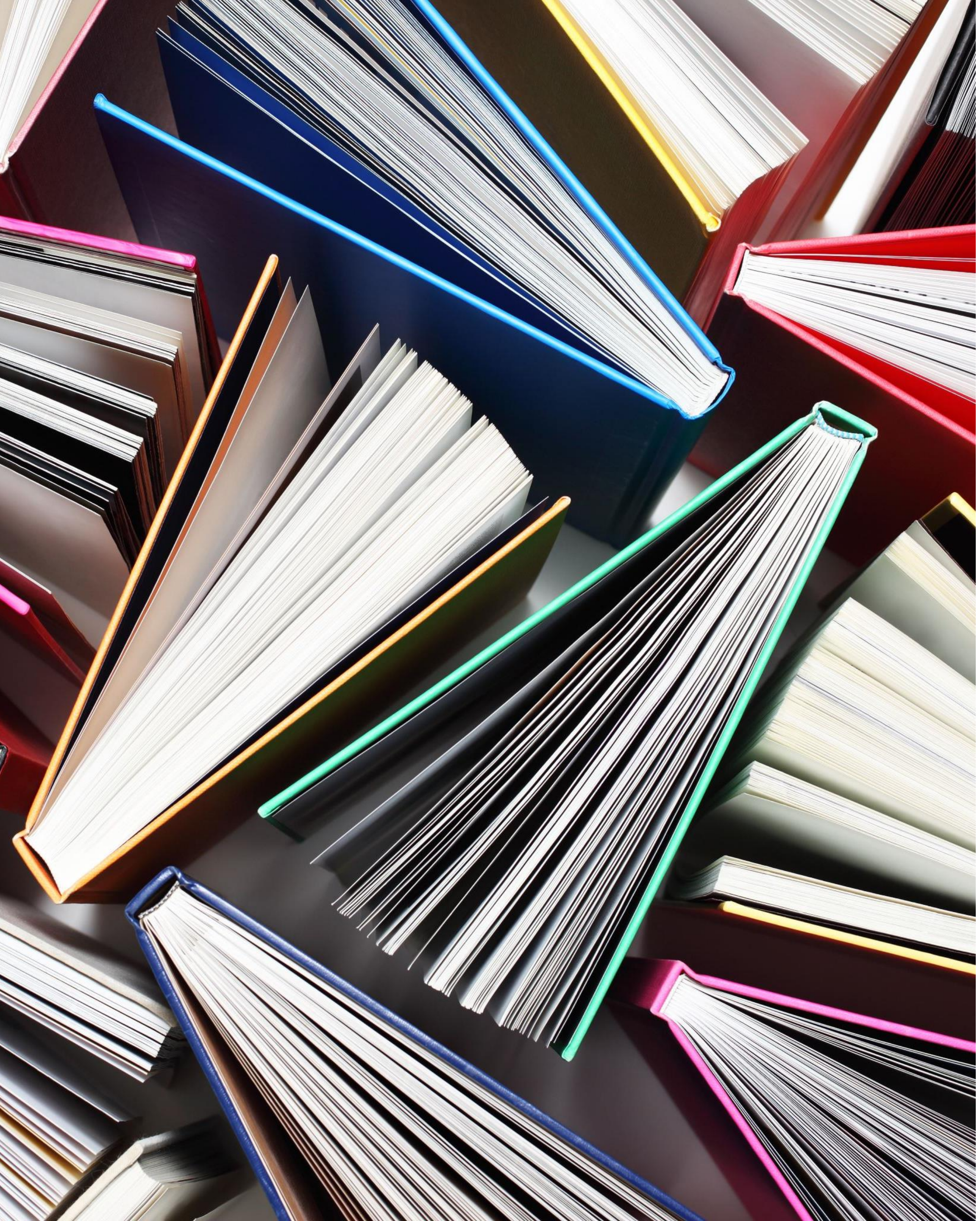

E.R. Love type left fractional integral inequalities

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Theorem 1 *(Hardy's Inequality, integral version [3, Theorem 327]) If f is a complex-valued function in $L^r(0,\infty)$, $\|\cdot\|$ is the $L^r(0,\infty)$ norm and $r>1$, then*

$$\left\| \frac{1}{x}\int_0^x f(t)\,dt \right\| \leq \frac{r}{r-1}\|f\|. \tag{1}$$

