E is always the universe set of parameters with respect to U unless otherwise specified.

Now, we recall some basic notions in soft set theory.

Definition 5. [19, 20]. *A pair (F, A) is called a soft set over U if $A \subseteq E$ and $F : A \longrightarrow P(U)$, such that $F(x) \neq \varnothing$, if $x \in A \subseteq E$ and $F(x) = \varnothing$ if $x \notin A$.*

Definition 6. [3]. *Let U be an initial universe set and E be a universe set of parameters. Let (F, A) and (G, B) be soft sets over a common universe set U and $A, B \subseteq E$. Then*

(1) *(F, A) is a subset of (G, B), denoted by $(F, A)\tilde{\subseteq}(G, B)$, if*
(i) $A \subseteq B$; (ii) $F(x) \subseteq G(x)$ for all $x \in A$,

(2) *(F, A) equals (G, B), denoted by $(F, A) = (G, B)$, if $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$.*

Definition 7. [19]. *Let (F, A) and (G, B) be two soft sets over a common universe U. The union of (F, A) and (G, B) is defined to be a soft set (H, C), where $C = A \cup B$ and H is defined as follows:*

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ F(x) \cup G(x) & \text{if } x \in A \cap B \end{cases}$$

We write $(H, C) = (F, A)\tilde{\cup}(G, B)$.

Definition 8. [19]. *Let (F, A) and (G, B) be two soft sets over a common universe U. The intersection of (F, A) and (G, B) is defined to be a soft set (H, C) satisfying the following conditions:*

(1) *$C = A \cap B$,*
(2) *$H(x) = F(x)$ or $G(x)$ for each $x \in C$ (as both are same set).*

We write $(H, C) = (F, A)\tilde{\cap}(G, B)$.

Definition 9. [19]. *Let (F , A) and (G, B) be soft sets over a common universe set U.*

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(1) $(F,A)\tilde{\wedge}(G,B)$ *is a soft set defined by* $(F,A)\tilde{\wedge}(G,B) = (H, A\times B)$*, where* $H(x,y) = F(x)\cap G(y)$ *for any* $x \in A$ *and* $y \in B$*, where* $\cap$ *is the intersection operation of sets.*

(2) $(F,A)\tilde{\vee}(G,B)$ *is a soft set defined by* $(F,A)\tilde{\vee}(G,B) = (K, A\times B)$*, where* $K(x,y) = F(x)\cup G(y)$ *for any* $x \in A$ *and* $y \in B$*, where* $\cup$ *is the union operation of sets.*

4. Soft Boolean Algebras

Let K be a Boolean algebra and A be a nonempty set. R will refer to an arbitrary binary relation between an element of A and an element of K; that is, R is a subset of $A \times K$ unless otherwise specified. A set-valued function $F : A \to P(K)$ can be define as $F(x) = \{y \in K : (x,y) \in R\}$ for all $x \in A$. Then the pair (F,A) is a soft set over K, which is derived from the relation R.

Definition 10. *Let* (F,A) *be a soft set over* K*. Then* (F,A) *is called a soft Boolean algebra over* K *if* $F(x)$ *is a subalgebra of* K *for all* $x \in A$*.*

Example 1. *Let* $K = \{1,2,3,6\}$ *be set of all divisors of* 6*. Consider* $x \wedge y = mcd(x,y)$ $(2\wedge 3 = 1, 2\wedge 6 = 2)$*,* $x \vee y = mcm(x,y)(2 \vee 3 = 6, 2\vee 6 = 6)$ *and* $\neg x = \frac{6}{x}(\neg 2 = 3)$*. Then the structure* $\langle K, \wedge, \vee, \neg, 1, 6\rangle$ *is a Boolean algebra under the relation "* $\preceq$ *" which is given by* $x \preceq y$ *if* $x = x\wedge y$ *or* $x \vee y = y$*.*

Let (F,A) *be a soft set over* K*, where* $A = K$ *and* $F : A \longrightarrow P(K)$ *is a set-valued function defined by*

$$F(x) = \left\{ y \in K : xRy \Longleftrightarrow \left\{ \begin{array}{ll} y \in \{1,6\} & \text{if } x \notin \{2,3\} \\ y & \text{if } x \in \{2,3\} \end{array} \right\} \right\}$$

Then $F(1) = F(6) = \{1,6\}$ *,* $F(2) = F(3) = \{1,2,3,6\}$*. Therefore,* $F(x)$ *is a subalgebra of* K *for all* $x \in A$*. Hence* (F,A) *is a soft Boolean algebra over* K*.*

Now, let (G,B) *be a soft set over* K*, where* $B = K$ *and* $G : B \longrightarrow P(K)$ *is a set-valued function defined by*

$$G(x) = \{y \in K : xR'y \Longleftrightarrow mcd(x,y) \le mcm(x,y)\}.$$

Then $G(1) = G(2) = G(3) = G(6) = \{1,2,3,6\} = K$*. Hence* $G(x)$ *is a subalgebra of* K *for all* $x \in B$*. Then* (G,B) *is a soft Boolean algebra over* K*.*

Example 2. *Let* $M = \{a,b,c\}$ *be a set and* $K = FUN(M, \{0,1\})$ *be the set of all functions from* M *and to* $\{0,1\}$*. Define the Boolean operations on* K *as follows for all* $f, g \in K$*:*

$(f\vee g)(x) = \max\{f(x), g(x)\}$ *,* $(f\wedge g)(x) = \min\{f(x), g(x)\}$ *and* $\neg f(x) = \{0,1\} - \{f(x)\}$ *for all* $x \in A$*. Then* K *together with these operations is a Boolean algebra and consists of elements* $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$ *defined by*

$$\begin{array}{llll} f_1(a) = 0 & f_2(a) = 1 & f_3(a) = 1 & f_4(a) = 1 \\ f_1(b) = 0 & f_2(b) = 1 & f_3(b) = 0 & f_4(b) = 1 \\ f_1(c) = 0 & f_2(c) = 1 & f_3(c) = 0 & f_4(c) = 0 \end{array}$$

$$\begin{array}{llll} f_5(a) = 1 & f_6(a) = 0 & f_7(a) = 0 & f_8(a) = 0 \\ f_5(b) = 0 & f_6(b) = 1 & f_7(b) = 0 & f_8(b) = 1 \\ f_5(c) = 1 & f_6(c) = 1 & f_7(c) = 1 & f_8(c) = 0 \end{array}$$

Let (F,A) be a soft set over K, where $A = M$ and $F : A \longrightarrow P(K)$ is a set-valued function defined by

$$F(x) = \left\{ f \in K : xRf \Longleftrightarrow \left\{ \begin{array}{ll} f(a) = f(c) & \text{if } x = a \\ f(b) = f(a) & \text{if } x = b \\ f(c) = f(b) & \text{if } x = c \end{array} \right\} \right\}$$

Then $F(a) = \{f_1, f_2, f_5, f_8\}$, $F(b) = \{f_1, f_2, f_4, f_7\}$ and $F(c) = \{f_1, f_2, f_3, f_6\}$. Therefore, $F(x)$ is a subalgebra of K for all $x \in A$. Hence (F,A) is a soft Boolean algebra over K.

Example 3. *Every Boolean algebra can be considered as a soft Boolean algebra.*

Theorem 1. *Let (F,A) and (G,B) be two soft Boolean algebras over K. If $A \cap B \neq \varnothing$, then $(F,A)\tilde{\cap}(G,B)$ is a soft Boolean algebra over K.*

Proof. By Definition 8, we can write $(F,A)\tilde{\cap}(G,B) = (H,C)$, where $C = A \cap B$ and $H(x) = F(x)$ or $G(x)$ for all $x \in C$. For a mapping $H : C \longrightarrow P(K)$, (H,C) is a soft set over K. Since (F,A) and (G,B) are soft Boolean algebras over K, there exists an equality such that $H(x) = F(x)$ or $H(x) = G(x)$ for all $x \in C$. But in either case, $H(x)$ is a subalgebra of K for all $x \in C$. Hence $(H,C) = (FA)\tilde{\cup}(G,B)$ is a soft Boolean algebra over K. □

Theorem 2. *Let (F,A) and (G,B) be two soft Boolean algebras over K. If A and B are disjoint, then $(F,A)\tilde{\cup}(G,B)$ is a soft Boolean algebra over K.*

Proof. By Definition 7, we can write $(F,A)\tilde{\cup}(G,B) = (H,C)$, where $C = A \cup B$ and for all $x \in C$,

$$H(x) = \begin{cases} F(x) & \text{if } x \in A - B \\ G(x) & \text{if } x \in B - A \\ F(x) \cup G(x) & \text{if } x \in A \cap B \end{cases}$$

Since $A \cap B = \varnothing$, either $x \in A - B$ or $x \in B - A$. Since (F,A) is a soft Boolean algebra over K, then $H(x) = F(x)$ is a subalgebra of K for $x \in A - B$. Similarty, since (G,B) is a soft Boolean algebra over K, then $H(x) = G(x)$ is a subalgebra of K for $x \in B - A$. Hence (H,C) is a soft Boolean algebra over K and so $(H,C) = (F,A)\tilde{\cup}(G,B)$ is a soft Boolean algebra over K. □

Remark 2. *Let K be a Boolean algebra and S be a non-empty family of subalgebras of K. Note that intersection of members of S is again a subalgebra of K. But this is not correct for union. So Theorem 2 does not hold in general if $A \cap B \neq \varnothing$.*

Theorem 3. *Let (F,A) and (G,B) be two soft Boolean algebras over K, then $(F,A)\tilde{\wedge}(G,B)$ is a soft Boolean algebra over K.*

Proof. By Definition 9, we have

$$(F,A)\tilde{\wedge}(G,B) = (H, A \times B)$$

where $H(x,y) = F(x) \cap G(y)$ for all $x \in A$ and $y \in B$. Since (F,A) and (G,B) are soft Boolean algebras over K, then $F(x)$ and $G(y)$ are subalgebras of K for all $x \in A$, $y \in B$ and so the intersection $F(x) \cap G(y)$ is also a subalgebra of K. Hence $H(x,y)$ is a subalgebra of K for all $x \in A$ and $y \in B$. Then $(F,A)\tilde{\wedge}(G,B) = (H, A \times B)$ is a soft Boolean algebra over K. □

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Theorem 4. *Let (F, A) be a soft Boolean algebra over K. If B is a subset of A, then $(F\mid_B, B)$ is a soft Boolean algebra over K.*

Proof. Since (F, A) is a soft Boolean algebra over K, then $F(x)$ is a subalgebra of K for all $x \in A$. Therefore, $F(x)$ is also a subalgebra of K for all $x \in B \subseteq A$. Hence $(F\mid_B, B)$ is a soft Boolean algebra over K. □

Proposition 1. *Let (G, B) be a soft set over K and (F_α, A_α) be a soft Boolean algebra over K for $\alpha \in \Lambda$, where Λ is an index set. Then*

$$\langle (G, B)\rangle = \cap \{(F_\alpha, A_\alpha) : (G, B) \subseteq (F_\alpha, A_\alpha)\ \},$$

where (F_α, A_α) is a soft Boolean algebra over K for $\alpha \in \Lambda$, is a soft Boolean algebra over K. We say that $\langle (G, B)\rangle$ is a soft Boolean algebra over K, which is generated by (G, B) .

Definition 11. *Let (F, A) and (G, B) be two soft Boolean algebras over K. Then (G, B) is a soft subalgebra of (F, A), denoted by $(G, B)\tilde{<}(F, A)$, if*

(1) *$B \subseteq A$,*
(2) *$G(x)$ is a subalgebra of $F(x)$ for all $x \in B$.*

Example 4. *Let (F, A) be a soft set over K and $A = K$, where K is the Boolean algebra given in Example 1 and $F : A \to P(K)$ is a set-valued function defined by*

$$F(x) = \{y \in K : xRy \Leftrightarrow mcd(x, y) \leq x\}.$$

Then $F(1) = F(2) = F(3) = F(6) = \{1, 2, 3, 6\} = K$ for all $x \in A$ and hence (F, A) is a soft Boolean algebra over K.

Let $B = \{1, 2, 3\}$ and $G : B \to P(K)$ be the set-valued function defined by

$$G(x) = \{1\} \cup \{y \in K : xR'y \Leftrightarrow x \prec y\},$$

where " $\prec$ " is the operation (strictly) defined on K for all $x \in B$. Then $G(1) = \{1, 2, 3, 6\}$, $G(2) = G(3) = \{1, 6\}$ and hence (G, B) is a soft Boolean algebra over K. Moreover, since $G(x)$ is a subalgebra of $F(x)$ for all $x \in B$, then (G, B) is a soft subalgebra of (F, A).

The class of all soft subalgebras of a soft Boolean algebra (F, A) is a complete lattice under the relation of being a soft subalgebra $(\tilde{<})$.

Theorem 5. *Let (F, A) and (G, A) be two soft Boolean algebras over K.*

(1) *If $G(x) \subseteq F(x)$ for all $x \in A$, then $(G, A)\tilde{<}(F, A)$.*
(2) *If (F, A) is a soft Boolean algebra over K such that $F(x) = \{\bar{0}, \bar{1}\}$ for all $x \in A$, then $(F, A)\tilde{<}(G, B)$ where (G, B) is any soft Boolean algebra over K.*

Proof. The proof is clear. □

Theorem 6. *(F, A) is a soft Boolean algebra over K and $\{(G_\alpha, B_\alpha) : \alpha \in \Lambda\}$ is a nonempty family of soft subalgebras of (F, A), where Λ is an index set. Then*

(1) *$\tilde{\cap}_{\alpha\in\Lambda} (G_\alpha, B_\alpha)$ is a soft subalgebra of (F, A),*
(2) *$\tilde{\wedge}_{\alpha\in\Lambda} (G_\alpha, B_\alpha)$ is a soft subalgebra of (F, A),*
(3) *$\tilde{\vee}_{\alpha\in\Lambda} (G_\alpha, B_\alpha)$ is a soft subalgebra of (F, A) if $B_\alpha \cap B_\beta = \varnothing$ for all $\alpha, \beta \in \Lambda$.*

Proof. The proof is clear. □

Definition 12. *Let (F, A) be a soft Boolean algabra over K. Then*

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(1) (F, A) *is said to be an identity soft Boolean algebra over* K *if* $F(x) = \{\bar{0}, \bar{1}\}$ *for all* $x \in A$*, where* $\bar{0}$ *and* $\bar{1}$ *is two distinguished elements of* K*.*
(2) (F, A) *is said to be an absolute soft Boolean algebra over* K *if* $F(x) = K$ *for all* $x \in A$*.*

Example 5. *Consider the Boolean algebra* $K = FUN(M, \{0, 1\})$ *which is given Example 2. Let* (F, A) *be a soft set over* K*,* $A = M$ *and* $F : A \to P(K)$ *be a set-valued function defined by*

$$F(x) = \{f \in K : xRf \Leftrightarrow f(x) \in \{0, 1\}\}$$

for all $x \in A$*. Then* $F(a) = F(b) = F(c) = K$ *and so* (F, A) *is an absolute soft Boolean algebra over* K*.*

Next, let (G, B) *be a soft set over* K*, where* $B = M$ *and* $G : B \to P(K)$ *is a set-valued function defined by*

$$G(x) = \{f \in K : xR'f \Leftrightarrow f(x) = f(y) \text{ for } x \neq y\}.$$

Then $G(a) = G(b) = G(c) = \{f_1, f_2\}$ *and hence* (G, B) *is an identity soft Boolean algebra over* K*.*

Definition 13. *Let* (F, A) *be a soft Boolean algebra over* K*. A soft set* (G, B) *over* K *is called a soft filter on* (F, A)*, denoted by* $(G, B)\tilde{\triangleright}(F, A)$*, if it satisfies the following conditions:*

(1) $B \subseteq A$,
(2) $G(x)$ *is a filter of* $F(x)$ *for all* $x \in B$*.*

Definition 14. *Let* (F, A) *be a soft Boolean algebra over* K*. A soft set* (G, B) *over* K *is called a soft ideal on* (F, A)*, denoted by* $(G, B)\tilde{\triangleleft}(F, A)$*, if it satisfies the following conditions:*

(1) $B \subseteq A$,
(2) $G(x)$ *is an ideal of* $F(x)$ *for all* $x \in B$*.*

Example 6. *Let* (F, A) *be soft Boolean algebra over* K*, which is given in Example 5. That is,* $A = M$ *and* $F : A \to P(K)$ *is the set-valued function defined by*

$$F(x) = \{f \in K : xRf \Leftrightarrow f(x) \in \{0, 1\}\}$$

such that $F(x) = K$ *for all* $x \in A$*.*

Let (G, B) *be a soft set over* K*,* $B = M$ *and* $G : B \to P(K)$ *be a set-valued function defined by*

$$G(x) = \{f \in K : xRf \Leftrightarrow f(x) = 1\}$$

for all $x \in B$*. Then* $G(a) = \{f_2, f_3, f_4, f_5\}$*,* $G(b) = \{f_2, f_4, f_6, f_8\}$ *and* $G(c) = \{f_2, f_5, f_6, f_7\}$*. So* $G(x)$ *is a filter of* $F(x)$ *for all* $x \in B$*. Hence* (G, B) *is a soft filter on* (F, A)*.*

Let $C = M$ *and* $H : C \to P(K)$ *be the set-valued function defined by*

$$H(x) = \{f \in K : xRf \Leftrightarrow f(x) = 0\}$$

for all $x \in C$*. Then* $H(a) = \{f_1, f_6, f_7, f_8\}$*,* $H(b) = \{f_1, f_3, f_5, f_7\}$ *and* $H(c) = \{f_1, f_3, f_4, f_8\}$*. So* $H(x)$ *is an ideal of* $F(x)$ *for all* $x \in C$*. Hence* (H, C) *is a soft ideal on* (F, A)*.*

Remark 3. (F, A) *be a soft Boolean algebra over* K*.*

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(1) *If* (G, B) *is a soft filter on* (F, A), *then*

$$(H, C) = \{F(x) - G(x) : G(x) \text{ is a filter of } F(x) \text{ for all } x \in B\}$$

is a soft ideal on (F, A).

(2) *If* (H, C) *is a soft ideal on* (F, A), *then*

$$(G, B) = \{F(x) - H(x) : H(x) \text{ is an ideal of } F(x) \text{ for all } x \in C\}$$

is a soft filter on (F, A). *If this is the case we say that* (G, B) *and* (H, C) *are dual to each other.*

Definition 15. *Let* (F, A) *be a soft Boolean algebra over* K *and* (G, B) *a soft set over* K *such that* $B \subseteq A$. *Then*

(1) (G, B) *is said to be identity soft ideal on* (F, A) *if* $G(x) = \{\bar{0}\}$ *for every* $x \in B$,

(2) (G, B) *is said to be identity soft filter on* (F, A) *if* $G(x) = \{\bar{1}\}$ *for every* $x \in B$.

Example 7. *Let* (F, A) *be a soft Boolean algebra over* K, *which is given in Example 2 and* (G, B) *be a soft set over* K. *Let* $B = M$ *and* $G : B \to P(K)$ *be a set-valued function defined by*

$$G(x) = \{f \in K : xRf \Leftrightarrow f(x) = f(y) = 0 \text{ for } x \neq y\}$$

for all $x \in B$. *Then* $G(a) = G(b) = G(c) = \{f_1\}$ *and so* (G, B) *is an identity soft ideal on* (F, A).

Let $C = M$ *and* $H : C \to P(K)$ *be the set-valued function defined by*

$$H(x) = \{f \in K : xRf \Leftrightarrow f(x) = f(y) = 1 \text{ for } x \neq y\}$$

for all $x \in C$. *Then* $H(a) = H(b) = H(c) = \{f_2\}$ *and so* (H, C) *is an identity soft filter on* (F, A).

Definition 16. *Let* (F, A) *be a soft Boolean algebra over* K. *Then*

(1) (F, A) *is called to be an atomic soft Boolean algebra over* K, *if* $F(x)$ *is an atomic subalgebra of* K *for all* $x \in A$,

(2) (F, A) *is called to be a complete soft Boolean algebra over* K, *if* $F(x)$ *is a complete subalgebra of* K *for all* $x \in A$.

Theorem 7. *Let* (F, A) *be a soft Boolean algebra over* K. *If* K *is a finite Boolean algebra, then* (F, A) *is a complete and atomic soft Boolean algebra over* K.

Proof. Let (F, A) be a soft Boolean algebra over K. Since K is a finite Boolean algebra, it is both complete and atomic. Therefore, we have that $F(x)$ is a complete and atomic subalgebra of K for all $x \in A$. Hence (F, A) is a complete and atomic soft Boolean algebra over K. □

Definition 17. *Let* K *and* L *be two Boolean algebras and* $f : K \longrightarrow L$ *be a mapping of Boolean algebras. If* (F, A) *and* (G, B) *are soft sets over* K *and* L, *respectively then*

(1) $(f(F), A)$ *is a soft set over* L, *where* $f(F) : A \longrightarrow P(L)$ *is defined by* $f(F)(x) = f(F(x))$ *for all* $x \in A$,

(2) $(f^{-1}(G), B)$ *is a soft set over* K, *where* $f^{-1}(G) : B \longrightarrow P(K)$ *is defined by* $f^{-1}(G)(y) = f^{-1}(G(y))$ *for all* $y \in B$.

Proposition 2. *Let* $f : K \longrightarrow L$ *be an onto Boolean homomorphism. Then*

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(1) *if* (F, A) *is a soft Boolean algebra over* K*, then* $(f(F), A)$ *is a soft Boolean algebra over* L*,*
(2) *if* (G, B) *is a soft Boolean algebra over* L*, then* $(f^{-1}(G), B)$ *is a soft Boolean algebra over* K *if it is non-null.*

Proof. Let $f : K \longrightarrow L$ be an onto Boolean homomorphism.

(1) Since $F(x)$ is a subalgebra of K and its homomorphic image is a subalgebra of L for all $x \in A$, it follows that $f(F)(x) = f(F(x))$ is a subalgebra of L. Hence $(f(F), A)$ is a soft Boolean algebra over L.
(2) Because of the fact that (G, B) is a soft Boolean algebra over L, we have $G(y)$ is a subalgebra of L for all $y \in B$. Since f is a Boolean homomorphism, its homomophic inverse image $f^{-1}(G(y))$ is also a subalgebra of K for all $y \in B$. Then $(f^{-1}(G), B)$ is a soft Boolean algebra over K.

□

Proposition 3. *Let* (F, A) *be a soft Boolean algebra over* K *and* (G, B) *be a soft subalgebra of* (F, A)*. If* f *is a Boolean homomorphism from* K *to* L*, then* $(f(G), B)$ *is a soft subalgebra of* $(f(F), A)$*.*

Proof. If (G, B) is a soft subalgebra of (F, A), then $B \subseteq A$ and $G(x)$ is a subalgebra of $F(x)$ for all $x \in B$. Since f is a Boolean homomorphism from K to L and homomorphic image of a subalgebra in K is a subalgebra in L, we have that $f(F(x))$ and $f(G(y))$ are subalgebras of L for all $x \in A$ and $y \in B$. Also, $f(G(y))$ is a subalgebra of $f(F(x))$ for all $y \in B$. Hence $(f(G), B)$ is a soft subalgebra of $(f(F), A)$. □

Proposition 4. *Let* (F, A) *be a soft Boolean algebra over* L *and* (G, B) *be a soft subalgebra of* (F, A)*. If* f *is a Boolean homomorphism from* K *to* L*, then* $(f^{-1}(G), B)$ *is a soft subalgebra of* $(f^{-1}(F), A)$*.*

Proof. The proof is made similar to Proposition 3. □

Proposition 5. *Let* (F, A) *be a soft Boolean algebra over* K *and* (G, B) *be a soft set over* K*. Suppose that* f *is a Boolean homomorphism from* K *to* L*. Then*

(1) *if* (G, B) *is a soft filter on* (F, A) *then* $(f(G), B)$ *is a soft filter on* $(f(F), A)$
(2) *if* (G, B) *is a soft ideal on* (F, A) *then* $(f(G), B)$ *is a soft ideal on* $(f(F), A)$*.*

Proof. The proof is made similar to Proposition 3. □

Theorem 8. *Let* $f : K \longrightarrow L$ *be a Boolean homomorphism. Suppose that* (F, A) *and* (G, B) *are two soft Boolean algebras over* K *and* L*, respectively.*

(1) *If* (F, A) *is an identity soft Boolean algebra over* K*, then* $(f(F), A)$ *is an identity soft Boolean algebra over* L*.*
(2) *If* f *is onto and* (F, A) *is an absolute soft Boolean algebra, then* $(f(F), A)$ *is an absolute soft Boolean algebra over* L*.*
(3) *If* $G(y) = f(K)$ *for all* $y \in B$*, then* $(f^{-1}(G), B)$ *is an absolute soft Boolean algebra over* K*.*
(4) *If* f *is injective and* (G, B) *is an identity soft Boolean algebra, then* $(f^{-1}(G), B)$ *is an identity soft Boolean algebra over* K*.*

Proof. Suppose that $f : K \longrightarrow L$ is a Boolean homomorphism.

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(1) Let (F,A) be an identity soft Boolean algebras over K. Then for all $x \in A$, we have $F(x) = \{\bar{0}_K, \bar{1}_K\}$, where $\bar{0}_K$ and $\bar{1}_K$ are two distinguished elements of K. Since f is a Boolean homomorphism, $f(F)(x) = f(F(x)) = f(\{\bar{0}_K, \bar{1}_K\}) = \{\bar{0}_L, \bar{1}_L\}$ for all $x \in A$. Then $(f(F), A)$ is an identity soft Boolean algebra over L.
(2) Let f be onto and (F,A) is absolute soft Boolean algebra. Then, $F(x) = K$ for all $x \in A$, and so $f(F)(x) = f(F(x)) = f(K) = L$ for all $x \in A$. Then $(f(F), A)$ is an absolute soft Boolean algebra over L.
(3) Let $G(y) = f(K)$ for all $y \in B$. Then $f^{-1}(G)(y) = f^{-1}(G(y)) = f^{-1}(f(K)) = K$ for all $y \in B$. Hence $(f^{-1}(G), B)$ is an absolute soft Boolean algebra over K.
(4) Let f be injective and (G,B) is an identity soft Boolean algebra over L. Then $G(y) = \{\bar{0}_L, \bar{1}_L\}$ for all $y \in B$ and so $f^{-1}(G)(y) = f^{-1}(G(y)) = f^{-1}(\{\bar{0}_L, \bar{1}_L\}) = \{\bar{0}_K, \bar{1}_K\}$. Then $(f^{-1}(G), B)$ is an identity soft Boolean algebra over K.

□

Definition 18. *Let (F,A) and (G,B) be two soft Boolean algebras over K and L respectively. Let $f : K \longrightarrow L$ and $g : A \longrightarrow B$. Then (f,g) is said to be a soft Boolean homomorphism if*

(1) *f is a Boolean homomorphism from K onto L,*
(2) *g is a mapping from A onto B,*
(3) *$f(F(x)) = G(g(x))$ for all $x \in A$.*

Then (F,A) is said to be soft Boolean homomorphic to (G,B) and it is denoted by $(F,A) \sim (G,B)$. If f is a Boolean isomorphism from K onto L and g is a bijection from A to B, then (f,g) is said to be a soft Boolean isomorphism. If there exists a such isomorphism, we say that (F,A) is soft Boolean isomorphic to (G,B) and denote by $(F,A) \simeq (G,B)$.

Example 8. *Let (G,B) be the soft Boolean algebra given in Example 1. For $C = \{a,b\}$, consider a Boolean algebra L consisting of all subsets of C. Define the set-valued function H by*

$$H(x) = \{X \in L : xRX \Leftrightarrow \{x\} \cup X \subseteq \{a,b\}\}.$$

Then $H(a) = H(b) = L$. Therefore, $H(x)$ is a subalgebra of L for all $x \in C$. Hence (H,C) is a soft Boolean algebra over L. Now, define $f : K \longrightarrow L$ by $f(1) = \varnothing$, $f(2) = \{a\}$, $f(3) = \{b\}$ and $f(6) = \{a,b\}$, and $g : B \longrightarrow C$ by $g(1) = a$, $g(6) = a$, $g(2) = b$ and $g(3) = b$. Then f is a Boolean homomorphism from K to L and g is a mapping from B to C. Moreover, we have $f(G(x)) = H(g(x))$ for all $x \in B$. So (G,B) is soft Boolean homomorphic to (H,C).

Theorem 9. *Let K and L be Boolean algebras and (F,A), (G,B) soft sets over K and L, respectively. If (F,A) is a soft Boolean algebra over K and $(F,A) \simeq (G,B)$, then (G,B) is a soft Boolean algebra over L.*

Proof. The proof is clear. □

Definition 19. *Let (F,A) and (G,B) be two soft Boolean algebras over K and L, respectively. The product of soft Boolean algebras (F,A) and (G,B) is defined as $(F,A) \times (G,B) = (H, A \times B)$, where $H(x,y) = F(x) \times G(y)$ for all $(x,y) \in A \times B$.*

Theorem 10. *Let (F, A) and (G, B) be two soft Boolean algebras over K and L, respectively. If it is non-null, then the product $(F, A) \times (G, B)$ is a soft Boolean algebra over $K \times L$.*

Proof. Let $(F, A) \times (G, B) = (H, A \times B)$, where $H(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$. Then by hypothesis, $(H, A \times B)$ is a non-null soft set over $K \times L$. Since $F(x)$ is a subalgebra of K and $G(y)$ is a subalgebra of L, it follows that $H(x, y)$ is a subalgebra of $K \times L$ for all $x \in A$ and $y \in B$. Therefore $(H, A \times B)$ is a soft Boolean algebra over $K \times L$. □

5. Conclusion

In this paper, we have introduced the concept of soft Boolean algebra and have studied some of their algebraic properties. Basic notions such as soft subalgebra, soft ideal, soft filter and soft Boolean homomorphism were introduced and their properties have been investigated. Many examples supporting the results obtained were also given . The aim of this article is to obtain next algebraic structure by combining the notion of soft set theory with Boolean algebra which is a well known algebra. We hope that ideas and methods developed in this paper will be a source of inspiration for further study.

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GENERATING FUNCTIONS FOR THE GENERALIZED BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. The main object of this study is to derive various families of multilinear and multilateral generating functions for the generalized bivariate Fibonacci and Lucas polynomials. Furthermore, we discuss some critical connections between the generalized bivariate Fibonacci, Lucas polynomials and the well-known polynomials and numbers, such as, bivariate and univariate Fibonacci and Lucas polynomials, the classical Fibonacci and Lucas numbers, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas and also the first and second kind Chebyshev polynomials.

1. Introduction

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by $(U_n) = (U_n(x,y))$ and $(V_n) = (V_n(x,y))$, are defined by

$$U_0 = 0,\ U_1 = 1,\ U_n = xU_{n-1} + yU_{n-2},\ (n \geq 2)$$

and

$$V_0 = 2,\ V_1 = x,\ V_n = xV_{n-1} + yV_{n-2},\ (n \geq 2).$$

It is established, see for example [7, 9], that

$$U_n(x,y) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} x^{n-2k-1} y^k, \tag{1}$$

$$V_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k. \tag{2}$$

In [3], G.P. Djordjević considered the partial derivative sequences of the generalized bivariate Fibonacci polynomials $U_{n,m}(x,y)$ and the generalized bivariate Lucas polynomials $V_{n,m}(x,y)$. These polynomials are defined by

$$U_{n,m} = xU_{n-1,m} + yU_{n-m,m}\ ,\ n \geq m,$$

with $U_{0,m} = 0,\ U_{n,m} = x^{n-1},\ n = 1, 2, ..., m-1$ and

$$V_{n,m} = xV_{n-1,m} + yV_{n-m,m}\ ,\ n \geq m,$$

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with $V_{0,m} = 2,\ V_{n,m} = x^n,\ n = 1, 2, ..., m-1$ and generated by

$$t\left(1 - xt - yt^m\right)^{-1} = \sum_{n=0}^{\infty} U_{n,m}\left(x, y\right) t^n \tag{3}$$

and

$$\left(2 - xt^{m-1}\right)\left(1 - xt - yt^m\right)^{-1} = \sum_{n=0}^{\infty} V_{n,m}\left(x, y\right) t^n. \tag{4}$$

Clearly, (3) and (4) yield the following explicit representations, respectively:

$$U_{n,m}\left(x, y\right) = \sum_{k=0}^{\left[\frac{n-1}{m}\right]} \binom{n-1-(m-1)k}{k} x^{n-1-mk} y^k \tag{5}$$

and

$$V_{n,m}\left(x, y\right) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk} y^k. \tag{6}$$

If $m = 2$, then polynomials $U_{n,m}\left(x, y\right)$ and $V_{n,m}\left(x, y\right)$ would reduce at once to the polynomials $U_n\left(x, y\right)$ and $V_n\left(x, y\right)$ given by (1) and (2), respectively.

The aim of this paper is to obtain various families of multilateral and multilinear generating functions for the generalized Fibonacci and Lucas polynomials. We also give their special cases for these polynomials. For the theory and applications of the various methods and techniques for deriving generating functions of special functions and polynomials, we may refer the interested reader to a recent treatise on the subject of generating functions [1, 2].

2. Bilinear and Bilateral Generating Functions

In this section, firstly we derive several families of bilinear and bilateral generating functions for the generalized bivariate Fibonacci Polynomials $U_{n,m}\left(x, y\right)$ which are generated by (3) and given explicitly by (5).

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s complex variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) z^k \tag{7}$$

$$(a_k \neq 0\,,\ \ \mu, \nu \in \mathbb{C})$$

and

$$\Theta_{n,m,p,\mu,\nu}(x, y; y_1, ..., y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k U_{n-pk,m}\left(x, y\right) \Omega_{\mu+\nu k}(y_1, ..., y_s) \zeta^k \tag{8}$$

$$n, p \in \mathbb{N}.$$

Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,m,p,\mu,\nu}(x, y; y_1, ..., y_s; \frac{\eta}{t^p}) t^n = \frac{t}{1 - xt - yt^m} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \tag{9}$$

provided that each member of (9) exists.

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Proof. For convenience, let S denote the first member of the assertion (9) of Theorem 2.1. Then, upon substituting for the polynomials

$$\Theta_{n,m,p,\mu,\nu}(x,y;y_1,...,y_s;\frac{\eta}{t^p})$$

from the definition (8) into the left-hand side of (9), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kU_{n-pk,m}\left(x,y\right)\Omega_{\mu+\nu k}(y_1,...,y_s\,)\eta^kt^{n-pk}\,. \tag{10}$$

Upon inverting the order of summation in (10), if we replace n by $n+pk$, we can write

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_k\,U_{n,m}\left(x,y\right)\Omega_{\mu+\nu k}(y_1,...,y_s\,)\eta^kt^n \\ &= \sum_{n=0}^{\infty}U_{n,m}\left(x,y\right)t^n\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,...,y_s\,)\eta^k \\ &= \frac{t}{1-xt-yt^m}\Lambda_{\mu,\nu}(y_1,...,y_s;\eta), \end{aligned}$$

which completes the proof of Theorem 2.1. □

The partial derivatives of $U_{n,m}\left(x,y\right)$ are defined by [3]

$$U_{n,m}^{(k,j)}\left(x,y\right)=\frac{\partial^{k+j}}{\partial x^k\partial y^j}U_{n,m}\left(x,y\right),\ k\geq 0,\ j\geq 0.$$

Let $k\geq 0,\ j\geq 0,\ r\geq 0$. Then, the generalized bivariate Fibonacci Polynomials $U_{n,m}\left(x,y\right)$ have the following relation [3]:

$$\begin{aligned} &\sum_{i_1+i_2+...+i_r=n}U_{i_1,m}^{(k,j)}\left(x,y\right)U_{i_2,m}^{(k,j)}\left(x,y\right)...U_{i_r,m}^{(k,j)}\left(x,y\right) \\ &= \frac{((k+j)!)^r}{(rk+rj+r-1)!}U_{n,m}^{(rk+r-1,rj)}\left(x,y\right). \end{aligned} \tag{11}$$

In precisely the same manner as described proof of Theorem 2.1 and using (11), we can prove the following result, immediately.

Theorem 2.2. *For a non-vanishing function $\Omega_{\mu}(y_1,...,y_s\,)$ of complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ let*

$$\begin{aligned} &\Lambda_{\mu,\nu}^{n,p}(x,y;y_1,...,y_s;z) \\ :&= \sum_{h=0}^{[n/p]}a_h\frac{((k+j)!)^r}{(rk+rj+r-1)!}U_{n-rph,m}^{(rk+r-1,rj)}\left(x,y\right)\Omega_{\mu+\nu h}(y_1,...,y_s\,)z^h, \end{aligned}$$

$$(a_h\neq 0\,;\quad n,k,j,r\in\mathbb{N}_0;\ \mathbb{N}_0=\mathbb{N}\cup\{0\})\,.$$

Then we have

$$\begin{aligned} &\sum_{i_1+i_2+...+i_r=n}\ \sum_{l=0}^{[i_1+i_2+...+i_r/p]}a_l\,U_{i_1-pl,m}^{(k,j)}\left(x,y\right)...U_{i_r-pl,m}^{(k,j)}\left(x,y\right)\Omega_{\mu+\nu l}(y_1,...,y_s\,)z^l \\ &\qquad = \Lambda_{\mu,\nu}^{n,p}(x_1,...,x_r;y_1,...,y_s;z) \end{aligned} \tag{12}$$

provided that each member of (12) exists.

Now we derive several families of bilinear and bilateral generating functions for the generalized bivariate Lucas Polynomials $V_{n,m}(x,y)$ which are generated by (4) and given explicitly by (6).

Theorem 2.3. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_s)$ of s complex variables $y_1,...,y_s$ $(s\in\mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1,...,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,...,y_s)z^k$$

$$(a_k\neq 0\,,\ \ \mu,\nu\in\mathbb{C})$$

and

$$\Theta_{n,m,p,\mu,\nu}(x,y;y_1,...,y_s;\zeta):=\sum_{k=0}^{[n/p]}a_kV_{n-pk,m}(x,y)\,\Omega_{\mu+\nu k}(y_1,...,y_s)\zeta^k$$

$$n,p\in\mathbb{N}.$$

Then we have

$$\sum_{n=0}^{\infty}\Theta_{n,m,p,\mu,\nu}(x,y;y_1,...,y_s;\frac{\eta}{t^p})t^n=\frac{2-xt^{m-1}}{1-xt-yt^m}\Lambda_{\mu,\nu}(y_1,...,y_s;\eta) \tag{13}$$

provided that each member of (13) exists.

Proof. In precisely the same manner as described proof of Theorem 2.1 and using the generating function (4) we can prove Theorem 2.3. □

By expressing the multivariable function $\Omega_{\mu+\nu k}(y_1,...,y_s)$ $(k\in\mathbb{N}_0\,,\ s\in\mathbb{N})$ in terms of simpler function of one and more variables, we can give further applications of Theorems 2.1, 2.2 and 2.3. For example, if we set

$$s=r\ \text{ and }\ \Omega_{\mu+\nu k}(y_1,...,y_r)=h_{\mu+\nu k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 2.1, where a multivariable extension of the Lagrange-Hermite polynomials $h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ are defined by means of the generating function [1]

$$\prod_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)t^n, \tag{14}$$

$$\left(\alpha_j\in\mathbb{C}\ (j=1,...,r)\ ;\ |t|<\min_{j\in\{1,...,r\}}\left\{|x_1|^{-1/j}\right\}\right),$$

then we obtain the following result which provides a class of bilateral generating functions for the multivariable Lagrange-Hermite polynomials and the generalized bivariate Fibonacci polynomials.

Corollary 2.4. *If $\Lambda_{\mu,\nu}(y_1,...,y_r;z):=\sum_{k=0}^{\infty}a_kh_{\mu+\nu k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)z^k$ where $(a_k\neq 0\,,\ \mu,\nu\in\mathbb{C})$; and*

$$\Theta_{n,m,p,\mu,\nu}(x,y;y_1,...,y_r;\zeta):=\sum_{k=0}^{[n/p]}a_kU_{n-pk,m}(x,y)\,h_{\mu+\nu k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\zeta^k$$

$$n,p\in\mathbb{N}.$$

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Then we have

$$\sum_{n=0}^{\infty}\Theta_{n,m,p,\mu,\nu}(x,y;y_1,...,y_s;\frac{\eta}{t^p})t^n=\frac{t}{1-xt-yt^m}\Lambda_{\mu,\nu}(y_1,...,y_s;\eta) \tag{15}$$

provided that each member of (15) exists.

Remark 2.1. *Using the generating relation (14) for the multivariable Lagrange-Hermite polynomials and taking $a_k=1$, $\mu=0$, $\nu=1$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}U_{n-pk,m}\left(x,y\right)h_k^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\eta^k t^{n-pk}\\ =\;&\frac{t}{1-xt-yt^m}\prod_{j=1}^{r}\left\{(1-y_j\eta^j)^{-\alpha_j}\right\},\end{aligned}$$

where

$$|\eta|<\min\left\{|y_1|^{-1},...,|y_r|^{-1/r}\right\}.$$

Choosing $s=2$ and $\Omega_{\mu+\nu k}(u_1,u_2)=U^{(k,j)}_{\mu+\nu k,m}\left(u_1,u_2\right)$, $(\mu,\nu\in\mathbb{N}_0)$, in Theorem 2.2 we obtain the following class of bilinear generating functions for the partial derivatives of the generalized bivariate Fibonacci polynomials $U_{n,m}\left(x,y\right)$.

Corollary 2.5. *If*

$$\begin{aligned}&\Lambda^{n,p}_{\mu,\nu}(x,y;u_1,u_2;z)\\ :=\;&\sum_{h=0}^{[n/p]}a_h\frac{((k+j)!)^r}{(rk+rj+r-1)!}U^{(rk+r-1,rj)}_{n-rph,m}\left(x,y\right)U^{(k,j)}_{\mu+\nu k,m}\left(u_1,u_2\right)z^h,\end{aligned}$$

where $a_h\neq 0$, $\mu,\nu\in\mathbb{N}_0$. Then we have

$$\begin{aligned}&\sum_{i_1+i_2+...+i_r=n}\;\sum_{l=0}^{[i_1+i_2+...+i_r/p]}a_l\,U^{(k,j)}_{i_{1-pl},m}\left(x,y\right)...U^{(k,j)}_{i_r-pl,m}\left(x,y\right)U^{(k,j)}_{\mu+\nu l,m}\left(u_1,u_2\right)z^l\\ &\qquad=\Lambda^{n,p}_{\mu,\nu}(x,y;u_1,u_2;z)\end{aligned} \tag{16}$$

provided that each member of (16) exists.

If we set

$$s=1\;\text{ and }\;\Omega_{\mu+\nu k}(u)=E_{\mu+\nu k}(u)$$

in Theorem 2.3, where the Euler polynomials $E_n\left(x\right)$ is defined by means of the generating function [4]

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n\left(x\right)\frac{t^n}{n!}, \tag{17}$$

then we obtain the following result which provides a class of bilateral generating functions for the Euler polynomials and the generalized bivariate Lucas polynomials.

Corollary 2.6. *If $\Lambda_{\mu,\nu}(u;z):=\sum\limits_{k=0}^{\infty}a_kE_{\mu+\nu k}(u)z^k$ where $(a_k\neq 0,\ \mu,\nu\in\mathbb{C})$; and*

$$\Theta_{n,m,p,\mu,\nu}(x,y;u;\zeta):=\sum_{k=0}^{[n/p]}a_kV_{n-pk,m}\left(x,y\right)E_{\mu+\nu k}(u)\zeta^k$$

$$n, p \in \mathbb{N}.$$

Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,m,p,\mu,\nu}(x, y; u; \frac{\eta}{t^p})t^n = \frac{2 - xt^{m-1}}{1 - xt - yt^m} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \tag{18}$$

provided that each member of (18) exists.

Remark 2.2. *Using the generating relation (17) for the Euler polynomials and taking* $a_k = \frac{1}{k!}$, $\mu = 0$, $\nu = 1$, *we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \frac{1}{k!} V_{n-pk,m}(x, y)\, E_k(u) \eta^k t^{n-pk} \\ = \; &\frac{2e^{u\eta}\left(2 - xt^{m-1}\right)}{\left(e^{\eta} + 1\right)\left(1 - xt - yt^m\right)}. \end{aligned}$$

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\nu k}(y_1, ..., y_s)$, ($s \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorems 2.1, 2.2 and 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the generalized bivariate Fibonacci and Lucas polynomials.

3. Further Consequences

In this section, we give some special cases of the results obtain in the previous section. Here, we only discuss some critical connections between the generalized bivariate Fibonacci, Lucas polynomials and the well-known polynomials and numbers, but we avoid their statements and proofs.

- **(a)** In the case of $m = 2$, our all results contain several families of multilateral and multilinear generating functions of the bivariate polynomials of Fibonacci and Lucas given in (1) and (2).

- **(b)** If we choose $m = 2$ and $y = 1$, then we get miscellaneous results for the families of multilinear and multilateral generating functions of the (univariate) Fibonacci and Lucas polynomials, respectively, by (see [8]):

$$f_n(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-k}{k} x^{n-1-2k},$$

$$l_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}.$$

- **(c)** If we take $m = 2$ and $x = y = 1$, then we now obtain some families of multilinear and multilateral generating functions for the well-known Fibonacci and Lucas numbers defined by (see, for example, [8])

$$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-k}{k},$$

$$L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k}.$$

(d) Of course, replacing x by $2x$ in (1) and (2) (and resp. replacing y by $2y$ in (1) and (2)) from **(a)**, we get various families of multilinear and multilateral generating functions for the Pell and Pell-Lucas polynomials (and resp. Jacobsthal and Jacobsthal-Lucas polynomials) [5, 6].

(e) By using the following relation (see [8]) between Fibonacci and the first kind Chebyshev polynomials $U_n(x)$

$$f_n(x) = ni^{n-1} U_{n-1}\left(-\frac{ix}{2}\right), \ \left(n \geq 1 \text{ and } i = \sqrt{-1}\right),$$

all results in **(b)** can be modified according to the polynomials $U_n(x)$.

(f) Similarly, with the help of the fact that

$$l_n(x) = 2i^n T_n\left(-\frac{ix}{2}\right), \ (n \geq 0),$$

where $T_n(x)$ denotes the second kind Chebyshev polynomials, the results in **(b)** are also valid for the polynomials $T_n(x)$. We should recall that the well-known properties of the first and second kind Chebyshev polynomials may be found in the books [10].

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Integral norms of $Q_{K,\omega}(p,q;n)$ spaces and weighted Bloch spaces

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Abstract

In this paper we characterize $Q_{K,\omega}(p,q;n)$ spaces of analytic functions on the unit disk in terms of nondecreasing functions. The relations between integral norms of $Q_{K,\omega}(p,q;n)$ spaces and the norms of weighted Bloch spaces $\mathcal{B}^{\alpha}_{\omega}$ are given. Further, we obtain similar criteria for the little weighted Bloch space of analytic functions.

1 Introduction

Let $\Delta = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [1, 2, 23, 26, 27, 28]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty\}.$$

Let $0 < q < \infty$. The Besov-type spaces

$$\mathbf{B^q} = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^q \left(1-|z|^2\right)^{q-2} (1-|\varphi_a(z)|^2)^2 d\sigma_z < \infty \right\}$$

are introduced and studied intensively by Stroethoff (cf. [32]), where $d\sigma_z$ is the Euclidean area element $dxdy$. Here, $\varphi_a(z)$ stands for the Möbius transformation of Δ given by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, where $a \in \Delta$. In 1994, Aulaskari and Lappan [14] introduced a class of holomorphic functions, the so called Q_p-spaces as follows:

$$Q_p = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^2 g^p(z,a) d\sigma_z < \infty \right\},$$

where $0 < p < \infty$ and the weight function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. The weight function $g(z,a)$ is actually Green's function in Δ with pole at $a \in \Delta$.

For $0 < p < \infty$, $-2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ (cf. [9, 11, 33]), if

$$\|f\|_{K,p,q} = \sup_{a\in\Delta} \int_{\Delta} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(g(z,a)) d\sigma_z < \infty.$$

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Recall that the analytic function

$$f(z)=\sum_{k}^{\infty}a_k z^{n_k}\quad (\text{with } \ n_k\in\mathbb{N}\ ; \text{ for all } \ k\in\mathbb{N}=\{1,2,3,\ldots\})$$

is said to belong to the Hadamard gap class (also known as lacunary series) if there exists a constant $c>1$ such that $\frac{n_{k+1}}{n_k}\geq c$ for all $k\in\mathbb{N}$ (see e.g. [4, 5, 22, 24]).
Now, given a reasonable function $\omega:(0,1]\to[0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ (see [16]) is defined as the set of all analytic functions f on Δ satisfying

$$(1-|z|)|f'(z)|\leq C\omega(1-|z|),\quad z\in\Delta,$$

for some fixed $C=C_f>0$. In the special case where $\omega\equiv 1,\mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.
Now, we introduce the following definitions:

Definition 1.1 *For a given reasonable function $\omega:(0,1]\to[0,\infty)$ and for $0<\alpha<\infty$. An analytic function f on Δ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}^\alpha_{\omega;n}$ if*

$$\|f\|_{\mathcal{B}^\alpha_{\omega;n}}=\sup_{z\in\Delta}\frac{(1-|z|)^{n\alpha}}{\omega(1-|z|)}|f^{(n)}(z)|<\infty;\quad n\in\mathbb{N}.$$

Definition 1.2 *For a given reasonable function $\omega:(0,1]\to[0,\infty)$ and for $0<\alpha<\infty$. An analytic function f on Δ is said to belong to the little weighted Bloch space $\mathcal{B}^\alpha_{\omega,n;0}$ if*

$$\|f\|_{\mathcal{B}^\alpha_{\omega;n,0}}=\lim_{|z|\to 1^-}\frac{(1-|z|)^{n\alpha}}{\omega(1-|z|)}|f^{(n)}(z)|=0;\quad n\in\mathbb{N}.$$

One should note that this class is different from the weighted Bloch space which studied in [36].
Throughout this paper and for some techniques we consider the case of $\omega\not\equiv 0$. Now, we introduce the following definition:

Definition 1.3 *For a nondecreasing function $K:[0,\infty)\to[0,\infty),0<p<\infty,\ -2<q<\infty$ and for a given reasonable function $\omega:(0,1]\to(0,\infty)$, an analytic function f in Δ is said to belong to the space $Q_{K,\omega}(p,q;n)$ if*

$$\|f\|^p_{K,\omega,p,q;n}\quad=\quad\sup_{a\in\Delta}\int_\Delta\left|f^{(n)}(z)\right|^p(1-|z|)^{np-p+q}\frac{K(g(z,a))}{\omega^p(1-|z|)}\,d\sigma_z<\infty;\quad n\in\mathbb{N}.$$

Remark 1.1 *It should be remarked that $Q_{K,\omega}(p,q;n)$ classes are more general than many classes of analytic functions. If $n=1$, we obtain the class $Q_{K,\omega}(p,q)$ as studied in [10, 29, 30]. If $n=1$, and $\omega\equiv 1$, we obtain $Q_K(p,q)$ type spaces (cf. [9, 11, 33]). If If $n=1$, $q=p=2$, and $\omega(t)=t$, we obtain Q_K spaces as studied recently in [17, 18, 34] and others. If If $n=1$, $q=p=2$, $\omega(t)=t$ and $K(t)=t^p$, we obtain Q_p spaces as studied in [14, 15] and others. If If $n=1$, $\omega\equiv 1$ and $K(t)=t^s$, then $Q_{K,\omega}=F(p,q,s)$ classes (cf. [8, 37, 38]).*

In this paper, we characterize the weighted Bloch space $\mathcal{B}^\alpha_{\omega;n}$ by $Q_{K,\omega}(p,q;n)$ spaces. One of the main results is a general Besov-type characterization for $\mathcal{B}^\alpha_{\omega;n}$ functions that extends and generalizes the Stroethoff's theorem [32]. Also, we extend and improve some results due to Essén et. al [18] using our new definitions.

2 Analytic $Q_{K,\omega}(p,q;n)$ classes

In this paper we show some relations betwen $Q_{K,\omega}(p,q;n)$ norms and $\mathcal{B}^\alpha_{\omega;n}$ norms for a nondecreasing function K, also we give a general way to construct different spaces $Q_{K,\omega_1}(p,q;n)$ and $Q_{K_2,\omega}(p,q;n)$ by

using some functions K_1 and K_2.
Before proving theorems we recall some few facts about the Möbius function φ_a. First, the function φ_a is easily seen as:

$$(\varphi_a \circ \varphi_a)(z) = z \quad \text{for all} \quad z \in \Delta$$

The following identity can be obtained by straight forward computation:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\ , \quad a, z \in \Delta.$$

A slightly different form in which we will apply the above identity is:

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)|\ , \quad a, z \in \Delta. \tag{1}$$

For a point $a \in \Delta$ and $0 < r < 1$, the pseudo-hyperbolic disc $\Delta(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $\Delta(a,r) = \varphi_a(r\Delta)$.
For $a \in \Delta$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by the equality $d\sigma_w = |\varphi_a'(z)|^2 d\sigma_z$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)} h(\varphi_a(w))d\sigma_w = \int_{\Delta(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 d\sigma_z\ . \tag{2}$$

We assume throughout this paper that

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r)^2}\,dr < \infty\ . \tag{3}$$

We need the following lemmas in the sequel.

Lemma 2.1 *Let $\alpha \in (0,\infty)$ and suppose that $f(z) = \sum\limits_{j=1}^{\infty} a_j z^{n_j}$ belongs to Hadamard gap class. Then $f \in \mathcal{B}^{\alpha}_{\omega;n}$ if and only if*

$$\sup_{j\in\mathbb{N}} |a_j| n_j^{1-\alpha} < \infty\ , \quad \textit{where}\ \ \mathbb{N} = \{1,2,3,\ldots\}\,.$$

Proof: The proof is very similar to the overspending result in [35] with simple modifications, so it will be omitted.

Lemma 2.2 *For a given reasonable function $\omega : (0,1] \to [0,\infty)$. Let $f_1(z), f_2(z)$ be analytic functions on Δ. Then,*

$$|f_1^{(n)}(z)| + |f_2^{(n)}(z)| \approx \frac{\omega(1-|z|)}{(1-|z|)}\ , \qquad z \in \Delta. \tag{4}$$

Proof: If $n = 1$, the proof is known from [29]. Now, we consider the case $1 < n < \infty$. For a large number $q \in \mathbb{N}$, choose a gap series:

$$f_1^{(n-1)}(z) = \sum_{j=0}^{\infty} z^{q^j}, \qquad z \in \Delta.$$

Then, apply lemma 2.1 to infer that $\frac{(1-|z|)|f_1^{(n)}(z)|}{\omega((1-|z|))} \le \lambda$ holds for all $z \in \Delta$, where λ is a constant. Furthermore, let us verify

$$\frac{(1-|z|)|f_1^{(n)}(z)|}{\omega((1-|z|))} \ge \lambda\ , \quad 1 - q^{-k} \le |z| \le 1 - q^{-(k+\frac{1}{2})}\ , \ \ k \in \mathbb{N}. \tag{5}$$

And

$$q^{-(k+\frac{1}{2})} \le 1 - |z| \le q^{-k} \Rightarrow \omega(q^{-(k+\frac{1}{2})}) \le \omega(1-|z|) \le \omega(q^{-k}).$$

Observe that for any $z \in \Delta$,

$$|f_1^{(n)}(z)| \geq q^k |z|^{q^k} - \sum_{j=0}^{k-1} q^j |z|^{q^j} - \sum_{k+1}^{\infty} q^j |z|^{q^j} = T_1 - T_2 - T_3.$$

Then, fix a z with $|z| \in [1 - q^{-k}, 1 - q^{-(k+\frac{1}{2})}], k \in \mathbb{N}$, and put $x = |z|^{q^k}$. Thus

$$(1 - q^{-k})^{q^k} \leq x \leq [(1 - q^{-(k+\frac{1}{2})})^{q^{k+\frac{1}{2}}}]^{q^{\frac{-1}{2}}}.$$

If q is large enough, then for $k \geq 1$ one has

$$\frac{1}{3} \leq x \leq (\frac{1}{2})^{q^{\frac{-1}{2}}}, \tag{6}$$

and hence $T_1 \geq \frac{q^k}{3}$. Since it is easy to establish $T_2 \leq \frac{1}{n-1} \sum_{j=0}^{k-1} q^j \leq \frac{q^k}{q-1}$, it remains to deal with the third term T_3. Noting that $|z|^{q^n(q-1)} \leq |z|^{q^{k+1}(q-1)}, \quad n \geq k+1$, namely, in T_3 the quotient of two successive terms is not greater than the ratio of the first two terms, one finds that the series of T_3 is controlled by the geometric series having the same first two terms. Accordingly (6) is applied to produce

$$\begin{aligned} T_3 &\leq q^{k+1} |z|^{q^{k+1}} \sum_{j=0}^{\infty} \left(q |z|^{q^{k+2} - q^{k+1}} \right)^j \\ &= \frac{q^{k+1} |z|^{q^{k+1}}}{1 - q |z|^{(q^{k+2} - q^{k+1})}} = q^k \frac{q x^q}{1 - q x^{q^2 - q}} \leq q^k \frac{q (\frac{1}{2})^{q^{\frac{1}{2}}}}{1 - q (\frac{1}{2}) q^{\frac{3}{2}} - q^{\frac{1}{2}}}. \end{aligned}$$

The preceding estimates for T_1, T_2 and T_3 imply

$$\begin{aligned} |f_1^{(n)}(z)| &\geq \frac{q^k}{4} \frac{\omega(1-|z|)}{\omega(1-|z|)} = \frac{q^{k+\frac{1}{2}}}{4q^{\frac{1}{2}}} \frac{\omega(1-|z|)}{\omega(1-|z|)} \\ &\geq \frac{\omega(1-|z|)}{4q^{\frac{1}{2}}(1-|z|) \times \omega(1-|z|)} \geq \frac{\omega(1-|z|)}{4q^{\frac{1}{2}} \omega(q^{-k}) \times (1-|z|)}; \quad \omega(q^{-k}) \not\to \infty. \end{aligned}$$

Reaching (5). In a completely similar manner one can prove that if q is a large natural number, for example $q = m^2$ where m is a large natural number, and if $f_2^{(n-1)}(z) = \sum\limits_{j=0}^{\infty} z^{q^j}, \quad z \in \Delta$, then $(1-|z|^2)|f_2^n(z)| \leq \lambda$ for all $z \in \Delta$ (owing to Lemma 2.1) and

$$\frac{(1-|z|)|f_1^{(n)}(z)|}{\omega((1-|z|))} \leq \lambda, \quad 1 - q^{-(k+\frac{1}{2})} \leq |z| \leq 1 - q^{-(k+1)}, \quad k \in \mathbb{N}. \tag{7}$$

Of course, (5) and (7) yield (4) unless it occurs that $f_1^{(n)}(z)$ and $f_2^{(n)}(z)$ have common zero in $\{z \in \Delta : |z| < 1 - q^{-1}\}$ in which case one can replace $f_2(z)$ with $f_2(\zeta z)$ for appropriate $\zeta \in \partial\Delta$, where $\partial\Delta$ is the boundary of the unit disk (note that $f^{(n)}(0) = 1$). Our lemma is therefore proved .

Using the same steps of Lemma 2.2, it is not hard to prove the following lemma.

Lemma 2.3 *Let $\omega : (0, 1] \to (0, \infty)$ and let $1 \leq \alpha < \infty$. Then there are two functions $f_1, \ f_2 \in \mathcal{B}^{\alpha}_{\omega;n}$ such that*

$$|f_1^{(n)}(z)| + |f_2^{(n)}(z)| \approx \frac{\omega(1-|z|)}{(1-|z|)^{\alpha}}, \quad z \in \Delta. \tag{8}$$

Proof: The proof is very similar to the proof of Lemma 2.2 and lemma 3.1 in [19], so it will be omitted.

Theorem 2.1 *Let $0 < p < \infty, \ -2 < q < \infty$. Then, for each non-decreasing function $K : [0, \infty) \to [0, \infty)$ and for a given reasonable non-decreasing function $\omega : (0, 1] \to (0, \infty)$ with $\omega(k\,t) \approx \omega(t), \quad k > 0$, we have that*

(i) $\quad Q_{K,\omega}(p, q; n) \subset \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$ *and*

(ii) $\quad Q_{K,\omega}(p, q; n) = \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$ *if and only if (3) holds.*

Proof: For a fixed $r \in (0,1)$ and $a \in \Delta$, let

$$E(a,r) = \left\{ z \in \Delta \ , \ |z-a| < r(1-|a|) \right\}.$$

We know that $E(a,r) \subset \Delta(a,r)$ and for any $z \in E(a,r)$, we have

$$(1-r)(1-|a|) \leq 1-|z| \leq (1+r)(1-|a|),$$

which means that $1-|z| \simeq 1-|a|$ for any $z \in E(a,r)$. Denote

$$F_{\omega,p,q;n}(f)(z) = \left|f^{(n)}(z)\right|^p \ \frac{(1-|z|)^{np-p+q}}{\omega^p(1-|z|)}$$

Then, we obtain

$$\begin{aligned} \int_\Delta F_{\omega,p,q;n}(f)(z)K\big(g(z,a)\big)\ d\sigma_z &\geq \int_{\Delta(a,r)} F_{\omega,p,q;n}(f)(z)K\big(g(z,a)\big)\ d\sigma_z \\ \geq \ K\left(\log\frac{1}{r}\right)&\int_{\Delta(a,r)} F_{\omega,p,q;n}(f)(z)\ d\,\sigma_z \\ \geq \ K\left(\log\frac{1}{r}\right)&\int_{E(a,r)} F_{\omega,p,q;n}(f)(z)\ d\sigma_z. \end{aligned}$$

For every $z \in E(a,r)$, we have that

$$(1-r)(1-|a|) \leq 1-|z| \leq (1+r)(1-|a|).$$

Now, since we assume that ω is non-decreasing, then we obtain that

$$\int_{E(a,r)} F_{\omega,p,q;n}(f)(z)\ d\sigma_z \geq \frac{(1-r)^{np-p+q}(1-|a|)^{np-p+q}}{\omega^p((1-r)(1-|a|))} \int_{E(a,r)} \left|f^{(n)}(z)\right|^p\ d\sigma_z.$$

Since $|f^{(n)}(z)|^p$ is a subharmonic function, then

$$\int_{E(a,r)} \left|f^{(n)}(z)\right|^p\ d\sigma_z \geq |E(a,r)|\ \times\ |f^{(n)}(a)\big|^p = r^2(1-|a|)^2|f^{(n)}(a)\big|^p.$$

Then we obtain

$$\begin{aligned} \int_\Delta F_{\omega,p,q;n}(f)(z)K\big(g(z,a)\big)\ d\sigma_z &\geq K\left(\log\frac{1}{r}\right)\frac{(1-r)^p(1-|a|)^{q+2}}{\omega^p((1-r)(1-|a|))}|f^{(n)}(a)\big|^p \\ \geq \ \pi r^2 K\left(\log\frac{1}{r}\right)&\frac{(1-r)^{np-p+q}(1-|a|)^{np-p+q+2}}{\omega^p(1-|a|)}\left|f^{(n)}(a)\right|^p \end{aligned}$$

If $f \in Q_{K,\omega}(p,q;n)$, then by the above estimate we have that

$$\sup_{a\in\Delta} \frac{(1-|a|)^{np-p+q+2}|f^{(n)}(z)|^p}{\omega^p(1-|a|)} < \infty.$$

The proof of (i) is therefore completed.

Now, we show that $\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}} \subset Q_{K,\omega}(p,q;n)$ provided that K satisfies condition (3). For $f \in \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$, we have that,

$$\begin{aligned} \int_\Delta F_{\omega,p,q;n}(f)(z)K\big(g(z,a)\big)\ d\sigma_z &\leq \big\|f\big\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \ \int_\Delta (1-|z|^2)^{-2}K\big(g(z,a)\big)\,d\,\sigma_z \\ = 2\pi\big\|f\big\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \ \int_0^1 K\left(\log\frac{1}{r}\right)&\frac{r}{(1-r^2)^2}\,dr < \infty, \end{aligned}$$

which shows that
$$\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}} \subset Q_{K,\omega}(p,q;n).$$
Now we assume that $\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}} = Q_{K,\omega}(p,q;n)$ and we verify that (3) holds. From Lemma 2.3, for f_1 and f_2 in $\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$, we have that
$$|f_1^{(n)}(z)| + |f_2^{(n)}(z)| \geq \frac{\omega(1-|z|)}{(1-|z|)^{\frac{np-p+q+2}{p}}}. \tag{9}$$
Then $f_1, f_2 \in Q_{K,\omega}(p,q;n)$ and
$$\begin{aligned}\infty &> \sup_{a\in\Delta}\int_\Delta \Big(\big|f_1^{(n)}(z)\big|^p + \big|f_2^{(n)}(z)\big|^p\Big)(1-|z|)^{np-p+q+2}\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z\\ &\geq \int_\Delta \Big(\big|f_1^{(n)}(z)\big| + \big|f_2^{(n)}(z)\big|\Big)^p (1-|z|)^{np-p+q+2}\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\, d\sigma_z \end{aligned}\tag{10}$$
From (9) and (10), we obtain
$$\int_\Delta \big(\big|f_1^{(n)}(z)\big|^p + \big|f_2^{(n)}(z)\big|^p\big)(1-|z|)^{np-p+q+2}\frac{K\big(g(z,0)\big)}{\omega^p(1-|z|)}\, d\sigma_z \approx 2\pi\int_0^1 K\Big(\log\frac{1}{r}\Big)\frac{r}{(1-r)^2}\, dr.$$
Thus (3) holds, and this completes the proof.

We say that $f \in Q_{K,\omega,0}(p,q;n)$ if
$$\lim_{|a|\to 1^-}\int_\Delta \big|f^{(n)}(z)\big|^p (1-|z|)^{np-p+q+2}\,\frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z = 0. \tag{11}$$
Now, we give the following result for $Q_{K,\omega,0}(p,q;n)$ and $\mathcal{B}^\alpha_{\omega,n;0}$ classes.

Theorem 2.2 *Let $0<p<\infty$, $-2<q<\infty$. Then, for each non-decreasing function $K:[0,\infty)\to[0,\infty)$ and for a given reasonable non-decreasing function $\omega:(0,1]\to(0,\infty)$ with $\omega(k\,t)\approx\omega(t)$, $k>0$, we have that*
(i) $Q_{K,\omega,0}(p,q;n) \subset \mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}$ *and*
(ii) $Q_{K,\omega,0}(p,q;n) = \mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}$, *if and only if (3) holds.*

Proof: Without loss of generality, we assume that $K(1)>0$. From the proof of Theorem 2.1, we have that
$$\begin{aligned}\pi(\tfrac{1}{e})^2 K(1)\tfrac{(1-|a|)^{np-p+q+2}}{\omega^p(1-|a|)}|f^{(n)}(a)|^p &\leq K(1)\int_{E(a)} F_{\omega,p,q;n}(f)(z)\, d\sigma_z\\ &\leq K(1)\int_{\Delta(a,\frac{1}{e})} F_{\omega,p,q;n}(f)(z)\, d\sigma_z\\ &\leq \int_\Delta F_{\omega,p,q;n}(f)(z) K\big(g(z,a)\big)\ \ d\sigma_z\ ,\end{aligned}$$
where
$$E(a) = \Big\{z\in\Delta\ ,\ |z-a| < \frac{1}{e}(1-|a|)\Big\}.$$
If $f \in Q_{K,\omega,0}(p,q;n)$, we obtain that
$$\lim_{|a|\to 1^-}\frac{(1-|a|)^{np-p+q+2}|f^{(n)}(a)|^p}{\omega^p(1-|a|)} = 0.$$

(ii) We only need to prove that $\mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}} \subset Q_{K,w,0}(p,q;n)$. Assume that

$$A = \int_0^1 K\left(\log\frac{1}{r}\right) \frac{r}{(1-r)^2}\,dr < \infty.$$

For a given $\epsilon > 0$ there exists an r_1, $0 < r_1 < 1$, such that

$$\int_{r_1}^1 K\left(\log\frac{1}{r}\right) \frac{r}{(1-r)^2}\,dr < \epsilon. \tag{12}$$

Then we have that,

$$\begin{aligned}
&\int_{\Delta\setminus\Delta(a,r_1)} \left|f^{(n)}(z)\right|^p (1-|z|)^{np-p+q}\frac{K(g(z,a))}{\omega^p(1-|z|)}\,d\sigma_z \\
\leq\;& \|f\|^p_{\mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}} \int_{\Delta\setminus\Delta(a,r_1)} \frac{K(g(z,a))}{(1-|z|)^2}\,d\sigma_z \\
=\;& \|f\|^p_{\mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}} \int_{r_1<|w|<1} K\left(\log\frac{1}{|w|}\right)\frac{1}{(1-|w|)^2}\,d\sigma_w \\
=\;& \|f\|^p_{\mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}} \int_{r_1}^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r)^2}\,dr \leq 2\pi\,\epsilon\,\|f\|^p_{\mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}}\,.
\end{aligned} \tag{13}$$

Similarly, if $f \in \mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}$, we obtain that

$$|f^{(n)}(\varphi_a(w))|^p \frac{(1-|\varphi_a(w)|^2)^{\frac{np-p+q+2}{p}}}{\omega^p(1-|\varphi_a(w)|)} \quad \longrightarrow 0$$

converges uniformly for $|w| \leq r$ if $|a| \to 1^-$, where r is fixed and $0 < r < 1$. Then, we obtain that

$$\begin{aligned}
&\lim_{|a|\to 1^-}\int_\Delta \left|f^{(n)}(z)\right|^p (1-|z|)^{np-p+q}\,\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z \\
&= \lim_{|a|\to 1^-}\int_{|w|<r} \frac{\left|f^{(n)}(\varphi_a(w))\right|^p (1-|\varphi_a(w)|)^{np-p+q}\,K\big(\log\frac{1}{|w|}\big)}{\omega^p(1-|\varphi_a(w)|)(1-|w|)^2}\,d\sigma_w \\
&\leq A \lim_{|a|\to 1^-}\sup_{|w|\leq r_1}\left|f^{(n)}(\varphi_a(w))\right|^p \frac{(1-|\varphi_a(w)|)^{np-p+q+2}}{\omega^p(1-|\varphi_a(w)|)} = 0.
\end{aligned} \tag{14}$$

By (13) and (14) it is easy to obtain that

$$\lim_{|a|\to 1^-}\int_\Delta \left|f^{(n)}(z)\right|^p (1-|z|)^{np-p+q}\,\frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\,d\sigma_z = 0. \tag{15}$$

Conversely, suppose that (3) does not hold; that is

$$\int_0^1 K\left(\log\frac{1}{r}\right) \frac{r}{(1-r)^2}\,dr = \infty.$$

Thus we find a continuous strictly decreasing function $g : [0,1) \longrightarrow [0,\infty)$ tending to zero at 1 such that

$$\int_0^1 K\left(\log\frac{1}{r}\right) \frac{g(r)}{(1-r)^2\omega^p(1-r)}\,r\,dr = \infty. \tag{16}$$

It is easy to see that

$$r^{2^{k+1}-2} \geq \exp\{-2^{k+2}(1+r)\}, \quad r \in [0.5, 1). \tag{17}$$

We know that for $\beta > 0$ that, $t^{2\beta}\exp\{-4t\}_{t=\frac{\beta}{2}} = \left(\frac{\beta}{2}\right)^{2\beta}\exp\{-2\beta\}$. Then, there exists an integer k for $\frac{3}{4} \le r < 1$ such that $\frac{\beta}{2} \le 2^k(1-r) < \frac{\beta+1}{2}$ and

$$\begin{aligned} 2^{\beta k}\exp\{-2^{k+2}(1-r)\} &= (1-r)^{-2\beta}\Big(2^k(1-r)\Big)^{2\beta}\exp\{-2^{k+2}(1-r)\} \\ &> \left(\frac{1+\beta}{2}\right)^{2\beta}(1-r)^{-2\beta}\exp\{-2(\beta+1)\}. \end{aligned} \tag{18}$$

For $\frac{3}{4} \le r < 1$ we define

$$f_0(z) = \sum_{k=0}^{\infty} a_k\, 2^{\frac{2k}{p}}\, z^{2^k},$$

where $a_k = g\big(1 - \frac{(p+1)}{p}\, 2^k\big),\ k = 0, 1, 2, \ldots$. By (17) and (18), we deduce that

$$\begin{aligned} M_2^2(r, f_0^{(n)}) &= \int_0^{2\pi} |f_0^{(n)}(r\, e^{i\theta})|^2\, d\theta = 2\pi\sum_{k=0}^{\infty} a_k^2\ (2^k-1)!\ 2^{\frac{2k(p+2)}{p}}\ z^{2^k-2} \\ \ge\ & 2\pi\big(g(r)\big)^{\frac{2}{p}}\ (2^k-1)!\ 2^{\frac{2k(q+2)}{p}}\exp\{-2^{k+2}(1-r)\} \ge \lambda\,\big(g(r)\big)^{\frac{2}{p}}\ (1-r)^{\frac{-2(q+2)}{p}}\ , \end{aligned} \tag{19}$$

where λ is a constant. Since f_0 is defined by a gap series with Hadamard condition, we have

$$M_2(r, f_0^{(n)}; \omega) \approx M_p(r, f_0^{(n)}; \omega), \quad \text{where} \quad M_p(r, f_0^{(n)}; \omega) = \left(\int_0^{2\pi} \frac{|f_0^{(n)}(r\, e^{i\theta})|^p}{\omega^p(1-r)}\, d\theta\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} &\sup_{a\in\Delta}\int_\Delta \left|f_0^{(n)}(z)\right|^p (1-|z|)^{np-p+q}\,\frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z \\ &\ge \int_0^1 M_p^p(r, f_0^{(n)})(1-r)^{np-p+q} K\left(\log\frac{1}{r}\right) r\, dr \\ &\approx \int_0^1 M_2^p(r, f_0^{(n)})(1-r)^{np-p+q} K\left(\log\frac{1}{r}\right) r\, dr \\ &\ge \int_{\frac{3}{4}}^1 K\left(\log\frac{1}{r}\right)\ \frac{g(r)}{(1-r)^2\omega^p(1-r)}\, r\ dr = \infty. \end{aligned}$$

This means that $f_0 \in \mathcal{B}^{\frac{q+2}{p}}_{\omega,;;0} \setminus Q_{K,w,0}(p, q; n)$, which is a contraction. Hence (3) holds. This completes the proof of our theorem.

3 Weights on $Q_{K,\omega}(p, q; n)$-spaces

The following result means that the kernel function K can be chosen as bounded.

Theorem 3.1 *Assume that $K(1) > 0$ and $K_1(r) = \inf\{K(r),\ K(1)\}$, then for $0 < p < \infty$, $-2 < q < \infty$, we have that*

$$Q_{K,w}(p, q; n) = Q_{K_1,w}(p, q; n).$$

Proof: Since $K_1 \le K$ and K_1 is nondecreasing, it is clear that $Q_{K,\omega}(p, q; n) \subset Q_{K_1,w}(p, q; n)$. It remains to prove that

$$Q_{K_1,\omega}(p, q; n) \subset Q_{K,\omega}(p, q; n).$$

We note that

$$g(z, a) > 1, \quad z \in \Delta(a, \frac{1}{e}) \quad \text{and}$$

$$g(z,a) \le 1, \quad z \in \Delta \setminus \Delta(a, \frac{1}{e}).$$

Thus $K(g(z,a)) = K_1(g(z,a))$ in $\Delta \setminus \Delta(a, \frac{1}{e})$. It suffices to deal with integrals over $\Delta(a, \frac{1}{e})$.
If $f \in Q_{K_1,\omega}(p,q;n)$ and f is a weighted Bloch function i.e, $f \in \mathcal{B}_{\omega;n}$, then by Theorem 2.1, it follows that

$$\begin{aligned}
&\int_{\Delta(a,\frac{1}{e})} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} \frac{K\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \\
&\le \|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \int_{\Delta(a,\frac{1}{e})} K\big(g(z,a)\big) \frac{1}{(1-|z|)^2}\, d\sigma_z \\
&= \|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \int_{\Delta(0,\frac{1}{e})} K\left(\log\frac{1}{|w|}\right) \frac{1}{(1-|z|)^2}\, d\sigma_w \le C\|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}}
\end{aligned}$$

Thus, $f \in Q_{K,\omega}(p,q;n)$ and Theorem 4.1 is proved.

Corollary 3.1 *Suppose that* $0 < p < \infty,\ -2 < q < \infty$ *and* $\omega : (0,1] \to (0,\infty)$. *Then* $f \in Q_{K,w}(p,q;n)$ *if and only if*

$$\sup_{a\in\Delta} \int_{\Delta} |f^{(n)}(z)|^p \,(1-|z|)^q \,\frac{K(1-|\varphi_a(z)|^2)}{\omega^p(1-|z|)}\, d\sigma_z < \infty.$$

For the application of the above results, we state the following lemma which is needed later.

Lemma 3.1 *Suppose that* $K : [0,\infty) \to [0,\infty),\ 0 < p < \infty,\ -2 < q < \infty$ *and* $\omega : (0,1] \to (0,\infty)$. *Then*
(i) $f \in \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\sup_{a\in\Delta} \int_{\Delta(a,R)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} \,\frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z < \infty, \tag{20}$$

(ii) $f \in \mathcal{B}_{\omega,n;0}^{\frac{np-p+q+2}{p}}$ *if and only if there exists* $R \in (0,1)$ *such that*

$$\lim_{|a|\to 1^-} \int_{\Delta(a,R)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} \frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z = 0. \tag{21}$$

Proof: (i) Assume $f \in \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$. For any $R \in (0,1)$ and $a \in \Delta$, we have

$$\begin{aligned}
&\int_{\Delta(a,R)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} \frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z \\
&= \int_{\Delta(0,R)} |f^{(n)}(\varphi_a(z))|^p \,\frac{(1-|\varphi_a(z)|^2)^{np-p+q+2}}{(1+|\varphi_a(z)|)^{np-p+q+2}} \,\frac{K\big(\frac{1}{|z|}\big)}{(1-|z|^2)^2\omega^p(1-|z|)}\, d\sigma_z \\
&\le \|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \int_{\Delta(0,R)} K\left(\log\frac{1}{|z|}\right) \frac{1}{(1-|z|^2)^2}\, d\sigma_z \\
&\le \lambda_1 \|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \quad ,
\end{aligned}$$

where $1 < (1+|\varphi_a(z)|)^{np-p+q+2} < 2^{np-p+q+2}$ and λ_1 is a constant. Conversely, suppose that (20) holds for some $R, 0 < R < 1$, by the proof of Theorem 2.1 (i) with $1-|a| \approx 1-|z|$ on $E(a,R)$; $a, z \in \Delta$, we obtain

$$\begin{aligned}
&\int_{\Delta(a,R)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} \,\frac{K(g(z,a))}{\omega^p(1-|z|)}\, d\sigma_z \\
&\ge K(\log\frac{1}{R}) \int_{\Delta(a,R)} |f^{(n)}(z)|^p \frac{(1-|z|)^{np-p+q}}{\omega^p(1-|z|)}\, d\sigma_z \\
&\ge \lambda_2\, K\left(\log\frac{1}{R}\right) \omega^{-p}(1-|a|) \int_{E(a,R)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q} d\sigma_z \\
&\ge \pi\lambda_2 R^2 K\left(\ \log\frac{1}{R}\right) \,\frac{(1-|a|)^{np-p+q}}{\omega^p(1-|a|)} |f^{(n)}(a)|^p \ ,
\end{aligned} \tag{22}$$

where λ_2 is a constant. The last inequality shows that $f \in \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$ The proof of (ii) is similar to proof (i) by taking the limit when $|a| \longrightarrow 1^-$ in (i), hence it can be omitted.

Theorem 3.2 *Let $0 < p < \infty$, $-2 < q < \infty$ and $\omega : (0,1] \to (0,\infty)$. Assume $K_1(r) \le K_2(r)$ for $r \in (0,1)$ and $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$. If the integral in (3) is divergent for K_2, then*

$$Q_{K_2,\omega}(p,q;n) \subsetneqq Q_{K_1,\omega}(p,q;n).$$

Proof: It is clear that $Q_{K_2,\omega}(p,q;n) \subset Q_{K_1,\omega}(p,q;n)$. Suppose that

$$Q_{K_2,\omega}(p,q;n) = Q_{K_1,\omega}(p,q;n).$$

By the open mapping theorem (see [25]), we know that the identity map from one of these spaces into the other one is continuous. Thus there exists a constant C such that

$$\|f\|_{K_{2,\omega}(p,q;n)} \le C\|f\|_{K_{1,\omega}(p,q;n)} .$$

Since $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$, then there exists $r_0 \in (0,1)$ such that

$$K_1(r) \le (2C)^{-1}K_2(r) \quad \text{for} \;\; 0 < r \le r_0.$$

Choose $t_0 = e^{-r_0}$ and we deduce that if $f \in Q_{K_2,\omega}(p,q;n)$, then

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{\Delta} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q}\frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \\
&\le C\sup_{a\in\Delta}\int_{\Delta(a,t_0)} |f^{(n)}(z)|^p \,(1-|z|)^{q}\; \frac{K_1\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \\
&+\frac{1}{2}\sup_{a\in\Delta}\int_{\Delta} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q}\; \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{\Delta} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q}\; \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \\
&\le 2C\sup_{a\in\Delta}\int_{\Delta(a,t_0)} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q}\; \frac{K_1\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z.
\end{aligned}$$

By Lemma 4.1 and for $f \in Q_{K_2,\omega}(p,q;n)$, there exists a constant C_1 such that

$$\sup_{a\in\Delta}\int_{\Delta} |f^{(n)}(z)|^p \,(1-|z|)^{np-p+q+2}\; \frac{K_2\big(g(z,a)\big)}{\omega^p(1-|z|)}\, d\sigma_z \le C_1\|f\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}}. \tag{23}$$

If $g \in \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$ and $g_r(z) = g(rz)$, $0 < r < 1$, then

$$\big\|g_r\big\|_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} \le \big\|g\big\|_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} .$$

Since $g_r \in Q_{K_2,\omega}(p,q;n)$, $0 < r < 1$, we can choose $f = g_r$ in the inequality (23). Using Fatou's lemma (see [31]), we deduce that

$$\sup_{a\in\Delta}\int_{\Delta} |g^{(n)}(z)|^p(1-|z|)^{np-p+q}\frac{K_2(g(z,a))}{\omega^p(1-|z|)}d\sigma_z \; < C_1\big\|g\big\|^p_{\mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}} .$$

We have proved that $g \in Q_{K_2,\omega}(p,q;n)$. It means that $Q_{K_2,\omega}(p,q;n) = \mathcal{B}_{\omega;n}^{\frac{np-p+q+2}{p}}$. It follows from Theorem 2.1 that the integral in (3) with $K = K_2$ must be convergent, a contradiction. We obtain that

$$Q_{K_2,\omega}(p,q;n) \subsetneqq Q_{K_1,\omega}(p,q;n).$$

Now, the proof of Theorem 4.2 is completed.

Remark 3.1 *It is still an open problem to extend the results of this paper in Clifford analysis. For more details on some classes of quaternion function spaces, we refer to ([3, 4, 5, 6, 7, 12, 13, 20, 21, 22]) and others.*

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On two Dimensional q-Bernoulli and q-Genocchi Polynomials: Properties and location of zeros

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Abstract

The main purpose of this paper is to investigate two dimensional generalized Genocchi polynomials based on the q-integers. The q-analogues of well-known formulas are derived. The q-analogue of the Srivastava–Pintér addition theorem is obtained. Furthermore we explore the shapes of the q-Genocchi numbers and the q-Genocchi polynomials. We describe the structure of the roots of the q-Genocchi polynomials for values of the index n using a computer.

1 Introduction

Throughout this paper, we always make use of the following notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N}, \quad (a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q|<1, \quad a \in \mathbb{C}.$$

The q-numbers and q-numbers factorial is defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (q \neq 1); \quad [0]_q! = 1; \quad [n]_q! = [1]_q [2]_q \ldots [n]_q \qquad n \in \mathbb{N}, \quad a \in \mathbb{C}$$

respectively. The q-polynomail coefficient is defined by

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}.$$

The q-analogue of the function $(x \oplus y)^n$ is defined by

$$(x \oplus y)_q^n := \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q x^k y^{n-k}, \quad n \in \mathbb{N}_0.$$

In the standard approach to the q-calculus two exponential function are used:

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\frac{1}{(1-(1-q)q^k z)}, \quad 0<|q|<1, \ |z|<\frac{1}{|1-q|},$$
$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}z^n}{[n]_q!} = \prod_{k=0}^{\infty}\left(1+(1-q)q^k z\right), \qquad 0<|q|<1, \ z \in \mathbb{C}.$$

From this form we easily see that $e_q(z) E_q(-z) = 1$. Moreover,

$$D_q e_q(z) = e_q(z), \qquad D_q E_q(z) = E_q(qz),$$

The above q-standard notation can be found in [1].

Carlitz has introduced the q-Bernoulli numbers and polynomials in [2]. Srivastava and Pintér proved some relations and theorems between the Bernoulli polynomials and Euler polynomials in [20]. They also gave some generalizations of these polynomials. In [10]-[22], Kim et al. investigated some properties of the q-Euler polynomials and Genocchi polynomials. They gave some recurrence relations. In [4], Cenkci et al. gave the q-extension of Genocchi numbers in a different manner. In [23], Kim gave a new concept for the q-Genocchi numbers and polynomials. In [25], Simsek et al. investigated the q-Genocchi zeta function and l-function by using generating functions and Mellin transformation. There are numerous recent studies on this subject by among many other authors: Cenkci et al. [4], [5], Choi et al [7], Cheon [6], Luo and Srivastava [13], [14], [15], Srivastava et al.[20], [26], Gabouary and Kurt B., [8], Kim et al. [24].

We propose the following definitions. We define the q-Bernoulli and the q-Genocchi polynomials of higher order in two variables x and y, using two q-exponential functions, which helps us easily prove some properties of these polynomials and q-analogue of the Srivastava and Pintér addition theorem.

Definition 1 *The q-Bernoulli numbers $\mathfrak{B}_{n,q}$ and polynomials $\mathfrak{B}_{n,q}(x,y)$ in x, y are defined by means of the generating functions:*

$$\frac{t}{e_q(t)-1} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,$$

$$\frac{t}{e_q(t)-1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.$$

Definition 2 *The q-Genocchi numbers $\mathfrak{G}_{n,q}$ and polynomials $\mathfrak{G}_{n,q}(x,y)$ in x, y are defined by means of the generating functions:*

$$\frac{2t}{e_q(t)+1} = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < \pi,$$

$$\frac{2t}{e_q(t)+1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.$$

It is obvious that

$$\mathfrak{B}_{n,q} = \mathfrak{B}_{n,q}(0,0), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q}(x,y) = B_n(x+y), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q} = B_n,$$
$$\mathfrak{G}_{n,q} = \mathfrak{G}_{n,q}(0,0), \quad \lim_{q\to 1^-} \mathfrak{G}_{n,q}(x,y) = G_n(x+y), \quad \lim_{q\to 1^-} \mathfrak{G}_{n,q} = G_n.$$

Here $B_n(x)$ and $G_n(x)$ denote the classical Bernoulli and Genocchi polynomials are defined by

$$\frac{t}{e^t-1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad \text{and} \qquad \frac{2t}{e^t+1} e^{tx} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.$$

The aim of the present paper is to obtain some results for the q-Genocchi polynomials (Properties of the q-Bernoulli polynomials is studied in [19]). The q-analogues of well-known results, for example, Srivastava and Pintér [20], can be derived from these q-identities. It should be mentioned that probabilistic proof the Srivastava-Pintér addition theorems were given recently in [26]. The formulas involving the q-Stirling numbers of the second kind, q-Bernoulli polynomials and q-Bernstein polynomials are also given. Furthermore some special cases are also considered.

2 Properties

In this section we shall provide some basic formulas for the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$ in order to obtain the main results of this paper in the next section.

The following elementary properties of the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$ are readily derived from Definition 2.

Property 1. *Summation formulas for the q-Genocchi polynomials:*

Lemma 3 *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{G}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{k,q} (x \oplus y)_q^{n-k}.$$

Proof. The proof is based on the following identity

$$\begin{aligned}
\left(\frac{2t}{e_q(t)+1}\right) e_q(tx) e_q(ty) &= \sum_{k=0}^{\infty} \mathfrak{G}_{k,q} \frac{t^k}{[k]_q!} \sum_{n=0}^{\infty} \frac{t^n x^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{t^n y^n}{[n]_q!} \\
&= \sum_{k=0}^{\infty} \mathfrak{G}_{k,q} \frac{t^k}{[k]_q!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{t^k x^k}{[k]_q!} . \frac{t^{n-k} y^{n-k}}{[n-k]_q!} \right) \\
&= \sum_{k=0}^{\infty} \mathfrak{G}_{k,q} \frac{t^k}{[k]_q!} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} \right) \frac{t^n}{[n]_q!} \\
&= \sum_{k=0}^{\infty} \mathfrak{G}_{k,q} \frac{t^k}{[k]_q!} \sum_{n=0}^{\infty} (x \oplus y)_q^{n-k} \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q} (x \oplus y)_q^{n-k} \frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Lemma 4 *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{G}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{k,q}(x) y^{n-k}.$$

Proof. The proof is based on the following identity

$$\begin{aligned}
\left(\frac{2t}{e_q(t)+1}\right) e_q(tx) e_q(ty) &= \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \frac{t^n y^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathfrak{G}_{k,q}(x) \frac{t^k}{[k]_q!} \frac{t^{n-k} y^{n-k}}{[n-k]_q!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{k,q}(x) y^{n-k} \frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Lemma 5 *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{G}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{k,q} x^{n-k}.$$

Proof. The proof is readily derived from Definition 2. ■

Property 2. *Difference equation:*

Lemma 6 *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{G}_{n,q}(x,1) + \mathfrak{G}_{n,q}(x,0) = 2[n]_q x^{n-1}.$$

Proof.

$$\begin{aligned}\mathfrak{G}_{n,q}(x,1) + \mathfrak{G}_{n,q}(x,0) &= \frac{2t}{e_q(t)+1} e_q(tx) e_q(t) + \frac{2t}{e_q(t)+1} e_q(tx) \\ &= \frac{2t}{e_q(t)+1} e_q(tx)(e_q(t)+1) \\ &= 2te_q(tx) = 2t\sum_{n=0}^{\infty} \frac{t^n x^n}{[n]_q!} = 2\sum_{n=1}^{\infty} \frac{t^n x^{n-1}}{[n-1]_q!} \\ &= 2\sum_{n=1}^{\infty} \frac{t^n x^{n-1}}{[n]_q!}[n]_q = \sum_{n=1}^{\infty} 2x^{n-1}[n]_q \frac{t^n}{[n]_q!}.\end{aligned}$$

■

Property 3. *Differential relation:*

Lemma 7 *For all $x, y \in \mathbb{C}$ we have*

$$D_{q,x}\mathfrak{G}_{n,q}(x) = [n]_q \mathfrak{G}_{n-1,q}(x).$$

Proof. It follows from the following relation

$$\begin{aligned}D_{q,x}\left(\frac{2t}{e_q(t)+1}\right) e_q(tx) &= \left(\frac{2t}{e_q(t)+1}\right) te_q(tx) \\ &= t\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}[n]_q \mathfrak{G}_{n-1,q}(x)\frac{t^n}{[n]_q!}.\end{aligned}$$

■

3 Explicit relationship between the q-Genocchi and the q-Bernoulli polynomials

In this section we prove an interesting relationship between the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$ and the q-Bernoulli polynomials. Here some q-analogues of known results will be given. We also obtain new formulas and their some special cases below.

