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On A System of Rational Difference Equations

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In this paper, we investigate behaviors of well-defined solutions of the following system

$$x_{n+1}=\frac{A_1y_{n-(3k-1)}}{B_1+C_1y_{n-(3k-1)}x_{n-(2k-1))}y_{n-(k-1)}},$$

$$y_{n+1}=\frac{A_2x_{n-(3k-1)}}{B_2+C_2x_{n-(3k-1)}y_{n-(2k-1))}x_{n-(k-1)}},$$

where $n\in\mathbb{N}_0$, $k\in\mathbb{Z}^+$ the coefficients A_1,A_2,B_1,B_2,C_1,C_2 and the initial conditions are arbitrary real numbers.
Keywords: System of difference equations, Asymptotic behavior, Periodicity, Closed form solution.
AMS Classification: 39A10

1 Introduction

There has been a great effort in studying periodic and asymptotic behaviors of solutions of difference equations (see e.g. [3,6,12,15,18,20-23,27,35,45,46]). Also, studying in system of difference equations has increased considerably (see, e.g. [5,7,8,16,17,19,28-30,32-34,37,38,40,43,47]).

Ozkan et al. [31] gave the solutions of the systems of the difference equations

$$\begin{aligned}x_{n+1}&=\frac{y_{n-2}}{-1\mp y_{n-2}x_{n-1}y_n},\\ y_{n+1}&=\frac{x_{n-2}}{-1\mp x_{n-2}y_{n-1}x_n},\\ z_{n+1}&=\frac{x_{n-2}+y_{n-2}}{-1\mp x_{n-2}y_{n-1}x_n},n\in\mathbb{N}_0.\end{aligned}\tag{1}$$

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In [39] it was showed that the system of difference equations, which is an extension of first and second equations of system (1) with respect to coefficients,

$$x_n = \frac{c_n y_{n-3}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3}},$$
$$y_n = \frac{\gamma_n x_{n-3}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3}}, n \in \mathbb{N}_0, \tag{2}$$

where the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial values $x_i, y_i, i \in \{1, 2, 3\}$ are real numbers, such that $c_n \neq 0, \gamma_n \neq 0, n \in \mathbb{N}_0$, can be solved in closed form, and for the case when all sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$ are constant it was described the asymptotic behavior of well-defined solutions of the system.

In [41] it was showed that an extension of system (2) with respect to indices

$$x_n = \frac{c_n y_{n-(2k-1)}}{a_n + b_n y_{n-(2k-1)} \prod_{i=1}^{k-1} y_{n-(2i-1)} x_{n-2i}},$$
$$y_n = \frac{\gamma_n x_{n-(2k-1)}}{\alpha_n + \beta_n x_{n-(2k-1)} \prod_{i=1}^{k-1} x_{n-(2i-1)} y_{n-2i}}, \tag{3}$$

where $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, n \in \mathbb{N}_0$, and the initial conditions $x_i, y_i, i \in \{1, 2, ...2k-1\}$ are real numbers, is solved in closed form, and the behavior of its well-defined solutions when all the sequences $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$ are constant was described. Related rational difference equations are studied, e.g. in [1,2,4,9-11,13,14,24-26,31,36,42,44,48].

In this paper we consider an other extension of system (2)

$$x_{n+1} = \frac{A_1 y_{n-(3k-1)}}{B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{A_2 x_{n-(3k-1)}}{B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{4}$$

where $n \in \mathbb{N}_0$, k is a positive integer, the initial conditions and the coefficients A_1, A_2, B_1 , B_2, C_1, C_2 are arbitrary real numbers. We will consider only well-defined solutions, that is, $B_1 + C_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)} \neq 0$ and $B_2 + C_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)} \neq 0, n = 0, 1, 2, ...$

2 Special Cases

2.1 The case $A_1 = 0$ or $A_2 = 0$

If $A_1 = 0$, we obtain directly $x_n = 0$ for $n > 0$. By using this, we get $y_n = 0$ for $n > 3k$. If $A_2 = 0$, we obtain directly $y_n = 0$ for $n > 0$. By using this, we get $x_n = 0$ for $n > 3k$. From now on both of A_1 and A_2 will be considered a non-zero real numbers.

System (4) is equivalent to the following system

$$x_{n+1} = \frac{y_{n-(3k-1)}}{b_1 + c_1 y_{n-(3k-1)} x_{n-(2k-1))} y_{n-(k-1)}},$$
$$y_{n+1} = \frac{x_{n-(3k-1)}}{b_2 + c_2 x_{n-(3k-1)} y_{n-(2k-1))} x_{n-(k-1)}}, \tag{5}$$

where $n \in \mathbb{N}_0$, $b_i = \frac{B_i}{A_i}$ and $c_i = \frac{C_i}{A_i}, i = 1, 2$. So, we will consider system (5) instead of system (4).

2.2 The case $b_1 = 0$ or $b_2 = 0$

If $b_1 = 0$, from the first equation of system (5), we have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $n > 0$. Using this, we obtain directly $y_n = \alpha x_{n-3k}$, $n \geq k$, where $\alpha = \frac{c_1}{b_2 c_1 + c_2}$. From this and by the change of variables

$$z_n = \frac{y_n}{x_{n-3k}}, w_n = \frac{y_{n-3k}}{x_n}, n \geq k, \tag{6}$$

system (5) can be transformed into the system

$$w_{n+1} = c - c b_2 z_{n-(k-1)}, z_{n+1} = \alpha, n \geq k - 1, \tag{7}$$

where $c = \frac{c_1}{c_2}$. The solutions are obtained easily as $z_n = w_n = \alpha$, $n \geq k$. This means every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

If $b_2 = 0$, we get immediately $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. From the first equation in system (5) and using this, we obtain $x_n = \beta y_{n-3k}$, $n \geq k$, where $\beta = \frac{c_2}{b_1 c_2 + c_1}$. The change of variables

$$u_n = \frac{x_n}{y_{n-3k}}, t_n = \frac{x_{n-3k}}{y_n}, n \geq k, \tag{8}$$

reduces system (5) to the system

$$t_{n+1} = \bar{c} - \bar{c} b_1 u_{n-(k-1)}, u_{n+1} = \beta, n \geq 2k - 1, \tag{9}$$

where $\bar{c} = \frac{c_2}{c_1}$. The solutions of this system $t_n = u_n = \beta$, $n \geq 2k - 1$, are obtained easily. So, every solution of system (5) is periodic with $6k$ periods, not necessarily prime period, such that $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n \geq 4k$.

Assume that $b_1 = 0$ and $b_2 = 0$. We have $x_{n-2k} y_{n-k} x_n = \frac{1}{c_1}$, $y_{n-2k} x_{n-k} y_n = \frac{1}{c_2}$, $n > 0$. Then, we get immediately $x_n = \frac{c_2}{c_1} y_{n-3k}$, $y_n = \frac{c_1}{c_2} y_{n-3k}$, $n > k$. Thus, we can write $x_n = x_{n-6k}$, $y_n = y_{n-6k}$, $n > 4k$.

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2.3 The case $c_1 = 0$ or $c_2 = 0$

If $c_1 = 0$, we have $x_n = \frac{1}{b_1} y_{n-3k}$, $n > 0$. From this and using the change of variables

$$v_n = \frac{1}{x_{n+3k} y_{n-k} x_n}, n > 0, \tag{10}$$

the second equation of system (5) implies the linear equation

$$v_{n+1} = b_1 b_2 v_{n-(2k-1)} + {b_1}^2 c_2, n = 0, 1, 2, ... \tag{11}$$

We can rewrite the equation (11) in the form of

$$v_{2kn+m} = b_1 b_2 v_{2k(n-1)+m} + b_1^2 c_2, \tag{12}$$

where $n \in \mathbb{N}_0$, $m = 1, 2, ..., k$. Considering the solution of a nonhomogeneous first order difference equation, we can give the solution of the equation (12) such that

$$v_{2kn+m} = (b_1 b_2)^n v_{m-2k} + b_1^2 c_2 \frac{1 - (b_1 b_2)^{n+1}}{1 - b_1 b_2}, n \geq 0. \tag{13}$$

when $b_1 b_2 \neq 1$. If $b_1 b_2 = 1$, the solution of the equation (12) can be written as

$$v_{2kn+m} = v_{m-2k} + (n+1) b_1^2 c_2, n \geq 0. \tag{14}$$

From (10), we have

$$x_{2kn+3k+m} = \frac{v_{2k(n-1)+m}}{v_{2kn+m}} x_{2kn-3k+m}.$$

Considering $x_n = \frac{1}{b_1} y_{n-3k}$, we obtain the solutions of system (5) as

$$x_{6kn+3k+m} = x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, y_{6kn+m} = b_1 x_{-3k+m} \prod_{r=0}^{n} \frac{v_{6kr-2k+m}}{v_{6kr+m}}, \tag{15}$$

$$x_{6kn+5k+m} = x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, y_{6kn++2k+m} = b_1 x_{-k+m} \prod_{r=0}^{n} \frac{v_{6kr+m}}{v_{6kr+2k+m}}, \tag{16}$$

$$x_{6kn+7k+m} = x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, y_{6kn+4k+m} = b_1 x_{k+m} \prod_{r=0}^{n} \frac{v_{6kr+2k+m}}{v_{6kr+4k+m}}, \tag{17}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that $c_2 = 0$. Then, we have $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$. From this and using the change of variable

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$$u_n = \frac{1}{y_{n+3k}y_{n+k}y_{n-k}}, n > 0, \tag{18}$$

the first equation of system (5) implies the linear equation

$$u_{n+1} = b_1 b_2 u_{n-(2k-1)} + b_2^2 c_1, n \geq 0. \tag{19}$$

By similar processes just as we did, we can rewrite the equation (19) as

$$u_{2kn+m} = b_1 b_2 u_{2k(n-1)+m} + b_2^2 c_1, n \geq 0, \tag{20}$$

where $m = 1, 2, ..., k$. We obtain the solution of the equation (20)

$$u_{2kn+m} = (b_1 b_2)^n u_{m-2k} + b_2^2 c_1 \frac{1-(b_1 b_2)^{n+1}}{1-b_1 b_2}, n \geq 0, \tag{21}$$

when $b_1 b_2 \neq 1$. When $b_1 b_2 = 1$, the solution of the equation (20)

$$u_{2kn+m} = u_{m-2k} + (n+1) b_2^2 c_1, n \geq 0. \tag{22}$$

From (18), we have

$$y_{n+3k} = \frac{1}{u_n y_{n+k} y_{n-k}}, n > 0,$$

and

$$y_{2kn+3k+m} = \frac{u_{2k(n-1)+m}}{u_{2kn+m}} y_{2kn-3k+m}.$$

Considering $y_n = \frac{1}{b_2} x_{n-3k}$, $n > 0$, we obtain the solutions of system (5) as

$$x_{6kn+m} = b_2 y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, y_{6kn+3k+m} = y_{-3k+m} \prod_{r=0}^{n} \frac{u_{6kr-2k+m}}{u_{6kr+m}}, \tag{23}$$

$$x_{6kn++2k+m} = b_2 y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, y_{6kn+5k+m} = y_{-k+m} \prod_{r=0}^{n} \frac{u_{6kr+m}}{u_{6kr+2k+m}}, \tag{24}$$

$$x_{6kn+4k+m} = b_2 y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, y_{6kn+7k+m} = y_{k+m} \prod_{r=0}^{n} \frac{u_{6kr+2k+m}}{u_{6kr+4k+m}}, \tag{25}$$

$n \geq 0$ and $m = 1, 2, ..., 2k$.

Suppose that both c_1 and c_2 are equal to zero. We get immediately $x_{n+1} = \frac{1}{b_1} y_{n-(3k-1)}$, $y_{n+1} = \frac{1}{b_2} x_{n-(3k-1)}$, $n \geq 0$. From this result, we obtain $x_{n+1} = \frac{1}{b_1 b_2} x_{n-(6k-1)}, y_{n+1} = \frac{1}{b_1 b_2} y_{n-(6k-1)}$, $n \geq 3k$. So, we have $x_{6kn+3k+m} = \frac{1}{b_1 b_2}$ $x_{6kn-3k+m}$, $y_{6kn+3k+m} = \frac{1}{b_1 b_2} y_{6kn-3k+m}$ and from this $x_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1}$ x_{-3k+m}, $y_{6kn+3k+m} = \left(\frac{1}{b_1 b_2}\right)^{n+1} y_{-3k+m}$, $n \geq 0$, $m = 1, 2, ..., 6k$.

3 Main Case

In this section, we will need the following results, given in the reference [16], in the proofs of our results.

Consider the first order Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, n = 0, 1, ..., \tag{26}$$

where the parameters and the initial condition x_0 are arbitrary real numbers.

Theorem 1 *The followings are true:*

1) *Eq.(26) has a prime period-2 solution if and only if $b + c = 0$.*

2) *Suppose $b + c = 0$. Then every solution $\{x_n\}$ of Eq. (26) with $x_0 \neq 0$ is periodic with period 2.*

Theorem 2 *Assume that $d \neq 0, bc - ad \neq 0$, $b + c \neq 0$ and $R = \frac{bc-ad}{(b+c)^2} < \frac{1}{4}$. Then the forbidden set F of Eq.(26) is given as follows:*

$$F = \left\{ \frac{b+c}{d} \left(\frac{\lambda_1 \lambda_2^n - \lambda_2 \lambda_1^n}{\lambda_2^n - \lambda_1^n} \right) - \frac{c}{d} : n \geq 1 \right\} .$$

For any well-defined solution $\{x_n\}$ of Eq. (26), we have

$$x_n = \frac{b+c}{d} \left(\frac{c_1 \lambda_1^{n+1} - c_2 \lambda_2^{n+1}}{c_1 \lambda_1^n - c_2 \lambda_2^n} \right) - \frac{c}{d},$$

for $n = 0, 1, ...,$ where $\lambda_1 = \frac{1-\sqrt{1-4R}}{2}$, $\lambda_2 = \frac{1+\sqrt{1-4R}}{2}, c_1 = \frac{\lambda_2(b+c)-(dx_0+c)}{(\lambda_2-\lambda_1)(b+c)}$ and $c_2 = \frac{(dx_0+c)-\lambda_1(b+c)}{(\lambda_2-\lambda_1)(b+c)}$.

Corollary 1 *Assume that the conditions in Theorem2 hold. Let $\{x_n\}$ be a well-defined solution of Eq. (26). Then*

$$lim_{n\to\infty} x_n = \frac{\lambda_2(b+c)-c}{d}.$$

Theorem 3 *Assume that $d \neq 0, bc - ad \neq 0$, $b + c \neq 0$ and $R = \frac{bc-ad}{(b+c)^2} = \frac{1}{4}$. Then the forbidden set F of Eq.(26) is given as follows:*

$$F = \left\{ \frac{n(b-c)-(b+c)}{2dn} : n \geq 1 \right\}.$$

For any well-defined solution $\{x_n\}$ of Eq. (26), we have

$$x_n = \frac{b+c}{d} \left(\frac{(b+c)+(n+1)(2dx_0+(c-b))}{2(b+c)+2n(2dx_0+(c-b))} \right) - \frac{c}{d},$$

for $n = 0, 1, ...$

Corollary 2 *Assume that the conditions in Theorem3 hold. Let $\{x_n\}$ be a well-defined solution of Eq.(26). Then*

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$$lim_{n\to\infty} x_n = \tfrac{b-c}{2d}.$$

Now we consider the system (5) with b_1, b_2, c_1, c_2 parameters and the initial conditions are non-zero real numbers. By the change of variables (6), the system (5) reduces to

$$z_{n+1} = \frac{w_{n-(k-1)}}{\frac{1}{\alpha} w_{n-(k-1)} - \gamma_2}, w_{n+1} = \frac{1}{\beta} - \gamma_1 z_{n-(k-1)}, n \geq k, \tag{27}$$

where $\gamma_1 = \frac{c_1 b_2}{c_2}, \gamma_2 = \frac{c_2 b_1}{c_1}, \alpha = \frac{c_1}{b_2 c_1 + c_2}, \beta = \frac{c_2}{b_1 c_2 + c_1}$. We can rewrite the system (27) such that

$$z_{n+1} = \frac{\frac{1}{\beta} - \gamma_1 z_{n-(2k-1)}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{n-(2k-1)}}, w_{n+1} = \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{n-(2k-1)} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{n-(2k-1)} - \gamma_2}, \tag{28}$$

$n \geq 2k$. Each of the equation in (28)is a $2k$th order Riccati difference equation. Furthermore, the equations in (28) can be rewritten such that

$$\begin{aligned} z_{2kn+1+i} &= \frac{\frac{1}{\beta} - \gamma_1 z_{2k(n-1)+1+i}}{\left(\frac{1}{\alpha\beta} - \gamma_2\right) - \frac{\gamma_1}{\alpha} z_{2k(n-1)+1+i}}, \\ w_{2kn+1+i} &= \frac{\left(\frac{1}{\alpha\beta} - \gamma_1\right) w_{2k(n-1)+1+i} - \frac{\gamma_2}{\beta}}{\frac{1}{\alpha} w_{2k(n-1)+1+i} - \gamma_2}, \end{aligned} \tag{29}$$

$n > 0$, $i = 0, 1, ..., (2k-1)$. Note that the equations in (29) are first order Riccati difference equation in variables z_{2kn+i}, w_{2kn+i}, for $i = 1, 2, ..., 2k$.

Theorem 4 *Assume that $b_1 b_2 = -1$ and $\{x_n, y_n\}$ is a well-defined solution of system (5). Then,*

$$\begin{aligned} x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}}, \\ y_{2kn+1+i} &= \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}, \end{aligned}$$

for $n \geq 4$, $i = 0, 1, ..., (2k-1)$.

***Proof* 1** *Consider system (29) and suppose that $b_1 b_2 = -1$. Then, we have*

$$\begin{aligned} \frac{1}{\alpha\beta} - \gamma_1 - \gamma_2 &= \frac{1}{\frac{c_1}{b_2 c_1 + c_2} \frac{c_2}{b_1 c_2 + c_1}} - \frac{c_1 b_2}{c_2} - \frac{c_2 b_1}{c_1} \\ &= \frac{b_1 b_2 c_1 c_2 + c_1^2 b_2 + c_2^2 b_1 + c_1 c_2 - c_1^2 b_2 - c_2^2 b_1}{c_1 c_2} \\ &= \frac{c_1 c_2 (b_1 b_2 + 1)}{c_1 c_2} \\ &= 0. \end{aligned}$$

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So, from Theorem1(2) we conclude that every solution of each equation in system (29) *is periodic with period* $4k$, *that is,*

$$z_{2kn+1+i} = z_{2k(n-2)+1+i}, w_{2kn+1+i} = w_{2k(n-2)+1+i}, \tag{30}$$

for $n \geq 2,\ i = 0, 1, ..., (2k-1)$.

From (6), we have

$x_n = \frac{z_{n-3k}}{w_n} x_{n-6k}, y_n = \frac{z_n}{w_{n-3k}} y_{n-6k}$, (31)

for $n \geq 4k$. *System* (31) *can be written such that*

$$x_{2kn+1+i} = \frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} x_{2kn+1+i-6k}, y_{2kn+1+i} = \frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} y_{2kn+1+i-6k} \tag{32}$$

for $n \geq 2,\ i = 0, 1, ..., (2k-1)$. *From* (6), (30) *and* (32), *we get*

$$\begin{aligned} x_{2kn+1+i} &= \frac{z_{2k(n-2)+1+i-3k}}{w_{2k(n-2)+1+i}} x_{2kn+1+i-6k} \\ &= \frac{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i-6k}}}{\frac{y_{2k(n-2)+1+i-3k}}{x_{2k(n-2)+1+i}}} x_{2kn+1+i-6k} \\ x_{2kn+1+i} &= \frac{x_{2k(n-2)+1+i} x_{2k(n-3)+1+i}}{x_{2k(n-5)+1+i}} \end{aligned} \tag{33}$$

and similarly

$y_{2kn+1+i} = \frac{y_{2k(n-2)+1+i} y_{2k(n-3)+1+i}}{y_{2k(n-5)+1+i}}$ (34)

for $n \geq 4,\ i = 0, 1, ..., (2k-1)$.

Theorem 5 *Assume that* $\{x_n, y_n\}$ *is a well-defined solution of system* (5). *Then the followings are true:*

i) *Assume that* $b_1 b_2 = 1$. *Then every solution converges to a periodic solution with period* $6k$.

ii) *Assume that* $b_1 b_2 \neq 1$. *Then,*

a) *If* $b_1 b_2 < -1$ *or* $b_1 b_2 > 1$, *then*

$$lim_{n\to\infty} \frac{x_n}{x_{n-6k}} = lim_{n\to\infty} \frac{y_n}{y_{n-6k}} = \frac{b_2 c_1 + c_2}{b_2 c_2 (b_1 b_2 c_1 + c_2 b_1 + c_1)}.$$

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b) *If* $-1 < b_1b_2 < 1$, *then every solution converges to a periodic solution with period* $6k$.

Proof 2

i) Consider system (29) with $\gamma_1 = \frac{c_1b_2}{c_2}, \gamma_2 = \frac{c_2b_1}{c_1}, \alpha = \frac{c_1}{b_2c_1+c_2}, \beta = \frac{c_2}{b_1c_2+c_1}$. *Suppose that* $b_1b_2 = 1$. *Then, we have*

$$\begin{aligned}\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} &= \frac{\gamma_1\gamma_2}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} \\ &= \frac{\frac{c_1b_2}{c_2}\frac{c_2b_1}{c_1}}{\left(-\frac{c_1b_2}{c_2}+\frac{1}{\frac{c_1}{b_2c_1+c_2}\frac{c_2}{b_1c_2+c_1}}-\frac{c_2b_1}{c_1}\right)^2} \\ &= \frac{b_1b_2}{(b_1b_2+1)^2} \\ &= \frac{1}{4}.\end{aligned} \tag{35}$$

Similarly, it can be seen that

$$\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} = \frac{\gamma_1\gamma_2}{(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2)^2} = \frac{1}{4}. \tag{36}$$

So, from (31), (32) and Theorem3, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} &= \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}} \\ &= \frac{\frac{-\gamma_1-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{2\left(-\frac{\gamma_1}{\alpha}\right)}}{\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)-(-\gamma_2)}{2\left(\frac{1}{\alpha}\right)}} = 1,\end{aligned}$$

$i = 0, 1, ..., (2k-1)$. *Thus, we have* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-6k}$ *and* $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n-6k}$. *So, theproof of* (i) *is finished.*

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ii)a) Assume that $b_1b_2 < -1$ *or* $b_1b_2 > 1$. *From* (35) *and* (36), *we get that*
$-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}\left(-\frac{\gamma_1}{\alpha}\right))\frac{1}{\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)^2} < \frac{1}{4}$
(36)

and
$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha})\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2} < \frac{1}{4}.$
So, from (31), (32) and Theorem2, we obtain that

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}} = \lim_{n\to\infty}\frac{z_{2kn+1+i-3k}}{w_{2kn+1+i}}$$

$$= \frac{\dfrac{\frac{1+\sqrt{1-4\frac{-\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha})}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2}}}{2}\left(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2\right)-\left(\frac{1}{\alpha\beta}-\gamma_2\right)}{\left(-\frac{\gamma_1}{\alpha}\right)}}{\dfrac{\frac{1+\sqrt{1-4\frac{\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}}}{2}\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)-(-\gamma_2)}{\frac{1}{\alpha}}}$$

$$= \frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)} \tag{37}$$

$$= \frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$$

and

$\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}} = \lim_{n\to\infty}\frac{z_{2kn+1+i}}{w_{2kn+1+i-3k}}$
$=(b_2c_1+c_2)\frac{1}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$,
$i = 0, 1, ..., (2k-1)$. *Thus, we have* $\lim_{n\to\infty}\frac{x_n}{x_{n-6k}} = \lim_{n\to\infty}\frac{y_n}{y_{n-6k}} = \frac{b_2c_1+c_2}{b_2c_2(b_1b_2c_1+c_2b_1+c_1)}$.

b) Assume that $-1 < b_1b_2 < 1$. *From (35) and (36), we get that*
(
-$\gamma_1\left(\frac{1}{\alpha\beta}-\gamma_2\right)-\frac{1}{\beta}(-\frac{\gamma_1}{\alpha}))\frac{1}{(-\gamma_1+\frac{1}{\alpha\beta}-\gamma_2)^2} < \frac{1}{4}$
and
(

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$\left(\frac{1}{\alpha\beta}-\gamma_1\right)(-\gamma_2)-\left(-\frac{\gamma_2}{\beta}\right)\frac{1}{\alpha}\right)\frac{1}{\left(\frac{1}{\alpha\beta}-\gamma_1-\gamma_2\right)^2}<\frac{1}{4}$. *So, from (31), (32), (37) and Theorem2, we obtain*

$$\lim_{n\to\infty}\frac{x_{2kn+1+i}}{x_{2kn+1+i-6k}}=\lim_{n\to\infty}\frac{y_{2kn+1+i}}{y_{2kn+1+i-6k}}$$

$$=\frac{\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)-\left(b_1b_2+1+\frac{c_1b_2}{c_2}\right)}{-\frac{c_1b_2}{c_2}\left(\frac{1+\left|\frac{b_1b_2-1}{b_1b_2+1}\right|}{2}(b_1b_2+1)+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2+\frac{c_1b_2}{c_2}}{\frac{c_1b_2}{c_2}\left(1+\frac{c_2b_1}{c_1}\right)}$$

$$=\frac{b_1b_2c_2+c_1b_2}{b_1b_2c_2+c_1b_2}=1,$$

$i=0,1,...,(2k-1)$. *So, we get immediately* $\lim_{n\to\infty}x_n=\lim_{n\to\infty}x_{n-6k}$ *and* $\lim_{n\to\infty}y_n=\lim_{n\to\infty}y_{n-6k}$ *and then the proof of is finished.*

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A Numerical Approach Based on Subdivision Schemes for Solving Non-Linear Fourth Order Boundary Value Problems

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Abstract

This paper presents an iterative collocation numerical approach based on interpolating subdivision schemes for the solution of non-linear fourth order boundary value problems involving ordinary differential equations. Numerical evidence suggests that the scheme converges to a smooth approximate solution of non-linear fourth order boundary value problem. The convergence of the approach is also discussed. Main purpose of this article is to explore and seek the applications of subdivision schemes in the field of physics and engineering.

Keywords: Boundary value problem, Subdivision schemes, Collocation algorithm, Approximation, interpolation

AMS Classification: 30E25; 65D07; 97N50

1 Introduction

Boundary value problems arise in several branches of physics and engineering. In recent years, there has been significant progress in solving problems associated with system of linear and nonlinear partial and ordinary differential equations involving boundary conditions. Two point nonlinear boundary value problems often cannot be solved by analytical methods. With increasing interest in finding solutions to nonlinear boundary value problems has come an increasing need for solution techniques.

In this paper, we consider the following type of nonlinear boundary value problem

$$y^{(iv)} = f(x, y, y') \tag{1.1}$$

with the boundary conditions

$$\begin{cases} y(a) = \alpha_1, & y'(a) = \alpha_2, \\ y(b) = \alpha_3, & y'(b) = \alpha_4, \end{cases} \tag{1.2}$$

where $\alpha_i, i = 1, 2, 3, 4$ are constants. We assume that the problem is well-posed.

A variety of methods have been introduced to solve these problem e.g., shooting methods, splines methods [2, 3, 4, 5, 6], finite difference methods, finite element methods, the collocation methods and other approximation methods. For discrete methods, like shooting and finite differences methods, only discrete approximate values of the unknown $y(x)$ can be obtained. For fitting curve to data we need further data processing techniques. For the case of spline interpolation or approximation methods the unknown function $y(x)$ is assumed to be piecewise polynomial which requires at least piecewise higher order differentiability of the function $f(x, y, y')$. To overcome these disadvantages, Qu and Agarwal

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[8, 9, 10] introduced the subdivision based algorithm for the solution of two point second order boundary value problems. Mustafa and Ejaz [14] solved third order boundary value problem by using subdivision technique. Higher order nonlinear problems have not been solved by subdivision techniques. This motivates us to solve fourth order boundary value problems by subdivision schemes based collocation iterative method. This paper introduces a numerical method based on subdivision technique for the solution of fourth order nonlinear boundary value problem.

In Section 2, some results about subdivision algorithms and basis function are given. In Section 3, a numerical method to solve (1.1) using refinable basis functions is formulated and its convergence properties studied. Error properties are given in Section 4. Numerical examples illustrating the feasibility of our proposed algorithm are given in Section 5.

2 Basis Functions and their Derivatives

Some useful results for the solution of non-linear boundary value problem are discussed in this section. Introduction to the basis functions of subdivision schemes that are used to construct the approximate solutions of proposed problem (1.1) is also part of this section.

2.1 Interpolating subdivision scheme

A mathematical formulation of binary subdivision scheme is defined as:

$$\begin{cases} P_{2i}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{-2j} P_{i+j}^{k} \\ P_{2i+1}^{k+1} = \sum\limits_{j\in\mathbb{Z}} a_{i-2j} P_{i+j}^{k} \end{cases} \tag{2.1}$$

The scheme is a stepwise interpolatory scheme if and only if the coefficient a_i satisfy $a_{2i} = \delta_i \quad \forall\ i \in \mathbb{Z}$. We consider the following binary interpolating subdivision scheme [7, 12, 13].

$$\begin{cases} p_{2i}^{k+1} = p_i^k, \\ p_{2i+1}^{k+1} = \frac{35}{65536}\left(p_{i-4}^k + p_{i+5}^k\right) - \frac{405}{65536}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{567}{16384}\left(p_{i-2}^k + p_{i+3}^k\right) \\ \qquad\qquad - \frac{2205}{16384}\left(p_{i-3}^k + p_{i+4}^k\right) + \frac{19845}{32768}\left(p_i^k + p_{i+1}^k\right). \end{cases} \tag{2.2}$$

The scheme (2.2) is C^4-continuous, having support length $(-9, 9)$ and approximation order is ten.

2.2 Basis functions

The basis functions is the limit function resulting from cardinal data, where all vertices of the polygon have value zero except for one. Let $\phi(x)$, $x \in \mathbb{R}$ be the fundamental solution of (2.2) and satisfies the two scale equations

$$\begin{aligned} \phi(x) = \phi(2x) + \frac{1}{65536}&[39690\{\phi(2x-1) + \phi(2x+1)\} - 8820\{\phi(2x-3) \\ &+\phi(2x+3)\} + 2268\{\phi(2x-5) + \phi(2x+5)\} - 405\{\phi(2x-7) \\ &+\phi(2x+7)\} + 35\{\phi(2x-9) + \phi(2x+9)\}], \quad x \in \mathbb{R} \end{aligned} \tag{2.3}$$

and

$$\phi(x) \in C^4, \quad \phi(x) = 0, \quad x \in]-8, 8[, \quad \phi(i) = \delta_0, \quad i \in \mathbb{Z}. \tag{2.4}$$

Furthermore, first derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(i)}(0) \quad = 0, & \phi^{(i)}(\pm 1) = \mp\frac{1914621952}{1159104017}, & \phi^{(i)}(\pm 2) = \pm\frac{530452796}{1159104017}, \\ \phi^{(i)}(\pm 3) = \mp\frac{1470464}{13780629}, & \phi^{(i)}(\pm 4) = \pm\frac{17297069}{1159104017}, & \phi^{(i)}(\pm 5) = \mp\frac{2772992}{5795520085}, \\ \phi^{(i)}(\pm 6) = \mp\frac{1127636}{10431936153}, & \phi^{(i)}(\pm 7) = \mp\frac{4096}{8113728119}, & \phi^{(i)}(\pm 8) = \mp\frac{5}{9272832136}. \end{cases} \tag{2.5}$$

Second derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(ii)}(0) = -\frac{2370618501415}{309077185968}, & \phi^{(ii)}(\pm1) = \frac{3265310153216}{676106344305}, & \phi^{(ii)}(\pm2) = -\frac{878265102572}{676106344305}, \\ \phi^{(ii)}(\pm3) = \frac{734063059456}{2028319032915}, & \phi^{(ii)}(\pm4) = -\frac{80883901277}{1352212688610}, & \phi^{(ii)}(\pm5) = \frac{214899200}{135221268861}, \\ \phi^{(ii)}(\pm6) = \frac{297875188}{405663806583}, & \phi^{(ii)}(\pm7) = \frac{64000}{19317324123}, & \phi^{(ii)}(\pm8) = \frac{4375}{618154371936}. \end{cases} \tag{2.6}$$

Third derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iii)}(0) = 0, & \phi^{(iii)}(\pm1) = \pm\frac{43317515008}{15295995855}, & \phi^{(iii)}(\pm2) = \mp\frac{121530512357}{61183983420}, \\ \phi^{(iii)}(\pm3) = \pm\frac{240606976}{566518365}, & \phi^{(iii)}(\pm4) = \mp\frac{5285889107}{244735933680}, & \phi^{(iii)}(\pm5) = \mp\frac{37414144}{3059199171}, \\ \phi^{(iii)}(\pm6) = \pm\frac{1090169}{453214692}, & \phi^{(iii)}(\pm7) = \mp\frac{21760}{437028453}, & \phi^{(iii)}(\pm8) = \mp\frac{2975}{13984910496}. \end{cases} \tag{2.7}$$

Fourth derivatives of $\phi(i)$ at $i \in [-8, 8]$ are

$$\begin{cases} \phi^{(iv)}(0) = \frac{33869667}{457408}, & \phi^{(iv)}(\pm1) = -\frac{5295054752}{89730585}, & \phi^{(iv)}(\pm2) = \frac{10404741119}{358922340}, \\ \phi^{(iv)}(\pm3) = -\frac{74879584}{9970065}, & \phi^{(iv)}(\pm4) = \frac{295020869}{2871378720}, & \phi^{(iv)}(\pm5) = \frac{9238624}{17946117}, \\ \phi^{(iv)}(\pm6) = -\frac{900187}{7976052}, & \phi^{(iv)}(\pm7) = \frac{71840}{17946117}, & \phi^{(iv)}(\pm8) = \frac{11225}{328157568}. \end{cases} \tag{2.8}$$

The above derivative values are found by using the left eigenvectors of the subdivision process (2.2). The detailed description about these left eigenvectors and derivatives can be found in [8, 11, 14]. The graphical representations of above mentioned derivatives are shown in Figure 1.

3 Description of Iterative Numerical Method

This section describes the method for the numerical solution of nonlinear boundary value problem (1.1). The detail of the method is given below:

3.1 The collocation method

In this subsection, the collocation method is constructed based on the interpolating subdivision scheme (2.2). Our numerical approach for nonlinear fourth order boundary value problem using collocation method based on subdivision scheme is to seek an approximate solution as

$$Z(x) = \sum_{i=-8}^{N+8} z_i \phi\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.1}$$

where N is the positive integer $N \geqslant 8$, $h = 1/N$ and $x_i = i/N = ih$ and $\{z_i\}$ are the unknowns to be determined for the solution of (1.1). In order to solve the problem, a collocation method $Z(x)$ is considered to be the solution of the above differential equation at $x = x_j$ and we substitute equation (3.1) into equation (1.1). This leads to

$$Z^{(iv)}(x_j) = f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N \tag{3.2}$$

and boundary conditions

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4 \tag{3.3}$$

From (3.1), we get

$$Z^{(iv)}(x) = \frac{1}{h^4} \sum_{i=-8}^{N+8} z_i \phi^{(iv)}\left(\frac{x - x_i}{h}\right), \quad 0 \leqslant x \leqslant 1 \tag{3.4}$$

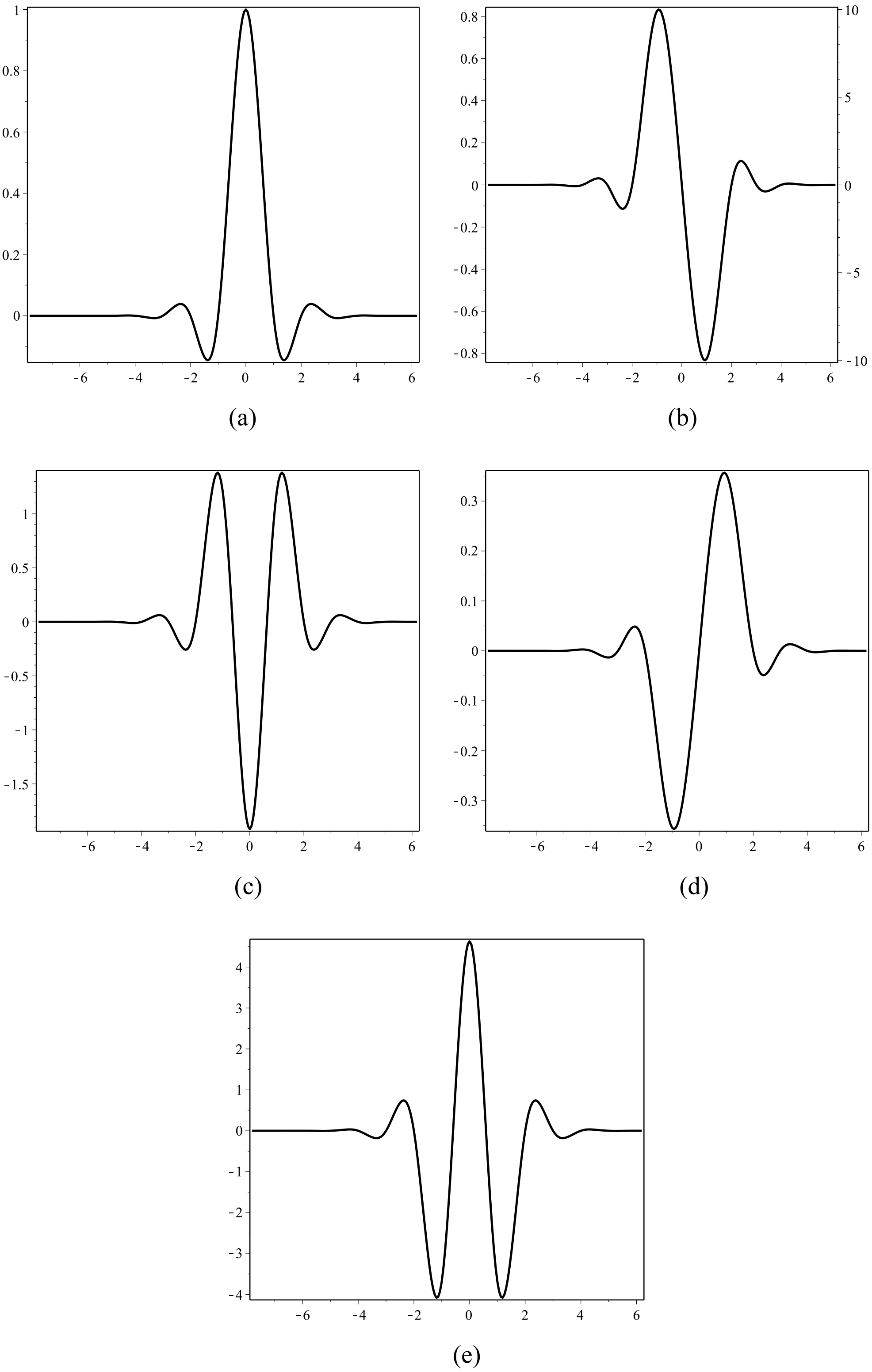

Figure 1: Graphical representation of: (a) basis functions, (b) first, (c) second, (d) third, (e) fourth derivatives of basis function

Substituting (3.4) into (3.2), we obtain

$$\sum_{i=-8}^{N+8} z_i \phi^{(iv)} \left(\frac{x_j - x_i}{h}\right) = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N.$$

This can be written as:

$$\sum_{i=-8}^{N+8} z_i \phi_{j-i}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N$$

Since $\phi_i^{(iv)} = \phi_{-i}^{(iv)}$, the above system of equations becomes

$$\sum_{i=-8}^{N+8} z_i \phi_{i-j}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)), \quad j = 0, 1, 2, \cdots, N. \tag{3.5}$$

The nonlinear system of equations (3.5) can be simply in following Theorems 1 and 2.

Theorem 1. *The nonlinear system of equations (3.5) for $j = 0$ becomes*

$$\sum_{i=-8}^{8} z_i \phi_i^{(iv)} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)). \tag{3.6}$$

Proof. By expanding (3.5) for $j = 0$, we obtain

$$\sum_{i=-8}^{N+8} z_i \phi_i^{(iv)} = h^4 f(x_0, Z(x_0), Z^{'}(x_0))$$

$$z_{-8}\phi_{-8}^{iv} + z_{-7}\phi_{-7}^{iv} + z_{-6}\phi_{-6}^{iv} + \cdots + z_7\phi_7^{iv} + z_8\phi_8^{iv} + z_9\phi_9^{iv} + \cdots + z_{N+7}\phi_{N+7}^{iv}$$
$$+ z_{N+8}\phi_{N+8}^{iv} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

Since $\phi^{iv}(i)$ exists only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation can be written as

$$z_{-8}\phi_{-8}^{iv} + z_{-7}\phi_{-7}^{iv} + z_{-6}\phi_{-6}^{iv} + \cdots + z_7\phi_7^{iv} + z_8\phi_8^{iv} = h^4 f(x_0, Z(x_0), Z^{'}(x_0)).$$

□

Theorem 2. *For $j = 1, 2, \cdots, N$, the nonlinear system of equations (3.5) becomes*

$$\sum_{i=-8+j}^{8+j} z_i \phi_{i-j}^{(iv)} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \tag{3.7}$$

Proof. By expanding (3.5), for $j = 1, 2, 3, \cdots, N$, we get

$$\begin{aligned} &z_{-8}\phi_{-8-j}^{iv} + z_{-7}\phi_{-7-j}^{iv} + z_{-6}\phi_{-6-j}^{iv} + \cdots + z_7\phi_{7-j}^{iv} + z_8\phi_{8-j}^{iv} + z_9\phi_{9-j}^{iv} + z_{10}\phi_{10-j}^{iv} \\ &+ \cdots + z_{N+6}\phi_{N+6-j}^{iv} + z_{N+7}\phi_{N+7-j}^{iv} + z_{N+8}\phi_{N+8-j}^{iv} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.8}$$

Substituting $j = 1$ in (3.8), it becomes

$$z_{-8}\phi_{-8-1}^{iv} + z_{-7}\phi_{-7-1}^{iv} + z_{-6}\phi_{-6-1}^{iv} + \cdots + z_7\phi_{7-1}^{iv} + z_8\phi_{8-1}^{iv} + z_9\phi_{9-1}^{iv} + z_{10}\phi_{10-1}^{iv}$$
$$+ \cdots + z_{N+6}\phi_{N+6-1}^{iv} + z_{N+7}\phi_{N+7-1}^{iv} + z_{N+8}\phi_{N+8-1}^{iv} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)).$$

This implies

$$z_{-8}\phi_{-9}^{iv} + z_{-7}\phi_{-8}^{iv} + z_{-6}\phi_{-7}^{iv} + \cdots + z_7\phi_6^{iv} + z_8\phi_7^{iv} + z_9\phi_8^{iv} + z_{10}\phi_9^{iv} + \cdots$$
$$+ z_{N+6}\phi_{N+5}^{iv} + z_{N+7}\phi_{N+6}^{iv} + z_{N+8}\phi_{N+7}^{iv} = h^4 f(x_1, Z(x_1), Z^{'}(x_1)).$$

Since $\phi^{iv}(i)$ is non-zero only for the interval for $i \in [-8, 8]$ and outside the interval it will be zero. Then above equation becomes

$$z_{-7}\phi^{iv}_{-8} + z_{-6}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_6 + z_8\phi^{iv}_7 + z_9\phi^{iv}_8 = h^4 f(x_1, Z(x_1), Z^{'}(x_1)). \tag{3.9}$$

For j=2, (3.8) becomes

$$\begin{aligned} &z_{-8}\phi^{iv}_{-8-2} + z_{-7}\phi^{iv}_{-7-2} + z_{-6}\phi^{iv}_{-6-2} + \cdots + z_7\phi^{iv}_{7-2} + z_8\phi^{iv}_{8-2} + z_9\phi^{iv}_{9-2} + z_{10}\phi^{iv}_{10-2} \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+6-2} + z_{N+7}\phi^{iv}_{N+7-2} + z_{N+8}\phi^{iv}_{N+8-2} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

This implies

$$\begin{aligned} &z_{-8}\phi^{iv}_{-10} + z_{-7}\phi^{iv}_{-9} + z_{-6}\phi^{iv}_{-8} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 \\ &+ \cdots + z_{N+6}\phi^{iv}_{N+4} + z_{N+7}\phi^{iv}_{N+5} + z_{N+8}\phi^{iv}_{N+6} = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \end{aligned}$$

By using the definition of ϕ^{iv}_i given in (2.8), above equation yields

$$z_{-6}\phi^{iv}_{-8} + z_{-5}\phi^{iv}_{-7} + \cdots + z_7\phi^{iv}_5 + z_8\phi^{iv}_6 + z_9\phi^{iv}_7 + z_{10}\phi^{iv}_8 = h^4 f(x_2, Z(x_2), Z^{'}(x_2)). \tag{3.10}$$

By using the similar pattern for $j = 1, 2$, we can find the expression for $j = 3, 4, \cdots N$

$$\begin{aligned} &z_{-8+j}\phi^{iv}_{-8-j} + z_{-7+j}\phi^{iv}_{-7-j} + z_{-6+j}\phi^{iv}_{-6-j} + \cdots + z_{7+j}\phi^{iv}_{7-j} + z_{8+j}\phi^{iv}_{8-j} + z_{9+j}\phi^{iv}_{9-j} \\ &+ \cdots + z_{N+6+j}\phi^{iv}_{N+6-j} + z_{N+7+j}\phi^{iv}_{N+7-j} + z_{N+8+j}\phi^{iv}_{N+8-j} = h^4 f(x_j, Z(x_j), Z^{'}(x_j)). \end{aligned} \tag{3.11}$$

□

The nonlinear system of equations (3.5) is equivalent to the following non-linear system of $N + 1$ equations with (N+17) unknowns $\{z_i\}$.

$$AZ = F(z) \tag{3.12}$$

where A is banded matrix of order $(N+1) \times (N+17)$, Z is the unknown vector of order $N+17$ and $F(z)$ is the vector of order $N + 1$ depends on z. The matrix A, vectors Z and $F(z)$ are given explicitly by

$$A = [\phi^{iv}_{pq}(q - p - 8)]_{(N+1)\times(N+17)} \tag{3.13}$$

where $p = 1, 2, 3 \cdots, N + 1$ and $q = 1, 2, 3, \cdots, N + 17$ represent the row and column respectively.

$$F(z) = \Big(h^4 f(x_0, Z(x_0), Z^{'}(x_0)), \cdots, h^4 f(x_N, Z(x_N), Z^{'}(x_N))\Big)^T \tag{3.14}$$

$$Z = (z_{-8}, z_{-7}, z_{-6}, \cdots, z_{N+6}, z_{N+7}, z_{N+8})^T \tag{3.15}$$

$$Z^{'}(x_j) = \sum_{i=-8}^{N+8} z_j \phi^{'}\left(\frac{x_j - x_i}{h}\right)$$

where $\phi'(i)$ is already defined in (2.5) with $\phi(i) = \phi_i$.

3.2 Boundary conditions at end points

For unique solution of the nonlinear system (3.5), we need sixteen more conditions. Four conditions can be attained from given boundary conditions for the nonlinear system of equations and remaining conditions are attained by using some extrapolation method. The details of the given boundary conditions and extrapolation method are given below:

3.2.1 Boundary Conditions

The given boundary conditions are

$$Z(0) = \alpha_1,\ Z^{'}(0) = \alpha_2,\ Z(N) = \alpha_3,\ Z^{'}(N) = \alpha_4$$

The approximation of derivative conditions at ends point is defined as:

$$\begin{aligned} Z'(0) = \left(\frac{N}{2520}\right)\{&-7381z_0 + 25200z_1 - 56700z_2 + 100800z_3 - 132300z_4 + 127008z_5 \\ &-88200z_6 + 43200z_7 - 14175z_8 + 28800z_9 - 252z_{10}\} + O(h^{10}) \end{aligned} \tag{3.16}$$

$$\begin{aligned} Z'(N) = \left(\frac{N}{2520}\right)\{&7381z_N - 25200z_{N-1} + 56700z_{N-2} - 100800z_{N-3} + 132300z_{N-4} \\ &-127008z_{N-5} + 88200z_{N-6} - 43200z_{N-7} + 14175z_{N-8} - 28800z_{N-9} \\ &+252z_{N-10}\} + O(h^{10}). \end{aligned} \tag{3.17}$$

3.2.2 Extrapolation Method

The remaining twelve conditions for the nonlinear systems (3.5) to obtain stable systems for the solution of (1.1) are obtained by using the following extrapolation method.
We define six conditions at left end points and six conditions at the right end points. Since subdivision scheme (2.2) reproduces nine degree (i.e. tenth order) polynomials, so we define boundary conditions of order ten for the solution of (3.5). For simplicity only left end points $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ are discussed and the values of right end points $z_{N+2}, z_{N+3}, z_{N+4}, z_{N+5}, z_{N+6}, z_{N+7}$ can be treated similarly.

The values $z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}$ can be determined by the polynomial $q(x)$ interpolating $(x_i, z_i),\ 2 \leq i \leq 7$. Precisely, we have

$$z_{-i} = q(-x_i),\ i = 2, 3, 4, 5, 6, 7$$

where

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} Z(x_{i-j}).$$

From (3.1), $Z_1(x_i) = z_i$, $i = 2, 3, 4, 5, 6, 7$ and replacing x_i by $-x_i$, we have

$$q(-x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{-i+j}.$$

Hence the following boundary conditions can be employed at the left end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{-i+j} = 0,\ i = 7, 6, 5, 4, 3, 2. \tag{3.18}$$

Similarly for the right end, we can define $z_i = q(-x_i)$, $i = N+2,\ N+3,\ N+4,\ N+5,\ N+6,\ N+7$ and

$$q(x_i) = \sum_{j=1}^{10} \binom{10}{j} (-1)^{j+1} z_{i-j}.$$

So we have the following boundary conditions at the right end

$$\sum_{j=0}^{10} \binom{10}{j} (-1)^{j} z_{i-j} = 0,\ i = N+2, N+3, N+4, N+5, N+6, N+7. \tag{3.19}$$

Finally, we obtain a new system of $(N+17)$ linear equations with $(N+17)$ unknowns $\{z_i\}$. The $N+1$ equations are obtained from (3.5), four equations from boundary conditions (3.3) and twelve from boundary conditions (3.18) and (3.19) for the numerical solution of proposed problem.
Hence the stable nonlinear system of equations is defined as:

$$BZ = R(z) \tag{3.20}$$

where the matrix B is given by

$$B = (C_0^T, A^T, C_1^T)^T \tag{3.21}$$

A is defined in (3.13), C_0, C_1 and the vector R(z) is defined as

$$C_0 = \left(\begin{array}{cccccccccccc} 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 \\ 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 & -10 \\ 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 & 45 \\ 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\ 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7381N}{2520} & \frac{25200N}{2520} & -\frac{56700N}{2520} & \frac{100800N}{2520} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right.$$

$$\left.\begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -10 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 45 & -10 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -120 & 45 & -10 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 210 & -120 & 45 & -10 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{132300N}{2520} & \frac{127008N}{2520} & -\frac{88200N}{2520} & \frac{43200N}{2520} & -\frac{14175N}{2520} & \frac{2800N}{2520} & -\frac{252N}{2520} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right) \tag{3.22}$$

The first six rows of C_0 are obtained from (3.18), second last row is obtained from (3.16) and last row is taken from given boundary conditions $Z_1(0)$ which is defined in (3.3) and

$$C_1 = \left(\begin{array}{cccccccccccccc} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & \frac{N}{10} & -\frac{10N}{9} & \frac{45N}{8} & -\frac{120N}{7} & 35N & -\frac{252N}{5} & \frac{105N}{2} & -40N & \frac{45N}{2} & -10N \\ 0 & 0 & \cdots & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 & -120 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 & 210 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 & -252 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 & 210 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 & -120 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 45 \end{array}\right.$$

$$\left.\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7381N}{2520} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 & 0 \\ 210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 & 0 \\ -252 & 210 & -120 & 45 & -10 & 1 & 0 & 0 & 0 \\ -120 & 210 & -252 & 210 & -120 & 45 & -10 & 1 & 0 \end{array}\right), \tag{3.23}$$

First row of C_1 is obtained from $Z_1(N)$ which is defined in (3.3), second row is obtained from (3.17) and the last six rows are obtained from (3.19), Z which is defined in (3.15) and R_1 is defined as

$$R(z) = (0, 0, 0, 0, 0, 0, Z'(0), Z(1), F^T(z), Z(1), Z'(1), 0, 0, 0, 0, 0, 0)^T, \tag{3.24}$$

where $F(z)$ is defined by (3.14).