Theorem 2 *(E.R. Love, [4]) If* $s \geq r \geq 1$, $0 \leq a < b \leq \infty$, γ *is real,* $\omega(x)$ *is decreasing and positive in* (a,b), $f(x)$ *and* $H(x,y)$ *are measurable and non-negative on* (a,b), $H(x,y)$ *is homogeneous of degree* -1,

$$(Hf)(x) = \int_a^x H(x,y)\, f(y)\, dy \tag{2}$$

and

$$\|f\|_r = \left(\int_a^b f(x)^r \, x^{\gamma-1} \omega(x) \, dx \right)^{\frac{1}{r}}, \tag{3}$$

then

$$\|Hf\|_r \leq C \, \|f\|_s \, , \tag{4}$$

where

$$C = \int_{\frac{a}{b}}^{1} H(1,t) \, t^{-\frac{\gamma}{r}} \left(\int_a^{bt} x^{\gamma-1} \omega(x) \, dx \right)^{\frac{1}{r}-\frac{1}{s}} dt. \tag{5}$$

Here $\frac{a}{b}$ *is to mean* 0 *if* $a = 0$ *or* $b = \infty$ *or both; and* bt *is to mean* ∞ *if* $b = \infty$.

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Theorem 3 *(E.R. Love, [4]) If* $p>0$, $q>0$, $p+q=r\geq 1$, $0\leq a<b\leq \infty$, $\gamma<r$, $\omega(x)$ *is decreasing and positive in* (a,b), $f(x)$ *is measurable and non-negative on* (a,b), I^{α} *is the left Riemann-Liouville operator of fractional integration defined by*

$$\left(I^{\alpha}f\right)(x)=\int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt \quad \text{for } \alpha>0, \tag{6}$$

$I^0 f(x)=f(x)$, *where* Γ *is the gamma function, and* $I^{\beta}f$ *is defined similarly for* $\beta\geq 0$, *then*

$$\int_a^b \left[\left(I^{\alpha}f\right)(x)\right]^p \left[\left(I^{\beta}f\right)(x)\right]^q x^{\gamma-\alpha p-\beta q-1}\omega(x)\,dx \leq C\int_a^b f(x)^r x^{\gamma-1}\omega(x)\,dx, \tag{7}$$

where

$$C=\left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\alpha+1-\frac{\gamma}{r}\right)}\right)^p \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\beta+1-\frac{\gamma}{r}\right)}\right)^q. \tag{8}$$

Also Theorem 2 implies Theorem 1 (by [4]), just take $a=0$, $b=\infty$, $\gamma=1$, $s=r>1$, $\omega(x)=1$ and $H(x,y)=\frac{1}{x}$.

2 Main Results

We start with a general result, see also [2].

Theorem 4 *(Reverse Minkowski integral inequality) Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be* σ*-finite measure spaces and let* $0 < p < 1$. *Here* f *is a nonnegative function on* $X \times Y$ *with* $f(x, y) > 0$ *for almost all* $x \in X$, *almost all* $y \in Y$ *and* $\int_Y (f(x, y))^p \, d\nu(y) < \infty$ *for almost all* $x \in X$.

Then

$$\left(\int_Y \left(\int_X f(x, y) \, d\mu(x) \right)^p d\nu(y) \right)^{\frac{1}{p}} \geq \int_X \left(\int_Y (f(x, y))^p \, d\nu(y) \right)^{\frac{1}{p}} d\mu(x), \quad (9)$$

if left-hand side is finite.