Theorem 8 *For $n \in \mathbb{N}_0$, the following relationship*

$$\mathfrak{G}_{n,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{[k+1]_q} m^{k-n+1}\left(\mathfrak{G}_{k+1,q}\left(x, \frac{1}{m}\right) - \mathfrak{G}_{k+1,q}(x)\right)\mathfrak{B}_{n=k,q}(my).$$

holds true between the q-Genocchi and the q-Bernoulli polynomials..

4

Proof. Using the following identity

$$\frac{2t}{e_q(t)+1}e_q(tx)\,e_q(ty)=\frac{2t}{e_q(t)+1}e_q(tx)\cdot\frac{e_q\left(\frac{t}{m}\right)-1}{t}\cdot\frac{t}{e_q\left(\frac{t}{m}\right)-1}\cdot e_q\left(\frac{t}{m}my\right)$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-1\right)\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\&=m\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q m^{k-n}\mathfrak{G}_{k,q}(x)-\mathfrak{G}_{n,q}(x)\right)\frac{t^{n-1}}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\&=m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k\end{array}\right]_q m^{k-n-1}\mathfrak{G}_{k,q}(x)-\mathfrak{G}_{n+1,q}(x)\right)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\&=m\sum_{n=0}^{\infty}\left(\mathfrak{G}_{n+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{G}_{n+1,q}(x)\right)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{1}{[k+1]_q}m^{k-n+1}\left(\mathfrak{G}_{k+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{G}_{k+1,q}(x)\right)\mathfrak{B}_{n=k,q}(my)\frac{t^n}{[n]_q!}.\end{aligned}$$

■

Corollary 9 *For $n\in\mathbb{N}_0$, $m\in\mathbb{N}$ the following relationship holds true.*

$$G_n(x+y)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\frac{2}{k+1}\left((k+1)y^k-G_{k+1,q}(y)\right)B_{n-k}(x), \tag{1}$$

$$G_n(x+y)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\frac{1}{m^{n-k-1}(k+1)}\left[2(k+1)G_k\left(y+\frac{1}{m}-1\right)-G_{k+1}\left(y+\frac{1}{m}-1\right)-G_{k+1}(y)\right]\times B_{n-k,q}(mx) \tag{2}$$

between the classical Genocchi polynomials and the classical Bernoulli polynomials.

Note that the formula (2) is new for the classical polynomials.

In terms of the q-Genocchi numbers $\mathfrak{G}_{k,q}$, by setting $y=0$ in Theorem 8, we obtain the following explicit relationship between the q-Genocchi polynomials $\mathfrak{G}_{k,q}$ of order α and the q-Bernoulli polynomials.

Corollary 10 *For $n\in\mathbb{N}_0$ the following relationship holds true.*

$$\mathfrak{G}_{n,q}(x,y)=\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{2}{[k+1]_q}\left[[k+1]_q q^{\frac{1}{2}k(k-1)}y^k-\mathfrak{G}_{k+1,q}(y)\right]\mathfrak{B}_{n-k,q}(x).$$

Corollary 11 *For $n\in\mathbb{N}_0$ the following relationship holds true.*

$$\begin{aligned}\mathfrak{G}_{n,q}(x)&=-\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{2}{[k+1]_q}\mathfrak{G}_{k+1,q}\mathfrak{B}_{n-k,q}(x),\\\mathfrak{G}_{n,q}&=-\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{2}{[k+1]_q}\mathfrak{G}_{k+1,q}\mathfrak{B}_{n-k,q}.\end{aligned}$$

Theorem 12 *For $n \in \mathbb{N}_0$, the following relationship*

$$\mathfrak{B}_{n,q}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q m^{k-n}\left[\frac{1}{[k+1]_q}\mathfrak{B}_{k+1,q}(x) + m^{-k}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q \frac{1}{[j+1]_q} m^j \mathfrak{B}_{j+1,q}(x)\right]\mathfrak{G}_{n-k,q}(my).$$

holds true between the q-Bernoulli and the q-Genocchi polynomials.

Proof. Using the following identity

$$\frac{t}{e_q(t)-1}e_q(tx)\,e_q(ty) = \frac{t}{e_q(t)-1}e_q(tx)\cdot\frac{2t}{e_q\left(\frac{t}{m}\right)+1}\cdot e_q\left(\frac{t}{m}my\right)\frac{e_q\left(\frac{t}{m}\right)+1}{2t}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!} \\ &\quad + \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\ &= I_1 + I_2.\end{aligned}$$

It is clear that

$$\begin{aligned}I_2 &= \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n+1,q}(x)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\frac{1}{[k+1]_q}\mathfrak{B}_{k+1,q}(x)\,\mathfrak{G}_{n-k,q}(my)\frac{t^n}{[n]_q!}.\end{aligned}$$

On the other hand

$$\begin{aligned}I_1 &= \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n+1,q}(x)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\left[\begin{array}{c} n \\ j \end{array}\right]_q m^{-n}\mathfrak{G}_{j,q}(my)\frac{t^n}{[n]_q!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q\frac{1}{[k+1]_q}m^{k-n}\mathfrak{B}_{k+1,q}(x)\sum_{j=0}^{n-k}\left[\begin{array}{c} n-k \\ j \end{array}\right]_q\mathfrak{G}_{j,q}(my)\frac{t^n}{[n]_q!}\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= I_1 + I_2 \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q m^{k-n}\left[\frac{1}{[k+1]_q}\mathfrak{B}_{k+1,q}(x) + m^{-k}\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q\frac{1}{[j+1]_q}m^j\mathfrak{B}_{j+1,q}(x)\right] \\ &\quad\times\mathfrak{G}_{n-k,q}(my)\frac{t^n}{[n]_q!}.\end{aligned}$$

■

Theorem 13 *The polynomials* $\mathfrak{B}_{n,q}(x,y)$ *and* $\mathfrak{G}_{n,q}(x,y)$ *satisfy the following relationship:*

$$\mathfrak{B}_{n,q}(x,y)=\sum_{\substack{k=1\\k\neq n}}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\frac{1}{[n+1]_q}\mathfrak{G}_{k,q}(x,y)\,\mathfrak{B}_{n+1-k,q}(x,y).$$

Proof. Comparing coefficients of $\frac{t^n}{[n]_q!}$ we get the desired identity:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{t}{e_q(t)-1}e_q(tx)\,e_q(ty)\\&=\frac{1}{2}\frac{2t}{e_q(t)+1}e_q(tx)\,e_q(ty)+\frac{1}{t}\frac{t}{e_q(t)-1}\frac{2t}{e_q(t)+1}e_q(tx)\,e_q(ty)\\&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\frac{1}{t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\mathfrak{G}_{k,q}(x,y)\,\mathfrak{B}_{n-k,q}(x,y)\frac{t^{n-1}}{[n]_q!}\\&=\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}\left[\begin{array}{c}n+1\\k\end{array}\right]_q\frac{1}{[n+1]_q}\mathfrak{G}_{k,q}(x,y)\,\mathfrak{B}_{n+1-k,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}$$

■

Theorem 14 *The polynomials* $\mathfrak{B}_{n,q}(x,y)$ *and* $\mathfrak{G}_{n,q}(x,y)$ *satisfy the following relationship:*

$$\mathfrak{G}_{n,q}(x,y)=-2\left(\sum_{k=1}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{1}{[k+1]_q}\mathfrak{G}_{k+1,q}\mathfrak{B}_{n-k,q}(x,y)+\frac{1}{[n+1]_q}\mathfrak{G}_{n+1,q}\right)$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!}&=\frac{1}{t}\frac{2t}{e_q(t)+1}(e_q(t)-1)\frac{t}{e_q(t)-1}e_q(tx)\,e_q(ty)\\&=\frac{1}{t}\left(2t-2\frac{2t}{e_q(t)+1}\right)\frac{t}{e_q(t)-1}e_q(tx)\,e_q(ty)\\&=\frac{1}{t}\left(2t-0-2t-2\sum_{n=2}^{\infty}\mathfrak{G}_{n,q}\frac{t^n}{[n]_q!}\right)\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\\&=-2\sum_{n=1}^{\infty}\mathfrak{G}_{n+1,q}\frac{t^n}{[n+1]_q!}\left(1+\sum_{n=1}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!}\right)\\&=-2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left[\begin{array}{c}n\\k\end{array}\right]_q\frac{1}{[k+1]_q}\mathfrak{G}_{k+1,q}\mathfrak{B}_{n-k,q}(x,y)+\frac{1}{[n+1]_q}\mathfrak{G}_{n+1,q}\right)\frac{t^n}{[n]_q!}\end{aligned}$$

■

4 Location of zeros of the q-Genocchi polynomials

In this section, we display the shapes of the q-Genocchi numbers and polynomials. Next, we investigate the zeros of the q-Genocchi polynomials using a computer.

Our numerical results for the approximate solutions of the real zeros of $\mathfrak{G}_{n,q}(x)$,$q=0.\overline{9}$, are shown in tables 1. The results were obtained using the Mathematica®software.

The shapes of the q-Genocchi numbers $\mathfrak{G}_{n,q}$ for $n=1,...,20$. $q=\frac{1}{2},0.9,0.9999$ are shown in figure 1, figure 2 and and figure 3.

7

Table 1: Approximate solutions of $\mathfrak{G}_{n,q}(x) = 0$

n	q	# of Real Roots	# of Complex Roots
20	0.5	3	17
	0.6	3	17
	0.7	3	17
	0.8	3	17
	0.9	3	17
	0.9999	7	13
	$\frac{9}{10}$	3	17
	1	7	13

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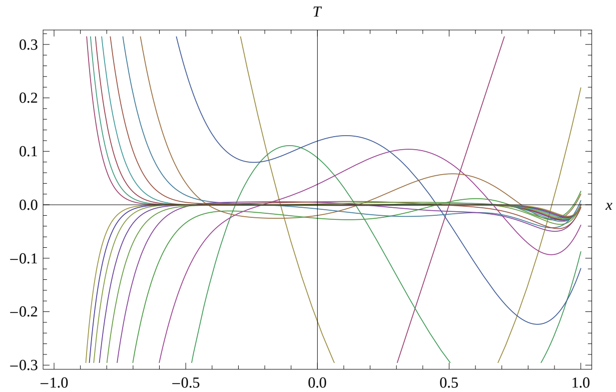

Figure 1: Shape of $\mathfrak{G}_{n,0.5}(x)$ within $x = [-1, 1]$

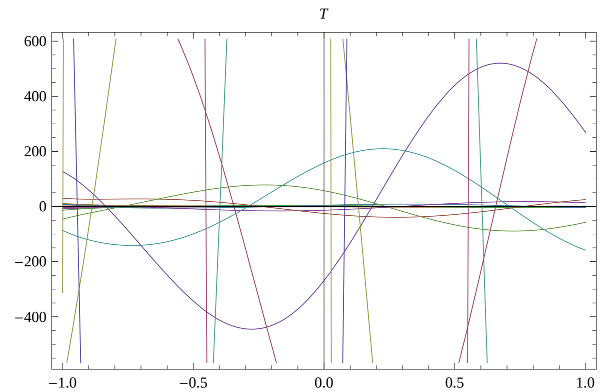

Figure 2: Shape of $\mathfrak{G}_{n,0.9}(x)$ within $x = [-1, 1]$

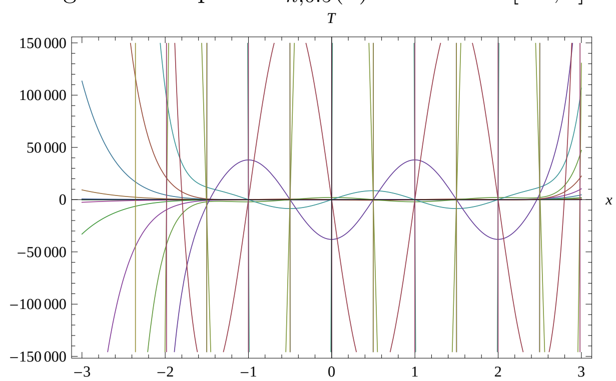

Figure 3: Shape of $\mathfrak{G}_{n,0.9999}$ within $x = [-3, 3]$

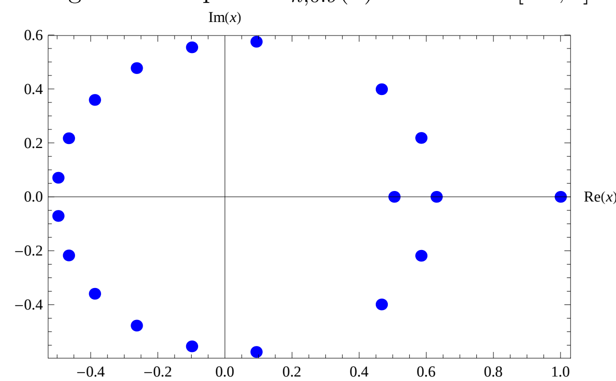

Figure 4: Zeros of $\mathfrak{G}_{20,0.5}(x)$

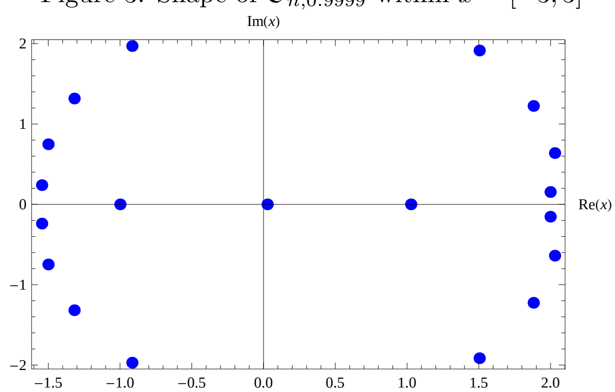

Figure 5: Zeros of $\mathfrak{G}_{20,0.9}(x)$

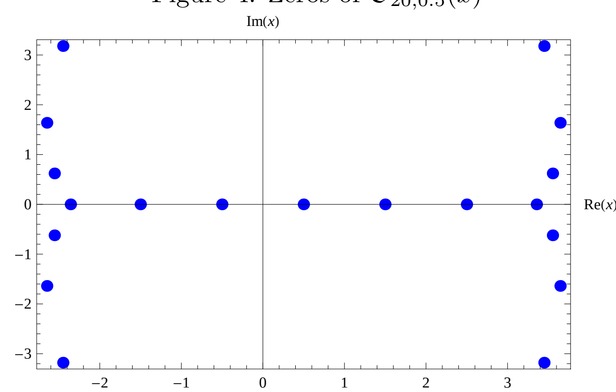

Figure 6: Zeros of $\mathfrak{G}_{20,0.9999}(x)$

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Existence results of sequential derivatives of nonlinear quantum difference equations with a new class of three-point boundary value problems conditions

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Abstract

In this paper, we study a new class of three-point boundary value problems of Sequential Derivatives of nonlinear q-difference equations. Our problems contain different numbers of q in derivatives and integrals. By using a variety of fixed point theorems (such as Banach's contraction principle and Krasnoselskii's fixed point theorem, some new existence and uniqueness results are obtained. Illustrative example is also presented.

Keywords: existence; q-derivative; q-integral; q-difference equation

(2010) Mathematics Subject Classifications: 34B10; 39A13.

1 Introduction

In 20^{th} century, the intensive works on q-difference equations by Jackson [1], Carmichael [2], Mason [3] and Adams [4] became more interesting in many areas of mathematics and applications, e.g. the applications to orthogonal polynomials and mathematical control theories, since years eighties. Although many researches related with q-calculus are raised (see [5]-[14],[16]-[28]),there are lack of works of studying of boundary value problem of q-difference equations. Recently, there are few researches concerning with the boundary value problem of nonlinear q-difference equations as follows.

In 2012, Ahmad et al. [24] proposed the boundary value problem of nonlinear second order q-difference equations with nonlocal boundary conditions given by

$$D_q^2 x(t) = f(t, x(t)), \quad t \in [0, T], \tag{1.1}$$

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$$\begin{cases} \alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \\ \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2), \end{cases} \tag{1.2}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and q is a constant that $q \in (0,1)$. By employing Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative, Ahmad et al derived the existence of solutions of the above problem.

Later, Yu and Wang [28] proposed the boundary value problem of nonlinear second order q-difference equation given by

$$\begin{cases} D_q^2 u(t) + f(t, u(t), D_q u(t)) = 0, \quad t \in [0,T], \\ D_q u(0) = 0, \quad D_q u(1) = \alpha u(1), \end{cases} \tag{1.3}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and $\alpha \neq 0$ is a fixed constant. They discussed the existence and uniqueness of solution of this problem by using Banach's contraction principle, the Leray-Schauder nonlinear alternative and Leray-Schauder continuation theorem.

Lately, Pongarm et al. [29] studied the sequential derivative of nonlinear q-difference equation with three-point boundary conditions. The problem is in the form

$$\begin{cases} D_q \left(D_p + \lambda\right) u(t) = f(t, u(t)), \quad t \in [0,T], \\ u(0) = 0, \quad u(T) = \alpha \displaystyle\int_0^\eta u(s) d_r s, \end{cases} \tag{1.4}$$

where $0 < p, q, r < 1$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $0 < \eta < T$ and λ, β are given constants. For this problem, the existence solution is discussed by employing Banach's contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder degree theory. Since the number of papers about the problems with different values of the q-numbers is sparse, there is a need for further study.

In this article, we consider the following nonlinear q-difference equation with three-point integral boundary condition given by

$$\begin{cases} D_q(D_p + \lambda) x(t) = f(t, x(t)), \qquad t \in [0,T], \\ x(\eta) = 0, \qquad \displaystyle\int_0^T x(s) d_r s = 0, \qquad 0 < \eta < T, \end{cases} \tag{1.5}$$

where $0 < p, q, r < 1$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$, and $\eta T(1+r) \neq T^2$.

The given problem consists of three different values of the q numbers, in q-derivatives and the q-integral. In addition, the value of function in an intermediate point is used. Our overall goal is to prove an existence and uniqueness of solutions of the problem (1.5) by employing Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 2, we briefly discuss about the basic definitions, some properties of q-difference and present a lemma that will be used throughout the paper. In Sections 3 and 4, we give the main results and example, respectively.

2 Preliminaries

The basic definitions and some properties of q-calculus [15, 18] are as follows.

Definition 2.1. *For $0 < q < 1$, we define the q-derivative of a real valued function f as*

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

The higher order q-derivatives are given by

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

where $D_q^0 f(t) = f(t)$

The definite q-integral of a function f defined on the interval $[0, T]$ is given by

$$I_q f\left(t\right) = \int_0^t f\left(s\right) d_q s = \sum_{n=0}^{\infty} t(1-q) q^n f\left(tq^n\right)$$

where last term is convergent series.

If $a \in [0, T]$, then

$$\int_a^b f(s) d_q s = I_q f(b) - I_q f(a) = (1-q) \sum_{n=0}^{\infty} q^n \left[bf\left(bq^n\right) - af\left(aq^n\right)\right].$$

We note that

$$D_q I_q f(x) = f(x),$$

while if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0).$$

The property of product rule and integration by parts formula are

$$D_q(gh)(t) = (D_q g(t)) h(t) + g(qt) D_q h(t),$$

$$\int_0^x f(t) D_q g(t) d_q t = \Big[f(t) g(t) \Big]_0^x - \int_0^x D_q f(t) g(qt) d_q t.$$

For reversing the order of integration is given by

$$\int_0^t \int_0^s f(r) d_q r d_q s = \int_0^t \int_{qr}^t f(r) d_q s d_q r.$$

In the limit $q \to 1$ the above results correspond to their counterparts in standard calculus.

Lemma 2.1. *Let $\eta T(1+r) \neq T^2$, $0 < p$, $q\, r < 1$ and λ be a constant. Then for any $h \in C[0,T]$, the boundary value problem*

$$D_q(D_p+\lambda)x(t) = f(t,x(t)), \qquad t \in [0,T], \tag{2.1}$$

$$x(\eta) = 0, \qquad \int_0^T x(s)d_r s = 0, \qquad 0 < \eta < T, \tag{2.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \int_0^t \int_0^s h(u)d_qud_ps - \lambda \int_0^t x(s)d_ps \\ &+ \frac{1}{\eta T(1+r) - T^2}\Bigg(\lambda T(t+rt-T)\int_0^\eta x(s)d_ps - T(t+rt-T)\int_0^\eta\int_0^s h(u)d_qud_ps \\ &- (1+r)\lambda(t-\eta)\int_0^T\int_0^v x(s)d_psd_rv + (1+r)(t-\eta)\int_0^T\int_0^v\int_0^s h(u)d_qud_psd_rv\Bigg) \end{aligned} \tag{2.3}$$

Proof. For $t \in [0,T]$, q-integrating (2.1) from 0 to t, we obtain

$$(D_p+\lambda)x(t) = \int_0^t h(s)d_qs + c_1. \tag{2.4}$$

Equation (2.4)) can be written as

$$D_px(t) = \int_0^t h(s)d_qs - \lambda x(t) + c_1. \tag{2.5}$$

For $t \in [0,T]$, q-integrating (2.5) from 0 to t, we obtain

$$x(t) = \int_0^t\int_0^s h(u)d_qud_ps - \lambda\int_0^t x(s)d_ps + c_1t + c_2. \tag{2.6}$$

In particular, for $t = \eta$, we get

$$x(\eta) = \int_0^\eta\int_0^s h(u)d_qud_ps - \lambda\int_0^\eta x(s)d_ps + c_1\eta + c_2. \tag{2.7}$$

Using the first condition of (2.2) with (2.7), we obtain

$$\eta c_1 + c_2 = \lambda\int_0^\eta x(s)d_ps - \int_0^\eta\int_0^s h(u)d_qud_ps. \tag{2.8}$$

Form (2.6), we take the r-integral of $x(t)$ from 0 to t, we obtain

$$\int_0^t x(s)d_rs = \int_0^t\int_0^v\int_0^s h(u)d_qud_psd_rv - \lambda\int_0^t\int_0^v x(s)d_psd_rv + c_1\frac{t^2}{1+r} + c_2t. \tag{2.9}$$

By substituting $t = T$ in (2.9) and employing the second condition of (2.2), we find that

$$\frac{T^2}{1+r}c_1 + Tc_2 = \lambda \int_0^T \int_0^v x(s)d_p s d_r v - \int_0^T \int_0^v \int_0^s h(u)d_q u d_p s d_r v. \tag{2.10}$$

From (2.9) and (2.10), we obtain the system of linear equations. Solving this system, we get

$$\begin{aligned} c_1 =& \frac{(1+r)t}{\eta T(1+r) - T^2}\Big(\lambda T \int_0^\eta x(s)d_p s - T \int_0^\eta \int_0^s h(u)d_q u d_p s \\ & - \lambda \int_0^T \int_0^v x(s)d_p s d_r v + \int_0^T \int_0^v \int_0^s h(u)d_q u d_p s d_r v\Big) \end{aligned}$$

and

$$\begin{aligned} c_2 = & -\frac{1+r}{\eta T(1+r) - T^2}\Big(\frac{\lambda T^2}{1+r} \int_0^\eta x(s)d_p s - \frac{T^2}{1+r} \int_0^\eta \int_0^s h(u)d_q u d_p s \\ & - \eta\lambda \int_0^T \int_0^v x(s)d_p s d_r v + \int_0^T \int_0^v \int_0^s h(u)d_q u d_p s d_r v\Big) \end{aligned}$$

After substituting c_1 and c_2 in (2.6), we get (2.3) as desired. Therefore the proof is completes. □

3 Main results

To accomplish the main results, We transform the boundary value problem (1.5) into a fixed point problem. From Lemma (2.1), We let $\mathcal{C} = C([0,T], \mathbb{R})$ denote the Banach space of all functions x. The norm id defined by $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. The operator $F : \mathcal{C} \to \mathcal{C}$ is define by

$$\begin{aligned} (Fx)(t) = & \int_0^t \int_0^s f(u, x(u))d_q u d_p s - \lambda \int_0^t x(s)d_p s \\ & + \frac{1}{\eta T(1+r) - T^2}\Big(\lambda T(t + rt - T) \int_0^\eta x(s)d_p s - T(t + rt - T)\times \\ & \int_0^\eta \int_0^s f(u, x(u))d_q u d_p s - (1+r)\lambda(t-\eta) \int_0^T \int_0^v x(s)d_p s d_r v \\ & + (1+r)(t-\eta) \int_0^T \int_0^v \int_0^s f(u, x(u))d_q u d_p s d_r v\Big) \end{aligned} \tag{3.1}$$

Based on Banach's fixed point theorem, we find that the problem (1.5) has solutions if and only if the operator F has fixed points.

Theorem 3.1. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous function satisfying the conditions*

(H_1) $|f(t,x) - f(t,y)| \le L|x-y|$, *for all* $t \in [0,T]$ *and* $x, y \in \mathbb{R}$,

(H_2) $\Phi + L\Omega < 1$,

where L is a Lipschitz constant, and

$$\begin{aligned} \Phi = &\ |\lambda|T + \frac{|\lambda|T(\eta r + T - \eta)}{|\eta(1+r) - T|} \\ \Omega = &\ \frac{T^2}{1+p} + \frac{\eta^2 rT}{|\eta(1+r)-T|(1+p)} + \frac{(1+r)(T-\eta)T^2}{|\eta(1+r)-T|(1+q)(1+r+r^2)} \end{aligned} \tag{3.2}$$

Then the boundary value problem (1.5) *has a unique solution.*

Proof. We transform the boundary value problem (1.5) into a fixed point problem $x = Fx$, where $F : \mathcal{C} \to \mathcal{C}$ is defined by (3.1). Assume that $\sup_{t\in[0,T]} |f(t,0)| = M$, and choose a constant R satisfied

$$R \ge \frac{M\Lambda}{1 - (\Phi + L\Omega)}.$$

Our goal is to show that $FB_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \le R\}$. For any $x \in B_R$, we have

$$\begin{aligned} &\|Fx\| \\ = &\sup_{t\in[0,T]} \Bigg| \int_0^t \int_0^s f(u,x(u))d_qud_ps - \lambda \int_0^t x(s)d_ps \\ &+ \frac{1}{\eta T(1+r) - T^2} \Bigg(\lambda T(t + rt - T) \int_0^\eta x(s)d_ps - T(t+rt-T) \times \\ &\int_0^\eta \int_0^s f(u,x(u))d_qud_ps - (1+r)\lambda(t-\eta) \int_0^T \int_0^v x(s)d_psd_rv \\ &+ (1+r)(t-\eta) \int_0^T \int_0^v \int_0^s f(u,x(u))d_qud_psd_rv \Bigg) \Bigg| \end{aligned}$$

$$
\begin{aligned}
\leq\ & \sup_{t\in[0,T]}\Bigg\{\int_0^t\int_0^s(|f(u,x(u))+f(u,0)|+|f(u,0)|)d_qud_ps-|\lambda|\int_0^t|x(s)|d_ps \\
&+\frac{1}{|\eta T(1+r)-T^2|}\Bigg(|\lambda T(t+rt-T)|\int_0^\eta|x(s)|d_ps+|T(t+rt-T)|\times \\
&\int_0^\eta\int_0^s(|f(u,x(u))-f(u,0)|+|f(u,0)|)d_qud_ps+|(1+r)\lambda(t-\eta)|\int_0^T\int_0^v|x(s)|d_psd_rv \\
&+|(1+r)(t-\eta)|\int_0^T\int_0^v\int_0^s(|f(u,x(u))-f(u,0)|+|f(u,0)|)d_qud_psd_rv\Bigg)\Bigg\} \\
\leq\ & \sup_{t\in[0,T]}\Bigg\{(L\|x\|+M)\int_0^t\int_0^s d_qud_ps+|\lambda|\|x(s)\|\int_0^t d_ps \\
&+\frac{1}{|\eta T(1+r)-T^2|}\Bigg(\|x(s)\||\lambda T(t+rt-T)|\int_0^\eta d_ps+(L\|x\|+M)|T(t+rt-T)|\times \\
&\int_0^\eta\int_0^s d_qud_ps+\|x(s)\||(1+r)\lambda(t-\eta)|\int_0^T\int_0^v d_psd_rv \\
&+(L\|x\|+M)|(1+r)(t-\eta)|\int_0^T\int_0^v\int_0^s d_qud_psd_rv\Bigg)\Bigg\} \\
=\ & \sup_{t\in[0,T]}\Bigg\{(L\|x\|+M)\frac{t^2}{1+p}+|\lambda|\|x(s)\|t+\frac{1}{|\eta T(1+r)-T^2|}\Bigg(\|x(s)\||\lambda T(t+rt-T)|\eta \\
&+(L\|x\|+M)|T(t+rt-T)|\frac{\eta^2}{1+p}+\|x(s)\||(1+r)\lambda(t-\eta)|\frac{T^2}{1+r} \\
&+(L\|x\|+M)|(1+r)(t-\eta)|\frac{T^3}{(1+q)(1+r+r^2)}\Bigg)\Bigg\} \\
=\ & \sup_{t\in[0,T]}\Bigg\{\|x\|\Bigg(|\lambda|t+\frac{\eta|\lambda T(t+rt-T)|+T(t-\eta)}{|\eta(1+r)-T|}\Bigg) \\
&+(L\|x\|+M)\Bigg(\frac{t^2}{1+p}+\frac{\eta^2|t+rt-T|}{|\eta(1+r)-T|(1+p)}+\frac{|(1+r)(t-\eta)|T^2}{|\eta(1+r)-T|(1+q)(1+r+r^2)}\Bigg)\Bigg\} \\
\leq\ & R\Bigg(|\lambda|T+\frac{|\lambda|(\eta r+T-\eta)}{|\eta(1+r)-T|}\Bigg) \\
&+(LR|+M)\Bigg(\frac{T^2}{1+p}+\frac{\eta^2 rT}{|\eta(1+r)-T|(1+p)}+\frac{|(1+r)(T-\eta)|T^2}{|\eta(1+r)-T|(1+q)(1+r+r^2)}\Bigg) \\
=\ & R\Phi+(LR+M)\Omega \\
\leq\ & R
\end{aligned}
$$

Therefore, $AB_R \subset B_R$.

We next show that F is a contraction. For any $x, y \in \mathcal{C}$ and for each $t \in [0,T]$,

we have

$$
\begin{aligned}
& \|(Fx)(t)-(Fy)(t)\| \\
= & \sup_{t\in[0,T]}\Big|(Fx)(t)-(Fy)(t)\Big| \\
= & \sup_{t\in[0,T]}\Big|\int_0^t\int_0^s (f(u,x(u))-f(u,y(u)))d_qud_ps-\lambda\int_0^t (x(s)-y(s))d_ps \\
& +\frac{1}{\eta T(1+r)-T^2}\Big(\lambda T(t+rt-T)\int_0^\eta (x(s)-y(s))d_ps-T(t+rt-T)\times \\
& \int_0^\eta\int_0^s (f(u,x(u))-f(u,y(u)))d_qud_ps-(1+r)\lambda(t-\eta)\int_0^T\int_0^v (x(s)-y(s))d_psd_rv \\
& +(1+r)(t-\eta)\int_0^T\int_0^v\int_0^s (f(u,x(u))-f(u,y(u)))d_qud_psd_rv\Big)\Big| \\
\leq & \sup_{t\in[0,T]}\Big\{L\|x-y\|\int_0^t\int_0^s d_qud_ps+|\lambda|\|x-y\|\int_0^t d_ps \\
& +\frac{1}{|\eta T(1+r)-T^2|}\Big(\|x-y\||\lambda T(t+rt-T)|\int_0^\eta d_ps+L\|x-y\||T(t+rt-T)|\times \\
& \int_0^\eta\int_0^s d_qud_ps+\|x-y\||(1+r)\lambda(t-\eta)|\int_0^T\int_0^v d_psd_rv \\
& +L\|x-y\||(1+r)(t-\eta)|\int_0^T\int_0^v\int_0^s d_qud_psd_rv\Big)\Big\} \\
\leq & \|x-y\|\left(|\lambda|T+\frac{|\lambda|(\eta r+T-\eta)}{|\eta(1+r)-T|}\right) \\
& +L\|x-y\|\left(\frac{T^2}{1+p}+\frac{\eta^2rT}{|\eta(1+r)-T|(1+p)}+\frac{|(1+r)(T-\eta)|T^2}{|\eta(1+r)-T|(1+q)(1+r+r^2)}\right) \\
= & (\Phi+L\Omega)\|x-y\|.
\end{aligned}
$$

Since $\Phi+L\Omega<1$, A is a contraction. Note that we complete this proof by using Banach's contraction mapping principle. □

Further, we consider the existence and uniqueness of a solution to the boundary value problem (1.5). We shall use the Krasnoselskii's fixed point theorem [31].

Theorem 3.2. *Let K be a bounded closed convex and nonempty subset of a Banach space X. Let A, B be operators such that:*

(i) $Ax+By\in K$ whenever $x,y\in K$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z\in K$ such that $z=Az+Bz$.

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold. In addition we suppose that:*

(H_3) $|f(t,x)| \leq \mu(t)$, *for all* $(t,x) \in [0,T] \times \mathbb{R}$, *with* $\mu \in L^1([0,T], \mathbb{R}^+)$.

If

$$\Phi + L\Omega < 1, \tag{3.3}$$

where Φ *and* Ω *is given by* (3.2), *then the boundary value problem* (1.5) *has at least one solution on* $[0,T]$.

Proof. Setting $\max_{t\in[0,T]} |\mu(t)| = \|\mu\|$ and choosing a constant

$$R \geq \frac{\|\mu\|\Omega}{1-\Phi}, \tag{3.4}$$

we consider $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$.

In view of Lemma 2.1, we define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on the ball B_R as

$$\begin{aligned}
(\mathcal{F}_1 x)(t) &= \int_0^t \int_0^s f(u,x(u))d_q u d_p s - \lambda \int_0^t x(s) d_p s, \\
(\mathcal{F}_2 x)(t) &= \frac{1}{\eta T(1+r) - T^2}\Bigg(\lambda T(t+rt-T)\int_0^\eta x(s)d_p s - T(t+rt-T) \times \\
&\quad \int_0^\eta \int_0^s f(u,x(u))d_q u d_p s - (1+r)\lambda(t-\eta)\int_0^T\int_0^v x(s) d_p s d_r v \\
&\quad +(1+r)(t-\eta)\int_0^T\int_0^v\int_0^s f(u,x(u))d_q u d_p s d_r v\Bigg)
\end{aligned}$$

For $x, y \in B_R$, by computing directly, we have

$$\begin{aligned}
& \|\mathcal{F}_1 x + \mathcal{F}_2 y\| \\
\leq\ & \|\mu\| \int_0^t \int_0^s d_q u d_p s + |\lambda|\|x\| \int_0^t d_p s \\
& + \frac{1}{|\eta T(1+r) - T^2|}\Bigg(\|y\| |\lambda T(t+rt-T)| \int_0^\eta d_p s + \|\mu\| |T(t+rt-T)| \times \\
& \int_0^\eta \int_0^s d_q u d_p s + \|y\| |(1+r)\lambda(t-\eta)| \int_0^T \int_0^v d_p s d_r v \\
& + \|\mu\| |(1+r)(t-\eta)| \int_0^T \int_0^v \int_0^s d_q u d_p s d_r v \Bigg) \\
\leq\ & R\Phi + \|\mu\|\Omega \\
\leq\ & R.
\end{aligned}$$

Therefore $\mathcal{F}_1 x + \mathcal{F}_2 y \in B_R$. The condition (3.3) implies that $\mathcal{F}_2$ is a contraction mapping. Next, we will show that $\mathcal{F}_1$ is compact and continuous. Continuity of f coupled with the assumption (H_3) implies that the operator $\mathcal{F}_1$ is continuous and uniformly bounded on B_R. We define $\sup_{(t,x)\in[0,T]\times B_R}|f(t,x)| = f_{\max} < \infty$. For $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ and $x \in B_R$, we have that

$$\begin{aligned}
\|\mathcal{F}_1 x(t_2) - \mathcal{F}_1 x(t_1)\| &= \sup_{(t,x)\in[0,T]\times B_R}\bigg|\int_0^{t_1}\int_0^s f(u,x(u))d_qud_ps - \lambda\int_0^{t_1}x(s)d_ps \\
&\quad -\int_0^{t_2}\int_0^s f(u,x(u))d_qud_ps + \lambda\int_0^{t_2}x(s)d_ps\bigg| \\
&= \sup_{(t,x)\in[0,T]\times B_R}\bigg|\int_{t_2}^{t_1}\int_0^s f(u,x(u))d_qud_ps - \lambda\int_{t_2}^{t_1}x(s)d_ps\bigg| \\
&\leq f_{max}\frac{|t_1^2 - t_2^2|}{1+p} + |\lambda|(t_1 - t_2)R.
\end{aligned}$$

In particular, if $t_1 - t_2 \to 0$ the right-hand side of the above inequality tends to zero. Thus, $\mathcal{F}_1$ is relatively compact on B_R. Hence, we can conclude by the Arzelá-Ascoli Theorem that $\mathcal{F}_1$ is compact on B_R. Therefore, all the assumptions of Theorem 3.3 are satisfied and the conclusion of Theorem 3.3 implies that the boundary value problem (1.5) has at least one solution on $[0,T]$. This completes the proof. □

4 Examples

In this section, we give some examples to illustrate our main results with . Consider the following boundary value problem of nonlinear second-order q-difference equations with three-point boundary conditions

$$\begin{cases} D_{\frac{1}{2}}(D_{\frac{1}{3}} + \dfrac{1}{100})x(t) = \dfrac{e^{-\sin^2 t}}{100 + e^{\cos^2 t}} \cdot \dfrac{|x(t)|}{1+|x(t)|}, & t \in [0,3], \\ x(2) = 0, \quad \displaystyle\int_0^3 x(s)d_{\frac{3}{4}}s = 0. \end{cases} \tag{4.1}$$

For this example, we have $q = 1/2$, $p = 1/3$, $r = 3/4$, $\lambda = 10$, $T = 3$, and $\eta = 2$. Form Theorem 3.3, we find that

$$\begin{aligned}
\Phi &= \frac{3}{100} + \frac{\frac{3}{100}(9/4+1)}{|3(7/4) - 3|} \approx 0.06 \\
\Omega &= \frac{9}{4/3} + \frac{9}{|3(7/4) - 3|(4/3)} + \frac{(7/4)9}{|3(7/4) - 3|(3/2)(37/16)} \approx 11.77
\end{aligned}$$

Since, $|f(t,x) - f(t,y)| \leq \frac{1}{101}|x - y|$, then (H_1) is satisfied with $L = \frac{1}{101}$. We can find that

$$\Phi + L\Omega \approx 0.177 < 1.$$

Hence, by Theorem 3.1, problem (4.1) with $f(t,x)$ has a unique solution on $[0,3]$.

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An iterative method for solving fourth-order boundary value problems of mixed type integro-differential equations

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Abstract

In this paper, reproducing kernel Hilbert space method is introduced as an efficient solver for fourth-order boundary value problems of mixed type integro-differential equations where two reproducing kernel functions are used throughout the evolution of the algorithm to obtain the required nodal values of the unknown variable. The solution methodology is based on generating the orthogonal basis from the obtained kernel function in the space $W_2^5[0,1]$. After that, the orthonormal basis is constructing in order to formulate and utilize the solution in the same space. In addition to that, an error estimation and bound based on the use of reproducing kernel theory has been carried out. Four numerical test problems including linear and nonlinear equations were analyzed to illustrate the procedure and confirm the performance of the proposed method. The numerical results show that the proposed algorithm is a robust and accurate procedure for solving fourth-order boundary value problems of mixed type integro-differential equations.

Keywords: Integro-differential equation; Reproducing kernel function; Iterative method

AMS Subject Classification: 34K28; 45J05; 47B32

1 Introduction

Most engineering and physical problems are governed by functional equations, for example, ordinary differential equations, integral equations, integro-differential equations (IDEs), and stochastic differential equations. Many mathematical formulation of physical phenomena contain IDEs with proper boundary conditions, these equations arises in fluid dynamics, biological models, and chemical kinetics, etc. [1–7]. In most cases, the equation is too complex to allow one to find an exact solution, where solution of such equations is always demand due to practical interests. Therefore, an efficient, reliable computer stimulation is required; it is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in mathematical, physical, and engineering problems solving has increased dramatically in recent years.

Today, computers and numerical methods provide an alternative for complicated calculations. Using computer power to obtain solutions directly, we can approach these calculations without recourse to simplifying assumptions or time-intensive techniques. Although analytical solutions are still extremely valuable both for problem solving and for providing insight, numerical methods represent alternatives that greatly enlarge our capabilities to confront and solve problems. As a result, more time is available for the use of creative skills. Thus, more emphasis can be placed on problem formulation and solution interpretation and the incorporation of total system.

Investigation about solvability of fourth-order boundary value problems (BVPs) of mixed type IDEs is scarce. Recently, many authors have discussed the numerical solvability for Volterra type by using some of the well-known methods. It is to be noted that the Volterra type is just a special case of the problem that we propose in this paper. However, the reader is asked to refer to [8–12] in order to know more details about these methods, including their kinds and history, their modification for use, their applications on the other problems, and their characteristics. In this paper, we introduce a novel method based on the use of reproducing kernel Hilbert space (RKHS) method for numerically approximating a solution of fourth-order BVPs of mixed type IDEs in which the given boundary conditions can be involved. The present method has the following characteristics:

1. The method is of global nature in terms of the solution obtained as well as its ability to solve other mathematical, physical, and engineering problems.

2. The present method is accurate, need less effort to achieve the results, and is developed especially for nonlinear case. However, if the problem comes nonlinear, then the RKHS method does not require discretization or perturbation and it does not make closure approximation.

3. In the proposed method, it is possible to pick any point in the interval of integration and as well the approximate solution and all its derivatives up to order four will be applicable.

4. The RKHS method does not require discretization of the variables, that is, time and space; it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time.

In the strict sense of the word, before applying a numerical method to the solution of IDEs, we must be certain that a solution exists. We are also interested in whether the solution is unique. It is worth stating that in many cases, since IDEs are often derived from problems in physical world, existence and uniqueness are often obvious for physical reasons. Notwithstanding this, a mathematical statement about existence and uniqueness is worthwhile. On the other hand, uniqueness would be of importance if, for instance, we wished to approximate the solution. If two solution passed through a point, then approximations could very well jump from one solution to the other with misleading consequences. Therefore, we assume that the fourth-order BVPs of mixed type IDEs to be solved numerically using RKHS method have unique solution on the given interval.

This paper is arranged in the following form: in the next section, a short introduction to reproducing kernel theory is presented. In Section 3, we state the problem and an algorithm solver is introduced. In Section 4, several reproducing kernel functions are constructed in order to apply the RKHS method. In Section 5, we formulate the problem and a theoretic basis of the method is introduced in the space $W_2^5[0,1]$. In Section 6, we will give the representation of exact and approximate solutions, also, an iterative method for solving the present problem numerically using RKHS method is described. In Section 7, we derive an error bound for the present method in order to capture the behavior of solution. Software libraries and numerical results are given in Section 8 in order to verify the mathematical simulation of the proposed algorithm. Finally, concluding remarks are presented in Section 9.

2 Preface to reproducing kernel theory

After a brief introduction to reproducing kernel theory, we view the elements of RKHS and discuss its properties, its applications, and its advantages. In particular, we focus on the spaces $W_2^5[0,1]$ and $W_2^1[0,1]$ among other reproducing kernel, because of their use in this paper, especially in constructing the needed reproducing kernel functions.

In functional analysis, a RKHS is a Hilbert space of functions in which pointwise evaluation is a continuous linear functional. Equivalently, they are spaces that can be defined by reproducing kernels. An abstract set is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for the number of elements.

Definition .1 [13] Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{C}$ is a reproducing kernel of the Hilbert space H if

1. For each $x \in E$, $K(\cdot, x) \in H$.

2. For each $x \in E$ and $\varphi \in H$, $\langle \varphi(\cdot), K(\cdot, x)\rangle = \varphi(x)$.

Remark .1 The condition (2) in Definition .1 is called "the reproducing property" which means that the value of the function φ at the point x is reproducing by the inner product of $\varphi(\cdot)$ with $K(\cdot, x)$. A Hilbert space which possesses a reproducing kernel is called a RKHS [13].

As a special case, the spaces $W_2^5[0,1]$ and $W_2^1[0,1]$ are complete Hilbert with some special properties. So, all the properties of the Hilbert space will be hold. Further, theses spaces possesses some special and better properties which could make some problems be solved easier. For instance, many problems studied in $L_2[0,1]$ space, which is a complete Hilbert space, requires large amount of integral computations, and such computations may be very difficult in some cases. Thus, the numerical integrals have to be calculated in the cost of losing some accuracy. However, the properties of $W_2^5[0,1]$ and $W_2^1[0,1]$ require no more integral computation for some functions, instead of computing some values of a function at some nodes. In fact, this simplification of integral computation not only improves the computational speed, but also improves the computational accuracy.

Reproducing kernel theory has important application in numerical analysis, differential equations, integral equations, probability and statistics, and so fourth [14–16]. Recently, a lot of research work has been devoted to the applications of RKHS method to a wide class of stochastic and deterministic problems involving operator equations, differential equations, and integral equations. The RKHS method was used by many authors to investigate several scientific applications side by side with their theory. The reader is kindly requested to go through [12–31] in order to know more details about RKHS method, including its history, its modification for use, its applications on the other problems, and its characteristics. On the other hand, the numerical solvability of other version of differential problems can be found in [32–35] and references therein.

3 Problem statement and numerical algorithm

Numerical methods tend to emphasize the implementation of algorithms. The aim of numerical methods is therefore to provide systematic methods for solving problems in a numerical form. The process of solving problems generally involves starting from an initial data, using high precision digital computers, following the steps in the algorithms, and finally obtaining the results. Often the numerical data and the methods used are approximate ones.

Let us consider the following fourth-order BVPs of mixed type IDEs described the ordinary functional equation:

$$u^{(4)}(x) = F\left(x, u'''(x), u''(x), u'(x), u(x)\right) + [Tu](x), \tag{1}$$

in which the mixed Fredholm-Volterra operator, $[Tu]$, is given as

$$[Tu](x) = \int_0^1 k_1(x,t) G_1(u'''(t), u''(t), u'(t), u(t))dt + \int_0^x k_2(x,t) G_2(u'''(t), u''(t), u'(t), u(t))dt,$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = \alpha_0,\ u(1) = \beta_0, \\ &u''(0) = \alpha_1,\ u''(1) = \beta_1, \end{aligned} \tag{2}$$

where $0 \le t < x \le 1$, α_i, β_i, $i = 0, 1$ are real finite constants, $u \in W_2^5[0,1]$ is an unknown function to be determined, $k_1(x,t), k_2(x,t)$ are continuous functions on $[0,1]^2$, $F(x, w_1, w_2, w_3, w_4), G_1(w_1, w_2, w_3, w_4), G_2(w_1, w_2, w_3, w_4)$ are continuous terms in $W_2^1[0,1]$ as $w_i = w_i(x) \in W_2^5[0,1]$, $0 \le x \le 1$, $-\infty < w_i < \infty$, $i = 1, 2, 3, 4$ and are depending on the problem discussed, and $W_2^1[0,1], W_2^5[0,1]$ are two reproducing kernel spaces.

The following is the main steps for formulating Eqs. (1) and (2) in order to apply the RKHS method. The steps in the algorithm are explained in more detail in the next sections.

Algorithm 1 To find a series representation of analytic and approximate solutions of Eqs. (1) and (2) using RKHS method, we do the following steps:

Step 1: Introduce new unknown function $v(x)$ as

$$v(x) = u(x) - \phi(x),$$

where $\phi(x)$ satisfies the requirements $\phi(0) = \alpha_0$ and $\phi_1(1) = \beta_0$. Similarly, $\phi''(x)$ satisfies the requirements $\phi''(0) = \alpha_1$ and $\phi''(1) = \beta_1$. Hence, one can obtain

$$\phi(x) = \tfrac{1}{6}(\beta_1 - \alpha_1)x^3 + \tfrac{1}{2}\alpha_1 x^2 + \left(\beta_0 - \alpha_0 - \tfrac{1}{6}\beta_1 - \tfrac{1}{3}\alpha_1\right)x + \alpha_0.$$

Step 2: The form of Eq. (1) with nonhomogeneous boundary conditions (2) can be equivalently reduced to the problem of finding a function $v(x)$ that satisfying the following equation:

$$v^{(4)}(x) = F\left(x, (v+\phi)'''(x), (v+\phi)''(x), (v+\phi)'(x), (v+\phi)(x)\right) + [T(v+\phi)](x), \tag{3}$$

subject to the homogeneous boundary conditions

$$\begin{aligned} &v(0) = 0,\ v(1) = 0, \\ &v''(0) = 0,\ v''(1) = 0. \end{aligned} \tag{4}$$

Step 3: Find the representation form of the two kernel functions $K_x(y)$ and $R_x(y)$ of the spaces $W_2^5[0,1]$ and $W_2^1[0,1]$, respectively.

Step 4: Construct the orthogonal function system $\psi_i(x)$ of the space $W_2^5[0,1]$ as $\psi_i(x) = L_y[K_x(y)]_{y=x_i}$.

Step 5: Construct the orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^5[0,1]$ as $\bar{\psi}_i(x) = \sum\limits_{k=1}^{i} \beta_{ik}\psi_k(x)$.

Step 6: The analytic solution $v(x)$ and the approximate solution $v_n(x)$ of Eqs. (3) and (4) are obtained, respectively, as

$$\begin{aligned} v(x) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} \times \\ &\left\{F\left(x_k, (v+\phi)'''(x_k), (v+\phi)''(x_k), (v+\phi)'(x_k), (v+\phi)(x_k)\right) + [T(v+\phi)](x_k)\right\}\bar{\psi}_i(x), \end{aligned}$$

$$\begin{aligned} v_n(x) &= \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} \times \\ &\left\{F\left(x_k, (v+\phi)'''(x_k), (v+\phi)''(x_k), (v+\phi)'(x_k), (v+\phi)(x_k)\right) + [T(v+\phi)](x_k)\right\}\bar{\psi}_i(x), \end{aligned}$$

where $\beta_{ik}, x_k, \bar{\psi}_i(x)$ are all given in the process of formulation.

Step 7: The analytic solution $u(x)$ and the approximate solution $u_n(x)$ of Eqs. (1) and (2) are obtained, respectively, as

$$u(x) = \phi(x_k) + \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\{F(x_k, u'''(x_k), u''(x_k), u'(x_k), u(x_k)) + [Tu](x_k)\right\}\bar{\psi}_i(x),$$

$$u_n(x) = \phi(x_k) + \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\left\{F(x_k, u'''(x_k), u''(x_k), u'(x_k), u(x_k)) + [Tu](x_k)\right\}\bar{\psi}_i(x).$$

4 Several reproducing kernel functions

In this section, we formulate two reproducing kernels in order to generating the orthogonal basis in the space $W_2^5[0,1]$. After that, an orthonormal basis is constructing in order to formulate and utilize the solution of Eqs. (3) and (4) using RKHS method in the same space.

To apply the RKHS method, we first define and construct a reproducing kernel space $W_2^5[0,1]$ in which every function satisfies the boundary conditions $z(0) = z''(0) = z(1) = z''(1) = 0$.

Definition .2 The inner product space $W_2^5[0,1]$ is defined as $W_2^5[0,1] = \{z(x) : z^{(i)}, i = 0,1,2,3,4$ are absolutely continuous real-valued functions on $[0,1]$, $z^{(5)} \in L^2[0,1]$, and $z(0) = z''(0) = z(1) = z''(1) = 0\}$. The inner product and the norm in $W_2^5[0,1]$ are given by

$$\langle z_1(x), z_2(x)\rangle_{W_2^5} = \sum_{i=0}^{2} z_1^{(i)}(0)\, z_2^{(i)}(0) + \sum_{i=0}^{1} z_1^{(i)}(1)\, z_2^{(i)}(1) + \int_0^1 z_1^{(5)}(x) z_2^{(5)}(x)dx, \tag{5}$$

and $||z||_{W_2^5} = \sqrt{\langle z(x), z(x)\rangle_{W_2^5}}$, respectively, where $z_1, z_2 \in W_2^5[0,1]$ and $L^2[0,1] = \left\{z : \int_0^1 z^2(x)\,dx < \infty\right\}$.

It is easy to see that $\langle z_1(x), z_2(x)\rangle_{W_2^5}$ satisfies all the requirements of the inner product. First, $\langle z_1(x), z_1(x)\rangle_{W_2^5} \geq 0$. Second, $\langle z_1(x), z_2(x)\rangle_{W_2^5} = \langle z_2(x), z_1(x)\rangle_{W_2^5}$. Third, $\langle \gamma z_1(x), z_2(x)\rangle_{W_2^5} = \gamma\langle z_1(x), z_2(x)\rangle_{W_2^5}$. Fourth, $\langle z_1(x)+z_2(x), z_3(x)\rangle_{W_2^5} = \langle z_1(x), z_3(x)\rangle_{W_2^5} + \langle z_2(x), z_3(x)\rangle_{W_2^5}$. It remains only to prove that $\langle z_1(x), z_1(x)\rangle_{W_2^5} = 0$ if and only if $z_1(x) = 0$. In fact, it is obvious that when $z_1(x) = 0$, then $\langle z_1(x), z_1(x)\rangle_{W_2^5} = 0$. On the other hand, if $\langle z_1(x), z_1(x)\rangle_{W_2^2} = 0$, then by Eq. (5), we have $\langle z_1(x), z_1(x)\rangle_{W_2^2} = \sum_{i=0}^{2}\left(z_1^{(i)}(0)\right)^2 + \sum_{i=0}^{1}\left(z_1^{(i)}(1)\right)^2 + \int_0^1 \left(z_1^{(5)}(x)\right)^2 dx = 0$. Therefore, $z_1(0) = z_1'(0) = z_1''(0) = 0$, $z_1(1) = z_1'(1) = 0$, and $z_1^{(5)}(x) = 0$. Then, we can obtain $z_1(x) = 0$.

Definition .3 [13] The Hilbert space $W_2^5[0,1]$ is called a reproducing kernel if for each fixed $x \in [0,1]$ and any $z(y) \in W_2^5[0,1]$, there exist $K(x,y) \in W_2^5[0,1]$ (simply $K_x(y)$) and $y \in [0,1]$ such that $\langle z(y), K_x(y)\rangle_{W_2^5} = z(x)$.

It is very important to obtain the representation form of the reproducing kernel function $K_x(y)$, because it is the basis of our algorithm. In the following theorem, we will give the method for obtaining the reproducing kernel function $K_x(y)$ in the space $W_2^5[0,1]$. After that, we construct the space $W_2^1[0,1]$ in order to define a linear bounded operator L as shown later in the next section.

Theorem .1 The Hilbert space $W_2^5[0,1]$ is a reproducing kernel and its reproducing kernel function $K_x(y)$ is given by

$$K_x(y) = \begin{cases} \sum_{i=0}^{9} p_i(x) y^i, & y \leq x, \\ \sum_{i=0}^{9} q_i(x) y^i, & y > x. \end{cases} \tag{6}$$

where $p_i(x)$ and $q_i(x)$, $i = 0, 2, ..., 9$ are unknown coefficients of $K_x(y)$ and are given as

$$\begin{aligned}
p_0(x) &= 0,\ p_1(x) = \frac{1}{725764}\left(362884x - 725782x^3 + 362903x^4 - 12x^7 + 9x^8 - 2x^9\right),\ p_2(x) = 0, \\
p_3(x) &= \frac{1}{43894206720}(-43895295360x \\
&\quad +87800025610x^3 - 43910173465x^4 + 10160696x^5 - 5806148x^7 + 1088673x^8 - 6x^9), \\
p_4(x) &= \frac{1}{87788413440}x(43896746880 \\
&\quad -87820346930x^2 + 43950816165x^3 - 30482088x^4 + 4354668x^6 - 1088709x^7 + 14x^8), \\
p_5(x) &= p_6(x) = 0, \\
p_7(x) &= -\frac{1}{21947103360}x\left(362880 - 2177292x + 2903074x^2 - 1088667x^3 + 12x^6 - 9x^7 + 2x^8\right), \\
p_8(x) &= \frac{1}{29262804480}x\left(-362884 + 725782x^2 - 362903x^3 + 12x^6 - 9x^7 + 2x^8\right), \\
p_9(x) &= \frac{1}{131682620160}\left(362882 - 362880x - 18x^3 + 21x^4 - 12x^7 + 9x^8 - 2x^9\right),
\end{aligned}$$

$$\begin{aligned}
q_0(x) &= \frac{1}{362880}x^9, \\
q_1(x) &= -\frac{1}{7315701120}\times \\
&\quad x\left(-3657870720 + 7315882560x^2 - 3658062240x^3 + 120960x^6 + 90721x^7 + 20160x^8\right), \\
q_2(x) &= \frac{1}{10080}x^7,
\end{aligned}$$

$$\begin{aligned}
q_3(x) &= -\frac{1}{43894206720}\times \\
&\quad x\left(43895295360-87800025610x^2+43910173465x^3+5806148x^6-1088673x^7+6x^8\right), \\
q_4(x) &= \frac{1}{87788413440}\times \\
&\quad x\left(43896746880-87820346930x^2+43950816165x^3+4354668x^6-1088709x^7+14x^8\right), \\
q_5(x) &= -\frac{1}{2880}x^4,\ q_6(x)=\frac{1}{4320}x^3, \\
q_7(x) &= -\frac{1}{21947103360}x\left(362880+2903074x^2-1088667x^3+12x^6-9x^7+2x^8\right), \\
q_8(x) &= \frac{1}{29262804480}x\left(362880+725782x^2-362903x^3+12x^6-9x^7+2x^8\right), \\
q_9(x) &= -\frac{1}{131682620160}x\left(362880+18x^2-21x^3+12x^6-9x^7+2x^8\right).
\end{aligned}$$

Proof. The proof of the completeness and reproducing property of $W_2^5[0,1]$ is similar to the proof in [17]. Let us now find out the expression form of the reproducing kernel function $K_x(y)$ in the space $W_2^5[0,1]$. Clearly, $\int_0^1 z^{(5)}(y)K_x^{(5)}(y)\,dy=\sum_{i=0}^{4}(-1)^{4-i}z^{(i)}(y)K_x^{(9-i)}(y)\mid_{y=0}^{y=1}+(-1)^3\int_0^1 z(y)K_x^{(10)}(y)\,dy$. Hence, $\langle z(y),K_x(y)\rangle_{W_2^5}=\sum_{i=0}^{2}z^{(i)}(0)K_x^{(i)}(0)+\sum_{i=0}^{1}z^{(i)}(1)K_x^{(i)}(1)+\sum_{i=0}^{4}(-1)^{4-i}z^{(i)}(y)K_x^{(9-i)}(y)\mid_{y=0}^{y=1}-\int_0^1 z(y)K_x^{(10)}(y)\,dy$. Since $K_x(y)\in W_2^5[0,1]$, it follows that $K_x(0)=K_x''(0)=K_x(1)=K_x''(1)=0$. Further, since $z(x)\in W_2^5[0,1]$, one obtains $z(0)=z''(0)=z(1)=z''(1)=0$. Thus, if $K_x^{(i)}(0)=K_x^{(i)}(1)=0$, $i=5,6$, $K_x'(0)+K_x^{(8)}(0)=0$, and $K_x'(1)-K_x^{(8)}(1)=0$, then $\langle z(y),K_x(y)\rangle_{W_2^5}=\int_0^1 z(y)\left(-K_x^{(10)}(y)\right)dy$. Now, for each $x\in[0,1]$, if $K_x(y)$ also satisfies $-K_x^{(10)}(y)=\delta(x-y)$, where δ is the dirac-delta function, then $\langle z(y),K_x(y)\rangle_{W_2^5}=z(x)$. Obviously, $K_x(y)$ is the reproducing kernel function of $W_2^5[0,1]$. Let us now utilizing the expression form of the reproducing kernel function $K_x(y)$. The characteristic equation of $-K_x^{(10)}(y)=\delta(y-x)$ is $\lambda^{10}=0$, and their characteristic values are $\lambda=0$ with 10 multiple roots. So, let the expression form of the reproducing kernel function $K_x(y)$ be as defined in Eq. (6). On the other aspect as well, let $K_x(y)$ satisfies $K_x^{(m)}(x+0)=K_x^{(m)}(x-0)$, $m=0,1,...,8$. Integrating $-K_x^{(10)}(y)=\delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon\to 0$, we have the jump degree of $K_x^{(9)}(y)$ at $y=x$ given by $K_x^{(9)}(x-0)-K_x^{(9)}(x+0)=1$. Through the last descriptions and by using MATHEMATICA 7.0 software package, the unknown coefficients $p_i(x)$ and $q_i(x)$, $i=0,2,...,9$ of Eq. (6) can be obtained as given in the theorem. This completes the proof. ■

Definition .4 [18] The inner product space $W_2^1[0,1]$ is defined as $W_2^1[0,1]=\{z(x):z$ is absolutely continuous real-valued function on $[0,1]$ and $z'\in L^2[0,1]\}$. The inner product and the norm in $W_2^1[0,1]$ are defined as $\langle z_1(x),z_2(x)\rangle_{W_2^1}=\int_0^1\left(z_1'(x)z_2'(x)+z_1(x)z_2(x)\right)dx$ and $||z||_{W_2^1}=\sqrt{\langle z(x),z(x)\rangle_{W_2^1}}$ respectively, where $z_1,z_2\in W_2^1[0,1]$ and $L^2[0,1]=\left\{z:\int_0^1 z^2(x)\,dx<\infty\right\}$.

Theorem .2 [18] The Hilbert space $W_2^1[0,1]$ is a complete reproducing kernel and its reproducing kernel function $R_x(y)$ can be written as

$$R_x(y)=\begin{cases} p_0(x)e^{y}+p_1(x)e^{-y}, & y\le x,\\ q_0(x)e^{y}+q_1(x)e^{-y}, & y>x.\end{cases}$$

where $p_i(x)$ and $q_i(x)$, $i=0,1$ are unknown coefficients of $R_x(y)$ and are given as

$$\begin{aligned}
p_0(x) &= \frac{1}{2\sinh(1)}\cosh(x-1), \\
p_1(x) &= \frac{1}{2\sinh(1)}\cosh(x-1),
\end{aligned}$$

$$\begin{aligned} q_0(x) &= \frac{1}{4\sinh(1)}\left(e^{x-1}+e^{-1-x}\right), \\ q_1(x) &= \frac{1}{4\sinh(1)}\left(e^{1-x}+e^{1+x}\right). \end{aligned}$$

In fact, it easy to see that $q_0(x)e^y + q_1(x)e^{-y} = p_0(y)e^x + p_1(y)e^{-x}$. As a result, the reproducing kernel function posses some important properties such as: it is symmetric, unique, and nonnegative. The reader is asked to refer to [12–31] in order to know more details about reproducing kernel function including its mathematical properties, types and kinds, applications, method of calculations, and others.

5 Problem formulation in the space $\mathbf{W}_2^5[0,1]$

Problem formulation is normally the most important part of the process. It is the selection of linear operator, orthogonal basis, and orthonormal basis. In this section, Eqs. (3) and (4) are first formulated as a differential linear operator based on the spaces $W_2^5[0,1]$ and $W_2^1[0,1]$. After that, the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ is presented.

In order to apply the RKHS method, as in [12, 13, 17–31], we firs define a differential linear operator L as $L: W_2^5[0,1] \to W_2^1[0,1]$ such that $Lv(x) = v^{(4)}(x)$. Thus, discretized form of Eqs. (3) and (4) can be obtained as follows:

$$Lv(x) = F\left(x, (v+\phi)'''(x), (v+\phi)''(x), (v+\phi)'(x), (v+\phi)(x)\right) + [T(v+\phi)](x), \tag{7}$$

subject to the two-point boundary conditions

$$\begin{aligned} &v(0)=0,\ v(1)=0, \\ &v''(0)=0,\ v''(1)=0, \end{aligned} \tag{8}$$

where v and ϕ are as given in Algorithm 1.

Theorem .3 The operator $L: W_2^5[0,1] \to W_2^1[0,1]$ is bounded and linear.

Proof. For boundedness, we need to prove $\|Lv(x)\|_{W_2^1}^2 \le M\|Lv(x)\|_{W_2^5}^2$, where M is a positive constant. From the definition of the inner product and the norm of $W_2^1[0,1]$, we have $\|(Lv)(x)\|_{W_2^1}^2 = \langle (Lv)(x), (Lv)(x)\rangle_{W_2^1} = \int_0^1 \left\{\left[(Lv)'(x)\right]^2 + [(Lv)(x)]^2\right\} dx$. By reproducing property of $K_x(y)$, we have $v(x) = \langle v(y), K_x(y)\rangle_{W_2^5}$, $(Lv)(x) = \langle v(y), (LK_x)(y)\rangle_{W_2^5}$, and $(Lv)'(x) = \langle v(y), (LK_x)'(y)\rangle_{W_2^5}$. Again, by Schwarz inequality, we get

$$|(Lv)(x)| = \left|\langle v(x), (LK_x)(x)\rangle_{W_2^5}\right| \le \|LK_x(x)\|_{W_2^5}\|v(x)\|_{W_2^5} = M_1\|v(x)\|_{W_2^5},\ M_1>0,$$

$$|(Lv)'(x)| = \left|\langle v(x), (LK_x)'(x)\rangle_{W_2^5}\right| \le \|(LK_x)'(x)\|_{W_2^5}\|v(x)\|_{W_2^5} = M_2\|v(x)\|_{W_2^5},\ M_2>0.$$

Thus, $\|(Lv)(x)\|_{W_2^1}^2 = \int_0^1 \left\{\left[(Lv)'(x)\right]^2 + [(Lv)(x)]^2\right\} dx \le (M_1^2+M_2^2)\|v(x)\|_{W_2^5}^2$ or $\|(Lv)(x)\|_{W_2^1} \le M\|v(x)\|_{W_2^5}$, where $M = \sqrt{M_1^2+M_2^2}$. The linearity part is obvious. This complete the proof. ■

After that, we construct an orthogonal function system of $W_2^5[0,1]$ as follows: put $\varphi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L_i^*\varphi(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and L^* is the adjoint operator of L. In terms of the properties of reproducing kernel function $K_x(y)$, one can obtains $\langle v(x), \psi_i(x)\rangle_{W_2^5} = \langle v(x), L^*\varphi_i(x)\rangle_{W_2^5} = \langle Lv(x), \varphi_i(x)\rangle_{W_2^1} = Lv(x_i)$, $i=1,2,\ldots$. In fact, the orthonormal function system $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ of the space $W_2^5[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as

$$\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \tag{9}$$

where β_{ik} are orthogonalization coefficients and are given as follows: $\beta_{ij} = \frac{1}{\|\psi_1\|}$ for $i=j=1$, $\beta_{ij} = \frac{1}{d_i}$ for

$i=j\neq 1$, and $\beta_{ij}=-\frac{1}{d_i}\sum\limits_{k=j}^{i-1}c_{ik}\beta_{kj}$ for $i>j$ such that $d_i=\sqrt{\|\psi_i\|^2-\sum\limits_{k=1}^{i-1}c_{ik}^2}$, $c_{ik}=\langle\psi_i,\overline{\psi}_k\rangle_{W_2^5}$, and $\{\psi_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^5[0,1]$.

Through the next theorem the subscript y by the operator L (L_y) indicates that the operator L applies to the function of y.

Theorem .4 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system of $W_2^5[0,1]$ and $\psi_i(x)=L_yK_x(y)|_{y=x_i}$.

Proof. Clearly, $\psi_i(x)=L_i^*\varphi(x)=\langle L_i^*\varphi(y),K_x(y)\rangle_{W_2^5}=\langle\varphi_i(y),L_yK_x(y)\rangle_{W_2^1}=L_yK_x(y)|_{y=x_i}\in W_2^5[0,1]$. Now, for each fixed $v(x)\in W_2^5[0,1]$, let $\langle v(x),\psi_i(x)\rangle_{W_2^5}=0$, $i=1,2,\ldots$. In other word, $\langle v(x),\psi_i(x)\rangle_{W_2^5}=\langle v(x),L^*\varphi_i(x)\rangle_{W_2^5}=\langle Lv(x),\varphi_i(x)\rangle_{W_2^1}=Lv(x_i)=0$. Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, therefore $Lv(x)=0$. It follows that $v(x)=0$ from the existence of L^{-1}. So, the proof of the theorem is complete. ∎

Lemma .1 If $v(x)\in W_2^5[0,1]$, then there exists $M>0$ such that $\left\|v^{(i)}(x)\right\|_C\leq M\|v(x)\|_{W_2^5}$, $i=0,1,2,3,4$, where $\|v(x)\|_C=\max\limits_{a\leq x\leq b}|v(x)|$.

Proof. For any $x,y\in[0,1]$, we have $v^{(i)}(x)=\left\langle v(y),K_x^{(i)}(y)\right\rangle_{W_2^5}$, $i=0,1,2,3,4$. By the expression of $K_x(y)$, it follows that $\left\|K_x^{(i)}(y)\right\|_{W_2^5}\leq M_i$, $i=0,1,2,3,4$. Thus, $\left|v^{(i)}(x)\right|=\left|\left\langle v(x),K_x^{(i)}(x)\right\rangle_{W_2^5}\right|\leq\left\|K_x^{(i)}(x)\right\|_{W_2^5}\|v(x)\|_{W_2^5}\leq M_i\|v(x)\|_{W_2^5}$, $i=0,1,2,3,4$. Hence, $\left\|v^{(i)}(x)\right\|_C\leq M\|v(x)\|_{W_2^5}$, $i=0,1,2,3,4$, where $M=\max\limits_{i=0,1,2,3,4}\{M_i\}$. ∎

6 Representation of exact and approximate solutions

In this section, we will give the representation form of exact and approximate solutions of Eqs. (3) and (4) in the space $W_2^5[0,1]$. After that, an iterative formulas of obtaining approximate solution is represented for both linear and nonlinear case.

Theorem .5 For each $v(x)\in W_2^5[0,1]$, the series $\sum\limits_{i=1}^{\infty}\left\langle v(x),\bar{\psi}_i(x)\right\rangle\bar{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^5[0,1]$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then the following are hold:

(i) The exact solution of Eqs. (7) and (8) could be represented by

$$\begin{aligned}v(x)&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\times\\&\left\{F\left(x_k,(v+\phi)'''(x_k),(v+\phi)''(x_k),(v+\phi)'(x_k),(v+\phi)(x_k)\right)+[T(v+\phi)](x_k)\right\}\bar{\psi}_i(x).\end{aligned}\tag{10}$$

(ii) The approximate solution of Eqs. (7) and (8)

$$\begin{aligned}v_n(x)&=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\times\\&\left\{F\left(x_k,(v+\phi)'''(x_k),(v+\phi)''(x_k),(v+\phi)'(x_k),(v+\phi)(x_k)\right)+[T(v+\phi)](x_k)\right\}\bar{\psi}_i(x),\end{aligned}\tag{11}$$

and its derivative up to order four are converging uniformly to the exact solution $v(x)$ and all its derivative as $n\to\infty$, respectively.

Proof. For the first part, let $v(x)$ be solution of Eqs. (7) and (8) in the space $W_2^5[0,1]$. Since $v(x)\in W_2^5[0,1]$, $\sum\limits_{i=1}^{\infty}\left\langle v(x),\bar{\psi}_i(x)\right\rangle\bar{\psi}_i(x)$ is the Fourier series about normal orthogonal system $\left\{\bar{\psi}_i(x)\right\}_{i=1}^{\infty}$, and $W_2^5[0,1]$ is the Hilbert space, then the series $\sum\limits_{i=1}^{\infty}\left\langle v(x),\bar{\psi}_i(x)\right\rangle\bar{\psi}_i(x)$ is convergent in the sense of $\|\cdot\|_{W_2^5}$. On the other aspect as well, using Eq. (9), we have

$$\begin{aligned}
v(x) &= \sum_{i=1}^{\infty} \left\langle v(x), \bar{\psi}_i(x) \right\rangle_{W_2^5} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle v(x), \psi_k(x) \right\rangle_{W_2^5} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle v(x), L^*\varphi_k(x) \right\rangle_{W_2^5} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Lv(x), \varphi_k(x) \right\rangle_{W_2^1} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \times \\
&\quad \left\langle F\left(x, (v+\phi)'''(x), (v+\phi)''(x), (v+\phi)'(x), (v+\phi)(x)\right) + [T(v+\phi)](x), \varphi_k(x) \right\rangle_{W_2^1} \bar{\psi}_i(x) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \times \\
&\quad \left\{ F\left(x_k, (v+\phi)'''(x_k), (v+\phi)''(x_k), (v+\phi)'(x_k), (v+\phi)(x_k)\right) + [T(v+\phi)](x_k) \right\} \bar{\psi}_i(x).
\end{aligned}$$

Therefore, the form of Eq. (10) is the exact solution of Eqs. (7) and (8).

For the second part, it easy to see that by Lemma .1, for any $x \in [0,1]$

$$\begin{aligned}
\left| v_n^{(i)}(x) - v^{(i)}(x) \right| &= \left| \left\langle v_n(x) - v(x), K_x^{(i)}(x) \right\rangle_{W_2^5} \right| \\
&\leq \left\| K_x^{(i)}(x) \right\|_{W_2^5} \left\| v_n(x) - v(x) \right\|_{W_2^5} \\
&\leq M_i \left\| v_n(x) - v(x) \right\|_{W_2^5}, \ i = 0,1,2,3,4,
\end{aligned}$$

where M_i, $i = 0,1,2,3,4$ are positive constants. Hence, if $\|v_n(x) - v(x)\|_{W_2^5} \to 0$ as $n \to \infty$, the approximate solution $v_n(x)$ and $v_n^{(i)}(x)$, $i = 0,1,2,3,4$ are converge uniformly to the exact solution $v(x)$ and all its derivative, respectively. So, the proof of the theorem is complete. ■

Next, we will mention the following remark about the exact and approximate solutions of Eqs. (3) and (4).

Remark .2 [12,13,17–31] In order to apply the RKHS technique for solve Eqs. (3) and (4), we define an initial guess approximation function as $v_0(x_1) = v(x_1) = 0$. On the other hand, we have the following two cases based on the form of Eq. (11) and the structure of the functions F, G_1, and G_2 in Eq. (3).

Case 1: If Eq. (3) is linear, then the approximate solution can be obtained directly as follows:

$$\begin{aligned}
v_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \{ & F\left(x_k, (v_{k-1}+\phi)'''(x_k), (v_{k-1}+\phi)''(x_k), (v_{k-1}+\phi)'(x_k), (v_{k-1}+\phi)(x_k)\right) \\
& + [T(v_{k-1}+\phi)](x_k) \} \bar{\psi}_i(x).
\end{aligned}$$

Case 2: If Eq. (3) is nonlinear, then the approximate solution can be obtained immediately as follows:

$$\begin{aligned}
v_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} \{ & F\left(x_k, (v_{n-1}+\phi)'''(x_k), (v_{n-1}+\phi)''(x_k), (v_{n-1}+\phi)'(x_k), (v_{n-1}+\phi)(x_k)\right) \\
& + [T(v_{n-1}+\phi)](x_k) \} \bar{\psi}_i(x).
\end{aligned}$$

The reader is asked to refer to [12,13,17–31] in order to know more details about these two case, including their derivation, their importance, and their relationship to the exact solution.

7 Error estimation and error bound

When solving practical problems, it is necessary to take into account all the errors of the measurements. Moreover, in accordance with the technical progress and the degree of complexity of the problem, it becomes necessary to improve the technique of measurement of quantities. Considerable errors of measurement become inadmissible in solving complicated mathematical, physical, and engineering problems. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique. In this section, we derive an error bounds for the present method and problem in order to capture behavior of the solution.

In the next theorem, we show that the error of approximate solution is monotone decreasing, while the next lemma is presented in order to prove the recent theorem.

Theorem .6 Let $\varepsilon_{s,n}^2 = ||v(x) - v_n(x)||^2_{W_2^3}$, where $v(x)$ and $v_n(x)$ are given by Eq. (10) and Eq. (11), respectively. Then, the sequence of numbers $\{\varepsilon_n\}$ are monotone decreasing in the sense of the norm of $W_2^5[0,1]$ and $\varepsilon_n \to 0$ as $n \to \infty$.

Proof. Since, $v(x) = \sum_{i=1}^{\infty} \langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \bar{\psi}_i(x)$ it obvious that

$$\varepsilon_n^2 = ||v(x) - v_n(x)||^2_{W_2^5} = \left\| \sum_{i=n+1}^{\infty} \langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \bar{\psi}_i(x) \right\|^2_{W_2^5} = \sum_{i=n+1}^{\infty} \left(\langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \right)^2,$$

$$\varepsilon_{n-1}^2 = ||v(x) - v_{n-1}(x)||^2_{W_2^5} = \left\| \sum_{i=n}^{\infty} \langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \bar{\psi}_i(x) \right\|^2_{W_2^5} = \sum_{i=n}^{\infty} \left(\langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \right)^2.$$

Clearly, $\varepsilon_{n-1} \geq \varepsilon_n$, and consequently $\{\varepsilon_n\}$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^5}$. On the other aspect as well, by Theorem .5, we know that $\sum_{i=1}^{\infty} \langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \bar{\psi}_i(x)$ is convergent in the sense of $\|\cdot\|_{W_2^5}$. Thus, we have $\varepsilon_n^2 = \sum_{i=n+1}^{\infty} \left(\langle v(x), \bar{\psi}_i(x) \rangle_{W_2^5} \right)^2 \to 0$ or $\varepsilon_n \to 0$. This complete the proof. ■

Lemma .2 Let $v(x)$ is the exact solution of Eqs. (7) and (8), $v_n(x)$ is the approximate solution of $v(x)$, and $T = \left\{ x_{k+1} = \frac{k}{2^i} : k = 0, 1, ..., 2^i \right\}$. Then, $Lv(x_k) = Lv_n(x_k)$, for $n = 2^i + 1$ and $x_k \in T$.

Proof. Set the projective operator $P_n : W_2^5[0,1] \to \{\sum_{m=1}^{n} c_m \psi_m(x), c_m \in \mathbb{R}\}$, Then, we have $Lv_n(x_k) = \langle v_n(\xi), L_{x_k} F_{x_k}(\xi) \rangle_{W_2^5} = \langle v_n(\xi), \psi_k(\xi) \rangle_{W_2^5} = \langle P_n v(\xi), \psi_k(\xi) \rangle_{W_2^5} = \langle v(\xi), P_n \psi_k(\xi) \rangle_{W_2^5} = \langle v(\xi), \psi_k(\xi) \rangle_{W_2^5} = \langle v(\xi), L_{x_k} F_{x_k}(\xi) \rangle_{W_2^5} = L_{x_k} \langle v(\xi), F_{x_k}(\xi) \rangle_{W_2^5} = L_{x_k} v(x_k) = Lv(x_k)$. ■

Theorem .7 Let $v(x)$ is the exact solution of Eqs. (7) and (8), $v_n(x)$ is the approximate solution of $v(x)$, and $T = \left\{ x_{k+1} = \frac{k}{2^i} : k = 0, 1, ..., 2^i \right\}$. Then, $|v(x) - v_n(x)| < \frac{M}{n}$, where M is the product of the sup of convergent basis $\left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \{F(x_k, (v+\phi)'''(\xi_k), (v+\phi)''(\xi_k), (v+\phi)'(\xi_k), (v+\phi)(\xi_k)) + [T(v+\phi)](\xi_k)\} \bar{\psi}_i(\xi) \right\|_{W_2^5}$ and the maximum of determinate function $\left\| \frac{\partial}{\partial \eta} K_\eta(\xi) \right\|_{W_2^5}$ about the variable in $[0,1]$.

Proof. Since $|v(x) - v_n(x)| = \left| L^{-1}(Lv(x) - Lv_n(x)) \right|$ and for every given $x \in [0,1]$, there is always $x_0 \in T$ satisfying $x_0 < x$ and $x - x_0 = \frac{1}{n}$. On the other hand, Lemma .2 and $x_0 \in T$ implying that $Lv(x_0) = Lv_n(x_0)$. So, we obtain

$$|Lv(x) - Lv_n(x)| = |(Lv(x) - Lv(x_0)) - (Lv_n(x) - Lv_n(x_0))|. \tag{12}$$

By applying the reproducing kernel properties $v(x) = \langle v(\xi), R_x(\xi) \rangle_{W_2^5}$ and $Lv(x) = \langle v(\xi), LK_x(\xi) \rangle_{W_2^5}$ to Eq.

(12), we conclude

$$\begin{aligned} Lv(x) - Lv_n(x) &= (Lv(x) - Lv(x_0)) - (Lv_n(x) - Lv_n(x_0)) \\ &= \langle v(\xi), LK_x(\xi) - LK_{x_0}(\xi)\rangle_{W_2^5} - \langle v_n(\xi), LK_x(\xi) - LR_{x_0}(\xi)\rangle_{W_2^5} \\ &= \langle v(\xi) - v_n(\xi), LK_x(\xi) - LK_{x_0}(\xi)\rangle_{W_2^5}. \end{aligned}$$

But on the other aspect as well, we have

$$\begin{aligned} |v(x) - v_n(x)| &= \left|L^{-1}(Lv(x) - Lv_n(x))\right| \\ &\leq \left|\left\langle v(\xi) - v_n(\xi), L^{-1}LK_x(\xi) - L^{-1}LK_{x_0}(\xi)\right\rangle_{W_2^5}\right| \\ &= \left|\langle v(\xi) - v_n(\xi), K_x(\xi) - K_{x_0}(\xi)\rangle_{W_2^5}\right| \\ &\leq ||v(\xi) - v_n(\xi)||_{W_2^5} ||K_x(\xi) - K_{x_0}(\xi)||_{W_2^5}. \end{aligned}$$

Here, we take the norm of $||K_x(\xi) - K_{x_0}(\xi)||_{W_2^5}$ for the variable ξ and the function $K_x(\xi)$ is derived on x in $[0,1]$. So, we have $K_x(\xi) - K_{x_0}(\xi) = \frac{\partial}{\partial\eta}K_\eta(\xi)(x - x_0)$. Hence, we can write

$$\begin{aligned} |v(x) - v_n(x)| &\leq ||v(\xi) - v_n(\xi)||_{W_2^5} \left|\left|\frac{\partial}{\partial\eta}K_\eta(\xi)(x - x_0)\right|\right|_{W_2^5} \\ &= \| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\{F\left(x_k, (v+\phi)'''(\xi_k), (v+\phi)''(\xi_k), (v+\phi)'(\xi_k), (v+\phi)(\xi_k)\right) \\ &\quad + [T(v+\phi)](\xi_k)\}\bar{\psi}_i(\xi) \|_{W_2^5} \| \frac{\partial}{\partial\eta}K_\eta(\xi) \|_{W_2^5} (x - x_0) \\ &= \frac{M}{n}. \end{aligned}$$

So, the proof of the theorem is complete. ■

8 Software libraries and numerical outcomes

Software packages have great capabilities for solving mathematical, physical, and engineering problems. Sometimes, it is very difficult to solve these problems analytically, so it is required to obtain an efficient approximate solution. Thus, some software mathematical packages such as MATHEMATICA or MAPLE can be helpful in visualizing the behavior of the solutions of such problems. Indeed, throughout the whole paper we used MATHEMATICA 7.0 software package for numerical experiment.

The object of the next algorithm is to implement a procedure to solve Eqs. (1) and (2) in numeric form in terms of their grid nodes based on the use of RKHS method.

Algorithm 2 To approximate the solution of Eqs. (1) and (2), we do the following steps:

Input: The endpoints of $[0,1]$; the integers n and N; the kernel functions $K_x(y)$ and $R_x(y)$; the differential operator L; the function F; the operator $[Tu]$.

Output: Approximate solution $u_n(x)$ or $u_n^N(x)$ of Eqs. (1) and (2).

Step 1: Fixed x in $[0,1]$ and set $y \in [0,1]$;

If $y \leq x$ then set $K_x(y) = \sum_{i=0}^{9} p_i(x)y^i$;

else set $K_x(y) = \sum_{i=0}^{9} q_i(x)y^i$;

For $i = 1, 2, ..., n$ do the following:

Set $x_i = \frac{i-1}{n-1}$;

Set $\psi_i(x) = L_y\left[K_x(y)\right]_{y=x_i}$;

Output: the orthogonal function system $\psi_i(x)$.

Step 2: For $i = 2, 3..., n$ and $j = 1, 2..., i$ do the following:

Set $d_i = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}$;

If $j \neq i$ then set $\beta_{ij} = -\frac{1}{d_i}\sum_{k=j}^{i-1} c_{ik}\beta_{kj}$;

else set $\beta_{ij} = \frac{1}{d_i}$;

else set $\beta_{11} = \frac{1}{\|\psi_1\|}$;

Output: the orthogonalization coefficients β_{ij}.

Step 3: For $i = 2, 3..., n$ and $k = 1, 2..., i-1$ do the following:

Set $\bar{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$;

Set $c_{ik} = \langle \psi_i, \overline{\psi}_k \rangle_{W_2^5}$;

Output: the orthonormal function system $\bar{\psi}_i(x)$.

Step 4: Set $v_0(x_1) = v(x_1) = 0$;

For $i = 1, 2, ...n$ do the following:

If F and $[Tu]$ are linear then set $B_i = \sum_{k=1}^{i} \beta_{ik}\{F(x_k, (v_{k-1}+\phi)'''(x_k), (v_{k-1}+\phi)''(x_k),$

$(v_{k-1}+\phi)'(x_k), (v_{k-1}+\phi)(x_k)) + [T(v_{k-1}+\phi)](x_k)\}$;

set $v_i(x) = \sum_{i=1}^{i} B_i\bar{\psi}_i(x)$;

else for $i = 1, 2, ...N$ do the following:

set $x_i = \frac{i-1}{N-1}$;

set $B_i = \sum_{k=1}^{i} \beta_{ik}\{(Fx_k, (v_{n-1}+\phi)'''(x_k), (v_{n-1}+\phi)''(x_k),$

$(v_{n-1}+\phi)'(x_k), (v_{n-1}+\phi)(x_k))$

$+[T(v_{n-1}+\phi)](x_k)\}\bar{\psi}_i(x)$;

set $v_n^i(x) = \sum_{i=1}^{i} B_i\bar{\psi}_i(x)$;

Output: the approximate solution $v_n(x)$ or $v_n^N(x)$ of Eqs. (3) and (4).

Step 5: Use the transformation $u_n(x) = v_n(x) + \phi(x)$ or $u_n^N(x) = v_n^N(x) + \phi(x)$;

Output: the approximate solution $u_n(x)$ or $u_n^N(x)$ of Eqs. (1) and (2).

Step 6: Stop.

Next, we propose few numerical simulations for solving some specific examples of Eqs. (1) and (2). However, we apply the techniques described in the previous sections to some linear and nonlinear test examples in order to demonstrate the efficiency, accuracy, and applicability of the proposed method. Results obtained by the method are compared with the analytical solution of each example by computing the exact and relative errors and are found to be in good agreement with each other.

Problem .1 Consider the following linear equation:

$$u^{(4)}(x) = -u''(x) + \pi^4 u(x) + f(x) + [Tu](x),$$

in which the mixed operator is given as

$$[Tu](x) = -\int_0^1 x^2 t u'(t) dt + \int_0^x (x+1) t u(t) dt,$$

and subject to the boundary conditions

$$\begin{aligned} &u(0) = 0, u(1) = 0, \\ &u''(0) = 0, u''(1) = 0, \end{aligned}$$

where $0 \le t < x \le 1$ and $f(x)$ is chosen such that the exact solution is $u(x) = \sin(\pi x)$.

Using Algorithms 1 and 2, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$. The numerical results of approximate solution $u_n(x)$ of $u(x)$ are calculated at some selected grid points for $n = 36$ and are tabulated in Table 1.

Table 1. Numerical results for Problem 1: solutions and corresponding errors.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	0.4817536741017153	0.4817521115692216	1.56253×10^{-6}	3.24343×10^{-6}
0.32	0.8443279255020151	0.8443251824015761	2.74310×10^{-6}	3.24886×10^{-6}
0.48	0.9980267284282716	0.9980234824190575	3.24601×10^{-6}	3.25243×10^{-6}
0.64	0.9048270524660195	0.9048241075118995	2.94495×10^{-6}	3.25471×10^{-6}
0.80	0.5877852522924732	0.5877833384377027	1.91385×10^{-6}	3.25604×10^{-6}
0.96	0.1253332335643045	0.1253328254068720	4.08157×10^{-7}	3.25658×10^{-6}

Problem .2 Consider the following nonlinear equation:

$$u^{(4)}(x) = -x(u''(x))^2 - \cos(x)u'(x) - u(x) + 2\ln((u(x))^3) + f(x) + [Tu](x),$$

in which the mixed operator is given as

$$[Tu](x) = \int_0^1 x t u(x) u'(t) \ln(u(t)) dt - 9\int_0^x (x-t) u''(t) (u(t))^3 dt,$$

and subject to the boundary conditions

$$\begin{aligned} &u(0) = 1, u(1) = e, \\ &u''(0) = 1, u''(1) = e, \end{aligned}$$

where $0 \le t < x \le 1$ and $f(x)$ is chosen such that the exact solution is $u(x) = e^x$.