3.2.3 Non-singularity of a matrix

We can check the non-singularity of coefficient matrix B defined in (3.21) by different methods. We observe that the determinant of matrix B is non-zero for $N \leq 500$. Hence the non linear system of equations have a solution for $N \leq 500$. We also check the non singularity of matrix by finding eigenvalues up to $N \leq 500$ and we observe that all the eigenvalues are non-zero. Hence by [15] we conclude that the B is non-singular. For large $N > 500$ the matrix may or may not be singular.

3.3 Iterative algorithm and its convergence

An iterative algorithm and its convergence are described in this section.

3.3.1 Iterative algorithm based on basis function

The iterative algorithm based on basis function of the subdivision scheme (2.2) are as defined in the following three steps.

First step: Initial approximation

The initial approximation is important because the numerical solution depends on the initial approximation. We define the process for finding the initial approximation as follows:
Let initial approximate solution Z^0 be the solution of the following linear system

$$BZ^0 = F^0 \tag{3.25}$$

where

$$\begin{cases} F^0 = (0,0,0,0,0,0,y'(a),y(a),f_0,f_1,f_2,\cdots,f_N,y(b),y'(b),0,0,0,0,0,0)^T, \\ f_i = h^4 f(x_i,L_i,D), \quad i = 0,1,2,\cdots N \\ L_i = y(0) + ih\left(\frac{y(b)-y(a)}{b-a}\right) \\ D = y(b) - y(a). \end{cases} \tag{3.26}$$

F^0 is the initial linear approximation of the non-linear vector $R(z)$.

Second step: Numerical solution

The numerical solutions Z^* of the nonlinear system are obtained by using the simple iterative scheme

$$BZ^{(m+1)} = R(Z^m), \quad m = 0,1,2,3,\cdots \tag{3.27}$$

Third step: Stopping condition

The above iterative processes will terminate when the following condition is satisfied

$$||z^{(m)} - z^{(m-1)}|| \leq tol \tag{3.28}$$

where tolerance is supposed value i.e. $tol = 10^{-6}$. The convergence of the above iterative algorithm is guaranteed by the following proposition.

Theorem 3. *The successive solutions $\{Z^{(m)}\}$ generated by the iterative algorithm (3.27) linearly converges to the solution Z^* of the non-linear solution of the system (3.20) provided that the M_0 and M_1 are Lipschitz constants and step size h is small.*
i.e.

$$\|B^{-1}\| \leq \left(M_0 h^4 + \frac{4994220330463}{1460471061420} M_1 h^3\right). \tag{3.29}$$

Proof. Let Z^* and $Z^{(m)}$ be the solutions of the nonlinear system (3.20). Then by definition, for small h we have

$$BZ^* = R(Z^*), \tag{3.30}$$
$$BZ^{m+1} = R(Z^m). \tag{3.31}$$

Let the error vector be defined as $e^{(k)} = Z^k - Z^*$ at kth iteration which satisfies

$$\begin{aligned} BZ^{(m+1)} - BZ^* &= R(Z^k) - R(Z^*), \\ B(Z^{(m+1)} - Z^*) &= R(Z^k) - R(Z^*), \\ Be^{(k+1)} &= R(Z^k) - R(Z^*). \end{aligned} \tag{3.32}$$

For $i = 0, 1, 2, \cdots, N$

$$D_4 e_i^{(k+1)} = (F(Z^k) - F(Z^*))_i.$$

By mean value theorem, which is stated as "If a function $f(x, y, z)$ is continuously differentiable in an open set of $\mathbb{R}^3$ containing points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the line segment connecting them, then an equation

$$f(x_2, y_2, z_2) - f(x_1, y_1, z_1) = f_x^{'}(r, s, t)(x_2 - x_1) + f_y^{'}(r, s, t)(y_2 - y_1) + f_z^{'}(r, s, t)(z_2 - z_1)$$

is valid for the interior point (a, b, c) of the segment.", we have

$$D_4 e_i^{(k+1)} = f(x_i, Z_i^{(k)}, Z'^{(k)}) - f(x_i, Z_i^{(*)}, Z'^{(*)}).$$

The above equation can be written as (by using mean value theorem)

$$D_4 e_i^{(k+1)} = f_x^*(x_i - x_i) + f_y^*(Z_i^{(k)} - Z_i^{(*)}) + f_{y'}^*(Z'^{(k)} - Z'^{(*)})$$

by using the definition of error vector, we have

$$\begin{aligned} D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* e'^{(k)}, \\ D_4 e_i^{(k+1)} &= f_y^* e^{(k)} + f_{y'}^* D_1 e^{(k)} \end{aligned}$$

where D_4 and D_1 are the derivative difference operators defined as

$$\begin{aligned} D_1 f_i = \frac{1}{2920942122840h} &[1575(f_{i-8} - f_{i+8}) + 1474560(f_{i-7} - f_{i+7}) \\ &+315738080(f_{i-6} - f_{i+6}) + 1397587968(f_{i-5} - f_{i+5}) \\ &-43588613880(f_{i-4} - f_{i+4}) + 311679549440(f_{i-3} - f_{i+3}) \\ &-1336741045920(f_{i-2} - f_{i+2}) + 4824847319040(f_{i-1} - f_{i+1})] \end{aligned}$$

$$\begin{aligned} D_4 f_i = \frac{1}{183768238080h^4} &[392875(f_{i+8} - f_{i-8}) + 45977600(f_{i+7} - f_{i-7}) \\ &-1296269280(f_{i+6} - f_{i-6}) + 5912719360(f_{i+5} - f_{i-5}) \\ &+1180083476(f_{i+4} - f_{i-4}) - 86261280768(f_{i+3} - f_{i-3}) \\ &+332951715808(f_{i+2} - f_{i-2}) - 677767008256(f_{i+1} - f_{i-1}) \\ &+850467338370 f_i]. \end{aligned}$$

This implies

$$D_4 e_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}.$$

Since $e_i = e_{N-i} = 0, \quad i = 0, -1, -2, \cdots, -8$, we have

$$Be_i^{(k+1)} = h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}$$

This can be written as

$$e_i^{(k+1)} = B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)}).$$

By taking norm on both sides, we get

$$\|e_i^{(k+1)}\| = \|B^{-1}(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

This implies

$$\|e_i^{(k+1)}\| = \|B^{-1}\|\|(h^4 f_y^* e^{(k)} + h^3 f_{y'}^* D_1 e^{(k)})\|.$$

By using the definition of Lipschitz condition, we get

$$\|e^{(k+1)}\| \leq h^4 M_0 (b-a)\|B^{-1}\|\|e^k\| + h^3 M_1 \|D_1\|\|e^{(k)}\|.$$

This implies

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \leq \|B^{-1}\| \left(h^4 M_0 (b-a) + h^3 M_1 \|D_1\|\right),$$

which is equivalent to

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h^3 M_1 \|B^{-1}\|\|D_1\| \leq h M_1 \|B^{-1}\|\|D_1\|,$$

i-e

$$\frac{\|e_i^{(k+1)}\|}{\|e^{(k)}\|} \approx h M_1 \|B^{-1}\|\|D_1\|.$$

The results follows immediately from this inequality and the following fact

$$\|D_1\| = \frac{4994220330463}{1460471061420}. \tag{3.33}$$

A simple approximation of condition by omitting the quatric term is

$$h \leq \frac{1460471061420}{4994220330463} M_1^{-1} \left\|B^{-1}\right\|^{-1}. \tag{3.34}$$

This complete the proof. □

4 Error Estimation

From the approximation properties of the basis function $\phi(x)$, it is shown that the collocation method (3.1) with nonic precision treatments at the end points has at least power of approximation $O(h^3)$. Here we present our main results for error estimation. Proof of these results are similar to the proof of Proposition [14, 8].

Theorem 4. *Suppose the exact solution $y(x) \in C^4[0,1]$ and $\{z_i\}$ are obtained by (3.20) then absolute error by interpolating collocation algorithm is*

$$||err(x)||_\infty = ||Z^{(l)}(x) - y^{(l)}(x)||_\infty = O(h^{3-l}),\ l = 0, 1, 2, 3.$$

where l denotes the order of derivative.

Proof. Since the order of approximation of subdivision scheme (2.2) is ten so by direct calculation (fourth left eigenvector), we can find derivative of smooth function $y(x)$ as

$$y^{iv}(x_j) = \frac{2^4}{183768238080h^4}\{392875y(x_j-8h)+45977600y(x_j-7h)$$

$$-1296269280y(x_j-6h)+5912719360y(x_j-5h)+1180083476y(x_j-4h)$$

$$-86261280786y(x_j-3h)+332951715808y(x_j-2h)-677767008256y(x_j-h)$$

$$+850467338370y(x_j)-677767008256y(x_j+h)+332951715808y(x_j+2h)$$

$$-86261280786y(x_j+3h)+1180083476y(x_j+4h)+5912719360y(x_j+5h)$$

$$-1296269280y(x_j+6h)+45977600y(x_j+7h)+392875y(x_j+8h)\}+O(h^{10}).$$

This can be written as

$$y_j^{iv} = \frac{2^4}{183768238080h^4}\{392875y_{j-8}+45977600y_{j-7}-1296269280y_{j-6}$$

$$+5912719360y_{j-5}+1180083476y_{j-4}-86261280786y_{j-3}+332951715808y_{j-2}$$

$$-677767008256y_{j-1}+850467338370y_j-677767008256y_{j+1}+332951715808y_{j+2}$$

$$-86261280786y_{j+3}+1180083476y_{j+4}+5912719360y_{j+5}-1296269280y_{j+6}$$

$$+45977600y_{j+7}+392875y_{j+8}\}+O(h^{10}). \tag{4.1}$$

Similarly, we have

$$Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875z_{j-8}+45977600z_{j-7}-1296269280z_{j-6}$$

$$+5912719360z_{j-5}+1180083476z_{j-4}-86261280786z_{j-3}+332951715808z_{j-2}$$

$$-677767008256z_{j-1}+850467338370z_j-677767008256z_{j+1}+332951715808z_{j+2}$$

$$-86261280786z_{j+3}+1180083476z_{j+4}+5912719360z_{j+5}-1296269280z_{j+6}$$

$$+45977600z_{j+7}+392875z_{j+8}\}+O(h^{10}). \tag{4.2}$$

If we define error function $e(x) = Z(x) - y(x)$ and error vectors at the nodes by

$$e(x_j) = Z(x_j) - y(x_j + jh),\ -8 \le j \le N+8,$$

or equivalently $e_j = Z_j - y_j,\ -8 \le j \le N+8$, This implies

$$\begin{cases} e_j' = Z_j' - y_j', \\ e_j'' = Z_j'' - y_j'', \\ e_j''' = Z_j''' - y_j''' \\ e_j^{iv} = Z_j^{iv} - y_j^{iv}. \end{cases} \tag{4.3}$$

By subtracting (4.2) from (4.1), we get

$$y_j^{iv} - Z_j^{iv} = \frac{2^4}{183768238080h^4}\{392875(y_{j-8}-z_{j-8}) + 45977600(y_{j-7}-z_{j-7})$$

$$-1296269280(y_{j-6}-z_{j-6}) + 5912719360(y_{j-5}-z_{j-5}) + 1180083476(y_{j-4}-z_{j-4})$$

$$-86261280786(y_{j-3}-z_{j-3}) + 332951715808(y_{j-2}-z_{j-2}) - 677767008256(y_{j-1}-z_{j-1})$$

$$+850467338370(y_j-z_j) - 677767008256(y_{j+1}-z_{j+1}) + 332951715808(y_{j+2}-z_{j+2})$$

$$-86261280786(y_{j+3}-z_{j+3}) + 1180083476(y_{j+4}-z_{j+4}) + 5912719360(y_{j+5}-z_{j+5})$$

$$-1296269280(y_{j+6}-z_{j+6}) + 45977600(y_{j+7}-z_{j+7}) + 392875(y_{j+8}-z_{j+8})\} + O(h^{10}).$$

This implies

$$e_j^{iv} = \frac{2^4}{183768238080h^4}\{392875e_{j-8} + 45977600e_{j-7} - 1296269280e_{j-6}$$

$$+5912719360e_{j-5} + 1180083476e_{j-4} - 86261280786e_{j-3} + 332951715808e_{j-2}$$

$$-677767008256e_{j-1} + 850467338370e_j - 677767008256e_{j+1} + 332951715808e_{j+2}$$

$$-86261280786e_{j+3} + 1180083476e_{j+4} + 5912719360e_{j+5} - 1296269280e_{j+6}$$

$$+45977600e_{j+7} + 392875e_{j+8}\} + O(h^{10}). \tag{4.4}$$

From (1.1), (3.1), (4.3) and by assuming the tenth order boundary treatments at the end points, we have

$$e_j^{iv} = a_je_j + b_je_j^{'}, \quad 0 \le i \le N \tag{4.5}$$

and

$$e_j = \begin{cases} \max\limits_{0\le k\le 7}\{|e_k|\}O(h^{10}), & -8 \le i \le 0 \\ \max\limits_{N-3\le k\le N}\{|e_k|\}O(h^{10}), & N \le i \le N+8 \end{cases} \tag{4.6}$$

where $j = 0, 1, \cdots N$

$$a_j = f_y(t_j, y_j^*, y_j^{'*}), \quad b_j = f_{y'}(t_j, y_j^*, y_j^{'*}),$$

and

$$y_j^* = y_j + \theta_je_j, \quad y_j^{'*} = y_j' + \theta_je_j^{'}, \quad 0 \le \theta_j \le 1.$$

Using the results (4.4) and

$$\begin{aligned}&[1575(z_{i-8}-z_{i+8}) + 1474560(z_{i-7}-z_{i+7}) + 315738080(z_{i-6}-z_{i+6}) + 1397587968\\&(z_{i-5}-z_{i+5}) - 43588613880(z_{i-4}-z_{i+4}) + 311679549440(z_{i-3}-z_{i+3}) - 1336741045920\\&(z_{i-2}-z_{i+2}) + 4824847319040(z_{i-1}-z_{i+1})] = 2920942122840hZ^{'} + O(h^{10}),\end{aligned} \tag{4.7}$$

It can be conclude that relation (4.5)and (4.6) is equivalent to

$$(B + O(h^8) - O(h^4) - D_1O(h^3))E = O(h^{10})\|E\|,$$

where $E = (e_{-8}, e_{-7}, \cdots, e_7, e_8)$.

Hence for small h, the coefficient matrix $B + O(h)$, will be invertible, thus using the standard result from algebra and effect of $\|B^{-1}\|$, we have the following estimate

$$\|E\| \le \frac{\|B^{-1}\|}{1-O(h)}O(h^{10}) = O(h^3). \tag{4.8}$$

□

5 Results and Discussions

In this section, we test the proposed method on some nonlinear problems. Numerical results for each of the problems are presented in the tables. These values are very close to the true solutions and the values of the errors are also given in the table.

Example 1. *Consider the following non-linear boundary value problem [1]*

$$y^{iv} - 6\exp(-4y) = -12(1+x)^{-4}, \tag{5.1}$$

with boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = ln(2) = y'(1) = 0.5.$$

The exact solution of the problem (5.1) is $y = \ln(1+x)$. *Using the collocation method described in Section 3 for* $N = 10$, $h = 10^{-1}$ *and* $tol = 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 1. The maximum absolute error obtained by the proposed method is* 1.78×10^{-3}. *The graphical comparison between exact and approximate solutions is shown in Figure 2.*

Table 1: Numerical results of Example 1

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.0953101798	0.0950147533	0.0002954265
0.2	0.1823215568	0.1814496227	0.0008719341
0.3	0.2623642645	0.2609546573	0.0014096072
0.4	0.3364722366	0.3347370220	0.0017352146
0.5	0.4054651081	0.4036840381	0.0017810699
0.6	0.4700036292	0.4684459279	0.0015577013
0.7	0.5306282511	0.5294932609	0.0011349902
0.8	0.5877866649	0.5871580370	0.0006286279
0.9	0.6418538862	0.6416636708	0.0001902154
1.0	0.6931471806	0.6931471806	0

Example 2. *Consider the non-linear boundary value problem [1]*

$$y^{(iv)} = y^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \tag{5.2}$$

subject to the boundary conditions

$$y(0) = y'(0) = 0, y(1) = y'(1) = 1.$$

Using the collocation method described in Section 3 for $N = 10$, $h = 10^{-1}$ *and* $tol = 10^{-6}$ *with tenth order boundary treatment at end points. The numerical results are obtained after third iteration with the condition (3.28). The obtained numerical results for this problem are presented in Table 2. The maximum absolute error obtained by the proposed method is* 1.73×10^{-2}. *The graphical comparison between exact and approximate solutions is shown in Figure 3.*

6 Conclusion

This study has presented a numerical approach based on subdivision collocation algorithm for solving the numerical solution of nonlinear fourth order boundary value problems. The proposed iterative method

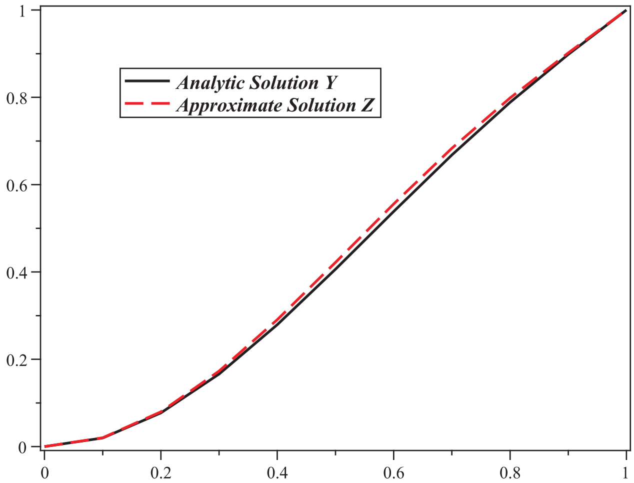

Figure 2: *Comparison of the analytic and approximate solution of Example 1.*

Table 2: Numerical results of Example 2

x_i	Analytic solution Y_i	Approximate solution Z_i	Error $= \|\|Y_i - Z_i\|\|_\infty$
0.0	0	0	0
0.1	0.01981	0.0202195	0.0004095
0.2	0.07712	0.0796952	0.0025752
0.3	0.16623	0.1728732	0.0066432
0.4	0.27904	0.2905995	0.0115595
0.5	0.40625	0.4219208	0.0156708
0.6	0.53856	0.5558846	0.0173246
0.7	0.66787	0.6833406	0.0154706
0.8	0.78848	0.7987412	0.0102612
0.9	0.89829	0.9019417	0.0036517
1.0	1.00000	1.0000000	0

has been applied on different nonlinear fourth order boundary value problems. Numerical results show that the accuracy of the approximate solution is $O(h^3)$. We have also observed that the accuracy of the solution can be improved by choosing different subdivision schemes with the proper adjustment of boundary conditions.

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Competing interests

The authors declare that they have no competing interests.

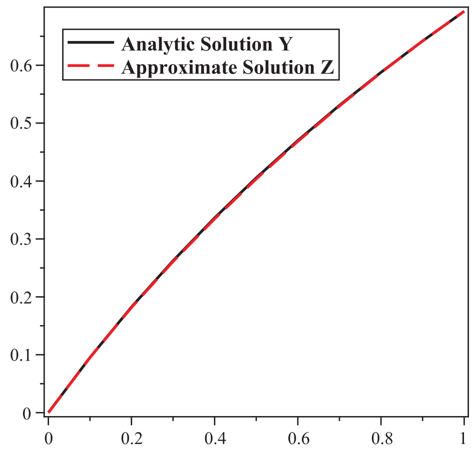

Figure 3: *Comparison of the analytic and approximate solution of Example 2.*

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Stability of Quintic Functional Equations in Random Normed Spaces

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Abstract. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y)=20[f(x+y)+f(x-y)]+90f(x)$$

in random normed spaces under the minimum t-norm.

1. **Introduction**

A classical question in stability of functional equations is as follows:

Under what conditions, is it true that a mapping which approximately satisfies a functional equation (ξ) *must be somehow close to an exact solution of* (ξ)?

We say the functional equation (ξ) is *stable* if any approximate solution of (ξ) is near to a true solution of (ξ).

The study of stability problem for functional equations is related to a question of Ulam [15] concerning the stability of group homomorphisms. The famous Ulam stability problem was partially solved by Hyers [9] for linear functional equation of Banach spaces. Subsequently, the result of Hyers theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. Cădariu and Radu [3] applied the *fixed point method* to investigation of the Jensen functional equation. They could present a short and a simple proof (different from the *direct method* initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation and for quadratic functional equation. Their methods are a powerful tool for studying the stability of several functional equations.

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[0]**Keywords**: Generalized Hyers-Ulam stability, quintic functional equation, random normed spaces, fixed point theorem.

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On the other hand, the theory of *random normed spaces* (briefly, RN-spaces) is important as a generalization of deterministic result of normed spaces and also in the study of random operator equations. The notion of an RN-space corresponds to the situations when we do not know exactly the norm of the point and we know only probabilities of passible values of this norm. The RN-spaces may provide us the appropriate tools to study the geometry of nuclear physics and have usefully application in quantum particle physics. A number of papers and research monographs have been published on generalizations of the stability of different functional equations in RN-spaces [5, 6, 10, 11, 16].

In the sequel, we use the definitions and notations of a random normed space as in [1, 13, 14].

A function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ is called a *distribution function* if it is nondecreasing and left-continuous, with $F(0) = 0$ and $F(+\infty) = 1$. The class of all probability distribution functions F with $F(0) = 0$ is denoted by Λ. D^+ is a subset of Λ consisting of all functions $F \in \Lambda$ for which $F(+\infty) = 1$, where $l^- F(x) = \lim_{t\to x^-} F(t)$. For any $a \geq 0$, ϵ_a is the element of D^+, which is defined by

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a. \end{cases}$$

Definition 1.1. ([13]) A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:
(1) T is commutative and associative;
(2) T is continuous;
(3) $T(a, 1) = a$ for all $a \in [0, 1]$;
(4) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous t-norms are as follows:

$$T_M(a, b) = \min\{a, b\}, \quad T_P(a, b) = ab, \quad T_L(a, b) = \max\{a + b - 1, 0\}.$$

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n) = T(x_1, \cdots, x_n)$ for each $n \geq 2$ and $T_{i=n}^\infty x_n$ is defined as $T_{i=1}^\infty x_{n+i}$ ([8]).

Definition 1.2. ([14]) Let X be a real linear space, μ be a mapping from X into D^+ (for any $x \in X$, $\mu(x)$ is denoted by μ_x) and T be a continuous t-norm. The triple (X, μ, T) is called a random normed space (briefly RN-space) if μ satisfies the following conditions:
(RN1) $\mu_x(t) = \epsilon_o(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X, \alpha \neq 0$ and all $t \geq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Example 1.1. Every normed space $(X, \|\cdot\|)$ defines a RN-space (X, μ, T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$ and T_M is the minimum t-norm. This space is called the *induced random normed space.*

Definition 1.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x}(t) > 1 - \lambda$$

whenever $n \geq N$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} \mu_{x_n - x} = 1$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for all $t > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x_m}(t) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) The RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. ([13]) *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence of X such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Cho et. al. [4] was introduced and proved the Hyers-Ulam-Rassias stability of the following quintic functional equations

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x) \tag{1.1}$$

for fixed $k \in \mathbb{Z}^+$ with $k \geq 3$ in quasi-β-normed spaces.

Remark 1.1. (1) If we put $x = y = 0$ in the equation (1.1), then $f(0) = 0$.

(2) $f(2^n x) = 2^{5n} f(x)$ for all $x \in X$ and $n \in \mathbb{Z}^+$.

(3) f is an odd mapping.

Throughout this paper, let X be a real linear space, (Z, μ', T_M) be an RN-space and (Y, μ, T_M) be a complete RN-space. For any mapping $f : X \to Y$, we define

$$\begin{aligned} &Df(x, y) \\ &= 2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) - 20[f(x+y) + f(x-y)] - 90f(x) \end{aligned}$$

for all $x, y \in X$. In this paper, using the direct and fixed point methods, we investigate the generalized Hyers-Ulam stability of the quintic functional equation:

$$2f(2x+y) + 2f(2x-y) + f(x+2y) + f(x-2y) = 20[f(x+y) + f(x-y)] + 90f(x)$$

in random normed spaces under the minimum t-norm.

2. Random stability of the functional equation (1.1)

In this section, we investigate the generalized Hyers-Ulam stability problem of the quintic functional equation (1.1) in RN-spaces in the sense of Scherstnev under the minimum t-norm T_M.

Theorem 2.1. *Let $\phi : X^2 \to Z$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(2x,2y)}(t) \geq \mu'_{\alpha\phi(x,y)}(t) \tag{2.1}$$

and $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ *for all* $x, y \in X$ *and* $t > 0$. *If* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *such that*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.2}$$

for all $x, y \in X$ *and* $t > 0$, *then there exists a unique quintic mapping* $Q : X \to Y$ *such that*

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\Big(2^2(2^5-\alpha)t\Big) \tag{2.3}$$

for all $x \in X$ *and* $t > 0$.

Proof. Letting $y = 0$ in (2.2), we get

$$\mu_{\frac{f(2x)}{2^5}-f(x)}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.4}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^n x$ in (2.4), we get

$$\mu_{\frac{f(2^{n+1}x)}{2^{5(n+1)}}-\frac{f(2^n x)}{2^{5n}}}(t) \geq \mu'_{\phi(x,0)}\Big(\Big(\frac{2^5}{\alpha}\Big)^n 128t\Big)$$

for all $x \in X$ and $t > 0$. Since $\frac{f(2^n x)}{2^{5n}} - f(x) = \sum_{j=0}^{n-1}\Big(\frac{f(2^{j+1}x)}{2^{5(j+1)}} - \frac{f(2^j x)}{2^{5j}}\Big)$,

$$\mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}\Big(\sum_{j=0}^{n-1}\frac{1}{128}\Big(\frac{\alpha}{2^5}\Big)^j t\Big) \geq T_M{}_{j=0}^{n-1}(\mu'_{\phi(x,0)}(t)) = \mu'_{\phi(x,0)}(t) \tag{2.5}$$

for all $x \in X$ and $t > 0$. Substituting x by $2^m x$ in (2.5), we get

$$\mu_{\frac{f(2^{n+m}x)}{2^{5(n+m)}}-\frac{f(2^m x)}{2^{5m}}}(t) \geq \mu'_{\phi(x,0)}\Big(\frac{t}{\sum_{j=m}^{n+m-1}(\frac{\alpha}{2^5})^j}\Big) \tag{2.6}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Since $\alpha < k^3$, the sequence $\{\frac{f(2^n x)}{2^{5n}}\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.6). Then we get

$$\mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\right),$$

and so, for any $\delta > 0$,

$$\begin{aligned}
&\mu_{Q(x)-f(x)}(\delta + t)\\
&\geq T_M\Big(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu_{\frac{f(2^n x)}{2^{5n}}-f(x)}(t)\Big)\\
&\geq T_M\left(\mu_{Q(x)-\frac{f(2^n x)}{2^{5n}}}(\delta), \mu'_{\phi(x,0)}\Big(\frac{128t}{\sum_{j=0}^{n-1}(\frac{\alpha}{2^5})^j}\Big)\right)
\end{aligned} \tag{2.7}$$

for all $x \in X$ and $t > 0$. Taking the limit as $n \to \infty$ in (2.7), we get

$$\mu_{Q(x)-f(x)}(\delta + t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right) \tag{2.8}$$

Since δ is arbitrary, by taking $\delta \to 0$ in (2.8), we have

$$\mu_{Q(x)-f(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right) \tag{2.9}$$

for all $x \in X$ and $t > 0$. Therefore, we conclude that the condition (2.3) holds.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.2), respectively, we have

$$\mu_{\frac{Df(2^n x, 2^n y)}{2^{5n}}}(t) \geq \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t)$$

for all $x, y \in X$ and $t > 0$. It follows from $\lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = 1$ that Q satisfies the equation (1.1), which implies that Q is a quintic mapping.

To prove the uniqueness of the quintic mapping Q, let us assume that there exists another mapping $\widetilde{Q} : X \to Y$ which satisfies (2.3). Fix $x \in X$. Then $Q(2^n x) = 2^{5n}Q(x)$ and $\widetilde{Q}(2^n x) = 2^{5n}\widetilde{Q}(x)$ for all $n \in \mathbb{Z}^+$. Thus it follows from (2.3) that

$$\begin{aligned} &\mu_{Q(x)-\widetilde{Q}(x)}(t) \\ &= \mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}(t) \\ &\geq T_M\left(\mu_{\frac{Q(2^n x)}{2^{5n}} - \frac{f(2^n x)}{2^{5n}}}\left(\frac{t}{2}\right), \mu_{\frac{f(2^n x)}{2^{5n}} - \frac{\widetilde{Q}(2^n x)}{2^{5n}}}\left(\frac{t}{2}\right)\right) \\ &\geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)\left(\frac{2^5}{\alpha}\right)^n t\right). \end{aligned} \tag{2.10}$$

Since $\lim_{n\to\infty}\left(2^2(2^5-\alpha)\left(\frac{2^5}{\alpha}\right)^n t\right) = \infty$, we have $\mu_{Q(x)-\widetilde{Q}(x)}(t) = 1$ for all $t > 0$. Thus the quintic mapping Q is unique. This completes the proof. □

Theorem 2.2. *Let $\phi : X^2 \to Z$ be a function such that, for some $2^5 < \alpha$,*

$$\mu'_{\phi(\frac{x}{2},\frac{y}{2})}(t) \geq \mu'_{\phi(x,y)}(\alpha t) \tag{2.11}$$

and $\lim_{n\to\infty} \mu'_{2^{5n}\phi(\frac{x}{2^n},\frac{y}{2^n})}(t) = 1$ for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.2), then there exists a unique cubic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.12}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.2) that

$$\mu_{f(x)-2^5 f(\frac{x}{2})}(t) \geq \mu'_{\phi(x,0)}\left(2^2\alpha t\right) \tag{2.13}$$

for all $x \in X$. Applying the triangle inequality and (2.13), we have

$$\mu_{f(x)-2^{5n} f(\frac{x}{2^n})}(t) \geq \mu'_{\phi(x,0)}\left(\frac{2^2\alpha t}{\sum_{j=m}^{n+m-1}\left(\frac{2^5}{\alpha}\right)^j}\right) \tag{2.14}$$

for all $x \in X$ and $m, n \in \mathbb{Z}$ with $n > m \geq 0$. Then the sequence $\{2^{5n}f(\frac{x}{2^n})\}$ is a Cauchy sequence in the complete RN-space (Y, μ, T_M) and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} 2^{5n} f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Then the mapping Q satisfies (1.1) and (2.12). The remaining assertion follows the similar proof method in Theorem 2.1. This complete the proof. □

Corollary 2.3. *Let θ be a nonnegative real number and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{\theta z_0}(t) \tag{2.15}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $C : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\theta z_0}(124t) \tag{2.16}$$

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for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = \theta z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof. □

Corollary 2.4. *Let $p, q \in \mathbb{R}$ be positive real numbers with $p, q < 5$ and z_0 be a fixed unit point of Z. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies*

$$\mu_{Df(x,y)}(t) \geq \mu'_{(\|x\|^p+\|y\|^q)z_0}(t) \tag{2.17}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\|x\|^p z_0}\Big(2^2(2^5 - 2^p)t\Big) \tag{2.18}$$

for all $x \in X$ and $t > 0$.

Proof. Let $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the proof follows from Theorem 2.1 by $\alpha = 2^p$. This completes the proof. □

Now, we give an example to illustrate that the quintic functional equation (1.1) is not stable for $r = 5$ in Corollary 2.4

Example 2.1. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^5, & \text{for} \quad |x| < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^{5n}}$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$\begin{aligned} &|2f(2x + y) + 2f(2x - y) + f(x + 2y) + f(x - 2y) - 20[f(x + y) + f(x - y)] - 90f(x)| \\ &\leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big) \end{aligned} \tag{2.19}$$

for all $x, y \in X$, but there do not exist a quintic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that

$$|f(x) - Q(x)| \leq d|x|^5$$

for all $x \in \mathbb{R}$. In fact, it is clear that f is bounded by $\frac{32}{31}$ on $\mathbb{R}$. If $|x|^5 + |y|^5 = 0$, then (2.19) is trivial. If $|x|^5 + |y|^5 \geq \frac{1}{32}$, then

$$|Df(x, y)| \leq \frac{136 \cdot 32}{31} \leq \frac{136 \cdot 32^2}{31}\Big(|x|^5 + |y|^5\Big).$$

Now, suppose that $0 < |x|^5 + |y|^5 < \frac{1}{32}$. Then there exists a positive integer $k \in Z^+$ such that

$$\frac{1}{32^{k+2}} \leq |x|^5 + |y|^5 < \frac{1}{32^{k+1}}$$

and so

$$32^k|x|^5 < \frac{1}{32}, \quad 32^k|y|^5 < \frac{1}{32},$$

$$2^n(2x + y),\ 2^n(2x - y),\ 2^n(x + 2y),\ 2^n(x - 2y),\ 2^n(x - y),\ 2^n x \in (-1, 1)$$

and

$$\begin{aligned}&\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\quad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)\\&=0\end{aligned}$$

for all $n=0,1,\cdots,k-1$. Thus we obtain

$$\begin{aligned}&|Df(x,y)|\\&\le\sum_{n=0}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\sum_{n=k}^{\infty}\frac{1}{2^{5n}}|\phi(2^n(2x+y))+2\phi(2^n(2x-y))+\phi(2^n(x+2y))\\&\qquad+\phi(2^n(x-2y))-20[\phi(2^n(x+y))+\phi(2^n(x-y))]-90\phi(2^nx)|\\&\le\frac{136\cdot 32^2}{31}\Big(|x|^5+|y|^5\Big).\end{aligned}$$

Therefore, f satisfies (2.19).

Now, we claim that the quintic functional equation (1.1) is not stable for $r=5$ in Corollary 2.4. Suppose on the contrary that there exists a quintic mapping $Q:\mathbb{R}\to\mathbb{R}$ and constant $d>0$ such that

$$|f(x)-Q(x)|\le d|x|^5$$

for all $x\in\mathbb{R}$. Since f is bounded and continuous for all $x\in\mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 2.1, Q must have $Q(x)=cx^5$ for all $x\in\mathbb{R}$. So, we obtain

$$|f(x)|\le(d+|c|)|x|^5 \tag{2.20}$$

for all $x\in\mathbb{R}$. Let $m\in\mathbb{Z}^+$ such that $m+1>d+|c|$.

If x is in $(0,2^{-m})$, then $2^nx\in(0,1)$ for $n=0,1,\cdots,m$. For this x, we have

$$f(x)=\sum_{n=0}^{\infty}\frac{\phi(2^n)}{2^{5n}}\ge\sum_{n=0}^{m}\frac{(2^nx)^5}{2^{5n}}=(m+1)x^5>(d+|c|)|x|^5,$$

which contradiction (2.20).

Remark 2.1. In Corollary 2.4, if we assume that

$$\phi(x,y)=\|x\|^r\|y\|^rz_0$$

or

$$\phi(x,y)=(\|x\|^r\|y\|^s+\|x\|^{r+s}+\|y\|^{r+s})z_0,$$

then we have Ulam-Gavuta-Rassias product stability and JMRassias mixed product-sum stability, respectively.

Next, we apply a fixed point method for the generalized Hyer-Ulam stability of the functional equation (1.1) in RN-spaces. The following Theorem will be used in the proof of Theorem 2.6.

Theorem 2.5. ([7]) *Suppose that (Ω, d) is a complete generalized metric space and $J : \Omega \to \Omega$ is a strictly contractive mapping with Lipshitz constant $L < 1$. Then, for each $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers $n \geq 0$ or there exists a natural number n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y*$ of J;*
(3) *$y*$ is the unique fixed point of J in the set $\Lambda = \{y \in \Omega : d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Lambda$.*

Theorem 2.6. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < \alpha < 2^5$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(2x,2y)}(\alpha t) \tag{2.21}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ such that

$$\mu_{D(x,y)}(t) \geq \mu'_{\phi(x,y)}(t) \tag{2.22}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,y)}\big(2^2(2^5-\alpha)t\big) \tag{2.23}$$

for all $x \in X$ and $t > 0$.

Proof. It follows from (2.22) that

$$\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t) \tag{2.24}$$

for all $x \in X$ and $t > 0$. Let $\Omega = \{g : X \to Y, g(x) = 0\}$ and the mapping d defined on Ω by

$$d(g,h) = \inf\{c \in [0,\infty) : \mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t),\ \forall x \in X\}$$

where, as usual, $\inf \emptyset = -\infty$. Then (Ω, d) is a generalized complete metric space (see [10]). Now, let us consider the mapping $J : \Omega \to \Omega$ defined by

$$Jg(x) = \frac{1}{2^5} g(2x)$$

for all $g \in \Omega$ and $x \in X$. Let g, h in Ω and $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) < c$. Then $\mu_{g(x)-h(x)}(ct) \geq \mu'_{\phi(x,0)}(t)$ for all $x \in X$ and $t > 0$ and so

$$\mu_{Jg(x)-Jh(x)}\Big(\frac{\alpha ct}{2^5}\Big) = \mu_{g(2x)-h(2x)}(\alpha ct) \geq \mu'_{\phi(x,0)}(t) \tag{2.25}$$

for all $x \in X$ and $t > 0$. Hence we have

$$d(Jg, Jh) \leq \frac{\alpha c}{2^5} \leq \frac{\alpha}{2^5} d(g,h)$$

for all $g, h \in \Omega$. Then J is a contractive mapping on Ω with the Lipschitz constant $L = \frac{\alpha}{2^5} < 1$. Thus it follows from Theorem 2.5 that there exists a mapping $Q : X \to Y$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega : d(f,g) < \infty\}$, such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{5n}}$$

for all $x \in X$ since $\lim_{n\to\infty} d(J^n f, Q) = 0$. Also, from $\mu_{f(x)-\frac{f(2x)}{2^5}}(t) \geq \mu'_{\phi(x,0)}(128t)$, it follows that $d(f, Jf) \leq \frac{1}{128}$. Therefore, using Theorem 2.5 again, we get

$$d(f,Q) \leq \frac{1}{1-L} d(f,Jf) \leq \frac{1}{2^2(2^5-\alpha)}.$$

This means that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(2^5-\alpha)t\right)$$

for all $x \in X$ and $t > 0$.

Also, replacing x and y by $2^n x$ and $2^n y$ in (2.22), respectively, we have

$$\mu_{DQ(x,y)}(t) \geq \lim_{n\to\infty} \mu'_{\phi(2^n x, 2^n y)}(2^{5n}t) = \lim_{n\to\infty} \mu'_{\phi(x,y)}\left(\left(\frac{2^5}{\alpha}\right)^n t\right) = 1$$

for all $x, y \in X$ and $t > 0$. By (RN1), the mapping Q is quintic.

To prove the uniqueness, let us assume that there exists a quintic mapping $Q' : X \to Y$ which satisfies (2.23). Then Q' is a fixed point of J in Ω_1. However, it follows from Theorem 2.5 that J has only one fixed point in Ω_1. Hence $Q = Q'$. This completes the proof. □

Theorem 2.7. *Let $\phi : X^2 \to D^+$ be a function such that, for some $0 < 2^5 < \alpha$,*

$$\mu'_{\phi(x,y)}(t) \leq \mu'_{\phi(\frac{x}{2},\frac{y}{2})}(\alpha t) \tag{2.26}$$

for all $x, y \in X$ and $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (2.22), then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\phi(x,0)}\left(2^2(\alpha - 2^5)t\right) \tag{2.27}$$

for all $x \in X$ and $t > 0$.

Proof. By a modification in the proofs of Theorem 2.2 and 2.6, we can easily obtain the desired results. This completes the proof. □

Now, we present a corollary that is an application of Theorem 2.6 and 2.7 in the classical case.

Corollary 2.8. *Let X be a Banach space, ϵ and p be positive real numbers with $p \neq 5$. Assume that $f : X \to X$ is a mapping with $f(0) = 0$ which satisfies*

$$\|Df(x,y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique quintic mapping $Q : X \to Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{\epsilon\|x\|^p}{2^2|2^5 - 2^p|}$$

for all $x \in X$ and $t > 0$.

Proof. Define $\mu : X \times \mathbb{R} \to \mathbb{R}$ by

$$\mu_x(t) = \begin{cases} \frac{t}{t+\|x\|}, & \text{if } t > 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then (X, μ, T_M) is a complete RN-space. Denote $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$ and $t > 0$. It follows from $\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$ that

$$\mu_{Df(x,y)}(t) \geq \mu'_{\phi(x,y)}(t)$$

for all $x, y \in X$ and $t > 0$, where $\mu' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\mu'_x(t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

is a random norm on $\mathbb{R}$. Then all the conditions of Theorems 2.6 and 2.7 hold and so there exists a unique quintic mapping $Q : X \to X$ such that

$$\begin{aligned} \frac{t}{t + \|Q(x) - f(x)\|} &= \mu_{Q(x)-f(x)}(t) \\ &\geq \mu'_{\phi(x,0)}\Big(2^2|2^5 - \alpha|t\Big) = \frac{2^2|2^5 - \alpha|t}{2^2|2^5 - \alpha|t + \epsilon\|x\|^p}. \end{aligned}$$

Therefore, we obtain the desired result, where $\alpha = 2^p$. This completes the proof. □

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Generalized composition operators on Zygmund type spaces and Bloch type spaces

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Abstract. In this paper, we investigate the boundedness and compactness of generalized composition operators on Zygmund type spaces and Bloch type spaces with normal weight.

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Keywords: Generalized composition operator, Bloch type space, Zygmund type space.

1 Introduction

Let k be a positive continuous function on $[0, 1)$. k is called normal, if there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that(see [12]),

$$\frac{k(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{k(r)}{(1-r)^a} = 0; \tag{1}$$

$$\frac{k(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{k(r)}{(1-r)^b} = \infty. \tag{2}$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let ω be normal on $[0, 1)$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space, denoted by $\mathcal{B}_\omega$, if

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(|z|)|f'(z)| < \infty.$$

It is easy to see that $\mathcal{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $\omega(t) = 1 - t^2$, we get the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$. See [19] for more information of the Bloch space.

Suppose μ is normal on $[0, 1)$. The Zygmund type space, denoted by $\mathscr{Z}_\mu$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty.$$

It is also easy to see that $\mathscr{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathscr{Z}_\mu}$. When $\mu(t) = 1 - t^2$, we get the Zygmund space (see [2, 8]).

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Throughout the paper, $S(\mathbb{D})$ denotes the set of analytic self-map of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by $(C_\varphi f)(z) = f(\varphi(z)),\quad f \in H(\mathbb{D})$. We refer the books [1, 19] for the theory of composition operators. Composition operators mapping into the Bloch space on $\mathbb{D}$ were studied in, for example, [1, 4, 11, 14, 15, 18]. See [5, 6, 9, 10] for some results of the composition operator mapping into the Zygmund space.

Motivated by the fact that weighted composition operators naturally come from isometries of some function spaces, for $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stević [9] defined the generalized composition operator, denoted by C_φ^g, as follows.

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

They characterized the boundedness and compactness of C_φ^g on the Zygmund space and the Bloch space in [9]. See, for example, [7, 13, 16] for the study of the operator C_φ^g.

In this paper, motivated by [9], we investigate the boundedness and compactness of the generalized composition operator C_φ^g on Zygmund type spaces and Bloch type spaces with normal weight.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Proof of main results

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

Lemma 1. [3] *Suppose μ is normal on $[0, 1)$. Then there exists $\mu_* \in H(\mathbb{D})$, such that*

(i) For any $t \in [0, 1)$, $\mu_*(t) \in R^+$, $\mu_*(t)$ is increasing on $[0, 1)$;

(ii) $\inf_{t\in[0,1)} \mu(t)\mu_*(t) > 0$; $\quad \sup_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty$.

In the rest of the paper, we will always use μ_* to denote the analytic function related to μ in Lemma 1. By a calculation, we get the following lemma.

Lemma 2. *Suppose μ is normal on $[0, 1)$. Then the following statements hold.*

(i) There exists a $\delta \in (0, 1)$, such that μ is decreasing on $[\delta, 1)$, $\lim_{t\to 1} \mu(t) = 0$.

(ii) For all $\alpha > 1, \beta \in (0, 1)$, when $t \in (0, 1)$, $s \in (\beta, 1)$,

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\mu(t)}dt \approx \int_0^s \frac{1}{\mu(t)}dt.$$

(iii) For any $z \in \mathbb{D}$, $\left|\int_0^z \mu_*(\eta)d\eta\right| \lesssim \int_0^{|z|} \mu_*(t)dt$. If $|\eta| \le |z|$, $\mu(|z|)|\mu_*(\eta)| < C$.

Proof. (i). By the definition of normal function, there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) hold. Since $\mu(t) = \frac{\mu(t)}{(1-t)^a}(1-t)^a$, we see that μ is decreasing on $[\delta, 1)$ and $\lim_{t\to 1}\mu(t) = 0$.

(ii). From $\lim_{t\to 1}\frac{1-t}{1-t^\alpha} = \frac{1}{\alpha} > 0$, for any $t \in [\delta^{\frac{1}{\alpha}}, 1)$,

$$1 > \frac{\mu(t)}{\mu(t^\alpha)} = \frac{\frac{\mu(t)}{(1-t)^b}}{\frac{\mu(t^\alpha)}{(1-t^\alpha)^b}}\frac{(1-t)^b}{(1-t^\alpha)^b} > \frac{(1-t)^b}{(1-t^\alpha)^b} > C.$$

So when $t \in (0, 1)$, $\mu(t) \approx \mu(t^\alpha)$. By Lemma 1, when $t \in (0, 1)$, $\mu(t) \approx \frac{1}{\mu_*(t)}$ is obvious.

When $s \in (\beta, 1)$,

$$\begin{aligned}\int_0^{s^\alpha}\frac{1}{\mu(t)}dt &= \int_0^{\beta^\alpha}\frac{1}{\mu(t)}dt + \int_{\beta^\alpha}^{s^a}\frac{1}{\mu(t)}dt = C + \int_\beta^s\frac{\alpha t^{\alpha-1}}{\mu(t^\alpha)}dt\\ &\approx \int_0^\beta\frac{1}{\mu(t)}dt + \int_\beta^s\frac{1}{\mu(t)}dt = \int_0^s\frac{1}{\mu(t)}dt.\end{aligned}$$

(iii). Since μ_* is analytic, we see that (iii) holds. The proof is completed. □

Lemma 3.[17] *Suppose μ is normal on $[0, 1)$. Then for all $z \in \mathbb{D}$ and $f \in \mathcal{B}_\mu$,*

$$|f(z)| < G_\mu(z)\|f\|_{\mathcal{B}_\mu},\ where\ G_\mu(z) = 1 + \int_0^{|z|}\frac{1}{\mu(t)}dt.$$

Remark 1. From the definitions of $\mathcal{Z}_\mu$ and $\mathcal{B}_\mu$, for all $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\mu$,

$$|f'(z)| \le G_\mu(z)\|f'\|_{\mathcal{B}_\mu} \le G_\mu(z)\|f\|_{\mathcal{Z}_\mu}.$$

Lemma 4. [17] *Suppose that μ is normal on $[0, 1)$ such that $\int_0^1\frac{1}{\mu(t)}dt < \infty$. If $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|f_n(z)| = 0.$$

The relationship between Zygmund type spaces and Bloch type spaces was established as follows.

Lemma 5. *Suppose that μ is normal on $[0, 1)$. Let $\mu_+(t) = (1-t)\mu(t)$. Then*

(i) μ_+ is normal on $[0, 1)$, $\lim_{|z|\to 1}G_{\mu_+}(z) = \infty$.

(ii) $\mathcal{B}_\mu = \mathcal{Z}_{\mu_+}$ and $\|\cdot\|_{\mathcal{B}_\mu} \approx \|\cdot\|_{\mathcal{Z}_{\mu_+}}$.

Proof. (i) Obviously, μ_+ is normal on $[0, 1)$. Since μ is normal,there exist positive numbers a and b, $0 < a < b$, and $\delta \in [0, 1)$ such that (1) and (2) holds. Then

$$\int_0^1\frac{1}{\mu_+(t)}dt > \int_\delta^1\frac{1}{(1-t)^{a+1}}\frac{(1-t)^a}{\mu(t)}dt > \frac{(1-\delta)^a}{\mu(\delta)}\int_\delta^1\frac{1}{(1-t)^{1+a}}dt = +\infty,$$

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as desired.

(ii) First we prove that $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$. For all $f \in \mathscr{Z}_{\mu_+}$, we have

$$\mu(|z|)|f'(z) - f'(0)| = \mu(|z|)\left|\int_0^z f''(\eta)d\eta\right| \le \|f\|_{\mathscr{Z}_{\mu+}} \int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt. \tag{3}$$

If $|z| \le \delta$, $\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt < C$. If $|z| > \delta$,

$$\begin{aligned}\int_0^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt &= \left(\int_0^{\delta} \frac{\mu(|z|)}{\mu(t)(1-t)}dt + \int_\delta^{|z|} \frac{\mu(|z|)}{\mu(t)(1-t)}dt\right)\\ &\le \left(C + \int_\delta^{|z|} \frac{\frac{\mu(|z|)}{(1-|z|)^a}}{\frac{\mu(t)}{(1-t)^a}} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right)\\ &\le \left(C + \int_\delta^{|z|} \frac{(1-|z|)^a}{(1-t)^{a+1}}dt\right) \le C.\end{aligned}$$

From Lemma 2, $\mu(t)$ is bounded on $[0, 1)$. By (3),

$$\mu(|z|)|f'(z)| \le C\|f\|_{\mathscr{Z}_{\mu+}} + \mu(|z|)|f'(0)| \lesssim \|f\|_{\mathscr{Z}_{\mu+}} + |f'(0)| \le 2\|f\|_{\mathscr{Z}_{\mu+}}.$$

Therefore $\|f\|_{\mathcal{B}_\mu} \lesssim \|f\|_{\mathscr{Z}_{\mu+}}$ and $\mathscr{Z}_{\mu_+} \subseteq \mathcal{B}_\mu$.

Next we prove that $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. For any $f \in \mathcal{B}_\mu$, by Cauchy's formula,

$$|f''(z)| \le \frac{2}{1-|z|} \max_{|\eta - z| = \frac{1-|z|}{2}} |f'(\eta)| \le \frac{2}{1-|z|} \max_{|\eta| = \frac{1+|z|}{2}} |f'(\eta)| \le \frac{2\|f\|_{\mathcal{B}_\mu}}{\mu(\frac{1+|z|}{2})(1-|z|)}.$$

If $|z| \le \delta$, $\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} < C$ is obvious. When $\delta < |z| < 1$,

$$\frac{\mu(|z|)}{\mu(\frac{1+|z|}{2})} = 2^b \frac{\frac{\mu(|z|)}{(1-|z|)^b}}{\frac{\mu(\frac{1+|z|}{2})}{(1-\frac{1+|z|}{2})^b}} < 2^b.$$

So $\|f\|_{\mathscr{Z}_{\mu+}} \lesssim \|f\|_{\mathcal{B}_\mu}$ and hence $\mathcal{B}_\mu \subseteq \mathscr{Z}_{\mu_+}$. The proof is completed. □

To study the compactness, we need the following lemma, which can be proved in a standard way (see, for example, Proposition 3.11 in [1]).

Lemma 6. *Suppose that $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, X, Y are Bloch type spaces or Zygmund type spaces. If $C_\varphi^g : X \to Y$ is bounded, then $C_\varphi^g : X \to Y$ is a compact operator if and only if whenever $\{f_n\}$ is bounded in X and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$,* $\lim_{n\to\infty} \|C_\varphi^g f_n\|_Y = 0$.

3 The boundness and compactness of $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

Theorem 1. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \qquad \textit{and} \qquad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{4}$$

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Proof. Suppose that (4) holds. For any $f \in \mathscr{Z}_\mu$, by Lemma 3 and Remark 1, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \omega(|z|)|(C_\varphi^g f)''(z)| &\le \sup_{z\in\mathbb{D}} \omega(|z|)\left|f''(\varphi(z))\varphi'(z)g(z)\right| + \sup_{z\in\mathbb{D}} \omega(|z|)\left|f'(\varphi(z))g'(z)\right| \\ &\le \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)}\|f\|_{\mathscr{Z}_\mu} + \sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \\ &<\infty,\end{aligned}$$

and $|(c_\varphi^g f)(0)| + |(C_\varphi^g f)'(0)| = |f'(\varphi(0))g(0)| \le |g(0)|G_\mu(\varphi(0))\|f\|_{\mathscr{Z}_\mu} < \infty$. Hence $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded.