Theorem 5 *Let* $0 < r < 1$, $0 < a < b < \infty$, $f(x)$ *and* $H(x,y)$ *are measurable and non-negative on* (a,b), $(a,b)^2$, *respectively,* $H(x,y)$ *is homogeneous of degree* -1,

$$(Hf)(x) = \int_a^x H(x,y)\, f(y)\, dy, \tag{11}$$

and suppose that

$$\|Hf\|_{r,[a,b]} = \left(\int_a^b (Hf)(x)^r\, dx\right)^{\frac{1}{r}} < \infty, \tag{12}$$

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and $\|f\|_{r,[a,b]}$ *is defined similarly.*

We assume that $H(1,t)\,f(x,t)>0$, *for almost all* $t\in\left[\frac{a}{x},1\right]$, *for almost all* $x\in[a,b]$, *and* $\frac{H(1,t)}{t^{\frac{1}{r}}}\|f\|_{r,[a,bt]}<\infty$, *for almost all* $t\in\left[\frac{a}{b},1\right]$.

Then

$$\|Hf\|_{r,[a,b]}\geq\int_{\frac{a}{b}}^{1}\frac{H(1,t)}{t^{\frac{1}{r}}}\|f\|_{r,[a,bt]}\,dt. \tag{13}$$

We give a reverse left fractional inequality.

Corollary 6 *(to Theorem 5) Let $0 < r < p$ with $r < 1$, $0 < a < b < \infty$, f is measurable and non-negative on (a, b) such that $f(x) > 0$ almost everywhere on $[a, b]$. Here I^{α} is the left fractional Riemann-Liouville integral operator of order $\alpha > 0$, defined by*

$$\left(I^{\alpha} f\right)(x) = \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\, dt, \quad I^0 f(x) = f(x), \tag{17}$$

$\forall\ x \in [a, b]$, and suppose that $\left\| x^{-\alpha} I^{\alpha} f(x) \right\|_{r,[a,b]} < \infty$. We assume that $\|f\|_{r,[a,bt]} < \infty$, for almost all $t \in \left[\frac{a}{b}, 1\right]$. Then

$$\int_a^b \left(I^{\alpha} f(x)\right)^p (f(x))^{r-p} x^{-\alpha p} dx \geq \frac{1}{\Gamma(\alpha)^p} \left(\int_{\frac{a}{b}}^1 \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \|f\|_{r,[a,bt]}\, dt \right)^p \|f\|_r^{r-p}. \tag{18}$$

Next we present a reverse Opial type [5] inequality.

Corollary 7 *(to Corollary 6) Let* $0 < r < p$ *with* $r < 1$, $m \in \mathbb{N}$, $0 < a < b < \infty$, $f : [a, b] \to \mathbb{R}$ *such that* $f^{(m-1)}$ *is an absolutely continuous function over* $[a, b]$, *where* $f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0$, *and* $f^{(m)}$ *is non-negative, with* $f^{(m)}(x) > 0$ *almost everywhere over* $[a, b]$. *We assume that* $\|x^{-m} f(x)\|_{r,[a,b]} < \infty$, *and* $\left\|f^{(m)}\right\|_{r,[a,bt]} < \infty$, *a.e. for* $t \in \left[\frac{a}{b}, 1\right]$. *Then*

$$\int_a^b (f(x))^p \left(f^{(m)}(x)\right)^{r-p} x^{-mp} dx \geq$$

$$\frac{1}{((m-1)!)^p} \left(\int_{\frac{a}{b}}^1 \frac{(1-t)^{m-1}}{t^{\frac{1}{r}}} \left\|f^{(m)}\right\|_{r,[a,bt]} dt \right)^p \left\|f^{(m)}\right\|_r^{r-p}. \quad (21)$$

We need

Definition 8 *Let* $\alpha > 0$, $n = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *is the ceiling),* $f \in AC^n([a,b])$ *(i.e.* $f^{(n-1)}$ *is absolutely continuous function). The left Caputo fractional derivative is given by*

$$D_{*a}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt \qquad (23)$$

and exists almost everywhere for x *in* $[a,b]$, $D_{*a}^{0} f = f$, *see [1], p. 394.*

We mention

Corollary 9 *([1], p. 395) Let* $\alpha > 0$, $n = \lceil \alpha \rceil$, $f \in AC^n([a,b])$, *and* $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. *Then*

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} D_{*a}^{\alpha} f(t)\, dt = I^{\alpha} D_{*a}^{\alpha} f(x), \quad \forall\, x \in [a,b]. \tag{24}$$

We give a reverse left fractional Opial type [1] inequality.