Using Algorithms 1 and 2, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$. The numerical results of approximate solution $u_n^N(x)$ of $u(x)$ are calculated at some selected grid points for $N = 36$, $n = 2$ and are tabulated in Table 2.

Table 2. Numerical results for Problem 2: solutions and corresponding errors.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	1.1735108709918103	1.1735115633003046	6.92308×10^{-7}	5.89946×10^{-7}
0.32	1.3771277643359572	1.3771289810084700	1.21667×10^{-6}	8.83486×10^{-7}
0.48	1.6160744021928934	1.6160758551178427	1.45292×10^{-6}	8.99046×10^{-7}
0.64	1.8964808793049515	1.8964822199979947	1.34069×10^{-6}	7.06937×10^{-7}
0.80	2.2255409284924680	2.2255418182404440	8.89748×10^{-7}	3.99790×10^{-7}
0.96	2.6116964734231180	2.6116966661583056	1.92735×10^{-7}	7.37969×10^{-8}

Problem .3 Consider the following nonlinear equation:

$$u^{(4)}(x) = -xu'''(x) - u'(x)e^{u(x)} - (u(x))^2 + f(x) + [Tu](x),$$

in which the mixed operator is given as

$$[Tu](x) = \int_0^1 e^x(t-1)\sinh(u(x))dt + \int_0^x \cosh(x)t^3 e^{u(t)}dt,$$

and subject to the boundary conditions

$$u(0) = 0, u(1) = \ln(2),$$
$$u''(0) = 1, u''(1) = -\tfrac{1}{4},$$

where $0 \le t < x \le 1$ and $f(x)$ is chosen such that the exact solution is $u(x) = \ln(x+1)$.

Using Algorithms 1 and 2, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$. The numerical results of approximate solution $u_n^N(x)$ of $u(x)$ are calculated at some selected grid points for $N = 36$, $n = 2$ and are tabulated in Table 3.

Table 3. Numerical results for Problem 3: solutions and corresponding errors.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	0.1484200051182732	0.1484150657910304	4.93933×10^{-6}	3.32794×10^{-5}
0.32	0.2776317365982796	0.2776238623910086	7.87421×10^{-6}	2.83621×10^{-5}
0.48	0.3920420877760237	0.3920336245177083	8.46326×10^{-6}	2.15876×10^{-5}
0.64	0.4946962418361071	0.4946891488543930	7.09298×10^{-6}	1.43381×10^{-5}
0.80	0.5877866649021191	0.5877822986892660	4.36621×10^{-6}	7.42823×10^{-6}
0.96	0.6729444732424258	0.6729435645496804	9.08693×10^{-7}	1.35032×10^{-6}

Problem .4 Consider the following nonlinear equation:

$$u^{(4)}(x) = -u'''(x) + 3u''(x) - \sin(u'(x)) - \frac{1}{3}(u(x))^3 + f(x) + [Tu](x),$$

in which the mixed operator is given as

$$[Tu](x) = -\frac{1}{4}\int_0^1 \ln(u(t))u'(t)dt + \frac{1}{2}\int_0^x \ln(x)u''(t)(u(t))^2 dt,$$

and subject to the boundary conditions

$$u(0) = 1, u(1) = \cosh(1),$$
$$u''(0) = 1, u''(1) = \cosh(1),$$

where $0 \le t < x \le 1$ and $f(x)$ is chosen such that the exact solution is $u(x) = \cosh(x)$.

Using Algorithms 1 and 2, taking $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$. The numerical results of approximate solution $u_n^N(x)$ of $u(x)$ are calculated at some selected grid points for $N = 36$, $n = 2$ and are tabulated in Table 4.

Table 4. Numerical results for Problem 4: solutions and corresponding errors.

x	Exact solution	Approximate solution	Absolute error	Relative error
0.16	1.0128273299790107	1.0128278915466450	5.61568×10^{-7}	5.54455×10^{-7}
0.32	1.0516384007048240	1.0516393762289420	9.75524×10^{-7}	9.27623×10^{-7}
0.48	1.1174288969995172	1.1174300474164940	1.15042×10^{-6}	1.02952×10^{-6}
0.64	1.2118866516740000	1.2118877015831520	1.04991×10^{-6}	8.66343×10^{-7}
0.80	1.3374349463048447	1.3374356375116260	6.91207×10^{-7}	5.16815×10^{-7}
0.96	1.4972946796991150	1.4972948288491459	1.49150×10^{-7}	9.96130×10^{-8}

The influence of the number of nodes, N, on the absolute error function $\left|u(x) - u_n^N(x)\right|$ of RKHS method is explored next. Figure 1 gives the relevant data for Problem 2, where the number of nodes covers the range from 9 to 72 in multiple of 2 in which $n = 2$. It is observed that the increase in the number of node results in a reduction in the absolute error and correspondingly an improvement in the accuracy of the obtained solution. This goes in agreement with the known fact, the error is monotone decreasing, where more accurate solutions are achieved using an increase in the number of nodes. On the other hand, the cost to be paid while going in this direction is the rapid increase in the number of iterations required for convergence. For instance, while increasing the number of nodes from 9 to 18 to 36 to 72, the maximum absolute error jumps from almost 2.5×10^{-5} to 6×10^{-6} to 1.4×10^{-6} to 3.5×10^{-7}, i.e. 0.24 to 0.233 to 0.25 multiplication factor.

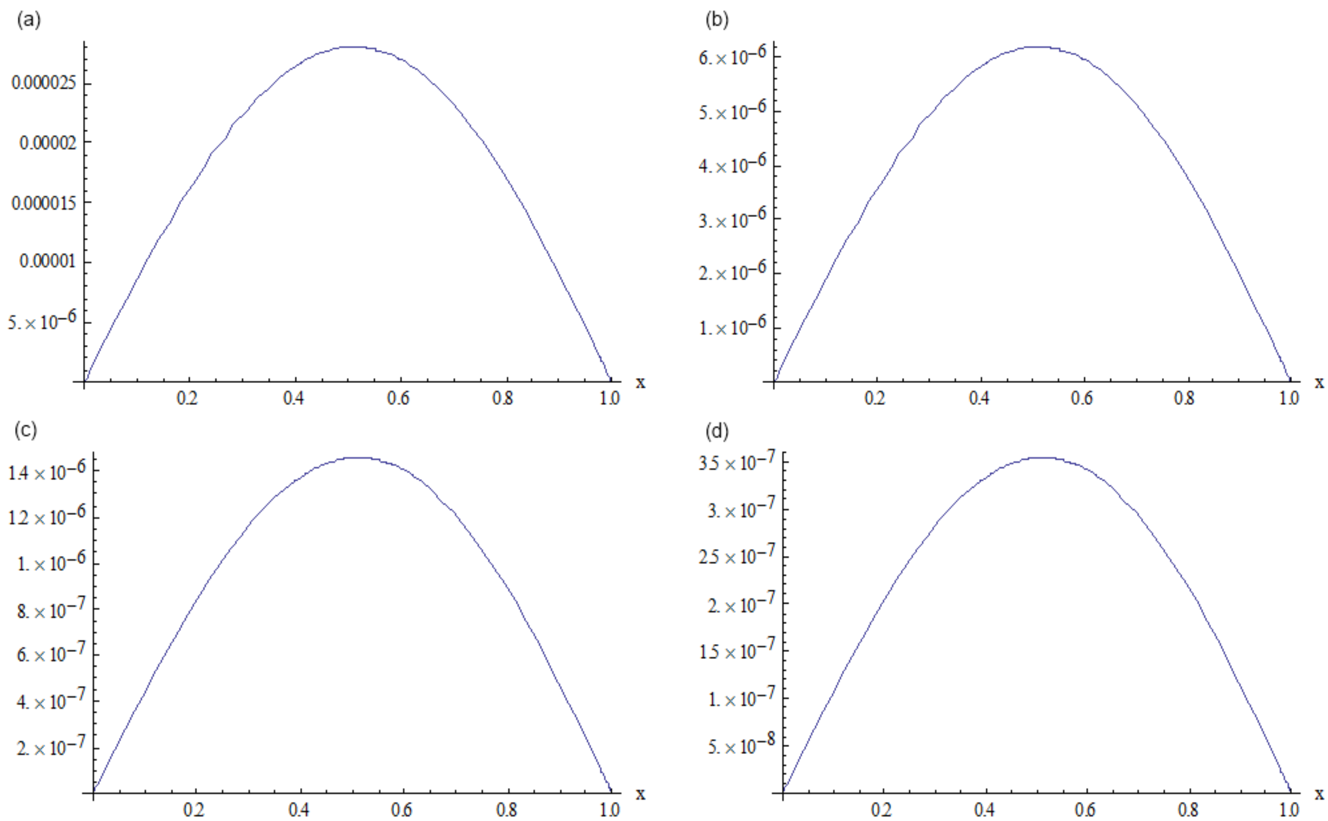

Figure 1: The influence of the number of nodes on the absolute error function of Problem 2 for: (a) $\left|u(x) - u_2^9(x)\right|$; (b) $\left|u(x) - u_2^{18}(x)\right|$; (c) $\left|u(x) - u_2^{36}(x)\right|$; (d) $\left|u(x) - u_2^{72}(x)\right|$.

It is to be noted that the accuracy of certain node is inversely proportional to its distance (number of nodes) from the boundaries. From the last mentioned sub figures, we see that we can achieve a very good approximation with the exact solution in uniform multiplication factors.

Next, according to fact that it is possible to pick any point in $[0, 1]$ and as well the approximate solutions and all their derivatives up to order four will be applicable; some numerical values of $u^{(m)}(x)$, $m = 0, 1, 2, 3, 4$, where $x_i = \frac{i-1}{N-1}$, $i = 1, 2, ..., N$ in which $N = 36$, $n = 2$ at some selected grid nodes in $[0, 1]$ for Problem 4 are tabulated in Table 5.

Table 5. Approximate numerical values of $u^{(m)}(x)$, $m = 0, 1, 2, 3, 4$ for Problem 4: selected nodes in $[0, 1]$.

m	$x = 0.16$	$x = 0.48$	$x = 0.64$	$x = 0.96$
0	1.0128273299790107	1.1174288969995172	1.2118866516740000	1.4972946796991150
1	0.1606834378652106	0.4986425689588153	0.6845916538580700	1.1144011777719312
2	1.0128256744654371	1.1174207910724487	1.2118775439320457	1.4972933355230342
3	0.1606750325962580	0.4985853351414680	0.6845395334998298	1.1143922879515298
4	1.0127516387810787	1.1171994146708600	1.2115152792084836	1.4972589410677763

9 Concluding remarks

In this paper, we introduce an algorithm for solving fourth-order BVPs of mixed type IDEs by using RKHS method. The approximate solution obtained by the present method is uniformly convergent while the series solution methodology can be applied to much more complicated nonlinear problems. In the proposed algorithm, the solution $u(x)$ and the approximate solution $u_n(x)$ are represented in the form of rabidly convergent series in the space $W_2^5[0,1]$. Moreover, the approximate solution and all its derivatives up to order four converge uniformly to the exact solution and all its derivatives, respectively. Additionally, we note that not only a computational method is presented but also we proved that the error of the approximate solution is monotone decreasing in the sense of the norm of $W_2^5[0,1]$. The method is shown to be of good convergence, simple in principle, and easy to program.

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AN AQCQ-FUNCTIONAL EQUATION IN NORMED 2-BANACH SPACES

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Abstract. In this paper, we prove the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in normed 2-Banach spaces.

1. Introduction and preliminaries

In the 1960's, Gähler [18, 19] introduced the concept of linear 2-normed spaces.

Definition 1.1. Let X be a real linear space with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \to \mathbb{R}_{\geq 0}$ be a function satisfying the following properties:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha| \, \|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$
for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2-*norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called a *linear* 2-*normed space*. Sometimes the condition (d) called the *triangle inequality*.

We introduce a basic property of linear 2-normed spaces.

Lemma 1.2. ([35]) *Let* $(X, \|\cdot,\cdot\|)$ *be a linear* 2*-normed space. If* $x \in X$ *and* $\|x, y\| = 0$ *for all* $y \in X$*, then* $x = 0$.

In the 1960's, Gähler and White [20, 47, 48] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *Cauchy sequence* if

$$\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$$

for all $y \in X$.

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *convergent sequence* if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Triangle inequality implies the following lemma.

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Lemma 1.5. ([35]) *For a convergent sequence* $\{x_n\}$ *in a linear* 2*-normed space* X,

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for all $y \in X$.

Definition 1.6. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space*.

Definition 1.7. A 2-Banach space X is called a *normed* 2*-Banach space* if X is a normed space with norm $\| \cdot \|$.

The stability problem of functional equations originated from a question of Ulam [46] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [4] for additive mappings and by Th.M. Rassias [37] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [17] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [38] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [16] following the same approach as in Th.M. Rassias [37], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Th.M. Rassias and Šemrl [39] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [8], and Hyers, Isac and Th.M. Rassias [22]).

In 1982, J.M. Rassias [36] followed the innovative approach of the Th.M. Rassias' theorem [37] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [45] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [7] proved the Hyers-Ulam stability of the quadratic functional equation.

In [24], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [27], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 3, 5, 9, 11, 12, 14, 15, 23, 25, 26, 28, 29, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44]).

In this paper, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y) \\ &- 6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \end{aligned} \tag{1.3}$$

in 2-normed spaces.

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x).$$

It was shown in [13, Lemma 2.2] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in [10, Lemma 2.1] that $g(x) := f(2x)-4f(x)$ and $h(x) := f(2x)-16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

Throughout this paper, let X be a normed space and Y a normed 2-Banach space.

2. Hyers-Ulam stability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces: odd mapping case

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces for an odd mapping case.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ by

$$\begin{aligned} Df(x,y) &:= f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x) \\ &\quad -f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned}$$

for all $x, y \in X$.

Theorem 2.1. *Let $\theta \in [0,\infty)$, $p, q \in (0,\infty)$ with $p+q > 1$ and let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y), w\| \le \theta \|x\|^p \|y\|^q \|w\| \tag{2.1}$$

for all $x, y \in X$ and all $w \in Y$. Then there is a unique additive mapping $A : X \to Y$ such that

$$\|f(2x)-8f(x)-A(x), w\| \le \frac{4+2^p}{2^{p+q}-2}\theta \|x\|^{p+q}\|w\| \tag{2.2}$$

for all $x \in X$ and all $w \in Y$.

Proof. Letting $x = y$ in (2.1), we get

$$\|f(3y)-4f(2y)+5f(y), w\| \le \theta \|y\|^{p+q}\|w\| \tag{2.3}$$

for all $y \in X$ and all $w \in Y$.

Replacing x by $2y$ in (2.1), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y), w\| \leq 2^p\theta\|y\|^{p+q}\|w\| \tag{2.4}$$

for all $y \in X$ and all $w \in Y$.

By (2.3) and (2.4),

$$\begin{aligned} &\|f(4y) - 10f(2y) + 16f(y), w\| \quad (2.5)\\ &\quad \leq \|4(f(3y) - 4f(2y) + 5f(y)), w\| + \|f(4y) - 4f(3y) + 6f(2y) - 4f(y), w\| \\ &\quad = 4\|f(3y) - 4f(2y) + 5f(y), w\| + \|f(4y) - 4f(3y) + 6f(2y) - 4f(y), w\| \\ &\quad \leq (4 + 2^p)\theta\|y\|^{p+q}\|w\| \end{aligned}$$

for all $y \in X$ and all $w \in Y$. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 8f(x)$ in (2.5), we get

$$\left\|g(x) - 2g\left(\frac{x}{2}\right), w\right\| \leq \frac{4 + 2^p}{2^{p+q}}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$. Hence

$$\left\|2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right), w\right\| \leq 2\sum_{j=l}^{m-1} \frac{4 + 2^p}{2^{p+q}} \frac{2^j}{2^{(p+q)j}}\theta\|x\|^{p+q}\|w\| \tag{2.6}$$

for all nonnegative integers m and l with $m > l$, all $x \in X$ and all $w \in Y$. It follows from (2.6) that the sequence $\{2^k g(\frac{a}{2^k})\}$ is Cauchy for each $x \in X$. Since Y is 2-Banach space, the sequence $\{2^k g(\frac{a}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{j\to\infty} 2^j g\left(\frac{x}{2^j}\right)$$

for all $x \in X$. That is,

$$\lim_{j\to\infty} \left\|2^j g\left(\frac{x}{2^j}\right) - A(x), w\right\| = 0$$

for all $x \in X$ and all $w \in Y$.

Moreover, letting $0 = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.2).

By (2.1),

$$\|DA(x, y), w\| = \lim_{k\to\infty} \left\|2^k Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right), w\right\| \leq \lim_{k\to\infty} \frac{(2^p + 8)2^k}{2^{(p+q)k}}\theta\|x\|^p\|y\|^q\|w\| = 0$$

and so $\|DA(x, y), w\| = 0$ for all $x, y \in X$ and all $w \in Y$. Hence $DA(x, y) = 0$ for all $x, y \in X$. Since $g : X \to Y$ is odd, $A : X \to Y$ is odd. So the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned} \|A(x) - T(x), w\| &= \left\|2^l A\left(\frac{x}{2^l}\right) - 2^l T\left(\frac{x}{2^l}\right), w\right\| \\ &\leq \left\|2^l\left(A\left(\frac{x}{2^l}\right) - g\left(\frac{x}{2^l}\right)\right), w\right\| + \left\|2^l\left(T\left(\frac{x}{2^l}\right) - g\left(\frac{x}{2^l}\right)\right), w\right\| \\ &\leq 2\frac{4 + 2^p}{2^{p+q} - 2}\frac{2^l}{2^{(p+q)l}}\theta\|x\|^{p+q}\|w\|, \end{aligned}$$

which tends to zero as $l \to \infty$ for all $x \in X$ and all $w \in Y$. So we can conclude that $A(a) = T(a)$ for all $a \in X$. This proves the uniqueness of A.

Therefore, $A : X \to Y$ is a unique additive mapping satisfying (2.2), as desired. □

Theorem 2.2. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q < 1$ and let $f : X \to Y$ be an odd mapping satisfying* (2.1)*. Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x), w\| \leq \frac{4 + 2^p}{2 - 2^{p+q}}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 8f(x)$ in (2.5), we get

$$\left\|g(x) - \frac{1}{2}g\left(2x\right), w\right\| \leq \frac{4 + 2^p}{2}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q > 3$ and let $f : X \to Y$ be an odd mapping satisfying* (2.1)*. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x), w\| \leq \frac{4 + 2^p}{2^{p+q} - 8}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 2f(x)$ in (2.5), we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right), w\right\| \leq \frac{4 + 2^p}{2^{p+q}}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q < 3$ and let $f : X \to Y$ be an odd mapping satisfying* (2.1)*. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x), w\| \leq \frac{4 + 2^p}{8 - 2^{p+q}}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 2f(x)$ in (2.5), we get

$$\left\|g(x) - \frac{1}{8}g\left(2x\right), w\right\| \leq \frac{4 + 2^p}{8}\theta\|x\|^{p+q}\|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Now we prove the superstability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces for an odd mapping case.

Theorem 2.5. *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ with $r \neq 1$ and let $f : X \to Y$ be an odd mapping such that*

$$\|Df(x, y), w\| \leq \theta\|x\|^p\|y\|^q\|w\|^r \tag{2.7}$$

for all $x, y \in X$ and all $w \in Y$. Then $f : X \to Y$ is realized as the sum of an additive mapping and a cubic mapping.

Proof. Replacing w by sw in (2.7) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\|Df(x,y), sw\| \le \theta \|x\|^p \|y\|^q \|sw\|^r$$

and so

$$\|Df(x,y), w\| \le \theta \|x\|^p \|y\|^q \|w\|^r \frac{|s|^r}{|s|} \tag{2.8}$$

for all $x, y \in X$, all $w \in Y$ and all $s \in \mathbb{R} \setminus \{0\}$.

If $r > 1$, then the right side of (2.8) tends to $\|f(x+y+z), w\|$ as $s \to 0$.

If $r < 1$, then the right side of (2.8) tends to $\|f(x+y+z), w\|$ as $s \to +\infty$.

Thus

$$\|Df(x,y), w\| = 0$$

for all $x, y \in X$ and all $w \in Y$. By [13, Lemma 2.2], $f : X \to Y$ is realized as the sum of an additive mapping and a cubic mapping. □

3. Hyers-Ulam stability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces: even mapping case

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces for an even mapping case.

Theorem 3.1. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q > 2$ and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there is a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 16f(x) - Q(x), w\| \le \frac{4 + 2^p}{2^{p+q} - 4} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Letting $x = y$ in (2.1), we get

$$\|f(3y) - 6f(2y) + 15f(y), w\| \le \theta \|y\|^{p+q} \|w\| \tag{3.1}$$

for all $y \in X$ and all $w \in Y$.

Replacing x by $2y$ in (2.1), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y), w\| \le 2^p \theta \|y\|^{p+q} \|w\| \tag{3.2}$$

for all $y \in X$ and all $w \in Y$.

By (3.1) and (3.2),

$$\begin{aligned} &\|f(4y) - 20f(2y) + 64f(y), w\| \\ &\quad \le \|4(f(3y) - 6f(2y) + 15f(y)), w\| + \|f(4y) - 4f(3y) + 4f(2y) + 4f(y), w\| \\ &\quad = 4\|f(3y) - 6f(2y) + 15f(y), w\| + \|f(4y) - 4f(3y) + 4f(2y) + 4f(y), w\| \\ &\quad \le (4 + 2^p)\theta \|y\|^{p+q} \|w\| \end{aligned} \tag{3.3}$$

for all $y \in X$ and all $w \in Y$. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 16f(x)$ in (3.3), we get

$$\left\| g(x) - 4g\left(\frac{x}{2}\right), w \right\| \le \frac{4 + 2^p}{2^{p+q}} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q < 2$ and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there is a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 16f(x) - Q(x), w\| \leq \frac{4 + 2^p}{4 - 2^{p+q}} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 16f(x)$ in (3.3), we get

$$\left\| g(x) - \frac{1}{4} g\left(2x\right), w \right\| \leq \frac{4 + 2^p}{4} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.3. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q > 4$ and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there is a unique quartic mapping $R : X \to Y$ such that*

$$\|f(2x) - 4f(x) - R(x), w\| \leq \frac{4 + 2^p}{2^{p+q} - 16} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 4f(x)$ in (3.3), we get

$$\left\| g(x) - 16 g\left(\frac{x}{2}\right), w \right\| \leq \frac{4 + 2^p}{2^{p+q}} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.4. *Let $\theta \in [0, \infty)$, $p, q \in (0, \infty)$ with $p + q < 4$ and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there is a unique quartic mapping $R : X \to Y$ such that*

$$\|f(2x) - 4f(x) - R(x), w\| \leq \frac{4 + 2^p}{16 - 2^{p+q}} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 16f(x)$ in (3.3), we get

$$\left\| g(x) - \frac{1}{16} g\left(2x\right), w \right\| \leq \frac{4 + 2^p}{16} \theta \|x\|^{p+q} \|w\|$$

for all $x \in X$ and all $w \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Now we prove the superstability of the AQCQ-functional equation (1.3) in normed 2-Banach spaces for an even mapping case.

Theorem 3.5. *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ with $r \neq 1$ and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then $f : X \to Y$ is realized as the sum of a quadratic mapping and a quartic mapping.*

Proof. By the same reasoning as in the proof of Theorem 2.5, one can obtain

$$\|Df(x, y), w\| = 0$$

for all $x, y \in X$ and all $w \in Y$. By [10, Lemma 2.1], $f : X \to Y$ is realized as the sum of a quadratic mapping and a quartic mapping. □

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then f_o is odd and f_e is even. f_o, f_e satisfy the functional equation (1.3). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x)$. Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

We summarize the above results as follows.

Theorem 3.6. *Let $\theta \in [0,\infty)$ and $p,q \in (0,\infty)$ with $p+q > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $Q : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $R : X \to Y$ such that*

$$\begin{aligned} &\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}Q(x) - \frac{1}{6}C(x) - \frac{1}{12}R(x), w \right\| \\ &\quad \le \left(\frac{4+2^p}{6(2^{p+q}-2)} + \frac{4+2^p}{12(2^{p+q}-4)} + \frac{4+2^p}{6(2^{p+q}-8)} + \frac{4+2^p}{12(2^{p+q}-16)} \right) \theta \|x\|^{p+q} \|w\| \end{aligned}$$

for all $x \in X$ and all $w \in Y$.

Theorem 3.7. *Let $\theta \in [0,\infty)$ and $p,q \in (0,\infty)$ with $p+q < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $Q : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $R : X \to Y$ such that*

$$\begin{aligned} &\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}Q(x) - \frac{1}{6}C(x) - \frac{1}{12}R(x), w \right\| \\ &\quad \le \left(\frac{4+2^p}{6(2-2^{p+q})} + \frac{4+2^p}{12(4-2^{p+q})} + \frac{4+2^p}{6(8-2^{p+q})} + \frac{4+2^p}{12(16-2^{p+q})} \right) \theta \|x\|^{p+q} \|w\| \end{aligned}$$

for all $x \in X$ and all $w \in Y$.

Theorem 3.8. *Let $\theta \in [0,\infty)$, $p,q,r \in (0,\infty)$ with $r \neq 1$ and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then $f : X \to Y$ is realized as the sum of an additive mapping, a quadratic mapping, a cubic mapping and a quartic mapping.*

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REFINED GENERAL QUADRATIC EQUATION WITH FOUR VARIABLES AND ITS STABILITY RESULTS

HARK-MAHN KIM AND SOON LEE

ABSTRACT. In this article, we establish the general solution of a functional equation

$$rf(\frac{-x+y+z+w}{s})+rf(\frac{x-y+z+w}{s})+rf(\frac{x+y-z+w}{s})+rf(\frac{x+y+z-w}{s})$$
$$=tf(x)+tf(y)+tf(z)+tf(w)$$

and present the generalized Hyers–Ulam stability of the equation.

1. INTRODUCTION

In 1940, S.M. Ulam [13] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms: Let G_1 be a group and G_2 a metric group with metric $\varphi(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\varphi(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\varphi(f(x), h(x)) < \varepsilon$ for all $x \in G_1$?

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. D.H. Hyers [6] showed that if $\varepsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

In 1950 T. Aoki [1] and in 1951 D.G. Bourgin [2] provided a generalized the Hyers theorem for additive mapping and in 1978 Th.M. Rassias [11] generalized the Hyers theorem for liner mapping by allowing the Cauchy difference to be unbounded. Let $f : X \to Y$ be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\varepsilon \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

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for all $x, y \in X$. Then Th.M. Rassias proved that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in X$. And then, the result of Th.M. Rassias theorem has been generalized by P. Găvruta [5] by allowing the Cauchy difference to be a generalized control function.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

which may be originated from this parallelogram equality, is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A Hyers–Ulam stability problem for the quadratic functional equation was proved by F. Skof [12] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. P.W. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In S. Czerwik [4] proved the Hyers–Ulam stability of the quadratic functional equation. In the last decade, S. Lee and K. Jun [8] and S. Lee and C. Park [9] have proved the Hyers–Ulam stability of quadratic type functional equation with three variables.

C. Park [10] has investigated the Hyers–Ulam stability of the following functional equation, which has exactly quadratic mappings as solutions up to $f(0)$,

$$\begin{aligned} rf(\frac{x+y+z+w}{s}) + rf(\frac{x+y-z-w}{s}) + rf(\frac{x-y+z-w}{s}) \\ +rf(\frac{x-y-z+w}{s}) = tf(x) + tf(y) + tf(z) + tf(w) \end{aligned}$$

for all $x, y, z, w \in X$ under the assumption of an even mapping $f : X \to Y$ with $f(0) = 0$. Recently, the authors [7] have established the general solution of the above functional equation and then improved the Hyers–Ulam stability of the equation without the even condition and $f(0) = 0$. In this paper, we are going to establish the general solution of the following modified functional equation, which has exactly quadratic and additive mappings as solutions up to $f(0)$,

$$\begin{aligned} rf(\frac{-x+y+z+w}{s}) + rf(\frac{x-y+z+w}{s}) + rf(\frac{x+y-z+w}{s}) \\ +rf(\frac{x+y+z-w}{s}) = tf(x) + tf(y) + tf(z) + tf(w) \end{aligned} \tag{1.1}$$

for fixed nonzero real numbers r, s, t, and then investigate the Hyers–Ulam stability of the functional equation for mappings $f : X \to Y$ between normed spaces.

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2. General solution of the functional equation.

First of all, we solve the general solution of the equation (1.1) in the class of mappings between linear spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies the equation*

$$\begin{aligned} rf(\frac{-x+y+z+w}{s}) + rf(\frac{x-y+z+w}{s}) + rf(\frac{x+y-z+w}{s}) \\ +rf(\frac{x+y+z-w}{s}) = tf(x) + tf(y) + tf(z) + tf(w) \end{aligned}$$

for all $x, y, z, w \in X$, then $f(x) = Q(x) + A(x) + f(0)$, where Q is quadratic and A is additive, and $f(0) = 0$ if $r \neq t$.

Proof. Let f be a solution of the equation (1.1). Now, letting $x = y = z = w := 0$ in (1.1), one has $f(0) = 0$ if $r \neq t$. First, we prove the case $r \neq t$. Let $f_e(x) := \dfrac{f(x) + f(-x)}{2}$ be an even part of f and $f_o(x) := \dfrac{f(x) - f(-x)}{2}$ be an odd part of f. Then, we see that f_e, f_o are also solutions of the equation (1.1). Putting $y = z = w := 0$ in (1.1) for the even mapping f_e, we have

$$4rf_e(\frac{x}{s}) = tf_e(x) \tag{2.1}$$

for all $x \in X$, which yields

$$\begin{aligned} f_e(-x+y+z+w) + f_e(x-y+z+w) + f_e(x+y-z+w) \\ +f_e(x+y+z-w) = 4f_e(x) + 4f_e(y) + 4f_e(z) + 4f_e(w) \end{aligned} \tag{2.2}$$

for all $x, y, z, w \in X$. Putting $z = w := 0$ in (2.2), we deduce

$$f_e(x+y) + f_e(x-y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$. So $f_e := Q$ is quadratic.

Putting $y = z = w := 0$ in (1.1) for the odd mapping f_o, we have the relation $2rf_o(\frac{x}{s}) = tf_o(x)$ for all $x \in X$. Thus, it follows that

$$\begin{aligned} f_o(-x+y+z+w) + f_o(x-y+z+w) + f_o(x+y-z+w) \\ +f_o(x+y+z-w) = 2f_o(x) + 2f_o(y) + 2f_o(z) + 2f_o(w) \end{aligned} \tag{2.3}$$

for all $x, y, z, w \in X$. Putting $z = w := 0$ in (2.3), one conclude

$$f_o(x+y) = f_o(x) + f_o(y),$$

and so, $f_o := A$ is additive. Therefore, $f(x) = f_e(x) + f_o(x) = Q(x) + A(x)$, where Q is quadratic and A is additive.

Next, we prove the case $r = t$. Let $f(x) - f(0) = \tilde{f}(x),\ x \in X$. In this case, $\tilde{f}(0) = 0$, and we see the functional equation

$$\begin{aligned}\tilde{f}(\frac{-x+y+z+w}{s}) + \tilde{f}(\frac{x-y+z+w}{s}) + \tilde{f}(\frac{x+y-z+w}{s}) \\ +\tilde{f}(\frac{x+y+z-w}{s}) = \tilde{f}(x) + \tilde{f}(y) + \tilde{f}(z) + \tilde{f}(w)\end{aligned} \tag{2.4}$$

for all $x, y, z, w \in X$. It follows that $\tilde{f}_e(x) := \dfrac{\tilde{f}(x) + \tilde{f}(-x)}{2}$ and $\tilde{f}_o(x) = \dfrac{\tilde{f}(x) - \tilde{f}(-x)}{2}$ also satisfy the equation (2.4). Putting $y = z = w := 0$ in (2.4) for the even mapping $\tilde{f}_e$, we have $4\tilde{f}_e(\frac{x}{s}) = \tilde{f}_e(x)$, and so,

$$\begin{aligned}\tilde{f}_e(-x+y+z+w) + \tilde{f}_e(x-y+z+w) + \tilde{f}_e(x+y-z+w) \\ +\tilde{f}_e(x+y+z-w) = 4[\tilde{f}_e(x) + \tilde{f}_e(y) + \tilde{f}_e(z) + \tilde{f}_e(w)]\end{aligned} \tag{2.5}$$

for all $x, y, z, w \in X$. Thus, $\tilde{f}_e := Q$ is quadratic. Similarly, putting $y = z = w := 0$ in (2.4) for the odd mapping $\tilde{f}_o$, we have $2\tilde{f}_o(\frac{x}{s}) = \tilde{f}_o(x)$, and so, we get

$$\begin{aligned}\tilde{f}_o(-x+y+z+w) + \tilde{f}_o(x-y+z+w) + \tilde{f}_o(x+y-z+w) \\ +\tilde{f}_o(x+y+z-w) = 2[\tilde{f}_o(x) + \tilde{f}_o(y) + \tilde{f}_o(z) + \tilde{f}_o(w)]\end{aligned} \tag{2.6}$$

for all $x, y, z, w \in X$. Thus, we conclude that $\tilde{f}_o(x+y) = \tilde{f}_o(x) + \tilde{f}_o(y)$, and hence, $\tilde{f}_o = A$ is additive. Therefore, $f(x) - f(0) = \tilde{f}(x) = \tilde{f}_e(x) + \tilde{f}_o(x) = Q(x) + A(x)$, where Q is quadratic and A is additive. □

Remark 2.2. If $r = t$ and $s \neq 2$ is a rational number, and if f is a solution of the equation (1.1), then we note that $\tilde{f}_o(x) := A(x) \equiv 0$ identically, and $\tilde{f}_e(x) := Q(x) \equiv 0$ identically. Hence, $f(x) = f(0)$ must be a constant solution.

If $r = t$ and $s = 2$, and if f is a solution of the equation (1.1) with $f(0) = 0$, then the equation

$$\begin{aligned}f(\frac{-x+y+z+w}{2}) + f(\frac{x-y+z+w}{2}) + f(\frac{x+y-z+w}{2}) \\ +f(\frac{x+y+z-w}{2}) = f(x) + f(y) + f(z) + f(w), \quad x, y, z, w \in X\end{aligned}$$

yields

$$2f(\frac{x+y}{2}) + f(\frac{x-y}{2}) + f(\frac{-x+y}{2}) = f(x) + f(y),$$

$$\Leftrightarrow 2f(u) + f(v) + f(-v) = f(u+v) + f(u-v),$$

which is well-known Drygas functional equation with general solution $f(x) = Q(x) + A(x),\ x \in X$. Therefore, if $r = t$ and $s = 2$, and if f is a solution of the equation (1.1), then $f(x) = Q(x) + A(x) + f(0)$ is a general solution of the equation.

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If $r \neq t$ and $s = 2$, and if f is a solution of the equation (1.1), then we see that $f_e(x) = 0,\ x \in X$ identically and $f_o(x) = 0,\ x \in X$ identically. Thus, $f(x) = f(0) = 0$ must be a constant solution.

We remark that if a mapping $f : X \to Y$ satisfies the equation

$$\begin{aligned} &f(-x+y+z+w) + f(x-y+z+w) + f(x+y-z+w) \\ &+f(x+y+z-w) = 4[f(x)+f(y)+f(z)+f(w)] \end{aligned}$$

for all $x, y, z, w \in X$, then (i) $f(0) = 0$; (ii) $f(-x) = f(x)$; (iii) $f(x+y) + f(x-y) = 2[f(x) + f(y)]$ for all $x, y \in X$, and thus, f is quadratic.

3. Stability of the functional equation for even mappings.

We now prove the Hyers–Ulam stability of the functional equation for even mappings $f : X \to Y$ with some regularity conditions. Given a mapping $f : X \to Y$ and a function $\varphi : X^4 \to \mathbb{R}^+ := [0, \infty)$, we set for notational convenience

$$\begin{aligned} \|Df(x,y,z,w)\| &\leq \varphi(x,y,z,w), \qquad (3.1)\\ Df(x,y,z,w) &:= rf(\frac{-x+y+z+w}{s}) + rf(\frac{x-y+z+w}{s}) + rf(\frac{x+y-z+w}{s}) \\ &\quad +rf(\frac{x+y+z-w}{s}) - [tf(x)+tf(y)+tf(z)+tf(w)] \end{aligned}$$

for all $x, y, z, w \in X$. From now on, we assume that that X and Y are a normed linear space with norm $\|\cdot\|$ and a Banach space with norm $\|\cdot\|$, respectively.

Theorem 3.1. *Assume that an even mapping $f : X \to Y$ satisfies the functional inequality (3.1) and*

$$\Phi_1(x,y,z,w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z, 2^i w)}{4^i} < \infty, \qquad (3.2)$$

$$\left(\Phi_2(x,y,z,w) := \sum_{i=1}^{\infty} 4^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, \frac{w}{2^i}) < \infty, \quad \textit{resp.}\right)$$

for all $x, y, z, w \in X$. Then, there exists a unique quadratic mapping $Q_1 : X \to Y$, $(Q_2 : X \to Y,\ \textit{resp.})$, defined as $Q_1(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{4^n},\ x \in X$,

$$\left(Q_2(x) = \lim_{n\to\infty} 4^n [f(\frac{x}{2^n}) - f(0)],\ x \in X,\ \textit{resp.}\right)$$

such that

$$\|f(x) - f(0) - Q_1(x)\| \leq \frac{1}{4|t|}[\Phi_1(x,x,x,x) + \Phi_1(2x,0,0,0)], \qquad (3.3)$$

$$\left(\|f(x) - f(0) - Q_2(x)\| \leq \frac{1}{4|t|}[\Phi_2(x,x,x,x) + \Phi_2(2x,0,0,0)],\ \textit{resp.}\right)$$

for all $x \in X$.

Proof. First, we observe that

$$\begin{aligned} 4 \mid r-t \mid \|f(0)\| &\leq \varphi(0,0,0,0), & (3.4)\\ \|4rf(\frac{2x}{s}) - 4tf(x)\| &\leq \varphi(x,x,x,x), & (3.5)\\ \|2rf(\frac{2x}{s}) - 2tf(x) + 2(r-t)f(0)\| &\leq \varphi(x,x,0,0), & (3.6)\\ \|4rf(\frac{x}{s}) - tf(x) - 3tf(0)\| &\leq \varphi(x,0,0,0) & (3.7)\end{aligned}$$

for all $x \in X$. By using (3.5) and (3.6), we have

$$\|4(r-t)f(0)\| \leq \varphi(x,x,x,x) + 2\varphi(x,x,0,0) \tag{3.8}$$

for all $x \in X$. By using (3.5) and (3.7), one has

$$\|f(2x) - 4f(x) + 3f(0)\| \leq \frac{1}{|t|}[\varphi(x,x,x,x) + \varphi(2x,0,0,0)] \tag{3.9}$$

for all $x \in X$. Let $f(x) - f(0) := \tilde{f}(x),\ x \in X.$ Then one obtains from (3.9)

$$\|\tilde{f}(2x) - 4\tilde{f}(x)\| \leq \frac{1}{|t|}[\varphi(x,x,x,x) + \varphi(2x,0,0,0)] \tag{3.10}$$

for all $x \in X$. Thus, we can prove by triangle inequality

$$\|\frac{\tilde{f}(2^n x)}{4^n} - \tilde{f}(x)\| \leq \frac{1}{4|t|}\sum_{i=0}^{n-1}\frac{1}{4^i}[\varphi(2^i x, 2^i x, 2^i x, 2^i x) + \varphi(2^{i+1}x,0,0,0)] \tag{3.11}$$

for all $x \in X$. Now, it follows from the last inequality that for all nonnegative integers n, m with $n > m \geq 0$

$$\begin{aligned} &\|\frac{\tilde{f}(2^n x)}{4^n} - \frac{\tilde{f}(2^m x)}{4^m}\| \qquad (3.12)\\ &\leq \sum_{i=m}^{n-1}\|\frac{\tilde{f}(2^{i+1}x)}{4^{i+1}} - \frac{\tilde{f}(2^i x)}{4^i}\|\\ &\leq \frac{1}{4|t|}\sum_{i=m}^{n-1}\frac{1}{4^i}[\varphi(2^i x, 2^i x, 2^i x, 2^i x) + \varphi(2^{i+1}x,0,0,0)]\end{aligned}$$

for all $x \in X$, of which the right-hand side approaches 0 as m tends to infinity. This shows that the sequence $\{\frac{\tilde{f}(2^n x)}{4^n}\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{\tilde{f}(2^n x)}{4^n}\}$ converges in Y for all $x \in X$, and so one can define a mapping $Q_1 : X \to Y$ by

$$Q_1(x) = \lim_{n\to\infty}\frac{\tilde{f}(2^n x)}{4^n} = \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$$

for all $x \in X$.

Now, it follows from (3.1) and the definition of Q_1 that

$$\begin{aligned}\|DQ_1(x,y,z,w)\| &= \lim_{n\to\infty} \frac{\|Df(2^n x, 2^n y, 2^n z, 2^n w)\|}{4^n} \\ &\leq \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z, 2^n w)}{4^n} = 0\end{aligned}$$

for all $x, y, z, w \in X$. Thus, the mapping Q_1 is quadratic by Lemma 2.1. Moreover, if we let $n \to \infty$ in (3.11), we get the desired approximation (3.3).

To prove the uniqueness, let $Q^{'} : X \to Y$ be another quadratic mapping satisfying (3.3). Then, we have

$$\begin{aligned}\| Q_1(x) - Q^{'}(x) \| &= \frac{1}{4^n}\|Q_1(2^n x) - Q^{'}(2^n x)\| \\ &\leq \frac{1}{4^n}\Big(\|Q_1(2^n x) - f(2^n x) + f(0)\| + \|f(2^n x) - f(0) - Q^{'}(2^n x)\|\Big) \\ &\leq \frac{1}{4^n \cdot 2|t|}\Big(\Phi_1(2^n x, 2^n x, 2^n x, 2^n x) + \Phi_1(2^{n+1}x, 0, 0, 0)\Big),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So one can conclude that $Q_1(x) = Q^{'}(x)$ for all $x \in X$. This proves the uniqueness. □

In the following, we consider another stability results of the functional equation (1.1) by using the similar manner to the reference [7].

Theorem 3.2. *Assume that an even mapping $f : X \to Y$ satisfies the functional inequality (3.1) and the condition (3.2). Then, there exists a unique quadratic mapping $Q_1 : X \to Y$, ($Q_2 : X \to Y$, resp.), defined as $Q_1(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{4^n}$, $x \in X$*

$$\left(Q_2(x) = \lim_{n\to\infty} 4^n[f(\frac{x}{2^n}) - \frac{(4r-t)}{3t}f(0)],\ x \in X,\ \textit{resp.}\right)$$

such that

$$\|f(x) - \frac{(4r-t)}{3t}f(0) - Q_1(x)\| \leq \frac{1}{4|t|}[2\Phi_1(x,x,0,0) + \Phi_1(2x,0,0,0)], \tag{3.13}$$

$$\left(\|f(x) - \frac{(4r-t)}{3t}f(0) - Q_2(x)\| \leq \frac{1}{4|t|}[2\Phi_2(x,x,0,0) + \Phi_2(2x,0,0,0)],\ \textit{resp.}\right)$$

for all $x \in X$, where $f(0) = 0$ if $r \neq t$.

Proof. Associating (3.6) with (3.7), one has

$$\|f(2x) - 4f(x) + \frac{(4r-t)}{t}f(0)\| \leq \frac{1}{|t|}[2\varphi(x,x,0,0) + \varphi(2x,0,0,0)],$$

which yields

$$\|\frac{\tilde{f(2x)}}{4} - \tilde{f}(x)\| \leq \frac{1}{4|t|}[2\varphi(x,x,0,0) + \varphi(2x,0,0,0)], \tag{3.14}$$

where $\tilde{f}(x) := f(x) - \frac{(4r-t)}{3t} f(0),\ x \in X$. It follows from (3.14) that

$$\|\tilde{f}(x) - \frac{\tilde{f}(2^n x)}{4^n}\| \le \frac{1}{4|t|} \sum_{i=0}^{n-1} [\frac{2\varphi(2^i x, 2^i x, 0, 0) + \varphi(2^{i+1} x, 0, 0, 0)}{4^i}]$$

for all $x \in X$.

The rest of proof is similarly verified by the same argument as that of Theorem 3.1. □

In Theorem 3.2, we remark that $Q_2(0) = 0$ by definition if $r = t$, and also, $Q_2(0) = 0$ if $r \neq t$ because $f(0) = 0 = \varphi(0,0,0,0)$ by the convergence of $\Phi_2(0,0,0,0)$.

Corollary 3.3. *Let δ, θ be nonnegative real numbers and $p \neq 2$ be a positive real number. Assume that an even mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y,z,w)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$, where $\delta = 0$ when $p > 2$. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - f(0) - Q(x)\| \ \le \ \frac{1}{|t|}\Big[\frac{2\delta}{3} + \frac{4\theta\|x\|^p}{|4-2^p|} + \frac{2^p\theta\|x\|^p}{|4-2^p|}\Big]$$

for all $x \in X$, where $f(0) = 0$ if $r \neq t$ and $p > 2$.

4. Stability of the functional equation for odd mappings.

We now prove the Hyers–Ulam stability of the functional equation for odd mappings $f : X \to Y$ with some regularity conditions.

Theorem 4.1. *Assume that an odd mapping $f : X \to Y$ satisfies the functional inequality (3.1) and*

$$\Phi_3(x,y,z,w) = \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z, 2^i w)}{2^i} < \infty \tag{4.1}$$

$$\Big(\Phi_4(x,y,z,w) = \sum_{i=1}^{\infty} 2^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, \frac{w}{2^i}) < \infty, \ \textit{resp.}\Big)$$

for all $x, y, z, w \in X$. Then, there exists a unique additive mapping $A_3 : X \to Y$, $(A_4 : X \to Y,\ \textit{resp.})$, defined as $A_3(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},\ x \in X$,

$$\Big(A_4(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}),\ x \in X,\ \textit{resp.}\Big)$$

such that

$$\|f(x) - A_3(x)\| \le \frac{1}{4|t|}[\Phi_3(x,x,x,x) + 2\Phi_3(2x,0,0,0)], \tag{4.2}$$

$$\Big(\|f(x) - A_4(x)\| \le \frac{1}{4|t|}[\Phi_4(x,x,x,x) + 2\Phi_4(2x,0,0,0)],\ \textit{resp.}\Big)$$

for all $x \in X$.

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Proof. First, we note that

$$\|4rf(\frac{2x}{s}) - 4tf(x)\| \leq \varphi(x, x, x, x), \tag{4.3}$$

$$\|2rf(\frac{2x}{s}) - 2tf(x)\| \leq \varphi(x, x, 0, 0), \tag{4.4}$$

$$\|2rf(\frac{2x}{s}) - tf(2x)\| \leq \varphi(2x, 0, 0, 0) \tag{4.5}$$

for all $x \in X$. By using (4.3) and (4.5), one has

$$\|f(2x) - 2f(x)\| \leq \frac{1}{2|t|}[\varphi(x, x, x, x) + 2\varphi(2x, 0, 0, 0)] \tag{4.6}$$

for all $x \in X$. Thus, by using triangle inequality and (4.6), one can prove the useful inequality

$$\|\frac{f(2^n x)}{2^n} - f(x)\| \leq \frac{1}{4|t|}\sum_{i=0}^{n-1}\frac{1}{2^i}[\varphi(2^i x, 2^i x, 2^i x, 2^i x) + 2\varphi(2^{i+1}x, 0, 0, 0)] \tag{4.7}$$

for all $x \in X$.

Applying the same argument as in Theorem 3.1, one can obtain the desired results. □

In the following, we consider another stability results of the functional equation (1.1) by using the similar way to the reference [7].

Theorem 4.2. *Assume that an odd mapping $f : X \to Y$ satisfies the functional inequality (3.1) and the condition (4.1). Then, there exists a unique additive mapping $A_3 : X \to Y$, $(A_4 : X \to Y,$ resp.$)$, defined as $A_3(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x),\ x \in X$,*

$$\Big(A_4(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}),\ x \in X,\ \textit{resp.}\Big)$$

such that

$$\|f(x) - A_3(x)\| \leq \frac{1}{2|t|}[\Phi_3(x, x, 0, 0) + \Phi_3(2x, 0, 0, 0)], \tag{4.8}$$

$$\Big(\|f(x) - A_4(x)\| \leq \frac{1}{2|t|}[\Phi_4(x, x, 0, 0) + \Phi_4(2x, 0, 0, 0)],\ \textit{resp.}\Big)$$

for all $x \in X$.

Proof. Associating (4.4) with (4.5), one has

$$\|f(2x) - 2f(x)\| \leq \frac{1}{|t|}[\varphi(x, x, 0, 0) + \varphi(2x, 0, 0, 0)], \tag{4.9}$$

which yields

$$\|f(x) - \frac{f(2^n x)}{2^n}\| \leq \frac{1}{2|t|}\sum_{i=0}^{n-1}[\frac{\varphi(2^i x, 2^i x, 0, 0) + \varphi(2^{i+1}x, 0, 0, 0)}{2^i}]$$

for all $x \in X$.

The rest of proof is similarly verified by the same argument as that of Theorem 4.1. □

Corollary 4.3. *Let δ, θ be nonnegative real numbers and $p \neq 1$ be a positive real number. Assume that an odd mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y,z,w)\| \leq \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$, where $\delta = 0$ when $p > 1$. Then, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|t|}\Big[\frac{3\delta}{2} + \frac{2\theta\|x\|^p}{|2-2^p|} + \frac{2^p\theta\|x\|^p}{|2-2^p|}\Big]$$

for all $x \in X$.

5. Stability of the functional equation for general mappings.

Finally, we now prove the Hyers–Ulam stability of the functional equation (1.1) for general mappings $f : X \to Y$ with some regularity conditions.

Theorem 5.1. *Assume that a mapping $f : X \to Y$ satisfies the functional inequality (3.1) and*

$$\Phi_3(x,y,z,w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z, 2^i w)}{2^i} < \infty \tag{5.1}$$

for all $x, y, z, w \in X$. Then, there exist a unique quadratic mapping $Q_1 : X \to Y$ and a unique additive mapping $A_3 : X \to Y$ such that

$$\begin{aligned}
&\|\frac{f(x)+f(-x)}{2} - f(0) - Q_1(x)\| \\
&\leq \frac{1}{8|t|}[\Phi_1(x,x,x,x) + \Phi_1(-x,-x,-x,-x) + \Phi_1(2x,0,0,0) + \Phi_1(-2x,0,0,0)], \\
&\|\frac{f(x)-f(-x)}{2} - A_3(x)\| \\
&\leq \frac{1}{8|t|}[\Phi_3(x,x,x,x) + \Phi_3(-x,-x,-x,-x) + 2\Phi_3(2x,0,0,0) + 2\Phi_3(-2x,0,0,0)], \\
&\|f(x) - f(0) - A_3(x) - Q_1(x)\| \qquad\qquad (5.2)\\
&\leq \frac{1}{8|t|}[\Phi_1(x,x,x,x) + \Phi_1(-x,-x,-x,-x) + \Phi_1(2x,0,0,0) + \Phi_1(-2x,0,0,0) \\
&\qquad +\Phi_3(x,x,x,x) + \Phi_3(-x,-x,-x,-x) + 2\Phi_3(2x,0,0,0) + 2\Phi_3(-2x,0,0,0)]
\end{aligned}$$

for all $x \in X$.

Proof. Let $f_e(x) := \frac{f(x)+f(-x)}{2}, x \in X$, be the even part of f and $f_o(x) := \frac{f(x)-f(-x)}{2}, x \in X$, the odd part of f. Then, it is easy to see that

$$\|Df_e(x,y,z,w)\| \leq \frac{1}{2}[\varphi(x,y,z,w) + \varphi(-x,-y,-z,-w)], \tag{5.3}$$

$$\|Df_o(x,y,z,w)\| \leq \frac{1}{2}[\varphi(x,y,z,w) + \varphi(-x,-y,-z,-w)] \tag{5.4}$$

for all $x, y, z, w \in X$. Now, applying Theorem 3.1 to the functional inequality (5.3), we see that there exists a unique quadratic mapping $Q_1 : X \to Y$, defined by $Q_1(x) = \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{4^n \cdot 2}$, $x \in X$, such that

$$\begin{aligned} &\|f_e(x) - f_e(0) - Q_1(x)\| \\ &\quad\le \frac{1}{8|t|}[\Phi_1(x,x,x,x) + \Phi_1(-x,-x,-x,-x) + \Phi_1(2x,0,0,0) + \Phi_1(-2x,0,0,0)] \end{aligned} \tag{5.5}$$

for all $x \in X$.

On the other hand, applying Theorem 3.1 to the functional inequality (5.4), one obtains that there exists a unique additive mapping $A_3 : X \to Y$, defined by $A_3(x) = \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}$, $x \in X$, such that

$$\begin{aligned} &\|f_o(x) - A_3(x)\| \\ &\quad\le \frac{1}{8|t|}[\Phi_3(x,x,x,x) + \Phi_3(-x,-x,-x,-x) + 2\Phi_3(2x,0,0,0) + 2\Phi_3(-2x,0,0,0)] \end{aligned} \tag{5.6}$$

for all $x \in X$. Hence, we lead to the desired approximation (5.2). □

Theorem 5.2. *Assume that a mapping $f : X \to Y$ satisfies the functional inequality (3.1) and*

$$\begin{aligned} \Phi_4(x,y,z,w) &:= \sum_{i=1}^{\infty} 2^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, \frac{w}{2^i}\right) < \infty, \\ \Phi_1(x,y,z,w) &:= \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 2^i z, 2^i w)}{4^i} < \infty \end{aligned} \tag{5.7}$$

for all $x, y, z, w \in X$. Then, there exist a unique quadratic mapping $Q_1 : X \to Y$ and a unique additive mapping $A_4 : X \to Y$ such that

$$\begin{aligned} &\left\|\frac{f(x) + f(-x)}{2} - f(0) - Q_1(x)\right\| \\ &\quad\le \frac{1}{8|t|}[\Phi_1(x,x,x,x) + \Phi_1(-x,-x,-x,-x) + \Phi_1(2x,0,0,0) + \Phi_1(-2x,0,0,0)], \\ &\left\|\frac{f(x) - f(-x)}{2} - A_4(x)\right\| \\ &\quad\le \frac{1}{8|t|}[\Phi_4(x,x,x,x) + \Phi_4(-x,-x,-x,-x) + 2\Phi_4(2x,0,0,0) + 2\Phi_4(-2x,0,0,0)], \\ &\|f(x) - f(0) - A_4(x) - Q_1(x)\| \qquad (5.8) \\ &\quad\le \frac{1}{8|t|}[\Phi_1(x,x,x,x) + \Phi_1(-x,-x,-x,-x) + \Phi_1(2x,0,0,0) + \Phi_1(-2x,0,0,0) \\ &\qquad + \Phi_4(x,x,x,x) + \Phi_4(-x,-x,-x,-x) + 2\Phi_4(2x,0,0,0) + 2\Phi_4(-2x,0,0,0)] \end{aligned}$$

for all $x \in X$, where $f(0) = 0$ if $r \neq t$.

Proof. Applying Theorem 3.1 to the functional inequality (5.3), we see that there exists a unique quadratic mapping $Q_1 : X \to Y$, defined by $Q_1(x) = \lim_{n\to\infty} \dfrac{f(2^n x) + f(-2^n x)}{4^n \cdot 2}$, $x \in X$, such that

$$\begin{aligned} &\|f_e(x) - f_e(0) - Q_1(x)\| \qquad (5.9)\\ &\le \frac{1}{8|t|}[\Phi_1(x, x, x, x) + \Phi_1(-x, -x, -x, -x) + \Phi_1(2x, 0, 0, 0) + \Phi_1(-2x, 0, 0, 0)] \end{aligned}$$

for all $x \in X$. In particular, it follows from the convergence of the series $\Phi_4(0, 0, 0, 0)$ that $0 = \varphi(0, 0, 0, 0)$, and so $f(0) = f_e(0) = 0$ if $r \neq t$.

In turn, applying Theorem 4.1 to the inequality (5.4), one obtains that there exists a unique additive mapping $A_4 : X \to Y$, defined by $A_4(x) = \lim_{n\to\infty} \frac{2^n}{2}[f(\dfrac{x}{2^n}) - f(\dfrac{-x}{2^n})]$, $x \in X$, such that

$$\begin{aligned} &\|f_o(x) - A_4(x)\| \qquad (5.10)\\ &\le \frac{1}{8|t|}[\Phi_4(x, x, x, x) + \Phi_4(-x, -x, -x, -x) + 2\Phi_4(2x, 0, 0, 0) + 2\Phi_4(-2x, 0, 0, 0)] \end{aligned}$$

for all $x \in X$. Hence, we lead to the desired approximation (5.8). □

Theorem 5.3. *Assume that a mapping $f : X \to Y$ satisfies the functional inequality (3.1) and*

$$\Phi_2(x, y, z, w) := \sum_{i=1}^{\infty} 4^i \varphi(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, \frac{w}{2^i}) < \infty \qquad (5.11)$$

for all $x, y, z, w \in X$. Then, there exist a unique quadratic mapping $Q_2 : X \to Y$ and a unique additive mapping $A_4 : X \to Y$ such that

$$\begin{aligned} &\|\frac{f(x) + f(-x)}{2} - f(0) - Q_2(x)\|\\ &\le \frac{1}{8|t|}[\Phi_2(x, x, x, x) + \Phi_2(-x, -x, -x, -x) + \Phi_2(2x, 0, 0, 0) + \Phi_2(-2x, 0, 0, 0)],\\ &\|\frac{f(x) - f(-x)}{2} - A_4(x)\|\\ &\le \frac{1}{8|t|}[\Phi_4(x, x, x, x) + \Phi_4(-x, -x, -x, -x) + 2\Phi_4(2x, 0, 0, 0) + 2\Phi_4(-2x, 0, 0, 0)],\\ &\|f(x) - f(0) - A_4(x) - Q_2(x)\| \qquad (5.12)\\ &\le \frac{1}{8|t|}[\Phi_2(x, x, x, x) + \Phi_2(-x, -x, -x, -x) + \Phi_2(2x, 0, 0, 0) + \Phi_2(-2x, 0, 0, 0)\\ &\quad +\Phi_4(x, x, x, x) + \Phi_4(-x, -x, -x, -x) + 2\Phi_4(2x, 0, 0, 0) + 2\Phi_4(-2x, 0, 0, 0)] \end{aligned}$$

for all $x \in X$, where $f(0) = 0$ if $r \neq t$.

Proof. Applying Theorem 3.1 to the functional inequality (5.3), we see that there exists a unique quadratic mapping $Q_2 : X \to Y$, defined by $Q_2(x) = \lim_{n\to\infty} \frac{4^n}{2}[f(\dfrac{x}{2^n}) + f(\dfrac{-x}{2^n})]$,

$x \in X$, such that

$$\begin{aligned} &\|f_e(x) - f_e(0) - Q_2(x)\| \\ &\leq \frac{1}{8|t|}[\Phi_2(x,x,x,x) + \Phi_2(-x,-x,-x,-x) + \Phi_2(2x,0,0,0) + \Phi_2(-2x,0,0,0)] \end{aligned} \tag{5.13}$$

for all $x \in X$.

On the other hand, applying Theorem 4.1 to the functional inequality (5.4), one obtains that there exists a unique additive mapping $A_4 : X \to Y$, defined by $A_4(x) = \lim_{n\to\infty} 2^{n-1}[f(\frac{x}{2^n}) - f(\frac{-x}{2^n})]$, $x \in X$, such that

$$\begin{aligned} &\|f_o(x) - A_4(x)\| \\ &\leq \frac{1}{8|t|}[\Phi_4(x,x,x,x) + \Phi_4(-x,-x,-x,-x) + 2\Phi_4(2x,0,0,0) + 2\Phi_4(-2x,0,0,0)] \end{aligned} \tag{5.14}$$

for all $x \in X$. Hence, we lead to the desired approximation (5.12). □

Now, applying main Theorems 5.1, 5.2 and 5.3, we obtain the following corollary for each three cases $0 < p < 1, 1 < p < 2$ and $p > 2$ concerning the stability of equation (1.1).

Corollary 5.4. *Let δ, θ be nonnegative real numbers and $p \neq 1, 2$ be a positive real number. Assume that a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y,z,w)\| \leq \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x, y, z, w \in X$, where $\delta = 0$ when $p > 1$. Then, there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} \|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\| &\leq \frac{1}{|t|}\Big[\frac{2\delta}{3} + \frac{4\theta\|x\|^p}{|4-2^p|} + \frac{2^p\theta\|x\|^p}{|4-2^p|}\Big], \\ \|\frac{f(x)-f(-x)}{2} - A(x)\| &\leq \frac{1}{|t|}\Big[\frac{3\delta}{2} + \frac{2\theta\|x\|^p}{|2-2^p|} + \frac{2^p\theta\|x\|^p}{|2-2^p|}\Big], \\ \|f(x) - f(0) - A(x) - Q(x)\| &\leq \frac{1}{|t|}\Big[\frac{13\delta}{6} + \frac{4\theta\|x\|^p}{|4-2^p|} + \frac{2^p\theta\|x\|^p}{|4-2^p|} + \frac{2\theta\|x\|^p}{|2-2^p|} + \frac{2^p\theta\|x\|^p}{|2-2^p|}\Big] \end{aligned}$$

for all $x \in X$, where $f(0) = 0$ if $r \neq t$ and $p > 2$.

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HYERS-ULAM STABILITY OF A CLASS OF DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. In this paper we prove the Hyers-Ulam stability of a class of differential equations of second order which includes Euler differential equation and second order linear differential equations with constant coefficients.

1. Introduction

In 1940, Ulam [20] discussed the question concerning the stability of homomorphisms as follows: let G_1 be a group and G_2 be a metric group with a metric $d(\cdot,\cdot)$. For a given $\varepsilon > 0$, is there a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \delta$ for all $x \in G_1$? The question of Ulam was answered by Hyers in [7] for the case both G_1 and G_2 are Banach spaces. See [3, 5, 6, 17, 18, 19] for the stability problems of functional equations.

Definition 1.1. Let $I \subset \mathbb{R}$ be an open interval. We say that the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0y(t) + h(t) = 0 \qquad (1.1)$$

has the Hyers-Ulam stability, if for any function $f : I \to \mathbb{R}$ satisfying the differential inequality

$$|a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0y(t) + h(t)| \le \varepsilon$$

for all $t \in I$ and for some $\varepsilon > 0$, there exists a solution $g : I \to \mathbb{R}$ of (1.1) such that $|f(t) - g(t)| \le K(\varepsilon)$ for any $t \in I$, where $K(\varepsilon)$ is a constant depending only on ε.

The first result concerning the Hyers-Ulam stability of ordinary differential equations was due to Alsina and Ger, see [2] (see also [15, 16]). In fact, their result dealt with the Hyers-Ulam stability of linear differential equations of first order. The result of Alsina and Ger has been generalized by many mathematicians (Ref. [8, 9, 10, 12, 13]).

The Hyers-Ulam stability of second order linear differential equations has been investigated in [4] and [14]. Furthermore, Abdollahpour and Najati [1] proved the Hyers-Ulam stability of the third order differential equation $y^{(3)}(t)+\alpha y''(t)+\beta y'(t)+\gamma y(t) = f(t)$.

The aim of this paper is to investigate the Hyers-Ulam stability of the differential equation

$$\left(\frac{1}{h'(x)}\right)^2 y''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) y'(x) + \beta y(x) = f(x). \qquad (1.2)$$

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More precisely, the problem, we will deal with the following. Let $\varepsilon > 0$ be fixed and $[a, b] \subset \mathbb{R}$. $h \in C^2[a, b]$ be a function for which either $h'(x) > 0$ or $h'(x) < 0$ holds for all $x \in [a, b]$ and let $f \in C[a, b]$. Assume that for the unknown function $y \in C^2[a, b]$

$$\left| \left(\frac{1}{h'(x)}\right)^2 y''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) y'(x) + \beta y(x) - f(x) \right| < \varepsilon$$

holds for all $x \in [a, b]$. Is it true there exist a constant $K(\varepsilon)$ depending only on ε and a solution $g \in C^2[a, b]$ of equation (1.2) such that for any $x \in [a, b]$

$$|y(x) - g(x)| \leq \varepsilon$$

is satisfied?

In the subsequent section we will answer affirmatively this problem.

2. Hyers-Ulam stability of the differential equation $(\frac{1}{h'(x)})^2 y''(x) + (\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}) y'(x) + \beta y(x) = f(x)$

Throughout this section, a and b are real numbers with $-\infty < a < b < +\infty$.

Theorem 2.1. *The differential equation*

$$\left(\frac{1}{h'(x)}\right)^2 y''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) y'(x) + \beta y(x) = f(x)$$

has the Hyers-Ulam stability, where $y, h \in C^2[a, b]$, $f \in C[a, b]$ and $h'(x) > 0$ (or $h'(x) < 0$) for all $x \in [a, b]$.

Proof. Suppose that λ, μ are the (real or complex) roots of $m^2 + \alpha m + \beta = 0$ with $p = \Re\mu$ and $q = \Re\lambda$. Here $\Re$ denotes the real part. Let $\varepsilon > 0$ and $y, h \in C^2[a, b]$ with

$$\left| \left(\frac{1}{h'(x)}\right)^2 y''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) y'(x) + \beta y(x) - f(x) \right| \leq \varepsilon.$$

Let $g(x) = \frac{1}{h'(x)} y'(x) - \lambda y(x)$ for all $x \in [a, b]$. It is clear that

$$\begin{aligned} &\left| \frac{1}{h'(x)} g'(x) - \mu g(x) - f(x) \right| \\ &= \left| \left(\frac{1}{h'(x)}\right)^2 y''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) y'(x) + \beta y(x) - f(x) \right| \leq \varepsilon. \end{aligned}$$

We define the function $z : [a, b] \to \mathbb{R}$ by

$$z(x) = e^{\mu[h(x) - h(b)]} g(b) - e^{\mu h(x)} \int_x^b h'(t) f(t) e^{-\mu h(t)}\, dt, \quad x \in [a, b].$$

Then

$$z'(x) = \mu h'(x) z(x) + f(x) h'(x), \quad x \in [a, b]. \tag{2.1}$$

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Also we have

$$\begin{aligned}
|z(x)-g(x)| =& \left| e^{\mu[h(x)-h(b)]}g(b) - g(x) - e^{\mu h(x)} \int_x^b h'(t)f(t)e^{-\mu h(t)}\ dt \right| \\
=& |e^{\mu h(x)}| \left| e^{-\mu h(b)}g(b) - e^{-\mu h(x)}g(x) - \int_x^b h'(t)f(t)e^{-\mu h(t)}\ dt \right| \\
=& e^{ph(x)} \left| \int_x^b [e^{-\mu h(t)}g(t)]'\ dt - \int_x^b h'(t)f(t)e^{-\mu h(t)}\ dt \right| \\
=& e^{ph(x)} \left| \int_x^b e^{-\mu h(t)}[g'(t) - \mu h'(t)g(t) - h'(t)f(t)]\ dt \right| \\
\le& e^{ph(x)} \int_x^b |h'(t)e^{-\mu h(t)}| \left| \frac{1}{h'(t)}g'(t) - \mu g(t) - f(t) \right| dt \\
\le& \varepsilon e^{ph(x)} \int_x^b |h'(t)| e^{-ph(t)}\ dt
\end{aligned}$$

for all $x \in [a,b]$. Therefore if $h'(x) > 0$ then

$$|z(x)-g(x)| \le \begin{cases} \dfrac{1-e^{-p[h(b)-h(a)]}}{p}\varepsilon & \text{if } p \ne 0; \\ (h(b)-h(a))\varepsilon & \text{if } p = 0 \end{cases} \tag{2.2}$$

for all $x \in [a,b]$ and if $h'(x) < 0$ then

$$|z(x)-g(x)| \le \begin{cases} \dfrac{e^{-p[h(b)-h(a)]}-1}{p}\varepsilon & \text{if } p \ne 0; \\ (h(a)-h(b))\varepsilon & \text{if } p = 0 \end{cases} \tag{2.3}$$

for all $x \in [a,b]$.

Now, we define

$$u(x) = y(b)e^{\lambda[h(x)-h(b)]} - e^{\lambda h(x)} \int_x^b h'(t)z(t)e^{-\lambda h(t)}\ dt, \quad x \in [a,b].$$

Due to the definition of function u, we immediately obtain that $u \in C^2[a,b]$ and

$$u'(x) = \lambda h'(x)u(x) + z(x)h'(x). \tag{2.4}$$

Then

$$\frac{u''(x)h'(x) - h''(x)u'(x)}{[h'(x)]^2} = \lambda u'(x) + z'(x).$$

It follows from (2.1) and (2.4) that

$$\left(\frac{1}{h'(x)}\right)^2 u''(x) + \left(\frac{\alpha}{h'(x)} - \frac{h''(x)}{(h'(x))^3}\right) u'(x) + \beta u(x) = f(x), \quad x \in [a,b].$$

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Furthermore, for the function u, the inequality

$$\begin{aligned}
|y(x)-u(x)| =& \left|y(x)-y(b)e^{\lambda[h(x)-h(b)]}+e^{\lambda h(x)}\int_x^b h'(t)z(t)e^{-\lambda h(t)}\,dt\right| \\
=& |e^{\lambda h(x)}|\left|y(x)e^{-\lambda h(x)}-y(b)e^{-\lambda h(b)}+\int_x^b h'(t)z(t)e^{-\lambda h(t)}\,dt\right| \\
=& e^{qh(x)}\left|\int_x^b h'(t)z(t)e^{-\lambda h(t)}\,dt-\int_x^b [e^{-\lambda h(t)}y(t)]'\,dt\right| \\
=& e^{qh(x)}\left|\int_x^b h'(t)e^{-\lambda h(t)}[z(t)-\frac{1}{h'(t)}y'(t)+\lambda y(t)]\,dt\right| \\
\le& e^{qh(x)}\int_x^b |h'(t)e^{-\lambda h(t)}||z(t)-g(t)|\,dt \\
=& e^{qh(x)}\int_x^b e^{-qh(t)}|h'(t)||z(t)-g(t)|\,dt
\end{aligned}$$

is also valid for all $x\in[a,b]$. It follows from (2.2) that

$$|y(x)-u(x)|\le\begin{cases}
\dfrac{[1-e^{-p[h(b)-h(a)]}][1-e^{-q[h(b)-h(a)]}]}{pq}\varepsilon & \text{if } p,q\neq 0;\\[2ex]
\dfrac{[1-e^{-p[h(b)-h(a)]}](h(b)-h(a))}{p}\varepsilon & \text{if } p\neq 0,\ q=0;\\[2ex]
\dfrac{[1-e^{-q[h(b)-h(a)]}](h(b)-h(a))}{q}\varepsilon & \text{if } p=0,\ q\neq 0;\\[2ex]
(h(b)-h(a))^2\varepsilon & \text{if } p,q=0
\end{cases}$$

and (2.3) implies that

$$|y(x)-u(x)|\le\begin{cases}
\dfrac{[e^{-p[h(b)-h(a)]}-1][e^{-q[h(b)-h(a)]}-1]}{pq}\varepsilon & \text{if } p,q\neq 0;\\[2ex]
\dfrac{[e^{-p[h(b)-h(a)]}-1](h(a)-h(b))}{p}\varepsilon & \text{if } p\neq 0,\ q=0;\\[2ex]
\dfrac{[e^{-q[h(b)-h(a)]}-1](h(a)-h(b))}{q}\varepsilon & \text{if } p=0,\ q\neq 0;\\[2ex]
(h(a)-h(b))^2\varepsilon & \text{if } p,q=0
\end{cases}$$

for all $x\in[a,b]$. This completes the proof. □

In [11], Li and Shen proved that a second order differential equation with constant coefficients has the Hyers-Ulam stability, if its characteristic equation has two positive roots. It is necessary to mention that our next result is more general than their result, because in our result, there is no restriction on roots of characteristic equation.

Corollary 2.2. *The second order differential equation with constant coefficients* $y''(x)+\alpha y'(x)+\beta y(x)=f(x)$ *has the Hyers-Ulam stability.*

Proof. It is enough to take $h(x) = x$ and replace α by $\alpha + 1$ in Theorem 2.1. □

Corollary 2.3. *If $0 < a < b$ or $a < b < 0$ then the second order Euler differential equation $x^2y''(x) + \alpha xy'(x) + \beta y(x) = f(x)$ has the Hyers-Ulam stability.*

Proof. It is enough to take $h(x) = Ln(|x|)$ and replace α by $\alpha + 1$ in Theorem 2.1. □

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An iterative algorithm based on the hybrid steepest descent method for strictly pseudocontractive mappings

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Abstract

In this paper, we consider a general iterative method based on the hybrid steepest descent method for finding fixed points of a strictly pseudocontractive mapping in a Hilbert space. Utilizing weaker control conditions than previous ones, we establish the strong convergence of the sequence generated by the proposed iterative method to a fixed point of the mapping, which is the unique solution of a certain variational inequality.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C. We denote by $F(S)$ the set of fixed points of S.

The class of pseudocontractive mappings is one of the most important classes of mappings among nonlinear mappings. We recall that a mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C.$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive. The mapping T is also said to be pseudocontractive if $k = 1$ and T is said to be strongly pseudocontractive if there exists a constant $\nu \in (0, 1)$ such that $T - \nu I$ is pseudocontractive. Clearly, the class of k-strictly

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pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings. Also we remark that the class of strongly pseudocontractive mappings is independent of the class of k-strictly pseudocontractive mappings (see [1]). Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [2–7] and the references therein.

In 2010, by combining Yamada's method [8] and Marino and Xu's method [9], Tian [10] considered the following explicit iterative scheme for the nonexpansive mapping S:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) S x_n, \quad \forall n \geq 0, \tag{1.1}$$

where $F : H \to H$ is a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$ (i.e., $\|Fx - Fy\| \leq \rho\|x - y\|$ and $\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$, $x, y \in H$, respectively), $V : H \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$, $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau = \mu(\eta - \frac{\mu\rho^2}{2})$. In particular, by using control conditions (i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ on $\{\alpha_n\}$, he proved that the sequence $\{x_n\}$ generated by (1.1) converges strongly to a fixed point $\widetilde{x}$ of S, which is the unique solution of the following variational inequality related to the operator F:

$$\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - p\rangle \leq 0, \quad \forall p \in F(S). \tag{1.2}$$

His results improved the results of Tian [11] from the case of the contractive mapping f a constant $\alpha \in (0, 1)$ to the case of a Lipschitzian mapping V with a constant $l \geq 0$.

In 2011, Ceng *et al.* [12] also considered the following explicit iterative schemes for the nonexpansive mapping S:

$$x_{n+1} = P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) S x_n], \quad \forall n \geq 0, \tag{1.3}$$

where P_C is the metric projection of H ont C; $F : C \to H$ is a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$; $V : C \to H$ is an l-Lipschitzian mapping with a constant $l \geq 0$; $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$. In particular, by using the same control conditions on $\{\alpha_n\}$ as in Tian [11,10], they proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a fixed point $\widetilde{x}$ of S, which is the unique solution of the variational inequality (1.2). Their results also improved the results of Tian [11] from the case of the contractive mapping f with a constant $\alpha \in (0, 1)$ to the case of a Lipschitzian mapping V with a constant $l \geq 0$, and extended the range $0 < \gamma\alpha < \tau = \mu(\eta - \frac{\mu\rho^2}{2})$ in [10, Theorem 3.1 and Theorem 3.2] to the case of range $0 < \gamma l < \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.

In this paper, motivated by the above-mentioned results, we consider the following explicit iterative scheme for a k-strictly pseudocontractive mapping T for some $0 \leq k < 1$:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T_n x_n, \quad \forall n \geq 0, \tag{1.4}$$

where $T_n : H \to H$ is a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n)Tx$, $\forall x \in H$, with $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$. By using weaker control conditions than previous ones, we establish the strong convergence of the sequence generated by the proposed scheme (1.4) to a fixed point of T, which is a solution of the variational inequality (1.2), where the constraint set is $F(T)$. The results in this paper improve and develop the corresponding results given in [3,4,6,9–14] and references therein.

2

2 Preliminaries and Lemmas

Throughout this paper, when $\{x_n\}$ is a sequence in E, $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and that for $x \in H$,

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x,\ y \in H.$$

Let LIM be a Banach limit. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ for every $a = \{a_n\} \in \ell^\infty$. The following properties are well-known:

(i) for all $n \geq 1, a_n \leq c_n$ implies $LIM_n(a_n) \leq LIM_n(c_n)$,

(ii) $LIM_n(a_{n+N}) = LIM_n(a_n)$ for any fixed positive integer N,

(iii) $\liminf_{n\to\infty} a_n \leq LIM_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$.

The following lemma was given in [15].

Lemma 2.2. *Let $a \in \mathbb{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $LIM_n(a_n) \leq a$ for all Banach limit LIM. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

We also need the following lemmas for the proof of our main results.

Lemma 2.3 ([16])**.** *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n\delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\beta_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \beta_n |\delta_n| < \infty$,

(iii) $r_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} r_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([17])**.** *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to H$ be a k-strictly pseudocontractive mapping on C. Then the following hold:*

(i) *The fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

(ii) $F(P_C T) = F(T)$.

(iii) *If we define a mapping $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for all $x \in C$. then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(T) = F(S)$.*

The following lemmas can be easily proven, and therefore, we omit the proofs (see [5,8]).

Lemma 2.5. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and $F : H \to H$ be a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu F - \gamma V$ is strongly monotone with a constant $\mu\eta - \gamma l$.

Lemma 2.6. *Let H be a real Hilbert space H. Let $F : H \to H$ be a ρ-Lipschitzian and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t < \varsigma \leq 1$. Then $S := \varsigma I - t\mu F : H \to H$ is a contractive mapping with a constant $\varsigma - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.*

Finally, we recall that the sequence $\{x_n\}$ in H is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \quad \text{that is,} \quad x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3 Main results

Throughout the rest of this paper, we always assume as follows: Let H be a real Hilbert space. Let $T : H \to H$ be a k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ for some $0 \leq k < 1$, let $F : H \to H$ be a ρ-Lipschitzain and η-strongly monotone operator with constants $\rho > 0$ and $\eta > 0$, and let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$. Let $T_n : H \to H$ be a mapping defined by $T_n x = \lambda_n x + (1 - \lambda_n)Tx$, $\forall x \in H$, where $0 \leq k \leq \lambda_n \leq \lambda < 1$ and $\lim_{n\to\infty} \lambda_n = \lambda$. By Lemma 2.4, T_n is nonexpansive.

In this section, we consider the following explicit scheme which generates a sequence in an explicit way:

$$x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu F)T_n x_n, \quad \forall n \geq 0, \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ and $x_0 \in H$ is an arbitrary initial guess, and establish strong convergence of this sequence to a fixed point $\widetilde{x}$ of T, which is the unique solution of the variational inequality:

$$\langle (\mu F - \gamma V)\widetilde{x}, \widetilde{x} - p \rangle \leq 0, \quad \forall p \in F(T). \tag{3.2}$$

First, we consider the following scheme that generates a net $\{x_t\}_{t\in(0,1)}$ in an implicit way:

$$x_t = t\gamma V x_t + (I - t\mu F)T_t x_t, \tag{3.3}$$

where $T_t x = \lambda_t x + (1 - \lambda_t)Tx$, $\forall x \in H$, with $0 \leq k \leq \lambda_t \leq \lambda < 1$ and $\lim_{t\to 0} \lambda_t = \lambda$.

Indeed, for $t \in (0,1)$, consider a mapping $Q_t : H \to H$ defined by

$$Q_t x = t\gamma V x + (I - t\mu F)T_t x, \quad \forall x \in H.$$

It is easy to see that Q_t is a contractive mapping with constant $1 - t(\tau - \gamma l)$. Indeed, by Lemma 2.6, we have

$$\begin{aligned} \|Q_t x - Q_t y\| &\leq t\gamma \|Vx - Vy\| + \|(I - t\mu F)T_t x - (I - t\mu F)T_t y\| \\ &\leq t\gamma l \|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - t(\tau - \gamma l))\|x - y\|. \end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation (3.3).

By utilizing the same method as in [10,12], we obtain the following theorem for strong convergence of the net $\{x_t\}$ as $t \to 0$, which guarantees the existence of solutions of the variational inequality (3.2).

Theorem 3.1. *The net $\{x_t\}$ defined via (3.3) converges strongly to a fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the variational inequality (3.2), equivalently, we have $P_{F(T)}(I - \mu F + \gamma V)\widetilde{x}) = \widetilde{x}$.*

Proof. We first show the uniqueness of a solution of the variational inequality (3.2), which is indeed a consequence of the strong monotonicity of $\mu F - \gamma V$. In fact, noting that $0 \leq \gamma l < \tau$ and $\mu\eta \geq \tau \Longleftrightarrow \rho \geq \eta$, it follows from Lemma 2.5 that

$$\langle (\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y \rangle \geq (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Suppose that $\widetilde{x} \in F(T)$ and $\widehat{x} \in F(T)$ both are solutions to (3.2). Then we have

$$\langle (\mu F - \gamma V)\widetilde{x}, \widetilde{x} - \widehat{x} \rangle \le 0 \tag{3.4}$$

and

$$\langle (\mu F - \gamma V)\widehat{x}, \widehat{x} - \widetilde{x} \rangle \le 0. \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\langle (\mu F - \gamma V)\widetilde{x} - (\mu F - \gamma V)\widehat{x}, \widetilde{x} - \widehat{x} \rangle \le 0.$$

The strong monotonicity of $\mu F - \gamma V$ implies that $\widetilde{x} = \widehat{x}$ and the uniqueness is proved.

Next, we prove that $x_t \to \widetilde{x}$ as $t \to 0$. Observing $F(T) = F(T_t)$ by Lemma 2.4, from (3.3), we write, for given $p \in F(T)$,

$$x_t - p = t(\gamma V x_t - \mu F p) + (I - t\mu F)T_t x_t - (I - t\mu F)p$$

to derive that

$$\begin{aligned} \|x_t - p\|^2 &= t\langle \gamma V x_t - \mu F p, x_t - p \rangle + \langle (I - t\mu F)T_t x_t - (I - t\mu F)T_t p, x_t - p \rangle \\ &\le (1 - t\tau)\|x_t - p\|^2 + t\langle \gamma V x_t - \mu F p, x_t - p \rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_t - p\|^2 &\le \frac{1}{\tau}\langle \gamma V x_t - \mu F p, x_t - p \rangle \\ &\le \frac{1}{\tau}[\gamma l \|x_t - p\|^2 + \langle \gamma V p - \mu F p, x_t - p \rangle]. \end{aligned}$$

Therefore

$$\|x_t - p\|^2 \le \frac{1}{\tau - \gamma l}\langle \gamma V p - \mu F p, x_t - p \rangle. \tag{3.6}$$

From (3.6), it follows that

$$\|x_t - p\| \le \frac{1}{\tau - \gamma l}\|\gamma V p - \mu F p\|,$$

and so $\{x_t\}$, $\{V x_t\}$, $\{T x_t\}$, $\{T_t x_t\}$, $\{F x_t\}$ and $\{F T_t x_t\}$ are bounded. As a consequence, it follows that

$$\lim_{t \to 0} \|(I - T_t)x_t\| = \lim_{t \to 0} t\|\gamma V x_t - \mu F T_t x_t\| = 0. \tag{3.7}$$

Since $\{x_t\}$ is bounded as $t \to 0$, we show that if $\{t_n\}$ is a subsequence in (0,1) such that $t_n \to 0$ and $x_{t_n} \rightharpoonup x^*$, then $x^* \in F(T)$. To this end, define $S : H \to H$ by $Sx = \lambda x + (1 - \lambda)Tx$, $\forall x \in H$. Then S is nonexpansive with $F(S) = F(T)$ by Lema 2.4. Notice that

$$\begin{aligned} \|Sx_{t_n} - x_{t_n}\| &\le \|Sx_{t_n} - T_{t_n}x_{t_n}\| + \|T_{t_n}x_{t_n} - x_{t_n}\| \\ &\le |\lambda - \lambda_{t_n}|\|x_{t_n} - Tx_{t_n}\| + \|x_{t_n} - T_{t_n}x_{t_n}\|. \end{aligned}$$

By (3.7) and $\lambda_{t_n} \to \lambda$, we have

$$\lim_{n \to \infty} \|Sx_{t_n} - x_{t_n}\| = 0.$$

Hence, if $x^* \neq Sx^*$, then, by Opial condition, we have

$$\begin{aligned}\liminf_{n\to\infty} \|x_{t_n} - x^*\| &< \liminf_{n\to\infty} \|x_{t_n} - Sx^*\| \\ &\leq \liminf_{n\to\infty} (\|x_{t_n} - Sx_{t_n}\| + \|Sx_{t_n} - Sx^*\|) \\ &\leq \liminf_{n\to\infty} \|x_{t_n} - x^*\|,\end{aligned}$$

which is a contradiction. So $x^* \in F(S) = F(T)$. Thus, by replacing p with x^* in (3.6), we get $x_{t_n} \to x^*$.

Finally, we prove that x^* is a solution of the variational inequality (3.2). Since

$$x_t = t\gamma V x_t + (I - t\mu F)T_t x_t,$$

we have

$$(\mu F - \gamma V)x_t = -\frac{1}{t}(I - T_t)x_t + \mu(Fx_t - FT_t x_t).$$

From $T_t p = p$ for $p \in F(T)$, it follows that

$$\begin{aligned}\langle(\mu F - \gamma V)x_t, x_t - p\rangle = & -\frac{1}{t}\langle(I - T_t)x_t - (I - T_t)p, x_t - p\rangle \\ & + \mu\langle Fx_t - FT_t x_t, x_t - p\rangle \\ & \leq \mu\langle Fx_t - FT_t x_t, x_t - p\rangle\end{aligned} \tag{3.8}$$

since $I - T_t$ is monotone (i.e., $\langle x - y, (I - T_t)x - (I - T_t)y\rangle \geq 0, \quad x, y \in H$, which is due to the nonexpansivity of T_t). Now replacing t in (3.8) with t_n and noticing that $Fx_{t_n} - FT_{t_n}x_{t_n} \to Fx^* - Fx^* = 0$ as $n \to \infty$ for $x^* \in F(T)$, we obtain

$$\langle(\mu F - \gamma V)x^*, x^* - p\rangle \leq 0.$$

That is, $x^* \in F(T)$ is a solution of the variational inequality (3.2); hence $x^* = \widetilde{x}$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (at $t \to 0$) equals $\widetilde{x}$. Therefore, $x_t \to \widetilde{x}$ as $t \to 0$.

The variational inequality (3.2) can be rewritten as

$$\langle(I - \mu F + \gamma V)\widetilde{x} - \widetilde{x}, \widetilde{x} - p\rangle \geq 0, \quad \forall p \in F(T).$$

By reminding Lemma 2.4 and (2.1), this is equivalent to the fixed point equation

$$P_{F(T)}(I - \mu F + \gamma V)\widetilde{x}) = \widetilde{x}. \qquad \square$$

Remark 3.1. 1) Theorem 3.1 improves the case of the nonexpansive mapping S in Theorem 3.1 of Tian [10] (and Ceng *et al.* [12]) to the case of the k-strictly pseudocontractive mapping T.

2) Theorem 3.1 includes the corresponding results of Tian [11], Marino and Xu [9], Moudafi [13] and Xu [14] as some special cases.

First of all, we give the following result in order to establish strong convergence of the sequence generated by the explicit scheme (3.1).

Theorem 3.2. *Let $\{x_n\}$ be the sequence generated iteratively by the scheme* (3.1) *and let LIM be a Banach limit. If $\{\alpha_n\}$ satisfies the following condition:*

(C1) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

then

$$LIM_n(\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n\rangle) \leq 0,$$

where $\widetilde{x} = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by*

$$x_t = t\gamma V x_t + (I - t\mu F)Sx_t, \tag{3.9}$$

and $Sx = \lambda x + (1-\lambda)Tx$, $\forall x \in H$, *with* $0 \leq k \leq \lambda < 1$.

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n\tau < 1$ for all $n \geq 0$.

Let $\{x_t\}$ be the net generated by (3.9). By Theorem 3.1 with $\lambda_t = \lambda$ for $t \in (0,1)$ and Lemma 2.4, there exists $\lim_{t\to 0} x_t \in F(T)$. Denote it by $\widetilde{x}$. Moreover $\widetilde{x}$ is the unique solution of the variational inequality (3.2). By (3.9), we have

$$\begin{aligned}\|x_t - x_{n+1}\| &= \|t\gamma V x_t + (I - t\mu F)Sx_t - x_{n+1}\| \\ &= \|(I - t\mu F)Sx_t - (I - t\mu F)x_{n+1} + t(\gamma V x_t - \mu F x_{n+1})\|.\end{aligned}$$

Applying Lemma 2.1 and Lemma 2.6, we have

$$\|x_t - x_{n+1}\|^2 \leq (1 - t\tau)^2\|Sx_t - x_{n+1}\|^2 + 2t\langle \gamma V x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle. \tag{3.10}$$

From the proof of Theorem 3.1, we know that $\{x_t\}$, $\{Vx_t\}$, $\{Tx_t\}$, $\{Sx_t\}$, $\{FSx_t\}$ and $\{Fx_t\}$ are bounded.