Conversely, suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. From $z, z^2 \in \mathscr{Z}_\mu$, we see that

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)| < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \omega(|z|)|\varphi'(z)g(z)| < \infty. \tag{5}$$

Therefore

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty \quad \text{and} \quad \sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty. \tag{6}$$

For any $\xi \in \mathbb{D}$, if $|\varphi(\xi)| > \frac{1}{2}$, let $a = \varphi(\xi)$ and

$$p_a(z) = \int_0^{\overline{a}z}\left(\int_0^{t^2} \mu_*(\eta)d\eta\right)^2 dt - \int_0^{\overline{a}z}\left(\int_0^{\frac{t^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2 dt,$$

$$q_a(z) = \frac{p_a(z)}{\int_0^{|a|} \mu_*(\eta)d\eta}. \tag{7}$$

Then

$$p_a'(z) = \overline{a}\left(\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right)^2 - \overline{a}\left(\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right)^2,$$

$$p_a''(z) = 4\overline{a}^3 z\mu_*(\overline{a}^2 z^2)\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta - \frac{6\overline{a}^4 z^2}{|a|^2}\mu_*\left(\frac{(\overline{a}z)^3}{|a|^2}\right)\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta.$$

By Lemmas 1 and 2,

$$\mu(|z|)|p_a''(z)| \lesssim \left|\int_0^{(\overline{a}z)^2} \mu_*(\eta)d\eta\right| + \left|\int_0^{\frac{(\overline{a}z)^3}{|a|^2}} \mu_*(\eta)d\eta\right| \lesssim \int_0^{|a|} \mu_*(\eta)d\eta$$

So

$$\|q_a\|_{\mathscr{Z}_\mu} = q_a(0) + q_a'(0) + \sup_{z\in\mathbb{D}} \mu(|z|)|q_a''(z)| < C. \tag{8}$$

Hence, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)} \approx \omega(|\xi|)|(C_\varphi^g q_a)''(\xi)| \le \|C_\varphi^g q_a\|_{\mathscr{Z}_\omega} < \|q_a\|_{\mathscr{Z}_\mu}\|C_\varphi^g\| < \infty. \tag{9}$$

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From (6) and (9), we see that the second inequality in (4) holds.

Let $f_a(z) = \int_0^{\overline{a}z} \int_0^{\eta} \mu_*(w)dwd\eta$. Then

$$f_a'(z) = \overline{a} \int_0^{\overline{a}z} \mu_*(w)dw, \ \ f_a''(z) = \overline{a}^2 \mu_*(\overline{a}z), \ \|f_a\|_{\mathscr{Z}_\mu} \leq C.$$

By Lemma 2, when $|\varphi(\xi)| > \frac{1}{2}$,

$$\begin{aligned} \omega(|\xi|)|g'(\xi)| \int_0^{|a|} \frac{1}{\mu(t)} dt \approx & \omega(|\xi|) \left| (C_\varphi^g f_a)''(\xi) - f_a''(\varphi(\xi))\varphi'(\xi)g(\xi) \right| \\ \leq & \|C_\varphi^g f_a\|_{\mathscr{Z}_\omega} + \sup_{\xi \in \mathbb{D}} \omega(|\xi|)\mu_*(|\varphi(\xi)|^2)|\varphi'(\xi)g(\xi)| \qquad (10) \\ \lesssim & \|f_a\|_{\mathscr{Z}_\mu} \|C_\varphi^g\| + \sup_{\xi \in \mathbb{D}} \frac{\omega(|\xi|)|\varphi'(\xi)g(\xi)|}{\mu(|\varphi(\xi)|)}. \end{aligned}$$

From (6) and (10), we see that the first inequality in (4) holds. The proof is completed. □

Theorem 2. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$ such that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. Then the following statements hold:*

(i) When $\lim_{|z| \to 1} G_\mu(z) < \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)| \to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{11}$$

(ii) When $\lim_{|z| \to 1} G_\mu(z) = \infty$, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\lim_{|\varphi(z)| \to 1} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0. \tag{12}$$

Proof. Because $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, (5) holds.

(i). Suppose (11) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } |\varphi(z)| > \delta. \tag{13}$$

Let $\{f_n\} \subset \mathscr{Z}_\mu$ be bounded and converge to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 4 and Cauchy estimate,

$$\lim_{n \to \infty} \sup_{z \in \mathbb{D}} |f_n'(z)| = 0, \quad \text{and} \quad \lim_{n \to \infty} \sup_{|z| \leq \delta} |f_n''(z)| = 0. \tag{14}$$

From Remark 1, (5) and $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathscr{Z}_\mu} < \infty$,

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &= |(C_\varphi^g f_n)'(0)| + \sup_{z \in \mathbb{D}} \omega(|z|) \left| f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z) \right| \\ &\lesssim |f_n'(\varphi(0))| + \sup_{|\varphi(z)| \leq \delta} |f_n''(\varphi(z))| + \sup_{|\varphi(z)| > \delta} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} + \sup_{z \in \mathbb{D}} |f_n'(z)|. \end{aligned}$$

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By (13) and (14), $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. Using Lemma 6, we see that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Suppose $\{z_n\} \subset \mathbb{D}$ is a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and

$$r_n(z) = \mu(|a_n|) \int_0^{\overline{a_n} z} \int_0^t \mu_*^2(\eta) d\eta dt.$$

From Lemma 2, $\{r_n\}$ is bounded in $\mathscr{Z}_\mu$ and $r_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$ when $n \to \infty$. By Lemmas 4 and 6, we have

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |r_n'(z)| = 0 \text{ and } \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} = 0. \tag{15}$$

Using Lemma 2, (5) and (15),

$$\begin{aligned} \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} &\approx \lim_{n\to\infty} \omega(|z_n|) \left|(C_\varphi^g r_n)''(z_n) - r_n'(a_n)g'(z_n)\right| \\ &\le \lim_{n\to\infty} \|C_\varphi^g r_n\|_{\mathscr{Z}_\omega} + \lim_{n\to\infty} \omega(|z_n|) \left|r_n'(a_n)g'(z_n)\right| = 0, \end{aligned}$$

which implies that $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

(ii). Suppose (12) holds. For any $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$\omega(|z|)|g'(z)|G_\mu(\varphi(z)) < \varepsilon \quad \text{and} \quad \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \tag{16}$$

when $|\varphi(z)| > \delta$. Let $\{f_n\}$ be a bounded sequence in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Cauchy estimate,

$$\lim_{n\to\infty} \sup_{|\varphi(w)|\le\delta} |f_n'(\varphi(w))| = 0, \ \lim_{n\to\infty} \sup_{|\varphi(w)|\le\delta} |f_n''(\varphi(w))| = 0. \tag{17}$$

From Lemma 3, Remark 1 and (5),

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathscr{Z}_\omega} &= |(C_\varphi^g f_n)'(0)| + \sup_{z\in\mathbb{D}} \omega(|z|) \left|f_n''(\varphi(z))\varphi'(z)g(z) + f_n'(\varphi(z))g'(z)\right| \\ &\lesssim |f_n'(\varphi(0))| + \sup_{|\varphi(z)|\le\delta} |f_n''(\varphi(z))| + \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \\ &\quad \sup_{|\varphi(z)|>\delta} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} \|f_n\|_{\mathscr{Z}_\mu} + \sup_{|\varphi(z)|>\delta} \omega(|z|)|g'(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \end{aligned}$$

By (16) and (17), we see that $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathscr{Z}_\mu} = 0$. From Lemma 6, $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact.

Conversely, suppose that $C_\varphi^g : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $q_n = q_{a_n}$, where q_a is defined in (7). By (8), $\{q_n\}$ is bounded in $\mathscr{Z}_\mu$. Obviously, $q_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0$. By (9),

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} \approx \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g q_n)''(z_n)| \le \lim_{n\to\infty} \|C_\varphi^g q_n\|_{\mathscr{Z}_\omega} = 0,$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

Let

$$k_n(z) = \frac{\int_0^{\overline{a_n}z} \left(\int_0^t \mu_*(s)ds\right)^2 dt}{\int_0^{|a_n|} \mu_*(s)ds}. \tag{18}$$

Then

$$k_n'(z) = \frac{\overline{a_n}\left(\int_0^{\overline{a_n}z} \mu_*(s)ds\right)^2}{\int_0^{|a_n|} \mu_*(s)ds}, k_n''(z) = \frac{2(\overline{a_n})^2\mu_*(\overline{a_n}z)\int_0^{\overline{a_n}z} \mu_*(s)ds}{\int_0^{|a_n|} \mu_*(s)ds}.$$

By Lemma 2, $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 6, $\lim\limits_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} = 0$. By Lemma 2,

$$\begin{aligned}
& \lim_{n\to\infty} \omega(|z_n|)|g'(z_n)| \int_0^{|\varphi(z_n)|} \mu_*(s)ds \\
\lesssim\ & \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathscr{Z}_\omega} + 2\lim_{n\to\infty} \left|\varphi'(z_n)g(z_n)\omega(|z_n|)\mu_*(|\varphi(z_n)|^2)\right| \\
\lesssim\ & \lim_{n\to\infty} \frac{\omega(|z_n|)|\varphi'(z_n)g(z_n)|}{\mu(|\varphi(z_n)|)} = 0,
\end{aligned}$$

which implies that $\lim\limits_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$. The proof is completed. □

Theorem 3. *Suppose $g \in H(\mathbb{D}), \varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup\limits_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z)) < \infty$.

(iii) $\sup\limits_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) < \infty$ *and* $\sup\limits_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} < \infty$.

Proof. (ii)⇒(i). Suppose that *(ii)* holds. For any $f \in \mathscr{Z}_\mu$, using Remark 1,

$$\|C_\varphi^g f\|_{\mathcal{B}_\omega} = \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)f'(\varphi(z))| \le \sup_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f\|_{\mathscr{Z}_\mu} \lesssim \|f\|_{\mathscr{Z}_\mu} < \infty.$$

So $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded.

(ii)⇒(i). Suppose $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded. Then

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)| = \|C_\varphi^g z\|_{\mathcal{B}_\omega} < \infty. \tag{19}$$

For all $\eta \in \mathbb{D}$, let $u_a(z) = \int_0^{\overline{a}z}\int_0^t \mu_*(s)dsdt$, where $a = \varphi(\eta)$. By Lemma 2, $\sup_{\eta\in\mathbb{D}} \|u_a\|_{\mathscr{Z}_\mu} < \infty$. Thus $\sup_{\eta\in\mathbb{D}} \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < \infty$. When $|\varphi(\eta)| > \frac{1}{2}$,

$$\omega(|\eta|)|g(\eta)| \int_0^{|\varphi(\eta)|} \frac{1}{\mu(s)}ds \approx \omega(|\eta|)|(C_\varphi^g u_a)'(\eta)| \le \|C_\varphi^g u_a\|_{\mathcal{B}_\omega} < C. \tag{20}$$

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By (19) and (20),

$$\sup_{\eta\in\mathbb{D}} \omega(|\eta|)|g(\eta)|G_\mu(\varphi(\eta)) < \infty.$$

By Lemma 5 and Theorem 1, *(i)*⇔ *(iii)*. The proof is completed. □

Theorem 4. *Suppose* $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω *and* μ *are normal on* $[0, 1)$ *such that* $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is bounded. Then the following statements hold.*

(i) If $\lim_{|z|\to1} G_\mu(z) < \infty$, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) if $\lim_{|z|\to1} G_\mu(z) = \infty$, *then the following statements are equivalent.*

(a) $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ *is compact.*

(b) $\lim_{|\varphi(z)|\to1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$.

(c) $\lim_{|\varphi(z)|\to1} \omega_+(|z|)|g'(z)|G_\mu(\varphi(z)) = 0$ *and* $\lim_{|\varphi(z)|\to1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu(|\varphi(z)|)} = 0$.

Proof. Since $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is bounded, (19) holds.

(i). Suppose $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Then $\{f'_n\}$ is also bounded in $\mathcal{B}_\mu$ and $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Lemma 4, $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(z)| = 0$. Using (19),

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|(C^g_\varphi f_n)'(z)| = \lim_{n\to\infty} \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f'_n(\varphi(z))| \lesssim \lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(\varphi(z))| = 0.$$

Thus $\lim_{n\to\infty} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(ii). *(b)*⇒*(a)*. Assume that $\lim_{|\varphi(z)|\to1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$. Then for any $\varepsilon > 0$, there exists a $\delta \in (0, 1)$, such that

$$\omega(|z|)|g(z)|G_\mu(\varphi(z)) < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Suppose that $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then $f'_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By (19) and Remark 1,

$$\begin{aligned} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|g(z) f'_n(\varphi(z))| \\ &\le \sup_{|\varphi(z)|\le\delta} \omega(|z|)|g(z) f'_n(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z) f'_n(\varphi(z))| \\ &\lesssim \sup_{|\varphi(z)|\le\delta} |f'_n(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|g(z)|G_\mu(\varphi(z))\|f_n\|_{\mathscr{Z}_\mu} \\ &\lesssim \sup_{|\varphi(z)|\le\delta} |f'_n(\varphi(z))| + \varepsilon, \end{aligned}$$

which implies that $\lim_{n\to\infty} \|C^g_\varphi f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C^g_\varphi : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact.

(a)⇒(b). Assume that $C_\varphi^g : \mathscr{Z}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim\limits_{n\to\infty} |\varphi(z_n)| = 1$. From the proof of Theorem 2, we see that $\{k_n\}$ is bounded in $\mathscr{Z}_\mu$ and $k_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemmas 1, 2 and 6,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)| \left(\int_0^{|a_n|^2} \frac{1}{\mu(s)} ds\right)^2}{\int_0^{|a_n|} \frac{1}{\mu(s)} ds} \\ \approx\ &\lim_{n\to\infty} \omega(|z_n|) \left|(C_\varphi^g k_n)'(z_n)\right| \le \lim_{n\to\infty} \|C_\varphi^g k_n\|_{\mathcal{B}_\omega} = 0, \end{aligned}$$

which implies $\lim\limits_{n\to\infty} \omega(|z_n|)|g(z_n)G_\mu(\varphi(z_n))| = 0$. So $\lim\limits_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_\mu(\varphi(z)) = 0$.

Using Lemma 5 and Theorem 2, we see that *(a)⇔(c)*. The proof is completed. □

4 The boundness and compactness of $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega(\mathcal{B}_\omega)$

From Lemma 5, Theorems 1 and 2, notice that $\lim\limits_{|z|\to 1} G_{\mu_+}(z) = \infty$, we have the following theorems.

Theorem 5. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty \quad \textit{and} \quad \sup_{z\in\mathbb{D}} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty.$$

Theorem 6. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathscr{Z}_\omega$ is bounded. Then C_φ^g is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \omega(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$$

Theorem 7. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0, 1)$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is bounded.*

(ii) $\sup\limits_{z\in\mathbb{D}} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \infty$.

(iii) $\sup\limits_{z\in\mathbb{D}} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) < \infty$ *and* $\sup\limits_{z\in\mathbb{D}} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} < \infty$.

(iv) $\sup\limits_{z\in\mathbb{D}} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) < \infty$.

Proof. *(ii)⇔(i).* By Lemma 3 and taking the function $f(z) = \int_0^z \mu_*(\eta)d\eta \in \mathcal{B}_\mu$, we can get the desired result. Since the proof is similar to the proof of Theorem 1, we omit the details.

By Lemma 5, Theorems 1 and 3, we see that *(i)⇔ (iii)* and *(i)⇔ (iv)* hold. The proof is completed. □

Theorem 8. *Suppose $g \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$ such that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ *is compact.*

(ii) $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0.$

(iii) $\lim_{|\varphi(z)|\to 1} \omega_+(|z|)|g'(z)|G_{\mu_+}(\varphi(z)) = 0, \ \lim_{|\varphi(z)|\to 1} \frac{\omega_+(|z|)|\varphi'(z)g(z)|}{\mu_+(|\varphi(z)|)} = 0.$

(iv) $\lim_{|\varphi(z)|\to 1} \omega(|z|)|g(z)|G_{\mu_+}(\varphi(z)) = 0.$

Proof. Since $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is bounded, we get that $\sup_{z\in\mathbb{D}} \omega(|z|)|g(z)| = \|C_\varphi^g z\|_{\mathcal{B}_\omega} < \infty$.

(ii)⇒(i). Suppose $\lim_{|\varphi(z)|\to 1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} = 0$. For any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} < \varepsilon, \text{ when } \delta < |\varphi(z)| < 1.$$

Let $\{f_n\}$ be bounded in $\mathcal{B}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. From Cauchy estimate, $f_n' \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 3,

$$\begin{aligned}
\|C_\varphi^g f_n\|_{\mathcal{B}_\omega} &= \sup_{z\in\mathbb{D}} \omega(|z|)|f_n'(\varphi(z))g(z)| \\
&\le \sup_{|\varphi(z)|\le\delta} \omega(|z|)|f_n'(\varphi(z))g(z)| + \sup_{\delta<|\varphi(z)|<1} \omega(|z|)|f_n'(\varphi(z))g(z)| \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \frac{\omega(|z|)|g(z)|}{\mu(|\varphi(z)|)} \|f_n\|_{\mathcal{B}_\mu} \\
&\lesssim \sup_{|\varphi(z)|\le\delta} |f_n'(\varphi(z))| + \varepsilon,
\end{aligned}$$

which implies that $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. By Lemma 6, $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact.

(i)⇒(ii). Suppose that $C_\varphi^g : \mathcal{B}_\mu \to \mathcal{B}_\omega$ is compact. Let $\{z_n\} \subset \mathbb{D}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let $a_n = \varphi(z_n)$ and $f_n(z) = \mu(|a_n|) \int_0^{\overline{a_n}z} \mu_*^2(\eta) d\eta$. Then $\{f_n\}$ is bounded in $\mathcal{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 6, $\lim_{n\to\infty} \|C_\varphi^g f_n\|_{\mathcal{B}_\omega} = 0$. Therefore

$$\lim_{n\to\infty} \frac{\omega(|z_n|)|g(z_n)|}{\mu(|\varphi(z_n)|)} = \lim_{n\to\infty} \omega(|z_n|)|(C_\varphi^g f_n)'(z_n))| = 0,$$

which implies that (ii) holds.

By Lemma 5, Theorems 2 and 4, *(i)⇔(iii)* and *(i)⇔(iv)* hold. The proof is completed. □

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Convergence and error estimates for the series solutions of higher-order differential equations

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Abstract

Little work on the convergence and error estimates of approximate series solutions exists in the literature. For general nth-order linear differential equations with initial conditions, a rigorous proof of convergence for the series solutions given by the homotopy analysis method is first presented in this paper. Furthermore, an upper bound for the absolute error of these approximations is obtained.

1 Introduction

Higher-order differential equations arise in various branches of science and engineering. However, unlike numerical solutions, little work on the convergence and error estimates of the approximate series solutions to these equations can be found in the literature.

Consider general nth-order linear differential equations with initial conditions

$$\begin{cases} L[u(x)] = f(x), \\ u^{(i)}(x_0) = A_i, \quad i = 0, \cdots, n-1 \end{cases} \tag{1}$$

where

$$L := \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x),$$

$p_i(x)$, $i = 0, 1, \cdots, n-1$ and $f(x)$ are continuous in some neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0. The main purpose of this paper is to present a rigorous proof of convergence for the series solutions given by the homotopy analysis method and to establish an upper bound for the absolute error of these approximations.

The homotopy analysis method (or HAM) [1, 2] is a popular analytic approach for seeking series solutions to differential equations and related problems. It has been applied to solve many problems in different fields of science and engineering [3, 4, 5, 6, 7, 8, 9, 10].

For the sake of easy reference, the homotopy analysis method is briefly described as follows. Given a (usually nonlinear) problem

$$\mathcal{N}[u(x)] = 0,\ x \in \Omega, \tag{2}$$

one first constructs a zeroth-order deformation equation

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] = q\, c_0\, \mathcal{N}[\phi(x;q)], \tag{3}$$

where $\mathcal{L}$ is an auxiliary linear operator, $u_0(x)$ an initial guess satisfying the given initial/boundary conditions, and $c_0 \neq 0$ the convergence-control parameter. At $q = 0$ and $q = 1$, one has

$$\phi(x;0) = u_0(x),\ \phi(x;1) = u(x), \tag{4}$$

respectively. To seek a series solution, one expands $\phi(x;q)$ into a Taylor series at $q = 0$

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x;c_0) q^m. \tag{5}$$

Assuming that c_0 is properly chosen so that the series (5) converges at $q = 1$, then

$$u(x) = \phi(x;1) = \sum_{m=0}^{+\infty} u_m(x;c_0) \tag{6}$$

must be one of the solutions to the given problem as shown in [1], where $u_m(x;c_0)$ is governed by the mth-order deformation equation

$$\mathcal{L}[u_m(x;c_0) - \chi_m u_{m-1}(x;c_0)] = \frac{c_0}{(m-1)!} \left. \frac{\partial^{m-1}\mathcal{N}[\phi(x;q)]}{\partial q^{m-1}} \right|_{q=0}, \tag{7}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

In practice, one can only calculate an Nth-order approximation

$$\psi_N(x;c_0) = \sum_{m=0}^{N} u_m(x,c_0) \tag{8}$$

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of which the higher-order terms $u_m(x; c_0), m \geq 1$ are calculated via (7).

To obtain an accurate approximation (8), the optimal value c_0 is determined by minimizing the average residual error

$$\mathbf{E}(c_0) = \frac{1}{M} \sum_{j=1}^{M} \left(\mathcal{N}[\psi_N(x_j; c_0)]\right)^2, \tag{9}$$

where $x_1, x_2 \cdots, x_M \in \Omega$ are sample points.

For the problem (1), a rigorous proof of convergence for the series solutions given by the HAM is presented in Section 2. Moreover, an approach is also given for determining the valid region of c_0 that ensures the convergence of (6), and for obtaining an upper bound for the absolute error of the Nth-order approximation (8). In Section 3, two examples are given to illustrate the procedure and to demonstrate how accurate and convergent series solutions can be obtained. Some concluding remarks are given in the last section.

2 Convergence and error estimates

Let $F(x)$ be continuous on $[x_0 - \delta, x_0 + \delta]$. Denote

$$\|F(\cdot)\| := \max_{x \in [x_0 - \delta, x_0 + \delta]} |F(x)|.$$

For the initial value problem (1), it is assumed for the rest of the paper that the neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0 is sufficiently small so that

$$1 - \sum_{k=1}^{n} \frac{\delta^k}{(k-1)!} \|p_{n-k}(\cdot)\| > 0. \tag{10}$$

First, one constructs the zeroth-order deformation equation (3) with an initial guess $u_0(x)$ satisfying the initial conditions in (1)

$$u^{(i)}(x_0) = A_i, \quad i = 0, \cdots, n-1. \tag{11}$$

The linear operator and nonlinear operator are set out below

$$\mathcal{L}[\phi(x; q)] = \frac{\partial^n \phi(x; q)}{\partial x^n}, \quad \mathcal{N}[\phi(x; q)] = L[\phi(x; q)] - f(x).$$

Following the procedure outlined in Section 1, one obtains a series solution

$$u(x; c_0) = \sum_{m=0}^{+\infty} u_m(x; c_0) \tag{12}$$

and an Nth-order approximation

$$\psi_N(x;c_0) = \sum_{m=0}^{N} u_m(x;c_0) \tag{13}$$

to the initial value problem (1). For these solutions, one has the following convergence theorem:

Theorem 1. *For* $c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n} \frac{\delta^k}{k!} \|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^{n} \frac{\delta^k}{k!} \|p_{n-k}(\cdot)\|} \right]$*, the series solution (12) converges to the true solution* $u(x)$ *on* $[x_0-\delta, x_0+\delta]$*, where* ϵ *is a small number satisfying* $0 < \epsilon < 1 - \sum_{k=1}^{n} \frac{\delta^k}{k!} \|p_{n-k}(\cdot)\|$*.* $K \|E_N(\cdot;c_0)\|$ *is an upper bound for the absolute error of the* N*th-order approximation (13), where* $K = \frac{\delta^n}{(n-1)! \left[1-\sum_{k=1}^{n} \frac{\delta^k}{(k-1)!} \|p_{n-k}(\cdot)\|\right]}$ *and* $E_N(x;c_0) = \frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$.

Proof: Let

$$u(x) = \sum_{m=0}^{N} u_m(x;c_0) + R_N(x;c_0). \tag{14}$$

Then the series (12) converges to $u(x)$ on $[x_0-\delta, x_0+\delta]$ if and only if

$$\lim_{N\to\infty} \|R_N(\cdot;c_0)\| = 0. \tag{15}$$

To achieve this goal, we substitute (14) into the differential equation in (1), which gives a differential equation satisfied by $R_N(x;c_0)$,

$$L[R_N(x;c_0)] = -S_N(x;c_0) + f(x), \tag{16}$$

where

$$S_N(x;c_0) = \sum_{m=0}^{N} L[u_m(x;c_0)].$$

Since the initial guess $u_0(x)$ satisfies the initial conditions in (1), one sets

$$u_m(x_0;c_0) = 0, u_m'(x_0;c_0) = 0, \cdots, u_m^{(n-1)}(x_0;c_0) = 0 \quad \text{for } m \geq 1. \tag{17}$$

Consequently the initial conditions for the remainder $R_N(x;c_0)$ are

$$R_N(x_0;c_0) = R_N'(x_0;c_0) = \cdots = R_N^{(n-1)}(x_0;c_0) = 0. \tag{18}$$

Next, we want to simplify $S_N(x;c_0)$ in (16). Note that the mth-order deformation equation (7) becomes

$$u_m^{(n)}(x;c_0) - \chi_m u_{m-1}^{(n)}(x;c_0) = c_0[L(u_{m-1}(x;c_0)) - (1-\chi_m)f(x)]. \tag{19}$$

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Adding both sides of (19) from $m = 1$ to $N + 1$ yields

$$\frac{1}{c_0} u_{N+1}^{(n)}(x; c_0) = S_N(x; c_0) - f(x). \tag{20}$$

Consequently (16) becomes

$$L[R_N(x; c_0)] = -\frac{1}{c_0} u_{N+1}^{(n)}(x; c_0). \tag{21}$$

Our next goal is to estimate $\|R_N(\cdot; c_0)\|$. Noticing (18), one has

$$\begin{aligned} R_N(x; c_0) &= \int_{x_0}^{x} R'_N(t_1; c_0)dt_1 = \int_{x_0}^{x}\int_{x_0}^{t_2} R''_N(t_1; c_0)dt_1 dt_2 = \cdots \\ &= \int_{x_0}^{x}\int_{x_0}^{t_{n-1}} \cdots \int_{x_0}^{t_2} R_N^{(n-1)}(t_1; c_0)dt_1 \cdots dt_{n-2}dt_{n-1}, \end{aligned}$$

which implies

$$\left\| R_N^{(k)}(\cdot; c_0) \right\| \le \frac{\delta^{n-1-k}}{(n-1-k)!} \left\| R_N^{(n-1)}(\cdot; c_0) \right\|, \quad k = 0, 1, \cdots, n-2. \tag{22}$$

Using the initial condition $R_N^{(n-1)}(x_0; c_0) = 0$ and integrating (21) from x_0 to x give

$$\begin{aligned} R_N^{(n-1)}(x; c_0) = -\Bigg(&\int_{x_0}^{x} p_{n-1}(t) R_N^{(n-1)}(t; c_0)dt + \cdots + \int_{x_0}^{x} p_0(t) R_N(t; c_0)dt \\ &+ \frac{1}{c_0}\int_{x_0}^{x} u_{N+1}^{(n)}(t; c_0)dt\Bigg). \end{aligned}$$

Following a similar reasoning as above, one arrives at

$$\left\| R_N^{(n-1)}(\cdot; c_0) \right\| \le \sum_{k=1}^{n} \frac{\delta^k}{(k-1)!} \|p_{n-k}(\cdot)\| \cdot \left\| R_N^{(n-1)}(\cdot; c_0) \right\| + \frac{\delta \left\| u_{N+1}^{(n)}(\cdot; c_0) \right\|}{|c_0|}.$$

In view of (10), one has

$$\left\| R_N^{(n-1)}(\cdot; c_0) \right\| \le \frac{\delta}{1 - \sum_{k=1}^{n} \frac{\delta^k}{(k-1)!} \|p_{n-k}(\cdot)\|} \frac{\left\| u_{N+1}^{(n)}(\cdot; c_0) \right\|}{|c_0|}. \tag{23}$$

Combining (22) and (23), one finally achieves

$$\|R_N(\cdot;c_0)\| \le \frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]}\frac{\left\|u_{N+1}^{(n)}(\cdot;c_0)\right\|}{|c_0|}.$$

Consequently

$$\|R_N(\cdot;c_0)\| \le K\,\|E_N(\cdot;c_0)\|\,, \tag{24}$$

where

$$K=\frac{\delta^n}{(n-1)!\left[1-\sum_{k=1}^{n}\frac{\delta^k}{(k-1)!}\|p_{n-k}(\cdot)\|\right]} \quad \text{and} \quad E_N(x;c_0)=\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}.$$

Therefore, we have proved that $K\,\|E_N(\cdot;c_0)\|$ is an upper bound for the absolute error of the Nth-order approximation (13) on $[x_0-\delta, x_0+\delta]$. Our next goal is to prove that, if

$$c_0 \in \left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}, -\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right],$$

then

$$\lim_{N\to\infty}\|R_N(\cdot;c_0)\|=0. \tag{25}$$

First we figure out an expression for $E_N(x;c_0)$. The following lemma can be proved by mathematical induction.

Lemma 2. *Noticing* $E_N(x;c_0)=\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}$*, one has*

$$E_N(x;c_0)=\sum_{k=0}^{N}\binom{N}{k}a_k(x)c_0^k, \tag{26}$$

where $a_0(x)=L[u_0(x;c_0)]-f(x)$*, and for* $0\le k\le N-1$*,*

$$\begin{aligned} a_{k+1}(x) &= a_k(x)+p_{n-1}(x)\int_{x_0}^{x}a_k(t_1)dt_1+p_{n-2}(x)\int_{x_0}^{x}\int_{x_0}^{t_2}a_k(t_1)dt_1dt_2 \\ &\quad +\cdots+p_0(x)\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2}a_k(t_1)dt_1\cdots dt_{n-1}dt_n. \end{aligned} \tag{27}$$

Based on Lemma 2, one can determine the relation between $E_N(x;c_0)$ and $E_{N+1}(x;c_0)$. In view of (17), (26) and (27), one has

$$\begin{aligned}
&E_{N+1}(x;c_0)\\
&=(1+c_0)\frac{u_{N+1}^{(n)}(x;c_0)}{c_0}+p_{n-1}(x)u_{N+1}^{(n-1)}(x;c_0)+\cdots+p_0(x)u_{N+1}(x;c_0)\\
&=(1+c_0)E_N(x;c_0)+c_0p_{n-1}(x)\int_{x_0}^{x}E_N(t_1;c_0)dt_1+\cdots\\
&+c_0p_0(x)\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_2}E_N(t_1;c_0)dt_1\cdots dt_{n-1}dt_n,
\end{aligned}$$

which implies

$$\|E_{N+1}(\cdot;c_0)\|\le\left[|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right]\|E_N(\cdot;c_0)\|.$$

Next, let

$$w(\delta,c_0)=|1+c_0|+|c_0|\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|. \tag{28}$$

Case I. If $c_0\in\left[-1,-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right]$, then (28) becomes

$$w(\delta,c_0)=1+c_0\left(1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\le 1-\epsilon.$$

Case II. If $c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-1\right)$, then (28) becomes

$$w(\delta,c_0)=-1-c_0\left(1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|\right)\le 1-\epsilon.$$

One thus concludes that if

$$c_0\in\left[-\frac{2-\epsilon}{1+\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|},-\frac{\epsilon}{1-\sum_{k=1}^{n}\frac{\delta^k}{k!}\|p_{n-k}(\cdot)\|}\right], \tag{29}$$

then

$$w(\delta,c_0)\le 1-\epsilon<1.$$

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Thus $E_N(x; c_0)$ is a contraction mapping

$$\|E_{N+1}(\cdot; c_0)\| \le w(\delta, c_0) \|E_N(\cdot; c_0)\| \le (1-\epsilon) \|E_N(\cdot; c_0)\| .$$

Iteration then yields

$$\|E_N(\cdot; c_0)\| \le (1-\epsilon)^N \|E_0(\cdot; c_0)\| ,$$

Consequently

$$\lim_{N\to\infty} \|E_N(\cdot; c_0)\| = 0 \quad \text{and} \quad \lim_{N\to\infty} \|R_N(\cdot; c_0)\| = 0,$$

and the theorem has finally been proved. ♠

3 Examples

In this section, we apply the approach to investigate some initial value problems.

3.1 Buckling of a cantilever bar

A cantilever bar of length l is free at the upper end and is built-in at the bottom. The axial load P is supposed to be uniformly distributed along the bar axis. The deflection w satisfies the differential equation

$$\frac{d^3w}{dx^3} + \frac{P}{EI}(l-x)\frac{dw}{dx} = 0, \tag{30}$$

where EI is the bending rigidity. The initial conditions are

$$w(0) = 0, \quad \left.\frac{dw}{dx}\right|_{x=0} = 0, \quad \left.\frac{d^2w}{dx^2}\right|_{x=0} = 1. \tag{31}$$

A simple way to reduce the order of the equation is to set $u(x) = dw/dx$, but a more interesting way is as follows.

We make a change of variable

$$z = \frac{2}{3}\sqrt{\frac{P}{EI}}(l-x)^{\frac{3}{2}}. \tag{32}$$

Table 1: Upper bound for the absolute error of the Nth-order approximation.

N	2nd order	3rd order	4th order	5th order	6th order
$K\,\|E_N(\cdot;-1)\|$	1.190E-2	1.146E-3	8.188E-5	5.794E-6	3.800E-7

Step by step differentiation yields

$$\begin{aligned}\frac{dw}{dx} &= -\frac{dw}{dz}\sqrt[3]{\frac{3P}{2EI}}z,\\ \frac{d^2w}{dx^2} &= \left(\frac{3P}{2EI}\right)^{\frac{2}{3}}\left(\frac{1}{3}z^{-\frac{1}{3}}\frac{dw}{dz}+z^{\frac{2}{3}}\frac{d^2w}{dz^2}\right),\\ \frac{d^3w}{dx^3} &= \frac{3P}{2EI}\left(\frac{1}{9}z^{-1}\frac{dw}{dz}-\frac{d^2w}{dz^2}-z\frac{d^3w}{dz^3}\right).\end{aligned} \tag{33}$$

Introducing this in (30) and using the notation

$$\frac{dw}{dz}=u, \tag{34}$$

we get

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(1-\frac{1}{9z^2}\right)u=0. \tag{35}$$

It turns out that (35) is a differential equation of Bessel type. For computational purposes, one sets $l=1, EI=1$, and $P=\left(\frac{1000\pi}{1122}\right)^2$, the buckling critical load (see [11]). Then the initial conditions become

$$u(z_0)=0, \quad u'(z_0)=\frac{1}{P}, \tag{36}$$

where $z_0=\frac{1000\pi}{1683}$. Now let us solve the initial value problem (35)-(36).

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (35)-(36).

The radius $\delta=\frac{3}{5}$ of the neighborhood of $z_0=\frac{1000\pi}{1683}$ is determined by the condition (10). Then a valid region $[-1.211828346+0.6059141731\,\epsilon, -2.860401756\,\epsilon]$ of c_0 is obtained by means of (29), where $0<\epsilon<0.3496012397$.

It follows from Theorem 1 that, for $c_0\in[-1.211828346+0.6059141731\,\epsilon, -2.860401756\,\epsilon]$, the series solution (12) converges on $\left[\frac{1000\pi}{1683}-\frac{3}{5}, \frac{1000\pi}{1683}+\frac{3}{5}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0=-1$. Notice that $-1\in[-1.211828346+$

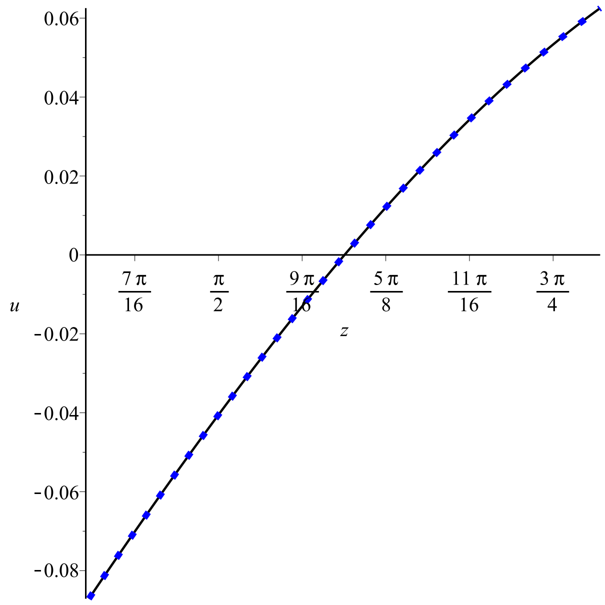

Figure 1: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

$0.6059141731\,\epsilon, -2.860401756\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\|E_N(\cdot;-1)\|$ for the absolute error of the Nth-order approximation (13) when $N = 2, 3, 4, 5$, and 6 on $\left[\frac{1000\pi}{1683} - \frac{3}{5}, \frac{1000\pi}{1683} + \frac{3}{5}\right]$ are calculated, as shown in Table 1. It is very accurate as shown in Figure 1.

3.2 Third-order equation with variable coefficients

Consider the third-order initial value problem (see [12])

$$\begin{aligned} &u^{(3)}(x) + xu''(x) + x^{\frac{2}{3}}u'(x) + x^{\frac{1}{3}}u(x) = 0, \\ &u(1) = 1,\ u'(1) = 0,\ u''(1) = 1. \end{aligned} \tag{37}$$

This initial value problem does not have a closed-form solution.

Following the procedure outlined at the beginning of Section 2, one obtains a series solution (12) and an Nth-order approximation (13) to the problem (37).

The radius $\delta = \frac{9}{20}$ of the neighborhood of $x_0 = 1$ is determined by the condition (10). Then a valid region $[-1.111481567{+}0.5557407833\,\epsilon, -4.985046404\,\epsilon]$ of c_0 is obtained by means of (29), where $0 < \epsilon < 0.2005999381$.

Table 2: Upper bound for the absolute error of the Nth-order approximation.

N	5th order	10th order	15th order	20th order
$K\,\Vert E_N(\cdot;-1)\Vert$	4.000E-4	1.116E-8	1.347E-13	1.017E-18

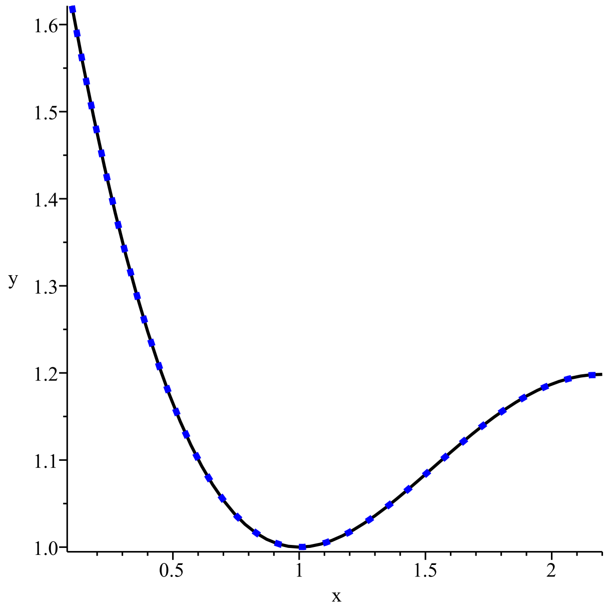

Figure 2: The solid line: numerical solution; the dot line: the 5th-order HAM approximation.

It follows from Theorem 1 that, for $c_0 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$, the series solution (12) converges on $\left[\frac{11}{20}, \frac{29}{20}\right]$.

To get an accurate approximation, the optimal value of c_0 is determined by minimizing the averaged residual error (9) of the 5th-order approximation (13). It turns out that $c_0 = -1$. Notice that $-1 \in [-1.111481567 + 0.5557407833\,\epsilon, -4.985046404\,\epsilon]$. So the corresponding series solution (12) does converge.

The upper bounds $K\,\Vert E_N(\cdot;-1)\Vert$ for the absolute error of the Nth-order approximation (13) when $N = 5, 10, 15$ and 20 on $\left[\frac{11}{20}, \frac{29}{20}\right]$ are calculated, as shown in Table 2. It is very accurate as shown in Figure 2.

4 Conclusion

For general nth-order linear differential equations with initial conditions, we have presented a rigorous proof of convergence for the series solutions given

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by the HAM for the first time. Furthermore, we have proposed an approach for seeking convergent series solutions to these problems. Some outstanding features of the approach include the determination of a valid region of the convergence-control parameter for ensuring convergence, and the calculation of an upper bound for the absolute error of an approximation.

Some issues still deserve further investigations. For example, how can one enlarge the neighborhood $[x_0 - \delta, x_0 + \delta]$ of x_0 so that Theorem 1 is still valid? How can one extend the techniques to investigate the convergence of nonlinear problems? It is believed that substantial work has to be done for dealing with these issues.

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ON THE GENERALIZED Z-ALGORITHM FOR THE NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. For any d-dimensional neutral stochastic functional differential equation with infinite delay and m-dimensional Brownian motion, we introduce a sequence of approximate equations and offer sufficient conditions such that the approximate solutions converge with probability one to the solution of the given equation. This iterative method called the generalized Z-algorithm is a generalization to many well-known analytic iterative method.

1. Introduction

The neutral stochastic functional differential equation, abbreviated as NSFDE, which was introduced by Kolmanovskii and Nosov [5], has been received much more attention in recent years. The existence and uniqueness theorems of the solution to the NSFDE with finite delay can be seen in Mao's books [6]. Recently, the existence of the solution to NSFDEs with infinite delay has been established in [1, 8, 9] by the classic Picard iteration argument. In 2010, S. Janković, M. Vasilova and M. Krstić [4] utilized successfully a general analytic method called the Z-algorithm to verify the existence of the solutions to NSFDEs with finite delay.

Actually, the Z-algorithm method could be backed to works [10, 11] in which Zuber posed an analytic iterative method for solving the Cauchy problem of the ordinary differential equation $X' = f(t, X)$ with $X(t_0) = X_0$. In fact, Zuber considered the approximate equations $X'_{n+1} = f_n(t; X_{n+1})$ with $X_{n+1}(t_0) = X_0$ for $n = 0, 1, \cdots$. It was showed in [10] that if $\sum_{n=1}^{\infty} \sup_{|t-t_0|<\varepsilon} |f(t, X_n(t)) - f_n(t, X_n(t))| < 1$, then the sequence of the solutions $\{X_n\}$ converges to the solution X of the initial equation, uniformly in an interval around t_0. Here we remark that, by the advantages of the Z-algorithm method, if we choose the functions $\{f_n\}$ good enough so that the approximate equations can be effectively solved, then the solution X of the initial equation can be effectively approximated. Later, Janković

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Keywords. Neutral stochastic functional differential system; Infinite delay; Approximate solution; Z-algorithm.

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*Key words and phrases.*Neutral stochastic functional differential system; Infinite delay; Approximate solution; Z-algorithm.

[2, 3] applied an analogous analytic method to the following stochastic differential equation of the Itô type,

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t), \quad t \in [0, T] \tag{1.1}$$
$$X(0) = X_0, \tag{1.2}$$

by comparing its solution with the solutions to the related equations

$$dX_{n+1}(t) = a_n(t, X_{n+1}(t))dt + b_n(t, X_{n+1}(t))dB(t), \quad t \in [0, T] \tag{1.3}$$
$$X_{n+1}(0) = X_0, \tag{1.4}$$

for $n = 0, 1, ...$ and some suitable functions $\{a_n\}$ and $\{b_n\}$. It was shown in [2] that, if

$$\sum_{n=1}^{\infty} \sup_{t,X}\{|a(t, X) - a_n(t, X)| + |b(t, X) - b_n(t, X)|\} < \infty,$$

then the solutions $X_{n+1}(t)$ of (1.3) converge to the solution $X(t)$ of (1.1) a.s. uniformly in $[0, T]$ as $n \to \infty$.

Motivated by the works mentioned above, we will discuss in this paper the existence of the Z-algorithm approximate solutions to NSFDEs with infinite delay. Our main theorem will be stated and shown in the next section, see Theorem 2.4; and some comments will be given in section 3. Here we need to give the notations and definitions which will be used in the paper.

Let $|\cdot|$ denote the Euclidean norm in $\mathbf{R}^d$. If A is a vector or a matrix, its transpose is denoted by A^T; The trace norm of a matrix A is represented by $|A| = \sqrt{trace(A^T A)}$. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t \geq t_0}$ satisfying the usual conditions, i.e., it is right continuous and $\mathscr{F}_{t_0}$ contains all P-null sets. Assume that $B(t)$ is a m-dimensional Brownian motion defined on $(\Omega, \mathscr{F}, P)$, that is $B(t) = \big(B_1(t), B_2(t), ..., B_m(t)\big)^T$ and each $B_i(t)$ is a standard Brownian motion for $i = 1, 2, \cdots, n$. Let $BC((-\infty, 0]; \mathbf{R}^d)$ denote the family of bounded continuous $\mathbf{R}^d$-value functions φ defined on $(-\infty, 0]$ with the norm $||\varphi|| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$.

In this paper, we consider the following d-dimensional NSFDEs with infinite delay,

$$d\,[X(t) - D(X_t)] = f(t, X_t)dt + g(t, X_t)dB(t), \quad t_0 \leq t \leq T, \tag{1.5}$$

where

$$X_t = \big\{X(t + \theta) : -\infty < \theta \leq 0\big\}$$

can be regarded as a $BC((-\infty, 0]; \mathbf{R}^d)$-valued stochastic process; and we assume that D is a vector-value function from $BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^d$, and f is a Borel measurable function from $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^d$, and g is a matrix-value Borel measurable function from $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$ to $\mathbf{R}^{md}$. Here one notes that the last term, $g(t, X_t)dB(t)$, in equation (1.5) can be rewritten as

$$g_1(t, X_t)dB_1(t) + g_2(t, X_t)dB_2(t) + \cdots + g_m(t, X_t)dB_m(t).$$

The initial value is assumed to be

$$X_{t_0} = \{\xi(\theta) : -\infty < \theta \leq 0\} \tag{1.6}$$

where $\xi(\theta)$ is an $\mathscr{F}_{t_0}$-measurable $BC((-\infty, 0]; \mathbf{R}^d)$-value random variable such that $\mathbf{E}||\xi||^2 < \infty$.

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Definition 1.1. $\mathbf{R}^d$-value stochastic process $X(t)$ defined on $(-\infty, T]$ is called the solution of (1.5) with initial value (1.6), if $X(t)$ has the following properties:

(1) $X(t)$ is continuous and $\{X(t) : -\infty \le t \le T\}$ is $\mathscr{F}_t$-adapted;
(2) $f(t, X_t) \in \mathscr{L}^1([t_0, T]; \mathbf{R}^d)$ and $g(t, X_t) \in \mathscr{L}^2([t_0, T]; \mathbf{R}^{d\times m})$;
(3) $X_{t_0} = \xi$, and for each $t_0 \le t \le T$,

$$X(t) = \xi(0) + D(X_t) - D(\xi) + \int_{t_0}^t f(s, X_s)ds + \int_{t_0}^t g(s, X_s)dB(s), \quad a.s. \tag{1.7}$$

We say that the solution of (1.5) with initial value (1.6), $X(t)$, is the unique solution, if for any other solution $\bar{X}(t)$ distinguishable with $X(t)$ we have

$$P\left\{X(t) = \bar{X}(t) : -\infty < t \le T\right\} = 1$$

Here we remarked that the existence and uniqueness of the solution to the equation (1.5) were established by the well-known Picard iteration in [1, 8, 9] under the following assumptions:

(M1) There exists a positive constant K such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t, \varphi) - f(t, \psi)| + |g(t, \varphi) - g(t, \psi)| \le K||\varphi - \psi||$$

(M2) There exists a positive constant $\bar{K}$ such that, for all $\varphi \in BC((-\infty, 0]; \mathbf{R}^d)$ and $t \in [t_0, T]$,

$$|f(t, \varphi)| + |g(t, \varphi)| \le \bar{K}(1 + ||\varphi||)$$

(M3) There exists a constant $k \in (0, 1)$ such that, for all $\varphi, \psi \in BC((-\infty, 0]; \mathbf{R}^d)$,

$$|D(\varphi) - D(\psi)| \le k||\varphi - \psi||$$

Recall the stochastic integral equation (1.7), we introduce the following sequence of the related equations:

$$\begin{aligned} X^{n+1}(t) - D_n(X_t^{n+1}) &= \xi^{n+1}(0) - D_n(\xi^{n+1}) + \int_{t_0}^t f_n(s, X_s^{n+1})ds \\ &\quad + \int_{t_0}^t g_n(s, X_s^{n+1})dB(s) \end{aligned} \tag{1.8}$$

for $t \in [t_0, T]$ and $n = 0, 1, \cdots$, where $X_t^{n+1} = \{X^{n+1}(t + \theta) : -\infty < \theta \le 0\}$ are $BC((-\infty, 0]; \mathbf{R}^d)$-value stochastic processes, $\xi^{n+1} = X_{t_0}^{n+1}$ are the initial conditions. The functions D_n, f_n, g_n will be chosen late such that

$$D_n(X) \to D(X),\ f_n(t, X) \to f(t, X),\ g_n(t, X) \to g(t, X), \text{ as } n \to \infty,$$

uniformly in $[t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d)$.

Let $X(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.7) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X(t)|^p < \infty$. Let $X^{n+1}(t)$ be the unique $\mathscr{F}_t$-adapted solution to the equation (1.8) satisfying $\mathbf{E}\sup_{t\in(-\infty,T]} |X^{n+1}(t)|^p < \infty$ for $n = 0, 1, \cdots$.

We will use the following notations in this paper,

$$\begin{aligned} \gamma_n &= \mathbf{E}||\xi - \xi^n||^p, \\ \delta_n &= \mathbf{E} \sup_{t\in[t_0,T]} |D(X_t^n) - D_n(X_t^n)|^p, \\ \varepsilon_n &= \mathbf{E} \sup_{t\in[t_0,T]} \left[|f(t, X_t^n) - f_n(t, X_t^n)|^p + |g(t, X_t^n) - g_n(t, X_t^n)|^p\right]. \end{aligned}$$

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We take the initial iteration to be $X^0(t) = \xi(0)$ a.s., for $t \in [t_0, T]$, and $X^0_{t_0} = \xi$. We will use the following conditions:

$$\sum_{n=0}^{\infty} \gamma_n < \infty, \quad \sum_{n=0}^{\infty} \delta_n < \infty, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty \tag{1.9}$$

2. The main theorem and its proof

At first we give three lemmas, which are useful for our investigation.

Lemma 2.1. *For any $X, Y \in \mathbf{R}^d$ and $\theta \in (0,1)$, we have*

$$|X+Y|^p \le \frac{|X|^p}{(1-\theta)^{p-1}} + \frac{|Y|^p}{\theta^{p-1}}, \quad p \ge 1$$

The proof of Lemma 2.1 can be found in [6]

Lemma 2.2. *Let $u, v : [a,b] \to \mathbf{R}_+$ be continuous functions and L be a positive constant, if*

$$u(t) \le v(t) + L\int_a^t u(s)ds,$$

then for all $t \in [a,b]$ we have

$$u(t) \le v(t) + L\int_a^t e^{L(t-s)} v(s)ds.$$

Especially, if $v(t) = M$ is a constant, then $u(t) \le Me^{L(t-a)}$.

Lemma 2.2 is a special case of the Gronwall lemma, so we omit its proof here.