Corollary 10 *(to Corollary 6) Let* $0 < r < p$ *with* $r < 1$, $0 < a < b < \infty$, $\alpha > 0$, $n = \lceil \alpha \rceil$, $f \in AC^n([a,b])$, *and* $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. *Assume here that* $D_{*a}^{\alpha} f$ *is non-negative over* (a,b) *such that* $D_{*a}^{\alpha} f > 0$ *almost everywhere on* $[a,b]$. *Suppose that* $\left\| x^{-\alpha} f(x) \right\|_{r,[a,b]} < \infty$ *and* $\left\| D_{*a}^{\alpha} f \right\|_{r,[a,bt]} < \infty$, *for almost all* $t \in \left[\frac{a}{b}, 1\right]$. *Then*

$$\int_a^b (f(x))^p \left(D_{*a}^{\alpha} f(x)\right)^{r-p} x^{-\alpha p} dx \geq$$

$$\frac{1}{\Gamma(\alpha)^p} \left(\int_{\frac{a}{b}}^1 \frac{(1-t)^{\alpha-1}}{t^{\frac{1}{r}}} \left\| D_{*a}^{\alpha} f \right\|_{r,[a,bt]} dt \right)^p \left\| D_{*a}^{\alpha} f \right\|_r^{r-p}. \tag{25}$$

Definition 11 *([1], pp. 7-8) Let* $\nu > 0$, $n := [\nu]$ $[\cdot]$ *the integral part, and* $\alpha := \nu - n$ $(0 < \alpha < 1)$*;* $x, x_0 \in [a,b] \subset \mathbb{R}$ *such that* $x \geq x_0$, x_0 *is fixed. Let* $f \in C([a,b])$ *and define*

$$\left(J_{\nu}^{x_0} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} f(t)\, dt, \quad x_0 \leq x \leq b, \tag{26}$$

the left Riemann-Liouville integral. We define the subspace $C_{x_0}^{\nu}([a,b])$ *of* $C^n([a,b])$:

$$C_{x_0}^{\nu}([a,b]) := \left\{ f \in C^n([a,b]) : J_{1-\alpha}^{x_0} D^n f \in C^1([x_0,b]) \right\}. \tag{27}$$

For $f \in C_{x_0}^{\nu}([a,b])$ *we define the left generalized* ν*-fractional derivative of* f *over* $[x_0,b]$ *as*

$$D_{x_0+}^{\nu} f := D J_{1-\alpha}^{x_0} f^{(n)} \quad (f^{(n)} := D^n f). \tag{28}$$

Notice $D_{x_0+}^{\nu} f \in C([x_0,b])$.

We also need

Theorem 12 *(from Theorem 2.1, p. 8 of [1]) Let* $f \in C_{x_0}^{\nu}([a,b])$, $x_0 \in [a,b]$ *fixed.*

(i) If $\nu \geq 1$, *and* $f^{(i)}(x_0) = 0$, $i = 0, 1, ..., n-1$, *then* $f(x) = \left(J_{\nu}^{x_0} D_{x_0+}^{\nu} f\right)(x)$, *all* $x \in [a,b] : x \geq x_0$.

(ii) If $0 < \nu < 1$, *then* $f(x) = \left(J_{\nu}^{x_0} D_{x_0+}^{\nu} f\right)(x)$, *all* $x \in [a,b] : x \geq x_0$.