Now we show that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\mu Fp - \gamma Vp\|}{\tau - \gamma l}\}$ for all $n \geq 0$ and all $p \in F(T)$. Indeed, let $p \in F(T)$. Noticing $p = T_n p$, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \alpha_n\mu F)T_n x_n - (I - \alpha_n \mu F)T_n p\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\|\gamma V x_n - \mu F p\| \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n(\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) \\ &\leq [1 - (\tau - \gamma l)\alpha_n]\|x_n - p\| + (\tau - \gamma l)\alpha_n \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}\right\}.\end{aligned}$$

Using an induction, we have $\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma l}\}$. Hence $\{x_n\}$ is bounded, and so are $\{Vx_n\}$, $\{Tx_n\}$ $\{T_n x_n\}$, $\{FT_n x_n\}$, and $\{Fx_n\}$. As a consequence of condition (C1), we get

$$\|x_{n+1} - T_n x_n\| = \alpha_n\|\gamma V x_n - \mu F T_n x_n\| \to 0 \quad (n \to \infty).$$

From definitions of S and T_n with $\lim_{n\to\infty} \lambda_n = \lambda$, we deduce

$$\begin{aligned}\|Sx_t - x_{n+1}\| &\leq \|Sx_t - Sx_n\| + \|Sx_n - T_n x_n\| + \|T_n x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + |\lambda - \lambda_n|\|x_n - Tx_n\| + \|T_n x_n - x_{n+1}\| \\ &\leq \|x_t - x_n\| + |\lambda - \lambda_n|K_1 + \|T_n x_n - x_{n+1}\| \\ &= \|x_t - x_n\| + e_n,\end{aligned},$$

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where $K_1 = \sup\{\|x_n - Tx_n\| : n \geq 0\}$ and $e_n = |\lambda - \lambda_n|K_1 + \|x_{n+1} - T_n x_n\| \to 0$ as $n \to \infty$. Also observing that F is η-strongly monotone, we have

$$\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle \geq \mu\eta\|x_t - x_n\|^2 \geq \tau\|x_t - x_n\|^2. \tag{3.11}$$

So, by combining (3.10) and (3.11), we obtain

$$\begin{aligned}
\|x_t - x_{n+1}\|^2 &\leq (1 - t\tau)^2(\|x_t - x_n\| + e_n)^2 \\
&\quad + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\
&\leq (t^2\tau - 2t)\tau\|x_t - x_n\|^2 + \|x_t - x_n\|^2 \\
&\quad + (1 - t\tau)^2 e_n(2\|x_t - x_n\| + e_n) \\
&\quad + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\
&\leq (t^2\tau - 2t)\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + e_n(K_2 + e_n) + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle \\
&\quad + 2t\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle \\
&= t^2\tau\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle + \|x_t - x_n\|^2 \\
&\quad + e_n(K_2 + e_n) + 2t\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle \\
&\quad + 2t(\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle - \langle \mu F x_t - \mu F x_n, x_t - x_n\rangle),
\end{aligned} \tag{3.12}$$

where $K_2 = \sup\{2\|x_t - x_n\| : t, n \geq 0\}$. Applying the Banach limit LIM to (3.12) together with $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}
LIM_n(\|x_t - x_{n+1}\|^2) &\leq t^2\tau LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) + LIM_n(\|x_t - x_n\|^2) \\
&\quad + 2tLIM_n(\langle \gamma V x_t - \mu F x_t, x_t - x_{n+1}\rangle) \\
&\quad + 2t[LIM_n(\langle \mu F x_t - \mu F x_{n+1}, x_t - x_{n+1}\rangle) \\
&\qquad - LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle)].
\end{aligned} \tag{3.13}$$

Using the property $LIM_n(a_n) = LIM_n(a_{n+1})$ of Banach limit in (3.12), we obtain

$$\begin{aligned}
&LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_n\rangle) = LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_{n+1}\rangle) \\
&\leq \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) \\
&\quad + \frac{1}{2t}[LIM_n(\|x_t - x_n\|^2) - LIM_n(\|x_t - x_n\|^2)] \\
&\quad + [LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) - LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle)] \\
&= \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle).
\end{aligned} \tag{3.14}$$

Since

$$\begin{aligned}
t\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle &\leq t\mu\rho\|x_t - x_n\|^2 \\
&\leq t\mu\rho(\|x_t - p\| + \|p - x_n\|)^2 \\
&\leq t\mu\rho\left(\frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l} + \|x_0 - p\|\right)^2 \to 0 \quad (\text{as } t \to 0),
\end{aligned} \tag{3.15}$$

we conclude from (3.14) and (3.15) that

$$\begin{aligned}
LIM_n(\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n\rangle\rangle) &\leq \limsup_{t\to 0} LIM_n(\langle \mu F x_t - \gamma V x_t, x_t - x_n\rangle) \\
&\leq \limsup_{t\to 0} \frac{t\tau}{2} LIM_n(\langle \mu F x_t - \mu F x_n, x_t - x_n\rangle) \leq 0.
\end{aligned}$$

This completes the proof. $\square$

Now, using Theorem 3.2, we establish strong convergence of the sequence generated by the explicit scheme (3.1) to a fixed point $\widetilde{x}$ of T, which is the unique solution of the variational inequality (3.2).

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated iteratively by the scheme* (3.1)*, where $\{\alpha_n\}$ satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$.

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to $\widetilde{x} \in F(T)$, where is the unique solution of the variational inequality (3.2)*.*

Proof. First, note that from the condition (C1), without loss of generality, we assume that $\alpha_n \tau < 1$ and $\frac{2\alpha_n(\tau-\gamma l)}{1-\alpha_n\gamma l} < 1$ for all $n \geq 0$.

Let x_t be defined by (3.9), that is, $x_t = t\gamma V x_t + (I - t\mu F)Sx_t$ for $0 < t < 1$, where $Sx = \lambda x + (1-\lambda)Tx$, $\forall x \in H$, with $0 \leq k \leq \lambda < 1$, and let $\lim_{t\to 0} x_t := \widetilde{x} \in F(S) = F(T)$ (by using Theorem 3.1 and Lemma 2.4). Then $\widetilde{x}$ is the unique solution of the variational inequality (3.2).

We divides the proof several steps:

Step 1. We see that $\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Fp\|}{\tau-\gamma l}\right\}$ for all $n \geq 0$ and all $p \in F(T)$ as in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{T_n x_n\}$, $\{FT_n x_n\}$ and $\{Vx_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n\rangle \leq 0$. To this end, put

$$a_n := \langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n)\rangle, \quad \forall n \geq 0.$$

Then Theorem 3.2 implies that $LIM_n(a_n) \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(\widetilde{x} - x_{n_j+1}) = w - \lim_{j\to\infty}(\widetilde{x} - x_{n_j}) = (\widetilde{x} - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, (\widetilde{x} - x_{n_j+1}) - (\widetilde{x} - x_{n_j})\rangle = 0.$$

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Then Lemma 2.2 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_n)\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. By using (3.1), we have

$$x_{n+1} - \widetilde{x} = \alpha_n(\gamma V x_n - \mu F\widetilde{x}) + (I - \alpha_n\mu F)T_n x_n - (I - \alpha_n\mu F)\widetilde{x}.$$

Applying Lemma 2.1 and Lemma 2.6, we obtain

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^2 &= \|(I - \alpha_n\mu F)T_n x_n - (I - \alpha_n\mu F)\widetilde{x} + \alpha_n(\gamma V x_n - \mu F\widetilde{x})\|^2 \\
&\le \|(I - \alpha_n\mu F)T_n x_n - (I - \alpha_n\mu F)T_n\widetilde{x}\|^2 \\
&\quad + 2\alpha_n\langle \gamma V x_n - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - \widetilde{x}\|^2 + 2\alpha_n\langle \gamma V x_n - \gamma V\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\quad + 2\alpha_n\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - \widetilde{x}\|^2 + \alpha_n\gamma l(\|x_n - \widetilde{x}\|^2 + \|x_{n+1} - \widetilde{x}\|^2) \\
&\quad + 2\alpha_n\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle.
\end{aligned} \tag{3.15}$$

It then follows from (3.15) that

$$\begin{aligned}
\|x_{n+1} - \widetilde{x}\|^2 &\le \frac{(1 - \alpha_n\tau)^2 + \alpha_n\gamma l}{1 - \alpha_n\gamma l}\|x_n - \widetilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma l}\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle \\
&\le \left(1 - \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n\gamma l}\right)\|x_n - \widetilde{x}\|^2 \\
&\quad + \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n\gamma l}\left(\frac{1}{\tau - \gamma l}\langle \gamma V\widetilde{x} - \mu F\widetilde{x}, x_{n+1} - \widetilde{x}\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma l)}K_3\right),
\end{aligned} \tag{3.16}$$

where $K_3 = \sup\{\|x_n - \widetilde{x}\|^2 : n \ge 0\}$. Put

$$\beta_n = \frac{2\alpha_n(\tau - \gamma l)}{1 - \alpha_n\gamma l} \quad \text{and} \quad \delta_n = \frac{1}{\tau - \gamma l}\langle \mu F\widetilde{x} - \gamma V\widetilde{x}, \widetilde{x} - x_{n+1}\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma l)}K_3.$$

From (C1), (C2) and Step 2, it follows that $\beta_n \to 0$, $\sum_{n=0}^{\infty}\beta_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.16) reduces to

$$\|x_{n+1} - \widetilde{x}\|^2 \le (1 - \beta_n)\|x_n - \widetilde{x}\|^2 + \beta_n\delta_n,$$

from Lemma 2.3 with $r_n = 0$, we conclude that $\lim_{n\to\infty}\|x_n - \widetilde{x}\| = 0$. This completes the proof. □

Corollary 3.1. *Let $\{x_n\}$ be the sequence generated iteratively by the scheme* (3.1)*, where $\{\alpha_n\}$ satisfies the following conditions:*

(C1) $\{\alpha_n\} \subset (0, 1)$ *and* $\lim_{n\to\infty}\alpha_n = 0$.

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$.

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $\widetilde{x} \in F(T)$, where is the unique solution of the variational inequality (3.2).

Remark 3.2. If $\{\alpha_n\}$ and $\{\lambda_n\}$ in Corollary 3.1 satisfy conditions (C1), (C2),

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or

(C4) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ or, equivalently, $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = 0$; or,

(C5) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition); and

(C6) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then the sequence $\{x_n\}$ generated by (3.1) is asymptotically regular. Now we give only the proof in case when $\{\alpha_n\}$ satisfies the conditions (C1), (C2), (C5) and (C6). By Step 1 in the proof of Theorem 3.3, there exists a constant $K_4 > 0$ such that for all $n \ge 0$,

$$\mu\|FT_nx_n\| + \gamma\|Vx_n\| \le K_4.$$

Next, we notice that

$$\begin{aligned}\|T_nx_n - T_{n-1}x_{n-1}\| &\le \|T_nx_n - T_nx_{n-1}\| + \|T_nx_{n-1} - T_{n-1}x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - Tx_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|K_5,\end{aligned}$$

where $K_5 = \sup\{\|x_n - Tx_n\| : n \ge 0\}$. So, we obtain, for all $n \ge 0$,

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &= \|(I - \alpha_n\mu F)T_nx_n - (I - \alpha_n\mu F)T_{n-1}x_{n-1} + \mu(\alpha_n - \alpha_{n-1})FT_{n-1}x_{n-1} \\ &\qquad + \gamma[\alpha_n(Vx_n - Vx_{n-1}) + Vx_{n-1}(\alpha_n - \alpha_{n-1})]\| \\ &\le (1 - \alpha_n\tau)\|T_nx_n - T_{n-1}x_{n-1}\| + \mu|\alpha_n - \alpha_{n-1}|\|FT_{n-1}x_{n-1}\| \\ &\quad + \gamma[\alpha_n[l\|x_n - x_{n-1}\| + \|Vx_{n-1}\||\alpha_n - \alpha_{n-1}] \\ &\le (1 - \alpha_n(\tau - \gamma l))\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|K_5 + |\alpha_n - \alpha_{n-1}|K_4 \\ &\le (1 - \alpha_n(\tau - \gamma l))\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|K_5 + (o(\alpha_n) + \sigma_{n-1})K_4.\end{aligned} \tag{3.17}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\beta_n = \alpha_n(\tau - \gamma l)$, $\beta_n\delta_n = o(\alpha_n)K_4$ and $r_n = \sigma_{n-1}K_4 + |\lambda_n - \lambda_{n-1}|K_5$, from (3.17) we have

$$s_{n+1} \le (1 - \beta_n)s_n + \beta_n\delta_n + r_n.$$

Hence, by (C1), (C2), (C5), (C6) and Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following:

Corollary 3.2. *Let $\{x_n\}$ be the sequence generated iteratively by the scheme* (3.1), *where $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the conditions* (C1), (C2), (C5) *and* (C6) *(or the conditions* (C1), (C2), (C3) *and* (C6), *or the conditions* (C1), (C2), (C4) *and* (C6)*). Then $\{x_n\}$ converges strongly to $\widetilde{x} \in F(T)$, where is the unique solution of the variational inequality* (3.2).

Remark 3.3. 1) Theorem 3.3 extends Theorem 3.2 of Tian [10] and Ceng *et al.* [12] in the following ways:

(a) The nonexpansive mapping S in [10, 12, Theorem 3.2] is extended to the case of a k-strictly pseudocontractive mapping T.

(b) The condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ in [10, 12, Theorem 3.2] is relaxed to the weak asymptotic regularity on $\{x_n\}$.

2) Theorem 3.3 also generalizes the corresponding results of Cho *et al.* [3], Jung [6] and Marino and Xu [9] in following aspects:

(a) A strongly positive bounded linear operator A in [3,6,9] is extended to the case of a ρ-Lipschitzian and η-strongly monotone operator F. (In fact, from the definitions, it follows that a strongly positive bounded linear operator A (i.e., there exists a constant $\overline{\gamma} > 0$ with the property: $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad x \in H$) is a $\|A\|$-Lipschitzian and $\overline{\gamma}$-strongly monotone operator).

(b) The contractive mapping f with a constant $\alpha \in (0, 1)$ in [3,6,9] is extended to the case of a Lipschizian mapping V with a constant $l \geq 0$.

(c) The nonexpansive mapping S in [3,9] is extended to the case of a k-strictly pseudo-contractive mapping T.

(d) The condition $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ in [3,9] is weakened to the weak asymptotic regularity on $\{x_n\}$.

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BE-ALGEBRAS WITH ORDER REVERSING INVOLUTION

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Abstract. The notions of a filter's radical and extended filter are introduced in BE-algebras. Then some properties of filter's radical and extended filter are obtained. Using a special set $x^{-1} * F$, we give an equivalent condition for a filter to be prime.

1. Introduction

In [6], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [4, 5] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [2] fuzzified the concept of BE-algebras and investigated some of their properties. Y. B. Jun and S. S. Ahn ([7]) provided several degrees in defining a fuzzy filter and a fuzzy implicative filter. It was a generalization of a fuzzy filter.

In this paper, we introduce the notions of a filter's radical and an extended filter in BE-algebras. Some properties of a filter's radical and an extended filter are obtained. Using a special set $x^{-1} * F$, we obtain an equivalent condition for a filter to be a prime filter.

2. Preliminaries

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* ([6]) if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*).

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Note that $x * x = 1$ for all $x \in X$. It is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. A mapping $f : X \to Y$ of BE-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. A homomorphism f of BE-algebras is called an *epimorphism* if f is onto. Note that if f is a homomorphism of BE-algebras, then $f(1) = 1$.

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Proposition 2.1([6]). *Let $(X;*,1)$ be a self distributive BE-algebra. Then the following hold: for any $x,y,z\in X$,*

(i) *if $x\le y$, then $z*x\le z*y$ and $y*z\le x*z$,*
(ii) $y*z\le (z*x)*(y*z)$,
(iii) $y*z\le (x*y)*(x*z)$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies Proposition 2.1(iii). If a BE-algebra X is transitive, then Proposition 2.1(i) holds ([7]).

Definition 2.2. Let X be a BE-algebra and let $\emptyset\ne F\subseteq X$ of a BE-algebra X. F is called a *filter* ([6]) of X if

(F1) $1\in F$;
(F2) if $x*y, x\in F$, then $y\in F$.

F is an *implicative filter* ([7]) of X if (F1) and

(F3) if $x*(y*z)\in F$ and $x*y\in F$, then $x*z\in F$.

Note that every implicative filter is a filter in a BE-algebra.

Proposition 2.3. *Let X be a BE-algebra and let F be a filter of X. If $x\le y$ and $x\in F$ for any $y\in F$, then $y\in F$.*

Definition 2.4. Let X be a BE-algebra X and $\emptyset\ne A\subseteq X$. If B is the least filter containing A in X, then B is called the *filter generated by* A and denoted by $(A]$.

It is trivial to verify that

$$(A]=\cap\{B|A\subseteq B\subseteq X, B \text{ is a filter}\}.$$

In what follows, $(\{a\}]$ is denoted by $(a]$ and $[a_1,a_2,\cdots,a_n,x]:=a_1*(a_2*(\cdots(a_n*x)\cdots))$. Specially, $[a,x]^0:=x, [a,x]^1:=a*x$, and $[a,x]^n:=\underbrace{a*(a*(\cdots*(a}_{n}*x)\cdots))$ $(n\ge 2)$.

Proposition 2.5([8]). *Let X be a transitive BE-algebra and $\emptyset\ne A\subseteq X$. Then*

$$(A]=\{x\in X|\exists a_1,\cdots a_n\in A, n\in\mathbb{N} \text{ such that } [a_1,a_2,\cdots,a_n,x]=1\}.$$

Definition 2.6([1]). Let X be a BE-algebra. X is said to be *commutative* if the following identity holds

(C) $(x*y)*y=(y*x)*x$, i.e., $x\vee y=y\vee x$ where $x\vee y=(y*x)*x$,

for all $x,y\in X$.

Theorem 2.7([1]). *If $(X;*,1)$ is a commutative BE-algebra X, then it is a semilattice with respect to $\vee$.*

3. BE-algebra with order reversing involution

Definition 3.1. A BE-algebra $(X;*,1)$ is said to have an *order reversing involution* " $'$ " if

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(i) if $x \leq y$, then $y' \leq x'$ and $(x')' = x$;
(ii) 0 is the smallest element of X;
(iii) $x * y = y' * x'$

for all $x, y \in X$, where $0' := 1$.

In what follows, let X denote a BE-algebra with order reversing involution $'$ unless otherwise specified.

Proposition 3.2. *Suppose that X is a transitive BE-algebra with order reversing involution $'$. Then the following hold:*

(i) $x * 0 = x'$;
(ii) $0 * x = 1$;
(iii) $x \leq y \Leftrightarrow y * z \leq x * z$.

for any $x, y, z \in X$.

Proof. (i) By Definition 3.1(iii), we have $x * 0 = 0' * x' = 1 * x' = x'$.
(ii) Let $x \in X$. Using Definition 3.1(iii) and (BE2), we get $0 * x = x' * 0' = x' * 1 = 1$.
(iii) By Proposition 2.1(i), $x \leq y$ imply $y * z \leq x * z$. Conversely, suppose that $y * z \leq x * z$ for all $x, y, z \in X$. By Definition 3.1(i), we have $x * y = y' * x' = (y * 0) * (x * 0) = 1$, proving that $x \leq y$. □

Theorem 3.3. *Let X be a transitive BE-algebra and let $a, b, x \in X$.*

(i) *If $a \geq b$, then $[a, x]^n \leq [b, x]^n$ for any $n \in \mathbb{N}$;*
(ii) *If $n, m \in \mathbb{N}$ with $n \geq m$, then $[a, x]^n \geq [a, x]^m$;*
(iii) *$[a, x]^n \geq x$ for any $n \in \mathbb{N}$.*

Proof. These conditions are trivial when $n = 0$ or $m = 0$.
(i) We use induction on n to show $[a, x]^n \leq [b, x]^n$. If $n = 1$, then $a \geq b$ imply $[a, x]^1 = a * x \leq [b, x]^1 = b * x$. For $n > 1$, assume that $[a, x]^m \leq [b, x]^m$ for any $m < n$. Then

$$[a, x]^n = a * [a, x]^{n-1} \leq a * [b, x]^{n-1} \leq b * [b, x]^{n-1} = [b, x]^n.$$

(ii) Suppose that $n = m + p$. Then $p \geq 0$. We use induction on p to show $[a, x]^{m+p} \geq [a, x]^m$. If $p = 0$, then $[a, x]^{m+p} \geq [a, x]^m$ holds. For $p > 1$, assume that $[a, x]^{m+q} \geq [a, x]^m$ for any $q < p$. It follows that

$$[a, x]^{m+p} = a * [a, x]^{m+(p-1)} \geq a * [a, x]^m \geq [a, x]^m.$$

(iii) The proof is similar to (i). □

Theorem 3.4. *Let X be a transitive BE-algebra and let $a, b \in X$. Then*

(i) *$(a \vee b] \subseteq (a] \cup (b]$;*
(ii) *if $a \leq b$, then $(b] \subseteq (a]$.*

Proof. (i) For any $x \in (a \vee b]$, there exists $n \in \mathbb{N}^+$ such that $[a \vee b, x]^n = 1$. Since $a \leq a \vee b$ and $b \leq a \vee b$, by Theorem 3.3(i), $[a \vee b, x]^n \leq [a, x]^n$ and $[a \vee b, x]^n \leq [b, x]^n$. Hence $[a, x]^n = 1$ and $[b, x]^n = 1$, i.e., $x \in (a] \cap (b]$. Therefore $x \in (a] \cap (b]$. Thus $(a \vee b] \subseteq (a] \cap (b]$.

(ii) If $x \in (b]$, then there exists $n \in \mathbb{N}^+$ such that $[b,x]^n = 1$. Since $a \leq b$ imply $[b,x]^n \leq [a,x]^n$, we have $[a,x]^n = 1$. Hence $x \in (a]$. □

Proposition 3.5. *Let X_1, X_2 be BE-algebras with order reversing involution $'$ and $f : X_1 \to X_2$ be a homomorphism of BE-algebras. If $f(0) = 0$, then $f(x') = (f(x))'$ for any $x \in X$.*

Proof. For any $x \in X_1$, we have $f(x') = f(x * 0) = f(x) * f(0) = f(x) * 0 = (f(x))'$. □

Let X be a BE-algebra with an order reversing involution $'$. For any $x, y \in X$, we define a binary operation " $\oplus$ " as follows:

$$x \oplus y := x' * y.$$

For any $a \in X$ and $n \in \mathbb{N}$, we denote

$$(n+1)a := a \oplus (na).$$

Proposition 3.6. *Let X be a BE-algebra with order reserving $'$. Then for any $a, b \in X$ and $m, n \in \mathbb{N}^+$, we have*

(i) *$a, b \leq a \oplus b$;*
(ii) *if $m \leq n$, then $ma \leq na$.*

Proof. (i) By (BE4) and (BE2), we have $b * (a \oplus b) = b * (a' * b) = a' * (b * b) = a' * 1 = 1$ and so $b \leq a \oplus b$. Since $a * (a \oplus b) = a * (a' * b) = a * (b' * (a')') = a * (b' * a) = b' * (a * a) = b' * 1 = 1$, we obtain $a \leq a \oplus b$.
(ii) Using (i), we obtain $na = ma \oplus (n-m)a \geq ma$. This completes the proof. □

Proposition 3.7. *Let X be a transitive BE-algebra with order reserving involution $'$. Then, for any $a, b, c \in X$, $a \leq b$ implies $a \oplus c \leq b \oplus c$.*

Proof. Let $a \leq b$ for any $a, b \in X$. By Definition 3.1(iii), we have $b' \leq a'$. Since X is transitive, we obtain $a' * c \leq b' * c$ for any $c \in X$, proving that $a \oplus c \leq b \oplus c$. □

Let $f : X_1 \to X_2$ be a homomorphism of BE-algebras with order reserving involution $'$. We define the *dual kernel* of f denoted by $DKer\ f$, as follows:

$$DKer\ f := \{x \in X_1 | f(x) = 1_2\}.$$

4. The filter's radical in BE-algebras

Definition 4.1. Let J be a filter of X. A subset $A := \{x \in X | \exists n \in \mathbb{N} \text{ such that } nx \in J\}$ is called a *filter's radical* of J and we denote it by $\sqrt{J}$.

Example 4.2. Let $X := \{0, a, b, c, d, 1\}$ be a set with $0' = 1, a' = c, b' = d, c' = a, d' = b, 1' = 0$. Define a binary operation $*$ as follows:

$*$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(X; *, ', 1, 0)$ is a BE-algebra with order reversing involution $'$. Since $\{1\}$ is a filter of X and $2b = 1, na = a, nc = c, nd = a$ for all $n \in \mathbb{N}$, we obtain $\sqrt{\{1\}} = \{b, 1\}$.

Theorem 4.3. *Let J be a filter of a BE-algebra X. Then $J \subseteq \sqrt{J}$.*

Proof. Let $x \in J$. Then $1x = x \in J$, i.e., $\exists 1 \in \mathbb{N}$ such that $1x \in J$. Hence $x \in \sqrt{J}$. Thus $J \subseteq \sqrt{J}$. □

Note that $1 \in \sqrt{J}$ whenever J is a filter of a BE-algebra X.

Theorem 4.4. *Let J_1 and J_2 be filters of a BE-algebra X with an order reversing involution $'$. Then the following hold:*

(i) *if $J_1 \subseteq J_2$, then $\sqrt{J_1} \subseteq \sqrt{J_2}$;*
(ii) $\sqrt{J_1} \cap \sqrt{J_2} = \sqrt{J_1 \cap J_2}$;
(iii) $\sqrt{J_1} \cup \sqrt{J_2} \subseteq \sqrt{(J_1 \cup J_2]}$;
(iv) $\sqrt{J_1} \subseteq \sqrt{(\sqrt{J_1}]}$.

Proof. (i) Let $x \in \sqrt{J_1}$. Then there exists $n \in \mathbb{N}$ such that $nx \in J_1 \subseteq J_2$. Hence $x \in \sqrt{J_2}$. Therefore $\sqrt{J_1} \subseteq \sqrt{J_2}$.
(ii) Let $x \in \sqrt{J_1} \cap \sqrt{J_2}$. Then there exist $m, n \in \mathbb{N}$ such that $mx \in J_1$ and $nx \in J_2$ and so $m, n \leq m+n$. By Proposition 3.6(ii), we have $mx \leq (m+n)x$ and $n \leq (m+n)x$. Since $mx \in J_1$ and J_1 is a filter of X, we get $(m+n)x \in J_1$. Since $nx \in J_2$ and J_2 is a filter of X, we obtain $(m+n)x \in J_2$. Hence $x \in \sqrt{J_1 \cap J_2}$. Therefore $\sqrt{J_1} \cap \sqrt{J_2} \subseteq \sqrt{J_1 \cap J_2}$.

Since J_1, J_2 are filters of X, $J_1 \cap J_2$ is a filter of X. Since $J_1 \cap J_2 \subseteq J_1$ and $J_1 \cap J_2 \subseteq J_2$, by (i) we get $\sqrt{J_1 \cap J_2} \subseteq \sqrt{J_1}$ and $\sqrt{J_1 \cap J_2} \subseteq \sqrt{J_2}$. Hence $\sqrt{J_1 \cap J_2} \subseteq \sqrt{J_1} \cap \sqrt{J_2}$. Thus we have $\sqrt{J_1} \cap \sqrt{J_2} = \sqrt{J_1 \cap J_2}$.
(iii) Let $x \in \sqrt{J_1} \cup \sqrt{J_2}$. Then $x \in \sqrt{J_1}$ or $x \in \sqrt{J_2}$. Hence there exists $n \in \mathbb{N}$ such that $nx \in J_1$ or there exists $m \in \mathbb{N}$ such that $mx \in J_2$. In any case, we have $n \in \mathbb{N}$ such that $nx \in J_1 \cup J_2 \subseteq (J_1 \cup J_2]$. Thus $x \in \sqrt{(J_1 \cup J_2]}$, i.e., $\sqrt{J_1} \cup \sqrt{J_2} \subseteq \sqrt{(J_1 \cup J_2]}$.
(iv) It follows immediately from Theorem 4.3 and Theorem 4.4(i). □

Corollary 4.5. *Let $J_1, J_2, \cdots, J_n$ be an implicative filter of a BE-algebra X. If $\sqrt{J_1} = \sqrt{J_2} = \cdots = \sqrt{J_n}$, then $\sqrt{J_1} = \sqrt{J_1 \cap \cdots J_n}$.*

Proof. Using Theorem 4.4(ii), we have $\sqrt{J_1} = \sqrt{J_1} \cap \cdots \cap \sqrt{J_n} = \sqrt{J_1 \cap \cdots \cap J_n}$. □

The relations between the generated filters and their radicals are discussed as follows.

Proposition 4.6. *Let X be a transitive BE-algebra and let $a, b \in X$. Then the following hold:*

(i) *if $a \leq b$, then $\sqrt{(b]} \subseteq \sqrt{(a]}$;*
(ii) $\sqrt{(a \vee b]} \subseteq \sqrt{(a] \cap (b]}$.

Proof. (i) If $a \leq b$, then using Theorem 3.4(ii) we have $(b] \subseteq (a]$. By Theorem 4.4(i), we have $\sqrt{(b]} \subseteq \sqrt{(a]}$.
(ii) Since $(a \vee b] \subseteq (a] \cap (b]$, by Theorem 4.4(i), we obtain $\sqrt{(a \vee b]} \subseteq \sqrt{(a] \cap (b]}$. □

Theorem 4.7. *Let J be an implicative filter of a transitive BE-algebra X with order reversing involution $'$. Then, for any $n \in \mathbb{N}$ and for any $a \in X$, $na \in J$ implies $a \in J$.*

Proof. Use an induction on n. If $n = 1$, then it is trivial. Suppose the conclusion holds for $p < n+1$ and $(n+1)a \in J$. Then $a' * (a' * (n-1)a) = a' * na = (n+1)a \in J$. Since $a' * a' = 1 \in J$ and J is an implicative filter of X, we have $na = a' * (n-1)a \in J$. By assumption, $a \in J$. □

Theorem 4.8. *Let J be an implicative filter of a transitive BE-algebra X with order reversing involution $'$. Then $J = \sqrt{J}$.*

Proof. By Theorem 4.3, $J \subseteq \sqrt{J}$. Let $x \in \sqrt{J}$. Then there exists $n \in \mathbb{N}$ such that $nx \in J$. Using Theorem 4.7, we have $x \in J$. Hence $\sqrt{J} \subseteq J$. Thus $J = \sqrt{J}$. □

Theorem 4.9. *Let X_1, X_2 be BE-algebras with order reversing involution $'$. Let $f : X_1 \to X_2$ be a homomorphism of BE-algebras. If $f(0) = 0$, then for any $n \in \mathbb{N}$, $f(nx) = nf(x)$ for any $x \in X_1$.*

Proof. We use the induction on n to prove the conclusion. If $n = 1$, the conclusion is trivial. Now suppose that $n > 1$ and the conclusion holds for n. Then, by Proposition 3.5, we obtain

$$\begin{aligned} f((n+1)x) =& f(x \oplus nx) = f(x' * nx) \\ =& f(x') * f(nx) = (f(x))' * nf(x) \\ =& f(x) \oplus nf(x) = (n+1)f(x), \end{aligned}$$

i.e., the conclusion holds for $n + 1$. □

Theorem 4.10. *Let $(X_1, *_1, {}'_1, 0_1, 1_1)$ and $(X_2, *_2, {}'_2, 0_2, 1_2)$ be BE-algebras with order reversing involutions ${}'_i$. Let $f : X_1 \to X_2$ be an epimorphism of BE-algebras. If J is a filter of X_1 such that $DKer\ f \subseteq J$, then $f(J)$ is a filter of X_2.*

Proof. Suppose that J is a filter of X_1 such that $DKer\ f \subseteq J$. Then $1_1 \in J$ and so $1_2 = f(x) *_2 f(x) = f(x *_1 x) = f(1_1) \in f(J)$.

Assume that $y *_2 z \in f(J)$ and $y \in f(J)$ for any $y, z \in X_2$. Then there exist $x, w \in J$ and $v \in X_1$ such that $f(x) = y *_2 z, f(w) = y$ and $f(v) = z$. Hence $f(x *_1 (w *_1 v)) = f(x) *_2 (f(w *_1 v)) = (y *_2 z) *_2 (f(w) *_2 f(v)) = (y *_2 z) *_2 (y *_2 z) = 1_2$ and so $x *_1 (w *_1 v) \in DKer\ f$. Using $DKer\ f \subseteq J$, we have $x *_1 (w *_1 v) \in J$. Since $x \in J$ and J is a filter of X, we get $w *_1 v \in J$. Using $w \in J$, we obtain $v \in J$. Therefore $z = f(v) \in f(J)$. Thus $f(J)$ is a filter of X_2. □

Proposition 4.11. *Let $(X_1, *_1, {}'_1, 0_1, 1_1)$ and $(X_2, *_2, {}'_2, 0_2, 1_2)$ be BE-algebras with order reversing involutions ${}'_i$. Let $f : X_1 \to X_2$ be an epimorphism of BE-algebras with $f(0) = 0$. If J is a filter of X_1 such that $DKer\ f \subseteq J$, then $f(\sqrt{J}) = \sqrt{f(J)}$.*

Proof. Let J be a filter of X_1 such that $DKer\ f \subseteq J$. By Theorem 4.10, $f(J)$ is a filter of X_2. If $y \in f(\sqrt{J})$, then there exists $x \in \sqrt{J}$ such that $f(x) = y$. Since $x \in \sqrt{J}$, there exists $n \in \mathbb{N}$ such that $nx \in J$. Using Theorem 4.9, we have $f(nx) = nf(x) = ny \in f(J)$. Hence $y \in \sqrt{f(J)}$. Therefore $f(\sqrt{J}) \subseteq \sqrt{f(J)}$.

Conversely, let $y \in \sqrt{f(J)}$. Then there exists $n \in \mathbb{N}$ such that $ny \in f(J)$. Since f is an epimorphism, there exists $x \in X_1$ such that $f(x) = y$. Hence $f(nx) = nf(x) = ny \in f(J)$ and so there exists $z \in J$ such that $f(z) = f(nx)$. Therefore $f(z *_1 nx) = f(z) *_2 f(nx) = f(z) *_2 f(z) = 1_2$ and so $z * nx \in DKer\ f \subseteq J$. Since $z \in J$ and J is a filter of X_1, we obtain $nx \in J$. Hence $x \in \sqrt{J}$ and so $y = f(x) \in f(\sqrt{J})$. Therefore $\sqrt{f(J)} \subseteq f(\sqrt{J})$. This completes the proof. □

Theorem 4.12. *Let $(X_1, *_1, {}'_1, 0_1, 1_1)$ and $(X_2, *_2, {}'_2, 0_2, 1_2)$ be BE-algebras with order reversing involutions ${}'_i$. Let $f : X_1 \to X_2$ be a homomorphism of BE-algebras. If J is a filter of X_2, then $f^{-1}(J)$ is a filter of X_1. Furthermore, if $f(0_1) = 0_2$, then $f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$.*

Proof. Let J be a filter of X_2. Let $x *_1 y, x \in f^{-1}(J)$. Then $f(x *_1 y) = f(x) *_2 f(y) \in J$ and $f(x) \in J$. Since J is a filter of X_2, we have $f(y) \in J$ and so $y \in f^{-1}(J)$. Since $f(1_1) = 1_2 \in J$, we obtain $1_1 \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is a filter of X_1.

We assume that $f(0_1) = 0_2$. Let $x \in f^{-1}(\sqrt{J})$. Then $f(x) \in \sqrt{J}$ and so there exists $n \in \mathbb{N}$ such that $nf(x) = f(nx) \in J$. Hence $nx \in f^{-1}(J)$. Therefore $x \in \sqrt{f^{-1}(J)}$. Thus $f^{-1}(\sqrt{J}) \subseteq \sqrt{f^{-1}(J)}$.

Conversely, let $x \in \sqrt{f^{-1}(J)}$. Then there exists $n \in \mathbb{N}$ such that $nx \in f^{-1}(J)$. Hence $f(nx) = nf(x) \in J$ and so $f(x) \in \sqrt{J}$. Therefore $x \in f^{-1}(\sqrt{J})$. Thus $\sqrt{f^{-1}(\sqrt{J})} \subseteq f^{-1}(\sqrt{J})$. □

5. The extended filter in BE-algebras

For any non-empty subset F of X and $x \in X$, we define

$$x^{-1} * F := \{y \in X | x \vee y \in F\}.$$

Note that if F is a filter of X, then $1 \in x^{-1} * F$.

Proposition 5.1. *Let X be a transitive commutative BE-algebra. If F is a filter of X, then $x^{-1} * F$ is a filter of X containing F.*

Proof. Let $y \in x^{-1} * F$ and $y * z \in x^{-1} * F$. Then $x \vee y \in F$ and $x \vee (y * z) \in F$. Now $(x \vee y) * (x \vee z) = ((y * x) * x)) * ((z * x) * x) \geq (z * x) * (y * x) \geq y * z$ and $(x \vee y) * (x \vee z) \geq x \vee z \geq x$. It follows form Theorem 2.6 that $x \vee (y * z) \leq (x \vee y) * (x \vee z)$ so that $(x \vee y) * (x \vee z) \in F$. Using the fact F is a filter of X and $x \vee y \in F$, we get $x \vee z \in F$, i.e., $z \in x^{-1} * F$. Thus $x^{-1} * F$ is a filter of X.

Let $y \in F$. Since $y \leq x \vee y$, it follows that $x \vee y \in F$, i.e., $y \in x^{-1} * F$. Hence $F \subseteq x^{-1} * F$. This completes the proof. □

Proposition 5.2. *Let F and G be filters of a BE-algebra X. Then the following hold: for any $x, y \in X$,*

(i) $x^{-1} * F = X$ *if and only if* $x \in F$*;*
(ii) $F \subseteq G$ *imply* $x^{-1} * F \subseteq y^{-1} * F$*;*
(iii) $x^{-1} * (F \cap G) = (x^{-1} * F) \cap (x^{-1} * G)$ *and* $x^{-1} * (F \cup G) = (x^{-1} * F) \cup (x^{-1} * G)$*.*

Proof. (i) Let $x \in F$. Since $x \leq (y * x) * x = x \vee y$, we have $x \vee y \in F$ for all $y \in X$, i.e., $y \in x^{-1} * F$. Thus $x^{-1} * F = X$. Conversely, assume that $x^{-1} * F = X$. Then $x \vee y \in F$ for all $y \in X$. In particular $x = x \vee x \in F$.
(ii) Assume that $F \subseteq G$. If $z \in x^{-1} * F$, then $x \vee z \in F \subseteq G$, i.e., $x \vee z \in G$. Hence $z \in x^{-1} * G$. Thus $x^{-1} * F \subseteq x^{-1} * G$.
(iii) For any $x, z \in X$ we have

$$\begin{aligned} z \in x^{-1} * (F \cap G) &\Leftrightarrow x \vee z \in F \cap G \\ &\Leftrightarrow x \vee z \in F \text{ and } x \vee z \in G \\ &\Leftrightarrow z \in x^{-1} * F \text{ and } z \in x^{-1} * G \\ &\Leftrightarrow z \in (x^{-1} * F) \cap (x^{-1} * G) \end{aligned}$$

and

$$\begin{aligned} z \in x^{-1} * (F \cup G) &\Leftrightarrow x \vee z \in F \cup G \\ &\Leftrightarrow x \vee z \in F \text{ or } x \vee z \in G \\ &\Leftrightarrow z \in x^{-1} * F \text{ or } z \in x^{-1} * G \\ &\Leftrightarrow z \in (x^{-1} * F) \cup (x^{-1} * G). \end{aligned}$$

This completes the proof. □

Definition 5.3. A proper filter of a BE-algebra X is said to be *prime* if for any $x, y \in X$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.

Proposition 5.4. *Let P and F be filters of X such that $F \subseteq P$. If P is prime, then $x^{-1} * F \subseteq P$ for all $x \in X \setminus P$.*

Proof. Let $z \in x^{-1} * F$ for all $x \in X \setminus P$. Then $x \vee z \in F \subseteq P$. Since P is prime and $x \notin P$, we have $z \in P$, i.e., $x^{-1} * F \subseteq P$. □

Proposition 5.5. *If P is a prime filter of X, then $X \setminus P$ is $\vee$-closed, i.e., $x \vee y \in X \setminus P$ whenever $x \in X \setminus P$ and $y \in X \setminus P$.*

Proof. Straightforward. □

Theorem 5.6. *Let X be a transitive commutative BE-algebra. A filter P of X is prime if and only if $x^{-1} * P = P$ for all $X \setminus P$.*

Proof. Suppose that P is a prime filter of X. Let $x \in X \setminus P$. The conclusion $P \subseteq x^{-1} * P$ follows from Proposition 5.1. If $y \in x^{-1} * P$, then $x \vee y \in P$. Since P is a prime filter of X, we have $y \in P$. Hence $x^{-1} * P \subseteq P$. Thus $x^{-1} * P = P$.

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Conversely, assume that $x^{-1} * P = P$ for all $x \in X \setminus P$. Let $y \vee z \in P$ and $z \notin P$. By the hypothesis, $z^{-1} * P = P$. Hence $y \in z^{-1} * P = P$. Thus P is prime. □

Definition 5.7. Let P be a filter of a BE-algebra X. P is called a *primary filter* of X if $x \vee y \in P$ and $x \notin P \Rightarrow \exists n \in \mathbb{N}$ such that $ny \in P$, for any $x, y \in X$.

Theorem 5.8. *Let X be a transitive commutative BE-algebra and let J be a filter of X. If $x^{-1} * J = J, \forall x \notin J$, then J is a primary filter of X.*

Proof. By Definition 5.7, a prime filter is a primary filter. Suppose that $x^{-1} * J = J, \forall x \notin J$. Using Theorem 5.6, J is a prime filter of X. Hence J is a primary filter of X. □

Theorem 5.9. *Let X be a transitive commutative BE-algebra. If J is a filter of X, then $J = \cap_{x \in X} x^{-1} * J$.*

Proof. By Proposition 5.1, $J \subseteq x^{-1} * J$ for all $x \in X$. Hence $J \subseteq \cap_{x \in X} x^{-1} * J$.

Assume that $y \in \cap_{x \in X} x^{-1} * J$. Then $y \in y^{-1} * J$ and so $y \in J$. Hence $\cap_{x \in X} x^{-1} * J \subseteq J$. This completes the proof. □

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SYMMETRY p-ADIC INVARIANT INTEGRAL ON $\mathbb{Z}_p$ FOR q-EULER POLYNOMIALS

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Abstract. In this paper, we investigate several further interesting properties of symmetry for the p-adic fermionic integral on $\mathbb{Z}_p$. By using this symmetry of fermionic p-adic integral on $\mathbb{Z}_p$, we give some relations of symmetry between the power sum q-polynomials and q-Euler polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$; if $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$. We use the notation for q-number as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1} [x]_q = x$. As is well known, the Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1-13]).

When $x=0$, $E_n = E_n(0)$ are called the Euler numbers.

From (1), we note that

$$(E+1)^n + E_n = 2\delta_{0,n}, \quad (n \geq 0),$$

and

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l,$$

(see [1, 4, 7, 9]).

Recently, Kim considered a q-extension of Euler polynomials (called q-Euler polynomials) as follows :

$$2\sum_{m=0}^{\infty} (-1)^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \tag{2}$$

(see [6]).

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From (2), we can derive

$$E_{n,q}(x) = \frac{2}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^l} q^{lx} = 2\sum_{m=0}^{\infty} (-1)^m [m+x]_q^n . \tag{3}$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called q-Euler numbers.
By (2), we easily get

$$(qE_q + 1)^n + E_{n,q} = 2\delta_{n,0},$$

with the usual convention about replacing E_q^n by $E_{n,q}$.
Let $\mathscr{C}(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$.
For $f \in \mathscr{C}(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows :

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x , \tag{4}$$

(see [4, 5]).
From (4), we note that

$$\int_{\mathbb{Z}_p} f(x+n)\, d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{5}$$

whre $n \in \mathbb{N}$ (see [1, 3, 4, 5]).
By (5), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{6}$$

From (6), we have

$$\int_{\mathbb{Z}_p} (x+y)^n\, d\mu_{-1}(y) = E_n(x), \quad (n \geq 0). \tag{7}$$

In [4], some relations of symmetry between the power sum polynomials and Euler polynomials were given by (7) and finding a q-extension of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for q-Euler polynomials was remained as an open question.

In this paper, we investigate several further interesting properties of symmetry for the p-adic fermionic p-adic integral on $\mathbb{Z}_p$, and give some relations of symmetry between the power sum q-polynomials and q-Euler polynomials.

Recently, several authors have studied the identities of symmetry and q-extensions of Euler polynomials (see [1-13]).

2. Identities of symmetry of the q-Euler polynomials

From (4) and (5), we can derive the following equation :

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = 2\sum_{m=0}^{\infty} (-1)^m e^{[m+x]_q t}. \tag{8}$$

By (2) and (8), we get

$$\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_{-1}(y)\,\frac{t^n}{n!}=\sum_{n=0}^{\infty}E_{n,q}(x)\,\frac{t^n}{n!}. \tag{9}$$

By comparing coefficients on the both sides of (9), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_{-1}(y)&=E_{n,q}(x)\\&=2\sum_{m=0}^{\infty}(-1)^m[m+x]_q^n\\&=\frac{2}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\frac{1}{1+q^l}.\end{aligned} \tag{10}$$

Let w_1, $w_2\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$ and $w_2\equiv 1\pmod 2$.
Then we observe that

$$\begin{aligned}&\int_{\mathbb{Z}_p}e^{[w_1]_q\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}t}d\mu_{-1}(y)\\&=\int_{\mathbb{Z}_p}e^{[w_1w_2x+w_2j+w_1y]_qt}d\mu_{-1}(y)\\&=\lim_{N\to\infty}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1y]_qt}(-1)^y\\&=\lim_{N\to\infty}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}e^{[w_1w_2x+w_2j+w_1(i+w_2y)]_qt}(-1)^{i+w_2y}.\end{aligned} \tag{11}$$

From (11), we note that

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\int_{\mathbb{Z}_p}e^{[w_1]_q\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}t}d\mu_{-1}(y)\\&=\lim_{N\to\infty}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}(-1)^{i+j+y}e^{[w_1w_2(x+y)+w_2j+w_1i]_qt}.\end{aligned} \tag{12}$$

By the same method as (12), we get

$$\begin{aligned}&\sum_{j=0}^{w_2-1}(-1)^j\int_{\mathbb{Z}_p}e^{[w_2]_q\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}t}d\mu_{-1}(y)\\&=\lim_{N\to\infty}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1}(-1)^{i+j+y}e^{[w_1w_2(x+y)+w_1j+w_2i]_qt}.\end{aligned} \tag{13}$$

Therefore, by (12) and (13), we obtain the following theorem.

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Theorem 1. *For* w_1, $w_2 \in \mathbb{N}$ *with* $w_1 \equiv 1 \pmod 2$ *and* $w_2 \equiv 1 \pmod 2$, *we have*

$$\sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} e^{[w_1]_q \left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}} t} d\mu_{-1}(y)$$
$$= \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} e^{[w_2]_q \left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}} t} d\mu_{-1}(y).$$

Corollary 2. *For* $n \geq 0$, *we have*

$$[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} \left[w_2x + \frac{w_2}{w_1}j + y\right]_{q^{w_1}}^n d\mu_{-1}(y)$$
$$= [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} \left[w_1x + \frac{w_1}{w_2}j + y\right]_{q^{w_2}}^n d\mu_{-1}(y).$$

By (9) and Corollary 2, we obtain the following theorem.

Theorem 3. *For* $n \geq 0$ *and* w_1, $w_2 \in \mathbb{N}$ *with* $w_1 \equiv 1 \pmod 2$ *and* $w_2 \equiv 1 \pmod 2$, *we have*

$$[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j E_{n,q^{w_1}}\left(w_2x + \frac{w_2}{w_1}j\right)$$
$$= [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j E_{n,q^{w_2}}\left(w_1x + \frac{w_1}{w_2}j\right).$$

From (10), we note that

$$\text{(14)} \qquad \int_{\mathbb{Z}_p} \left[w_2x + \frac{w_2}{w_1}j + y\right]_{q^{w_1}}^n d\mu_{-1}(y)$$
$$= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j} \int_{\mathbb{Z}_p} [w_2x + y]_{q^{w_1}}^{n-i} d\mu_{-1}(y)$$
$$= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j} E_{n-i,q^{w_1}}(w_2x).$$

Thus, by (14), we get

$$\text{(15)} \qquad [w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \int_{\mathbb{Z}_p} \left[w_2x + \frac{w_2}{w_1}j + y\right]_{q^{w_1}}^n d\mu_{-1}(y)$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i \sum_{j=0}^{w_1-1} (-1)^j [j]_{q^{w_2}}^i q^{w_2(n-i)j} E_{n-i,q^{w_1}}(w_2x)$$
$$= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i S_{n,i,q^{w_2}}(w_1) E_{n-i,q^{w_1}}(w_2x),$$

where $S_{n,i,q}(w_1) = \sum_{j=0}^{w_1-1} (-1)^j q^{(n-i)j} [j]_q^i$.

By the same method as (15), we get

$$\begin{aligned} &[w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \int_{\mathbb{Z}_p} \left[w_1 x + \frac{w_1}{w_2} j + y\right]_{q^{w_2}}^n d\mu_{-1}(y) \\ &= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} S_{n,i,q^{w_1}}(w_2) E_{n-i,q^{w_2}}(w_1 x). \end{aligned} \tag{16}$$

Therefore, by Corollary 2, (15) and (16), we obtain the following theorem.

Theorem 4. *For $n \geq 0$ and w_1, $w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$ and $w_2 \equiv 1 \pmod 2$, we have*

$$\begin{aligned} &\sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^i E_{n-i,q^{w_1}}(w_2 x) S_{n,i,q^{w_2}}(w_1) \\ &= \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_2}}(w_1 x) S_{n,i,q^{w_1}}(w_2), \end{aligned}$$

where $S_{n,i,q}(w) = \sum_{j=0}^{w-1} (-1)^j q^{(n-i)j} [j]_q^i$.

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Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials

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Abstract

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In this paper, we consider Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials $S_n^{(r,k)}(x|a_1,\dots,a_r)$ whose generating function is given by

$$\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^\infty S_n^{(r,k)}(x|a_1,\dots,a_r)\frac{t^n}{n!}\,, \tag{1}$$

where $r\in\mathbb{Z}_{>0}$, $k\in\mathbb{Z}$, $a_1,\dots,a_r\neq 0$, and

$$\mathrm{Li}_k(x)=\sum_{m=1}^\infty\frac{x^m}{m^k}$$

is the kth polylogarithm function. $S_n^{(r,k)}(x|a_1,\dots,a_r)$ will be called Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials. When $S_n^{(r,k)}(a_1,\dots,a_r) = S_n^{(r,k)}(0|a_1,\dots,a_r)$ will be called Barnes' multiple Bernoulli and poly-Bernoulli mixed-type numbers.

Recall that, for every integer k, the poly-Bernoulli polynomials $B_n^{(k)}(x)$ are defined by the generating function as

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{xt}=\sum_{n=0}^\infty B_n^{(k)}(x)\frac{t^n}{n!} \tag{2}$$

([14], *Cf.*[4]). Also, recall that the Barnes' multiple Bernoulli polynomials $B_n(x|a_1,\dots,a_r)$ are defined by the generating function as

$$\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}e^{xt}=\sum_{n=0}^\infty B_n(x|a_1,\dots,a_r)\frac{t^n}{n!}\,, \tag{3}$$

where $a_1,\dots,a_r\neq 0$ (see [1-14])).

In this paper, we consider Barnes' multiple Bernoulli and poly-Bernoulli mixed-type polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^\infty\frac{a_k}{k!}t^k\middle|a_k\in\mathbb{C}\right\}\,. \tag{4}$$

2

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\,\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{5}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{6}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{7}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{8}$$

([12, Theorem 2.2.5]). Thus, by (8), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{9}$$

Sheffer sequences are characterized in the generating function ([12, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([12, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{10}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{11}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \tag{12}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([12, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0)\,.$$

Then we have ([12, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle\,. \tag{13}$$

3 Main results

We now note that $B_n^{(k)}(x)$, $B_n(x|a_1, \ldots, a_r)$ and $S_n^{(r,k)}(x|a_1, \ldots, a_r)$ are the Appell sequences for

$$g_k(t) = \frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, \quad g_r(t) = \frac{\prod_{j=1}^r (e^{a_j t} - 1)}{t^r}, \quad g_{r,k}(t) = \frac{\prod_{j=1}^r (e^{a_j t} - 1)}{t^r} \frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}\,.$$

So,

$$B_n^{(k)}(x) \sim \left(\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t\right)\,, \tag{14}$$

$$B_n(x|a_1, \ldots, a_r) \sim \left(\frac{\prod_{j=1}^r (e^{a_j t} - 1)}{t^r}, t\right)\,, \tag{15}$$

$$S_n^{(r,k)}(x|a_1, \ldots, a_r) \sim \left(\frac{\prod_{j=1}^r (e^{a_j t} - 1)}{t^r} \frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t\right)\,. \tag{16}$$

In particular, we have

$$tB_n^{(k)}(x) = \frac{d}{dx}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x)\,, \tag{17}$$

$$\begin{aligned} tB_n(x|a_1,\dots,a_r) &= \frac{d}{dx}B_n(x|a_1,\dots,a_r) \\ &= nB_{n-1}(x|a_1,\dots,a_r)\,, \end{aligned} \tag{18}$$

$$\begin{aligned} tS_n^{(r,k)}(x|a_1,\dots,a_r) &= \frac{d}{dx}S_n^{(r,k)}(x|a_1,\dots,a_r) \\ &= nS_{n-1}^{(r,k)}(x|a_1,\dots,a_r)\,. \end{aligned} \tag{19}$$

Notice that

$$\frac{d}{dx}\mathrm{Li}_k(x) = \frac{1}{x}\mathrm{Li}_{k-1}(x)\,.$$

3.1 Explicit expressions

Write $B_n(a_1,\dots,a_r) := B_n(0|a_1,\dots,a_r)$ and $S_n^{(r,k)}(a_1,\dots,a_r) := S_n^{(r,k)}(0|a_1,\dots,a_r)$. Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \ge 1)$ with $(n)_0 = 1$.

Theorem 1

$$S_n^{(r,k)}(x|a_1,\dots,a_r) = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}(a_1,\dots,a_r)B_l^{(k)}(x)\,, \tag{20}$$

$$= \sum_{l=0}^{n}\binom{n}{l}B_{n-l}^{(k)}B_l(x|a_1,\dots,a_r)\,, \tag{21}$$

$$= \sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{n}{l}\frac{1}{(m+1)^k}B_{n-l}(a_1,\dots,a_r)(x-j)^l\,, \tag{22}$$

$$\begin{aligned} &= \sum_{l=0}^{n}\left(\sum_{j=l}^{n}\sum_{m=0}^{n-j}(-1)^{n-m-j}\binom{n}{j}\binom{j}{l}\right. \\ &\qquad \left.\times\frac{m!}{(m+1)^k}S_2(n-j,m)B_{j-l}(a_1,\dots,a_r)\right)x^l\,, \end{aligned} \tag{23}$$

$$= \sum_{j=0}^{n}\binom{n}{j}S_{n-j}^{(r,k)}(a_1,\dots,a_r)x^j\,. \tag{24}$$

Proof. By (1), (2) and (3), we have

$$
\begin{aligned}
S_n^{(r,k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{i=0}^{\infty} S_i^{(r,k)}(y|a_1,\dots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \Big| x^n \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)} \Big| \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} x^n \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)} \Big| \sum_{l=0}^{\infty} B_l^{(k)}(y)\frac{t^l}{l!} x^n \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)} \Big| \sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(y) x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(y) \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(y) \left\langle \sum_{i=0}^{\infty} B_i(a_1,\dots,a_r)\frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(y) B_{n-l}(a_1,\dots,a_r)\,.
\end{aligned}
$$

So, we get (20).

We also have

$$\begin{aligned}
S_n^{(r,k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{i=0}^{\infty} S_i^{(r,k)}(y|a_1,\dots,a_r)\frac{t^i}{i!}\Big| x^n \right\rangle \\
&= \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big| \frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)} e^{yt} x^n \right\rangle \\
&= \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big| \sum_{l=0}^{\infty} B_l(y|a_1,\dots,a_r)\frac{t^l}{l!} x^n \right\rangle \\
&= \left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big| \sum_{l=0}^{n} B_l(y|a_1,\dots,a_r)\binom{n}{l} x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} B_l(y|a_1,\dots,a_r)\left\langle \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} B_l(y|a_1,\dots,a_r)\left\langle \sum_{i=0}^{\infty} B_i^{(k)}\frac{t^i}{i!}\Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l} B_l(y|a_1,\dots,a_r) B_{n-l}^{(k)} .
\end{aligned}$$

Thus, we get (21).
In [7] we obtained that

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(x-j)^n .$$

So,

$$\begin{aligned}
S_n^{(r,k)}(x|a_1,\dots,a_r) &= \frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n \\
&= \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)}(x-j)^n \\
&= \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{l=0}^{n}\binom{n}{l} B_{n-l}(a_1,\dots,a_r)(x-j)^l \\
&= \sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{n}{l}\frac{1}{(m+1)^k} B_{n-l}(a_1,\dots,a_r)(x-j)^l ,
\end{aligned}$$

which is the identity (22).
In [7] we obtained that

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j} m! S_2(n-j,m)\right) x^j ,$$

where $S_2(l,m)$ are the Stirling numbers of the second kind, defined by

$$(e^t-1)^m = m!\sum_{l=m}^{\infty} S_2(l,m)\frac{t^l}{l!}\,.$$

Thus,

$$\begin{aligned}
S_n^{(r,k)}(x|a_1,\dots,a_r) &= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j}m!S_2(n-j,m)\right)\frac{t^r}{\prod_{i=1}^{r}(e^{a_it}-1)}x^j\\
&= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j}m!S_2(n-j,m)\right)B_j(x|a_1,\dots,a_r)\\
&= \sum_{j=0}^{n}\left(\sum_{m=0}^{n-j}\frac{(-1)^{n-m-j}}{(m+1)^k}\binom{n}{j}m!S_2(n-j,m)\right)\sum_{l=0}^{j}\binom{j}{l}B_{j-l}(a_1,\dots,a_r)x^l\\
&= \sum_{l=0}^{n}\left(\sum_{j=l}^{n}\sum_{m=0}^{n-j}(-1)^{n-m-j}\binom{n}{j}\binom{j}{l}\frac{m!}{(m+1)^k}S_2(n-j,m)B_{j-l}(a_1,\dots,a_r)\right)x^l\,,
\end{aligned}$$

which is the identity (23).
By (11) with (16), we have

$$\begin{aligned}
\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle &= \left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^j\Big|x^n\right\rangle\\
&= (n)_j\left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big|x^{n-j}\right\rangle\\
&= (n)_j\left\langle \sum_{i=0}^{\infty}S_i^{(r,k)}(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-j}\right\rangle\\
&= (n)_jS_{n-j}^{(r,k)}(a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get (24). ∎

3.2 Sheffer identity

Theorem 2

$$S_n^{(r,k)}(x+y|a_1,\dots,a_r) = \sum_{j=0}^{n}\binom{n}{j}S_j^{(r,k)}(x|a_1,\dots,a_r)y^{n-j}\,. \tag{25}$$

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Proof. By (16) with

$$\begin{aligned} p_n(x) &= \frac{\prod_{j=1}^{r}(e^{a_j t}-1)}{t^r}\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}S_n^{(r,k)}(x|a_1,\dots,a_r) \\ &= x^n \sim (1,t)\,, \end{aligned}$$

using (12), we have (25). ∎

3.3 Recurrence

Theorem 3

$$\begin{aligned} S_{n+1}^{(r,k)}&(x|a_1,\dots,a_r) = xS_n^{(r,k)}(x|a_1,\dots,a_r) \\ &- \frac{1}{n+1}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n+1}{l}(-a_j)^{n+1-l}B_{n+1-l}S_l^{(r,k)}(x|a_1,\dots,a_r) \\ &- \frac{1}{n+1}\left(S_{n+1}^{(r+1,k)}(x|a_1,\dots,a_r,1) - S_{n+1}^{(r+1,k-1)}(x|a_1,\dots,a_r,1)\right)\,, \end{aligned} \tag{26}$$

where B_n is the nth ordinary Bernoulli number.

Proof. By applying

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x)$$

([12, Corollary 3.7.2]) with (16), we get

$$S_{n+1}^{(r,k)}(x|a_1,\dots,a_r) = \left(x - \frac{g'_{r,k}(t)}{g_{r,k}(t)}\right)S_n^{(r,k)}(x|a_1,\dots,a_r)\,.$$

Now,

$$\begin{aligned} &\frac{g'_{r,k}(t)}{g_{r,k}(t)} = (\ln g_{r,k}(t))' \\ &= \left(\sum_{j=1}^{r}\ln(e^{a_j t}-1) - r\ln t + \ln(1-e^{-t}) - \ln \mathrm{Li}_k(1-e^{-t})\right)' \\ &= \sum_{j=1}^{r}\frac{a_j e^{a_j t}}{e^{a_j t}-1} - \frac{r}{t} + \frac{e^{-t}}{1-e^{-t}}\left(1 - \frac{\mathrm{Li}_{k-1}(1-e^{-t})}{\mathrm{Li}_k(1-e^{-t})}\right) \\ &= \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(a_j t e^{a_j t} - e^{a_j t}+1)}{t\prod_{j=1}^{r}(e^{a_j t}-1)} + \frac{t}{e^t-1}\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t\mathrm{Li}_k(1-e^{-t})}\,. \end{aligned}$$

Since

$$\begin{aligned} \frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(a_j t e^{a_j t} - e^{a_j t}+1)}{\prod_{j=1}^{r}(e^{a_j t}-1)} &= \frac{\frac{1}{2}\left(\sum_{j=1}^{r} a_1\cdots a_{j-1}a_j^2 a_{j+1}\cdots a_r\right)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots} \\ &= \frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t + \cdots \end{aligned}$$

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is a series with order≥ 1 and

$$\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}}=\left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)t+\cdots$$

is a delta series, we have

$$\begin{aligned}
&S_{n+1}^{(r,k)}(x|a_1,\dots,a_r)=xS_n^{(r,k)}(x|a_1,\dots,a_r)-\frac{g'_{r,k}(t)}{g_{r,k}(t)}S_n^{(r,k)}(x|a_1,\dots,a_r)\\
&=xS_n^{(r,k)}(x|a_1,\dots,a_r)-\frac{g'_{r,k}(t)}{g_{r,k}(t)}\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}x^n\\
&=xS_n^{(r,k)}(x|a_1,\dots,a_r)\\
&\quad-\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\frac{\sum_{j=1}^r\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^r(e^{a_jt}-1)}x^n\\
&\quad-\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{t}{e^t-1}\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})}x^n\,.
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{\sum_{j=1}^r\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{t\prod_{j=1}^r(e^{a_jt}-1)}x^n\\
&=\frac{\sum_{j=1}^r\prod_{i\neq j}(e^{a_it}-1)(a_jte^{a_jt}-e^{a_jt}+1)}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{x^{n+1}}{n+1}\\
&=\frac{1}{n+1}\sum_{j=1}^r\frac{a_jte^{a_jt}-e^{a_jt}+1}{e^{a_jt}-1}x^{n+1}\\
&=\frac{1}{n+1}\sum_{j=1}^r\left(\frac{a_jte^{a_jt}}{e^{a_jt}-1}-1\right)x^{n+1}\\
&=\frac{1}{n+1}\sum_{j=1}^r\left(\sum_{l=0}^\infty\frac{(-1)^lB_la_j^l}{l!}t^l-1\right)x^{n+1}\\
&=\frac{1}{n+1}\sum_{j=1}^r\left(\sum_{l=0}^{n+1}\binom{n+1}{l}(-a_j)^lB_lx^{n+1-l}-x^{n+1}\right)\\
&=\frac{1}{n+1}\sum_{j=1}^r\sum_{l=1}^{n+1}\binom{n+1}{l}(-a_j)^lB_lx^{n+1-l}\\
&=\frac{1}{n+1}\sum_{j=1}^r\sum_{l=0}^{n}\binom{n+1}{l}(-a_j)^{n+1-l}B_{n+1-l}x^l\,.
\end{aligned}$$

Also,

$$\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})}x^n=\frac{1}{n+1}\frac{\mathrm{Li}_k(1-e^{-t})-\mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}}x^{n+1}\,.$$

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Thus, we get the identity (26). ∎

3.4 One more relation

Theorem 4

$$\begin{aligned} S_n^{(r,k)}(x|a_1,\dots,a_r) &= xS_{n-1}^{(r,k)}(x|a_1,\dots,a_r) \\ &+ \sum_{m=1}^{n} \frac{(-1)^{m-1}\binom{n-1}{m-1}B_m}{m} \sum_{j=1}^{r} a_j^m S_{n-m}^{(r,k)}(x|a_1,\dots,a_r) \\ &+ \frac{1}{n}S_n^{(r+1,k-1)}(x|a_1,\dots,a_r,1) - \frac{1}{n}S_n^{(r+1,k)}(x|a_1,\dots,a_r,1)\,. \end{aligned} \tag{27}$$

Proof. For $n \geq 1$ we have

$$\begin{aligned} S_n^{(r,k)}(y|a_1,\dots,a_r) &= \left\langle \sum_{l=0}^{\infty} S_l^{(r,k)}(y|a_1,\dots,a_r)\frac{t^l}{l!} \middle| x^n \right\rangle \\ &= \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^n \right\rangle \\ &= \left\langle \partial_t \left(\frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \left(\partial_t \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} (\partial_t e^{yt}) \middle| x^{n-1} \right\rangle \\ &= yS_{n-1}^{(r,k)}(y|a_1,\dots,a_r) \\ &\quad + \left\langle \left(\partial_t \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \right) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle . \end{aligned}$$

Observe that

$$\begin{aligned}
\partial_t\left(\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\right) &= \frac{rt^{r-1}-t^r\sum_{j=1}^r\frac{a_je^{a_jt}}{e^{a_jt}-1}}{\prod_{j=1}^r(e^{a_jt}-1)}\\
&= \frac{t^{r-1}}{\prod_{j=1}^r(e^{a_jt}-1)}\left(r-\sum_{j=1}^r\frac{a_jte^{a_jt}}{e^{a_jt}-1}\right)\\
&= \frac{t^{r-1}}{\prod_{j=1}^r(e^{a_jt}-1)}\left(r-\sum_{j=1}^r\frac{-a_jt}{e^{-a_jt}-1}\right)\\
&= \frac{t^{r-1}}{\prod_{j=1}^r(e^{a_jt}-1)}\left(r-\sum_{j=1}^r\sum_{m=0}^\infty\frac{(-a_j)^mB_mt^m}{m!}\right)\\
&= \frac{t^{r-1}}{\prod_{j=1}^r(e^{a_jt}-1)}\left(r-\sum_{m=0}^\infty\left(\sum_{j=1}^r(-a_j)^m\right)\frac{B_mt^m}{m!}\right)\\
&= \frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\sum_{m=1}^\infty\left(\sum_{j=1}^r a_j^m\right)\frac{(-1)^{m-1}B_m}{m!}t^{m-1}\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\langle\left(\partial_t\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\right)\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^{n-1}\right\rangle\\
&= \left\langle\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|\sum_{m=1}^n\left(\sum_{j=1}^r a_j^m\right)\frac{(-1)^{m-1}B_m}{m!}t^{m-1}x^{n-1}\right\rangle\\
&= \sum_{m=1}^n\frac{(-1)^{m-1}\binom{n-1}{m-1}B_m}{m}\sum_{j=1}^r a_j^m\left\langle\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^{n-m}\right\rangle\\
&= \sum_{m=1}^n\frac{(-1)^{m-1}\binom{n-1}{m-1}B_m}{m}S_{n-m}^{(r,k)}(y|a_1,\dots,a_r)\sum_{j=1}^r a_j^m\,.
\end{aligned}$$

Since

$$\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\left(\frac{1}{2^{k-1}}-\frac{1}{2^k}\right)t+\cdots$$

is a delta series, we have

$$\begin{aligned}
&\left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right) e^{yt} \middle| x^{n-1} \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{e^{-t}\big(\mathrm{Li}_{k-1}(1-e^{-t}) - \mathrm{Li}_k(1-e^{-t})\big)}{(1-e^{-t})^2} e^{yt} \middle| x^{n-1} \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{t}{e^t-1} \frac{\mathrm{Li}_{k-1}(1-e^{-t}) - \mathrm{Li}_k(1-e^{-t})}{t(1-e^{-t})} e^{yt} \middle| x^{n-1} \right\rangle \\
&= \left\langle \frac{t^{r+1}}{\prod_{j=1}^r (e^{a_j t}-1)(e^t-1)} \frac{\mathrm{Li}_{k-1}(1-e^{-t}) - \mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| \frac{x^n}{n} \right\rangle \\
&= \frac{1}{n} \left\langle \frac{t^{r+1}}{\prod_{j=1}^r (e^{a_j t}-1)(e^t-1)} \frac{\mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^n \right\rangle \\
&\quad - \frac{1}{n} \left\langle \frac{t^{r+1}}{\prod_{j=1}^r (e^{a_j t}-1)(e^t-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^n \right\rangle \\
&= \frac{1}{n} S_n^{(r+1,k-1)}(y|a_1,\dots,a_r,1) - \frac{1}{n} S_n^{(r+1,k)}(y|a_1,\dots,a_r,1)\,.
\end{aligned}$$

Therefore, we obtain the desired result. ∎

Remark. After simple modification, Theorem 4 becomes

$$\begin{aligned}
S_{n+1}^{(r,k)}(x|a_1,\dots,a_r) &= x S_n^{(r,k)}(x|a_1,\dots,a_r) \\
&+ \sum_{l=1}^{n+1} \frac{(-1)^{l-1}\binom{n}{l-1} B_l}{l} \sum_{j=1}^r a_j^l S_{n+1-l}^{(r,k)}(x|a_1,\dots,a_r) \\
&+ \frac{1}{n+1} S_{n+1}^{(r+1,k-1)}(x|a_1,\dots,a_r,1) - \frac{1}{n+1} S_{n+1}^{(r+1,k)}(x|a_1,\dots,a_r,1)\,.
\end{aligned}$$

which is the same as the above recurrence formula (26) upon replacing n by $n-1$.

3.5 Relations with poly-Bernoulli numbers and Barnes' multiple Bernoulli numbers

Theorem 5

$$\begin{aligned}
&\sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} S_m^{(r,k)}(a_1,\dots,a_r) \\
&\qquad = \sum_{l=0}^n \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_m^{(k-1)} B_{n-l}(a_1,\dots,a_r)\,. \quad (28)
\end{aligned}$$

Proof. We shall compute

$$\left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \mathrm{Li}_k(1-e^{-t}) \middle| x^{n+1} \right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
&\left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \mathrm{Li}_k(1-e^{-t}) \middle| x^{n+1} \right\rangle = \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| (1-e^{-t}) x^{n+1} \right\rangle \\
&= \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n+1} - (x-1)^{n+1} \right\rangle \\
&= \sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} \left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^m \right\rangle \\
&= \sum_{m=0}^n \binom{n+1}{m} (-1)^{n-m} S_m^{(r,k)}(a_1, \dots, a_r) \,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} \mathrm{Li}_k(1-e^{-t}) \middle| x^{n+1} \right\rangle = \left\langle \mathrm{Li}_k(1-e^{-t}) \middle| \frac{t^r}{\prod_{j=1}^r (e^{a_j t}-1)} x^{n+1} \right\rangle \\
&= \left\langle \mathrm{Li}_k(1-e^{-t}) \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \left\langle \int_0^t \left(\mathrm{Li}_k(1-e^{-s}) \right)' ds \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \left\langle \int_0^t e^{-s} \frac{\mathrm{Li}_{k-1}(1-e^{-s})}{1-e^{-s}} ds \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \left\langle \int_0^t \left(\sum_{j=0}^\infty \frac{(-s)^j}{j!} \right) \left(\sum_{m=0}^\infty \frac{B_m^{(k-1)}}{m!} s^m \right) ds \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \left\langle \sum_{l=0}^\infty \left(\sum_{m=0}^l (-1)^{l-m} \binom{l}{m} B_m^{(k-1)} \right) \frac{1}{l!} \int_0^t s^l ds \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \sum_{l=0}^n \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} \frac{B_m^{(k-1)}}{(l+1)!} \left\langle t^{l+1} \middle| B_{n+1}(x|a_1, \dots, a_r) \right\rangle \\
&= \sum_{l=0}^n \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} \frac{B_m^{(k-1)}}{(l+1)!} (n+1)_{l+1} B_{n-l}(a_1, \dots, a_r) \\
&= \sum_{l=0}^n \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_m^{(k-1)} B_{n-l}(a_1, \dots, a_r) \,.
\end{aligned}$$

Here, $B_{n-l}(a_1, \dots, a_r) = B_{n-l}(0|a_1, \dots, a_r)$. Thus, we get (28). ∎

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3.6 Relations with the Stirling numbers of the second kind and the falling factorials

Theorem 6

$$S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^{n}\left(\sum_{l=m}^{n}S_2(l,m)\binom{n}{l}S_{n-l}^{(r,k)}(a_1,\dots,a_r)\right)(x)_m\,. \tag{29}$$

Proof. For (16) and $(x)_n\sim(1,e^t-1)$, assume that $S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^{n}C_{n,m}(x)_m$. By (13), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{1}{\frac{\prod_{j=1}^{r}(e^{a_jt}-1)}{t^r}\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}}(e^t-1)^m\middle|x^n\right\rangle\\&=\frac{1}{m!}\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle|(e^t-1)^mx^n\right\rangle\\&=\frac{1}{m!}\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle|m!\sum_{l=m}^{n}S_2(l,m)\frac{t^l}{l!}x^n\right\rangle\\&=\sum_{l=m}^{n}S_2(l,m)\binom{n}{l}\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle|x^{n-l}\right\rangle\\&=\sum_{l=m}^{n}S_2(l,m)\binom{n}{l}S_{n-l}^{(r,k)}(a_1,\dots,a_r)\,.\end{aligned}$$

Thus, we get the identity (29). ∎

3.7 Relations with the Stirling numbers of the second kind and the rising factorials

Theorem 7

$$S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^{n}\left(\sum_{l=m}^{n}S_2(l,m)\binom{n}{l}S_{n-l}^{(r,k)}(-m|a_1,\dots,a_r)\right)(x)^{(m)}\,. \tag{30}$$

Proof. For (16) and $(x)^{(n)}=x(x+1)\cdots(x+n-1)\sim(1,1-e^{-t})$, assume that $S_n^{(r,k)}(x|a_1,\dots,a_r)=$

$\sum_{m=0}^{n} C_{n,m}(x)^{(m)}$. By (13), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{1}{\frac{\prod_{j=1}^{r}(e^{a_j t}-1)}{t^r}\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}}(1-e^{-t})^m \middle| x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-mt} \middle| (e^t-1)^m x^n \right\rangle \\
&= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle e^{-mt} \middle| \frac{t^r}{\prod_{j=1}^{r}(e^{a_j t}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}x^{n-l} \right\rangle \\
&= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l}\left\langle e^{-mt} \middle| S_{n-l}^{(r,k)}(x|a_1,\dots,a_r) \right\rangle \\
&= \sum_{l=m}^{n} S_2(l,m)\binom{n}{l} S_{n-l}^{(r,k)}(-m|a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get the identity (30). ∎

3.8 Relations with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see e.g. [6]).

Theorem 8

$$S_n^{(r,k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left(\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}S_{n-m}^{(r,k)}(j|a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda)\,. \tag{31}$$

Proof. For (16) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right),$$

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assume that $S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^n C_{n,m}H_m^{(s)}(x|\lambda)$. By (13), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^s\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^m\Big|x^n\right\rangle\\
&=\frac{1}{m!(1-\lambda)^s}\left\langle(e^t-\lambda)^s\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big|t^mx^n\right\rangle\\
&=\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^s\binom{s}{j}(-\lambda)^{s-j}\left\langle e^{jt}\Big|\frac{t^r}{\prod_{j=1}^r(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}x^{n-m}\right\rangle\\
&=\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^s\binom{s}{j}(-\lambda)^{s-j}S_{n-m}^{(r,k)}(j|a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get the identity (31). ∎

3.9 Relations with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [12, Section 2.2]).

Theorem 9

$$S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^n\left(\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)S_{n-m-l}^{(r,k)}(a_1,\dots,a_r)\right)\mathfrak{B}_m^{(s)}(x)\,. \tag{32}$$

Proof. For (16) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right)\,,$$

assume that $S_n^{(r,k)}(x|a_1,\dots,a_r)=\sum_{m=0}^{n}C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (13), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\left(\frac{e^t-1}{t}\right)^s\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}t^m\Big|x^n\right\rangle\\
&=\binom{n}{m}\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big|\left(\frac{e^t-1}{t}\right)^s x^{n-m}\right\rangle\\
&=\binom{n}{m}\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big|\sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)t^lx^{n-m}\right\rangle\\
&=\binom{n}{m}\sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)(n-m)_l\left\langle\frac{t^r}{\prod_{j=1}^{r}(e^{a_jt}-1)}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\Big|x^{n-m-l}\right\rangle\\
&=\binom{n}{m}\sum_{l=0}^{n-m}\frac{s!}{(l+s)!}S_2(l+s,s)(n-m)_lS_{n-m-l}^{(r,k)}(a_1,\dots,a_r)\\
&=\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}S_2(l+s,s)S_{n-m-l}^{(r,k)}(a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we get the identity (32). ∎

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FUNCTIONAL INEQUALITIES ASSOCIATED WITH INNER PRODUCT PRESERVING MAPPINGS

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Abstract. In this paper, we prove the Hyers-Ulam stability of inner product preserving mappings in Hilbert spaces for the following additive functional equation

$$f(ax + by) = af(x) + bf(y).$$

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the question of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equation is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$ and some $\epsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$.

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Rassias [22] introduced the following inequality: Assume that there exist constants $\lambda \geq 0$ and $p \in [0,1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \lambda(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Rassias [22] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\lambda}{2 - 2^p}\|x\|^p$$

for all $x \in X$. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [11] generalized the Rassias' result.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [27] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain X is replace by an Abelian group. In [5], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [10] generalized the stability result as follows: Let G be an abelian group, E a Banach space. Assume that a mapping $f : G \to E$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varphi(x,y)$$

for all $x, y \in G$, where $\varphi : G \times G \to [0,\infty)$ is a function such that

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}}\varphi(2^i x, 2^i y) < \infty$$

for all $x \in G$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found in ([6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26]).

Let X and Y be complex Hilbert spaces. An additive mapping $f : X \to Y$ is called *an inner product preserving mapping* if f satisfies the orthogonality equation

$$\langle f(x), f(y)\rangle = \langle x, y\rangle$$

for all $x, y \in X$. The inner product preserving mapping problem has been investigated in several papers (see [1, 2, 3]).

In this paper, we prove the Hyers-Ulam stability of inner product preserving mappings in Hilbert spaces for the following additive function equation.

$$f(ax+by) = af(x) + bf(y), \quad a,b \in \mathbb{R} \setminus \{0\}. \tag{1}$$

Throughout this paper, assume that X and Y are complex Hilbert spaces, and that $a,b \in \mathbb{R}\setminus\{0\}$ with $|a|<1$ or $|b|>1$ or $a=b=1$.

2. Hyers-Ulam stability of (1) in Hilbert spaces

We prove the Hyers-Ulam stability of inner product preserving mappings in Hilbert spaces for the additive function equation (1) when $|a|<1$ or $|b|>1$.

Theorem 2.1. *Let $|a|<1$ and let $f: X \to Y$ be a mapping with $f(0)=0$ for which there exists a function $\phi : X\times X \to [0,\infty)$ such that*

$$\widetilde{\phi}(x,y) := \sum_{j=0}^{\infty} |a^j| \phi\left(\frac{x}{a^j}, \frac{y}{a^j}\right) < \infty, \tag{2.1}$$

$$\|f(ax+by) - af(x) - bf(y)\| \le \phi(x,y), \tag{2.2}$$

$$|\langle f(x), f(y)\rangle - \langle x,y\rangle| \le \phi(x,y) \tag{2.3}$$

for all $x,y\in X$. Then there exists a unique inner product preserving mapping $I: X\to Y$ such that

$$\|f(x) - I(x)\| \le \widetilde{\phi}\left(\frac{x}{a}, 0\right) \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y=0$ in (2.2), we obtain

$$\|f(ax) - af(x)\| \le \phi(x,0).$$

Then

$$\left\|f(x) - af\left(\frac{x}{a}\right)\right\| \le \phi\left(\frac{x}{a}, 0\right). \tag{2.5}$$

It follows from (2.5) that

$$\begin{aligned}\left\|a^l f\left(\frac{x}{a^l}\right) - a^m f\left(\frac{x}{a^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|a^j f\left(\frac{x}{a^j}\right) - a^{j+1} f\left(\frac{x}{a^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} |a^j| \phi\left(\frac{x}{a^{j+1}}, 0\right)\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It means that the sequence $\{a^n f\left(\frac{x}{a^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{a^n f(\frac{x}{a^n})\}$ converges. We define the mapping $I: X\to Y$ by $I(x) = \lim_{n\to\infty}\{a^n f(\frac{x}{a^n})\}$ for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$, we get (2.4).

We show that $I(x)$ is an additive mapping.

$$\begin{aligned}
\|I(x+y)-I(x)-I(y)\| &= \lim_{n\to\infty}|a^n|\left\|f\left(\frac{x+y}{a^n}\right)-f\left(\frac{x}{a^n}\right)-f\left(\frac{y}{a^n}\right)\right\| \\
&\le \lim_{n\to\infty}|a^n|\left\{\left\|f\left(\frac{x+y}{a^n}\right)-af\left(\frac{x}{aa^n}\right)-bf\left(\frac{y}{ba^n}\right)\right\|\right. \\
&\left. + \left\|af\left(\frac{x}{a^na}\right)-f\left(\frac{x}{a^n}\right)\right\|+\left\|bf\left(\frac{y}{ba^n}\right)-f\left(\frac{y}{a^n}\right)\right\|\right\} \\
&\le \lim_{n\to\infty}|a^n|\left\{\phi\left(\frac{x}{aa^n},\frac{y}{ba^n}\right)+\phi\left(\frac{x}{a^n},0\right)+\phi\left(0,\frac{y}{a^n}\right)\right\} \\
&= 0.
\end{aligned}$$

It follows from (2.1) and (2.3) that

$$\begin{aligned}
\left|\left\langle a^nf\left(\frac{x}{a^n}\right),a^nf\left(\frac{y}{a^n}\right)\right\rangle-\langle x,y\rangle\right| &= |a^{2n}|\cdot\left|\left\langle f\left(\frac{x}{a^n}\right),f\left(\frac{y}{a^n}\right)\right\rangle-\left\langle\frac{x}{a^n},\frac{y}{a^n}\right\rangle\right| \\
&\le |a^{2n}|\phi\left(\frac{x}{a^n},\frac{y}{a^n}\right)\le|a^n|\phi\left(\frac{x}{a^n},\frac{y}{a^n}\right),
\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x,y\in X$.

$$\langle I(x),I(y)\rangle=\lim_{n\to\infty}\left\langle a^nf\left(\frac{x}{a^n}\right),a^nf\left(\frac{y}{a^n}\right)\right\rangle=\langle x,y\rangle$$

for all $x,y\in X$.

It only remains to show that the mapping $I:X\to Y$ is unique. Let g be another additive mapping satisfying (2.4). Then

$$\begin{aligned}
\|g(x)-I(x)\| &= |a^n|\left\|g\left(\frac{x}{a^n}\right)-I\left(\frac{y}{a^n}\right)\right\| \\
&\le |a^n|\left(\left\|g\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right\|+\left\|f\left(\frac{x}{a^n}\right)-I\left(\frac{y}{a^n}\right)\right\|\right) \\
&\le 2\sum_{i=1}^{\infty}|a^{i+n}|\widetilde{\phi}\left(\frac{x}{a^{i+n}},0\right)
\end{aligned}$$

for all $x\in X$ and $n\in\mathbb{N}$. Thus from $n\to\infty$, one establishes

$$g(x)-I(x)=0$$

for all $x\in X$. This completes the proof of uniqueness. □

Corollary 2.2. *Let $|a|<1$ and let $f:X\to Y$ be a mapping for which there exist constants $\theta\ge0$ and $r\in[0,1)$ such that*

$$\|f(ax+by)-af(x)-bf(y)\|\le\theta(\|x\|^r+\|y\|^r), \tag{2.6}$$

$$|\langle f(x),f(y)\rangle-\langle x,y\rangle|\le\theta(\|x\|^r+\|y\|^r) \tag{2.7}$$

for all $x,y\in X$. Then there exists a unique inner product preserving mapping $I:X\to Y$ such that

$$\|f(x)-I(x)\|\le\frac{\theta}{|a|^r-|a|}\|x\|^r$$

for all $x \in X$.

Proof. Defining $\phi(x,y) = \theta(\|x\| + \|y\|)$ and applying Theorem 2.1, we get the desired result. □

Theorem 2.3. *Let* $|b| > 1$ *and let* $f : X \to Y$ *be a mapping with* $f(0) = 0$, (2.2) *and* (2.3) *for which there exists a function* $\phi : X \times X \to [0, \infty)$ *such that*

$$\sum_{j=0}^{\infty} |b^{2j}| \phi\left(\frac{x}{b^j}, \frac{y}{b^j}\right) < \infty \tag{2.8}$$

for all $x, y \in X$. *Then there exists a unique inner product preserving mapping* $I : X \to Y$ *such that*

$$\|f(x) - I(x)\| \le \widetilde{\phi}\left(0, \frac{x}{b}\right) \tag{2.9}$$

for all $x \in X$, *where*

$$\widetilde{\phi}(x,y) := \sum_{j=0}^{\infty} |b^j| \phi\left(\frac{x}{b^j}, \frac{y}{b^j}\right)$$

for all $x, y \in X$.

Proof. Letting $x = 0$ and replacing y by x in (2.2), we obtain

$$\|f(bx) - bf(x)\| \le \phi(0, x),$$

an so

$$\begin{aligned}\left\|b^l f\left(\frac{x}{b^l}\right) - b^m f\left(\frac{x}{b^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|b^j f\left(\frac{x}{b^j}\right) - b^{j+1} f\left(\frac{x}{b^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} |b^j| \phi\left(0, \frac{x}{b^{j+1}}\right)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{b^n f\left(\frac{x}{b^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{b^n f(\frac{x}{b^n})\}$ converges. We define the mapping $I : X \to Y$ by $I(x) = \lim_{n\to\infty}\{b^n f(\frac{x}{b^n})\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.9).

It follows from (2.3) and (2.8) that

$$\begin{aligned}\left|\left\langle b^n f\left(\frac{x}{b^n}\right), b^n f\left(\frac{y}{b^n}\right)\right\rangle - \langle x, y\rangle\right| &= |b^{2n}| \cdot \left|\left\langle f\left(\frac{x}{b^n}\right), f\left(\frac{y}{b^n}\right)\right\rangle - \left\langle \frac{x}{b^n}, \frac{y}{b^n}\right\rangle\right| \\ &\le |b^{2n}| \phi\left(\frac{x}{b^n}, \frac{y}{b^n}\right),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in X$.

$$\langle I(x), I(y)\rangle = \lim_{n\to\infty} \left\langle b^n f\left(\frac{x}{b^n}\right), b^n f\left(\frac{y}{b^n}\right)\right\rangle = \langle x, y\rangle$$

for all $x, y \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $|b| > 1$ and let $f : X \to Y$ be a mapping for which there exist constants $\theta \geq 0$ and $r > 2$ satisfying (2.6) and (2.7). Then there exists a unique inner product preserving mapping $I : X \to Y$ such that*

$$\|f(x) - I(x)\| \leq \frac{\theta}{|b|^r - |b|}\|x\|^r$$

for any $x \in X$.

Proof. Defining $\phi(x, y) = \theta(\|x\| + \|y\|)$ and applying Theorem 2.3, we get the desired result. □

Now we prove the Hyers-Ulam stability of inner product preserving mappings in Hilbert spaces for the additive function equation (1) when $a = b = 1$.

Theorem 2.5. *Let $f : X \to Y$ be a mapping for which there exists a function $\phi : X \times X \to [0, \infty)$ satisfying (2.3) and*

$$\widetilde{\phi}(x, y) := \sum_{j=1}^{\infty} 4^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2.10}$$

$$\|f(x + y) - f(x) - f(y)\| \leq \phi(x, y) \tag{2.11}$$

for all $x, y \in X$. Then there exists a unique inner product preserving mapping $I : X \to Y$ such that

$$\|f(x) - I(x)\| \leq \frac{1}{2}\widetilde{\phi}(x, x) \tag{2.12}$$

for all $x \in X$, where

$$\widetilde{\phi}(x, y) := \sum_{j=1}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (2.11), we obtain

$$\|f(2x) - 2f(x)\| \leq \phi(x, x).$$

Then

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \phi\left(\frac{x}{2}, \frac{x}{2}\right). \tag{2.13}$$

It follows from (2.13) that

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 2^j \phi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. We define the mapping $I : X \to Y$ by $I(x) = \lim_{n\to\infty}\{2^n f(\frac{x}{2^n})\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.12).