Lemma 2.3. *Let $p \ge 2$, $t \in [t_0, T]$, and let $X_t, X^n_t \in BC((-\infty, 0]; \mathbf{R}^d)$ be the stochastic process mentioned above, then for any $r \in [t_0, t]$ we have*

$$||X_r - X^n_r||^p \le ||\xi - \xi^n||^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p.$$

Proof. From the definition of the norm in $BC((-\infty, 0]; \mathbf{R}^d)$, we have

$$\begin{aligned}
||X_r - X^n_r||^p &= \sup_{\theta\in(-\infty,0]} |X(r+\theta) - X^n(r+\theta)|^p \\
&= \sup_{u\in(-\infty,r]} |X(u) - X^n(u)|^p \\
&\le \sup_{u\in(-\infty,t_0]} |X(u) - X^n(u)|^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p \\
&= ||\xi - \xi^n||^p + \sup_{u\in[t_0,r]} |X(u) - X^n(u)|^p.
\end{aligned}$$

The lemma is proved. □

Now we state our main theorem.

Theorem 2.4. *Let $p \ge 2$ and $\mathbf{E}||\xi||^p < \infty$, $\mathbf{E}||\xi^n||^p < \infty$, $n = 0, 1, ...$ Assume that the functions D, f, g, D_n, f_n, g_n satisfy the Lipschitz conditions (M1) and (M3) with constants $K > 0$ and $k \in \left(0, 1/(3\cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ for any $n = 0, 1, ...$ If the condition (1.9) is vilid. Then, the sequence of solutions $\{X^{n+1}(t) : t \in (-\infty, T], n = 0, 1, ...\}$ of the equations (1.8) converges uniformly in $[t_0, T]$, with probability one as $n \to \infty$, to the solution $\{X(t), t \in (-\infty, T]\}$ of equation (1.7).*

Proof. From (1.7) and (1.8), for all $r \in [t_0, T]$, we have

$$\begin{aligned} X(r) - X^{n+1}(r) &= \xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1}) \\ &\quad + D(X_r) - D_n(X_r^{n+1}) \\ &\quad + \int_{t_0}^{r} [f(s, X_s) - f_n(s, X_s^{n+1})] ds \\ &\quad + \int_{t_0}^{r} [g(s, X_s) - g_n(s, X_s^{n+1})] dB(s). \end{aligned} \tag{2.1}$$

Since $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$ and $p > 1$, we can choose θ such that

$$(3k^p)^{\frac{1}{4(p-1)}} \leq \theta < \frac{1}{2}. \tag{2.2}$$

Hence, for $t_0 \leq r \leq t \leq T$, we see from Lemma 2.1 that

$$\mathbf{E} \sup_{r \in [t_0, t]} |X(r) - X^{n+1}(r)|^p \leq \frac{I_1}{(1-\theta)^{p-1}} + \frac{J_1(t)}{\theta^{2(p-1)}} + \frac{J_2(t)}{(\theta(1-\theta))^{p-1}}, \tag{2.3}$$

where

$$\begin{aligned} I_1 &= \mathbf{E}|\xi(0) - \xi^{n+1}(0) - D(\xi) + D_n(\xi^{n+1})|^p \\ J_1(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left|D(X_r) - D_n(X_r^{n+1})\right|^p \\ J_2(t) &= \mathbf{E} \sup_{r \in [t_0, t]} \left| \int_{t_0}^{r} \left[f(s, X_s) - f_n(s, X_s^{n+1})\right] ds \right. \\ &\qquad \left. + \int_{t_0}^{r} \left[g(s, X_s) - g_n(s, X_s^{n+1})\right] dB(s) \right|^p \end{aligned}$$

Next we will give the estimates for I_1, J_1 and J_2, respectively. According to the condition (M3), we can deduce that

$$\begin{aligned} I_1 &\leq 2^{p-1}[\mathbf{E}|\xi(0) - \xi^{n+1}(0)|^p + \mathbf{E}|D(\xi) - D_n(\xi^{n+1})|^p] \\ &\leq 2^{p-1}\gamma_{n+1} + 8^{p-1}\mathbf{E}\big[|D(\xi) - D(\xi^n)|^p + |D(\xi^n) - D_n(\xi^n)|^p \\ &\qquad + |D_n(\xi^n) - D_n(\xi)|^p + |D_n(\xi) - D_n(\xi^{n+1})|^p\big] \\ &\leq 2^{p-1}\gamma_{n+1} + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p + 8^{p-1}\delta_n + 8^{p-1}k^p\mathbf{E}||\xi - \xi^n||^p \\ &\qquad + 8^{p-1}k^p\mathbf{E}||\xi - \xi^{n+1}||^p \\ &= 2^{p-1}\gamma_{n+1} + 8^{p-1}(2k^p\gamma_n + k^p\gamma_{n+1} + \delta_n). \end{aligned} \tag{2.4}$$

For $J_1(t)$, from Lemma 2.1 and the condition (M3), we get

$$\begin{aligned} &|D(X_r) - D_n(X_r^{n+1})|^p \\ &\leq \frac{1}{(1-\theta)^{p-1}}\Big[\frac{1}{(1-\theta)^{p-1}}|D(X_r) - D(X_r^n)|^p + \frac{1}{\theta^{p-1}}|D(X_r^n) - D_n(X_r^n)|^p\Big] \\ &\quad + \frac{1}{\theta^{p-1}}\Big[\frac{1}{(1-\theta)^{p-1}}|D_n(X_r^n) - D_n(X_r)|^p + \frac{1}{\theta^{p-1}}|D_n(X_r) - D_n(X_r^{n+1})|^p\Big] \\ &\leq k^p\Big[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}\Big]||X_r - X_r^n||^p + \frac{\delta_n}{(\theta(1-\theta))^{p-1}} \\ &\quad + \frac{k^p}{\theta^{2(p-1)}}||X_r - X_r^{n+1}||^p. \end{aligned}$$

Lemma 2.3 yields that

$$\mathbf{E}\sup_{r\in[t_0,t]}||X_r - X_r^n||^p \le \gamma_n + \mathbf{E}\sup_{r\in[t_0,t]}|X(r) - X^n(r)|^p$$

Therefore,

$$\begin{aligned} J_1(t) \quad &\le \quad k^p\Big[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}\Big]\Big[\gamma_n + \mathbf{E}\sup_{r\in[t_0,t]}|X(r) - X^n(r)|^p\Big] \\ &+ \quad \frac{k^p}{\theta^{2(p-1)}}\big[\gamma_{n+1} + \mathbf{E}\sup_{r\in[t_0,t]}|X(r) - X^{n+1}(r)|^p\big] + \frac{\delta_n}{(\theta(1-\theta))^{p-1}}. \end{aligned} \tag{2.5}$$

For $J_2(t)$, we can decompose it into two party

$$\begin{aligned} J_2(t) \quad &\le \quad 2^{p-1}\left[\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[f(s,X_s) - f_n(s,X_s^{n+1})]ds\Big|^p\right. \\ &\qquad \left. + \mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s) - g_n(s,X_s^{n+1})]dB(s)\Big|^p\right] \\ &=: \quad 2^{p-1}[J_{21}(t) + J_{22}(t)]. \end{aligned}$$

To estimate $J_{21}(t)$, using the Hölder inequality and the condition (M1), we get that

$$\begin{aligned} &J_{21}(t) \\ &\le 4^{p-1}\mathbf{E}\left[\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[f(s,X_s) - f(s,X_s^n)]ds\Big|^p + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[f(s,X_s^n) - f_n(s,X_s^n)]ds\Big|^p\right. \\ &\quad \left. + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[f_n(s,X_s^n) - f_n(s,X_s)]ds\Big|^p + \sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[f_n(s,X_s) - f_n(s,X_s^{n+1})]ds\Big|^p\right] \\ &\le 4^{p-1}\Big[K^p(t-t_0)^{p-1}\int_{t_0}^{t}\mathbf{E}||X_s - X_s^n||^p ds + (t-t_0)^p\varepsilon_n \\ &\quad + K^p(t-t_0)^{p-1}\int_{t_0}^{t}\mathbf{E}||X_s - X_s^n||^p ds + K^p(t-t_0)^{p-1}\int_{t_0}^{t}\mathbf{E}||X_s - X_s^{n+1}||^p ds\Big] \\ &\le 4^{p-1}(T-t_0)^{p-1}\left[2K^p\int_{t_0}^{t}\mathbf{E}||X_s - X_s^n||^p ds\right. \\ &\qquad\qquad \left. + K^p\int_{t_0}^{t}\mathbf{E}||X_s - X_s^{n+1}||^p ds + (t-t_0)\varepsilon_n\right]. \end{aligned}$$

Hence by lemma 2.3 we obtain that

$$\begin{aligned} J_{21}(t) \quad &\le \quad 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^{t}\big(\gamma_n + \mathbf{E}\sup_{r\in[t_0,s]}|X(r) - X^n(r)|^p\big)ds \\ &\qquad + K^p\int_{t_0}^{t}\big(\gamma_{n+1} + \mathbf{E}\sup_{r\in[t_0,s]}|X(r) - X^{n+1}(r)|^p\big)ds + (t-t_0)\varepsilon_n\Big] \\ &= \quad 4^{p-1}(T-t_0)^{p-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r) - X^n(r)|^p ds \\ &\qquad + K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r) - X^{n+1}(r)|^p ds + (t-t_0)(2K^p\gamma_n + K^p\gamma_{n+1} + \varepsilon_n)\Big]. \end{aligned}$$

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In order to estimate $J_{22}(t)$, we use Cauchy inequality to get that

$$\begin{aligned}J_{22}(t) \quad &\leq \quad 4^{p-1}\Big[\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\Big|^p \\ &\qquad +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s^n)-g_n(s,X_s^n)]dB(s)\Big|^p \\ &\qquad +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g_n(s,X_s^n)-g_n(s,X_s)]dB(s)\Big|^p \\ &\qquad +\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g_n(s,X_s)-g_n(s,X_s^{n+1})]dB(s)\Big|^p\Big]\end{aligned}$$

Applying the Burkhölder-Davis-Gundy inequality and the Hölder inequality to the first Itô integral, we obtain that

$$\begin{aligned}&\mathbf{E}\sup_{r\in[t_0,t]}\Big|\int_{t_0}^{r}[g(s,X_s)-g(s,X_s^n)]dB(s)\Big|^p \leq c_p\mathbf{E}\Big[\int_{t_0}^{t}|g(s,X_s)-g(s,X_s^n)|^2ds\Big]^{\frac{p}{2}} \\ &\leq c_p(t-t_0)^{\frac{p}{2}-1}\int_{t_0}^{t}\mathbf{E}\big|g(s,X_s)-g(s,X_s^n)\big|^p ds\end{aligned}$$

where c_p is a universal constant, more precisely, $c_p=[p^{p+1}/2(p-1)^{p-1}]^{\frac{p}{2}}$ for $p>2$ and $c_p=4$ for $p=2$. The other Itô integrals can be estimated analogously, Thus according to Lemma 2.3 and the condition (M1), we get that

$$\begin{aligned}&J_{22}(t) \\ &\leq 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[\int_{t_0}^{t}\mathbf{E}|g(s,X_s)-g(s,X_s^n)|^p ds+\int_{t_0}^{t}\mathbf{E}|g(s,X_s^n)-g_n(s,X_s^n)|^p ds \\ &\quad +\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s^n)-g_n(s,X_s)|^p ds+\int_{t_0}^{t}\mathbf{E}|g_n(s,X_s)-g_n(s,X_s^{n+1})|^p ds\Big] \\ &\leq 4^{p-1}c_p(t-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^n||^p ds+K^p\int_{t_0}^{t}\mathbf{E}||X_s-X_s^{n+1}||^p ds+(t-t_0)\varepsilon_n\Big] \\ &\leq 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}(\gamma_n+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p)ds \\ &\quad +K^p\int_{t_0}^{t}(\gamma_{n+1}+\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p)ds+(t-t_0)\varepsilon_n\Big] \\ &= 4^{p-1}c_p(T-t_0)^{\frac{p}{2}-1}\Big[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p ds \\ &\quad +K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p ds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Big]\end{aligned}$$

Therefore,

$$\begin{aligned}J_2(t) \leq \quad & C\Bigg[2K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^n(r)|^p ds \\ &+K^p\int_{t_0}^{t}\mathbf{E}\sup_{r\in[t_0,s]}|X(r)-X^{n+1}(r)|^p ds+(t-t_0)(2K^p\gamma_n+K^p\gamma_{n+1}+\varepsilon_n)\Bigg]\end{aligned} \tag{2.6}$$

with the positive constant $C = 8^{p-1}\Big[(T-t_0)^{p-1} + c_p(T-t_0)^{\frac{p}{2}-1}\Big]$.

Introduce the following notations

$$u_n(t) = \mathbf{E}\sup_{r\in[t_0,t]}|X(r)-X^n(r)|^p, \quad t\in[t_0,T], \quad n=0,1,\cdots$$

Then from the inequalities (2.4), (2.5) and (2.6), we can easily obtain

$$u_{n+1}(t) \le \frac{k^p}{\theta^{4(p-1)}}u_{n+1}(t)+\bar{\alpha}\int_{t_0}^t u_{n+1}(s)ds+2\bar{\alpha}\int_{t_0}^t u_n(s)ds+\bar{\beta}u_n(t)+(t-t_0)\mu_n+\nu_n$$

where $\bar{\beta} = \frac{k^p}{\theta^{2(p-1)}}[\frac{1}{(1-\theta)^{2(p-1)}} + \frac{1}{(\theta(1-\theta))^{p-1}}]$, $\bar{\alpha}$ is a positive constant, and $\mu_n = a_1\gamma_n + a_2\gamma_{n+1} + a_3\varepsilon_n$, $\nu_n = b_1\gamma_n + b_2\gamma_{n+1} + b_3\delta_n$, here a_i, b_i (i=1,2,3) are generic positive constants. Recall the inequality (2.2) we have $k^p < 3k^p \le \theta^{4(p-1)}$ and so $1-\frac{k^p}{\theta^{4(p-1)}} > 0$, then one can see that

$$u_{n+1}(t) \le \alpha\int_{t_0}^t u_{n+1}(s)ds + 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n$$

where α, λ are generic positive constants and $\beta = \frac{k^p}{\theta^{4(p-1)}-k^p}[(\frac{\theta}{1-\theta})^{2(p-1)}+(\frac{\theta}{1-\theta})^{(p-1)}]$. This and Lemma 2.2 yield that

$$\begin{aligned}
u_{n+1}(t) \quad &\le \quad 2\alpha\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \lambda(t-t_0)\mu_n + \lambda\nu_n \\
&\quad +\alpha\int_{t_0}^t e^{\alpha(t-s)}\Big[2\alpha\int_{t_0}^s u_n(r)dr + \beta u_n(s) + \lambda(s-t_0)\mu_n + \lambda\nu_n\Big]ds \\
&= \quad 2\alpha\int_{t_0}^t u_n(s)ds + 2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds + \alpha\beta\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds \\
&\quad +\alpha\lambda\mu_n\int_{t_0}^t(s-t_0)e^{\alpha(t-s)}ds + \alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds + \beta u_n(t) \\
&\quad +\lambda(t-t_0)\mu_n + \lambda\nu_n.
\end{aligned}$$

By direct computation we note that

$$\begin{aligned}
&2\alpha^2\int_{t_0}^t e^{\alpha(t-s)}\int_{t_0}^s u_n(r)drds = 2\alpha\int_{t_0}^t(e^{\alpha(t-s)}-1)u_n(s)ds, \\
&\alpha\lambda\mu_n\int_{t_0}^t(s-t_0)e^{\alpha(t-s)}ds = -\lambda t\mu_n - \frac{\lambda}{\alpha}\mu_n + \frac{\lambda e^{\alpha(t-t_0)}}{\alpha}\mu_n + \lambda t_0\mu_n, \\
&\alpha\lambda\nu_n\int_{t_0}^t e^{\alpha(t-s)}ds = -\lambda\nu_n + \lambda e^{\alpha(t-t_0)}\nu_n.
\end{aligned}$$

Therefor we have

$$\begin{aligned}
u_{n+1}(t) \quad &\le \quad \alpha(2+\beta)\int_{t_0}^t e^{\alpha(t-s)}u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(t-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(t-t_0)}\nu_n \\
&\le \quad \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t u_n(s)ds + \beta u_n(t) + \frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\mu_n + \lambda e^{\alpha(T-t_0)}\nu_n.
\end{aligned}$$

Thus, for every $m = 1, 2, ...,$ we have

$$\sum_{n=0}^m u_n(t) - u_0(t) \quad \le \quad \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^t\sum_{n=0}^m u_n(s)ds + \beta\sum_{n=0}^m u_n(t)$$

$$+\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n.$$

Hence,

$$\begin{aligned}(1-\beta)\sum_{n=0}^{m}u_n(t)\leq \alpha(2+\beta)e^{\alpha(T-t_0)}\int_{t_0}^{t}\sum_{n=0}^{m}u_n(s)ds\\ +\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\end{aligned}$$

Noting $0<\frac{\theta}{1-\theta}<1$ and $\theta^{4(p-1)}-k^p\geq 2k^p$, one has that $\beta<\frac{2k^p}{\theta^{4(p-1)}-k^p}\leq 1$. From this and by Lemma 2.2, we get that

$$\begin{aligned}\sum_{n=0}^{m}u_n(t) &\leq \frac{1}{1-\beta}\Big[\frac{\lambda e^{\alpha(T-t_0)}-\lambda}{\alpha}\sum_{n=0}^{m}\mu_n+\lambda e^{\alpha(T-t_0)}\sum_{n=0}^{m}\nu_n+u_0(T)\Big]\\ &\quad\cdot e^{\frac{\alpha(2+\beta)e^{\alpha(T-t_0)}}{1-\beta}(t-t_0)}\end{aligned}$$

The condition (1.9) implies that $\sum_{n=0}^{\infty}\mu_n<\infty$ and $\sum_{n=0}^{\infty}\nu_n<\infty$. Therefore,

$$\sum_{n=0}^{\infty}u_n(T)<\infty.$$

Recall that

$$\begin{aligned}\mathbf{E}\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|^p &\leq \mathbf{E}||\xi-\xi^n||^p+\mathbf{E}\sup_{t\in[t_0,T]}|X(t)-X^n(t)|^p\\ &= \gamma_n+u_n(T),\end{aligned}$$

then from the condition (1.9) and Doob's martingale inequality [7], we find for an arbitrary $\varepsilon>0$ that,

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{P}\Big\{\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\geq\varepsilon\Big\} &\leq \frac{1}{\varepsilon^p}\sum_{n=0}^{\infty}\mathbf{E}|X(T)-X^n(T)|^p\\ &\leq \frac{1}{\varepsilon^p}\Big[\sum_{n=0}^{\infty}\gamma_n+\sum_{n=0}^{\infty}u_n(T)\Big]\\ &< \infty.\end{aligned}$$

Hence, by applying the Borel-Cantelli lemma, we conclude that for an arbitrary small $\varepsilon>0$, there exists a positive integer $n_0=n_0(\omega)$ such that

$$\sup_{t\in(-\infty,T]}|X(t)-X^n(t)|\leq\varepsilon,\qquad n\geq n_0.$$

This shows that the sequence $\{X^n(t),\ t\in(-\infty,T],\ n=0,1,...\}$ converges with probability one to the solution $\{X(t),\ t\in(-\infty,T]\}$. The proof is complete. □

3. Comments and examples

In [1, 8, 9], the proof of the existence of the solution to the equation (1.5) is based on the well-known Picard iteration, which establishes the iteration on the solution. However, the Z-algorithm method iterates for the whole equation. The Z-algorithm is a more general algorithm and can be applied to discuss more equations. Many historically and well-known analytic methods are its special cases, for example,

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Picard iteration, Newton-Kantorovich method and Chaplygin methods of chords and tangents.

Next, we give some examples to illustrate Theorem 2.4.

Example 1: Let $\{\xi^n, n = 0, 1, ...\}$ satisfy $\sum_{n=0}^{\infty} \mathbf{E}||\xi - \xi^n||^p < \infty$ and let D_n, f_n, g_n be defined in the following way: for $n = 0, 1, ...$, $X \in BC((-\infty, 0]; \mathbf{R}^d)$ and for every fixed $t \in [t_0, T]$,

$$D_n(X; X_t^n) := \varphi_n(X - X_t^n) + D(X_t^n) \tag{3.1}$$

$$f_n(t, X; X_t^n) := \psi_n(X - X_t^n) + f(t, X_t^n) \tag{3.2}$$

$$g_n(t, X; X_t^n) := \theta_n(X - X_t^n) + g(t, X_t^n), \tag{3.3}$$

where

$$\begin{aligned} &\varphi_n : BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \varphi_n(0) = 0 \\ &\psi_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^d,\ \psi_n(t, 0) \equiv 0 \\ &\theta_n : [t_0, T] \times BC((-\infty, 0]; \mathbf{R}^d) \to \mathbf{R}^{d+m},\ \theta_n(t, 0) \equiv 0. \end{aligned}$$

The functions φ_n, ψ_n, θ_n satisfy the conditions (M1)-(M3) with constants $K > 0$ and $k \in \left(0, 1/(3 \cdot 2^{4(p-1)})^{\frac{1}{p}}\right)$. Obviously,

$$\begin{aligned} &D_n(X_t^n; X_t^n) - D(X_t^n) \equiv 0 \\ &f_n(t, X_t^n; X_t^n) - f(t, X_t^n) \equiv 0 \\ &g_n(t, X_t^n; X_t^n) - g(t, X_t^n) \equiv 0. \end{aligned}$$

This shows that the condition (1.9) is satisfied. Thus theorem 2.4 is obtained.

Example 2: In particular, we linearize the equation (3.1) by: for $n = 0, 1, ...,$

$$D_n(X; X_t^n) := (X - X_t^n) \cdot \varphi_n + D(X_t^n) \tag{3.4}$$

$$f_n(t, X; X_t^n) := (X - X_t^n) \cdot \psi_n + f(t, X_t^n) \tag{3.5}$$

$$g_n(t, X; X_t^n) := (X - X_t^n) \cdot \theta_n + g(t, X_t^n), \tag{3.6}$$

where $\theta_n = (\theta_{1n}, \theta_{2n}, ..., \theta_{mn})$ and φ_n, ψ_n, $\theta_{in} (i = 1, 2, ..., m)$ are scalar sequences. We can easily see that all conditions of Theorem 2.4 are satisfied. So Theorem 2.4 succeed.

Example 3: More specifically, we assume that $\xi^n = \xi$ a.s. and $\varphi_n = \psi_n = \theta_n = 0$ in equation (3.4) for all $n = 0, 1, ...$, then we obtain the Picard iteration. Of course, in this case, Theorem 2.4 is successful. Therefore, the Picard iteration is a special Z-algorithm.

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Conflict of interest:

The authors declared that they have no conflicts of interest to this work. We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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An improved generalized parameterized inexact Uzawa method for singular saddle point problems *

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Abstract

In this paper, based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and Wang [*Applied Mathematics and Computation*, 219(2013) 4225-4231], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems. Moreover, theoretical analysis shows that the semi-convergence of the IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the improved generalized parameterized inexact Uzawa method (IGPIU) with appropriate parameters improve the convergence of iteration method efficiently when solving singular saddle point problems from the classic incompressible steady state Stokes problems.

Key words: Krylov subspace methods; Generalized saddle point matrices; Minimal polynomial; Preconditioners.

MSC: 65F10; 65F15

1 Introduction

Consider a singular saddle point problem

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1}$$

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where $A \in R^{m\times m}, B \in R^{m\times n}, m \geq n$. The matrix A is symmetric positive matrix and the matrix and B is a rank-deficient matrix. Systems of the form (1) arise in a variety of scientific and engineering applications and have attracted a lot of research, see [1-7] for a comprehensive survey. When A is symmetric positive definite and B is of full column rank, we refer the reader to [2,7-18] for many efficient iterative methods and [19] for a survey.

For large, sparse or structure matrices, iterative method is an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1}b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [20].

Generally speaking, the matrix B is full column rank, but not always. If B is rank-deficient, how to effectively solve the singular saddle point problem (1) is important in both scientific computing and engineering applications. For solving the rank-deficient saddle point problems, Ma and Zheng et al. [17,21] presented the parameterized Uzawa method. Bai et al. [22-23] studied the PHSS iteration method. Fischer et al. [24] considered the preconditioned minimum residual (PMINRES) method. Wu et al. [7] discussed the preconditioned conjugate gradient (PCG) method. Zhang and Wang [17] introduced the generalized parameterized inexact Uzawa (GPIU) method.

In this paper, based on the generalized parameterized inexact Uzawa (GPIU) method presented by Zhang and Wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Similar to the proving process of section 3 in [17], theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, one numerical example presented shows correctness and availability of our theory about the improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems.

This paper is organized as follows. In Section 2, we will present the improved generalized parameterized inexact uzawa method (IGPIU) for singular saddle point problems (1). The semi-convergence of the IGPIU method are discussed in Section 3. Moreover, our methods are the generalization of known literature. Some numerical examples are given to demonstrate the efficiency of the IGPIU method in Section 4. Finally, conclusions are made in Section 5.

2 An improved generalized parameterized inexact uzawa method (IGPIU)

Recently, for singular saddle point problems (1), Zhang and Wang [17] make the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{M} - \mathcal{N},$$

where

$$\mathcal{M} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & Q_2 \end{pmatrix}, \mathcal{N} = \begin{pmatrix} P - A & -B \\ Q_1 & Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix.

To construct the improved generalized parameterized inexact Uzawa method (IGPIU), if we can add one parameter in the above splitting, then we may change the parameter to improve the performance of presented method. Hence, we propose the following splitting

$$\mathcal{A} := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{L} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}, \mathcal{U} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix}$$

$P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ is an arbitrary matrix. Based the generalized parameterized inexact Uzawa (GPIU) iteration method presented by Zhang and Wang [17], we consider an improved generalized parameterized inexact uzawa method (IGPIU) for solving the singular saddle point (1).

$$\begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} p \\ -q \end{pmatrix}, \tag{2}$$

or equivalently,

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega}Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega}Q_2^{-1}Q_1\left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{3}$$

The iteration matrix of the IGPIU method (2) or (3) is given by

$$\mathcal{T} = \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix}^{-1} \begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} = I - \mathcal{L}^{-1}\mathcal{A}. \tag{4}$$

The IGPIU method: Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be prescribed symmetric positive definite matrices and $Q_1 \in R^{n\times m}$ be an arbitrary matrix. Given initial vector $x^{(0)} \in R^m$ and $y^{(0)} \in R^n$ and the relaxation parameter ω with $\omega \neq 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{(x^{(k)^T}, y^{(k)^T})^T\}$ is convergent, compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + P^{-1}\left[p - Ax^{(k)} - By^{(k)}\right], \\ y^{(k+1)} = y^{(k)} + \frac{1}{\omega}Q_2^{-1}\left[B^T x^{(k+1)} - q\right] - \frac{1}{\omega}Q_2^{-1}Q_1\left[x^{(k+1)} - x^{(k)}\right]. \end{cases} \tag{5}$$

Remark 2.1. It is obvious that when choosing $\omega = 1$, then the IGPIU method reduces to the GPIU method [17]. Hence, we may change the parameter to improve the performance of presented method.

3 The semi-convergence of IGPIU method

In this section, we discuss the semi-convergence of the IGPIU method for solving the singular saddle point problem (1). We first reveal some basic concepts and notations.

Denote $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ as the spectrum and spectral radius of the matrix $\mathcal{A}$, respectively. The *rank* and *index* of the matrix $\mathcal{A}$ are denoted by *rank*$(\mathcal{A})$ and *index*$(\mathcal{A})$, respectively.

Assume that the matrix $\mathcal{A}$ can be split into $\mathcal{A} = \mathcal{M} - \mathcal{N}$ with $\mathcal{M}$ nonsingular. Then we can construct a splitting iteration method:

$$x^{(k+1)} = \mathcal{T}x^{(k)} + \mathcal{M}^{-1}c, k = 0, 1, 2, ... \tag{6}$$

where $\mathcal{T} = \mathcal{M}^{-1}\mathcal{N}$ is the iteration matrix.

It is well known that any of the following three conditions is necessary and sufficient for guaranteeing the semi-convergence of the iteration method (6) for the singular linear systems $\mathcal{A}X = c$ (see [17,22]):

(a) The spectral radius of the iteration matrix $\mathcal{T}$ is equal to one , i.e., $\rho(\mathcal{T}) = 1$;

(b) The elementary divisor associated with $\lambda \in \sigma(\mathcal{A})$ is linear when $\lambda = 1$, i.e., $rank((I-\mathcal{T})^2) = rank(I - \mathcal{T})$, or equivalently, $index(I - \mathcal{T}) = 1$;

(c) If $\lambda \in \sigma(\mathcal{T})$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\mathcal{V}(\mathcal{T}) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\} < 1$.

When iteration scheme (6) is semi-convergent, $\mathcal{V}(\mathcal{T})$ is said to be the semi-convergence factor. As usual, the splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ and the corresponding iteration matrix $\mathcal{T}$ are called as semi-convergent if the iteration (6) is semi-convergent. Next we study the semi-convergence of the IGPIU iteration (5). To get the semi-convergence conditions, the following lemmas are used.

Lemma 3.1. **[25]** *Consider the quadratic equation $x^2 - \delta x + \eta = 0$, where δ and η are real numbers. Both roots of the equation are less than one in modulus if and only if $|\eta| < 1$ and $|\delta| < 1 + \eta$.*

Lemma 3.2. **[17]** *Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite and $B \in R^{m\times n}$ be of column rank-deficient, with $m \geq n$. Suppose that λ is an eigenvalue of the iteration matrix $\mathcal{T}$ and $(u^T, v^T)^T \in R^{m+n}$ is the corresponding eigenvector. Then $\lambda = 1$ if and only if $u = 0$.*

Theorem 3.3. *Let $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ be symmetric positive definite and $B \in R^{m\times n}$ be of column rank-deficient, with $m \geq n$. Suppose that $\lambda \neq 1$ is an eigenvalue of the iteration matrix $\mathcal{T}$ and $(u^T, v^T)^T \in R^{m+n}$ is the corresponding eigenvector. Then λ satisfies the following quadratic equation:*

$$\lambda^2 + \frac{\beta + \gamma - 2\omega\alpha - \tau}{\alpha}\lambda + \frac{\alpha + \tau - \omega\beta}{\alpha} = 0,$$

where

$$\alpha = \frac{u^*Pu}{u^*u} > 0, \beta = \frac{u^*Au}{u^*u} > 0, \gamma = \frac{u^*BQ_2^{-1}B^Tu}{u^*u} \geq 0, \tau = \frac{u^*BQ_2^{-1}Q_1u}{u^*u}.$$

Proof. Firstly, since $\lambda \neq 1$, we know $u \neq 0$ from Lemma 3.2. By (4) we have

$$\begin{pmatrix} P - A & -B \\ Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} P & 0 \\ -B^T + Q_1 & \omega Q_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{7}$$

or equivalently

$$\begin{cases} [(1-\lambda)P - A]\, u = Bv, \\ \left[(1-\lambda)Q_1 + \lambda B^T\right] u = \omega(\lambda - 1)Q_2 v. \end{cases} \tag{8}$$

Because Q_2 is symmetric positive definite and $\lambda \neq 1$, from the second equation in (8), we obtain that $v = \frac{1}{\omega}(-Q_2^{-1}Q_1 + \frac{\lambda}{\lambda-1}Q_2^{-1}B^T)u$, which together with the first equation in (8), result in

$$\omega\lambda^2 Pu + \lambda\omega(Au + BQ_2^{-1}B^Tu - 2\omega Pu - BQ_2^{-1}Q_1u) + \omega(Pu + BQ_2^{-1}Q_1u - \omega Au) = 0. \tag{9}$$

4

Since $u \neq 0$, by left multiplying u^* and with the positive definiteness of $P(u^*Pu \neq 0)$, we have

$$\lambda^2 + \tfrac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\lambda + \tfrac{\alpha+\tau-\omega\beta}{\alpha} = 0. \tag{10}$$

Thus, the proof is completed.

Theorem 3.4. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric. Then $\sigma(\mathcal{T}) < 1$ holds if and only if one of the following conditions hold:*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Proof. Since P is symmetric positive definite and $BQ_2^{-1}Q_1$ is symmetric. By Lemma 3.1 and Eq. (10), we know that the spectral radius of the IGPIU iteration matrix is less than one in modulus if and only if

$$\begin{cases} \left|\frac{\alpha+\tau-\omega\beta}{\alpha}\right| < 1, \\ \left|\frac{\beta+\gamma-2\omega\alpha-\tau}{\alpha}\right| < 1 + \frac{\alpha+\tau-\omega\beta}{\alpha}. \end{cases} \tag{11}$$

or equivalently,

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Thus, the proof is completed. $\diamond$

Theorem 3.5. **[17]** *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$, then index$(I-\mathcal{T}) = 1$. Here and in the sequel, $\mathcal{N}(\bullet)$ and $\mathcal{R}(\bullet)$ is used to represent the null space and range space of the corresponding matrix, respectively.*

Combining Theorem 3.4 and Theorem 3.5 , we immediately obtain the following sufficient conditions for the convergence result of the IGPIU method for solving singular saddle point problem (1).

Theorem 3.6. *Assume that $A \in R^{m\times m}$ is symmetric positive definite, $B \in R^{m\times n}$ is rank-deficient, $P \in R^{m\times m}$ and $Q_2 \in R^{n\times n}$ are symmetric positive definite and $Q_1 \in R^{n\times m}$ is an arbitrary matrix such that $BQ_2^{-1}Q_1$ is symmetric and $\mathcal{N}(B) \cap \mathcal{R}(Q_2^{-1}B^TA^{-1}B) = \{0\}$. Then the IGPIU method for solving singular saddle point problem (1) is semi-convergent if $\omega, \tau, \beta, \gamma, \alpha$ satisfy one of the following conditions*

$$\omega > 0, \tau < \omega\beta, 0 < \frac{\gamma}{2} < (1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta.$$

Remark 3.1. From the Theorem 3.5, it is not difficult to find that when $\omega > 1, 2\alpha > \beta$ or $0 < \omega < 1, 2\alpha < \beta$, then $(1+\omega)\alpha + \tau - \frac{1+\omega}{2}\beta > 2\alpha + \tau - \beta$. Hence, under these conditions, the range of γ is wider and we will have more space of parameters range.

Remark 3.2. It is obvious that when choosing $\omega = 1, P = \frac{1}{\xi}A, Q_1 = 0,$ and $\mathrm{Q}_2 = \frac{1}{\zeta}\mathrm{Q}$, Q is an approximate matrix to the Schur complement $B^TA^{-1}B$, then the IGPIU method reduces to the PIU method in [10,17].

Remark 3.3. Some choices of the parameter matrices P, Q_1 and Q_2 are given in Table 1 [17]. When choosing different parameter matrices P, Q_1 and Q_2, we may immediately obtain a series of iterative methods for solving singular saddle problem (1).

Table 1: Some choices of the parameter matrices P, Q_1 and Q_2.

Case	P	Q_1	Q_2
I	$\frac{1}{\xi}A$	0	$\frac{1}{\zeta}I_n$
II	$\frac{1}{\xi}diag(A)$	0	$\frac{1}{\zeta}I_n$
III	$\frac{1}{\xi}tridiag(A)$	0	$\frac{1}{\zeta}I_n$
IV	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}I_n$
V	$\frac{1}{\xi}A$	$-\frac{\theta}{\zeta}B^T$	$\frac{1}{\zeta}diag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$
VI	$\frac{1}{\xi}A$	$-\theta Q_2 B^T$	$\frac{1}{\zeta}tridiag(\hat{B}^T P^{-1}\hat{B}, \tilde{B}^T\tilde{B})$

4 Numerical examples

In this section, we give numerical experiments to demonstrate the conclusions drawn above. The numerical experiments were done by using MATLAB 7.1 and the matrix of the numerical experiments were generated by IFISS software. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least seven orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-7}\|b\|_2$).

We consider the classic incompressible steady state Stokes problems:

$$\begin{cases} -\Delta u + gradp = f, \text{ in } \Omega, \\ -divu = 0, \text{ in } \Omega, \end{cases}$$

with suitable boundary condition on $\partial\Omega$. It is known that many discretization schemes for the above Stokes problems will lead to generalized saddle point problems of the form (1). Here, we get the test problem (leak-lid driven cavity) by using IFISS software written by David Silvester, Howard Elman and Alison Ramage. We take a finite element subdivision based on 32×32 uniform grids of square elements. The mixed finite element used is the bilinear-constant velocity-pressure: $Q_1 - P_0$ pair with stabilization. $Q_1 - P_0$ finite element subdivision is shown in Figure 1. The stabilization parameter is chosen to $\frac{1}{4}$. We get the (1,1) block A of the coefficient matrix corresponding to the discretization of the conservative term. Since the matrix B produced by the software is rank deficient, so $\mathcal{A}$ is singular matrix. In our experiment, we choose uniform grids 8×8, 16×16.

In Tables 2, when choosing different parameters, we show iteration counts, relative residual and computing time about the GPIU and the IGPIU methods for solving singular saddle problem (1), where IT, RES and CPU are the iteration numbers, relative residual and computing time about the GPIU and the IGPIU methods, respectively. Moreover, we also show the corresponding reduction of residual 2-norm and eigenvalues distributions about two methods for different parameters. Figures 2 $\sim$ 5 show the reduction of residual 2-norm with Case I, II, III and IV of Table 2. Figures 6 $\sim$ 9 show the eigenvalues distributions with Case I, II, III and IV of Table 2. Figures 10 and 11 show the reduction of residual 2-norm with uniform grids 16×16 and Cases I, II. Figures 12 and 13 show the eigenvalues distributions with uniform grids 16×16 and Cases I, II.

Remark 3.1. From Table 2, Figures 2 $\sim$ 5, 10 and 11 , it is very easy to get that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. By numerical experiments for many times, we can find that, when $0.75 \leq \omega \leq 1.05$ the IGPIU method is very efficient. For Case II, when $\omega = 1.05$ the IGPIU method is little efficient. Hence, we suggest that, the selection range of the parameters may be $0.75 \leq \omega \leq 1$.

Remark 3.2. From Figures 6 $\sim$ 9,12 and 13, we may find that the eigenvalue distribution about the GPIU method has the same spectral clustering compared with the IGPIU method when choosing suitable parameters.

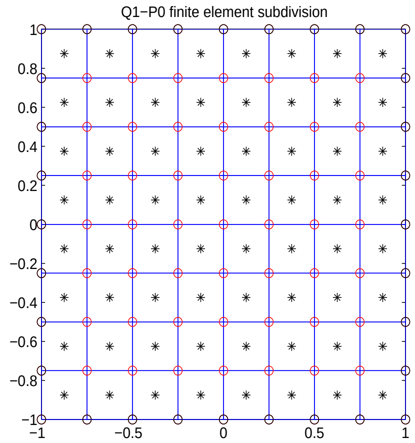

Figure 1: $Q_1 - P_0$ finite element subdivision

Table 2: numerical results of different parameters about GPIU and IGPIU methods for solving singular saddle problem (1). Here, uniform grids are 8×8.

Case	$(\xi, \theta, \zeta, \omega)$	IT	RES	CPU
Case I	$(0.8, 0, 10, 1)$	124	9.6146×10^{-8}	1.776
	$(0.8, 0, 10, 0.85)$	100	9.9265×10^{-8}	1.437
	$(0.8, 0, 10, 0.75)$	103	9.9106×10^{-8}	1.468
	$(0.8, 0, 10, 1.05)$	79	9.8442×10^{-8}	1.156
Case II	$(0.8, 0, 10, 1)$	451	8.8682×10^{-8}	0.813
	$(0.8, 0, 10, 0.85)$	435	9.6783×10^{-8}	0.812
	$(0.8, 0, 10, 0.75)$	439	9.6727×10^{-8}	0.797
	$(0.8, 0, 10, 1.05)$	462	7.7405×10^{-8}	0.844
Case III	$(0.8, 0, 10, 1)$	375	7.2718×10^{-8}	2.797
	$(0.8, 0, 10, 0.85)$	356	9.7272×10^{-8}	2.672
	$(0.8, 0, 10, 0.75)$	354	8.3278×10^{-8}	2.703
	$(0.8, 0, 10, 1.05)$	373	7.3365×10^{-8}	2.829
Case IV	$(0.8, 0, 10, 1)$	141	9.3439×10^{-8}	1.984
	$(0.8, 0, 10, 0.85)$	124	9.3478×10^{-8}	1.734
	$(0.8, 0, 10, 0.75)$	118	9.91×10^{-8}	1.625
	$(0.8, 0, 10, 1.05)$	139	9.913×10^{-8}	1.907

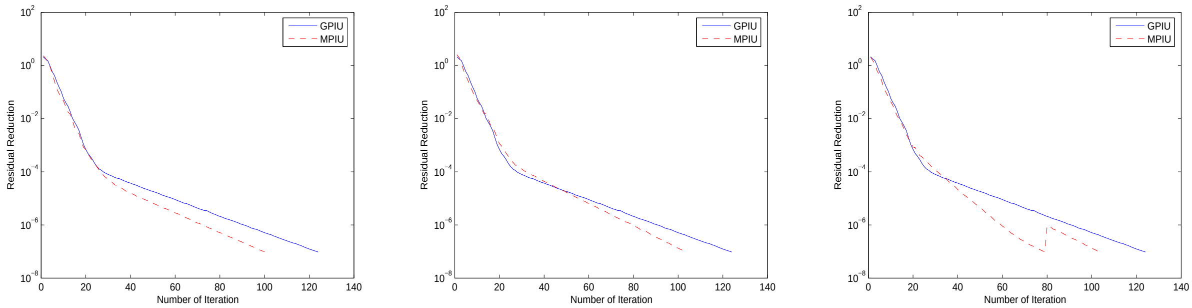

Figure 2: Reduction of residual 2-norm with Case I of Table 2. The left figure shows that the first line parameters (GPIU) of Case I compare with the second line parameters (IGPIU) of Case I; The middle figure shows that the first line parameters (GPIU) of Case I compare with the third line parameters (IGPIU) of Case I; The right figure shows that the first line parameters (GPIU) of Case I compare with the forth line parameters (IGPIU) of Case I.

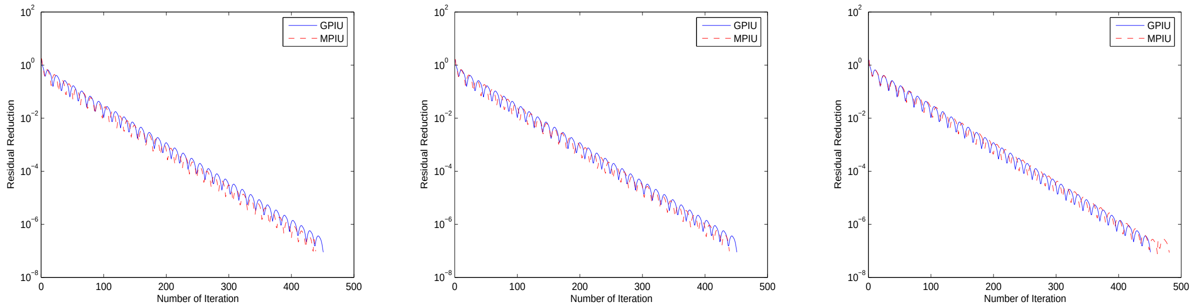

Figure 3: Reduction of residual 2-norm with Case II of Table 2. The left figure shows that the first line parameters (GPIU) of Case II compare with the second line parameters (IGPIU) of Case II; The middle figure shows that the first line parameters (GPIU) of Case II compare with the third line parameters (IGPIU) of Case II; The right figure shows that the first line parameters (GPIU) of Case II compare with the forth line parameters (IGPIU) of Case II.

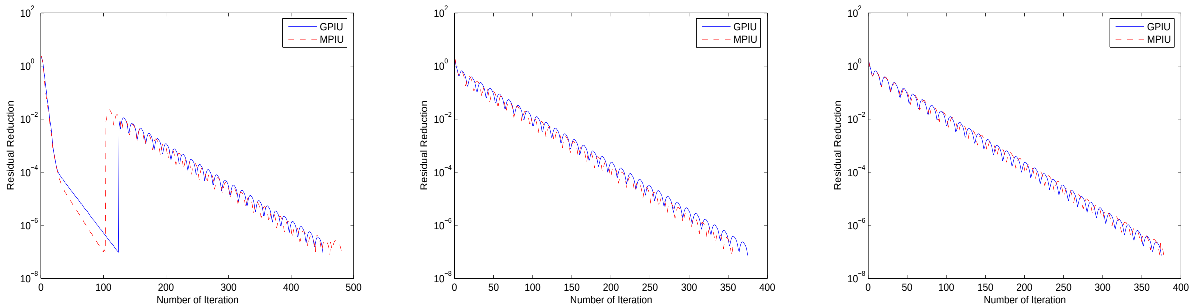

Figure 4: Reduction of residual 2-norm with Case III of Table 2. The left figure shows that the first line parameters (GPIU) of Case III compare with the second line parameters (IGPIU) of Case III; The middle figure shows that the first line parameters (GPIU) of Case III compare with the third line parameters (IGPIU) of Case III; The right figure shows that the first line parameters (GPIU) of Case III compare with the forth line parameters (IGPIU) of Case III.

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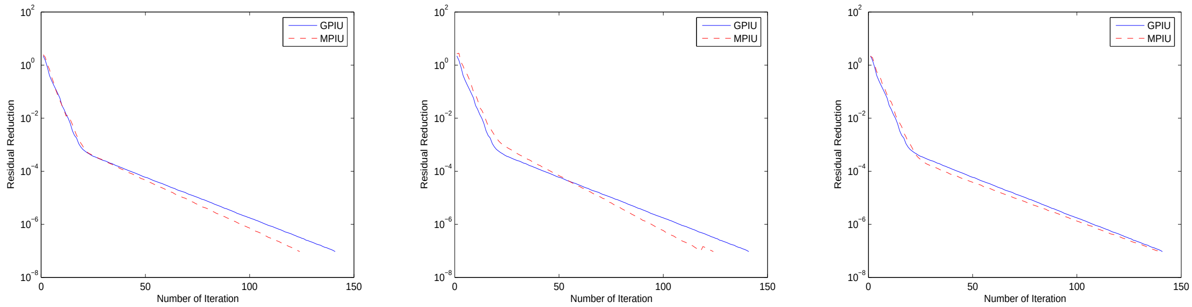

Figure 5: Reduction of residual 2-norm with Case IV of Table 2. The left figure shows that the first line parameters (GPIU) of Case IV compare with the second line parameters (IGPIU) of Case IV; The middle figure shows that the first line parameters (GPIU) of Case IV compare with the third line parameters (IGPIU) of Case IV; The right figure shows that the first line parameters (GPIU) of Case IV compare with the forth line parameters (IGPIU) of Case IV.

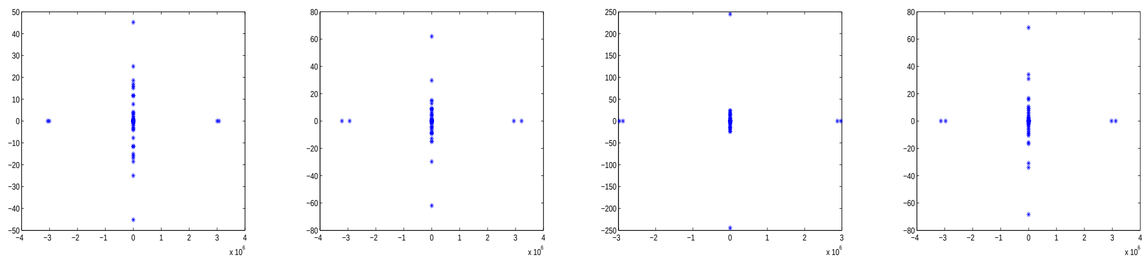

Figure 6: Eigenvalues distributions with Case I of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case I; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case I; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case I.

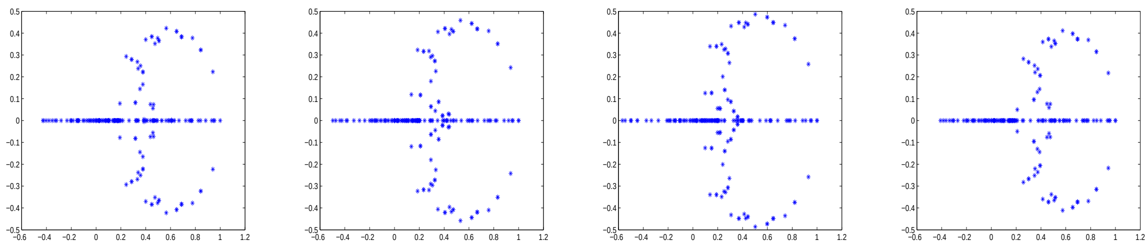

Figure 7: Eigenvalues distributions with Case II of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case II; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case II; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case II.

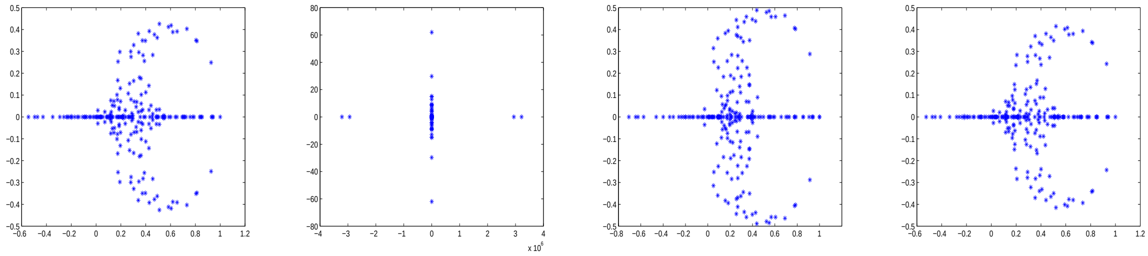

Figure 8: Eigenvalues distributions with Case III of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case III; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case III; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case III; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case III.

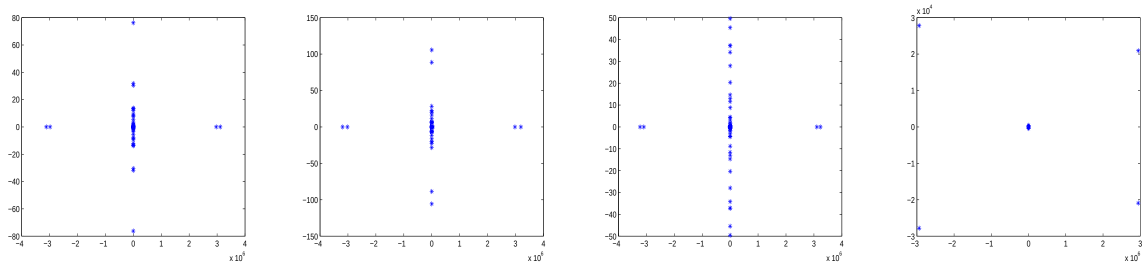

Figure 9: Eigenvalues distributions with Case IV of Table 2. The first figure shows eigenvalues distributions for the first line parameters (GPIU) of Case IV; The second figure shows eigenvalues distributions for the second line parameters (IGPIU) of Case IV; The third figure shows eigenvalues distributions for the third line parameters (IGPIU) of Case IV; The forth figure shows eigenvalues distributions for the forth line parameters (IGPIU) of Case IV.