That is, in both cases we have

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-t)^{\nu-1} \left(D_{x_0+}^{\nu} f\right)(t)\, dt, \quad x_0 \leq x \leq b. \tag{29}$$

If $x_0 = a$, *we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{a}^{x} (x-t)^{\nu-1} \left(D_{a+}^{\nu} f\right)(t)\, dt = \left(J_{\nu}^{a} D_{a+}^{\nu} f\right)(x), \quad \text{all } a \leq x \leq b. \tag{30}$$

We give another reverse left fractional Opial type inequality.

Corollary 13 *(to Corollary 6) Let* $0 < r < p$ *with* $r < 1$, $0 < a < b < \infty$, $\nu > 0$, $n = [\nu]$; $f \in C_a^\nu([a, b])$, *such that* $f^{(i)}(a) = 0$, $i = 0, 1, ..., n-1$ *for only the case of* $\nu \geq 1$. *Assume here that* $D_{a+}^\nu f$ *is non-negative over* (a, b) *such that* $D_{a+}^\nu f > 0$ *almost everywhere on* $[a, b]$. *Suppose that* $\|x^{-\nu} f(x)\|_{r,[a,b]} < \infty$ *and* $\|D_{a+}^\nu f\|_{r,[a,bt]} < \infty$, *for almost all* $t \in \left[\frac{a}{b}, 1\right]$. *Then*

$$\int_a^b (f(x))^p \left(D_{a+}^\nu f(x)\right)^{r-p} x^{-\nu p} dx \geq$$

$$\frac{1}{\Gamma(\nu)^p} \left(\int_{\frac{a}{b}}^1 \frac{(1-t)^{\nu-1}}{t^{\frac{1}{r}}} \left\|D_{a+}^\nu f\right\|_{r,[a,bt]} dt \right)^p \left\|D_{a+}^\nu f\right\|_r^{r-p}. \qquad (31)$$

We need the following representation result.

Theorem 14 *([1], p. 395) Let* $\nu \geq \overline{\gamma}+1$, $\overline{\gamma} \geq 0$. *Call* $n = \lceil \nu \rceil$, $m := \lceil \overline{\gamma} \rceil$. *Assume* $f \in AC^n([a,b])$, *such that* $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, *and* $D^{\nu}_{*a} f \in L_{\infty}(a,b)$. *Then* $D^{\overline{\gamma}}_{*a} f \in C([a,b])$, $D^{\overline{\gamma}}_{*a} f(x) = I^{m-\overline{\gamma}} f^{(m)}(x)$, *and*

$$D^{\overline{\gamma}}_{*a} f(x) = \frac{1}{\Gamma(\nu-\overline{\gamma})} \int_a^x (x-t)^{\nu-\overline{\gamma}-1} D^{\nu}_{*a} f(t)\, dt = \left(I^{\nu-\overline{\gamma}} D^{\nu}_{*a} f\right)(x), \quad (32)$$

$\forall\ x \in [a,b]$.

Remark 15 *(to Theorem 14) By Corollary 9 we also have*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D^{\nu}_{*a} f(t)\, dt = \left(I^{\nu} D^{\nu}_{*a} f\right)(x), \quad (33)$$

$\forall\ x \in [a,b]$.

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Theorem 16 *If* $p > 0,\ q > 0,\ p + q = r \geq 1,\ 0 \leq a < b < \infty,\ \gamma < r,$ $\omega(x)$ *is decreasing and positive in* (a, b). *Let* $\nu \geq \overline{\gamma} + 1,\ \overline{\gamma} \geq 0,$ *call* $n = \lceil \nu \rceil,$ $f \in AC^n([a, b]) : f^{(k)}(a) = 0,\ k = 0, 1, ..., n - 1;\ D^{\nu}_{*a} f \in L_{\infty}(a, b),$ *with* $D^{\nu}_{*a} f \geq 0$ *over* (a, b). *Then*

$$\int_a^b (f(x))^p \left(D^{\overline{\gamma}}_{*a} f(x)\right)^q x^{\gamma - \nu p - (\nu - \overline{\gamma}) q - 1} \omega(x)\, dx \leq$$

$$C \int_a^b \left(\left(D^{\nu}_{*a} f\right)(x)\right)^r x^{\gamma - 1} \omega(x)\, dx, \tag{34}$$

where

$$C = \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu + 1 - \frac{\gamma}{r}\right)}\right)^p \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma} + 1 - \frac{\gamma}{r}\right)}\right)^q. \tag{35}$$