It follows from (2.3) and (2.10) that

$$\begin{aligned}\left|\left\langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right)\right\rangle - \langle x, y\rangle\right| &= |2^{2n}| \cdot \left|\left\langle f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)\right\rangle - \left\langle \frac{x}{2^n}, \frac{y}{2^n}\right\rangle\right| \\ &\le |2^{2n}|\phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right),\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x, y \in X$.

$$\langle I(x), I(y)\rangle = \lim_{n\to\infty}\left\langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right)\right\rangle = \langle x, y\rangle$$

for all $x, y \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $f : X \to Y$ be a mapping for which there exist constants $\theta \ge 0$ and $r > 2$ satisfying* (2.7) *and*

$$\|f(x + y) - f(x) - f(y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.14}$$

for all $x, y \in X$. Then there exists a unique inner product preserving mapping $I : X \to Y$ such that

$$\|f(x) - I(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. Defining $\phi(x, y) = \theta(\|x\| + \|y\|)$ and applying Theorem 2.5, we get the desired result. □

Theorem 2.7. *Let $f : X \to Y$ be a mapping for which there exists a function $\phi : X\times X \to [0, \infty)$ satisfying* (2.3), (2.11) *and*

$$\widetilde{\phi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\phi\left(2^j x, 2^j y\right)$$

for all $x, y \in X$. Then there exists a unique inner product preserving mapping $I : X \to Y$ such that

$$\|f(x) - I(x)\| \le \frac{1}{2}\widetilde{\phi}(x, x)$$

for all $x \in X$.

Proof. It follows from (2.13) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\phi(x, x).$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.5. □

Corollary 2.8. *Let $f : X \to Y$ be a mapping for which there exist constants $\theta \geq 0$ and $r \in (0,1)$ satisfying* (2.7) *and* (2.14)*. Then there exists a unique inner product preserving mapping $I : X \to Y$ such that*

$$\|f(x) - I(x)\| \leq \frac{2\theta}{2-2^r}\|x\|^r$$

for all $x \in X$.

Proof. Defining $\phi(x,y) = \theta(\|x\| + \|y\|)$ and applying Theorem 2.7, we get the desired result. □

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STABILITY AND SUPERSTABILITY OF (f_r, f_s)-DOUBLE DERIVATIONS IN QUASI-BANACH ALGEBRAS

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Abstract. In this paper, the following functional equation

$$\sum_{k=2}^{n}\Big[\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} f\Big(\sum_{\substack{i=1,\\ i\neq i_1,\cdots,i_{n-k+1}}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r}x_{i_r}\Big)\Big]$$
$$+f\Big(\sum_{i=1}^{n} a_i x_i\Big) = a_1\cdot 2^{n-1} f(x_1) \tag{0.1}$$

is considered ($n \geq 2$), and the Hyers-Ulam stability and superstability of (f_r, f_s)-double derivations on quasi-Banach algebras associated with the functional equation (0.1) are proved.

1. **Introduction and preliminaries**

The stability of functional equations theory discusses and studies about solutions of functional equations and analyzes the relationships between approximate and exact solutions of the functional equations. Actually, we say a functional equation is *stable*, if one can find an exact solution for any approximate solution of the functional equation. Subsequently, the concept of superstability has a near nature to the stability sense. In other words, it happens when any approximate solution is also an exact solution that in such situation the functional equation is called *superstable*.

In 1940, the most preliminary kind of stability problems was proposed by Ulam [37]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, Hyers [14] formulated and proved the Ulam's problem for the Cauchy's functional equation on Banach spaces. The result of Hyers was generalized by Aoki [3] for additive mappings and by Rassias [32] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [13] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping.

It seems that considering stability problems concerning derivations returns to 1994 by Šemrl [34] who had worked on derivations between operator algebras and afterwards Jun and Park [16]. They had investigated approximate derivations on Banach algebras

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$C^n[0,1]$. More stability results in many kind of derivations can be found in (cf. [2, 4, 7, 8, 10, 24, 25, 26, 28, 36]).

At present, the theory of stability is quickly being deployed by numerous mathematicians. They pose and investigate various stability problems including different functional equations, derivations and homomorphisms in various spaces and structures. For more epochal information and various aspects about the stability theory, the readers can refer to monographs (cf. [5, 6, 9, 11, 12, 15, 18, 19, 20, 21, 22, 23, 27, 29, 30, 31, 33, 35, 38]).

Now we give briefly some useful definitions, preliminary and fundamental results of quasi-Banach spaces.

Definition 1.1. ([17]) Let $\mathcal{X}$ be a real linear space. A function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a *quasi-norm* (valuation) if it satisfies the following conditions:
($\mathcal{N}_1$) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if only if $x = 0$;
($\mathcal{N}_2$) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$;
($\mathcal{N}_3$) There is a constant $\mathcal{K} \geq 1$ such that $\|x+y\| \leq \mathcal{K}(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.
In this case, the pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* and the smallest possible $\mathcal{K}$ is called the modulus of concavity of $\|\cdot\|$.

A complete quasi-normed space is a *quasi-Banach space*.

Definition 1.2. ([17]) Let $0 < p \leq 1$ be a real number. A quasi-normed space $(\mathcal{X}, \|\cdot\|)$ is called a *p-normed space* if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$.

Definition 1.3. ([1]) Let $\mathcal{X}$ be an algebra and $(\mathcal{X}, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed algebra* if there exist a constant $\mathcal{C} > 0$ such that

$$\|xy\| \leq \mathcal{C}\|x\|\|y\|$$

for all $x, y \in \mathcal{X}$. In addition, if the quasi-norm $\|\cdot\|$ is a p-norm, then the quasi-normed algebra $(\mathcal{X}, \|\cdot\|)$ is called a *p-normed algebra*.

Definition 1.4. ([24]) Let $\mathcal{X}$ be an algebra and $f_2, f_3 : \mathcal{X} \to \mathcal{X}$ be linear mappings. A linear mapping $f_1 : \mathcal{X} \to \mathcal{X}$ is called an (f_2, f_3)-*double derivation* if

$$f_1(xy) = f_1(x)y + xf_1(y) + f_2(x)f_3(y) + f_3(x)f_2(y)$$

for all $x, y \in \mathcal{X}$. By an f_2-double derivation we mean an (f_2, f_2)-double derivation. It is clear that f_1 is an (f_2, f_3)-double derivation if and only if f_1 is an (f_3, f_2)-double derivation.

Now, consider the functional equation (0.1). This equation which is called the general n-dimensional additive functional equation, was introduced by Khodaei and Rassias [17]. In order to investigate (0.1), throughout this paper $a_1, \cdots, a_n$ (with $n \geq 2, a_1 > 1$), are fixed positive integers and $\mathcal{X}$ will be also a p-Banach algebra with p-norm $\|\cdot\|$, as well as the integers $1 \leq j, r, s \leq 3$ are assumed with $j \neq r \neq s$.

2. Hyers-Ulam stability of (f_r, f_s)-double derivations on quasi-Banach algebras

In this section, we prove the Hyers-Ulam stability of (f_r, f_s)-double derivations on quasi-Banach algebras associated with the functional equation (0.1).

First of all, for convenience, for given mappings $f_j : \mathcal{X} \to \mathcal{X}$, we define the difference operators:

$$D_\lambda f_j(x_1, \cdots, x_n) := \sum_{k=2}^{n} \Big[\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n}$$

$$f_j\Big(\sum_{\substack{i=1, \\ i \neq i_1, \cdots, i_{n-k+1}}}^{n} a_i \lambda x_i - \sum_{r=1}^{n-k+1} a_{i_r} \lambda x_{i_r} \Big) \Big] + f_j\Big(\sum_{i=1}^{n} a_i \lambda x_i \Big) - a_1 \cdot 2^{n-1} \lambda f_j(x_1),$$

$$D f_j, f_r, f_s(x, y) := f_j(xy) - f_j(x)y - x f_j(y) - f_r(x) f_s(y) - f_s(x) f_r(y)$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$.

Lemma 2.1. ([17]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (0.1) if and only if $f : \mathcal{X} \to \mathcal{Y}$ is additive.*

From now on, $0 < p \leq 1$ is a real number.

Theorem 2.2. *Let $\phi : \mathcal{X}^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{i=0}^{\infty} \left[\frac{1}{{a_1}^i} \phi(a_1^i x_1, \cdots, a_1^i x_n) \right]^p < +\infty \tag{2.1}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying the inequalities

$$\max_j \Big\{ \big\| D_\lambda f_j(x_1, x_2, \cdots, x_n) \big\| \Big\} \leq \phi(x_1, x_2, \cdots, x_n), \tag{2.2}$$

$$\max_{j,r,s} \Big\{ \big\| D_{f_j, f_r, f_s}(x, y) \big\| \Big\} \leq \phi(x + y, \cdots, x + y) \tag{2.3}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. Then unique $(\mathcal{F}_r, \mathcal{F}_s)$-double derivations $\mathcal{F}_j : \mathcal{X} \to \mathcal{X}$ defined by the limits

$$\mathcal{F}_j(x) = \lim_{m \to \infty} \frac{1}{a_1^m} f_j(a_1^m x)$$

exist and satisfy the inequalities

$$\Big\| \mathcal{F}_j(x) - f_j(x) \Big\| \leq \frac{1}{a_1 \cdot 2^{n-1}} \left[\tilde{\phi}(x, 0, \cdots, 0) \right]^{\frac{1}{p}} \tag{2.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x_1 = x$, $x_2 = \cdots = x_n = 0$ and $\lambda = 1$ in (2.2), we get

$$\big\| D_1 f_1(x, 0, \cdots, 0) \big\| \leq \phi(x, 0, \cdots, 0)$$

for all $x \in \mathcal{X}$. From this inequality and the fact that

$$2^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 1 + \sum_{i=1}^{n-1} \binom{n-1}{i},$$

we have

$$\left\| f_1(x) - \frac{1}{a_1} f_1(a_1 x) \right\| \leq \frac{1}{a_1 \cdot 2^{n-1}} \phi\big(x, 0, \cdots, 0\big)$$

for all $x \in \mathcal{X}$. Replacing x by $a_1^i x$ and then dividing both sides by a_1^i, we get

$$\left\| \frac{1}{a_1^i} f_1(a_1^i x) - \frac{1}{a_1^{i+1}} f_1(a_1^{i+1} x) \right\| \leq \frac{1}{{a_1}^{i+1} \cdot 2^{n-1}} \phi\big(a_1^i x, 0, \cdots, 0\big) \tag{2.5}$$

for all $x \in \mathcal{X}$ and all nonnegative integers i. Assume that m, t are positive integers with $m > t$. Since $\mathcal{X}$ is a p-Banach space, it follows from (2.5) that

$$\begin{aligned} \left\| \frac{1}{a_1^m} f_1(a_1^m x) - \frac{1}{a_1^t} f_1(a_1^t x) \right\|^p &\leq \sum_{i=t}^{m-1} \left\| \frac{1}{a_1^{i+1}} f_1(a_1^{i+1} x) - \frac{1}{a_1^i} f_1(a_1^i x) \right\|^p \\ &\leq \frac{1}{(a_1 \cdot 2^{n-1})^p} \sum_{i=t}^{m-1} \left[\frac{1}{{a_1}^i} \phi\left(a_1^i x, 0, \cdots, 0\right) \right]^p \end{aligned} \tag{2.6}$$

for all $x \in \mathcal{X}$. Now by the condition (2.1), we deduce that the right-hand side tends to zero as $t, m \to \infty$, and this implies that the sequence $\left\{ \frac{1}{a_1^m} f_1(a_1^m x) \right\}$ is Cauchy. Since $\mathcal{X}$ is complete, the sequence $\left\{ \frac{1}{a_1^m} f_1(a_1^m x) \right\}$ converges in $\mathcal{X}$, and therefore we can define for all $x \in \mathcal{X}$ the mapping $\mathcal{F}_1 : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{F}_1(x) = \lim_{m \to \infty} \frac{1}{a_1^m} f_1(a_1^m x).$$

Now we claim that $\mathcal{F}_1 : \mathcal{X} \to \mathcal{X}$ is $\mathbb{R}$-linear. In order to verify that, we first show that $\mathcal{F}_1$ is additive. It follows from (2.2) and (2.1) that

$$\begin{aligned} \left\| D_1 \mathcal{F}_1\big(x_1, x_2, \cdots, x_n\big) \right\|^p &= \left\| \lim_{m \to \infty} \frac{1}{a_1^m} D_1 f_1\big(a_1^m x_1, a_1^m x_2, \cdots, a_1^m x_n\big) \right\|^p \\ &\leq \lim_{m \to \infty} \left[\frac{1}{a_1^m} \phi\big(a_1^m x_1, a_1^m x_2, \cdots, a_1^m x_n\big) \right]^p = 0 \end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$. So $D_1 \mathcal{F}_1(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, which means $\mathcal{F}_1 : \mathcal{X} \to \mathcal{X}$ satisfies the functional equation (0.1). Therefore, Lemma 2.1 clarifies that the mapping $\mathcal{F}_1$ is additive. So

$$\begin{aligned} a_1 \mathcal{F}_1(x) &= \mathcal{F}_1(a_1 x) = \lim_{m \to \infty} \frac{1}{a_1^m} f_1(a_1^{m+1} x), \\ \mathcal{F}_1(x) &= \lim_{m \to \infty} \frac{1}{a_1^{m+1}} f_1(a_1^{m+1} x) \end{aligned}$$

for all $x \in \mathcal{X}$. For $x_1 = a_1^m x$ and $x_2 = \cdots = x_n = 0$, it follows from (2.2) that

$$\left\| a_1^{-(m+1)} f_1(a_1^{m+1} \lambda x) - a_1^{-m} \lambda f_1(a_1^m x) \right\|^p \leq \frac{1}{a_1^p \cdot 2^{(n-1)p}} \left[\frac{1}{a_1^m} \phi\big(a_1^m x, 0, \cdots, 0\big) \right]^p$$

for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. By (2.1), the right-hand side tends to zero as $m \to \infty$. Hence

$$\left\|\mathcal{F}_1(\lambda x) - \lambda \mathcal{F}_1(x)\right\|^p = 0$$

for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. So $\mathcal{F}_1(\lambda x) = \lambda \mathcal{F}_1(x)$ for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$, and thus $\mathcal{F}_1$ is $\mathbb{R}$-linear.

Putting $t = 0$ and passing the limit $m \to \infty$ in (2.6), we get the inequality (2.4) for $j = 1$.

Let $\mathcal{F}_1' : \mathcal{X} \to \mathcal{X}$ be another $\mathbb{R}$-linear mapping satisfying (2.4). Then we have

$$\begin{aligned} \left\|\mathcal{F}_1(x) - \mathcal{F}_1'(x)\right\|^p &\leq \frac{1}{a_1^{mp}}\Big(\left\|\mathcal{F}_1(a_1^m x) - f_1(a_1^m x)\right\|^p + \left\|\mathcal{F}_1'(a_1^m x) - f_1(a_1^m x)\right\|^p\Big) \\ &\leq \frac{2}{a_1^{mp}} \cdot \frac{1}{(a_1.2^{n-1})^p}\tilde{\phi}\big(a_1^m x, 0, \cdots, 0\big) \\ &= \frac{2}{(a_1.2^{n-1})^p} \cdot \sum_{i=m}^{\infty}\left[\frac{1}{a_1^i}\phi\big(a_1^i x, 0, \cdots, 0\big)\right]^p \end{aligned}$$

for all $x \in \mathcal{X}$. By (2.1), the right-hand side tends to zero as $m \to \infty$, which signifies the uniqueness of $\mathcal{F}_1$.

By a similar method, one can easily show that the unique and $\mathbb{R}$-linear mappings $\mathcal{F}_2 : \mathcal{X} \to \mathcal{X}$ and $\mathcal{F}_3 : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{F}_2(x) = \lim_{m\to\infty}\frac{1}{a_1^m}f_2(a_1^m x), \qquad \mathcal{F}_3(x) = \lim_{m\to\infty}\frac{1}{a_1^m}f_3(a_1^m x)$$

exist and satisfy (2.4) for all $x \in \mathcal{X}$.

To end the proof, it is just necessary to show that $\mathcal{F}_1$ is an $(\mathcal{F}_2, \mathcal{F}_3)$-double derivation. It follows from (2.3) that

$$\left\|D_{f_1,f_2,f_3}(x,y)\right\| = \left\|D_{f_1,f_3,f_2}(x,y)\right\| \leq \phi\big(x+y, \cdots, x+y\big)$$

for all $x, y \in \mathcal{X}$. We know $a_1 > 1$, so ${a_1}^{mp} > 1$ and $\big({a_1}^{mp}\big)^2 > {a_1}^{mp}$. Therefore, the last inequality and the condition (2.1) imply that

$$\begin{aligned} &\left\|\mathcal{F}_1(xy) - \mathcal{F}_1(x)y - x\mathcal{F}_1(y) - \mathcal{F}_2(x)\mathcal{F}_3(y) - \mathcal{F}_3(x)\mathcal{F}_2(y)\right\|^p \\ &\quad = \lim_{m\to\infty}\frac{1}{{a_1}^{2mp}}\Big\|f_1\left(a_1^{2m}xy\right) - f_1\left(a_1^m x\right)a_1^m y - a_1^m x f_1\left(a_1^m y\right) \\ &\qquad\qquad - f_2\left(a_1^m x\right)f_3\left(a_1^m y\right) - f_3\left(a_1^m x\right)f_2\left(a_1^m y\right)\Big\|^p \\ &\quad = \lim_{m\to\infty}\frac{1}{{a_1}^{2mp}}\left\|D_{f_1,f_3,f_2}\left(a_1^m x, a_1^m y\right)\right\|^p \\ &\quad < \lim_{m\to\infty}\left[\frac{1}{{a_1}^m}\phi\Big(a_1^m(x+y), \cdots, a_1^m(x+y)\Big)\right]^p = 0 \end{aligned}$$

for all $x, y \in \mathcal{X}$. Hence

$$\mathcal{F}_1(xy) = \mathcal{F}_1(x)y + x\mathcal{F}_1(y) + \mathcal{F}_2(x)\mathcal{F}_3(y) + \mathcal{F}_3(x)\mathcal{F}_2(y)$$

for all $x, y \in \mathcal{X}$.

Similarly we can show that $\mathcal{F}_2$ and $\mathcal{F}_3$ are respectively $(\mathcal{F}_1, \mathcal{F}_3)$-double derivation and $(\mathcal{F}_1, \mathcal{F}_2)$-double derivation and so the proof is complete. □

Theorem 2.3. *Let $\phi : \mathcal{X}^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \cdots, x_n) := \sum_{i=0}^{\infty} \left[a_1^i \phi\Big(\frac{x_1}{a_1^i}, \cdots, \frac{x_n}{a_1^i}\Big)\right]^p < +\infty,$$

$$\lim_{m\to\infty} \left[{a_1}^{2m} \phi\Big(\frac{x}{a_1^m}, \cdots, \frac{x}{a_1^m}\Big)\right]^p = 0 \tag{2.7}$$

for all $x, x_1, \cdots, x_n \in \mathcal{X}$. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying (2.2) *and* (2.3). *Then unique $(\mathcal{F}_r, \mathcal{F}_s)$-double derivations $\mathcal{F}_j : \mathcal{X} \to \mathcal{X}$ defined by the limits*

$$\mathcal{F}_j(x) = \lim_{m\to\infty} a_1^m f_j\Big(\frac{x}{a_1^m}\Big)$$

exist and satisfy the inequalities

$$\Big\|\mathcal{F}_j(x) - f_j(x)\Big\| \le \frac{1}{2^{n-1}} \left[\tilde{\phi}\Big(\frac{x}{a_1}, 0, \cdots, 0\Big)\right]^{1/p} \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x by $\frac{x}{a_1^{2i+1}}$ in (2.5) and then multiplying by a_1^{2i+1}, we obtain

$$\left\| a_1^{i+1} f_1\Big(\frac{x}{a_1^{i+1}}\Big) - a_1^i f_1\Big(\frac{x}{a_1^i}\Big)\right\| \le \frac{a_1^i}{2^{n-1}} \phi\Big(\frac{x}{a_1^{i+1}}, 0, \cdots, 0\Big)$$

for all $x \in \mathcal{X}$. Now by the same method which was done in the proof of Theorem 2.2, we can assert that the sequence $\left\{a_1^m f_1\big(\frac{x}{a_1^m}\big)\right\}$ is Cauchy and convergent in $\mathcal{X}$ and the unique and $\mathbb{R}$-linear mappings $\mathcal{F}_j(x) = \lim_{m\to\infty} a_1^m f_j\big(\frac{x}{a_1^m}\big)$ exist and satisfy (2.8).

The inequality (2.3) and the condition (2.7) imply that

$$\begin{aligned}\Big\|D_{\mathcal{F}_1,\mathcal{F}_3,\mathcal{F}_2}(x, y)\Big\|^p &= \lim_{m\to\infty} {a_1}^{2mp} \Big\|D_{f_1,f_3,f_2}\Big(\frac{x}{a_1^m}, \frac{y}{a_1^m}\Big)\Big\|^p \\ &\le \lim_{m\to\infty} \left[{a_1}^{2m} \phi\Big(\frac{x+y}{a_1^m}, \cdots, \frac{x+y}{a_1^m}\Big)\right]^p = 0\end{aligned}$$

for all $x, y \in \mathcal{X}$. This shows that $\mathcal{F}_1$ is an $(\mathcal{F}_2, \mathcal{F}_3)$-double derivation.

The arguments for $j = 2, 3$ are also the same as for $j = 1$ and so we will omit them. □

Corollary 2.4. *Let δ be a nonnegative real number and q be a positive real number such that $q < 1$ or $q > 2$. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying the inequalities*

$$\begin{aligned}\max_j \Big\{\big\|D_\lambda f_j(x_1, x_2, \cdots, x_n)\big\|\Big\} &\le \delta\left(\|x_1\|^q + \cdots + \|x_n\|^q\right), \\ \max_{j,r,s} \Big\{\big\|D_{f_j,f_r,f_s}(x, y)\big\|\Big\} &\le n\delta\|x+y\|^q\end{aligned}$$

for all $x, y, x_1, \cdots, x_n \in \mathcal{X}$, and all $\lambda \in \mathbb{R}$. Then unique $(\mathcal{F}_r, \mathcal{F}_s)$-double derivations $\mathcal{F}_j : \mathcal{X} \to \mathcal{X}$ exist and satisfy the inequalities

$$\Big\|\mathcal{F}_j(x) - f_j(x)\Big\| \leq \frac{\delta\|x\|^q}{2^{n-1} \cdot \left|a_1^p - a_1^{pq}\right|^{\frac{1}{p}}}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\phi(x_1, \cdots, x_n) := \delta\left(\|x_1\|^q + \cdots + \|x_n\|^q\right)$ and applying Theorem 2.2 for the case $q < 1$, and Theorem 2.3 for the case $q > 2$, we get the result. □

3. Superstability of (f_r, f_s)-double derivations on quasi-Banach algebras

In this section, we prove the superstability of (f_r, f_s)-double derivations associated with the functional equation (0.1).

Initially, we improve Lemma 2.1 to a stronger statement and afterwards we use it for the proof of superstability theorem of this section.

Lemma 3.1. *Let $n \geq 3$ be a fixed integer and $f : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$\begin{aligned} \Big\|2^{n-1}a_1\lambda f(x_1) - \sum_{k=2}^{n}\Big[\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n} \\ f\Big(\sum_{\substack{i=1,\\ i\neq i_1,\cdots,i_{n-k+1}}}^{n} a_i\lambda x_i - \sum_{r=1}^{n-k+1} a_{i_r}\lambda x_{i_r}\Big)\Big]\Big\| \quad \leq \quad \Big\|f\Big(\sum_{i=1}^{n} a_i\lambda x_i\Big)\Big\| \qquad (3.1) \end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. Then f is $\mathbb{R}$-linear.

Proof. First, assume that $n \geq 4$.

Putting $x_4 = \cdots = x_n = 0$ in (3.1), we get

$$\begin{aligned} \Big\|2^{n-1}\lambda a_1 f(x_1) - \Big(2^{n-3} - 1\Big)f\big(\lambda[a_1x_1 + a_2x_2 + a_3x_3]\big) - 2^{n-3} \\ f\big(\lambda[a_1x_1 - a_2x_2 + a_3x_3]\big) - 2^{n-3}f\big(\lambda[a_1x_1 + a_2x_2 - a_3x_3]\big) \\ -2^{n-3}f\big(\lambda[a_1x_1 - a_2x_2 - a_3x_3]\big)\Big\| \\ \leq \Big\|f\big(\lambda[a_1x_1 + a_2x_2 + a_3x_3]\big)\Big\| \qquad (3.2) \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. Letting $x_1 = x_2 = x_3 = 0$ and $\lambda = 1$ in (3.2), we obtain

$$\Big\|\Big((a_1 - 1)2^{n-1} + 1\Big)f(0)\Big\| \leq \Big\|f(0)\Big\|.$$

So $f(0) = 0$, (since $(a_1 - 1)2^{n-1} > 0$). Putting $\lambda = 1$ and substituting x_1, x_2, x_3 by $x/a_1, -x/a_2, 0$ and then by $x/a_1, -x/2a_2, -x/2a_3$ in (3.2), respectively, we get

$$\begin{aligned} a_1 \cdot 2^{n-1}f\Big(\frac{x}{a_1}\Big) &= 2^{n-2}f(2x), \\ a_1 \cdot 2^{n-1}f\Big(\frac{x}{a_1}\Big) &= 2^{n-2}f(x) + 2^{n-3}f(2x) \end{aligned}$$

for all $x \in \mathcal{X}$. Thus $f(2x) = 2f(x)$ and $f(x) = a_1 f\left(\frac{x}{a_1}\right)$ for all $x \in \mathcal{X}$. Letting $\lambda = 1$, $x_1 = x + y/a_1$, $x_2 = -x/a_2$ and $x_3 = -y/a_3$ in (3.2), we have

$$a_1 \cdot 2^{n-1} f\Big(\frac{x+y}{a_1}\Big) - 2^{n-3} f(2x) - 2^{n-3} f(2y) - 2^{n-3} f(2x+2y) = 0$$

for all $x, y \in \mathcal{X}$, which implies that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$.

Finally, letting $x_1 = x/a_1$, $x_2 = -x/a_2$ and $x_3 = 0$ in (3.2), we obtain

$$a_1 \lambda \cdot 2^{n-1} f\Big(\frac{x}{a_1}\Big) - 2^{n-2} f(2\lambda x) = 0$$

for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. So $f(\lambda x) = \lambda f(x)$ for all $x \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$, and therefore f is $\mathbb{R}$-linear.

Now assume that $n = 3$ in (3.1). Then

$$\begin{aligned} \Big\| 4a_1 \lambda f(x_1) - f\big(\lambda[a_1x_1 - a_2x_2 - a_3x_3]\big) \\ - f\big(\lambda[a_1x_1 + a_2x_2 - a_3x_3]\big) - f\big(\lambda[a_1x_1 - a_2x_2 + a_3x_3]\big)\Big\| \\ \le \Big\| f\big(\lambda[a_1x_1 + a_2x_2 + a_3x_3]\big)\Big\| \end{aligned}$$

for all $x_1, x_2, x_3 \in \mathcal{X}$ and all $\lambda \in \mathbb{R}$. As it is obvious that we can get an inequality similar to the normed inequality (3.2) for the case $n \ge 4$, one can easily get the desired result. $\square$

Theorem 3.2. *Let $\phi : \mathcal{X}^n \to [0, \infty)$, $n \ge 3$, be a function such that*

$$\lim_{l\to\infty} t^{-2l} \phi(t^l x, \cdots, t^l x) = 0$$

for all $x \in \mathcal{X}$, where $t \neq 1$ is a real number. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying (3.1) *and* (2.3). *Then the mappings $f_j : \mathcal{X} \to \mathcal{X}$ are (f_r, f_s)-double derivations.*

Proof. Since (3.1) holds for the mappings $f_j : \mathcal{X} \to \mathcal{X}$, Lemma 3.1 asserts that the mappings f_j are $\mathbb{R}$-linear. So it follows from (2.3) and the assumption on ϕ that

$$\begin{aligned} &\Big\| f_j(xy) - f_j(x)y - x f_j(y) - f_r(x) f_s(y) - f_s(x) f_r(y) \Big\|^p \\ &= \lim_{l\to\infty} \frac{1}{t^{2lp}} \Big\| f_j\left(t^{2l}xy\right) - f_j\left(t^l x\right) t^l y - t^l x f_j\left(t^l y\right) \\ &\qquad - f_r\left(t^l x\right) f_s\left(t^l y\right) - f_s\left(t^l x\right) f_r\left(t^l y\right) \Big\|^p \\ &= \lim_{l\to\infty} \frac{1}{t^{2lp}} \Big\| D_{f_j, f_k, f_i}\left(t^l x, t^l y\right) \Big\|^p \\ &\le \left[\lim_{l\to\infty} \frac{1}{t^{2l}} \phi\Big(t^l(x+y), \cdots, t^l(x+y)\Big)\right]^p = 0^p \end{aligned}$$

for all $x, y \in \mathcal{X}$, which implies that the mappings $f_j : \mathcal{X} \to \mathcal{X}$ are (f_r, f_s)-double derivations, as desired. $\square$

Corollary 3.3. *Let δ be a nonnegative real number and $q_1, \cdots, q_n$ be positive real numbers such that $q_1, \cdots, q_n > 2$ or $q_1, \cdots, q_n < 2$. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying* (3.1) *and the inequalities*

$$\max_{j,r,s}\left\{\left\|D_{f_j,f_r,f_s}(x,y)\right\|\right\} \leq \delta\Big(\|x+y\|^{q_1} + \cdots + \|x+y\|^{q_n}\Big)$$

for all $x, y \in \mathcal{X}$. Then the mappings $f_j : \mathcal{X} \to \mathcal{X}$ are (f_r, f_s)-double derivations.

Proof. The proof follows from Theorem 3.2 by taking $\phi(x_1, \cdots, x_n) := \delta\left(\|x_1\|^{q_1} + \cdots + \|x_n\|^{q_n}\right)$ with $t > 1$ for the case $q_1, \cdots, q_n < 2$ and with $t < 1$ for the case $q_1, \cdots, q_n > 2$. $\square$

Corollary 3.4. *Let δ be a nonnegative real number and $q_1, \cdots, q_n$ be positive real numbers such that $q_1 + \cdots + q_n \neq 2$. Suppose that $f_j : \mathcal{X} \to \mathcal{X}$, $j = 1, 2, 3$, are mappings satisfying* (3.1) *and the inequality*

$$\max_{j,r,s}\left\{\left\|D_{f_j,f_r,f_s}(x,y)\right\|\right\} \leq \delta\|x+y\|^{q_1+\cdots+q_n}$$

for all $x, y \in \mathcal{X}$. Then the mappings $f_j : \mathcal{X} \to \mathcal{X}$ are (f_r, f_s)-double derivations.

Proof. The proof follows from Theorem 3.2 by taking $\phi(x_1, \cdots, x_n) := \delta\left(\|x_1\|^{q_1} \cdots \|x_p\|^{q_n}\right)$ with $t > 1$ for the case $q_1 + \cdots + q_n < 2$ and with $t < 1$ for the case $q_1 + \cdots + q_n > 2$. $\square$

The obtained results in this section can be simpler. Indeed, one can set $q_1 = \cdots = q_n = q$ in two last corollaries and get the better statements.

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THE FIXED POINT METHOD FOR PERTURBATION OF BIHOMOMORPHISMS AND BIDERIVATIONS IN NORMED 3-LIE SYSTEMS: REVISITED

CHOONKIL PARK, JUNG RYE LEE, EON WHA SHIM, AND DONG YUN SHIN*

Abstract. Shokri et al. [11] proved the Hyers-Ulam stability of bihomomorphisms and biderivations on normed 3-Lie systems by using the fixed point method.

Under the conditions in the main theorems of [11, Section 2], we can show that the related mappings must be zero.

In this paper, we correct the statements of the results in [11, Section 2], and prove the corrected theorems.

1. Introduction and preliminaries

The stability problems of functional equations and inequalities has been studied in many mathmaticians (see [3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15]).

Shokri et al. [11] defined bihomomorphisms and biderivations.

Definition 1.1. ([11]) Let A and B be normed Lie triple systems. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a *bihomomorphism* if it satisfies

$$\begin{aligned} H([x,y,z],w) &= [H(x,w),H(y,w),H(z,w)], \\ H(x,[y,z,w]) &= [H(x,y),H(x,z),H(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$.

Note that if we replace w by $2w$ in the first equality of the definition of bihomomorphism then $2H([x,y,z],w) = 8[H(x,w),H(y,w),H(z,w)]$ and so $H([x,y,z],w) = 0$ for all $x,y,z,w \in A$. Similarly, one can show that $H(x,[y,z,w]) = 0$ for all $x,y,z,w \in A$. Thus we correct the definition of bihomomorphism as follows.

Definition 1.2. Let A and B be normed Lie triple systems. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a *bihomomorphism* if it satisfies

$$\begin{aligned} H([x,y,z],w^3) &= [H(x,w),H(y,w^*),H(z,w)], \\ H(x^3,[y,z,w]) &= [H(x,y),H(x^*,z),H(x,w)] \end{aligned}$$

for all $x, y, z, w \in A$.

Definition 1.3. ([11]) Let A and B be normed Lie triple systems. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *biderivation* if it satisfies

$$\begin{aligned} \delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\ \delta(x,[y,z,w],w) &= [\delta(x,y),z,w] + [y,\delta(x,z),w] + [y,z,\delta(x,w)] \end{aligned}$$

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for all $x, y, z, w \in A$.

The w-variable of the left side in the first equality is $\mathbb{C}$-linear and the x-variable of the left side in the second equality is $\mathbb{C}$-linear. But the w-variable of the right side in the first equality is not $\mathbb{C}$-linear and the x-variable of the right side in the second equality is not $\mathbb{C}$-linear. Thus we correct the definition of biderivation as follows.

Definition 1.4. Let A and B be normed Lie triple systems. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *biderivation* if it satisfies

$$\begin{aligned}\delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)],\\ \delta(x,[y,z,w],w) &= [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)]\end{aligned}$$

for all $x, y, z, w \in A$.

All the mappings T and δ, given in [11, Section 2], satisfy the bi-additive functional equation (1.1) in [11]. Letting $x = z = 0$ in (1.1), we get $f(y,-w) = f(y,w)$ for all y, w. Thus f is not bi-additive. So the results of [11, Section 2] are meaningless.

In this paper, we will replace the equation (1.1), given in [11], by

$$f(x+y,z+w) + f(x+y,z-w) = 2f(x,z) + 2f(y,z). \tag{1}$$

Moreover, we correct the statements of the results in [11, Section 2], and prove the corrected theorems.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5. [2] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y,y^*) \le \frac{1}{1-L} d(y,Jy)$ for all $y \in Y$.*

Throughout this paper, assume that A is a normed Lie triple system and B is a Banach Lie triple system.

2. Hyers-Ulam stability of bihomomorphisms in Banach Lie triple systems

For a given mapping $f : A \times A \to B$, we define

$$\begin{aligned}D_{\lambda,\mu} f(x,y,z,w) &= f(\lambda x + \lambda y, \mu z + \mu w) + f(\lambda x + \lambda y, \mu z - \mu w)\\ &- 2\lambda\mu f(x,z) - 2\lambda\mu f(y,z)\end{aligned}$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$.

From now on, assume that $f(0,z) = f(x,0) = 0$ for all $x, z \in A$.

We need the following lemmas to obtain the main results.

Lemma 2.1. ([1]) *Let $f : A \times A \to B$ be a bi-additive mapping such that $f(\lambda x, \mu y) = \lambda\mu f(x, y)$ for all $x, y \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

Lemma 2.2. *Let $f : A \times A \to B$ be a mapping satisfying $D_{\lambda,\mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \to B$ is $\mathbb{C}$-bilinear.*

Proof. Letting $\lambda = \mu = 1$ in $D_{\lambda,\mu} f(x, y, z, w) = 0$, we get (1). Letting $y = 0$ in (1), we get $f(x, z + w) + f(x, z - w) = 2f(x, z)$ for all $x, z, w \in A$. Letting $w = 0$ in (1), we get $2f(x + y, z) = 2f(x, z) + 2f(y, z)$ for all $x, y, z \in A$. So f is bi-additive.

Letting $y = w = 0$ in $D_{\lambda,\mu} f(x, y, z, w) = 0$, we get $2f(\lambda x, \mu z) = 2\lambda\mu f(x, z)$ for all $x, z \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. By Lemma 2.1, the mapping $f : A\times A \to B$ is $\mathbb{C}$-bilinear. $\square$

Theorem 2.3. *Let $f : A \times A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to [0, \infty)$ such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\| \leq \varphi(x, y, z, w), \tag{2}$$

$$\begin{aligned}&\|f([x, y, z], w^3) - [f(x, w), f(y, w^*), f(z, w)]\| \\ &+\|f(x^3, [y, z, w]) - [f(x, y), f(x^*, z), f(x, w)]\| \leq \varphi(x, y, z, w)\end{aligned} \tag{3}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. If there exists an $L < 1$ such that $\varphi(x, y, z, w) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right)$ for all $x, y, z, w \in A$, then there exists a unique bihomomorphism $H : A \times A \to B$ such that

$$\|f(x, z) - H(x, z)\| \leq \frac{1}{4 - 4L}\varphi(x, x, z, z) \tag{4}$$

for all $x, z \in A$.

Proof. Letting $\lambda = \mu = 1$, $y = x$ and $w = z$ in (2), we get

$$\|f(2x, 2z) - 4f(x, z)\| \leq \varphi(x, x, z, z) \tag{5}$$

and so

$$\left\|f(x, z) - \frac{1}{4}f(2x, 2z)\right\| \leq \frac{1}{4}\varphi(x, x, z, z) \tag{6}$$

for all $x, z \in A$.

Consider the set

$$S := \{h : A \times A \to B\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \{\mu \in \mathbb{R}_+ : \|g(x, z) - h(x, z)\| \leq \mu\varphi (x, x, z, z)\,,\ \forall x, z \in A\}\,,$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [8]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x, z) := \frac{1}{4}g\,(2x, 2z)$$

for all $x, z \in A$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x, z) - h(x, z)\| \leq \varepsilon\varphi\,(x, x, z, z)$$

for all $x \in A$. Hence

$$\|Jg(x,z) - Jh(x,z)\| = \left\| \frac{1}{4}g\left(2x, 2z\right) - \frac{1}{4}h\left(2x, 2z\right)\right\| \leq \varepsilon L \varphi\left(x, x, z, z\right)$$

for all $x, z \in A$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \varepsilon L$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (6) that $d(f, Jf) \leq \frac{1}{4}$.

By Theorem 1.5, there exists a mapping $H : A \to B$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H\left(2x, 2z\right) = 4H(x, z) \tag{7}$$

for all $x, z \in A$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in S : d(h, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x,z) - H(x,z)\| \quad \leq \quad \mu\varphi\left(x, x, z, z\right)$$

for all $x, z \in A$;

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x, 2^n z\right) = H(x, z) \tag{8}$$

for all $x, z \in A$;

(3) $d(f, H) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{4 - 4L}.$$

This implies that the inequality (4) holds true.

It follows from (2) and (8) that

$$\begin{aligned} \|D_{\lambda,\mu}H(x,y,z,w)\| &= \left\|\frac{1}{4^n}D_{\lambda,\mu}f(2^n x, 2^n y, 2^n z, 2^n w)\right\| \\ &\leq \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. By Lemma 2.2, the mapping $H : A \times A \to B$ is $\mathbb{C}$-bilinear.

Now let $T : A \times A \to B$ be another bi-additive mapping satisfying (4). Then

$$\begin{aligned} &\|H(x,z) - T(x,z)\| = \frac{1}{4^n}\|H(2^n x, 2^n z) - T(2^n, 2^n z)\| \\ &\leq \frac{1}{4^n}\left(\|H(2^n x, 2^n z) - f(2^n, 2^n z)\| + \|f(2^n x, 2^n z) - T(2^n, 2^n z)\|\right) \\ &\leq \frac{2}{4^n}\varphi(2^n x, 2^n x, 2^n z, 2^n z), \end{aligned}$$

which tends to zero as $n \to \infty$. This proves the uniqueness of H.

It follows from (3) that

$$\begin{aligned}
&\|H([x,y,z],w^3) - [H(x,w),H(y,w^*),H(z,w)]\| \\
&\quad + \|H(x^3,[y,z,w]) - [H(x,y),H(x^*,z),H(x,w)]\| \\
&= \lim_{n\to\infty} \frac{1}{64^n}\big(\|f([2^n x,2^n y,2^n z],8^n w^3) \\
&\quad - [f(2^n x,2^n w),f(2^n y,2^n w^*),f(2^n z,2^n w)]\| \\
&\quad + \|f(8^n x^3,[2^n y,2^n z,2^n w]) \\
&\quad - [f(2^n x,2^n y),f(2^n x^*,2^n z),f(2^n x,2^n w)]\|\big) \\
&\le \lim_{n\to\infty} \frac{1}{64^n}\varphi(2^n,2^n y,2^n z,2^n w) \le \lim_{n\to\infty}\frac{1}{4^n}\varphi(2^n,2^n y,2^n z,2^n w) = 0
\end{aligned}$$

for all $x,y,z,w \in A$. So

$$H([x,y,z],w^3) = [H(x,w),H(y,w^*),H(z,w)]$$

and

$$H(x^3,[y,z,w]) = [H(x,y),H(x^*,z),H(x,w)]$$

for all $x,y,z,w \in A$.

Therefore, the mapping H is a unique bi-homomorphism satisfying (4). □

Corollary 2.4. Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \to B$ be a mapping such that

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \tag{9}$$

$$\begin{aligned}
&\|f([x,y,z],w^3) - [f(x,w),f(y,w^*),f(z,w)]\| \\
&\quad + \|f(x^3,[y,z,w]) - [f(x,y),f(x^*,z),f(x,w)]\| \\
&\qquad \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)
\end{aligned} \tag{10}$$

for all $\lambda,\mu \in \mathbb{T}^1$ and all $x,y,z,w \in A$. Then there exists a unique bihomomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\| \le \frac{2\theta}{4-2^p}(\|x\|^p + \|y\|^p)$$

for all $x,y \in A$.

Proof. Defining $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and letting $L = 2^{p-2}$ in Theorem 2.3, we obtain the desired result. □

Similarly, one can obtain the following.

Theorem 2.5. *Let $f : A \times A \to B$ be a mapping for which there exists a function $\varphi : A^4 \to [0,\infty)$ satisfying (2) and (3). If there exists an $L < 1$ such that $\varphi(x,y,z,w) \le \frac{L}{64}\varphi(2x,2y,2z,2w)$ for all $x,y,z,w \in A$, then there exists a unique bihomomorphism $H : A \times A \to B$ such that*

$$\|f(x,z) - H(x,z)\| \le \frac{L}{64-64L}\varphi(x,x,z,z) \tag{11}$$

for all $x,z \in A$.

Note that $\frac{L}{64}\varphi(2x,2y,2z,2w) \le \frac{L}{4}\varphi(2x,2y,2z,2w)$ for all $x, y, zw \in A$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := 4g\left(\frac{x}{2}, \frac{z}{2}\right)$$

for all $x, z \in A$.

It follows from (5) that

$$\left\|f(x,z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right) \le \frac{L}{64}\varphi\left(x,x,z,z\right) \le \frac{L}{4}\varphi\left(x,x,z,z\right)$$

for all $x, z \in A$. Thus $d(f, Jf) \le \frac{L}{4}$. So

$$d(f, H) \le \frac{L}{4-4L}.$$

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. Let p and θ be positive real numbers with $p > 6$, and let $f : A\times A \to B$ be a mapping satisfying (9) and (10). Then there exists a unique bihomomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\| \le \frac{2\theta}{2^p - 4}(\|x\|^p + \|y\|^p)$$

for all $x, y \in A$.

Proof. Defining $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and letting $L = 2^{2-p}$ in Theorem 2.5, we obtain the desired result. □

3. Hyers-Ulam stability of biderivations on Banach Lie triple systems

In this section, we prove the Hyers-Ulam stability of biderivations on Banach Lie triple systems.

Theorem 3.1. *Let $f : A \times A \to A$ be a mapping such that*

$$\|D_{\lambda,\mu}f(x,y,z,w)\| \;\le\; \varphi(x,y,z,w), \tag{12}$$

$$\begin{aligned} &\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\ &\quad+\|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\| \\ &\quad\le \varphi(x,y,z,w) \end{aligned} \tag{13}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Assume that there exists an $L < 1$ such that $\varphi(x,y,z,w) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right)$ for all $x, y, z, w \in A$. If the mapping $f : A \times A \to A$ satisfies

$$\lim_{n\to\infty}\frac{1}{4^n}f(2^n x, 2^n z) = \lim_{n\to\infty}\frac{1}{16^n}f(8^n x, 2^n z) = \lim_{n\to\infty}\frac{1}{16^n}f(2^n x, 8^n z) \tag{14}$$

for all $x, z \in A$, then there exists a unique biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \le \frac{1}{4-4L}\varphi(x,x,z,z) \tag{15}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, we get a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ given by $\delta(x,z) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n z)$ satisfying (15).

It follows from (13) and (14) that

$$\begin{aligned}
&\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\
&\quad + \|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\
&\quad = \lim_{n\to\infty} \frac{1}{16^n}(\|f(8^n[x,y,z],2^n w) - [f(2^n x,2^n w),2^n y,2^n z] \\
&\qquad\quad - [2^n x, f(2^n y, 2^n w^*), 2^n z] - [2^n x, 2^n y, f(2^n z, 2^n w)]\| \\
&\qquad + \|f(2^n x, 8^n[y,z,w]) - [f(2^n x, 2^n y), 2^n z, 2^n w] \\
&\qquad\quad - [2^n y, f(2^n x^*, 2^n z), 2^n w] - [2^n y, 2^n z, f(2^n x, 2^n w)]\|) \\
&\quad \le \lim_{n\to\infty} \frac{1}{16^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) \le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0
\end{aligned}$$

for all $x,y,z,w \in A$. So

$$\delta([x,y,z],w) = [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)]$$

and

$$\delta(x,[y,z,w]) = [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)]$$

for all $x,y,z,w \in A$.

Therefore, the mapping $\delta : A\times A \to A$ is a biderivation satisfying (15), as desired. □

Corollary 3.2. Let p and θ be positive real numbers with $p < 2$, and let $f : A\times A \to A$ be a mapping satisfying (14) and

$$\|D_{\lambda,\mu} f(x,y,z,w)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p), \tag{16}$$

$$\begin{aligned}
&\|f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w^*),z] - [x,y,f(z,w)]\| \\
&\quad + \|f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x^*,z),w] - [y,z,f(x,w)]\| \\
&\quad \le \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)
\end{aligned} \tag{17}$$

for all $\lambda,\mu \in \mathbb{T}^1$ and all $x,y,z,w \in A$. Then there exists a unique biderivation $\delta : A \times A \to A$ such that

$$\|f(x,y) - \delta(x,y)\| \le \frac{2\theta}{4-2^p}(\|x\|^p + \|y\|^p)$$

for all $x,y \in A$.

Proof. Defining $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and letting $L = 2^{p-2}$ in Theorem 3.1, we obtain the desired result. □

Theorem 3.3. *Let $f : A\times A \to A$ be a mapping satisfying* (12) *and* (13). *Assume that there exists an $L < 1$ such that $\varphi(x,y,z,w) \le \frac{L}{16}\varphi(2x,2y,2z,2w)$ for all $x,y,z,w \in A$. If the mapping $f : A\times A \to A$ satisfies*

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n},\frac{z}{2^n}\right) = \lim_{n\to\infty} 16^n f\left(\frac{x}{8^n},\frac{z}{2^n}\right) = \lim_{n\to\infty} 16^n f\left(\frac{x}{2^n},\frac{z}{8^n}\right) \tag{18}$$

for all $x,z \in A$, then there exists a unique biderivation $\delta : A\times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \le \frac{L}{4-4L}\varphi(x,x,z,z) \tag{19}$$

for all $x, z \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, we get a unique $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ given by $\delta(x,z) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{z}{2^n}\right)$ satisfying (19).

It follows from (13) and (18) that

$$\begin{aligned}
&\|\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w^*),z] - [x,y,\delta(z,w)]\| \\
&\quad + \|\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x^*,z),w] - [y,z,\delta(x,w)]\| \\
&= \lim_{n\to\infty} 16^n \left(\left\| f\left(\frac{[x,y,z]}{8^n}, \frac{w}{2^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{w}{2^n}\right), \frac{y}{2^n}, \frac{z}{2^n}\right] \right.\right. \\
&\qquad \left. - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{w^*}{2^n}\right), \frac{z}{2^n}\right] - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{w}{2^n}\right)\right] \right\| \\
&\quad + \left\| f\left(\frac{x}{2^n}, \frac{[y,z,w]}{8^n}\right) - \left[f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), \frac{z}{2^n}, \frac{w}{2^n}\right] - \right. \\
&\qquad \left.\left. \left[\frac{y}{2^n}, f\left(\frac{x^*}{2^n}, \frac{z}{2^n}\right), \frac{w}{2^n}\right] - \left[\frac{y}{2^n}, \frac{z}{2^n}, f\left(\frac{x}{2^n}, \frac{w}{2^n}\right)\right] \right\| \right) \\
&\le \lim_{n\to\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0
\end{aligned}$$

for all $x, y, z, w \in A$. So

$$\delta([x,y,z],w) = [\delta(x,w),y,z] + [x,\delta(y,w^*),z] + [x,y,\delta(z,w)]$$

and

$$\delta(x,[y,z,w]) = [\delta(x,y),z,w] + [y,\delta(x^*,z),w] + [y,z,\delta(x,w)]$$

for all $x, y, z, w \in A$.

Therefore, the mapping $\delta : A\times A \to A$ is a biderivation satisfying (19), as desired. □

Corollary 3.4. Let θ and p be positive real numbers with $p > 4$ and let $f : A \times A \to A$ be a mapping satisfying (16), (17) and (18). Then there exists a unique biderivation $\delta : A \times A \to A$ such that

$$\|f(x,z) - \delta(x,z)\| \le \frac{2\theta}{2^p - 4}(\|x\|^p + \|z\|^p) \tag{20}$$

for all $x, z \in A$.

Proof. Defining $\varphi(x,y,z,w) := \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and letting $L = 2^{2-p}$ in Theorem 3.3, we obtain the desired result. □

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Dynamics of some Rational Difference Equations

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Abstract

The main goal of this paper is to investigate the qualitative behavior of the solutions for the following rational difference equation:

$$x_{n+1} = \frac{\alpha + \sum_{i=0}^{k} a_{2i} x_{n-2i}}{\beta + \sum_{i=0}^{k} b_{2i+1} x_{n-2i-1}}, \qquad n = 0,\ 1,\ 2,\ ...$$

where $\alpha,\ \beta,\ a_i,\ b_i \in (0,\infty),\ i = 0,1,...,k$; with the initial conditions $x_0,\ x_{-1},\ ...,\ x_{-2k}, x_{-2k-1} \in (0,\infty)$. We determine the equilibrium points of the considered equation and then study their local stability. Also we study the boundedness and the permanence of the solutions. Finally we investigate the global asymptotically stable of the equilibrium points.

Keywords: permanence, global stability, difference equations.
Mathematics Subject Classification: 39A10

1 Introduction

Rational difference equations is an important class of difference equations where they have many applications, for example, the difference equation

$$x_{n+1} = \frac{a+bx_n}{c+x_n}, \qquad n \geq 0,$$

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has applications in Optics and Mathematical Biology and is known in the literature as the Riccati difference equation. The equation

$$x_{n+1} = \frac{1+x_n}{x_{n-1}}, \qquad n \geq 0,$$

was discovered by Lyness [12] while he was working on a problem in Number Theory. Also this equation has many applications in geometry (see Leech [10]) and in frieze patterns (see Conway and Coxeter [4]). Also, we believe that the results about rational difference equations are of paramount importance in their on right and offer prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one, so, there has been a great interest in studying the qualitative properties of rational difference equations by several authors such as Kulenovic and Ladas [9] presented some known results and derived several new ones on the boundedness, the global stability, and the periodicity of solutions of all rational difference equations of the form

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}.$$

Also, Camouzis and Ladas [2] presented the global character of solutions of the third-order rational difference equation. They presented a summary of the recent work and a large number of open problems and conjectures on the third order rational recursive sequence of the form

$$x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}+\delta x_{n-2}}{A+Bx_n+Cx_{n-1}+Dx_{n-2}}.$$

Li and Sun [11] investigated the periodic character, invariant intervals and global stability of all positive solutions of the recursive sequence

$$x_{n+1} = \frac{px_n+x_{n-k}}{q+x_{n-k}}.$$

Kocic et al. [8] examined the periodicity and oscillating properties of the positive solutions as well as the global attractivity of the nonnegative equilibrium of the difference equation

$$x_{n+1} = \frac{a+bx_n}{d+x_{n-k}}.$$

In [6], the author studied the boundedness, the existence of prime period to solutions and the global attracitvity of solutions of the following recursive sequences

$$x_{n+1} = \frac{ax^{l_0}_{n-k_0}x^{l_1}_{n-k_1}...x^{l_i}_{n-k_i}+bx^{s_0}_{n-r_0}x^{s_1}_{n-r_1}...x^{s_j}_{n-r_j}}{cx^{l_0}_{n-k_0}x^{l_1}_{n-k_1}...x^{l_i}_{n-k_i}+dx^{s_0}_{n-r_0}x^{s_1}_{n-r_1}...x^{s_j}_{n-r_j}}, \qquad n \geq 0, \tag{1}$$

$$y_{n+1} = \frac{\alpha_0 y_n+\alpha_1 y_{n-1}+...+\alpha_t y_{n-t}}{\beta_0 y_n+\beta_1 y_{n-1}+...+\beta_t y_{n-t}}, \qquad n \geq 0. \tag{2}$$

Some related results to rational difference equations can be found in [1,3,5,13] and the references therein.

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Let I be an interval of real numbers and let $F : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$y_{n+1} = F\left(y_n, y_{n-1}, ..., y_{n-k}\right), \quad n = 0, 1, ..., \tag{3}$$

with $y_{-k}, ..., y_0 \in I$.
Recall that the point $\overline{y} \in I$ is called an equilibrium point of Eq.(3) if

$$F(\overline{y}, \overline{y}, ..., \overline{y}) = \overline{y}.$$

That is, $y_n = \overline{y}$ for $n \geq 0$, is a solution of Eq.(3), or equivalently, $\overline{y}$ is a fixed point of F.
Let $\overline{y}$ be an equilibrium point of Eq.(3). Then the linearized equation of Eq.(3) about $\overline{y}$ is given by

$$w_{n+1} = \sum_{i=0}^{k} p_i w_{n-i}, \quad n = 0, 1, ..., \tag{4}$$

where $p_i = \frac{\partial f}{\partial y_{n-i}}(\overline{y}, ..., \overline{y}), i = 0, 1, 2..., k$ and the characteristic equation of Eq.(4) is

$$\lambda^{(k+1)} - p_1\lambda^k - p_2\lambda^{(k-1)} - ... - p_k\lambda - p_{(k+1)} = 0.$$

Theorem A [8]: Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$u_{n+1} + pu_n + qu_{n-k} = 0, \quad n = 0, 1, ...$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$u_{n+k} + p_1 u_{n+k-1} + ... + p_k u_n = 0, \quad n = 0, 1, ... \tag{5}$$

where $p_1, p_2, ..., p_k$ and $k \in \{1, 2, ...\}$. Then Eq.(5) is asymptotically stable provided that $\sum_{i=1}^{k} |p_i| < 1$.

Theorem B [7]: Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of Eq.(3), and suppose that there exist constants $A \in I$ and $B \in I$ such that $A \leq y_n \leq B$ for all $n \geq -k$. Let ℓ_0 be a limit point of the sequence $\{y_n\}_{n=-k}^{\infty}$. Then the following statements are true:
(i) There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of Eq.(3), called a full limiting sequence of $\{y_n\}_{n=-k}^{\infty}$, such that $L_0 = \ell_0$, and such that for every $N \in \{...,-1,0,1,...\}$ L_N is a limit point of $\{y_n\}_{n=-k}^{\infty}$.
(ii) For every $i_0 \leq -k$, there exists a subsequence $\{y_{r_i}\}_{i=0}^{\infty}$ of $\{y_n\}_{n=-k}^{\infty}$ such that $L_N = \lim_{i\to\infty} y_{r_i+N}$ for every $N \geq i_0$.

Theorem C [9]: Let $[p, q]$ be an interval of real numbers and assume that

$$g : [p, q]^3 \to [p, q],$$

is a continuous function satisfying the following properties :
(a) $g(x, y, z)$ is non-decreasing in y and z in $[p, q]$ for each $x \in [p, q]$, and is non-increasing in $x \in [p, q]$ for each y and z in $[p, q]$;
(b) If $(m, M) \in [p, q] \times [p, q]$ is a solution of the system

$$M = g(m, M, M) \quad and \quad m = g(M, m, m),$$

then $m = M$. Then Eq.(5) has a unique equilibrium $\overline{x} \in [p, q]$ and every solution of Eq.(5) converges to $\overline{x}$.
In this paper we study the boundedness character and investigate the global stability for the solutions of the following difference equation:

$$x_{n+1} = \frac{\alpha + \sum\limits_{i=0}^{[\frac{k+1}{2}]} a_{2i} x_{n-2i}}{\beta + \sum\limits_{i=0}^{[\frac{k+1}{2}]} b_{2i+1} x_{n-2i-1}}, \qquad n = 0,\ 1,\ 2,\ .., \tag{6}$$

where $\alpha,\ \beta \in [0, \infty),\ a_i,\ b_i \in (0, \infty)$ for $i = 0, 1, ..., k$; and $k \in \{10, 1, 2, ...\}$ with the initial conditions $x_0,\ x_{-1}, ..., x_{-2k},\ x_{-2k-1} \in (0,\infty)$.

2 Local Stability of Eq.(6)

In this section we discuss the local stability of the equilibrium points of Eq.(6). Let $A = \sum\limits_{i=0}^{[\frac{k+1}{2}]} a_{2i}$ and $B = \sum\limits_{i=0}^{[\frac{k+1}{2}]} b_{2i+1}$, then the following statements are true:
(i) At $\alpha = 0$ and $\beta = 0$, Eq.(6) has the equilibrium point $\overline{x} = 0$ and the unique positive equilibrium point $\overline{x} = \frac{A}{B}$.
(ii) At $\alpha = 0$ and $\beta < A$, Eq.(6) has the equilibrium point $\overline{x} = 0$ and the positive equilibrium point $\overline{x} = \frac{A-\beta}{B}$.
(iii) At $\alpha = 0$ and $\beta \geq A$, Eq.(6) has the unique equilibrium point $\overline{x} = 0$.
(iv) At $\alpha \neq 0$ and $\beta = 0$, Eq.(6) has the unique positive equilibrium point $\overline{x} = \frac{A+\sqrt{A^2+4\alpha B}}{2B}$.
(v) At $\alpha \neq 0$ and $\beta \neq 0$, Eq.(6) has the unique positive equilibrium point $\overline{x} = \frac{A-B+\sqrt{(A-B)^2+4\alpha B}}{2B}$.
The following theorem deals with the local stability of the positive equilibrium point of Eq.(6).

Theorem 1 *The equilibrium point $\overline{x} = \frac{A-B+\sqrt{(A-B)^2+4\alpha B}}{2B}$ of Eq.(6) is locally stable if $A < \beta$.*

Proof. The linearized equation of Eq.(6) about the equilibrium point $\overline{x} = \frac{A-B+\sqrt{(A-B)^2+4\alpha B}}{2B}$ is given by

$$z_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial x_{n-i}} z_{n-i} = \sum_{i=0}^{k} \frac{a_{2i}}{\beta+B\overline{x}} z_{n-2i} - \sum_{i=0}^{k} \frac{b_{2i+1}(\alpha+A\overline{x})}{(\beta+B\overline{x})^2} z_{n-2i-1}$$

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for $n = 0, 1, ...$ and the associated characteristic equation about $\overline{x}$ is

$$F(\lambda) = \lambda^{2k+2} - \sum_{i=0}^{k} \tfrac{a_{2i}}{\beta+B\overline{x}} \lambda^{2(k-i)+1} + \sum_{i=0}^{k} \tfrac{b_{2i+1}(\alpha+A\overline{x})}{(\beta+B\overline{x})^2} \lambda^{2(k-i)} = 0.$$

Then it follows by Theorem A that $\overline{x}$ is locally stable if

$$\tfrac{A}{\beta+B\overline{x}} + \tfrac{B(\alpha+A\overline{x})}{(\beta+B\overline{x})^2} < 1 \Longleftrightarrow \tfrac{A}{\beta+B\overline{x}} + \tfrac{B\overline{x}}{\beta+B\overline{x}} < 1 \Longleftrightarrow A < \beta.$$

This ends the proof of the theorem.

3 Boundedness of Solutions

Here we study the permanence of Eq.(6).

Theorem 2 *Assume that $A < \beta$. Then every solution of Eq.(6) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6). Then

$$x_{n+1} = \frac{\alpha + \sum\limits_{i=0}^{[\frac{k+1}{2}]} a_{2i}x_{n-2i}}{\beta + \sum\limits_{i=0}^{[\frac{k+1}{2}]} b_{2i+1}x_{n-2i-1}} \le \tfrac{\alpha}{\beta} + \tfrac{1}{\beta} \sum_{i=0}^{[\frac{k+1}{2}]} a_{2i}x_{n-2i}.$$

Then $\limsup\limits_{n\longrightarrow\infty} x_n \le \frac{\alpha}{\beta-A} \overset{def}{=} M$. Thus $x_n \le M$ for all $n \ge 1$.
Now we wish to find a constant $m > 0$ such that $x_n \ge m$ for all $n \ge 1$. The change of variables $x_n = \frac{1}{y_n}$, gives Eq.(6) in the form

$$\frac{1}{y_{n+1}} = \frac{\alpha + \frac{a_0}{y_n} + \frac{a_2}{y_{n-2}} + ... + \frac{a_{2k}}{y_{n-2k}}}{\beta + \frac{b_1}{y_{n-1}} + \frac{b_3}{y_{n-3}} + ... + \frac{b_{2k+1}}{y_{n-2k-1}}},$$

or in the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{\beta + \frac{b_1}{y_{n-1}} + \frac{b_3}{y_{n-3}} + ... + \frac{b_{2k+1}}{y_{n-2k-1}}}{\alpha + \frac{a_0}{y_n} + \frac{a_2}{y_{n-2}} + ... + \frac{a_{2k}}{y_{n-2k}}} \le \frac{\beta + \frac{b_1}{y_{n-1}} + \frac{b_3}{y_{n-3}} + ... + \frac{b_{2k+1}}{y_{n-2k-1}}}{\alpha} \\ &\le \frac{\beta + M\sum\limits_{i=0}^{k} b_{2i+1}}{\alpha} \le \frac{\beta + MB}{\alpha} \qquad \text{for all } n \ge 1. \end{aligned}$$

Thus we obtain

$$x_n = \frac{1}{y_n} \ge \frac{\alpha}{\beta + BM} \overset{def}{=} m \quad \text{for all} \quad n \ge 1.$$

Therefor we see that

$$m \le x_n \le M \qquad \text{for all} \quad n \ge 1.$$

Then the proof is so complete.

Theorem 3 *Assume that $A < \beta$. Then every solution of Eq.(6) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6).
Set $H = \max\{x_{-2k-1}, x_{-2k}, ..., x_0, \frac{\alpha}{\beta - A}\}$, then it follows from (6) that

$$x_1 = \frac{\alpha + \sum_{i=0}^{[\frac{k+1}{2}]} a_{2i}x_{-2i}}{\beta + \sum_{i=0}^{[\frac{k+1}{2}]} b_{2i+1}x_{-2i-1}} \leq \frac{\alpha + H\sum_{i=0}^{[\frac{k+1}{2}]} a_{2i}}{\beta} \leq \frac{\alpha + AH}{\beta} \leq H.$$

It follows by induction that $x_n \leq H$ for all $n \geq 0$. Now we wish to find a constant $h > 0$ such that $x_n \geq h$ for all $n \geq 1$.
Again it follows from Eq.(7) that

$$x_1 = \frac{\alpha + \sum_{i=0}^{[\frac{k+1}{2}]} a_{2i}x_{-2i}}{\beta + \sum_{i=0}^{[\frac{k+1}{2}]} b_{2i+1}x_{-2i-1}} \geq \frac{\alpha}{\beta + H\sum_{i=0}^{[\frac{k+1}{2}]} b_{2i+1}} \geq \frac{\alpha}{\beta + BH} \stackrel{def}{=} h.$$

Then it follows by induction that $x_n \geq h$ for all $n \geq 1$.

4 Global Stability of the Equilibrium Points

Theorem 4 *Let $\alpha = 0$ and assume that $A < \beta$. Then every nonnegative solution of Eq.(6) converges to the unique equilibrium point of Eq.(6) $\overline{x} = 0$.*

Proof. It follows by Theorem B that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(6) with

$$I = I_0 = \lim_{n\to\infty} \inf\ x_n \ \leq\ \lim_{n\to\infty} \sup\ x_n = S_0 = S,$$

where

$$I_n,\ S_n \in [I,S]\ ,\quad n = 0, -1, ...\ .$$

Since $\overline{x} = 0$ is the unique nonnegative equilibrium point of Eq.(6), then it suffices to show that $S = 0$. Suppose for the sake of contradiction that $S > 0$.Then it follows from Eq.(6) that

$$S = \frac{a_0S_{-1} + a_2S_{-3} + ... + a_{2k}S_{-2k-1}}{\beta + b_1S_{-2} + b_3S_{-4} + ... + b_{2k+1}S_{-2k-1}} \leq \frac{AS}{\beta + BI},$$

and so

$$0 \leq BSI \leq (A - \beta)S < 0,$$

which is a contradiction. Hence

$$\lim_{n\to\infty} x_n = \overline{x} = 0.$$

which is true by the hypothesis of the theorem and this completes the proof.

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Theorem 5 *Assume that $A < \beta$, then the equilibrium point*

$$\overline{x} = \frac{A - B + \sqrt{(A-B)^2 + 4\alpha B}}{2B}$$

of Eq.(6) is global attractor of the solutions of Eq.(6).

Proof. It was shown in Theorem 2 that $m \leq x_n \leq M$ for all $n \geq 1$.Then it follows again by Theorem B that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(6) with

$$I = I_0 = \lim_{n\to\infty} \inf\ x_n \ \leq \ \lim_{n\to\infty} \sup\ x_n \ = S_0 = S,$$

where

$$I_n,\ S_n \in [I,S]\ , \quad n = 0, -1,... \ .$$

It suffices to show that $I \ \geq \ S$.
Now it follows from Eq.(6) that

$$I = \frac{\alpha + a_0 I_{-1} + a_2 I_{-3} + ... + a_{2k} I_{-2k-1}}{\beta + b_1 I_{-2} + b_1 I_{-2} a_2 I_{-3} + ... + a_{2k} I_{-2k-1}} \geq \frac{\alpha + AI}{\beta + BS}$$

and so

$$(\beta - A)I + BSI \geq \alpha. \tag{7}$$

Similarly, we see from Eq.(6) that

$$S = \frac{\alpha + a_0 S_{-1} + a_2 S_{-3} + ... + a_{2k} S_{-2k-1}}{\beta + b_1 S_{-2} + b_3 S_{-4} + ... + b_{2k+1} S_{-2k-1}} \leq \frac{\alpha + AS}{\beta + BI},$$

and so

$$(\beta - A)S + BSI \leq \alpha. \tag{8}$$

Then we obtain from relations (7) and (8) that

$$(\beta - A)S + BSI \leq \alpha \leq (\beta - A)I + BSI,$$

thus

$$(\beta - A)(I - S) \geq 0$$

and since $(\beta - A) > 0$, then we should have $I \geq S$.This completes the proof.
We give the following two theorems which is a minor modification of Theorem A.0.2 in [9].

Theorem 6 *Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b]^{k+1} \to [a, b]$, is a continuous function satisfying the following properties:*

(i) $f(x_0, x_1, ..., x_k)$ is non-decreasing in its arguments x_{2t} with $t \in \left\{0, 1, 2, ..., [\frac{k+1}{2}]\right\}$ for each x_r $(r \neq 2t)$ in $[a, b]$ and non-increasing in its arguments x_{2r+1} with $r \in \left\{0, 1, 2, ..., [\frac{k+1}{2}]\right\}$ for all x_t $(t \neq r)$ in $[a, b]$.

(ii) If $(M,m) \in [a,b] \times [a,b]$ is a solution of the system

$$M = f(M,m,M,m,...,M,m,...,M,m), \; m = f(m,M,m,M,...,m,M...,m,M),$$

implies $m = M$. Then the difference equation $x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k})$ has a unique equilibrium $\overline{x} \in [a,b]$ and every solution of the equation converges to $\overline{x}$.

Proof. Set

$$m_0 = a \quad and \quad M_0 = b,$$

and for each $i = 1, 2, ...$ set

$$m_i = f(m_{i-1}, M_{i-1}, m_{i-1}, M_{i-1}, ..., m_{i-1}, M_{i-1}, m_{i-1}, ..., m_{i-1}, M_{i-1}),$$

and

$$M_i = f(M_{i-1}, m_i, M_{i-1}, m_i, ..., M_{i-1}, m_{i-1}, M_{i-1}, ..., M_{i-1}, m_{i-1}).$$

Now observe that for each $i \geq 0$,

$$a = m_0 \leq m_1 \leq ... \leq m_i \leq ... \leq M_i \leq ... \leq M_1 \leq M_0 = b,$$

and

$$m_i \leq x_p \leq M_i \quad for \quad p \geq (k+1)i + 1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad and \quad M = \lim_{i\to\infty} M_i.$$

Then

$$M \geq \lim_{i\to\infty} \sup x_i \geq \lim \inf_{i\to\infty} x_i \geq m$$

and by the continuity of f,

$$M = f(M,m,M,m,...,M,m,..,M,m) \; and \; m = f(m,M,m,M,...,m,M,...,m,M).$$

In view of (ii),

$$m = M = \overline{x},$$

from which the result follows.

Theorem 7 *Let $[a,b]$ be an interval of real numbers and assume that $f : [a,b]^{k+1} \to [a,b]$, is a continuous function satisfying the following properties:*

(i) $f(x_0, x_1, ..., x_k)$ is non-increasing in its arguments x_{2t} with $t \in \{0, 1, 2, ..., [\frac{k+1}{2}]\}$ for each x_r $(r \neq 2t)$ in $[a,b]$ and non-decreasing in its arguments x_{2r+1} with $r \in \{0, 1, 2, ..., [\frac{k+1}{2}]\}$ for all x_t $(t \neq r)$ in $[a,b]$.
(ii) If $(M,m) \in [a,b] \times [a,b]$ is a solution of the system

$$M = f(m,M,m,M,...,m,M,...,m,M), \; m = f(M,m,M,m,...,M,m...,M,m),$$

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implies

$$m = M.$$

Then the difference equation $x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k})$ has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of the equation converges to $\overline{x}$.

Proof. As the proof of Theorem 6 and will be omitted.

Theorem 8 *Assume that $A \neq \beta$. Then every nonnegative solution of Eq.(6) converges to the unique equilibrium point of Eq.(6) $\overline{x} = \frac{A-B+\sqrt{(A-B)^2+4\alpha B}}{2B}$.*

Proof. Rewrite Eq. (6) in the following form

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}) = \frac{\alpha + \sum\limits_{i=0}^{[\frac{k+1}{2}]} a_{2i} x_{n-2i}}{\beta + \sum\limits_{i=0}^{[\frac{k+1}{2}]} b_{2i+1} x_{n-2i-1}}, \quad n = 0,\ 1,\ 2,\ \dots .$$

Then the function f satisfies the hypotheses (i) of Theorem 6. Now consider the system

$M = f(M, m, M, m, ..., M, m) = \frac{\alpha+AM}{\beta+Bm}$, $m = f(m, M, m, M, ..., m, M) = \frac{\alpha+Am}{\beta+BM}$.

Thus it is easy to see that $m = M$. Therefore the function f satisfies the hypotheses (ii) of Theorem 6 too. Then it follows by Theorem 6 that the equilibrium point $\overline{x} = \frac{A-B+\sqrt{(A-B)^2+4\alpha B}}{2B}$ of Eq.(6) is a global attractor of the solutions of Eq.(6).

Theorem 9 *Let $\alpha = 0$ and $A < \beta$. Then every nonnegative solution of Eq.(6) converges to the unique equilibrium point of Eq.(6) $\overline{x} = 0$.*

Proof. The proof is similar to the proof of Theorem 8 and will be omitted.

Theorem 10 *Let $\alpha = 0$ and $\beta = 0$. Then every positive solution of Eq.(6) converges to the unique positive equilibrium point of Eq.(6) $\overline{x} = \frac{A}{B}$.*

Proof. The proof is similar to the proofs of Theorem 5 and Theorem 8 and will be omitted.

Theorem 11 *Let $\alpha = 0$ and $\beta < A$. Then every nonnegative solution of Eq.(6) converges to the unique positive equilibrium point of Eq.(6) $\overline{x} = \dfrac{A-\beta}{B}$.*

Proof. The proof is similar to the proofs of Theorem 5 and Theorem 8 and will be omitted.

Theorem 12 *Let $\alpha \neq 0$ and $\beta = 0$. Then every positive solution of Eq.(6) converges to the unique positive equilibrium point of Eq.(6) $\overline{x} = \dfrac{A + \sqrt{A^2 + 4\alpha B}}{2B}$.*

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Proof. The proof is similar to the proofs of Theorem 5 and Theorem 8 and will be omitted.

Remark: It is easy to obtain -by using Theorem B and Theorem 7- similar results for the following difference equation

$$y_{n+1} = \frac{\delta + \sum\limits_{i=0}^{[\frac{k+1}{2}]} p_{2i+1} y_{n-2i-1}}{\gamma + \sum\limits_{i=0}^{[\frac{k+1}{2}]} q_{2i} y_{n-2i}}.$$

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Generalized integration operators from Hardy spaces to Zygmund-type spaces *†

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Abstract Let $H(\mathbb{D})$ denote the space of all holomorphic functions on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Let φ be a holomorphic self-map of $\mathbb{D}$, n be a positive integer and $g \in H(\mathbb{D})$. In this paper, we investigate the boundedness and compactness of a generalized integration operator

$$I_{g,\varphi}^{(n)} = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d(\zeta)$$

from Hardy spaces to the Zygmund-type spaces $\mathcal{Z}_\mu$.

1 Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$.

For $0 < r < 1$, $f \in H(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\theta})\right|^p d\theta\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \max_{0\le\theta\le 2\pi}\left|f(re^{i\theta})\right|.$$

For $0 < p \le \infty$, the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$, for which

$$\|f\|_p = \sup_{0\le r<1} M_p(r,f) < \infty.$$

It is well known that with norm $\|f\|_p$ the H^p space is a Banach space if $1 \le p \le \infty$, for $0 < p < 1$, H^p space is a nonlocally convex topological vector space, and $d(f,g) = \|f-g\|_p^p$ is a complete metric for it. For more information about the H^p space, one may see these books, for example, [4, 5].

Let μ be a weight, that is, μ is a positive continuous function on $\mathbb{D}$. The Zygmund-type $\mathcal{Z}_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

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With the norm $\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$, it becomes a Banach space. The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ is a subspace of $\mathcal{Z}_\mu$ consisting of those $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z|\to 1} \mu(z)|f''(z)| = 0.$$

When $\mu(z) = (1 - |z|^2)$, the Zygmund-type space becomes the Zygmund space $\mathcal{Z}$ ([7]), while the little Zygmund-type space $\mathcal{Z}_{\mu,0}$ becomes the little Zygmund space $\mathcal{Z}_0$.

Let φ be an analytic self-map of $\mathbb{D}$, then the composition operator on $H(\mathbb{D})$ is given by

$$C_\varphi f = f \circ \varphi.$$

Composition operators acting on various spaces of analytic functions have been the object for recent years, especially the problems of relating operator-theoretic properties of C_φ to function the theoretic properties of φ. See the book of Cowen and MacCluer ([3]) and Shapiro ([12]) for discussions of composition operators on classical spaces of analytic functions.

Assume that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map of the unit disk $\mathbb{D}$, for $f \in H(\mathbb{D})$, define

$$I_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta, z \in \mathbb{D}.$$

This operator is called Riemann-Stieltjes operator (or Extended-Ces*à*ro operator). Ch. Pommerenke ([11]) initiated the study of Riemann-Stieltjes operator I_g on H^2 , where he showed that I_g is bounded on H^2 if and only if g is in $BMOA$. This was extended to other Hardy spaces H^p $(1 \le p < \infty)$ in [1] and [2] where compactness of I_g on H^p and Schatten class membership of I_g on H^2 was also completely characterized in terms of the symbol g.

In this paper, we consider an integration operator $I_{g,\varphi}^{(n)}$ which is defined as

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta, z \in \mathbb{D}.$$

This operator is called the generalized integration operator, which was introduced in [13] and studied in [6, 13, 14, 20]. Also, the operator $I_{g,\varphi}^{(n)}$ is a generalization of the Riemann- Stieltjes operator I_g induced by g. In fact, the operator $I_{g,\varphi}^{(n)}$ can induce many known operators. For example, when $n = 1$, $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently studied by S. Stević, S. Li, X. Zhu and W. Yang in [8, 15, 19]. When $n = 1$ and $g(z) = \varphi'(z)$, we obtain the composition operator C_φ defined as $C_\varphi f = f \circ \varphi + f(\varphi(0))$, $f \in H(\mathbb{D})$. Let $\mathcal{D}$ be the differentiation operator, $n = m + 1$ and $g(z) = \varphi'(z)$, then we get the operator $C_\varphi \mathcal{D}^m f(z) = f^{(m)}(\varphi(z)) - f^{(m)}(\varphi(0))$ which was studied in [10, 18].

In [13], S. D. Sharma and A. Sharmat have characterized the boundedness and compactness of generalized integration operators $I_{g,\varphi}^{(n)}$ from Bloch type spaces to weighted $BMOA$ spaces by using logarithmic Carleson measure characterization of the weighted $BMOA$ spaces. In [20], X. Zhu characterized the boundedness and compactness of generalized integration operators from H^∞ to Zygmund-type spaces. In [6], Z. He and G. Cao have characterized the boundedness and compactness of generalized integration operators between Bloch-type spaces and $F(p, q, s)$ spaces. Motivated by the results [6, 13, 14, 20], this paper is devoted to investigating the boundedness and compactness of generalized integration operators from Hardy spaces H^p $(0 < p < \infty)$ to Zygmund-type spaces.

Through out this paper, we will use the letter C to denote a generic positive constant that can change its value at each occurrence.

2 Auxiliary results

Here we quote some auxiliary results which will be used in the proofs of the main results in this paper.

LEMMA 2.1 ([4]) *For $p>1$, there exists a constant $C(p)$ such that*

$$\int_0^{2\pi} \frac{d\theta}{|1-z|^p} \le \frac{C(p)}{(1-|z|^2)^{p-1}}\ , \quad for\ every\ z\in\mathbb{D}.$$

LEMMA 2.2 ([4, 5, 16]) *Suppose that $0<p<\infty$, $f\in H^p$, then*

$$\left|f^{(n)}(z)\right| \le C\frac{\parallel f\parallel_p}{(1-|z|^2)^{1/p+n}},$$

for every $z\in\mathbb{D}$ and all nonnegative $n=0,1,2,\cdots$.

The following criterion for the compactness is a useful tool and it follows from standard arguments, for example, [3, Proposition 3.11].

LEMMA 2.3 *Assume that n be a nonnegative integer and φ be a holomorphic self-map of $\mathbb{D}$, $0<p<\infty$, μ be a weight. Then $I_{g,\varphi}^{(n)}: H^p\to\mathcal{Z}_\mu$ is compact if and only if $I_{g,\varphi}^{(n)}: H^p\to\mathcal{Z}_\mu$ is bounded and for any bounded sequence $\{f_k\}$ in H^p which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, we have $\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu}\to 0$ as $k\to\infty$.*

The following lemma was proved in [7] similar to the corresponding lemma in [9].

LEMMA 2.4 *A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1}\lim_{f\in K}\mu(z)|f''(z)|=0.$$

LEMMA 2.5 *Assume that $0<p<\infty$, then for $a,b>0$*

$$(a+b)^p\le C\left(a^p+b^p\right).$$

3 Boundedness and compactness of $I_{g,\varphi}^{(n)}$ from H^p $(0<p<\infty)$ spaces to Zygmund-type spaces

In this section, we study the boundedness and compactness of $I_{g,\varphi}^{(n)}: H^p\to\mathcal{Z}_\mu$.

THEOREM 3.1. *Let $g\in H(\mathbb{D})$, n be a nonnegative integer and φ be a holomorphic self-map of $\mathbb{D}$, $0<p<\infty$, μ be a weight. Then $I_{g,\varphi}^{(n)}: H^p\to\mathcal{Z}_\mu$ is bounded if and only if the following conditions are satisfied*

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n}}<\infty, \tag{3.1}$$

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} < \infty. \tag{3.2}$$

Proof. Assume that (3.1) and (3.2) hold. Then for every $z \in \mathbb{D}$ and $f \in H^p$, by Lemma 2.2 we have

$$\begin{aligned}
&\mu(z)\left|(I^{(n)}_{g,\varphi}f)''(z)\right| \\
&= \mu(z)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right| \\
&\le \mu(z)|g(z)\varphi'(z)||f^{(n+1)}(\varphi(z))| + \mu(z)|g'(z)||f^{(n)}(\varphi(z))| \\
&\le C\|f\|_p \frac{\mu(z)|g(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1/p+n+1}} + C\|f\|_p \frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{1/p+n}} < \infty,
\end{aligned} \tag{3.3}$$

On the other hand , we have

$$|(I^{(n)}_{g,\varphi}f)(0)| = 0 \tag{3.4}$$

and

$$|(I^{(n)}_{g,\varphi}f)'(0)| = |f^{(n)}(\varphi(0))g(0)| \le C\frac{|g(0)|}{(1-|\varphi(0)|^2)^{1/p+n}}\|f\|_p. \tag{3.5}$$

Applying conditions (3.3), (3.4) and (3.5), we deduce that the operator $I^{(n)}_{g,\varphi} : H^p \to \mathcal{Z}_\mu$ is bounded.

Conversely we suppose that $I^{(n)}_{g,\varphi} : H^p \to \mathcal{Z}_\mu$ is bounded, that is there exists a constant C such that

$$\|I^{(n)}_{g,\varphi}f\|_{\mathcal{Z}_\mu} \le C\|f\|_p$$

for all $f \in H^p$. For $f(z) = \frac{z^n}{n!} \in H^p$, we have that

$$\sup_{z\in\mathbb{D}} \mu(z)|g'(z)| < \infty. \tag{3.6}$$

Let $f(z) = \frac{z^{n+1}}{(n+1)!} \in H^p$, we have that

$$\sup_{z\in\mathbb{D}} \mu(z)|\varphi'(z)g(z) + \varphi(z)g'(z)| < \infty. \tag{3.7}$$

By (3.6), (3.7) and the boundedness of the function $\varphi(z)$, we get

$$\sup_{z\in\mathbb{D}} \mu(z)|g(z)\varphi'(z)| < \infty. \tag{3.8}$$

For a fixed $\omega \in \mathbb{D}$, set

$$f_\omega(z) = (1/p+n+2)\frac{1-|\varphi(\omega)|^2}{(1-z\overline{\varphi(\omega)})^{1/p+1}} - (1/p+1)\frac{(1-|\varphi(\omega)|^2)^2}{(1-z\overline{\varphi(\omega)})^{1/p+2}}. \tag{3.9}$$

From Lemma 2.1 and Lemma 2.5 we have

$$
\begin{aligned}
\|f_\omega\|_p &= \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f_\omega(re^{i\theta})|^p\right)^{1/p}\\
&\le C\sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|(1/p+n+2)\frac{1-|\varphi(\omega)|^2}{(1-\overline{\varphi(\omega)}re^{i\theta})^{1/p+1}}\right|^p\right)^{1/p}\\
&+C\sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|(1/p+1)\frac{(1-|\varphi(\omega)|^2)^2}{(1-\overline{\varphi(\omega)}re^{i\theta})^{1/p+2}}\right|^p\right)^{1/p}\\
&\le C\sup_{0\le r<1}\left((1-|\varphi(\omega)|^2)^p\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\overline{\varphi(\omega)}re^{i\theta}|^{p+1}}\right)^{1/p}\\
&+C\sup_{0\le r<1}\left((1-|\varphi(\omega)|^2)^{2p}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\overline{\varphi(\omega)}re^{i\theta}|^{2p+1}}\right)^{1/p}\\
&\le C\left((1-|\varphi(\omega)|^2)^p\frac{C(p)}{(1-|\varphi(\omega)|^2)^p}\right)^{1/p}+C\left((1-|\varphi(\omega)|^2)^{2p}\frac{C(p)}{(1-|\varphi(\omega)|^2)^{2p}}\right)^{1/p}\\
&\le C,
\end{aligned}
$$

hence $f_\omega\in H^p$ and $\sup_{\omega\in\mathbb{D}}\|f_\omega\|_p\le C<\infty$.

On the other hand we get that

$$
\begin{aligned}
f_\omega^{(n)}(z) &= \left((1/p+n+2)\prod_{j=1}^{n}(1/p+j)\right)\frac{(1-|\varphi(\omega)|^2)\overline{\varphi(\omega)}^n}{(1-z\overline{\varphi(\omega)})^{1/p+n+1}}\\
&-\left((1/p+1)\prod_{j=2}^{n+1}(1/p+j)\right)\frac{(1-|\varphi(\omega)|^2)^2\overline{\varphi(\omega)}^n}{(1-z\overline{\varphi(\omega)})^{1/p+n+2}},
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
f_\omega^{(n+1)}(z) &= \left((1/p+n+2)\prod_{j=1}^{n+1}(1/p+j)\right)\frac{(1-|\varphi(\omega)|^2)\overline{\varphi(\omega)}^{n+1}}{(1-z\overline{\varphi(\omega)})^{1/p+n+2}}\\
&-\left((1/p+1)\prod_{j=2}^{n+2}(1/p+j)\right)\frac{(1-|\varphi(\omega)|^2)^2\overline{\varphi(\omega)}^{n+1}}{(1-z\overline{\varphi(\omega)})^{1/p+n+3}}.
\end{aligned}
\tag{3.11}
$$

From (3.10) and (3.11), we have $f_\omega^{(n+1)}(\varphi(\omega))=0$ and

$$
f_\omega^{(n)}(\varphi(\omega))=\left(\prod_{j=1}^{n}(1/p+j)\right)\frac{\overline{\varphi(\omega)}^n}{(1-|\varphi(\omega)|^2)^{1/p+n}}.
$$

Hence

$$\begin{aligned}
C &\geq \|I_{g,\varphi}^{(n)} f_\omega\|_{\mathcal{Z}_\mu} \\
&\geq \sup_{z\in\mathbb{D}} \mu(z) \left|(I_{g,\varphi}^{(n)} f_\omega)''(z)\right| \\
&\geq \sup_{z\in\mathbb{D}} \mu(z) \left|f_\omega^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f_\omega^{(n)}(\varphi(z))g'(z)\right| \\
&\geq \mu(|\omega|) \left|f_\omega^{(n+1)}(\varphi(\omega))\varphi'(\omega)g(\omega) + f_\omega^{(n)}(\varphi(\omega))g'(\omega)\right| \\
&= \left(\prod_{j=1}^{n}(1/p+j)\right) \frac{\mu(|\omega|)|g'(\omega)||\overline{\varphi(\omega)}|^n}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n}}. \qquad (3.12)
\end{aligned}$$

By (3.12), we have

$$\begin{aligned}
\sup_{\frac{1}{2}<|\varphi(\omega)|<1} \frac{\mu(|\omega|)|g'(\omega)|}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n}} &\leq 2^n \sup_{\frac{1}{2}<|\varphi(\omega)|<1} \left(\prod_{j=1}^{n}(1/p+j)\right) \frac{\mu(|\omega|)|g'(\omega)||\overline{\varphi(\omega)}|^n}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n}} \\
&\leq 2^n \sup_{\omega\in\mathbb{D}} \left(\prod_{j=1}^{n}(1/p+j)\right) \frac{\mu(|\omega|)|g'(\omega)||\overline{\varphi(\omega)}|^n}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n}} \\
&\leq C < \infty. \qquad (3.13)
\end{aligned}$$

And from (3.6), we obtain that

$$\begin{aligned}
\sup_{|\varphi(\omega)|\leq\frac{1}{2}} \frac{\mu(|\omega|)|g'(\omega)|}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n}} &\leq (\frac{4}{3})^{1/p+n} \sup_{|\varphi(\omega)|\leq\frac{1}{2}} \mu(|\omega|)|g'(\omega) \\
&\leq (\frac{4}{3})^{1/p+n} \sup_{\omega\in\mathbb{D}} \mu(|\omega|)|g'(\omega)| \leq C < \infty. \qquad (3.14)
\end{aligned}$$

Thus combing (3.13) with (3.14) we get the condition (3.1).