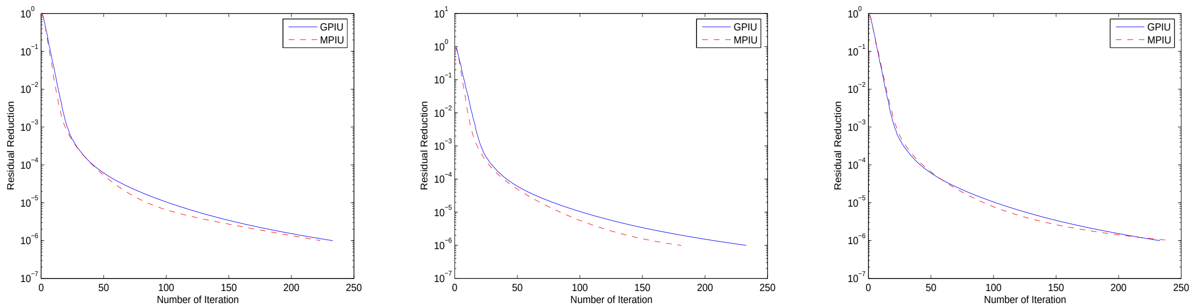

Figure 10: Reduction of residual 2-norm with uniform grids 16×16 and Case I. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case I; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case I compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

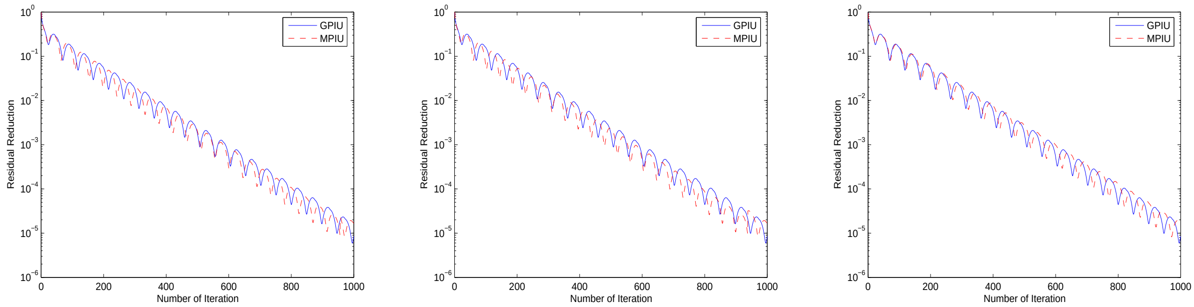

Figure 11: Reduction of residual 2-norm with uniform grids 16×16 and Case II. The left figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The middle figure shows that parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The right figure shows that the parameters $(1, 0, 10, 1)$ (GPIU) of Case II compare with the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

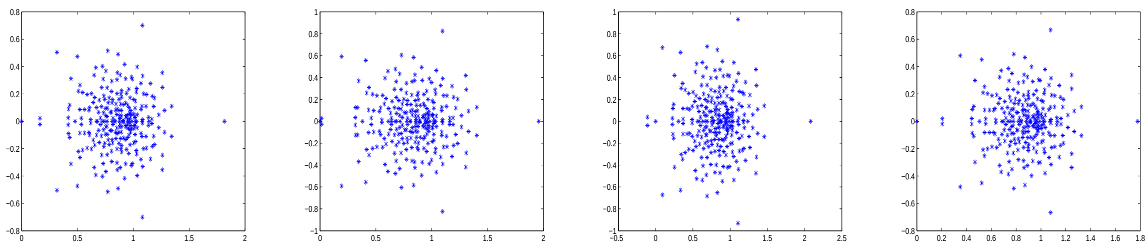

Figure 12: Eigenvalues distributions with uniform grids 16×16 and Case I. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case I; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) if Case I; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case I; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case I.

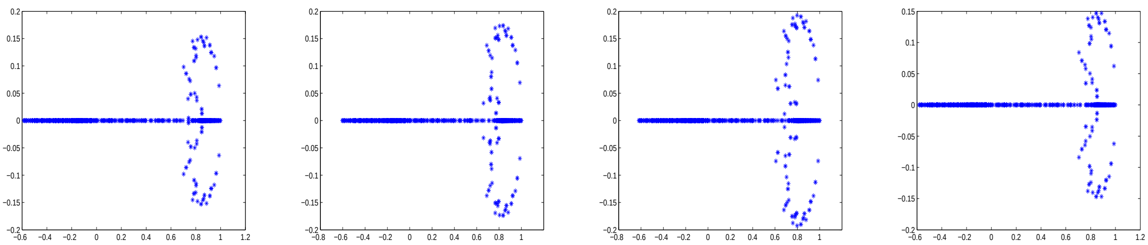

Figure 13: Eigenvalues distributions with uniform grids 16×16 and Case II. The first figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1)$ (GPIU) of Case II; The second figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.85)$ (IGPIU) of Case II; The third figure shows eigenvalues distributions for the parameters $(1, 0, 10, 0.75)$ (IGPIU) of Case II; The forth figure shows eigenvalues distributions for the parameters $(1, 0, 10, 1.05)$ (IGPIU) of Case II.

5 Conclusions

Based on the generalized parameterized inexact Uzawa method (GPIU) presented by Zhang and wang [17], we introduce and study an improved generalized parameterized inexact Uzawa method (IGPIU) for singular saddle point problems (1). Moreover, theoretical analysis shows that the semi-convergence of IGPIU method can be guaranteed by suitable choices of the iteration parameters. Finally, numerical experiments are carried out, which show that the IGPIU method is in general better than the GPIU method when choosing suitable parameters. Moreover, we also may find that the eigenvalue distribution about GPIU method has the same spectral clustering compared with IGPIU method when choosing suitable parameters.

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IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we study linear differential equations arising from Bessel polynomials and their applications. From these linear differential equations, we give some new and explicit identities for Bessel polynomials.

1. Introduction

As is well known, the Bessel differential equation is given by

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\alpha^2\right)y=0,\quad (\text{see [17]}). \tag{1.1}$$

for an arbitrary complex number α.

The Bessel functions of the first kind $J_\alpha(x)$ are defined by the solution of (1.1).

For $n\in\mathbb{Z}$, $J_n(x)$ are sometimes also called cylinder function or cylindrical harmonics.

It is known that

$$J_n(x)=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!\,(n+l)!}\left(\frac{x}{2}\right)^{2l+n},\quad (\text{see [1, 16, 17]}). \tag{1.2}$$

The generating function of Bessel functions is given by

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=-\infty}^{\infty}J_n(x)\,t^n, \tag{1.3}$$

and $J_n(x)$ can be also represented by the contour integral as

$$J_n(x)=\frac{1}{2\pi i}\oint e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}t^{-n-1}dt,\quad (\text{see [17]}), \tag{1.4}$$

where the contour encloses the origin and is traversed in a counterclockwise direction.

The Bessel polynomials are defined by the solution of the differential equation

$$x^2\frac{d^2y}{dx^2}+2(x+1)\frac{dy}{dx}-n(n+1)y=0,\quad (\text{see [1–6, 15, 16]}). \tag{1.5}$$

Indeed, the solutions of (1.5) are given by

$$\begin{aligned} y_n(x)&=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k \\ &=\sqrt{\frac{2}{\pi x}}e^{\frac{1}{x}}K_{-n-\frac{1}{2}}\left(\frac{1}{x}\right),\quad (\text{see [1, 15–17]}), \end{aligned} \tag{1.6}$$

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where

$$K_\nu(z) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{\left(t^2+z^2\right)^{\nu+\frac{1}{2}}} dt.$$

We note that $y_n(x)$ are very similar to the modified spherical Bessel function of the second kind.

The first few are given as

$$\begin{aligned} &y_0(x) = 1, \quad y_1(x) = x+1, \quad y_2(x) = 3x^2+3x+1, \\ &y_3(x) = 15x^3+15x^2+6x+1, \\ &y_4(x) = 105x^4+105x^3+45x^2+10x+1, \quad \ldots. \end{aligned}$$

Carlitz reverse Bessel polynomials are defined by

$$p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right), \quad (n \in \mathbb{N} \cup \{0\}), \quad (\text{see } [4, 15]). \tag{1.7}$$

These polynomials are also given by the generating function as

$$e^{x\left(1-\sqrt{1-2t}\right)} = \sum_{n=0}^\infty p_n(x) \frac{t^n}{n!}. \tag{1.8}$$

The explicit formulas for them are

$$\begin{aligned} p_n(x) &= \sum_{k=1}^n \frac{(2n-k-1)!}{2^{n-k}(k-1)!(n-k)!} x^k \\ &= (2n-3)!! x \, {}_1F_1(1-n; 2-2n; 2x), \quad (\text{see } [1, 15, 16]), \end{aligned} \tag{1.9}$$

where

$$n!! = \begin{cases} n(n-2)\cdots 5\cdot 3\cdot 1 & \text{if } n>0 \text{ odd}, \\ n(n-2)\cdots 6\cdot 4\cdot 2 & \text{if } n>0 \text{ even}, \\ 1 & \text{if } n=-1, 0, \end{cases}$$

and

$$\begin{aligned} {}_1F_1(a; b; z) &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \\ &= \sum_{k=0}^\infty \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)}\frac{z^k}{k!} \\ &= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \end{aligned}$$

The first few polynomials are

$$\begin{aligned} &p_1(x) = x, \\ &p_2(x) = x^2+x, \\ &p_3(x) = x^3+3x^2+3x, \\ &p_4(x) = x^4+6x^3+15x^2+15x, \cdots. \end{aligned}$$

Recently, several authors have studied non-linear differential equations related to special polynomials (see [7–14]).

The reverse Bessel polynomials are used in the design of Bessel electronic filters.

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In this paper, we consider linear differential equations arising from Carlitz reverse Bessel polynomials and give some new and explicit identities for Bessel polynomials.

2. Identities involving Bessel polynomials arising from linear differential equations

Let us put

$$F = F(t,x) = e^{x\left(1-\sqrt{1-2t}\right)}. \tag{2.1}$$

Thus, by (2.1), we get

$$F^{(1)} = \frac{d}{dt}F(t,x) = x(1-2t)^{-\frac{1}{2}}F, \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= \left(x(1-2t)^{-\frac{3}{2}} + x^2(1-2t)^{-1}\right)F, \end{aligned} \tag{2.3}$$

$$\begin{aligned} F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= \left(3x(1-2t)^{-\frac{5}{2}} + 3x^2(1-2t)^{-2} + x^3(1-2t)^{-\frac{3}{2}}\right)F, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} F^{(4)} &= \frac{dF^{(3)}}{dt} \\ &= \left(15x(1-2t)^{-\frac{7}{2}} + 15x^2(1-2t)^{-3} + 6x^3(1-2t)^{-\frac{5}{2}} + x^4(1-2t)^{-2}\right)F. \end{aligned} \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right)F, \end{aligned} \tag{2.6}$$

where $N = 1, 2, 3, \ldots$.

From (2.6), we note that

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt}F^{(N)} \\ &= \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)\left(-\frac{i}{2}\right)(1-2t)^{-\frac{i}{2}-1}(-2)\right)F \\ &\quad + \sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}F^{(1)} \end{aligned} \tag{2.7}$$

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$$
\begin{aligned}
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}}\right) F \\
&\quad + \left(\sum_{i=N}^{2N-1} a_{i-N}(N,x)(1-2t)^{-\frac{i}{2}}\right) x(1-2t)^{-\frac{1}{2}} F \\
&= \left(\sum_{i=N}^{2N-1} i a_{i-N}(N,x)(1-2t)^{-\frac{i+2}{2}}\right) F + \left(\sum_{i=N}^{2N-1} x a_{i-N}(N,x)(1-2t)^{-\frac{i+1}{2}}\right) F \\
&= \Big\{ x a_0(N,x)(1-2t)^{-\frac{N+1}{2}} + (2N-1) a_{N-1}(N,x)(1-2t)^{-\frac{2N+1}{2}} \\
&\quad + \sum_{i=N+1}^{2N-1} \left((i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x)\right)(1-2t)^{-\frac{i+1}{2}} \Big\} F.
\end{aligned}
$$

By replacing N by $N+1$ in (2.6), we get

$$
\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=N+1}^{2N+1} a_{i-N-1}(N+1,x)(1-2t)^{-\frac{i}{2}}\right) F \\
&= \left(\sum_{i=N}^{2N} a_{i-N}(N+1,x)(1-2t)^{-\frac{i+1}{2}}\right) F.
\end{aligned} \tag{2.8}
$$

By comparing the coefficients on both sides (2.7) and (2.8), we have

$$a_0(N+1,x) = x a_0(N,x), \tag{2.9}$$

$$a_N(N+1,x) = (2N-1) a_{N-1}(N,x), \tag{2.10}$$

and

$$a_{i-N}(N+1,x) = (i-1) a_{i-N-1}(N,x) + x a_{i-N}(N,x), \tag{2.11}$$

where $N+1 \le i \le 2N-1$.

From (2.2) and (2.6), we can derive the following equation (2.11):

$$x(1-2t)^{-\frac{1}{2}} F = F^{(1)} = a_0(1,x)(1-2t)^{-\frac{1}{2}} F. \tag{2.12}$$

Thus, by (2.12), we have

$$a_0(1,x) = x. \tag{2.13}$$

From (2.9), we note that

$$a_0(N+1,x) = x a_0(N,x) = x^2 a_0(N-1,x) = \cdots = x^N a_0(1,x) = x^{N+1}, \tag{2.14}$$

and, by (2.10), we see

$$
\begin{aligned}
a_N(N+1,x) &= (2N-1) a_{N-1}(N,x) \\
&= (2N-1)(2N-3) a_{N-2}(N-1,x) \\
&\ \ \vdots \\
&= (2N-1)(2N-3)\cdots 3 \cdot 1 a_0(1,x) \\
&= (2N-1)!!x.
\end{aligned} \tag{2.15}
$$

The matrix $(a_i(j,x))_{0\le i\le N-1, 1\le j\le N}$ is given by

$$\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & N \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \end{array} & \begin{bmatrix} x & x^2 & x^3 & x^4 & \cdots & x^N \\ & 1!!x & & & & \\ & & 3!!x & & & \\ & & & 5!!x & & \\ & & & & \ddots & \\ & 0 & & & & (2N-3)!!x \end{bmatrix}
\end{array}$$

From (2.11), we obtain

$$\begin{aligned}
&a_1(N+1,x) \\
&= Na_0(N,x) + xa_1(N,x) \\
&= Na_0(N,x) + x(N-1)a_0(N-1,x) + x^2a_1(N-1,x) \\
&\vdots \\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x) + x^{N-1}a_1(2,x) \\
&= \sum_{i=0}^{N-2} x^i(N-i)a_0(N-i,x) + x^{N-1}x \\
&= \sum_{i=0}^{N-1} x^i(N-i)a_0(N-i,x),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&a_2(N+1,x) \\
&= (N+1)a_1(N,x) + xa_2(N,x) \\
&= (N+1)a_1(N,x) + xNa_1(N-1,x) + x^2a_2(N-1,x) \\
&\vdots \\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x) + x^{N-2}a_2(3,x) \\
&= \sum_{i=0}^{N-3} x^i(N+1-i)a_1(N-i,x) + 3x^{N-2}a_1(2,x) \\
&= \sum_{i=0}^{N-2} x^i(N+1-i)a_1(N-i,x),
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&a_3(N+1,x) \\
&= (N+2)a_2(N,x) + xa_3(N,x) \\
&= (N+2)a_2(N,x) + x(N+1)a_2(N-1,x) + x^2a_3(N-1,x)
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\vdots\\
&=\sum_{i=0}^{N-4} x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right)+5x^{N-3}a_{2}\left(3,x\right)\\
&=\sum_{i=0}^{N-3} x^{i}\left(N-i+2\right)a_{2}\left(N-i,x\right).
\end{aligned}$$

Continuing this process, we get

$$a_{j}\left(N+1,x\right)=\sum_{i=0}^{N-j} x^{i}\left(N-i+j-1\right)a_{j-1}\left(N-i,x\right), \tag{2.19}$$

where $j=1,2,\ldots,N-1$.

Now, we give explicit expressions for $a_{j}\left(N+1,x\right)$ $\left(j=1,2,\ldots,N-1\right)$. From (2.14) and (2.16), we can easily derive the following equation:

$$\begin{aligned}
a_{1}\left(N+1,x\right)&=\sum_{i_1=0}^{N-1} x^{i_1}\left(N-i_1\right)a_{0}\left(N-i_1,x\right)\\
&=x^{N}\sum_{i_1=0}^{N-1}\left(N-i_1\right).
\end{aligned}\tag{2.20}$$

By (2.17), (2.18) and (2.19), we get

$$\begin{aligned}
a_{2}\left(N+1,x\right)&=\sum_{i_2=0}^{N-2} x^{i_2}\left(N-i_2+1\right)a_{1}\left(N-i_2,x\right)\\
&=x^{N-1}\sum_{i_2=0}^{N-2}\sum_{i_1=0}^{N-2-i_2}\left(N-i_2+1\right)\left(N-i_2-i_1-1\right),
\end{aligned}\tag{2.21}$$

$$\begin{aligned}
a_{3}\left(N+1,x\right)&=\sum_{i_3=0}^{N-3} x^{i_3}\left(N-i_3+2\right)a_{2}\left(N-i_3,x\right)\\
&=x^{N-2}\sum_{i_3=0}^{N-3}\sum_{i_2=0}^{N-3-i_3}\sum_{i_1=0}^{N-3-i_3-i_2}\left(N-i_3+2\right)\left(N-i_3-i_2\right)\\
&\quad\times\left(N-i_3-i_2-i_1-2\right),
\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}
a_{4}\left(N+1,x\right)&=\sum_{i_4=0}^{N-4} x^{i_4}\left(N-i_4+3\right)a_{3}\left(N-i_4,x\right)\\
&=x^{N-3}\sum_{i_4=0}^{N-4}\sum_{i_3=0}^{N-4-i_4}\\
&\quad\times\sum_{i_2=0}^{N-4-i_4-i_3}\sum_{i_1=0}^{N-4-i_4-i_3-i_2}\left(N-i_4+3\right)\left(N-i_4-i_3+1\right)\\
&\qquad\times\left(N-i_4-i_3-i_2-1\right)\left(N-i_4-i_3-i_2-i_1-3\right).
\end{aligned}\tag{2.23}$$

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Continuing this process, we get

(2.24)

$$\begin{aligned}&a_j\left(N+1,x\right)\\&=x^{N-j+1}\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots-i_2}\prod_{k=1}^{j}\left(N-i_j-\cdots-i_k-\left(j-\left(2k-1\right)\right)\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $N\in\mathbb{N}$, the linear differential equations*

$$F^{(N)}=\left(\frac{d}{dt}\right)^N F\left(t,x\right)=\left(\sum_{i=N}^{2N-1}a_{i-N}\left(N,x\right)\left(1-2t\right)^{-\frac{i}{2}}\right)F$$

has a solution $F=F\left(t,x\right)=e^{x\left(1-\sqrt{1-2t}\right)}$, where

$$\begin{aligned}&a_0\left(N,x\right)=x^N,\quad a_{N-1}\left(N,x\right)=\left(2n-3\right)!!x,\\&a_j\left(N,x\right)=x^{N-j}\sum_{i_j=0}^{N-j-1}\sum_{i_{j-1}=0}^{N-j-1-i_j}\cdots\sum_{i_1=0}^{N-j-1-i_j-\cdots-i_2}\\&\qquad\times\left(\prod_{k=1}^{j}\left(N-i_j-i_{j-1}-\cdots-i_k-\left(j-\left(2k-2\right)\right)\right)\right).\end{aligned}$$

Recall the the reverse Bessel polynomials $p_k\left(x\right)$ are given by the generating function as

$$\begin{aligned}F&=F\left(t,x\right)=e^{x\left(1-\sqrt{1-2t}\right)}\\&=\sum_{k=0}^{\infty}p_k\left(x\right)\frac{t^k}{k!}.\end{aligned}\tag{2.25}$$

Thus, by (2.25), we get

$$\begin{aligned}F^{(N)}&=\left(\frac{d}{dt}\right)^N F\left(t,x\right)\\&=\sum_{k=N}^{\infty}p_k\left(x\right)\left(k\right)_N\frac{t^{k-N}}{k!}\\&=\sum_{k=0}^{\infty}p_{k+N}\left(x\right)\left(k+N\right)_N\frac{t^k}{\left(k+N\right)!}\\&=\sum_{k=0}^{\infty}p_{k+N}\left(x\right)\frac{t^k}{k!}.\end{aligned}\tag{2.26}$$

On the other hand, by Theorem 1, we get

$$\begin{aligned}F^{(N)}&=\left(\sum_{i=N}^{2N-1}a_{i-N}\left(N,x\right)\left(1-2t\right)^{-\frac{i}{2}}\right)F\\&=\sum_{i=N}^{2N-1}a_{i-N}\left(N,x\right)\left(\sum_{l=0}^{\infty}\left(-\frac{i}{2}\right)_l\frac{\left(-2t\right)^l}{l!}\right)\left(\sum_{m=0}^{\infty}p_m\left(x\right)\frac{t^m}{m!}\right)\end{aligned}\tag{2.27}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x) \right\} \frac{t^k}{k!}.$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $k \in \mathbb{N} \cup \{0\}$, and $N \in \mathbb{N}$, we have*

$$p_{k+N}(x) = \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left(\frac{i}{2} + l - 1 \right)_l p_{k-l}(x),$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

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On existence and comparison results for solutions to stochastic functional differential equations in the G-framework

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Abstract

With the advancement in stochastic calculus, stochastic differential equations have now become very common in different fields such as engineering, population dynamics, physics, system sciences, ecological sciences, medicine and financial mathematics. In several stochastic dynamic systems, one assumes that the future state of the system does not depend on its past states. However, under close analysis, it becomes evident that most realistic models would contain some of the past states of the system, and one would require stochastic functional differential equations in order to study such systems. This paper presents the existence theory for stochastic functional differential equations in the G-framework (in short G-SFDEs). The comparison theorem has been developed in a bid to obtain the required results. It is ascertained that the G-SFDEs, whose coefficients may be discontinuous functions, have more than one continuous and bounded solutions.

Key words: Existence, G-Brownian motion, Stochastic functional differential equations, discontinuous coefficients.

1 Introduction

In the last twenty years, the greater requirement for tools and procedure of stochastic calculus has been recorded in different scientific fields. In the study of financial markets, it has acquired the state of an essential element, projected in dynamic phenomena of routine changes in share and stock prices. Stochastic calculus has its applications in engineering, as well as in filtering and control theory, and even in physics, when it deals with the effect of random changes on different physical phenomena. In Biology, its main usage is in modeling the achievement of stochastic changes in reproduction on populations processes. The idea of G-Brownian motion, which is a new

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stochastic process, was given by a Chinese mathematician Shige Peng in 2006 [12]. This theory opened a new era in stochastic calculus and financial mathematics. This type of motion has a newer construction as it does not depend on a specific probability space. This motion explains the ancient Brownian motion in an extraordinary way. In the framework of a sublinear expectation (called as G-expectation), he established the associated Itô's calculus. During his research on stochastic calculus, Peng set up the existence and uniqueness of solutions for stochastic differential equations driven by G-Brownian motion in short (G-SDEs) with Lipschitz continuous coefficients [12, 13]. Then F. Gao generalized the associated Itô's calculus and the existence theory of G-SDEs with Lipschitz continuity condition using the concept of G-capacity and quasi-sure analysis [6]. Y. Ren and L. Hu proved the existence and uniqueness of solutions for G-SDEs under the Carathéodory conditions, while later on, X. Bai and Y. Lin extended the theory for G-SDEs to the integral Lipschitz conditions [1]. In the G-frame, stochastic functional differential equations were introduced by Ren, Bi and Sakthivel [14]. Then studied by Faizullah[4]. He used the Cauchy-Maruyama approximation scheme to establish the existence-and-uniqueness theorem for SFDEs in the G-frame with linear growth condition as well as Lipschitz continuity condition [4]. In a different manner, this paper explores the existence theory for SFDEs in the G-frame, whose coefficients may not be continuous. This is the generalization of the previous work by Faizullah, Mukhtar and Rana [5]. We consider stochastic functional differential equations in the G-framework of the following type

$$dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \lambda(t, Y_t)dB(t),\ 0 \le t \le T. \tag{1.1}$$

Recall that $Y_t = \{Y(t+\theta) : -\delta \le \theta \le 0, \delta > 0\}$ is a bounded continuous stochastic process from $[-\tau, 0]$ to $\mathbb{R}$ where at time t, the value of stochastic process is denoted by $Y(t)$ [4]. Also, Y_t indicates the collection of continuous bounded real-valued functions ψ defined on $[-\delta, 0]$ with norm $\|\psi\| = \sup_{-\delta\le\theta\le 0} \mid \psi(\theta) \mid$. Let κ, λ and μ are Borel measurable functions from $[0, T] \times BC([-\tau, 0]; \mathbb{R})$ to $\mathbb{R}$. We define the initial data of equation (1.1) as follows;

$$\begin{gathered} Y_{t_0} = \zeta = \{\zeta(\theta) : -\tau < \theta \le 0\}\ is\ \mathcal{F}_0 - measurable,\ BC([-\tau, 0]; \mathbb{R}) -\ valued \\ random\ variable\ so\ that\ \zeta \in M_G^2\left([-\tau, 0]; \mathbb{R}\right). \end{gathered} \tag{1.2}$$

The integral form of problem (1.1) is given as the following

$$Y(t) = \zeta(0) + \int_0^t \kappa(s, Y_s)ds + \int_0^t \lambda(s, Y_s)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s)dB(s).$$

The G-SFDE (1.1) admit at most solution $Y(t) \in M_G^2\left([-\tau, T]; \mathbb{R}\right)$ if all its coefficients gratify the linear growth condition as well as Lipschitz condition. [4, 14]. On the other hand, in this article we assume that the coefficients κ and λ may be discontinuous functions. The solution to problem 1.1 with initial data 1.2 is a real valued stochastic process $Y(t)$, $t \in [-\tau, T]$ if it holds the following characteristics

(**a**) For every $t \in [0, T]$, $Y(t)$ is $\mathcal{F}_t$-adapted as well as path-wise continuous.

(**b**) $\kappa(t, Y_t), \lambda(t, Y_t) \in \mathcal{L}^1([o, T]; \mathbb{R})$ and $\mu(t, Y_t) \in \mathcal{L}^2([o, T]; \mathbb{R})$;

(**c**) $Y_0 = \zeta$ and $dY(t) = \kappa(t, Y_t)dt + \lambda(t, Y_t)d\langle B, B\rangle(t) + \mu(t, Y_t)dB(t)$ q.s. for each $t \in [0, T]$.

The rest of the paper is organized as follows. Some basic definitions and notions are given in the subsequent section. Section 3 presents an important results known as the comparison theorem. The final section develops the existence theorem with possible discontinuous coefficients.

2 Preliminary Concerns

This section presents some basic notions and results, which are used in forthcoming research work of this paper [2, 3, 6, 13].

2.1 Sublinear Expectation

Suppose that Ω (sample space) is a grand set and H be a family of linear and real valued functions described on Ω. Suppose that H fulfil $k \in H$, for any constant k and $|Y| \in H$ if $Y \in H$. H containing the stochastic variables.

Definition 2.1. A functional E, where $E : H \to R$, is known as a G-expectation or sublinear expectation if

(**1**) E is monotonic, that is, if $Y \geq Z$ for all $Y, Z \in H \Rightarrow E[Y] \geq E[Z]$.

(**2**) E is constant conserving, that is, $E[k] = k \quad k \in H$.

(**3**) E is sub-additive, that is, if $E[Y + Z] \leq E[Y] + E[Z]$, for each $Y, Z \in H$.

(**4**) E is positive homogeneous, that is, $E[bY] = b[y]$ for $b \geq 0$.

the space given by triple (Ω, H, E) is said to be sublinear expectation space. And E is nonlinear expectation if it satisfies the above two conditions. Sublinear expectation is also able to state the supremum of linear expectation

Definition 2.2. G-Brownian motion A d-dimensional process $(B_t)_{t \geq 0}$, define on $(\Omega, C_{l,lip}(H), E)$, is known as G-Browmain motion, if the following conditions are hold.

(**1**) $B_0(w) = 0$.

(**2**) The increment $B_{t+r} - B_t$ is G-normally distributed for any $t, r \geq 0$.

(**3**) $B_{t+r} - B_t$ is independent from $B_{t_1}, B_{t_2}, ...B_{t_n}$ for any $n \in N,\ t, r \geq 0$ and $0 \leq t_1 \leq t_2 \leq, ... \leq t_n \leq t$.

2.2 Ito's integral of G-Brownian motion

Definition 2.3. If $T \in R^+$, a partition π_T of the interval $[0, T]$ is

$$\pi_T = \{t_0, t_1, ..., t_N\},$$

since

$$\rho(\pi_T) = max\{|t_{\epsilon+1} - t_\epsilon| : \epsilon = 0, 1, ...N - 1\},$$

where

$$0 = t_0 \leq t_1 \leq, ... \leq t_N = T,$$

3

we customize $\pi_T^N = \{t_0^N, t_1^N, ...t_N^N\}$ to represent a sequence of partition of $[0, T]$ since

$$\lim_{N\to\infty} \rho(\pi_T^N) = 0.$$

Let $p \geq 1$. Suppose the following sort of processes of a partition

$$\pi_T = \{t_0, t\pi_1, ..., t_N\}.$$

We take,

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

where $\xi_m \in L_G^p(\omega_{tm})$, for all $m = 0, 1, 2, ...N-1$. The group of these process is represented by $M_G^{p,0}(0,T)$.

Definition 2.4. Let $\eta \in M_G^{1,0}(0,T)$ with

$$\eta_t(\omega) = \sum_{m=0}^{N-1} \xi_m(\omega) I_{[t_m, t_{m+1})}(t)$$

it can be written as,

$$\int_0^T \eta_t(\omega) dt = \sum_{m=0}^{N-1} \xi_m(\omega)(t_{m+1} - t_m)$$

Definition 2.5. For every $p \geq 1$, we represent by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \{E[\int_0^T |\eta_t|^p dt]\}^{1/p},$$

where for $1 \leq p \leq q$, $M_G^p(0,T) \supset M_G^q(0,T)$.

Definition 2.6. For every $\eta \in M_G^p(0,T)$ of the arrangement

$$\eta_t(w) = \sum_{\epsilon=0}^{N-1} \xi_\epsilon(w) I[t_\epsilon, t_{\epsilon+1})(t),$$

it can be written as,

$$I(\eta) = \int_0^T \eta_t dB_t = \sum_{\epsilon=0}^{N-1} \xi_\epsilon (B_{t_{\epsilon+1}} - B_{t_\epsilon}).$$

Lemma 2.7. *Let a function* $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$, *then it can be continuously extended to* $I : M_G^2(0,T) \to L_G^2(\Omega_T)$. *Moreover,*

$$E[\int_0^T \eta_t dB_t] = 0,$$

$$E[(\int_0^T \eta_t dB_t)^2] \leq \sigma^2 E[\int_0^T \eta_t^2 dt].$$

4

2.3 (Peng's quadratic variation process $\langle B\rangle_t$)

Definition 2.8. A 1-dimensional G-quadratic variation process is introduced as follows. Let $\pi_t^N, N = 1, 2, ...,$ be a sequence of the partition $[0, T]$ then

$$\begin{aligned} B_t^2 &= \sum_{\epsilon=0}^{N-1}(B^2_{tN_{\epsilon+1}} - B^2_{tN_\epsilon}) \\ &= \sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{i+\epsilon}} - B_{tN_\epsilon}) + \sum_{\epsilon=0}^{N-1}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon})^2. \end{aligned}$$

Taking limit $\mu(\pi_t^N) \to 0$

$$\sum_{\epsilon=0}^{N-1} 2B_{tN_\epsilon}(B_{tN_{\epsilon+1}} - B_{tN_\epsilon}) \quad \textit{converges to} \quad 2\int_0^t B_s dB_s,$$

and we have

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_s dB_s.$$

Definition 2.9. Let $\mathcal{P}$ be a (weakly compact) collection of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$ then the capacity $\hat{c}(.)$ associated to $\mathcal{P}$ is defined by

$$\hat{c}(B) = \sup_{P\in\mathcal{P}} P(B), \quad B \in \mathcal{B}(\Omega),$$

where $\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω. A set B is said to be polar if its capacity is zero, that is, $\hat{c}(B) = 0$ and a statement holds quasi-surely in short (q.s.) if it holds except on a polar set.

3 An important result

In this section, we establish an important result known as comparison theorem. First, we assume two stochastic functional integral equations given as follows.

$$Y(t) = \zeta_1(0) + \int_{t_0}^t \kappa_1(s, Y_s)ds + \int_{t_0}^t \lambda_1(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0, T], \quad (3.1)$$

$$Y(t) = \zeta_2(0) + \int_{t_0}^t \kappa_2(s, Y_s)ds + \int_{t_0}^t \lambda_2(s, Y_s)d\langle B, B\rangle(s) + \int_{t_0}^t \mu(s, Y_s)dB(s), \qquad t \in [0, T]. \quad (3.2)$$

Theorem 3.1. *Let Y^1 and Y^2 are the respective unique solutions of equations* (3.1) *and* (3.2). *Suppose that $\kappa_1(s, Y_s) \le \kappa_2(s, Y_s)$ and $\lambda_1(s, Y_s) \le \lambda_2(s, Y_s)$ are componentwise for every $t \in [t_0, T]$, $y \in BC([-\tau, 0]; \mathbb{R}^d)$ and $\zeta^1 \le \zeta^2$. Also, let the coefficients κ_1, λ_1 or κ_2, λ_2 are increasing functions. Then for every $t > 0$, $Y^1 \le Y^2$ q.s.*

Proof. Suppose that κ_2 and λ_2 are increasing and consider the problem

$$\begin{aligned} Z(t) = \zeta_2(0) + \int_{t_0}^{t} \kappa_2(s, \max\{{Y_s}^1, Z_s\})ds + \int_{t_0}^{t} \lambda_2(s, \max\{{Y_s}^1, Y_s\})d\langle B, B\rangle(s) \\ + \int_{t_0}^{t} \mu(s, \max\{{Y_s}^1, Z_s\})dB(s), \ \ t_0 \le t \le T, \end{aligned} \tag{3.3}$$

where the function $x \to \max\{y, z\}$ satisfies the growth condition $|\max\{y, z\}| \le |y| + |z|$ and the Lipschitz condition with constant one. It follows that all coefficients of the above equation 3.3 gratify the growth condition as well as Lipschitz condition. Thus problem 3.3 admit the only one solution say $Z(t)$. Now one has to show that $Z(t) \ge Y_s^1$ q.s. First define stopping times δ_1 and δ_2 as follows. More details on stopping times can be found in [9, 10, 11].

$$\begin{aligned} \delta_1 &= \inf\{t \in [t_0, T] : Y_s^1 - Z(t) > 0\} \ \ where \ \ \delta_1 < T, \\ \delta_2 &= \inf\{t \in [\tau_1, T] : Y_s^1 - Z(t) < 0\}. \end{aligned}$$

Contrary assume that $(\delta_1, \delta_2) \subset [t_0, T]$ be an arbitrary interval, such that $Z(\delta_1) = Y^1(\delta_1) = \zeta^*(0)$ and $Z(t) \le Y^1(t)$ for every $t \in (\delta_1, \delta_2)$. Then,

$$\begin{aligned} Z(t) - Y^1(t) = \zeta^*(0) &+ \int_{\delta_1}^{t} \kappa_2(s, \max\{{Y_s}^1, Z_s\})ds + \int_{\delta_1}^{t} \lambda_2(s, \max\{{Y_s}^1, Z_s\})d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} \mu(s, \max\{{Y_s}^1, Z_s\})dB(s) - \zeta^*(0) - \int_{\delta_1}^{t} \kappa_1(s, {Y_s}^1)ds \\ &- \int_{\delta_1}^{t} \lambda_1(s, {Y_s}^1)d\langle B, B\rangle(s) - \int_{\delta_1}^{t} \mu(s, {Y_s}^1)dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, \max\{{Y_s}^1, Z_s\}) - \kappa_1(s, {Y_s}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, \max\{{Y_s}^1, Z_s\}) - \lambda_1(s, {Y_s}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, \max\{{Y_s}^1, Z_s\}) - \mu(s, {Y_s}^1)]dB(s), \ \ t \in (\delta_1, \delta_2). \end{aligned}$$

But the assumption $Z(t) \le Y^1(t)$ gives $\max[Y^1, Z] = Y^1$. So, we have

$$\begin{aligned} Z(t) - Y^1(t) = &\int_{\delta_1}^{t} [\kappa_2(s, {Y_s}^1) - \kappa_1(s, {Y_s}^1)]ds \\ &+ \int_{\delta_1}^{t} [\lambda_2(s, {Y_s}^1) - \lambda_1(s, {Y_s}^1)]d\langle B, B\rangle(s) \\ &+ \int_{\delta_1}^{t} [\mu(s, {Y_s}^1) - \mu(s, {Y_s}^1)]dB(s), \end{aligned}$$

which gives $Z(t) \ge Y^1(t)$ because $\kappa_2(t, y) \ge \kappa_1(t, y)$ and $\lambda_2(t, y) \ge \lambda_1(t, y)$. This gives contradiction. So, the supposition $Z(t) \le Y^1(t)$ for every $t \in (\delta_1, \delta_2)$ is not true. Thus $Z(t) \ge Y^1(t)$ q.s. and hence $\max\{Y^1, Z\} = Z$. It follows that $Z = Y^2 \ge Y^1$ because problem (3.3) admit a single solution Y^2. The proof is complete. □

4 Existence of solutions to SFDEs in the G-framework

Next, we assume that the coefficients κ and λ are not continuous. However, they are increasing, left continuous and $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ for every $(t,y) \in [0,T] \times BC([-\delta,0];\mathbb{R})$. Assume a sequence of problems given as follows.

$$Y^l(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1})ds + \int_0^t \lambda(s, Y_s^l)d\langle B, B\rangle(s) + \int_0^t \mu(s, Y_s^l)dB(s), \ t \in [0,T], \tag{4.1}$$

where $Y^0 = L_t$, L_t is the unique solution of the equation given by

$$L_t = \zeta + \int_0^t \mu(s, L_s)dB(s), \tag{4.2}$$

where $t \in [0,T]$. By our supposition $\kappa(t,y) \geq 0$, $\lambda(t,y) \geq 0$ and comparison result we obtain $Y^1 \geq L_t$. Thus, one can see that the sequence $\{Y^l : l \geq 1\}$ is increasing. In the following lemma we show that Y^l is bounded.

Lemma 4.1. *Let $Y^l(t)$ denotes a solution of equation (4.1). Then*

$$E\left(\sup_{-\delta \leq s \leq T} |Y^l(s)|^2\right) \leq K,$$

where $K = C_6 e^{C_5 T}$, $C_6 = E[\|\zeta\|] + C_4$, $C_5 = 4(C_1 + C_2 + C_3)$, $C_4 = 4[E|\zeta|^2 + C_1 T + C_2 T + C_3 T]$, C_1, C_2 and C_3 are positive constants.

Proof. Define the following stopping time, for any $l \geq 1$

$$\delta_m = T \wedge inf\{t \in [t_0, T] : \|Y_t^l\| \geq m\}.$$

We get $\delta_m \uparrow T$ and define $Y^{l,m}(t) = Y^l(t \wedge \delta_m)$ for $t \in (-\tau, T)$. Next we proceed as follows.

$$Y^{l,m}(t) = \zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m})I_{[o,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m})I_{[o,\delta_m]}d\langle B, B\rangle_s + \int_0^t \mu(s, Y_s^{l,m})I_{[o,\delta_m]}dB_s.$$

$$\begin{aligned}
|Y^{l,m}(t)|^2 &= |\zeta(0) + \int_0^t \kappa(s, Y_s^{l-1,m})I_{[0,\delta_m]}ds + \int_0^t \lambda(s, Y_s^{l,m})I_{[0,\delta_m]}d\langle B, B\rangle_s \\
&\quad + \int_0^t \mu(s, Y_s^{l,m})I_{[0,\delta_m]}dB_s|^2 \\
&\leq 4|\zeta(0)|^2 + 4|\int_0^t \kappa(s, Y_s^{l-1,m})I_{[0,\delta_m]}ds|^2 + 4|\int_0^t \lambda(s, Y_s^{l,m})I_{[0,\delta_m]}d\langle B, B\rangle_s|^2 \\
&\quad + 4|\int_0^t \mu(s, Y_s^{l,m})I_{[0,\delta_m]}dB_s|^2
\end{aligned}$$

By taking G-expectation on both sides, using the linear growth condition and Burkholder-Davis-Gundy inequalities [6, 13] we proceed as follows

$$
\begin{aligned}
E[|Y^{l,m}(t)|^2] &\leq 4E|\zeta(0)|^2 + 4C_1\int_0^t [1+E|Y_s^{l-1,m}|^2]ds + 4C_2\int_0^t [1+E|Y_s^{l,m}|^2]ds| \\
&+ 4C_3\int_0^t [1+E|Y_s^{l,m}|^2]ds \\
&\leq 4E|\zeta(0)|^2 + 4C_1\int_0^t ds + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t dt + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
&+ 4C_3\int_0^t ds + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds \\
&= 4E|\xi(0)|^2 + 4C_1T + 4C_1\int_0^t E|Y_s^{l-1,m}|^2 ds + 4C_2T + 4C_2\int_0^t E|Y_s^{l,m}|^2 ds \\
&+ 4C_3T + 4C_3\int_0^t E|Y_s^{l,m}|^2 ds.
\end{aligned}
$$

For any $j \in \mathbb{N}$ we get,

$$\max_{1\leq l\leq j} E[|Y^{l,m}(t)|^2] \leq C_4 + 4C_1\int_0^t \max_{1\leq l\leq j} E|Y_s^{l-1,m}|^2 ds + 4C_2\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds + 4C_3\int_0^t \max_{1\leq l\leq j} E|Y_s^{l,m}|^2 ds,$$

where $C_4 = 4[E|\zeta|^2 + C_1T + C_2T + C_3T]$. Hence by Doob's martingale inequality we get for any $l, m \in \mathbb{N}$

$$E[\sup_{0\leq s\leq t} |Y^{l,m}(s)|^2] \leq C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds, \tag{4.3}$$

where $C_5 = 4(C_1 + C_2 + C_3)$. One can observe the fact [11],

$$\sup_{-\delta\leq s\leq t} |Y^{l,m}(v)|^2 \leq \|\zeta\| + \sup_{0\leq s\leq t} |Y^{l,m}(s)|^2,$$

and hence 4.3 gives

$$
\begin{aligned}
E[\sup_{-\delta\leq s\leq t} |Y^{l,m}(s)|^2] &\leq E[\|\zeta\|] + C_4 + C_5\int_0^t E|Y_s^{l,m}|^2 ds \\
&\leq C_6 + C_5\int_0^t E[\sup_{-\delta\leq q\leq s} |Y^{l,m}(q)|^2] ds,
\end{aligned}
$$

where $C_6 = E[\|\zeta\|] + C_4$. Finally, taking $m \to \infty$ and by the Gronwall's inequality we get,

$$E[\sup_{-\delta\leq s\leq t} |Y^l(s)|^2] \leq C_6 e^{C_5 t}.$$

Letting $t = T$ we have

$$E[\sup_{-\delta\leq s\leq T} |Y^l(s)|^2] \leq K,$$

where $K = C_6 e^{C_5 T}$. Hence, the proof stands completed. □

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Theorem 4.2. *Let the coefficients $\kappa(t,y)$ and $\lambda(t,y)$ are increasing in the second variable y and left continuous. For all $(t,y) \in [0,T] \times BC([-\tau,0];\mathbb{R})$, $\kappa(t,y) \geq 0$ and $\lambda(t,y) \geq 0$. Then there exists at least one solution $Y(t) \in M_G^2([-\tau,T];\mathbb{R})$ to problem* (1.1).

Proof. Theorem 3.1 follows that the sequence $\{Y^l\}$ is increasing. On the other hand, Lemma 4.1 shows that $\{Y^l\}$ is a bounded sequence in the norm $\mathbb{L}^2$. Thus dominated convergence theorem yields that Y^n converges in $\mathbb{L}^2$. Let Y be the limit of Y^l. Then for almost all w, we have

$$\kappa(t,Y^l(t)) \to \kappa(t,Y(t)) \ as \ l \to \infty,$$
$$\lambda(t,Y^l(t)) \to \lambda(t,Y(t)) \ as \ l \to \infty.$$

Also

$$|\kappa(t,Y^l(t))| \leq K(1+\sup_l |Y^l(t)|) \in L^1([t_0,T]),$$
$$|\lambda(t,Y^l(t))| \leq K(1+\sup_l |Y^l(t)|) \in L^1([t_0,T]).$$

Since $\langle B \rangle$ is continuous, so, for uniformly in t and almost all w

$$\int_0^t \kappa(s,Y^l(s))ds \to \int_0^t \kappa(s,Y(s))ds, \ l \to \infty,$$
$$\int_0^t \lambda(s,Y^l(s))\langle B,B\rangle(s) \to \int_0^t \lambda(s,Y(s))\langle B,B\rangle(s), \ l \to \infty.$$

Since G-integral is continuous we get,

$$\sup_{0\leq t\leq T}\left|\int_0^t \mu(s,Y^l(s))dB(s) - \int_0^t \mu(s,Y(s))dB(s)\right| \to 0 \ (q.s), \ l \to \infty.$$

Obviously, the sequence Y^l converges uniformly to Y in t, hence Y is continuous. Taking limits $l \to \infty$ on both sides of equation (4.1), we obtain that Y is the solution to G-SFDE (1.1) with initial condition (1.2). □

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Interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and their calculations[†]

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Abstract: It is necessary to assume additivity and independent among decision making criteria for traditional multiple decision making (MDM) in which the weights given by decision makers based on a additive measure. However, most criteria have inter-dependent or interactive characteristics in the real decision making problems. Furthermore, with respect to multiple attribute group decision making (MAGDM) problems in which the attribute weights and the expert weights take the form of real numbers and the attribute values take the form of interval-valued intuitionistic sets, we propose interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations in this paper. First, we introduce some concepts of fuzzy measure, interval-valued intuitionistic sets and Archimedean t-norm. Then, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are obtained, and the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are presented under intuitionistic fuzzy environment. Finally, as fuzzy Choquet integral operators, some aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are given.

Keywords: Intuitionistic sets; Fuzzy Choquet integral operators; Archimedean t-norm.

1. Introduction

Multiple attribute decision making (MADM) problem is an important research topic in decision theory. Because the objects are fuzzy and uncertain, the attributes involved in decision problems are not always expressed as real numbers, and some better suited to be denoted by fuzzy numbers, such as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, linguistic numbers on uncertain linguistic variables, and intuitionistic fuzzy numbers. Because Zadeh initially proposed the basic model of fuzzy decision making based on the theory of fuzzy mathematics, fuzzy MADM has been receiving more and more attention. We also notice that the main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate the aggregation operators proposed in the practice.

The fuzzy set (FS) theory proposed by Zadeh [1] was a very good tool to research the fuzzy MADM problems, the fuzzy set is used to character the fuzziness just by membership degree. Different from fuzzy set, there is another parameter: non-membership degree in intuitionistic fuzzy set (IFS) which is proposed by Atanassov [2, 3]. Clearly, the IFS can describe and character the fuzzy essence of the objective world more accurately [2] than the fuzzy set, and has received more and more attention since its appearance. Later, Atanassov and Gargov [4, 5] further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are interval numbers rather real numbers.

Base on Archimedean t-conorm and t-norm [6, 7], and the aggregation functions for the classical fuzzy sets (FSs), Beliakov et al. gave some operations about intuitionistic fuzzy sets, proposed two general concepts for constructing other types of aggregation operators for intuitionistic fuzzy sets (IFSs)

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extending the existing methods and showed that the operators obtained by using the Lukasiewicz t-norm are consistent with the ones on ordinary FSs. We can find above aggregation operators are all based on different relationships of the aggregated arguments, which can provide more choices for the decision makers.

As an aggregation function, it is well-known that Choquet integral [8] based on non-additive fuzzy measure, is a kind of non-additive and non-linear integral, and has been successfully used for handling information fusion and decision making problems (MCDM). The main characteristic of this aggregation function is that it is able to flexibly describe the relative importance of decision criteria as well as their interactions. There are many works on the Choquet integral of single-valued functions, set-valued functions and studied their mathematical properties. It is of interest to combine the Choquet integral and the IFS theory or MCDM under intuitionistic fuzzy environment, because, by doing this, we cannot only deals with the imprecise and uncertain decision information but also efficiently take into account the various interactions among the decision criteria. The intuitionistic fuzzy-valued Choquet integral, the combination of the Choquet integral and the IFS theory, can also act an aggregation tool employed in MCDM as well as other multicriteria analysis field. In this paper, we propose the interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and discuss their calculations. First, we introduced some concepts of fuzzy measure and interval-valued intuitionistic sets based on Archimedean t-norm. Then, interval-valued intuitionistic weighted average(geometric) operator based on Archimedean t-norm, interval-valued intuitionistic ordered weighted average operator based on Archimedean t-norm are developed.

The rest of this study is organized as follows. In section 2, we recall the definitions of intuitionistic fuzzy set、Archimedean t-norm and Choquet integral. In section 3, the representations and transformations of Archimedean t-norm and Archimedean t-conorm are proposed and inveastigated, and some of its properties are investigated in detail by means of the representation theorem. In section 4, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm is presented under intuitionistic fuzzy environment. In section 5, an aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are defined and discussed.

2. Definitions and preliminaries

A fuzzy measure on X is a set function $\mu : P(X) \rightarrow [0,1]$ such that
(i) $\mu(\emptyset) = 0, \mu(X) = 1$;
(ii) $A, B \subseteq X$, $A \subseteq B$ implies $\mu(A) \leqslant \mu(B)$.

Definition 2.1. Let $A, B \in P(X), A \cap B = \varnothing$. If fuzzy measure g satisfies the following conditions:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$$

and $\lambda \in (-1, \infty)$.

Especially if $\lambda = 0$, then g is an additive measure, which means there is no interaction between coalitions A and B .

Let $X = \{x_1, x_2, ...x_n\}$ be a attribute index set, if $i, j = 1, 2, ..., n$ and $i \neq j, x_i \cap x_j = \varnothing, \bigcup_{i=1}^n x_i = X$, then

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^n [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i=1}^n g(x_i) & \lambda = 0, \end{cases} \tag{1}$$

From Eq. (1), for the $A \in P(X)$, g can be expressed by

$$g(X) = \begin{cases} \frac{1}{\lambda}(\prod_{i\in A} [1 + \lambda g(x_i)] - 1) & \lambda \neq 0, \\ \sum_{i\in A} g(x_i) & \lambda = 0, \end{cases} \tag{2}$$

For x_i, $g(x_i)$ is called a fuzzy measure function, and it indicates the importance degree of x_i.

From $g(X) = 1$, we know λ is determined by $\lambda + 1 = \prod_{i=1}^n (1 + \lambda g(x_i))$.

Definition 2.2. Let f be a positive real-valued function on X, the discrete Choquet integral of f with respect to a fuzzy measure μ on X is defined as

$$C_\mu(f(x_{(1)}), ..., f(x_{(n)})) = \sum_{i=1}^n f(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)}) \leqslant \cdots \leqslant f(x_{(n)})$. $A_{(i)} = (i, \ldots, n)$, and $A_{(n+1)} = \emptyset$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following four conditions [8,9]:
1) $T(1,x) = x$, for all x.
2) $T(x,y) = T(y,x)$, for all x and y.
3) $T(x, T(y,z)) = T(T(x,y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $T(x,y) \leqslant T(x', y')$, $x, y, x', y' \in [0,1]$.

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following four conditions[8,9]:
1) $S(0,x) = x$, for all x.
2) $S(x,y) = S(y,x)$, for all x and y.
3) $S(x, S(y,z)) = S(S(x,y), z)$, for all x, y and z.
4) $x \leqslant x', y \leqslant y'$ implies $S(x,y) \leqslant S(x', y')$, $x, y, x', y' \in [0,1]$.

Definition 2.3 [8,9]. A t-norm function $T(x,y)$ is called Archimedean t-norm if it is continuous and $T(x,x) < x$ for all $x \in [0,1]$. An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for $x, y \in (0,1)$.

A t-conorm function $S(x,y)$ is called Archimedean t-conorm if it is continuous and $S(x,x) > x$ for all $x \in [0,1]$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $x, y \in (0,1)$.

Definition 2.4. Let X be in a given domain. Then,

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$$

is called an interval-valued intuitionistic fuzzy set ($IVIFS$), where $\mu_A : X \to I \subset [0,1]$, $\nu_A : X \to J \subset [0,1]$ and I, J are closed intervals in $[0,1]$, the following condition is met: $\sup \mu_A(x) + \sup \nu_A(x) \leqslant 1$, $x \in X$. The intervals $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x on X.

Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are, respectively, denoted by $\mu_A^L(x)$, $\mu_A^U(x)$,$\nu_A^L(x)$,$\nu_A^U(x)$. We can denote by

$$A = \{\langle x, [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]]\rangle | x \in X\},$$

where $0 \leqslant \mu_A^U(x) + \nu_A^U(x) \leqslant 1$, $x \in X$, $\mu_A^L(x) \geqslant 0$ and $\nu_A^L(x) \geqslant 0$.

Simply, we write $A = \langle [\mu_A^L(x), \mu_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]$.

For each element x, we can compute its hesitation interval of x as:

$$\pi_A(x) = [\pi_A^L(x), \pi_A^U(x)] = [1 - \nu_A^U(x) - \mu_A^U(x), 1 - \nu_A^L(x) - \mu_A^L(x)].$$

3. The representations and transformations of Archimedean t-norm and Archimedean t-conorm

Definition 3.1. A mapping $N : [0,1] \to [0,1]$ is called negation operator, if N is decreasing and $N(0) = 1$, $N(1) = 0$. Especially, we have

(i) If $N(x) = 1 - x$, it is called standard negation operator.

(ii) $\forall x \in [0,1]$, if $N(N(x)) = x$, then it is called cyclotron negation operator. Obviously, cyclotron negation operator is continuous and strictly increasing.