Theorem 17 *If $p > 0$, $q > 0$, $p + q = r \geq 1$, $0 \leq a < b < \infty$, $\gamma < r$, $\omega(x)$ is decreasing and positive in (a, b). Let $\nu \geq \overline{\gamma_i} + 1$, $\overline{\gamma_i} \geq 0$, $i = 1, 2$, call $n = \lceil \nu \rceil$, $f \in AC^n([a, b]) : f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$; $D^{\nu}_{*a} f \in L_{\infty}(a, b)$, with $D^{\nu}_{*a} f \geq 0$ over (a, b). Then*

$$\int_a^b \left(D^{\overline{\gamma_1}}_{*a} f(x)\right)^p \left(D^{\overline{\gamma_2}}_{*a} f(x)\right)^q x^{\gamma-(\nu-\overline{\gamma_1})p-(\nu-\overline{\gamma_2})q-1} \omega(x)\, dx \leq$$

$$C^* \int_a^b \left(\left(D^{\nu}_{*a} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{37}$$

where

$$C^* = \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_1}+1-\frac{\gamma}{r}\right)}\right)^p \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_2}+1-\frac{\gamma}{r}\right)}\right)^q. \tag{38}$$

Proof. Use of Theorem 14 and similar to Theorem 16. ∎

Corollary 18 *All as in Theorem 16. Then*

$$\int_a^b \left(D_{*a}^{\overline{\gamma}} f(x)\right)^p \left(\left(D_{*a}^{\nu} f\right)(x)\right)^q x^{\gamma-(\nu-\overline{\gamma})p-1} \omega(x)\, dx \leq$$

$$\overline{C} \int_a^b \left(\left(D_{*a}^{\nu} f\right)(x)\right)^r x^{\gamma-1} \omega(x)\, dx, \tag{39}$$

where

$$\overline{C} = \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma}+1-\frac{\gamma}{r}\right)} \right)^p .$$

Proof. By Theorems 3, 14. ■

We need

Remark 19 *(see [1], p. 26) Let* $\nu \geq \overline{\gamma}+1$, $\overline{\gamma} \geq 0$, $n = [\nu]$, $x_0 \in [a,b]$ *fixed,* $f \in C^{\nu}_{x_0}([a,b]) : f^{(i)}(x_0) = 0$, $i = 0, 1, ..., n-1$. *Then*

$$\left(D^{\overline{\gamma}}_{x_0+} f\right)(x) = \frac{1}{\Gamma(\nu-\overline{\gamma})} \int_{x_0}^{x} (x-t)^{(\nu-\overline{\gamma})-1} \left(D^{\nu}_{x_0+} f\right)(t)\, dt, \qquad (40)$$

which is continuous in x *on* $[x_0, b]$.

Theorem 20 *If* $p > 0$, $q > 0$, $p + q = r \geq 1$, $0 \leq a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* (a,b). *Let* $\nu \geq \overline{\gamma} + 1$, $\overline{\gamma} \geq 0$, $n = [\nu]$, $f \in C_a^{\nu}([a,b]) : f^{(i)}(a) = 0$, $i = 0, 1, ..., n-1$. *Assume that* $D_{a+}^{\nu} f \geq 0$ *on* (a,b). *Then*

$$\int_a^b \left(D_{a+}^{\overline{\gamma}} f(x) \right)^p \left(D_{a+}^{\nu} f(x) \right)^q x^{\gamma - (\nu - \overline{\gamma}) p - 1} \omega(x)\, dx \leq$$

$$C_1 \int_a^b \left(D_{a+}^{\nu} f(x) \right)^r x^{\gamma - 1} \omega(x)\, dx, \tag{41}$$

where

$$C_1 = \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma} + 1 - \frac{\gamma}{r}\right)} \right)^p.$$