For a fixed $\omega \in \mathbb{D}$, set

$$h_\omega(z) = (1/p+n+1)\frac{1-|\varphi(\omega)|^2}{(1-z\overline{\varphi(\omega)})^{1/p+1}} - (1/p+1)\frac{(1-|\varphi(\omega)|^2)^2}{(1-z\overline{\varphi(\omega)})^{1/p+2}}, \qquad (3.15)$$

we get that

$$\begin{aligned}
h_\omega^{(n)}(z) &= \left((1/p+n+1)\prod_{j=1}^{n}(1/p+j)\right) \frac{(1-|\varphi(\omega)|^2)\overline{\varphi(\omega)}^n}{(1-z\overline{\varphi(\omega)})^{1/p+n+1}} \\
&\quad - \left((1/p+1)\prod_{j=2}^{n+1}(1/p+j)\right) \frac{(1-|\varphi(\omega)|^2)^2\overline{\varphi(\omega)}^n}{(1-z\overline{\varphi(\omega)})^{1/p+n+2}}, \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
h_\omega^{(n+1)}(z) &= \left((1/p+n+1)\prod_{j=1}^{n+1}(1/p+j)\right) \frac{(1-|\varphi(\omega)|^2)\overline{\varphi(\omega)}^{n+1}}{(1-z\overline{\varphi(\omega)})^{1/p+n+2}} \\
&\quad - \left((1/p+1)\prod_{j=2}^{n+2}(1/p+j)\right) \frac{(1-|\varphi(\omega)|^2)^2\overline{\varphi(\omega)}^{n+1}}{(1-z\overline{\varphi(\omega)})^{1/p+n+3}}. \qquad (3.17)
\end{aligned}$$

From Lemma 2.1 and Lemma 2.5 we obtain that $h_\omega \in H^p$ and $\sup_{\omega\in\mathbb{D}} \|h_\omega\|_p \le C < \infty$ with a direct calculation. From (3.16) and (3.17), we have $h_\omega^{(n)}(\varphi(\omega)) = 0$ and

$$h_\omega^{(n+1)}(\varphi(\omega)) = -\left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\overline{\varphi(\omega)}^{n+1}}{(1-|\varphi(\omega)|^2)^{1/p+n+1}}.$$

Hence

$$\begin{aligned} C &\ge \|I_{g,\varphi}^{(n)}h_\omega\|_{\mathcal{Z}_\mu} \\ &\ge \sup_{z\in\mathbb{D}}\mu(z)\left|(I_{g,\varphi}^{(n)}h_\omega)''(z)\right| \\ &\ge \sup_{z\in\mathbb{D}}\mu(z)\left|h_\omega^{(n+1)}(\varphi(z))\varphi'(z)g(z)+h_\omega^{(n)}(\varphi(z))g'(z)\right| \\ &\ge \mu(|\omega|)\left|h_\omega^{(n+1)}(\varphi(\omega))\varphi'(\omega)g(\omega)+h_\omega^{(n)}(\varphi(\omega))g'(\omega)\right| \\ &= \left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\mu(|\omega|)|g(\omega)\varphi'(\omega)||\overline{\varphi(\omega)}|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n+1}}. \end{aligned} \tag{3.18}$$

By (3.18), we have

$$\begin{aligned} &\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\frac{\mu(|\omega|)|g(\omega)\varphi'(\omega)|}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n+1}} \\ &\le 2^{n+1}\sup_{\frac{1}{2}<|\varphi(\omega)|<1}\left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\mu(|\omega|)|g(\omega)\varphi'(\omega)||\overline{\varphi(\omega)}|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n+1}} \\ &\le 2^{n+1}\sup_{\omega\in\mathbb{D}}\left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\mu(|\omega|)|g(\omega)\varphi'(\omega)||\overline{\varphi(\omega)}|^{n+1}}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n+1}} \\ &\le C<\infty. \end{aligned} \tag{3.19}$$

And from (3.8), we obtain that

$$\begin{aligned} &\sup_{|\varphi(\omega)|\le\frac{1}{2}}\frac{\mu(|\omega|)|g(\omega)\varphi'(\omega)|}{\left(1-|\varphi(\omega)|^2\right)^{1/p+n+1}} \\ &\le (\frac{4}{3})^{1/p+n+1}\sup_{|\varphi(\omega)|\le\frac{1}{2}}\mu(|\omega|)|g(\omega)\varphi'(\omega)| \\ &\le (\frac{4}{3})^{1/p+n+1}\sup_{\omega\in\mathbb{D}}\mu(|\omega|)|g(\omega)\varphi'(\omega)| \\ &\le C<\infty. \end{aligned} \tag{3.20}$$

Thus combing (3.19) with (3.20) we get the condition (3.2).

THEOREM 3.2. *Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be a holomorphic self-map of $\mathbb{D}$, $0 < p < \infty$, μ be a weight. Then $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is compact if and only if the following* *conditions are satisfied,*

$$M_1 := \sup_{z\in\mathbb{D}} \mu(z)|g'(z)| < \infty, \tag{3.21}$$

$$M_2 := \sup_{z\in\mathbb{D}} \mu(z)|g(z)\varphi'(z)| < \infty, \tag{3.22}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n}} = 0 \tag{3.23}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} = 0. \tag{3.24}$$

Proof. Assume that (3.21), (3.22), (3.23), and (3.24) hold. From (3.21) and (3.23) it is easy to see that (3.1) hold and from (3.22), (3.24) we get (3.2) hold. Hence $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is bounded, by Theorem 3.1. For any bounded sequence $\{f_k\}$ in H^p with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 2.3, to show that

$$\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} \to 0, \quad as\ k \to \infty.$$

We assume that $\|f_k\|_p \le 1$. From (3.23) and (3.24) we have for any $\varepsilon > 0$, there exists $\rho \in (0,1)$, when $\rho < |\varphi(z)| < 1$, such that

$$\frac{\mu(z)|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n}} < \varepsilon, \tag{3.25}$$

and

$$\frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} < \varepsilon. \tag{3.26}$$

Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that $f_k^{(n)}$, $f_k^{(n+1)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. There exists a $K_0 \in \mathbb{N}$ such that when $k > K_0$, from (3.21), (3.22) and Lemma 2.2, we have

$$\begin{aligned}
&|(I_{g,\varphi}^{(n)} f_k)(0)| + |(I_{g,\varphi}^{(n)} f_k)'(0)| + \sup_{|\varphi(z)|\le\rho} \mu(z)|(I_{g,\varphi}^{(n)} f_k)''(z)| \\
&\le |f_k^{(n)}(\varphi(0))||g(0)| + \sup_{|\varphi(z)|\le\rho} \mu(z)|g(z)\varphi'(z)||f_k^{(n+1)}(\varphi(z))| \\
&+ \sup_{|\varphi(z)|\le\rho} \mu(z)|g'(z)||f_k^{(n)}(\varphi(z))| \\
&\le |f_k^{(n)}(\varphi(0))||g(0)| + M_2 \sup_{|\varphi(z)|\le\rho} |f_k^{(n+1)}(\varphi(z))| + M_1 \sup_{|\varphi(z)|\le\rho} |f_k^{(n)}(\varphi(z))| \\
&< C\varepsilon.
\end{aligned} \tag{3.27}$$

From (3.25), (3.26), (3.27) and Lemma 2.2 we have

$$\begin{aligned}
\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} &= \left|(I_{g,\varphi}^{(n)} f_k)(0)\right| + \left|(I_{g,\varphi}^{(n)} f_k)'(0)\right| + \sup_{z\in\mathbb{D}} \mu(z) \left|(I_{g,\varphi}^{(n)} f_k)''(z)\right| \\
&\le \left(\left|I_{g,\varphi}^{(n)} f_k)(0)\right| + \left|(I_{g,\varphi}^{(n)} f_k)'(0)\right| + \sup_{\{z\in\mathbb{D}:\ |\varphi(z)|\le\rho\}} \mu(z) \left|(I_{g,\varphi}^{(n)} f_k)''(z)\right|\right) \\
&\quad + \sup_{\{z\in\mathbb{D}:\ \rho<|\varphi(z)|<1\}} \mu(z) \left|(I_{g,\varphi}^{(n)} f_k)''(z)\right| \\
&< C\varepsilon + \sup_{\{z\in\mathbb{D}:\ \rho<|\varphi(z)|<1\}} \frac{\mu(z)|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n}} C\|f\|_p \\
&\quad + \sup_{\{z\in\mathbb{D}:\ \rho<|\varphi(z)|<1\}} \frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} C\|f\|_p \\
&< C\varepsilon + C\varepsilon + C\varepsilon \\
&= 3C\varepsilon,
\end{aligned} \tag{3.28}$$

when $K > K_0$. It follows that the operator $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is compact.

Conversely. Assume that $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is compact. Then it is clear that $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is bounded, and hence (3.21) and (3.22) hold from the proof of Theorem 3.1. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$.

We can use the test functions

$$f_k(z) = f_{z_k}(z) = (1/p+n+2)\frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^{1/p+1}} - (1/p+1)\frac{(1-|\varphi(z_k)|^2)^2}{(1-z\overline{\varphi(z_k)})^{1/p+2}}.$$

By a direct calculation, we may easily prove that f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ and $\sup_{k\in\mathbb{N}} \|f_k\|_p \le C < \infty$. Then f_k is a bounded sequence in H^p which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.3, we have

$$\lim_{k\to\infty} \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} = 0. \tag{3.29}$$

Note that $f_k^{(n+1)}(\varphi(z_k)) = 0$,

$$f_k^{(n)}(\varphi(z_k)) = \left(\prod_{j=1}^{n}(1/p+j)\right) \frac{\overline{\varphi(z_k)}^n}{(1-|\varphi(z_k)|^2)^{1/p+n}}.$$

From (3.12) and using the compactness of $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ we obtain

$$\left(\prod_{j=1}^{n}(1/p+j)\right) \frac{\mu(|z_k|)|g'(z_k)||\overline{\varphi(z_k)}|^n}{\left(1-|\varphi(z_k)|^2\right)^{1/p+n}} \le \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} \to 0\ , \quad as\ k \to \infty. \tag{3.30}$$

From (3.30) and $|\varphi(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty} \frac{\mu(|z_k|)|g'(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{1/p+n}} = 0,$$

and consequently (3.23) holds.

Next, let

$$h_k(z) = h_{z_k}(z) = (1/p+n+1)\frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^{1/p+1}} - (1/p+1)\frac{(1-|\varphi(z_k)|^2)^2}{(1-z\overline{\varphi(z_k)})^{1/p+2}}.$$

By a direct calculation, we obtain that $h_k \rightrightarrows 0$ on compact subsets of $\mathbb{D}$ and $\sup_{k\in\mathbb{N}}\|h_k\|_p \leq C < \infty$. Then h_k is a bounded sequence in H^p which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.3, we have

$$\lim_{k\to\infty}\|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}_\mu} = 0. \tag{3.31}$$

Note that

$$h_k^{(n+1)}(\varphi(z_k)) = -\left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\overline{\varphi(z_k)}^{n+1}}{(1-|\varphi(z_k)|^2)^{1/p+n+1}}\ ,\quad h_k^{(n)}(\varphi(z_k)) = 0.$$

From (3.18) and using the compactness of $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ we obtain

$$\left(\prod_{j=1}^{n+1}(1/p+j)\right)\frac{\mu(|z_k|)\,|g(z_k)\varphi'(z_k)|\,|\overline{\varphi(z_k)}|^{n+1}}{\left(1-|\varphi(z_k)|^2\right)^{1/p+n+1}} \leq \|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}_\mu} \to 0\ ,\ as\ k\to\infty. \tag{3.32}$$

$|\varphi(z_k)| \to 1$ implies that

$$\lim_{k\to\infty}\frac{\mu(|z_k|)\,|g(z_k)\varphi'(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{1/p+n+1}} = 0,$$

(3.24) holds.

Theorem 3.3. *Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be a holomorphic self-map of $\mathbb{D}$, $0 < p < \infty$, μ be a weight. Then $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu,0}$ is compact if and only if the following conditions are satisfied,*

$$\lim_{|z|\to 1}\frac{\mu(z)|g'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n}} = 0 \tag{3.33}$$

and

$$\lim_{|z|\to 1}\frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} = 0. \tag{3.34}$$

Proof. Suppose that (3.33) and (3.34) hold. It is clear that (3.1)and (3.2) hold. by Theorem 3.1 we have $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_\mu$ is bounded. On the other hand for any $f \in H^p$ we have

$$\begin{aligned}
&\mu(z)\left|(I_{g,\varphi}^{(n)}f)''(z)\right| \\
&= \mu(z)\left|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)\right| \\
&\leq \mu(z)|g(z)\varphi'(z)||f^{(n+1)}(\varphi(z))| + \mu(z)|g'(z)||f^{(n)}(\varphi(z))| \\
&\leq C\|f\|_p\frac{\mu(z)|g(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{1/p+n+1}} + C\|f\|_p\frac{\mu(z)|g'(z)|}{(1-|\varphi(z)|^2)^{1/p+n}} \to 0\ , as\ |z|\to 1. \quad (3.35)
\end{aligned}$$

it follows that $I_{g,\varphi}^{(n)} f \in \mathcal{Z}_{\mu,0}$. Hence $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu,0}$ is bounded. Taking the supremum in inequality (3.35) over all $f \in H^p$ such that $\|f\|_p \le 1$ and letting $|z| \to 1$, yields

$$\lim_{|z|\to 1} \sup_{\|f\|_p \le 1} \mu(z) \left| (I_{g,\varphi}^{(n)} f)''(z) \right| = 0.$$

Hence, by Lemma 2.4 we see that the operator $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu,0}$ is compact.

Now Assume that $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu,0}$ is compact. Firstly, it is obvious $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu,0}$ is bounded. By Theorem 3.1 taking function $f(z) = \frac{z^n}{n!} \in H^p$, we have that

$$\lim_{|z|\to 1} \mu(z)|g'(z)| = 0. \tag{3.36}$$

Let $f(z) = \frac{z^{n+1}}{(n+1)!} \in H^p$, we have that

$$\lim_{|z|\to 1} \mu(z)|\varphi'(z)g(z) + \varphi(z)g'(z)| = 0. \tag{3.37}$$

By (3.36), (3.37) and the boundedness of the function $\varphi(z)$, we get

$$\lim_{|z|\to 1} \mu(z)|g(z)\varphi'(z)| = 0. \tag{3.38}$$

Since $I_{g,\varphi}^{(n)} : H^p \to \mathcal{Z}_{\mu}$ is compact, by Theorem 3.2 we have (3.23) and (3.24) hold.

It from (3.23) follows that for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|g'(z)|}{\left(1 - |\varphi(z)|^2\right)^{1/p+n}} < \varepsilon, \tag{3.39}$$

when $\delta < |\varphi(z)| < 1$. Using (3.26) we see that there exists $\tau \in (0,1)$ such that

$$\mu(z)|g'(z)| \le \varepsilon \inf_{t\in[0,\delta]} (1 - t^2)^{1/p+n}, \tag{3.40}$$

when $\tau < |z| < 1$. Therefore when $\tau < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (3.39), we have

$$\frac{\mu(z)|g'(z)|}{\left(1 - |\varphi(z)|^2\right)^{1/p+n}} < \varepsilon. \tag{3.41}$$

On the other hand, when $\tau < |z| < 1$ and $|\varphi(z)| \le \delta$, by (3.40), we obtain

$$\frac{\mu(z)|g'(z)|}{\left(1 - |\varphi(z)|^2\right)^{1/p+n}} \le \frac{\mu(z)|g'(z)|}{\inf_{t\in[0,\delta]} (1 - t)^{1/p+n}} < \varepsilon. \tag{3.42}$$

From (3.41) and (3.42) we have

$$\lim_{|z|\to 1} \frac{\mu(z)|g(z)|}{\left(1 - |\varphi(z)|^2\right)^{1/p+n}} = 0,$$

thus (3.33) holds.

From (3.24), it follows that for every $\varepsilon > 0$, there exists $\gamma \in (0,1)$ such that

$$\frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} < \varepsilon, \tag{3.43}$$

when $\gamma < |\varphi(z)| < 1$. Using (3.38) we see that there exists $\eta \in (0,1)$ such that

$$\mu(z)|g(z)\varphi'(z)| \le \varepsilon \inf_{t\in[0,\gamma]} (1-t^2)^{1/p+n+1}, \tag{3.44}$$

when $\eta < |z| < 1$. Therefore when $\eta < |z| < 1$ and $\gamma < |\varphi(z)| < 1$, by (3.43), we have

$$\frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} < \varepsilon. \tag{3.45}$$

On the other hand, when $\eta < |z| < 1$ and $|\varphi(z)| \le \gamma$, by (3.44), we obtain

$$\frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} \le \frac{\mu(z)|g(z)\varphi'(z)|}{\inf\limits_{t\in[0,\gamma]} (1-t)^{1/p+n+1}} < \varepsilon. \tag{3.46}$$

From (3.45), (3.46) we have

$$\lim_{|z|\to 1} \frac{\mu(z)|g(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p+n+1}} = 0,$$

we obtain (3.34) holds, the proof is completed.

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Approximation Properties of the Modification of Durrmeyer Type q-Baskakov Operators Which Preserve x^2

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Abstract. In this paper, we introduce a new kind of modification of Durrmeyer type q-Baskakov operators which preserve x^2 based on the concept of q-integer. We investigate the moments and central moments of the operators by computation, obtain a local approximation theorem and also get the pointwise convergence rate theorem and a weighted approximation theorem.

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1 Introduction

In recent years, the applications of q-integers in the approximation theory is one of the main area of research. After q-Bernstein polynomials were introduced by Phillips [12] in 1997, many researchers have studied in this field, we mention some of them as [2]-[4], [11]-[16].

In 2010, Aral and Gupta [2] introduced the Durrmeyer type q-Baskakov operators as

$$D^*_{n,q}(f;x) = [n-1]_q \sum_{k=0}^{\infty} p^*_{n,k}(q;x) \int_0^{\infty/A} p^*_{n,k}(q;t) f(t) d_q t, \tag{1}$$

where, $p^*_{n,k}(q;x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{x^k}{(1+x)_q^{n+k}}$ for every $n \in \mathbb{N}$, $q \in (0,1)$, $x \in [0,\infty)$ and for every real valued continuous and bounded function f on $[0,\infty)$. Apparently, these operators reproduce only constant functions. In 2012, Cai and Zeng [3] introduced a new

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modification of Durrmeyer type q-Baskakov operators $\widetilde{D_{n,q}}$ as follows:

$$\widetilde{D_{n,q}}(f;x) = [n-1]_q \sum_{k=1}^{\infty} \widetilde{p_{n,k}}(q;t) f(t) d_q t + \frac{[n]_q + (1+x)_q^n - 1}{[n]_q (1+x)_q^n} f(0), \tag{2}$$

where, $\widetilde{p_{n,k}}(q;x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{2k^2+1}{4}} \frac{x^k}{(1+x)_q^{n+k}}$, which reproduce not only constant functions but also linear functions. They establish direct and local approximation theorems of operators $\widetilde{D_{n,q}}$ and obtain the estimates on the rate of convergence and weighted approximation properties.

Since the types of operators which preserve linear functions and preserve x^2 are important in approximation theory, in the present paper, we will introduce a new modification of Durrmeyer type q-Baskakov operators which will be defined by equality (6). The advantage of these new operators is that they reproduce not only constant functions but also x^2.

Firstly, we recall some concepts of q-calculus. All of the results can be found in [8, 10]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ...; \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \geq k \geq 0).$$

The q-Jackson integrals and the q-improper integrals are defined as (see [9, 11])

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \quad a > 0,$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0, \tag{3}$$

provided the sums converge absolutely.

The q-analog $E_q(x)$ of the exponential function is given as

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!} = (1 + (1-q)x)_q^{\infty}, \quad |q| < 1,$$

where

$$(1+x)_q^\infty = \prod_{j=0}^{\infty}(1+q^j x).$$

The q-Gamma integral is defined as (see [11])

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx) d_q x, \quad t > 0,$$

which satisfies the following functional equations:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \qquad \Gamma_q(1) = 1.$$

The q-Beta integral is defined as

$$B_q(t;s) = K(A;t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x, \tag{4}$$

where $K(x;t)=\frac{1}{x+1}x^t\left(1+\frac{1}{x}\right)_q^t (1+x)_q^{1-t}$ and $(1+x)_q^\tau = (1+x)(1+qx)...(1+q^{\tau-1}x),\ \tau > 0\ (\tau = t+s)$.
In particular for any positive integer n

$$K(x;n) = q^{\frac{n(n-1)}{2}}, \quad K(x;0) = 1 \ \ and \ \ B_q(t;s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}, \tag{5}$$

(see [4]).
For $f \in C[0,\infty)$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the new modification of the Durrmeyer type q-Baskakov operators $D_{n,q}(f,x)$ as

$$\begin{aligned} D_{n,q}(f;x) = {} & [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} p_{n+2,k-2}(q;t) f(t) d_q t \\ & + \frac{q^2[n-1]_q\left(1+[n+1]_q x\right) + [2]_q (1+x)_q^{n+1}}{[n+1]_q q (1+x)_q^{n+1}} f(0), \end{aligned} \tag{6}$$

where

$$p_{n,k}(q;x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2+1}{2}} \frac{x^k}{(1+x)_q^{n+k}}. \tag{7}$$

2 Some Preliminary Results

In this section we give the following lemmas, which are need to prove our theorems:

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned} D_{n,q}(1;x) &= 1, && (8) \\ D_{n,q}(t;x) &= x - \frac{\left[q^2[n-1]_q + [2]_q (1+x)_q^{n+1}\right] x}{[n+1]_q (1+x)_q^{n+1}} - \frac{q^2[n-1]_q}{[n+1]_q [n]_q}\left[1 - \frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right] && (9) \\ &\doteq x - A_{n,q}(x) - B_{n,q}(x), && (10) \\ D_{n,q}(t^2;x) &= x^2. && (11) \end{aligned}$$

Proof. Since

$$\begin{aligned}
& [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q;x) \int_0^{\infty/A} p_{n+2,k-2}(q;t) d_q t \\
= \;& [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q;x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{(k-2)^2+1}{2}} \int_0^{\infty/A} \frac{t^{k-2}}{(1+t)_q^{n+k}} d_q t \\
= \;& [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q;x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{k^2-4k+5}{2}} \frac{B_q(k-1;n+1)}{K(A;k-1)} \\
= \;& [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q;x) \frac{[n+k-1]_q!}{[k-2]_q![n+1]_q!} q^{\frac{k^2-4k+5}{2}} \frac{[k-2]_q![n]_q!}{[n+k-1]_q! q^{\frac{(k-1)(k-2)}{2}}} \\
= \;& \frac{q^2[n-1]_q}{[n+1]_q} \sum_{k=2}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}}.
\end{aligned}$$

by Euler's identity (see [1], Chap. 10, Coroll. 10.2.2), it is clear that

$$\sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} = 1,$$

and using $\frac{q^2[n-1]_q}{[n+1]_q} = 1 - \frac{[2]_q}{[n+1]_q}$, we obtain

$$\begin{aligned}
& D_{n,q}(1;x) \\
= \;& \left(1 - \frac{[2]_q}{[n+1]_q}\right)\left(1 - \frac{1}{(1+x)_q^n} - \frac{[n]_q x}{(1+x)_q^{n+1}}\right) + \frac{q^2[n-1]_q\,(1+[n+1]_q x)\,[2]_q(1+x)_q^{n+1}}{[n+1]_q(1+x)_q^{n+1}} \\
= \;& 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
D_{n,q}(t;x) &= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \int_0^{\infty/A} p_{n+2,k-2}(q,t) t d_q t \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{k^2-4k+5}{2}} \int_0^{\infty/A} \frac{t^{k-1}}{(1+t)_q^{n+k}} d_q t \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{k^2-4k+5}{2}} \frac{B_q(k,n)}{K(A,k)} \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \frac{[n+k-1]_q!}{[k-2]_q![n+1]_q!} q^{\frac{k^2-4k+5}{2}} \frac{[k-1]_q![n-1]_q!}{[n+k-1]_q! q^{\frac{k(k-1)}{2}}} \\
&= \sum_{k=2}^{\infty} \frac{[n+k-1]_q!}{[k]_q![n-1]_q!} q^{\frac{k^2-3k+6}{2}} \frac{[k-1]_q[n-1]_q}{[n+1]_q[n]_q} \frac{x^k}{(1+x)_q^{n+k}},
\end{aligned}$$

since $[k-1]_q = [k]_q - q^{k-1}$, we get

$$\begin{aligned}
D_{n,q}(t;x) &= \sum_{k=2}^{\infty} \frac{[n+k-1]_q!}{[k]_q![n-1]_q!} q^{\frac{k^2-3k+6}{2}} \frac{[k]_q[n-1]_q}{[n+1]_q[n]_q} \frac{x^k}{(1+x)_q^{n+k}} \\
&\quad - \sum_{k=2}^{\infty} \frac{[n+k-1]_q!}{[k]_q![n-1]_q!} q^{\frac{k^2-3k+6}{2}} \frac{q^{k-1}[n-1]_q}{[n+1]_q[n]_q} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \frac{q^2[n-1]_q x}{[n+1]_q} \sum_{k=2}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k+1}} \\
&\quad - \frac{q^2[n-1]_q}{[n+1]_q[n]_q} \sum_{k=2}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \frac{q^2[n-1]_q x}{[n+1]_q} \left[1 - \frac{1}{(1+x)_q^{n+1}}\right] - \frac{q^2[n-1]_q}{[n+1]_q[n]_q} \left[1 - \frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right] \\
&= x - \frac{\left[q^2[n-1]_q + [2]_q(1+x)_q^{n+1}\right] x}{[n+1]_q(1+x)_q^{n+1}} - \frac{q^2[n-1]_q}{[n+1]_q[n]_q} \left[1 - \frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right].
\end{aligned}$$

Finally,

$$\begin{aligned}
D_{n,q}(t^2;x) &= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \int_0^{\infty/A} p_{n+2,k-2}(q,t) t^2 d_q t \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{k^2-4k+5}{2}} \int_0^{\infty/A} \frac{t^k}{(1+t)_q^{n+k}} d_q t \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q q^{\frac{k^2-4k+5}{2}} \frac{B_q(k+1,n-1)}{K(A,k+1)} \\
&= [n-1]_q \sum_{k=2}^{\infty} p_{n,k}(q,x) \frac{[n+k-1]_q!}{[k-2]_q![n+1]_q!} q^{\frac{k^2-4k+5}{2}} \frac{[k]_q![n-2]_q!}{[n+k-1]_q! q^{\frac{k(k+1)}{2}}} \\
&= \sum_{k=2}^{\infty} p_{n,k}(q;x) q^{\frac{5-5k}{2}} \frac{[k]_q[k-1]_q}{[n]_q[n+1]_q} \\
&= \sum_{k=2}^{\infty} \frac{[n+k-1]_q!}{[k-2]_q![n+1]_q!} q^{\frac{(k-2)(k-3)}{2}} \frac{x^k}{(1+x)_q^{n+k}} \\
&= \sum_{k=2}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^{k+2}}{(1+x)_q^{n+k+2}} \\
&= x^2,
\end{aligned}$$

We obtain the desired result. □

Remark 2.2. *Let $n \in \mathbb{N}$ and $x \in [0, \infty)$, then for every $q \in (0,1)$, by Lemma 2.1, we have*

$$D_{n,q}(1+t^2;x) = 1 + x^2. \tag{12}$$

Lemma 2.3. *For every $q \in (0,1)$ and $x \in [0,\infty)$, we have*

$$D_{n,q}\left((t-x)^2;x\right) \leq 2x^2\left[\frac{[2]_q}{[n+1]_q}+\frac{q^2[n-1]_q}{[n+1]_q(1+x)_q^{n+1}}\right]+\frac{2q^2[n-1]_q x}{[n+1]_q[n]_q} \doteq \beta_{n,q}(x). \quad (13)$$

Proof. Since $D_{n,q}\left((t-x)^2;x\right) = D_{n,q}\left(t^2;x\right) - 2xD_{n,q}(t;x) + x^2$ and Lemma 2.1, We have

$$\begin{aligned} & D_{n,q}\left((t-x)^2;x\right) \\ = \; & 2x^2 - 2x\left\{x - \frac{\left[q^2[n-1]_q + [2]_q(1+x)_q^{n+1}\right]x}{[n+1]_q(1+x)_q^{n+1}} - \frac{q^2[n-1]_q}{[n+1]_q[n]_q}\left[1-\frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right]\right\} \\ = \; & 2x^2\left[\frac{[2]_q}{[n+1]_q}+\frac{q^2[n-1]_q}{[n+1]_q(1+x)_q^{n+1}}\right]+\frac{2q^2[n-1]_q x}{[n+1]_q[n]_q}\left[1-\frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right] \\ \leq \; & 2x^2\left[\frac{[2]_q}{[n+1]_q}+\frac{q^2[n-1]_q}{[n+1]_q(1+x)_q^{n+1}}\right]+\frac{2q^2[n-1]_q x}{[n+1]_q[n]_q}. \end{aligned}$$

Thus the result holds. □

Remark 2.4. *Let sequence $q = \{q_n\}$ satisfies $q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$, then for any fixed $x \in [0,\infty)$, by Lemma 2.3, we have*

$$\lim_{n\to\infty} D_{n,q_n}\left((t-x)^2;x\right) = 0. \quad (14)$$

3 Local approximation

In this section we establish direct and local approximation theorems in connection with the operators $D_{n,q}(f;x)$.

We denote the space of all real valued continuous bounded functions f defined on the interval $[0,\infty)$ by $C_B[0,\infty)$. The norm $||\cdot||$ on the space $C_B[0,\infty)$ is given by $||f|| = \sup\{|f(x)| : x \in [0,\infty)\}$.

Further let us consider Peetre's $K-$functional:

$$K_2(f;\delta) = \inf_{g\in W^2}\left\{||f-g|| + \delta||g''||\right\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

For $f \in C_B[0,\infty)$, the modulus of continuity of second order is defined by

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

by [5, p.177] there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2\left(f;\sqrt{\delta}\right), \quad \delta > 0. \quad (15)$$

For $f \in C_B[0,\infty)$, the modulus of continuity is defined by

$$\omega(f;\delta) = \sup_{0<h\leq\delta}\sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Our first result is a direct local approximation theorem for the operators $D_{n,q}(f;x)$.

Theorem 3.1. *For $q \in (0,1)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B[0,\infty)$, we have*

$$|D_{n,q}(f,x)-f(x)| \leq C\omega_2\left(f;\sqrt{\beta_{n,q}(x)+(A_{n,q}(x)+B_{n,q}(x))^2}\right)+\omega\left(f;A_{n,q}(x)+B_{n,q}(x)\right), \tag{16}$$

where C is a positive constant, $\beta_{n,q}(x)$, $A_{n,q}(x)$ and $B_{n,q}(x)$ are defined in (13), (9) *and* (10).

Proof. For $x \in (0,\infty]$, we define the auxiliary operators $\overline{D_{n,q}}(f;x)$

$$\overline{D_{n,q}}(f;x) = D_{n,q}(f;x) - f\left(x - A_{n,q}(x) - B_{n,q}(x)\right) + f(x), \tag{17}$$

where, $A_{n,q}(x)$ and $B_{n,q}(x)$ are defined in (9) and (10). Obviously, we have

$$\overline{D_{n,q}}(t-x;x) = 0. \tag{18}$$

Let $g \in W^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad x,t \in [0,\infty).$$

Using (18), we get

$$\overline{D_{n,q}}(g;x) = g(x) + \overline{D_{n,q}}\left(\int_x^t (t-u)g''(u)du;x\right),$$

hence, we have

$$\begin{aligned}
& \left|\overline{D_{n,q}}(g;x) - g(x)\right| \\
= & \left|D_{n,q}\left(\int_x^t (t-u)g''(u)du;x\right)\right| + \left|\int_{x-A_{n,q}(x)-B_{n,q}(x)}^x \left\{u - [x - A_{n,q}(x) - B_{n,q}(x)]\right\} g''(u)du\right| \\
\leq & D_{n,q}\left(\left|\int_x^t (t-u)|g''(u)|du\right|;x\right) + \int_{x-A_{n,q}(x)-B_{n,q}(x)}^x |u - [x - A_{n,q}(x) - B_{n,q}(x)]|\, g''(u)|du \\
\leq & \left[\beta_{n,q}(x) + (A_{n,q}(x) + B_{n,q}(x))^2\right] ||g''||,
\end{aligned}$$

where, $\beta_{n,q}(x)$, $A_{n,q}(x)$ and $B_{n,q}(x)$ are defined in (13), (9) and (10).

On the other hand, using (17), (15) and Lemma 2.1, we have

$$\left|\overline{D_{n,q}(f;x)}\right| \leq |D_{n,q}(f;x)| + 2||f|| \leq ||f||D_{n,q}(1;x) + 2||f|| \leq 3||f||. \tag{19}$$

Thus,

$$\begin{aligned}
& |D_{n,q}(f;x) - f(x)| \\
\leq & |\overline{D_{n,q}}(f-g;x) - (f-g)(x)| + |\overline{D_{n,q}}(g;x) - g(x)| + |f\left(x - A_{n,q}(x) - B_{n,q}(x)\right) - f(x)| \\
\leq & 4||f-g|| + \left\{\beta_{n,q}(x) + [A_{n,q}(x) + B_{n,q}(x)]^2\right\} ||g''|| + \left|f\left(x - A_{n,q}(x) - B_{n,q}(x)\right) - f(x)\right|.
\end{aligned}$$

Hence taking infimum on the right hand side over all $g \in W^2$, we get

$$|D_{n,q}(f;x) - f(x)| \leq 4K_2\left(f;\beta_{n,q}(x) + [A_{n,q}(x) + B_{n,q}(x)]^2\right) + \omega\left(f;A_{n,q}(x) + B_{n,q}(x)\right).$$

By (15), for every $q \in (0,1)$, we have

$$|D_{n,q}(f,x) - f(x)| \leq C\omega_2\left(f;\sqrt{\beta_{n,q}(x) + [A_{n,q}(x) + B_{n,q}(x)]^2}\right) + \omega\left(f;A_{n,q}(x) + B_{n,q}(x)\right),$$

where, $\beta_{n,q}(x)$, $A_{n,q}(x)$ and $B_{n,q}(x)$ are defined in (13), (9) and (10). This completes the proof of Theorem 3.1. □

4 Rate of convergence

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where M_f is a constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $||f||_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of f on the closed interval $[0,a], (a > 0)$ by

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,a]} |f(t) - f(x)|.$$

Obviously, for function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.1. *Let $f \in C_{x^2}[0,\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then we have*

$$||D_{n,q}(f) - f||_{C[0,a]} \leq 6M_f(1+a^2)\beta_{n,q}(a) + 2\omega_{a+1}\left(f;\sqrt{\beta_{n,q}(a)}\right), \tag{20}$$

where, $\beta_{n,q}(a)$ is defined in (13).

Proof. For $x \in [0,a]$ and $t > a+1$, we have

$$|f(t) - f(x)| \leq M_f(2 + x^2 + t^2) \leq M_f[2 + 3x^2 + 2(t-x)^2],$$

hence, we obtain

$$|f(t) - f(x)| \leq 6M_f(1+a^2)(t-x)^2. \tag{21}$$

For $x \in [0,a]$ and $t \leq a+1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f;|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta), \quad \delta > 0. \tag{22}$$

From (21) and (22), we get

$$|f(t) - f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{23}$$

For $x \in [0, a]$ and $t \geq 0$, by Schwarz's inequality and Lemma 2.3, we have

$$\begin{aligned} & |D_{n,q}(f;x) - f(x)| \\ \leq \; & D_{n,q}(|f(t) - f(x)|; x) \\ \leq \; & 6M_f(1+a^2)D_{n,q}((t-x)^2;x) + \omega_{a+1}(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{D_{n,q}((t-x)^2;x)}\right) \\ \leq \; & 6M_f\left(1+a^2\right)\beta_{n,q}(a) + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\beta_{n,q}(a)}\right), \end{aligned}$$

where, $\beta_{n,q}(a)$ is defined in (13). By taking $\delta = \sqrt{\beta_{n,q}(a)}$, we get the assertion of Theorem 4.1. □

5 Weighted approximation

Now we will discuss the weighted approximation theorems.

Theorem 5.1. *Let the sequence $q = \{q_n\}$ satisfies $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$, for $f \in C^*_{x^2}[0, \infty)$, we have*

$$\lim_{n\to\infty} ||D_{n,q_n}(f) - f||_{x^2} = 0. \tag{24}$$

Proof. By using the Korovkin theorem in [7], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} ||D_{n,q_n}(t^v;x) - x^v||_{x^2}, \quad v = 0, 1, 2. \tag{25}$$

Since $D_{n,q_n}(1;x) = 1$ and $D_{n,q_n}(t^2;x) = x^2$, (24) holds true for $v = 0$ and $v = 2$.
Finally, for $v = 1$, we have

$$\begin{aligned} & ||D_{n,q_n}(t;x) - x||_{x^2} \\ = \; & \sup_{x\in[0,\infty)} \frac{|D_{n,q_n}(t;x) - x|}{1+x^2} \\ = \; & \left\{1 - \frac{q^2[n-1]_q}{[n+1]_q}\left[1 - \frac{1}{(1+x)_q^{n+1}}\right]\right\} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ & + \frac{q^2[n-1]_q}{[n+1]_q[n]_q}\left[1 - \frac{1+[n+1]_q x}{(1+x)_q^{n+1}}\right] \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ \leq \; & 1 - \frac{q^2[n-1]_q}{[n+1]_q}\left[1 - \frac{1}{(1+x)_q^{n+1}}\right] + \frac{q^2[n-1]_q}{[n+1]_q[n]_q}, \end{aligned}$$

since $\lim_{n\to\infty} q_n = 1$, we get $\lim_{n\to\infty} \frac{{q_n}^2[n-1]_{q_n}}{[n+1]_{q_n}} = 1$, $\lim_{n\to\infty} \frac{1}{(1+x)_q^{n+1}} = 0$ and $\lim_{n\to\infty} \frac{q_n^2[n-1]_{q_n}}{[n+1]_{q_n}[n]_{q_n}} = 0$, so the second condition of (25) holds for $v = 1$ as $n \to \infty$, then the proof of Theorem 5.1 is completed. □

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Qualitative behavior of two systems of second-order rational difference equations *

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Abstract

In this paper, we study the qualitative behavior of two systems of second-order rational difference equations. More precisely, we study the local asymptotic stability and instability of equilibrium points, global character of equilibrium points and rate of convergence of these systems. Some numerical examples are given to verify our theoretical results.

Keywords and phrases: Rational difference equations, stability, global character, rate of convergence.

2010 AMS Mathematics subject classifications: 39A10, 40A05.

1 Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations. For basic theory and applications of difference equations, we refer interested readers to [1, 2, 3, 4, 5]. Moreover, in [9, 7, 8, 10, 11, 12], dynamics of some difference equations is given. In Refs. [16, 17, 18, 19, 20], qualitative behavior of some biological models is discussed. Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For some results in this area, for example:

Gibbons *et al*. [10] investigated the global asymptotic stability of the difference equation:

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n},$$

where $\beta > 0$ and $\alpha, \gamma \geq 0$.

Bajo and Liz [11] investigated the global behavior of difference equation:

$$x_{n+1} = \frac{x_{n-1}}{a + b x_{n-1} x_n},$$

for all values of real parameters $a,\ b$.

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To be motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of following two systems of second-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma y_n y_{n-1}}, \; y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_n x_{n-1}}, \; n = 0, 1, \cdots, \tag{1}$$

where the parameters $\alpha, \ \beta, \ \gamma, \ \alpha_1, \ \beta_1, \ \gamma_1$ and initial conditions $x_0, \ x_{-1}, \ y_0, \ y_{-1}$ are positive real numbers, and

$$x_{n+1} = \frac{a y_{n-1}}{b + c x_n x_{n-1}}, \; y_{n+1} = \frac{a_1 x_{n-1}}{b_1 + c_1 y_n y_{n-1}}, \; n = 0, 1, \cdots, \tag{2}$$

where the parameters $a, \ b, \ c, \ a_1, \ b_1, \ c_1$ and initial conditions $x_0, \ x_{-1}, \ y_0, \ y_{-1}$, are positive real numbers.

Let us consider four-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} &= g(x_n, x_{n-1}, y_n, y_{n-1}), \; n = 0, 1, \cdots, \end{aligned} \tag{3}$$

where $f : I^2 \times J^2 \to I$ and $g : I^2 \times J^2 \to J$ are continuously differentiable functions and $I, \ J$ are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$. Along with system (3) we consider the corresponding vector map $F = (f, x_n, x_{n-1}, g, y_n, y_{n-1})$. An equilibrium point of (3) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \bar{y}, \bar{y},) \\ \bar{y} &= g(\bar{x}, \bar{x}, \bar{y}, \bar{y}) \end{aligned}$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (3).*

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-1, 0\}$ $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\|.\|$ is the usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be unstable if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-1}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \eta$ and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called asymptotic global attractor if it is a global attractor and stable.

Definition 2. *Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map*

$$F = (f, x_n, x_{n-1}, g, y_n, y_{n-1}),$$

where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and F_J is the Jacobian matrix of the system (3) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1. *[3] For the system $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$ of difference equations such let $\bar{X}$ be a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside an open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has norm greater than one, then $\bar{X}$ is unstable.*

Lemma 2. *[4] Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$ is a system of difference equations and $\bar{X}$ is the equilibrium point of this system. The characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lies inside the open unit disk $|\lambda|$ if and only if $\Delta_k > 0$ for $k = 0, 1, \cdots$, where Δ_k is the principal minor of order k of the $n \times n$ matrix*

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ a_0 & a_2 & a_4 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}. \tag{4}$$

The following result gives the rate of convergence of solution of a system of difference equations

$$X_{n+1} = (A + B(n))\, X_n, \tag{5}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{6}$$

as $n \to \infty$,where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}$$

Proposition 1. *(Perron's Theorem)[13] Suppose that condition (6) holds. If X_n is a solution of (5) , then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n} \tag{7}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2. *[13] Suppose that condition (6) holds . If X_n is a solution of (5) , then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{8}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2 On the system $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma y_n y_{n-1}}$, $y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1+\gamma_1 x_n x_{n-1}}$

In this section, we shall investigate the qualitative behavior of the system (1). Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (1), then for $\alpha > \beta$ and $\alpha_1 > \beta_1$ system (1) has following two equilibrium points $P_0 = (0, 0)$, $P_1 = \left(\sqrt{\frac{\alpha_1-\beta_1}{\gamma_1}}, \sqrt{\frac{\alpha-\beta}{\gamma}}\right)$.

To construct corresponding linearized form of the system (1) we consider the following transformation:

$$(x_n, x_{n-1}, y_n, y_{n-1}) \mapsto (f, f_1, g, g_1), \tag{9}$$

where $f = \frac{\alpha x_{n-1}}{\beta+\gamma y_n y_{n-1}}$, $f_1 = x_n$, $g = \frac{\alpha_1 y_{n-1}}{\beta_1+\gamma_1 x_n x_{n-1}}$, $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (12) is given by

$$F_J(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & \frac{\alpha}{\beta+\gamma\bar{y}^2} & -\frac{\alpha\gamma\bar{x}\bar{y}}{\left(\beta+\gamma\bar{y}^2\right)^2} & -\frac{\alpha\gamma\bar{x}\bar{y}}{\left(\beta+\gamma\bar{y}^2\right)^2} \\ 1 & 0 & 0 & 0 \\ -\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{\left(\beta_1+\gamma_1\bar{x}^2\right)^2} & -\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{\left(\beta_1+\gamma_1\bar{x}^2\right)^2} & 0 & \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem 1. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then every solution (x_n, y_n) of the system (1) is bounded.*

Proof. It is easy to verify that

$$0 \leq x_n \leq \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-1}, \text{ if } n = 2m+1,$$

$$0 \leq x_n \leq \left(\frac{\alpha}{\beta}\right)^{m+1} x_0, \text{ if } n = 2m+2,$$

$$0 \leq y_n \leq \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-1}, \text{ if } n = 2m+1,$$

$$0 \leq y_n \leq \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_0, \text{ if } n = 2m+2,$$

Taking $\delta_1 = \max\{x_{-1}, x_0\}$ and $\delta_2 = \max\{y_{-1}, y_0\}$. Then, $0 \leq x_n < \delta_1$ and $0 \leq y_n < \delta_2$ for all $n = 0, 1, 2, \cdots$. □

Theorem 2. *For the equilibrium point P_0 of the system (1) following results hold*
(i) If $\alpha < \beta$ and $\alpha_1 < \beta_1$, then equilibrium point P_0 is locally asymptotically stable.
(ii) If $\alpha > \beta$ or $\alpha_1 > \beta_1$, then equilibrium point P_0 is unstable.

Proof. (i) The linearized system of (1) about the equilibrium point $(0, 0)$ is given by

$$X_{n+1} = F_J(0, 0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$, and $F_J(0, 0) = \begin{pmatrix} 0 & \frac{\alpha}{\beta} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0, 0)$ is given by

$$P(\lambda) = \lambda^4 - (\frac{\alpha}{\beta} + \frac{\alpha_1}{\beta_1})\lambda^2 + \frac{\alpha\alpha_1}{\beta\beta_1}. \tag{10}$$

The roots of $P(\lambda)$ are $\lambda = \pm\sqrt{\frac{\alpha}{\beta}}$, $\lambda = \pm\sqrt{\frac{\alpha_1}{\beta_1}}$. Since all eigenvalues of Jacobian matrix $F_J(0, 0)$ about $(0, 0)$ lie in open unit dick $|\lambda| < 1$. Hence, the equilibrium point $(0, 0)$ is locally asymptotically stable.

(ii) It is easy to see that if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then there exists at least one root λ of Equation (**??**) such that $|\lambda| > 1$. Hence, by Lemma 1 if $\alpha > \beta$ or $\alpha_1 > \beta_1$, then $(0, 0)$ is unstable. □

Theorem 3. *If $\alpha > \beta$ and $\alpha_1 > \beta_1$, then positive equilibrium point*

$$(\bar{x}, \bar{y}) = \left(\sqrt{\frac{\alpha_1 - \beta_1}{\gamma_1}}, \sqrt{\frac{\alpha - \beta}{\gamma}}\right)$$

of the system (1) is unstable.

Proof. The linearized system of (1) about the equilibrium point P_1 is given by

$$X_{n+1} = F_J(P_1)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $F_J(P_1) = \begin{pmatrix} 0 & 1 & A & A \\ 1 & 0 & 0 & 0 \\ B & B & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(P_1)$ is given by

$$P(\lambda) = \lambda^4 - (2 + AB)\lambda^2 - 2AB\lambda + 1 - AB, \tag{11}$$

where $A = -\frac{1}{\alpha}\sqrt{\frac{\gamma(\alpha-\beta)(\alpha_1-\beta_1)}{\gamma_1}}$ and $B = -\frac{1}{\alpha_1}\sqrt{\frac{\gamma_1(\alpha-\beta)(\alpha_1-\beta_1)}{\gamma}}$. It is clear that not all of $\Delta_k > 0$ for $k = 1, 2, 3, 4$. Hence by Lemma 2, the positive equilibrium point

$$(\bar{x}, \bar{y}) = \left(\sqrt{\frac{\alpha_1 - \beta_1}{\gamma_1}}, \sqrt{\frac{\alpha - \beta}{\gamma}}\right)$$

is locally unstable. □

Theorem 4. *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then the equilibrium point $P_0 = (0, 0)$ of Equation (1) is globally asymptotically stable.*

Proof. For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 2 $(0, 0)$ is locally asymptotically stable. From Theorem 1, every positive solution (x_n, y_n) of the system (1) is bounded. Now, it is sufficient to prove that (x_n, y_n) is decreasing. From system (1) one has

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_{n-1}}{\beta + \gamma y_n y_{n-1}} \\ &\leq \frac{\alpha x_{n-1}}{\beta} < x_{n-1}. \end{aligned}$$

This implies that $x_{2n+1} < x_{2n-1}$ and $x_{2n+3} < x_{2n+1}$. Hence, the subsequences $\{x_{2n+1}\}$, $\{x_{2n+2}\}$ are decreasing, i.e., the sequence $\{x_n\}$ is decreasing. Similarly, one has

$$\begin{aligned} y_{n+1} &= \frac{\alpha_1 y_{n-1}}{\beta_1 + \gamma_1 x_n x_{n-1}} \\ &\leq \frac{\alpha_1 y_{n-1}}{\beta_1} < y_{n-1}. \end{aligned}$$

This implies that $y_{2n+1} < y_{2n-1}$ and $y_{2n+3} < y_{2n+1}$. Hence, the subsequences $\{y_{2n+1}\}$, $\{y_{2n+2}\}$ are decreasing, i.e., the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

2.1 Rate of Convergence

We investigate the rate of convergence of a solution that converges to the equilibrium point P_0 of the system (1).

Let $\{(x_n, y_n)\}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has from the system (1)

$$x_{n+1} - \bar{x} = \sum_{i=0}^{1} A_i \left(x_{n-i} - \bar{x}\right) + \sum_{i=0}^{1} B_i \left(y_{n-i} - \bar{y}\right),$$

$$y_{n+1}-\bar{y}=\sum_{i=0}^{1}C_i\left(x_{n-i}-\bar{x}\right)+\sum_{i=0}^{1}D_i\left(y_{n-i}-\bar{y}\right).$$

Set $e_n^1=x_n-\bar{x}$ and $e_n^2=y_n-\bar{y}$, one has

$$e_{n+1}^1=\sum_{i=0}^{1}A_ie_{n-i}^1+\sum_{i=0}^{1}B_ie_{n-i}^2,$$

$$e_{n+1}^2=\sum_{i=0}^{1}C_ie_{n-i}^1+\sum_{i=0}^{1}D_ie_{n-i}^2.$$

where $A_0=0$, $A_1=\frac{\alpha}{\beta+\gamma y_ny_{n-1}}$, $B_0=-\frac{\alpha\gamma\bar{x}y_{n-1}}{(\beta+\gamma y_ny_{n-1})(\beta+\gamma\bar{y}^2)}$, $B_1=-\frac{\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma y_ny_{n-1})(\beta+\gamma\bar{y}^2)}$,
$C_0=-\frac{\alpha_1\gamma_1\bar{y}x_{n-1}}{(\beta_1+\gamma_1x_nx_{n-1})(\beta_1+\gamma_1\bar{x}^2)}$, $C_1=-\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1x_nx_{n-1})(\beta_1+\gamma_1\bar{x}^2)}$, $D_0=0$, $D_1=\frac{\alpha_1}{\beta_1+\gamma_1x_nx_{n-1}}$.

Taking the limits, we obtain $\lim_{n\to\infty}A_0=0$, $\lim_{n\to\infty}A_1=\frac{\alpha}{\beta+\gamma\bar{y}^2}$, $\lim_{n\to\infty}B_0=\lim_{n\to\infty}B_1=-\frac{\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma\bar{y}^2)^2}$, $\lim_{n\to\infty}C_0=\lim_{n\to\infty}C_1=-\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x}^2)^2}$, $\lim_{n\to\infty}D_0=0$, $\lim_{n\to\infty}D_1=\frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^2}$. So, the limiting system of error terms can be written as

$$E_{n+1}=KE_n,$$

where $E_n=\begin{pmatrix}e_n^1\\ e_{n-1}^1\\ e_n^2\\ e_{n-1}^2\end{pmatrix}$ and

$$K=\begin{pmatrix}0 & \frac{\alpha}{\beta+\gamma\bar{y}^2} & -\frac{\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma\bar{y}^2)^2} & -\frac{\alpha\gamma\bar{x}\bar{y}}{(\beta+\gamma\bar{y}^2)^2}\\ 1 & 0 & 0 & 0\\ -\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x}^2)^2} & -\frac{\alpha_1\gamma_1\bar{x}\bar{y}}{(\beta_1+\gamma_1\bar{x}^2)^2} & 0 & \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^2}\\ 0 & 0 & 1 & 0\end{pmatrix},$$

which is similar to linearized system of (2) about the equilibrium point $(\bar{x},\bar{y})$.

Using proposition (1), one has following result.

Theorem 5. *Assume that $\{(x_n,y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty}x_n=\bar{x}$, and $\lim_{n\to\infty}y_n=\bar{y}$, where $(\bar{x},\bar{y})=(0,0)$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty}(\|E_n\|)^{\frac{1}{n}}=|\lambda F_J(\bar{x},\bar{y})|,\ \lim_{n\to\infty}\frac{\|E_{n+1}\|}{\|E_n\|}=|\lambda F_J(\bar{x},\bar{y})|,$$

where $\lambda F_J(\bar{x},\bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x},\bar{y})$ about $(0,0)$.

3 On the system $x_{n+1}=\frac{ay_{n-1}}{b+cx_nx_{n-1}},\ y_{n+1}=\frac{a_1x_{n-1}}{b_1+c_1y_ny_{n-1}}$

In this section, we shall investigate the qualitative behavior of the system (2). Let $(\bar{x},\bar{y})$ be an equilibrium point of the system (2), then system (2) has a unique equilibrium point $(0,0)$. To construct corresponding linearized form of the system (2) we consider the following transformation:

$$(x_n,x_{n-1},y_n,y_{n-1})\mapsto(f,f_1,g,g_1),\tag{12}$$

where $f = \frac{ay_{n-1}}{b+cx_nx_{n-1}}$, $f_1 = x_n$, $g = \frac{a_1x_{n-1}}{b_1+ c_1y_ny_{n-1}}$, $g_1 = y_n$. The Jacobian matrix about the fixed point $(\bar{x}, \bar{y})$ under the transformation (12) is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} -\frac{ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} & -\frac{ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} & 0 & \frac{a}{b+c\bar{x}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1+c_1\bar{y}^2} & -\frac{a_1c_1\bar{x}\bar{y}}{(b_1+c_1\bar{y}^2)^2} & -\frac{a_1c_1\bar{x}\bar{y}}{(b_1+c_1\bar{y}^2)^2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem 6. *Let $\{(x_n, y_n)\}$ be positive solution of system (2), then for every $m \geq 1$ the following results hold.*

$$(i)\ \ 0 \leq x_n \leq \begin{cases} \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m y_{-1}, & if\ \ n = 4m+1, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m y_0, & if\ \ n = 4m+2, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m x_{-1}, & if\ \ n = 4m+3, \\ \left(\frac{a}{b}\right)^{m+1}\left(\frac{a_1}{b_1}\right)^m y_0, & if\ \ n = 4m+4. \end{cases}$$

$$(ii)\ \ 0 \leq y_n \leq \begin{cases} \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_{-1}, & if\ \ n = 4m+1, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} x_0, & if\ \ n = 4m+2, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} y_{-1}, & if\ \ n = 4m+3, \\ \left(\frac{a}{b}\right)^{m}\left(\frac{a_1}{b_1}\right)^{m+1} y_0, & if\ \ n = 4m+4. \end{cases}$$

Theorem 7. *For the equilibrium point P_0 of the system (2) following results hold*
(i) If $a < b$ and $a_1 < b_1$, then equilibrium point P_0 is locally asymptotically stable.
(ii) If $a > b$ or $a_1 > b_1$, then equilibrium point P_0 is unstable.

Proof. (i) The linearized system of (2) about the equilibrium point $(0, 0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$, and $F_J(0,0) = \begin{pmatrix} 0 & 0 & 0 & \frac{a}{b} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

The characteristic polynomial of $F_J(0, 0)$ is given by

$$P(\lambda) = \lambda^4 - \frac{aa_1}{bb_1}. \tag{13}$$

The roots of $P(\lambda)$ are $\lambda = \pm(\frac{aa_1}{bb_1})^{\frac{1}{4}}$, $\lambda = \pm\iota(\frac{aa_1}{bb_1})^{\frac{1}{4}}$. Since all eigenvalues of Jacobian matrix $F_J(0, 0)$ about $(0, 0)$ lie in open unit dick $|\lambda| < 1$. Hence, the equilibrium point $(0, 0)$ is locally asymptotically stable.

(ii) It is easy to see that if $a > b$ or $a_1 > b_1$, then there exists at least one root λ of Equation (**??**) such that $|\lambda| > 1$. Hence, by Lemma 1 if $a > b$ or $a_1 > b_1$, then $(0, 0)$ is unstable. □

Theorem 8. *Let $a < b$ and $a_1 < b_1$, then the equilibrium point $P_0 = (0, 0)$ of Equation (2) is globally asymptotically stable.*

Proof. For $a < b$ and $a_1 < b_1$, from Theorem 7 $(0, 0)$ is locally asymptotically stable. From Theorem 6, it is easy to show that every positive solution (x_n, y_n) of the system (2) is bounded. Now, it is sufficient to prove that (x_n, y_n) is decreasing. From system (2) one has

$$\begin{aligned} x_{n+1} &= \frac{ay_{n-1}}{b+cx_nx_{n-1}}, \\ &\leq \frac{ay_{n-1}}{b} < y_{n-1}. \end{aligned}$$

This implies that $x_{4n+1} < y_{4n-1}$ and $x_{4n+5} < y_{4n+3}$. Also

$$\begin{aligned} y_{n+1} &= \frac{a_1x_{n-1}}{b_1+\ c_1y_ny_{n-1}}, \\ &\leq \frac{ax_{n-1}}{b} < x_{n-1}. \end{aligned}$$

This implies that $y_{4n+1} < x_{4n-1}$ and $y_{4n+5} < x_{4n+3}$. so $x_{4n+5} < y_{4n+3} < x_{4n+1}$ and $y_{4n+5} < x_{4n+3} < y_{4n+1}$. Hence, the subsequences

$$\{x_{4n+1}\},\ \{x_{4n+2}\},\ \{x_{4n+3}\},\ \{x_{4n+4}\}$$

and

$$\{y_{4n+1}\},\ \{y_{4n+2}\},\ \{y_{4n+3}\},\ \{y_{4n+4}\}$$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

3.1 Rate of Convergence

We investigate the rate of convergence of a solution that converges to the equilibrium point P_0 of the system (2).

Let $\{(x_n, y_n)\}$ be any solution of the system (2) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has from the system (2)

$$x_{n+1} - \bar{x} = \sum_{i=0}^{1} A_i \left(x_{n-i} - \bar{x}\right) + \sum_{i=0}^{1} B_i \left(y_{n-i} - \bar{y}\right),$$

$$y_{n+1} - \bar{y} = \sum_{i=0}^{1} C_i \left(x_{n-i} - \bar{x}\right) + \sum_{i=0}^{1} D_i \left(y_{n-i} - \bar{y}\right).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = \sum_{i=0}^{1} A_i e_{n-i}^1 + \sum_{i=0}^{1} B_i e_{n-i}^2,$$

$$e_{n+1}^2 = \sum_{i=0}^{1} C_i e_{n-i}^1 + \sum_{i=0}^{1} D_i e_{n-i}^2.$$

where $A_0 = -\frac{acx_{n-1}\bar{y}}{(b+cx_nx_{n-1})(b+c\bar{x}^2)}$, $A_1 = -\frac{ac\bar{x}\bar{y}}{(b+cx_nx_{n-1})(b+c\bar{x}^2)}$, $B_0 = 0$, $B_1 = \frac{a}{b+cx_nx_{n-1}}$, $C_0 = 0$, $C_1 = \frac{a_1}{b_1+\ c_1y_ny_{n-1}}$, $D_0 = -\frac{a_1\ c_1\bar{x}y_{n-1}}{(\ b_1+\ c_1y_ny_{n-1})(\ b_1+\ c_1\bar{y}^2)}$, $D_1 = -\frac{a_1\ c_1\bar{x}\bar{y}}{(\ b_1+\ c_1y_ny_{n-1})(\ b_1+\ c_1\bar{y}^2)}$.

Taking the limits, we obtain $\lim_{n\to\infty} A_0 = \lim_{n\to\infty} A_1 = -\frac{ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2}$, $\lim_{n\to\infty} B_0 = 0$, $\lim_{n\to\infty} B_1 = \frac{a}{b+c\bar{x}^2}$, $\lim_{n\to\infty} C_0 = 0$, $\lim_{n\to\infty} C_1 = \frac{a_1}{b_1+\ c_1\bar{y}^2}$, $\lim_{n\to\infty} D_0 = \lim_{n\to\infty} D_1 = -\frac{a_1\ c_1\bar{x}\bar{y}}{(\ b_1+\ c_1\bar{y}^2)^2}$. So, the limiting system of error terms can be written as

$$G_{n+1} = MG_n,$$

where $G_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix}$ and

$$M = \begin{pmatrix} -\frac{ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} & -\frac{ac\bar{x}\bar{y}}{(b+c\bar{x}^2)^2} & 0 & \frac{a}{b+c\bar{x}^2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{a_1}{b_1+ c_1\bar{y}^2} & -\frac{a_1\ c_1\bar{x}\bar{y}}{(\ b_1+\ c_1\bar{y}^2)^2} & -\frac{a_1\ c_1\bar{x}\bar{y}}{(\ b_1+\ c_1\bar{y}^2)^2} \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is similar to linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Using proposition (1), one has following result.

Theorem 9. *Assume that $\{(x_n, y_n)\}$ be a positive solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$, and $\lim_{n\to\infty} y_n = \bar{y}$, where $(\bar{x}, \bar{y}) = (0, 0)$. Then, the error vector E_n of every solution of (1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} (\|E_n\|)^{\frac{1}{n}} = |\lambda F_J(\bar{x}, \bar{y})|, \ \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda F_J(\bar{x}, \bar{y})|,$$

where $\lambda F_J(\bar{x}, \bar{y})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{x}, \bar{y})$ about $(0, 0)$.

4 Examples

In order to verify our theoretical results and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to the systems of nonlinear difference equations (1) and (2). All plots in this section are drawn with mathematica.

Example 1. *Consider the system (1) with initial conditions $x_{-1} = 1.2,\ x_0 = 2.9,\ y_{-1} = 1.7,\ y_0 = 1.8$. Moreover, choosing the parameters $\alpha = 15.5,\ \beta = 16,\ \gamma = 0.008,\ \alpha_1 = 18,\ \beta_1 = 19,\ \gamma_1 = 0.002$. Then, the system (1) can be written as:*

$$x_{n+1} = \frac{15.5x_{n-1}}{16 + 0.008y_n y_{n-1}},\ y_{n+1} = \frac{18y_{n-1}}{19 + 0.002x_n x_{n-1}},\ n = 0, 1, \cdots, \tag{14}$$

and with initial conditions $x_{-1} = 1.2,\ x_0 = 2.9,\ y_{-1} = 1.7,\ y_0 = 1.8$.

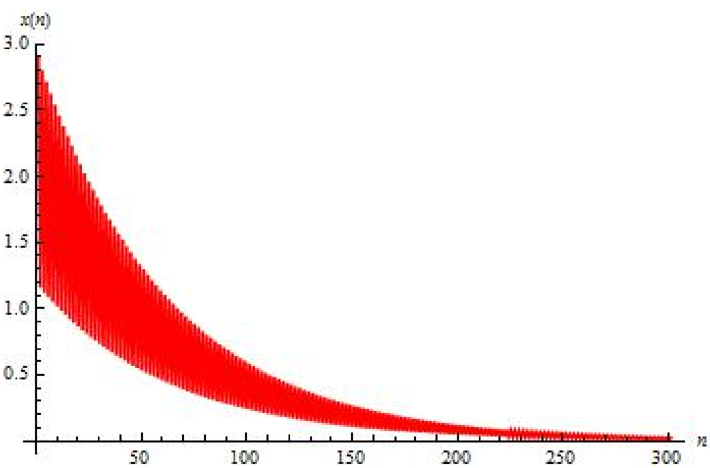
Moreover, in Fig. 1 the plot of x_n is shown in Fig. 1a, the plot of y_n is shown in Fig. 1b and an attractor of the system (14) is shown in Fig. 1c.

Example 2. *Consider the system (1) with initial conditions $x_{-1} = 1.2,\ x_0 = 3.2,\ y_{-1} = 1.7,\ y_0 = 0.8$. Moreover, choosing the parameters $\alpha = 55,\ \beta = 60,\ \gamma = 1.4,\ \alpha_1 = 17,\ \beta_1 = 19,\ \gamma_1 = 0.3$. Then, the system (1) can be written as:*

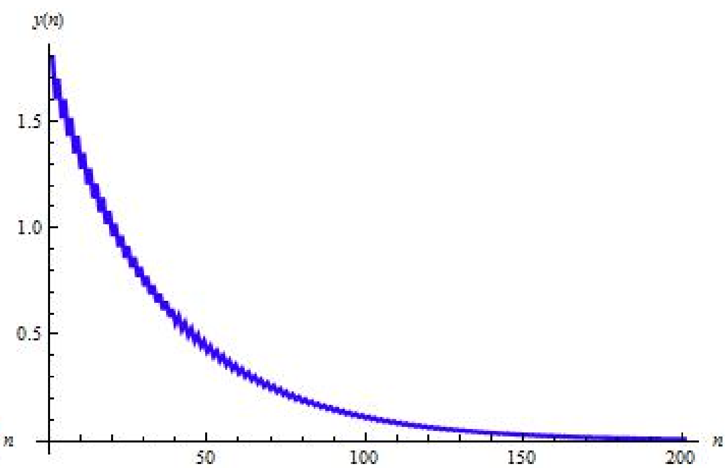
$$x_{n+1} = \frac{55x_{n-1}}{60 + 1.4y_n y_{n-1}},\ y_{n+1} = \frac{17y_{n-1}}{19 + 0.3x_n x_{n-1}},\ n = 0, 1, \cdots, \tag{15}$$

and with initial conditions $x_{-1} = 1.2,\ x_0 = 3.2,\ y_{-1} = 1.7,\ y_0 = 0.8$.

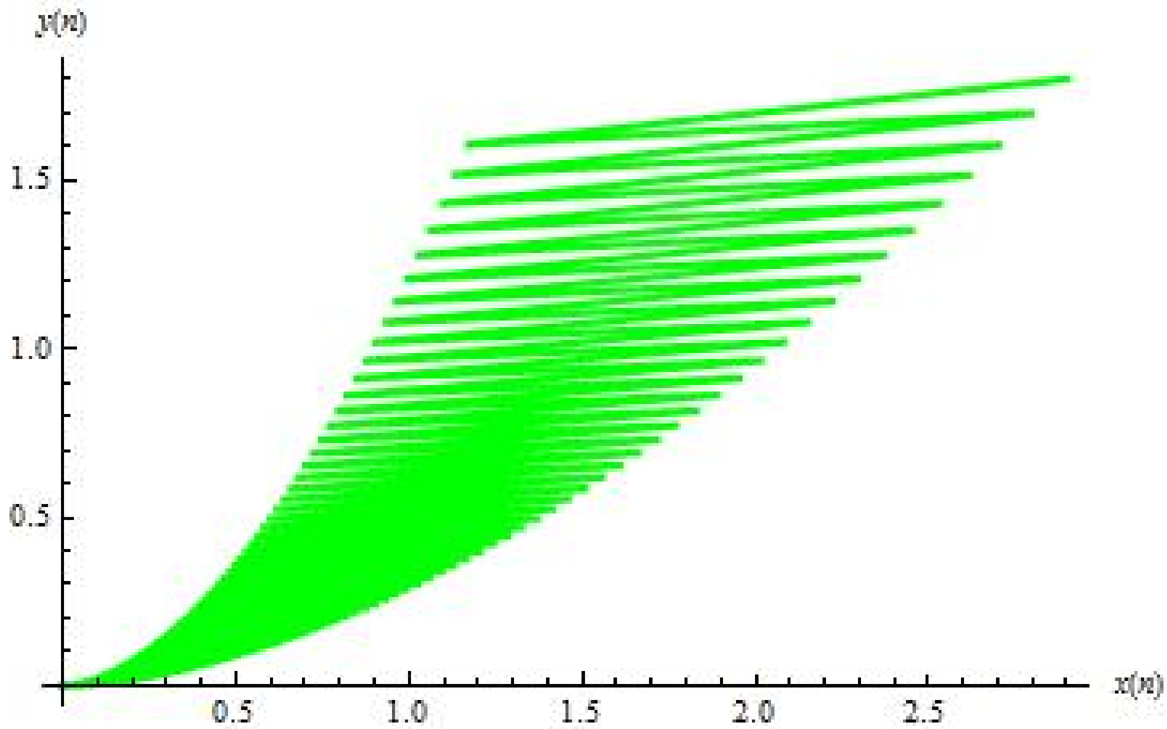
Moreover, in Fig. 2 the plot of x_n is shown in Fig. 2a, the plot of y_n is shown in Fig. 2b and an attractor of the system (15) is shown in Fig. 2c.



(a) Plot of x_n for the system (14)

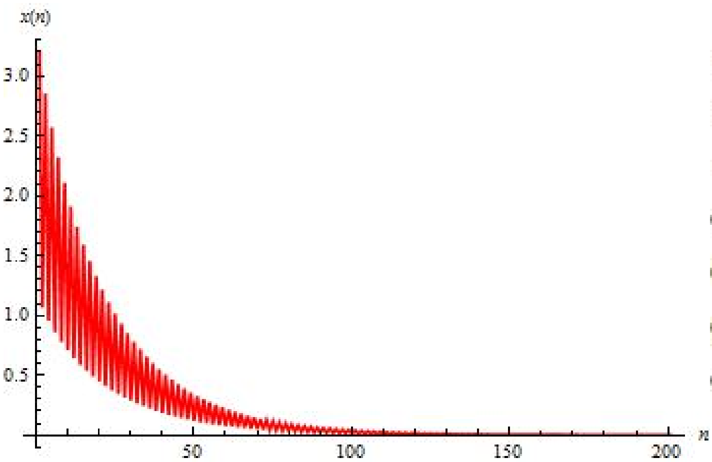


(b) Plot of y_n for the system (14)

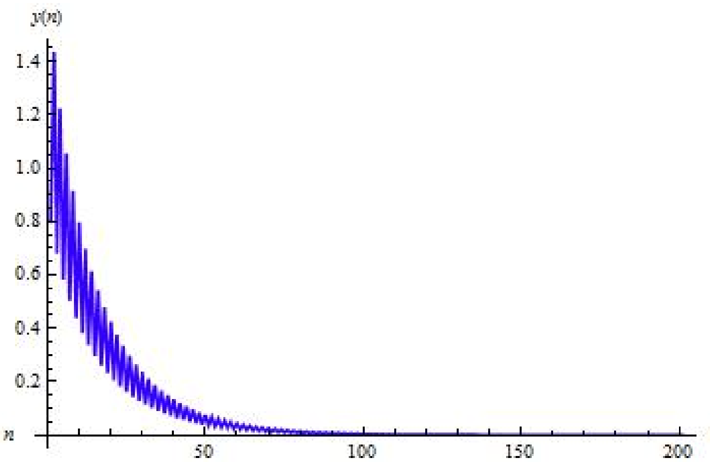


(c) An attractor of the system (14)

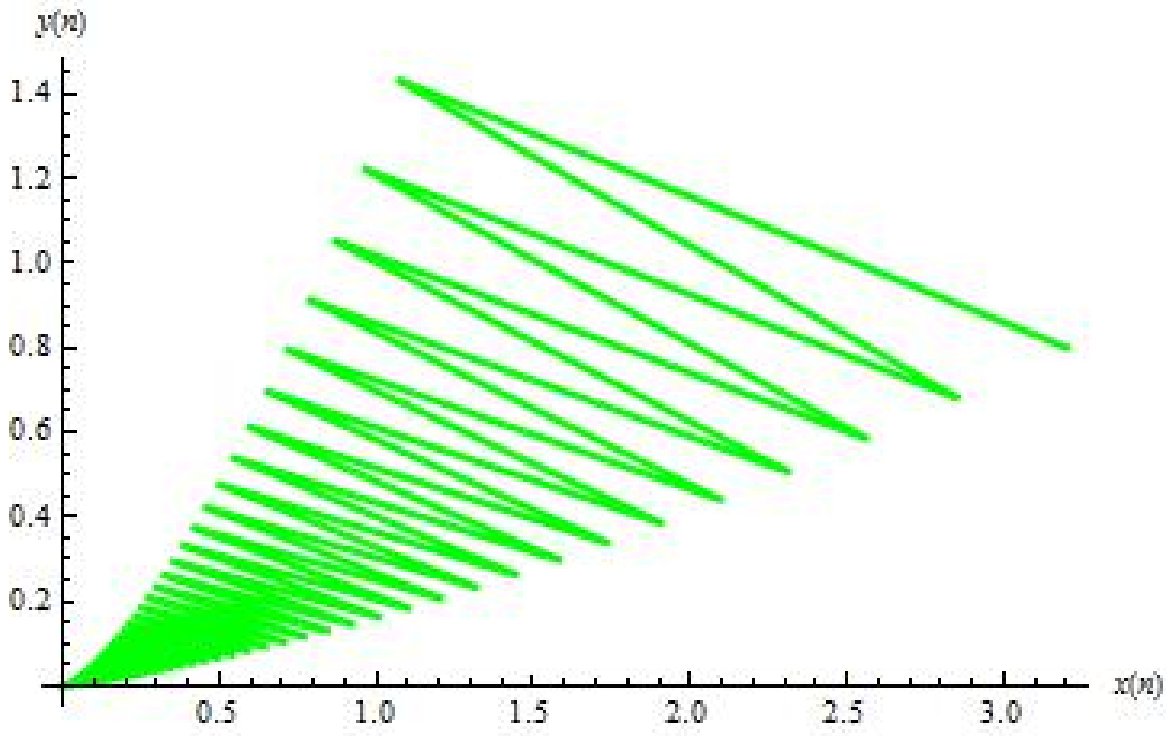
Figure 1: Plots for the system (14)



(a) Plot of x_n for the system (15)



(b) Plot of y_n for the system (15)



(c) An attractor of the system (15)

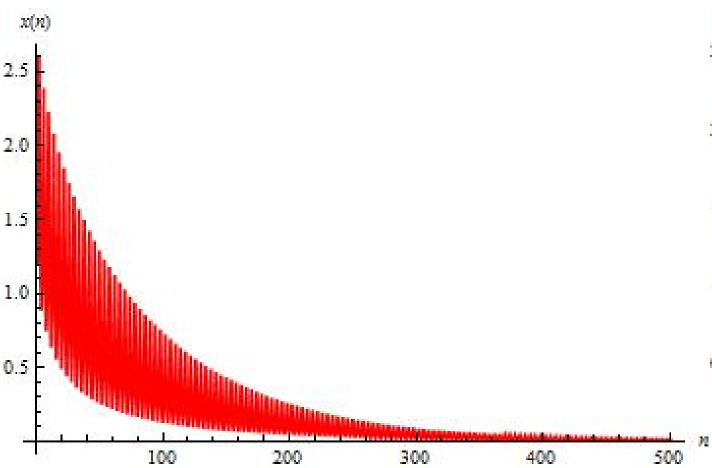
Figure 2: Plots for the system (15)

Example 3. *Consider the system (2) with initial conditions* $x_{-1} = 1.1,\ x_0 = 1.2,\ y_{-1} = 2.7,\ y_0 = 1.8$. *Moreover, choosing the parameters* $a = 128,\ b = 129,\ c = 3,\ \ a_1 = 115,\ \ b_1 = 119,\ \ c_1 = 2$. *Then, the system (2) can be written as:*

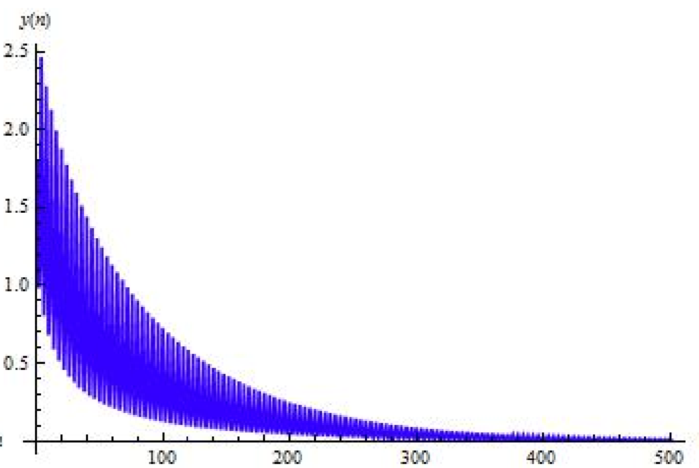
$$x_{n+1} = \frac{128y_{n-1}}{129 + 3x_n x_{n-1}},\ y_{n+1} = \frac{115x_{n-1}}{119 + 2y_n y_{n-1}},\ n = 0, 1, \cdots, \tag{16}$$

and with initial conditions $x_{-1} = 1.1,\ x_0 = 1.2,\ y_{-1} = 2.7,\ y_0 = 1.8$.

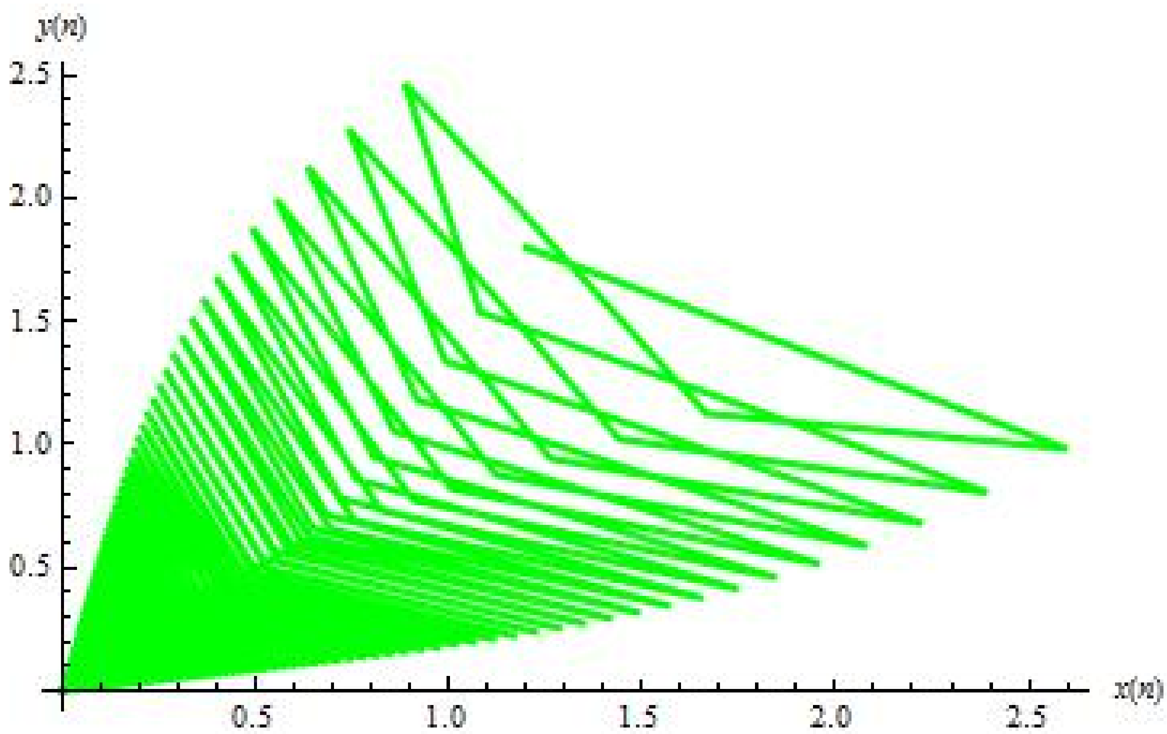
Moreover, in Fig. 3 the plot of x_n *is shown in Fig. 3a, the plot of* y_n *is shown in Fig. 3b and an attractor of the system (16) is shown in Fig. 3c.*



(a) Plot of x_n for the system (16)



(b) Plot of y_n for the system (16)



(c) An attractor of the system (16)

Figure 3: Plots for the system (16)

Conclusion

In the paper, we have investigated the qualitative behavior of two four-dimensional discrete dynamical systems. Each system has only one equilibrium point which is stable under some restriction to parameters. The most important finding here is that the unique equilibrium point $(0, 0)$ can be a global asymptotic attractor for the systems (1) and (2). Moreover, we have determined the rate of convergence of a solution that converges to the equilibrium point $(0, 0)$ of the systems (1) and (2). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

Acknowledgements

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Strong differential subordination results using a generalized Sălăgean operator and Ruscheweyh operator

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Abstract

In the present paper we study the operator using the extended generalized Sălăgean operator and extended Ruscheweyh operator, denote by $DR_\lambda^m f(z,\zeta)$, the Hadamard product of the extended generalized Sălăgean operator $D_\lambda^m f(z,\zeta)$ and extended Ruscheweyh operator $R^m f(z,\zeta)$, given by $DR_\lambda^m f(z,\zeta) : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta)$ and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U\times\overline{U}) : f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$. We obtain several strong differential subordinations regarding the operator $DR_\lambda^m f(z,\zeta)$.

Keywords: strong differential subordination, univalent function, convex function, differential operator, best dominant, extended generalized Sălăgean operator, extended Ruscheweyh operator, convolution product
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U\times\overline{U})$ the class of analytic functions in $U\times\overline{U}$.

Let $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U\times\overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, with $\mathcal{A}_{1\zeta}^* = \mathcal{A}_\zeta^*$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U\times\overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Definition 1.1 *([2]) For $f \in \mathcal{A}_\zeta^*$, $\lambda \ge 0$ and $m \in \mathbb{N}$, the extended generalized Sălăgean operator D_λ^m is defined by $D_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned}
D_\lambda^0 f(z,\zeta) &= f(z,\zeta)\\
D_\lambda^1 f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f_z'(z,\zeta) = D_\lambda f(z,\zeta), \dots\\
D_\lambda^{m+1} f(z,\zeta) &= (1-\lambda) D_\lambda^m f(z,\zeta) + \lambda z \left(D_\lambda^m f(z,\zeta)\right)_z' = D_\lambda\left(D_\lambda^m f(z,\zeta)\right), \quad z \in U,\ \zeta \in \overline{U}.
\end{aligned}$$

Remark 1.1 *([2]) If $f \in \mathcal{A}_\zeta^*$ and $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $D_\lambda^m f(z,\zeta) = z + \sum_{j=2}^\infty [1 + (j-1)\lambda]^m a_j(\zeta) z^j$, for $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.2 *([1]) For $f \in \mathcal{A}_\zeta^*$, $m \in \mathbb{N}$, the extended Ruscheweyh derivative R^m is defined by $R^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$,*

$$\begin{aligned}
R^0 f(z,\zeta) &= f(z,\zeta)\\
R^1 f(z,\zeta) &= z f_z'(z,\zeta), \dots\\
(m+1) R^{m+1} f(z,\zeta) &= z \left(R^m f(z,\zeta)\right)_z' + m R^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.
\end{aligned}$$

Remark 1.2 *([1]) If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^\infty a_j(\zeta) z^j$, then $R^m f(z,\zeta) = z + \sum_{j=2}^\infty \frac{(m+j-1)!}{m!(j-1)!} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Definition 1.3 *([3]) Let $\lambda \ge 0$, $m \in \mathbb{N}$. Denote by DR_λ^m the extended operator given by $DR_\lambda^m : \mathcal{A}_\zeta^* \to \mathcal{A}_\zeta^*$, $DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.3 *([3]) If $f \in \mathcal{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j^2(\zeta) z^j$, then*
$DR_\lambda^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \left\{ [1+(j-1)\lambda]^m \frac{(m+j-1)!}{m!(j-1)!} \right\} a_j^2(\zeta) z^j,\ z \in U,\ \zeta \in \overline{U}.$
This operator was studied also in [4] and [9].

Remark 1.4 *For $\lambda = 1$ we obtain the Hadamard product of the Sălăgean operator and Ruscheweyh operator, which was studied in [5], [6], [7], [8].*

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [12] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [13].

Definition 1.4 *([13]) Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.5 *([13]) (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.4 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*
(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

We have need the following lemmas to study the strong differential subordinations.

Lemma 1.1 *([1]) Let $g(z,\zeta)$ be a convex function in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and let $h(z,\zeta) = g(z,\zeta) + n\alpha z g_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $\alpha > 0$ and n is a positive integer. If $p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots$, $z \in U$, $\zeta \in \overline{U}$, is holomorphic in $U \times \overline{U}$ and $p(z,\zeta) + \alpha z p_z'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and this result is sharp.*

Lemma 1.2 *([1]) Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and $p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $g(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$ is convex and it is the best dominant.*

2 Main results

Extending the results obtained in [10] and [11] to the class $\mathcal{A}_\zeta^*$, we obtain the following theorems:

Theorem 2.1 *Let $g(z,\zeta)$ be a convex function, $g(0,\zeta) = 1$ and let h be the function $h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta} g_z'(z,\zeta)$, for $z \in U$, $\zeta \in \overline{U}$. If $\lambda, \delta \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^{\delta-1} \left(DR_\lambda^m f(z,\zeta)\right)_z' \prec\prec h(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{2.1}$$

then $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and this result is sharp.

Proof. Consider $p(z,\zeta) = \left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta = \left(\frac{z+\sum_{j=2}^{\infty}\left\{[1+(j-1)\lambda]^m \frac{(m+j-1)!}{m!(j-1)!}\right\} a_j^2(\zeta) z^j}{z}\right)^\delta = 1 + p_\delta(\zeta) z^\delta + p_{\delta+1}(\zeta) z^{\delta+1}$ $+ \ldots$, $z \in U$, $\zeta \in \overline{U}$. We deduce that $p \in \mathcal{H}^*[1,\delta,\zeta]$.

Differentiating with respect to z we obtain $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^{\delta-1} \left(DR_\lambda^m f(z,\zeta)\right)_z' = p(z,\zeta) + \frac{1}{\delta} z p_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Then (2.1) becomes $p(z,\zeta) + \frac{1}{\delta} z p_z'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + \frac{z}{\delta} g_z'(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.2 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1 + \frac{z h''_{z^2}(z,\zeta)}{h_z'(z,\zeta)}\right) > -\frac{1}{2}$, *$z \in U$, $\zeta \in \overline{U}$, and $h(0,\zeta) = 1$, $\zeta \in \overline{U}$. If $\lambda, \delta \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^{\delta-1} \left(DR_\lambda^m f(z,\zeta)\right)_z' \prec\prec h(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{2.2}$$

then $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\delta}{z^\delta} \int_0^z h(t,\zeta) t^{\delta-1} dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta=\left(\frac{z+\sum_{j=2}^\infty\left\{[1+(j-1)\lambda]^m\frac{(m+j-1)!}{m!(j-1)!}\right\}a_j^2(\zeta)z^j}{z}\right)^\delta=$ $\left(1+\sum_{j=2}^\infty\left\{[1+(j-1)\lambda]^m\frac{(m+j-1)!}{m!(j-1)!}\right\}a_j^2(\zeta)z^{j-1}\right)^\delta=1+\sum_{j=\delta+1}^\infty p_j(\zeta)z^{j-1}$, for $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[1,\delta,\zeta]$.

Differentiating with respect to z, we obtain $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^{\delta-1}\left(DR_\lambda^m f(z,\zeta)\right)'_z=p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ and (2.2) becomes $p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ i.e. $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta\prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$, $z\in U$, $\zeta\in\overline{U}$ and q is the best dominant. ■

Corollary 2.3 *Let $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\le\beta<1$, $z\in U$, $\zeta\in\overline{U}$. If $\delta,\lambda\ge0$, $m\in\mathbb{N}$, $f\in\mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^{\delta-1}\left(DR_\lambda^m f(z,\zeta)\right)'_z\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},\tag{2.3}$$

then $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where q is given by $q(z,\zeta)=(2\beta-\zeta)+\frac{2(\zeta-\beta)\delta}{z^\delta}\int_0^z\frac{t^{\delta-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.2 and considering $p(z,\zeta)=\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta$.the strong differential subordination (2.3) becomes $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.2 for $\gamma=\delta$, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, i.e. $\left(\frac{DR_\lambda^m f(z,\zeta)}{z}\right)^\delta\prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt=$ $\frac{\delta}{z^\delta}\int_0^z t^{\delta-1}\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=\frac{\delta}{z^\delta}\int_0^z\left[(2\beta-\zeta)t^{\delta-1}+2(\zeta-\beta)\frac{t^{\delta-1}}{1+t}\right]dt=(2\beta-\zeta)+\frac{2(\zeta-\beta)\delta}{z^\delta}\int_0^z\frac{t^{\delta-1}}{1+t}dt$, $z\in U$, $\zeta\in\overline{U}$. ■

Theorem 2.4 *Let g be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda,\delta\ge0$, $m\in\mathbb{N}$, $z\in U$, $\zeta\in\overline{U}$, $f\in\mathcal{A}_\zeta^*$ and the strong differential subordination*

$$z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}-2\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}\right]\prec\prec h(z,\zeta)\tag{2.4}$$

holds, where $z\in U$, $\zeta\in\overline{U}$, then $z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and this result is sharp.

Proof. Consider $p(z,\zeta)=z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}$ and we obtain $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}+$ $\frac{z^2}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}-2\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}\right]$.

Relation (2.4) becomes $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ i.e. $z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. ■

Theorem 2.5 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, *$z\in U$, $\zeta\in\overline{U}$, and $h(0,\zeta)=1$. If $\lambda,\delta\ge0$, $m\in\mathbb{N}$, $z\in U$, $\zeta\in\overline{U}$, $f\in\mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}-2\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}\right]\prec\prec h(z,\zeta),\tag{2.5}$$

$z\in U$, $\zeta\in\overline{U}$, then $z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}$, $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[1,1,\zeta]$. Differentiating with respect to z, we obtain $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z\frac{\delta+1}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}+\frac{z^2}{\delta}\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}-2\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}\right]$, $z\in U$, $\zeta\in\overline{U}$, and (2.5) becomes $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $z\frac{DR_\lambda^m f(z,\zeta)}{\left(DR_\lambda^{m+1}f(z,\zeta)\right)^2} \prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$, $z\in U$, $\zeta\in\overline{U}$, and q is the best dominant. ■

Theorem 2.6 *Let g be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda,\delta\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{DR_\lambda^m f(z,\zeta)}-\left(\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\right)^2\right]\prec\prec h(z,\zeta), \tag{2.6}$$

$z\in U$, $\zeta\in\overline{U}$, holds, then $z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. This result is sharp.

Proof. Let $p(z,\zeta)=z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}$. We deduce that $p\in\mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{DR_\lambda^m f(z,\zeta)}-\left(\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\right)^2\right]$, $z\in U$, $\zeta\in\overline{U}$.

Using the notation in (2.6), the strong differential subordination becomes $p(z,\zeta)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+\frac{z}{\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and this result is sharp. ■

Theorem 2.7 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, *$z\in U$, $\zeta\in\overline{U}$ and $h(0,\zeta)=1$. If $\lambda,\delta\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and satisfies the strong differential subordination*

$$z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{DR_\lambda^m f(z,\zeta)}-\left(\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\right)^2\right]\prec\prec h(z,\zeta), \tag{2.7}$$

$z\in U$, $\zeta\in\overline{U}$, then $z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $p(z,\zeta)+\frac{z}{\delta}p'_z(z,\zeta)=z^2\frac{\delta+2}{\delta}\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}+\frac{z^3}{\delta}\left[\frac{\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{DR_\lambda^m f(z,\zeta)}-\left(\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\right)^2\right]$, $z\in U$, $\zeta\in\overline{U}$, and (2.7) becomes $p(z)+\frac{1}{\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $z^2\frac{\left(DR_\lambda^m f(z,\zeta)\right)'_z}{DR_\lambda^m f(z,\zeta)}\prec\prec q(z,\zeta)=\frac{\delta}{z^\delta}\int_0^z h(t,\zeta)t^{\delta-1}dt$, $z\in U$, $\zeta\in\overline{U}$, and q is the best dominant. ■

Theorem 2.8 *Let g be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda\geq 0$, $m\in\mathbb{N}$, $f\in\mathcal{A}^*_\zeta$ and the strong differential subordination*

$$1-\frac{DR_\lambda^m f(z,\zeta)\cdot\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{\left[\left(DR_\lambda^m f(z,\zeta)\right)'_z\right]^2}\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U} \tag{2.8}$$

holds, then $\frac{DR_\lambda^m f(z,\zeta)}{z\left(DR_\lambda^m f(z,\zeta)\right)'_z}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. This result is sharp.