(iii) For each negation operator, T and S are dual with respect to $N(x)$ if and only if $T(N(x), N(y)) = N(S(x,y))$.

It is well known [9] that a strict Archimedean t-norm is expressed via its additive generator g as $T(x,y) = g^{-1}(g(x) + g(y))$, and similarly, applied to its dual t-conorm $S(x,y) = h^{-1}(h(x) + h(y))$ with $h(t) = g(N(t))$. We notice that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0,1] \to [0, +\infty)$ such that $g(1) = 0$. If we assign specific forms to the function g, then some well-known t-conorms and t-norms can be obtained. Let me emphasize that the results (1-4) were shown in [11], however, considering that the representation of the negation operator is always restricted by the policy mak- ers' historical knowledge, perceptual judgement and other factors in the game playing, benefit groups' voting or decision making process, we could define the negation operator by means of the fuzzy logic non-portal operators in this paper and calculate Archimedean t-norm and Archimedean t-conorm as results (5-8) as follows.

Theorem 3.1 Let $T(x,y)$ be Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm. Then we have the following statements:

If $N(x)=1-x$, i.e. $h(t)=g(1-t)$), then the following are valid:

(1) Let $g(t)=-\log t$, then $h(t)=-\log(1-t), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=1-\exp^{-t}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A(x,y)=x\cdot y,\quad S^A(x,y)=x+y-xy.$$

(2) Let $g(t)=\log(\frac{2-t}{t})$, then $h(t)=\log(\frac{2-(1-t)}{1-t}), g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=1-\frac{2}{\exp^t+1}$, and we get Einstein t-conorm and t-norm [10]:

$$T^E(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E(x,y)=\frac{x+y}{1+xy}.$$

(3) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t)}{1-t}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=1-\frac{\gamma}{\exp^t+\gamma-1}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S^H_\gamma(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(4) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}, h^{-1}(t)=1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, and we have Frank t-conorm and t-norm [10] as follows:

$$T^F_\gamma(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S^F_\gamma(x,y)=1-\log_\gamma(1+\frac{(\gamma^{1-x}-1)(\gamma^{1-y}-1)}{\gamma-1})\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S^F_\gamma(x,y)=S^A_\gamma(x,y)$ and $\lim_{\gamma\to1}T^F_\gamma(x,y)=T^A_\gamma(x,y)$.

If $N(x)=1-x^2$, i.e. $h(t)=g(1-t^2)$ then the following are also valid:

(5) Let $g(t)=-\log t$, then $h(t)=-\log(1-t^2), g^{-1}(t)=\exp^{-t}, h^{-1}(t)=\sqrt{1-\exp^{-t}}$, and Algebraic t-conorm and t-norm [10] are obtained as follows:

$$T^A_2(x,y)=xy\ ,\ S^A_2(x,y)=\sqrt{1-(1-x^2)(1-y^2)}.$$

(6) Let $g(t)=\log(\frac{2-t}{t})$, then we have $h(t)=\log\frac{1+t^2}{1-t^2}, g^{-1}(t)=\frac{2}{\exp^t+1}, h^{-1}(t)=\sqrt{\frac{\exp^t-1}{\exp^t+1}}$, and we get Einstein t-conorm and t-norm [10] are obtained as follows:

$$T^E_2(x,y)=\frac{xy}{1+(1-x)(1-y)}\ ,\ S^E_2(x,y)=\sqrt{\frac{x^2+y^2}{1+x^2y^2}}/$$

(7) Let $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then we have $h(t)=\log(\frac{\gamma+(1-\gamma)(1-t^2)}{1-t^2}), g^{-1}(t)=\frac{\gamma}{\exp^t+\gamma-1}$, $h^{-1}(t)=\sqrt{1-\frac{\gamma}{\exp^t+\gamma-1}}$, and Hamacher t-conorm and t-norm [10] are obtained as follows:

$$T^H_{2\gamma}(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)}\ ,\ \gamma>0,$$

$$S_{2\gamma}^H(x,y)=\sqrt{\frac{x^2+y^2-x^2y^2-(1-\gamma)x^2y^2}{1-(1-\gamma)x^2y^2}}\ ,\ \gamma>0.$$

Especially, if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Algebraic t-conorm and t-norm respectively; if $\gamma=1$, then Hamacher t-conorm and t-norm reduce to the Einstein t-conorm and t-norm respectively.

(8) Let $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then $h(t)=\log(\frac{\gamma-1}{\gamma^{1-t^2}-1}), g^{-1}(t)=\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}$, $h^{-1}(t)=\sqrt{1-\frac{\log(\frac{\gamma-1+\exp^t}{\exp^t})}{\log\gamma}}$,
and we have Frank t-conorm and t-norm [10] as follows:

$$T_{2\gamma}^F(x,y)=\log_\gamma(1+\frac{(\gamma^x-1)(\gamma^y-1)}{\gamma-1})\ ,\ \gamma>1,$$

$$S_{2\gamma}^F(x,y)=\sqrt{1-\log_\gamma(1+\frac{(\gamma^{1-x^2}-1)(\gamma^{1-y^2}-1)}{\gamma-1})}\ ,\ \gamma>1.$$

Especially, if $\gamma\to1$, then we have

$$\lim_{\gamma\to1}g(t)=\lim_{\gamma\to1}\log(\frac{\gamma-1}{\gamma^t-1})=\lim_{\gamma\to1}\log(\frac{1}{t\gamma^{t-1}-1})=-\log t.$$

which indicates that $\lim_{\gamma\to1}S_\gamma^F(x,y)=S_\gamma^A(x,y)$ and $\lim_{\gamma\to1}T_\gamma^F(x,y)=T_\gamma^A(x,y)$.

4. The operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued intuitionistic fuzzy sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, and $\lambda\geqslant0$. We can define the operational rules about $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ based on Archimedean t-norm as follows

(1) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[S(\mu^L(\alpha_1),\mu^L(\alpha_2)),S(\mu^U(\alpha_1),\mu^U(\alpha_2))],[T(\nu^L(\alpha_1),\nu^L(\alpha_2)),T(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[h^{-1}(h(\mu^L(\alpha_1))+h(\mu^L(\alpha_2))),h^{-1}(h(\mu^U(\alpha_1))+h(\mu^U(\alpha_2)))],$
$[g^{-1}(g(\nu^L(\alpha_1))+g(\nu^L(\alpha_2))),g^{-1}(g(\nu^U(\alpha_1))+g(\nu^U(\alpha_2)))]\rangle;$

(2) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[T(\mu^L(\alpha_1),\mu^L(\alpha_2)),T(\mu^U(\alpha_1),\mu^U(\alpha_2))],[S(\nu^L(\alpha_1),\nu^L(\alpha_2)),S(\nu^U(\alpha_1),\nu^U(\alpha_2))]\rangle$
$=\langle[g^{-1}(g(\mu^L(\alpha_1))+g(\mu^L(\alpha_2))),g^{-1}(g(\mu^U(\alpha_1))+g(\mu^U(\alpha_2)))],$
$[h^{-1}(h(\nu^L(\alpha_1))+h(\nu^L(\alpha_2))),h^{-1}(h(\nu^U(\alpha_1))+h(\nu^U(\alpha_2)))]\rangle;$

(3) $\lambda\widetilde{\alpha}_1=\langle[h^{-1}(\lambda h(\mu^L(\alpha_1))),h^{-1}(\lambda h(\mu^U(\alpha_1)))],[g^{-1}(\lambda g(\nu^L(\alpha_1))),g^{-1}(\lambda g(\nu^U(\alpha_1)))]\rangle;$

(4) $\widetilde{\alpha}_1^\lambda=\langle[g^{-1}(\lambda g(\mu^L(\alpha_1))),g^{-1}(\lambda g(\mu^U(\alpha_1)))],[h^{-1}(\lambda h(\nu^L(\alpha_1))),h^{-1}(\lambda h(\nu^U(\alpha_1)))]\rangle.$

Obviously, the above operational result is still an the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm. According Theorem 3.1 and Definition 4.1, we have Theorem 4.1 and Theorem 4.2, the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm are obtained as follows.

Theorem 4.1. Let $\widetilde{\alpha}_i=\langle[\mu^L(\alpha_i),\mu^U(\alpha_i)],[\nu^L(\alpha_i),\nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be two interval-valued fuzzy intuitionistic sets, $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. Then the following operational rules based on Archimedean t-norm are hold:

(1) If $g(t)=-\log t$, then [9]

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle;$

(ii) $\widetilde{\alpha}_1\otimes\widetilde{\alpha}_2=\langle[\mu^L(\alpha_1)\mu^L(\alpha_2),\mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2),\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1=\langle[s_{\lambda\times\theta(\alpha_1)},s_{\lambda\times\tau(\alpha_1)}],[1-(1-\mu^L(\alpha_1))^\lambda,1-(1-\mu^U(\alpha_1))^\lambda],[(\nu^L(\alpha_1))^\lambda,(\nu^U(\alpha_1))^\lambda]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda=\langle[s_{(\theta(\alpha_1))^\lambda},s_{(\tau(\alpha_1))^\lambda}],[(\mu^L(\alpha_1))^\lambda,(\mu^U(\alpha_1))^\lambda],[1-(1-\nu^L(\alpha_1))^\lambda,1-(1-\nu^U(\alpha_1))^\lambda]\rangle.$

(2) If $g(t)=\log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1\oplus\widetilde{\alpha}_2=\langle[\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)}{1+\mu^L(\alpha_1)\mu^L(\alpha_2)},\frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)}{1+\mu^U(\alpha_1)\mu^U(\alpha_2)}],[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))},\frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}], [\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)}{1+\nu^L(\alpha_1)\nu^L(\alpha_2)}, \frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)}{1+\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\frac{(1+\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+\mu^L(\alpha_1))^\lambda+(1-\mu^L(\alpha_1))^\lambda}, \frac{(1+\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+\mu^U(\alpha_1))^\lambda+(1-\mu^U(\alpha_1))^\lambda}], [\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+\nu^L(\alpha_1)^\lambda}, \frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+\nu^U(\alpha_1)^\lambda}]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\frac{2(\mu^L(\alpha_1))^\lambda}{(2-\mu^L(\alpha_1))^\lambda+\mu^L(\alpha_1)^\lambda}, \frac{2(\mu^U(\alpha_1))^\lambda}{(2-\mu^U(\alpha_1))^\lambda+\mu^U(\alpha_1)^\lambda}], [\frac{(1+\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+\nu^L(\alpha_1))^\lambda+(1-\nu^L(\alpha_1))^\lambda}, \frac{(1+\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+\nu^U(\alpha_1))^\lambda+(1-\nu^U(\alpha_1))^\lambda}]\rangle.$

(3) If $g(t)=\log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma>0$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2)-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)}{1-(1-\gamma)\mu^L(\alpha_1)\mu^L(\alpha_2)}, \frac{\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2)-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}{1-(1-\gamma)\mu^U(\alpha_1)\mu^U(\alpha_2)}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\frac{\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2)-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)}{1-(1-\gamma)\nu^L(\alpha_1)\nu^L(\alpha_2)}, \frac{\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2)-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}{1-(1-\gamma)\nu^U(\alpha_1)\nu^U(\alpha_2)}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\frac{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda-(1-\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^L(\alpha_1))^\lambda+(\gamma-1)(1-\mu^L(\alpha_1))^\lambda}, \frac{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda-(1-\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\mu^U(\alpha_1))^\lambda+(\gamma-1)(1-\mu^U(\alpha_1))^\lambda}],$
$[\frac{\gamma(\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^\lambda+(\gamma-1)(\nu^L(\alpha_1))^\lambda}, \frac{\gamma(\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^\lambda+(\gamma-1)(\nu^U(\alpha_1))^\lambda}]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\frac{\gamma(\mu^L(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^\lambda+(\gamma-1)(\mu^L(\alpha_1))^\lambda}, \frac{\gamma(\mu^U(\alpha_1))^\lambda}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^\lambda+(\gamma-1)(\mu^U(\alpha_1))^\lambda}],$
$[\frac{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda-(1-\nu^L(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^L(\alpha_1))^\lambda+(\gamma-1)(1-\nu^L(\alpha_1))^\lambda}, \frac{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda-(1-\nu^U(\alpha_1))^\lambda}{(1+(\gamma-1)\nu^U(\alpha_1))^\lambda+(\gamma-1)(1-\nu^U(\alpha_1))^\lambda}]\rangle.$

(4) If $g(t)=\log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma>1$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}), 1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)(\gamma^{\nu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)(\gamma^{\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)(\gamma^{\mu^L(\alpha_2)}-1)}{\gamma-1}), \log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)(\gamma^{\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}), 1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [1-\log_\gamma(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-\log_\gamma(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[\log_\gamma(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle [\log_\gamma(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), \log_\gamma(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})],$
$[1-\log_\gamma(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}}), 1-\log_\gamma(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)^\lambda}{(\gamma-1)^{\lambda-1}})]\rangle.$

Theorem 4.2. Let $\widetilde{\alpha}_i = \langle [\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i=1,2)$ be be two interval-valued intuitionistic fuzzy sets , $T(x,y)$ Archimedean t-norm and $S(x,y)$ its dual Archimedean t-conorm, $N(x)=1-x$. Then the following operational rules based on Archimedean t-norm valid:

(1) If $g(t)=-\log t$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 =$
$\langle [\sqrt{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2-(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}, \sqrt{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2-(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}],$
$[\nu^L(\alpha_1)\nu^L(\alpha_2), \nu^U(\alpha_1)\nu^U(\alpha_2)]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\mu^L(\alpha_1)\mu^L(\alpha_2), \mu^U(\alpha_1)\mu^U(\alpha_2)],$
$[\sqrt{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2-(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}, \sqrt{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2-(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{1-(1-(\mu^L(\alpha_1))^2)^\lambda}, \sqrt{1-(1-(\mu^U(\alpha_1))^2)^\lambda}], [(\nu^L(\alpha_1))^\lambda, (\nu^U(\alpha_1))^\lambda]\rangle;$

(iv) $\widetilde{\alpha}_1^\lambda = \langle [(\mu^L(\alpha_1))^\lambda, (\mu^U(\alpha_1))^\lambda], [\sqrt{1-(1-(\nu^L(\alpha_1))^2)^\lambda}, \sqrt{1-(1-(\nu^U(\alpha_1))^2)^\lambda}]\rangle.$

(2) If $g(t)=\log(\frac{2-t}{t})$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2}{1+(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}}, \sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2}{1+(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{1+(1-\nu^L(\alpha_1))(1-\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{1+(1-\nu^U(\alpha_1))(1-\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{1+(1-\mu^L(\alpha_1))(1-\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{1+(1-\mu^U(\alpha_1))(1-\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2}{1+(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}}, \sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2}{1+(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{\frac{(1+(\mu^L(\alpha_1))^2)^\lambda-(1-(\mu^L(\alpha_1))^2)^\lambda}{(1+(\mu^L(\alpha_1))^2)^\lambda+(1-(\mu^L(\alpha_1))^2)^\lambda}}, \sqrt{\frac{(1+(\mu^U(\alpha_1))^2)^\lambda-(1-(\mu^U(\alpha_1))^2)^\lambda}{(1+(\mu^U(\alpha_1))^2)^\lambda+(1-(\mu^U(\alpha_1))^2)^\lambda}}],$
$[\frac{2(\nu^L(\alpha_1))^\lambda}{(2-\nu^L(\alpha_1))^\lambda+(\nu^L(\alpha_1))^\lambda}, \frac{2(\nu^U(\alpha_1))^\lambda}{(2-\nu^U(\alpha_1))^\lambda+(\nu^U(\alpha_1))^\lambda}]\rangle;$

(iv) $\widetilde{\alpha}_1^{\lambda} = \langle [\frac{2(\mu^L(\alpha_1))^{\lambda}}{(2-\mu^L(\alpha_1))^{\lambda}+(\mu^L(\alpha_1))^{\lambda}}, \frac{2(\mu^U(\alpha_1))^{\lambda}}{(2-\mu^U(\alpha_1))^{\lambda}+(\mu^U(\alpha_1))^{\lambda}}],$
$[\sqrt{\frac{(1+(\nu^L(\alpha_1))^2)^{\lambda}-(1-(\nu^L(\alpha_1))^2)^{\lambda}}{(1+(\nu^L(\alpha_1))^2)^{\lambda}+(1-(\nu^L(\alpha_1))^2)^{\lambda}}}, \sqrt{\frac{(1+(\nu^U(\alpha_1))^2)^{\lambda}-(1-(\nu^U(\alpha_1))^2)^{\lambda}}{(1+(\nu^U(\alpha_1))^2)^{\lambda}+(1-(\nu^U(\alpha_1))^2)^{\lambda}}}]\rangle.$

(3) If $g(t) = \log(\frac{\gamma+(1-\gamma)t}{t})$, $\gamma > 0$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \langle [\sqrt{\frac{(\mu^L(\alpha_1))^2+(\mu^L(\alpha_2))^2-(\mu^L(\alpha_1))^2(\mu^L(\alpha_1))^2-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}{1-(1-\gamma)(\mu^L(\alpha_1))^2(\mu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\mu^U(\alpha_1))^2+(\mu^U(\alpha_2))^2-(\mu^U(\alpha_1))^2(\mu^U(\alpha_1))^2-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}{1-(1-\gamma)(\mu^U(\alpha_1))^2(\mu^U(\alpha_2))^2}}],$
$[\frac{\nu^L(\alpha_1)\nu^L(\alpha_2)}{\gamma+(1-\gamma)(\nu^L(\alpha_1)+\nu^L(\alpha_2)-\nu^L(\alpha_1)\nu^L(\alpha_2))}, \frac{\nu^U(\alpha_1)\nu^U(\alpha_2)}{\gamma+(1-\gamma)(\nu^U(\alpha_1)+\nu^U(\alpha_2)-\nu^U(\alpha_1)\nu^U(\alpha_2))}]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\frac{\mu^L(\alpha_1)\mu^L(\alpha_2)}{\gamma+(1-\gamma)(\mu^L(\alpha_1)+\mu^L(\alpha_2)-\mu^L(\alpha_1)\mu^L(\alpha_2))}, \frac{\mu^U(\alpha_1)\mu^U(\alpha_2)}{\gamma+(1-\gamma)(\mu^U(\alpha_1)+\mu^U(\alpha_2)-\mu^U(\alpha_1)\mu^U(\alpha_2))}],$
$[\sqrt{\frac{(\nu^L(\alpha_1))^2+(\nu^L(\alpha_2))^2-(\nu^L(\alpha_1))^2(\nu^L(\alpha_1))^2-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}{1-(1-\gamma)(\nu^L(\alpha_1))^2(\nu^L(\alpha_2))^2}},$
$\sqrt{\frac{(\nu^U(\alpha_1))^2+(\nu^U(\alpha_2))^2-(\nu^U(\alpha_1))^2(\nu^U(\alpha_1))^2-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}{1-(1-\gamma)(\nu^U(\alpha_1))^2(\nu^U(\alpha_2))^2}}]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{\frac{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^{\lambda}-(1-(\mu^L(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\mu^L(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\mu^L(\alpha_1))^2)^{\lambda}}}, \sqrt{\frac{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^{\lambda}-(1-(\mu^U(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\mu^U(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\mu^U(\alpha_1))^2)^{\lambda}}}],$
$[\frac{\gamma(\nu^L(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\nu^L(\alpha_1)))^{\lambda}+(\gamma-1)(\nu^L(\alpha_1))^{\lambda}}, \frac{\gamma(\nu^U(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\nu^U(\alpha_1)))^{\lambda}+(\gamma-1)(\nu^U(\alpha_1))^{\lambda}}]\rangle;$

(iv) $\widetilde{\alpha}_1^{\lambda} = \langle [\frac{\gamma(\mu^L(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\mu^L(\alpha_1)))^{\lambda}+(\gamma-1)(\mu^L(\alpha_1))^{\lambda}}, \frac{\gamma(\mu^U(\alpha_1))^{\lambda}}{(1+(\gamma-1)(1-\mu^U(\alpha_1)))^{\lambda}+(\gamma-1)(\mu^U(\alpha_1))^{\lambda}}],$
$[\sqrt{\frac{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^{\lambda}-(1-(\nu^L(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\nu^L(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\nu^L(\alpha_1))^2)^{\lambda}}}, \sqrt{\frac{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^{\lambda}-(1-(\nu^U(\alpha_1))^2)^{\lambda}}{(1+(\gamma-1)(\nu^U(\alpha_1))^2)^{\lambda}+(\gamma-1)(1-(\nu^U(\alpha_1))^2)^{\lambda}}}]\rangle.$

(4) If $g(t) = \log(\frac{\gamma-1}{\gamma^t-1})$, $\gamma > 1$, then

(i) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 =$
$\langle [\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)(\gamma^{1-(\mu^L(\alpha_2))^2}-1)}{\gamma-1})}, \sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)(\gamma^{1-(\mu^U(\alpha_2))^2}-1)}{\gamma-1})}],$
$[\log_{\gamma}(1+\frac{(\gamma^{1-\nu^L(\alpha_1)}-1)(\gamma^{1-\nu^L(\alpha_2)}-1)}{\gamma-1}), \log_{\gamma}(1+\frac{(\gamma^{1-\nu^U(\alpha_1)}-1)(\gamma^{1-\nu^U(\alpha_2)}-1)}{\gamma-1})]\rangle;$

(ii) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \langle [\log_{\gamma}(1+\frac{(\gamma^{1-\mu^L(\alpha_1)}-1)(\gamma^{1-\mu^L(\alpha_2)}-1)}{\gamma-1}), \log_{\gamma}(1+\frac{(\gamma^{1-\mu^U(\alpha_1)}-1)(\gamma^{1-\mu^U(\alpha_2)}-1)}{\gamma-1})],$
$[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)(\gamma^{1-(\nu^L(\alpha_2))^2}-1)}{\gamma-1})}, \sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)(\gamma^{1-(\nu^U(\alpha_2))^2}-1)}{\gamma-1})}]]\rangle;$

(iii) $\lambda\widetilde{\alpha}_1 = \langle [\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^L(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}, \sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\mu^U(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}],$
$[\log_{\gamma}(1+\frac{(\gamma^{\nu^L(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}), \log_{\gamma}(1+\frac{(\gamma^{\nu^U(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})]\rangle;$

(iv) $\widetilde{\alpha}_1^{\lambda} = \langle [\log_{\gamma}(1+\frac{(\gamma^{\mu^L(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}}), \log_{\gamma}(1+\frac{(\gamma^{\mu^U(\alpha_1)}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})],$
$[\sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^L(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}, \sqrt{1-\log_{\gamma}(1+\frac{(\gamma^{1-(\nu^U(\alpha_1))^2}-1)^{\lambda}}{(\gamma-1)^{\lambda-1}})}]\rangle.$

Theorem 4.3. Let $\widetilde{\alpha}_i$ $(i = 1, 2)$ be be two interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. We can easily prove the the following statements:

(1) $\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \oplus \widetilde{\alpha}_1$;
(2) $\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 = \widetilde{\alpha}_2 \otimes \widetilde{\alpha}_1$;
(3) $\lambda(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2) = \lambda\widetilde{\alpha}_1 \oplus \lambda\widetilde{\alpha}_2,\ \lambda > 0$;
(4) $\lambda_1\widetilde{\alpha}_1 \oplus \lambda_2\widetilde{\alpha}_1 = (\lambda_1 + \lambda_2)\widetilde{\alpha}_1,\ \lambda_1, \lambda_2 > 0$;
(5) $\widetilde{\alpha}_1^{\lambda_1} \otimes \widetilde{\alpha}_1^{\lambda_2} = (\widetilde{\alpha}_1)^{\lambda_1+\lambda_2},\ \lambda_1, \lambda_2 > 0$;
(6) $\widetilde{\alpha}_1^{\lambda} \otimes \widetilde{\alpha}_2^{\lambda} = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2)^{\lambda},\ \lambda > 0$.

According Theorem 3.1 and Definition 4.1, Theorem 4.3 is easy to prove.

5. Aggregating of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm

Definition 5.1. Let $\widetilde{\alpha}_1 = \langle [\mu^L(\alpha_1), \mu^U(\alpha_1)], [\nu^L(\alpha_1), \nu^U(\alpha_1)]\rangle$ be an interval-valued fuzzy intuitionistic sets. An expected value $E(\widetilde{\alpha}_1)$ of $\widetilde{\alpha}_1$ can be represented as follows

$$E(\widetilde{\alpha}_1) = \frac{1}{2} \times (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + 1 - \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2})$$

$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + 2 - \nu^L(\alpha_1) - \nu^U(\alpha_1))/4.$$

An accuracy function $H(\widetilde{\alpha}_1)$ can be represented as follows

$$H(\widetilde{\alpha}_1) = (\frac{\mu^L(\alpha_1) + \mu^U(\alpha_1)}{2} + \frac{\nu^L(\alpha_1) + \nu^U(\alpha_1)}{2})$$
$$= (\mu^L(\alpha_1) + \mu^U(\alpha_1) + \nu^L(\alpha_1) + \nu^U(\alpha_1))/4.$$

Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2)$ be two interval-valued fuzzy intuitionistic sets. Then

(1) If $E(\widetilde{\alpha}_1) > E(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

(2) If $E(\widetilde{\alpha}_1) = E(\widetilde{\alpha}_2)$, then:

If $H(\widetilde{\alpha}_1) > H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 \succ \widetilde{\alpha}_2$.

If $H(\widetilde{\alpha}_1) = H(\widetilde{\alpha}_2)$, then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

Based on the the above operational rules, we propose weighted average (geometric) operator, ordered weighted average (geometric) operator and hybrid average (geometric) operator for interval-valued intuitionistic fuzzy sets based on Archimedean t-norm in this part.

Definition 5.2. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \dots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. We define interval-valued intuitionistic fuzzy weighted average operator based on Archimedean t-norm as follows: $ATS - IVIFWA : \Omega^n \to \Omega$,

$$ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) = \sum_{j=1}^{n} \mu_j \widetilde{\alpha}_j,$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then $ATS - IVIFWA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic average operator based on Archimedean t-norm $(ATS - IVIFAA)$:

$$ATS - IVIFAA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) = \frac{1}{n}(\widetilde{\alpha}_1 \oplus \widetilde{\alpha}_2 \oplus \dots \oplus \widetilde{\alpha}_n).$$

Similarly, we could define interval-valued intuitionistic fuzzy weighted geometric average operator based on Archimedean t-norm, $ATS - IVIFWGA : \Omega^n \to \Omega$, as follows

$$ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) = \prod_{j=1}^{n} (\widetilde{\alpha}_j)^{\mu_j},$$

Specifically, if $\mu = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then $ATS - IVIFWGA$ operator degenerates interval-valued intuitionistic fuzzy arithmetic geometric average operator based on Archimedean t-norm $(ATS - IVIFGA)$:

$$ATS - IVIFGA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) = (\widetilde{\alpha}_1 \otimes \widetilde{\alpha}_2 \otimes \dots \otimes \widetilde{\alpha}_n)^{\frac{1}{n}}.$$

where Ω is the set of all interval-valued fuzzy intuitionistic sets, and $\mu = (\mu_1, \mu_2, \dots, \mu_n)^T$ is the weighted vector of $\widetilde{\alpha}_j (j = 1, 2, \dots, n)$, μ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, $A_{(j)} = (j, \dots, n)$ with $A_{(n+1)} = \emptyset$.

Theorem 5.1. Let $\widetilde{\alpha}_i = \langle[\mu^L(\alpha_i), \mu^U(\alpha_i)], [\nu^L(\alpha_i), \nu^U(\alpha_i)]\rangle$ $(i = 1, 2, \dots, n)$ be a collection of interval-valued intuitionistic fuzzy sets, $T(x, y)$ Archimedean t-norm and $S(x, y)$ its dual Archimedean t-conorm, $N(x) = 1 - x$. Then, the result aggregated by Definition 5.1 is still an intuitionistic fuzzy set, and

$(i) ATS - IVIFWA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) =$
$\langle[h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\mu^U(\alpha_j)))], [g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\nu^U(\alpha_j)))]\rangle$.

$(ii) ATS - IVIFWGA_\mu(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \dots, \widetilde{\alpha}_n) =$
$\langle[g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^L(\alpha_j))), g^{-1}(\sum_{j=1}^{n} \mu_j g(\mu^U(\alpha_j)))], [h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^L(\alpha_j))), h^{-1}(\sum_{j=1}^{n} \mu_j h(\nu^U(\alpha_j)))]\rangle$,
where $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is a fuzzy measure on X with $\mu_j \in [0, 1]$, $\mu_j = \mu(A_{(j)}) - \mu(A_{(j+1)})$, and $\sum_{j=1}^{n} \mu_j = 1$, the parentheses used for indices represent a permutation on X such that $\widetilde{\alpha}_1 \leqslant \widetilde{\alpha}_2 \leqslant \dots \leqslant \widetilde{\alpha}_n$, $A_{(j)} = (j, ..., n)$, $A_{(n+1)} = \varnothing$.

Theorem 5.1 can be proven by mathematical induction. The steps in the proof are as follows:

Proof. We only prove that (i) holds. the proof of (ii) is similar.

(1) When $n=1$, obviously, it is right.

(2) When $n=2$,

$\mu_1\widetilde{\alpha}_1=\langle[h^{-1}(\mu_1 h(\mu^L(\alpha_1))),h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],[g^{-1}(\mu_1 g(\nu^L(\alpha_1))),g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle$.

$\mu_2\widetilde{\alpha}_2=\langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))),h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],[g^{-1}(\mu_2 g(\nu^L(\alpha_2))),g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle$.

$$\begin{aligned}
&ATS-ATS-IVIFWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2)=\mu_1\widetilde{\alpha}_1\oplus\mu_2\widetilde{\alpha}_2\\
&=[h^{-1}(\mu_1 h(\mu^L(\alpha_1))),h^{-1}(\mu_1 h(\mu^U(\alpha_1)))],[g^{-1}(\mu_1 g(\nu^L(\alpha_1))),g^{-1}(\nu_1 g(\nu^U(\alpha_1)))]\rangle\\
&\oplus\langle[h^{-1}(\mu_2 h(\mu^L(\alpha_2))),h^{-1}(\mu_2 h(\mu^U(\alpha_2)))],[g^{-1}(\mu_2 g(\nu^L(\alpha_2))),g^{-1}(\nu_2 g(\nu^U(\alpha_2)))]\rangle=\\
&\langle[h^{-1}(h(h^{-1}(\mu_1 h(\mu^L(\alpha_1))))+h(h^{-1}(\mu_2 h(\mu^L(\alpha_2))))),h^{-1}(h(h^{-1}(\mu_1 h(\mu^U(\alpha_1))))+h(h^{-1}(\mu_2 h(\mu^U(\alpha_2)))))],\\
&[g^{-1}(g(g^{-1}(\mu_1 g(\nu^L(\alpha_1))))+g(g^{-1}(\mu_2 g(\nu^L(\alpha_2))))),g^{-1}(g(g^{-1}(\mu_1 g(\nu^U(\alpha_1))))+g(g^{-1}(\mu_2 g(\nu^U(\alpha_2)))))]\rangle=\\
&\langle[h^{-1}(\textstyle\sum_{j=1}^2\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^2\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^2\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^2\mu_j g(\nu^U(\alpha_j)))]\rangle.
\end{aligned}$$

Therefore, when $n=2$, the conclusion is right.

(3) Suppose when $n=k$, the conclusion is right, i.e.

$$\begin{aligned}
&ATS-IVIFWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_k)=\\
&\langle[h^{-1}(\textstyle\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j)))]\rangle.
\end{aligned}$$

Then, when $n=k+1$,

$$\begin{aligned}
&ATS-IVIULWA_\mu(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_k,\widetilde{\alpha}_{k+1})=\\
&\langle[h^{-1}(\textstyle\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j)))]\rangle\\
&\oplus\langle[h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))),h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1})))],[g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))),g^{-1}(\nu_{k+1}g(\nu^U(\alpha_{k+1})))]\rangle=\\
&\langle[h^{-1}(h(h^{-1}(\textstyle\sum_{j=1}^k\mu_j h(\mu^L(\alpha_j))))+h(h^{-1}(\mu_{k+1}h(\mu^L(\alpha_{k+1}))))),\\
&h^{-1}(h(h^{-1}(\textstyle\sum_{j=1}^k\mu_j h(\mu^U(\alpha_j))))+h(h^{-1}(\mu_{k+1}h(\mu^U(\alpha_{k+1})))))],\\
&[g^{-1}(g(g^{-1}(\textstyle\sum_{j=1}^k\mu_j g(\nu^L(\alpha_j))))+g(g^{-1}(\mu_{k+1}g(\nu^L(\alpha_{k+1}))))),\\
&g^{-1}(g(g^{-1}(\textstyle\sum_{j=1}^k\mu_j g(\nu^U(\alpha_j))))+g(g^{-1}(\mu_{k+1}g(\nu^U(\alpha_{k+1})))))]\rangle=\\
&\langle[h^{-1}(\textstyle\sum_{j=1}^{k+1}\mu_j h(\mu^L(\alpha_j))),h^{-1}(\sum_{j=1}^{k+1}\mu_j h(\mu^U(\alpha_j)))],[g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^L(\alpha_j))),g^{-1}(\sum_{j=1}^{k+1}\mu_j g(\nu^U(\alpha_j)))]\rangle.
\end{aligned}$$

So, when $n=k+1$, the conclusion is right, too.

According to steps (1), (2) and (3), we can conclude the conclusion is right for all n.

6. Conclusions

The main technologies in multiple attribute decision making, whether the situation is certain or vague, are how to define and calculate aggregation operators proposed in the practice. In this study we only discussed and investigated the operational rules of interval-valued intuitionistic fuzzy sets based on Archimedean t-norm, and the aggregating of interval-valued intuitionistic fuzzy sets based on Archimed -ean t-norm. In order to do this we also obtained the representations and transformations of Archimedean t-norm and Archimedean t-conorm. Based on these operators proposed in this note, we could make multiple attribute group decision making problems easily. Limited to the length of this paper it can not be discussed. However, it will be our main work in the future.

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Approximate bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras

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Abstract. In this paper, we generalize the concept of homomorphisms and derivations in intuitionistic fuzzy normed algebras for 2-dimensional functional equations. Furthermore, we investigate the Hyers-Ulam stability bi-homomorphisms and bi-derivations in intuitionistic fuzzy ternary normed algebras concerning a 2-dimensional bi-additive functional equation.

1. **Introduction and preliminaries**

We say a functional equation (ζ) is stable if any function g satisfying the equation (ζ) approximately is near to true solution of (ζ). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it. The stability problem of functional equations originated from a question of Ulam [37] in 1940, concerning the stability of group homomorphisms. We are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The case of approximately additive mappings was solved by Hyers [11] under the assumption that G_1 and G_2 are Banach spaces. In 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [28]. In 1991, Gajda [8] answered the question for the case $p > 1$, which was raised by Rassias. For more information on functional equations, see [18, 25, 26, 27, 32, 34, 35].

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. This new theory was introduced by Zadeh [38], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [5], chaos control [7], computer programming [9], nonlinear dynamical systems [10], fuzzy physics [12], fuzzy topology [31], fuzzy stability [13, 14, 15, 16, 24], nonlinear operators [20], statistical convergence [21, 23], etc. The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [29]. In [30], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces(see [3, 4, 6, 19]).

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Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N1)$ $N(x,c) = 0$ for $c \leqslant 0$;

$(N2)$ $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

$(N3)$ $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

$(N4)$ $N(x+y, s+t) \geqslant \min\{N(x,s), N(y,t)\}$;

$(N5)$ $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

$(N6)$ For $x \neq 0, N(x,.)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x,t)$ as the truth value of the statement the norm of x is less than or equal to the real number t.

The stability problem for a 2-dimensional bi-additive functional equation was proved by Bae and Park [1] for mappings $f : X \times X \to Y$, where X is a real normed space and Y is a Banach space.

In this paper, we determine some stability results of bi-homomorphism and bi-derivation concerning the 2-dimensional bi-additive functional equation

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x,z) - 2f(y,w) \tag{1.1}$$

in intuitionistic fuzzy ternary normed algebras. It has been discussed that $f(x,y) = ax^2 + by^2$ is a solution of (1.1) (see [2]).

We recall some notations and basic definitions used in this paper.

We use the definition of intuitionistic fuzzy normed spaces given in [17, 22, 29] to investigate some stability results for the functional equation (1.1) in the intuitionistic fuzzy normed vector space setting.

Definition 1.1. ([33]) *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* t*-norm if it satisfies the following conditions:*

(a) *is commutative and associative;*

(b) *is continuous;*

(c) $a * 1 = a$ *for all* $a \in [0,1]$*;*

(d) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0,1]$.

Definition 1.2. ([33]) *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* t*-conorm if it satisfies the following conditions:*

(a) *is commutative and associative;*

(b) *is continuous;*

(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*

(d) $a \diamond b \leqslant c \diamond d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for all* $a, b, c, d \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [29] have introduced the concept of intuitionistic fuzzy normed space.

Definition 1.3. ([22, 29]) *The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous* t*-norm,* $\diamond$ *is a continuous* t*-conorm, and* μ, ν *are fuzzy sets on* $X \times (0,\infty)$ *satisfying the following conditions: for every* $x, y \in X$ *and* $s, t > 0$,

(i) $\mu(x,t) + \nu(x,t) \leqslant 1$, (ii) $\mu(x,t) > 0$, (iii) $\mu(x,t) = 1$ *if and only if* $x = 0$, (iv) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (v) $\mu(x,t) * \mu(y,s) \leqslant \mu(x+y, t+s)$, (vi) $\mu(x,.) : (0,\infty) \to [0,1]$ *is continuous,* (vii) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$, $(viii)$ $\nu(x,t) < 1$, (ix) $\nu(x,t) = 0$ *if and only if* $x = 0$, (x) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ *for each* $\alpha \neq 0$, (xi) $\nu(x,t) \diamond \nu(y,s) \geqslant \nu(x+y, t+s)$, (xii) $\nu(x,.) : (0,1) \to [0,1]$ *is continuous,* $(xiii)$ $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

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Definition 1.4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy convergent *to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - L, t) = 0$ for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$. A sequence $\{x_n\}$ is said to be* intuitionistic fuzzy Cauchy sequence *if $\lim_{k\to\infty} \mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_{k+p} - x_k, t) = 0$ for all $p \in \mathbb{N}$ and all $t > 0$. Then IFNS $(X, \mu, \nu, *, \diamond)$ is said to be* complete *if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamond)$ and $(X, \mu, \nu, *, \diamond)$ is also called an* intuitionistic fuzzy Banach space.

The concepts of convergent sequence and Cauchy sequence in an intuitionistic fuzzy normed space are studied in [29].

Definition 1.5. *Let X be a ternary algebra with $[\cdot,\cdot,\cdot]$ and $(X, \mu, \nu, *, \diamond)$ be an IFNS.*
(1) *The intuitionistic fuzzy normed space $(X, \mu, \nu, *, \diamond)$ is called an intuitionistic fuzzy ternary normed algebra if*

$$\mu([x, y, z], stu) \geqslant \mu(x, s) * \mu(y, t) * \mu(z, u)$$
$$\nu([x, y, z], stu) \geqslant \nu(x, s) * \nu(y, t) * \nu(z, u)$$

for all $x, y, z \in X$ and $s, t, u > 0$.

(2) *A complete intuitionistic fuzzy ternary normed algebra is called an intuitionistic fuzzy ternary Banach algebra.*

Definition 1.6. *Let X be a ternary normed (Banach) algebra and (Y, μ, ν) an intuitionistic fuzzy ternary Banach algebra.*

(1) *A bi-additive mapping $H : X \times X \to Y$ is called a ternary bi-homomorphism if*

$$H([x, y, z], [w, w, w]) = [H(x, w), H(y, w), H(z, w)],$$
$$H([x, x, x], [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in X$.

(2) *A bi-additive mapping $\delta : X \times X \to X$ is called a ternary bi-derivation if*

$$\delta([x, y, z], w) = [\delta(x, w), y, z] + [x, \delta(y, w), z] + [x, y, \delta(z, w)],$$
$$\delta(x, [y, z, w]) = [\delta(x, y), z, w] + [y, \delta(x, z), w] + [y, z, \delta(x, w)]$$

for all $x, y, z, w \in X$.

2. Bi-homomorphisms in intuitionistic fuzzy ternary normed algebras

We begin with a Hyers-Ulam type theorem in intuitionistic fuzzy ternary normed algebras to approximate bi-homomorphism associated to the functional equation (1.1). For notational convenience, given a function $f : X \times X \to Y$, we define the difference operator

$$D_q f(x, y, z, w) = f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w)$$

Lemma 2.1. ([36, Theorem 3.1]) *Let X be a linear space and let (Z, μ', ν') be an IFNS. Let $\varphi : X^4 \to Z$ be a mapping such that, for some $0 < \alpha < 4$.*

$$\begin{cases} \mu'(\varphi(2x, 2y, 2z, 2w), t) \geqslant \mu'(\alpha\varphi(x, y, z, w), t), \\ \nu'(\varphi(2x, 2y, 2z, 2w), t) \leqslant \nu'(\alpha\varphi(x, y, z, w), t), \end{cases} \tag{2.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a mapping satisfying $f(0, 0) = 0$ and

$$\begin{cases} \mu(D_q f(x, y, z, w), t) \geqslant \mu'(\varphi(x, y, z, w), t), \\ \nu(D_q f(x, y, z, w), t) \leqslant \nu'(\varphi(x, y, z, w), t) \end{cases} \tag{2.2}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ satisfying (1.1) *such that*

$$\begin{cases} \mu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \geqslant *^{\infty}\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^{\infty} \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(H(x,y) - f(x,y), t\Big) \\ \quad \leqslant \diamond^{\infty}\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^{\infty} \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{2.3}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^{\infty}a := a * a * \cdots$ and $\diamond^{\infty}a := a \diamond a \diamond \cdots$ for all $a \in [0, 1]$.

Theorem 2.2. *Let X be a ternary algebra and let (Z, μ', ν) be an IFNS. Let $\varphi : X^4 \to Z$ be a mapping satisfying* (2.1)*. Let (Y, μ, ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a mapping satisfying $f(0, 0) = 0$,* (2.2) *and*

$$\begin{cases} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \geqslant \mu'(\varphi(x,y,z,w), t), \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\nu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \leqslant \nu'(\varphi(x,y,z,w), t) \end{cases} \tag{2.4}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (1.1) *and* (2.3)*.*

Proof. In Lemma 2.1, the mapping $H : X \times X \to Y$ was defined by $H(x, y) = \lim_{n\to\infty} \frac{f(2^n x, 2^n y)}{4^n}$ for all $x, z \in X$.

From (2.4) and definition of H, it follows that

$$\begin{aligned} &\mu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \mu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \\ &\quad = \mu\Big(\frac{f([2^n x, 2^n y, 2^n z],[2^n w, 2^n w, 2^n w])}{64^n} - \Big[\frac{f(2^n x, 2^n w)}{4^n}, \frac{f(2^n y, 2^n w)}{4^n}, \frac{f(2^n z, 2^n w)}{4^n}\Big], t\Big) \\ &\quad + \mu\Big(\frac{f([2^n x, 2^n x, 2^n x],[2^n x, 2^n y, 2^n z])}{64^n} - \Big[\frac{f(2^n x, 2^n y)}{4^n}, \frac{f(2^n x, 2^n z)}{4^n}, \frac{f(2^n x, 2^n w)}{4^n}\Big], t\Big) \\ &\quad \geqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^n w), 4^{3n}t) \geqslant \mu'(\varphi(x,y,z,w), \frac{4^{3n}}{\alpha^n}t) \to 1 \end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in X$ and all $t > 0$, and similarly

$$\begin{aligned} &\nu(H([x,y,z],[w,w,w]) - [H(x,y), H(y,w), H(z,w)], t) \\ &\quad + \nu(H([x,x,x],[y,z,w]) - [H(x,y), H(x,z), H(x,w)], t) \leqslant 0 \end{aligned}$$

for all $x, y, z, w \in X$ and all $t > 0$. So we conclude that

$$\begin{aligned} H([x,y,z],[w,w,w]) &= [H(x,w), H(y,w), H(z,w)], \\ H([x,x,x],[y,z,w]) &= [H(x,y), H(x,z), H(x,w)] \end{aligned}$$

for all $x, y, z, w \in X$. □

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Corollary 2.3. *Let p be a nonnegative real number with $p < 2$, X be a ternary normed algebra with norm $\|.\|$, (Z,μ',ν') be an intuitionistic fuzzy ternary normed algebra, (Y,μ,ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$ and*

$$\begin{cases} \mu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(f([x,y,z],[w,w,w]) - [f(x,y), f(y,w), f(z,w)], t) \\ \quad +\mu(f([x,x,x],[y,z,w]) - [f(x,y), f(x,z), f(x,w)], t) \\ \quad \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{cases} \tag{2.5}$$

and

$$\begin{cases} \mu(D_q f(x,y), t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(D_q f(x,y), t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{cases} \tag{2.6}$$

for all $x, y, z, w \in X$ and $t > 0$, then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that

$$\begin{cases} \mu(H(x,y) - f(x,y), t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \\ \nu(H(x,y) - f(x,y), t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

Lemma 2.4. ([36, Theorem 3.3]) *Let X be a linear space and let (Z,μ',ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping such that, for some $\alpha > 4$,*

$$\begin{cases} \mu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \geqslant \mu'(\varphi(x,y,z,w), \alpha t), \\ \nu'\left(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t\right) \leqslant \nu'(\varphi(x,y,z,w), \alpha t), \end{cases} \tag{2.7}$$

for all $x, y, z, w \in X$ and all $t > 0$. Let (Y,μ,ν) be an intuitionistic fuzzy Banach space and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0,0) = 0$. Then there exists a unique bi-additive mapping $H : X \times X \to Y$ such that

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \geqslant *^\infty\, \mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty\, \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) \\ *^\infty\, \mu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \mu(H(x,y) - f(x,y), t) \leqslant \diamond^\infty\, \nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \diamond^\infty\, \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big) \\ \diamond^\infty\, \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big) \end{aligned} \tag{2.9}$$

for all $x, y \in X$ and all $t > 0$.

Theorem 2.5. *Let X be a ternary algebra and let (Z,μ',ν') be an IFNS. Let $\varphi : X \times X \times X \times X \to Z$ be a mapping satisfying (2.7). Let (Y,μ,ν) be an intuitionistic fuzzy ternary Banach algebra and let $f : X \times X \to Y$ be a φ-approximately bi-additive mapping in the sense of (2.2) and (2.4) with $f(0,0) = 0$. Then there exists a unique bi-homomorphism $H : X \times X \to Y$ satisfying (2.8) and (2.9).*

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.6. *Let p be a nonnegative real number with $p > 2$, X be a ternary normed algebra with norm $\|.\|$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (Y, μ, ν) be a complete intuitionistic fuzzy ternary normed algebra, and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (2.5) and (2.6). then there exists a unique bi-homomorphism $H : X \times X \to Y$ such that*

$$\begin{cases} \mu(H(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\|+\|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\Big((\|x\|+\|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \\ \nu(H(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\|+\|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\Big((\|x\|+\|y\|)z_0, \frac{(2^p-4)t}{8}\Big) \end{cases}$$

for all $x, y \in X$ and $t > 0$.

3. Bi-derivations on intuitionistic fuzzy ternary normed algebras

In this section, we investigate generalized Hyers-Ulam stability of bi-derivations on intuitionistic fuzzy ternary normed algebrasfor the functional equation (1.1).

Theorem 3.1. *Let X be an intuitionistic fuzzy ternary Banach algebra and let (Z, μ', ν') be an IFNS. Let $f : X \times X \to X$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ such that, for some $0 < \alpha < 4$ satisfying (2.1), (2.2) and*

$$\begin{cases} \mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \geqslant \mu'(\varphi(x,y,z,w),t), \\ \nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\ \qquad +\nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\ \quad \leqslant \nu'(\varphi(x,y,z,w),t) \end{cases} \tag{3.1}$$

for all $x, y, z, w \in X$ and all $t > 0$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ satisfying (1.1) such that

$$\begin{cases} \mu\Big(\delta(x,y) - f(x,y),t\Big) \\ \geqslant *^\infty\mu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) *^\infty \mu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big), \\ \nu\Big(\delta(x,y) - f(x,y),t\Big) \\ \leqslant \diamond^\infty\nu'\Big(\varphi(x,x,y,-y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(4-\alpha)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(4-\alpha)}{8}t\Big) \end{cases} \tag{3.2}$$

for all $x, y, z, w \in X$ and all $t > 0$, where $*^\infty a := a * a * \cdots$ and $\diamond^\infty a := a \diamond a \diamond \cdots$ for all $a \in [0,1]$.

Proof. By the same argument as in the proof of Theorem 2.2, there exists a unique bi-additive mapping $\delta : X \times X \to X$ satisfying (3.2). The mapping δ is given by

$$\delta(x,y) = \lim_{n\to\infty} \frac{1}{4} f(2^n x, 2^n y)$$

for all $x, y \in X$. It follows from (3.1) that

$$\begin{aligned}
&\mu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \mu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) \\
&\quad = \mu\Big(\frac{1}{4^{3n}} f(2^{3n}[x,y,z],2^{3n}w) - \Big[\frac{1}{4^n} f(2^n x, 2^n w), y, z\Big] \\
&\qquad - \Big[x, \frac{1}{4^n} f(2^n x, 2^n w), z\Big] - \Big[x, y, \frac{1}{4^n} f(2^n z, 2^n w)\Big], t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}} f(2^{3n}x, 2^{3n}[y,z,w]) - \Big[\frac{1}{4^n} f(2^n x, 2^n y), z, w\Big] \\
&\qquad - \Big[y, \frac{1}{4^n} f(2^n x, 2^n z), w\Big] - \Big[y, z, \frac{1}{4^n} f(2^n x, 2^n w)\Big], t\Big) \\
&\quad = \mu\Big(\frac{1}{4^{3n}} f([2^n x, 2^n y, 2^n z], 2^{3n} w) - \frac{1}{4^{3n}}[f(2^n x, 2^{3n} w), 2^n y, 2^n z] \\
&\qquad - \frac{1}{4^{3n}}[2^n x, f(2^n y, 2^{3n} w), 2^n z] - \frac{1}{4^{3n}}[2^n x, 2^n y, f(2^n z, 2^{3n} w)], t\Big) \\
&\qquad + \mu\Big(\frac{1}{4^{3n}} f(2^{3n} x, [2^n y, 2^n z, 2^n w]) - \frac{1}{4^{3n}}[f(2^{3n} x, 2^n y), 2^n z, 2^n w] \\
&\qquad - \frac{1}{4^{3n}}[2^n y, f(2^{3n} x, 2^n z), 2^n w] - \frac{1}{4^{3n}}[2^n y, 2^n z, f(2^{3n} x, 2^n w)], t\Big) \\
&\qquad \leqslant \mu'(\varphi(2^n x, 2^n y, 2^n z, 2^{3n} w), 4^{3n} t) + \mu'(\varphi(2^{3n} x, 2^n y, 2^n z, 2^n w), 4^{3n} t)) \\
&\qquad \leqslant 2\mu'\Big(\varphi(x,y,z,w), \frac{4^{3n} t}{\alpha^{3n}}\Big) \longrightarrow 1
\end{aligned}$$

as $n \to \infty$ for all $x, y, z, w \in \mathcal{A}$. Similarly, we obtain

$$\begin{aligned}
&\nu(\delta([x,y,z],w) - [\delta(x,w),y,z] - [x,\delta(y,w),z] - [x,y,\delta(z,w)],t) \\
&\quad + \nu(\delta(x,[y,z,w]) - [\delta(x,y),z,w] - [y,\delta(x,z),w] - [y,z,\delta(x,w)],t) = 0
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. Thus

$$\begin{aligned}
\delta([x,y,z],w) &= [\delta(x,w),y,z] + [x,\delta(y,w),z] + [x,y,\delta(z,w)], \\
\delta(x,[y,z,w]) &= [\delta(x,y),z,w] + [y,\delta(x,z),w] + [y,z,\delta(x,w)]
\end{aligned}$$

for all $x, y, z, w \in \mathcal{A}$. So we conclude that δ is a unique bi-derivation satisfying (3.2). $\square$

Corollary 3.2. *Let p be a nonnegative real number with $p < 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra, and let $z_0 \in Z$. If $f : X \to X$ is a mapping with $f(0,0) = 0$ such that*

$$\begin{cases}
\mu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad + \mu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t) \\
\nu(f([x,y,z],w) - [f(x,w),y,z] - [x,f(y,w),z] - [x,y,f(z,w)],t) \\
\qquad + \nu(f(x,[y,z,w]) - [f(x,y),z,w] - [y,f(x,z),w] - [y,z,f(x,w)],t) \\
\qquad \geqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0, t)
\end{cases} \tag{3.3}$$

and

$$\begin{cases} \mu(D_q f(x,y),t) \geqslant \mu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \\ \nu(D_q f(x,y),t) \leqslant \nu'((\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0, t) \end{cases} \tag{3.4}$$

for all $x,y,z,w \in X$ and $t > 0$, then there exists a unique bi-derivation $\delta : X \times X \to X$ such that

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(4-2^p)t}{8}\right) \end{cases}$$

for all $x,y \in X$ and $t > 0$.