Theorem 22 *Let $0 \leq a < b \leq \infty$, γ is real, $\omega(x)$ is decreasing and positive in (a,b), $f(x)$ and $H_k(x,y)$ $(k = 1, ..., n)$ are measurable and non-negative on (a,b), $H_k(x,y)$ is homogeneous of degree -1,*

$$(H_k f)(x) = \int_a^x H_k(x,y) f(y)\, dy, \quad k = 1, ..., n. \tag{48}$$

Let $p_k > 0 : \sum_{k=1}^{n} p_k = r \geq 1$. Then

$$\int_a^b \prod_{k=1}^{n} (H_k f(x))^{p_k} x^{\gamma-1} \omega(x)\, dx \leq \widehat{C} \left(\int_a^b f(x)^r x^{\gamma-1} \omega(x)\, dx \right), \tag{49}$$

where

$$\widehat{C} = \prod_{k=1}^{n} \left(\int_{\frac{a}{b}}^{1} H_k(1,t)\, t^{-\frac{\gamma}{r}} dt \right)^{p_k}. \tag{50}$$

Theorem 23 *Here* $p_k > 0 : \sum_{k=1}^{n} p_k = r \geq 1$. *Let* $0 \leq a < b \leq \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* (a,b), $f(x)$ *is measurable and non-negative on* (a,b), I^{α_k} *is the left Riemann-Liouville operator of fractional integration defined by*

$$\left(I^{\alpha_k} f\right)(x) = \int_a^x \frac{(x-t)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f(t)\, dt, \quad \textit{for } \alpha_k > 0, \tag{51}$$

and

$$I^{\alpha_k} f(x) = f(x), \ \textit{for } \alpha_k = 0;\ k = 1, \ldots, n.$$

Then

$$\int_a^b \prod_{k=1}^{n} \left(\left(I^{\alpha_k} f\right)(x)\right)^{p_k} x^{\gamma - \sum_{k=1}^{n} \alpha_k p_k - 1} \omega(x)\, dx \leq \widetilde{C} \left(\int_a^b f(x)^r x^{\gamma - 1} \omega(x)\, dx\right), \tag{52}$$

where

$$\widetilde{C} = \prod_{k=1}^{n} \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\alpha_k + 1 - \frac{\gamma}{r}\right)}\right)^{p_k}. \tag{53}$$

Next we give general left fractional direct Opial type integral inequalities.

Theorem 24 *Here* $p_j > 0 : \sum_{j=1}^{N} p_j = r \geq 1$. *Let* $0 \leq a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* (a, b). *Let* $\nu \geq \overline{\gamma_j} + 1$, $\overline{\gamma_j} \geq 0$, $j = 2, ..., N$, $n = \lceil \nu \rceil$,

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$f \in AC^n([a,b]) : f^{(k)}(a) = 0,\ k = 0,1,...,n-1,$ *and* $D_{*a}^{\nu} f \in L_\infty(a,b),$ *with* $D_{*a}^{\nu} f \geq 0$ *over* (a,b). *Then*

$$\int_a^b (f(x))^{p_1} \prod_{j=2}^N \left(D_{*a}^{\overline{\gamma_j}} f(x)\right)^{p_j} x^{\gamma - \nu p_1 - \sum_{j=2}^N (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leq$$

$$\left[\left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu+1-\frac{\gamma}{r}\right)}\right)^{p_1} \prod_{j=2}^N \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_j}+1-\frac{\gamma}{r}\right)}\right)^{p_j}\right]$$

$$\left(\int_a^b \left(D_{*a}^{\nu} f(x)\right)^r x^{\gamma-1} \omega(x)\, dx\right). \tag{55}$$