Proof. Let $p(z,\zeta)=\frac{DR_\lambda^m f(z,\zeta)}{z\left(DR_\lambda^m f(z,\zeta)\right)'_z}$. We deduce that $p\in\mathcal{H}^*[1,1,\zeta]$. Differentiating with respect to z, we obtain $1-\frac{DR_\lambda^m f(z,\zeta)\cdot\left(DR_\lambda^m f(z,\zeta)\right)''_{z^2}}{\left[\left(DR_\lambda^m f(z,\zeta)\right)'_z\right]^2}=p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using the notation in (2.8), the strong differential subordination becomes $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ i.e. $\frac{DR_\lambda^m f(z,\zeta)}{z\left(DR_\lambda^m f(z,\zeta)\right)'_z}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and this result is sharp. ■

Theorem 2.9 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right) > -\frac{1}{2}$, $z \in U$, $\zeta \in \overline{U}$, *and* $h(0,\zeta)=1$. *If* $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ *and satisfies the strong differential subordination*

$$1-\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2}}{\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.9}$$

then $\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *where* $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$ *is convex and it is the best dominant.*

Proof. Let $p(z)=\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z}$, $z \in U$, $\zeta \in \overline{U}$, $p \in \mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $1-\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2}}{\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2}=p(z,\zeta)+zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and (2.9) becomes $p(z,\zeta)+zp'_z(z,\zeta) \prec\prec h(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$ i.e. $\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$, $z \in U$, $\zeta \in \overline{U}$, and q is the best dominant. ■

Corollary 2.10 *Let* $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$, *where* $0 \leq \beta < 1$. *If* $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ *and satisfies the strong differential subordination*

$$1-\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2}}{\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.10}$$

then $\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *where q is given by* $q(z,\zeta)=(2\beta-\zeta)+2(\zeta-\beta)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$. *The function q is convex and it is the best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z,\zeta)=\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z}$, the strong differential subordination (2.10) becomes $p(z,\zeta)+zp'_z(z,\zeta) \prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.2 for $\gamma=1$, we have $p(z,\zeta) \prec\prec q(z,\zeta)$, i.e. $\frac{DR^m_\lambda f(z,\zeta)}{z\left(DR^m_\lambda f(z,\zeta)\right)'_z} \prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=\frac{1}{z}\int_0^z\left[(2\beta-\zeta)+\frac{2(\zeta-\beta)}{1+t}\right]dt=(2\beta-\zeta)+2(\zeta-\beta)\frac{\ln(1+z)}{z}$, $z \in U$, $\zeta \in \overline{U}$. ■

Theorem 2.11 *Let g be a convex function such that* $g(0,\zeta)=0$ *and let h be the function* $h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. *If* $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ *and the strong differential subordination*

$$\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2+DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U} \tag{2.11}$$

holds, then $\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)'_z}{z} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. *This result is sharp.*

Proof. Let $p(z,\zeta)=\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)'_z}{z}$. We deduce that $p \in \mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2+DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2}=p(z,\zeta)+zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Using the notation in (2.11), the strong differential subordination becomes $p(z,\zeta)+zp'_z(z,\zeta) \prec\prec h(z,\zeta)=g(z,\zeta)+zg'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$ i.e. $\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)'_z}{z} \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and this result is sharp. ■

Theorem 2.12 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right) > -\frac{1}{2}$, $z \in U$, $\zeta \in \overline{U}$ *and* $h(0,\zeta)=0$. *If* $\lambda \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}^*_\zeta$ *and satisfies the strong differential subordination*

$$\left[\left(DR^m_\lambda f(z,\zeta)\right)'_z\right]^2+DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)''_{z^2} \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.12}$$

then $\frac{DR^m_\lambda f(z,\zeta)\cdot\left(DR^m_\lambda f(z,\zeta)\right)'_z}{z} \prec\prec q(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, *where* $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$ *is convex and it is the best dominant.*

Proof. Let $p(z,\zeta)=\frac{DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))'_z}{z}$, $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $\left[(DR_\lambda^m f(z,\zeta))'_z\right]^2+DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))''_{z^2}=p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and (2.12) becomes $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))'_z}{z}\prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$, and q is the best dominant. ■

Corollary 2.13 *Let $h(z)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\le\beta<1$. If $\lambda\ge0$, $m\in\mathbb{N}$, $f\in\mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$\left[(DR_\lambda^m f(z,\zeta))'_z\right]^2+DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))''_{z^2}\prec\prec h(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},\tag{2.13}$$

then $\frac{DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))'_z}{z}\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where q is given by $q(z,\zeta)=(2\beta-\zeta)+2(\zeta-\beta)\frac{\ln(1+z)}{z}$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 2.12 and considering $p(z,\zeta)=\frac{DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))'_z}{z}$, the strong differential subordination (2.13) becomes $p(z,\zeta)+zp'_z(z,\zeta)\prec\prec h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.2 for $\gamma=1$, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, i.e. $\frac{DR_\lambda^m f(z,\zeta)\cdot(DR_\lambda^m f(z,\zeta))'_z}{z}\prec\prec q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt=\frac{1}{z}\int_0^z\left[(2\beta-\zeta)+\frac{2(\zeta-\beta)}{1+t}\right]dt=(2\beta-\zeta)+2(\zeta-\beta)\frac{\ln(1+z)}{z}$, $z\in U$, $\zeta\in\overline{U}$. ■

Theorem 2.14 *Let g be a convex function such that $g(0,\zeta)=0$ and let h be the function $h(z,\zeta)=g(z,\zeta)+\frac{z}{1-\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $\lambda\ge0$, $\delta\in(0,1)$, $m\in\mathbb{N}$, $f\in\mathcal{A}_\zeta^*$ and the strong differential subordination*

$$\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m+1}f(z,\zeta)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}-\delta\frac{(DR_\lambda^m f(z,\zeta))'_z}{DR_\lambda^m f(z,\zeta)}\right)\prec\prec h(z,\zeta),\tag{2.14}$$

$z\in U$, $\zeta\in\overline{U}$, holds, then $\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. This result is sharp.

Proof. Let $p(z,\zeta)=\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta$. We deduce that $p\in\mathcal{H}^*[1,1,\zeta]$. Differentiating with respect to z, we obtain $\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m+1}f(z,\zeta)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}-\delta\frac{(DR_\lambda^m f(z,\zeta))'_z}{DR_\lambda^m f(z,\zeta)}\right)=p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using the notation in (2.14), the strong differential subordination becomes $p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+\frac{z}{1-\delta}g'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

By using Lemma 1.1, we have $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$,and this result is sharp. ■

Theorem 2.15 *Let h be a holomorphic function which satisfies the inequality* Re $\left(1+\frac{zh''_{z^2}(z,\zeta)}{h'_z(z,\zeta)}\right)>-\frac{1}{2}$, $z\in U$, $\zeta\in\overline{U}$ *and $h(0,\zeta)=1$. If $\lambda\ge0$, $\delta\in(0,1)$, $m\in\mathbb{N}$, $f\in\mathcal{A}_\zeta^*$ and satisfies the strong differential subordination*

$$\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m+1}f(z,\zeta)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}-\delta\frac{(DR_\lambda^m f(z,\zeta))'_z}{DR_\lambda^m f(z,\zeta)}\right)\prec\prec h(z,\zeta),\tag{2.15}$$

$z\in U$, $\zeta\in\overline{U}$, then $\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t,\zeta)t^{-\delta}dt$ is convex and it is the best dominant.

Proof. Let $p(z,\zeta)=\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta$, $z\in U$, $\zeta\in\overline{U}$, $p\in\mathcal{H}^*[0,1,\zeta]$. Differentiating with respect to z, we obtain $\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\frac{DR_\lambda^{m+1}f(z,\zeta)}{1-\delta}\left(\frac{\left(DR_\lambda^{m+1}f(z,\zeta)\right)'_z}{DR_\lambda^{m+1}f(z,\zeta)}-\delta\frac{(DR_\lambda^m f(z,\zeta))'_z}{DR_\lambda^m f(z,\zeta)}\right)=p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and (2.15) becomes $p(z,\zeta)+\frac{1}{1-\delta}zp'_z(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Using Lemma 1.2, we have $p(z,\zeta)\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{DR_\lambda^{m+1}f(z,\zeta)}{z}\cdot\left(\frac{z}{DR_\lambda^m f(z,\zeta)}\right)^\delta\prec\prec q(z,\zeta)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t,\zeta)t^{-\delta}dt$, $z\in U$, $\zeta\in\overline{U}$, and q is the best dominant. ■

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On some differential sandwich theorems using a generalized Sălăgean operator and Ruscheweyh operator

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Abstract

In this work we define a new operator using the generalized Sălăgean operator and Ruscheweyh operator. Denote by $DR_\lambda^{m,n}$ the Hadamard product of the generalized Sălăgean operator D_λ^m and Ruscheweyh operator R^n, given by $DR_\lambda^{m,n} : \mathcal{A} \to \mathcal{A}$, $DR_\lambda^{m,n} f(z) = (D_\lambda^m * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + ...,$ $z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The purpose of this paper is to introduce sufficient conditions for subordination and superordination involving the operator $DR_\lambda^{m,n}$ and also to obtain sandwich-type results.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.
2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$. Denote by $K = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$, the class of normalized convex functions in U.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [15] obtained conditions h, q and ψ for which the following implication holds $h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z)$.

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution product) of $f(z)$ and $g(z)$, written as $(f * g)(z)$, is defined by $f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

Definition 1.1 *(Al Oboudi [7]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \ldots \\ D_\lambda^m f(z) &= (1-\lambda) D_\lambda^{m-1} f(z) + \lambda z \left(D_\lambda^m f(z)\right)' = D_\lambda \left(D_\lambda^{m-1} f(z)\right), \quad z \in U. \end{aligned}$$

Remark 1.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^m f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m a_j z^j$, for $z \in U$.*

Remark 1.2 *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [18].*

Definition 1.2 *(Ruscheweyh [17]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z), \ldots \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of M. Darus, K. Al-Shaqs [14], Shanmugam, Ramachandran, Darus and Sivasubramanian [19].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.3 *[16] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim\limits_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1.1 *[16] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$. If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 1.2 *[13] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $\operatorname{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ for $z \in U$ and $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and $\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.*

2 Main results

Definition 2.1 *Let $\lambda \geq 0$ and $n, m \in \mathbb{N}$. Denote by $DR_\lambda^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the generalized Sălăgean operator D_λ^m and the Ruscheweyh operator R^n, $DR_\lambda^{m,n} f(z) = (D_\lambda^m * R^n) f(z)$, for any $z \in U$ and each nonnegative integers m, n.*

Remark 2.1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $DR_\lambda^{m,n} f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, for $z \in U$.*

This operator was studied in [12].

Remark 2.2 *For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [1] of the Sălăgean operator S^n and Ruscheweyh derivative R^n, which was studied in [2], [3].*

Remark 2.3 *For $m = n$ we obtain the Hadamard product DR_λ^n [4] of the generalized Sălăgean operator D_λ^n and Ruscheweyh derivative R^n, which was studied in [5], [6], [8], [9], [10], [11].*

Using simple computation one obtains the next result.

Proposition 2.1 *For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$DR_\lambda^{m+1,n} f(z) = (1-\lambda) DR_\lambda^{m,n} f(z) + \lambda z \left(DR_\lambda^{m,n} f(z)\right)' \tag{2.1}$$

and

$$z \left(DR_\lambda^{m,n} f(z)\right)' = (n+1) DR_\lambda^{m,n+1} f(z) - n DR_\lambda^{m,n} f(z). \tag{2.2}$$

Proof. We have $DR_{\lambda}^{m+1,n}f(z)=z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m+1}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j=$
$z+\sum_{j=2}^{\infty}[(1-\lambda)+\lambda j][1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j=$
$z+(1-\lambda)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j+\lambda\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}ja_j^2z^j=$
$(1-\lambda)DR_{\lambda}^{m,n}f(z)+\lambda z\left(DR_{\lambda}^{m,n}f(z)\right)'$, and
$(n+1)DR_{\lambda}^{m,n+1}f(z)-nDR_{\lambda}^{m,n}f(z)=$
$(n+1)z+(n+1)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j)!}{(n+1)!(j-1)!}a_j^2z^j-nz-n\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j=$
$z+(n+1)\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{n+j}{n+1}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j-n\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}a_j^2z^j=$
$z+\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{m}\frac{(n+j-1)!}{n!(j-1)!}ja_j^2z^j=z\left(DR_{\lambda}^{m,n}f(z)\right)'$. ∎

We begin with the following

Theorem 2.2 *Let* $\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\in\mathcal{H}(U)$, $z\in U$, $f\in\mathcal{A}$, $m,n\in\mathbb{N}$, $\lambda\geq 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$\operatorname{Re}\left(1+\frac{\alpha}{\mu}+\frac{2\beta}{\mu}q(z)+\frac{zq''(z)}{q'(z)}\right)>0,\quad z\in U, \tag{2.3}$$

for $\alpha,\beta,\mu,\in\mathbb{C}$,$\mu\neq 0$, $z\in U$, *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z):=\left(\frac{1-\lambda(n+1)}{\lambda}\mu+\alpha\right)\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}+ \tag{2.4}$$

$$+\mu(n+1)[1-\lambda(n+2)]\frac{DR_{\lambda}^{m,n+1}f(z)}{DR_{\lambda}^{m,n}f(z)}+\lambda\mu(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m,n}f(z)}+(\beta-\frac{\mu}{\lambda})\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^2.$$

If q *satisfies the following subordination*

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha q(z)+\beta(q(z))^2+\mu zq'(z), \tag{2.5}$$

for, $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$ *then*

$$\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\prec q(z),\quad z\in U, \tag{2.6}$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z):=\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$. Differentiating this function, with respect to z,we get $zp'(z)=\frac{z\left(DR_{\lambda}^{m+1,n}f(z)\right)'}{DR_{\lambda}^{m,n}f(z)}-\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\frac{z\left(DR_{\lambda}^{m,n}f(z)\right)'}{DR_{\lambda}^{m,n}f(z)}$. By using the identity (2.1) and (2.2), we obtain

$$zp'(z)=\frac{1-\lambda(n+1)}{\lambda}\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}+(n+1)[1-\lambda(n+2)]\frac{DR_{\lambda}^{m,n+1}f(z)}{DR_{\lambda}^{m,n}f(z)}+$$

$$\lambda(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m,n}f(z)}-\frac{1}{\lambda}\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^2+\lambda(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m,n}f(z)}-\frac{1}{\lambda}\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^2 \tag{2.7}$$

By setting $\theta(w):=\alpha w+\beta w^2$ and $\phi(w):=\mu$, $\alpha,\beta,\mu\in\mathbb{C}$, $\mu\neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$. Also, by letting $Q(z)=zq'(z)\phi(q(z))=\mu zq'(z)$,we find that $Q(z)$ is starlike univalent in U. Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta(q(z))^2+\mu zq'(z)$, $z\in U$. If we derive the function Q, with respect to z, perform calculations, we have $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\operatorname{Re}\left(1+\frac{\alpha}{\mu}+\frac{2\beta}{\mu}q(z)+\frac{zq''(z)}{q'(z)}\right)>0$.

By using (2.7), we obtain $\alpha p(z)+\beta(p(z))^2+\mu zp'(z)=\left(\frac{1-\lambda(n+1)}{\lambda}\mu+\alpha\right)\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}+\mu(n+1)[1-\lambda(n+2)]\cdot\frac{DR_{\lambda}^{m,n+1}f(z)}{DR_{\lambda}^{m,n}f(z)}+\lambda\mu(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m,n}f(z)}+(\beta-\frac{\mu}{\lambda})\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^2$. By using (2.5), we have $\alpha p(z)+\beta(p(z))^2+\mu zp'(z)\prec\alpha q(z)+\beta(q(z))^2+\mu zq'(z)$. Therefore, the conditions of Lemma 1.1 are met, so we have $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\prec q(z)$, $z\in U$, and q is the best dominant. ∎

Corollary 2.3 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $m,n \in \mathbb{N}$, $\lambda \ge 0$, $z \in U$. *Assume that (2.3) holds. If* $f \in \mathcal{A}$ *and* $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \mu\frac{(A-B)z}{(1+Bz)^2}$, *for* $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \ne 0$, $-1 \le B < A \le 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.4), then* $\frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \prec \frac{1+Az}{1+Bz}$ *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.2 we get the corollary. ■

Corollary 2.4 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $m,n \in \mathbb{N}$, $\lambda \ge 0$, $z \in U$. *Assume that (2.3) holds. If* $f \in \mathcal{A}$ *and* $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^\gamma + \beta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \mu\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1}$, *for* $\alpha,\mu,\beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\mu \ne 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.4), then* $\frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma$, *and* $\left(\frac{1+z}{1-z}\right)^\gamma$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.2 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \le 1$. ■

Theorem 2.5 *Let* q *be convex and univalent in* U, *such that* $q(0) = 1$, $m,n \in \mathbb{N}$, $\lambda \ge 0$. *Assume that*

$$\operatorname{Re}\left(\frac{q'(z)}{\mu}(\alpha + 2\beta q(z))\right) > 0, \text{ for } \alpha,\mu,\beta \in \mathbb{C},\ \mu \ne 0,\ z \in U. \tag{2.8}$$

If $f \in \mathcal{A}$, $\frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* U, *where* $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.4), then*

$$\alpha q(z) + \beta(q(z))^2 + \mu z q'(z) \prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z), \quad z \in U, \tag{2.9}$$

implies

$$q(z) \prec \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)}, \quad z \in U, \tag{2.10}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z) := \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)}$, $z \in U$, $z \ne 0$, $f \in \mathcal{A}$. By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \ne 0$, $w \in \mathbb{C}\backslash\{0\}$. Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)}{\mu}(\alpha + 2\beta q(z))$, it follows that $\operatorname{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = \operatorname{Re}\left(\frac{q'(z)}{\mu}(\alpha + 2\beta q(z))\right) > 0$, for $\mu,\xi,\beta \in \mathbb{C}$, $\mu \ne 0$.

By using (2.9) we obtain $\alpha q(z) + \beta(q(z))^2 + \mu z q'(z) \prec \alpha q(z) + \beta(q(z))^2 + \mu z q'(z)$. Using Lemma 1.2, we have $q(z) \prec p(z) = \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.6 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $m,n \in \mathbb{N}$, $\lambda \ge 0$. *Assume that (2.8) holds. If* $f \in \mathcal{A}, \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha\frac{1+Az}{1+Bz} + \beta\left(\frac{1+Az}{1+Bz}\right)^2 + \mu\frac{(A-B)z}{(1+Bz)^2} \prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\mu,\beta \in \mathbb{C}$, $\mu \ne 0$, $-1 \le B < A \le 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.4), then* $\frac{1+Az}{1+Bz} \prec \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)}$ *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$ in Theorem 2.5 we get the corollary. ■

Corollary 2.7 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $m,n \in \mathbb{N}$, $\lambda \ge 0$. *Assume that (2.8) holds. If* $f \in \mathcal{A}, \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha\left(\frac{1+z}{1-z}\right)^\gamma + \beta\left(\frac{1+z}{1-z}\right)^{2\gamma} + \mu\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1} \prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\mu,\beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\mu \ne 0$, *where* $\psi_\lambda^{m,n}$ *is defined in (2.4), then* $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)}$ *and* $\left(\frac{1+z}{1-z}\right)^\gamma$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.5 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \le 1$. ■

Combining Theorem 2.2 and Theorem 2.5, we state the following sandwich theorem.

Theorem 2.8 *Let q_1 and q_2 be analytic and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$, with $zq_1'(z)$ and $zq_2'(z)$ being starlike univalent. Suppose that q_1 satisfies (2.3) and q_2 satisfies (2.8). If $f \in \mathcal{A}$, $\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ is as defined in (2.4) univalent in U, then $\alpha q_1(z) + \beta(q_1(z))^2 + \mu z q_1'(z) \prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z) \prec \alpha q_2(z) + \beta(q_2(z))^2 + \mu z q_2'(z)$, for $\alpha,\mu,\beta \in \mathbb{C}$, $\mu \neq 0$, implies $q_1(z) \prec \frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \prec q_2(z)$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \le B_2 < B_1 < A_1 < A_2 \le 1$, we have the following corollary.

Corollary 2.9 *Let $m,n \in \mathbb{N}$, $\lambda \ge 0$. Assume that (2.3) and (2.8) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \frac{1+A_1 z}{1+B_1 z} + \beta\left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \mu \frac{(A_1-B_1)z}{(1+B_1 z)^2} \prec \psi_\lambda^{m,n}(\alpha,\beta,\mu;z)$ $\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \beta\left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \mu \frac{(A_2-B_2)z}{(1+B_2 z)^2}$, for $\alpha,\mu,\beta \in \mathbb{C}$, $\mu \neq 0$, $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, where $\psi_\lambda^{m,n}$ is defined in (2.4), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \prec \frac{1+A_2 z}{1+B_2 z}$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

Theorem 2.10 *Let $\left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $\delta \in \mathbb{C}, \delta \neq 0$, $m,n \in \mathbb{N}$, $\lambda \ge 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0)=1$, $z \in U$. Assume that*

$$\operatorname{Re}\left(\frac{\alpha+\beta}{\beta} + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{2.11}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\psi_\lambda^{m,n}(\alpha,\beta;z) := \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \left[\alpha + \delta\beta \frac{1-\lambda(n+1)}{\lambda} + \right.$$

$$\left. \delta\beta(n+1)\left[1-\lambda(n+2)\right] \frac{DR_\lambda^{m,n+1} f(z)}{DR_\lambda^{m+1,n} f(z)} + \delta\beta\lambda(n+1)(n+2) \frac{DR_\lambda^{m,n+2} f(z)}{DR_\lambda^{m+1,n} f(z)} - \frac{\delta\beta}{\lambda} \frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right] \tag{2.12}$$

If q satisfies the following subordination

$$\psi_\lambda^{m,n}(\alpha,\beta;z) \prec \alpha q(z) + \beta z q'(z), \tag{2.13}$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then

$$\left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \prec q(z), \quad z \in U,\ \delta \in \mathbb{C},\ \delta \neq 0, \tag{2.14}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$. We have $zp'(z) = \delta z \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \frac{DR_\lambda^{m,n} f(z)}{DR_\lambda^{m+1,n} f(z)} \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)' = \delta \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \frac{DR_\lambda^{m,n} f(z)}{DR_\lambda^{m+1,n} f(z)} \left(\frac{z\left(DR_\lambda^{m+1,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)} - \frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}\right)$.

By using the identity (2.1) and (2.2), we obtain $zp'(z) = \delta \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \frac{DR_\lambda^{m,n} f(z)}{DR_\lambda^{m+1,n} f(z)} \left[\left(\frac{1-\lambda(n+1)}{\lambda}\right) \frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)} \right.$ $\left. +(n+1)\left[1-\lambda(n+2)\right] \frac{DR_\lambda^{m,n+1} f(z)}{DR_\lambda^{m,n} f(z)} + \lambda(n+1)(n+2) \frac{DR_\lambda^{m,n+2} f(z)}{DR_\lambda^{m,n} f(z)} - \frac{1}{\lambda}\left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^2\right]$ so, we obtain

$$zp'(z) = \delta \left(\frac{DR_\lambda^{m+1,n} f(z)}{DR_\lambda^{m,n} f(z)}\right)^\delta \left[\frac{1-\lambda(n+1)}{\lambda} + \right.$$

$$(n+1)\left[1-\lambda(n+2)\right]\frac{DR_{\lambda}^{m,n+1}f(z)}{DR_{\lambda}^{m+1,n}f(z)}+\lambda(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m+1,n}f(z)}-\frac{1}{\lambda}\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\Bigg] \tag{2.15}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U. Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta zq'(z)$.We have $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re}\left(\frac{\alpha+\beta}{\beta} + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.15), we obtain $\alpha p(z) + \beta zp'(z) = \left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta}\Big[\alpha + \delta\beta\frac{1-\lambda(n+1)}{\lambda} + \delta\beta(n+1)\left[1-\lambda(n+2)\right]\cdot \frac{DR_{\lambda}^{m,n+1}f(z)}{DR_{\lambda}^{m+1,n}f(z)} + \delta\beta\lambda(n+1)(n+2)\frac{DR_{\lambda}^{m,n+2}f(z)}{DR_{\lambda}^{m+1,n}f(z)} - \frac{\delta\beta}{\lambda}\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\Big]$.

By using (2.13), we have $\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z)$. From Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta} \prec q(z)$, $z \in U, \delta \in \mathbb{C}, \delta \neq 0$ and q is the best dominant. ■

Corollary 2.11 *Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \leq B < A \leq 1$, $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda}^{m,n}(\alpha,\beta;z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Bz)^2}$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda}^{m,n}$ is defined in (2.12), then $\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 *Let $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \beta\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1}$, for $\alpha,\beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_{\lambda}^{m,n}$ is defined in (2.12), then $\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ■

Theorem 2.13 *Let q be convex and univalent in U such that $q(0) = 1$. Assume that*

$$\operatorname{Re}\left(\frac{\alpha}{\beta}q'(z)\right) > 0, \text{ for } \alpha,\beta \in \mathbb{C},\ \beta \neq 0. \tag{2.16}$$

If $f \in \mathcal{A}$, $\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda}^{m,n}(\alpha,\beta;z)$ is univalent in U, where $\psi_{\lambda}^{m,n}(\alpha,\beta;z)$ is as defined in (2.12), then

$$\alpha q(z) + \beta zq'(z) \prec \psi_{\lambda}^{m,n}(\alpha,\beta;z) \tag{2.17}$$

implies

$$q(z) \prec \left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta}, \quad \delta \in \mathbb{C},\ \delta \neq 0,\ z \in U, \tag{2.18}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z) := \left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta}$, $z \in U$, $z \neq 0$, $\delta \in \mathbb{C}, \delta \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$. By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta}q'(z)$, it follows that $\operatorname{Re}\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = \operatorname{Re}\left(\frac{\alpha}{\beta}q'(z)\right) > 0$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.17) we obtain $\alpha q(z) + \beta zq'(z) \prec \alpha q(z) + \beta zq'(z)$, $z \in U$. From Lemma 1.2, we have $q(z) \prec p(z) = \left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta}$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q is the best subordinant. ■

Corollary 2.14 *Let $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (2.16) holds. If $f \in \mathcal{A}$, $\left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta} \in \mathcal{H}[q(0),1] \cap Q$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and $\alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda}^{m,n}(\alpha,\beta;z)$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda}^{m,n}$ is defined in (2.12), then $\frac{1+Az}{1+Bz} \prec \left(\frac{DR_{\lambda}^{m+1,n}f(z)}{DR_{\lambda}^{m,n}f(z)}\right)^{\delta}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}, m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.16) holds. If* $f \in \mathcal{A}, \left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \beta\frac{2\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma-1} \prec \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z)$, *for* $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.12), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta}$, $\delta \in \mathbb{C}$, $\delta \neq 0$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$, *for all* $z \in U$. *Suppose that* q_1 *satisfies (2.11) and* q_2 *satisfies (2.16). If* $f \in \mathcal{A}$, $\left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$, $\delta \in \mathbb{C}, \delta \neq 0$ *and* $\psi_{\lambda}^{m,n}(\alpha, \beta; z)$ *is as defined in (2.12) univalent in* U, *then* $\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda}^{m,n}(\alpha, \beta; z) \prec \alpha q_2(z) + \beta z q_2'(z)$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, *implies* $q_1(z) \prec \left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta} \prec q_2(z)$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, *and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 *Let* $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (2.11) and (2.16) hold for* $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ *and* $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, *respectively. If* $f \in \mathcal{A}$, $\left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta} \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\alpha\frac{1+A_1 z}{1+B_1 z} + \beta\frac{(A_1 - B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha\frac{1+A_2 z}{1+B_2 z} + \beta\frac{(A_2 - B_2)z}{(1+B_2 z)^2}$, $z \in U$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (2.4), then* $\frac{1+A_1 z}{1+B_1 z} \prec \left(\frac{DR_{\lambda}^{m+1,n} f(z)}{DR_{\lambda}^{m,n} f(z)}\right)^{\delta} \prec \frac{1+A_2 z}{1+B_2 z}$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, *hence* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

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Subalgebras of BCK/BCI-algebras based on (α, β)-type fuzzy sets

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Abstract. The notion of (ε, δ)-characteristic fuzzy sets is introduced. Given a subalgebra F of a BCK/BCI-algebra X, conditions for the (ε, δ)-characteristic fuzzy set in X to be an $(\in, \in \vee q)$-fuzzy subalgebra, an $(\in, q)$-fuzzy subalgebra, an $(\in, \in \wedge q)$-fuzzy subalgebra, a (q, q)-fuzzy subalgebra, a $(q, \in)$-fuzzy subalgebra, a $(q, \in \vee q)$-fuzzy subalgebra and a $(q, \in \wedge q)$-fuzzy subalgebra are provided. Using the notions of (α, β)-fuzzy subalgebra $\mu_F^{(\varepsilon,\delta)}$, conditions for the F to be a subalgebra of X are investigated where (α, β) is one of $(\in, \in \vee q)$, $(\in, \in \wedge q)$, $(\in, q)$, $(q, \in \vee q)$, $(q, \in \wedge q)$, $(q, \in)$ and (q, q).

1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set is given in [7] which played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Several authors [3, 4, 5, 8] have studied the concept of (α, β)-fuzzy subalgebras in BCK/BCI-algebras, which is an important and useful generalization of the well-known concepts, called fuzzy subalgebras.

In this paper, we introduce the notion of (ε, δ)-characteristic fuzzy sets in BCK/BCI-algebras. Given a subalgebra F of a BCK/BCI-algebra X, we provide conditions for the (ε, δ)-characteristic fuzzy set in X to be an $(\in, \in \vee q)$-fuzzy subalgebra, an $(\in, q)$-fuzzy subalgebra, an $(\in, \in \wedge q)$-fuzzy subalgebra, a (q, q)-fuzzy subalgebra, a $(q, \in)$-fuzzy subalgebra, a $(q, \in \vee q)$-fuzzy subalgebra and a $(q, \in \wedge q)$-fuzzy subalgebra. Using the notions of (α, β)-fuzzy subalgebra $\mu_F^{(\varepsilon,\delta)}$, we investigate conditions for the F to be a subalgebra of X where (α, β) is one of $(\in, \in \vee q)$, $(\in, \in \wedge q)$, $(\in, q)$, $(q, \in \vee q)$, $(q, \in \wedge q)$, $(q, \in)$ and (q, q).

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2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [2] and [6] for further information regarding BCK/BCI-algebras.

A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [7] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t\, q\, \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set μ. To say that $x_t \in\vee q\; \mu$ (resp. $x_t \in\wedge q\; \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if it satisfies:

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \tag{2.1}$$

for all $x, y \in X$.

Proposition 2.1 ([4]). *Let X be a BCK/BCI-algebra. A fuzzy set μ in X is a fuzzy subalgebra of X if and only if the following assertion is valid.*

$$x_t \in \mu,\; y_s \in \mu \implies (x * y)_{\min\{t,s\}} \in \mu \tag{2.2}$$

for all $x, y \in X$ and $t, s \in (0, 1]$.

3. Subalgebras of BCK/BCI-algebras based on (α, β)-type fuzzy sets

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Let F be a non-empty subset of X and $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \delta$. Define a fuzzy set $\mu_F^{(\varepsilon,\delta)}$ in X as follows:

$$\mu_F^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in F, \\ \delta & \text{otherwise.} \end{cases}$$

We say that $\mu_F^{(\varepsilon,\delta)}$ is an (ε,δ)-*characteristic fuzzy set* in X over F. In particular, the $(1,0)$-characteristic fuzzy set $\mu_F^{(1,0)}$ in X over F is the characteristic function χ_F of F.

Theorem 3.1. *For any non-empty subset F of X, the following are equivalent:*

(1) *F is a subalgebra of X.*
(2) *The fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of X for all $\varepsilon,\delta\in[0,1]$ with $\varepsilon>\delta$.*

Proof. Assume that F is a subalgebra of X and let $\varepsilon,\delta\in[0,1]$ be such that $\varepsilon>\delta$. Let $x,y\in X$. If $x,y\in F$, then $x*y\in F$ and so

$$\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon=\min\left\{\mu_F^{(\varepsilon,\delta)}(x),\mu_F^{(\varepsilon,\delta)}(y)\right\}.$$

If $x\notin F$ or $y\notin F$, then $\mu_F^{(\varepsilon,\delta)}(x)=\delta$ or $\mu_F^{(\varepsilon,\delta)}(y)=\delta$. Hence

$$\mu_F^{(\varepsilon,\delta)}(x*y)\geq\delta=\min\left\{\mu_F^{(\varepsilon,\delta)}(x),\mu_F^{(\varepsilon,\delta)}(y)\right\}.$$

Therefore $\mu_F^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of X for all $\varepsilon,\delta\in[0,1]$ with $\varepsilon>\delta$.

Conversely, suppose that (2) is valid. Let $x,y\in F$. Then $\mu_F^{(\varepsilon,\delta)}(x)=\varepsilon$ and $\mu_F^{(\varepsilon,\delta)}(y)=\varepsilon$. It follows that $\mu_F^{(\varepsilon,\delta)}(x*y)\geq\min\left\{\mu_F^{(\varepsilon,\delta)}(x),\mu_F^{(\varepsilon,\delta)}(y)\right\}=\varepsilon$. Thus $x*y\in F$, and therefore F is a subalgebra of X. □

Definition 3.2 ([4]). A fuzzy set μ in X is said to be an (α,β)-*fuzzy subalgebra* of X, where $\alpha,\beta\in\{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha\neq\,\in\wedge q$, if it satisfies the following condition:

$$x_{t_1}\alpha\,\mu,\ y_{t_2}\alpha\,\mu\ \Rightarrow\ (x*y)_{\min\{t_1,t_2\}}\,\beta\,\mu. \tag{3.1}$$

for all $x,y\in X$ and $t_1,t_2\in(0,1]$.

Lemma 3.3 ([4]). *A fuzzy set μ in X is an $(\in,\in\vee q)$-fuzzy subalgebra of X if and only if it satisfies:*

$$(\forall x,y\in X)\,(\mu(x*y)\geq\min\{\mu(x),\mu(y),0.5\})\,. \tag{3.2}$$

Theorem 3.4. *If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in,\in\vee q)$-fuzzy subalgebra of X for all $\varepsilon,\delta\in[0,1]$ with $\varepsilon>\delta$.*

Proof. Assume that F is a subalgebra of X and let $\varepsilon,\delta\in[0,1]$ such that $\varepsilon>\delta$. For any $x,y\in X$, if $x,y\in F$, then $x*y\in F$ and so

$$\mu_F^{(\varepsilon,\delta)}(x*y)=\varepsilon\geq\min\left\{\mu_F^{(\varepsilon,\delta)}(x),\mu_F^{(\varepsilon,\delta)}(y),0.5\right\}.$$

If $x\notin F$ or $y\notin F$, then $\mu_F^{(\varepsilon,\delta)}(x)=\delta$ or $\mu_F^{(\varepsilon,\delta)}(y)=\delta$. Hence

$$\mu_F^{(\varepsilon,\delta)}(x*y)\geq\delta\geq\min\left\{\mu_F^{(\varepsilon,\delta)}(x),\mu_F^{(\varepsilon,\delta)}(y),0.5\right\}.$$

It follows from Lemma 3.3 that $\mu_F^{(\varepsilon,\delta)}$ is an $(\in,\in\vee q)$-fuzzy subalgebra of X for all $\varepsilon,\delta\in[0,1]$ with $\varepsilon>\delta$. □

Corollary 3.5. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is an $(\in,\in\vee q)$-fuzzy subalgebra of X.*

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Proof. The necessity is by taking $\varepsilon = 1$ and $\delta = 0$ in Theorem 3.4.

Conversely, suppose that the characteristic function χ_F of F is an $(\in, \in\vee q)$-fuzzy subalgebra of X. Let $x, y \in F$. Then $\chi_F(x) = 1 = \chi_F(y)$, which implies from (3.2) that

$$\chi_F(x * y) \geq \min\{\chi_F(x), \chi_F(y), 0.5\} = \min\{1, 0.5\} = 0.5.$$

Hence $x * y \in F$, and therefore F is a subalgebra of X. $\square$

We consider the converse of Theorem 3.4.

Theorem 3.6. *For any $\varepsilon, \delta \in [0, 1]$ such that $\delta < \varepsilon \leq 0.5$, if the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of X then F is a subalgebra of X.*

Proof. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$. Using Lemma 3.3, we have

$$\mu_F^{(\varepsilon,\delta)}(x * y) \geq \min\left\{\mu_F^{(\varepsilon,\delta)}(x), \mu_F^{(\varepsilon,\delta)}(y), 0.5\right\} = \min\{\varepsilon, 0.5\} = \varepsilon,$$

and so $x * y \in F$. Therefore F is a subalgebra of X. $\square$

Theorem 3.7. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \delta$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of X whenever if any element t in $(0, 1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta < t$ and $1 - t < \varepsilon$.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) \geq t_1 > \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) \geq t_2 > \delta$. It follows that $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x, y \in F$. Since F is a subalgebra of X, we have $x * y \in F$. Hence $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$, and thus $\mu_F^{(\varepsilon,\delta)}(x * y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$ which shows that $(x * y)_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Therefore $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of X. $\square$

Theorem 3.8. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \delta$. If $\varepsilon + \delta \leq 1$ and the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Assume that $\varepsilon + \delta \leq 1$ and the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of X. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$. Hence $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_F^{(\varepsilon,\delta)}$, which implies that $\mu_F^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. Therefore $\mu_F^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$, and thus $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$, that is, $x * y \in F$. Consequently, F is a subalgebra of X. $\square$

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.7 and 3.8, then we have the following corollary.

Corollary 3.9. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is an $(\in, q)$-fuzzy subalgebra of X.*

Theorem 3.10. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \delta$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a (q, q)-fuzzy subalgebra of X whenever if any element t in $(0, 1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta \leq 1 - t < \varepsilon$.*

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Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) + t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y) + t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x) > 1 - t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1 - t_2 \geq \delta$. It follows that $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$ and so that $x, y \in F$. Since F is a subalgebra of X, we have $x * y \in F$ and so $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus

$$\mu_F^{(\varepsilon,\delta)}(x * y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1,$$

that is, $(x * y)_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. This shows that $\mu_F^{(\varepsilon,\delta)}$ is a (q, q)-fuzzy subalgebra of X. □

Theorem 3.11. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon + \delta \leq 1$. If the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a (q, q)-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_F^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a (q, q)-fuzzy subalgebra of X, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Hence $\mu_F^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$, and therefore $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$. This proves that $x * y \in F$, and F is a subalgebra of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.10 and 3.11, then we have the following corollary.

Corollary 3.12. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is a (q, q)-fuzzy subalgebra of X.*

Theorem 3.13. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \delta$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of X whenever if any element t in $(0, 1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta \leq 1 - t$ and $t < \varepsilon$.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1}\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) + t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y) + t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x) > 1 - t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1 - t_2 \geq \delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x, y \in F$. Since F is a subalgebra of X, we have $x * y \in F$ and thus

$$\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{t_1, t_2\},$$

that is, $(x * y)_{\min\{t_1,t_2\}} \in \mu_F^{(\varepsilon,\delta)}$. This shows that $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of X. □

Theorem 3.14. *Let $\varepsilon, \delta \in [0, 1]$ such that $\varepsilon > \max\{\delta, 0.5\}$. If the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_F^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of X, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_F^{(\varepsilon,\delta)}$ and so that $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$, that is, $x * y \in F$. Therefore F is a subalgebra of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.13 and 3.14, then we have the following corollary.

Corollary 3.15. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is a $(q, \in)$-fuzzy subalgebra of X.*

Theorem 3.16. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \delta$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of X whenever if any element t in $(0,1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta < t$ and $1 - t < \varepsilon$.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \in \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \in \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) \geq t_1 > \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) \geq t_2 > \delta$, which imply that $x, y \in F$ and $\varepsilon \geq \min\{t_1, t_2\}$. Since F is a subalgebra of X, we have $x * y \in F$. Hence $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{t_1, t_2\}$, i.e., $(x * y)_{\min\{t_1,t_2\}} \in \mu_F^{(\varepsilon,\delta)}$. Now, $\mu_F^{(\varepsilon,\delta)}(x * y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$ and so $(x * y)_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Therefore $(x * y)_{\min\{t_1,t_2\}} \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$, and consequently $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of X. □

Theorem 3.17. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \delta$. If $\varepsilon + \delta \leq 1$ and the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Assume that $\varepsilon + \delta \leq 1$ and the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of X. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_F^{(\varepsilon,\delta)}$. Hence $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$, that is, $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_F^{(\varepsilon,\delta)}$ and $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Hence $\mu_F^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_F^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. If $\mu_F^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and thus $x * y \in F$. If $\mu_F^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$, then $\mu_F^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so $\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon$, which shows that $x * y \in F$. Therefore F is a subalgebra of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.16 and 3.17, then we have the following corollary.

Corollary 3.18. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is an $(\in, \in\wedge q)$-fuzzy subalgebra of X.*

Theorem 3.19. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \delta$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q)$-fuzzy subalgebra of X under the condition that if any element t in $(0,1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta \leq 1 - t$ and $t < \varepsilon$.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1}\, q\, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2}\, q\, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) + t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y) + t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x) > 1 - t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1 - t_2 \geq \delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$ and $\varepsilon > \max\{1 - t_1, 1 - t_2\}$, and so $x, y \in F$. Since F is a subalgebra of X, we have $x * y \in F$ and thus

$$\mu_F^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{t_1, t_2\},$$

that is, $(x * y)_{\min\{t_1,t_2\}} \in \mu_F^{(\varepsilon,\delta)}$. Now, $\mu_F^{(\varepsilon,\delta)}(x * y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$, and so $(x * y)_{\min\{t_1,t_2\}}\, q\, \mu_F^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{t_1,t_2\}} \in\wedge q\, \mu_F^{(\varepsilon,\delta)}$, and $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q)$-fuzzy subalgebra of X. □

Theorem 3.20. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \max\{\delta, 0.5\}$. If the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q)$-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_F^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\wedge\, q)$-fuzzy subalgebra of X, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge\, q \, \mu_F^{(\varepsilon,\delta)}$ and so that $\mu_F^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$. Hence $x*y \in F$ and F is a subalgebra of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.19 and 3.20, then we have the following corollary.

Corollary 3.21. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is a $(q, \in\wedge\, q)$-fuzzy subalgebra of X.*

Theorem 3.22. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \delta$. Assume that if any element t in $(0,1]$ satisfies $x_t \in \mu_F^{(\varepsilon,\delta)}$ for $x \in X$ then $\delta \leq 1-t$. If F is a subalgebra of X, then the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $x_{t_1} \, q \, \mu_F^{(\varepsilon,\delta)}$ and $y_{t_2} \, q \, \mu_F^{(\varepsilon,\delta)}$. Then $\mu_F^{(\varepsilon,\delta)}(x) + t_1 > 1$ and $\mu_F^{(\varepsilon,\delta)}(y) + t_2 > 1$, which imply that $\mu_F^{(\varepsilon,\delta)}(x) > 1 - t_1 \geq \delta$ and $\mu_F^{(\varepsilon,\delta)}(y) > 1 - t_2 \geq \delta$. Hence $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, and so $\varepsilon > \max\{1-t_1, 1-t_2\}$ and $x, y \in F$. Since F is a subalgebra of X, we have $x*y \in F$ and thus $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon$ which implies that $\mu_F^{(\varepsilon,\delta)}(x*y) + \min\{t_1, t_2\} = \varepsilon + \min\{t_1, t_2\} > 1$, i.e., $(x*y)_{\min\{t_1,t_2\}} \, q \, \mu_F^{(\varepsilon,\delta)}$. It follows that $(x*y)_{\min\{t_1,t_2\}} \in\vee\, q \, \mu_F^{(\varepsilon,\delta)}$. Therefore $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subalgebra of X. □

Theorem 3.23. *Let $\varepsilon, \delta \in [0,1]$ such that $\varepsilon > \max\{\delta, 0.5\}$ and $\varepsilon + \delta \leq 1$. If the fuzzy set $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subalgebra of X, then F is a subalgebra of X.*

Proof. Let $x, y \in F$. Then $\mu_F^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_F^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_F^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_F^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_F^{(\varepsilon,\delta)}$. Since $\mu_F^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q)$-fuzzy subalgebra of X, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\vee\, q \, \mu_F^{(\varepsilon,\delta)}$, that is, $\mu_F^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$ or $\mu_F^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$. If $\mu_F^{(\varepsilon,\delta)}(x*y) \geq \varepsilon$, then $x*y \in F$. If $\mu_F^{(\varepsilon,\delta)}(x*y) + \varepsilon > 1$, then $\mu_F^{(\varepsilon,\delta)}(x*y) > 1 - \varepsilon \geq \delta$ and so $\mu_F^{(\varepsilon,\delta)}(x*y) = \varepsilon$. Thus $x*y \in F$, and therefore F is a subalgebra of X. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.22 and 3.23, then we have the following corollary.

Corollary 3.24. *A non-empty subset F of X is a subalgebra of X if and only if the characteristic function χ_F of F is a $(q, \in\vee\, q)$-fuzzy subalgebra of X.*

Conclusions

We have introduced the notion of (ε, δ)-characteristic fuzzy sets in BCK/BCI-algebras. Given a subalgebra F of a BCK/BCI-algebra X, we have provided conditions for the (ε, δ)-characteristic fuzzy set in X to be an $(\in, \in\vee q)$-fuzzy subalgebra, an $(\in, q)$-fuzzy subalgebra, an $(\in, \in\wedge q)$-fuzzy subalgebra, a (q, q)-fuzzy subalgebra, a $(q, \in)$-fuzzy subalgebra, a $(q, \in\vee q)$-fuzzy subalgebra and a $(q, \in\wedge q)$-fuzzy subalgebra. Using the notions of (α, β)-fuzzy subalgebra $\mu_F^{(\varepsilon,\delta)}$, we have investigated conditions for the

F to be a subalgebra of X where (α, β) is one of $(\in, \in \vee q)$, $(\in, \in \wedge q)$, $(\in, q)$, $(q, \in \vee q)$, $(q, \in \wedge q)$, $(q, \in)$ and (q, q).

In the consecutive research, we will discuss the following items:

(1) Given a subalgebra F of a BCK/BCI-algebra X, we will provide conditions for the (ε, δ)-characteristic fuzzy set in X to be an $(\in \vee q, \in \vee q)$-fuzzy subalgebra, an $(\in \vee q, \in)$-fuzzy subalgebra, an $(\in \vee q, \in \wedge q)$-fuzzy subalgebra, and an $(\in \vee q, q)$-fuzzy subalgebra.

(2) Using the notions of (α, β)-fuzzy subalgebra $\mu_F^{(\varepsilon,\delta)}$ where (α, β) is one of $(\in \vee q, \in \vee q)$, $(\in \vee q, \in)$, $(\in \vee q, \in \wedge q)$ and $(\in \vee q, q)$, we investigate conditions for the F to be a subalgebra of X.

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Existence results for nonlinear fractional integrodifferential equations with antiperiodic type integral boundary conditions

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Abstract: This paper investigates the existence of solutions for a class of nonlinear boundary value problems involving fractional integrodifferential equations of fractional order $\alpha \in (2, 3]$ with antiperiodic type integral boundary conditions. Our results are based on contraction mapping principle and Krasnoselskii fixed point theorem. As an application, an interesting example is presented to illustrate the main results.

Keywords: Fractional integrodifferential equations; antiperiodic boundary conditions; integral boundary conditions; fixed point theorem

2010 Mathematics Subject Classification: 34A08, 34B18.

1 Introduction

In the last few decades, the topic of fractional differential equations has gained a considerable attention and it has emerged as a popular field of research due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, etc. Therefore, there have been many papers and books dealing with the theoretical development of fractional calculus and the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations; for examples and details, one can see [1, 2, 3, 4, 5, 6, 7] and references along these lines. For instance, Ahmad and Sivasundaram [8] proved the existence and uniqueness of solutions for a four-point nonlocal boundary value problem of nonlinear integro-differential equations of fractional order $q \in (1, 2]$ by applying some standard fixed point theorems. Ahmad and Ntouyas [9, 10] studied the existence and uniqueness results for a class of nonlocal boundary value problems of nonlinear differential equations and inclusions of fractional order with strip conditions by using a variety of fixed point theorems.

Recently, antiperiodic and/or integral boundary value problems of fractional differential equations occur in the mathematic modeling of a variety of physical processes and have been studied by a number of authors. For examples and details of antiperiodic and/or integral fractional boundary conditions, see [11, 12, 13, 14, 15, 16] and the references therein. For example, Ahmad and Nieto [17] obtained the existence and uniqueness results for antiperiodic boundary value problem for nonlinear fractional differential equation of order $q \in (1, 2]$ by applying some standard fixed point principles. By using Schauder's fixed point theorem and the contraction mapping principle, Wang and Liu [18] considered the existence and uniqueness results for antiperiodic fractional boundary value problem with fractional derivative. In [19], the authors investigated the following boundary value problem for a nonlinear fractional integrodifferential equation with integral boundary conditions

$$
\begin{gathered}
{}^cD^q x(t) = f(t, x(t), (\chi x)(t)), \quad t \in [0, 1], \quad q \in (1, 2], \\
\alpha x(0) + \beta'(0) = \int_0^1 q_1(x(s))ds, \quad \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s))ds,
\end{gathered}
$$

where $f : [0, 1] \times X \times X \to X$, $(\chi x)(t) = \int_0^t \gamma(t, s)x(s)ds$ for $\gamma : [0, T] \times [0, T] \to [0, +\infty)$, $q_1, q_2 : X \to X$

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and $\alpha > 0$ and $\beta \geq 0$ are real numbers. The authors established sufficient conditions for the existence of solutions for the above boundary value problems.

In [20], Wang et al. studied the following fractional boundary value problem with antiperiodic boundary conditions

$$\begin{gathered}
{}^cD^\alpha x(t) = f(t, x(t)), \quad t \in [0, T], \quad \alpha \in (2, 3], \\
x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T), \quad {}^cD^q x(0) = -{}^cD^q x(T),
\end{gathered}$$

where $0 < p < 1 < q < 2$, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Some existence and uniqueness results are obtained contraction mapping principle and Leray-Schauder's fixed point theorem.

In [21], Chai concerned with the following antiperiodic boundary value problems of fractional differential equations

$$\begin{gathered}
{}^cD^\alpha x(t) = f(t, x(t), {}^cD^{\alpha_1}(t), {}^cD^{\alpha_2}(t)), \quad t \in (0, 1), \quad \alpha \in (2, 3], \\
x(0) = -x(T), \quad t^{\beta_1 - 1c}D^{\beta_1} x(t)|_{t\to 0} = -t^{\beta_1 - 1c}D^{\beta_1} x(t)|_{t=1}, \quad t^{\beta_2 - 1c}D^{\beta_2} x(t)|_{t\to 0} = -t^{\beta_2 - 1c}D^{\beta_2} x(t)|_{t=1},
\end{gathered}$$

where $2 < \alpha \leq 3$, $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2$, $0 < \beta_1 < 1 < \beta_2 < 2$, and f is a given continuous function. The author obtained some existence results by applying the Banach contraction mapping principle and the Leray-Schauder degree theory.

In [22], Alsaedi discussed existence of solutions for the following integrodifferential equations of fractional order with antiperiodic boundary conditions

$$\begin{gathered}
{}^cD^q x(t) = f(t, x(t), (\chi x)(t)), \quad t \in [0, T], \quad q \in (1, 2], \\
x(0) = -x(T), \quad x'(0) = -x'(T),
\end{gathered}$$

where $f : [0, 1] \times X \times X \to X$, $(\chi x)(t) = \int_0^t \gamma(t, s)x(s)ds$ for $\gamma : [0, 1] \times [0, 1] \to [0, +\infty)$.

In [23], Ahmad et al. considered the existence and uniqueness of the solutions for a new class of boundary value problems of nonlinear fractional differential equations with non-separated type integral boundary conditions

$$\begin{gathered}
{}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \quad q \in (1, 2], \\
x(0) - \lambda_1 x(T) = \mu_1 \int_0^T g(s, x(s))ds, \quad x'(0) - \lambda_2 x'(T) = \mu_2 \int_0^T h(s, x(s))ds,
\end{gathered}$$

where $f, g, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \mu_j \in \mathbb{R}$ $(\lambda_j \neq 0)$, $j = 1, 2$.

Ahmad and Ntouyas [24] and Ahmad et al. [25] study a boundary value problem of fractional differential eqations and inclusions with anti-periodic type integral boundary conditions given by

$$\begin{gathered}
{}^cD^\alpha x(t) = f(t, x(t)) \ (\in F(t, x(t))), \quad t \in [0, T], \quad \alpha \in (2, 3], \\
x^{(j)}(0) - \lambda_j x^{(j)}(T) = \mu_j \int_0^T g_j(s, x(s))ds, \quad j = 0, 1, 2,
\end{gathered}$$

where $x^{(j)}(\cdot)$ denotes jth derivative of $x(\cdot)$ with $x^{(0)} = x(\cdot)$, $g_j : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \mu_j \in \mathbb{R}$ $(\lambda_j \neq 0)$, respectively.

In this paper, motivated greatly by the above mentioned works, we consider the following boundary value problem for a nonlinear fractional integrodifferential equation of fractional order $\alpha \in (2, 3]$ with antiperiodic type integral boundrary conditions

$$\begin{gathered}
{}^cD^\alpha u(t) = f(t, u(t), (\phi u)(t), (\psi u)(t)), \quad t \in \mathscr{J} = [0, T] \quad (T > 0), \\
u(0) = \mu_0 u(T) + \nu_0 \int_0^T g_0(s, u(s))ds, \quad w_i(0) = \mu_i w_i(T) + \nu_i \int_0^T g_i(s, u(s))ds, \quad i = 1, 2,
\end{gathered} \tag{1.1}$$

where ${}^cD^\alpha$ is the standard Caputo fractional derivative of fractional order α, $0 < \alpha_1 < 1 < \alpha_2 < 2$, $w_i(t) = t^{\alpha_i - i c}D^{\alpha_i} u(t)$, $w_i(0) = \lim_{t\to 0^+} w_i(t)$, $w_i(T) = [w_i(t)]_{t=T}$, $\mu_0, \nu_0, \mu_i, \nu_i \in \mathbb{R}$ $(\mu_0, \mu_i \neq 1)$, $i = 1, 2$, the nonlinear function $f : \mathscr{J} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $g_i : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions for $i = 0, 1, 2$, and for $\gamma, \delta : [0, T] \times [0, T] \to [0, \infty)$,

$$(\phi u)(t) = \int_0^t \gamma(t, s, u(s))ds, \quad (\psi u)(t) = \int_0^T \delta(t, s, u(s))ds.$$

Let $C = C(\mathscr{J}, \mathbb{R})$ be Banach space of all continuous functions from $\mathscr{J} \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm denoted by $\|\varphi\| = \sup\{|\varphi(t)| : s \in \mathscr{J}\}$.

2 Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of the problem (1.1). These materials can be found in the recent literature, see [27, 29, 30].

Definition 2.1. For at least n-times continuously differentiable function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2. The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,$$

provided that such integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order α for a function f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \quad n = [\alpha] + 1,$$

provided that the right-hand side of the previous equation is pointwise defined on $(0, \infty)$.

Lemma 2.4 ([26, 28])**.** *Let $\alpha > 0$, then the fractional differential equation*

$${}^cD^\alpha u(t) = 0$$

has a unique solution given by the expression

$$u(t) = \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j.$$

Lemma 2.5 ([26, 29])**.** *Let $\alpha > 0$, then*

$$I^{\alpha c}D^\alpha u(t) = u(t) - \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j.$$

In view of Lemma 2.5, it follows that

$$I^{\alpha c}D^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$ $(n = [\alpha] + 1)$.

Lemma 2.6. *For any $y \in C[0,T]$, the unique solution of the linear fractional boundary value problem*

$$\begin{gathered} {}^cD^\alpha u(t) = y(t), \quad t \in [0,T], \quad \alpha \in (2,3], \\ u(0) = \mu_0 u(T) + \nu_0 \int_0^T g_0(s,u(s))ds, \quad w_i(0) = \mu_i w_i(T) + \nu_i \int_0^T g_i(s,u(s))ds, \end{gathered} \tag{2.2}$$

is given by

$$u(t) = \int_0^T G(t,s)y(s)ds + \nu_0\xi_0 \int_0^T g_0(s,u(s))ds + \nu_1\eta_1(t) \int_0^T g_1(s,u(s))ds + \nu_2\eta_2(t) \int_0^T g_2(s,u(s))ds,$$

where $G(t,s)$ is Green's function given by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1} + \mu_0\xi_0(T-s)^{\alpha-1}}{\Gamma(\alpha)} \\ +\dfrac{\mu_1\eta_1(t)T^{\alpha_1-1}(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} + \dfrac{\mu_2\eta_2(t)T^{\alpha_2-2}(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)}, & s \le t, \\ \dfrac{\mu_0\xi_0(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{\mu_1\eta_1(t)T^{\alpha_1-1}(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} + \dfrac{\mu_2\eta_2(t)T^{\alpha_2-2}(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)}, & t \le s, \end{cases} \tag{2.3}$$

$$\eta_1(t) = \xi_1[\mu_0 T + (1-\mu_0)t],$$

$$\eta_2(t) = \xi_2[\mu_0[2\mu_1 + (1-\mu_1)(2-\alpha_1)]T^2 + 2\mu_1 T(1-\mu_0)t + (1-\mu_0)(1-\mu_1)(2-\alpha_1)t^2],$$

$$\xi_0 = \frac{1}{1-\mu_0}, \quad \xi_1 = \frac{\Gamma(2-\alpha_1)}{(1-\mu_0)(1-\mu_1)}, \quad \xi_2 = \frac{\Gamma(3-\alpha_2)}{2(1-\mu_0)(1-\mu_1)(1-\mu_2)(2-\alpha_1)}.$$

Proof. Using (2.1), for some constants $b_0, b_1, b_2 \in \mathbb{R}$, we have

$$u(t) = I^\alpha y(t) + b_0 + b_1 t + b_2 t^2. \tag{2.4}$$

Using the facts that ${}^cD^{\alpha_1} b = 0$ (b is constant and $0 < \alpha_1 < 1$), ${}^cD^{\alpha_1} t = \frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)}$, ${}^cD^{\alpha_1} t^2 = \frac{2t^{2-\alpha_1}}{\Gamma(3-\alpha_1)}$, and ${}^cD^{\alpha_1} I^\alpha y(t) = I^{\alpha-\alpha_1} y(t)$, we obtain

$${}^cD^{\alpha_1} u(t) = I^{\alpha-\alpha_1} y(t) + \frac{b_1}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + \frac{2b_2}{\Gamma(3-\alpha_1)} t^{2-\alpha_1}. \tag{2.5}$$

In view of ${}^cD^{\alpha_2} t = 0$ ($1 < \alpha_2 < 2$), ${}^cD^{\alpha_2} t^2 = \frac{2t^{2-\alpha_2}}{\Gamma(3-\alpha_2)}$, and ${}^cD^{\alpha_2} I^\alpha y(t) = I^{\alpha-\alpha_2} y(t)$, we get

$${}^cD^{\alpha_2} u(t) = I^{\alpha-\alpha_2} y(t) + \frac{2b_2}{\Gamma(3-\alpha_2)} t^{2-\alpha_2}. \tag{2.6}$$

From (2.5) and (2.6), we have

$$t^{\alpha_1-1\,c}D^{\alpha_1} u(t) = t^{\alpha_1-1} I^{\alpha-\alpha_1} y(t) + \frac{b_1}{\Gamma(2-\alpha_1)} + \frac{2b_2}{\Gamma(3-\alpha_1)} t, \tag{2.7}$$

$$t^{\alpha_2-2\,c}D^{\alpha_2} u(t) = t^{\alpha_2-2} I^{\alpha-\alpha_2} y(t) + \frac{2b_2}{\Gamma(3-\alpha_2)}. \tag{2.8}$$

From [**?**], we can know that

$$\lim_{t\to 0^+} t^{\alpha_1-1} I^{\alpha-\alpha_1} y(t) = 0, \qquad \lim_{t\to 0^+} t^{\alpha_2-2} I^{\alpha-\alpha_2} y(t) = 0. \tag{2.9}$$

Applying the boundary conditions for problem (2.2) and (2.9) in (2.4), (2.7) and (2.8), we find that

$$\begin{aligned}
b_0 &= \frac{\mu_0}{1-\mu_0} I^\alpha y(T) + \frac{\mu_0\mu_1\Gamma(2-\alpha_1)T^{\alpha_1}}{(1-\mu_0)(1-\mu_1)} I^{\alpha-\alpha_1} y(T) \\
&\quad + \frac{\mu_0\mu_2\Gamma(3-\alpha_2)[2\mu_1+(1-\mu_1)(2-\alpha_1)]T^{\alpha_2}}{2(1-\mu_0)(1-\mu_1)(1-\mu_2)(2-\alpha_1)} I^{\alpha-\alpha_2} y(T) + \frac{\nu_0}{1-\mu_0}\int_0^T g_0(s,u(s))ds \\
&\quad + \frac{\mu_0\nu_1\Gamma(2-\alpha_1)T}{(1-\mu_0)(1-\mu_1)}\int_0^T g_1(s,u(s))ds + \frac{\mu_0\nu_2\Gamma(3-\alpha_2)[2\mu_1+(1-\mu_1)(2-\alpha_1)]T^2}{2(1-\mu_0)(1-\mu_1)(1-\mu_2)(2-\alpha_1)}\int_0^T g_2(s,u(s))ds, \\
b_1 &= \frac{\mu_1\Gamma(2-\alpha_1)T^{\alpha_1-1}}{1-\mu_1} I^{\alpha-\alpha_1} y(T) + \frac{\mu_1\mu_2\Gamma(3-\alpha_2)T^{\alpha_2-1}}{(1-\mu_1)(1-\mu_2)(2-\alpha_1)} I^{\alpha-\alpha_2} y(T) \\
&\quad + \frac{\nu_1\Gamma(2-\alpha_1)}{1-\mu_1}\int_0^T g_1(s,u(s))ds + \frac{\mu_1\nu_2\Gamma(3-\alpha_2)T}{(1-\mu_1)(1-\mu_2)(2-\alpha_1)}\int_0^T g_2(s,u(s))ds, \\
b_2 &= \frac{\mu_2\Gamma(3-\alpha_2)T^{\alpha_2-2}}{2(1-\mu_2)} I^{\alpha-\alpha_2} y(T) + \frac{\nu_2\Gamma(3-\alpha_2)}{2(1-\mu_2)}\int_0^T g_2(s,u(s))ds.
\end{aligned}$$

Thus, the unique solution of (2.2) is

$$\begin{aligned}
u(t) &= I^\alpha y(t) + \mu_0\xi_0 I^\alpha y(T) + \mu_1\eta_1(t)T^{\alpha_1-1}I^{\alpha-\alpha_1}y(T) + \mu_2\eta_2(t)T^{\alpha_2-2}I^{\alpha-\alpha_2}y(T) \\
&\quad + \nu_0\xi_0\int_0^T g_0(s,u(s))ds + \nu_1\eta_1(t)\int_0^T g_1(s,u(s))ds + \nu_2\eta_2(t)\int_0^T g_2(s,u(s))ds \\
&= \int_0^T G(t,s)y(s)ds + \nu_0\xi_0\int_0^T g_0(s,u(s))ds + \nu_1\eta_1(t)\int_0^T g_1(s,u(s))ds + \nu_2\eta_2(t)\int_0^T g_2(s,u(s))ds,
\end{aligned}$$

where $G(t,s)$ is given by (2.3). The proof is completed. □

3 Main results

For the sake of simplicity, we always consider the boundary value problem (1.1) together with the following assumptions.

(H_1) There exist positive constants $\mathscr{L}_i$ and M_i $(i = 0, 1, 2)$ such that

$$\|g_i(t,u) - g_i(t,v)\| \le \mathscr{L}_i\|x-y\|, \quad \|g_i(t,u)\| \le M_i, \quad \forall t \in \mathscr{J}, \quad u, v \in \mathbb{R}.$$

(H_2) There exist continuous function $L_i : [0,T] \to \mathbb{R}^+ = [0,\infty)$ $(i = 1, 2, 3)$ such that

$$\|f(t,u,\phi u,\psi u) - f(t,v,\phi v,\psi v)\| \le L_1(t)\|u-v\| + L_2(t)\|\phi u - \phi v\| + L_3(t)\|\psi u - \psi v\|, \quad \forall t \in \mathscr{J}, \quad u, v \in \mathbb{R}.$$

(H_3) There exist continuous functions $p, q : [0,T] \to \mathbb{R}^+$ such that

$$\left\|\int_0^t (\gamma(t,s,u(s)) - \gamma(t,s,v(s)))ds\right\| \le p(t)\|u-v\|, \quad \forall t \in \mathscr{J}, \quad u, v \in \mathbb{R},$$

$$\left\|\int_0^T (\delta(t,s,u(s)) - \delta(t,s,v(s)))ds\right\| \le q(t)\|u-v\|, \quad \forall t \in \mathscr{J}, \quad u, v \in \mathbb{R}.$$

(H_4) $\rho = (\mathscr{L}_0|\nu_0\xi_0| + \mathscr{L}_1\lambda_1|\nu_1| + \mathscr{L}_2\lambda_2|\nu_2|)T + \mathscr{L}_3\lambda_3 < 1$, where

$$\lambda_1 = \sup\{|\eta_1(t)| : t \in \mathscr{J}\}, \quad \lambda_2 = \sup\{|\eta_2(t)| : t \in \mathscr{J}\},$$

$$\mathscr{L}_3 = \sup\{M(t) = L_1(t) + L_2(t)p(t) + L_3(t)q(t) : t \in \mathscr{J}\}.$$

$$\lambda_3 = \frac{(1+|\mu_0\xi_0|)T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda_1|\mu_1|T^{\alpha-1}}{\Gamma(\alpha-\alpha_1+1)} + \frac{\lambda_2|\mu_2|T^{\alpha-2}}{\Gamma(\alpha-\alpha_2+1)},$$

(H_5) $\|f(t,u(t),(\phi u)(t),(\psi u)(t))\| \le \omega(t)$, for all $(t,u,\phi u,\psi u) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, where $\omega \in L^1([,T],\mathbb{R}^+)$.

Theorem 3.1. *Assume that (H_1)-(H_4) hold. Then the boundary value problem (1.1) has a unique solution on $\mathscr{J}$.*

Proof. Define $F: C \to C$ by

$$\begin{aligned}(Fu)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))ds\\ &+\mu_0\xi_0\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),(\phi u)(s),(\psi u)(s))ds\\ &+\mu_1\eta_1(t)T^{\alpha_1-1}\int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} f(s,u(s),(\phi u)(s),(\psi u)(s))ds\\ &+\mu_2\eta_2(t)T^{\alpha_2-2}\int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)} f(s,u(s),(\phi u)(s),(\psi u)(s))ds\\ &+\nu_0\xi_0\int_0^T g_0(s,u(s))ds+\nu_1\eta_1(t)\int_0^T g_1(s,u(s))ds+\nu_2\eta_2(t)\int_0^T g_2(s,u(s))ds, \quad t\in\mathscr{J}.\end{aligned}$$

Let us set

$$\sup\{L_2(t): t\in\mathscr{J}\} = \bar{L}_2, \quad \sup\{L_3(t): t\in\mathscr{J}\} = \bar{L}_3, \quad \sup\left\{\left\|\int_0^t \gamma(t,\tau,0)d\tau\right\| : t\in\mathscr{J}\right\} = M_3,$$

$$\sup\left\{\left\|\int_0^t \delta(t,\tau,0)d\tau\right\| : t\in\mathscr{J}\right\} = M_4, \quad \sup\{\|f(t,0,0,0)\| : t\in\mathscr{J}\} = M_5.$$

According to the assumptions (H_2) and (H_3), we have

$$\begin{aligned}\|(\phi u)(s)\| &\le \left\|\int_0^s (\gamma(s,\tau,u(\tau))-\gamma(s,\tau,0))d\tau\right\| + \left\|\int_0^s \gamma(s,\tau,0)d\tau\right\|\\ &\le p(s)\|u\| + \left\|\int_0^s \gamma(s,\tau,0)d\tau\right\| \le p(s)\|u\|+M_3,\\ \|(\psi u)(s)\| &\le \left\|\int_0^T (\delta(s,\tau,u(\tau))-\delta(s,\tau,0))d\tau\right\| + \left\|\int_0^T \delta(s,\tau,0)d\tau\right\|\\ &\le q(s)\|u\| + \left\|\int_0^T \delta(s,\tau,0)d\tau\right\| \le q(s)\|u\|+M_4.\end{aligned}$$

From the two above inequalities, we get

$$\begin{aligned}\|f(s,u(s),(\phi u)(s),(\psi u)(s))\| &\le \|f(s,u(s),(\phi u)(s),(\psi u)(s)) - f(s,0,0,0)\| + \|f(s,0,0,0)\|\\ &\le L_1(s)\|u(s)\| + L_2(s)\|(\phi u)(s)\| + L_3(s)\|(\psi u)(s))\| + \|f(s,0,0,0)\|\\ &\le (L_1(s)+L_2(s)p(s)+L_3(s)q(s))\|u\| + (L_2(s)M_3+L_3(s)M_4)+M_5\\ &\le M(s)\|u\| + (\bar{L}_2M_3+\bar{L}_3M_4)+M_5 = M(s)\|u\|+M^*,\end{aligned}$$

where $M^* = (\bar{L}_2M_3+\bar{L}_3M_4)+M_5$. And consider $B_r = \{u\in C: \|u\|\le r\}$, where $r \ge \rho_1/(1-\rho)$, with

$$\rho_1 = (M_0|\nu_0\xi_0| + M_1\lambda_1|\nu_1| + M_2\lambda_2|\nu_2|)T + \lambda_3 M^*,$$

and ρ is given by the assumption (H_4). Now we show that $FB_r \subset B_r$. For $u \in B_r$, we have

$$\begin{aligned}
\|(Fu)(t)\| \leq\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,u(s),(\phi u)(s),(\psi u)(s))\|ds \\
&+|\mu_0\xi_0|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\|f(s,u(s),(\phi u)(s),(\psi u)(s))\|ds \\
&+|\mu_1\eta_1(t)|T^{\alpha_1-1}\int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)}\|f(s,u(s),(\phi u)(s),(\psi u)(s))\|ds \\
&+|\mu_2\eta_2(t)|T^{\alpha_2-2}\int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)}\|f(s,u(s),(\phi u)(s),(\psi u)(s))\|ds \\
&+|\nu_0\xi_0|\int_0^T \|g_0(s,u(s))\|ds+|\nu_1\eta_1(t)|\int_0^T \|g_1(s,u(s))\|ds+|\nu_2\eta_2(t)|\int_0^T \|g_2(s,u(s))\|ds \\
\leq\ & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(M(s)\|u\|+M^*)ds+|\mu_0\xi_0|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}(M(s)\|u\|+M^*)ds \\
&+|\mu_1\eta_1(t)|T^{\alpha_1-1}\int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)}(M(s)\|u\|+M^*)ds \\
&+|\mu_2\eta_2(t)|T^{\alpha_2-2}\int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)}(M(s)\|u\|+M^*)ds \\
&+|\nu_0\xi_0|\int_0^T [\|g_0(s,u(s))-g_0(s,0)\|+\|g_0(s,0)\|]ds \\
&+|\nu_1\eta_1(t)|\int_0^T [\|g_1(s,u(s))-g_1(s,0)\|+\|g_1(s,0)\|]ds \\
&+|\nu_2\eta_2(t)|\int_0^T [\|g_2(s,u(s))-g_2(s,0)\|+\|g_2(s,0)\|]ds \\
\leq\ & (\mathscr{L}_3 r+M^*)\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds+|\mu_0\xi_0|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right. \\
&\left.+|\mu_1\eta_1(t)|T^{\alpha_1-1}\int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)}ds+|\mu_2\eta_2(t)|T^{\alpha_2-2}\int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)}ds\right) \\
&+|\nu_0\xi_0|(\mathscr{L}_0 r+M_0)T+|\nu_1\eta_1(t)|(\mathscr{L}_1 r+M_1)T+|\nu_2\eta_2(t)|(\mathscr{L}_2 r+M_2)T \\
\leq\ & (\mathscr{L}_3 r+M^*)\left(\frac{(1+|\mu_0\xi_0|)T^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda_1|\mu_1|T^{\alpha-1}}{\Gamma(\alpha-\alpha_1+1)}+\frac{\lambda_2|\mu_2|T^{\alpha-2}}{\Gamma(\alpha-\alpha_2+1)}\right) \\
&+|\nu_0\xi_0|(\mathscr{L}_0 r+M_0)T+\lambda_1|\nu_1|(\mathscr{L}_1 r+M_1)T+\lambda_2|\nu_2|(\mathscr{L}_2 r+M_2)T \\
=\ & [(\mathscr{L}_0|\nu_0\xi_0|+\mathscr{L}_1\lambda_1|\nu_1|+\mathscr{L}_2\lambda_2|\nu_2|)T+\mathscr{L}_3\lambda_3]r \\
&+(M_0|\nu_0\xi_0|+M_1\lambda_1|\nu_1|+M_2\lambda_2|\nu_2|)T+\lambda_3 M^* \leq \rho r+\rho_1 \leq r.
\end{aligned}$$

Now, for u, v and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
& \|(Fu)(t) - (Fv)(t)\| \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u(s), (\phi u)(s), (\psi u)(s)) - f(s, v(s), (\phi v)(s), (\psi v)(s))\| ds \\
& + |\mu_0 \xi_0| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \|f(s, u(s), (\phi u)(s), (\psi u)(s)) - f(s, v(s), (\phi v)(s), (\psi v)(s))\| ds \\
& + |\mu_1 \eta_1(t)| T^{\alpha_1 - 1} \int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} \|f(s, u(s), (\phi u)(s), (\psi u)(s)) - f(s, v(s), (\phi v)(s), (\psi v)(s))\| ds \\
& + |\mu_2 \eta_2(t)| T^{\alpha_2 - 2} \int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)} \|f(s, u(s), (\phi u)(s), (\psi u)(s)) - f(s, v(s), (\phi v)(s), (\psi v)(s))\| ds \\
& + |\nu_0 \xi_0| \int_0^T \|g_0(s, u(s)) - g_0(s, v(s))\| ds + |\nu_1 \eta_1(t)| \int_0^T \|g_1(s, u(s)) - g_1(s, v(s))\| ds \\
& + |\nu_2 \eta_2(t)| \int_0^T \|g_2(s, u(s)) - g_2(s, v(s))\| ds \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} M(s) \|u - v\| ds + |\mu_0 \xi_0| \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} M(s) \|u - v\| ds \\
& + |\mu_1 \eta_1(t)| T^{\alpha_1 - 1} \int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} M(s) \|u - v\| ds \\
& + |\mu_2 \eta_2(t)| T^{\alpha_2 - 2} \int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)} M(s) \|u - v\| ds \\
& + |\nu_0 \xi_0| \int_0^T \mathscr{L}_0 \|u - v\| ds + |\nu_1 \eta_1(t)| \int_0^T \mathscr{L}_1 \|u - v\| ds + |\nu_2 \eta_2(t) \int_0^T \mathscr{L}_2 |\|u - v\| ds \\
\leq & \mathscr{L}_3 \left(\frac{(1 + |\mu_0 \xi_0|) T^\alpha}{\Gamma(\alpha + 1)} + \frac{\lambda_1 |\mu_1| T^{\alpha-1}}{\Gamma(\alpha - \alpha_1 + 1)} + \frac{\lambda_2 |\mu_2| T^{\alpha-2}}{\Gamma(\alpha - \alpha_2 + 1)} \right) \|u - v\| \\
& + (M_0 |\nu_0 \xi_0| + M_1 \lambda_1 |\nu_1| + M_2 \lambda_2 |\nu_2|) T \|u - v\| \\
= & [(M_0 |\nu_0 \xi_0| + M_1 \lambda_1 |\nu_1| + M_2 \lambda_2 |\nu_2|) T + \mathscr{L}_3 \lambda_3] \|u - v\| = \rho \|u - v\|.
\end{aligned}$$

Observe that ρ depends only on the parameters involved in the problem. As $\rho < 1$ (H_4), therefore F is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), it follows that problem (1.1) has a unique solution on $[0, T]$. □

Our next existence results is based on Krasnoselskii fixed point theorem [31].

Theorem 3.2. (Krasnoselskii). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be two operators such that* (i) *$Ax + By \in M$ whenever $x, y \in M$;* (ii) *A is compact and continuous;* (iii) *B is a contraction mapping. Then there exisz $\in M$ such that $z = Az + Bz$.*

Theorem 3.3. *Assume that (H_1)-(H_3) and (H_5) hold. Then the boundary value problem (1.1) has at least one solution on $\mathscr{J}$ provided*

$$\mathscr{L}_3 \left(\frac{|\mu_0 \xi_0| T^\alpha}{\Gamma(\alpha + 1)} + \frac{\lambda_1 |\mu_1| T^{\alpha-1}}{\Gamma(\alpha - \alpha_1 + 1)} + \frac{\lambda_2 |\mu_2| T^{\alpha-2}}{\Gamma(\alpha - \alpha_2 + 1)} \right) + (\mathscr{L}_0 |\nu_0 \xi_0| + \mathscr{L}_1 \lambda_1 |\nu_1| + \mathscr{L}_2 \lambda_2 |\nu_2|) T < 1.$$

Proof. Let us fix

$$r \geq (M_0 |\nu_0 \xi_0| + M_1 \lambda_1 |\nu_1| + M_2 \lambda_2 |\nu_2|) T + \lambda_3 \|\omega\|_{L^1}.$$

and consider $B_r = \{u \in \mathscr{S} : \|u\| \leq r\}$. We define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi u)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds, \\
(\Psi u)(t) &= \mu_0\xi_0 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \\
&+\mu_1\eta_1(t)T^{\alpha_1-1} \int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \\
&+\mu_2\eta_2(t)T^{\alpha_2-2} \int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \\
&+\nu_0\xi_0 \int_0^T g_0(s, u(s))ds + \nu_1\eta_1(t) \int_0^T g_1(s, u(s))ds + \nu_2\eta_2(t) \int_0^T g_2(s, u(s))ds.
\end{aligned}$$

Let us observe that if $u, v \in B_r$, we find that

$$\begin{aligned}
\|(\Phi u)(t) + (\Psi v)(t)\| &= \Bigg\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \\
&+\mu_0\xi_0 \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s), (\phi v)(s), (\psi v)(s))ds \\
&+\mu_1\eta_1(t)T^{\alpha_1-1} \int_0^T \frac{(T-s)^{\alpha-\alpha_1-1}}{\Gamma(\alpha-\alpha_1)} f(s, v(s), (\phi v)(s), (\psi v)(s))ds \\
&+\mu_2\eta_2(t)T^{\alpha_2-2} \int_0^T \frac{(T-s)^{\alpha-\alpha_2-1}}{\Gamma(\alpha-\alpha_2)} f(s, v(s), (\phi v)(s), (\psi v)(s))ds \\
&+\nu_0\xi_0 \int_0^T g_0(s, v(s))ds + \nu_1\eta_1(t) \int_0^T g_1(s, v(s))ds + \nu_2\eta_2(t) \int_0^T g_2(s, v(s))ds \Bigg\| \\
&\leq \|\omega\|_{L^1} \left(\frac{(1+|\mu_0\xi_0|)T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda_1|\mu_1|T^{\alpha-1}}{\Gamma(\alpha-\alpha_1+1)} + \frac{\lambda_2|\mu_2|T^{\alpha-2}}{\Gamma(\alpha-\alpha_2+1)} \right) \\
&+(M_0|\nu_0\xi_0| + M_1\lambda_1|\nu_1| + M_2\lambda_2|\nu_2|)T \\
&= (M_0|\nu_0\xi_0| + M_1\lambda_1|\nu_1| + M_2\lambda_2|\nu_2|)T + \lambda_3\|\omega\|_{L^1} \leq r.
\end{aligned}$$

Thus, $\Phi u + \Psi v \in B_r$. It follows from the assumption (H_1)-(H_3) that Ψ is a contraction mapping for

$$\mathscr{L}_3 \left(\frac{|\mu_0\xi_0|T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda_1|\mu_1|T^{\alpha-1}}{\Gamma(\alpha-\alpha_1+1)} + \frac{\lambda_2|\mu_2|T^{\alpha-2}}{\Gamma(\alpha-\alpha_2+1)} \right) + (\mathscr{L}_0|\nu_0\xi_0| + \mathscr{L}_1\lambda_1|\nu_1| + \mathscr{L}_2\lambda_2|\nu_2|)T < 1.$$

Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as $\|\Phi u\| \leq \|\omega\|_{L^1}/\Gamma(\alpha+1)$. Now we prove the compactness of the operator Φ. In view of (H_2), we define

$$\sup_{(t,u,\phi u,\psi u)\in\Omega} \|f(t, u, \phi u, \psi u)\| = f_{max}, \quad \Omega = \mathscr{J} \times B_r \times B_r \times B_r,$$

and consequently, for $t_1, t_2 \in \mathscr{J}$ with $t_1 < t_2$, we have

$$\begin{aligned}
\|(\Phi u)(t_2) - (\Phi u)(t_1)\| &\leq \left\| \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \right\| \\
&+ \left\| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), (\phi u)(s), (\psi u)(s))ds \right\| \\
&\leq \frac{f_{max}}{\Gamma(\alpha+1)} |2(t_2-t_1)^\alpha + t_1^\alpha - t_2^\alpha|,
\end{aligned}$$

which is independent of x. So Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 3.2 are satisfied and the conclusion of Theorem 3.2 implies that the boundary value problem (1.1) has at least one solution on $\mathscr{J}$. □

4 An example

Consider the following nonlinear fractional integrodifferential equations boundary value problem

$$\begin{aligned} {}^cD^{5/2}u(t) &= \frac{1}{10} + \frac{1}{(5+t)^2}\frac{|u(t)|}{1+|u(t)|} + \int_0^t \frac{|u(s)|}{25e^t+s}ds + \int_0^1 \frac{|u(s)|e^{-t}}{25+|u(s)|^2+s}ds, \quad t \in \mathscr{J} = [0,1],\\ u(0) &= \frac{1}{2}u(1) + \int_0^1 \frac{|u(s)|}{10+|u(s)|+s}ds, \quad \lim_{t\to 0^+}[t^{-1/2c}D^{1/2}u(t)] = \frac{1}{3}D^{1/2}u(1) + \int_0^1 \frac{1}{4+|u(s)|+s}ds, \\ &\lim_{t\to 0^+}[t^{-1/2c}D^{3/2}u(t)] = \frac{1}{4}D^{3/2}u(1) + \int_0^1 \frac{|u(s)|}{17+|u(s)|+s}ds. \end{aligned} \tag{4.1}$$

Here $T=1$, $\alpha = 5/2$, $\alpha_1 = 1/2$, $\alpha_2 = 3/2$, $\mu_0 = 1/2$, $\mu_1 = 1/3$, $\mu_2 = 1/4$, $\nu_0 = \nu_1 = \nu_2 = 1$, and

$$f(t,x,y,z) = \frac{1}{10} + \frac{1}{(5+t)^2}\frac{x}{10+x} + y + z, \quad \gamma(t,s,u) = \frac{|u|}{25e^t+s}, \quad \delta(t,s,u) = \frac{|u|e^{-t}}{25+|u|^2+s},$$

$$g_0(s,u) = \frac{|u|}{10+|u|+s}, \quad g_1(s,u) = \frac{1}{4+|u|+s}, \quad g_2(s,u) = \frac{|u|}{17+|u|+s}.$$

As

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le \frac{1}{(5+t)^2}\|x_1-x_2\| + \|y_1-y_2\| + \|z_1-z_2\|,$$

$$\|g_0(s,u)-g_0(s,v)\| \le \frac{1}{10}\|u-v\|, \quad \|g_0(s,u)\| \le 1, \quad \|g_1(s,u)-g_1(s,v)\| \le \frac{1}{16}\|u-v\|, \quad \|g_0(s,u)\| \le \frac{1}{4},$$

$$\|g_2(s,u)-g_2(s,v)\| \le \frac{1}{17}\|u-v\|, \quad \|g_0(s,u)\| \le 1, \quad \left\|\int_0^t \left(\frac{|u(s)|}{25e^t+s} - \frac{|v(s)|}{25e^t+s}\right)ds\right\| \le \frac{1}{25e^t}\|u-v\|,$$

$$\left\|\int_0^1 \left(\frac{|u(s)|e^{-t}}{25+|u(s)|^2+s} - \frac{|v(s)|e^{-t}}{25+|v(s)|^2+s}\right)ds\right\| \le \frac{1}{25e^t}\|u-v\|,$$

therefore, (H_1)-(H_3) are satisfied with $\mathscr{L}_0 = 1/10$, $\mathscr{L}_1 = 1/16$, $\mathscr{L}_2 = 1/17$, $L_1(t) = 1/(5+t)^2$, $L_2(t) = L_3(t) = 1$, $p(t) = q(t) = 1/(25e^t)$. Further, $\xi_0 = 2$, $\xi_1 = 3\sqrt{\pi}/2$, $\xi_2 = 4\sqrt{\pi}/3$, $\lambda_1 = \sup\{|\eta_1(t)| = (3\sqrt{\pi}/4)|1+t| : t \in \mathscr{J}\} = 3\sqrt{\pi}/2$, $\lambda_2 = \sup\{|\eta_2(t)| = (2\sqrt{\pi}/9)|3t^2+2t+5| : t \in \mathscr{J}\} = 20\sqrt{\pi}/9$, $\lambda_3 = 16/(15\sqrt{\pi}) + 29\sqrt{\pi}/36$, $\mathscr{L}_3 = \sup\{M(t) = L_1(t) + L_2(t)p(t) + L_3(t)q(t) = 1/(5+t)^2 + 2/(25e^t) : t \in \mathscr{J}\} = 3/25$, and

$$\rho = (\mathscr{L}_0|\nu_0\xi_0| + \mathscr{L}_1\lambda_1|\nu_1| + \mathscr{L}_2\lambda_2|\nu_2|)T + \mathscr{L}_3\lambda_3 = \frac{1}{5} + \frac{16}{125\sqrt{\pi}} + \frac{39307\sqrt{\pi}}{122400} \approx 0.841414 < 1.$$

Thus, by Theorem 3.1, the boundary value problem (4.1) has a unique solution on $[0,1]$.

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IDENTITIES OF SYMMETRY FOR HIGHER-ORDER q-BERNOULLI POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

ABSTRACT. Recently, the higher-order Carlitz's q-Bernoulli polynomials are represented as q-Volkenborn integral on $\mathbb{Z}_p$ by Kim. A question was asked in [13] as to finding the extended formulae of symmetries for Bernoulli polynomials which are related to Carlitz q-Bernoulli polynomials. In this paper, we give some new identities of symmetry for the higher-order Carlitz's q-Bernoulli polynomials which are derived from multivariate q-Volkenborn integrals on $\mathbb{Z}_p$. We note that they are a partial answer to that question.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. Let q be an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient,

$$F_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Z}_p \quad \text{by } F_f(x,y) = \frac{f(x) - f(y)}{x-y},$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. If f is uniformly differentiable on $\mathbb{Z}_p$, we denote this property by $f \in UD(\mathbb{Z}_p)$.

For $f \in UD(\mathbb{Z}_p)$, the q-Volkenborn integral is defined by Kim to be

$$(1) \qquad I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x,$$

where $[x]_q = \frac{1-q^x}{1-q}$, (see [12, 13, 14]).

From (1), we note that

$$(2) \qquad qI_q(f_1) = I_q(f) + (q-1) f(0) + \frac{q-1}{\log q} f'(0)$$

where $f_1(x) = f(x+1)$.

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$(3) \qquad \frac{t}{e^t-1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

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When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

By (3), we get

$$(B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases} \text{ and } B_0 = 1. \tag{4}$$

In [3], Carlitz defined q-Bernoulli numbers as follows :

$$\beta_{0,q} = 1, \qquad q\left(q\beta_q + 1\right)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{5}$$

with the usual convention about replacing β_q^i by $\beta_{i,q}$.

From (4) and (5), we note that $\lim_{q\to 1} \beta_{n,q} = B_n$.

The q-Bernoulli polynomials are given by

$$\begin{aligned} \beta_{n,q}(x) &= \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l} \\ &= \left(q^x \beta_q + [x]_q\right)^n, \quad (n \geq 0), \quad (\text{see } [2, 3, 4, 14]). \end{aligned} \tag{6}$$

In [12], Kim proved that Carlitz q-Bernoulli polynomials can be written by q-Volkenborn integral on $\mathbb{Z}_p$ as follows :

$$\begin{aligned} \beta_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l \, d\mu_q(x). \end{aligned} \tag{7}$$

Thus, by (7), we get

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x), \quad (n \geq 0). \tag{8}$$

From (2), we note that

$$q \int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \begin{cases} q-1 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases} \tag{9}$$

By (7), (8) and (9), we see that

$$\beta_{0,q} = 1, \quad q\left(q\beta_q + 1\right)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases} \tag{10}$$

Let

$$\begin{aligned} I_1(f) = \lim_{q\to 1} I_q(f) &= \int_{\mathbb{Z}_p} f(x) \, d\mu_1(x) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see } [12, 23, 24]). \end{aligned} \tag{11}$$

Then, by (2), we get

$$I_1(f_1) - I_1(f) = f'(0). \tag{12}$$

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Let us take $f(x) = e^{tx}$. Then we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_1(x) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{13}$$

and

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{14}$$

For $r \in \mathbb{N}$, the higher-order Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \underbrace{\left(\frac{t}{e^t - 1}\right) \times \cdots \times \left(\frac{t}{e^t - 1}\right)}_{r-\text{times}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}. \tag{15}$$

By (14), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x+y_1+\cdots+y_r)} d\mu_1(y_1) \cdots d\mu_1(y_r) &= \left(\frac{t}{e^t - 1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{16}$$

In [3, 4], Carlitz introduced the q-extension of higher-order Bernoulli polynomials as follows :

$$\beta_{n,q}^{(r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{l+1}{[l+1]_q}\right)^r, \tag{17}$$

where $n \geq 0$ and $r \in \mathbb{N}$.