Theorem 3.3. *Let X be an intuitionistic fuzzy ternary Banach algebra and let $(Z < \mu', \nu,)$ be an IFNS. Let $f : X \times X \to Y$ be a mapping with $f(0,0) = 0$ for which there exists a mapping $\varphi : X \times X \times X \times X \to Z$ satisfying (2.1), (2.7) and (3.1) for some $\alpha > 4$. Then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\mu(\delta(x,y) - f(x,y),t) \geqslant *^\infty \mu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) *^\infty \mu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$*^\infty \mu\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

and

$$\mu(\delta(x,y) - f(x,y),t) \leqslant \diamond^\infty \nu'\Big(\varphi(x,x,y,-y), \frac{(\alpha-4)}{8}t\Big) \diamond^\infty \nu'\Big(\varphi(x,-x,y,y), \frac{(\alpha-4)}{8}t\Big)$$
$$\diamond^\infty \nu'\Big(\varphi(0,x,0,y), \frac{(\alpha-4)}{8}t\Big)$$

for all $x,y \in X$.

Proof. The proof is similar to the proof of Theorems 2.5 and 3.1. □

Corollary 3.4. *Let p be a nonnegative real number with $p > 2$, (Z, μ', ν') be an intuitionistic fuzzy ternary normed algebra, (X, μ, ν) be a complete intuitionistic fuzzy ternary Banach algebra and let $z_0 \in Z$. If $f : X \to Y$ is a mapping satisfying $f(0,0) = 0$, (3.3) and (3.4), then there exists a unique bi-derivation $\delta : X \times X \to X$ such that*

$$\begin{cases} \mu(\delta(x,y) - f(x,y),t) \geqslant *^2\mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \mu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\right) \\ \nu(\delta(x,y) - f(x,y),t) \leqslant *^2\nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{16}\right) * \nu'\left((\|x\| + \|y\|)z_0, \frac{(2^p-4)t}{8}\right) \end{cases}$$

for all $x,y \in X$ and $t > 0$.

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ON NEW REFINEMENTS AND APPLICATIONS OF EFFICIENT QUADRATURE RULES USING N-TIMES DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, new efficient quadrature rules are established using a newly developed special type of kernel for n-times differentiable mappings, having five steps. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

1. Introduction

In 1938, Ostrowski [13] first announced his inequality for different differentiable mappings, which is given below:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (I° is the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) i.e. $\|f'\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In 1976, Milovanovic et. al [11], proved a generalization of Ostrowski's inequality for n-time differentiable mappings. Up till now, a large number of research papers and books have been written on inequalities and their applications (see for instance [2]-[5], [8] and [14]-[16]). In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities like Ostrowski are very helpful for this purpose. Ostrowski type inequalities have immediate applications in numerical integration, optimization theory, statistics, and integral operator theory.

We indicate another inequality called Grüss inequality [11] which is stated as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals, which is given below.

Theorem 2. *Let $f,\ g : [a,b] \to \mathbb{R}$ be integrable functions such that $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$, for some constants $\varphi,\ \Phi,\ \gamma,\ \Gamma$ and $x \in [a,b]$. Then*

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx.\frac{1}{b-a}\int_a^b g(x)dx \right| \tag{1.2}$$
$$\le \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).$$

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Dragomir et. al [4] combined Ostrowski and Grüss inequality to give a new inequality which they named Ostrowski-Grüss type inequality. Dragomir [3], Liu [6], Alomari [1] and Liu et. al [8] established some companions of ostrowski type integral inequalities.

Recently, Liu [7] proved the following companions of ostrowski type inequalities for 3-step kernels.

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be differentiable in* (a,b). *If* $f' \in L^1[a,b]$, *and* $\gamma \leq f'(x) \leq \Gamma$, *for all* $x \in [a,b]$, *then for all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] (S-\gamma) \tag{1.3}$$

and

$$\begin{aligned} &\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \left[\frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right] (\Gamma - S). \end{aligned} \tag{1.4}$$

More recently, Qayyum et. al [9]-[10] proved companions of Ostrowski inequality for 5-step linear and quadratic kernels but in this paper, we establish our results for 5-step kernel for n-times differentiable mappings. In this paper, new ontrowski inequalities are extended. Using these inequalities, some efficient quadrature rules are established. Some previous inequalities are also recaptured as special cases of our main inequalities. At the end, efficiency of the newly developed quadrature rules are discussed.

2. Derivation of Ostrowski inequalities using 5-step kernel

We will start our work by introducing a new Peano kernel defined by $P(x,.) : [a,b] \to \mathbb{R}$

$$P_n(x,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & t \in \left(a, \frac{a+x}{2}\right], \\ \frac{1}{n!}\left(t - \frac{3a+b}{4}\right)^n, & t \in \left(\frac{a+x}{2}, x\right], \\ \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n, & t \in (x, a+b-x], \\ \frac{1}{n!}\left(t - \frac{a+3b}{4}\right)^n, & t \in \left(a+b-x, \frac{a+2b-x}{2}\right], \\ \frac{1}{n!}(t-b)^n, & t \in \left(\frac{a+2b-x}{2}, b\right], \end{cases} \tag{2.1}$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

The following lemma is the main tool to prove the main results.

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *be an n-times differentiable function such that* $f^{(n-1)}(x)$ *for* $n \in N$ *is absolutely continuous on* $[a,b]$ *then*

$$\begin{aligned} &\frac{1}{b-a} \int_a^b P_n(x,t) f^{(n)}(t)dt \\ &= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \left[\frac{1}{2^{k+1}} \left\{ (x-a)^{k+1} - \left(x - \frac{a+b}{2} \right)^{k+1} \right\} f^{(k)}\left(\frac{a+x}{2} \right) \right. \end{aligned} \tag{2.2}$$

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$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^{n}}{b-a}\int_a^b f(t)dt,$$

for all $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. The proof of (2.2) is established using mathematical induction.

Take $n=1$,

$$L.H.S \text{ of } (2.2)=\int_a^b P_1(x,t)f'(t)dt. \tag{2.3}$$

After integrating by parts, we get

$$\frac{1}{b-a}\int_a^b P_1(x,t)f'(t)dt \tag{2.4}$$
$$=\frac{1}{4}\left[f\left(\frac{a+x}{2}\right)+f(x)+f(a+b-x)+f\left(\frac{a+2b-x}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt.$$

We have

$$L.H.S=\int_a^b P_1(x,t)f'(t)dt.$$

Equation (2.3), is identical to the $R.H.S$ of (2.2).

Assume that (2.2) is true for n.

$$\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt$$
$$=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+(-1)^{n+1}\int_a^b f(t)dt,$$

where

$$P_{n+1}(x,t)=\begin{cases}\frac{1}{(n+1)!}\left(t-a\right)^{n+1}, & t\in\left(a,\frac{a+x}{2}\right],\\ \frac{1}{(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}, & t\in\left(\frac{a+x}{2},x\right],\\ \frac{1}{(n+1)!}\left(t-\frac{a+b}{2}\right)^{n+1}, & t\in\left(x,a+b-x\right],\\ \frac{1}{(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right],\\ \frac{1}{(n+1)!}\left(t-b\right)^{n+1}, & t\in\left(\frac{a+2b-x}{2},b\right].\end{cases}$$

After integeration by parts, we get

$$\begin{aligned}
&\int_a^b P_{n+1}(x,t)f^{(n+1)}(t)dt\\
&=\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\right.\\
&-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\
&-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\
&\left.+(-1)^{n}\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)+\left(\frac{x-a}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{n!}\left[\int_a^{\frac{a+x}{2}}\left(t-a\right)^{n}f^{(n)}(t)dt+\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^{n}f^{(n)}(t)dt\right.\\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^{n}f^{(n)}(t)dt+\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{3a+b}{4}\right)^{n}f^{(n)}(t)dt\\
&\left.+\int_{\frac{a+2b-x}{2}}^{b}\left(t-b\right)^{n}f^{(n)}(t)dt\right]
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{(n+1)!}\left[\frac{1}{2^{n+1}}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(x)\right.\\
&-\frac{1}{2^{n+1}}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+x}{2}\right)+(-1)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(a+b-x)\\
&-\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}(x)+\left(\frac{-1}{2}\right)^{n+1}\left(x-\frac{a+b}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\\
&-(-1)^{n+1}\left(x-\frac{3a+b}{4}\right)^{n+1}f^{(n)}(a+b-x)\\
&\left.-\left(\frac{-1}{2}\right)^{n+1}\left(x-a\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b P_n(x,t)f^{(n)}(t)dt
\end{aligned}$$

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$$= \frac{1}{(n+1)!}\left[\left\{(x-a)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}\frac{1}{2^{n+1}}f^{(n)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{n+1}-\left(x-\frac{a+b}{2}\right)^{n+1}\right\}f^{(n)}(x)$$
$$+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}(-1)^{n+1}f^{(n)}(a+b-x)$$
$$\left.+\left\{\left(x-\frac{a+b}{2}\right)^{n+1}-(x-a)^{n+1}\right\}\left(\frac{-1}{2}\right)^{n+1}f^{(n)}\left(\frac{a+2b-x}{2}\right)\right]$$

$$+\sum_{k=0}^{n-1}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+(-1)^{n+1}\int_a^b f(t)dt$$

$$=\sum_{k=0}^{n}\frac{(-1)^{n+k+2}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+(-1)^{n+1}\int_a^b f(t)dt.$$

This completes the proof of lemma 1. □

Now we will present our results by imposing three different conditions on $f^{(n)}$ and $f^{(n+1)}$.

3. **Case A:** When $f^{(n)}\in L^1[a,b]$

Theorem 4. *Let $f\colon [a,b]\to\mathbb{R}$ be an n-times differentiable function on (a,b), $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $\gamma\le f^{(n)}(t)\le\Gamma$, $\forall\ t\in[a,b]$, then for* *all* $x \in \left[a, \frac{a+b}{2}\right]$, *we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\left[\times\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + (1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{3.2}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + (1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq \delta(x)(b-a)(\Gamma-S),$$

where

$$S = \frac{f^{\ (n-1)}(b) - f^{\ (n-1)}(a)}{b-a}, \tag{3.3}$$

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$$\delta(x)=\max\left\{\left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^{n}-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{n!}\left(x-\frac{3a+b}{4}\right)^{2}-\frac{\lambda(x)}{b-a}\right|,\right.$$
$$\left.\left|\frac{1}{n!}\left(x-\frac{a+b}{2}\right)^{n}-\frac{\lambda(x)}{b-a}\right|,\left|\frac{1}{4n!}\left(x-\frac{a+b}{2}\right)^{2}-\frac{\lambda(x)}{b-a}\right|,\frac{\lambda(x)}{b-a}\right\}$$

and

$$\lambda(x)=\frac{1}{(n+1)!}\left[(1+(-1)^{n})\left\{\frac{1}{2^{n+1}}(x-a)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Proof. Let

$$\frac{1}{b-a}\int_{a}^{b}P_{n}(x,t)dt \tag{3.4}$$
$$=\frac{1}{(b-a)(n+1)!}\left[(1+(-1)^{n})\left\{\frac{1}{2^{n+1}}(x-a)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Using (3.4), we get

$$\frac{1}{b-a}\int_{a}^{b}P_{n}(x,t)f^{(n)}(t)dt-\frac{1}{(b-a)^{2}}\int_{a}^{b}P_{n}(x,t)dt\int_{a}^{b}f^{(n)}(t)dt \tag{3.5}$$
$$=\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right.$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^{n}}{b-a}\int_{a}^{b}f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^{2}}\frac{1}{(n+1)!}$$
$$\left[\times\frac{1}{2^{n+1}}(1+(-1)^{n})(x-a)^{n+1}+(1+(-1)^{n})\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$
$$\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right].$$

Denote the L.H.S of (3.5) by $R_n(x)$. If $C \in R$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b \left(f^{(n)}(t) - C\right)\left[P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right]dt. \tag{3.6}$$

Furthermore, we have

$$|R_n(x)| \leq \frac{1}{b-a}\max_{t\in[a,b]}\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|\int_a^b \left|f^{(n)}(t) - C\right|dt. \tag{3.7}$$

Now

$$\left|P_n(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| \tag{3.8}$$
$$= \max\left\{\begin{array}{c} \left|\frac{1}{n!}\left(\frac{x-a}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{n!}\left(x - \frac{3a+b}{4}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \\ \left|\frac{1}{n!}\left(x - \frac{a+b}{2}\right)^n - \frac{\lambda(x)}{b-a}\right|, \left|\frac{1}{4n!}\left(x - \frac{a+b}{2}\right)^2 - \frac{\lambda(x)}{b-a}\right|, \frac{\lambda(x)}{b-a}\end{array}\right\} = \delta(x),$$

where

$$\lambda(x) = \frac{1}{(n+1)!}\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1} + \left(x - \frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left. + \left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x - \frac{a+b}{2}\right)^{n+1}\right].$$

We also have

$$\int_a^b \left|f^{(n)}(t) - \gamma\right|dt = (S-\gamma)(b-a), \tag{3.9}$$

$$\int_a^b \left|f^{(n)}(t) - \Gamma\right|dt = (\Gamma - S)(b-a). \tag{3.10}$$

Using (3.4) to (3.10) and using $C = \gamma$ and $C = \Gamma$ in (3.7), we can obtain (3.1) and (3.2). □

Remark 1. *If we substitute $n = 2$ in (3.1) and (3.2), we get Qayyum et. al result proved in* [9].

Corollary 1. *Substitution of $x = a$ in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right. \tag{3.11}$$
$$\left. + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$
$$\leq \delta(a)(b-a)(S-\gamma)$$

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and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a)+\left(1+\frac{1}{2^{k+1}}+\frac{(-1)^k}{4^{k+1}}\right)f^{(k)}(b)\right\}\right. \tag{3.12}$$

$$\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)(1+(-1)^n)\right|$$

$$\leq \delta(a)(b-a)(\Gamma-S).$$

Corollary 2. *Substitution of* $x=\frac{a+b}{2}$ *in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{3.13}$$

$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$\left.-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$

$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)+\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{3.14}$$

$$\left.+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\}+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$\left.-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right|$$

$$\leq \delta\left(\frac{a+b}{2}\right)(b-a)(\Gamma-S).$$

Corollary 3. *Substitution of* $x=\frac{3a+b}{4}$ *in (3.1) and (3.2) gives*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{\left(1+(-1)^k\right)}{2^{k+1}}f^{(k)}\left(\frac{7a+b}{8}\right)+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)\right.\right. \tag{3.15}$$

$$\left.+f^{(k)}\left(\frac{a+3b}{4}\right)+\frac{1}{2^{k+1}}\left(1+(-1)^k\right)f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$

$$\left.-\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}\left\{1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right\}\right|$$

$$\leq \delta\left(\frac{3a+b}{4}\right)(b-a)(S-\gamma)$$

and

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right. \tag{3.16}$$
$$+(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)$$
$$\left.+\frac{1}{2^{k+1}}\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+7b}{8}\right)\right]+\frac{(-1)^n}{b-a}\int_a^b f(t)dt$$
$$\left.-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{1+(-1)^n+\frac{1}{2^n}+\frac{(-1)^n}{2^n}\right\}\right|$$
$$\leq \delta\left(\frac{3a+b}{4}\right)(b-a)(\Gamma-S).$$

4. **Case B:** When $f^{(n+1)}\in L^2[a,b]$

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be an n-times differentiable function on (a,b), $f^{(n+1)}\in L^2[a,b]$, then for all $x\in\left[a,\frac{a+b}{2}\right]$, we have*

$$\left|\sum_{k=0}^{n-1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{f^{(k)}\left(\frac{a+x}{2}\right)}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}\right.\right. \tag{4.1}$$
$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)$$
$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)$$
$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$
$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$
$$\times\left[(1+(-1)^n)\left\{\frac{1}{2^{n+1}}(x-a)^{n+1}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right\}\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$
$$\leq\frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$
$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$
$$-\frac{1}{(b-a)(n+1)!}\left\{(1+(-1)^n)\left(\frac{(x-a)^{n+1}}{2^{n+1}}+\left(x-\frac{3a+b}{4}\right)^{n+1}\right)\right.$$
$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.$$

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Proof. Substitute $C = f^{(n)}\left(\frac{a+b}{2}\right)$, in $R_n(x)$ given in (3.5) and use the Cauchy Inequality, then we get

$$\begin{aligned} &|R_n(x)| \qquad (4.2)\\ &\leq \frac{1}{b-a}\int_a^b \left|f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right|\left|P^{(n)}(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right| dt \\ &\leq \frac{1}{b-a}\left[\int_a^b \left(f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt\right]^{\frac{1}{2}} \\ &\times \left[\int_a^b \left(P^{(n)}(x,t) - \frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

Use the Diaz-Metcalf inequality [12] or [17], to get

$$\int_a^b \left(f^{(n)}(t) - f^{(n)}\left(\frac{a+b}{2}\right)\right)^2 dt \leq \frac{(b-a)^2}{\pi^2}\left\|f^{(n+1)}\right\|_2^2.$$

Therefore, using the above relations, we obtain (4.1). □

Corollary 4. *Substitution of $x = a$ in (4.1) gives*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b)\right\}\right. \\ &\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!}\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1+(-1)^n)\right| \\ &\qquad \leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)} - \frac{(1+(-1)^n)^2(b-a)^{2n+1}}{2^{2n+2}(n+1)!}\right]^{\frac{1}{2}}. \end{aligned}$$

Corollary 5. *Substitution of $x = \frac{a+b}{2}$ in (4.1) gives*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\frac{(b-a)^{k+1}}{4^{k+1}}\left\{f^{(k)}\left(\frac{3a+b}{4}\right)\right.\right. \qquad (4.3)\\ &\left.+\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\} \\ &\left.+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)\frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!}(1+(-1)^n)\right| \\ &\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\left[\frac{1}{(n!)^2(2n+1)}\frac{4}{4^{2n+1}}(b-a)^{2n+1}\right. \\ &\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{2(b-a)^{n+1}}{4^{n+1}}(1+(-1)^n)\right\}^2\right]^{\frac{1}{2}}. \end{aligned}$$

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Corollary 6. *Substitution of* $x = \frac{3a+b}{4}$ *in (4.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left\{1+(-1)^k\right\}}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right) \right.\right. \tag{4.4}$$

$$+ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)$$

$$\left.+\frac{1}{2^{k+1}}\left(1+(-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a}\int_a^b f(t)dt - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)$$

$$\left.\times \frac{1}{(n+1)!}\frac{(b-a)^{n-1}}{4^{n+1}}\left\{1+(-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right\}\right|$$

$$\leq \frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2 \left[\frac{1}{(n!)^2(2n+1)}\frac{(b-a)^{2n+1}}{4^{2n+1}}\left\{\frac{4}{2^{2n+1}} - 2\right\}\right.$$

$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\frac{(b-a)^{n+1}}{4^{n+1}}\left\{(1+(-1)^n)\left(2+\frac{1}{2^{n+1}}\right)\right\}^2\right]^{\frac{1}{2}}.$$

5. Case C: When $f^{(n)} \in L^2[a,b]$.

Theorem 6. *Let* $f : [a;b] \to R$ *be an n-times differentiable function on* (a,b)*, with* $f^{(n)} \in L^2[a,b]$*. Then, we have*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1} - \left(x - \frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}\left(\frac{a+x}{2}\right)\right.\right. \tag{5.1}$$

$$+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1} - \left(x-\frac{a+b}{2}\right)^{k+1}\right\} f^{(k)}(x)$$

$$+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - \left(x-\frac{3a+b}{4}\right)^{k+1}\right\} f^{(k)}(a+b-x)$$

$$\left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1} - (x-a)^{k+1}\right\} f^{(k)}\left(\frac{a+2b-x}{2}\right)\right]$$

$$+\frac{(-1)^n}{b-a}\int_a^b f(t)dt - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{(b-a)^2}\frac{1}{(n+1)!}$$

$$\times\left[\frac{1}{2^{n+1}}\left(1-(-1)^{n+1}\right)(x-a)^{n+1} + \left(1-(-1)^{n+1}\right)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$

$$\left.\left.+\left(\frac{-1}{2^{n+1}} + \frac{(-1)^{n+1}}{2^{n+1}} + (-1)^{n+1} - 1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right]\right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{(x-a)^{2n+1}}{2^{2n}} + 2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$

$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$

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$$-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(x-a)^{n+1}}{2^{n+1}}(1+(-1)^n)+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}\right.$$

$$\left.\left.+\left(\frac{-1}{2^{n+1}}-\frac{(-1)^n}{2^{n+1}}-(-1)^n-1\right)\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}},$$

for all $x\in\left[a,\frac{a+b}{2}\right]$, *where*

$$\sigma\left(f^{(n)}\right) \tag{5.2}$$

$$=\left\|f^{(n)}\right\|_2^2-\frac{\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)^2}{b-a}=\left\|f^{(n)}\right\|_2^2-k^2(b-a),$$

where S *is as defined in Theorem 4.*

Proof. Let $R_n(x)$ is defined as in (3.5). If we choose $C=\frac{1}{b-a}\int_a^b f^{(n)}(s)ds$ in (3.6) and use the Cauchy inequality and (3.5), then we get

$$|R_n(x)|$$

$$\leq\frac{1}{b-a}\int_a^b\left|f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right|\left|P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right|dt$$

$$\leq\frac{1}{b-a}\left[\int_a^b\left(f^{(n)}(t)-\frac{1}{b-a}\int_a^b f^{(n)}(s)ds\right)^2dt\right]^{\frac{1}{2}}$$

$$\times\left[\int_a^b\left(P_n(x,t)-\frac{1}{b-a}\int_a^b P^{(n)}(x,s)ds\right)^2dt\right]^{\frac{1}{2}}$$

$$=\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\int_a^b(P_n(x,t))^2-\frac{1}{b-a}\left(\int_a^b P^{(n)}(x,t)dt\right)^2\right]^{\frac{1}{2}}$$

$$\leq\frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\left\{\frac{1}{2^{2n}}(x-a)^{2n+1}+2\left(x-\frac{3a+b}{4}\right)^{2n+1}\right.\right.$$

$$\left.-\left(\frac{1}{2^{2n}}+2\right)\left(x-\frac{a+b}{2}\right)^{2n+1}\right\}$$

$$-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{1}{2^{n+1}}(1+(-1)^n)(x-a)^{n+1}\right.$$

$$+(1+(-1)^n)\left(x-\frac{3a+b}{4}\right)^{n+1}+\left(\frac{-1}{2^{n+1}}+\frac{(-1)^{n+1}}{2^{n+1}}+(-1)^{n+1}-1\right)$$

$$\left.\left.\times\left(x-\frac{a+b}{2}\right)^{n+1}\right\}^2\right]^{\frac{1}{2}}.$$

Hence theorem is completed. □

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Corollary 7. *Substitution of $x = a$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left\{(-1)^k f^{(k)}(a) + \left(1 + \frac{1}{2^{k+1}} + \frac{(-1)^k}{4^{k+1}}\right) f^{(k)}(b)\right\}\right. \tag{5.3}$$

$$\left. + \frac{(-1)^n}{b-a} \int_a^b f(t)dt - \frac{(b-a)^{n-1}}{2^{n+1}(n+1)!} \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right)(1 + (-1)^n)\right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a} \left[\frac{-1}{(n!)^2 (2n+1)} \frac{(b-a)^{2n+1}}{4^{3n+1}}\right.$$

$$\left. - \frac{1}{b-a} \frac{1}{(n+1)!} \left\{\frac{(b-a)^{n+1}}{2^{n+1}} (1 + (-1)^n)\right\}^2\right]^{\frac{1}{2}}.$$

Corollary 8. *Substitution of $x = \frac{a+b}{2}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left\{f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1 + (-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right)\right.\right. \tag{5.4}$$

$$\left. + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right\} + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$\left. - \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \frac{2(b-a)^{n-1}}{4^{n+1}(n+1)!} (1 + (-1)^n)\right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a} \left[\frac{(b-a)^{2n+1}}{2^{2n}(n!)^2(2n+1)} \left(1 + \frac{1}{2^{2n+1}}\right) - \frac{(b-a)^{2n+1}(1 + (-1)^n)^2}{4^{2n+1}(n+1)!}\right]^{\frac{1}{2}}.$$

Corollary 9. *Substitution of $x = \frac{3a+b}{4}$ in (5.1) gives*

$$\left|\sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(k+1)!} \frac{(b-a)^{k+1}}{4^{k+1}} \left[\frac{\left(1 + (-1)^k\right)}{2^{k+1}} f^{(k)}\left(\frac{7a+b}{8}\right)\right.\right. \tag{5.5}$$

$$+ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right)$$

$$\left. + \frac{1}{2^{k+1}} \left(1 + (-1)^k\right) f^{(k)}\left(\frac{a+7b}{8}\right)\right] + \frac{(-1)^n}{b-a} \int_a^b f(t)dt$$

$$- \left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \cdot \frac{1}{(n+1)!} \frac{(b-a)^{n-1}}{4^{n+1}}$$

$$\left. \times \left(1 + (-1)^n + \frac{1}{2^n} + \frac{(-1)^n}{2^n}\right)\right|$$

$$\leq \frac{\sqrt{\sigma\left(f^{(n)}\right)}}{b-a}\left[\frac{1}{(n!)^2(2n+1)}\frac{2(b-a)^{2n+1}}{4^{2n+1}}\left(\frac{1}{2^{2n}}+1\right)\right.$$
$$\left.-\frac{1}{b-a}\frac{1}{(n+1)!}\left\{\frac{(b-a)^{n+1}}{4^{n+1}}\left(1+(-1)^n\right)\left(1+\frac{1}{2^n}\right)\right\}^2\right]^{\frac{1}{2}}.$$

Remark 2. *By choosing $n=1$ in case A, B and C, we get all results obtained in* [10].

Remark 3. *By choosing $n=2$ in case A, B and C, we get all results obtained in* [9].

6. Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving higher order derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 5.

$$Q_{n,1}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}}{2^{k+1}(k+1)!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\frac{(b-a)^n}{2^{n+1}(n+1)!}\left(1+(-1)^n\right),$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(b-a)^{k+2}(-1)^k}{4^{k+1}(k+1)!}\left[f^{(k)}\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+\left\{1+(-1)^k\right\}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^k f^{(k)}\left(\frac{a+3b}{4}\right)\right]$$
$$+\left(f^{(n-1)}(b)-f^{(n-1)}(a)\right)\times\frac{2}{4^{n+1}}\frac{(b-a)^n}{(n+1)!}\left((-1)^n+1\right),$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$
$$\approx \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\frac{(b-a)^{k+2}}{4^{k+1}}\left[\frac{1}{2^{k+1}}\left(1+(-1)^k\right)\left(f^{(k)}\left(\frac{7a+b}{8}\right)+f^{(k)}\left(\frac{a+7b}{8}\right)\right)\right.$$
$$\left.+\left\{(-1)^k f^{(k)}\left(\frac{3a+b}{4}\right)+f^{(k)}\left(\frac{a+3b}{4}\right)\right\}\right]$$
$$+\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]\times\frac{(b-a)^n}{4^{n+1}(n+1)!}\left((-1)^n+1\right)\left(\frac{1}{2^n}+1\right).$$

Performance of the efficient quadrature rules

	Method	$n:\quad Q_{n,1}(f)$	$n:\quad Q_{n,2}(f)$	$n:\quad Q_{n,3}(f)$	Exact Value
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.30117	**4:** 0.301172	**4:** 0.30117	0.301169
	Error:	1.1381×10^{-6}	3.08726×10^{-6}	1.38925×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909328	**4:** 0.909324	**4:** 0.909327	0.909331
	Error:	2.33999×10^{-6}	7.13925×10^{-6}	3.21638×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**5:** 0.793022	**4:** 0.793031	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	2.9641×10^{-7}	1.33626×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**11:** 1.46266	**7:** 1.46265	**6:** 1.46266	1.46265
	Error:	5.8789×10^{-6}	2.29707×10^{-6}	5.20247×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**11:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	2.13363×10^{-6}	1.73918×10^{-6}	
7.	$\int_0^1 f_7(x)dx$	**6:** 1.34146	**4:** 1.34137	**4:** 1.34147	1.34147
	Error:	1.4808×10^{-7}	5.42574×10^{-7}	2.44601×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**9:** 0.62977	**5:** 0.629762	**4:** 0.629774	0.629769
	Error:	1.18074×10^{-6}	6.3567×10^{-6}	5.647×10^{-6}	

Table:

$$\begin{aligned} &f_1(x) = x^2 + x + 2, && f_2(x) = x\sin x, \\ &f_3(x) = e^x \sin x\ , && f_4(x) = x^2 + \sin x, \\ &f_5(x) = e^{x^2}, && f_6(x) = e^x \cos\left(e^x - 2x\right), \\ &f_7(x) = x + \cos x, && f_8(x) = \log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]. \end{aligned} \tag{6.1}$$

From the above table, we observe that all three quadrature rules show exact value of the integral of f_1 for $n = 2$. For any polynomial of degree k, $n = k + 1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

The integral of f_5 , $Q_{n,3}(f)$ report an error of the order of 10^{-6} for $n = 6$ while the other two quadrature rules give a similar error for $n = 7$ and $n = 11$. Similarly for all other functions $Q_{n,3}(f)$ report errors of the order of 10^{-6} or 10^{-7} for relatively smaller values of n as compared to the other two quadrature rules. Specifically, $Q_{n,3}(f)$ give an excellent estimate for the integrals of f_5 and f_8 at $n = 6$ and $n = 4$ respectively. In general $Q_{n,3}(f)$ gave better results as compared to the rest of the quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,3}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 10.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,3}(f)$ took less than a second.

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Based on this analysis, we can conjecture that $Q_{n,3}(f)$ is the most efficient quadrature rule, while $Q_{n,2}(f)$ comes second in terms of performance. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

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Duality in multiobjective nonlinear programming under generalized second order $(F, b, \phi, \rho, \theta)-$ univex functions

Falleh R. Al-Solamy, Meraj Ali Khan

Abstract

In the present paper, second order duality for multiobjective nonlinear programming are investigated under the second order generalized $(F, b, \phi, \rho, \theta)-$ univex functions. The weak, strong and converse duality theorems are proved. Further, we also illustrated an example of $(F, b, \phi, \rho, \theta)-$ univex functions. Results obtained in this paper extend some previously known results of multiobjective nonlinear programming in the literature.

Keywords: Duality, Multiobjective programming, Univex functions

Mathematics Subject Classification (2000): 90C32, 49K35, 49N15

1 Introduction

In recent years, the concept of convexity and generalized convexity is well known in optimization theory and plays a central role in mathematical economics, management science and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. In particular, the concept of generalized $(F, \rho)-$ convexity introduced by Preda [8]. In [9, 13], the concept of $V - \rho$-invexity and $(F, \alpha, \rho, d)-$ convexity were introduced respectively. Zhang and Mond [12] extended the class of $(F, \rho)-$ convex functions to second oder $(F, \rho)-$ convex functions and obtained the duality results for Mangasarian type, Mond-Weir type and general Mond-Weir type multiobjective dual problems. Motivated by Liang et al. [13] and Aghezzaf [2], I. Ahmad and Z. Husain [5] introduced second order $(F, \alpha, \rho, d)-$ convex functions and their generalization and they investigate weak, strong and strict converse duality theorems for second order Mond Weir type Multiobjective dual. Bector et al. [15] generalized the notion of convex function to univex functions. Rueda et al. [16] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions. Mishra [8] obtained optimality results and saddle point results for multiobjective programs under generalized type I univex functions. Recently, Zalmai [14] introduced the notion of second order $(F, b, \phi, \rho, \theta)-$ univex functions and he called these functions $(F, b, \phi, \rho, \theta)-$sounivex functions, these function generalize the second order $(F, \alpha, \rho, d)-$convex functions defined by Ahmad and Husain [5].

The concept of second order duality in nonlinear programming problems was first introduced by Mangasarian [11]. One significant practical application of second order dual over first order is that it may provide tighter bounds for value of objective function because there ae more parameters involved, several researchers [1, 4, 7, 21] considered second order dual models for multiobjective

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programming. In this paper , we formulate second order dual model and investigate weak, strong and strict converse duality theorems under $(F, b, \phi, \rho, \theta)-$ sounivexity assumptions. Further, an example have been constructed, which shows the existence of $(F, b, \phi, \rho, \theta)-$ sounivex functions.

2 Notations and Preliminaries

We consider the following multiobjective nonlinear programming problem:

$$(\mathbf{P}) \qquad \text{Minimize} \qquad f(x),$$

$$\text{subject to} \qquad g(x) \leqq 0, \quad x \in X, \tag{1}$$

where $f = (f_1, f_2, \ldots, f_k) : X \to R^k$, $g = (g_1, g_2, \ldots g_m) : X \to R^m$ are assumed to be twice differentiable function over X, an open subset of R^n.

Definition 2.1. A function $\mathcal{F} : X \times X \times R^n \to R$, where $X \subseteq R^n$ is said to be sublinear in its third argument, if $\forall \ x, \bar{x} \in X$,

(i) $\mathcal{F}(x, \bar{x}; a_1 + a_2) \leq \mathcal{F}(x, \bar{x}; a_1) + \mathcal{F}(x, \bar{x}; a_2), \ \forall \ a_1, a_2 \in R^n$,

(ii) $\mathcal{F}(x, \bar{x}; \alpha a) = \alpha \mathcal{F}(x, \bar{x}; a), \ \forall \ \alpha \in R_+, a \in R^n$.

Definition 2.2. A point $\bar{x} \in S$ is said to efficient solution of (P), if there exists no other feasible point x such that $f(x) \leq f(\bar{x})$ for each $x, \bar{x} \in X$.

Let $u \in R^n$ and assume that the function $f : X \to R$ is twice differentiable at u.

Definition 2.3. [14] The function f is said to be (strictly) $(F, b, \phi, \rho, \theta)-$ sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p)(>) \geqq F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p])$$

$$+\rho(x, u) \|\theta(x, u)\|^2,$$

where $\|.\|^2$ is a norm on R^n.

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$sounivex at u. Now we define generalized $(F, b, \phi, \rho, \theta)-$ sounivex functions

Definition 2.4. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X (x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2} p^t \nabla^2 f(u) p) < 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u) p]) < -\rho(x, u) \|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.5. A twice differentiable function f, over X is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \geqq -\rho(x, u)\|\theta(x, u)\|^2$$

$$\Rightarrow \quad \phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) > 0,$$

or equivalently

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) < -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be strictly $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components is strictly f_i is $(F, b, \phi, \rho, \theta)-$pseudo sounivex at u.

Definition 2.6. A twice differentiable function f, over X is said to be $(F, b, \phi, \rho, \theta)-$ quasi sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leqq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leqq -\rho(x, u)\|\theta(x, u)\|^2.$$

A twice differentiable vector function $f : X \to R^k$ is said to be $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u.

Definition 2.7. A twice differentiable function f, over X is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u if there exist functions $b : X \times X \to (0, \infty)$, $\phi : R \to R$, $\rho : X \times X \to R$, $\theta : X \times X \to R^n$, and a sublinear function $F(x, u, .) : R^n \to R$ such that for each $x \in X(x \neq u)$ and $p \in R^n$,

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0$$

$$\Rightarrow \quad F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) \leq -\rho(x, u)\|\theta(x, u)\|^2$$

A twice differentiable vector function $f : X \to R^k$ is said to be strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u, if each of its components f_i is strong $(F, b, \phi, \rho, \theta)-$ pseudo sounivex at u.

Every $(F, b, \phi, \rho, \theta)-$ sounivex function need not to be second order $(F, \alpha, \rho, d)-$ convex, definded in [5]. To show this, consider the following example.

Example 2.1. Let $f : X = (0, \infty) \to R$ be defined as $f(x) = -x^2 - x$. Let $\phi(t) = -t$, $b(x, u) = x - u$, $\rho = -10$, $\theta(x, u) = \frac{u+2}{2}$ and sublinear function is defined as $F(x, u, a) = a(x - u) + x$

$$F(x, u; b(x, u)[\nabla f(u) + \nabla^2 f(u)p]) = -(x^2 - u^2)(2u + 1 + 2p) + x - 10\|\frac{u+2}{2}\|^2,$$

at $u = 0$,

$$F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) = -x^2(1 + 2p) + x - 10$$

and

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p = -x^2 - x + u^2 + u - p^2,$$

at $u = 0$

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) = x^2 + x + p^2,$$

and it is easy to see that

$$\phi(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p])$$

$$= x^2 + p^2 + x^2(1 + 2p) + 10 \geq 0$$

for all $x \in R$ and $-1 \leq p < \infty$, so the function is $(F, b, \rho, \phi, \theta)-$ sounivex at $x = 0$, but at $p = -1, x = 10$

$$(f(x) - f(0) + \frac{1}{2}p^t\nabla^2 f(0)p) - F(x, 0; b(x, 0)[\nabla f(0) + \nabla^2 f(0)p]) < 0$$

Hence, the function is not $(F, \alpha, \rho, d)-$convex at $x = 0$.

Now we have following Kuhn-Tucker type necessary conditions, which will be useful to prove the strong duality theorem.

Theorem 2.1. (*Kuhn-Tucker type necessary conditions*) Assume that x^* is an efficient solution for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda^* \in R^k$ and $y^* \in R^m$, such that

$$\lambda^{*t}\nabla f(x^*) + y^{*t}\nabla g(x^*) = 0,$$

$$y^{*t}\nabla g(x^*) = 0,$$

$$y^* \geqq 0,$$

$$\lambda^* \geq 0.$$

3 Second order Mond-Weir type duality

In this section, we consider the following Mond-Weir second order dual associated with multiobjective problem (P) and establish weak, strong and strict converse duality theorems under generalized $(F, b, \rho, \phi, \theta)-$ sounivexity

(**MD**) Maximize

$$f(u) - \frac{1}{2}p^t\nabla^2 f(u)p$$

Subject to

$$\nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p + \nabla\lambda^t g(u) + \nabla^2\lambda^t g(u)p = 0, \tag{2}$$

$$y^t g(u) - \frac{1}{2}p^t\nabla^2 y^t g(u)p \geqq 0, \tag{3}$$

$$y \geqq 0, \tag{4}$$

$$\lambda \geq 0, \tag{5}$$

where λ is a $k-$dimensional vector, and y is an $m-$dimensional vector.

Theorem 2 (*weak duality*) Suppose that for all feasible solutions x in (P) and all feasible solutions (u, y, λ, p) in MD

(i) $y^t g(0)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at u,

(ii) $\lambda > 0$, and $f(.)$ is strong $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at u with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leq 0 \Rightarrow \phi(u) \leq 0$ and $v \leqq 0 \Rightarrow \phi(v) \leqq 0$, for all $u, v \in R^n$.

Then the following can not hold

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p. \tag{6}$$

Proof. Now suppose contrary to the result that (6) holds, i.e.,

$$f(x) \leq f(u) - \frac{1}{2}p^t\nabla^2 f(u)p,$$

or

$$f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p \leq 0,$$

then by assumption (iii)

$$\phi(f(x) - f(u) + \frac{1}{2}p^t\nabla^2 f(u)p) \leq 0, \tag{7}$$

which by virtue of assumption (ii) leads

$$F(x, u, b_1(x, u)\{\nabla f(u) + \nabla^2 f(u)p\}) \leq -\rho_1\|\theta(x, u)\|^2. \tag{8}$$

On multiplying (8) by $\lambda > 0$ and using sublinearity of F with $b_1(x, u) > 0$, we have

$$F(x, u, \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p) < -b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{9}$$

The first dual constraint and sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)p) \geqq -F(x, u; \nabla\lambda^t f(u) + \nabla^2\lambda^t f(u)p).$$

Applying (9) in above inequality, we have

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u)) > b_1^{-1}(x, u)\rho_1\lambda\|\theta(x, u)\|^2. \tag{10}$$

Let x be any feasible solution in (P) and (u, y, λ, p) be any feasible solution in (MD). Then we have

$$y^t g(x) \leqq 0 \leqq y^t g(u) - \frac{1}{2} p^t \nabla^2 y^t g(u) p), \tag{11}$$

by assumption (iii), (11) yields

$$\phi(y^t g(x) - y^t g(u) + \frac{1}{2} p^t \nabla^2 y^t g(u) p) \leqq 0. \tag{12}$$

Using $(F, b, \phi, \rho, \theta)$-quasi sounivexity of $y^t g(.)$, we have

$$F(x, u; b(x, u)\{\nabla y^t g(u) + \nabla^2 y^t g(u) p)\}) \leqq -\rho \|\theta(x, u)\|^2. \tag{13}$$

Since $b(x, u) > 0$, the above inequality with the sublinearity of F give

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq -b^{-1} \rho \|\theta(x, u)\|^2. \tag{14}$$

Now using the assumption $b^{-1}\rho + b_1^{-1}\rho_1 \lambda \geqq 0$, the above inequality yields

$$F(x, u; \nabla y^t g(u) + \nabla^2 y^t g(u) p) \leqq b_1^{-1} \rho_1 \lambda \|\theta(x, u)\|^2. \tag{15}$$

Which contradict (10), hence (6) can not hold.

Theorem 3 (*Strong duality*). Let $\bar{x}$ be an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is a feasible for (MD) and the corresponding values of (P) and (MD) are equal. If in addition, the assumptions of weak duality (Theorem 2) hold for all feasible solutions of (P) and (MD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of (MD).

Proof. Since $\bar{x}$ is an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 1, there exist $\bar{y} \in R^m$ and $\bar{\lambda} \in R^k$, such that

$$\bar{\lambda}^t \nabla f(\bar{x}) + \bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y}^t \nabla g(\bar{x}) = 0,$$

$$\bar{y} \geqq 0,$$

$$\bar{\lambda} \geq 0.$$

Therefore $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible for (MD) and the corresponding values of (P) and (MD) are equal. The efficiency of this feasible solution for (MD) thus follows from weak duality (Theorem 2).

Theorem 4 (*Strict converse duality*) Let $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, \bar{p})$ be the efficient solution of (P) and (MD), respectively such that

$$\bar{\lambda}^t f(\bar{x}) = \bar{\lambda}^t f(\bar{u}) - \frac{1}{2} \bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u}) \bar{p}. \tag{16}$$

Suppose

(i) $y^t g(.)$ is $(F, b, \phi, \rho, \theta)-$quasi sounivex at $\bar{u}$,

(ii) $\bar{\lambda}^t f(.)$ be $(F, b_1, \phi, \rho_1, \theta)-$ pseudo sounivex at $\bar{u}$ with $b^{-1}\rho + b_1^{-1}\rho_1\lambda \geqq 0$,

(iii) $u \leqq 0 \Rightarrow \phi(u) \leqq 0$ and $v < 0 \Rightarrow \phi(v) < 0$, for all $u, v \in R^n$.

Then $\bar{x} = \bar{u}$, that is $\bar{u}$ is an efficient solution.

Proof. We assume that $\bar{x} \neq \bar{u}$ and reach a contradiction, since $\bar{x}$ and $(\bar{u}, \bar{y}, \bar{\lambda}, p)$ are respectively the feasible solution of (P) and (MD), we have

$$\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p} \leqq 0. \tag{17}$$

Using the assumption (iii), we have

$$\phi(\bar{y}^t g(\bar{x}) - \bar{y}^t g(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq 0. \tag{18}$$

By $(F, b, \phi, \rho, \theta)-$quasi sounivexity of $\bar{y}^t g(.)$ at $\bar{u}$, we get

$$F(\bar{x}, \bar{u}; b(\bar{x}, \bar{u})\{\nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p})\}) \leqq -\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{19}$$

Since $b(\bar{x}, \bar{u}) > 0$, the inequality (19) along with the sublinearity of F, imply

$$F(\bar{x}, \bar{u}; \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}) \leqq -b^{-1}(x, u)\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{20}$$

The first dual constraint and sublinearity of F imply

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -F(\bar{x}, \bar{u}, \nabla \bar{y}^t g(\bar{u}) + \nabla^2 \bar{y}^t g(\bar{u})\bar{p}).$$

Applying (20) and $b^{-1}\rho + b_1^{-1}\rho \geqq 0$ in above inequality, we get

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) \geqq -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{21}$$

Suppose (16) does not holds, then we have

$$\bar{\lambda}^t f(\bar{x}) < \bar{\lambda}^t f(\bar{u}) - \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p},$$

now using assumption (iii)

$$\phi(\bar{\lambda}^t f(\bar{x}) - \bar{\lambda}^t f(\bar{u}) + \frac{1}{2}\bar{p}^t \nabla^2 \bar{\lambda}^t f(\bar{u})\bar{p}) < 0.$$

Now by the assumption (ii), the above inequality gives

$$F(\bar{x}, \bar{u}; b_1(\bar{x}, \bar{u})(\nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p})) < -\rho\|\theta(\bar{x}, \bar{u})\|^2,$$

or

$$F(\bar{x}, \bar{u}; \nabla \bar{\lambda}^t f(\bar{u}) + \nabla^2 \bar{y}^t f(\bar{u})\bar{p}) < -b_1^{-1}(\bar{x}, \bar{u})\rho\|\theta(\bar{x}, \bar{u})\|^2. \tag{22}$$

Which contradict (21). Hence result.

4 Conclusion

In this paper anew concept of generalized invex functions is introduced. Under this generalized invexity we establish weak, strong and converse duality theorems. These duality relations lead to duality in nonlinear fractional programming problems.

5 Authors contributions

Both the authors contributed equally to writing of this paper and the final manuscript is read and approved by the authors.

6 Competing interests

The author declare that they have no competing interests.

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STABILITY OF FRACTIONAL DIFFERENTIAL EQUATION WITH BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we prove the Hyers-Ulam stability of a fractional differential equation of order $\alpha \in (1, 2)$ with certain boundary conditions.

1. INTRODUCTION

The recent concentric area in the research world of mathematics is fractional differential equations. The concept of fractional derivative is not new and is very much as old as classical differential equations. In recent years, many authors disscussed and proved the existence results of fractional differential equations using various methods. For example, one can refer the monographs of Kilbas et al. [10], Miller and Ross [14], Podulbny [20], Diethelm et al. [4, 5], Benchora [2] and so on. Obviously, the differential equations of fractional order has been proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find many applications in electromagnetic, control, electrochemistry, etc. (see [6, 7]).

At the same instance, the stability concept is more devoloped in the research world of mathematics, particularly in functional equations. But the analysis of stability concepts of fractional differential equations has been very slow and there are only countable number of works. In 2009, Li [12], first proposed the Mittag-Leffler stability and in 2010 [13], the fractional Lyapunovs second method. In the next year, Li and Zhang [11] have been given a brief overview on the stability of the fractional differential equations. However, there are only few works available on the local stability and Mittag-Leffler stability for fractional differential equations and very rare works on the Ulam stability of fractional differential equations.

In 2011, Wang [24] carried out a pioneering work on the Hyers-Ulam stability and data dependence for fractional differential equations with Caputo derivative. Wang [25] proved the Hyers-Ulam stability of fractional differential equation of order $0 < \alpha < 1$ via a generalized fixed point approach, by adopting some part idea of Wang et al. [24], Cadariu and Radu [3] and Jung [9] in the next year. Particularly, there are very rare works on the Hyers-Ulam stability of fractional differential equations with boundary conditions. Recently, Rabha [8], Muniyappan and Rajan [16] had given Ulam stabilities with boundary conditions in the interval $(0, 1)$. For more information on functional equations and their stability problems, see [15, 17, 18, 19, 21, 22, 23].

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In this paper, the Hyers-Ulam stability of the following fractional boundary value problem is proved.

$$^{C}D^{\alpha}y(t) = F(t, y(t)), \quad 1 < \alpha < 2 \tag{1.1}$$

$$y(0) = y_0 \quad y(T) = y_T \tag{1.2}$$

This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Hyers-Ulam stability of the above fractional boundary value problem is proved.

2. Preliminaries

Throughout this paper, we assume that Y is a normed space and $I = [0, T]$ is a given interval.

Definition 2.1. ([2]) The fractional order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the gamma function.

Definition 2.2. ([2]) For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h, is defined by

$$({}^{c}D_{a+}^{\alpha} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. ([1]) A function $y \in C^2(I, \mathbb{R})$ is said to be a solution of (1.1)-(1.2) if y satisfies the equation ${}^{c}D^{\alpha}y(t) = F(t, y(t))$ on I, and the condition $y(0) = y_0$ and $y(T) = y_T$

Lemma 2.4. ([1]) *Let $1 < \alpha < 2$ and let $F : I \to \mathbb{R}$ be continuous. A function $y \in C^2(I, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

if and only if y is a solution of the fractional boundary value problem

$${}^{c}D^{\alpha}y(t) = F(t, y(t)), t \in [0, T]$$

$$y(0) = y_0 \quad y(T) = y_T$$

Definition 2.5. ([25]) The fractional differential equation (1.1) is Hyers-Ulam stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|{}^{c}D^{\alpha}y(t) - F(t, y(t))\| \leq \epsilon$$

for all $t \in I$ and for some $\epsilon > 0$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation 1.1 such that

$$\|f(t) - f_0(t)\| \leq K\epsilon$$

for all $t \in I$.

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Definition 2.6. ([25]) The fractional differential equation (1.1) is Hyers-Ulam-Rassias stable if there exists a continuously differentiable function $f : I \to Y$ satisfying the inequality

$$\|{}^{c}D^{\alpha}y(t) - F(t, y(t))\| \leq \varphi(t)$$

for all $t \in I$, there exists a solution $f_0 : I \to Y$ of the fractional differential equation (1.1) such that

$$\|f(t) - f_0(t)\| \leq \Phi(t)$$

for all $t \in I$. where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly.

Definition 2.7. ([25]) For a nonempty set X, a function $d : X \times X \to [0, \infty]$ is called generalized metric on X if and only if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Such a space is called a generalized complete metric space.

Theorem 2.8. ([3]) *Let (X,d) be a generalised complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$, If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k) < \infty$ for some $x \in X$, then the following are true:*

1. *The sequence $\{\Lambda^n x\}$ converges to a fixed end point x^* of Λ*
2. *x^* is the unique fixed point of Λ in $X^* = \left\{y \in X | d(\Lambda^k x, y) < \infty\right\}$;*
3. *If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$*

3. Hyers-Ulam stability

In this section, we first investigate the Hyers-Ulam stability of the fractional differential equation (1.1) with boundary condition (1.2) via Theorem 2.8.

Theorem 3.1. *Let $I = [0, T]$ be a closed interval. Let $K, P,$ and L be positive constants with $0 < KPL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the standard Lipschitz condition*

$$|F(t, y) - F(t, z)| \leq L\,|y - z| \tag{3.1}$$

for all $t \in I$ and $y, z \in \mathbb{R}$. If a continuously differential function $y : I \to \mathbb{R}$ satisfies

$$|{}^{c}D^{\alpha}y(t) - F(t, y(t))| \leq \varphi(t) \tag{3.2}$$

for all $t \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left|\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \varphi(\tau) d\tau\right| \leq K\varphi(t) \tag{3.3}$$

for all $t \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(s, y(s)) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} F(s, y(s)) ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.4}$$

and

$$|y(t) - y_0(t)| \leq \frac{K}{1 - KPL} \varphi(t) \tag{3.5}$$

for all $t \in I$.