Proof. By Theorem 23, use of Theorem 14 and (33). ∎

Theorem 25 *All as in Theorem 24. Then*

$$\int_a^b \left(D_{*a}^{\nu} f(x)\right)^{p_1} \prod_{j=2}^N \left(D_{*a}^{\overline{\gamma_j}} f(x)\right)^{p_j} x^{\gamma - \sum_{j=2}^N (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leq$$

$$\left(\prod_{j=2}^N \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_j}+1-\frac{\gamma}{r}\right)}\right)^{p_j}\right) \left(\int_a^b \left(D_{*a}^{\nu} f(x)\right)^r x^{\gamma-1} \omega(x)\, dx\right). \tag{56}$$

Corollary 26 *All as in Theorem 24, and* $\overline{\gamma_2} = \overline{\gamma_3} = ... = \overline{\gamma_\lambda}$, $\overline{\gamma_{\lambda+1}} = \overline{\gamma_{\lambda+2}} = ... = \overline{\gamma_\mu}$, *and* $\overline{\gamma_{\mu+1}} = \overline{\gamma_{\mu+2}} = ... = \overline{\gamma_N}$. *Then*

$$\int_a^b \left(D_{*a}^{\nu} f(x)\right)^{p_1} \left(D_{*a}^{\overline{\gamma_\lambda}} f(x)\right)^{\sum_{j=2}^{\lambda} p_j} \left(D_{*a}^{\overline{\gamma_\mu}} f(x)\right)^{\sum_{j=\lambda+1}^{\mu} p_j} \left(D_{*a}^{\overline{\gamma_N}} f(x)\right)^{\sum_{j=\mu+1}^{N} p_j}$$

$$x^{\gamma-(\nu-\overline{\gamma_\lambda})\left(\sum_{j=2}^{\lambda} p_j\right)-(\nu-\overline{\gamma_\mu})\left(\sum_{j=\lambda+1}^{\mu} p_j\right)-(\nu-\overline{\gamma_N})\left(\sum_{j=\mu+1}^{N} p_j\right)-1} \omega(x)\, dx \leq$$

$$\left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_\lambda}+1-\frac{\gamma}{r}\right)}\right)^{\sum_{j=2}^{\lambda} p_j} \left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_\mu}+1-\frac{\gamma}{r}\right)}\right)^{\sum_{j=\lambda+1}^{\mu} p_j}$$

$$\left(\frac{\Gamma\left(1-\frac{\gamma}{r}\right)}{\Gamma\left(\nu-\overline{\gamma_N}+1-\frac{\gamma}{r}\right)}\right)^{\sum_{j=\mu+1}^{N} p_j} \left(\int_a^b \left(D_{*a}^{\nu} f(x)\right)^r x^{\gamma-1} \omega(x)\, dx\right). \qquad (57)$$

Theorem 27 *Here* $p_j > 0 : \sum_{j=1}^{N} p_j = r \geq 1$. *Let* $0 \leq a < b < \infty$, $\gamma < r$, $\omega(x)$ *is decreasing and positive in* (a, b). *Let* $\nu \geq \overline{\gamma_j} + 1$, $\overline{\gamma_j} \geq 0$, $j = 2, ..., N$, $n = [\nu]$, $f \in C_a^{\nu}([a, b]) : f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$. *Assume that* $D_{a+}^{\nu} f \geq 0$ *on* (a, b). *Then*

$$\int_a^b \left(D_{a+}^{\nu} f(x)\right)^{p_1} \prod_{j=2}^{N} \left(D_{a+}^{\overline{\gamma_j}} f(x)\right)^{p_j} x^{\gamma - \sum_{j=2}^{N} (\nu - \overline{\gamma_j}) p_j - 1} \omega(x)\, dx \leq$$

$$\left(\prod_{j=2}^{N} \left(\frac{\Gamma\left(1 - \frac{\gamma}{r}\right)}{\Gamma\left(\nu - \overline{\gamma_j} + 1 - \frac{\gamma}{r}\right)}\right)^{p_j}\right) \left(\int_a^b \left(D_{a+}^{\nu} f(x)\right)^r x^{\gamma - 1} \omega(x)\, dx\right). \qquad (58)$$

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THANK YOU!