Note that $\lim_{q \to 1} \beta_{n,q}^{(r)}(x) = B_n^{(r)}(x)$.

From (16), we note that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + y_1 + \cdots + y_r)^n d\mu_1(y_1) \cdots d\mu_1(y_r) = B_n^{(r)}(x), \tag{18}$$

where $n \geq 0$ and $r \in \mathbb{N}$.

In this paper, we consider q-extensions of (17) which are related to higher-order Carlitz's q-Bernoulli polynomials. The purpose of this paper is to give some new and interesting identities of symmetry for the higher-order Carlitz's q-Bernoulli polynomials which are derived from multivariate q-Volkenborn integral on $\mathbb{Z}_p$.

2. Identities of symmetry for higher-order q-Bernoulli polynomials

In the sense of q-extension of (18), we observe the following equation (19)

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + y_1 + \cdots + y_r]_q^n d\mu_q(y_1) \cdots d\mu_q(y_n) \\ =&\frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{l+1}{[l+1]_q}\right)^r. \end{aligned} \tag{19}$$

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Thus, by (17) and (19), we get

$$\beta_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + y_1 + \cdots + y_r]_q^n \, d\mu_q(y_1) \cdots d\mu_q(y_r), \tag{20}$$

where $n \geq 0$ and $r \in \mathbb{N}$.

Let us consider the generating function of $\beta_{n,q}^{(r)}(x)$ as follows :

$$\sum_{n=0}^{\infty} \beta_{n,q}^{(r)}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[x+y_1+\cdots+y_r]_q t} d\mu_q(y_1) \cdots d\mu_q(y_r). \tag{21}$$

For w_1, $w_2 \in \mathbb{N}$, we have

(22)

$$\begin{aligned}
&\frac{1}{[w_1]_q^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l\right]_q t} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
&= \lim_{N\to\infty} \left(\frac{1}{[w_1]_q [p^N]_{q^{w_1}}} \right)^r \sum_{y_1, \cdots, y_r=0}^{p^N-1} e^{\left[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l\right]_q t} q^{w_1(y_1+\cdots+y_r)} \\
&= \lim_{N\to\infty} \left(\frac{1}{[w_1]_q [w_2 p^N]_{q^{w_1}}} \right)^r \\
&\quad \times \sum_{y_1, \cdots, y_r=0}^{w_2 p^N-1} e^{\left[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l\right]_q t} q^{w_1(y_1+\cdots+y_r)} \\
&= \lim_{N\to\infty} \left(\frac{1}{[w_1 w_2 p^N]_q} \right)^r \\
&\quad \times \sum_{i_1, \cdots, i_r=0}^{w_2-1} \sum_{y_1, \cdots, y_r=0}^{p^N-1} e^{\left[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 i_l + w_1 w_2 y_l)\right]_q t} q^{w_1 \sum_{l=1}^r (i_l + w_2 y_l)}.
\end{aligned}$$

Thus, by (22), we get

$$\begin{aligned}
&\frac{1}{[w_1]_q^r} \sum_{j_1, \cdots, j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
&\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 y_l)\right]_q t} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
&= \lim_{N\to\infty} \left(\frac{1}{[w_1 w_2 p^N]_q} \right)^r \sum_{j_1, \cdots, j_r=0}^{w_1-1} \sum_{i_1, \cdots, i_r=0}^{w_2-1} \\
&\quad \times \sum_{y_1, \cdots, y_r=0}^{p^N-1} e^{\left[w_1 w_2 x + \sum_{l=1}^r (w_2 j_l + w_1 i_l + w_1 w_2 y_l)\right]_q t} q^{\sum_{l=1}^r (w_1 i_l + w_2 j_l + w_1 w_2 y_l)}.
\end{aligned} \tag{23}$$

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By the same method as (23), we get

$$\begin{aligned}(24)\qquad &\frac{1}{[w_2]_q^r}\sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_1w_2x+\sum_{l=1}^r(w_1j_l+w_2y_l)\right]_q t} d\mu_{q^{w_2}}(y_1)\cdots d\mu_{q^{w_2}}(y_l)\\ =&\lim_{N\to\infty}\left(\frac{1}{[w_1w_2p^N]_q}\right)^r \sum_{j_1,\cdots,j_r=0}^{w_2-1}\sum_{i_1,\cdots,i_r=0}^{w_1-1}\\ &\times\sum_{y_1,\cdots,y_r=0}^{p^N-1} e^{\left[w_1w_2x+\sum_{l=1}^r(w_1j_l+w_2i_l+w_1w_2y_l)\right]_q t} q^{\sum_{l=1}^r(w_2i_l+w_1j_l+w_1w_2y_l)}.\end{aligned}$$

Therefore, by (23), we obtain the following theorem.

Theorem 1. *For $w_1,\, w_2\in\mathbb{N}$, we have*

$$\begin{aligned}&\frac{1}{[w_1]_q^r}\sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2\sum_{l=1}^r j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_1w_2x+\sum_{l=1}^r(w_2j_l+w_1y_l)\right]_q t} d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\ =&\frac{1}{[w_2]_q^r}\sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{\left[w_1w_2x+\sum_{l=1}^r(w_1j_l+w_2y_l)\right]_q t} d\mu_{q^{w_2}}(y_1)\cdots d\mu_{q^{w_2}}(y_r).\end{aligned}$$

It is easy to show that

$$\begin{aligned}(25)\qquad &[w_1w_2x+w_2(j_1+\cdots+j_r)+w_1(y_1+\cdots+y_r)]_q\\ =&[w_1]_q\left[w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)+(y_1+\cdots+y_r)\right]_{q^{w_1}}.\end{aligned}$$

Therefore, by (20), Theorem 1 and (25), we obtain the following corollary, and theorem.

Corollary 2. *For $n\geq 0$ and $w_1,\, w_2\in\mathbb{N}$, we have*

$$\begin{aligned}&[w_1]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2\sum_{l=1}^r j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_r)+y_1+\cdots+y_r\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\ =&[w_2]_q^{n-r}\sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1\sum_{l=1}^r j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_r)+y_1+\cdots+y_r\right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1)\cdots d\mu_{q^{w_2}}(y_r).\end{aligned}$$

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Theorem 3. *For $n \geq 0$ and $w_1, w_2 \in \mathbb{N}$, we have*

$$\begin{aligned}&[w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2(j_1+\cdots+j_r)} \beta_{n,q^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_r) \right) \\ &= [w_2]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1(j_1+\cdots+j_r)} \beta_{n,q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2}(j_1+\cdots+j_r) \right).\end{aligned}$$

Remark. Let $w_2 = 1$. Then we have

$$\beta_{n,q}^{(r)}(w_1 x) = [w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{j_1+\cdots+j_r} \beta_{n,q^{w_1}}^{(r)} \left(x + \frac{j_1+\cdots+j_r}{w_1} \right).$$

By (20), we see that

(26)

$$\begin{aligned}&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1}(j_1+\cdots+j_r) + y_1 + \cdots + y_r \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^r j_l} \\ &\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \sum_{l=1}^r y_l \right]_{q^{w_1}}^{n-i} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\ &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1+\cdots+j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^r j_l} \beta_{n-i,q^{w_1}}^{(r)}(w_2 x).\end{aligned}$$

From Corollary 2 and (26), we have

$$\begin{aligned}
(27)\qquad & [w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
&= \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^r j_l} \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \\
&\times [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^r j_l} \beta_{n-i,q^{w_1}}^{(r)}(w_2 x) \\
&= \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \beta_{n-i,q^{w_1}}^{(r)}(w_2 x) \\
&\times \sum_{j_1,\cdots,j_r=0}^{w_1-1} [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i+1)\sum_{l=1}^r j_l} \\
&= \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i,q^{w_1}}^{(r)}(w_2 x) \\
&\times \sum_{j_1,\cdots,j_r=0}^{w_1-1} [j_1 + \cdots + j_r]_{q^{w_2}}^{n-i} q^{w_2(i+1)\sum_{l=1}^r j_l} \\
&= \sum_{i=0}^n \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i,q^{w_1}}^{(r)}(w_2 x) T_{n,i}^{(r)}(w_1 | q^{w_2}),
\end{aligned}$$

where

$$(28)\qquad T_{n,i}^{(r)}(w|q) = \sum_{j_1,\cdots,j_r=0}^{w-1} [j_1 + \cdots + j_r]_q^{n-i} q^{(i+1)\sum_{l=1}^r j_l}.$$

By the same method as (28), we get

$$\begin{aligned}
(29)\qquad & [w_2]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^r j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_l) \\
&= \sum_{i=0}^n \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i,q^{w_2}}^{(r)}(w_1 x) T_{n,i}^{(r)}(w_2 | q^{w_1}).
\end{aligned}$$

Therefore, by Corollary 2, (27) and (29), we obtain the following theorem.

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Theorem 4. *For $n \geq 0$ and r, w_1, $w_2 \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{n-i}[w_2]_q^{i-r}\beta_{i,q^{w_2}}^{(r)}(w_1x)T_{n,i}^{(r)}(w_2|q^{w_1})\\ =&\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^{n-i}[w_1]_q^{i-r}\beta_{i,q^{w_1}}^{(r)}(w_2x)T_{n,i}^{(r)}(w_1|q^{w_2}),\end{aligned}$$

where

$$T_{n,i}^{(r)}(w|q)=\sum_{j_1,\cdots,j_r=0}^{w-1}[j_1+\cdots+j_r]_q^{n-i}q^{(i+1)\sum_{l=1}^{r}j_l}.$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+y_1+\cdots+y_r]_q^n q^{\sum_{l=1}^{r}(h-l)y_l}d\mu_q(y_1)\cdots d\mu_q(y_r)\\ =&\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{q^{xj}}{(1-q)^n}\lim_{N\to\infty}\frac{1}{[p^N]_q^r}\sum_{y_1,\cdots,y_r=0}^{p^N-1}q^{j\sum_{l=1}^{r}y_l}q^{\sum_{l=1}^{r}(h-l+1)y_l}\\ =&\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{q^{xj}}{(1-q)^n}\frac{(j+h)(j+h-1)\cdots(j+h-r+1)}{[j+h]_q[j+h-1]_q\cdots[j+h-r+1]_q}\\ =&\frac{1}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{xj}\frac{\binom{j+h}{r}}{\binom{j+h}{r}_q}\frac{r!}{[r]_q!},\end{aligned}$$

where $\binom{x}{r}_q=\frac{[x]_q[x-1]_q\cdots[x-r+1]_q}{[r]_q!}=\frac{[x]_q[x-1]_q\cdots[x-r+1]_q}{[r]_q[r-1]_q\cdots[2]_q[1]_q}$.

From (18), we can also define q-extensions of higher-order Bernoulli polynomials as follows :

$$\beta_{n,q}^{(h,r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+y_1+\cdots+y_r]_q^n q^{\sum_{l=1}^{r}(h-l)y_l}d\mu_q(y_1)\cdots d\mu_q(y_r),\tag{30}$$

where $n \geq 0$ and $h \in \mathbb{Z}$, $r \in \mathbb{N}$.

Let w_1, $w_2 \in \mathbb{N}$. Then we see that

(31)

$$\begin{aligned}&\frac{1}{[w_1]_q^r}\sum_{j_1,\cdots,j_r=0}^{w_1-1}q^{w_2\sum_{l=1}^{r}(h-l+1)j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{w_1\sum_{l=1}^{r}(h-l)y_l}e^{\left[w_1w_2x+\sum_{l=1}^{r}(w_2j_l+w_1y_l)\right]_q t}d\mu_{q^{w_1}}(y_1)\cdots d\mu_{q^{w_1}}(y_r)\\ =&\frac{1}{[w_2]_q^r}\sum_{j_1,\cdots,j_r=0}^{w_2-1}q^{w_1\sum_{l=1}^{r}(h-l+1)j_l}\\ &\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{w_2\sum_{l=1}^{r}(h-l)y_l}e^{\left[w_1w_2x+\sum_{l=1}^{r}(w_1j_l+w_2y_l)\right]_q t}d\mu_{q^{w_2}}(y_1)\cdots d\mu_{q^{w_2}}(y_r).\end{aligned}$$

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From (31), we have

(32)

$$\begin{aligned}
&[w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^{r}(h-l+1)j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^{r}(h-l)y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
=&[w_2]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^{r}(h-l+1)j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_2 \sum_{l=1}^{r}(h-l)y_l} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r),
\end{aligned}$$

where $n \geq 0$ and $r \in \mathbb{N}$, $h \in \mathbb{Z}$.

Therefore, by (30) and (32), we obtain the following theorem.

Theorem 5. *For $n \geq 0$, $h \in \mathbb{Z}$ and $w_1, w_2 \in \mathbb{N}$, we have*

$$\begin{aligned}
&[w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^{r}(h-l+1)j_l} \beta_{n,q^{w_1}}^{(h,r)} \left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right) \\
=&[w_2]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^{r}(h-l+1)j_l} \beta_{n,q^{w_2}}^{(h,r)} \left(w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_r) \right).
\end{aligned}$$

From (30), we can derive the following equation :

(33)

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^{r}(h-l)y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
=&\sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^{r} j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^{r}(h-r)y_l} \left[w_2 x + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^{n-i} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
=&\sum_{i=0}^{n} \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^{r} j_l} \beta_{n-i,q^{w_1}}^{(h,r)}(w_2 x).
\end{aligned}$$

By (33), we get

(34)

$$\begin{aligned}
&[w_1]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^{r}(h-l+1)j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_1 \sum_{l=1}^{r}(h-l)y_l} \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^{n} d\mu_{q^{w_1}}(y_1) \cdots d\mu_{q^{w_1}}(y_r) \\
=& \sum_{j_1,\cdots,j_r=0}^{w_1-1} q^{w_2 \sum_{l=1}^{r}(h-l+1)j_l} \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} [j_1+\cdots+j_r]_{q^{w_2}}^i \\
&\times q^{w_2(n-i)\sum_{l=1}^{r} j_l} \beta_{n-i,q^{w_1}}^{(h,r)}(w_2 x) \\
=& \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i-r} \beta_{n-i,q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1,\cdots,j_r=0}^{w_1-1} [j_1+\cdots+j_r]_{q^{w_2}}^i \\
&\times q^{w_2 \sum_{l=1}^{r}(n+h-l-i+1)j_l} \\
=& \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i,q^{w_1}}^{(h,r)}(w_2 x) T_{n,i}^{(h,r)}(w_1|q^{w_2}),
\end{aligned}$$

where

$$T_{n,i}^{(h,r)}(w|q) = \sum_{j_1,\cdots,j_r=0}^{w-1} [j_1+\cdots+j_r]_q^{n-i} q^{\sum_{l=1}^{r}(i+h-l+1)j_l}. \tag{35}$$

By the same method as (34), we see that

(36)

$$\begin{aligned}
&[w_2]_q^{n-r} \sum_{j_1,\cdots,j_r=0}^{w_2-1} q^{w_1 \sum_{l=1}^{r}(h-l+1)j_l} \\
&\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{w_2 \sum_{l=1}^{r}(h-l)y_l} \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q^{w_1}}^{n} d\mu_{q^{w_2}}(y_1) \cdots d\mu_{q^{w_2}}(y_r) \\
=& \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i,q^{w_2}}^{(h,r)}(w_1 x) T_{n,i}^{(h,r)}(w_2|q^{w_1}).
\end{aligned}$$

Therefore, by (34) and (36), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, $h \in \mathbb{Z}$ and r, w_1, $w_2 \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} [w_2]_q^{n-i} [w_1]_q^{i-r} \beta_{i,q^{w_1}}^{(h,r)}(w_2 x) T_{n,i}^{(h,r)}(w_1|q^{w_2}) \\
=& \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^{n-i} [w_2]_q^{i-r} \beta_{i,q^{w_2}}^{(h,r)}(w_1 x) T_{n,i}^{(h,r)}(w_2|q^{w_1}),
\end{aligned}$$

where

$$T_{n,i}^{(h,r)}(w|q) = \sum_{j_1,\cdots,j_r=0}^{w-1} [j_1+\cdots+j_r]_q^{n-i} q^{\sum_{l=1}^{r}(h+i-l+1)j_l}.$$

Remark. A p-adic approach to identities of symmetry for Carlitz's q-Bernoulli polynomials has been studied in [10].

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FUZZY STABILITY OF FUNCTIONAL EQUATIONS IN MATRIX FUZZY NORMED SPACES

CHOONKIL PARK, DONG YUN SHIN*, AND JUNG RYE LEE

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of the Cauchy additive functional equation and the quadratic functional equation in matrix fuzzy normed spaces.

1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [65] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [19]).

The proof given in [65] appealed to the theory of ordered operator spaces [12]. Effros and Ruan [20] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [55] and Haagerup [27] (as modified in [18]).

The stability problem of functional equations originated from a question of Ulam [71] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [28] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [59] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [26] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [60] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [25] following the same approach as in Th.M. Rassias [59], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [25], as well as by Th.M. Rassias and Šemrl [64] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [16], Hyers, Isac and Th.M. Rassias [29]).

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

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is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [70] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [13] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [14] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 21, 30, 32, 33, 38, 39, 40, 41, 42, 43, 49, 53, 58, 61, 62, 63, 68, 69]).

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [3, 24, 35, 37, 46, 72]. Following Cheng and Mordeson [8], Bag and Samanta [3] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [36] and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 46, 47] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

Definition 1.1. [3, 46, 47, 48] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \le 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \ne 0$;

(N_4) $N(x + y, s + t) \ge \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(N_6) for $x \ne 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed space*.

Definition 1.2. [3, 46, 47, 48] (1) Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

(2) Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

We introduce the concept of matrix fuzzy normed space.

Definition 1.3. Let (X, N) be a fuzzy normed space. (1) $(X, \{N_n\})$ is called a *matrix fuzzy normed space* if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \ge N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \ne 0$.

(2) $(X, \{N_n\})$ is called a *matrix fuzzy Banach space* if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

Example 1.4. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $N_n(x,t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then*

$$N_k(AxB, t) = \frac{t}{t + \|AxB\|_k} \geq \frac{t}{t + \|A\| \cdot \|x\|_n \cdot \|B\|} = \frac{\frac{t}{\|A\|\cdot\|B\|}}{\frac{t}{\|A\|\cdot\|B\|} + \|x\|_n}$$

for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\| \cdot \|B\| \neq 0$. So $(X, \{N_n\})$ is a matrix fuzzy normed space.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.5. [5, 17] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [31] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 23, 34, 45, 50, 51, 54, 56]).

Throughout this paper, let $(X, \{N_n\})$ be a matrix fuzzy normed space and $(Y, \{N_n\})$ a matrix fuzzy Banach space.

In Section 2, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces by using fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the quadratic functional equation in matrix fuzzy normed spaces by using fixed point method.

2. Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces

Using the fixed point method, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix fuzzy normed spaces.

We will use the following notations:
$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbb{R})$ is that j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{R})$ is that (i,j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero.

Lemma 2.1. *Let $(X, \{N_n\})$ be a matrix fuzzy normed space.*
(1) *$N_n(E_{kl} \otimes x, t) = N(x,t)$ for all $t > 0$ and $x \in X$.*
(2) *For all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^n t_{ij}$,*

$$\begin{aligned} N(x_{kl}, t) &\geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i,j = 1,2,\cdots,n\}, \\ N(x_{kl}, t) &\geq N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i,j = 1,2,\cdots,n\right\}. \end{aligned}$$

(3) *$\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_{ijn} = x_{ij}$ for $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.*

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $N_n(E_{kl} \otimes x, t) \geq N(x,t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $N_n(E_{kl} \otimes x, t) \leq N(x,t)$. So $N(E_{kl} \otimes x, t) = N(x,t)$.

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geq N_n\left([x_{ij}], \frac{t}{\|e_k\|\cdot\|e_l\|}\right) = N_n([x_{ij}], t)$.

$$\begin{aligned} N_n([x_{ij}], t) &= N_n\left(\sum_{i,j=1}^n E_{ij} \otimes x_{ij}, t\right) \geq \min\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i,j = 1,2,\cdots,n\} \\ &= \min\{N(x_{ij}, t_{ij}) : i,j = 1,2,\cdots,n\}, \end{aligned}$$

where $t = \sum_{i,j=1}^n t_{ij}$. So $N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i,j = 1,2,\cdots,n\right\}$.

(3) By $N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t}{n^2}\right) : i,j = 1,2,\cdots,n\right\}$, we obtain the result. □

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = f(a+b) - f(a) - f(b),$$

$$Df_n([x_{ij}], [y_{ij}]) := f_n([x_{ij} + y_{ij}]) - f_n([x_{ij}]) - f_n([y_{ij}])$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq \frac{\alpha}{2}\varphi(2a, 2b) \tag{2.1}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})} \tag{2.2}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $A(a) := N\text{-}\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij})} \tag{2.3}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$. Then (2.2) is equivalent to

$$N(f(a+b) - f(a) - f(b), t) \geq \frac{t}{t + \varphi(a, b)} \tag{2.4}$$

for all $t > 0$ and $a, b \in X$.

Letting $b = a$ in (2.4), we get

$$N(f(2a) - 2f(a), t) \geq \frac{t}{t + \varphi(a, a)} \tag{2.5}$$

and so

$$N\left(f(a) - 2f\left(\frac{a}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{a}{2}, \frac{a}{2}\right)} \geq \frac{t}{t + \frac{\alpha}{2}\varphi(a, a)} \tag{2.6}$$

for all $t > 0$ and $a \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : N(g(a) - h(a), \mu t) \geq \frac{t}{t + \varphi(a, a)}, \ \forall a \in X, \forall t > 0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [44, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(a) - h(a), \varepsilon t) \geq \frac{t}{t + \varphi(a, a)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(a) - Jh(a), \alpha\varepsilon t) &= N\left(2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right), \alpha\varepsilon t\right) = N\left(g\left(\frac{a}{2}\right) - h\left(\frac{a}{2}\right), \frac{\alpha}{2}\varepsilon t\right) \\ &\geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \varphi\left(\frac{a}{2}, \frac{a}{2}\right)} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}\varphi(a, a)} = \frac{t}{t + \varphi(a, a)} \end{aligned}$$

for all $a \in X$ and $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for all $g, h \in S$.

It follows from (2.4) that $d(f, Jf) \leq \frac{\alpha}{2}$.

By Theorem 1.5, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right) = A(a)$$

for all $a \in X$.

(3) $d(f,A) \le \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f,A) \le \frac{\alpha}{2-2\alpha}. \tag{2.7}$$

By (2.2),

$$N\left(2^l f\left(\frac{a+b}{2^l}\right) - 2^l f\left(\frac{a}{2^l}\right) - 2^l f\left(\frac{b}{2^l}\right), 2^l t\right) \ge \frac{t}{t + \varphi\left(\frac{a}{2^l}, \frac{b}{2^l}\right)}$$

for all $a, b \in X$ and $t > 0$. So

$$N\left(2^l f\left(\frac{a+b}{2^l}\right) - 2^l f\left(\frac{a}{2^l}\right) - 2^l f\left(\frac{b}{2^l}\right), t\right) \ge \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b)}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{2^l}}{\frac{t}{2^l} + \frac{\alpha^l}{2^l}\varphi(a,b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$N\left(A(a+b) - A(a) - A(b), t\right) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $A(a+b) - A(a) - A(b) = 0$. So the mapping $A : X \to Y$ is additive.

By Lemma 2.1 and (2.7),

$$\begin{aligned} N_n(f_n([x_{ij}]) - A_n([x_{ij}]), t) &\ge \min\left\{N\left(f(x_{ij}) - A(x_{ij}), \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n\right\} \\ &\ge \min\left\{\frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\varphi(x_{ij}, x_{ij})} : i, j = 1, 2, \cdots, n\right\} \\ &\ge \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij})} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (2.3), as desired. □

Corollary 2.3. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \ge \frac{t}{t + \sum_{i,j=1}^n \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{2.8}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $A(a) := N\text{-}\lim_{l\to\infty} 2^l f\left(\frac{a}{2^l}\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f_n([x_{ij}]) - A_n([x_{ij}]), t\right) \ge \frac{(2-2^r)t}{(2-2^r)t + n^2 \cdot 2^r \sum_{i,j=1}^n \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping satisfying (2.2) for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \le 2\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. Then $A(a) := N\text{-}\lim_{l\to\infty} \frac{1}{2^l} f\left(2^l a\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f_n([x_{ij}]) - A_n([x_{ij}]), t\right) \ge \frac{2(1-\alpha)t}{2(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij})}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (2.5) that $d(f, Jf) \le \frac{1}{2}$. So

$$d(f, A) \le \frac{1}{2 - 2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying (2.8). Then $A(a) := N\text{-}\lim_{l\to\infty} \frac{1}{2^l} f\left(2^l a\right)$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f_n([x_{ij}]) - A_n([x_{ij}]), t\right) \ge \frac{(2^r - 2)t}{(2^r - 2)t + n^2 \cdot 2^r \sum_{i,j=1}^{n} \theta\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

3. Hyers-Ulam stability of the quadratic functional equation in matrix fuzzy normed spaces

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic functional equation in matrix fuzzy normed spaces.

For a mapping $f : X \to Y$, define $Df : X^2 \to Y$ and $Df_n : M_n(X^2) \to M_n(Y)$ by

$$Df(a,b) = f(a+b) + f(a-b) - 2f(a) - 2f(b),$$

$$Df_n([x_{ij}], [y_{ij}]) := f_n([x_{ij} + y_{ij}]) + f_n([x_{ij} - y_{ij}]) - 2f_n([x_{ij}]) - 2f_n([y_{ij}])$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \le \frac{\alpha}{4}\varphi(2a, 2b) \tag{3.1}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \ge \frac{t}{t + \sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})} \tag{3.2}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $A(a) := N\text{-}\lim_{l\to\infty} 4^l f\left(\frac{a}{2^l}\right)$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f_n([x_{ij}]) - Q_n([x_{ij}]), t) \ge \frac{4(1-\alpha)t}{4(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij})} \tag{3.3}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $n = 1$. Then (3.2) is equivalent to

$$N(f(a+b) + f(a-b) - 2f(a) - 2f(b), t) \ge \frac{t}{t + \varphi(a,b)} \tag{3.4}$$

for all $t > 0$ and $a, b \in X$.

Letting $b = a$ in (3.4), we get

$$N(f(2a) - 4f(a), t) \ge \frac{t}{t + \varphi(a,a)} \tag{3.5}$$

and so

$$N\left(f(a) - 4f\left(\frac{a}{2}\right), t\right) \ge \frac{t}{t + \varphi\left(\frac{a}{2}, \frac{a}{2}\right)} \ge \frac{t}{t + \frac{\alpha}{4}\varphi(a,a)} \tag{3.6}$$

for all $t > 0$ and $a \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 4g\left(\frac{a}{2}\right)$$

for all $a \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(a) - h(a), \varepsilon t) \ge \frac{t}{t + \varphi(a,a)}$$

for all $a \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(a) - Jh(a), \alpha\varepsilon t) &= N\left(4g\left(\frac{a}{2}\right) - 4h\left(\frac{a}{2}\right), \alpha\varepsilon t\right) = N\left(g\left(\frac{a}{4}\right) - h\left(\frac{a}{4}\right), \frac{\alpha}{4}\varepsilon t\right) \\ &\ge \frac{\frac{\alpha t}{4}}{\frac{\alpha t}{4} + \varphi\left(\frac{a}{2}, \frac{a}{2}\right)} \ge \frac{\frac{\alpha t}{4}}{\frac{\alpha t}{4} + \frac{\alpha}{4}\varphi(a,a)} = \frac{t}{t + \varphi(a,a)} \end{aligned}$$

for all $a \in X$ and $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le \alpha\varepsilon$. This means that

$$d(Jg, Jh) \le \alpha d(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that $d(f, Jf) \le \frac{\alpha}{4}$.

By Theorem 1.5, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{a}{2}\right) = \frac{1}{4}Q(a)$$

for all $a \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$N\text{-}\lim_{l\to\infty} 4^l f\left(\frac{a}{2^l}\right) = Q(a)$$

for all $a \in X$.

(3) $d(f, Q) \le \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{\alpha}{4 - 4\alpha}. \tag{3.7}$$

By (3.2),

$$N\left(4^l f\left(\frac{a+b}{2^l}\right) + 4^l f\left(\frac{a-b}{2^l}\right) - 2\cdot 4^l f\left(\frac{a}{2^l}\right) - 2\cdot 4^l f\left(\frac{b}{2^l}\right), 4^l t\right) \ge \frac{t}{t + \varphi\left(\frac{a}{2^l}, \frac{b}{2^l}\right)}$$

for all $a, b \in X$ and $t > 0$. So

$$N\left(4^l f\left(\frac{a+b}{2^l}\right) + 4^l f\left(\frac{a-b}{2^l}\right) - 2\cdot 4^l f\left(\frac{a}{2^l}\right) - 2\cdot 4^l f\left(\frac{b}{2^l}\right), t\right) \ge \frac{\frac{t}{4^l}}{\frac{t}{4^l} + \frac{\alpha^l}{4^l}\varphi(a, b)}$$

for all $a, b \in X$ and $t > 0$. Since $\lim_{l\to\infty} \frac{\frac{t}{4^l}}{\frac{t}{4^l} + \frac{\alpha^l}{4^l}\varphi(a,b)} = 1$ for all $a, b \in X$ and $t > 0$,

$$N\left(Q(a+b) + Q(a-b) - 2Q(a) - 2Q(b), t\right) = 1$$

for all $a, b \in X$ and $t > 0$. Thus $Q(a+b) + Q(a-b) - 2Q(a) - 2Q(b) = 0$. So the mapping $Q : X \to Y$ is quadratic.

By Lemma 2.1 and (3.7),

$$\begin{aligned} N_n(f_n([x_{ij}]) - Q_n([x_{ij}]), t) &\ge \min\left\{N\left(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}\right) : i, j = 1, 2, \cdots, n\right\} \\ &\ge \min\left\{\frac{4(1-\alpha)t}{4(1-\alpha)t + n^2\alpha\varphi(x_{ij}, x_{ij})} : i, j = 1, 2, \cdots, n\right\} \\ &\ge \frac{4(1-\alpha)t}{4(1-\alpha)t + n^2\alpha\sum_{i,j=1}^n \varphi(x_{ij}, x_{ij})} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $Q : X \to Y$ is a unique quadratic mapping satisfying (3.3), as desired. $\square$

Corollary 3.2. *Let r, θ be positive real numbers with $r < 2$. Let $f : X \to Y$ be a mapping satisfying*

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \ge \frac{t}{t + \sum_{i,j=1}^n \theta(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{3.8}$$

for all $t > 0$ *and* $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. *Then* $A(a) := N\text{-}\lim_{l\to\infty} 4^l f\left(\frac{a}{2^l}\right)$ *exists for each* $a \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) \geq \frac{2(4-2^r)t}{2(4-2^r)t + n^2 \cdot 2^r \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{r-2}$ and we get the desired result. □

Theorem 3.3. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (3.2) *for which there exists a function* $\varphi : X^2 \to [0, \infty)$ *such that there exists an* $\alpha < 1$ *with*

$$\varphi(a,b) \leq 4\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right)$$

for all $a, b \in X$. *Then* $Q(a) := N\text{-}\lim_{l\to\infty} \frac{1}{4^l} f\left(2^l a\right)$ *exists for each* $a \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) \geq \frac{4(1-\alpha)t}{4(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij})}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 4g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (3.5) that $d(f, Jf) \leq \frac{1}{4}$. So

$$d(f, Q) \leq \frac{1}{4 - 4\alpha}.$$

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let* r, θ *be positive real numbers with* $r > 2$. *Let* $f : X \to Y$ *be a mapping satisfying* (3.8). *Then* $Q(a) := N\text{-}\lim_{l\to\infty} \frac{1}{4^l} f\left(2^l a\right)$ *exists for each* $a \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N\left(f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) \geq \frac{2(2^r-4)t}{2(2^r-4)t + n^2 \cdot 2^r \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{2-r}$ and we get the desired result. □

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On the stability of multi-additive mappings in non-Archimedean normed spaces

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Abstract. We establish some new stability results concerning multi-additive functional equation in non-Archimedean normed spaces. The results improve some recent results. Some applications of our result will be illustrated. In particular, we will see that some results about stability of multi-additive mappings in real normed spaces are not valid in non-Archimedean normed spaces.

Keywords: Stability; Multi-additive mapping; Non-Archimedean normed space; Fixed point.
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1. Introduction

In 1897, Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. During the last three decades, the theory of non-Archimedean spaces has gained the interest of physicists for their research, in particular, in problems deriving from quantum physics, p-adic strings and superstrings [10, 18]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of approach. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be a no unit vector in a non-Archimedean normed space [10]. These facts show that the non-Archimedean framework is of special interest.

A basic question in the theory of functional equations is the following: when is it true that a function, which approximately satisfies a functional equation must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [12]). The first stability problem concerning group homomorphisms was raised by Ulam [17] in 1940 and affirmatively solved by Hyers [7]. The result of Hyers was generalized by Th.M. Rassias [14] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y) = f(x+y)-[f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p+\|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [6], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$ by following Th.M. Rassias' approach. Furthermore for an extensive account of methods and results concerning Hyers-Ulam stability of additive, multi-additive, multi-Jensen mappings and functional equations in a single variable as well as in several variables we refer the reader to ([1–4, 8, 9, 11, 15, 16, 19–24]) and references therein.

In this paper, we determine some results concerning the stability of the multi-additive mappings in the non-Archimedean normed spaces. The presented results correspond to some outcomes from [1] and sometimes are their slight generalizations.

2. Preliminaries

We recall some basic facts concerning non-Archimedean space and some basic results.

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r|+|s|, \quad \forall r,s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality.

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Definition 2.1. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot|:\mathbb{K}\to\mathbb{R}$ such that for any $r,s\in\mathbb{K}$ we have

(i) $|r|\geq 0$ and equality holds if and only if $r=0$;
(ii) $|rs|=|r||s|$;
(iii) $|r+s|\leq\max\{|r|,|s|\}$.

The condition (iii) is called the strong triangle inequality. Clearly, $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. We always assume in addition that $|\cdot|$ is non trivial, i.e., that

(iv) there is an $r_0\in\mathbb{K}$ such that $|r_0|\neq 0,1$.

The most important examples of non-Archimedean spaces are p-adic numbers.

Example 2.2. Let p be a prime number. For any nonzero rational number x, there exists a unique integer n_x such that $x=\frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p:=p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y)=|x-y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x=\sum_{k\geq n_x}^{\infty}a_kp^k$, where $|a_k|\leq p-1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\geq n_x}^{\infty}a_kp^k|_p=p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [18]). Note that if $p>2$, then $|2^n|_p=1$ for each integer n but $|2|_2<1$.

Throughout this paper, we assume that the base field is a non-Archimedean field and hence we can call it simply a field. Moreover, $\mathbb{N}$ stands for the set of all positive integers.

Definition 2.3. Let $\mathscr{X}$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:\mathscr{X}\to[0,\infty)$ is a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\|$ for all $r\in\mathbb{K}$ and $x\in\mathscr{X}$;
(iii) the strong triangle inequality

$$\|x+y\|\leq\max\{\|x\|,\|y\|\},\quad\forall x,y\in\mathscr{X}.$$

Then $(\mathscr{X},\|\cdot\|)$ is called a non-Archimedean normed space.

Definition 2.4. Let $\mathscr{X}$ be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in $\mathscr{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x\in\mathscr{X}$ such that $\lim\limits_{n\to\infty}\|x_n-x\|=0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim\limits_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ in $\mathscr{X}$ is said to be a Cauchy sequence if $\lim\limits_{n\to\infty}\|x_{n+p}-x_n\|=0$ for all $p=1,2,\ldots$. Due to the fact that

$$\|x_n-x_m\|\leq\max\{\|x_{j+1}-x_j\|:\quad m\leq j\leq n-1\}\quad(n>m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean normed space.

It is known that every convergent sequence in a non-Archimedean normed space is a Cauchy sequence. If every Cauchy sequence in $\mathscr{X}$ converges, then the non-Archimedean normed space $\mathscr{X}$ is called a non-Archimedean Banach space.

Definition 2.5. Let $\mathscr{X}$ denotes a linear space and $\mathscr{Y}$ represents a complete non-Archimedean normed space and $n\geq 1$ is an integer. A function $f:\mathscr{X}^n\to\mathscr{Y}$ is called a multi-additive mapping, if f is additive in each variable:

$$f(x_1,\ldots,x_{i-1},x_i+x_i',x_{i+1},\ldots,x_n)=f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)+f(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)$$

for all $i=1,2,\ldots,n$ and all $x_1,\ldots,x_{i-1},x_i,x_i',x_{i+1},\ldots,x_n\in\mathscr{X}$. Some basic facts on such mappings can be found for instance in [1, 2, 13], where their application to the representation of polynomial functions is also presented.

Let Ω be a set. A function $d:\Omega\times\Omega\to[0,\infty]$ is called a generalized metric on Ω if d satisfies the following conditions:

(a) $d(x,y)=0$ if and only if $x=y$;
(b) $d(x,y)=d(y,x)$ for all $x,y\in\Omega$;
(c) $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in\Omega$.

It will later on the following fixed point alternative theorem (cf. [5]) will be useful.

Theorem 2.6. *Let (Ω, d) be a complete generalized metric space and $J : \Omega \to \Omega$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$, that is*

$$d(Jx, Jy) \leq Ld(x, y) \qquad \forall\, x, y \in \Omega.$$

Then, for each given $x \in \Omega$, either

$$d(J^n x, J^{n+1}x) = \infty \qquad \forall\, n \geq 0,$$

or

$$d(J^n x, J^{n+1}x) < \infty, \qquad \forall\, n \geq n_0,$$

for some natural number n_0. Actually if the second alternative holds, then the sequence $\{J^n x\}$ is convergent to a fixed point x^* of J and

(1) *x^* is the unique fixed point of J in the set $\Omega^* = \{y \in \Omega : \ d(J^{n_0}x, y) < \infty\}$;*

(2) *$d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Omega^*$.*

3. Non-Archimedean stability of the multi-additive mapping: a direct method

Let $\mathscr{X}$ be a linear space over a non-Archimedean field $\mathbb{K}$ with a valuation $|\cdot|$ and $\mathscr{Y}$ be a complete non-Archimedean normed space over $\mathbb{K}$. For the given mapping $f : \mathscr{X}^n \to \mathscr{Y}$ and every $i \in \{1, 2, \dots, n\}$, we define the difference operator

$$\begin{aligned} &D_i f(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n) \\ &\quad := f(x_1, \dots, x_{i-1}, x_i + x_i', x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n) \end{aligned}$$

for all $x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n \in \mathscr{X}$.

Theorem 3.1. *Let $\mathscr{X}$ be a linear space over a non-Archimedean field $\mathbb{K}$ with a valuation $|\cdot|$ and $\mathscr{Y}$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $n \in \mathbb{N}$ and for every $i \in \{1, 2, \dots, n\}$, $\varphi_i : \mathscr{X}^{n+1} \to [0, \infty)$ be a function. Let for some natural number $k \in \mathbb{K}$,*

$$\begin{cases} \lim_{m\to\infty} |k|^m \varphi_i(x_1/k^m, \dots, x_i, x_i', x_{i+1}, \dots, x_n) = 0, \\ \qquad \vdots \\ \lim_{m\to\infty} |k|^m \varphi_i(x_1, \dots, x_{i-2}, x_{i-1}/k^m, x_i, x_i', x_{i+1}, \dots, x_n) = 0, \\ \lim_{m\to\infty} |k|^m \varphi_i(x_1, \dots, x_{i-1}, x_i/k^m, x_i'/k^m, x_{i+1}, \dots, x_n) = 0, \\ \lim_{m\to\infty} |k|^m \varphi_i(x_1, \dots, x_i, x_i', x_{i+1}/k^m, x_{i+2}, \dots, x_n) = 0, \\ \qquad \vdots \\ \lim_{m\to\infty} |k|^m \varphi_i(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_{n-1}, x_n/k^m) = 0 \end{cases} \tag{3.1}$$

for all $x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n \in \mathscr{X}$ and let for each $(x_1, \dots, x_n) \in \mathscr{X}^n$ the limit

$$\lim_{m\to\infty} \max\{|k|^{s+1} \max\{\varphi_i(x_1, \dots, x_{i-1}, x_i/k^{s+1}, jx_i/k^{s+1}, x_{i+1}, \dots, x_n) : 1 \leq j \leq k-1\} : 0 \leq s < m\},$$

denoted by $\tilde{\varphi}_i(x_1, \dots, x_n)$, exists. Suppose that $f : \mathscr{X}^n \to \mathscr{Y}$ be a mapping satisfying

$$\|D_i f(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n)\| \leq \varphi_i(x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n) \tag{3.2}$$

for all $x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n \in \mathscr{X}$ and $i \in \{1, \dots, n\}$. Then for every $i \in \{1, \dots, n\}$ there exists a multi-additive mapping $F_i : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_1, \dots, x_n) - F_i(x_1, \dots, x_n)\| \leq \frac{1}{|k|} \tilde{\varphi}_i(x_1, \dots, x_n) \tag{3.3}$$

for all $x_1, \dots, x_n \in \mathscr{X}$. For every $i \in \{1, \dots, n\}$ the function F_i is given by

$$F_i(x_1, \dots, x_n) := \lim_{j\to\infty} k^j f(x_1, \dots, x_{i-1}, x_i/k^j, x_{i+1}, \dots, x_n) \tag{3.4}$$

for all $x_1, \dots, x_n \in \mathscr{X}$, and if, in addition,

$$\lim_{p\to\infty} |k|^p \tilde{\varphi}_i(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n) = 0,$$

then F_i is the unique multi-additive mapping satisfying (3.3).

Proof. Fix $j \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. Letting $x_i' = x_i$ in (3.2), we get

$$\|f(x_1, \dots, x_{i-1}, 2x_i, x_{i+1}, \dots, x_n) - 2f(x_1, \dots, x_n)\| \leq \varphi_i(x_1, \dots, x_i, x_i, x_{i+1}, \dots, x_n) \tag{3.5}$$

for all $x_1, \dots, x_n \in \mathscr{X}$. Letting $x_i' = 2x_i$ in (3.2), we get

$$\begin{aligned} &\|f(x_1, \dots, x_{i-1}, 3x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, 2x_i, x_{i+1}, \dots, x_n)\| \\ &\leq \varphi_i(x_1, \dots, x_i, 2x_i, x_{i+1}, \dots, x_n) \end{aligned} \tag{3.6}$$

for all $x_1, \dots, x_n \in \mathscr{X}$. Similarly, putting $x_i' = jx_i$ ($3 \leq j \leq k-1$) in (3.2), we get

$$\begin{aligned} &\|f(x_1, \dots, x_{i-1}, (j+1)x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, jx_i, x_{i+1}, \dots, x_n)\| \\ &\leq \varphi_i(x_1, \dots, x_i, jx_i, x_{i+1}, \dots, x_n) \end{aligned} \tag{3.7}$$

for all $x_1, \dots, x_n \in \mathscr{X}$. By (3.5)–(3.7), we see that

$$\begin{aligned} &\|f(x_1, \dots, x_{i-1}, kx_i, x_{i+1}, \dots, x_n) - kf(x_1, \dots, x_n)\| \\ &\leq \max\{\varphi_i(x_1, \dots, x_i, jx_i, x_{i+1}, \dots, x_n) : \ \ 1 \leq j \leq k-1\} \end{aligned} \tag{3.8}$$

for all $x_1, \dots, x_n \in \mathscr{X}$, and so

$$\begin{aligned} &\|k^{m-1} f(x_1, \dots, x_{i-1}, x_i/k^{m-1}, x_{i+1}, \dots, x_n) - k^m f(x_1, \dots, x_{i-1}, x_i/k^m, x_{i+1}, \dots, x_n)\| \\ &\leq |k|^{m-1} \max\{\varphi_i(x_1, \dots, x_{i-1}, x_i/k^m, jx_i/k^m, x_{i+1}, \dots, x_n) : \ \ 1 \leq j \leq k-1\}. \end{aligned} \tag{3.9}$$

for all $x_1, \dots, x_n \in \mathscr{X}$ and $m = 0, 1, \dots$. By (3.1), it follows that $\{k^m f(x_1, \dots, x_{i-1}, x_i/k^m, x_{i+1}, \dots, x_n)\}$ is a Cauchy sequence in the complete non-Archimedean space $\mathscr{Y}$. This sequence is convergent and we define $F_i : \mathscr{X}^n \to \mathscr{Y}$ by

$$F_i(x_1, \dots, x_n) := \lim_{m \to \infty} k^m f(x_1, \dots, x_{i-1}, x_i/k^m, x_{i+1}, \dots, x_n) \tag{3.10}$$

for all $x_1, \dots, x_n \in \mathscr{X}$. Using (3.9), one can show that

$$\begin{aligned} &\|f(x_1, \dots, x_n) - k^m f(x_1, \dots, x_{i-1}, x_i/k^m, x_{i+1}, \dots, x_n)\| \\ &= \left\| \sum_{s=1}^{m} [k^{s-1} f(x_1, \dots, x_{i-1}, x_i/k^{s-1}, x_{i+1}, \dots, x_n) - k^s f(x_1, \dots, x_{i-1}, x_i/k^s, x_{i+1}, \dots, x_n)] \right\| \\ &\leq \tfrac{1}{|k|} \max\{|k|^{s+1} \max\{\varphi_i(x_1, \dots, x_{i-1}, x_i/k^{s+1}, jx_i/k^{s+1}, x_{i+1}, \dots, x_n) : 1 \leq j \leq k-1\} : 0 \leq s < m\}, \end{aligned} \tag{3.11}$$

by taking limit as $m \to \infty$ of both sides of (3.11), one can obtain the inequality (3.3).

Now, we will show that for every $i \in \{1, \dots, n\}$ the mapping F_i is multi-additive. By (3.2), we have

$$\left\{ \begin{aligned} &\|k^j D_s f(x_1, \dots, x_{s-1}, x_s, x_s', x_{s+1}, \dots, x_{i-1}, x_i/k^j, x_{i+1}, \dots, x_n)\| \\ &\quad \leq |k|^j \varphi_s(x_1, \dots, x_{s-1}, x_s, x_s', x_{s+1}, \dots, x_{i-1}, x_i/k^j, x_{i+1}, \dots, x_n), && \text{for } \ s < i, \\ &\|k^j D_i f(x_1, \dots, x_{i-1}, x_i/k^j, x_i'/k^j, x_{i+1}, \dots, x_n)\| \\ &\quad \leq |k|^j \varphi_i(x_1, \dots, x_{i-1}, x_i/k^j, x_i'/k^j, x_{i+1}, \dots, x_n), && \text{for } \ s = i, \\ &\|k^j D_s f(x_1, \dots, x_{i-1}, x_i/k^j, x_{i+1}, \dots, x_{s-1}, x_s, x_s', x_{s+1}, \dots, x_n)\| \\ &\quad \leq |k|^j \varphi_s(x_1, \dots, x_{i-1}, x_i/k^j, x_{i+1}, \dots, x_{s-1}, x_s, x_s', x_{s+1}, \dots, x_n), && \text{for } \ s > i \end{aligned} \right. \tag{3.12}$$

for all $x_1, \dots, x_i, x_i', x_{i+1}, \dots, x_n \in \mathscr{X}$, $x_s' \in \mathscr{X}$ $(s \in \{1, 2, \dots, n\} \backslash \{i\})$ and all $j \in \mathbb{N}$. Letting $j \to \infty$ in the above inequalities and using (3.1), we see that the mapping F_i is multi-additive. To prove the uniqueness of the mapping F_i, assume that $F_i' : \mathscr{X}^n \to \mathscr{Y}$ is another multi-additive mapping satisfying (3.3). Then we have

$$\begin{aligned} &\|F_i(x_1, \dots, x_n) - F_i'(x_1, \dots, x_n)\| \\ &= |k|^p \|F_i(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n) - F_i'(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n)\| \\ &\leq \max\{|k|^p \|f(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n) - F_i(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n)\|, \\ &\qquad |k|^p \|f(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n) - F_i'(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n)\|\} \\ &\leq 2|k|^{p-1} \tilde{\varphi}_i(x_1, \dots, x_{i-1}, x_i/k^p, x_{i+1}, \dots, x_n) \end{aligned}$$

for all $p \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathscr{X}$. If

$$\lim_{p\to\infty} |k|^p \tilde{\varphi}_i(x_1, \ldots, x_{i-1}, x_i/k^p, x_{i+1}, \ldots, x_n) = 0,$$

we may conclude that $F_i(x_1, \ldots, x_n) = F_i'(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in \mathscr{X}$, and the proof is complete. □

The proof of the following result is similar to that in Theorem 3.1, hence it is omitted.

Theorem 3.2. *Let $\mathscr{X}$ be a linear space over a non-Archimedean field $\mathbb{K}$ with a valuation $|\cdot|$ and $\mathscr{Y}$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $n \in \mathbb{N}$ and for every $i \in \{1, 2, \ldots, n\}$, $\varphi_i : \mathscr{X}^{n+1} \to [0, \infty)$ be a function. Assume for some natural number $k \in \mathbb{K}$,*

$$\begin{cases} \lim\limits_{m\to\infty} \frac{1}{|k|^m}\varphi_i(k^m x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n) = 0, \\ \qquad\vdots \\ \lim\limits_{m\to\infty} \frac{1}{|k|^m}\varphi_i(x_1, \ldots, x_{i-2}, k^m x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_n) = 0, \\ \lim\limits_{m\to\infty} \frac{1}{|k|^m}\varphi_i(x_1, \ldots, x_{i-1}, k^m x_i, k^m x_i', x_{i+1}, \ldots, x_n) = 0, \\ \lim\limits_{m\to\infty} \frac{1}{|k|^m}\varphi_i(x_1, \ldots, x_i, x_i', k^m x_{i+1}, x_{i+2}, \ldots, x_n) = 0, \\ \qquad\vdots \\ \lim\limits_{m\to\infty} \frac{1}{|k|^m}\varphi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_{n-1}, k^m x_n) = 0 \end{cases}$$

for all $x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathscr{X}$ and let for each $(x_1, \ldots, x_n) \in \mathscr{X}^n$ the limit

$$\lim_{m\to\infty} \max\left\{\frac{1}{|k|^s}\max\{\varphi_i(x_1, \ldots, x_{i-1}, k^s x_i, jk^s x_i, x_{i+1}, \ldots, x_n) : 1 \le j \le k-1\} : 0 \le s < m\right\},$$

denoted by $\tilde{\varphi}_i(x_1, \ldots, x_n)$, exists. Suppose that $f : \mathscr{X}^n \to \mathscr{Y}$ is a mapping satisfying

$$\|D_i f(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n)\| \le \varphi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathscr{X}$ and $i \in \{1, \ldots, n\}$. Then for every $i \in \{1, \ldots, n\}$ there exists a multi-additive mapping $F_i : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_1, \ldots, x_n) - F_i(x_1, \ldots, x_n)\| \le \frac{1}{|k|}\tilde{\varphi}_i(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathscr{X}$. For every $i \in \{1, \ldots, n\}$ the function F_i is given by

$$F_i(x_1, \ldots, x_n) := \lim_{j\to\infty} \frac{1}{k^j} f(x_1, \ldots, x_{i-1}, k^j x_i, x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathscr{X}$, and if, in addition,

$$\lim_{p\to\infty} \frac{1}{|k|^p}\tilde{\varphi}_i(x_1, \ldots, x_{i-1}, k^p x_i, x_{i+1}, \ldots, x_n) = 0,$$

then F_i is the unique multi-additive mapping satisfying (3.3).

Corollary 3.3. *Let $\mathbb{K}$ be a non-Archimedean field, $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a non-Archimedean normed space over $\mathbb{K}$, $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $k \in \mathbb{N}$ with $|k| < 1$, $\varepsilon, \delta \ge 0, 0 < r_i < 1 (1 \le i \le n)$, and $f : \mathscr{X}^n \to \mathscr{Y}$ be a mapping such that*

$$\|D_i f(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n)\|_{\mathscr{Y}} \le \varepsilon + \delta[\|x_1\|_{\mathscr{X}}^{r_1} \cdots \|x_{i-1}\|_{\mathscr{X}}^{r_{i-1}}(\|x_i\|_{\mathscr{X}}^{r_i} + \|x_i'\|_{\mathscr{X}}^{r_i})\|x_{i+1}\|_{\mathscr{X}}^{r_{i+1}} \cdots \|x_n\|_{\mathscr{X}}^{r_n}]$$

for all $x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathscr{X}$ and every $i \in \{1, 2, \ldots, n\}$. Then there exists a unique multi-additive mapping $F_i : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_1, x_2, \ldots, x_n) - F_i(x_1, x_2, \ldots, x_n)\|_{\mathscr{Y}} \le \varepsilon + \frac{2\delta}{|k|^{r_i}}(\|x_1\|_{\mathscr{X}}^{r_1} \cdot \|x_2\|_{\mathscr{X}}^{r_2} \cdot \cdots \cdot \|x_n\|_{\mathscr{X}}^{r_n})$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{X}$.

Corollary 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a non-Archimedean normed space over $\mathbb{K}$, $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $k \in \mathbb{N}$ with $|k| < 1$, $\varepsilon, \delta \ge 0, r_i, s_i > 0(1 \le$*

$i \le n$) with $\lambda_i := r_i + s_i < 1$, and $f : \mathscr{X}^n \to \mathscr{Y}$ be a mapping such that

$$\begin{aligned}&\|D_i f(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n)\|_{\mathscr{Y}}\\ &\quad \le \varepsilon + \delta\{\|x_1\|_{\mathscr{X}}^{\lambda_1} \cdots \|x_{i-1}\|_{\mathscr{X}}^{\lambda_{i-1}}[\|x_i\|_{\mathscr{X}}^{r_i}\|x_i'\|_{\mathscr{X}}^{s_i} + (\|x_i\|_{\mathscr{X}}^{r_i+s_i} + \|x_i'\|_{\mathscr{X}}^{r_i+s_i})]\|x_{i+1}\|_{\mathscr{X}}^{\lambda_{i+1}} \cdots \|x_n\|_{\mathscr{X}}^{\lambda_n}\}\end{aligned}$$

for all $x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathscr{X}$ and every $i \in \{1, 2, \ldots, n\}$. Then there exists a unique multi-additive mapping $F_i : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_1, x_2, \ldots, x_n) - F_i(x_1, x_2, \ldots, x_n)\|_{\mathscr{Y}} \le \varepsilon + \frac{3\delta}{|k|^{\lambda_i}}(\|x_1\|_{\mathscr{X}}^{\lambda_1} \cdot \|x_2\|_{\mathscr{X}}^{\lambda_2} \cdot \cdots \cdot \|x_n\|_{\mathscr{X}}^{\lambda_n})$$

for all $x_1, x_2, \ldots, x_n \in \mathscr{X}$.

Remark. Theorems 3.1 and 3.2 are generalized versions of Theorem 2.1 in [11].

In [2], Ciepliński proved the following:

Theorem 3.5. *Let $\mathscr{X}$ be a commutative semigroup and $\mathscr{Y}$ be a Banach space. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1, 2, \ldots, n\}$, $\varphi_i : \mathscr{X}^{n+1} \to [0, \infty)$ is a mapping such that for any $(x_1, x_2, \ldots, x_n) \in \mathscr{X}^{n+1}$ we have*

$$\begin{aligned}\tilde{\varphi}_i(x_1, \ldots, x_{n+1}) := \; &\sum_{j=0}^{\infty} \tfrac{1}{2^j}[\varphi_i(2^j x_1, x_2, \ldots, x_{n+1}) + \cdots + \varphi_i(x_1, \ldots, x_{i-2}, 2^j x_{i-1}, x_i, \ldots, x_{n+1})\\ &+ \tfrac{1}{2}\varphi_i(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_{i+1}, x_{i+2}, \ldots, x_{n+1}) + \varphi_i(x_1, \ldots, x_{i+1}, 2^j x_{i+2}, x_{i+3}, \ldots, x_{n+1})\\ &+ \cdots + \varphi_i(x_1, \ldots, x_n, 2^j x_{n+1}) < \infty.\end{aligned}$$

If $f : \mathscr{X}^n \to \mathscr{Y}$ is a function satisfying

$$\begin{aligned}&\|f(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)\|\\ &\quad \le \varphi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n), \quad (x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n) \in \mathscr{X}^{n+1}, i \in \{1, \ldots, n\},\end{aligned}$$

then for every $i \in \{1, \ldots, n\}$ there exists a multi-additive mapping $F_i : \mathscr{X}^n \to \mathscr{Y}$ such that for any $(x_1, \ldots, x_n) \in \mathscr{X}^n$ we have

$$\|f(x_1, \ldots, x_n) - F_i(x_1, \ldots, x_n)\| \le \tilde{\varphi}_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n).$$

For every $i \in \{1, \ldots, n\}$ the function F_i is given by

$$F_i(x_1, \ldots, x_n) := \lim_{j\to\infty} \frac{1}{2^j} f(x_1, \ldots, x_{i-1}, 2^j x_i, x_{i+1}, \ldots, x_n), \quad (x_1, \ldots, x_n) \in \mathscr{X}^n.$$

The following example shows that the same result of Theorem 3.5 is not true in non-Archimedean normed spaces and the assumption $|k| < 1$ cannot be omitted in Corollaries 3.3 and 3.4. This example is a modification of the example of [22].

Example 3.6. Let $p > 2$ be a prime number and $f : \mathbb{Q}_p^2 \to \mathbb{Q}_p$ be defined by $f(x_1, x_2) = 2$. Since, $|2^j|_p = 1$ for all $j \in \mathbb{Z}$, then for $\varphi_1(x_1, x_2, x_3) = \varphi_2(x_1, x_2, x_3) = 1$ (the case when $k = 2, \varepsilon = 1$ and $\delta = 0$ is considered),

$$|D_1 f(x_1, x_1', x_2)|_p = |D_2 f(x_1, x_2, x_2')|_p = |2|_p = 1 \le \varphi_1(x_1, x_2, x_3) = \varphi_2(x_1, x_2, x_3), \qquad x_1, x_1', x_2, x_2' \in \mathbb{Q}_p.$$

However

$$|2^j f(x_1/2^j, x_2) - 2^{j+1} f(x_1/2^{j+1}, x_2)|_p = |2^{j+1}|_p = 1$$

and

$$|2^j f(x_1, x_2/2^j) - 2^{j+1} f(x_1, x_2/2^{j+1})|_p = |2^{j+1}|_p = 1$$

for all $x_1, x_2 \in \mathbb{Q}_p$ and $j \in \mathbb{N}$. Hence neither $\{2^j f(x_1/2^j, x_2)\}$ nor $\{2^j f(x_1, x_2/2^j)\}$ is a Cauchy sequence. Hence these sequences are not convergent in $\mathbb{Q}_p$.

4. A fixed point approach to the stability

In [2], Ciepliński reduce the system of n Cauchy equations to a single functional equation (see [2, Theorem 2]).

Theorem 4.1. *Assume that $n \in \mathbb{N}$ and let $\mathscr{X}$ be a commutative semigroup with the identity element 0 and $\mathscr{Y}$ be a linear space. A mapping $f : \mathscr{X}^n \to \mathscr{Y}$ is multi-additive if and only if*

$$f(x_{11}+x_{12},\ldots,x_{n1}+x_{n2}) = \sum_{i_1,\ldots,i_n\in\{1,2\}} f(x_{1i_1},\ldots,x_{ni_n}) \tag{4.1}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$.

Next, we will show the Hyers-Ulam stability of Eq.(4.1) by using the fixed point method. For the given mapping $f : \mathscr{X}^n \to \mathscr{Y}$, we define the difference operator

$$Df(x_{11},x_{12},\ldots,x_{n1},x_{n2}) := f(x_{11}+x_{12},\ldots,x_{n1}+x_{n2}) - \sum_{i_1,\ldots,i_n\in\{1,2\}} f(x_{1i_1},\ldots,x_{ni_n})$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$.

Theorem 4.2. *Let $\mathscr{X}$ be a linear space over a non-Archimedean field $\mathbb{K}$ with a valuation $|\cdot|$ and $\mathscr{Y}$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $0 \le L < 1$ and $\varphi : \mathscr{X}^{2n} \to [0,\infty)$ be a mapping such that*

$$\varphi(x_{11},x_{12},\ldots,x_{n1},x_{n2}) \le \frac{L}{|2|^n}\varphi(2x_{11},2x_{12},\ldots,2x_{n1},2x_{n2}) \tag{4.2}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$. If $f : \mathscr{X}^n \to \mathscr{Y}$ is a function satisfying

$$\|Df(x_{11},x_{12},\ldots,x_{n1},x_{n2})\| \le \varphi(x_{11},x_{12},\ldots,x_{n1},x_{n2}) \tag{4.3}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$, then there exists a unique multi-additive mapping $F : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_{11},x_{21},\ldots,x_{n1}) - F(x_{11},x_{21},\ldots,x_{n1}\| \le \frac{L}{|2|^n(1-L)}\varphi(x_{11},x_{11},\ldots,x_{n1},x_{n1}) \tag{4.4}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. The function F is given by

$$F(x_{11},x_{21},\ldots,x_{n1}) := \lim_{m\to\infty} 2^{mn} f(x_{11}/2^m, x_{21}/2^m, \ldots, x_{n1}/2^m) \tag{4.5}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$.

Proof. It follows from (4.2) that

$$\lim_{j\to\infty} |2|^{nj}\varphi(x_{11}/2^j, x_{12}/2^j, \ldots, x_{n1}/2^j, x_{n2}/2^j) = 0 \tag{4.6}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$.

Consider the set

$$\Omega := \{g \mid \quad g : \mathscr{X}^n \to \mathscr{Y}\}$$

and introduce the generalized metric on Ω defined by

$$\begin{aligned} d(g,h) := \inf\{c > 0| \quad & \|g(x_{11},x_{21},\ldots,x_{n1}) - h(x_{11},x_{21},\ldots,x_{n1})\| \\ & \le c\varphi(x_{11},x_{11},\ldots,x_{n1},x_{n1}), \quad \forall\, x_{11},x_{21},\ldots,x_{n1} \in \mathscr{X}\}. \end{aligned}$$

It is easy to show that (Ω, d) is a complete generalized metric space. Now we consider the mapping $J : \Omega \to \Omega$ such that

$$Jg(x_{11},x_{21},\ldots,x_{n1}) = 2^n g(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2)$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}^n$. Let $g, h \in \Omega$ be given such that $d(g,h) < \beta$, by the definition,

$$\|g(x_{11},x_{21},\ldots,x_{n1}) - h(x_{11},x_{21},\ldots,x_{n1})\| \le \beta\varphi(x_{11},x_{11},\ldots,x_{n1},x_{n1})$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. Hence

$$\begin{aligned} &\|Jg(x_{11},x_{21},\ldots,x_{n1}) - Jh(x_{11},x_{21},\ldots,x_{n1})\| \\ &\quad = \|2^n g(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2) - 2^n h(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2)\| \\ &\quad = |2|^n \|g(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2) - h(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2)\| \\ &\quad \le |2|^n \beta\varphi((x_{11}/2, x_{11}/2, \ldots, x_{n1}/2, x_{n1}/2) \\ &\quad \le \beta L\varphi(x_{11},x_{11},\ldots,x_{n1},x_{n1}) \end{aligned}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. By the definition it follows that $d(Jg, Jh) \leq \beta L$. Therefore,

$$d(Jg, Jh) \leq L d(g, h), \quad \text{for all} \quad g, h \in \Omega.$$

This means that J is a strictly contractive self-mapping of Ω with Lipschitz constant L.

Putting $x_{i2} = x_{i1}$ for $i \in \{1, 2, \ldots, n\}$ in (4.3), we get

$$\|f(2x_{11}, 2x_{21}, \ldots, 2x_{n1}) - 2^n f(x_{11}, x_{21}, \ldots, x_{n1})\| \leq \varphi(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1})$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. Hence

$$\begin{aligned} \|2^n f(x_{11}/2, x_{21}/2, \ldots, x_{n1}/2) - f(x_{11}, x_{21}, \ldots, x_{n1})\| &\leq \varphi(x_{11}/2, x_{11}/2, \ldots, x_{n1}/2, x_{n1}/2) \\ &\leq \tfrac{L}{|2|^n} \varphi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \end{aligned}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. Therefore $d(Jf, f) \leq L/|2|^n$.

By Theorem 2.6, there exists a mapping $F : \mathscr{X}^n \to \mathscr{Y}$ such that

(1) F is a fixed point of J, that is,

$$F(2x_{11}, 2x_{21}, \ldots, 2x_{n1}) = 2^n F(x_{11}, x_{21}, \ldots, x_{n1}) \tag{4.7}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. The mapping F is a unique fixed point of J in the set $\Delta = \{g \in \Omega | \ d(f, g) < \infty\}$. This implies that F is a unique mapping satisfying (4.7) such that there exists $c \in (0, \infty)$ satisfying

$$\|F(x_{11}, x_{21}, \ldots, x_{n1}) - f(x_{11}, x_{21}, \ldots, x_{n1})\| \leq c \cdot \varphi(x_{11}, x_{11}, \ldots, x_{n1}, x_{n1})$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$.

(2) $d(J^m f, F) \to 0$ as $m \to \infty$. This implies the equality

$$F(x_{11}, x_{21}, \ldots, x_{n1}) := \lim_{m \to \infty} 2^{mn} f(x_{11}/2^m, x_{21}/2^m, \ldots, x_{n1}/2^m) \tag{4.8}$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$.

(3) $d(f, F) \leq d(f, Jf)/(1 - L)$, which implies the inequality

$$d(f, F) \leq L/[|2|^n (1 - L)].$$

Thus inequality (4.4) holds.

It follows from (4.3), (4.6), and (4.8) that

$$\begin{aligned} &\|F(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) - \sum_{i_1, \ldots, i_n \in \{1,2\}} F(x_{1i_1}, \ldots, x_{ni_n})\| \\ &= \lim_{j \to \infty} |2|^{nj} \|f((x_{11} + x_{12})/2^j, \ldots, (x_{n1} + x_{n2})/2^j) - \sum_{i_1, \ldots, i_n \in \{1,2\}} f(x_{1i_1}/2^j, \ldots, x_{ni_n}/2^j)\| \\ &\leq \lim_{j \to \infty} |2|^{nj} \varphi(x_{11}/2^j, x_{12}/2^j, \ldots, x_{n1}/2^j, x_{n2}/2^j) = 0 \end{aligned}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$. Therefore Theorem 4.1 now shows that F is multi-additive. This completes the proof. □

Corollary 4.3. *Let $\mathbb{K}$ be a non-Archimedean field, $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a non-Archimedean normed space over $\mathbb{K}$, $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $\varepsilon, \delta \geq 0, 0 < r < 1, |2| < 1$, and $f : \mathscr{X}^n \to \mathscr{Y}$ be a mapping such that*

$$\|Df(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\|_{\mathscr{Y}} \leq \varepsilon + \delta(\|x_{11}\|_{\mathscr{X}}^{nr} + \|x_{12}\|_{\mathscr{X}}^{nr} + \cdots + \|x_{n1}\|_{\mathscr{X}}^{nr} + \|x_{n2}\|_{\mathscr{X}}^{nr})$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$. Then there exists a unique multi-additive mapping $F : \mathscr{X}^n \to \mathscr{Y}$ such that

$$\|f(x_{11}, x_{21}, \ldots, x_{n1}) - F(x_{11}, x_{21}, \ldots, x_{n1})\|_{\mathscr{Y}} \leq \frac{1}{|2|^{nr} - |2|^n} [\varepsilon + 2\delta(\|x_{11}\|_{\mathscr{X}}^{nr} + \|x_{21}\|_{\mathscr{X}}^{nr} + \cdots + \|x_{n1}\|_{\mathscr{X}}^{nr})]$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$. The function F is given by

$$F(x_{11}, x_{12}, \ldots, x_{n1}) := \lim_{m \to \infty} 2^{mn} f(x_{11}/2^m, \ldots, x_{n1}/2^m)$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}$.

Corollary 4.4. *Let $\mathbb{K}$ be a non-Archimedean field, $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a non-Archimedean normed space over $\mathbb{K}$, $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ be a complete non-Archimedean normed space over $\mathbb{K}$, $\varepsilon, \delta \geq 0$ and $r, s > 0$ with $\lambda := r + s < 1$.*

Assume also that $f : \mathscr{X}^n \to \mathscr{Y}$ *be a mapping satisfying*

$$\|Df(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\|_{\mathscr{Y}} \le \varepsilon + \delta \sum_{k=1}^{n} [\|x_{k1}\|_{\mathscr{X}}^{nr} \cdot \|x_{k2}\|_{\mathscr{X}}^{ns} + (\|x_{k1}\|_{\mathscr{X}}^{n\lambda} + \|x_{k2}\|_{\mathscr{X}}^{n\lambda})]$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in \mathscr{X}$. *Then there exists a unique multi-additive mapping* $F : \mathscr{X}^n \to \mathscr{Y}$ *such that*

$$\|f(x_{11}, x_{21}, \ldots, x_{n1}) - F(x_{11}, x_{21}, \ldots, x_{n1})\| \le \frac{1}{|2|^{n\lambda} - |2|^n} [\varepsilon + 3\delta(\|x_{11}\|_{\mathscr{X}}^{n\lambda} + \|x_{21}\|_{\mathscr{X}}^{n\lambda} + \cdots + \|x_{n1}\|_{\mathscr{X}}^{n\lambda})]$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}^n$. *The function* F *is given by*

$$F(x_{11}, x_{12}, \ldots, x_{n1}) := \lim_{m\to\infty} 2^{mn} f(x_{11}/2^m, \ldots, x_{n1}/2^m)$$

for all $x_{11}, x_{21}, \ldots, x_{n1} \in \mathscr{X}^n$.

The following example shows that the assumption $|2| < 1$ cannot be omitted in Corollaries 4.3 and 4.4.

Example 4.5. Let $p > 2$ be a prime number and $f : \mathbb{Q}_p^2 \to \mathbb{Q}_p$ be defined by $f(x_1, x_2) = 2$. Since, $|2^j|_p = 1$ for all $j \in \mathbb{Z}$, then for $\varepsilon = 1$ and $\delta = 0$,

$$|Df(x_{11}, x_{12}, x_{21}, x_{22})|_p = |6|_p = |3|_p \le 1 = \varepsilon, \quad x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{Q}_p.$$

However,

$$|4^j f(x_1/2^j, x_2/2^j) - 4^{j+1} f(x_1/2^{j+1}, x_2)/2^{j+1}|_p = |7 \cdot 2^{2j}|_p = |7|_p$$

for all $x_1, x_2 \in \mathbb{Q}_p$ and $j \in \mathbb{N}$. Hence $\{4^j f(x_1/2^j, x_2/2^j)\}$ is not convergent in $\mathbb{Q}_p$.

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Notes on Harmonic Functions for which the second Dilatation is α - spiral

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Abstract

In this study, we consider, $f = h + \overline{g}$ harmonic functions in the unit disc D. By applying S. S. Miller and P. M. Mocanu result, we obtain a new subclass of harmonic functions , such as $S^*_{HPST}(\alpha, \beta)$ We introduce this new class as defined in the following form,

$$S^*_{HPST}(\alpha,\beta) = \Big\{ f = h(z) + \overline{g(z)} | f \in S_H, h(z) \in S^*, \\ \operatorname{Re}\left(e^{i\alpha} \frac{g'(z)}{h'(z)} \right) > \beta, |\alpha| < \frac{\pi}{2}, 0 \leq \beta < Re(be^{i\alpha}) \Big\} \tag{0.1}$$

We also use subordination principle , study on distortion theorems, some numerical examples and coefficient inequalities of this class.

1 Introduction

A planar harmonic mapping in the unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ is a complex-valued harmonic function f which maps $\mathbb{D}$ onto some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected, the mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, as usual, we call h the analytic part of f and g the co-analytic part of f. An elegant and complete account of the theory of planar harmonic mapping is given in Duren's monograph [3].

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Lewy [5] proved in 1936 that the harmonic function f is locally univalent in a simply connected domain $\mathbb{D}_1$ if and only if its Jacobien

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0$$

is different from zero in $\mathbb{D}_1$. In view of this result, locally univalent harmonic mappings in the unit disc are either sense-reversing if

$$|g'(z)| > |h'(z)|$$

in $\mathbb{D}_1$ or sense-preserving if

$$|g'(z)| < |h'(z)|$$

in $\mathbb{D}_1$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. However, since f is sense-preserving if and only if $\overline{f}$ is sense-reserving, all the results obtained in this article regarding sense-preserving harmonic mappings can be adapted to sense-reversing ones. Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc and the second-complex dilatation $w(z) = \frac{g'(z)}{h'(z)}$ has the property $|w(z)| < 1$ in $\mathbb{D}$, therefore we can take $h(z) = z + a_2z^2 + \cdots$, $g(z) = b_1z + b_2z^2 + \cdots$. Thus the class of all harmonic mappings being sense-preserving in the unit disc can be defined by

$$S_H = \Big\{ f = h(z) + \overline{g(z)} \,|\, h(z) = z + a_2z^2 + \cdots, g(z) = b_1z + b_2z^2 + \cdots, \\ f \text{ sense-preserving} \Big\}$$

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. Denote by P, the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ which are regular in $\mathbb{D}$ such that

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$ for all $z \in \mathbb{D}$.
Next, let S^* denote the family of functions $s(z) = z + c_2z^2 + c_3z^3 + \cdots$ which are regular in $\mathbb{D}$ such that

$$z\frac{s'(z)}{s(z)} = p(z) \tag{1.2}$$

for some $p(z) \in P$ for all $z \in \mathbb{D}$.
Let $s_1(z) = z + \alpha_2 z^2 + \alpha_3 z_3 + \cdots$ and $s_2(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$. Then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$, then $s_1(\mathbb{D}) \subset s_2(\mathbb{D})$.
Now, we consider the following class of harmonic mappings in the plane

$$S^*_{HPST}(\alpha, \beta) = \Big\{ f = h(z) + \overline{g(z)} | f \in S_H, h(z) \in S^*, \\ \operatorname{Re}\left(e^{i\alpha} \frac{g'(z)}{h'(z)} \right) > \beta, |\alpha| < \frac{\pi}{2}, 0 \leq \beta < Re(be^{i\alpha}) \Big\} \tag{1.3}$$

In the present paper we will investigate the class $S^*_{HPST}(\alpha, \beta)$.
We will need the following lemma and theorem in the sequel:

Theorem 1.1. *([4]) Let $h(z)$ be an element of S^*, then*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2},$$

for all $|z| = r < 1$.

$$\frac{1-r}{(1+r)^3} \leq |h'(z)| \leq \frac{1+r}{(1-r)^3}$$

These inequalities are sharp because the extremal function is $h(z) = \frac{z}{(1-z)^2}$.

Lemma 1.2. *([6]) Let $M(z)$ and $N(z)$ be regular in D with $M(0) = N(0) = 0$, and let γ be real. If $N(z)$ maps D onto a (possibly many-sheeted) domain which is starlike with respect to the origin, then*

$$Re(\frac{M'(z)}{N'(z)}) > \gamma \Rightarrow Re(\frac{M(z)}{N(z)}) > \gamma$$

2 Main Results

Theorem 2.1. *Let $g(z)$ and $h(z)$ be analytic in D with $g(0) = h(0) = 0$, and let $0 \leq \beta < Re(e^{i\alpha} b_1)$ and $|\alpha| < \frac{\pi}{2}$. If $h(z)$ maps D onto a (possibly many sheeted) domain which is starlike with respect to the origin, then*

$$Re(e^{i\alpha} \frac{g'(z)}{h'(z)}) > \beta (z \in D) \Rightarrow Re(e^{i\alpha} \frac{g(z)}{h(z)}) > \beta (z \in D)$$

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Proof. If we take $M(z) = g(z)$ and $N(z) = e^{-i\alpha}h(z)$, then $M(z)$ and $N(z)$ are analytic in D and $M(0) = N(0) = 0$.
Also,

$$Re(\frac{zN'(z)}{N(Z)}) = Re(\frac{zh'(z)}{h(z)}) > 0, (z \in D)$$

Thus, appliying Lemma 1.2, we prove the theorem. □

Example 2.2. *Let us consider the function $g(z)$ such that $e^{i\alpha}b_1 > 0$, $\beta = 0$ and $h(z) = \frac{z}{(1-z)^2}$.*
Now , we consider

$$e^{i\alpha}\frac{g'(z)}{h'(z)} = e^{i\alpha}b_1\frac{1+z}{1-z}$$

Then, we easily see that,

$$Re(e^{i\alpha}\frac{g'(z)}{h'(z)}) > 0, (z \in D).$$

For such $g(z)$, we have that

$$g'(z) = \frac{b_1(1+z)^2}{(1-z)^4}.$$

Thus, we obtain that

$$g(z) = b_1\int_0^z \frac{(1+t)^2}{(1-t)^4}dt = \frac{b_1 z(3+z^2)}{3(1-z)^3}$$

Using the above $g(z)$ and $h(z) = \frac{z}{(1-z)^2}$, we see that

$$Re(e^{i\alpha}\frac{g(z)}{h(z)}) = Re(\frac{e^{i\alpha}b_1(3+z^2)}{3(1-z)}) > 0, (z \in D)$$

Theorem 2.3. *If $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, then*

$$\left|\frac{g(z)}{h(z)} - \frac{|b_1|\, e^{i(\phi-\alpha)}(1 + r^2e^{-i2\phi}) - 2r^2\beta.e^{i\alpha}}{1-r^2}\right| \leq \frac{2r(|b_1|\cos\phi - \beta)}{1-r^2}$$

for $|z| = r < 1$, where $\phi = \alpha + \arg(b_1)$.

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Proof. For $f(z) \in S^*_{HPST}(\alpha, \beta)$, Theorem 1 gives us that

$$Re(e^{i\alpha}\frac{g(z)}{h(z)}) > \beta(z \in D)$$

Let us define

$$p(z) = e^{i\alpha}\frac{g(z)}{h(z)}$$

and

$$\phi(z) = \frac{p(z) - \beta - Imp(0)}{Rep(0) - \beta}$$

Then, we see that

$$p(0) = b_1 e^{i\alpha} = |b_1|\, e^{i\alpha}(\phi = \alpha + \arg(b_1))$$

and that $\phi(z)$ is analytic in D, $\phi(0) = 1$, and $Re\phi(z) > 0$, $(z \in D)$. Therefore, $\phi(z)$ is Caratheodory function. It follows from the above that,

$$\phi(z) \prec \frac{1+z}{1-z},$$

that is,

$$\phi(z) = \frac{1 + w(z)}{1 - w(z)}$$

where $w(z)$ is analytic in D, $w(0) = 0$ and $|w(z)| < 1$, $(z \in D)$. Therefore, using Schwarz lemma, we have that

$$|w(z)| = \left|\frac{\phi(z) - 1}{\phi(z) + 1}\right| \leq r(|z| = r < 1)$$

Note that

$$\left|\phi(z) - \frac{1 + r^2}{1 - r^2}\right| \leq \frac{2r}{1 - r^2}$$

and

$$\phi(z) = \frac{e^{i\alpha}\frac{g(z)}{h(z)} - \beta - i\,|b_1|\,\phi}{|b_1|\cos\phi - \beta}$$

Therefore, we have that

$$\left|e^{i\alpha}\frac{g(z)}{h(z)} - \frac{|b_1|\, e^{i\phi}(1 + r^2 e^{-i2\phi}) - 2r^2\beta}{1 - r^2}\right| \leq \frac{2r(|b_1|\cos\phi - \beta)}{1 - r^2}$$

Since $\phi = \alpha + \arg(b_1)$, we obtain that

$$\left| \frac{g(z)}{h(z)} - \frac{|b_1| e^{i(\phi-\alpha)}(1 + r^2 e^{-i2\phi}) - 2r^2\beta . e^{-i\alpha}}{1 - r^2} \right| \leq \frac{2r(|b_1| \cos\phi - \beta)}{1 - r^2}.$$

□

Corollary 2.4. *Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$; if $\arg(b_1) = -\alpha$, then $\phi = 0$. Therefore, we have that*

$$\left| \frac{b_1(1-r) - 2r\beta}{1+r} \right| \leq \left| \frac{g(z)}{h(z)} \right| \leq \left| \frac{b_1(1+r) - 2r\beta}{1-r} \right| \tag{2.1}$$

Proof. This is a simple consequence of Theorem 2.3. □

Corollary 2.5. *Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$; if $\beta = 0$, then we have*

$$\frac{|b_1| (1-r)}{1+r} \leq \left| \frac{g(z)}{h(z)} \right| \leq \frac{|b_1| (1+r)}{1-r} \tag{2.2}$$

Proof. This is a simple consequence of Theorem 2.3. □

Corollary 2.6. *Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, then*

$$\frac{r[|b_1| (1-r) - 2r\beta]}{(1+r^3)} \leq |g(z)| \leq \frac{r[|b_1| (1+r) - 2r\beta]}{(1-r^3)} \tag{2.3}$$

$$\frac{(1-r)[|b_1| (1-r) - 2r\beta]}{(1+r)^4} \leq |g'(z)| \leq \frac{(1+r)[|b_1| (1+r) - 2r\beta]}{(1-r)^4} \tag{2.4}$$

for all $|z| = r < 1$.

Proof. Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, then by using Theorem 1.1, we can write

$$|h(z)| \frac{|b_1| (1-r)2r\beta}{(1+r)} \leq |g(z)| \leq |h(z)| \frac{|b_1| (1+r)2r\beta}{(1-r)}$$

$$|h'(z)| \frac{|b_1| (1-r)2r\beta}{(1+r)} \leq |g'(z)| \leq |h'(z)| \frac{|b_1| (1+r)2r\beta}{(1-r)}$$

Therefore we can take the result easliy. □

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Corollary 2.7. *Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, if $\beta = 0$, then we have*

$$\frac{r\,|b_1|\,(1-r)}{(1+r)^3} \leq |g(z)| \leq \frac{r\,|b_1|\,(1+r)}{(1-r)^3}$$

$$\frac{|b_1|\,(1-r)^2}{(1+r)^4} \leq |g'(z)| \leq \frac{|b_1|\,(1+r)^2}{(1-r)^4}$$

Proof. This is a simple consequence of Corollary 2.6. □

Corollary 2.8. *Let $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, then*

$$F(|b_1|, -r, \beta) \leq J_{f(z)} \leq F(|b_1|, r, \beta)$$

where

$$F(|b_1|, r, \beta) = \frac{[1 - |b_1| - r(1 + |b_1| + 2\beta)][1 + |b_1| + r(-1 - |b_1| + 2\beta)]}{(1+r)^6}$$

Proof. Since

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 = |h'(z)|^2\,(1 - |w(z)|^2)$$

Using Corollary 2.6 and Theorem 2.3 we get the result easily. □

Theorem 2.9. *If $f(z) = h(z) + \overline{g(z)}$ is in the class $S^*_{HPST}(\alpha, \beta)$, then*

$$|b_n - b_1 a_n| \leq \frac{(n-1)(2n-1)}{3}(|b_1| \cos(\alpha + arg(b_1)) - \beta) \tag{2.5}$$

$$(n = 2, 3, 4, ...)$$

Proof. Since

$$Re(e^{i\alpha}\frac{g'(z)}{h'(z)}) > \beta, (z \in D)$$

for $f(z) \in S^*_{HPST}(\alpha, \beta)$, we see that

$$\phi(z) = \frac{p(z) - \beta - iImp(0)}{Rep(0) - \beta}$$

is the Caratheodory function, where

$$p(z) = e^{i\alpha}\frac{g'(z)}{h'(z)}$$

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and $p(0) = b_1 e^{i\alpha} = |b_1| e^{i(\alpha+\arg(b_1))}$ If we write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + ...,$$

we have that $|c_n|\}leq 2$, $(1, 2, 3, ...)$. Note that

$$p(z) = (Rep(0) - \beta)\phi(z) + \beta + iImp(0),$$

that is, that

$$e^{i\alpha} g'(z) = h'(z)[(Rep(0) - \beta)\phi(z) + \beta + iImp(0)]$$

It follows that

$$e^{i\alpha} b_1 + \sum_{n=2}^{\infty} n b_n z^{n-1}$$

$$= (1 + \sum_{n=2}^{\infty} n a_n z^{n-1})[(Rep(0) - \beta)(1 + \sum_{n=1}^{\infty} c_n z^n) + \beta + iImp(0)]$$

Considering the coefficient for z^{n-1}, we have that

$$me^{i\alpha} b_m = (Rep(b_1 e^{i\alpha}) - \beta)(c_{n-1} + 2a_2 c_{n-2} + 3a_3 c_{n-3} + ... + (n-1)a_{n-1} c_1) + b_1 e^{i\alpha} n a_n$$

This shows us that,

$$(b_n - b_1 a_n)e^{i\alpha} = \frac{1}{n}(Rep(b_1 e^{i\alpha}) - \beta)(c_{n-1} + 2a_2 c_{n-2} + 3a_3 c_{n-3} + ... + (n-1)a_{n-1} c_1). \tag{2.6}$$

Since $|a_n| \leq n$, $(n = 2, 3, 4, ...)$ for $h(z) \in S^*$ and $|c_n| \leq 2$, $(n = 1, 2, 3, ...)$, we obtain that

$$\begin{aligned} |b_n - b_1 a_n| &\leq \frac{2}{n}(|b_1| \cos(\alpha + \arg(b_1)) - \beta)(1 + 2|a_2| + 3|a_3| + ... + (n-1)|a_{n-1}|) \\ &\leq \frac{2}{n}(|b_1| \cos(\alpha + \arg(b_1)) - \beta)(1^2 + 2^2 + 3^2 + ... + (n-1)^2) \\ &= \frac{(n-1)(2n-1)}{3}(|b_1| \cos(\alpha + \arg(b_1)) - \beta) \end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.10. *If* $f(z) = h(z) + \overline{g(z)}$ *is in the class* $S^*_{HPST}(\alpha, \beta)$ *with* $h(z) \in K$, *then*

$$|b_n - b_1 a_n| \leq (n-1)(|b_1| \cos(\alpha + \arg(b_1)) - \beta), (n = 2, 3, 4, ...)$$

Proof. This is a simple consequence of Theorem 2.9. □

Theorem 2.11. *Let $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in D. Also, let $h(z)$ is starlike with respect to the origin in D, and*

$$f(z) = h(z) + \overline{g(z)}.$$

If $h(z)$ and $g(z)$ satisfy,

$$\sum_{n=2}^{\infty} n[\left|(1-\beta)a_n + e^{i\alpha}b_n\right| + \left|(1+\beta)a_n - e^{i\alpha}b_n\right|]$$

$$\leq \left|1-\beta+e^{i\alpha}b_1\right| - \left|1+\beta-e^{i\alpha}b_1\right|$$

for some real α ($|\alpha| < \frac{\pi}{2}$) and for some real β, ($0 \leq \beta \leq Re(b_1 e^{i\alpha})$, then $f(z) \in S^*_{HPST}(\alpha, \beta)$.

Proof. Let $p(z) = e^{i\alpha}\frac{g'(z)}{h'(z)}$ Then, if $p(z)$ satisfies,

$$\left|\frac{1-(p(z)-\beta)}{1+(p(z)-\beta)}\right| < 1$$

then $Rep(z) > \beta$, so that, $f(z) \in S^*_{HPST}(\alpha, \beta)$. It follows that

$$|1+(p(z)-\beta)| - |1-(p(z)-\beta)|$$

$$= \frac{1}{|h'(z)|}[\left|(1-\beta)h'(z) + e^{i\alpha}g'(z)\right| - \left|(1+\beta)h'(z) - e^{i\alpha}g'(z)\right|]$$

$$= \frac{1}{|h'(z)|}[\left|(1-\beta+e^{i\alpha}b_1) + \sum_{n=2}^{\infty} n((1-\beta)a_n + e^{i\alpha}b_n)z^{n-1}\right| - \left|(1+\beta-e^{i\alpha}b_1)\right.$$

$$\left. + \sum_{n=2}^{\infty} n((1+\beta)a_n - e^{i\alpha}b_n)z^{n-1}\right]$$

$$\geq \frac{1}{|h'(z)|}[\left|1-\beta+e^{i\alpha}b_1\right| - \sum_{n=2}^{\infty} n\left|(1-\beta)a_n + e^{i\alpha}b_n\right||z|^{n-1}$$

$$- \left|1+\beta-e^{i\alpha}b_1\right| - \sum_{n=2}^{\infty}\left|(1+\beta)a_n - e^{i\alpha}b_n\left|z\right|^{n-1}\right|]$$

$$> \frac{1}{|h'(z)|}[\left|1-\beta+e^{i\alpha}b_1\right| - \left|1+\beta-e^{i\alpha}b_1\right|]$$

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$$-\sum_{n=2}^{\infty} n\left[\left|(1-\beta)a_n + e^{i\alpha}b_n\right| + \left|(1+\beta)a_n - e^{i\alpha}b_n\right|\right]$$

for $z \in D$. Therefore, if $f(z)$ satisfies,

$$\sum_{n=2}^{\infty} n\left[\left|(1-\beta)a_n + e^{i\alpha}b_n\right| + \left|(1+\beta)a_n - e^{i\alpha}b_n\right|\right]$$

$$\leq \left|1-\beta+e^{i\alpha}b_1\right| - \left|1+\beta-e^{i\alpha}b_1\right|,$$

then $f(z) \in S^*_{HPST}(\alpha, \beta)$. $\square$

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