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}. \tag{3.6}$$

Similar to [9, Theorem 3.1], one can introduce a generalised complete metric on X as follows

$$d(f,g) = \inf\{C \in [0,\infty] |\, |f(t) - g(t)| \le C\varphi(t) \quad \text{for all} \quad t \in I\}. \tag{3.7}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,y(s))ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s,y(s))ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \tag{3.8}$$

for all $f \in X$.
It is easy to see that Λ is well defined, since F and f are continuous functions.
To acheive our aim, we need to prove that Λ is strictly contractive on X.
For any $f, g \in X$, let $C_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, that is, by (3.7), we have

$$|f(t) - g(t)| \le C_{fg}\varphi(t) \tag{3.9}$$

for all $t \in I$. It then follows from (3.1), (3.3), (3.7), (3.8) and (3.9) that

$$\begin{aligned}
|(\Lambda f)t - (\Lambda g)t| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s,f(s)) - F(s,g(s))|\, ds \\
&\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s,f(s)) - F(s,g(s))|\, ds \\
&\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)|\, ds + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)|\, ds \\
&\le \frac{L}{\Gamma(\alpha)} C_{fg} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds + \frac{tL}{T\Gamma(\alpha)} C_{fg} \int_0^T (T-s)^{\alpha-1} \varphi(s) ds \\
&\le KPLC_{fg}\varphi(t)
\end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$. That is,

$$d(\Lambda f, \Lambda g) \le KPLC_{fg}.$$

Hence we can conclude that

$$d(\Lambda f, \Lambda g) \le KPLd(f,g)$$

for all $f, g \in X$, where we note that $0 < KPL < 1$.
It follows from (3.6) and (3.8) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$\begin{aligned}
|(\Lambda g_0)(t) - g_0(t)| &= \Big|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,y(s))ds \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s,y(s))ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T - g_0(t)\Big| \\
&\le C\varphi(t)
\end{aligned}$$

for all $t \in I$, since $f(t, g_0(t))$ and $g_0(t)$ are bounded on I and $min_{t\in I}\varphi(t) > 0$.
Thus (3.7) implies that

$$d(\Lambda g_0, g_0) < \infty.$$

Therefore, according to Theorem 2.8, there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X, d) and $\Lambda y_0 = y_0$, that is, y_0 satisfies (3.4) for every $t \in I$.
we will now verify that $\{g \in X/d(g_0, g) < \infty\} = X$.
For any $g \in X$, since g and g_0 are bounded on I and $min_{t\in I}\varphi(t) > 0$, there exists a constant $0 < C_g < \infty$ such that $|g_0(t) - g(t)| \le C_g\varphi(t)$
Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is $\{g \in X/d(g_0, g) < \infty\} = X$.
Hence in view of Theorem 2.8, we conclude that y_0 is the unique continuous function with the property (3.4). On the other hand, it follows from (3.2) that

$$-\varphi(t) \le^c D^\alpha_{a+}y(t) - F(t, y(t)) \le \varphi(t)$$

for all $t \in I$.
If we integrate each term in the above inequality and substitute the boundary conditions, then we obtain

$$|y(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s, y(s))ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}F(s, y(s))ds - \left(\frac{t}{T} - 1\right)y_0 + \frac{t}{T}y_T|$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\varphi(t)ds$$

for all $t \in I$.
Thus by (3.3) and (3.8) we get

$$|y(t) - (\Lambda y)(t)| \le K\varphi(t)$$

for each $t \in I$, which implies that

$$d(y, \Lambda y) \le K. \tag{3.10}$$

Finally, Theorem 2.8 and (3.10) imply that

$$d(y, y_0) \le \frac{1}{1 - KPL}d(y, \Lambda y) \le \frac{K}{1 - KPL}.$$

□

Now, we will prove the Hyers-Ulam stability of the (1.1) with boundary condition (1.2)

Theorem 3.2. *Let $I = [0, T]$ be a closed interval. Let $r > 0$ be a positive constant with $0 \le t, T \le r$ and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies a Lipschitz condition (3.1) for all $t \in I$ and $y, z \in \mathbb{R}$, where L is a constant with $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfying the differential inequality*

$$\left|{}^cD^\alpha_{a+}y(t) - F(t, y(t))\right| \le \epsilon \tag{3.11}$$

for all $t \in I$ and for some $\epsilon \ge 0$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ satisfying (3.4) and

$$|y(t) - y_0(t)| \le \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon \tag{3.12}$$

for all $t \in I$.

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Proof. First, we define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} | f \text{ is continuous}\}$$

and introduce a generalized complete metric on X as follows

$$d(f,g) = \inf\{C \in [0,\infty] | \, |f(t) - g(t)| \le C \quad \text{for all} \quad t \in I\}$$

Define an operator $\Lambda : X \to X$ by

$$(\Lambda f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s,y(s))ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} F(s,y(s))ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T$$

for all $f \in X$.

We now assert that Λ is strictly contractive on X.

For all $f, g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, that is, let us assume that

$$|f(t) - g(t)| \le C_{fg} \tag{3.13}$$

for any $t \in I$. Moreover, it follows from (3.1), (3.8) and (3.13) that

$$\begin{aligned} |(\Lambda f)t - (\Lambda g)t| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s,f(s)) - F(s,g(s))| \, ds \\ &\quad + \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |F(s,f(s)) - F(s,g(s))| \, ds \\ &\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s) - g(s)| \, ds \\ &\quad + \frac{tL}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s) - g(s)| \, ds \\ &\le LC_{fg} \left[\frac{r^\alpha}{\alpha\Gamma(\alpha)} + \frac{tr^\alpha}{T\alpha\Gamma(\alpha)}\right] \\ &\le \frac{LC_{fg} r^\alpha}{\Gamma(\alpha+1)} \left[\frac{t+T}{T}\right] \\ &\le \frac{LPC_{fg} r^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

for all $t \in I$, where $P = \left(1 + \frac{t}{T}\right)$, that is

$$d(\Lambda f, \Lambda g) \le \frac{LPr^\alpha}{\Gamma(\alpha+1)} C_{fg}.$$

Thus it follows that

$$d(\Lambda f, \Lambda g) \le \frac{LPr^\alpha}{\Gamma(\alpha+1)} d(f,g)$$

for all $f, g \in X$, and we note that $0 < \frac{LPr^\alpha}{\Gamma(\alpha+1)} < 1$.

Analogously to the proof of Theorem 3.1, we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0) < \infty$.

Therefore, Theorem 2.8 implies that there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $\Lambda^n g_0 \to y_0$ in (X,d) as $n \to \infty$, and such that $y_0 = \Lambda y_0$, that is, y_0 satisfies the equation (3.4) for all $t \in I$.

If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with $|g_0(t) - g(t)| \leq C$ for all $t \in I$. This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently, $\{g \in X | d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 2.8, y_0 is a unique continuous function with property (3.4). Furtheremore, it follows from (3.11) that

$$-\epsilon \leq^c D^\alpha_{a+} y(t) - F(t, y(t)) \leq \epsilon$$

for all $t \in I$. If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$|(\Lambda y)(t) - y(t)| \leq \frac{r^\alpha}{\Gamma(\alpha+1)}\epsilon$$

for all $t \in I$, that is, it holds that $d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha+1)}\epsilon$.
It now follows from Theorem 2.8 that

$$d(y, y_0) \leq \frac{1}{1 - \frac{LPr^\alpha}{\Gamma(\alpha+1)}} d(\Lambda y, y) \leq \frac{r^\alpha}{\Gamma(\alpha+1) - LPr^\alpha}\epsilon, \tag{3.14}$$

which implies the validity of (3.12) for each $t \in I$ □

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Bernstein-Stancu type operators which preserve polynomials

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Abstract

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. This idea motivated us to modify the sequence of linear Bernstein Stancu type operators. Using numerical examples we show that these operators present a better degree of approximation than the original ones. In this note the modified Bernstein Stancu operators are studied in regard to uniform convergence and global smoothness preservation.

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1 Introduction

In 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem [3] were introduced the classical Bernstein operators $B_n : C[0,1] \to C[0,1]$, defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad \text{where } p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Lemma 1.1. *The Bernstein operators verify the following identities*

(i) $B_n(e_0;x) = 1$,

(ii) $B_n(e_1;x) = x$,

(iii) $B_n(e_2;x) = \frac{x}{n}(1+xn-x)$, *where* $e_i(t) = t^i$, $i = 0, 1, \ldots$.

In the last years there is an increasing interest in modifying linear operators so that the new versions reproduce some basic functions. King [12] consider for the first time this kind of modification for the Bernstein operators and proved that the modified operators reproduce the functions $e_i(x) = x^i$ for $i = 0, 2$ and approximate each continuous function on $[0,1]$ with an order of approximation at least as good as that of the classic Bernstein whenever $0 \le x < \frac{1}{3}$. Using the same type of technique introduced by King or new methods many authors published new results in regard with this subject. Cárdenas-Morales et al. [4] extended this result considering a family of sequences of operators $B_{n,\alpha}$ that preserve e_0 and $e_2+\alpha e_1$ with $\alpha \in [0,\infty)$. Gonska et al. [11] studied the sequence $V_n^\tau : C[0,1] \to C[0,1]$ defined by

$$V_n^\tau f := (B_n f) \circ (B_n \tau)^{-1} \circ \tau,$$

where τ is a continuous strictly increasing function defined on $[0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$. Note that if $\tau = \frac{e_2+\alpha e_1}{1+\alpha}$, then $V_n^\tau = B_{n,\alpha}$ and the operators V_n^τ preserve e_0 and τ. In [5], the authors inspired by the above ideas consider the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, τ being any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. Note that the Korovkin set $\{1, e_1, e_2\}$ is generalized to $\{1, \tau, \tau^2\}$ and these operators present a better degree of approximation than B_n.

Since the modified operators present a better degree of approximation than the original ones leads to an interesting area of research, so that generalized Bernstein-Durrmeyer operators and their approximation properties were studied in [1] and [6]. Also, the modified Szasz operators were considered recently in [2].

2 Bernstein-Stancu operators

In 1968, Stancu [15] proposed the sequence of positive linear operators $S_n^{<\alpha>} : C[0,1] \to C[0,1]$ depending on a non-negative parameter α given by

$$S_n^{<\alpha>}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{<\alpha>}(x),\ x \in [0,1], \tag{2.1}$$

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where

$$p_{n,k}^{<\alpha>}(x) = \binom{n}{k} \frac{x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}}{1^{[n,-\alpha]}}$$

and $t^{[n,h]} := t(t-h)\cdots(t-\overline{n-1}h)$ is the n^{th} factorial power of t with increment h.

For $\alpha = 0$ these operators reduce to the classical Bernstein operators.

The values of the test function by Bernstein-Stancu operators were given by Stancu [15] as follows

Lemma 2.1. *If $x \in [0,1]$, then*
(i) $S_n^{<\alpha>}(e_0; x) = 1$,
(ii)$S_n^{<\alpha>}(e_1; x) = x$,
(iii)$S_n^{<\alpha>}(e_2; x) = \frac{1}{1+\alpha}\left(\frac{x(1-x)}{n} + x(x+\alpha)\right)$.

Recently, in [13] Miclăuş proposed a new technique to obtain the values of the test function, without using properties of Bernstein operators.

It is well known the following form of Bernstein operators using the divided difference

$$B_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] x^k. \tag{2.2}$$

Starting with the form (2.2) of the Bernstein operators, the following Stancu type operators are constructed in [7]-[8]:

$$C_n : C[0,1] \to \Pi_n,$$

$$C_n(f;x) = \sum_{k=0}^{n} \frac{k!}{n^k}\binom{n}{k}\mathbf{m}_{k,n}\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \quad f \in C[0,1], \tag{2.3}$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators and Π_n is the linear space of all real polynomials of degree $\leq n$.

Let $\mathbf{m}_{0,n} = 1$, $\lim_{n\to\infty}\mathbf{m}_{1,n} = 1$ and $\mathbf{m}_{k,n} = \frac{(a_n)_k}{k!}$, $a_n \in (0,1]$. For this special case of real sequence $(m_{k,n})_{k=0}^{\infty}$ the Bernstein-Stancu operators C_n were written in the Bernstein basis as follows (see [7], Theorem 10):

$$C_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) C_{k,n}[f], \tag{2.4}$$

where

$$C_{k,n}[f] = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)(a_n)_j(1-a_n)_{k-j}.$$

We remark that $a_n \in (0,1]$ leads to C_n linear positive operators.

The coefficients $C_{k,n}[f]$ can be written as follows

$$C_{k,n}[f] = \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n) f\left(\frac{j}{n}\right).$$

Therefore,

$$C_{k,n}[f] = S_k^{<1>}(\tilde{f}; a_n), \quad \text{where } \tilde{f}(t) = f\left(t\frac{k}{n}\right).$$

Lemma 2.2. ([7]) *The Bernstein-Stancu operators C_n verify the following identities*

(i) $C_n(e_0; x) = 1$,

(ii) $C_n(e_1; x) = a_n x$,

(iii) $C_n(e_2; x) = x^2 + \frac{x(1-x)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2 + a_n)\right) x^2$.

Let

$$\begin{aligned} \mu_{n,m}(x) &= C_n\left((t-x)^m; x\right) \\ &= \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - x\right)^m, \quad n, m \in \mathbb{N}, \end{aligned}$$

be the central moment operators.

Lemma 2.3. ([7]) *The central moment operators verify*

(i) $\mu_{n,2}(x) = \frac{x(1-x)}{n} a_n + x^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,

(ii) $\mu_{n,4}(x) = x^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right]x^3 + \left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right]x^2$
$+\frac{a_n}{n^3}x + \frac{(a_n)_4}{n^4}\binom{n}{4} - \frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n$.

In [7], Cleciu obtained the following Voronovskaya type theorem:

Theorem 2.4. ([7]) *Suppose that $x_0 \in [0,1]$ and $f''(x_0)$ exists. If $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then*

$$\lim_{n\to\infty} n\left[f(x_0) - C_n(f; x_0)\right] = -\frac{x_0(1-x_0)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0)\right] L.$$

3 Modified Bernstein-Stancu operators

In this section we deal with Bernstein-Stancu type generalization of (2.4). We investigate its sharp preserving and convergence properties.

We define the modified Bernstein-Stancu operators as follows:

$$C_n^{\tau}(f; x) = \sum_{k=0}^{n} p_{n,k}^{\tau}(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(f \circ \tau^{-1}\right)\left(\frac{j}{n}\right), \tag{3.1}$$

where $p_{n,k}^{\tau}(x) = \binom{n}{k}\tau(x)^k\left(1 - \tau(x)\right)^{n-k}$ and τ is any function that is continuously differentiable ∞ times on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$.

Note that these operators are positive and linear and for the case $\tau(x) = x$, these operators (3.1) reduce to the Bernstein-Stancu operators defined by Cleciu [7]-[8].

Lemma 3.1. *The modified operators C_n^τ verify*
(i) $C_n^\tau e_0 = 1$,
(ii) $C_n^\tau \tau = a_n \tau$,
(iii) $C_n^\tau \tau^2 = \tau^2 + \frac{\tau(1-\tau)}{n} a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)\tau^2$.

Let

$$\begin{aligned}\mu_{n,m}^\tau(x) &= C_n^\tau\left((\tau(t)-\tau(x))^m; x\right)\\ &= \sum_{k=0}^{n} p_{n,k}^\tau(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left(\frac{j}{n} - \tau(x)\right)^m, \quad n, m \in \mathbb{N},\end{aligned}$$

be the central moment operators.

Lemma 3.2. *The central moment operators verify*
(i) $\mu_{n,0}^\tau(x) = 1$,
(ii) $\mu_{n,1}^\tau(x) = (a_n - 1)\tau(x)$,
(iii) $\mu_{n,2}^\tau(x) = \frac{\tau(x)(1-\tau(x))}{n} a_n + \tau(x)^2(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)$,
(iv) $\mu_{n,4}(x) = \tau(x)^4 + \left[\frac{6(a_n)_3}{n^4}\binom{n}{3} - \frac{12(a_n)_2}{n^3}\binom{n}{2} + \frac{6a_n}{n}\right]\tau(x)^3$
$+\left[\frac{7(a_n)_2}{n^4}\binom{n}{2} - \frac{4a_n}{n^2}\right]\tau(x)^2 + \frac{a_n}{n^3}\tau(x) + \frac{(a_n)_4}{n^4}\binom{n}{4}$
$-\frac{4(a_n)_3}{n^3}\binom{n}{3} + \frac{6(a_n)_2}{n^2}\binom{n}{2} - 4a_n$.

Lemma 3.3. *For all $n \in \mathbb{N}$ we have*

$$\mu_{n,2}^\tau(x) \le \delta_{n,\tau}^2(x) \quad \textit{for all } x \in [0,1],$$

where $\delta_{n,\tau}^2(x) := \frac{a_n}{n}\varphi_\tau^2(x) + (1-a_n)$ and $\varphi_\tau^2(x) := \tau(x)(1-\tau(x))$.

Proof. We have

$$\begin{aligned}|\mu_{n,2}^\tau(x)| &= \left|\frac{\tau(x)(1-\tau(x))a_n}{n}\right| + \left|\tau^2(x)(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right)\right|\\ &\le \varphi_\tau^2(x)\frac{a_n}{n} + (1-a_n) = \delta_{n,\tau}^2(x).\end{aligned}$$

□

Lemma 3.4. *If $f \in C[0,1]$, then $\|C_n^\tau f\| \le \|f\|$, where $\|\cdot\|$ is the uniform norm on $C[0,1]$.*

Proof. From the definition of the operator C_n^τ and using Lemma 3.1 it follows

$$\begin{aligned}|C_n^\tau(f;x)| &\le \sum_{k=0}^{n} p_{n,k}^\tau(x) \sum_{j=0}^{k} p_{k,j}^{<1>}(a_n)\left|\left(f\circ\tau^{-1}\right)\left(\frac{j}{n}\right)\right|\\ &\le \|f\circ\tau^{-1}\| C_n^\tau(e_0;x) = \|f\|.\end{aligned}$$

□

Theorem 3.5. *Let $f \in C[0,1]$, $a_n \in (0,1]$ and $\lim_{n\to\infty} a_n = 1$. Then $C_n^\tau f$ converges to f as n tends to infinity, uniformly on $[0,1]$.*

Proof. Using the well known Korovkin theorem and Lemma 3.1 and the fact that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$ it follows the uniform convergence of the operators C_n^τ. □

Let ω be the usual modulus of continuity of $f \in C[0,1]$ which is defined as

$$\omega(f;\delta) = \sup_{|h|\le\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Proposition 3.6. *Let $f \in C[0,1]$ with modulus of continuity $\omega(f,\cdot)$. Then*

$$|C_n^\tau(f;x) - f(x)| \le \left(1 + \frac{\mu_{n,2}^\tau(x)}{\delta^2}\right)\omega(f,\delta)$$

for $\delta > 0$ and $x \in [0,1]$.

Example 3.7. If we choose $\tau(x) = \frac{x^2+x}{2}$, we have $\tau(x)(1-\tau(x)) \le x(1-x)$ for all $x \in [0,1/2]$ and this inequality leads to $\mu_{n,2}^\tau(x) \le \mu_{n,2}(x)$. Therefore, the modified operators C_n^τ presents an order of approximation better than C_n in that interval.

Example 3.8. Now using a graphical example we try to illustrate these approximation processes. Let $f(x) = \sin(9x)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = 1/2$. For $n = 20$, the approximation to the function f by C_n and C_n^τ is shown in the Figure 1.

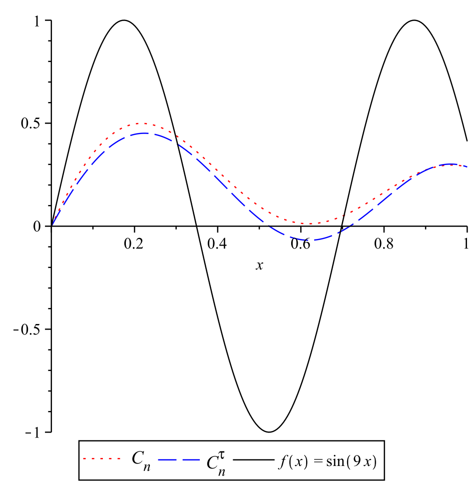

Figure 1. Approximation process by C_n and C_n^τ

Example 3.9. Let us take $f(x) = \log(x+1)$, $\tau(x) = \frac{x^2+x}{2}$ and $a_n = \frac{1}{2}$. In the Table 1 we computed the error of approximation for C_n and C_n^τ at the point $x_0 = 0.8$.

Table 1. Error of approximation for C_n and C_n^τ

n	$\lvert C_n(f;x_0) - f(x_0)\rvert$	$\lvert C_n^\tau(f;x_0) - f(x_0)\rvert$
5	0.2800807097	0.2613318434
10	0.2762200954	0.2502212648
15	0.2749367804	0.2465367167
20	0.2742959594	0.2447029941
25	0.2739117553	0.2436063038
30	0.2736557443	0.2428768564
35	0.2734729425	0.2423567117
40	0.2733358757	0.2419671158
45	0.2732292893	0.2416644116
50	0.2731440335	0.2414224523

From the above results it follows that C_n^τ converge faster than C_n to the function $f(x) = \log(x+1)$ at the point $x_0 = 0.8$. Also, the approximation to the function f by C_n and C_n^τ is shown in the Figure 2.

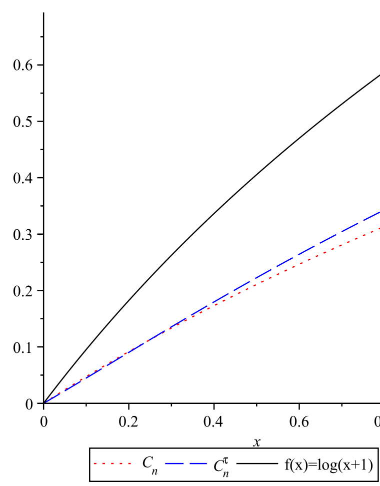

Figure 2. Approximation process by C_n and C_n^τ

4 Voronovskaya type theorem

Let $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, be a positive linear operator and $L_n e_0 = e_0$. Acar et al. [1] defined a general operator $K_n : C[0,1] \to C[0,1]$ by

$$K_n g := \left(L_n(g \circ \tau^{-1})\right) \circ \tau, \; n \geq 1.$$

The authors obtained the following Voronovskaya type formula for the modified operators K_n.

Theorem 4.1. ([1]) *Let $f \in C[0,1]$ with $f''(x)$ finite for $x \in [0,1]$. If there exists $\alpha, \beta \in C[0,1]$ such that*

$$\lim_{n\to\infty} n(L_n(f,x) - f(x)) = \alpha(x)f''(x) + \beta(x)f'(x),$$

then we have

$$\lim_{n\to\infty} n\left(K_n(g,t)-g(t)\right) = \frac{\alpha(\tau(t))}{\tau'(t)^2}g''(t) + \left(\frac{\beta(\tau(t))}{\tau'(t)} - \frac{\alpha(\tau(t))\tau''(t)}{\tau'(t)^3}\right)g'(t)$$

for $g \in C[0,1]$ *with* $g''(x)$ *finite for* $x \in [0,1]$.

Using Theorem 2.4 and Theorem 4.1 we obtain a Voronovskaya type theorem for C_n^τ.

Theorem 4.2. *Let* $f \in C^2[0,1]$. *If* $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ *and* $L := \lim_{n\to\infty} n(1-a_n)$ *exists, then*

$$\lim_{n\to\infty} n\left(C_n^\tau(f,x)-f(x)\right) = \frac{\alpha(\tau(x))}{\tau'(x)^2}f''(x) + \left(\frac{\beta(\tau(x))}{\tau'(x)} - \frac{\alpha(\tau(x))\tau''(x)}{\tau'(x)^3}\right)f'(x)$$

uniformly on $[0,1]$ *with* $\alpha(x) = -\frac{x(1-x)}{2} - \frac{x^2}{4}L$ *and* $\beta(x) = xL$.

5 Local Approximation

Let

$$W^2[0,1] = \left\{g \in C[0,1] : g' \in C[0,1]\right\}.$$

For $f \in C[0,1]$ and $\delta > 0$, the Peetre's K-functional [14] is defined by

$$K_2(f;\delta) = \inf_{g\in W^2[0,1]}\left\{\|f-g\| + \delta\|g\|_{W^2[0,1]}\right\},$$

where

$$\|f\|_{W^2[0,1]} = \|f\| + \|f'\| + \|f''\|.$$

Throughout this paper we assume that $\inf_{x\in[0,1]}\tau'(x) \geq a$, $a \in \mathbb{R}^+$.

Theorem 5.1. *Let* $a_n \in (0,1)$ *and* $\lim_{n\to\infty} a_n = 1$ *and* $f \in C[0,1]$. *For the operator* $C_n^\tau(f;\cdot)$, *there exists absolute constant* $C > 0$ *such that*

$$|C_n^\tau(f;x) - f(x)| \leq CK_2\left(f;\delta_{n,\tau}^2(x)\right) + \omega\left(f;\frac{1}{a}(1-a_n)\tau(x)\right).$$

Proof. Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then by Taylor's expansion, we get

$$\begin{aligned} g(t) &= \left(g\circ\tau^{-1}\right)(\tau(t)) \\ &= \left(g\circ\tau^{-1}\right)(\tau(x)) + \left(g\circ\tau^{-1}\right)'(\tau(x))\left(\tau(t)-\tau(x)\right) \\ &\quad + \int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(g\circ\tau^{-1}\right)''(u)du. \end{aligned} \tag{5.1}$$

If we consider the change of variable $u = \tau(y)$, it follows

$$\int_{\tau(x)}^{\tau(t)}\left(\tau(t)-u\right)\left(g\circ\tau^{-1}\right)''(u)du = \int_x^t\left(\tau(t)-\tau(y)\right)\left(g\circ\tau^{-1}\right)''(\tau(y))\,\tau'(y)dy,$$

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but

$$\left(g\circ\tau^{-1}\right)''(\tau(y))=\frac{1}{\tau'(y)}\cdot\frac{g''(y)\tau'(y)-g'(y)\tau''(y)}{(\tau'(y))^2},$$

therefore

$$\begin{aligned}&\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\left(g\circ\tau^{-1}\right)''(u)du\\&=\int_x^t(\tau(t)-\tau(y))\frac{g''(y)}{\tau'(y)}dy-\int_x^t(\tau(t)-\tau(y))\frac{g'(y)\tau''(y)}{(\tau'(y))^2}dy\\&=\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}\tag{5.2}$$

From (5.1) and (5.2) we can write

$$\begin{aligned}g(t)&=g(x)+\left(g\circ\tau^{-1}\right)'(\tau(x))(\tau(t)-\tau(x))+\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\\&\quad-\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}\tag{5.3}$$

We define

$$\tilde{C_n^\tau}(f;x)=C_n^\tau(f;x)+f(x)-\left(f\circ\tau^{-1}\right)(a_n\tau(x)).$$

From Lemma 3.1 it follows

$$\tilde{C_n^\tau}(e_0;x)=1 \text{ and } \tilde{C_n^\tau}(\tau;x)=C_n^\tau(\tau;x)+\tau(x)-a_n\tau(x)=\tau(x).$$

Now applying $\tilde{C_n^\tau}$ to both side of the relation (5.3) we can write

$$\begin{aligned}\tilde{C_n^\tau}(g;x)&=g(x)+C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\right)\\&\quad-\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^2}du\\&\quad-C_n^\tau\left(\int_{\tau(x)}^{\tau(t)}(\tau(t)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du\right)\\&\quad+\int_{\tau(x)}^{a_n\tau(x)}(a_n\tau(x)-u)\frac{g'(\tau^{-1}(u))\tau''(\tau^{-1}(u))}{[\tau'(\tau^{-1}(u))]^3}du.\end{aligned}$$

Since $\inf_{x\in[0,1]}\tau'(x)\geq a, a\in\mathbb{R}^+$ and τ is strictly increasing on the interval $(0,1)$, we obtain

$$\begin{aligned}\left|\tilde{C_n^\tau}(\tau;x)-g(x)\right|&\leq\frac{1}{2}\mu_{n,2}^\tau(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\quad+\frac{1}{2}\left(a_n\tau(x)-\tau(x)\right)^2\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\frac{1}{2}\left[\delta_{n,\tau}^2(x)+\tau^2(x)(1-a_n)^2\right]\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right]\\&\leq\delta_{n,\tau}^2(x)\left[\frac{\|g''\|}{a^2}+\frac{\|g'\|\cdot\|\tau''\|}{a^3}\right].\end{aligned}$$

By Lemma 3.4, it follows

$$|\tilde{C_n^\tau}(g;x)| \le |C_n^\tau(g;x)| + |g(x)| + \left|\left(g\circ\tau^{-1}\right)(a_n\tau(x))\right| \le 3\|g\|.$$

For $f \in C[0,1]$ and $g \in W_2[0,1]$, we can write

$$\begin{aligned}
&|C_n^\tau(f;x) - f(x)| \\
&= \left|\tilde{C_n^\tau}(f;x) - f(x) + \left(f\circ\tau^{-1}\right)(a_n\tau(x)) - f(x)\right| \\
&\le |\tilde{C_n^\tau}(f-g;x)| + |\tilde{C_n^\tau}(g;x) - g(x)| + |g(x) - f(x)| \\
&\quad + \left|\left(f\circ\tau^{-1}\right)(a_n\tau(x)) - \left(f\circ\tau^{-1}\right)(\tau(x))\right| \\
&\le 4\|f-g\| + \frac{\delta_{n,\tau}^2(x)}{a^2}\|g''\| + \frac{\delta_{n,\tau}^2(x)}{a^3}\|\tau''\|\,\|g'\| + \omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).
\end{aligned}$$

Let $C := \max\left\{4, \frac{1}{a^2}, \frac{\|\tau''\|}{a^3}\right\}$. Then

$$|C_n^\tau(f;x) - f(x)| \le C\left\{\|f-g\| + \delta_{n,\tau}^2(x)\|g\|_{W^2[0,1]}\right\} + \omega\left(f\circ\tau^{-1};(1-a_n)\tau(x)\right).$$

Using the following result (see [1]) $\omega\left(f\circ\tau^{-1};t\right) \le \omega\left(f;\frac{t}{a}\right)$, the theorem is proved. □

To describe our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [9]. Let $\varphi_\tau(x) := \sqrt{\tau(x)(1-\tau(x))}$ and $f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_{\varphi_\tau}(f;t) = \sup_{0<h\le t}\left\{\left|f\left(x+\frac{h\varphi_\tau(x)}{2}\right) - f\left(x-\frac{h\varphi_\tau(x)}{2}\right)\right|, x\pm\frac{h\varphi_\tau(x)}{2}\in[0,1]\right\}. \quad (5.4)$$

Further, the corresponding K-functional to (5.4) is defined by

$$K_{\varphi_\tau}(f;t) = \inf_{g\in W_{\varphi_\tau}[0,1]}\{||f-g|| + t||\varphi_\tau g'||\}\ (t>0), \quad (5.5)$$

where $W_{\varphi_\tau}[0,1] = \{g : g \in AC[0,1], \|\varphi_\tau g'\| < \infty\}$ and $AC[0,1]$ is the class of all absolutely continuous functions on $[0,1]$. It is well known ([9], p.11) that there exists a constant $C > 0$ such that

$$K_{\varphi_\tau}(f;t) \le C\omega_{\varphi_\tau}(f;t). \quad (5.6)$$

Now, we establish a direct approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 5.2. *Let $f \in C[0,1]$ and $\varphi_\tau(x) = \sqrt{\tau(x)(1-\tau(x))}$, then for every $x \in (0,1)$, we have*

$$|C_n^\tau(f;x) - f(x)| \le \tilde{C}\omega_{\varphi_\tau}\left(f;\frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)}\right),$$

where $\tilde{C}$ is a constant independent of n and x.

Proof. Using the representation

$$g(t) = \left(g \circ \tau^{-1}\right)(\tau(t)) = \left(g \circ \tau^{-1}\right)(\tau(x)) + \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du,$$

we get

$$|C_n^\tau(g;x) - g(x)| = \left| C_n^\tau\left(\int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du \right) \right|. \tag{5.7}$$

But,

$$\begin{aligned}\left| \int_{\tau(x)}^{\tau(t)} \left(g \circ \tau^{-1}\right)'(u)du \right| &= \left| \int_x^t \frac{g'(y)}{\tau'(y)} \tau'(y)dy \right| = \left| \int_x^t \frac{\varphi_\tau(y)}{\varphi_\tau(y)} \cdot \frac{g'(y)}{\tau'(y)} \tau'(y)dy \right| \\ &\leq \frac{\|\varphi_\tau g'\|}{a} \left| \int_x^t \frac{\tau'(y)}{\varphi_\tau(y)} dy \right|, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned}\left| \int_x^t \frac{\tau'(y)}{\varphi_\tau(y)} dy \right| &\leq \left| \int_x^t \left(\frac{1}{\sqrt{\tau(y)}} + \frac{1}{\sqrt{1-\tau(y)}} \right) \tau'(y)dy \right| \\ &\leq 2\left(\left| \sqrt{\tau(t)} - \sqrt{\tau(x)} \right| + \left| \sqrt{1-\tau(t)} - \sqrt{1-\tau(x)} \right| \right) \\ &= 2\left|\tau(t) - \tau(x)\right| \left(\frac{1}{\sqrt{\tau(t)} + \sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(t)} + \sqrt{1-\tau(x)}} \right) \\ &< 2|\tau(t) - \tau(x)| \left(\frac{1}{\sqrt{\tau(x)}} + \frac{1}{\sqrt{1-\tau(x)}} \right) \\ &\leq \frac{2\sqrt{2}|\tau(t) - \tau(x)|}{\varphi_\tau(x)}. \end{aligned} \tag{5.9}$$

From relations (5.7)-(5.9) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|C_n^\tau(g;x) - g(x)| &\leq 2\sqrt{2} \frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)} C_n^\tau\left(|\tau(t) - \tau(x)|; x\right) \\ &\leq 2\sqrt{2} \frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)} \left[C_n^\tau\left((\tau(t) - \tau(x))^2; x \right) \right]^{1/2} \\ &\leq 2\sqrt{2} \frac{\|\varphi_\tau g'\|}{a\varphi_\tau(x)} \delta_{n,\tau}(x). \end{aligned} \tag{5.10}$$

Using Lemma 3.4 and (5.10) it follows

$$\begin{aligned}|C_n^\tau(f;x) - f(x)| &\leq |C_n^\tau(f-g;x)| + |f(x) - g(x)| + |C_n^\tau(g;x) - g(x)| \\ &\leq C\left\{ \|f-g\| + \frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)} \|\varphi_\tau g'\| \right\}, \end{aligned}$$

where $C = \max\left\{2, \frac{2\sqrt{2}}{a}\right\}$.

Taking infimum on the right hand side of the above inequality over all $g \in W_{\varphi_\tau}[0,1]$, we get

$$|C_n^\tau(f;x) - f(x)| \leq CK_{\varphi_\tau}\left(f; \frac{\delta_{n,\tau}(x)}{\varphi_\tau(x)} \right).$$

Using the relation (5.6) this theorem is proven. □

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A NOTE ON HERMITE POLYNOMIALS

TAEKYUN KIM AND DAE SAN KIM

Abstract. In this paper, we consider linear differential equations satisfied by the generating function for Hermite polynomials and derive some new identities involving those polynomials.

1. Introduction

The Hermite polynomials form a Sheffer sequence and are given by the generating function

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n, \quad (\text{see [1–8, 10, 13, 14]}). \tag{1.1}$$

By using Taylor series, we get

$$\begin{aligned}H_n(x)&=\left[\left(\frac{\partial}{\partial t}\right)^n e^{\left(2xt-t^2\right)}\right]_{t=0}\\&=\left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}\\&=(-1)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0}\\&=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}, \quad (n\geq 0), \quad (\text{see [1–15, 18]}).\end{aligned}$$

The Hermite polynomials can be represented by the Contour integral as follows:

$$H_n(z)=\frac{n!}{2\pi i}\oint e^{-t^2+2tz}t^{-n-1}dt, \tag{1.2}$$

where the Contour encloses the origin and is traversed in a counterclockwise direction (see [2, 8, 11, 13]).

The probabilists' Hermite polynomials are given by the generating function

$$\begin{aligned}H_n^*(x)&=(-1)^n e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}\\&=\left(x-\frac{d}{dx}\right)^n\cdot 1, \quad (\text{see [10]}).\end{aligned} \tag{1.3}$$

The physicists' Hermite polynomials are also given by

$$\begin{aligned}H_n(x)&=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\\&=\left(2x-\frac{d}{dx}\right)^n\cdot 1 \quad (\text{see [20]}).\end{aligned} \tag{1.4}$$

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Thus, by (1.3) and (1.4), we get

$$H_n(x) = 2^{\frac{n}{2}} H_n^*\left(\sqrt{2}x\right), \quad H_n^*(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right), \tag{1.5}$$

where $n \geq 0$ (see [9, 11, 12, 15, 18]).

The first several Hermite polynomials are $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^2 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$, $H_5(x) = 32x^5 - 160x^3 + 120x$, $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$, ...

The probabilists' Hermite polynomials are solutions of the differential equation:

$$\left(e^{-\frac{x^2}{2}} u'\right)' + \lambda e^{-\frac{1}{2}x^2} u = 0,$$

where λ is a constant, with the boundary conditions that u should be polynomially bounded at infinity.

The generating function of the probabilists' Hermite polynomials is given by

$$e^{xt - \frac{t^2}{2}} = \sum_{n=0}^{\infty} H_n^*(x) \frac{t^n}{n!}, \quad (\text{see } [12, 15, 18]). \tag{1.6}$$

The Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν form an Appell sequence and are defined by the generating function

$$\sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!} t^k = e^{xt - \frac{\nu t^2}{2}}, \quad (\text{see } [12]). \tag{1.7}$$

Thus, by (1.7), we get

$$x^{2m+1} = \sum_{l=0}^{m} \binom{2m+1}{2l+1} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l+1}^{(\nu)}(x), \tag{1.8}$$

and

$$x^{2m} = \sum_{l=0}^{m} \binom{2m}{2l} \frac{(2m-2l)!}{(m-l)!} \left(\frac{\nu}{2}\right)^{m-l} H_{2l}^{(\nu)}(x), \quad (\text{see } [12]). \tag{1.9}$$

The Hermite polynomials have been studied in probability, combinatorics, numerical analysis, finite element methods, physics and system theory (see [1–15, 18]).

Recently, Kim has studied nonlinear differential equations arising from Frobenius-Euler numbers and polynomials.

In this paper, we consider linear differential equations arising from Hermite polynomials of variance ν and give some new and explicit identities for those polynomials.

2. Hermite polynomials of variance ν

Let

$$F = F(t : x, \nu) = e^{xt - \frac{\nu t^2}{2}}. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned} F^{(1)} &= \frac{d}{dt} F(t : x, \nu) \\ &= (x - \nu t) e^{xt - \frac{\nu t^2}{2}} \\ &= (x - \nu t) F, \end{aligned} \tag{2.2}$$

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$$F^{(2)} = \frac{d}{dt}F^{(1)} = \left(-\nu + (x - t\nu)^2\right)F, \tag{2.3}$$

$$F^{(3)} = \frac{d}{dt}F^{(2)} = \left(-3\nu\left(x - \nu t\right) + \left(x - \nu t\right)^3\right)F, \tag{2.4}$$

and

$$F^{(4)} = \frac{d}{dt}F^{(3)} = \left(3\nu^2 - 6\nu\left(x - \nu t\right)^2 + \left(x - \nu t\right)^4\right)F. \tag{2.5}$$

Continuing this process, we set

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F\left(t : x, \nu\right) \\ &= \left(\sum_{i=0}^{N} a_i\left(N, \nu\right)\left(x - \nu t\right)^i\right)F, \end{aligned} \tag{2.6}$$

where $N \in \mathbb{N} \cup \{0\}$.

From (2.6), we have

$$\begin{aligned} F^{(N+1)} &= \frac{d}{dt}F^{(N)} \\ &= \sum_{i=0}^{N} a_i\left(N, \nu\right) i\left(x - \nu t\right)^{i-1}\left(-\nu\right)F \\ &\quad + \sum_{i=0}^{N} a_i\left(N, \nu\right)\left(x - \nu t\right)^i F^{(1)}. \end{aligned} \tag{2.7}$$

By (2.2) and (2.7), we easily get

$$\begin{aligned} F^{(N+1)} = \Big\{ &-\nu a_1\left(N, \nu\right) + a_N\left(N, \nu\right)\left(x - \nu t\right)^{N+1} + a_{N-1}\left(N, \nu\right)\left(x - \nu t\right)^N \\ &+ \sum_{i=1}^{N-1}\left(-\left(i+1\right)\nu a_{i+1}\left(N, \nu\right) + a_{i-1}\left(N, \nu\right)\right)\left(x - \nu t\right)^i \Big\} F. \end{aligned} \tag{2.8}$$

By replacing N by $(N + 1)$ in (2.6), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i\left(N + 1, \nu\right)\left(x - \nu t\right)^i\right)F. \tag{2.9}$$

From (2.8) and (2.9), we can derive the following equations:

$$a_0\left(N + 1, \nu\right) = -\nu a_1\left(N, \nu\right), \tag{2.10}$$

$$a_N\left(N + 1, \nu\right) = a_{N-1}\left(N, \nu\right), \tag{2.11}$$

$$a_{N+1}\left(N + 1, \nu\right) = a_N\left(N, \nu\right) \tag{2.12}$$

and

$$a_i\left(N + 1, \nu\right) = -\left(i + 1\right)\nu a_{i+1}\left(N, \nu\right) + a_{i-1}\left(N, \nu\right), \tag{2.13}$$

where $1 \leq i \leq N - 1$.

It is not difficult to show that

$$F = F^{(0)} = a_0\left(0, \nu\right)F. \tag{2.14}$$

Thus, by (2.14), we get

$$a_0\left(0, \nu\right) = 1. \tag{2.15}$$

From (2.2) and (2.6), we note that

$$(x-\nu t)\,F = F^{(1)} = \left(a_0\left(1,\nu\right)+a_1\left(1,\nu\right)\left(x-\nu t\right)\right)F. \tag{2.16}$$

Thus, by comparing the coefficients on both sides of (2.16), we get

$$a_0\left(1,\nu\right)=0,\quad a_1\left(1,\nu\right)=1. \tag{2.17}$$

From (2.11), (2.12), (2.15) and (2.17), we have

$$a_N\left(N+1,\nu\right)=a_{N-1}\left(N,\nu\right)=\cdots=a_0\left(1,\nu\right)=0, \tag{2.18}$$

and

$$a_{N+1}\left(N+1,\nu\right)=a_N\left(N,\nu\right)=\cdots=a_1\left(1,\nu\right)=1. \tag{2.19}$$

Therefore, we obtain the following theorem.

Theorem 1. *The linear differential equations*

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F\left(t:x,\nu\right) \\ &= \left(\sum_{i=0}^{N} a_i\left(N,\nu\right)\left(x-\nu t\right)^i\right)F, \quad \left(N\in\mathbb{N}\cup\{0\}\right) \end{aligned}$$

has a solution $F=F\left(t:x,\nu\right)=e^{xt-\frac{\nu t^2}{2}}$, *where*

$$\begin{aligned} a_0\left(N,\nu\right) &= -\nu a_1\left(N-1,\nu\right), \\ a_{N-1}\left(N,\nu\right) &= a_{N-2}\left(N-1,\nu\right)=\cdots=a_1\left(2,\nu\right)=a_0\left(1,\nu\right)=0, \\ a_N\left(N,\nu\right) &= a_{N-1}\left(N-1,\nu\right)=\cdots=a_1\left(1,\nu\right)=a_0\left(0,\nu\right)=1, \end{aligned}$$

and

$$a_i\left(N,\nu\right)=-\left(i+1\right)\nu a_{i+1}\left(N-1,\nu\right)+a_{i-1}\left(N-1,\nu\right), \quad \left(1\le i\le N-2\right).$$

Example.

(1) $N=3,\ i=1$. By (2.13), we get

$$\begin{aligned} a_1\left(3,\nu\right) &= -2\nu a_2\left(2,\nu\right)+a_0\left(2,\nu\right) \\ &= -2\nu-\nu=-3\nu. \end{aligned}$$

(2) $N=4,\ 1\le i\le 2$. By (2.13), we have

$$a_1\left(4,\nu\right)=0,\quad a_2\left(4,\nu\right)=-6\nu.$$

(3) $N=5,\ 1\le i\le 3$. By (2.13), we get

$$a_1\left(5,\nu\right)=15\nu^2,\quad a_2\left(5,\nu\right)=0,\quad a_3\left(5,\nu\right)=-10\nu.$$

(4) $N=6,\ 1\le i\le 4$. From (2.13), we have

$$a_1\left(6,\nu\right)=0,\quad a_2\left(6,\nu\right)=45\nu^2,\quad a_3\left(6,\nu\right)=0,\quad a_4\left(6,\nu\right)=-15\nu.$$

Thus, we obtain the following result.

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Remark. The matrix $(a_i(j,\nu))_{0\le i,j\le 6}$ is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -\nu & 0 & 3\nu^2 & 0 & -15\nu^3 \\ & 1 & 0 & -3\nu & 0 & 15\nu^2 & 0 \\ & & 1 & 0 & -6\nu & 0 & 45\nu^2 \\ & & & 1 & 0 & -10\nu & 0 \\ & & & & 1 & 0 & -15\nu \\ & & 0 & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix} \end{array}.$$

From (1.7), we note that

$$\begin{aligned} F = F(t:x,\nu) &= e^{xt-\frac{\nu t^2}{2}} \\ &= \sum_{k=0}^{\infty} H_k^{(\nu)}(x)\frac{t^k}{k!}. \end{aligned} \tag{2.20}$$

Thus, by (2.20), we get

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t:x,\nu) \\ &= \sum_{k=N}^{\infty} H_k^{(\nu)}(x)(k)_N \frac{t^{k-N}}{k!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}(x)(k+N)_N \frac{t^k}{(n+k)!} \\ &= \sum_{k=0}^{\infty} H_{k+N}^{(\nu)}(x)\frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By Theorem 1, we easily get

$$\begin{aligned} &F^{(N)} \\ &= \left(\sum_{i=0}^{N} a_i(N,\nu)(x-\nu t)^i\right) F \\ &= \sum_{i=0}^{N} a_i(N,\nu) \sum_{m=0}^{\infty} (i)_m x^{i-m}(-\nu)^m \frac{t^m}{m!} \sum_{l=0}^{\infty} H_l^{(\nu)}(x)\frac{t^l}{l!} \\ &= \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=0}^{k} \binom{k}{l} (i)_{k-l}(-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x) \right\} \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left\{ \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l}(-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x) \right\} \frac{t^k}{k!}. \end{aligned} \tag{2.22}$$

Therefore, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2. *For $k, N \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned} & H_{k+N}^{(\nu)}(x) \\ & = \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x). \end{aligned}$$

It is easy to show that

$$H_{k+1}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right) H_k^{(\nu)}(x). \tag{2.23}$$

Thus, by (2.23), we have

$$H_{k+N}^{(\nu)}(x) = \left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x), \quad (N \in \mathbb{N} \cup \{0\}). \tag{2.24}$$

From Theorem 2, we note that

$$\begin{aligned} & \left(x - \nu \frac{\partial}{\partial x}\right)^N H_k^{(\nu)}(x) \\ & = \sum_{i=0}^{N} a_i(N,\nu) \sum_{l=\max\{0,k-i\}}^{k} \binom{k}{l} (i)_{k-l} (-\nu)^{k-l} x^{i+l-k} H_l^{(\nu)}(x), \end{aligned} \tag{2.25}$$

where $\frac{\partial}{\partial x} x - x \frac{\partial}{\partial x} =$ identity.

Now, we observe explicit determination of $a_i(j,\nu)$.

From (2.12) and (2.13), we can derive the following equations:

$$a_N(N,\nu) = 1, \tag{2.26}$$

$$\begin{aligned} a_{N-2}(N,\nu) &= -(N-1)\nu a_{N-1}(N-1,\nu) + a_{N-3}(N-1,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad + a_{N-4}(N-2,\nu) \\ &\vdots \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) + a_0(2,\nu) \\ &= -(N-1)\nu a_{N-1}(N-1,\nu) - (N-2)\nu a_{N-2}(N-2,\nu) \\ &\quad - \cdots - 2\nu a_2(2,\nu) - \nu a_1(1,\nu) \\ &= -\nu \sum_{i=1}^{N-1} i a_i(i,\nu), \end{aligned} \tag{2.27}$$

$$\begin{aligned} a_{N-4}(N,\nu) &= -(N-3)\nu a_{N-3}(N-1,\nu) + a_{N-5}(N-1,\nu) \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \\ &\quad + a_{N-6}(N-2,\nu) \\ &\vdots \\ &= -(N-3)\nu a_{N-3}(N-1,\nu) - (N-4)\nu a_{N-4}(N-2,\nu) \end{aligned} \tag{2.28}$$

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$$
\begin{aligned}
&-\cdots-2\nu a_2(4,\nu)+a_0(4,\nu)\\
&=-(N-3)\nu a_{N-3}(N-1,\nu)-(N-4)\nu a_{N-4}(N-2,\nu)\\
&\quad-\cdots-2\nu a_2(4,\nu)-\nu a_1(3,\nu)\\
&=-\nu\sum_{i=0}^{N-3} i a_i(i+2,\nu),
\end{aligned}
$$

and

$$
\begin{aligned}
a_{N-6}(N,\nu)&=-(N-5)\nu a_{N-5}(N-1,\nu)+a_{N-7}(N-1,\nu)\\
&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\
&\quad+a_{N-8}(N-2,\nu)\\
&\ \vdots\\
&=-(N-5)\nu a_{N-5}(N-1,\nu)-(N-6)\nu a_{N-6}(N-2,\nu)\\
&\quad-\cdots-2\nu a_2(6,\nu)-\nu a_1(5,\nu)\\
&=-\nu\sum_{i=1}^{N-5} i a_i(i+4,\nu).
\end{aligned}
\tag{2.29}
$$

Continuing in this fashion, for l with $1\le l\le\left[\frac{N-1}{2}\right]$,

$$
a_{N-2l}(N,\nu)=-\nu\sum_{i=1}^{N-2l+1} i a_i(i+2l-2,\nu). \tag{2.30}
$$

By (2.26), (2.27), (2.28), (2.29) and (2.30), we get

$$
a_{N-2}(N,\nu)=-\nu\sum_{i_1=1}^{N-1} i_1, \tag{2.31}
$$

$$
\begin{aligned}
a_{N-4}(N,\nu)&=-\nu\sum_{i_2=1}^{N-3} i_2 a_{i_2}(i_2+2,\nu)\\
&=(-\nu)^2\sum_{i_2=1}^{N-3}\sum_{i_1=1}^{i_2+1} i_2 i_1,
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
a_{N-6}(N,\nu)&=-\nu\sum_{i_3=1}^{N-5} i_3 a_{i_3}(i_3+4,\nu)\\
&=(-\nu)^3\sum_{i_3=1}^{N-5}\sum_{i_2=1}^{i_3+1}\sum_{i_1=1}^{i_2+1} i_3 i_2 i_1,
\end{aligned}
\tag{2.33}
$$

and

$$
a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1} i_l\cdot i_{l-1}\cdots i_1, \tag{2.34}
$$

where $1\le l\le\left[\frac{N-1}{2}\right]$.

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By (2.11) and (2.13), we easily get

(2.35)

$$a_{N-1}(N,\nu)=a_{N-2}(N-1,\nu)=a_{N-3}(N-2,\nu)=\cdots=a_0(1,\nu)=0,$$

(2.36)

$$\begin{aligned}a_{N-3}(N,\nu)&=-(N-2)\nu a_{N-2}(N-1,\nu)+a_{N-4}(N-1,\nu)\\&=a_{N-4}(N-1,\nu)\\&\ \ \vdots\\&=a_0(3,\nu)=-\nu a_1(2,\nu)=-\nu a_0(1,\nu)=0,\end{aligned}$$

(2.37)

$$\begin{aligned}a_{N-5}(N,\nu)&=-(N-4)\nu a_{N-4}(N-1,\nu)+a_{N-6}(N-1,\nu)=a_{N-6}(N-1,\nu)\\&\ \ \vdots\\&=a_0(5,\nu)=-\nu a_1(4,\nu)=0,\end{aligned}$$

(2.38)

$$\begin{aligned}a_{N-7}(N,\nu)&=-(N-6)\nu a_{N-6}(N-1,\nu)+a_{N-8}(N-1,\nu)\\&\ \ \vdots\\&=a_0(7,\nu)=-\nu a_1(6,\nu)=0,\end{aligned}$$

and

$$a_{N-(2l-1)}(N,\nu)=0,\quad \left(1\le l\le\left[\frac{N}{2}\right]\right). \tag{2.39}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $N\in\mathbb{N}\cup\{0\}$, we have*

$$a_{N-2l}(N,\nu)=(-\nu)^l\sum_{i_l=1}^{N-2l+1}\sum_{i_{l-1}=1}^{i_l+1}\cdots\sum_{i_1=1}^{i_2+1}i_li_{l-1}\cdots i_1,$$

where $1\le l\le\left[\frac{N-1}{2}\right]$.

Also,

$$a_{N-(2l-1)}(N,\nu)=0,\quad \text{if } 1\le l\le\left[\frac{N}{2}\right].$$

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